

# Zero Dynamics for Networks of Waves <sup>★</sup>

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## Abstract

The zero dynamics of infinite-dimensional systems can be difficult to characterize. The zero dynamics of boundary control systems are particularly problematic. In this paper the zero dynamics of port-Hamiltonian systems are studied. A complete characterization of the zero dynamics for a port-Hamiltonian systems with invertible feedthrough as another port-Hamiltonian system on the same state space is given. It is shown that the zero dynamics for any port-Hamiltonian system with commensurate wave speeds are a well-posed system, and are also a port-Hamiltonian system. Examples include wave equations with uniform wave speed on a network. A constructive procedure for calculation of the zero dynamics that can be used for very large system order is provided.

*Key words:* Port-Hamiltonian system, distributed parameter systems, boundary control, zero dynamics, networks, coupled wave equations.

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## 1 Introduction

The zeros of a system are well-known to be important to controller design; see for instance, the textbooks [6,22,25]. For example, the poles of a system controlled with a constant feedback gain move to the zeros of the open-loop system as the gain increases. Furthermore, regulation is only possible if the zeros of the system do not coincide with the poles of the signal to be tracked.

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Another example is sensitivity reduction - arbitrary reduction of sensitivity is only possible if all the zeros are in the left half-plane. Right half-plane zeros restrict the achievable performance; see for example, [6].

There are a number of definitions of zero dynamics. The most fundamental is that the zero dynamics are the dynamics of the system obtained by choosing the input  $u$  so that the output  $y$  is identically zero. This will only be possible for initial conditions in some subspace of the original state space. This definition applies to nonlinear and linear finite-dimensional systems [10]. For systems with linear ordinary differential equation models, the eigenvalues of the zero dynamics correspond to the invariant zeros, and if the realization is minimal, these are also the zeros of the transfer function. The inverse of the input-output map of a linear finite-dimensional system without right-hand-plane zeros can be approximated by a stable system. Such systems are said to be

minimum-phase, and they are typically easier to control than non-minimum phase systems.

However, many systems are modeled by delay or partial differential equations. This leads to an infinite-dimensional state space, and also an irrational transfer function. The calculation of zero dynamics for finite-dimensional systems, both linear and non-linear, is closely related to the construction of the Byrnes-Isidori form [10]. However, no such extension exists for general infinite-dimensional systems. The notion of minimum-phase as a system with an approximately invertible input-output map can be extended to infinite-dimensional systems. Minimum-phase infinite-dimensional systems are those for which the transfer function is an outer function, see [11]. A detailed study of conditions for second-order systems to be minimum-phase can be found in [11].

As for finite-dimensional systems, the zero dynamics are important for a number of approaches to controller design. Results on adaptive control and on high-gain feedback control of infinite-dimensional systems, see e.g. [18–21,26], require the system to be minimum-phase. Moreover, the sensitivity of an infinite-dimensional minimum-phase system can be reduced to an arbitrarily small level and stabilizing controllers exist that achieve arbitrarily high gain or phase margin [8].

Since the zeros of infinite-dimensional systems are often not accurately calculated by numerical approximations [4,5,9,17] it is useful to obtain an understanding of their behaviour in the original infinite-dimensional context. For infinite-dimensional systems with bounded control and observation, the zero dynamics have been calculated, although they are not always well-posed [23,24,34].

There are few results for zero dynamics for partial differential equations with boundary control and point observation. In [2,3] the zero dynamics are found for a class of parabolic systems defined on an interval with collocated boundary control and observation. This was extended to the heat equation on an arbitrary region with collocated control and observation in [27]. In [15] the invariant zeros for a class of systems with analytic semigroup that includes boundary control/point sensing are defined and analysed.

The zero dynamics of an important class of boundary control systems, port-Hamiltonian systems [13,16,32,31] or systems of linear conservation laws [1], are established in this paper. Such models are derived using Hamilton's Principle. Many situations of interest, in particular waves and vibrations, can be described in a port-Hamiltonian framework. The approach used here follows [13]. Both the control  $u$  and the measurement  $y$  are defined in terms of boundary conditions. In some cases the  $(u, y)$  pairing does not define a passive system, unlike traditional port-Hamiltonian systems [31]

where this pairing is always power flowing across the boundary. A complete characterization of the zero dynamics for port-Hamiltonian systems with commensurate wave speeds is obtained. For any port-Hamiltonian system with invertible feedthrough, the zero dynamics are another port-Hamiltonian system on the same state space. Port-Hamiltonian systems with commensurate wave speeds can be written as a coupling of scalar systems with the same wave speed. The zero dynamics are shown to be well-posed for such systems, and are in fact a new port-Hamiltonian system. This result echoes earlier results for zero dynamics of finite-dimensional Hamiltonian systems [25, chap. 12][29,30]. Preliminary versions of Proposition 7 (for constant coefficients), Theorem 12 (with an outline of the proof) and Example 3 appeared in [12].

A constructive procedure for exact calculation of the zero dynamics of a port-Hamiltonian system based on linear algebra is provided. This algorithm can be used on large networks, and does not use any approximation of the system of partial differential equations.

## 2 Infinite-dimensional Port-Hamiltonian Systems

Consider systems on a one-dimensional (spatial) domain of the form

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}(\zeta)x(\zeta, t)), \quad \zeta \in (0, 1), t \geq 0 \quad (1)$$

$$x(\zeta, 0) = x_0(\zeta), \quad \zeta \in (0, 1) \quad (2)$$

$$0 = W_{B,1} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \quad t \geq 0 \quad (3)$$

$$u(t) = W_{B,2} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \quad t \geq 0 \quad (4)$$

$$y(t) = W_C \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \quad t \geq 0, \quad (5)$$

where  $P_1$  is an Hermitian invertible  $n \times n$ -matrix,  $\mathcal{H}(\zeta)$  is a positive  $n \times n$ -matrix for a.e.  $\zeta \in (0, 1)$  satisfying  $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0, 1; \mathbb{C}^{n \times n})$ , and  $W_B := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$  is a  $n \times 2n$ -matrix of rank  $n$ . Such systems are said to be *port-Hamiltonian*, see [16,32,13], or systems of linear conservation laws [1]. Here,  $x(\cdot, t)$  is the state of the system at time  $t$ ,  $u(t)$  represents the input of the system at time  $t$  and  $y(t)$  the output of the system at time  $t$ .

A different representation of port-Hamiltonian systems, the diagonalized form, will be used. The matrices  $P_1 \mathcal{H}(\zeta)$  possess the same eigenvalues counted according to their multiplicity as the matrix  $\mathcal{H}^{1/2}(\zeta)P_1 \mathcal{H}^{1/2}(\zeta)$ , and as  $\mathcal{H}^{1/2}(\zeta)P_1 \mathcal{H}^{1/2}(\zeta)$  is diagonalizable the matrix

$P_1\mathcal{H}(\zeta)$  is diagonalizable as well. Moreover, by our assumptions, zero is not an eigenvalue of  $P_1\mathcal{H}(\zeta)$  and all eigenvalues are real, that is, there exists an invertible matrix  $S(\zeta)$  such that

$$P_1\mathcal{H}(\zeta) = S^{-1}(\zeta) \underbrace{\text{diag}(p_1(\zeta), \dots, p_k(\zeta), n_1(\zeta), \dots, n_l(\zeta))}_{=: \Delta(\zeta)} S(\zeta).$$

Here  $p_1(\zeta), \dots, p_k(\zeta) > 0$  and  $n_1(\zeta), \dots, n_l(\zeta) < 0$ . In the remainder of this article it is assumed that  $S$  and  $\Delta$  are continuously differentiable on  $(0, 1)$ . Introducing the new state vector

$$z(\zeta, t) = \begin{bmatrix} z_+(\zeta, t) \\ z_-(\zeta, t) \end{bmatrix} = S(\zeta)x(\zeta, t), \quad \zeta \in [0, 1],$$

with  $z_+(\zeta, t) \in \mathbb{C}^k$  and  $z_-(\zeta, t) \in \mathbb{C}^l$ , and writing

$$\Delta(\zeta) = \begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{bmatrix},$$

where  $\Lambda(\zeta)$  is a positive definite  $k \times k$ -matrix and  $\Theta(\zeta)$  is a negative definite  $l \times l$ -matrix, the system (1)–(5) can be equivalently written as

$$\frac{\partial z}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta} (\Delta(\zeta)z(\zeta, t)) + S(\zeta) \frac{S^{-1}(\zeta)}{d\zeta} \Delta(\zeta)z(\zeta, t) \quad (6)$$

$$z(\zeta, 0) = z_0(\zeta), \quad \zeta \in (0, 1) \quad (7)$$

$$\begin{bmatrix} 0 \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} K_{0+} & K_{0-} \\ K_{u+} & K_{u-} \end{bmatrix}}_K \begin{bmatrix} \Lambda(1)z_+(1, t) \\ \Theta(0)z_-(0, t) \end{bmatrix} + \underbrace{\begin{bmatrix} L_{0+} & L_{0-} \\ L_{u+} & L_{u-} \end{bmatrix}}_L \begin{bmatrix} \Lambda(0)z_+(0, t) \\ \Theta(1)z_-(1, t) \end{bmatrix}, \quad (8)$$

$$y(t) = \underbrace{\begin{bmatrix} K_{y+} & K_{y-} \end{bmatrix}}_{K_y} \begin{bmatrix} \Lambda(1)z_+(1, t) \\ \Theta(0)z_-(0, t) \end{bmatrix} + \underbrace{\begin{bmatrix} L_{y+} & L_{y-} \end{bmatrix}}_{L_y} \begin{bmatrix} \Lambda(0)z_+(0, t) \\ \Theta(1)z_-(1, t) \end{bmatrix}, \quad (9)$$

where  $t \geq 0$  and  $\zeta \in (0, 1)$ .

Defining  $A$

$$Af = -(\Delta f)' + S(S^{-1})'\Delta f, \\ D(A) = \left\{ \Delta f \in H^1(0, 1; \mathbb{C}^n) \mid \begin{bmatrix} 0 \\ 0 \end{bmatrix} = K \begin{bmatrix} \Lambda(1)f_+(1) \\ \Theta(0)f_-(0) \end{bmatrix} + L \begin{bmatrix} \Lambda(0)f_+(0) \\ \Theta(1)f_-(1) \end{bmatrix} \right\}$$

the system (6)–(9) with  $u \equiv 0$  can be written in abstract form,

$$\dot{z}(t) = Az(t).$$

The resolvent operator of  $A$  is compact, and thus the spectrum of  $A$  contains only eigenvalues.

Next, consider well-posedness of the control system (6)–(9), or equivalently of system (1)–(5). Well-posedness means that for every initial condition  $z_0 \in L^2(0, 1; \mathbb{C}^n)$  and every input  $u \in L^2_{\text{loc}}(0, \infty; \mathbb{C}^p)$  the unique mild solution  $z$  of the system (6)–(8) exists such that the state and the output (9) lie in the spaces  $X := L^2(0, 1; \mathbb{C}^n)$   $L^2_{\text{loc}}(0, \infty; \mathbb{C}^m)$ , respectively. See [13] for the precise definition and further results on well-posedness of port-Hamiltonian systems. To characterize well-posedness, define the matrices

$$K = \begin{bmatrix} K_0 \\ K_u \end{bmatrix} = \begin{bmatrix} K_{0+} & K_{0-} \\ K_{u+} & K_{u-} \end{bmatrix}, \quad L = \begin{bmatrix} L_0 \\ L_u \end{bmatrix} = \begin{bmatrix} L_{0+} & L_{0-} \\ L_{u+} & L_{u-} \end{bmatrix}.$$

**Theorem 1** [35], [13, Thm. 13.2.2 and 13.3.1]. *The following are equivalent*

- (1) *The system (6)–(9) is well-posed on  $L^2(0, 1; \mathbb{C}^n)$ ;*
- (2) *For every initial condition  $z_0 \in L^2(0, 1; \mathbb{C}^n)$ , the partial differential equation (6)–(8) with  $u = 0$  possesses a unique mild solution on the state space  $L^2(0, 1; \mathbb{C}^n)$ . Furthermore, this solution depends continuously on the initial condition;*
- (3) *The matrix  $K$  is invertible.*

**Example 2** *As an illustration, consider a small network of three tubes or ducts  $i = 1 \dots 3$  with flux density  $p_i$  and charge density  $q_i$ . Alternatively, these equations model a network of transmission lines; in this case  $p_i$  is flux and  $q_i$  is current. For simplicity of exposition, set physical parameters to 1.*

$$\frac{\partial p_i}{\partial t} = -\frac{\partial q_i}{\partial \xi}, \\ \frac{\partial q_i}{\partial t} = -\frac{\partial p_i}{\partial \xi}, \quad i = 1 \dots 3. \quad (10)$$

*The end of tube 1 is connected to the start of tubes 2 and 3, the end of tube 2 is connected to the start of tube 1, and the end of tube 3 is open. With control of flow at the start of tube 1 and observation of flow at the end of tube*

3, this yields the boundary conditions

$$\begin{aligned}
0 &= p_1(0, t) - p_2(1, t) \\
0 &= p_1(1, t) - p_2(0, t) \\
0 &= -p_2(0, t) + p_3(0, t) \\
0 &= q_1(1, t) + q_2(0, t) + q_3(0, t) \\
0 &= p_3(1, t), \\
u(t) &= -q_1(0, t) - q_2(1, t), \\
y(t) &= q_3(1, t).
\end{aligned} \tag{11}$$

With state  $x = [p_1 \ p_2 \ p_3 \ q_1 \ q_2 \ q_3]^T$ , and defining

$$P_1 = \begin{bmatrix} 0_{3 \times 3} & -I_3 \\ -I_3 & 0_{3 \times 3} \end{bmatrix}, \quad \mathcal{H} = I_6,$$

this system of PDEs (10) with the boundary conditions (11) is in the form (1)-(9). (If the physical constants were not 1, the only change would be that the matrix  $\mathcal{H}$  would have the parameters on the diagonal.)

To obtain a diagonal form (6) of the PDE, define the new state variables

$$z_{+i} = p_i - q_i, \quad z_{-i} = p_i + q_i, \quad i = 1 \dots 3$$

so that

$$\begin{bmatrix} z_+ \\ z_- \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}}_S \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

The PDE now has the form (6) with  $\Lambda = I_3$ ,  $\Theta = -I_3$ . The boundary conditions (11) are now written

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}}_K \begin{bmatrix} z_{+1}(1, t) \\ z_{+2}(1, t) \\ z_{+3}(1, t) \\ -z_{-1}(0, t) \\ -z_{-2}(0, t) \\ -z_{-3}(0, t) \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}}_L \begin{bmatrix} z_{+1}(0, t) \\ z_{+2}(0, t) \\ z_{+3}(0, t) \\ -z_{-1}(1, t) \\ -z_{-2}(1, t) \\ -z_{-3}(1, t) \end{bmatrix}.$$

The matrix  $K$  is invertible so the control system is well-posed.

**Example 3** Consider two coupled wave equations on  $(0, 1)$

$$\frac{\partial^2 w_1}{\partial t^2} = \frac{\partial^2 w_1}{\partial \zeta^2} \tag{12}$$

$$\frac{\partial^2 w_2}{\partial t^2} = 4 \frac{\partial^2 w_2}{\partial \zeta^2} \tag{13}$$

$$\frac{\partial w_1}{\partial t}(1, t) = 0 \tag{14}$$

$$\frac{\partial w_2}{\partial t}(1, t) = 0 \tag{15}$$

$$\frac{\partial w_1}{\partial t}(0, t) - \frac{\partial w_2}{\partial t}(0, t) = 0 \tag{16}$$

$$a \frac{\partial w_1}{\partial \zeta}(0, t) + b \frac{\partial w_2}{\partial \zeta}(0, t) = u(t) \tag{17}$$

with  $|a| + |b| > 0$ . In order to write this system as a port-Hamiltonian system, define

$$x = \left[ \frac{\partial w_1}{\partial t} \quad \frac{\partial w_1}{\partial \zeta} \quad \frac{\partial w_2}{\partial t} \quad \frac{\partial w_2}{\partial \zeta} \right]^*.$$

Then the system can be written

$$\frac{\partial x}{\partial t}(\zeta, t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x(\zeta, t) \right)$$

with boundary conditions

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}b \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u(t) \end{bmatrix}.$$

Alternatively, diagonalize  $P_1\mathcal{H}$  and define

$$\begin{aligned} z_{+1}(\zeta, t) &= w_{1t}(\zeta, t) + w_{1\zeta}(\zeta, t) \\ z_{+2}(\zeta, t) &= w_{2t}(\zeta, t) + 2w_{2\zeta}(\zeta, t) \\ z_{-1}(\zeta, t) &= w_{1t}(\zeta, t) - w_{1\zeta}(\zeta, t) \\ z_{-2}(\zeta, t) &= w_{2t}(\zeta, t) - 2w_{2\zeta}(\zeta, t). \end{aligned}$$

The partial differential equation becomes

$$\frac{\partial}{\partial t} \begin{bmatrix} z_+(\zeta, t) \\ z_-(\zeta, t) \end{bmatrix} = \frac{\partial}{\partial \zeta} \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} z_+(\zeta, t) \\ z_-(\zeta, t) \end{bmatrix} \right),$$

with boundary conditions

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 0 \\ u(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{a}{2} & \frac{b}{8} \end{bmatrix} \begin{bmatrix} z_{+1}(1, t) \\ 2z_{+2}(1, t) \\ -z_{-1}(0, t) \\ -2z_{-2}(0, t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & -\frac{1}{2} & 0 & 0 \\ \frac{a}{2} & \frac{b}{8} & 0 & 0 \end{bmatrix} \begin{bmatrix} z_{+1}(0, t) \\ 2z_{+2}(0, t) \\ -z_{-1}(1, t) \\ -2z_{-2}(1, t) \end{bmatrix}. \end{aligned}$$

By Theorem 1 this is a well-posed system if and only if  $2a \neq -b$ .

In the port-Hamiltonian formulation, the importance of connections between subsystems and the overall boundary conditions to well-posedness of the control system is clear. Well-posedness of a port-Hamiltonian system can be established by a simple check of the rank of the matrix  $K$  in the definition of the boundary conditions.

For the remainder of this paper it is assumed that  $K$  is invertible so that the control system is well-posed.

For port-Hamiltonian systems, well-posedness implies that the system (6)–(9) is also regular, that is, the transfer function  $G(s)$  possesses a limit over the real line, see [35] or [13, Section 13.3]. Writing

$$K_y K^{-1} = \begin{bmatrix} * & D \end{bmatrix} \quad (18)$$

with  $D \in \mathbb{C}^{m \times p}$  the feedthrough operator, this limit of  $G(s)$  over the real axis is  $D$ , see [13, Theorem 13.3.1].

### 3 Zero dynamics for port-Hamiltonian systems

Now we define zero dynamics for port-Hamiltonian systems.

**Definition 4** Consider the system (1)–(5) on the state space  $X = L^2(0, 1; \mathbb{C}^n)$ . The zero dynamics of (1)–(5) are the pairs  $(z_0, u) \in X \times L^2_{\text{loc}}(0, \infty; \mathbb{C}^p)$  for which the mild solution of (1)–(5) satisfies  $y = 0$ . The largest output nulling subspace is

$$V^* = \{z_0 \in X \mid \text{there is a function } u \in L^2_{\text{loc}}(0, \infty; \mathbb{C}^p) : \text{the mild solution of (1)–(5) satisfies } y = 0\}.$$

Thus,  $V^*$  is the space of initial conditions for which there exists a control  $u$  that “zeroes” the output. As system (1)–(5) is equivalent to system (6)–(9) we can equivalently study the largest nulling subspace of (6)–(9). Setting  $y = 0$  in (9) reveals that the zero dynamics are described by

$$\begin{aligned} \frac{\partial z}{\partial t}(\zeta, t) &= \frac{\partial}{\partial \zeta} (\Delta(\zeta)z(\zeta, t)) + S(\zeta) \frac{S^{-1}(\zeta)}{d\zeta} \Delta(\zeta)z(\zeta, t) \\ z(\zeta, 0) &= z_0(\zeta), \quad \zeta \in (0, 1) \end{aligned} \quad (19)$$

$$\begin{aligned} 0 &= \begin{bmatrix} K_0 \\ K_y \end{bmatrix} \begin{bmatrix} \Lambda(1)z_+(1, t) \\ \Theta(0)z_-(0, t) \end{bmatrix} \\ &+ \begin{bmatrix} L_0 \\ L_y \end{bmatrix} \begin{bmatrix} \Lambda(0)z_+(0, t) \\ \Theta(1)z_-(1, t) \end{bmatrix}, \end{aligned} \quad (21)$$

$$u(t) = K_u \begin{bmatrix} \Lambda(1)z_+(1, t) \\ \Theta(0)z_-(0, t) \end{bmatrix} + L_u \begin{bmatrix} \Lambda(0)z_+(0, t) \\ \Theta(1)z_-(1, t) \end{bmatrix}, \quad (22)$$

where  $t \geq 0$  and  $\zeta \in (0, 1)$ . Note that system (19)–(22) is still in the format of a port-Hamiltonian system, but even regarding (22) as the (new) output, it needs not to be a well-posed port-Hamiltonian system since the new “ $K$ -matrix”,  $\begin{bmatrix} K_0 \\ K_y \end{bmatrix}$  can have rank less than  $n$ . The zero dynamics are a well-posed dynamical system if the system (19)–(22) with state-space  $V^*$ , no input and output  $u$  is well-posed.

The eigenvalues of the zero dynamics of the system are closely related to the invariant and transmission zeros of the system. For simplicity only the single-input single-output case is considered ( $p = m = 1$ ).

**Definition 5** [27, 4] A complex number  $\lambda \in \mathbb{C}$  is an invariant zero of the system (6)–(9) on the state space  $X = L^2(0, 1; \mathbb{C}^n)$ , if there exist  $z \in H^1(0, 1; \mathbb{C}^n)$  and  $u \in \mathbb{C}$  such that

$$\begin{aligned} \lambda z(\zeta) &= \frac{\partial}{\partial \zeta} (\Delta(\zeta)z(\zeta)) + S(\zeta) \frac{S^{-1}(\zeta)}{d\zeta} \Delta(\zeta)z(\zeta), \\ 0 &= \begin{bmatrix} K_0 \\ K_y \end{bmatrix} \begin{bmatrix} \Lambda(1)z_+(1) \\ \Theta(0)z_-(0) \end{bmatrix} + \begin{bmatrix} L_0 \\ L_y \end{bmatrix} \begin{bmatrix} \Lambda(0)z_+(0) \\ \Theta(1)z_-(1) \end{bmatrix}, \\ u &= K_u \begin{bmatrix} \Lambda(1)z_+(1) \\ \Theta(0)z_-(0, t) \end{bmatrix} + L_u \begin{bmatrix} \Lambda(0)z_+(0, ) \\ \Theta(1)z_-(1, t) \end{bmatrix}, \end{aligned}$$

**Definition 6** A complex number  $s \in \mathbb{C}$  is a transmission zero of the system (6)–(9) if the transfer function satisfies  $G(s) = 0$ .

If  $\lambda \in \rho(A)$ , where  $\rho(A)$  denotes the resolvent set of  $A$ , then  $\lambda$  is an invariant zero if and only if  $\lambda$  is a transmission zero [13, Theorem 12.2.1]. Moreover, if the zero dynamics is well-posed, then the spectrum of the corresponding generator equals the set of invariant zeros of the system (6)–(9).

If the feedthrough operator of the original system is invertible, then the zero dynamics system is well-posed on the entire state space, and is also a port-Hamiltonian system.

**Proposition 7** Assume that the system has the same number of inputs as outputs. Then the zero dynamics are well-posed on the entire state space if and only if the feedthrough operator  $D$  of the original system is invertible.

**Proof:** This was proven in [12] in the case of a constant coefficient matrix  $\mathcal{H}$ . The proof presented here is more complete, and includes the generalization to variable coefficients. The feedthrough operator  $D$  of the original system is given by  $[*D] = K_y K^{-1}$  (see (18)). It will first be shown that invertibility of  $D$  is equivalent to invertibility of the “ $K$ -matrix” of equation (21):

$$\tilde{K} := \begin{bmatrix} K_0 \\ K_y \end{bmatrix}.$$

If  $D$  is singular, then there is  $u \neq 0$  in the kernel of  $D$ , and

$$K_y K^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix} = 0.$$

Combining this with the fact that  $K_0 K^{-1} = \begin{bmatrix} I & 0 \end{bmatrix}$ ,

$$\tilde{K} K^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} K_0 \\ K_y \end{bmatrix} K^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix} = 0.$$

Thus  $\tilde{K}$  is singular. Assume next that  $\tilde{K}$  is singular. Thus there exists non-zero  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  such that

$$\begin{bmatrix} K_0 \\ K_y \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (23)$$

This implies that

$$K \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K_0 \\ K_u \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ z \end{bmatrix},$$

where  $z \neq 0$ , since  $K$  is invertible. Thus

$$Dz = K_y K^{-1} \begin{bmatrix} 0 \\ z \end{bmatrix} = K_y \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

and thus  $D$  is not invertible.

Assume now that  $D$  is invertible, then by the above equivalence with the invertibility of  $\tilde{K}$  and Theorem 1 for every initial condition there exists a solution of (19)–(21). Since  $z$  is now determined,  $u$  is determined by (22). Now it is straightforward to see that the functions  $z$  and  $u$  satisfy (6)–(8) and the corresponding output  $y$  satisfies  $y = 0$ .

If for every  $z_0 \in L^2(0, 1; \mathbb{C}^n)$  there exists a solution of (19)–(22), then the functions  $z$  and  $u$  satisfy (6)–(8). Since  $K$  is invertible, the solution depends continuously on the initial condition. By construction,  $z$  is the solution of the homogeneous equation (19)–(21), and Theorem 1 implies the invertibility of  $\tilde{K}$ .  $\square$

The energy associated with a port-Hamiltonian system is

$$E(t) = \int_0^1 x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta. \quad (24)$$

The following proposition shows that for passive port-Hamiltonian systems (1)–(5) the zero dynamics are well-posed on the entire state space.

**Corollary 8** Assume that the system (1)–(5) has the same number of inputs as outputs and that along classical solutions  $\dot{E}(t) \leq u(t)^T y(t)$ , then the zero dynamics are well-posed on the entire state space and the feedthrough operator is invertible.

**Proof:** Consider the system (1)–(5) in which we set  $y(t) \equiv 0$ . Together with (3) this imposes  $n$  boundary conditions. Furthermore, we know from the power balance,

$$\dot{E}(t) \leq u(t)^T y(t) \quad (25)$$

that  $\dot{E} \leq 0$ . From [13, Theorem 7.1.5, Lemma 7.2.1, and Theorem 7.2.4] we conclude that this homogeneous PDE generates a contraction semigroup on the whole state space. Hence by Proposition 7 we find that the feedthrough is invertible.  $\square$

**Example 9** (Example 3 cont.) As output for the system select

$$y(t) = \frac{\partial w_1}{\partial t}(0, t). \quad (26)$$

The boundary conditions for the zero dynamics are (14)–(16) plus

$$\frac{\partial w_1}{\partial t}(0, t) = 0.$$

In the diagonal representation this is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{\tilde{K}} \begin{bmatrix} z_{+1}(0, t) \\ 2z_{+2}(0, t) \\ -z_{+1}(1, t) \\ -2z_{+2}(1, t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_{+1}(1, t) \\ 2z_{+2}(1, t) \\ -z_{+1}(0, t) \\ -2z_{+2}(0, t) \end{bmatrix}.$$

The matrix  $\tilde{K}$  has full rank and so the zero dynamics are defined on the original state space.

The transfer function for this system can be found by solving

$$\begin{aligned} s^2 \hat{w}_1(\zeta, s) &= \frac{\partial^2 \hat{w}_1}{\partial \zeta^2}(\zeta, s) \\ s^2 \hat{w}_2(\zeta, s) &= 4 \frac{\partial^2 \hat{w}_2}{\partial \zeta^2}(\zeta, s) \\ \hat{w}_1(1, s) &= 0 \\ \hat{w}_2(1, s) &= 0 \\ \hat{w}_1(0, s) - \hat{w}_2(0, s) &= 0 \\ a \frac{\partial \hat{w}_1}{\partial \zeta}(0, s) + b \frac{\partial \hat{w}_2}{\partial \zeta}(0, s) &= \hat{u}(s), \end{aligned}$$

with

$$\hat{y}(s) = s \hat{w}_1(0, s),$$

where the  $\hat{\cdot}$  denotes the Laplace transforms. The solution of the differential equation with the first two boundary conditions is

$$\hat{w}_1 = \alpha \sinh(s(\zeta - 1)), \quad \hat{w}_2 = \sinh(s/2(\zeta - 1)).$$

Using the other boundary conditions leads to the transfer function

$$G(s) = \frac{-2 \sinh(s/2) \sinh(s)}{b \sinh(s) \cosh(s/2) + 2a \cosh(s) \sinh(s/2)}.$$

Hence the feedthrough is  $\frac{-2}{b+2a}$ . The system is well-posed if and only if  $b + 2a \neq 0$  and in this case the inverse system is also well-posed. The zeros of  $G$  are all imaginary, and so the system is minimum phase [11]. Alternatively, calculation of the eigenvalues with  $\frac{\partial w_1}{\partial t}(0, t) = 0$  leads to the same conclusion.

The energy of this model is

$$E(t) = \frac{1}{2} \sum_{i=1}^4 \int_0^1 z_i(t)^2 dt.$$

Differentiating with respect to time, substitution of the differential equation, and integration by parts in the spatial variable yields

$$\dot{E}(t) = \left. \frac{\partial w_1}{\partial t}(\zeta, t) \frac{\partial w_1}{\partial \zeta}(\zeta, t) \right|_{\zeta=0}^1 + 4 \left. \frac{\partial w_2}{\partial t}(\zeta, t) \frac{\partial w_2}{\partial \zeta}(\zeta, t) \right|_{\zeta=0}^1.$$

Applying the boundary conditions (14)–(16) and (26) leads to

$$\dot{E}(t) = y(t) \left( -\frac{\partial w_1}{\partial \zeta}(0, t) - 4 \frac{\partial w_2}{\partial \zeta}(0, t) \right).$$

Thus, if  $a = -1$ ,  $b = -4$  in the boundary condition (17), then the control system satisfies  $\dot{E}(t) \leq u(t)^T y(t)$ .

It is very common though for the feedthrough to be non-invertible. This more challenging situation is considered in the next two sections.

#### 4 Commensurate constant wave speed

In this section, the following class of port-Hamiltonian systems is considered:

$$\frac{\partial}{\partial t} z(\zeta, t) = -\lambda_0 \frac{\partial}{\partial \zeta} z(\zeta, t), \quad (27)$$

$$z(\zeta, 0) = z_0(\zeta), \quad \zeta \in (0, 1) \quad (28)$$

$$\begin{bmatrix} 0 \\ u(t) \end{bmatrix} = -\lambda_0 \underbrace{\begin{bmatrix} K_0 \\ K_u \end{bmatrix}}_K z(0, t) - \lambda_0 \underbrace{\begin{bmatrix} L_0 \\ L_u \end{bmatrix}}_L z(1, t), \quad (29)$$

$$y(t) = -\lambda_0 K_y z(0, t) - \lambda_0 L_y z(1, t), \quad (30)$$

where  $\lambda_0$  is a scalar. If  $\mathcal{H}$  is constant, then (6)–(9) is of the form (27)–(30) with  $-\lambda_0$  replaced by a diagonal (constant) and invertible matrix  $\Delta$ . On the diagonal of the matrix  $\Delta$  are the possible different wave speeds of the system. If the ratio of any pair of diagonal entries of  $\Delta$  is rational, then the system (6)–(9) can be equivalently written in form (27)–(30) by dividing the intervals to adjust the propagation periods, that is, we divide the intervals in a series of intervals. This is a standard procedure and is illustrated in Example 10. The following simple reflection makes positive wave speeds into negative wave speed, while keeping the same absolute speed

$$\tilde{z}_k(\zeta, t) := z_k(1 - \zeta, t).$$

It is good to remark that the system (27)–(30) will in general have larger matrices than the original system

(6)–(9). However, for simplicity, still denote the size by  $n$ .

**Example 10** Consider the following system with commensurable wave speeds

$$\begin{aligned}\frac{\partial z_1}{\partial t} &= -\frac{\partial z_1}{\partial \zeta}, \\ \frac{\partial z_2}{\partial t} &= -\frac{1}{2} \frac{\partial z_2}{\partial \zeta},\end{aligned}$$

with  $\zeta \in [0, 1]$ ,  $t \geq 0$  and

$$\begin{aligned}\begin{bmatrix} 0 \\ u(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z(0, t) + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z(1, t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} z(0, t) + \begin{bmatrix} 0 & 0 \end{bmatrix} z(1, t).\end{aligned}$$

This system has not a uniform wave speed, but can be written equivalently as a system with one wave speed. To reach this goal, split the second equation in two and obtain the following equivalent system

$$\begin{aligned}\frac{\partial z_1}{\partial t} &= -\frac{\partial z_1}{\partial \zeta}, \\ \frac{\partial z_{2a}}{\partial t} &= -\frac{\partial z_{2a}}{\partial \zeta}, \\ \frac{\partial z_{2b}}{\partial t} &= -\frac{\partial z_{2b}}{\partial \zeta},\end{aligned}$$

with  $\zeta \in [0, 1]$ ,  $t \geq 0$ ,  $z_{2b}(\zeta, t) = z_2(\zeta/2, t)$  and  $z_{2a}(\zeta, t) = z_2((1 + \zeta)/2, t)$  and

$$\begin{aligned}\begin{bmatrix} 0 \\ 0 \\ u(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_K z(0, t) + \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_L z(1, t) \\ y(t) &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{K_y} z(0, t) + \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_{L_y} z(1, t).\end{aligned}$$

This transformation also works if  $\mathcal{H}(\zeta)$  is diagonal a.e.  $\zeta \in (0, 1)$  and the ratio of the numbers  $\tau_i := \int_0^1 \frac{1}{\mathcal{H}(\zeta)_{ii}} d\zeta$  are pairwise rational [28].

It is now shown that the zero dynamics can be well-posed through the input and output equations.

It is well-known that the solution of (27) is given by  $z(\zeta, t) = f(1 - \zeta + \lambda_0 t)$  for  $t \geq 0$  and some function  $f$ .

Using this fact, we write the system (27)–(30) equivalently as

$$f(t) = z_0(1 - t), \quad t \in [0, 1], \quad (31)$$

$$\begin{bmatrix} 0 \\ u(t) \end{bmatrix} = -\lambda_0 K f(1 + \lambda_0 t) - \lambda_0 L f(\lambda_0 t), \quad t \geq 0, \quad (32)$$

$$y(t) = -\lambda_0 K_y f(1 + \lambda_0 t) - \lambda_0 L_y f(\lambda_0 t), \quad t \geq 0. \quad (33)$$

Since the system is well-posed, the matrix  $K$  is invertible (Theorem 1). Thus, equivalently,

$$f(t) = z_0(1 - t), \quad (34)$$

$$f(1 + \lambda_0 t) = -K^{-1} L f(\lambda_0 t) - \lambda_0^{-1} K^{-1} \begin{bmatrix} 0 \\ u(t) \end{bmatrix}, \quad (35)$$

$$y(t) = (\lambda_0 K_y K^{-1} L - \lambda_0 L_y) f(\lambda_0 t) + K_y K^{-1} \begin{bmatrix} 0 \\ u(t) \end{bmatrix}. \quad (36)$$

Defining

$$\begin{aligned}A_d &= -K^{-1} L, \quad B_d = -\lambda_0^{-1} K^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}, \\ C_d &= -\lambda_0 K_y A_d - \lambda_0 L_y, \quad D_d = -\lambda_0 K_y B_d,\end{aligned} \quad (37)$$

equation (35)–(36) can be written as

$$\begin{aligned}f(1 + \lambda_0 t) &= A_d f(\lambda_0 t) + B_d u(t), \\ y(t) &= C_d f(\lambda_0 t) + D_d u(t).\end{aligned}$$

Define for  $j \in \mathbb{N}$  the functions  $z_d(j) \in L^2(0, 1; \mathbb{C}^n)$ ,  $u_d(j) \in L^2(0, 1; \mathbb{C}^p)$ , and  $y_d(j) \in L^2(0, 1; \mathbb{C}^m)$  by  $z_d(0)(\zeta) := z_0(1 - \zeta)$ ,  $z_d(j)(\zeta) = f(j + \zeta)$  for  $j \geq 1$  and

$$u_d(j)(\zeta) = u\left(\frac{j + \zeta}{\lambda_0}\right), \quad y_d(j)(\zeta) = y\left(\frac{j + \zeta}{\lambda_0}\right), \quad j \in \mathbb{N}.$$

Thus equations (27)–(30) can be equivalently rewritten as

$$z_d(j + 1)(\zeta) = A_d z_d(j)(\zeta) + B_d u_d(j)(\zeta) \quad (38)$$

$$(z_d(0))(\zeta) = z_0(1 - \zeta) \quad (39)$$

$$y_d(j)(\zeta) = C_d z_d(j)(\zeta) + D_d u_d(j)(\zeta) \quad (40)$$

This representation is very useful, not only for the zero dynamic, but also for other properties like stability.

**Theorem 11** [14, Corollary 3.7] *The system (27)–(30) is exponentially stable if and only if the spectral radius of  $A_d$  satisfies  $r(A_d) < 1$  or equivalently if  $\sigma_{max}(A_d) < 1$ .*

Further sufficient conditions for exponential stability can be found in [1, 7, 13]. In particular, exponential stability is implied by the condition  $KK^* - LL^* > 0$ , [1,



Thm. 3.2] and [13, Lemma 9.1.4]. However, the condition  $KK^* - LL^* > 0$  is in general not necessary, see [13, Example 9.2.1].

It will now be shown that the zero dynamics of systems of the form (27)–(30) are again a port-Hamiltonian system, but with possibly a smaller state, that is, instead of  $L^2(0, 1; \mathbb{C}^n)$  the state space will be  $L^2(0, 1; \mathbb{C}^k)$  with  $0 \leq k \leq n$ . First, it is shown that the problem of determining the zero dynamics for (27)–(30) can be transformed into determining the zero dynamics for the finite-dimensional discrete-time system described by the matrices  $A_d, B_d, C_d$  and  $D_d$ .

**Theorem 12** *Let  $z_0 \in L^2(0, 1; \mathbb{C}^n)$ . Then the following are equivalent.*

- (1) *There exists an input  $u \in L^2_{loc}(0, \infty; \mathbb{C}^p)$  such that the output  $y$  of (27)–(30) with initial condition  $z(\cdot, 0) = z_0$  is identically zero;*
- (2)  *$z_0 \in L^2(0, 1; V_d^*)$ , where  $V_d^* \subseteq \mathbb{C}^n$  is the largest output nulling subspace of the discrete-time system  $\Sigma(A_d, B_d, C_d, D_d)$  with state space  $\mathbb{C}^n$  given by*

$$\begin{aligned} w(j+1) &= A_d w(j) + B_d u(j), & (41) \\ y(j) &= C_d w(j) + D_d u(j). \end{aligned}$$

*In particular, the largest output nulling subspace  $V^*$  of (27)–(30) is given by  $V^* = L^2(0, 1; V_d^*)$ .*

**PROOF.** The system (27)–(30) can be equivalently written as as (38)–(40). In these equations the input, state and output were still spatially dependent. However, the time axis has been split as  $[0, \infty) = \cup_{j \in \mathbb{N}} [j, j+1]$ . Thus condition 1. is equivalent to

- 1' There exists a sequence  $(u_d(j))_{j \in \mathbb{N}} \subseteq L^2(0, 1; \mathbb{C}^m)$  and a set  $\Omega \subset (0, 1)$  whose complement has measure zero such that for every  $\zeta \in \Omega$ ,

$$\begin{aligned} z_d(j+1)(\zeta) &= A_d z_d(j)(\zeta) + B_d u_d(j)(\zeta), & (42) \\ (z_d(0))(\zeta) &= z_0(1 - \zeta). \\ 0 &= C_d z_d(j)(\zeta) + D_d u_d(j)(\zeta), \end{aligned}$$

Clearly, condition 1' implies that  $z_0(\zeta) \in V_d^*$  a.e., where  $V_d^*$  denotes the largest output nulling subspace of the finite-dimensional system (41). Since trivially  $z_0 \in L^2(0, 1; V_d^*)$ , condition 2 follows.

The system  $(A_d, B_d, C_d, D_d)$  is a finite-dimensional discrete-time system. Let  $V_d^* \subseteq \mathbb{C}^n$  indicate the largest output nulling subspace. Then there exists a matrix  $\mathcal{K}$  such that the output-nulling control is given by  $u_d(j) = \mathcal{K} z_d(j)$ , see [33]. Referring now to (42), if  $z_0 \in L^2(0, 1; V_d^*)$  then the output-nulling control

$(u_d(j))_{j \in \mathbb{N}}$  for system (42) satisfies  $u_d(j) \in L^2(0, 1; \mathbb{C}^p)$ . Condition 2 thus implies condition 1'.

For many partial differential equation systems, the largest output nulling subspace is not closed and the zero dynamics are not well-posed, [24,34]. However, for systems of the form (27)–(30) the largest output nulling subspace is closed, and the zero dynamics are well-posed. The following theorem provides a characterization of the largest output nulling subspace of  $\Sigma(A_d, B_d, C_d, D_d)$  and hence of the zero dynamics for the original partial differential equation. The proof can be found in [12].

**Theorem 13** *Define  $E = -\begin{bmatrix} K_0 \\ K_y \end{bmatrix}$ ,  $F = \begin{bmatrix} L_0 \\ L_y \end{bmatrix}$ . The initial condition  $v_0$  lies in the largest output nulling subspace  $V_d$  of  $\Sigma(A_d, B_d, C_d, D_d)$  if and only if there exists a sequence  $\{v_k\}_{k \geq 1} \subset \mathbb{C}^n$  such that*

$$E v_{k+1} = F v_k, \quad k \geq 0. \quad (43)$$

*Furthermore, the largest output nulling subspace  $V_d^*$  satisfies  $V_d^* = \cap_{k \geq 0} V^k$ , where  $V^0 = \mathbb{C}^n$ ,  $V^{k+1} = V^k \cap F^{-1} E V^k$ .*

Thus in addition to the well-known  $V^*$ -algorithm for finite-dimensional systems, see [1, p. 91], Theorem 13 provides an alternative algorithm. It remains to show that the system restricted to the output nulling subspace is again port-Hamiltonian.

**Theorem 14** *For the port-Hamiltonian system (27)–(30) the zero dynamics is well-posed, and the dynamics restricted to the largest output nulling subspace is a port-Hamiltonian system without inputs.*

**PROOF.** By Theorem 12, the largest output nulling subspace  $V^*$  of (27)–(30) is given by  $V^* = L^2(0, 1; V_d^*)$ . If  $V_d^* = \{0\}$ , then there is nothing to prove, and so assume that  $V_d^*$  is a non-trivial subspace of  $\mathbb{C}^n$ . It is well-known that there exists a matrix  $F_d$  such that [33]

$$(A_d + B_d F_d) V_d^* \subset V_d^*.$$

Therefore, using Theorem 12 and (38)–(40), it is easy to see that for the choice  $u_d(j)(\zeta) := F_d z_d(j)(\zeta)$  the output  $y_d(j)(\zeta)$  is zero provided the initial condition  $z_0$  lies in  $L^2(0, 1; V_d^*)$ . Using the definition of the  $A_d, B_d, C_d, D_d, u_d$  and  $z_d$ , it follows that for  $z_0 \in L^2(0, 1; V_d^*)$

there exists a function  $f$  satisfying

$$f(t) = z_0(1-t), \quad t \in [0, 1], \quad (44)$$

$$f(1 + \lambda_0 t) = -K^{-1}L f(\lambda_0 t)$$

$$-\lambda_0^{-1}K^{-1} \begin{bmatrix} 0 \\ F_d f(\lambda_0 t) \end{bmatrix}, \quad t \geq 0, \quad (45)$$

$$0 = (\lambda_0 K_y K^{-1}L - \lambda_0 L_y) f(\lambda_0 t)$$

$$+ K_y K^{-1} \begin{bmatrix} 0 \\ F_d f(\lambda_0 t) \end{bmatrix}, \quad t \geq 0. \quad (46)$$

Equations (45)-(46) can be equivalent written as

$$0 = -\lambda_0 K_{\text{ext}} f(1 + \lambda_0 t) - \lambda_0 L_{\text{ext}} f(\lambda_0 t), \quad (47)$$

with

$$K_{\text{ext}} = \begin{bmatrix} K \\ K_y \end{bmatrix} \quad (48)$$

and some matrix  $L_{\text{ext}}$ . Since  $z_0 \in L^2(0, 1; V_d^*)$ , for all  $t$  and almost all  $\zeta \in [0, 1]$ ,  $f(\zeta + \lambda_0 t) \in V_d^*$ . Thus,  $K_{\text{ext}}$  and  $L_{\text{ext}}$  can be restricted to  $V_d^*$  and equation (47) can equivalently be written with matrices  $K_{\text{ext}}|_{V_d^*}$  and  $L_{\text{ext}}|_{V_d^*}$ . Since  $K$  is part of the the matrix  $K_{\text{ext}}$ , the matrix  $K_{\text{ext}}|_{V_d^*}$  has rank equal to the dimension of  $V_d^*$ . Let  $P$  be the projection onto the range of  $K_{\text{ext}}|_{V_d^*}$ . This leads to

$$0 = -\lambda_0 P K_{\text{ext}}|_{V_d^*} f(1 + \lambda_0 t) - \lambda_0 P L_{\text{ext}}|_{V_d^*} f(\lambda_0 t). \quad (49)$$

Define  $K_{V_d^*} := P K_{\text{ext}}|_{V_d^*}$  and  $L_{V_d^*} := P L_{\text{ext}}|_{V_d^*}$ . The above equation is the solution of the partial differential equation

$$\frac{\partial}{\partial t} z(\zeta, t) = -\lambda_0 \frac{\partial}{\partial \zeta} z(\zeta, t), \quad (50)$$

$$0 = -\lambda_0 K_{V_d^*} z(0, t) - \lambda_0 L_{V_d^*} z(1, t) \quad (51)$$

on the state space  $L^2(0, 1; V_d^*)$ . Since  $K_{V_d^*}$  is invertible, Theorem 1 implies that this system is a well-posed port-Hamiltonian system.

In the following section a second method to obtain the zero dynamics for systems with one dimensional input and output spaces is developed. The advantage of this method is that a transformation to a discrete system is not needed and non-constant wave speed is possible.

## 5 Zero dynamics of port-Hamiltonian systems with commensurate wave speed

In this section the zero dynamics of systems of the form (27)-(30) with one dimensional input and output spaces

and (possibly) non-constant wave speed are defined. The class of systems considered has the form

$$\frac{\partial}{\partial t} z(\zeta, t) = -\frac{\partial}{\partial \zeta} (\lambda_0(\zeta) z(\zeta, t)) \quad (52)$$

$$0 = K_0(\lambda_0(0)z(0, t)) + L_0(\lambda_0(1)z(1, t)) \quad (53)$$

$$u(t) = K_u(\lambda_0(0)z(0, t)) + L_u(\lambda_0(1)z(1, t)) \quad (54)$$

$$y(t) = K_y z(0, t) + L_y z(1, t). \quad (55)$$

Here  $K_0, L_0 \in \mathbb{C}^{(n-1) \times n}$ ,  $K_u, K_y, L_u, L_y \in \mathbb{C}^{1 \times n}$  and  $\lambda_0 \in L^\infty(0, 1)$  satisfying  $0 < m \leq \lambda_0(\zeta) \leq M$  for almost every  $\zeta \in (0, 1)$  and constants  $m, M > 0$ . If  $P_1 \mathcal{H}$  is a diagonal matrix, then (6)–(9) is of the form (52)–(55) with  $-\lambda_0(\zeta)$  replaced by a diagonal and invertible matrix  $\Delta$ . On the diagonal of the matrix  $\Delta$  are the possible different wave speeds of the system. If the ratio of any pair of diagonal entries of  $\Delta$  is rational, then the system (6)–(9) can be equivalently written in form (52)–(55) by dividing the intervals to adjust the propagation periods. It will be assumed throughout this section that the port-Hamiltonian system (52)–(55) is a well posed linear system with state space  $L^2(0, 1; \mathbb{C}^n)$  or equivalently that the matrix  $\begin{bmatrix} K_0 \\ K_u \end{bmatrix}$  is an invertible  $n \times n$ -matrix, see Theorem 1. The corresponding generator  $A$  of the  $C_0$ -semigroup of the homogeneous system is given by [13]

$$\begin{aligned} A f &= -(\lambda_0 f)', \\ D(A) &= \{ \lambda_0 f \in H^1(0, 1; \mathbb{C}^n) \mid \\ &\quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_0 \\ K_u \end{bmatrix} (\lambda_0 f)(0) + \begin{bmatrix} L_0 \\ L_u \end{bmatrix} (\lambda_0 f)(1) \}. \end{aligned}$$

Denote by  $G(s)$  the transfer function of the port-Hamiltonian system (52)–(55). Since the port-Hamiltonian system is assumed to be well-posed, there exists a right half plane

$$\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \text{Re } s > \alpha\}$$

such that  $G : \mathbb{C}_\alpha \rightarrow \mathbb{C}$  is an analytic and bounded function. Define

$$p := \int_0^1 \lambda_0^{-1}(s) ds.$$

Moreover, using [13, Theorem 12.2.1] for  $s \in \rho(A)$ , where  $\rho(A)$  denotes the resolvent set of  $A$ , and  $u \in \mathbb{C}$  the number  $G(s)u$  is (uniquely) determined by

$$0 = (K_0 + L_0 e^{-sp})v, \quad (56)$$

$$u = (K_u + L_u e^{-sp})v, \quad (57)$$

$$G(s)u = (K_y + L_y e^{-sp})v \quad (58)$$

for some  $v \in \mathbb{C}^n$ .

**Lemma 15** *There exists  $\mu \in \mathbb{R}$  such that, for  $s \in \mathbb{C}_\mu$ ,  $G(s) = 0$  if and only if the matrix  $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_y + L_y e^{-sp} \end{bmatrix}$  is not invertible.*

**PROOF.** Since the matrix  $\begin{bmatrix} K_0 \\ K_u \end{bmatrix}$  is invertible and  $A$  generates a  $C_0$ -semigroup there is a  $\mu \in \mathbb{R}$  such that  $\rho(A) \subseteq \mathbb{C}_\mu$  and

$$\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_u + L_u e^{-sp} \end{bmatrix} = \begin{bmatrix} K_0 \\ K_u \end{bmatrix} + \begin{bmatrix} L_0 \\ L_u \end{bmatrix} e^{-sp}$$

is invertible for  $s \in \mathbb{C}_\mu$ .

Assume now  $G(s) = 0$  for some  $s \in \mathbb{C}_\mu$ . Then (56)–(58) imply that there exists  $v \in \mathbb{C}^n$  such that

$$\begin{aligned} 0 &= (K_0 + L_0 e^{-sp})v, \\ 1 &= (K_u + L_u e^{-sp})v, \\ 0 &= (K_y + L_y e^{-sp})v. \end{aligned}$$

Because  $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_u + L_u e^{-sp} \end{bmatrix}$  is invertible, it yields  $v \neq 0$ . Thus  $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_y + L_y e^{-sp} \end{bmatrix}$  is not invertible.

Conversely, assume that for some  $s \in \mathbb{C}_\mu$ ,  $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_y + L_y e^{-sp} \end{bmatrix}$  is not invertible. Then there exists a non-zero vector  $v \in \mathbb{C}^n \setminus \{0\}$  such that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_y + L_y e^{-sp} \end{bmatrix} v.$$

Set  $u := (K_u + L_u e^{-sp})v$ . Since  $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_u + L_u e^{-sp} \end{bmatrix}$  is invertible, it follows that  $u \neq 0$ . However,  $G(s)u = 0$  by (56)–(58), which implies  $G(s) = 0$ .

**Theorem 16** *Suppose that  $G(s) \not\equiv 0$ . Then the zero dynamics of the port-Hamiltonian system (52)–(55) are again a well-posed port-Hamiltonian system with wave speed  $-\lambda_0$  and possibly a smaller state space. More precisely, there exists  $k \in \{0, \dots, n\}$  such that the zero dynamics is described by the port-Hamiltonian system*

$$\begin{aligned} \frac{\partial}{\partial t} w(\zeta, t) &= -\frac{\partial}{\partial \zeta} (\lambda_0(\zeta) w(\zeta, t)) \\ 0 &= K_w(\lambda_0(0) w(0, t)) + L_w(\lambda_0(1) w(1, t)). \end{aligned}$$

with state space  $L^2(0, 1; \mathbb{C}^k)$  and the  $k \times k$ -matrix  $K_w$  is invertible.

**PROOF.** The zero dynamics are defined by the equa-

tions

$$\frac{\partial}{\partial t} z(\zeta, t) = -\frac{\partial}{\partial \zeta} (\lambda_0(\zeta) z(\zeta, t)) \quad (59)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_0 \\ K_y \end{bmatrix} (\lambda_0(0) z(0, t)) + \begin{bmatrix} L_0 \\ L_y \end{bmatrix} (\lambda_0(1) z(1, t)). \quad (60)$$

Since there is one input and one output, and  $\text{rank} \begin{bmatrix} K_0 \\ K_u \end{bmatrix} = n$ , the rank of the matrix  $\begin{bmatrix} K_0 \\ K_y \end{bmatrix}$  equals  $n-1$  or  $n$ .

If  $\text{rank} \begin{bmatrix} K_0 \\ K_y \end{bmatrix} = n$ , that is, this matrix is invertible, then the zero dynamics is well-posed on the whole state space  $L^2(0, 1; \mathbb{C}^n)$ , see Proposition 7. Theorem 1 implies that the zero dynamics are well-posed on the state space  $L^2(0, 1; \mathbb{C}^n)$ . Thus  $k = n$  and the theorem is proved.

Suppose next that  $\text{rank} \begin{bmatrix} K_0 \\ K_y \end{bmatrix} = n-1$ . Then  $K_y$  is a linear combination of the rows of  $K_0$  and there is an invertible transformation, a row reduction, so that (60) is equivalent to

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ 0 & 0 \end{bmatrix} (\lambda_0(0) z(0, t)) + \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} (\lambda_0(1) z(1, t)). \quad (61)$$

Here  $K_{11}, L_{11} \in \mathbb{C}^{(n-1) \times (n-1)}$  and  $L_{22} \in \mathbb{C}$ . Since  $\text{rank}[K_{11} \ K_{12}] = n-1$ , column transformations lead to a representation where the matrix  $K_{11}$  is invertible. Assume now that this has been done.

Since  $K_{11}$  is invertible, and  $G$  is not equivalently zero, Lemma 15, implies that there exists  $s_0 \in \mathbb{C}$  such that both  $T_1 := K_{11} + L_{11} e^{-s_0 p}$  and

$$T := \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} := \begin{bmatrix} K_{11} + L_{11} e^{-s_0 p} & K_{12} + L_{12} e^{-s_0 p} \\ L_{21} e^{-s_0 p} & L_{22} e^{-s_0 p} \end{bmatrix} \quad (62)$$

are invertible. Defining the Schur complement of  $T$  with respect to  $T_1$ ,

$$S = T_4 - T_3 T_1^{-1} T_2,$$

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ T_3 T_1^{-1} & I \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & T_1^{-1} T_2 \\ 0 & I \end{bmatrix}.$$

Since  $T_1$  and  $T$  are invertible,  $S$  is invertible and

$$T^{-1} := \begin{bmatrix} T_1^{-1} + T_1^{-1} T_2 S^{-1} T_3 T_1^{-1} & -T_1^{-1} T_2 S^{-1} \\ -S^{-1} T_3 T_1^{-1} & S^{-1} \end{bmatrix}.$$

We define the matrices

$$\begin{aligned} K_w &:= K_{11}(T_1^{-1} + T_1^{-1}T_2S^{-1}T_3T_1^{-1}) - K_{12}S^{-1}T_3T_1^{-1} \\ L_w &:= L_{11}(T_1^{-1} + T_1^{-1}T_2S^{-1}T_3T_1^{-1}) - L_{12}S^{-1}T_3T_1^{-1} \\ K_{w12} &:= K_{12}S^{-1} - K_{11}T_1^{-1}T_2S^{-1}. \end{aligned}$$

Thus it yields

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & 0 \end{bmatrix} T^{-1} = \begin{bmatrix} K_w & K_{w12} \\ 0 & 0 \end{bmatrix}.$$

Here  $K_{w12}$  is a  $(n-1) \times 1$ -matrix and  $\text{rank } K_w \geq n-2$ . Now applying the state transformation  $\tilde{z} = Tz$ , the equations (61) are equivalent to

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} K_{11} & K_{12} \\ 0 & 0 \end{bmatrix} T^{-1}(\lambda_0(0)\tilde{z}(0,t)) \\ &+ \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} T^{-1}(\lambda_0(1)\tilde{z}(1,t)) \\ &= \begin{bmatrix} K_w & K_{w12} \\ 0 & 0 \end{bmatrix} (\lambda_0(0)\tilde{z}(0,t)) \\ &+ \begin{bmatrix} L_w & L_{12}S^{-1} - L_{11}T_1^{-1}T_2S^{-1} \\ 0 & e^{s_0p} \end{bmatrix} (\lambda_0(1)\tilde{z}(1,t)). \end{aligned}$$

Also the system of partial differential equations (59) are equivalent to

$$\frac{\partial}{\partial t} \tilde{z}(\zeta, t) = -\frac{\partial}{\partial \zeta} (\lambda_0(\zeta)\tilde{z}(\zeta, t)). \quad (63)$$

Thus, the transformed partial differential equation is identical to the original. The general solution

$$\tilde{z}_n(\zeta, t) = \frac{c}{\lambda_0(\zeta)} e^{\int_0^\zeta \lambda_0^{-1}(s) ds - t}$$

and the boundary condition  $\tilde{z}_n(1, t) = 0$  imply that  $\tilde{z}_n \equiv 0$ . Define

$$w := \begin{bmatrix} \tilde{z}_1 \\ \vdots \\ \tilde{z}_{n-1} \end{bmatrix}. \quad (64)$$

The zero dynamics is described by the reduced port-Hamiltonian system

$$\begin{aligned} \frac{\partial w}{\partial t} &= -\frac{\partial}{\partial \zeta} (\lambda_0 w), \\ 0 &= K_w(\lambda_0(0)w(0, t)) + L_w(\lambda_0(1)w(1, t)). \end{aligned}$$

The reduced system is well-posed on  $L^2(0, 1; \mathbb{C}^{n-1})$  if and only  $K_w$  is invertible; that is,  $K_w$  has rank  $n-1$ . If  $K_w$  is invertible, then the theorem is proved.

Now suppose that  $\text{rank } K_w = n-2$ . As in the first part, elementary row and column transformations can be used to put the boundary conditions for the reduced system into the form, again indicating the state variables by  $w$ ,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ 0 & 0 \end{bmatrix} (\lambda_0(0)w(0, t)) \\ + \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix} (\lambda_0(1)w(1, t)).$$

where  $\tilde{K}_{11}$  is invertible. Define

$$\tilde{T}(s) = K_w + L_w e^{-sp}.$$

In order to repeat the above procedure, a complex number  $s$  such that  $\tilde{T}$  and  $\tilde{K}_{11} + \tilde{L}_{11}e^{-sp}$  are both invertible is needed. Set  $s = s_0$ . Define

$$X = T_1^{-1} + T_1^{-1}T_2S^{-1}T_3T_1^{-1}.$$

Recalling that  $T_1 = K_{11} + e^{-s_0p}L_{11}$ ,  $T_2 = K_{12} + e^{-s_0p}L_{12}$ ,

$$\begin{aligned} K_w + L_w e^{-s_0p} &= K_{11}X - K_{12}S^{-1}T_3T_1^{-1} \\ &\quad + e^{-s_0p}L_{11}X - e^{-s_0p}L_{12}S^{-1}T_3T_1^{-1} \\ &= T_1X - T_2S^{-1}T_3T_1^{-1} \\ &= I + T_2S^{-1}T_3T_1^{-1} - T_2S^{-1}T_3T_1^{-1} \\ &= I. \end{aligned}$$

Thus, with  $s = s_0$ ,  $\tilde{T}(s)$  is invertible. Define

$$f_w : \mathbb{C}_\alpha \rightarrow \mathbb{C}, \quad f_w(s) = \det[\tilde{T}(s)].$$

and so  $f_w(s_0) = 1$ . Since  $f_w$  is analytic, there is a sequence  $s_n$ ,  $\text{Res}_n \rightarrow \infty$  with  $f(s_n) \neq 0$ . Choose then  $s_w$  so that  $\tilde{K}_{11} + \tilde{L}_{11}e^{-s_w p}$  is invertible. Repeating the previous procedure leads to a port-Hamiltonian system with state-space  $L^2(0, 1; \mathbb{C}^{n-2})$ . Since each iteration leads to a state-space with fewer number of state variables, this procedure is guaranteed to converge within  $n$  steps.

Since the zero dynamics are a well-posed dynamical system, the following result is immediate.

**Corollary 17** *The invariant zeroes are contained in a left-hand-plane.*

One consequence of calculating the zero dynamics using the original port-Hamiltonian form is that it is easy to obtain the input  $u$  that zeroes the output. Suppose only one state space reduction in Theorem 16 is needed. The

state space of the zero dynamics is  $L^2(0, 1; \mathbb{C}^{n-1})$ . From (54) and (62)

$$\begin{aligned} u(t) &= K_u \lambda_0(0) z(0, t) + L_u \lambda_0(1) z(1, t) \\ &= K_u \lambda_0(0) T^{-1} \tilde{z}(0, t) + L_u \lambda_0(1) T^{-1} \tilde{z}(1, t). \end{aligned}$$

In the zero dynamics,  $\tilde{z}_n \equiv 0$ . Defining  $\tilde{K}_u$  to be the first  $n-1$  columns of  $K_u \lambda_0(0) T^{-1}$  and defining  $\tilde{L}_u$  similarly, the zeroing input is

$$u(t) = \tilde{K}_u w(0, t) + \tilde{L}_u w(1, t)$$

where  $w$  is defined in (64). For the situation where more than one state space reduction is needed, the calculation is similar, except that a transformation matrix  $T$  is needed for each reduction.

## 6 Computation

Theorem 4.2 leads to a characterization of the zero dynamics as a port-Hamiltonian system of smaller dimension. Moreover, the proof is constructive and can be used in an algorithm to calculate the zero dynamics using standard linear algebra algorithms, see the box on the following page. Zero dynamics can be calculated exactly for large system order; that is those with a large number of nodes. Furthermore, Theorem 11 can be used to check stability. Several examples are now presented to illustrate the calculation of zero dynamics.

**Example 18** Consider the system from Example 10, written in the equal wave speed form. For zero dynamics,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_K z(0, t) + \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_L z(1, t). \quad (65)$$

The rank of  $K = 2$  and so the zero dynamics are defined on a smaller state space than the original. Applying one iteration of the algorithm yields (with  $s_0 = 0$ )

$$TP = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad K_w = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad L_w = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

The last row of the transformation matrix  $TP$  indicates that for zero dynamics

$$z_1 + z_{2a} \equiv 0 \quad (66)$$

and the first two rows define the remaining state variables:

$$\tilde{z}_1 = 2z_1 + z_{2b}, \quad \tilde{z}_2 = z_{2a} - z_{2b}.$$

### Algorithm: Calculation of Zero Dynamics

The data are: wave speed  $p = \int_0^1 \frac{1}{\lambda_0(\xi)} d\xi$ , boundary condition matrices  $K_0, L_0$ , and output matrices  $K_y, L_y$ . The dimension of the system is  $n$ , the number of columns in  $K_0$ . Define

$$K = \begin{bmatrix} K_0 \\ K_y \end{bmatrix}.$$

If  $K$  is invertible the zero dynamics are well-posed with  $n$  state variables. Otherwise do the following calculations.

- (1) Perform LU-decomposition of  $K$ :  $P_{\ell u} K = M_{\ell} M_u$  where  $M_{\ell}$  is lower triangular,  $M_u$  is upper triangular and  $P_{\ell u}$  is a permutation matrix.
- (2) If necessary, permute last column of  $M_u$  with earlier column, so that rank of top left  $n-1$  block is  $n-1$ ; call the permutation matrix  $P$ . Partition  $M_u P$  and  $M_{\ell}^{-1} P_{\ell u} L P$  similarly as

$$M_u = \begin{bmatrix} K_{11} & K_{12} \\ [0 \dots 0] & 0 \end{bmatrix}, \quad M_{\ell}^{-1} P_{\ell u} L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}.$$

- (3) Define the matrices  $T_1 = K_{11} + L_{11} e^{-ps_0}$  and

$$T = \begin{bmatrix} T_1 & K_{12} + L_{12} e^{-ps_0} \\ L_{21} e^{-ps_0} & L_{22} e^{-ps_0} \end{bmatrix} \text{ for } s_0 \text{ so that both}$$

matrices are invertible. (The existence of such an  $s_0$  is guaranteed if the transfer function is not identically zero. A simple way find a suitable  $s_0$  is to start with  $s_0 = 0$  and then increase by an arbitrary amount until both matrices are invertible.)

- (4) Decompose  $T^{-1}$  using the same decomposition as for  $K_u$  and construct the inverse of  $T$  using the Schur complement. Letting  $X$  be the solution of  $T_3 = XT_1$ , define

$$S = (T_4 - XT_2)^{-1}.$$

(Note  $S$  is a scalar.) Only the 2 left blocks of  $T^{-1}$  are needed:

$$(T^{-1})_{11} = T_1^{-1}(I + T_2 S X), \quad (T^{-1})_{21} = -S X.$$

- (5) The boundary matrices for the reduced system are  $K_w = K_{11}(T^{-1})_{11} + K_{12}(T^{-1})_{21}$   
 $L_w = L_{11}(T^{-1})_{11} + L_{12}(T^{-1})_{21}$ .
- (6) The new variables are  $\tilde{z}_1 \dots \tilde{z}_{n-1}$  where  $\tilde{z} = TPz$ , the differential equation is

$$\frac{\partial}{\partial t} \tilde{z}(\zeta, t) = -\frac{\partial}{\partial \zeta} (\lambda_0(\xi) \tilde{z}(\zeta, t))$$

and the boundary conditions are

$$K_w \lambda_0(0) \tilde{z}(0, t) + L_w \lambda_0(1) \tilde{z}(1, t).$$

If rank  $K_w = n-1$ , the algorithm is complete. If not, return to the first step with  $K = K_w$ ,  $L = L_w$  and repeat the process.

Since rank  $K_w = 1$  another iteration of the algorithm is needed. This leads to

$$(TP)_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad K_{w2} = 1, \quad L_{w2} = 0.$$

Thus,

$$\tilde{z}_1 + \tilde{z}_2 \equiv 0,$$

which, along with (66), implies that

$$z_1 \equiv 0, \quad z_{2a} \equiv 0.$$

This leads to

$$\frac{\partial}{\partial t} z_{2b}(\zeta, t) = -\frac{\partial}{\partial \zeta} \tilde{z}_{2b}(\zeta, t), \quad z_{2b}(1, t) = 0.$$

The only solution to this equation is the zero function and so the zero dynamics are empty. There is no control  $u$  that zeros the output. This reflects the fact that the system has, regarding the network as pipes, pipe 1 closed with both inlet and outlet connected to the end of pipe 2. The control is applied at the start of pipe 2. Not only is the system unstable, but there is no control that can zero the measurement  $z_1(0, t)$ .

### Example 19

$$\frac{\partial x_i}{\partial t} = -\frac{\partial x_i}{\partial \zeta}, \quad i = 1, 2, 3.$$

with

$$\begin{bmatrix} 0 \\ 0 \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_K x(0, t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(1, t) \quad (67)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} x(0, t) + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(1, t).$$

The rank of  $K$  in (67) is 3 and so the system is well-posed. The transfer function is not identically zero.

Zero dynamics require

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(0, t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x(1, t). \quad (68)$$

Applying the algorithm yields (with  $s_0 = 0$ )

$$TP = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad K_w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_w = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The third row of  $TP$  implies that  $z_1 \equiv 0$ . The reduced states are

$$\tilde{z}_2 = -z_1 + z_2 = z_2, \quad \tilde{z}_3 = z_1 - z_3 = -z_3.$$

Since  $K_w$  does not have full rank. the algorithm needs to be repeated; but with  $K_w, L_w$  as the boundary matrices. This yields

$$(TP)_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (K_w)_2 = \begin{bmatrix} 1 \end{bmatrix}, \quad (L_w)_2 = \begin{bmatrix} 0 \end{bmatrix}.$$

Thus  $\tilde{z}_2 = z_2 \equiv 0$  and  $\tilde{z}_3(0) = -z_3(0) = 0$ .

This example is simple enough to do by hand. The original equations (68) are already row-reduced, and imply  $x_1 \equiv 0$ . The reduced system must have  $x_2 \equiv 0$ .

Either calculation leads to one non-zero equation, for  $x_3$  with the boundary condition

$$x_3(0, t) = 0.$$

The system equations (67) imply that in order to achieve this,  $u(t) = x_3(1, t)$ .

**Example 20** Consider a larger system with  $n = 10$ . Suppose the wave speed  $\lambda_0$  is such that  $-\int_0^1 \lambda_0(\xi) d\xi = -1$ . The entries in the boundary matrices are zero, except that

$$\begin{aligned} K_0(1, 2) &= 1, & K_0(1, 9) &= -3, & K_0(2, 3) &= 1, \\ K_0(2, 2) &= -1, & K_0(3, 6) &= 1, & K_0(3, 10) &= 2, \\ K_0(4, 1) &= -5, & K_0(4, 6) &= 2, & K_0(5, 10) &= 6, \\ K_0(5, 9) &= -4, & K_0(6, 8) &= 4, & K_0(6, 1) &= -2, \\ K_0(7, 6) &= 1, & K_0(7, 7) &= 3, & K_0(8, 3) &= -2, \\ K_0(8, 8) &= 1, & K_0(8, 5) &= -5, & K_0(9, 1) &= 1, \\ K_0(9, 6) &= 5, & K_0(9, 9) &= -1 \\ K_u(1, 4) &= 1; \\ L_y(1, 2) &= 1, & L_y(1, 4) &= -2. \end{aligned}$$

Since

$$\text{rank} \begin{bmatrix} K_0 \\ K_u \end{bmatrix} = 10$$

this system is well-posed. Also, the transfer function  $G$  is not identically 0; in particular  $G(0) \neq 0$ . Applying the

algorithm with  $s_0 = 0$  yields

$$TP = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.5 & 0 & 0 & 0.5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 5.4 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0.1852 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -0.1481 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1852 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_w = I_9, \quad L_w = 0_{9 \times 9}.$$

For zero dynamics,  $z_2 - 2z_4 \equiv 0$  and the zeroing input is

$$u(t) = K_u TPz(0, t) = -2.5z_5(0, t) + 0.5z_8(0, t) - 3z_9(0, t).$$

## 7 Conclusions

In this paper, zero dynamics were formally defined for port-Hamiltonian systems. If the feedthrough operator is invertible, then the zero dynamics are again a port-Hamiltonian system of the same order. In general, however, the feedthrough operator is not invertible. For many infinite-dimensional systems, where the feedthrough is not invertible, the zero dynamics are not well-posed. It has been shown in this paper that provided the system can be rewritten as a network of waves with the same speed, the zero dynamics are always well-posed, and are a port-Hamiltonian system. Furthermore, a numerical method to construct the zero dynamics using the original partial differential equation has been described. Finite-dimensional approximations, which can be inaccurate in calculation of zeros, are not needed. The approach applies to systems with commensurate but non-equal wave speeds, and this generalization will be explored in future work. The extension to multi-input multi-output systems also needs to be established.

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