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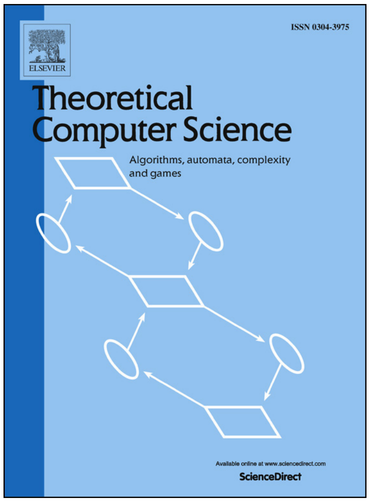
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Subword complexity and power avoidance

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Abstract

We begin a systematic study of the relations between subword complexity of infinite words and their power avoidance. Among other things, we show that
– the Thue-Morse word has the minimum possible subword complexity over all overlap-free binary words and all $(\frac{7}{3})$-power-free binary words, but not over all $(\frac{7}{3})^+$-power-free binary words;
– the twisted Thue-Morse word has the maximum possible subword complexity over all overlap-free binary words, but no word has the maximum subword complexity over all $(\frac{7}{3})$-power-free binary words;
– if some word attains the minimum possible subword complexity over all square-free ternary words, then one such word is the ternary Thue word;
– the recently constructed 1-2-bonacci word has the minimum possible subword complexity over all *symmetric* square-free ternary words.

Keywords: combinatorics on words, subword complexity, power-free word, critical exponent, Thue-Morse word

1 Introduction

Two major themes in combinatorics on words are *power avoidance* and *subword complexity* (also called *factor complexity* or just *complexity*). In power avoidance, the main goals are to construct infinite words avoiding various kinds of repetitions (see, e.g., [31]), and to count or estimate the number of length-n finite words avoiding these repetitions (see, e.g., [42]). In subword complexity, the main goal is to find explicit formulas for, or estimate, the number of distinct blocks of length n appearing in a given infinite word (see, e.g., [14]). In this paper we combine these two themes, beginning a systematic study of infinite binary and ternary words satisfying power avoidance restrictions. We follow two interlaced lines of research. First, given a power avoidance restriction, we study the set of infinite words satisfying this restriction, focusing on upper and lower bounds on their subword complexity, and examples of words of "large" and "small" complexity. Second, given a subword complexity restriction, we seek lower bounds on the powers avoided by infinite words of restricted complexity, and for words attaining these bounds. We also tried to cover the remaining blank spots with open questions and conjectures. Most of the results are gathered in Table 1; precise definitions are given below in Section 1.1.

The paper is organized as follows. After giving definitions, we study α-power-free infinite binary words for $\alpha \leq 7/3$ in Section 2. The number of distinct blocks of length n in this case is quite small, and all such infinite words are strongly related to the Thue-Morse word. We provide answers for most (but not all) questions about the complexity of these words. Next, in Section 3, we study low-complexity α-power-free infinite binary and ternary words for the case where α is large enough to provide sets of length-n blocks of size exponential in n. Here we leave more blank spots, but still obtain significant results, especially about square-free ternary words. Finally, in Section 4 we briefly consider high-complexity infinite words and relate the existence of words of "very high" complexity to an old problem of Restivo and Salemi.

1.1 Definitions and notation

Throughout we let Σ_k denote the k-letter alphabet $\{0, 1, \ldots, k-1\}$. By Σ_k^* we mean the set of all finite words over Σ_k, including the empty word ε. By Σ_k^ω we mean the set of all one-sided right-infinite words over Σ_k; throughout the paper they are referred to as "infinite words". The length of a finite word w is denoted by $|w|$.

If $x = uvw$, for possibly empty words u, v, w, x, then we say that u is a *prefix* of x, w is a *suffix* of x, and v is a *subword* (or *factor*) of x. A prefix (resp., suffix, factor) v of x is *proper* if $v \neq x$. A factor v of x can correspond to different factorizations of x: $x = u_1vw_1 = u_2vw_2 = \cdots$. Each factorization corresponds to an *occurrence* of v in x; the *position* of an occurrence is the length of the prefix of x preceding v. Thus occurrences of v (and, in particular, occurrences of letters) are linearly ordered by their positions, and we may speak about the "first" or "next" occurrence. An infinite word is *recurrent* if every factor has infinitely many occurrences.

Table 1: Infinite power-avoiding binary and ternary words of small and large complexity. Question marks indicate conjectured results.

Avoidance restriction	Small complexity	Large complexity
	Binary words	
symmetric overlap-free, symmetric $(\frac{7}{3})$-power-free	minimum: Thue-Morse word (Cor 4)	maximum: Thue-Morse word (Thm 18)
overlap-free	minimum: Thue-Morse word (Cor 4)	maximum: twisted Thue-Morse word (Thm 5)
$(\frac{7}{3})$-power-free	minimum: Thue-Morse word (Cor 4)	maximum: no (Thm 16); upper bound: $<4n$ (Thm 15)
$(\frac{7}{3})^+$-power-free	Thue-Morse word is not minimum (Thm 33)	exponential (Thm 38)
$(\frac{5}{2})^+$-power-free	minimum(?): $2n$ (new word—Thm 37)	exponential (Thm 38)
$(\frac{5+\sqrt{5}}{2})$-power-free	minimum: $n+1$ ([7], Fibonacci word); for $n+O(1)$ see Thm 35	exponential (Thm 38)
	Ternary words	
symmetric $(\frac{7}{4})^+$-power-free	minimal(?) growth const: 12 (Arshon word, Thm 27)	
symmetric $(\frac{5+\sqrt{5}}{4})$-power-free	minimal growth const: 6 (1-3-bonacci word, Rem 24)	
symmetric square-free	minimum: $6n-6$ (1-2-bonacci word, Thm 23)	
$(\frac{7}{4})^+$-power-free		exponential (Thm 38)
square-free	minimum(?): ternary Thue word (Thm 20)	exponential (Thm 38)
$(\frac{5}{2})$-power-free	minimum(?): $2n+1$ (new word, Thm 31)	exponential (Thm 38)

Every map $f : \Sigma_k \to \Sigma_m^*$ $(k, m \geq 1)$ can be uniquely extended to all finite and infinite words over Σ_k by setting $f(a_0a_1\cdots) = f(a_0)f(a_1)\cdots$, where $a_i \in \Sigma_k$ for all i. Such extended maps are called *morphisms*. A morphism is a *coding* if it maps letters to letters.

The Thue-Morse word

$$\mathbf{t} = t_0t_1t_2\cdots = 0110100110010110\cdots$$

is a well-studied infinite word with many equivalent definitions (see, e.g., [1]). The first is that t_i counts the number of 1's, modulo 2, in the binary expansion of i. The second is that $\mathbf{t}$ is the fixed point, starting with 0, of the morphism μ sending 0 to 01 and 1 to 10.

A *language* is a set of finite words over Σ_k. A language L is said to be *factorial* if $x \in L$ implies that every factor of x is also in L. If $\mathbf{u}$ is a one-sided or two-sided infinite word, then by $\mathrm{Fac}(\mathbf{u})$ we mean the factorial language of all finite factors of $\mathbf{u}$. We call a language $L \subseteq \Sigma_k^*$ *symmetric* if it is invariant under every permutation of the underlying alphabet; more precisely, if $f(L) = L$ for every bijective coding $f : \Sigma_k^* \to \Sigma_k^*$. An infinite word $\mathbf{u}$ is *symmetric* if $\mathrm{Fac}(\mathbf{u})$ is symmetric. For example, the Thue-Morse word is symmetric.

The so-called *subword complexity* or *factor complexity* of an infinite word $\mathbf{x}$ is the function $p_{\mathbf{x}}(n)$ that maps n to the number of distinct subwords (factors) of length n in $\mathbf{x}$. If the context is clear, we just write $p(n)$. A more general notion is the *growth function* (or *combinatorial complexity*, or *census function*) of a language L; it is the function $p_L(n)$ counting the number of words in L of length n. Thus, $p_{\mathbf{x}}(n) = p_{\mathrm{Fac}(\mathbf{x})}(n)$. These complexity functions can be roughly classified by their *growth rate* $\limsup_{n\to\infty}(p(n))^{1/n}$; for factorial languages, the lim sup can be replaced by lim, as was observed, e.g., in [20]. Exponential (resp., subexponential) words and languages have growth rate > 1 (resp., 1); growth rate 0 implies a finite language. Infinite words constructed by some regular procedure (e.g., those generated by morphisms) usually have small, often linear, complexity (see, e.g., [2]).

We say that an infinite word has *minimum* (resp., *maximum*) subword complexity in a set of words S if $p_{\mathbf{u}}(n) \le p_{\mathbf{v}}(n)$ (resp., $p_{\mathbf{u}}(n) \ge p_{\mathbf{v}}(n)$) for every word $\mathbf{v} \in S$ and every $n \ge 0$.

An integer power of a nonempty word x is a word of the form $x^n = \overbrace{xx\cdots x}^{n}$; by x^ω we mean the infinite word $xxx\cdots$. Integer powers can be generalized to fractional powers as follows: by x^α, for a real number $\alpha \ge 1$, we mean the prefix of length $\lceil \alpha|x| \rceil$ of the infinite word x^ω. If u is a finite word and x is the shortest word such that u is a prefix of x^ω, then the ratio $|u|/|x|$ is called the *exponent* of u and denoted by $\exp(u)$. The *critical* (or *local*) *exponent* of a finite or infinite word $\mathbf{u}$ is the supremum of the exponents of its factors. Thus, for example, the French word `contentent` (as in *ils se contentent*) has a suffix that is the $(3+\sqrt{5})/2 = 2.61803\cdots$'th power of the word `nte`, as well as the $(8/3)$'th power of this word; the exponent of the word `contentent` is 1, and its critical exponent is $8/3$.

We say a finite or infinite word *is α-power-free* or *avoids α-powers* if it has no factors that are β-powers for $\beta \ge \alpha$. Similarly, a finite or infinite word *is α^+-power-free* or *avoids α^+-powers* if it has no factors that are β-powers for $\beta > \alpha$. In what follows, we use only the term "α-power", assuming that α is either a number or a "number with a +". We write $L_{k,\alpha}$ for the language of all finite k-ary α-power-free words. The criterion for a language $L_{k,\alpha}$ to be infinite is $\alpha \ge RT(k)^+$, where the values of the *repetition threshold* $RT(k)$ are $RT(2) = 2$ [46], $RT(3) = 7/4$ [12], $RT(4) = 7/5$ [28], and $RT(k) = k/(k-1)$ for $k \ge 5$ (the crucial steps were done in [6, 10, 33]). The growth functions of the languages $L_{k,\alpha}$ also were studied in a number of papers; see the survey [42] for details. For our study, we need the following rough classification of growth functions of infinite power-free languages [18, 22, 34, 44, 47]: the languages $L_{2,\alpha}$ for $2^+ \le \alpha \le 7/3$ have polynomial growth functions, while all other infinite power-free languages are conjectured to have exponential growth functions. This conjecture has been proved for all power-free languages over the alphabets of even size up to 10 and of odd size up to 101. The "polynomial plateau" of binary power-free languages possesses several distinctive properties due to their intimate connection to the Thue-Morse word (see, e.g., [42, Section 2.2]).

An infinite word is called *periodic* if it has a suffix x^ω for some nonempty word x; otherwise, it is called *aperiodic*. Obviously, all power-free words are aperiodic. The minimum subword complexity of an aperiodic word is $n+1$, reached by the class of *Sturmian words* [25].

A finite word x from a language L is *right-extendable* in L if for every integer n there is a word v such that $|v| > n$ and $xv \in L$. Left-extendability is defined in a dual way. Further, x is *two-sided extendable* in L if for every integer n there are words u, v such that $|u|, |v| > n$ and $uxv \in L$. We write

$$\begin{aligned} \mathsf{rext}(L) &= \{x \in L \mid x \text{ is right-extendable in } L\} \\ \mathsf{ext}(L) &= \{x \in L \mid x \text{ is two-sided extendable in } L\} \end{aligned}$$

Note that all factors of an infinite word $\mathbf{u}$ are right-extendable in $\mathrm{Fac}(\mathbf{u})$. It is known known [39] that for every language L the languages $\mathsf{ext}(L)$ and $\mathsf{rext}(L)$ have the same growth rate as L.

For a word u over $\Sigma_2 = \{0, 1\}$, we say that we *flip* a letter a when we replace it with $1 - a$. The word $\overline{u}$ obtained from u by flipping all letters is the *complement* of u.

2 Minimum and maximum subword complexity in small languages

The 2^+-power-free words are commonly called *overlap-free* due to the following equivalent characterization: *a word w contains an α-power with $\alpha > 2$ if and only if two different occurrences of some factor in w overlap*. It has been known since Thue [46] that the Thue-Morse word $\mathbf{t}$ is overlap-free. The morphism μ satisfies the following very strong property.

Lemma 1 ([37]). *For every real $\alpha > 2$, an arbitrary word $u \in \Sigma_2$ avoids α-powers iff the word $\mu(u)$ does.*

Moreover, all (7/3)-power-free (in particular, overlap-free) binary words can be expressed in terms of the morphism μ. Below is the "infinite" version of a well-known result proved by Restivo and Salemi [34] for overlap-free words and extended by Karhumäki and Shallit [18] to all $(\frac{7}{3})$-power-free words. The second statement of this lemma and the uniqueness in the general case were proved in [32].

Lemma 2. *Let $\mathbf{u}$ be an infinite (7/3)-power-free binary word, $k \geq 0$ be an integer. Then $\mathbf{u}$ is uniquely representable in the form*

$$\mathbf{u} = x_o\mu(x_1\mu(\cdots x_k\mu(\mathbf{v})\cdots)) = x_0\mu(x_1)\cdots\mu^k(x_k)\mu^{k+1}(\mathbf{v}), \tag{1}$$

where $\mathbf{v}$ *is also an infinite* $(7/3)$*-power-free binary word,* $x_0, \ldots, x_k \in \{\varepsilon, 0, 1, 00, 11\}$*. Moreover, for every* $i \geq 1$ *the condition* $|x_i| = 2$ *implies either* $|x_{i-1}| = 0$*, or* $|x_{i-1}| = \cdots = |x_0| = 1$*, or* $|x_{i-1}| = \cdots = |x_j| = 1$*,* $|x_{j-1}| = 0$ *for some* $j \in \{1, \ldots, i-1\}$*.*

The factorization (1) implies that an infinite (7/3)-power-free binary word contains the words $\mu^k(0)$ and $\mu^k(1)$ as factors, for all $k \geq 0$. So we immediately get two corollaries of Lemma 2.

Corollary 3. *Every* $(7/3)$*-power-free infinite binary word contains, as factors, all elements of* $\mathrm{Fac}(\mathbf{t})$*. In particular, this is true of every overlap-free word.*

Corollary 4. *The Thue-Morse word* $\mathbf{t}$ *has the minimum subword complexity among all binary* $(7/3)$*-power-free (in particular, overlap-free) infinite words.*

The subword complexity $p_{\mathbf{t}}$ of the Thue-Morse sequence $\mathbf{t}$ has been known since the independent work of Brlek [5], de Luca and Varricchio [24], and Avgustinovich [4]. For $n \geq 2$ it is as follows:

$$p_{\mathbf{t}}(n+1) = \begin{cases} 4n - 2^i, & \text{if } 2^i \leq n \leq 3 \cdot 2^{i-1}; \\ 2n + 2^{i+1}, & \text{if } 3 \cdot 2^{i-1} \leq n \leq 2^{i+1}. \end{cases} \tag{2}$$

We now consider the analogue of the Thue-Morse sequence, where for $n \geq 0$ we count the number of 0's, mod 2, (instead of the number of 1's, mod 2) in the binary representation of n. By convention, we assume that the binary expansion of 0 is ε. We call this word

$$\begin{aligned} \mathbf{t}' = 001001101001011001101001100101101001011 \cdots = \\ 00\mu(1)\mu^2(0) \cdots \mu^{2n}(0)\mu^{2n+1}(1) \cdots \end{aligned} \tag{3}$$

the *twisted Thue-Morse word.* The word $\mathbf{t}'$ was mentioned in [36] and has appeared previously in the study of overlap-free and (7/3)-power-free words [13]. It is the image, under the coding $\{0, 2\} \to 0$, $1 \to 1$, of the fixed point of the morphism $0 \to 02$, $1 \to 21$, $2 \to 12$, and is known to be overlap-free.

We now state one of our main results. The proof follows after a series of preliminary statements.

Theorem 5. *The twisted Thue-Morse word* $\mathbf{t}'$ *has maximum subword complexity among all overlap-free infinite binary words, and is the unique word with this property, up to complement.*

Remark 6. The word $\mathbf{t}'$ has linear subword complexity, as is proved below, and so contains, as factors, only a small fraction of all right-extendable overlap-free words: the number of such words has superlinear growth (see [21]). This fact is explained in a broader context in Theorem 15.

Remark 7. The uniqueness of $\mathbf{t}'$, stated in Theorem 5, differs strikingly from the situation with minimum complexity, described by Corollary 4. Namely, the language of factors $\mathrm{Fac}(\mathbf{t})$, and thus the subword complexity $p_{\mathbf{t}}(n)$, is shared by a continuum of infinite words. The explicit construction of all such words can be found in [37]. (Precisely, Section 2 of [37] describes two-sided infinite words with the language of factors $\mathrm{Fac}(\mathbf{t})$, but all their suffixes have exactly the same factors.)

A word w is *minimal forbidden* for a factorial language L if $w \notin L$, while all proper factors of w belong to L. Every overlap-free infinite word $\mathbf{u}$ having a factor not in $\mathrm{Fac}(\mathbf{t})$ contains a minimal forbidden word x for $\mathrm{Fac}(\mathbf{t})$; moreover, every such word x appearing in $\mathbf{u}$ is right-extendable in the language $L_{2,2^+}$. These forbidden words are classified in the following lemma.

Lemma 8. *Let a_k be the last letter of $\mu^k(0)$.*

1. *The minimal forbidden words for* $\mathrm{Fac}(\mathbf{t})$ *are exactly the following words and their complements:*

   (a) 000;

   (b) $r_k = a_k\mu^k(010)0,\ k \geq 0$;

   (c) $s_k = a_k\mu^k(101)0,\ k \geq 0$.

2. *Among these, only the words r_k and their complements are right-extendable for $L_{2,2^+}$.*

Statement 1 is Proposition 1 of [38]; alternatively, it can be proved automatically using the `Walnut` prover [26]. Statement 2 follows from Proposition 3.7 and Example 1 of [43]. However, it is useful to provide some intuition. Let a binary word u of length ≥ 2 have the form $a\mu(v)b$, where $v \in \Sigma_2^*$, $a, b \in \{0, 1, \varepsilon\}$ (if u has two different representations of this form, choose the one with $a = \varepsilon$). We take an "approximate" μ-preimage, replacing u with $\bar{a}vb$ (here $\bar{\varepsilon} = \varepsilon$), and repeat the procedure while possible, denoting the final result by $\tilde{u}$. If $u \in \mathrm{Fac}(\mathbf{t})$, we will eventually arrive at $\tilde{u} \in \{01, 10\}$; if $u \notin \mathrm{Fac}(\mathbf{t})$, we will stop at some other word (often at u itself). Proposition 3.7 of [43] says that $\tilde{u}$ is (right, left, or two-sided) extendable for $L_{2,2^+}$ iff u is. One has $\tilde{r}_k = 00100$, which is extendable, e.g., to $\mathbf{t}'$, and $\tilde{s}_k = 000$, which is even not overlap-free.

Lemma 9. *Let $\mathbf{u}$ be an infinite overlap-free binary word and let $k \geq 0$. If $\mathbf{u} = xr_k\cdots$ or $\mathbf{u} = x\bar{r}_k\cdots$ for some word x, then $|x| \leq 2^k - 1$. In particular, the words $r_k, \bar{r}_k$ have, in total, at most one occurrence in $\mathbf{u}$.*

Proof. We prove the first statement by induction on k. Consider the base case $k = 0$. We have $r_0 = 00100$. Since $\mathbf{u}$ is overlap free, the letter following xr_0 in $\mathbf{u}$ is 1. On the other hand, if x is nonempty, it must end with either 0 or 1, and in both cases xr_01 has an overlap. So x must be empty.

Now the induction step. Assume the claimed result is true for $k' < k$; we prove it for k. Let $\mathbf{u} = xr_k\mathbf{v} = xa_k\mu^k(010)0\mathbf{v}$ and assume $|x| \geq 2^k$. We also have $\mathbf{u} = x_0\mu(\mathbf{u}')$ for some

$x_0 \in \{\varepsilon, 0, 00, 1, 11\}$ and some infinite word $\mathbf{u}'$ by Lemma 2. Note that $|x_0| \le 2 \le |x|$, and thus the prefix 0110 of $\mu^k(010)$ must be parsed as $\mu(01)$. Hence $0\mathbf{v}$ equals $\mu(0\mathbf{v}')$ for some infinite word $\mathbf{v}'$, which is overlap-free by Lemma 1. Next, $x = y\bar{a}_k$ for some nonempty word y. Indeed, $|xa_k| > |x_0|$, so the last two letters of xa_k should form the block $\mu(\bar{a}_k)$. Thus

$$\mathbf{u} = y\bar{a}_k a_k \mu^k(010)\mu(0\mathbf{v}') = y\mu(\bar{a}_k\mu^{k-1}(010)0\mathbf{v}').$$

Compare this to $\mathbf{u} = x_0\mu(\mathbf{u}')$. The uniqueness of factorization in Lemma 2 implies $y = x_0\mu(x')$ for some word x' (possibly empty). Observing that $\bar{a}_k = a_{k-1}$, we finally write

$$\mathbf{u} = x_0\mu(\mathbf{u}'), \quad \text{where } \mathbf{u}' = x'a_{k-1}\mu^{k-1}(010)0\mathbf{v}'.$$

Applying the inductive hypothesis to $\mathbf{u}'$, we obtain $|x'| \le 2^{k-1}-1$. Having $|x| = |x_0|+2|x'|+1$ and $|x| \ge 2^k$, we obtain $|x'| = 2^{k-1} - 1$. Since $|x'|$ has the maximum possible length, the inductive hypothesis guarantees that both words $0\mathbf{u}'$ and $1\mathbf{u}'$ contain overlaps. Then the word $\mathbf{u}'$ has some prefix uu that ends with 0, and also some prefix vv that ends with 1 (e.g., $\mathbf{u}'$ can be a word of the form $001001\cdots$). Hence $\mu(\mathbf{u}')$ has the prefix $\mu(u)\mu(u)$ that ends with 1, and the prefix $\mu(v)\mu(v)$ that ends with 0. This means that both words $0\mu(\mathbf{u}')$ and $1\mu(\mathbf{u}')$ contain overlaps, and thus $x_0 = \varepsilon$. So we have $|x| = 2|x'| + 1 = 2^k - 1$. This contradicts our assumption $|x| \ge 2^k$ and thus proves the inductive step.

For the second statement, observe that $|r_k| = |\bar{r}_k| = 3 \cdot 2^k + 2$, which is much bigger than $|x|$. Since the occurrences of a factor cannot overlap, $\mathbf{u}$ has at most one occurrence of each of the factors r_k, $\bar{r}_k$. Let $\mathbf{u}$ contain both, with r_k starting earlier. Then $\bar{r}_k$ must contain the suffix $\mu^k(10)0$ of r_k, which is not the case: the word $\bar{r}_k = \bar{a}_k\mu^k(101)1$, being overlap free, contains only one occurrence of $\mu^k(10)$, and this occurrence is followed by $\mu^k(1)$ which begins with 1. So $\mathbf{u}$ contains at most one of r_k, $\bar{r}_k$. □

Recall that a factor v of $\mathbf{u}$ is (right) $\mathbf{u}$-*special* if both $v0$ and $v1$ are factors of $\mathbf{u}$. The set of all special factors of $\mathbf{u}$ is denoted by $\mathsf{Spec}(\mathbf{u})$. We will omit $\mathbf{u}$ when it is clear from the context. We use the following familiar fact:

Lemma 10. *The number $D_{\mathbf{u}}(n) = \#(\mathsf{Spec}(\mathbf{u}) \cap \Sigma_2^n)$ is the first difference of the subword complexity of $\mathbf{u}$: $D_{\mathbf{u}}(n) = p_{\mathbf{u}}(n+1) - p_{\mathbf{u}}(n)$.*

Proof. Consider the function mapping every word from $\mathrm{Fac}(\mathbf{u})$ of length $n+1$ to its prefix of length n. Each special factor of $\mathbf{u}$ of length n has two preimages, while each non-special factor of length n has a single preimage. □

Corollary 11. *For all $n \ge 1$ and every infinite binary word $\mathbf{u}$ we have*

$$p_{\mathbf{u}}(n) = 2 + \sum_{1 \le i < n} D_{\mathbf{u}}(i). \tag{4}$$

Since $D_{\mathbf{u}}(0) = 1$ for every binary word, below we restrict our attention to $\{D_{\mathbf{u}}(n)\}$ for $n \geq 1$. For example,

$$D_{\mathbf{t}}(n) = \begin{cases} 4, & \text{if } n = 2^k + i \text{ for some } k \geq 1,\ i > 0,\ i \leq 2^{k-1}; \\ 2, & \text{otherwise,} \end{cases}$$

as was first computed in [5]. As an infinite word over $\{2, 4\}$, this sequence looks like

$$224244224444222244444444222222224 \cdots \tag{5}$$

where each subsequent block of equal letters is twice the size of the previous block of the same letter (except for the second block of 2's).

Let $\mathbf{u}$ be overlap-free. By Corollary 4, all $\mathbf{t}$-special factors are $\mathbf{u}$-special, so $D_{\mathbf{u}}(n) \geq D_{\mathbf{t}}(n)$ for all n. We call a $\mathbf{u}$-special factor *irregular* if it is not $\mathbf{t}$-special.

Lemma 12. *For an overlap-free infinite binary word* $\mathbf{u}$*, all* $\mathbf{u}$*-special factors are Thue-Morse factors.*

Proof. Every word from $\mathrm{Fac}(\mathbf{u}) \setminus \mathrm{Fac}(\mathbf{t})$ contains r_k or $\bar{r}_k$ by Lemma 8; so it occurs in $\mathbf{u}$ only once by Lemma 9 and thus is not $\mathbf{u}$-special. Hence $\mathsf{Spec}(\mathbf{u}) \subseteq \mathrm{Fac}(\mathbf{t})$. □

Proof of Theorem 5. Let $\mathbf{u}$ be an overlap-free infinite binary word. We show the following four facts:

(i) at every position in $\mathbf{u}$, the first occurrence of at most one irregular $\mathbf{u}$-special factor begins;

(ii) all irregular $\mathbf{u}$-special factors have different lengths;

(iii) for every $k \geq 0$, there exist at most 2^k irregular $\mathbf{u}$-special factors of length $\leq 2^{k+2}$; the length of each factor is in the range $[3{\cdot}2^i{+}1..2^{i+2}]$ for some $i \leq k$;

(iv) for every $k \geq 0$, there exist exactly 2^k irregular $\mathbf{t}'$-special factors of length $\leq 2^{k+2}$; their lengths are given by (6) below.

(i) Let v be an irregular $\mathbf{u}$-special factor. By Lemma 12, $v \in \mathrm{Fac}(\mathbf{t})$; but either $v0$ or $v1$ is not a Thue-Morse factor by definition of irregularity. W.l.o.g., $v0 \notin \mathrm{Fac}(\mathbf{t})$. Then some suffix of $v0$ is a minimal forbidden word for $\mathrm{Fac}(\mathbf{t})$. By Lemma 8, this suffix equals r_k for some $k \geq 0$. So we can write $v0 = v'r_k$ and $\mathbf{u} = xv'r_k\mathbf{u}'$; by Lemma 9, $|xv'| < 2^k$. In particular, $|x| < |v|$. Thus, the first occurrence of v in $\mathbf{u}$ is at the position $|x|$ and is followed by 0 ($v0 \notin \mathrm{Fac}(\mathbf{t})$), while all other occurrences of v are followed by 1 (and $v1 \in \mathrm{Fac}(\mathbf{t})$). Now assume that some proper prefix w of v is also an irregular $\mathbf{u}$-special factor. Since $v \in \mathrm{Fac}(\mathbf{t})$, the occurrence of w at the position $|x|$ is not the first occurrence of w in $\mathbf{u}$. Hence the first occurrences of each two irregular $\mathbf{u}$-special factors begin in different positions.

(ii) Take an irregular $\mathbf{u}$-special factor v such that $v0 = v'r_k$ and another $\mathbf{u}$-special factor w such that $w0 = w'r_i$ or $w1 = w'\bar{r}_i$. Note that $3 \cdot 2^k + 1 \leq |v| \leq 4 \cdot 2^k$ since $|v'| < 2^k$ by Lemma 9. The same calculation applies for w, so $i \neq k$ implies $|w| \neq |v|$. By Lemma 9, $\mathbf{u}$ has a unique occurrence of r_k and no occurrences of $\bar{r}_k$. Hence $i = k$ implies that $v0$ and $w0$ end at the same position of $\mathbf{u}$, so $|v| = |w|$ only if $v = w$.

(iii) Let v be an irregular $\mathbf{u}$-special factor of length $\leq 2^{k+2}$. As above, we can write w.l.o.g. $v0 = v'r_i$ for some $i \leq k$ and some word v' of length $< 2^i$; then $|v| = 3 \cdot 2^i + 1 + |v'|$, as required. (Note that $i > k$ implies $|v| > 3 \cdot 2^i > 2^{k+2}$.) Further, $\mathbf{u} = xv'r_i\mathbf{u}'$. Then the first occurrence of v in $\mathbf{u}$ is at the position $|x| \in [0..2^i{-}1]$. Thus, there are 2^k possible positions for the first occurrence of v; the reference to (i) finishes the proof.

(iv) From (3) it is easy to see that for every k the prefix of $\mathbf{t}'$ of length $5{\cdot}2^{2k}$ equals $v\mu^{2k}(0100)$, where $v = 0 = \mu^{2k}(0)$ for $k = 0$ and v has the common suffix $\mu^{2k-1}(1)$ with the word $\mu^{2k}(0)$ for $k > 0$. Similarly, the prefix of $\mathbf{t}'$ of length $5 \cdot 2^{2k+1}$ equals $v\mu^{2k+1}(1011)$, where v has the common suffix $\mu^{2k}(0)$ with the word $\mu^{2k+1}(1)$. According to the above description of the irregular special factors, $\mathbf{t}'$ has first occurrences of irregular special factors beginning at each position:

Factor	Position	Length
0010	0	4
0100110	1	7
10011010010110	2	14
0011010010110	3	13
$\vdots$	$\vdots$	$\vdots$
$r\mu^{2k}(010), r$ is a suffix of $\mu^{2k-1}(0)$	$2^{2k-1} + i,\ 0 \leq i < 2^{2k-1}$	$2^{2k+2} - 2^{2k-1} - i$
$r\mu^{2k+1}(101), r$ is a suffix of $\mu^{2k}(1)$	$2^{2k} + i,\ 0 \leq i < 2^{2k}$	$2^{2k+3} - 2^{2k} - i$

(6)

Let $\mathsf{Irr}(\mathbf{u})$ be the sequence of lengths of irregular $\mathbf{u}$-special factors in increasing order; by (ii), each length corresponds to a single factor. To finish the proof, we compare $\mathsf{Irr}(\mathbf{u})$ to $\mathsf{Irr}(\mathbf{t}')$ using (iii) and (iv):

$\mathsf{Irr}(\mathbf{u})$	$\mathsf{Irr}(\mathbf{t}')$
at most 1 of $\{4\}$	4
at most 2 of $\{4, 7, 8\}$	4, 7
at most 4 of $\{4, 7, 8, 13, 14, 15, 16\}$	4, 7, 13, 14
$\vdots$	$\vdots$
at most 2^k of length $\leq 2^{k+2}$	exactly 2^k, minimal possible

Hence for every $m \geq 0$ the m'th element of $\mathsf{Irr}(\mathbf{u})$ is greater than or equal to the m'th element of $\mathsf{Irr}(\mathbf{t}')$. Since $\mathbf{u}$ and $\mathbf{t}'$ have the same regular special factors, one has, for every n,

$$\sum_{1 \leq i < n} D_{\mathbf{u}}(i) \leq \sum_{1 \leq i < n} D_{\mathbf{t}'}(i),$$

The inequlities $p_{\mathbf{t}'}(n) \geq p_{\mathbf{u}}(n)$ for all $n \geq 0$ now follow from (4). It remains to note that the sequence $\mathsf{Irr}(\mathbf{t}')$ determines an overlap-free word up to the complement: there is only one way, shown in (6), to associate the lengths of irregular special factors to the positions of their first occurrences. Therefore, there are no words of complexity $p_{\mathbf{t}'}(n)$ except for $\mathbf{t}'$ and $\bar{\mathbf{t}}'$. The theorem is proved. □

As a consequence of the proof, we can determine a closed form for the subword complexity of $\mathbf{t}'$.

Proposition 13. *The number of special* $\mathbf{t}'$*-factors of length* $n > 0$ *is given by the formula*

$$D_{\mathbf{t}'}(n) = \begin{cases} 4, & 2^{k+1} < n \leq 3 \cdot 2^k \textit{ for some } k \geq 0; \\ 3, & n = 4 \textit{ or } 3 \cdot 2^k < n \leq 7 \cdot 2^{k-1} \textit{ for some } k \geq 1; \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. The proposition states that the sequence $\{D_{\mathbf{t}'}(n)\}$ $(n \geq 1)$ can be written as the following word over $\{2, 3, 4\}$:

$$224344324444332244444444333322224 \cdots$$

Comparing this to (5), we see that some 2's have been changed to 3's. This means an additional $\mathbf{t}'$-special factor for each corresponding length, and this factor must be irregular. According to (6), the set of all positions of 3's indeed coincides with the set of lengths of irregular $\mathbf{t}'$-special factors, thus proving the proposition. □

Corollary 14. *The maximum factor complexity of a binary overlap-free infinite word is the factor complexity of the twisted Thue-Morse word* $\mathbf{t}'$ *and is given, for* $n \geq 4$*, by the formula*

$$p_{\mathbf{t}'}(n+1) = \begin{cases} 4n - 3 \cdot 2^{i-2}, & \textit{if } 2^i \leq n \leq 3 \cdot 2^{i-1}; \\ 3n + 3 \cdot 2^{i-2}, & \textit{if } 3 \cdot 2^{i-1} \leq n \leq 7 \cdot 2^{i-2}; \\ 2n + 5 \cdot 2^{i-1}, & \textit{if } 7 \cdot 2^{i-2} \leq n \leq 2^{i+1}. \end{cases} \tag{7}$$

Proof. Immediate from Theorem 5, Proposition 13, and formula (4). □

A table of the first few values of the subword complexity of $\mathbf{t}'$ follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$p_{\mathbf{t}'}(n)$	1	2	4	6	10	13	17	21	24	26	30	34	38	42	45	48	50	52	56

2.1 Beyond overlap-free words

Now we turn to the case of α-power-free infinite binary words for arbitrary α from the interval $[2^+, \frac{7}{3}]$; in particular, all such words are $\left(\frac{7}{3}\right)$-power free. By Corollary 4, the Thue-Morse word is a word of minimum complexity. Theorem 15 below shows that the asymptotic growth of subword complexity belongs to a very small range. Theorem 16 demonstrates that there is no (7/3)-power-free infinite binary word of maximum complexity.

Theorem 15. *Every $\left(\frac{7}{3}\right)$-power-free infinite binary word $\mathbf{u}$ has linear subword complexity. Moreover, for every $n > 0$ one has $p_{\mathbf{u}}(n) < \frac{6}{5} \cdot p_{\mathbf{t}}(n)$.*

Proof. First assume that $\mathbf{u} = \mu^m(\mathbf{v})$ for some word $\mathbf{v}$ and some $m \geq 0$. Then $\mathbf{v}$ is $\left(\frac{7}{3}\right)$-power free by Lemma 1. According to Lemma 8, the shortest word that can be a factor of $\mathbf{v}$, but does not occur in $\mathbf{t}$, is either 00100 or 11011. Hence every factor of $\mathbf{u}$ that is contained in four consecutive blocks of the form $\mu^m(a)$, $a \in \Sigma_2$, is a factor of $\mathbf{t}$. Thus the shortest factor of $\mathbf{u}$ that is not in $\mathbf{t}$ has the length at least $3 \cdot 2^m + 2$. (If $\mathbf{v}$ contains 00100, this factor is r_m from Lemma 8.) So we have

$$\mathbf{u} = \mu^m(\mathbf{v}) \Longrightarrow p_{\mathbf{u}}(n) = p_{\mathbf{t}}(n) \text{ for every } n = 0, 1, \ldots, 3{\cdot}2^m{+}1. \tag{8}$$

Now let $\mathbf{u}$ be arbitrary. Still, $\mathbf{u}$ satisfies (8) with $m = 0$, so $p_{\mathbf{u}}(n) = p_{\mathbf{t}}(n)$ for $n \leq 4$. So we take an arbitrary $n \geq 5$ and choose a unique integer $m \geq 1$ satisfying the condition $3 \cdot 2^{m-1} + 1 < n \leq 3 \cdot 2^m + 1$. Consider the factorization of $\mathbf{u}$ of type (1):

$$\mathbf{u} = x_0\mu(x_1\mu(\cdots x_{m-1}\mu(\mathbf{v})\cdots)) = x_0\mu(x_1)\cdots\mu^{m-1}(x_{m-1})\mu^m(\mathbf{v}).$$

By (8), all factors of $\mu^m(\mathbf{v})$ of length n are Thue-Morse factors, so we have

$$p_{\mathbf{u}}(n) - p_{\mathbf{t}}(n) \leq |x_0\mu(x_1)\cdots\mu^{m-1}(x_{m-1})| = \sum_{i=0}^{m-1} 2^i|x_i|.$$

This upper bound is a bit loose; to tighten it, consider x_{m-1}. By Lemma 2, $|x_{m-1}| \leq 2$. Let $|x_{m-1}| = 2$ (w.l.o.g., $x_{m-1} = 00$). Since by Lemma 1 the word $x_{m-1}\mu(\mathbf{v})$ is $\left(\frac{7}{3}\right)$-power free, $\mathbf{v}$ cannot begin with 0, 11, or 100 (this would lead to forbidden factors 000, 01010, or 0010010, respectively). Then $\mathbf{v}$ begins with 101, and

$$\mu^{m-1}(x_{m-1})\mu^m(\mathbf{v}) = \mu^{m-1}(0)\mu^{m-1}(0100110)\cdots$$

Since 0100110 is a Thue-Morse factor, so is $\mu^{m-1}(0100110)$. Hence the number of length n words in $\mathrm{Fac}(\mathbf{u})\setminus\mathrm{Fac}(\mathbf{t})$ is at most $2^{m-1} + \sum_{i=0}^{m-2} 2^i|x_i|$. Applying the second statement of Lemma 2 to x_{m-1}, we obtain $\sum_{i=0}^{m-2} 2^i|x_i| \leq \sum_{i=0}^{m-2} 2^i = 2^{m-1}{-}1$ (note that some of x_i's can have length 2, but only if $|x_j| = 0$ for some bigger j). Therefore,

$$p_{\mathbf{u}}(n) - p_{\mathbf{t}}(n) \leq 2^m - 1. \tag{9}$$

Next let $x_{m-1} = 0$ and let $\mathbf{v} = abc\cdots$, $a, b, c \in \{0, 1\}$. If $a = b = c$ or $a = b = 1$, then $\mathbf{u}$ is not (7/3)-power free. Otherwise, $1abc$ is a Thue-Morse factor, as well as the suffix $\mu^{m-1}(0)\mu^m(abc)$ of $\mu^m(1abc)$. Then all length-n factors of $\mathbf{u}$, that are not in $\mathrm{Fac}(\mathbf{t})$, begin in $\mathbf{u}$ on the left of $\mu^{m-1}(x_{m-1})$. But $\sum_{i=0}^{m-2} 2^i|x_i| \leq \sum_{i=0}^{m-2} 2^{i+1} = 2^m - 2$, so again we have (9). For the same reason we obtain (9) in the case $x_{m-1} = \varepsilon$.

Finally, we apply (2) to get

$$p_{\mathbf{t}}(n) \geq p_{\mathbf{t}}(3 \cdot 2^{m-1} + 2) = 2 \cdot (3 \cdot 2^{m-1} + 1) + 2^{m+1} = 5 \cdot 2^m + 2 \tag{10}$$

and compare (9) to (10) to obtain the required inequality. □

Theorem 16. *There is no* (7/3)*-power-free infinite binary word of maximum complexity.*

Proof. Along with giving an example, we provide some intuition about its appearance. From the proof of Theorem 15 we see that for arbitrary (7/3)-power-free infinite words $\mathbf{u}$,

(a) if $n \le 4$, then $p_{\mathbf{u}}(n) = p_{\mathbf{t}}(n)$;

(b) if $5 \le n \le 7$, we write $\mathbf{u} = x_0\mu(\mathbf{v})$ and see that $p_{\mathbf{u}}(n) \le p_{\mathbf{t}}(n) + 1$; the only length n factor of $\mathbf{u}$ that is possibly not in $\mathrm{Fac}(\mathbf{t})$ is the prefix of $\mathbf{u}$;

(c) if $8 \le n \le 13$, we write $\mathbf{u} = x_0\mu(x_1)\mu^2(\mathbf{v})$ and see that $\mathrm{Fac}(\mathbf{u}) \setminus \mathrm{Fac}(\mathbf{t})$ contains at most $2^2 - 1 = 3$ words if $|x_1| = 2$ (these words begin in $\mathbf{u}$ at positions 0, 1, 2) and at most $2^2 - 2 = 2$ words otherwise (these words begin in $\mathbf{u}$ at positions 0 and 1). First consider a (7/3)-power-free infinite word $\mathbf{u}_1$ starting with 0 and satisfying $p_{\mathbf{u}_1}(n) = p_{\mathbf{t}}(n) + 3$ for some $n \in [8..13]$. From the proof of Theorem 15 for $\mathbf{u}_1 = x_0\mu(x_1)\mu^2(\mathbf{v})$ we have, up to flipping the letters, $x_0 = 0$, $x_1 = 00$, $\mathbf{v} = 101\cdots$, so

$$\mathbf{u}_1 = 0\mu(00)\mu^2(101\cdots) = 0\,0101\,1001\,0110\,1001\cdots$$

One has $p_{\mathbf{u}_2}(8) = p_{\mathbf{t}}(8) + 1$ (the only additional factor 10110010 begins at position 2), $p_{\mathbf{u}_2}(9) = p_{\mathbf{t}}(9) + 2$ (the additional factors begin at positions 1 and 2), and $p_{\mathbf{u}_2}(n) = p_{\mathbf{t}}(n) + 3$ for $n = 10, 11, 12, 13$ (the additional factors begin at positions 0, 1, and 2); hence we proved that no word $\mathbf{u}$ has $p_{\mathbf{u}}(n) = p_{\mathbf{t}}(n) + 3$ for $n = 8, 9$. An example of a (7/3)-power-free word $\mathbf{u}_1$ is $\mathbf{u}_1 = 0\mu(\mathbf{t}')$; it is (7/3)-power free because $\mu(\mathbf{t}')$ is overlap free by Lemma 1 and does not begin with a square of period ≤ 3. (One can check that $0\mu(\mathbf{t}')$ has the only 2^+-power $0(010110)^2$ at position 0.) Now construct a word $\mathbf{u}_2$ having the property $p_{\mathbf{u}_1}(8) = p_{\mathbf{t}}(8)+2$. The two additional factors of length 8 begin at positions 0 and 1 (otherwise $\mathbf{u}_2$ would have three additional factors of length 9, which is impossible as mentioned above). Since $\mathbf{u}_2$ is (7/3)-power-free, it does not contain $s_0, \bar{s}_0$, and can contain $r_0, \bar{r}_0, s_1$, or $\bar{s}_1$ as a prefix only. So if $\mathbf{u}_2$ has the prefix awb with $a, b \in \{0, 1\}$, $|w| = 7$, and $aw, wb \notin \mathrm{Fac}(\mathbf{t})$, then $wb \in \{r_1, \bar{r}_1\}$ and aw has $r_0, \bar{r}_0, s_1$, or $\bar{s}_1$ as a prefix. A simple check reveals the only possibility:

$$\mathbf{u}_2 = 00\mu(1)\mu^2(01\cdots) = 00\,10\,0110\,1001\cdots$$

Since $|x_1| = 1$, we have $p_{\mathbf{u}_2}(n) = p_{\mathbf{t}}(n) + 2$ for $n = 8, 9, \ldots, 13$. Using the fact that the exponents of prefixes of $\mathbf{t}$ do not exceed 5/3 [43, Prop. 2.1(5)], it is easy to check that the word $\mathbf{u}_2 = 0010\mathbf{t}$ is (7/3)-power-free. Indeed, if $\mathbf{u}_2$ contains a (7/3)-power of period p, then $4 = |0010| \ge 2p/3$, so $p \le 6$ and it is enough to analyze the prefix of $\mathbf{t}$ of length $6 \cdot 5/3 = 10$. (A more detailed check shows that $\mathbf{u}_1$ contains the only 2^+-power $0(100110)^2$ at position 1).

Overall, we see that the word $\mathbf{u}_1$ reaches the maximum possible complexity for $n = 10$ but not for $n = 8$, while $\mathbf{u}_2$ reaches this maximum for $n = 8$ but not for $n = 10$. □

If there is no maximum subword complexity, it makes sense to look at some sort of "asymptotically maximal" complexity. For a function $y(n)$ of linear growth, let its *linear growth constant* be $\limsup_{n\to\infty} \frac{y(n)}{n}$. For overlap-free infinite binary words, the maximum subword complexity has linear growth constant 7/2, as can be easily derived from (7). From (2) we see that the linear growth constant for the Thue-Morse word is 10/3. By Theorem 15, this means that the linear growth constants of all (7/3)-power-free infinite binary words are bounded above by 4.

Open Question 17. *What is the maximum linear growth constant for the subword complexity of a* (7/3)*-power-free infinite binary word? Which words have such complexity?*

2.2 The symmetric case

Instead of studying all (7/3)-power-free words, we can restrict our attention to the symmetric ones.

Theorem 18. *The only possible subword complexity function of a* (7/3)*-power-free symmetric infinite binary word is the function* $p_{\mathbf{t}}(n)$.

Proof. Let $\mathbf{u}$ be a (7/3)-power-free infinite binary word such that $p_{\mathbf{u}}(n) \neq p_{\mathbf{t}}(n)$. By Corollary 4, $\mathbf{u}$ contains a factor that is not Thue–Morse. By Lemma 8, $\mathbf{u}$ contains one of the factors r_k, s_k, or their complements. Assume that $\mathbf{u}$ contains $r_k = a_k\mu^k(010)0$. Taking the representation (1) of $\mathbf{u}$ and observing that $\mu^k(01)$ has no factors of the form $\mu^k(a)$ except for the prefix and the suffix, we see that $\mathbf{u}$ has the suffix $\mu^k(0100\cdots)$. If this suffix contains the factor $\bar{r}_k = \bar{a}_k\mu^k(101)1$, then $\bar{r}_k$ occurs inside a factor of the form $\mu^k(b11011c)$. Hence $\mathbf{u}$ contains either $\mu^k(111)$ or $\mu^k(0110110)$, which is impossible because $\mathbf{u}$ is (7/3)-power free. Thus, $\bar{r}_k$ cannot occur in $\mathbf{u}$ to the right of an occurrence of r_k. Similarly, if $\mathbf{u}$ contains $s_k = a_k\mu^k(101)0$, then the suffix $\mu^k(1010\cdots)$ of $\mathbf{u}$ cannot contain $\bar{s}_k$ without containing the (5/2)-power $\mu^k(10101)$.

Repeating the same argument for the factors $\bar{r}_k, \bar{s}_k$ in $\mathbf{u}$ we conclude that $\mathbf{u}$ contains neither $r_k, \bar{r}_k$ simultaneously, nor $s_k, \bar{s}_k$ simultaneously. So $\mathbf{u}$ is not symmetric. Hence every (7/3)-power-free symmetric infinite binary word has subword complexity $p_{\mathbf{t}}(n)$. □

3 Small subword complexity in big languages

In this section we study binary and ternary words. Note the interconnection of the results over Σ_2 and Σ_3 through the encodings of words in both directions. To improve readability, we will denote words over Σ_3 by capital letters and other words by small letters.

Our study follows two related questions about small subword complexity:

- Given a pair (k, α), how small can the subword complexity of an α-power-free infinite k-ary word be?

- Given an integer k and a function $f(n)$, what is the smallest power that can be avoided by an infinite k-ary word with a subword complexity bounded above by $f(n)$?

3.1 Ternary square-free words

Among α-power-free infinite ternary words, the most interesting are the square-free (= 2-power-free) words, the existence of which was established by Thue [45], and in particular, the $(\frac{7}{4})^+$-power-free words, because $\alpha = (\frac{7}{4})^+$ is the minimal power that can be avoided by an infinite ternary word, as shown by Dejean [12].

Consider the ternary Thue word $\mathbf{T}$ [46], which is the fixed point of the morphism θ defined by $0 \to 012, 1 \to 02, 2 \to 1$:

$$\mathbf{T} = T_1T_2T_3\cdots = 012021012102012021020121012021012102012101202102\cdots$$

This word has critical exponent 2, which is not reached, so $\mathbf{T}$ is square-free. Also, $\mathbf{T}$ has two alternative definitions in terms of the Thue-Morse word. The first definition says that for all $i \geq 1$, T_i is the number of zeroes between the i'th and $(i{+}1)$'th occurrences of 1 in $\mathbf{t}$. The second definition is

$$T_i = \begin{cases} 0, & \text{if } t_{i-1}t_i = 01; \\ 1, & \text{if } t_{i-1} = t_i; \\ 2, & \text{if } t_{i-1}t_i = 10. \end{cases} \tag{11}$$

The definition (11) easily implies a bijection between the length-n factors of $\mathbf{t}$ and length-$(n{-}1)$ factors of $\mathbf{T}$ for all $n \geq 3$. Hence, $p_{\mathbf{T}}(n) = p_{\mathbf{t}}(n{+}1)$ for all $n \geq 2$, and one can use formula (2). In [15], the complexity of $\mathbf{T}$ was computed directly from the morphism θ.

Conjecture 19. The ternary Thue word $\mathbf{T}$ has the minimum subword complexity over all square-free ternary infinite words.

The above conjecture is supported by the following result, showing that $\mathbf{T}$ is the only candidate for a square-free word of minimum complexity.

Theorem 20. *If a word* $\mathbf{U}$ *has minimum subword complexity over all square-free ternary infinite words, then* $\mathrm{Fac}(\mathbf{U}) = \zeta(\mathrm{Fac}(\mathbf{T}))$*, where* ζ *is a bijective coding.*

Before proving this theorem and presenting further results, we need to recall an encoding technique introduced in [41] by the second author as a development of a particular case of Pansiot's encoding [28]. In what follows, a, b, c are unspecified pairwise distinct letters from Σ_3. Ternary square-free words contain three-letter factors of the form aba, called *jumps* (of one letter over another). Jumps occur quite often: if a square-free word $\mathbf{u}$ has a jump aba at position i, then the next jump in $\mathbf{u}$ occurs at one of the positions $i{+}2$ ($\mathbf{u} = \cdots abaca \cdots$), $i{+}3$ ($\mathbf{u} = \cdots abacbc \cdots$), or $i{+}4$ ($\mathbf{u} = \cdots abacbab \cdots$). Note that a jump at position $i{+}1$ would mean that $\mathbf{u}$ has the square $abab$ at position i, while no jump up to position $i{+}5$ would lead to the square $bacbac$ at position $i{+}1$. Also note that a jump in a square-free word can be uniquely reconstructed from the previous (or the next) jump and the distance between them. Thus,

($\star$) a square-free ternary word $\mathbf{u}$ can be uniquely reconstructed from the following information: the leftmost jump, its position, the sequence of distances between successive jumps, and, for finite words only, the number of positions after the last jump.

The property ($\star$) allows one to encode square-free words by walks in the weighted $K_{3,3}$ graph shown in Fig. 1. The weight of an edge is the number of positions between the positions of two successive jumps. A square-free word $\mathbf{u}$ is represented by the walk visiting the vertices in the order in which jumps occur when reading $\mathbf{u}$ left to right. If the leftmost jump occurs in $\mathbf{u}$ at position $i > 1$, then we add the edge of length $i{-}1$ to the beginning of the walk; in this case the walk begins at an edge, not a vertex. A symmetric procedure applies to the end of $\mathbf{u}$ if $\mathbf{u}$ is finite. By ($\star$), we can omit the vertices (except for the first one), keeping just the weights of edges and marking the "hanging" edges in the beginning and/or the end. Due to symmetry, we can omit even the first vertex, retaining all information about $\mathbf{u}$ up to renaming the letters. The result is a word over $\{1, 2, 3\}$ with two additional bits of information (whether the first/last letters are marked; for infinite words, only one bit is needed). This word is called a *codewalk* of $\mathbf{u}$ and denoted by $\mathsf{cwk}(\mathbf{u})$. For example, here is some prefix of $\mathbf{T}$ (with first letters of jumps written in boldface) and the corresponding prefix of its codewalk (the marked letter is underlined):

$$\begin{aligned}\mathbf{T} &= 01\mathbf{2}02\mathbf{1}0\mathbf{1}21\mathbf{0}201\mathbf{2}021\mathbf{0}20\mathbf{1}2\mathbf{1}01\mathbf{2}02\mathbf{1}0\mathbf{1}21\mathbf{0}20\mathbf{1}2\mathbf{1}01\mathbf{2}021\mathbf{0}20\ \cdots\\ \mathsf{cwk}(\mathbf{T}) &= \underline{2}\quad 2\quad 1\quad 2\quad 3\quad 3\quad 2\quad 1\quad 2\quad 2\quad 1\quad 2\quad 2\quad 1\quad 2\quad 3\qquad \cdots\end{aligned}$$

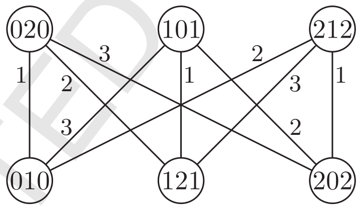


Figure 1: The graph of jumps in ternary square-free words. Vertices are jumps; two jumps that can follow each other in a square-free word are connected by an edge of weight i, where i is the number of positions between the positions of these jumps. Due to symmetry, the graph is undirected.

Codewalks are undefined for square-free words with no jumps (such words have length ≤ 5). Note that two words have the same codewalk if and only if they are images of each other under bijective codings and thus have the same structure, the same subword complexity, and the same properties related to power-freeness. A codewalk is *closed* if it corresponds to a closed walk without hanging edges in $K_{3,3}$; e.g., 212212 is closed and 212 is not.

Clearly, not all walks in the weighted $K_{3,3}$ graph correspond to square-free words. However, there is a strong connection between square-freeness of a word and forbidden factors in its codewalk, as the next lemma shows. (More restrictions can be added to statement 2 of this lemma, but we do not need them in our proofs.)

Lemma 21.

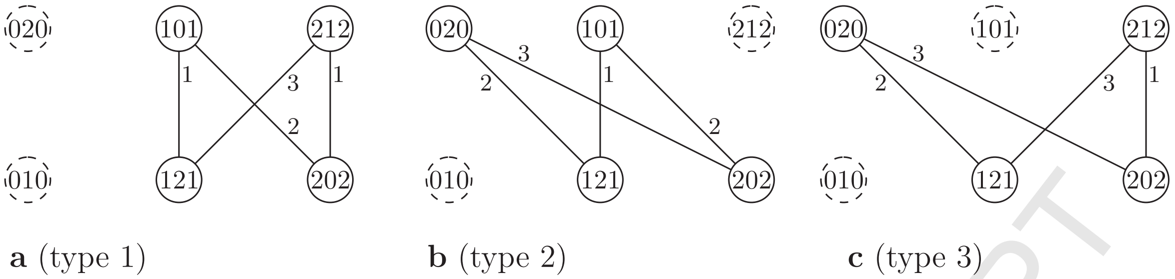
1. *If a codewalk has (a) no factors* $11, 222, 223, 322, 333$*, and (b) no factors of the form* $vxyv$*, where* $v \in \{1,2,3\}^*$*,* $x, y \in \{1,2,3\}$ *and the codewalk* vxy *is closed, then the word with this codewalk is square-free.*
2. *The codewalk of a square-free word contains (a) no proper factors* vxv *such that* $v \in \{1,2,3\}^*$*,* $x \in \{1,2,3\}$*, and* vx *is a closed codewalk; (b) no squares of closed codewalks; (c) no proper factors* 11*, no factors* 223 *except for the prefix, and no factors* 322 *except for the suffix.*

Proof. Statement 1 is proved in [41] as Lemma 4 (plus definitions), so we omit the proof. Statement 2 is close to the results of [41] but has no direct analogs there, so let us prove it.

The words with the codewalks 11, 223, and 322 are *abacaba*, *abacbcabacbab*, *abacbabcacbab*, respectively. Obviously, adding any letter from the left to the first two words, as well as adding any letter from the right to the first and third words, gives a square, so we have (c). For (a) and (b), let vx be a closed codewalk. A word with the codewalk vx looks like $abaUaba$ for some word U, because a closed codewalk begins and ends with the same jump. Then the word with the codewalk $vxvx$ is $W = (abaU)^2aba$, so we have (b). To get the word with the codewalk vxv, we should delete $(x+1)$ last letters of W. So if $x \in \{1,2\}$, then the resulting word contains a square, and if $x = 3$, then this word will contain a square if we add any letter on either side (the letter following or preceding a jump is uniquely determined by this jump). So (a) also holds. □

Proof of Theorem 20. Thue [46] showed that a square-free infinite ternary word contains all six factors of the form ab and all six factors of the form abc. As for the jumps, Thue proved that any two factors *from different parts of the* $K_{3,3}$ *graph in Fig. 1* can be absent. (This is an optimal result: if only one jump from some part is present, the codewalk is the product of factors 11, 22, 33 and then does not encode a square-free word by Lemma 21(2).) All three possible cases (up to symmetry) with two absent jumps are depicted in Fig. 2. We say that a square-free ternary word is of type i, $i \in \{1,2,3\}$, if it lacks two jumps connected by an edge of weight i (in [46], types 1 and 2 are switched). Note that $\mathbf{T}$ has no factors 010 and 212 and thus is of type 2.

Assume that a square-free infinite ternary word $\mathbf{U}$ has the minimum subword complexity among all such words. Observe that $\mathbf{U}$ is recurrent since no suffix of $\mathbf{U}$ has smaller complexity. Then by Lemma 21(2) $\mathsf{cwk}(\mathbf{U})$ has no factors $11, 222, 223, 322, 333$. Further, $\mathbf{U}$ avoids two jumps, for otherwise $p_{\mathbf{U}}(3) > p_{\mathbf{T}}(3)$. W.l.o.g., the two missing jumps are those indicated in Fig. 2, depending on the type of $\mathbf{U}$ (if this is not the case, we replace $\mathbf{U}$ with its image under an appropriate bijective coding). Note that all four remaining jumps occur in $\mathbf{U}$ infinitely often due to recurrence. Hence $\mathbf{U}$ has eight factors of length 4, containing a jump. Let us compute $p_{\mathbf{U}}(4)$. For this, we need to consider the factors without jumps. First, let $\mathbf{U}$ have type 1. Then 1021 and 2012 are factors of $\mathbf{U}$ because they are the only right extensions of 102 and 201, respectively. If the factor 0120 is absent, then it is impossible to move by



a (type 1) **b** (type 2) **c** (type 3)

Figure 2: Avoidance of two jumps in ternary square-free words. "Type" is the weight of the edge between the avoided jumps.

the edge of weight 2 from the vertex 101 to the vertex 202; since $\mathsf{cwk}(\mathbf{U})$ cannot contain 11 or 333, there is a unique extendable closed work, labeled by 1312, from the vertex 101 to itself. Visiting this vertex infinitely often, the codewalk of $\mathbf{U}$ must contain a square of 1312 in contradiction with Lemma 21(2). So the word 0120 must be a factor of $\mathbf{U}$. The same argument works for the factors 0210, 1201, 2102, which are responsible, respectively, for the moves from 202 to 101; from 212 to 121; and from 121 to 212. Thus $p_{\mathbf{U}}(4) = 14$.

If $\mathbf{U}$ has type 3, the argument is similar. Namely, the factors 2012 and 2102 occur in $\mathbf{U}$ as unique extensions of 201 and 210, respectively. The remaining four factors 0120, 0210, 1021, and 1201 are responsible for the moves by the edges of weight 3; if, say, 0120 is absent, one cannot move directly from 020 to 202; hence the only extendable closed walk from 020 to itself is labeled by 2313, because 11, 222, 223, 322, and 333 are forbidden by Lemma 21(2). Hence we again have $p_{\mathbf{U}}(4) = 14$.

The situation changes if $\mathbf{U}$ has type 2: due to recurrence, neither of the words 1021, 1201 occurs in $\mathbf{U}$ (1021 is followed by 0, and thus can be preceded neither by 0 nor by 2; similar for 1201). The remaining factors must present by the same argument as in the previous cases. So $\mathbf{U}$ has type 2 and $p_{\mathbf{U}}(4) = 12$. (Moreover, $\mathbf{T}$ and $\mathbf{U}$ have the same factors up to length 4.) To prove the theorem, it is enough to show that $\mathrm{Fac}(\mathbf{T}) \subseteq \mathrm{Fac}(\mathbf{U})$; the equality then follows by minimality of complexity of $\mathbf{U}$.

Consider the language C of all finite codewalks that can be read in the graph in Fig. 2b and correspond to square-free words. We define two sequences of codewalks by induction:

$$A_0 = 212, B_0 = 3,$$
$$A_{i+1} = B_i B_i A_i A_i A_i, B_{i+1} = B_i B_i A_i.$$

Below we prove the following three statements, which immediately imply the desired inclusion $\mathrm{Fac}(\mathbf{T}) \subseteq \mathrm{Fac}(\mathbf{U})$. We include in the preimage $\mathsf{cwk}^{-1}(X)$ only the words of type 2 avoiding the factors 010 and 212, as in Fig. 2b.

(i) $\mathrm{Fac}(\mathbf{U}) \supseteq \mathsf{cwk}^{-1}(A_i)$ for every $i \geq 0$;

(ii) $\mathsf{ext}(C) = \bigcup_{i\geq 0} \mathrm{Fac}(A_i)$;

(iii) $\mathrm{Fac}(\mathbf{T}) \subseteq \mathrm{Fac}(\mathsf{cwk}^{-1}(\mathsf{ext}(C)))$.

(i) The codewalks A_iA_i, A_iB_i, B_iA_i, and B_iB_i are closed for all i. This fact immediately follows by induction from the definition (for the base case, one may consult Fig. 2b).

We prove by induction that for each i some suffix of $\mathsf{cwk}(\mathbf{U})$ is an infinite product of blocks A_i and B_i. The base case follows from the fact that $\mathsf{cwk}(\mathbf{U})$ has no factors 11, 222, 223, and 322. Hence, as the codewalk reaches one of the vertices 020, 202 (see Fig. 2b) it infinitely proceeds between these two vertices with the paths labeled by A_0 and B_0; each path occurs infinitely many times because $\mathsf{cwk}(\mathbf{U})$ is aperiodic.

Now we proceed with the inductive step. First let $i = 1$. The factors $A_0B_0A_0$ and $B_0B_0B_0$ do not appear in the codewalks of square-free words by Lemma 21,2. Hence, B_0's always occur in pairs. Further, the factor $B_0A_0A_0B_0$ cannot appear far from the beginning of $\mathsf{cwk}(\mathbf{U})$, because otherwise it will be uniquely extended to $A_0B_0B_0A_0A_0B_0B_0A_0$, which is the square of a closed codewalk, impossible by Lemma 21,2. Observing that the factor $A_0A_0A_0A_0$ is also forbidden as the square of a closed codewalk, we see that there may be either one or three consecutive blocks A_0. Hence some suffix of $\mathsf{cwk}(\mathbf{U})$ can be partitioned into the blocks $B_0B_0A_0A_0A_0 = A_1$ and $B_0B_0A_0 = B_1$. As in the base case, both blocks must appear infinitely often to prevent periodicity. For the general case $i > 1$ the argument is essentially the same. The only difference is in proving the fact that $A_iB_iA_i$ and $B_iB_iB_i$ are forbidden: now B_i is a prefix of A_i by construction, so these two codewalks are extended to the right by B_i, which gives us the square of the closed codewalk A_iB_i (resp., B_iB_i). The inductive step is finished.

Thus we know that $\mathsf{cwk}(\mathbf{U})$ contains A_i (and even A_iA_i) for any i. Clearly, A_i is not closed, so it corresponds to a walk from the vertex 020 to 202 or vice versa. Hence $\mathsf{cwk}^{-1}(A_i)$ consists of two words, each one corresponds to a walk in one direction. But A_i's in the factor A_iA_i correspond to walks in opposite directions (the codewalk A_iA_i is closed), so both words from $\mathsf{cwk}^{-1}(A_i)$ are factors of $\mathbf{U}$. Statement (i) is proved.

(ii) As shown in the proof of (i), each A_i occurs infinitely often in $\mathsf{cwk}(\mathbf{U})$. Hence $\mathsf{ext}(C) \supseteq \bigcup_{i\geq 0} \mathrm{Fac}(A_i)$. For the reverse inclusion, first note the following property implied by the proof of (i). There is a function $f(n)$ such that for every $\mathbf{U}$ of type 2 its suffix, equal to the product of the blocks A_n and B_n, starts before the position $f(n)$. Let $V \in \mathsf{ext}(C)$ and let n be such that $|V| \leq |B_n|$. Since C is factorial, a two-sided infinite word, all finite factors of which belong to C, can be built by a standard procedure (start with V; choose $a, b \in \{0, 1, 2\}$ such that $aVb \in \mathsf{ext}(C)$ and replace V with aVb; repeat infinitely and take the limit). Now we take a suffix $\mathbf{W}$ of the constructed word such that $\mathbf{W}$ contains V at a position greater than $f(n)$. (Note that $\mathbf{W} = \mathsf{cwk}(\mathbf{U})$ for a square-free infinite word $\mathbf{U}$ of type 2.) Then V is a factor of one of the codewalks A_nA_n, A_nB_n, B_nA_n, B_nB_n. In each case, V is a factor of A_{n+2}, and we have the desired inclusion $\mathsf{ext}(C) \subseteq \bigcup_{i\geq 0} \mathrm{Fac}(A_i)$.

(iii) Since T is recurrent, its codewalk is recurrent as well, implying $\mathrm{Fac}(\mathsf{cwk}(\mathbf{T})) \subseteq \mathsf{ext}(C)$. The result now follows. □

Remark 22. The encoding (11) is such that if we replace $\mathbf{t}$ with any other word having the language $\mathrm{Fac}(\mathbf{t})$, we get a ternary word with the language $\mathrm{Fac}(\mathbf{T})$. Now Remark 7 and

Theorem 20 imply that the set of ternary infinite square-free words of minimum complexity either is empty or has the cardinality of the continuum.

For symmetric words, the square-free ternary infinite word of minimum subword complexity does exist. Recall that the Fibonacci word $\mathbf{f}$ is the fixed point of the binary morphism defined by $\phi(0) = 01, \phi(1) = 0$. We define the coding $\xi : (0 \to 2, 1 \to 1)$ and write $\mathbf{f}_{12} = \xi(\mathbf{f})$. Now consider the *1-2-bonacci word* $\mathbf{F}_{12} \in \Sigma_3^\omega$, which is the word beginning with 01 and having the codewalk $\mathbf{f}_{12}$. This word was introduced by Petrova [29], who proved that $\mathbf{F}_{12}$ has critical exponent 11/6 (reachable) and no length-5 factors of the form *abcab*. Also, $\mathbf{F}_{12}$ appeared to have a nice extremal property related to square-free partial words [17, Proposition 13].

Theorem 23. *The 1-2-bonacci word* $\mathbf{F}_{12}$ *has the minimum subword complexity over all symmetric square-free ternary infinite words. This complexity equals* $6n - 6$ *for all* $n \geq 2$*.*

Proof. Let $\mathbf{u}$ be a symmetric square-free ternary infinite word. Since $\mathbf{u}$ is aperiodic, it has a special factor of length n for each $n \geq 0$. (In the ternary case, a word v is called a $\mathbf{u}$-special factor if at least two of the words $v0, v1, v2$ are factors of $\mathbf{u}$.) If v is $\mathbf{u}$-special, then the word $\zeta(v)$, where ζ is any bijective coding, is $\mathbf{u}$-special as well, because of the symmetry of $\mathbf{u}$. Thus, there are at least six $\mathbf{u}$-special factors of length n for each $n \geq 2$. Together with the fact $p_{\mathbf{u}}(2) = 6$ mentioned above, this gives the lower bound for the complexity of $\mathbf{u}$: $p_{\mathbf{u}}(n) \geq 6n - 6$ for all $n \geq 2$. So we are going to prove that the 1-2-bonacci word is symmetric and its complexity matches this lower bound.

The codewalk $\mathbf{f}_{12}$ of the 1-2-bonacci word $\mathbf{F}_{12}$ has no 3's and thus corresponds to a walk in the subgraph of the $K_{3,3}$ graph (see Fig. 3). By definition of $\mathbf{F}_{12}$, this walk begins at the vertex 010. Let v be a factor of $\mathbf{f}_{12}$. Let us write $f_i = \xi(\phi^i(0))$ for all $i \geq 0$. Then there is i such that $f_i = uvw$ for some words u and w. Note that f_i occurs in $\mathbf{f}_{12}$ infinitely often, in particular, as a prefix and after each prefix f_{i+k}, where $k > 0$.

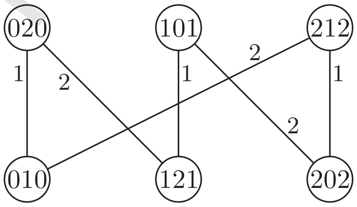


Figure 3: The graph of jumps in the 1-2-bonacci word.

We call two codewalks *equivalent* (and write $u \sim v$) if the corresponding walks, beginning in the same vertex, end in the same vertex. For example, $21221 \sim 2$, because the walk 1221 is closed. Similar to [29], we observe that

$$f_i \sim \begin{cases} 21, & \text{if } i \bmod 6 = 1; \\ 212, & \text{if } i \bmod 6 = 2; \\ 2, & \text{if } i \bmod 6 = 0 \text{ or } 3; \\ 12, & \text{if } i \bmod 6 = 4; \\ 1, & \text{if } i \bmod 6 = 5. \end{cases}$$

We note that the paths from the vertex 010, labeled by 21, 212, 2, 12, and 1, end in all five remaining vertices. Thus, when reading the codewalk of F_{12}, we read f_i starting from each vertex. Reading the word u from all vertices generates a bijection on the set of vertices; thus, we read v from each vertex. Therefore, $\mathbf{F}_{12}$ contains all six factors with the codewalk v. Since v is arbitrary, this means that $\mathbf{F}_{12}$ is symmetric.

Finally we compute the subword complexity of $\mathbf{F}_{12}$. Consider a $\mathbf{F}_{12}$-special factor V such that $|V| \geq 6$ and $Va, Vb \in \mathrm{Fac}(\mathbf{F}_{12})$. The codewalk of V has the form $xv\underline{1}$ for some $v \in \mathrm{Fac}(\mathbf{f}_{12})$, $x \in \{\varepsilon, \underline{1}, \underline{2}\}$. The factors Va and Vb of $\mathbf{F}_{12}$ have codewalks $xv1$ and $xv\underline{2}$. Hence both $v1$ and $v2$ are factors of $\mathbf{f}_{12}$, so v is $\mathbf{f}_{12}$-special. If U is another $\mathbf{F}_{12}$-special factor of length $|V|$, then, similarly, its codewalk is of the form $yu\underline{1}$ for some $\mathbf{f}_{12}$-special factor u and $y \in \{\varepsilon, \underline{1}, \underline{2}\}$. Since $\mathbf{f}_{12}$ is a Sturmian word, it has only one special factor of each length. But every suffix of a special factor is special, so w.l.o.g. u is a suffix of v.

Let $v = v'u$ and assume v' nonempty. The words U and V have suffixes of equal length encoded by $u\underline{1}$. Note that v' encodes at least two letters of V (just two if $v' = 1$), while y encodes at most two letters of U (exactly two if $y = \underline{2}$). But $\mathbf{f}_{12} \in \{1, 2\}^\omega$, so if $y = \underline{2}$ then $2u$ is $\mathbf{f}_{12}$-special and thus a suffix of v. Hence in this case the last letter in v' is 2, implying that v' encodes at least three letters of V. Therefore, $|V| > |U|$ in all cases, contradicting the assumption $v' \neq \varepsilon$. Then $v' = \varepsilon$, $u = v$, and $x = y$. Thus we proved that two $\mathbf{F}_{12}$-special factors of the same length have the same codewalk. Due to symmetry, this means that $\mathbf{F}_{12}$ has exactly six special factors of every length $n \geq 5$. Computing $p_{\mathbf{F}_{12}}(2) = 6$, $p_{\mathbf{F}_{12}}(3) = 12$, $p_{\mathbf{F}_{12}}(4) = 18$, $p_{\mathbf{F}_{12}}(5) = 24$, $p_{\mathbf{F}_{12}}(6) = 30$, we get $p_{\mathbf{F}_{12}}(n) = 6n - 6$ for all $n \geq 2$, as desired. □

Remark 24. As demonstrated in the proof of Theorem 23, the minimal linear growth constant of an aperiodic symmetric infinite ternary word is 6. However, there exist such words with linear growth constant 6 and critical exponent smaller than 11/6. An example of such word is the 1-3-bonacci word $\mathbf{F}_{13}$ obtained similar to the 1-2-bonacci word: take the Fibonacci word $\mathbf{f}$, replace all 0's with 3's to get the codewalk $\mathbf{f}_{13}$ and take the word with this codewalk as $\mathbf{F}_{13}$. The critical exponent of $\mathbf{F}_{13}$ is $\frac{5+\sqrt{5}}{4} \doteq 1.8090\cdots$ [29]; the fact that $\mathbf{F}_{13}$ is symmetric and the equality $p_{F_{13}}(n) = 6n$ for all $n \geq 5$ can be proved as in Theorem 23.

Remark 24 suggests the following question.

Open Question 25. *What is the minimal critical exponent of a symmetric infinite ternary word with linear growth constant 6?*

As to the case of $(\frac{7}{4})^+$-power-free words, our knowledge is quite limited. There are only two well-known such words: the Dejean word [12] and the Arshon word [3]. The former one is a fixed point of a morphism and possesses a partial symmetry: its language is stable under cyclic permutations of the alphabet but not under transpositions; e.g., this language contains only three of six possible factors with codewalks $12, 21, \underline{2}2, 2\underline{2}$. Due to such symmetry, the Dejean word seems to contain too many factors to have a minimal complexity over all $(\frac{7}{4})^+$-power-free words, so we do not study it here. The Arshon word $\mathbf{A}$, proved $(7/4)^+$-power free in [19], can be defined as follows: take the fixed point $0\cdots$ of the morphism $\rho : 0 \to 0\tilde{1}2, 1 \to 1\tilde{2}0, 2 \to 2\tilde{0}1,\ \tilde{0} \to \tilde{2}1\tilde{0}, \tilde{1} \to \tilde{0}2\tilde{1}, \tilde{2} \to \tilde{1}0\tilde{2}$ and apply the coding which deletes all $\sim$'s. The word $\mathbf{A}$ is recurrent by [2, Thm 10.8.6 + Thm 10.9.5] and symmetric (for any $k \geq 1$, having $\rho(a)$ one can obtain $\rho(b)$ applying a cyclic permutation of the alphabet, and $\rho(\tilde{a})$ applying a transposition on the alphabet and switching all $\sim$'s). So we consider $\mathbf{A}$ as a candidate word of minimal complexity over all symmetric $(\frac{7}{4})^+$-power-free words. We compute the subword complexity of $\mathbf{A}$ using the following characterization.

Lemma 26 ([30], Lemma 2)**.** *Let $\eta : 0 \to 010, 1 \to 011$ and $\xi : 0 \to 132, 1 \to 123$ be morphisms and let $\mathbf{h}$ be the fixed point of η. Then $\mathsf{cwk}(\mathbf{A}) = \underline{2}\xi(\mathbf{h})$.*

Theorem 27. *The subword complexity of the Arshon word equals $12n - 12$ for all $n \geq 9$.*

Proof. Let us prove that $p_{\mathbf{h}}(n) = 2n$ for every $n \geq 1$. We show by induction that $\mathrm{Fac}(\mathbf{h})$ has exactly two special words of each length $n \geq 1$. As usual, images of letters are called blocks. The base case $n \leq 2$ is checked by hand. For the induction step, note that all $u \in \mathrm{Fac}(\mathbf{h})$ with $|u| \geq 3$ have a unique factorization of the form $u = v_1\eta(v)v_2$, where v_1 is a proper suffix of a block, v_2 is a proper prefix of a block, and $v \in \mathrm{Fac}(\mathbf{h})$ (one or two of the words v, v_1, v_2 can be empty). If u is special, i.e., $u0, u1 \in \mathrm{Fac}(\mathbf{h})$, then $v_2 = 01$. Consider two cases.

If v_1 is empty, the condition $\eta(v)010, \eta(v)011 \in \mathrm{Fac}(\mathbf{h})$ is equivalent to $v0, v1 \in \mathrm{Fac}(\mathbf{h})$; hence u is special iff v is. The function that maps u to v is a bijection between the set of special factors of length $|u|$ and the set of special factors of smaller length $|v|$, which is two-element by the inductive hypothesis.

In the second case, v_1 is a non-empty suffix of $\eta(a)$, and such a letter a is unique. As above, we see that u is special iff the shorter word av is, and the reference to the inductive hypothesis finishes the proof.

Now turn to the Arshon word. Due to its symmetry, to prove the theorem it suffices to check directly that $p_{\mathbf{A}}(9) = 96$ and to show that for each $n \geq 9$ there are exactly two codewalks of $\mathbf{A}$-special factors of length n (recall that the codewalk determines the length of the word). For $n = 9$, such codewalks $\underline{1}3\underline{1}$ and $\underline{2}1\underline{2}$ can be found by hand; let V be an $\mathbf{A}$-special factor of length ≥ 10. Then $\mathsf{cwk}(V)$ ends with $\underline{1}$ or $\underline{2}$ (if it ends with $1, 2, 3$, or $\underline{3}$, then the letter following V is determined). Further, $\mathsf{cwk}(V)$ contains one of the factors $1, 2\underline{2}, \underline{2}2, 2\underline{3}, \underline{3}2, 23, 32$, which determines its position w.r.t. the blocks of ξ (see Lemma 26). So if $\mathsf{cwk}(V)$ ends with $\underline{1}, 2\underline{2}$, or $3\underline{2}$, then the next letter of V is determined by ξ. Hence $\mathsf{cwk}(V)$ ends with $1\underline{2}$ and can be parsed as $\mathsf{cwk}(V) = x'x1\underline{2}$, where x is a (possibly empty) product of ξ-blocks and x' is a proper suffix of a ξ-block (the first letter of x' can be underlined).

Next we observe that in the case $x' \in \{\varepsilon, \underline{1}, \underline{2}\}$ V is **A**-special iff $\xi^{-1}(x)$ is **h**-special; in the remaining case x' determines a ξ-block $\xi(a)$ and then V is **A**-special iff $a\xi^{-1}(x)$ is **h**-special. We call the obtained **h**-special word the *approximate ξ-preimage* of $\mathsf{cwk}(V)$. Note that if W is an **A**-special factor of length $|V|$, then the approximate ξ-preimages of $\mathsf{cwk}(V)$ and $\mathsf{cwk}(W)$ have equal positive length.

Now take an **h**-special word u and consider the (**A**-special) word U such that $\mathsf{cwk}(U) = \underline{2}\xi(u)1\underline{2}$. We observe that $|U| = 9(|u|+1)$ and u is the approximate ξ-preimage of $\mathsf{cwk}(U')$ for every suffix U' of U of length $> 9|u|$, including U itself. In total, we have proved that the approximate ξ-preimage defines a bijection between codewalks of **A**-special factors of any given length $n \geq 10$ and **h**-special factors of some other fixed length; so there are exactly two such codewalks, as required. □

If the answer to Open Question 25 is greater than 7/4, which looks plausible, then the Arshon word will be a candidate for a symmetric $(7/4)^+$-power-free word of minimal/minimum complexity. However, the growth constants between 6 and 12 cannot be excluded from consideration: note that the growth of the Thue-Morse word is strictly between the corresponding growth constants 2 and 4 for binary words.

Open Question 28. *Is there a symmetric $(\frac{7}{4})^+$-power-free infinite ternary word of minimum subword complexity? If yes, is the Arshon word an example of such word?*

Open Question 29. *What is the minimal linear growth constant of a $(\frac{7}{4})^+$-power-free infinite ternary word (in the general case and in the symmetric case)?*

3.2 Other ternary words

When squares are allowed in ternary words, we can build words of smaller complexity. Here we consider the special case of words with the complexity upper bounded by the function $2n+1$. Let

$$\mathbf{G} = 012020102012010201202\cdots \tag{12}$$

be the fixed point of the morphism γ defined by $0 \to 01, 1 \to 2, 2 \to 02$. The word **G** is a recoding of the sequence A287104 from Sloane's *Encyclopedia.*

Lemma 30. $p_{\mathbf{G}}(n) = 2n+1$ *for all* $n \geq 0$.

Proof. We use standard techniques (see, e.g., [8]), so we try to keep the proof short. Observe that $p_{\mathbf{G}}(2) = 5$: the factors 00, 11, 21, and 22 are forbidden. It is sufficient to prove that there are exactly two **G**-special words of length n for all $n \geq 1$. For $n \in \{1,2\}$ one can check that there is a unique special word ending in 0 and a unique such word ending in 1 (a **G**-special word cannot end in 2 because $21, 22 \notin \mathrm{Fac}(\mathbf{G})$). Let $V0$ be special; then $V01, V02 \in \mathrm{Fac}(\mathbf{G})$, implying that $\gamma(V)012, \gamma(V)0102 \in \mathrm{Fac}(\mathbf{G})$ and thus $\gamma(V)01$ is special. Similarly, if $U1$ is special, then $U10, U12 \in \mathrm{Fac}(\mathbf{G})$; $\gamma(U)201, \gamma(V)202 \in \mathrm{Fac}(\mathbf{G})$ and thus $\gamma(U)20$ is special. Since each suffix of a special word is special, there are special words of every length ending in 0 and in 1. Now assume that the lemma is false; then for some n one has $D_{\mathbf{G}}(n) > 2$, $D_{\mathbf{G}}(1) = \cdots = D_{\mathbf{G}}(n-1) = 2$.

Some case analysis is needed; all cases are similar, so we consider one of them. Assume that two special words of length n end with 0. Since their suffixes are special, and only one special word of length $n-1$ ends with 0, these two words are $aV0$ and $bV0$, where $a, b \in \Sigma_3$. Let $a = 0, b = 1$ (the other case is $a = 1, b = 2$). Then we can write $V = 2V'$. We have

$$\begin{aligned} 02V'01,\ 02V'02,\ 12V'01,\ 12V'02 &\in \mathrm{Fac}(\mathbf{G}) \quad \text{and hence} \\ 2\gamma^{-1}(V')0,\ 2\gamma^{-1}(V')2,\ 1\gamma^{-1}(V')0,\ 1\gamma^{-1}(V')2 &\in \mathrm{Fac}(\mathbf{G}) \end{aligned}$$

Then $2\gamma^{-1}(V'), 1\gamma^{-1}(V')$ are two special words of the same length $< n$, ending with the same letter 1; this is impossible by the choice of n. Studying all cases in the same way, we reach the same contradiction. Thus the lemma holds. □

Theorem 31. *The critical exponent of the word* $\mathbf{G}$ *is* $2 + \frac{1}{\lambda^2-1} = 2.4808627\cdots$, *where* $\lambda = 1.7548777\cdots$ *is the real zero of the polynomial* $x^3 - 2x^2 + x - 1$.

Proof. The critical exponent of $\mathbf{G}$ can be computed by Krieger's method [23]. We recall the necessary tools suitable for analyzing $\mathbf{G}$ specifically, rather than in full generality.

For a word $w \in \Sigma_k^*$, we let $|w|_a$ denote the number of occurrences of the letter a in w. The *Parikh vector* of w is the vector $\vec{P}(w) = (|w|_0, \ldots, |w|_{k-1})$. By *norm* of a vector we mean the sum of its coordinates; so $\|\vec{P}(w)\| = |w|$. If w is a prefix of x^ω for some word x, we say that w has *period* $|x|$. In this case, all factors of w of length $|x|$ share the same Parikh vector $\vec{P}(x)$, so we can speak about "Parikh vector of the period". If $|x|$ is the minimal period of w, we call x the *root* of w. The exponent of w then can be written as

$$\exp(w) = \frac{|w|}{|x|} = \frac{\|\vec{P}(w)\|}{\|\vec{P}(x)\|}.$$

The matrix A_f of a morphism $f : \Sigma_k^* \to \Sigma_m^*$ is a nonnegative integer $k \times m$ matrix, the i'th row of which is the Parikh vector of $f(i-1)$, where $i = 1, \ldots, k$. For example, the morphism γ has the matrix

$$A = A_\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

One has $\vec{P}(f(w)) = \vec{P}(w) \cdot A_f$. Note that the characteristic polynomial of A is $x^3 - 2x^2 + x - 1$, so the maximal (and unique) real eigenvalue of A is λ.

A *run* in a finite or infinite word w is an occurrence of a factor v of w such that (a) $\exp(v) \geq 2$ and (b) this occurrence cannot be extended in w to a longer factor with the same minimal period. For example, the word $\mathbf{G}$ has run 2020 at position 2 with period 2, run 02010201 at position 3 with period 4, and run 2010201201020120 at position 4 with period 7; see (12). For $\mathbf{G}$, as for every word containing squares, the critical exponent equals the supremum of exponents of its runs. Since $\mathbf{G}$ is a fixed point of a morphism, its runs can be grouped into infinite series in the following way:

— take a run V at position i with root X (let $\mathbf{G} = UV\cdots$, $|U| = i$);

— take the occurrence of $\gamma(V)$ at position $|\gamma(U)|$ and extend it to a run with period $|\gamma(X)|$;

— take the obtained run as V and repeat.

For example, the runs mentioned above form the beginning of a series:

$$\begin{array}{llllll}
2020 & \to & 02010201 & \to & \mathbf{2}01020120102012\mathbf{0} & \to \cdots \\
\text{at } 2 & & \text{at } 3 & & \text{at } 4 & \\
U = 01 & & \gamma(U) = 012 & & \gamma^2(U) = 01202 & \\
X = 20 & & \gamma(X) = 0201 & & \gamma^2(X) = 0102012 & \\
 & & \text{no extensions} & & \text{extended left by 2, right by 0} &
\end{array}$$

In the case of $\mathbf{G}$, the left (resp., right) extension is a longest common suffix (resp., prefix) of γ-images of corresponding letters. Thus, the left extension is either 2 (the common suffix of $\gamma(1)$ and $\gamma(2)$) or ε, and the right extension is either 0 (the common prefix of $\gamma(0)$ and $\gamma(2)$) or ε. Empty and non-empty extensions alternate on the left as well as on the right. Note that, for a run V, the first letter of X, the last letter of X, and the last letter of U are all distinct because $\mathbf{G}$ contains no squares of letters; in addition, $X[1]$ is preceded in $\mathbf{G}$ by two other letters, so $X[1] \neq 1$ because $\mathbf{G}$ contains no factor 21. This gives us the following picture of left extensions in a series of runs (X_m is the root of the m'th run in the series):

$$\xrightarrow{\gamma} \cdots 1\underbrace{0\cdots 2}_{X_m}\cdots \xrightarrow{\gamma} \cdots 1\underbrace{\mathbf{2}\cdots 0}_{X_{m+1}}\cdots \xrightarrow{\gamma} \cdots 2\underbrace{0\cdots 1}_{X_{m+2}}\cdots \xrightarrow{\gamma} \cdots 0\underbrace{\mathbf{2}\cdots 1}_{X_{m+3}}\cdots \xrightarrow{\gamma} \cdots 1\underbrace{0\cdots 2}_{X_{m+4}}\cdots \xrightarrow{\gamma}$$

The boldface 2's are non-empty left extensions; so empty and non-empty left extensions alternate. The picture for the right extensions is similar (X'_m is the length-$|X|$ suffix of the m'th run in the series, the boldface 0's are right extensions):

$$\xrightarrow{\gamma} \cdots\underbrace{2\cdots 1}_{X'_m}0\cdots \xrightarrow{\gamma} \cdots\underbrace{2\cdots \mathbf{0}}_{X'_{m+1}}1\cdots \xrightarrow{\gamma} \cdots\underbrace{0\cdots 1}_{X'_{m+2}}2\cdots \xrightarrow{\gamma} \cdots\underbrace{1\cdots \mathbf{0}}_{X'_{m+3}}2 \xrightarrow{\gamma} \cdots\underbrace{2\cdots 1}_{X'_{m+4}}0\cdots \xrightarrow{\gamma}$$

It is possible to compute the exponents of a run in a series in a uniform way. Namely, if one has a series $\{V_m\}_0^\infty$ such that V_0 has root X, then the exponent of each V_m can be computed by the following formula:

$$\exp(V_m) = \frac{\|\vec{P}(V_0)\cdot A^m + \sum_{i=1}^{m}\vec{P}_i \cdot A^{m-i}\|}{\|\vec{P}(X)\cdot A^m\|}\ , \tag{13}$$

where $\vec{P}_i$ is the sum of the Parikh vectors of the extensions which were added to $\gamma(V_{i-1})$ to get V_i. Consider the series introduced above, with $V_0 = 2020$. From the above description of extensions we have $\vec{P}_i = (0,0,0)$ for odd i and $\vec{P}_i = (1,0,1)$ for even i. Observing that $\vec{P}(X) = (1,0,1)$, $\vec{P}(V_0) = 2\vec{P}(x)$, we simplify (13) to get

$$\exp(V_m) = 2 + \frac{\|(1,0,1)\cdot\big(A^{m-2} + A^{m-4} + \cdots + A^{m-2\lfloor m/2\rfloor}\big)\|}{\|(1,0,1)\cdot A^m\|}\,. \tag{14}$$

In the same way, we consider the series $\{V'_m\}_0^\infty$ starting with $V'_0 = 201201$ at position 8. For this run, $\vec{P}_i = (1,0,0)$ for odd i and $\vec{P}_i = (0,0,1)$ for even i. Thus we get an analog of (14):

$$\begin{aligned}\exp(V'_m) &= 2 + \frac{\|(1,0,0)\cdot\big(A^{m-1} + A^{m-3} + \cdots\big) + (0,0,1)\cdot\big(A^{m-2} + A^{m-4} + \cdots\big)\|}{\|(1,1,1)\cdot A^m\|}\\ &= 2 + \frac{\|(1,1,1)\cdot\big(A^{m-2} + A^{m-4} + \cdots + A^{m-2\lfloor m/2\rfloor}\big)\| + [m \text{ is odd}]}{\|(1,1,1)\cdot A^m\|}\,. \end{aligned}\tag{15}$$

(The last equality stems from observing that $(1,0,0)A + (0,0,1) = (1,1,1)$; We use the Iverson bracket [..] to convert a boolean value into an integer.)

Further, assume that some other series $\{V''_m\}_0^\infty$ exists. Then V''_0 appeared through extension of some word $\gamma(ZaZ)$ on at least one side (given that $\mathbf{G}$ has no factor 11, a factor of the form $\gamma(ZabZ)$ needs at least three more letters to become a square). From the above description of extensions it is easy to check that either $ZaZ = 0\cdots120\cdots1$ and then $V''_0 = (201\cdots20)^2$ (the case $Z = \varepsilon$ leads to $V_0 = (20)^2$), or $ZaZ = 0\cdots010\cdots0$ and then $V''_0 = (201\cdots01)^2$ (the case $Z = 0$ leads to $V'_0 = (201)^2$); in particular, $V''_0 = X''_0X''_0$. Computing (13) for V''_m, we obtain the numerator as in (14) in the former case and as in (15) in the latter one. On the other hand, it can be checked by hand that the shortest valid option for the word ZaZ is 020120201, implying $\vec{P}(X''_0) \geq (4,2,3)$; so the denominator will be much bigger than in (14), (15). Hence $\exp(V''_m) < \min\{\exp(V_m), \exp(V'_m)\}$, so we can exclude from consideration all series of runs, except for $\{V_m\}_0^\infty$ and $\{V'_m\}_0^\infty$.

Let us compute the limit $\lim_{m\to\infty}\exp(V_m)$ using standard machinery of Perron–Frobenius theory; for details consult, e.g., [16, Ch. 13]. We describe the idea, omitting the plain calculus. For large m, the vector $(1,0,1)\cdot A^m$ is very close to the eigenvector of the matrix A corresponding to its maximal eigenvalue λ; so the multiplication of this vector by A corresponds, up to a small error, to its multiplication by λ. Next note that if we multiply the matrix in the numerator of (14) by $(A^2 - I)$, where I is the identity matrix, we obtain either $A^m - I$ or $A^m - A$, depending on the parity of m. Hence, as $m \to \infty$, the numerator multiplied by $(\lambda^2 - 1)$ approaches the denominator. Therefore,

$$\lim_{m\to\infty}\exp(V_m) = 2 + \frac{1}{\lambda^2 - 1},$$

as in the statement of the theorem. The same argument applied to (15) leads to the same limit for $\{\exp(V'_m)\}$. Thus it remains to show that for each m the exponents $\exp(V_m)$, $\exp(V'_m)$ are below this limit. This fact is computationally obvious (both sequences $\{\exp(V_m)\}$ and

$\{\exp(V'_m)\}$ monotonically increase up to the limits of precision of floating-point arithmetic), but its formal proof is rather tedious. For the sake of completeness, we provide it below.

We use (14), (15) to write, for $\vec{x} \in \{(1,0,1),(1,1,1)\}$,

$$C_m = \frac{X_m}{Y_m} = \frac{\|\vec{x}\cdot\left(A^{m-2}+A^{m-4}+\cdots+A^{m-2\lfloor m/2\rfloor}\right)\| + [m \text{ is odd}]\cdot[\vec{x}=(1,1,1)]}{\|\vec{x}\cdot A^m\|}. \tag{16}$$

We have to show that $C_m < \frac{1}{\lambda^2-1}$. We check this by hand for $m \le 2$ and assume $m \ge 3$ below. First we note that the two eigenvalues of A, except for λ, are conjugate complex numbers ν and ν^*; by Vieta's formulas, the product of all eigenvalues of A equals 1, so $|\nu| = 1/\sqrt{\lambda}$. Let us represent an arbitrary nonnegative vector $\vec{x} \ne \vec{0}$ in the basis of eigenvectors: $\vec{x} = x_1\vec{z}_\lambda + x_2\vec{z}_\nu + x_3\vec{z}_{\nu^*}$ ($\vec{z}_\lambda$ is a positive vector by Perron–Frobenius theorem [16, Ch. 13, Thm 2] and x_1 is positive). We define $D_{\vec{x}} = x_1\cdot\|\vec{z}_\lambda\|$ and $E_{\vec{x}} = |x_2|\cdot\|[\vec{z}_\nu]\| + |x_3|\cdot\|[\vec{z}_{\nu^*}]\|$; here $[\vec{z}]$ denotes the vector having components which are absolute values of the components of $\vec{z}$. Then $|\|\vec{x}\| - D_{\vec{x}}| < E_{\vec{x}}$. Furthermore one has, for any $m \ge 0$,

$$|\|\vec{x}\cdot A^m\| - \lambda^m\cdot D_{\vec{x}}| = |\lambda^m x_1\cdot\|\vec{z}_\lambda\| + \|\nu^m x_2\vec{z}_\nu + \nu^{*m}x_3\vec{z}_{\nu^*}\| - \lambda^m\cdot D_{\vec{x}}| < \lambda^{-m/2}\cdot E_{\vec{x}}, \tag{17}$$

thus getting a two-sided bound for Y_m. Next we show by induction that

$$X_m < \frac{D_{\vec{x}}}{\lambda^2-1}\cdot\lambda^m - \frac{\lambda E_{\vec{x}}}{\lambda-1}\cdot\lambda^{-m/2} \tag{18}$$

for every $m \ge 3$. The base cases $m = 3$ and $m = 4$ are checked directly; the approximate numerical values are as follows:

$$\begin{aligned}
\vec{z}_\lambda &\approx (1.3247, 0.7549, 1)\\
\vec{z}_\nu &\approx (-0.6624 - 0.5623i, -0.8774 + 0.7449i, 1)\\
\vec{z}_{\nu^*} &\approx (-0.6624 + 0.5623i, -0.8774 - 0.7449i, 1)
\end{aligned}$$

$\vec{x}$	$D_{\vec{x}}$	$E_{\vec{x}}$	(18), $m = 3$	(18), $m = 4$
$(1,0,1) \approx 0.7221\vec{z}_\lambda + (0.1389 + 0.2023i)\vec{z}_\nu + (0.1389 - 0.2023i)\vec{z}_{\nu^*}$	2.2239	1.4820	$4 < 4.2972$	$9 < 9.0231$
$(1,1,1) \approx 0.9566\vec{z}_\lambda + (0.0217 - 0.2121i)\vec{z}_\nu + (0.0217 + 0.2121i)\vec{z}_{\nu^*}$	2.9460	1.2876	$6 < 6.3682$	$12 < 12.463$

For the step case we note that $X_{m+2} = X_m + Y_m$ by (16) and use the upper bound for Y_m:

$$\begin{aligned}
X_{m+2} = X_m + Y_m &< \frac{D_{\vec{x}}}{\lambda^2-1}\cdot\lambda^m - \frac{\lambda E_{\vec{x}}}{\lambda-1}\cdot\lambda^{-m/2} + \lambda^m\cdot D_{\vec{x}} + \lambda^{-m/2}\cdot E_{\vec{x}}\\
&= \frac{D_{\vec{x}}}{\lambda^2-1}\cdot\lambda^{m+2} - \frac{\lambda E_{\vec{x}}}{\lambda-1}\cdot\lambda^{-(m+2)/2}
\end{aligned}$$

Finally, using the bounds (17), (18) we compute

$$C_m = \frac{X_m}{Y_m} < \frac{\dfrac{D_{\vec{x}}}{\lambda^2-1}\cdot\lambda^m - \dfrac{(\lambda^2+\lambda)E_{\vec{x}}}{\lambda^2-1}\cdot\lambda^{-m/2}}{\lambda^m\cdot D_{\vec{x}} - \lambda^{-m/2}\cdot E_{\vec{x}}} < \frac{1}{\lambda^2-1} - \frac{(\lambda^2+\lambda-1)E_{\vec{x}}}{(\lambda^2-1)D_{\vec{x}}}\cdot\lambda^{-3m/2},$$

implying $C_m < \frac{1}{\lambda^2-1}$ and thus $\exp(V_m), \exp(V'_m) < 2 + \frac{1}{\lambda^2-1}$ for all m. □

Conjecture 32.

1. The minimum subword complexity of a ternary $\left(\frac{5}{2}\right)^+$-power-free infinite word is $2n+1$;
2. Among all ternary infinite words with subword complexity bounded above by $2n+1$, the word $\mathbf{G}$ has the lowest possible critical exponent.

3.3 Binary words

When we switch from $(\frac{7}{3})$-power-free to $(\frac{7}{3})^+$-power-free infinite binary words, the Thue–Morse word loses its status as the word of minimum complexity. Let us give an example of a $(\frac{7}{3})^+$-power-free infinite word with subword complexity incomparable to $p_{\mathbf{t}}(n)$. Consider the morphism $g : \Sigma_3 \to \Sigma_2$ defined by the rules

$$\begin{aligned}0 &\to 01100100110\,1001\,0110\,1001\,1001,\\ 1 &\to 01100100110\,1001\,0110\,0110\,1001,\\ 2 &\to 01100100110\,1001\,1001\,0110\,1001.\end{aligned}$$

It maps square-free ternary words to $(\frac{7}{3})^+$-power-free binary words (see [37, Section 3]).

Theorem 33. *Let $\mathbf{g} = g(\mathbf{T})$, where the morphism g and the ternary Thue word $\mathbf{T}$ are defined above. Then $p_{\mathbf{g}}(n) < p_{\mathbf{t}}(n)$ for infinitely many values of n.*

Proof. Let $m = 2^k$ for some $k \geq 2$ and compare the number of factors of length $n = 27m - 7$ in $\mathbf{g}$ and $\mathbf{t}$. Note that images of letters under g differ only by the factor of length 8 at position 15. Hence a length-n factor of $\mathbf{g}$ contains exactly m such "identifying" factors (one of them, possibly, only partially). Thus every length-n factor of $\mathbf{g}$ is uniquely identified by its g-preimage of length m and its initial position inside the g-image of a letter. So $p_{\mathbf{g}}(n) \leq 27 \cdot p_{\mathbf{T}}(m) = 27 \cdot 3m = 81 \cdot 2^k$. By (2) we have $p_{\mathbf{t}}(n) = 2(n-1) + 2^{k+5} = 86 \cdot 2^k - 16$. Since $5 \cdot 2^k > 16$, we obtain $p_{\mathbf{g}}(n) < p_{\mathbf{t}}(n)$. □

Open Question 34. *What is the minimum value of α such that some α-power-free infinite binary word $\mathbf{u}$ has smaller complexity than the Thue-Morse word? (Recall that "smaller" means $p_{\mathbf{u}}(n) \leq p_{\mathbf{t}}(n)$ for all n and $p_{\mathbf{u}}(n) < p_{\mathbf{t}}(n)$ for some n.)*

It is known [7] that the critical exponent of a Sturmian word is at least $(5+\sqrt{5})/2 \doteq 3.61803\cdots$, and the minimum is reached by the Fibonacci word $\mathbf{f}$ defined above. This result can be slightly extended.

Theorem 35. *For all integer constants $c > 1$, the minimum critical exponent of an infinite binary word of subword complexity $n + c$ is $(5+\sqrt{5})/2$.*

Proof. Infinite binary words of subword complexity $n+c$ were characterized by Cassaigne [9, Proposition 8] as having the form $uf(\mathbf{v})$, where u is a finite word, $\mathbf{v}$ is a Sturmian word and f is a morphism. Clearly, for any word z one has $\exp(f(z)) \geq \lfloor \exp(z) \rfloor$. So it is enough to consider the case when $\mathbf{v}$ is 4-power-free. Fortunately, there are very few Sturmian words with such small critical exponents.

Recall some necessary facts about Sturmian words. Every Sturmian word $\mathbf{v}$ is *balanced*: for some irrational constant $\beta_{\mathbf{v}}$ and all $n > 0$, every factor of $\mathbf{v}$ of length n contains either $\lfloor \beta_{\mathbf{v}} n \rfloor$ or $\lceil \beta_{\mathbf{v}} n \rceil$ zeroes. An infinite *standard* word beginning with 0 is defined by a positive integer *directive* sequence $\{d_n\}$ as the limit of the sequence $\{s_n\}$ of finite words:

$$s_{-1} = 1,\ s_0 = 0,\ s_n = (s_{n-1})^{d_n} s_{n-2} \text{ for } n \geq 1. \tag{19}$$

Every Sturmian word has the same language as some standard word or its complement, so we can assume $\mathbf{v}$ to be standard. Theorem 4 of [7] implies that a standard word having $d_n > 1$ in its directive sequence for at least one large enough value of n, is not 4-power-free. (A closer analysis following the lines of [7, Prop. 15] shows that any $n \geq 3$ works.) The standard word satisfying $d_n = 1$ for all n is the Fibonacci word $\mathbf{f}$, and any standard word s satisfying $d_n = 1$ for all $n > n_0$ is, according to (19), the image of $\mathbf{f}$ under the morphism $0 \to s_{n_0}, 1 \to s_{n_0-1}$. Hence it is enough to estimate subword complexity for the words having the form $uf(\mathbf{f})$. Consider a sequence of runs in $\mathbf{f}$ with exponents converging to $(5+\sqrt{5})/2$. Since the limit is irrational, the lengths of these runs approach infinity. Due to the balance property, the exponents of morphic images of the runs tend to the same limit. Hence the critical exponent of $uf(\mathbf{f})$ is at least $(5+\sqrt{5})/2$. □

So, for the critical exponents smaller than $(5+\sqrt{5})/2$ we look at the infinite words with linear growth constant bigger than 1; the next natural candidate is 2. As Theorem 37 below shows, such words can have the critical exponent as small as 5/2. This gives an upper bound for the value of α in Open Question 34. Note that the gap between 5/2 and $(5+\sqrt{5})/2$ is quite big, so the words with the linear growth constant between 1 and 2 also can play a nontrivial role.

Remark 36. By backtracking, one can prove that the longest binary words avoiding 5/2-powers and with subword complexity $\leq 2n$ are of length 38. They are

$$00110011010011001001101001100100110010,$$
$$00110011010011001001101001100100110011,$$

and their reversals and complements.

Theorem 37. *Let $\tau : \Sigma_3 \to \Sigma_2$ be the morphism defined by $0 \to 0,\ 1 \to 01,\ 2 \to 011$ and $\mathbf{G}$ be the word defined in Section 3.2. Then*

$$\tau(\mathbf{G}) = 0010110011001001100101100100110010110011001\cdots$$

has the lowest critical exponent among all binary words with subword complexity $\leq 2n$. It (i) avoids $(\frac{5}{2})^+$-powers and (ii) has subword complexity exactly $2n$ for all $n > 0$.

Proof. In the proof we refer to the properties of the word $\mathbf{G}$ (Lemma 30 and Theorem 31) and their proofs. Recall that $\mathbf{G}$ has two special factors of each positive length n: one ends in 0 and the other ends in 1. If $V0$ (resp., $V1$) is $\mathbf{G}$-special, then $\tau(V)01$ (resp., $\tau(V)0$) is $\tau(\mathbf{G})$-special; since suffixes of special factors are special, $\tau(\mathbf{G})$ has, for each $n \geq 1$, length-n special factors ending in 0 and in 1. For (ii), we check by hand that for $n \leq 3$ the word $\tau(\mathbf{G})$ has just two special words. As in the proof of Lemma 30, we then assume that statement (ii) is false, choose n such that $D_{\tau(\mathbf{G})}(n) > 2$, $D_{\tau(\mathbf{G})}(n-1) = \cdots = D_{\tau(\mathbf{G})}(1) = 2$, and do case analysis. There are two $\tau(\mathbf{G})$-special length-n words ending with the same letter; these words have common suffix of length $n-1$ by the choice of n. Consider the case where this suffix equals $0v0$ for some $v \in \mathrm{Fac}(\tau(\mathbf{G}))$. From the condition

$$00v00, 00v01, 10v00, 10v01 \in \mathrm{Fac}(\tau(\mathbf{G}))$$

we conclude that $0v = \tau(V)$ for some $V \in \mathrm{Fac}(\mathbf{G})$, and all words $0V0, 0V2, 1V0, 1V2$ are factors of $\mathbf{G}$ (the second conclusion uses the fact that $\mathbf{G}$ has no factor 21). Then both $0V$ and $1V$ are $\mathbf{G}$-special, contradicting the fact that $\mathbf{G}$ has at most one special word of a given length ending with a given letter. The cases where the common suffix has the form $0v1, 1v0$, or $1v1$ are similar and imply the same contradiction. Hence $D_{\tau(\mathbf{G})}(n) = 2$ for all $n \geq 1$, and then $p_{\tau(\mathbf{G})}(n) = 2n$.

Let us prove (i). Since $\mathbf{G}$ has no factors 00, 11, 22, 21, 101, and 0202, the word $\tau(\mathbf{G})$ has no factors 000, 1010, 1101, 010010, and 00110011; it also has no factor 111 by definition. Hence the runs in $\tau(\mathbf{G})$ with periods ≤ 4 are 00, 11, 0101, 1001001, and 0110011001; consider the last one. It is a $(\frac{5}{2})$-power obtained by extending the image of the run V_0 in $\mathbf{G}$ by the common prefix of $\tau(1)$ and $\tau(2)$: $0110011001 = \tau(20201)$. From this point, we consider runs of periods ≥ 5. By definition of τ, every $u \in \mathrm{Fac}(\tau(\mathbf{G}))$ with $|u| \geq 5$ can be uniquely decomposed as $u = u_1\tau(V)u_2$, where $u_1 \in \{\varepsilon, 1, 11\}$, $u_2 \in \{\varepsilon, 0, 01\}$, and $u_2 = \varepsilon$ iff V ends with 2. If in addition u is a run, then $u_1 \neq 11$ and $u_2 \neq \varepsilon$ by the non-extendability condition. So if V has prefix X and the period $|X|$, then the period $|\tau(X)|$ is extended in u by 1 or 2 letters to the right and by 0 or 1 letters to the left. Thus if V is not a run, then $\exp(u) < \frac{5}{2}$. So we assume that V is a run, i.e., $V = W_m$ for some series $\{W_m\}_0^\infty$ of runs in $\mathbf{G}$. According to the analysis given in Theorem 31, if $\{W_m\}_0^\infty$ differs from $\{V_m\}_0^\infty$, $\{V'_m\}_0^\infty$ and $V = L\gamma^m(W_0)R$ for some $L, R \in \Sigma_3^*$, then either $V'_m = L\gamma^m(V'_0)R$ or $V_m = L\gamma^m(V_0)R$, and both V_0, V'_0 are subsequences of W_0. Hence $\exp(\tau(V)) < \min\{\exp(\tau(V_m)), \exp(\tau(V'_m))\}$ and the factor $\tau(V)$ has exactly the same period-preserving extension in $\tau(\mathbf{G})$ as either $\tau(V_m)$ or $\tau(V'_m)$. Since we are interested in maximum exponents, we can further assume $V = V_m$ or $V = V'_m$ for some m.

It is easy to see that $\|\vec{P}(x)\|$ equals the inner product of $\vec{P}(x)$ by the vector $(1, 1, 1)$ (we count each letter once for computing length). Now observe that $\|\vec{P}(\tau(x))\|$ equals the inner product of $\vec{P}(x)$ by the vector $(1, 2, 3)$ (computing the length of the τ-image of a word, we count 0's, 1's, and 2's in this word once, twice, and thrice, respectively). If v_m, v'_m are the runs in $\tau(\mathbf{G})$ obtained from the τ-images of the runs V_m and V'_m respectively, we observe that $\tau(V_m)$ is always extended by 2 letters, while $\tau(V'_m)$ is extended by 1 or 3 letters depending on the parity of m. So we get the following analogs of (14), (15) (the inner product is denoted by $\langle\cdot,\cdot\rangle$):

$$\exp(v_m) = 2 + \frac{\big\langle (1,0,1)\cdot\big(A^{m-2}+A^{m-4}+\cdots+A^{m-2\lfloor m/2\rfloor}\big),\ (1,2,3)\big\rangle + 2}{\big\langle (1,0,1)\cdot A^m,\ (1,2,3)\big\rangle}\ ,$$

$$\exp(v'_m) = 2 + \frac{\big\langle (1,1,1)\cdot\big(A^{m-2}+A^{m-4}+\cdots+A^{m-2\lfloor m/2\rfloor}\big),\ (1,2,3)\big\rangle + 1 + 3\cdot[m \text{ is odd}]}{\big\langle (1,1,1)\cdot A^m,\ (1,2,3)\big\rangle}\ .$$

Now we define X_m, Y_m, and C_m similar to (16) and proceed as in the proof of Theorem 31, replacing all norms with the inner products by the vector $(1,2,3)$. Namely, we define and compute $D_{\vec{x}}$ and $E_{\vec{x}}$, get the analog of (17) and prove the analog of (18) with the base cases $m=3, m=4$ for $\vec{x}=(1,1,1)$ and $m=3, m=6$ for $\vec{x}=(1,0,1)$. Using (18) for big values of m and checking the small values up to the base cases manually, we show that $C_m < \frac{1}{\lambda^2-1}$ for all m except for $m=0$, $\vec{x}=(1,0,1)$ (this is the exclusive case $\exp(v_0)=\frac{5}{2}$ mentioned above).

So we conclude that $\tau(\mathbf{G})$ has critical exponent $5/2$, as required, and moreover this exponent is reached solely by the factor 0110011001. □

4 Large subword complexity in big languages

If a language $L_{k,\alpha}$ has an exponential growth function, then it seems quite natural that there would be infinite α-power-free words over Σ_k having exponential subword complexity. For example, Currie and Rampersad [11, Prop. 9] gave an example of a square-free word over Σ_3 having exponential subword complexity.

Additional examples of such words can be provided using some standard techniques. Below we give the examples for the minimal binary and minimal ternary power-free languages of exponential growth.

Theorem 38.

(a) There is an infinite binary $(\frac{7}{3})^+$-power-free word having exponential subword complexity.

(b) There is an infinite ternary $(\frac{7}{4})^+$-power-free word having exponential subword complexity.

Proof. (a) First, create an infinite square-free word over Σ_4 with exponential subword complexity. For this, take an infinite square-free word $\mathbf{u}$ over Σ_3 and an infinite word $\mathbf{v}$ of exponential complexity over $\{2,3\}$. For each $i \geq 1$, replace the i'th occurrence of the symbol 2 in $\mathbf{u}$ with the i'th symbol of $\mathbf{v}$. The resulting word obviously satisfies the desired properties. Now apply the 21-uniform morphism $h : \Sigma_4^* \to \Sigma_2^*$ from [18, Lemma 8]. The lemma guarantees that the image of a square-free word is $(7/3)^+$-power-free, and every uniform injective morphism preserves the property of having exponential subword complexity.

(b) Start with an infinite $(7/5)^+$-power-free word over Σ_4, which exists by Pansiot's result [28]. As in (a), replace the occurrences of 3 in this word by an infinite word over $\{3,4\}$

with exponential subword complexity, getting an infinite $(7/5)^+$-power-free word over Σ_5 with exponential subword complexity. Now apply the 59-uniform morphism of Ochem [27, Thm 4.2]. The result is guaranteed to be $(7/4)^+$-free and to have exponential subword complexity. □

As usual, in such examples the growth rate of the subword complexity is barely above 1. Are there words having the subword complexity comparable to the growth function of the whole language? It turns out that this problem is closely related to an old problem by Restivo and Salemi [35, Problem 4]: *given two square-free words $u, v \in \Sigma_3^*$, provide an algorithm deciding whether there is a word $w \in \Sigma_3^*$ such that uwv is square-free.* The same decidability problem can be posed for any infinite language $L_{k,\alpha}$ (the solution does not need to be uniform with respect to α). The only solved case is the case of small binary languages (due to the existence of factorizations of type (1), it is easy to connect any right-extendable u to any left-extendable v by an appropriate Thue–Morse factor).

Clearly, the interesting part of the problem is formed by the case where u is right-extendable and v is left-extendable. To the best of our knowledge, there are no known tuples (k, α, u, v) such that u is right-extendable in $L_{k,\alpha}$, v is left-extendable in $L_{k,\alpha}$, and no word of the form uwv belongs to $L_{k,\alpha}$. For our purposes, we restrict ourselves to the consideration of two-sided extendable words.

We say that a language $L_{k,\alpha}$ *has the Restivo–Salemi property* if for every $u, v \in \mathsf{ext}(L_{k,\alpha})$ there is a word w such that $uwv \in \mathsf{ext}(L_{k,\alpha})$.

Theorem 39. *A power-free language $L_{k,\alpha}$ has the Restivo–Salemi property if and only if all words from $\mathsf{ext}(L_{k,\alpha})$ are factors of some α-power-free infinite recurrent k-ary word $\mathbf{u}$.*

An already mentioned result of [39] says that L and $\mathsf{ext}(L)$ have the same growth rate, so Theorem 39 implies the following.

Corollary 40. *If a power-free language $L_{k,\alpha}$ possesses the Restivo–Salemi property, then there is a symmetric α-power-free infinite recurrent k-ary word $\mathbf{u}$ with subword complexity having the same growth rate as $L_{k,\alpha}$.*

Proof of Theorem 39. For the forward implication we endow Σ_k^* with the radix order (the words are ordered by length, and the words of equal length are ordered lexicographically) and build the word $\mathbf{u}$ by induction. As the base case, we build the prefix $u_0 = 0$. For the inductive step, assume that the prefix u_n was constructed so far. Let v_n be the smallest in radix order word from $\mathsf{ext}(L_{k,\alpha})$ that is not a factor of u_n. Then we take w_n such that $u_n w_n v_n \in \mathsf{ext}(L_{k,\alpha})$ and put $u_{n+1} = u_n w_n v_n$. The resulting word $\mathbf{u}$ is α-power-free by construction. Further, every word $v \in \mathsf{ext}(L_{k,\alpha})$ is a factor of some u_n and thus of $\mathbf{u}$. Finally, for an arbitrary $v \in \mathsf{ext}(L_{k,\alpha})$ and every n, there is a word x such that $|x| > |u_n|$ and $xv \in \mathsf{ext}(L_{k,\alpha})$; since $\mathbf{u}$ contains the factor xv, there is an occurrence of v in $\mathbf{u}$ outside the prefix u_n. Hence v occurs in $\mathbf{u}$ infinitely many times. Thus $\mathbf{u}$ is recurrent and we proved this implication.

Now turn to the reverse implication. For arbitrary words $u, v \in \mathsf{ext}(L_{k,\alpha})$ each of them occurs in $\mathbf{u}$ infinitely often, so we can find a factor of the form uwv. This factor also occurs in

u infinitely often, allowing us to find arbitrarily long words x, y such that $xuwvy$ is a factor of **u**. Then $uwv \in \mathsf{ext}(L_{k,\alpha})$. Hence we proved the Restivo–Salemi property for $L_{k,\alpha}$. □

Remark 41. It is worth mentioning that for small binary languages, Theorem 39 works in an extremal form. Since $2^+ \leq \alpha \leq 7/3$ implies $\mathsf{ext}(L_{2,\alpha}) = \mathrm{Fac}(\mathbf{t})$, the language $L_{2,\alpha}$ trivially has the Restivo–Salemi property; as we know from Corollary 3, *all* α-power-free infinite binary words contain all words of $\mathsf{ext}(L_{2,\alpha})$ as factors.

The following conjecture is based on extensive numerical studies.

Conjecture 42. [40, Conjecture 1] All power-free languages satisfy the Restivo–Salemi property.

As an approach to Conjecture 42, we suggest the following.

Open Question 43. *Prove the converse of Corollary 40.*

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