Combinatorics of Grassmannian Decompositions

by

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Abstract

This thesis studies several combinatorially defined families of subsets of the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$, $Gr(k, n)$. We introduce and study a family of subsets called "basis shape loci" associated to transversal matroids. Additionally, we study the Deodhar and positroid decompositions of the Grassmannian.

A basis shape locus takes as input data a zero/nonzero pattern in an $n \times k$ matrix, which is equivalent to a specific presentation of a transversal matroid. The locus is defined to be the set of points in $Gr(k, n)$ which are the row space of a matrix with the prescribed zero/nonzero pattern. We show that this locus depends only on the transversal matroid, not on the specific presentation. When a transversal matroid is a positroid, the closure of its basis shape locus is exactly the positroid variety labelled by the matroid. We give a sufficient, and conjecturally necessary, condition for when a transversal matroid is a positroid.

Components in the Deodhar decomposition are indexed by Go-diagrams, certain fillings of Ferrers shapes with white stones, black stones, and pluses. Le-diagrams are a common combinatorial object indexing positroids; all Le-diagrams are Go-diagrams. We give a system of local flips on fillings of Ferrers shapes which may be used to turn arbitrary diagrams into Go-diagrams. When a Go-diagram is a Le-diagram, these flips are exactly the previously studied Le-moves. Using these local flips, we conjecture a combinatorial condition describing when one Deodhar component is contained in the closure of another within a Schubert cell. We define a variety containing and conjecturally equal to the closure of a Deodhar component and prove that this combinatorial criterion implies a containment of these varieties. We further show that there is no reasonable description of Go-diagrams in terms of forbidden subdiagrams by providing an injection from the set of valid Go-diagrams into the set of minimal forbidden subdiagrams. In lieu of such a description, we give an algorithmic characterization of Go-diagrams.

Finally, we use the above results to prove several corollaries about Wilson loop cells, which arise in the study of scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Notably, it was previously known that the matroid represented by a generic point in a Wilson loop cell is a positroid. We show that the closure of the Wilson loop cell agrees with the positroid variety labelled by this positroid.

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List of Symbols

$\mathcal{B}$	A matroid, presented as its set of bases.	25
$\mathcal{B}/I$	The matroid obtained from $\mathcal{B}$ by contracting I.	26
$\mathcal{B} \setminus I$	The matroid obtained from $\mathcal{B}$ by deleting I.	26
$\mathcal{B}\|_I$	The restriction of the matroid $\mathcal{B}$ to I.	26
$\mathcal{B} \oplus \mathcal{B}'$	The direct sum of the matroids $\mathcal{B}$ and $\mathcal{B}'$.	26
$\mathcal{B}^*$	The dual of the matroid $\mathcal{B}$.	26
$\mathcal{B}(\mathcal{S})$	The transversal matroid defined by the set system $\mathcal{S}$.	30
b^{in}	The set of boxes weakly to the right and weakly below b in a diagram.	7
b^{out}	The set of boxes strictly to the left or strictly above b in a diagram.	7
$c \preccurlyeq b$	b and c are boxes in the same diagram, and $c \in b^{in}$.	7
$\mathcal{D}, \mathcal{D}_{\mathbf{u},\mathbf{v}}$	The Deodhar component, labelled by the pair $\mathbf{u} \prec \mathbf{v}$.	17
$\Delta_I, \Delta_I(V)$	The I^{th} Plücker coordinate, of the point V.	12
Δ_σ	The Plücker coordinate on the flag manifold indexed by the permutatino σ	16
$\mathrm{ec}(\mathcal{B})$	The expected codimension of the matroid $\mathcal{B}$.	46
$\mathrm{ec}_{\mathcal{I}}(\mathcal{B})$	The expected codimension of the matroid $\mathcal{B}$ with respect to the collection of sets $\mathcal{I}$.	46
ε	The identity permutation.	6

$f(D,b)$	The diagram obtained from D by replacing all black stones in b^{in} and their partners with pluses.	86
$Fl(n)$	The full flag manifold.	15
$F(\mathcal{T})$	The flat of the transversal matroid $\mathcal{B}(\mathcal{S})$ defined by the subset $\mathcal{T}$ of $\mathcal{S}$.	32
$\Gamma_{\mathcal{S}}$	Bipartite graph associated to the set system $\mathcal{S}$.	30
$Gr(k,n)$	The Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$.	12
$Gr_{\geq 0}(k,n)$	The totally nonnegative Grassmannian.	13
I_b	A set labelling a box b in a Go-diagram.	17
I_λ	The set of vertical steps of the Ferrers shape λ.	7
$I \leq J$	Gale order on sets.	7
$I \leq_a J$	The a^{th} cyclic shift of the Gale order on sets.	59
J_b	Another set labelling a box b in a Go-diagram.	25
λ	A Ferrers shape.	7
$\ell(v)$	Length of the permutation v.	6
$\leq_a$	The a^{th} cyclic shift of the usual total order on $[n]$.	50
$L(\mathcal{S})$	The basis shape locus defined by the set system $\mathcal{S}$.	37
$L_{\geq 0}(\mathcal{S})$	The totally nonnegative part of the basis shape locus defined by the set system $\mathcal{S}$.	99
$M_{k,n}$	The set of $k \times n$ real matrices.	12
$M_{k,n}^{\text{rk } k}$	The set of full rank $k \times n$ real matrices.	12
$M_{\mathcal{S}}(\mathbf{x})$	Matrix associated to a set system.	37
$M_D(\mathbf{x},\mathbf{y}), M_{\mathcal{B}}(V)$	The Marsh-Rietsch matrix labelled by the Go-diagram D. The Marsh-Rietsch matrix labelled the matroid $\mathcal{B}$ representing the point V.	22

$MR(\mathcal{B})$	The Marsh-Rietsch cell (Deodhar component) labelled by the matroid $\mathcal{B}$. This notation is only used in Chapter 3 to avoid the overloading of calligraphic letters.	22
$\mu_{\mathcal{I}}$	Möbius function on the poset $\mathcal{I}$.	46
$[n]$	$\{1, 2, \ldots, n\}$.	7
$\binom{[n]}{k}$	The set of k element subsets of $[n]$.	7
$N(D)$	The Go-network obtained from the Go-diagram D.	19
$\mathrm{nmd}(\mathcal{S})$	The naive maximal dimension of the basis shape locus $L(\mathcal{S})$.	37
$\mathcal{P}, \mathcal{P}_{u,v}$	Positroid cell, labelled by u, v.	16
$\mathcal{P}_{\geq 0}$	The totally nonnegetative part of the positroid cell $\mathcal{P}$.	16
$\mathcal{P}([n])$	The power set of $[n]$.	46
$\mathbb{P}^N$	N-dimensional projective space.	12
π	Projection from $Gr(k, n+1)$ to $Gr(k, n) \cup Gr(k-1, n)$.	91
$\pi_{\geq 0}$	Projection from $Gr_{\geq 0}(k, n+1)$ to $Gr_{\geq 0}(k, n) \cup Gr_{\geq 0}(k-1, n)$.	95
$\pi^{-1}(M_{\mathcal{W}}(\mathbf{x}))$	Matrix obtained by appending a column to $M_{\mathcal{W}}(\mathbf{x})$.	102
$\mathrm{Prop}(v), \mathrm{Prop}(V)$	Propagator set of a vertex, set of vertices in a Wilson loop diagram.	99
$\mathbb{R}^n, \mathbb{R}^{[n]}$	n-dimensional real vector space, with a basis labelled by $[n]$.	12
$\mathcal{R}_{\lambda/\mu}, \mathcal{R}_{I,J}, \mathcal{R}_{u,v}$	Richardson cell labelled by the skew diagram μ/λ; sets I, J; Grassmannian permutations u, v.	15
$\mathrm{rk}(I)$	The rank of a subset I of the ground set in a matroid.	25
$r(M)$	The interval rank matrix of the matrix M.	61
$\mathcal{S} \preccurlyeq \mathcal{S}'$	Partial order on presentations of a transversal matroid.	31

s_b	The transposition labelling the box b in a diagram.	9
s_i	The simple transposition $(i, i+1)$ in $\mathfrak{S}_n$.	6
$\mathcal{S}_\lambda, \mathcal{S}_I, \mathcal{S}_v$	Schubert cell labelled by the Ferrers shape λ, set I, Grassmannian permutation v.	14
$\mathfrak{S}_n$	The symmetric group.	6
$\text{supp}(\mathbf{v})$	The support set of the vector $\mathbf{v}$.	37
$\tau(D)$	Tuple of permutations associated to a diagram D.	70
u_b^D, u_b	s_b if b contains a stone in the diagram D, ε otherwise.	10
$u_{b^{in}}^D, u_{b^{in}}$	The permutation obtained by multiplying the transpositions labelling all boxes containing stones in b^{in} in D in some valid reading order.	10
$u \leq v$	Bruhat order on permutations.	6
$\mathbf{u} \prec \mathbf{v}$	$\mathbf{u}$ is a distinguished subexpression of $\mathbf{v}$.	6
$\mathbf{v}$	An expression for the permutation v in the Coxeter generators.	6
$v_{(i)}$	Product of the initial i terms in an expression of the permutation v.	6
v_λ	The Grassmannian permutation in bijection with the Ferrers shape λ.	7
$V_\mathcal{B}$	The set of points in $Gr(k,n)$ representing the matroid $\mathcal{B}$.	27
$V(p), V(P)$	The support set of a propagator, set of propagators, in a Wilson loop diagram.	99
$W = (\mathcal{I}, n), \mathcal{W}$	A Wilson loop diagram, and the associated set system.	98

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Chapter 1

Introduction

This thesis studies several combinatorially defined families of subsets of the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$, $Gr(k,n)$. The most familiar families are the Schubert and Richardson decompositions of $Gr(k,n)$. These decompositions are both matroidal in the sense that cells may be defined by setting certain Plücker coordinates on the Grassmannian to zero, demanding that other Plücker coordinates do not vanish, and not specifying whether or not the remaining Plücker coordinates vanish. Other common matroidally defined decompositions of the Grassmannian include the positroid stratification, the Deodhar decomposition, and the GGSM decomposition (after after Gelfand, Goresky, MacPherson, and Serganova). These decompositions are all refinements of each other, with the Schubert decomposition being the coarsest, the Richardson being finer than the Schubert, the positroid finer than the Richardson, the Deodhar finer than the positroid, and the GGSM finer than the Deodhar. Geometrically, the cells in all of these decompositions aside from the GGSM decomposition are known to have nice properties. GGSM strata are famously poorly behaved, giving rise to "Murphy's law" type results [57].

Our focus will be on the finer, but still reasonable end of this hierarchy of decompositions, giving results about Deodhar components and positroids. Additionally, we define a new family of subsets of the Grassmannian associated to transversal matroids, which we call basis shape loci.

The study of totally positive matrices dates back to the 1930's, and was developed into the theory of total positivity in reductive groups and partial flag manifolds by Lusztig in the 1990's [43, 44]. In the case of $Gr(k,n)$, the totally positive part is the subset $Gr_{\geq 0}(k,n)$ where all Pücker coordinates have the same sign. Postnikov studied $Gr_{\geq 0}(k,n)$ from a combinatorial perspective in [53]. There, he showed that the intersection of a matroid's representation space with $Gr_{\geq 0}(k,n)$ is either empty or parameterized by $\mathbb{R}^m_{>0}$ for some m, giving a combinatorial realization of a parameterization studied by Marsh and Rietsch in [47]. This decomposition gives $Gr_{\geq 0}(k,n)$ the structure of a regular CW complex. The positive Grassmannian, the matroids and the cells appearing in this decomposition, and extensions of this decomposition to the entirety of $Gr(k,n)$ have all received significant attention in recent years. Though we will not attempt to give an exhaustive survey of the field, we highlight a few notable results.

In geometry, Knutson, Lam, and Speyer showed that Postnikov's stratification of $Gr_{\geq 0}(k,n)$ agrees with the stratification of $Gr_{\geq 0}(k,n)$ by projections of Richardson varieties from the full flag manifold [36, 37]. They showed that projections of Richardson varieties enjoy many of the nice geometric properties which hold for Richardson varieties: they are normal, Cohen–Macaulay, and have rational singularities. They further show that projected Richardson varieties are the only compatibly Frobenius split subvarieties of a (partial) flag variety. In [35], Knutson also noticed that certain varieties appearing during a shifting process in Vakil's geometric Littlewood-Richardson rule [56] were a special class of projected Richardson varieties. Topologically, Galashin, Karp, and Lam showed in [27] that $Gr_{\geq 0}(k,n)$ is homeomorphic to a ball.

In combinatorics, positroids, the class of matroids appearing in Postnikov's stratification, have received a significant amount of attention. Postnikov gave a plethora of combinatorical objects indexing positiods in [53], including decorated permutations, Grassmann necklaces, and planar bicolored graphs. Ardila, Rincón, and Williams used a connection between positroids and non-crossing partitions to give enumerative results about positroids in [7]. They further describe positroids as the class of matroids whose matroid polytope is defined by inequalities involving only cyclic interval of elements from the matroid's ground set. Deep connections between total positivity phenomena and cluster algebras have also been the subject of intense study [24].

In physics, a connection between the positive Grassmannian and scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory was described in [9]. A certain projection of $Gr_{\geq 0}(k,n)$ called the amplituhedron was defined in [11]; it was conjectured that volumes of the amplituhedron computed scattering amplitudes and that positroid cells related to the BCFW recurrence relation (after Britto, Cachazo, Feng, and Witten) projected to a triangulation of the amplituhedron. In [32] and [33] this triangulation was related to a classic triangulation of cyclic polytopes due to Rambau [54].

Deodhar components were originally introduced in [21] to help compute Kazhdan and Lusztig's R-polynomials, which may be used to recursively compute Kazhdan-Lusztig polynomials. Associated to any pair of permutations u and v with $u \leq v$ in Bruhat order is a Richardson cell $\mathcal{R}_{u,v}$ in the full flag manifold. The number of points in $\mathcal{R}_{u,v}$ in the flag manifold over the finite field $\mathbb{F}_q$ is a polynomial in q. This polynomial, $R_{u,v}(q)$, is the R-polynomial associated to u and v. Deodhar introduced a decomposition refining the Richardson decomposition in which each "component" is a product of affine spaces and tori. This allowed him to decompose $R_{u,v}(q)$ as a sum of terms of the form $q^{\ell}(q-1)^m$. While each summand becomes simple, this simplification comes at the cost of making components in the decomposition relatively complicated to define, and of sacrificing many nice geometric properties enjoyed by Richardson cells. Notably, the Deodhar decomposition is not a stratification; the closure of a Deodhar component is not in general a union of other Deodhar components.

The projection of the Deodhar decomposition from the flag manifold to the Grassmannian was studied in depth by Kodama and Williams in [39]. Their motivation was to study KP-solitons, systems of solitary waves satisfying the KP-hierarchy of differential equations. From any point in the Grassmannian, one may produce a solution to the KP-hierarchy. Asymptotically with time, a soliton solution to the KP-equations assumes a fixed shape. Kodama and Williams showed that this fixed shape depends only on which Deodhar component the point used to produce the solution lies in. In proving this, they developed a diagrammatic indexing set, Go-diagrams, for Deodhar components in the Grassmannian.

One does not expect the projection of the Deodhar decomposition to the Grassmannian to be any less wildly behaved than the Deodhar decomposition of the flag manifold. However, a reasonable question to ask is: When is one Deodhar component contained in the closure of another? We define a more combinatorially tractable variety which is conjecturally identical to the closure of a Deodhar component in a Schubert cell. We show that one of these varieties is contained in the closure of another when their Go-diagrams are related by a certain diagrammatic procedure. We conjecture that the same result holds for Deodhar components, and prove this fact for positroid varieties.

This diagrammatic procedure should be thought of as an extension of the "Le-game", studied by Lam and Williams in [41]. Their goal was to describe in terms of forbidden subdiagrams a combinatorial indexing set for cells stratifying the totally positive part of other cominiscule Grassmannians. They define a local procedure which recognizes a violation to a diagram indexing a positive cell, then locally removes this violation. They show that this procedure converges to a diagram indexing a positive cell, and are able to use it to completely describe sets of forbidden subdiagrams in types B and D (the type A case was already known). Our procedure behaves similarly, recognizing violations to a diagram indexing a Deodhar component, and locally removing these violations. Unlike Lam and Williams's work, this procedure does not yield a description of Go-diagrams in terms of forbidden subdiagrams. In fact, we are able to show that no reasonable description of this form can exist.

We define and study a further set of subvarieties of the Grassmannian, which we call basis shape loci. A basis shape locus takes as input a zero/nonzero pattern in a $k \times n$ matrix. The locus is defined to be the set of planes which are row spaces of matrices with the prescribed zero/nonzero pattern. Associated to a zero/nonzero pattern in a matrix is a transversal matroid, which is the matroid represented by a generic evaluation of the nonzero parameters. Transversal matroids are one of the older classes of matroids to receive significant study, for instance [15, 19]. We show that the closure of a basis shape locus depends only on the transversal matroid, not the specific zero/nonzero pattern. This theorem is proved by recognizing that if the zero/nonzero pattern has more nonzero entries than the minimum number across all patterns giving the same transversal matroid, then one of the nonzero entries may be set to zero without altering the closure of the basis shape locus. This fact provides a geometric realization of a well known combinatorial fact about transversal matroids [19].

We further show that when a transversal matroid is a positroid, the closure of its basis shape locus is exactly the associated positroid variety. As special cases, we see that all Schubert and Richardson varieties are closures of basis shape loci.

This theorem relating basis shape loci and positroid varieties leads us to ask when a transversal matroid is a positroid. We give a sufficient and conjecturally necessary condition. in terms of avoidance of a certain sub-arrangement of the nonzero entries of the matrix defining a transversal matroid, describing when a transversal matroid is a positroid. This condition is a direct generalization of the observation that it is not possible to fill in the stars in the matrices

$$\begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} * & 0 & * & * \\ 0 & * & 0 & * \end{pmatrix}$$

with nonzero numbers such that all 2×2 minors are nonnegative. This condition generalizes Ardila, Rincón, and Williams's result relating positroids and noncrossing partitions. We prove that this condition is necessary for a transversal matroid to be a positroid in many special cases, and offer computational evidence for this fact as well.

Finally, we apply our results about basis shape loci and Deodhar components to study subsets of $Gr(k, n)$ defined by Wilson loop diagrams, which arise in a program to compute scattering amplitudes in $\mathcal{N} = 4$ SYM theory. We prove that the cell associated to each Wilson loop diagram is $3k$-dimensional. Each Wilson loop diagram has an associated positroid, and we show that the closure of the Wilson loop cell is exactly the corresponding positroid variety. In joint work with Agarawala, we show that a certain vector bundle over a Wilson loop cell is a union of Deodhar components. We describe the boundary structure of Deodhar components appearing in this union.

Chapter 2 collects necessary background and proves a few preliminary technical results which will be used later. Combinatorial background includes the notion of distinguished subexpressions of permutations in the symmetric group; Go-diagrams, which are a diagrammatic mnemonic for these expressions; and matroidal background emphasizing positroids and transversal matroids. Geometric background includes the positroid and Deodhar decompositions of the Grassmannian, and an explanation of how to obtain explicit parameterizations of the cells in these decompositions using Go-diagrams. Chapter 3 introduces and studies basis shape loci. Chapter 4 studies the Deodhar decomposition and the Go-diagrams indexing cells in this decomposition. Chapter 5 applies results from the previous chapters to study Wilson loop cells.

Most of the work in this thesis previously appears in [45] and [46] by the author, and [6] by Agarwala and the author. Material in Chapter 3 and Section 5.2 is from [46]. Material in Chapter 4 aside from Section 4.4 is from [45]. Material from Section 4.4 and from Chapter 5 aside from Section 5.2 is from [6].

Chapter 2

Background

2.1 Permutations and Pipe Dreams

Let s_i denote the adjacent transposition $(i, i+1)$ in the symmetric group $\mathfrak{S}_n$. Italicized lowercase letters, v, will denote permutations and bold faced letters, $\mathbf{v}$, will denote specific expressions for permutations in the s_i's. A *subexpression* of $\mathbf{v}$ is a permutation obtained by replacing some of the factors in $\mathbf{v}$ by ε, the identity permutation in $\mathfrak{S}_n$. The terms "expression" and "word" will be used interchangeably, as will "subexpression" and "subword."

Given an expression $\mathbf{v} = v_1 v_2 \cdots v_m$, let $v_{(i)} = v_1 v_2 \cdots v_i$ denote the product of the initial i factors of $\mathbf{v}$. So, $v_{(0)} = \varepsilon$ and $v_{(m)} = v$.

The *length* of a permutation, $\ell(v)$, is the minimum number of letters in an expression of v. A word is *reduced* if $\ell(v_{(i+1)}) = \ell(v_{(i)}) + 1$ for every i. All reduced words for a permutation contain the same number of adjacent transpositions. The *Bruhat order* on permutations is the order given by setting $u \leq v$ if and only if some reduced word for u is a subword of some reduced word for v.

A subexpression $\mathbf{u}$ of $\mathbf{v}$ is *distinguished* if whenever $\ell(u_{(i)} v_{i+1}) < \ell(u_{(i)})$, one also has $u_{i+1} = v_{i+1}$, (i.e. $u_{i+1} \neq \varepsilon$). Write $\mathbf{u} \prec \mathbf{v}$ if $\mathbf{u}$ is a distinguished subexpression of $\mathbf{v}$. A subexpression $\mathbf{u}$ of $\mathbf{v}$ is *positive* if $\ell(u_{(i+1)}) \geq \ell(u_{(i)})$ for all i.

Example 2.1.1. Let $\mathbf{v} = s_1 s_2 s_1 s_3 s_2 s_1$. Then,

$$\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon, \ s_1 \varepsilon s_1 \varepsilon\varepsilon\varepsilon, \text{ and } s_1 \varepsilon\varepsilon\varepsilon\varepsilon s_1$$

are three subexpressions for the identity permutation in $\mathbf{v}$. The first is positive and distinguished, the second is distinguished but not positive, and the third is neither positive nor distinguished.

Lemma 2.1.2 (Lemma 3.5 in [47]). *Let $u \leq v$ be permutations and $\mathbf{v}$ be a reduced expression for v. Then, there is a unique positive distinguished subexpression for u in $\mathbf{v}$.*

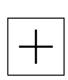
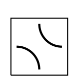
The Young subgroup $\mathfrak{S}_{n-k} \times \mathfrak{S}_k \subset \mathfrak{S}_n$ acts on a permutation $(v(1), v(2), \ldots, v(n))$ by letting $\mathfrak{S}_{n-k}$ act on $(v(1), v(2), \ldots, v(n-k))$ and $\mathfrak{S}_k$ act on $(v(n-k+1), v(n-k+2), \ldots, v(n))$. Any coset in the quotient $\mathfrak{S}_n/(\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ has a unique representative of the form $(i_1, i_2, \ldots, i_{n-k}, j_1, j_2, \ldots, j_k)$ where $i_1 < i_2 < \cdots < i_{n-k}$ and $j_1 < j_2 < \cdots < j_k$. These representatives are called *Grassmannian permutations*. Let $\binom{[n]}{k}$ be the collection of k element subsets of $[n] = \{1, 2, \ldots, n\}$. Grassmannian permutations are in bijection with subsets in $\binom{[n]}{k}$ sending $(i_1, i_2, \ldots, i_{n-k}, j_1, j_2, \ldots, j_k)$ to $\{j_1, j_2, \ldots, j_k\}$. Often, we will suppress curly braces and commas when writing sets to avoid unwieldy notation, writing $j_1 j_2 \ldots j_k$ to mean $\{j_1, j_2, \ldots, j_k\}$.

The Bruhat order on $\mathfrak{S}_n$ induces an order on the quotient $\mathfrak{S}_n/(\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ and thus a partial order on $\binom{[n]}{k}$. Concretely, if $I = \{i_1, i_2, \ldots, i_k\}$ and $J = \{j_1, j_2, \ldots, j_k\}$ with $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$, then $I \leq J$ if and only if $i_m \leq j_m$ for all m. This partial order is called the *Gale order* on sets.

A *Ferrers shape* is a collection of boxes obtained by taking a lattice path from the Northeast to Southwest corner of a $(n-k) \times k$ rectangle, then taking all boxes Northwest of this lattice path. The steps of the lattice path are labelled 1 to n starting at the Northeast corner. A box b has coordinates (i, j) if the vertical step of the boundary in the same row as b is labelled i and the horizontal step of the boundary in the same column as b is labelled j. For an illustration, see Example 2.1.3. Ferrers shapes contained in an $(n-k) \times k$ box are in bijection with subsets $\binom{[n]}{k}$ sending the Ferrers shape λ to the set I_λ of labels of the vertical steps in its boundary path. Composing bijections, to the Ferrers shape λ we also associate a Grassmannian permutation v_λ. Pictorially, the Gale order on $\binom{[n]}{k}$ translates to containment of Ferrers shapes. If $I \leq J$, then the Ferrers shape associated to J is contained in the Ferrers shape associated to I.

Given a box b in a Ferrers diagram D, b^{in} is the set of boxes in D weakly to the right and weakly below b. Additionally, set $b^{out} = D \setminus b^{in}$. We introduce a partial order on the boxes in a diagram, saying $c \preccurlyeq b$ if and only if $c \in b^{in}$.

A *pipe dream* is a filling of a Ferrers shape with crossing tiles and elbow pieces,

 and .

Think of this filling as a collection of pipes flowing from the Southeast boundary to the Northwest boundary. From a pipe dream, we obtain a permutation by labelling the edges along the North and West boundaries of the Ferrers shape such that for each pipe in the diagram, both ends of the pipe have the same label, then writing down the labels that appear along the Northwest boundary in order starting from the Northeast corner.

Say that two squares b and c in a pipe dream are a *crossing/uncrossing pair* if two pipes cross in b, flow to the Northwest, then next uncross in c. A pipe dream is *reduced* if it has no crossing/uncrossing pairs. Note that a crossing tile is a crossing if the label of the pipe entering from the bottom is larger than the label of the pipe entering from the right, and is an uncrossing otherwise. A certain class of pipe dreams was originally defined by Bergeron and Billey in [12], where they were called RC-graphs for "reduced word, compatible sequence." Diagrams of this type were later renamed pipe dreams by Knutson; we choose this terminology since the pipe dreams we consider will not in general be reduced. Another potential point of confusion: pipe dreams as introduced by Bergeron and Billey use an elbow piece which is a reflection of ours, and flow from Northeast to Southwest. We take our convention from [39].

Example 2.1.3. The pipe dream in Figure 2.1 gives the permutation $(2, 1, 3, 4, 5, 7, 6)$. The squares $(4, 5)$ and $(1, 7)$ form a crossing/uncrossing pair.

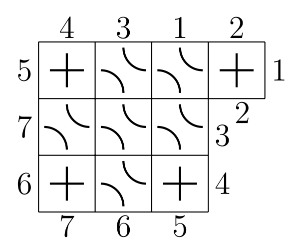


Figure 2.1: Example of a Pipe Dream.

A $\circ/+$-*diagram* is a filling of a Ferrers shape with white stones and pluses,

Pipe dreams are in bijection with $\circ/+$-diagrams, replacing the crossing tiles with white stones and the elbow pieces with pluses. This bijection is unfortunate, but is the standard convention in the literature.

A $\bullet/\circ/+$-diagram is a filling of a Ferrers shape with black stones, white stones, and pluses,

$$\boxed{\bullet}, \quad \boxed{\circ}, \text{ and } \boxed{+}.$$

We map $\bullet/\circ/+$-diagrams to pipe dreams by sending both the black and white stones to crossing tiles, and sending the pluses to elbow tiles. So, $\bullet/\circ/+$-diagrams are $\circ/+$-diagrams where the stones have been decorated to have two colors. Often, but not always, we will require that stones be colored black if and only if they are mapped to uncrossing tiles in the pipe dream. We state whether or not we make this assumption at the start of each section where $\bullet/\circ/+$-diagrams appear. For the remainder of this section, this assumption is not imposed.

Assign a transposition to each box in a Ferrers shape as follows. Assign the top left box of a Ferrers shape contained in a $k \times (n-k)$ box the simple transposition s_{n-k}. If the box to the left of b is assigned the transposition s_i, assign s_{i-1} to b, and if the box above b is assigned s_i assign s_{i+1} to b . Observe that for any box b, the permutation corresponding to the $\circ/+$ diagram where b is filled with a white stone and all other boxes are filled with pluses is exactly the transposition labelling b. This fact is illustrated in Figure 2.2. We use s_b to denote the simple transposition labelling the box b.

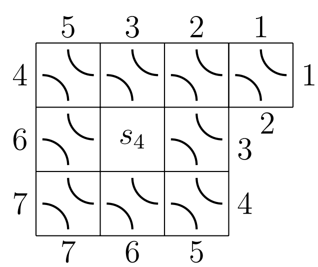


Figure 2.2: Simple transposition associated to a box in a Ferrers shape.

A *reading order* on a Ferrers shape λ containing m boxes is a filling of the boxes with the integers from 1 to m which is increasing upward and to the left. Reading the transpositions decorating the boxes of the Ferrers diagram in any reading order yields a reduced expression $\mathbf{v}$ for v_λ. Reading only the transpositions decorating boxes containing stones in either a $\circ/+$ or a $\bullet/\circ/+$-diagram in the same reading order gives a subexpression $\mathbf{u}$ of $\mathbf{v}$ for the permutation u given by the pipe dream associated to the diagram.

Theorem 2.1.4 (Proposition 4.5 in [41]). *Let D be a $\circ/+$ or a $\bullet/\circ/+$-diagram yielding the subword, word pair $\mathbf{u}, \mathbf{v}$ in some reading order.*

(i) The permutations v, coming from the Ferrers shape, and u, coming from the pipe dream depend only on D, not the choice of reading order.

(ii) Whether $\mathbf{u}$ is a distinguished subexpression of $\mathbf{v}$ depends only on D, not the choice of reading order.

(iii) Whether $\mathbf{u}$ is a positive subexpression of $\mathbf{v}$ depends only D, not on the choice of reading order.

This theorem is proved by noting that if the expressions $\mathbf{u}$ and $\mathbf{u}'$ are obtained by altering the reading order on the same diagram, then they are related by commutations of the Coxeter generators, and thus $u = u'$.

Let b be a box in the $\circ/+$ or $\bullet/\circ/+$-diagram D. Define $u^D_{b^{in}}$ to be the permutation obtained by multiplying the transpositions labelling all boxes containing stones in b^{in} in D in some valid reading order. As a corollary of Theorem 2.1.4, $u^D_{b^{in}}$ does not depend on the choice of reading order. If the diagram is clear from context, we will simply write $u_{b^{in}}$ instead of $u^D_{b^{in}}$. Let

$$u^D_b = \begin{cases} s_b & \text{if } b \text{ contains a stone in } D, \\ \varepsilon & \text{if } b \text{ contains a plus in } D. \end{cases}$$

We will simply write u_b instead of u^D_b if the diagram is clear from context.

Definition 2.1.5. Let D be a $\bullet/\circ/+$-diagram. Then, D is a *Go-diagram* if and only if for every box b in the diagram, b contains a black stone if and only if $\ell(u_{b^{in}} u_b s_b) < \ell(u_{b^{in}} u_b)$.

Diagrammatically, $u_{b^{in}} u_b$ is the permutation coming from the diagram where the filling of every box in $b^{out} \cup b$ has been changed to a plus. The permutation $u_{b^{in}} u_b s_b$ corresponds to the diagram where the filling of every box in b^{out} is changed to a plus and the filling of b is changed to a stone.

Let $\mathbf{u}, \mathbf{v}$ be the subword, word pair associated to the diagram D in some reading order. Theorem 2.1.4 implies that if D is a Go-diagram, then $\mathbf{u}$ is a distinguished subexpression of $\mathbf{v}$, and that this property is independent of the choice of reading order. If a box b has the property that

$$\ell(u_{b^{in}} u_b s_b) < \ell(u_{b^{in}} u_b),$$

but b is not filled with a black stone, we say that b *violates the distinguished property*. For example, in the pipe dream in Figure 2.1 the box $(3, 6)$ violates the distinguished property since

$$(s_1\varepsilon\varepsilon\varepsilon)(\varepsilon)(s_1) < (s_1\varepsilon\varepsilon\varepsilon)(\varepsilon).$$

Diagrammatically, one can see that this box violates the distinguished property since the pipes 4 and 5 are appear in this box with 4 entering from the bottom. So, if there were a crossing tile placed in $(3, 6)$ in the pipe dream, it would be an uncrossing.

Definition 2.1.6. Let D be a $\bullet / \circ /+$-diagram. Then, D is a ⅃-*diagram* if any of the following equivalent criteria hold:

(i) D is a Go-diagram and contains no black stones.

(ii) Let $\mathbf{u}, \mathbf{v}$ be the subword, word pair associated to D. Then, $\mathbf{u}$ is a positive distinguished subexpression of $\mathbf{v}$

(iii) D contains no black stones, and there is no box b in D containing a white stone such that there is a plus to the left of b in its row and above b in its column.

The equivalence between (i) and (ii) is immediate. The equivalence between (ii) and (iii) is Theorem 5.1 in [41]. The condition in (iii) is called the ⅃-*property* because the forbidden configuration in point (iii) forms a backwards L shape. Figure 4.3 illustrates a special case of this forbidden configuration.

Example 2.1.7. Consider the diagrams

○	+	+	1
+	+	○	2
5	4	3	

and

○	○	+	1
+	+	+	2
5	4	3	

.

The diagram on the left corresponds to the subexpression $s_2\varepsilon\varepsilon\varepsilon\varepsilon s_3$ of $s_2s_3s_4s_1s_2s_3$ and the diagram on the right corresponds to the subexpression $\varepsilon\varepsilon\varepsilon\varepsilon s_2s_3$. The diagram on the left is not a ⅃-diagram, which can be seen by noting that its subexpression is not distinguished or by noting that the box $(2, 3)$ contains a white stone, but has a plus both above it and to its left.

Proposition 2.1.8. *The locations of only the white stones or of only the pluses are enough to uniquely determine a Go-diagram.*

Proof. Given a Ferrers shape filled with white stones, we complete the filling in increasing order in the partial order on boxes. If c is some box in the diagram and all boxes in $c^{in} \setminus c$ have been filled, we may compute the permutation $u_{c^{in}} u_c$, which does not depend on the filling of c. Then, fill c with a plus if $\ell(u_{c^{in}} u_c s_c) > \ell(u_{c^{in}} u_c)$ and fill c with a black stone otherwise.

Given a Ferrers shape filled with pluses, construct a pipe dream by placing elbow tiles in the squares containing pluses and crossing tiles in all other squares. Then, for each square not containing a plus in the Ferrers shape, fill it with a white stone if the square is a crossing in the pipe dream and a black stone if it is an uncrossing. □

The following proposition follows directly from the definitions.

Proposition 2.1.9. *Let D be the Go-diagram associated to the distinguished subexpression $\mathbf{u} \prec \mathbf{v}$. Then,*

$$\begin{aligned} \ell(u) &= \#(\text{of } \circ\text{'s in } D) - \#(\text{of } \bullet\text{'s in } D) \\ &= \ell(v) - \#(\text{of } +\text{'s in } D) - 2 \cdot \#(\text{of } \bullet\text{'s in } D). \end{aligned}$$

2.2 Geometric Background

Let $M_{k,n}$ be the space of $k \times n$ real matrices and

$$M_{k,n}^{\mathrm{rk}\ k} = \{A \in M_{k,n} : \mathrm{rk}(A) = k\}.$$

Let

$$Gr(k,n) = \{V \subset \mathbb{R}^n : \dim(V) = k\}$$

be the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. Choosing a basis of $\mathbb{R}^n$, any point in $Gr(k,n)$ can be represented as the row span of a matrix in $M_{k,n}^{\mathrm{rk}\ k}$. If $A \in M_{k,n}^{\mathrm{rk}\ k}$ and $I \in \binom{[n]}{k}$, let $\Delta_I(A)$ be the determinant of the square matrix obtained by restricting A to the column set I. Two matrices A and B in $M_{k,n}^{\mathrm{rk}\ k}$ represent the same point in $Gr(k,n)$ if and only if there is some $X \in GL(k)$ such that $X \cdot A = B$. In this case, $\Delta_I(B) = \det(X)\Delta_I(A)$ for all $S \in \binom{[n]}{k}$. So, there is a map to the projective space $\mathbb{P}^{\binom{n}{k}-1}$,

$$\begin{array}{ccc} Gr(k,n) & \to & \mathbb{P}^{\binom{n}{k}-1} \\ V & \mapsto & (\Delta_{12\ldots k}(V) : \cdots : \Delta_{(n-k)\ldots(n-1)n}(V)). \end{array}$$

This map is an embedding of the Grassmannian into projective space called the *Plücker embedding*. For a point $V \in Gr(k,n)$, the collection

$$(\Delta_{12\dots k}(V) : \dots : \Delta_{(n-k)\dots(n-1)n}(V))$$

are called the *Plücker coordinates* of V. When there is no confusion, we will suppress the reference to V, simply using Δ_I to denote the I^{th} Plücker coordinate of V. For $I, J \in \binom{[n]}{k}$, $i \in I$ and $j \in J$, define $\mathrm{sgn}(I, i; J, j)$ by ordering the indices of I and J in increasing order, placing j in i's old position in I and i in j's old position in J, then taking -1 times the product of the signs of the two permutations need to rearrange to new sets so that their elements are in increasing order. The image of the Plücker embedding is a projective variety defined by the *Plücker relations*,

$$\Delta_I \Delta_J = \sum_{j \in J} \mathrm{sgn}(I, i; J, j) \Delta_{I \setminus i \cup j} \Delta_{J \setminus j \cup i}, \tag{2.1}$$

For example,

$$\Delta_{123}\Delta_{245} = \Delta_{223}\Delta_{145} - \Delta_{234}\Delta_{125} + \Delta_{235}\Delta_{124}$$

is a Plücker relation. Since Δ_{223} vanishes uniformly when viewed as a minor of a matrix and does not even exist when viewed as a coordinate on $\mathbb{P}^{\binom{n}{k}-1}$, this relation simplifies to

$$\Delta_{123}\Delta_{245} = \Delta_{235}\Delta_{124} - \Delta_{234}\Delta_{125}.$$

We will often only be concerned with determining whether certain Plücker coordinates vanish uniformly on a subset of $Gr(k,n)$, and will often be able to ignore signs in Plücker relations. For an example of this vanishing/non-vanishing arithmetic, if Δ_{234} and Δ_{124} vanish on some subset of the Grassmannian and Δ_{123} is nonvanishing on this subset, the Plücker relation above implies Δ_{245} also vanishes on the subset of the Grassmannian in question.

The *positive Grassmannian* also called the *totally nonnegative Grassmannian*, $Gr_{\geq 0}(k,n)$, is the subset of $Gr(k,n)$ where all nonzero Plücker coordinates have the same sign. For example,

$$\mathrm{span}\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \in Gr_{\geq 0}(2,4),$$

and

$$\mathrm{span}\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \notin Gr_{\geq 0}(2,4).$$

2.2.1 Decompositions of the Grassmannian

This section briefly introduces a hierarchy of decompositions of the Grassmannian associated to pairs of permutations. This thesis will be primarily concerned with the finer decompositions in this hierarchy, the positroid and Deodhar decompositions, which are given more detailed individual treatment in Sections 2.2.2 and 2.2.3.

Let $v \in \mathfrak{S}_n$ be the Grassmannian permutation associated to Ferrers shape λ. Associated to pairs of permutations u, v with various constraints imposed on u, there are several decompositions of the Grassmannian $Gr(k,n)$. The stricter the constraint imposed on u, the coarser the decomposition of the Grassmannian. Table 2.2 presents the common decompositions of $Gr(k,n)$ and the associated constraints on u, arranged from coarsest to finest.

Decomposition	Notation	Constraints on u
Schubert	$\mathcal{S}_v$	$u = \varepsilon$
Richardson	$\mathcal{R}_{u,v}$	u is Grassmannian, $u \leq v$
Positroid	$\mathcal{P}_{u,v}$	$u \leq v$
Deodhar	$\mathcal{D}_{\mathbf{u},\mathbf{v}}$	$\mathbf{u} \prec \mathbf{v}$

Table 2.1: Common decompositions of $Gr(k,n)$.

The Deodhar decomposition differs from the other decompositions in this list in that it doesn't just care that $u \leq v$, but how $\mathbf{u}$ is presented as a subword of $\mathbf{v}$.

All of the components in these decompositions have the feature that they can be described as subsets of the Grassmannian by setting certain Plücker coordinates to zero, demanding certain other Plücker coordinates be non-zero, and leaving the remaining Plücker coordinates unspecified. The coarser the decomposition, the more Plücker coordinates are left unspecified. From a combinatorial standpoint, all of the decompositions can be described as introducing decorations to the Ferrers shape λ.

The Schubert stratification remembers only λ. This should be viewed coming from the fact that the $\circ/+$-diagram corresponding to the positive distinguished subexpression of identity permutation in $\mathbf{v}$ is λ with every square with a plus. Let $I_\lambda \in \binom{[n]}{k}$ be the set associated to v. The Schubert cell $\mathcal{S}_v$ is defined by $\Delta_{I_\lambda} \neq 0$ and $\Delta_S = 0$ for all $S \not\geq I_\lambda$. In other words, $\mathcal{S}_v$ consists of points in $Gr(k,n)$ which are the row space of a matrix whose reduced row echelon form has pivot columns I_λ. Using our bijections between Grassmannian permutations, k-element subsets of $[n]$, and Ferrers shapes inside a $k \times (n-k)$ box, we will also use $\mathcal{S}_I$ or $\mathcal{S}_\lambda$ to denote Schubert cells. The set of generically nonzero Plücker coordinates on $\mathcal{S}_v$ is an up-set in the Gale order on $\binom{[n]}{k}$.

The Richardson stratification introduces another Ferrers shape μ contained inside λ. The pair, written as λ/μ is called a skew shape. This should be viewed coming from the fact that the $\circ/+$-diagram corresponding to the positive distinguished subexpression of u in $\mathbf{v}$ can be obtained by first drawing the shape μ associated u inside λ, then filling all squares in the skew shape λ/μ with pluses and filling all squares in μ with white stones. The Richardson cell $\mathcal{R}_{u,v}$ is defined by $\Delta_{I_\mu}, \Delta_{I_\lambda} \neq 0$ and $\Delta_S = 0$ for all $S \not\leq I_\mu$ and all $S \not\geq I_\lambda$. The set of generically nonzero Plücker coordinates on $\mathcal{R}_{u,v}$ is an interval in the Gale order on $\binom{[n]}{k}$. We will also use $\mathcal{R}_{I,J}$ and $\mathcal{R}_{\lambda/\mu}$ to denote Richardson strata.

The positroid and Deodhar decompositions will receive more detailed treatment in Sections 2.2.2 and 2.2.3. From the description of the cells in these decompositions, it is immediate that they refine each other.

Proposition 2.2.1. *Let $u, v \in \mathfrak{S}_n$, $u \leq v$, and let v be Grassmannian.*

(i) If u is Grassmannian, $\mathcal{R}_{u,v} \subseteq \mathcal{S}_v$.

(ii) If u' is the unique Grassmannian representative in the same equivalence class as u in $\mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_{n-k})$, then $\mathcal{P}_{u,v} \subseteq \mathcal{R}_{u',v}$.

(iii) If $\mathbf{u} \prec \mathbf{v}$, then $\mathcal{D}_{\mathbf{u},\mathbf{v}} \subseteq \mathcal{P}_{u,v}$.

2.2.2 The Positroid Decomposition

Positroid strata are in bijection with Ꞁ-diagrams. Positroids were originally defined to stratify positive Grassmannian. There have been several extensions of this stratification to the entire Grassmannian. When we say "positroid strata," we mean the stratification of the Grassmannian by projections of Richardson varieties in the full flag manifold, studied in [36].

The flag manifold, $Fl(n)$, is the collection of flags $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{R}^n$, where $\dim(V_i) = i$. There is a natural projection from $Fl(n)$ to $Gr(k, n)$, sending

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

to V_k. The flag manifold embeds into projective space via

$$\begin{aligned} Fl(n) &\hookrightarrow Gr(0,n) \times Gr(1,n) \times \cdots \times Gr(n,n) \\ &\hookrightarrow \mathbb{P}^{\binom{n}{0}-1} \times \mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n}-1} \\ &\hookrightarrow \mathbb{P}\left(\mathbb{R}^{\binom{n}{0}} \otimes \mathbb{R}^{\binom{n}{1}} \otimes \cdots \otimes \mathbb{R}^{\binom{n}{n}}\right) \\ &= \mathbb{P}^N, \end{aligned}$$

where the embedding on the first line is the product of the projections from $Fl(n)$ to $Gr(k,n)$, the embedding on the second line is the coordinatewise Plücker embedding and the embedding on the third line is the Segre embedding. For any point in the flag manifold, the coordinate $(\Delta_{I_0}, \Delta_{I_1}, \cdots, \Delta_{I_n})$ in the product of Plücker embeddings vanishes unless $I_0 \subset I_1 \subset \cdots \subset I_n$. These flags of sets are in bijection with permutations, identifying the permutation $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ with

$$\emptyset \subset \{\sigma(1)\} \subset \{\sigma(1), \sigma(2)\} \subset \cdots \subset \{\sigma(1), \sigma(2), \ldots, \sigma(n)\}.$$

Projecting out all but these coordinates, $Fl(n) \hookrightarrow \mathbb{P}^{n!-1}$. We denote these coordinates Δ_σ, for $\sigma \in \mathfrak{S}_n$, and call them Plücker coordinates for the flag manifold.

To a pair of permutations $u \leq v$, the associated Richardson cell is the subset of the flag manifold defined by

$$\begin{aligned} &\Delta_u, \Delta_v \neq 0, \\ &\Delta_w = 0 \text{ for all } w \notin [u,v]. \end{aligned}$$

Definition 2.2.2. Let $u \leq v$. The positroid cell $\mathcal{P}_{u,v}$ is the projection from $Fl(n)$ to $Gr(k,n)$ of the Richardson cell associated to $u \leq v$ in the flag manifold.

Multiple Richardson cells in the flag manifold project to the same positroid cell in the Grassmannian. [36] describes exactly when two Richardson cells project to the same positroid cell. The following theorem follows from Proposition 2.3 and Theorem 5.1 in [36].

Theorem 2.2.3. *Put an equivalence relation on Bruhat intervals, letting $[u,v] \sim [u',v']$ if and only if $\mathcal{P}_{u,v} = \mathcal{P}_{u',v'}$. Each equivalence class contains a unique interval $[u,v]$ such that v is a Grassmannian permutation.*

Positroid cells have many other descriptions. Another common description, also given in Theorem 5.1 of [36], is that positroid cells are intersections of cyclically shifted Schubert cells in the Grassmannian.

Positroid cells were originally introduced to study the positive Grassmannian. Positroid cells intersect the positive Grassmannian in their full dimension, and these intersections give $Gr_{\geq 0}(k,n)$ the structure of a CW-complex. A parameterization of $\mathcal{P}_{u,v} \cap Gr_{\geq 0}(k,n)$ is described in Section 2.2.4. This CW-complex was shown to be homeomorphic to a ball in [27]. If $\mathcal{P}$ is a positroid cell, $\mathcal{P}_{\geq 0}$ will denote its totally nonnegative part, $\mathcal{P} \cap Gr_{\geq 0}(k,n)$.

Theorem 2.2.4 (Theorem 3.5 in [53])**.** *Let $u, v \in \mathfrak{S}_n$, $u \leq v$, and v be Grassmannian. Then,*

$$\mathcal{P}_{u,v} \cap Gr_{\geq 0}(k,n) \cong \mathbb{R}_{>0}^{\dim(\mathcal{P}_{u,v})}.$$

These cells give $Gr_{\geq 0}(k,n)$ the structure of a CW-complex.

2.2.3 The Deodhar Decomposition

Deodhar components in the Grassmannian are another family of algebraic subsets obtained by setting some Plücker coordinates equal to zero and demanding other Plücker coordinates do not vanish. They are indexed by Go-diagrams, Definition 2.1.5. Our description of Deodhar components appears as Theorem 7.8 in [55]. Deodhar components were originally defined in [21].

Definition 2.2.5. Let b be a box in the Go-diagram D associated to the distinguished subword pair $\mathbf{u} \prec \mathbf{v}$. Define

$$I_b = u_{b^{in}} u_b (v_{b^{in}})^{-1} v\{n, n-1, \ldots, n-k+1\}. \tag{2.2}$$

Diagrammatically, I_b may be computed by:

- Changing the filling of all boxes in b^{out} to white stones,
- changing the filling of b to a plus,
- computing the pipe dream associated to this diagram, and
- setting I_b to be the labels of the pipes appearing along the left boundary of this pipe dream.

Another interpretation of the sets I_b, telling us how to recover the sets I_b given the Deodhar component, is given in Theorem 2.2.25, originally Theorem 1.17 in [6].

Example 2.2.6. A Go-diagram D is pictured on the left of Figure 2.3. In the Ferrers shape on the right, each box b is labelled with the set I_b from D.

+	●	+
●	+	○
+	○	+

456	245	234
146	145	134
126	125	124

Figure 2.3: Sets I_b labelling boxes of a Go-diagram.

Definition 2.2.7 (Theorem 7.8 in [55]). Let D be a Go-diagram of shape λ. Then, the *Deodhar component* $\mathcal{D}$ associated to D is the subset in $Gr(k,n)$ defined by:

- $\Delta_{I_b} = 0$ for all boxes $b \in D$ containing white stones.
- $\Delta_{I_b} \neq 0$ for all boxes $b \in D$ containing pluses.
- $\Delta_{I_\lambda} \neq 0$.
- $\Delta_S = 0$ for all $S \not\geq I_\lambda$.

Note that there is no constraint imposed on coordinates Δ_{I_b} where b contains a black stone. These coordinates are generically nonvanishing on the Deodhar component.

The following Theorem may be proved by combining Theorem 2.1.4 with the fact that reduced words for Grassmannian permutations do not admit braid moves.

Theorem 2.2.8 (Proposition 4.16 in [39]). *Let $\mathcal{R}_{u,v}$ be a Richardson cell in the Grammannian. The decomposition*

$$\mathcal{R}_{u,v} = \bigsqcup_{\mathbf{u} \preceq \mathbf{v}} \mathcal{D}_{\mathbf{u},\mathbf{v}}$$

does not depend on the choice of reduced word $\mathbf{v}$.

Remark 2.2.9. The situation in the Grassmannian differs from the situation in a general flag manifold, where different choices of $\mathbf{v}$ affect where in a Richardson cell a Deodhar component embeds.

Example 2.2.10. The Deodhar component associated to the Go-diagram in Figure 2.3 is the subset of $Gr(3,6)$ where

$$\Delta_{134}, \Delta_{125} = 0, \text{and}$$
$$\Delta_{123}, \Delta_{124}, \Delta_{126}, \Delta_{145}, \Delta_{234}, \Delta_{456} \neq 0.$$

In Chapter 4, where we study Go-diagrams and the Deodhar decomposition, we will use upper case letters to refer to Go-diagrams and calligraphic letters to refer to Deodhar components. When we want to make explicit reference to the distinguished subword pair associated to a Go-diagram, we will use the notation $\mathcal{D}_{\mathbf{u},\mathbf{v}}$.

Proposition 2.2.11. *Suppose that boxes b and c in a Go-diagram D share an edge. Then, I_b and I_c differ by a single element.*

Proof. Let $i_1 < i_2 < \cdots < i_k$ be the labels of the vertical steps of the boundary of D and $j_1 < j_2 < \cdots < j_{n-k}$ be the labels of the horizontal steps of the boundary of D and suppose $b = (i_\ell, j_m)$. Let p be the label of the pipe entering b from the bottom and q be the label of the pipe entering b from the left in the pipe dream associated to D. If $c = (i_\ell, j_{m+1})$, then $I_c = I_b \setminus p \cup j_{m+1}$. If $c = (i_{\ell-1}, j_m)$, then $I_c = I_b \setminus i_{\ell-1} \cup q$. □

2.2.4 Explicit Parameterizations

Deodhar components are homeomorphic to products of tori and affine spaces. This fact is shown in Deodhar's original paper introducing Deodhar components [21]. He gave a method for computing Kazhdan-Lusztig R-polynomials by counting points in Deodhar components over finite fields, and this homeomorphism reduced the point counting portion of this computation to an easy task. An explicit parameterization of Deodhar components was given in [47], and made combinatorial in terms of networks associated to Go-diagrams in [55]. Theorem 2.2.25 gives a way to recover the sets I_b from Definition 2.2.5 from these networks. Finally, in the case where a Go-diagram is a Ꞁ-diagram labelling a positroid $\mathcal{P}$, it was shown in [53] that the positive part of the parameterization is exactly $\mathcal{P}_{\geq 0}$.

Theorem 2.2.12 (Theorem 1.1 in [21])**.** *Let $\mathcal{D}$ be the Deodhar component labelled by the Go-diagram D. Then, $\mathcal{D}$ is homeomorphic to*

$$\mathbb{R}^{\#(\bullet' s\ in\ D)} \times (\mathbb{R} \setminus \{0\})^{\#(+'s\ in\ D)}.$$

Remark 2.2.13. In [21], this theorem is proved over an algebraically closed field. Using the explicit parameterizations of Deodhar components from [47], one see that this theorem holds over $\mathbb{R}$ as well.

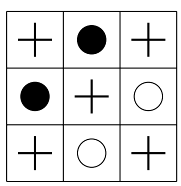
Definition 2.2.14 (Definition 3.2 in [55])**.** The *Go-network*, $N(D)$, associated to a Go-diagram D is built by:

- Placing a *boundary vertex* along each edge of D's southeast border.

- Placing an *internal vertex* for each plus or black stone in the diagram. We call these vertices +-vertices and •-vertices.
- From each internal vertex, drawing an edge right to the nearest +-vertex or boundary vertex.
- From each internal vertex, drawing an edge down to the nearest +-vertex or boundary vertex.
- Directing all edges left or down.

The vertical steps of D's boundary become sources in the Go-network and the horizontal steps become sinks.

Example 2.2.15. Figure 2.4 gives an example of a Go-diagram and its associated Go-network. The •-vertices have been drawn at a larger size to distinguish them.



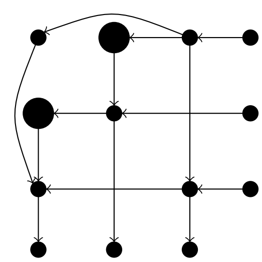


Figure 2.4: Go-network associated to a Go-diagram.

Definition 2.2.16. A *weighted Go-network* is a Go-network where the edges directed left toward +-vertices are weighted by elements of $\mathbb{R}^*$ and the edges directed left toward •-vertices are weighted by elements of $\mathbb{R}$. To a weighted Go-network, we associate the set of coordinates

$$\left\{ \Delta_J = \sum_{P:I\to J} \operatorname{sgn}(P) \prod_{p\in P} \prod_{e\in p} w(e) : J \subseteq [n] \right\}, \tag{2.3}$$

where sum is across all collections P of vertex disjoint paths from the set of sources I of the Go-network to J. Such a collection of paths is called a *flow*. The product is the product of the weights of all the edges appearing in all paths in P. The sign, $\operatorname{sgn}(P)$, is the sign of P viewed as a partial permutation. That is, $\text{sgn}(P) = (-1)^c$, where c is the number of edge crossings among the paths in P.

Example 2.2.17. Figure 2.5 gives a weighting of the Go-network from Example 2.2.15. The only collection of paths from sources $\{1,2,3\}$ to $\{1,2,3\}$ has no edges, so $\Delta_{123} = 1$.

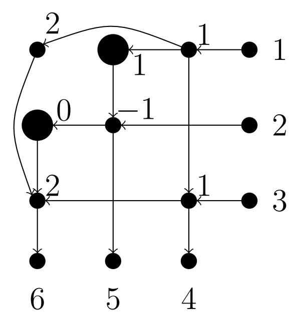


Figure 2.5: A weighted Go-network.

From $\{1,2,3\}$ to $\{2,3,6\}$, we need a path from 1 to 6. There are three such paths: the one along the top and left of the network has weight 2 and the one through the middle of the network has weight 0, and the one along the bottom row of the network has weight 2. So, $\Delta_{236} = 4$. There is no collection of vertex disjoint paths to $\{1,2,5\}$, so $\Delta_{125} = 0$. To $\{3,4,5\}$, there is one collection of vertex disjoint paths: a path from 1 to 4 of weight 1 and a path from 2 to 5 of weight -1. These paths have one edge crossing, introducing a factor of -1. So, $\Delta_{345} = 1$. Continuing in this fashion, one may compute:

$$\begin{array}{rclcrcl}
\Delta_{123} &=& 1, && \Delta_{234} &=& 1,\\
\Delta_{124} &=& 1, && \Delta_{235} &=& 1,\\
\Delta_{125} &=& 0, && \Delta_{236} &=& 4,\\
\Delta_{126} &=& 2, && \Delta_{245} &=& 1,\\
\Delta_{134} &=& 0, && \Delta_{246} &=& 2,\\
\Delta_{135} &=& -1, && \Delta_{256} &=& -2,\\
\Delta_{136} &=& 0, && \Delta_{345} &=& 1,\\
\Delta_{145} &=& -1, && \Delta_{346} &=& 0,\\
\Delta_{146} &=& 0, && \Delta_{356} &=& -2,\\
\Delta_{156} &=& 2, && \Delta_{456} &=& -2.
\end{array}$$

Theorem 2.2.18 (Theorem 3.16 in [55])**.** *The set of coordinates (2.3) of a weighted Go-network of shape $N(D)$ is the set of Plücker coordinates of some point in the Deodhar* *component $\mathcal{D}$. The map from weighted Go-networks to these coordinates is a bijection between weighted Go-networks of shape $N(D)$ and points in the Deodhar component $\mathcal{D}$.*

This theorem is a combinatorial realization of a theorem by Marsh and Reitsch, which provides a matrix parameterizing $\mathcal{D}$. We call this matrix the *Marsh-Reitsch matrix* of $\mathcal{D}$. Theorem 5.6 in [55] describes how to recover the Marsh-Reitsch matrix from the network $N(D)$.

Theorem 2.2.19 (Proposition 5.2 in [47]). *Let D be a Go-diagram with ℓ many black stones and m many pluses, and let $\mathcal{D}$ be the Deodhar component labelled by D. Then, there is a matrix $M_D(\mathbf{x}, \mathbf{y})$ in variables $(x_1, x_2, \ldots, x_\ell, y_1, y_2, \ldots, y_m)$ such that evaluating $\mathbf{x} \in \mathbb{R}^\ell$, $\mathbf{y} \in (\mathbb{R}^*)^m$ provides a parameterization of $\mathcal{D}$.*

When D is a ⅃-diagram, no collection of nonintersecting paths in $N(D)$ can cross. So, the sign term in (2.3) will always be positive. Thus, weighting all edges in $N(D)$ with positive edge weights will yield a point in the positive Grassmannian. This parameterization makes Theorem 2.2.12 explicit.

Theorem 2.2.20 (Theorem 5.13 in [39] and Theorem 6.5 in [53]). *Let D be a Go-diagram and let $\mathcal{D}$ be the Deodhar component labelled by D. Then, $\mathcal{D} \cap Gr_{\geq 0}(k, n) \neq \emptyset$ if and only if D is a ⅃-diagram. In this case, the map from weighted Go-networks of shape $N(D)$ with all positive edge weights to the Grassmannian is a bijection from $\mathbb{R}_{>0}^{\dim(\mathcal{D})}$ to $\mathcal{D} \cap Gr_{\geq 0}(k, n)$.*

Remark 2.2.21. In Chapter 3, calligraphic letters are reserved to denote set systems and matroids. Thus, we will notate the Deodhar component $\mathcal{D}$ by $MR(\mathcal{B})$ in this chapter to avoid overloading calligraphic letters. In this notation, MR stands for "Marsh-Rietsch" and $\mathcal{B}$ is a matroid. In that chapter, since we are using matroids rather than Go-diagrams as our indexing set of these cells, the matrix $M_D(\mathbf{x}, \mathbf{y})$ will be denoted $M_{\mathcal{B}}(\mathbf{x}, \mathbf{y})$.

The following corollary follows from Theorem 2.2.18.

Corollary 2.2.22. *Let D be a Go-diagram indexing the Deodhar component $\mathcal{D}$. The Plücker coordinate Δ_S vanishes uniformly on $\mathcal{D}$ if and only if there is no flow to S in $N(D)$.*

Proof. If there is not a flow to S, the sum (2.3) is empty, so Δ_S vanishes uniformly on $\mathcal{D}$. If there is a flow to S, choosing all of the edge weights to be algebraically independent yields a point in $\mathcal{D}$ where $\Delta_S \neq 0$. □

We will make reference to the following special case of this corollary.

Corollary 2.2.23. *Let D be a Go-diagram and suppose $b \in D$ contains a white stone. Then, there is not a flow to I_b in $N(D)$.*

For a point V in the Deodhar component $\mathcal{D}$ and a set $S \in \binom{[n]}{k}$, the corresponding Plücker coordinate, $\Delta_S(V)$, of V will not in general depend on the entire network. For instance, if $i_1 \in S$, then i_1 will not be involved in any flow determining $\Delta_S(V)$. So, the value of $\Delta_S(V)$ will not depend on the top row of the network. Similarly, if $j_{n-k} \notin S$, the value of $\Delta_S(V)$ will not depend on the leftmost column of the network. The following corollary generalizes this observation.

Corollary 2.2.24. *Let D be a Go-diagram inside the Ferrers shape with vertical steps $I = \{i_1, \dots, i_k\}$ and horizontal steps $J = \{j_1, \dots, j_{n-k}\}$, with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{n-k}$. Let V lie in the Deodhar component indexed by D, and let its associated weighted Go-network be $N(D)_V$. For $S \in \binom{[n]}{k}$ suppose the first ℓ elements of I are in S and last $n-k-m$ elements of J are not in S. That is,*

$$\{i_1, i_2, \dots, i_\ell\} \subseteq S, \text{ and}$$
$$\{j_{n-k}, \dots, j_{m+1}\} \cap S = \emptyset.$$

Then, the Plücker coordinate $\Delta_S(V)$ is determined by the restriction of $N(D)$ to the rows $i_{\ell+1}, i_{\ell+2}, \dots, i_k$ and columns $j_1, \dots, j_m$ of $N(D)_V$.

The network perspective gives an alternate algorithm for reading off the sets I_b defining equations for the Deodhar component from Definition 2.2.5. Suppose $b = (i_\ell, j_m)$ is in the row with vertical step i_ℓ and column with horizontal step j_m. The proof relies on interpreting $S = I_b \setminus j_m \cup i_\ell$ as the maximal element of $\binom{[n]}{k}$ in the Gale order that satisfies

$$i_1, i_2, \dots, i_\ell \in S,$$
$$j_{n-k}, j_{n-k-1}, \dots, j_m \notin S,$$

and such that Δ_S does not vanish uniformly on $\mathcal{D}$.

Theorem 2.2.25 (Theorem 1.17 in [6])**.** *Let b be a box in a Go-diagram D. Let D' be the Go-diagram obtained from D by:*

- *changing the filling of b to a plus,*
- *changing the filling of all boxes in b^{out} to white stones, and*

- *changing the filling of all boxes in b's row and b's column aside from b to white stones.*

Then, $I_b \in \binom{[n]}{k}$ *is the maximal set in the Gale order such that there is a collection of vertex disjoint paths flowing from the source nodes to* I_b *in* $N(D')$*, the Go-network associated to* D'*.*

Proof. Let D be a Go-diagram whose Ferrers shape has vertical steps $i_1 < \cdots < i_k$ and horizontal steps $j_1 < \cdots < j_{n-k}$. Let $b = (i_\ell, j_m)$ be a box in the Go-diagram D, and let D' be the diagram described in the theorem statement. In the pipe dream associated to D', any box in b^{out} or in b's row or column features a pipe $i_{\ell'}$ with $\ell' \leq \ell$ or a pipe $j_{m'}$ with $m' \geq m$. Pipes $i_{\ell'}$ with $\ell' \leq \ell$ and $j_{m'}$ with $m' \geq m$ cross any other pipe at most once in this pipe dream. So, none of these squares violate the distinguished property. For any box $(i_{\ell'}, j_{m'})$ with $\ell' > \ell$ and $m' < m$, $(i_{\ell'}, j_{m'})^{in}$ is identical in D and D'. So, none of these boxes violate the distinguished property and thus D' is a Go-diagram.

Let $M_b \in \binom{[n]}{k}$ be the maximal set in Gale order such that there is a collection of vertex disjoint paths flowing from the source nodes to M_b in the network $N(D')$. Let $M'_b = M_b \setminus j_m \cup i_\ell$. Then, M'_b is the maximal set in Gale order such that Δ_{M_b} does not vanish uniformly on $\mathcal{D}$, subject to the constraints

$$\begin{aligned} i_1, i_2, \ldots, i_\ell \in M'_b, \text{ and} \\ j_{n-k}, j_{n-k-1}, \ldots, j_m \notin M'_b. \end{aligned} \tag{2.4}$$

We show that $I_b = M'_b \setminus i_\ell \cup j_m$.

Let E be the diagram obtained by changing the filling of b to a white stone in D'. Using the same argument verifying that D' was a Go-diagram, we see that E is also a Go-diagram. Let $\mathcal{E}$ be the Deodhar component associated to E. The Go-network of E has no nodes in the rows $i_1, i_2, \ldots, i_\ell$ or the columns $j_{n-k}, j_{n-k-1}, \ldots, j_m$. So, Theorem 2.2.18 implies that Δ_S vanishes on $\mathcal{E}$ if $\{i_1, i_2, \ldots, i_\ell\} \nsubseteq S$ or $S \cap \{j_{n-k}, j_{n-k-1}, \ldots, j_m\} \neq \emptyset$. Let $\mathcal{V} \subseteq Gr(k, n)$ be the variety defined by

$$\Delta_S = 0 \text{ for all } S \text{ such that } \{i_1, i_2, \ldots, i_\ell\} \nsubseteq S \text{ or } S \cap \{j_{n-k}, j_{n-k-1}, \ldots, j_m\} \neq \emptyset.$$

Corollary 2.2.24 implies that $\mathcal{E} = \mathcal{D} \cap \mathcal{V}$. Let u', v' be the permutations associated to the diagram E. Proposition 2.2.1 says that $\Delta_{u'(n)u'(n-1)\ldots u'(n-k)} \neq 0$ on $\mathcal{E}$ and $\Delta_S = 0$ on $\mathcal{E}$ for all $S \nleq \{u'(n), u'(n-1), \ldots, u'(n-k)\}$. So,

$$\{u'(n), u'(n-1), \ldots, u'(n-k)\} = M'_b.$$

From Definition 2.2.5 and the diagrammatic description following, $M'_b = s_b I_b^E$. Turning our attention back to the diagram D', the transposition s_b in box b exchanges i_ℓ and j_m. So, $I_b = M'_b \setminus i_\ell \cup j_m = M_b$, as desired. □

The following proposition is an immediate consequence of Corollary 2.2.24, together with the fact that the Deodhar decomposition refines the Richardson decomposition, Proposition 2.2.1. This proposition will be used to verify the vanishing of certain Plücker coordinates in Section 4.2.

Proposition 2.2.26. *Suppose the vertical steps of the boundary of the Ferrers shape λ are $i_1 < i_2 < \cdots < i_k$ and the horizontal steps are $j_1 < j_2 < \cdots < j_{n-k}$. Let $b = (i_\ell, j_m)$ be a box in the Go-diagram D of shape λ indexing the Deodhar component $\mathcal{D}$. Let*

$$J_b = u_{b^{in}}(v_{b^{in}})^{-1}v\{n, n-1, \ldots, n-k+1\}. \tag{2.5}$$

If $S \in \binom{[n]}{k}$ is such that $i_1, i_2, \ldots, i_{\ell-1} \in S$, $j_{n-k}, j_{n-k-1}, \ldots, j_{m+1} \notin S$, and $S \not\leq J_b$, then Δ_S vanishes uniformly on $\mathcal{D}$.

Diagrammatically, the set J_b may be computed similarly to I_b, only b retains its original filling in the computation, rather than being changed to a plus.

2.3 Matroidal Background

Definition 2.3.1. A *matroid* is a collection of sets $\mathcal{B} \subseteq \binom{[n]}{k}$ such that for each $S, T \in \mathcal{B}$ and each $i \in S \setminus T$ there is some $j \in T \setminus S$ such that $S \setminus i \cup j$ and $T \setminus j \cup i$ are both in $\mathcal{B}$.

This definition is a combinatorial abstraction the Plücker exchange relations (2.1). For any $V \in Gr(k, n)$ the set

$$\left\{ I \in \binom{[n]}{k} : \Delta_I(V) \neq 0 \right\}$$

is a matroid, called the *matroid represented by V*.

We quickly review some terminology from matroid theory. For a more detailed introduction to matroids, see [17] or [34]. For a matroid $\mathcal{B}$, the sets in $\mathcal{B}$ are called *bases* of the matroid. The set $[n]$ is called the *ground set* of the matroid. A set $I \subseteq [n]$ is *independent* if it is contained in some element of $\mathcal{B}$. Otherwise, I is called *dependent*. The *rank* of I, $\mathrm{rk}(I)$, is the size of the largest independent subset of I. An element $x \in [n]$ is a *loop* if

$\text{rk}(x) = 0$ and is a *coloop* if $x \in B$ for all $B \in \mathcal{B}$. A set I is a *circuit* if it is dependent and all of its proper subsets are independent. A set I is a *cocircuit* if it is a minimal set intersecting every basis. The set I is a *flat* if for all $x \in [n] \setminus I$, $\text{rk}(I \cup x) = \text{rk}(I) + 1$. A flat is called a *cyclic flat* if it is a union of circuits.

Given $I \subseteq [n]$, the *restriction* of $\mathcal{B}$ to I, denoted $\mathcal{B}|_I$, is the collection of subsets $C \subseteq I$ of maximal size such that $C \subseteq B$ for some $B \in \mathcal{B}$. The *deletion* of I, denoted $\mathcal{B} \setminus I$, is the collection of subsets $C \subseteq [n] \setminus I$ of maximal size such that $C \subseteq B$ for some $B \in \mathcal{B}$. The *contraction* of I, denoted $\mathcal{B}/I$, is the collection of subsets $C \subseteq [n] \setminus I$ such that $C \cup D \in B$, where D is any maximal rank subset of I. The result of a restriction, deletion, or contraction of a matroid is another matroid, which is called a *minor* of $\mathcal{B}$. Note that the ground set $[n]$ is implicitly ordered $1 < 2 < \cdots < n$ and that the ground sets of $\mathcal{B}|_I$, $\mathcal{B} \setminus I$, and $\mathcal{B}/I$ naturally inherit orderings from this ordering on $[n]$. This feature is not always present when studying general matroids, but is a necessary feature when studying positivity phenomena.

The *dual* of a matroid, $\mathcal{B}^*$, is $\{[n] \setminus B : B \in \mathcal{B}\} \subseteq \binom{[n]}{n-k}$. The dual of a matroid is a matroid, and $\mathcal{B}^{**} = \mathcal{B}$. Deletion and contraction are dual operations in the sense that $(\mathcal{B} \setminus I)^* = \mathcal{B}^*/I$.

The *direct sum* of matroids $\mathcal{B}$ and $\mathcal{B}'$ on disjoint ground sets, denoted $\mathcal{B} \oplus \mathcal{B}'$, is the set of $B \cup B'$ where $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$. A matroid is *connected* if it cannot be written as the direct sum of two nontrivial matroids. If $\mathcal{B}$ is the direct sum of connected matroids, each constituent in this sum is called a *connected component* of $\mathcal{B}$.

For $S \subset [n]$, the *indicator vector* of S is the vector in $\mathbb{R}^{[n]}$ which has a 1 in the s^{th} coordinate if $s \in S$ and a 0 in the s^{th} coordinate if $s \notin S$. The *matroid polytope* of $\mathcal{B}$ is the convex hull of the indicator vectors of the sets in $\mathcal{B}$ in $\mathbb{R}^{[n]}$.

Theorem 2.3.2 (Proposition 2.6 in [23])**.** *If $\mathcal{B}$ is a rank k connected matroid on the ground set $[n]$, the matroid polytope of $\mathcal{B}$ is the subset of the simplex*

$$\left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^{[n]}_{\geq 0} : \sum_{i=1}^{n} x_i = k\right\}$$

defined by the inequalities

$$\sum_{i \in F} x_i \leq \text{rk}(F),$$

where F is some flat such that $\mathcal{B}|_F$ and $\mathcal{B}/F$ are both connected. Each such inequality defines a facet of the matroid polytope.

In light of this theorem, flats F of $\mathcal{B}$ such that $\mathcal{B}|_F$ and $\mathcal{B}/F$ are both connected are called *flacets*. All flacets containing more than one element are cyclic flats. To see this, note that if $|F| > 1$ and $\mathcal{B}|_F$ is connected, then for every $f \in F$ there must be some basis B_f of $\mathcal{B}|_F$ not containing f. Then, $f \cup B_f$ yields a circuit containing f. Then,

$$F = \bigcup_{f \in F} f \cup B_f.$$

The *matroid strata*, $V_{\mathcal{B}} \subseteq Gr(k,n)$, is the set of points in $Gr(k,n)$ representing $\mathcal{B}$. In the literature, $V_{\mathcal{B}}$ is also referred to as a thin Schubert cell or GGMS strata, after Gelfand, Goresky, MacPherson, and Serganova. All of these names are misleading, as the $V_{\mathcal{B}}$ are in general not cells and do not stratify the Grassmannian.

2.3.1 Matroids from Grassmannian Decompositions

The Schubert, Richardson, positroid, and Deodhar decompositions of the Grassmannian each give rise to a family of matroids obtained by looking at the matroid represented by a generic point in each cell in the decomposition. For example, the Schubert cell $\mathcal{S}_{13} \in Gr(2,4)$ is defined by $\Delta_{12} = 0, \Delta_{13} \neq 0$. At a generic point in $\mathcal{S}_{13}$, all other Plücker coordinates will be nonzero. So, the matroid associated to this cell is

$$\{13, 14, 23, 24, 34\}.$$

For each of these decompositions, if $\mathcal{B}$ is the matroid associated to a component, $V_{\mathcal{B}}$ is a dense subset of the component. The families of matroids associated to the common decompositions of the Grassmannian are given in the Table 2.2.

Decomposition	Family of Matroids
Schubert	Schubert matroids[1]
Richardson	Lattice path matroids
Positroid	Positroids
Deodhar	Deodroid

Table 2.2: Families of matroids from decompositions of $Gr(k,n)$.

[1]Schubert matroids are also called shifted matroids, nested matroids, and freedom matroids.

These classes of matroids can be described purely combinatorially in a fashion exactly identical to description of the components of these decompositions in Section 2.2.1. So, Schubert matroids are up-sets in the poset on $\binom{[n]}{k}$. Lattice path matroids are intervals in this poset. Schubert and lattice path matroids have both received a fair bit of attention in the structural combinatorics literature. The second half of the survey [16] is devoted to lattice path matroids.

Positroids are projections of Bruhat intervals from $\mathfrak{S}_n$ to $\mathfrak{S}_n/(\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$. Positroids will receive a more detailed description, and several other equivalent definitions in Section 2.3.2. To our knowledge, no one has yet considered the family of matroids represented by generic points in Deodhar components. We take this opportunity to give them a silly name.

Definition 2.3.3. A *Deodroid* is a matroid represented by a generic point in a Deodhar component $\mathcal{D}$ for some Go-diagram D.

We speculate some about the class of Deodroids in Section 2.4.

We saw in Section 2.2.1 that when $u, v \in \mathfrak{S}_n$ were both Grassmannian with $u \le v$, that they could be used to define either a Richardson cell $\mathcal{R}_{u,v}$ or a positroid cell $\mathcal{P}_{u,v}$. In this case, $\mathcal{R}_{u,v}$ and $\mathcal{P}_{u,v}$ are defined by setting the same set of Plücker coordinates to zero, then $\mathcal{P}_{u,v}$ imposes non-vanishing constraints on a larger set of Plücker coordinates than $\mathcal{R}_{u,v}$. However, at generic points in $\mathcal{R}_{u,v}$ or $\mathcal{P}_{u,v}$, the same set of coordinates will be non-vanishing. Hence, the lattice path matroid associated to $\mathcal{R}_{u,v}$ and the positroid associated to $\mathcal{P}_{u,v}$ will be the same matroid. The following proposition records this observation.

Proposition 2.3.4. *Let v be a Grassmannian permutation.*

(i) The lattice path matroid associated to the Richardson cell $\mathcal{R}_{u,v}$ is a Schubert matroid if and only if $u = \varepsilon$.

(ii) The positroid associated to the positroid cell $\mathcal{P}_{u,v}$ is a lattice path matroid if and only if u is Grassmannian.

(iii) The Deodroid associated to the Deodhar component $\mathcal{D}_{\mathbf{u},\mathbf{v}}$ is a positroid if and only if $\mathbf{u}$ is the positive distinguished subexpression for u in $\mathbf{v}$.

Given a $\bullet/\circ/+$-diagram representing $\mathbf{u} \prec \mathbf{v}$, there are easy graphical tests for when the diagram indexes a Schubert matroid, Richardson matroid or positroid. The diagram represents a positroid if and only there are no black stones. A diagram with no black stones represents a lattice path matroid if and only every box with a plus above it or to its left also contains a plus. The name “lattice path matroid” comes from the fact that all pluses in the diagram are contained in between boundary of the Ferrers shape λ and the lattice path obtained by drawing the Ferrers shape associated to u inside of λ. A diagram represents a Schubert matroid if and only if every square is filled with a plus.

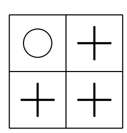
2.3.2 Positroids

This section provides background on positroids, the class of matroids. As is common in the literature, we will use the word “positroid” to refer to both a matroid and a component in the positroid decomposition of the Grassmannian, relying on context to clarify which is meant.

Definition 2.3.5. A *positroid* is a matroid which is representable by some point in the positive Grassmannian.

Given a Ⅎ-diagram D, there are several ways to compute the associated positroid. Using the characterization of positroids as projections of Bruhat intervals to $\mathfrak{S}_n/(\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$, we may compute the permutations u, v associated to D. Then, the positroid is the projection the interval $[u, v]$ from $\mathfrak{S}_n$ to $\mathfrak{S}_n/(\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$. Alternatively, the positroid associated D is the matroid represented by a generic point in the associated positroid cell. The network parameterization from Section 2.2.4 captures a dense subset of the positroid cell. Generically, a Plücker coordinate Δ_S is nonzero on this parameterized subset if and only if there is a flow from I_λ to S in $N(D)$. Hence, the positroid associated to D is the collection of all $S \in \binom{[n]}{k}$ such that there is a flow to S in $N(D)$.

Example 2.3.6. Consider the following Ⅎ-diagram.



The associated permutations are $u = (1, 3, 2, 4)$ from the pipe dream and $v = (3, 4, 1, 2)$ from the Ferrers shape. The Bruhat interval $[u, v]$ is given in Figure 2.6. The map from $\mathfrak{S}_4$ to $\binom{[4]}{2}$ sends any permutation to its last two entries. So, the matroid associated to this Ⅎ-diagram is

$$\{12, 13, 14, 23, 24\}.$$

The reader can verify that this is exactly the collection of sets there is a flow to in the associated ⅃-network.

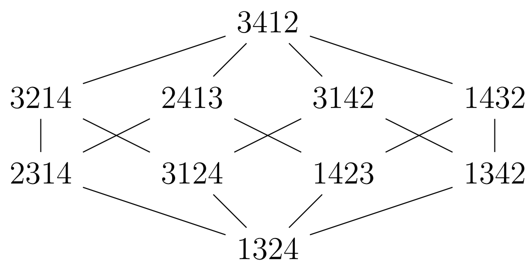


Figure 2.6: The Bruhat interval $[1324, 3412]$.

Given an $n \times k$ matrix M, cyclically shifting all the columns of M by 1 and multiplying the final column of the resulting matrix by $(-1)^{k-1}$ permutes the Plücker coordinates. Thus, the class of positroids is closed under cyclic rotation of the ground set.

Deleting any subset of M's columns evidently does not change the sign of the maximal minors that do not use these columns. So, the class of positroids is closed under restriction. Projecting the Bruhat interval $[u, v]$ to $\binom{[n]}{n-k}$ by taking the first $n-k$ entries of a permutation rather than projecting to $\binom{[n]}{k}$ yields the dual to the positroid obtained from $[u, v]$. Since the class of positroids is closed under restriction and duality, it is also closed under contraction.

Ardila, Rincón, and Williams characterize positroids in terms of their flacets and connected components in [7]. A *cyclic interval* in $[n]$ is an ordinary interval, or a set of the form $a, a+1, \ldots, n, 1, 2, \ldots, b$.

Theorem 2.3.7 (Proposition 5.6 in [7])**.** *A connected matroid is a positroid if and only if every flacet is a cyclic interval.*

Theorem 2.3.8 (Theorem 7.6 in [7])**.** *A matroid $\mathcal{B}$ is positroid if and only if each connected component of $\mathcal{B}$ is a positroid and the connected components of $\mathcal{B}$ form a noncrossing partition of $[n]$.*

When $\mathcal{B}$ is a positroid, it also has a Deodhar component associated to it. This Deodhar component contains the matroid stratum $V_{\mathcal{B}}$.

Theorem 2.3.9 (Corollary 7.9 in [55])**.** *Let $\mathcal{B}$ be a positroid and $\mathcal{D}$ be the associated Deodhar component. Then, $V_{\mathcal{B}} \subseteq \mathcal{D}$.*

2.3.3 Transversal Matroids

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ be a collection of subsets of $[n]$. The indexing of the elements of $\mathcal{S}$ does not matter, and $\mathcal{S}$ can possibly contain repeated sets. Let $\Gamma_{\mathcal{S}}$ be the bipartite graph where one part has nodes labelled $S_1, S_2, \ldots, S_k$, the other part has nodes labelled $1, 2, \ldots, n$, and there is an edge between S_i and j if and only if $j \in S_i$. For an illustration, see Example 2.3.10. A *matching* is a set of edges such that every vertex is incident to at most one edge in the matching. A set of vertices is *saturated* by the matching if every vertex in the set is incident to an edge in the matching. Let $\mathcal{B}(\mathcal{S}) \subseteq \binom{[n]}{k}$ consist of subsets B such that there is a matching saturating B in $\Gamma_{\mathcal{S}}$. The collection $\mathcal{B}(\mathcal{S})$ is a matroid. Matroids obtained in this way are called *transversal matroids*, and the set $\mathcal{S}$ is called a *presentation* of the transversal matroid $\mathcal{B}(\mathcal{S})$. If $\mathcal{B}$ is a transversal matroid with rank k, then $\mathcal{B} = \mathcal{B}(\mathcal{S})$ for some $\mathcal{S}$ with $|\mathcal{S}| = k$. We consider only these presentations, where $|\mathcal{S}| = \mathrm{rk}(\mathcal{B}(\mathcal{S}))$.

In general, a transversal matroid has multiple presentations. Presentations are partially ordered by $\mathcal{S} \preccurlyeq \mathcal{S}'$ if there is some ordering of the sets $S_1, \ldots, S_k$ and $S'_1, \ldots, S'_k$ in the set systems $\mathcal{S}$ and $\mathcal{S}'$ such that $S_i \subseteq S'_i$ for each i. A transversal matroid has a unique maximal presentation, and usually has several minimal presentations. For more detailed background on transversal matroids and their presentations than we provide here, see [19].

To avoid overly cluttered notation, we often suppress set braces when dealing with set systems. For instance, we will often write $\mathcal{S} \setminus S_1$ to mean $\mathcal{S} \setminus \{S_1\}$.

Example 2.3.10. Let $\mathcal{S}$ be the set system consisting of

$$\begin{aligned} S_1 &= \{1, 3, 4\}, \\ S_2 &= \{1, 2\}, \\ S_3 &= \{2, 3\}. \end{aligned}$$

The bipartite graph $\Gamma_{\mathcal{S}}$ is drawn in Figure 2.7. In $\Gamma_{\mathcal{S}}$, there is a matching saturating every element of $\binom{[4]}{3}$. So, $\mathcal{B}(\mathcal{S}) = \binom{[4]}{3}$. Let

$$\mathcal{S}' = (\mathcal{S} \setminus S_1) \cup \{3, 4\}.$$

So, $\Gamma_{\mathcal{S}'}$ is isomorphic to the graph obtained by deleting the edge from S_1 to 1. In $\Gamma_{\mathcal{S}'}$, there is also a matching saturating every element of $\binom{[4]}{3}$. So, $\mathcal{B}(\mathcal{S}') = \mathcal{B}(\mathcal{S})$, and $\mathcal{S}' \prec \mathcal{S}$.

The following theorem appears several times in the literature, for instance following from Theorems 3.2 and 3.7 in [19], or as Lemma 2 in [15].

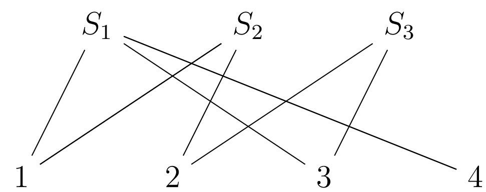


Figure 2.7: Example of the graph $\Gamma_{\mathcal{S}}$.

Theorem 2.3.11. *A presentation $\mathcal{S}$ of the transversal matroid $\mathcal{B}(\mathcal{S})$ is minimal if and only if each set in $\mathcal{S}$ is a distinct cocircuit of $\mathcal{B}(\mathcal{S})$.*

Theorem 2.3.12 (Theorem 2 in [15])**.** *Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ and $\mathcal{S}' = \{S'_1, \ldots, S'_k\}$ be minimal presentations of the transversal matroid $\mathcal{B}(\mathcal{S})$. Then, $\mathcal{B}(\mathcal{S})$ has a unique maximal presentation $\mathcal{M} = \{M_1, \ldots, M_k\}$. We may take the sets in the set systems $\mathcal{S}, \mathcal{S}'$ to be ordered such that $S_i \cup S'_i \subseteq M_i$ for all i. In this case, $|S_i| = |S'_i|$ for all i.*

If the graph $\Gamma_{\mathcal{S}}$ is disconnected, then the matroid $\mathcal{B}(\mathcal{S})$ is the direct sum of the transversal matroids coming from the connected components of $\Gamma_{\mathcal{S}}$. If $\mathcal{S}$ is a minimal presentation of $\mathcal{B}(\mathcal{S})$, then $\mathcal{B}(\mathcal{S})$ is a connected matroid if and only if $\Gamma_{\mathcal{S}}$ is connected.

Given $\mathcal{T} \subseteq \mathcal{S}$, the set

$$F(\mathcal{T}) = \{i \in [n] : i \notin S \quad \forall \quad S \notin \mathcal{T}\} \tag{2.6}$$

is a flat of the matroid $\mathcal{B}(\mathcal{S})$.

Lemma 2.3.13 (Corollary 2.8 in [16])**.** *Let F be a cyclic flat of the transversal matroid $\mathcal{B}(\mathcal{S})$. Then, $F = F(\mathcal{T})$ for some $\mathcal{T} \subseteq \mathcal{S}$ such that $|\mathcal{T}| = \mathrm{rk}(F)$. In particular, if $\mathcal{B}$ is connected, all flacets of $\mathcal{B}$ containing more than one element are of the form $F(\mathcal{T})$ with $|\mathcal{T}| = \mathrm{rk}(F(\mathcal{T}))$.*

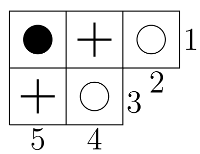
Given a set $I \subseteq [n]$, the restriction $\mathcal{B}(\mathcal{S})|_I$ is a transversal matroid and has the presentation $\mathcal{S}|_I = \{S \cap I : S \in \mathcal{S}\}$. Even if $\mathcal{S}$ is a minimal presentation of $\mathcal{B}(\mathcal{S})$, $\mathcal{S}|_I$ is not necessarily a minimal presentation of $\mathcal{B}(\mathcal{S})|_I$. However, if I is a cyclic flat of $\mathcal{B}(\mathcal{S})$, then $\mathcal{S}|_I$ will be a minimal presentation of $\mathcal{B}(\mathcal{S})|_I$, after possibly removing copies of the empty set from $\mathcal{S}|_I$.

The class of transversal matroids is not closed under contractions. However, for any flat of the form $F(\mathcal{T})$, if $\mathrm{rk}(F(\mathcal{T})) = |\mathcal{T}|$, then the contraction $\mathcal{B}(\mathcal{S})/F(\mathcal{T})$ is a transversal matroid on the ground set $[n]\backslash F(\mathcal{T})$ with presentation $\mathcal{S}\backslash\mathcal{T}$. If $\mathcal{S}$ is a minimal presentation of $\mathcal{B}(\mathcal{S})$, then $\mathcal{S}\setminus\mathcal{T}$ is a minimal presentation of $\mathcal{B}(\mathcal{S})/F(\mathcal{T})$. In particular, this fact holds if $F(\mathcal{T})$ is a cyclic flat.

2.4 Speculation: Deodroids

Let D be a Go-diagram and $\mathcal{B}$ be the associated Deodroid. So, every B such that Δ_B is not uniformly vanishing on the Deodhar component $\mathcal{D}$ is a basis of $\mathcal{B}$.

Example 2.4.1. Let D be the following Go-diagram.



The Deodhar component Deodhar component labelled by D, $\mathcal{D}$, is defined by

$$\Delta_{12}, \Delta_{14}, \Delta_{23} = 0$$
$$\Delta_{13}, \Delta_{15}, \Delta_{34} \neq 0.$$

The Plücker relation

$$\Delta_{14}\Delta_{23} = \Delta_{24}\Delta_{13} - \Delta_{34}\Delta_{12}$$

reduces to $\Delta_{24}\Delta_{13} = 0$. So, $\Delta_{24} = 0$ on $\mathcal{D}$ as well. Similarly, one may check that Δ_{25} also vanishes on $\mathcal{D}$, and that no other Plücker coordinates vanish uniformly on $\mathcal{D}$. So, the Deodroid associated to $\mathcal{D}$ is

$$\mathcal{B} = \{13, 15, 34, 35, 45\}.$$

The bases of a Deodroid have another perhaps more attractive description: From Corollary 2.2.22, the bases of $\mathcal{B}$ are sets B such that there is a vertex disjoint flow in $N(D)$ from the sources to B.

A common question to ask about matroids is whether or not they are closed under restriction and contraction. If a class is closed under minors, one may put a partial order on the class saying $\mathcal{B} \leq \mathcal{C}$ if $\mathcal{B} = (\mathcal{C}/J)|_I$ for some disjoint sets $I, J \subseteq [n]$. The set of all matroids is evidently closed under minors. For a minor closed class, it is common to look for minimal elements in the compliment of the class, called a set of excluded minors for the class. We record a special case of computing minors of Deodroids below, as well as some other simple facts, which are essentially matroidal versions of facts appealed to in the proof of Theorem 2.2.25. We conclude this section by conjecturing that the class of Deodroids is closed under minors, and conjecture a set of excluded minors for the class, (2.7). For positroids, several combinatorial and diagrammatic algorithms have been given to compute minors, for instance in [50]. If Deodroids are closed under minors, it would be useful to extend these algorithms to the class of Deodroids.

Another potentially interesting question one might ask about Deodroids is whether the class is closed under the action of any group on the ground set $\{1, \dots, n\}$. For instance, the class of positroids is closed under the action of the cyclic group on $\{1, \dots, n\}$ by rotation, but not under the group of all permutations.

Lemma 2.4.2. *Let D be Go-diagram whose Ferrers shape has vertical steps $I = \{i_1, \dots, i_k\}$ and horizontal steps $\{j_1, \dots, j_{n-k}\}$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{n-k}$ and let $\mathcal{B}$ be the associated Deodroid. Let E be the diagram obtained by changing the filling of all boxes in column j_{n-k} in D to white stones and let $\mathcal{B}'$ be the Deodroid associated to E. Then, $\mathcal{B} \setminus j_{n-k} = \mathcal{B}'$. Similarly, let F be the diagram obtained by changing the filling of all boxes in row i_1 to a white stone and $\mathcal{B}''$ be the Deodroid associated to F. Then, $\mathcal{B}/i_1 = \mathcal{B}''$.*

Proof. First consider the deletion $\mathcal{B} \setminus j_{n-k}$. Let E is the diagram obtained by changing the filling of all boxes in column j_{n-k} in D to a white stone and let $\mathcal{B}'$ be the Deodroid associated to E. As shown in the proof of Theorem 2.2.25, E is a valid Go-diagram. In the Go-network associated to E, there are no edges directed into the sink node j_{n-k}. So, $j_{n-k} \notin S$ for all $S \in \mathcal{B}(E)$. For any set $S \in \binom{[n] \setminus j_{n-k}}{k}$, Corollary 2.2.24 says that Δ_S depends only on the columns $j_1, \dots, j_{n-k-1}$ of the diagram. Since the restrictions of D and E to the columns $j_1, \dots, j_{n-k-1}$ are identical, Δ_S is generically nonvanishing on $\mathcal{D}$ if and only if Δ_S is generically non-vanishing on $\mathcal{E}$. So, $\mathcal{B}' = \{S \in \mathcal{B} : j_{n-k} \notin S\}$, as desired. The proof for the contraction $\mathcal{B}/i_1$ is similar and is left to the reader. □

Corollary 2.4.3. *Let D be a Go-diagram inside the Ferrers shape with vertical steps $I = \{i_1, \dots, i_k\}$ and horizontal steps $\{j_1, \dots, j_{n-k}\}$, with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{n-k}$ and let $\mathcal{B}$ be the associated Deodroid. Let E be the diagram obtained by changing the filling of all boxes in rows $i_1, i_2, \dots, i_\ell$ and columns $j_{n-k}, j_{n-k-1}, \dots, j_m$ to white stones. Then,*

$$\mathcal{B} \setminus j_{n-k}, j_{n-k-1}, \dots, j_m / i_1, i_2, \dots, i_\ell = \mathcal{B}',$$

where $\mathcal{B}'$ is the Deodroid associated to E.

Lemma 2.4.4. *Let D be a Go-diagram indexing the distinguished subword pair $\mathbf{u} \prec \mathbf{v}$ and let $\mathcal{B}$ be the Deodroid associated to D. Let*

$$\begin{aligned} I &= v\{n, n-1, \ldots, n-k\}, \text{ and} \\ J &= u\{n, n-1, \ldots, n-k\}. \end{aligned}$$

Then, $I, J \in \mathcal{B}$ and for all $B \in \mathcal{B}$, $I \leq B \leq J$.

Proof. Let $\mathcal{D}$ be the Deodhar component associated to D. The Deodhar decomposition refines the Richardson decomposition, so $\mathcal{D} \subseteq \mathcal{R}_{u,v}$. The Richardson cell $\mathcal{R}_{u,v}$ is defined by

$$\begin{aligned} \Delta_I, \Delta_J &\neq 0, \\ \Delta_B = 0 &\text{ for all } B \notin [I, J], \end{aligned}$$

where $[I, J]$ is the interval between I to J in Gale order. The inequalities above imply that $I, J \in \mathcal{B}$ and the equalities imply that $\mathcal{B} \subseteq [I, J]$. □

Remark 2.4.5. These facts are matroidal translations of ideas appearing in the proof of Theorem 2.2.25.

Conjecture 2.4.6. *The class of Deodroids is closed under restriction and contraction. Further, the matroids*

$$\begin{aligned} &\{12, 14, 23, 34\}, \textit{ and} \\ &\{12, 14, 23, 24, 34\} \end{aligned} \tag{2.7}$$

form a complete set of excluded minors for the class of Deodroids.

For intuition about why the first half of this conjecture should be true, note that if the matroid $\mathcal{B}$ is represented by the point in $V \in Gr(k, n)$, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ is the ordered basis used to compute the Plücker coordinates of V, then $\mathcal{B} \setminus n$ is the matroid represented by the projection of V onto its first $n-1$ coordinates and $\mathcal{B}/n$ is represented by the intersection of V with the hyperplane defined by $(x_n = 0)$ in $\mathbb{R}^n$. In the deletion, $V \cap \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1})$ is viewed as a plane in $\mathbb{R}^{n-1} \cong \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1})$ and in the contraction, $V \cap (x_n = 0)$ is viewed as a plane in the subspace $(x_n = 0)$. Deletions and contractions of other elements are obtained similarly. For a much more detailed geometric introduction to matroids, see [34].

Let V be in the Deodhar component $\mathcal{D}$ and represent the Deodroid $\mathcal{B}$. Let V' be obtained from V via projections and intersections with subspaces as described above. So,

V' represents some minor of $\mathcal{B}$ in the appropriate Grassmannian. Since the Grassmannian is the union of its Deodhar components, if V' lies in some Deodhar component $\mathcal{D}'$. However, V' is not necessarily guaranteed to represent $\mathcal{B}'$. The first part of the conjecture amounts to saying that if V was suitably generic, then V' represents the matroid $\mathcal{B}'$. That is, generic points in $\mathcal{D}$ get mapped to generic points in $\mathcal{D}'$ via the vector space maps realizing deletion and contraction.

For the second half of the conjecture, one can check that the two matroids in (2.7) are not Deodroids. Since every one element matroid is a Deodroid, these two matroids must then be excluded minors for the class, if the class is minor closed. Oh proved in [50] that the complete set of excluded minors for the class of positroids are the two matroids in (2.7) together with the matroid

$$\{12, 13, 14, 23, 34\}. \tag{2.8}$$

This matroid is different from the second matroid in (2.7) because we are considering matroids on an ordered ground set. This matroid is a Deodroid, represented by the Go-diagram

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+	○

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which leads us to the second half of the conjecture.

We caution that the class of Deodroids is not closed under order preserving duality. The dual of Deodroid (2.8) is the second matroid on our conjectured list of excluded minors (2.7). So, any proof of Conjecture 2.4.6 will have to treat the cases of contraction and restriction separately.

Chapter 3

Basis Shape Loci and the Positive Grassmannian

Throughout this chapter, $\mathcal{S} = \{S_1, \dots, S_k\}$ will be a family of subsets of $[n]$. The family $\mathcal{S}$ is unordered and may contain repeated sets. We say that S_i is the *support set* of a vector $\mathbf{v}_i$, $\mathrm{supp}(\mathbf{v}_i) = S_i$, if the j^{th} coordinate of $\mathbf{v}_i$ is nonzero if and only if $j \in S_i$ for all $j \in [n]$.

Definition 3.0.1. Given $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$, a family of subsets of $[n]$, the *basis shape locus* of $\mathcal{S}$ is the subset

$$L(\mathcal{S}) = \big\{\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) : \mathbf{v}_i \in \mathbb{R}^{[n]}, \mathrm{rk}(\mathbf{v}_1, \dots, \mathbf{v}_k) = k, \mathrm{supp}(\mathbf{v}_i) = S_i \text{ for all } i\big\}$$

of $Gr(k, n)$.

For $1 \le i \le k$, $1 \le j \le n$, let x_{ij} be algebraically independent, invertible variables. Let $M_{\mathcal{S}}(\mathbf{x})$ be the $k \times n$ matrix whose i, j entry is x_{ij} if $j \in S_i$ and is zero otherwise. So, any point in $L(\mathcal{S})$ may be obtained by evaluating the x_{ij} at nonzero real numbers in $M_{\mathcal{S}}(\mathbf{x})$, then taking the row span of the resulting matrix.

Also associated to $\mathcal{S}$ is a transversal matroid $\mathcal{B}(\mathcal{S})$. Background on transversal matroids is provided in Section 2.3.3. Theorem 3.1.9 shows that the closure $\overline{L(\mathcal{S})}$ depends only on the matroid $\mathcal{B}(\mathcal{S})$.

A naive upper bound on the dimension of $L(\mathcal{S})$ is

$$\mathrm{nmd}(\mathcal{S}) = -k + \sum_{i=1}^{k} |S_i|, \tag{3.1}$$

obtained by scaling each row of $M_{\mathcal{S}}(\mathbf{x})$ so that one of the entries is 1, then evaluating the other parameters freely. We call the parameter $\mathrm{nmd}(\mathcal{S})$ the *naive maximal dimension* of $L(\mathcal{S})$. Theorem 3.1.2 shows that

$$\dim(L(\mathcal{S})) = \mathrm{nmd}(\mathcal{S})$$

if and only if $\mathcal{S}$ is a minimal presentation for the transversal matroid $\mathcal{B}(\mathcal{S})$.

Example 3.0.2. Let $\mathcal{S}$ be the set system from Example 2.3.10 consisting of

$$\begin{aligned} S_1 &= \{1,3,4\}, \\ S_2 &= \{1,2\}, \\ S_3 &= \{2,3\}. \end{aligned}$$

Then,

$$M_{\mathcal{S}}(\mathbf{x}) = \begin{pmatrix} x_{11} & 0 & x_{13} & x_{14} \\ x_{21} & x_{22} & 0 & 0 \\ 0 & x_{32} & x_{33} & 0 \end{pmatrix}.$$

Here, $\mathrm{nmd}(\mathcal{S}) = 4$. However, we may eliminate a degree of freedom via

$$\begin{pmatrix} 1 & \frac{-x_{11}}{x_{21}} & \frac{x_{11}x_{22}}{x_{21}x_{32}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_{\mathcal{S}}(\mathbf{x}) = \begin{pmatrix} 0 & 0 & x_{13} + \frac{x_{11}x_{22}x_{33}}{x_{21}x_{32}} & x_{14} \\ x_{21} & x_{22} & 0 & 0 \\ 0 & x_{32} & x_{33} & 0 \end{pmatrix}. \tag{3.2}$$

We may not eliminate any more degrees of freedom from here, so the actual dimension of $L(\mathcal{S})$ is 3. Looking at the supports of this new matrix's row vectors gives a new set system $\mathcal{S}'$,

$$\begin{aligned} S'_1 &= \{3,4\}, \\ S'_2 &= \{1,2\}, \\ S'_3 &= \{2,3\}. \end{aligned}$$

In this case, the naive maximal dimension agrees with the actual dimension. As was seen in Example 2.3.10, $\mathcal{S}'$ is a minimal presentation the transversal matroid $\mathcal{B}(\mathcal{S})$.

Theorem 3.3.1 shows that if the transversal matroid $\mathcal{B}(\mathcal{S})$ is a positroid, then $\overline{L(\mathcal{S})}$ is exactly the positroid variety labelled by this matroid.

Theorem 3.4.3 gives a sufficient condition for testing when the matroid $\mathcal{B}(\mathcal{S})$ is a positroid. We conjecture that this condition is also necessary. Section 3.4 proves this conjecture in several special cases, including the case when $\overline{L(\mathcal{S})}$ is a Richardson variety, and the case where all sets in $\mathcal{S}$ have the same size.

Section 3.5 compares the subsets $L(\mathcal{S})$ we study to several similarly defined families of subsets of $Gr(k,n)$ including generalized Richardson varieties introduced by Billey and Coskun in [13], interval positroids introduced by Knutson in [35], and diagram varieties introduce by Liu in [42].

The motivation for this work partially comes from two different programs related to computing scattering amplitudes in $\mathcal{N} = 4$ SYM theory. The first computes the on-shell amplitudes as volumes of an object called the amplituhedron, a certain projection of $Gr_{\geq 0}(k,n)$. In [33], Karp, Williams, and Zhang give a program for triangulating the amplituhedron. Their program identifies certain cells, called BCFW cells, finds a basis with a prescribed support shape for any plane in these cells in $Gr_{\geq 0}(k,n)$, then uses these basis shapes and sign variation techniques to argue that the images of the BCFW cells are disjoint in the amplituhedron. The second program computes the total amplitude using a Wilson loop. This program identifies matrices of the form $M_{\mathcal{S}}(\mathbf{x})$ for a particular class of set systems $\mathcal{S}$, all of which are minimal presentations of positroids, then associates an integral to this family of matrices. Sections 3.6 and 5.2 discuss applications to these two programs, and draw connections between the basis shapes appearing in both of them.

3.1 Basis Shape Loci and Transversal Matroids

Let $\mathcal{S}$ be a set system and $L(\mathcal{S})$ be its basis shape locus.

Proposition 3.1.1. *A generic point in $L(\mathcal{S})$ represents the matroid $\mathcal{B}(\mathcal{S})$.*

Proof. Collections of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^{[n]}$ with $\text{supp}(\mathbf{v}_i) = S_i$ are in bijection with edge weightings of the bipartite graph $\Gamma_{\mathcal{S}}$, with the edge between S_i and j weighted by the j^{th} entry of $\mathbf{v}_i$. Given a set $I \in \binom{[n]}{k}$, the I^{th} Plücker coordinate of a point in $L(\mathcal{S})$ may be computed from the corresponding weighting of $\Gamma_{\mathcal{S}}$ by

$$\Delta_I = \sum_{M:\mathcal{S} \,-\, I} \text{sgn}(M) \prod_{e \in M} \text{wt}(e),$$

where the sum is across matchings from $\mathcal{S}$ to I in $\Gamma_{\mathcal{S}}$, $\text{sgn}(M)$ is the sign of M viewed as a permutation, and the product is across edges in the matching. Generically, the Plücker coordinate Δ_I is nonzero if and only if there is a matching in $\Gamma_{\mathcal{S}}$ saturating I. Thus, the matroid represented by a generic point in $L(\mathcal{S})$ is $\mathcal{B}(\mathcal{S})$. □

Theorem 3.1.2. *The following are equivalent:*

(i) $\dim(L(\mathcal{S})) = \operatorname{nmd}(\mathcal{S})$.

(ii) $\mathcal{S}$ *is a minimum presentation of* $\mathcal{B}(\mathcal{S})$.

(iii) For all $\mathcal{T} \subseteq \mathcal{S}$,

$$\left| \bigcup_{T \in \mathcal{T}} T \right| \geq \max_{T \in \mathcal{T}} (|T|) + |\mathcal{T}| - 1. \tag{3.3}$$

Remark 3.1.3. The equivalence of (ii) and (iii) in Theorem 3.1.2 are proved in Theorem 3 in [15]. There, condition (iii) is phrased as the equivalent condition that there is a matching of size $k-1$ from $\mathcal{S} \setminus T$ to $[n] \setminus T$ in $\Gamma_{\mathcal{S}}$ for all $T \in \mathcal{S}$. This equivalence is proved by noting that for any $\mathcal{T} \subseteq \mathcal{S} \setminus T$,

$$\begin{aligned} \left| \bigcup_{S \in \mathcal{T}} S \setminus T \right| &= \left| \bigcup_{S \in \mathcal{T} \cup T} S \setminus T \right| \\ &\geq \max_{S \in \mathcal{T} \cup T} (|S|) - |T| + |\mathcal{T} \cup T| - 1 \\ &\geq |\mathcal{T}|. \end{aligned}$$

These inequalities are exactly the inequalities from Hall's Matching Theorem.

Remark 3.1.4. The inequalities in point (iii) of Theorem 3.1.2 are similar to the *dragon marriage condition*. Using the marriage analogy for Hall's Theorem, the dragon marriage condition is the necessary and sufficient condition for a matching of brides and grooms to exist even if a dragon eats one of the brides. This analogy is credited to Postnikov. The inequality in this case is a polyamorous bride condition, where it is possible for every bride to get married even if one of the brides marries all of her potential suitors.

Taking the equivalence of (ii) and (iii) above, we establish a technical lemma which will be used to show the equivalence of (i) with the other two conditions.

Lemma 3.1.5. *Let* $\mathcal{S} = \{S_1, \ldots, S_k\}$ *be a minimal presentation of* $\mathcal{B}(\mathcal{S})$*. If* $|S| > 1$ *for some* $S \in \mathcal{S}$*, there is some* $S_i \in \mathcal{S}$ *and some* $s \in S_i$ *such that*

$$\mathcal{S}' = \{S_1, \ldots, S_i \setminus s, \ldots, S_k\}$$

is a minimal presentation of $\mathcal{B}(\mathcal{S}')$.

Proof. If there is a unique S of maximal size in $\mathcal{S}$, we may remove any element of S while preserving all the inequalities (3.3). Otherwise, suppose toward contradiction that for every $S \in \mathcal{S}$ of maximal size and every $s \in S$ that there is some $\mathcal{T}_{S,s} \subset \mathcal{S}$ such that $S \notin \mathcal{T}_{S,s}$ and

$$\left| \bigcup_{T \in \mathcal{T}_{S,s}} T \cup (S \setminus s) \right| < |\mathcal{T}_{S,s}| + |S|.$$

For each pair S, s, we choose $\mathcal{T}_{S,s}$ to be a maximal subset of $\mathcal{S}$ with these properties. The subsystems $\mathcal{T}_{S,s}$ are partially ordered by inclusion, and we may choose S, s such that $\mathcal{T}_{S,s}$ is minimal in this poset. That is, $\mathcal{T}_{S,s} \not\supset \mathcal{T}_{S',s'}$ for any other $S' \in \mathcal{S}$ of maximal size and any s'. Necessarily, $\mathcal{T}_{S,s}$ contains some set of size $|S|$.

Since $\mathcal{T}_{S,s} \cup S$ satisfies (3.3),

$$\left| \bigcup_{T \in \mathcal{T}_{S,s}} T \right| = |\mathcal{T}_{S,s}| + |S| - 1, \tag{3.4}$$

and

$$(S \setminus s) \subset \bigcup_{T \in \mathcal{T}_{S,s}} T. \tag{3.5}$$

The same holds for $\mathcal{T}_{S,t}$ for any $t \neq s$ in S. If there is some $S' \in \mathcal{T}_{S,s} \cap \mathcal{T}_{S,t}$ such that $|S'| = |S|$, then

$$\left| \bigcup_{T \in \mathcal{T}_{S,s} \cup \mathcal{T}_{S,t}} T \right| = |\mathcal{T}_{S,s} \cup \mathcal{T}_{S,t}| + |S'| - 1.$$

Since

$$S \subset \bigcup_{T \in \mathcal{T}_{S,s} \cup \mathcal{T}_{S,t}} T,$$

the set $\mathcal{T}_{S,s} \cup \mathcal{T}_{S,t} \cup S$ violates the inequality (3.3), contradicting the assumption that $\mathcal{S}$ was a minimal presentation. Thus,

$$\{S' \in \mathcal{T}_{S,s} \cap \mathcal{T}_{S,t} : |S'| = |S|\} = \emptyset.$$

Let $S' \in \mathcal{T}_{S,s}$ be of maximal size and $s' \in S'$. Since $\mathcal{T}_{S',s'}$ and $\mathcal{T}_{S',t'}$ likewise cannot share sets of maximal size for $s' \neq t'$, we may assume that $S \notin \mathcal{T}_{S',s'}$. By our assumption that $\mathcal{T}_{S,s}$ was chosen to be minimal in the poset on set systems ordered by containment, $\mathcal{T}_{S',s'} \not\subset \mathcal{T}_{S,s}$.

Applying (3.4) and (3.5) to $\mathcal{T}_{S',s'}$,

$$\left| \bigcup_{T \in \mathcal{T}_{S',s'}} T \cup S' \right| = |\mathcal{T}_{S',s'}| + |S'|.$$

Then, since $S' \in (\mathcal{T}_{S',s'} \cup S') \cap \mathcal{T}_{S,s}$,

$$\left| \bigcup_{T \in \mathcal{T}_{S,s} \cup \mathcal{T}_{S',s'}} T \right| = |\mathcal{T}_{S,s} \cup \mathcal{T}_{S',s'}| + |S'| - 1.$$

By the maximality of $\mathcal{T}_{S,s}$,

$$s \in \bigcup_{T \in \mathcal{T}_{S,s} \cup \mathcal{T}_{S',s'}} T.$$

So,

$$\left| \bigcup_{T \in \mathcal{T}_{S,s} \cup \mathcal{T}_{S',s'}} T \cup S \right| = |\mathcal{T}_{S,s} \cup \mathcal{T}_{S',s'}| + |S'| - 1,$$

violating the inequality (3.3). □

Example 3.1.6. This example illustrates that while one may always remove some element from a set in a minimal presentation of a transversal matroid to achieve a minimal presentation of a different transversal matroid, not every set of maximal size in a set system contains an element which may be removed to produce a minimal presentation. Consider the set system $\{12, 23, 34\}$, which is a minimal presentation of its transversal matroid. Removing 2 from the first set produces the set system $\{1, 23, 34\}$, which satisfies the inequalities (3.3) and is thus a minimal presentation of its transversal matroid. Likewise, we may remove 3 from the third set and obtain a set system satisfying the inequalities (3.3). Any other choice of element to remove produces a set system which is not a minimal presentation. For instance, $\{12, 3, 34\}$ is not a minimal presentation since the second set is a subset of the third.

Proof of Theorem 3.1.2. The equivalence of (ii) and (iii) are shown in Theorem 3 in [15]. We begin by showing that point (i) implies point (iii).

Let $M_{\mathcal{S}}(\mathbf{x})$ be the matrix from above, whose entries are algebraically independent invertible variables x_{ij} or zeros. Let $\mathbf{v}_i$ be the i^{th} row of $M_{\mathcal{S}}(\mathbf{x})$. Suppose one of the inequalities (3.3) is violated. So, there is some set $T \subseteq \{1, \ldots, k\}$ such that

$$\left| \bigcup_{j \in T} S_j \right| < \max_{j \in T}(|S_j|) + |T| - 1.$$

Take T to be such that no proper subset of T has this property. Let S_{i_1} be a set of maximum size in $\mathcal{T} = \{S_j : j \in T\}$ and let

$$a_1 \in S_{i_1} \cap \left(\bigcup_{j \in T \setminus i_1} S_j \right).$$

The minimality of T guarantees that $\Gamma_{\mathcal{T}}$ is connected, and thus such an a_1 must exist. Let $i_2 \in T \setminus i_i$ be such that $a_1 \in S_{i_2}$. Then,

$$\operatorname{supp}\left(\mathbf{v}_{i_1} - \frac{x_{i_1 a_1}}{x_{i_2 a_1}} \mathbf{v}_{i_2} \right) = (S_{i_1} \cup S_{i_2}) \setminus a_1.$$

Again from the minimality of T, these is some

$$a_2 \in (S_{i_1} \cup S_{i_2}) \cap (\cup_{j \in T \setminus i_1, i_2} S_j).$$

Say, $i_3 \in T \setminus i_1, i_2$ such that $a_2 \in S_{i_3}$. Then, there is some $c, d \in \mathbb{R}(\mathbf{x})$ such that

$$\operatorname{supp}\left(\mathbf{v}_{i_1} - \frac{x_{i_1 a_1}}{x_{i_2 a_1}} \mathbf{v}_{i_2} - c\left(\mathbf{v}_{i_3} - d\mathbf{v}_{i_2}\right) \right) = (S_{i_1} \cup S_{i_2} \cup S_{i_3}) \setminus a_1, a_2.$$

Continuing to eliminate variables in this way, we may replace $\mathbf{v}_{i_1}$ with some vector $\mathbf{v}'_{i_1}$ supported on

$$S'_1 = \bigcup_{j \in T} S_j \setminus \{a_1, \ldots, a_{|T|-1}\}$$

without altering $\overline{L(\mathcal{S})}$. Note that here we take the closure since the entries of $\mathbf{v}'_{i_1}$ are not nonzero variables. Rather, they are equations in the x_{ij} which might vanish at particular evaluations of the x_{ij} as in (3.2). Let

$$\mathcal{S}' = \mathcal{S} \setminus S_1 \cup S'_1.$$

Then,

$$|S_1'| = \left|\bigcup_{j \in T} S_j\right| - |T| + 1 < |S_1|.$$

Since $\overline{L(\mathcal{S})} = \overline{L(\mathcal{S}')}$,

$$\dim(L(\mathcal{S})) = \dim(L(\mathcal{S}')) \leq \mathrm{nmd}(\mathcal{S}') < \mathrm{nmd}(\mathcal{S}),$$

and thus point (i) implies point (iii).

We next establish that point (ii) implies point (i) by inducting on the dimension of $L(\mathcal{S})$. If $\dim(L(\mathcal{S})) = 0$, then exactly one Plücker coordinate Δ_I will be nonzero on $L(\mathcal{S})$. Say $I = \{i_1, \ldots, i_k\}$. Then, $\mathcal{S} = \{\{i_1\}, \ldots, \{i_k\}\}$ is the unique minimal presentation of $\mathcal{B}(\mathcal{S})$, and $\mathrm{nmd}(\mathcal{S}) = 0$ as well.

Suppose that $\dim(L(\mathcal{S})) > 0$ and that $\mathcal{S}$ is a minimal presentation of $\mathcal{B}(\mathcal{S})$. Lemma 3.1.5 says there is some $S_i \in \mathcal{S}$ and $s \in S_i$ such that $\mathcal{S}' = \{S_1, \ldots, S_i \setminus s, \ldots, S_k\}$ is a minimal presentation of $\mathcal{B}(\mathcal{S}')$.

We claim that $\dim(L(\mathcal{S})) > \dim(L(\mathcal{S}'))$. On a matroidal level, note that $\mathcal{B}(\mathcal{S}) \supset \mathcal{B}(\mathcal{S}')$. So, there is some Plücker coordinate Δ_I which vanishes on $L(\mathcal{S}')$, but is generically nonzero on $L(\mathcal{S})$. Pick a generic $V \in L(\mathcal{S}')$ represented by $M_{\mathcal{S}'}(\mathbf{y})$ for some $y_{ij} \in \mathbb{R}^*$. Perturbing $\mathbf{y}$, there is a $\dim(L(\mathcal{S}'))$-dimensional neighborhood around V. Both V and this neighborhood are in $\overline{L(\mathcal{S})}$ since they are obtained as a limit as the i, s entry of $M_{\mathcal{S}}(\mathbf{x})$ goes to zero. In $\overline{L(\mathcal{S})}$, we may perturb the i, s entry of $M_{\mathcal{S}}(\mathbf{y})$. Points obtained in this way are readily seen to be distinct from V's neighborhood in $L(\mathcal{S}')$, since $\Delta_I \neq 0$ at these points. So, $\dim(\overline{L(\mathcal{S})}) > \dim(\overline{L(\mathcal{S}')})$, as desired.

Finally, inducting on $\dim(L(\mathcal{S}))$,

$$\mathrm{nmd}(\mathcal{S}) \geq \dim(L(\mathcal{S})) > \dim(L(\mathcal{S}')) = \mathrm{nmd}(\mathcal{S}') = \mathrm{nmd}(\mathcal{S}) - 1.$$

Thus, $\mathrm{nmd}(\mathcal{S}) = \dim(L(\mathcal{S}))$, completing the proof. $\square$

The geometric content of this theorem is perhaps somewhat surprising. The locus $L(\mathcal{S})$ is the image of $\{M_{\mathcal{S}}(\mathbf{y}) : y_{ij} \in \mathbb{R}^*\}$ under quotient by left action of $GL(k)$. Fixing one of the y_{ij} to a generic parameter, we do not expect the dimension of the image of this family of matrices in the quotient to drop. However, setting some y_{ij} to zero, we should generally expect the dimension in the quotient to drop by one. Contrary to this expectation, the equivalence of (i) and (iii) together with the elimination argument used to prove that (i) implies (iii) show that if the set of matrices

$$\{M_{\mathcal{S}}(\mathbf{y}) : y_{ij} \in \mathbb{R}^*, \mathrm{rk}(M_{\mathcal{S}}(\mathbf{y})) = k\}$$

and the quotient $L(\mathcal{S})$ differ by more than a torus worth of symmetry, we may always set one of the y_{ij} to zero without altering $\overline{L(\mathcal{S})}$. This fact immediately implies the following combinatorial corollary.

Corollary 3.1.7. *Suppose that $\mathcal{S}$ is not a minimal presentation of $\mathcal{B}(\mathcal{S})$, then there is some $S_i \in \mathcal{S}$ and $s \in S_i$ such that*

$$\mathcal{B}(\{S_1, \ldots, S_i \setminus s, \ldots, S_k\}) = \mathcal{B}(\mathcal{S}).$$

Remark 3.1.8. This fact has been noted several times in the literature, for instance as Theorem 3.7 in [19]. While the geometric observation we may always set one of the y_{ij} to zero without altering $\overline{L(\mathcal{S})}$ immediately implies this corollary, Corollary 3.1.7 does not a priori imply this geometric analog.

Theorem 3.1.9. *Let $\mathcal{S}$ and $\mathcal{S}'$ be set systems. Then,*

$$\mathcal{B}(\mathcal{S}) = \mathcal{B}(\mathcal{S}') \quad \textit{if and only if} \quad \overline{L(\mathcal{S})} = \overline{L(\mathcal{S}')}.$$

Proof. If $\overline{L(\mathcal{S})} = \overline{L(\mathcal{S}')}$, Proposition 3.1.1 implies that $\mathcal{B}(\mathcal{S}) = \mathcal{B}(\mathcal{S}')$. Conversely, let $\mathcal{S}$ and $\mathcal{S}'$ be set systems such that $\mathcal{B}(\mathcal{S}) = \mathcal{B}(\mathcal{S}')$. It suffices to show that $\overline{L(\mathcal{S})} = \overline{L(\mathcal{S}')}$ in the case where $\mathcal{S}'$ is the unique maximal presentation of $\mathcal{B}(\mathcal{S})$ guaranteed by Theorem 2.3.12. The variable elimination argument used to show show point (i) implies point (iii) in Theorem 3.1.2 shows that we can set variables in $M_{\mathcal{S}'}(\mathbf{x})$ to zero without altering the closure $\overline{L(\mathcal{S}')}$ until we arrive at the matrix $M_{\mathcal{S}}(\mathbf{x})$. Hence, $\overline{L(\mathcal{S})} = \overline{L(\mathcal{S}')}$. □

3.2 Dimension Computations

This section is dedicated to proving the following theorem, which will be used later to show that $\overline{L(\mathcal{S})} = \overline{V_{\mathcal{B}(\mathcal{S})}}$ in the case where $\mathcal{B}(\mathcal{S})$ is a positroid.

Theorem 3.2.1. *Let $\mathcal{S}$ be a set system such that $\mathcal{B}(\mathcal{S})$ is a positroid. Then,*

$$\dim(L(\mathcal{S})) = \dim(V_{\mathcal{B}(\mathcal{S})}).$$

Evidently, $\dim(L(\mathcal{S})) \leq \dim(V_{\mathcal{B}(\mathcal{S})})$. In [25], Ford introduces a notion of expected codimension of a matroid variety, and proves that positroid varieties achieve their expected codimension. We will prove that for a transversal matroid, Ford's expected codimension agrees with $\operatorname{codim}(L(\mathcal{S}))$. Since the notion of expected codimension of a matroid variety is only used to prove Theorem 3.2.1, our exposition of this concept will be terse. For full details, see [25].

Definition 3.2.2. Let $\mathcal{B}$ be a rank k matroid on the ground set $[n]$ and let $\mathcal{I}$ be some collection of subsets of $[n]$. For $I \in \mathcal{I}$, let

$$c(I) = |I| - \operatorname{rk}(I),$$

and

$$b_{\mathcal{I}}(I) = \sum_{J \in \mathcal{I}} (k - \operatorname{rk}(J))\mu_{\mathcal{I}}(I, J), \tag{3.6}$$

where $\mu_{\mathcal{I}}$ is the Möbius function on the poset obtained by ordering the elements of $\mathcal{I}$ by containment. Then, the *expected codimension of* $\mathcal{B}$ *with respect to* $\mathcal{I}$ is

$$\operatorname{ec}_{\mathcal{I}}(\mathcal{B}) = \sum_{I \in \mathcal{I}} c(I) b_{\mathcal{I}}(I).$$

The *expected codimension of* $\mathcal{B}$ is

$$\operatorname{ec}(\mathcal{B}) = \operatorname{ec}_{\mathcal{P}([n])}(\mathcal{B}),$$

where $\mathcal{P}([n])$ is the power set of $[n]$.

Ford asks which $\mathcal{I} \subset \mathcal{P}([n])$ have the property that $\operatorname{ec}_{\mathcal{I}}(\mathcal{B}) = \operatorname{ec}(\mathcal{B})$. He observes that if $\mathcal{I}' \subset \mathcal{I}$ has the property that $b_{\mathcal{I}}(I) = 0$ for all $I \in \mathcal{I}'$, then

$$\operatorname{ec}_{\mathcal{I}}(\mathcal{B}) = \operatorname{ec}_{\mathcal{I} \setminus \mathcal{I}'}(\mathcal{B}).$$

Identifying particular classes of sets with this property and repeatedly applying this observation yields the following theorem.

Theorem 3.2.3 (Theorem 3.6 in [25])**.** *Let $\mathcal{B}$ be a connected matroid and suppose $\mathcal{I}$ is a collection of subsets of $[n]$ containing every set I such that $\mathcal{B}|_I$ and $\mathcal{B}/I$ are connected (i.e. the set of flacets of $\mathcal{B}$) and such that whenever $I \in \mathcal{I}$, all J such that $\mathcal{B}|_J$ is a connected component of $\mathcal{B}|_I$ are in $\mathcal{I}$ as well. Then, $\operatorname{ec}_{\mathcal{I}}(\mathcal{B}) = \operatorname{ec}(\mathcal{B})$.*

Lemma 3.2.4 (Proposition 3.7 in [25])**.** *Let* $\mathcal{B}_1, \mathcal{B}_2$ *be matroids. Then,*

$$\mathrm{ec}(\mathcal{B}_1 \oplus \mathcal{B}_2) = \mathrm{ec}(\mathcal{B}_1) + \mathrm{ec}(\mathcal{B}_2).$$

Theorem 3.2.5 (Theorem 4.7 in [25])**.** *If* $\mathcal{B}$ *is a positroid,* $\mathrm{ec}(\mathcal{B}) = \mathrm{codim}(\overline{V_{\mathcal{B}}})$.

Theorem 3.2.6. *Let* $\mathcal{S}$ *be a presentation of the transversal matroid* $\mathcal{B}(\mathcal{S})$*. Then,*

$$\mathrm{codim}(L(\mathcal{S})) = \mathrm{ec}(\mathcal{B}(\mathcal{S})).$$

Proof. Since $\dim(L(\mathcal{S}))$ and $\mathrm{ec}(\mathcal{B}(\mathcal{S}))$ both behave the same way under direct sums of matroids, we may suppose $\mathcal{B}(\mathcal{S})$ is connected. Since $\dim(L(\mathcal{S}))$ is invariant under the presentation of the matroid, we may suppose $\mathcal{S}$ is a minimal presentation of $\mathcal{B}(\mathcal{S})$.

Let $\mathcal{I}$ be the collection of all individual elements of $[n]$ together with all flats of the form $F(\mathcal{T})$, from (2.6), such that $\mathrm{rk}(F(\mathcal{T})) = |\mathcal{T}|$. From Lemma 2.3.13, $\mathcal{I}$ contains all flacets of $\mathcal{B}(\mathcal{S})$. If $\mathrm{rk}(F(\mathcal{T})) = |\mathcal{T}|$, the connected components of $F(\mathcal{T})$ are either coloops or sets of the form $F(\mathcal{T}')$ for some $\mathcal{T}' \subsetneq \mathcal{T}$ with $\mathrm{rk}(F(\mathcal{T}')) = |\mathcal{T}'|$. So, the collection $\mathcal{I}$ satisfies the hypotheses of Theorem 3.2.3.

We claim that for any $F(\mathcal{T}) \in \mathcal{I}$,

$$b_{\mathcal{I}}(F(\mathcal{T})) = \begin{cases} 1 & \text{if } |\mathcal{T}| = k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Evidently,

$$\begin{aligned} b_{\mathcal{I}}(F(\mathcal{S})) &= \sum_{J \in \mathcal{I}} (k - \mathrm{rk}(J)) \mu_{\mathcal{I}}(F(\mathcal{S}), J) \\ &= (k - \mathrm{rk}(F(\mathcal{S}))) \mu_{\mathcal{I}}(F(\mathcal{S}), F(\mathcal{S})) \\ &= 0. \end{aligned}$$

Since $\mathcal{S}$ is a minimal presentation, Hall's Matching Theorem and point (iii) of Theorem 3.1.2 imply there is matching in $\Gamma_{\mathcal{S}}$ saturating $[n] \setminus S_i$ for all $S_i \in \mathcal{S}$. So, $F(\mathcal{S} \setminus S_i) \in \mathcal{I}$ for all $S_i \in \mathcal{S}$. Then, $b_{\mathcal{I}}(F(\mathcal{S} \setminus S_i)) = 1$ for all $S_i \in \mathcal{S}$.

Let $F(\mathcal{T}) \in \mathcal{I}$ such that $|\mathcal{T}| < k-1$ and suppose (3.7) holds for all $F(\mathcal{T}')$ with $|\mathcal{T}'| > |\mathcal{T}|$. Applying Möbius inversion to (3.6),

$$k - \mathrm{rk}(F(\mathcal{T})) = \sum_{F(\mathcal{T}') \supseteq F(\mathcal{T})} b_{\mathcal{I}}(F(\mathcal{T}')).$$

Since $\mathcal{T} \subset \mathcal{S} \backslash S_i$ for each $S_i \in \mathcal{S} \backslash \mathcal{T}$, and $F(\mathcal{S} \backslash S_i) \in \mathcal{I}$ for each S_i, our inductive hypothesis reduces the sum on the right hand side to

$$k - |\mathcal{T}| + b_{\mathcal{I}}(F(\mathcal{T})).$$

Since $F(\mathcal{T}) \in \mathcal{I}$, $\mathrm{rk}(F(\mathcal{T})) = |\mathcal{T}|$ and so

$$b_{\mathcal{I}}(F(\mathcal{T})) = 0.$$

Thus, (3.7) holds for all $F(\mathcal{T}) \in \mathcal{I}$ inductively. Since $\mathcal{B}(\mathcal{S})$ is connected, for all $j \in [n]$, $\mathrm{rk}(j) = 1$ and thus $c(j) = 0$. Then,

$$\begin{aligned}
\mathrm{ec}(\mathcal{B}(\mathcal{S})) &= \mathrm{ec}_{\mathcal{I}}(\mathcal{B}(\mathcal{S})) \\
&= \sum_{S_i \in \mathcal{S}} c(F(\mathcal{S} \setminus S_i)) b_{\mathcal{I}}(F(\mathcal{S} \setminus S_i)) + \sum_{j \in [n]} c(j) b_{\mathcal{I}}(j) \\
&= \sum_{S_i \in \mathcal{S}} (|F(\mathcal{S} \setminus S_i)| - (k-1)) + \sum_{j \in [n]} c(j) b_{\mathcal{I}}(j) \\
&= \sum_{S_i \in \mathcal{S}} (n - |S_i|) - k(k-1).
\end{aligned}$$

From Theorem 3.1.2,

$$\dim(L(\mathcal{S})) = \sum_{S_i \in \mathcal{S}} |S_i| - k.$$

The dimension of $Gr(k, n)$ is $k(n-k)$. So,

$$\begin{aligned}
\mathrm{codim}(L(\mathcal{S})) &= k(n-k) - \left(\sum_{S_i \in \mathcal{S}} |S_i| - k \right) \\
&= \mathrm{ec}(\mathcal{B}(\mathcal{S})).
\end{aligned}$$ □

Proof of Theorem 3.2.1. Let $\mathcal{S}$ be a set system such that $\mathcal{B}(\mathcal{S})$ is a positroid. Combining Theorems 3.2.5 and 3.2.6,

$$\dim(V_{\mathcal{B}(\mathcal{S})}) = k(n-k) - \mathrm{ec}(\mathcal{B}(\mathcal{S})) = \dim(L(\mathcal{S})).$$ □

3.3 Basis Shape Loci and Positroids

Theorem 3.3.1. *Let $\mathcal{S}$ be a set system. If $\mathcal{B}(\mathcal{S})$ is a positroid, then $\overline{L(\mathcal{S})} = \overline{V_{\mathcal{B}(\mathcal{S})}}$, the positroid variety labelled by $\mathcal{B}(\mathcal{S})$.*

Proof. Suppose that $\mathcal{B}(\mathcal{S})$ is a positroid. Since $V_{\mathcal{B}} \cap L(\mathcal{S})$ is dense in $L(\mathcal{S})$ by Proposition 3.1.1, $\overline{L(\mathcal{S})} \subseteq \overline{V_{\mathcal{B}(\mathcal{S})}}$.

Using Theorem 3.1.9, we may suppose that $\mathcal{S}$ is a minimal presentation of $\mathcal{B}(\mathcal{S})$. Let $V \in L(\mathcal{S})$ be the row space of a matrix $M_{\mathcal{S}}(\mathbf{y})$ obtained by evaluating the x_{ij} at algebraically independent real numbers y_{ij} in $M_{\mathcal{S}}(\mathbf{x})$. Since V is generic, it represents the matroid $\mathcal{B}(\mathcal{S})$, and thus V is in the Marsh-Reitsch cell $MR(\mathcal{B}(\mathcal{S}))$. Let $M_{\mathcal{B}(\mathcal{S})}(V)$ be the Marsh-Reitsch matrix representing V, guaranteed by Theorem 2.2.19. Then, there is some $G \in Gl(k)$ such that

$$G \cdot M_{\mathcal{B}(\mathcal{S})}(V) = M_{\mathcal{S}}(\mathbf{y}).$$

The Marsh-Rietsch matrix has exactly $\dim(V_{\mathcal{B}})$ many free parameters. From Theorem 3.2.1, $\dim(L(\mathcal{S})) = \dim(V_{\mathcal{B}(\mathcal{S})})$ as well. So, $M_{\mathcal{B}(\mathcal{S})}(V)$ must have been obtained by evaluating the entries of the Marsh-Rietsch matrix at algebraically independent parameters. Then, the change of basis matrix G provides a basis of shape $\mathcal{S}$ for a generic point in $MR(\mathcal{B}(\mathcal{S}))$. So, $L(\mathcal{S}) \cap MR(\mathcal{B}(\mathcal{S}))$ is dense in $MR(\mathcal{B}(\mathcal{S}))$ and thus

$$\overline{V_{\mathcal{B}(\mathcal{S})}} = \overline{MR(\mathcal{B}(\mathcal{S}))} \subseteq \overline{L(\mathcal{S})}.$$ □

Remark 3.3.2. In the case where $\mathcal{B}(\mathcal{S})$ is not a positroid, little is known about $\overline{L(\mathcal{S})}$. It would be interesting to see how $\overline{L(\mathcal{S})}$ compares to the GGMS variety $\overline{V_{\mathcal{B}(\mathcal{S})}}$ in this case.

3.4 When is a Transversal Matroid a Positroid?

This section addresses the question of characterizing when a transversal matroid is a positroid. We call matroids which are transversal matroids and positroids *transversal positroids*.

Example 3.4.1. Let $\mathcal{S}$ be the set system

$$S_1 = \{1, 3, 4\},$$
$$S_2 = \{2, 4\}.$$

Any point in $L(\mathcal{S})$ is the row span of a unique matrix of the form

$$\begin{pmatrix} 1 & 0 & y_{13} & y_{14} \\ 0 & 1 & 0 & y_{24} \end{pmatrix},$$

where $y_{ij} \in \mathbb{R}^*$. Consider $L(\mathcal{S}) \cap Gr_{\geq 0}(k,n)$. For any point in this intersection, since $\Delta_{14} > 0$, we must have $y_{24} > 0$. Since $\Delta_{34} > 0$, $y_{13} > 0$ as well. However, this restriction forces $\Delta_{23} < 0$. Thus, $L(\mathcal{S}) \cap Gr_{\geq 0}(k,n) = \emptyset$. If $\mathcal{B}(\mathcal{S})$ is a positroid $\overline{L(\mathcal{S})} = \overline{V_{\mathcal{B}(\mathcal{S})}}$. However, when $\mathcal{B}$ is a positroid, $\dim(V_{\mathcal{B}} \cap Gr_{\geq 0}(k,n)) = \dim(V_{\mathcal{B}})$, a contradiction.

The fundamental obstruction illustrated in this example is that the set S_1 *crosses* S_2, in the sense of Definition 3.4.2 below.

For any $a \in [n]$, let $\leq_a$ denote that a^{th} cyclic shift of the usual total order on $[n]$. So,

$$a <_a a+1 <_a \cdots <_a n <_a 1 <_a \cdots <_a a-1. \tag{3.8}$$

Definition 3.4.2. In a set system $\mathcal{S}$, the set S_i *crosses* S_j if there are $a, b, c, d \in [n]$ such that:

(i) $a <_a b <_a c <_a d$,

(ii) $a, c \in S_i$, $a, c \notin S_j$,

(iii) $b, d \in S_j$, $b \notin S_i$.

The set system $\mathcal{S}$ is *noncrossing* if there is no pair $S_i, S_j \in \mathcal{S}$ with S_i crossing S_j.

Note that this definition is not symmetric. It is possible for the set S_i to cross S_j without S_j crossing S_i.

Theorem 3.4.3. *Suppose that $\mathcal{S}$ is a minimal presentation of $\mathcal{B}(\mathcal{S})$ and that no set crosses another in $\mathcal{S}$. Then, $\mathcal{B}(\mathcal{S})$ is a positroid.*

Proof. Suppose that $\mathcal{B}(\mathcal{S})$ is not a positroid. From Theorem 2.3.8, $\mathcal{B}(\mathcal{S})$ could fail to be a positroid either by having some connected component which is not a positroid, or having its connected components form a crossing partition. Since $\mathcal{S}$ is a minimal presentation, the connected components of $\mathcal{B}(\mathcal{S})$ correspond to the connected components of $\Gamma_{\mathcal{S}}$. Evidently, the existence of crossing connected components necessitates the existence of crossing sets in $\mathcal{S}$. So, we may suppose $\mathcal{B}(\mathcal{S})$ is connected.

Theorem 2.3.7 says that there is some flacet F of $\mathcal{B}(\mathcal{S})$ which is not a cyclic interval. Since flacets are cyclic flats, Lemma 2.3.13 implies $F = F(\mathcal{T})$ for some $\mathcal{T} \subset \mathcal{S}$ with $\text{rk}(\mathcal{T}) = |\mathcal{T}|$. Since F is a cyclic flat, $\mathcal{T}|_F$ is a minimal presentation of $\mathcal{B}(\mathcal{S})|_F$. Then, $\mathcal{B}(\mathcal{S})|_F$ is connected if and only if $\Gamma_{\mathcal{T}|_F}$ is connected. So, there must be some $S_i \in \mathcal{T}$ with $a, c \in S_i$, where a and c are not contained in a cyclic interval of F.

Since $\text{rk}(F) = |\mathcal{T}|$, the contraction $\mathcal{B}(\mathcal{S})/F$ is a transversal matroid with presentation $\mathcal{S}\setminus\mathcal{T}$, and this presentation is minimal. Then, there must be some $S_j \in \mathcal{S}\setminus\mathcal{T}$ with $b, d \in S_j$ where b and d are in different cyclic intervals of $[n] \setminus F$.

From the definition of $F(\mathcal{T})$, $a, c \notin S_j$. Since $\mathcal{S}$ is a minimal presentation, point (iii) of Theorem 3.1.2 implies $S_j \nsubseteq S_i$. So, we may take at least one of b or d to not be in S_i. Then, S_i crosses S_j in $\mathcal{S}$. □

Remark 3.4.4. The special case of Theorem 3.4.3 where $\mathcal{S}$ is the set system associated to a Wilson loop diagram (see Definition 5.1.1) appears as Theorem 3.38 in [2]. The proof above is a direct generalization of their argument.

We conjecture that the noncrossing condition is also necessary for a transversal matroid to be a positroid.

Conjecture 3.4.5. *Let $\mathcal{B}$ be a transversal matroid which is also a positroid. Then, there is a noncrossing minimal presentation $\mathcal{S}$ of $\mathcal{B}$.*

Remark 3.4.6. In the case of transversal positroids, Conjecture 3.4.5 is a strengthening of Theorem 2.3.8, saying that the connected components of a positroid must form a noncrossing partition. Since $\mathcal{S}$ is a minimal presentation in this conjecture, connected components of $\mathcal{B}(\mathcal{S})$ correspond to connected components of $\Gamma_{\mathcal{S}}$. Evidently, a crossing of connected components in $\Gamma_{\mathcal{S}}$ implies that $\mathcal{S}$ is crossing.

We will show that Conjecture 3.4.5 holds in the case where $\overline{V_{\mathcal{B}}}$ is a Richardson variety, and the case where all sets in a minimal presentation of the matroid $\mathcal{B}$ have the same size. A strengthening of this conjecture, providing an algorithmic method of producing a noncrossing minimal presentation is given in Conjecture 3.4.22. This strengthened conjecture holds in the special cases mentioned above and has received extensive computational verification.

Proposition 3.4.7. *Let $I, J \in \binom{[n]}{k}$ with $I \leq J$ and let $\mathcal{B}(I, J)$ be the lattice path matroid defined by this pair of sets. Then, $\mathcal{B}(I, J)$ is a transversal matroid with a noncrossing minimal presentation.*

Proof. Let $I = \{i_1, i_2, \ldots, i_k\}$ and $J = \{j_1, j_2, \ldots, j_k\}$ with $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. For $1 \le \ell \le k$, let

$$S_\ell = \{m : i_\ell \le m \le j_\ell\}.$$

It is well known, for instance Section 4 of [16], that $\mathcal{B}(I, J)$ is a transversal matroid and that $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ is a minimal presentation of $\mathcal{B}(I, J)$. Since every set in this presentation is an interval, no two sets cross each other. □

The following corollary follows immediately from Theorem 3.3.1 and the fact that all lattice path matroids are transversal matroids.

Corollary 3.4.8. *All Schubert and Richardson varieties in $Gr(k, n)$ are closures of basis shape loci.*

Proposition 3.4.9. *Let $\mathcal{B}$ be a transversal matroid such that all sets in a minimal presentation of $\mathcal{B}$ have the same size. If $\mathcal{B}$ is a positroid, then it has a minimal presentation which is noncrossing.*

We require some technical machinery to prove this proposition. This machinery is presented in a way that is functional even when not all sets in $\mathcal{S}$ have the same size, allowing us to give a strengthened version of Conjecture 3.4.5. Theorems 3.4.14 and 3.4.15 provide a procedure of pivoting between different minimal presentations of the matroid $\mathcal{B}(\mathcal{S})$. Each pivot replaces a single set in the set system $\mathcal{S}$ with a different set while preserving the matroid $\mathcal{B}(\mathcal{S})$. Lemma 3.4.20 shows that if the set system $\mathcal{S}$ is crossing and $\mathcal{B}(\mathcal{S})$ is a positroid, that we are able to perform a pivot that removes this crossing. Finally, we show that in the case where all sets in $\mathcal{S}$ have the same size that we are able to use these pivots to simultaneously remove all crossings in the set system.

Definition 3.4.10. Call a subsystem $\mathcal{T} \subseteq \mathcal{S}$ *exact* if

$$\left| \bigcup_{T \in \mathcal{T}} T \right| = |\mathcal{T}| + \max_{T \in \mathcal{T}} (|T|) - 1. \tag{3.9}$$

That is, the inequality (3.3) from point (iii) of Theorem 3.1.2 holds with equality for $\mathcal{T}$.

Lemma 3.4.11. *Let $\mathcal{S}$ be a minimal presentation of $\mathcal{B}(\mathcal{S})$, and let $\mathcal{T}$ be an exact subsystem of $\mathcal{S}$. Then,* $\operatorname{rk}(F(\mathcal{S} \setminus \mathcal{T})) = |\mathcal{S} \setminus \mathcal{T}|$.

Proof. Let $F = F(\mathcal{S} \setminus \mathcal{T})$, the flat from (2.6). For any $\mathcal{S}' \subset \mathcal{S} \setminus \mathcal{T}$,

$$\begin{aligned}\left|\bigcup_{S\in\mathcal{S}'} S \cap F\right| &= \left|\bigcup_{S\in\mathcal{S}'\cup\mathcal{T}} S\right| - \left|\bigcup_{S\in\mathcal{T}} S\right| \\ &\geq \left(|\mathcal{S}'\cup\mathcal{T}| + \max_{S\in\mathcal{S}'\cup\mathcal{T}}(|S|) - 1\right) - \left(|\mathcal{T}| + \max_{S\in\mathcal{T}}(|S|) - 1\right) \\ &\geq |\mathcal{S}'|.\end{aligned}$$

Here, the positive part of the inequality on the second line comes from (3.3), and the negative part comes from the assumption $\mathcal{T}$ was exact. Then, Hall's Matching Theorem implies there is a matching in $\Gamma_{\mathcal{S}}$ from $\mathcal{S}\setminus\mathcal{T}$ to F saturating $\mathcal{S}\setminus\mathcal{T}$. So, $\mathrm{rk}(F) = |\mathcal{S}\setminus\mathcal{T}|$. □

Lemma 3.4.12. *Let $\mathcal{S}$ be a minimal presentation of $\mathcal{B}(\mathcal{S})$. Let $\mathcal{T}$ and $\mathcal{T}'$ be exact subsystems of $\mathcal{S}$ with $S \in \mathcal{T}, \mathcal{T}'$ such that*

$$|S| = \max_{T\in\mathcal{T}\cup\mathcal{T}'}(|T|).$$

Then, $\mathcal{T}\cup\mathcal{T}'$ is an exact subsystem.

Proof. Note that

$$\begin{aligned}\left|\bigcup_{T\in\mathcal{T}\cup\mathcal{T}'} T\right| &\leq \left|\bigcup_{T\in\mathcal{T}} T\right| + \left|\bigcup_{T\in\mathcal{T}'} T\right| - \left|\bigcup_{T\in\mathcal{T}\cap\mathcal{T}'} T\right| \\ &= |\mathcal{T}| + |\mathcal{T}'| + 2|S| - 2 - \left|\bigcup_{T\in\mathcal{T}\cap\mathcal{T}'} T\right| \\ &\leq |\mathcal{T}| + |\mathcal{T}'| + 2|S| - 2 - (|\mathcal{T}\cap\mathcal{T}'| + |S| - 1) \\ &= |\mathcal{T}\cup\mathcal{T}'| + |S| - 1.\end{aligned}$$

Here, the equality on the second line comes from the assumption that $\mathcal{T}$ and $\mathcal{T}'$ are exact, and the inequality on the third line comes from (3.3). Then, (3.3) implies that $\mathcal{T}\cup\mathcal{T}'$ is exact. □

Definition 3.4.13. We say that the set systems $\mathcal{S}$ and $\mathcal{S}'$ are related by a *pivot* if they both satisfy the inequalities (3.3) and

$$\mathcal{S}' = \mathcal{S} \setminus S \cup \{S \setminus a \cup b\},$$

and there is some exact $\mathcal{T} \subseteq \mathcal{S}$ containing S such that $|S| = \max_{T\in\mathcal{T}} (|T|)$,

$$a \in S \cap T, \quad \text{and} \quad b \in T \setminus S$$

for some $T \in \mathcal{T}$.

Theorem 3.4.14. *Let $\mathcal{S}$ and $\mathcal{S}'$ be set systems satisfying the conditions of Theorem 3.1.2. If $\mathcal{S}$ and $\mathcal{S}'$ are related by a pivot, then $\overline{L(\mathcal{S})} = \overline{L(\mathcal{S}')}$ and thus $\mathcal{B}(\mathcal{S}) = \mathcal{B}(\mathcal{S}')$.*

Proof. Let $M_{\mathcal{S}}(\mathbf{x})$ be the matrix of indeterminates associated to $\mathcal{S}$ and let $\mathbf{v}_i$ be the i^{th} row vector of $M_{\mathcal{S}}(\mathbf{x})$.

If $\mathcal{S}$ and $\mathcal{S}'$ are related by a pivot replacing S by $S \setminus a \cup b$, an argument identical to the proof of Theorem 3.1.2 showing (iii) implies (i) shows there is some linear combination

$$\mathbf{v}_i' = \mathbf{v}_i + \sum_{j\in T\setminus i} c_j \mathbf{v}_j$$

such that $\text{supp}(\mathbf{v}_i') = S_i \setminus a \cup b$ and

$$\text{span}\left(\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \setminus \mathbf{v}_i \cup \mathbf{v}_i'\right) = L(\mathcal{S}'),$$

and thus $\overline{L(\mathcal{S})} = \overline{L(\mathcal{S}')}$. Then, Proposition 3.1.1 implies that $\mathcal{B}(\mathcal{S}) = \mathcal{B}(\mathcal{S}')$. □

Theorem 3.4.15. *Let $\mathcal{S}$ be an exact system and let $S \in \mathcal{S}$ be a set of maximal size. Then,*

$$\left\{ S' \in \binom{[n]}{|S|} \ : \ \mathcal{B}(\mathcal{S} \setminus S \cup S') = \mathcal{B}(\mathcal{S}) \right\} = \mathcal{B}^*(\mathcal{S} \setminus S),$$

the dual of the transversal matroid defined by $\mathcal{S} \setminus S$. Using the exact system $\mathcal{S}$, S may be pivoted to any element in this set.

Proof. Let $S' \in \binom{[n]}{|S|}$ and suppose that $S' \notin \mathcal{B}^*(\mathcal{S} \setminus S)$. So, $[n] \setminus S'$ is not a basis of $\mathcal{B}(\mathcal{S} \setminus S)$. Using Remark 3.1.3, $\mathcal{S} \setminus S \cup S'$ violates the conditions of Theorem 3.1.2. Since $\mathcal{S} \setminus S \cup S'$ is not a minimal presentation of its transversal matroid, Theorem 2.3.12 implies that $\mathcal{B}(\mathcal{S} \setminus S \cup S') \neq \mathcal{B}(\mathcal{S})$.

Now, let $S' \in \mathcal{B}^*(\mathcal{S} \setminus S)$. Let $T \in \binom{[n]}{|S|}$ be some set obtained from S by a series of pivots using the exact system $\mathcal{S}$. Among all such sets, suppose T maximizes $|T \cap S'|$. By Theorem 3.4.14, $\mathcal{B}(\mathcal{S}) = \mathcal{B}(\mathcal{S} \setminus S \cup T)$, so the previous paragraph implies $T \in \mathcal{B}^*(\mathcal{S} \setminus S)$.

Suppose there is some $b \in T \setminus S'$. Then, the matroid basis exchange axioms imply there is some $a \in S' \setminus T$ such that

$$([n] \setminus (T \setminus b \cup a)) \in \mathcal{B}^*(\mathcal{S} \setminus S).$$

Then, the bipartite graph $\Gamma_{\mathcal{S} \setminus S}$ has an alternating path

$$b = a_0 - S_{\alpha_0} - a_1 - S_{\alpha_1} - \cdots - S_{\alpha_m} - a_{m+1} = a,$$

with $a_\ell, a_{\ell+1} \in S_{\alpha_\ell}$ for $0 \leq i \leq m$. Let $T_{m+1} = T \setminus b \cup a$, and for $m \geq i \geq 0$ let

$$T_\ell = T_{\ell+1} \setminus a_{\ell+1} \cup a_\ell.$$

The set T_ℓ is obtained from $T_{\ell+1}$ via a pivot using the set S_{α_ℓ}, and $T_0 = T$. Then, $T \setminus b \cup a$ is reachable from S by a series of pivots, contradicting the assumption that T was chosen to maximize $|T \cap S'|$. Thus, S' is reachable from S by a series of pivots. □

Remark 3.4.16. Ardila and Ruiz give a method of pivoting between different presentations of a transversal matroid in [8]. The pivoting procedure of Definition 3.4.13 differs from Ardila and Ruiz's in that it only passes between minimal presentations of the transversal matroid, while theirs might use non-minimal presentations. In Lemma 4.4, they prove that any two presentations of a transversal matroid are related by a series of their pivots. We conjecture that a similar fact holds for the pivots described in Definition 3.4.13.

Conjecture 3.4.17. *Let $\mathcal{S}$ and $\mathcal{S}'$ be minimal presentations of the same transversal matroid. Then, $\mathcal{S}'$ is reachable from $\mathcal{S}$ via a sequence of the pivots described in Definition 3.4.13.*

The following series of lemmas show that if $\mathcal{B}(\mathcal{S})$ is a positroid and $\mathcal{S}$ is a crossing minimal presentation, then there is an exact subsystem of $\mathcal{S}$, which may be used to perform a pivot removing this crossing.

Lemma 3.4.18. *Suppose that $\mathcal{S}$ is a minimal presentation of $\mathcal{B}(\mathcal{S})$ and that S_i crosses S_j in $\mathcal{S}$ with $a, c \in S_i$ and $b, d \in S_j$ witnessing this crossing. If there is a matching of size $k-2$ in $\Gamma_{\mathcal{S}}$ from $\mathcal{S} \setminus \{S_i, S_j\}$ to $[n] \setminus (S_j \cup \{a, c\})$, then*

$$\dim(L(\mathcal{S})) > \dim(L(\mathcal{S}) \cap Gr_{\geq 0}(k, n)).$$

Hence, $\mathcal{B}(\mathcal{S})$ is not a positroid.

Proof. Suppose there is a matching in $\Gamma_{\mathcal{S}}$ from $\mathcal{S} \setminus \{S_i, S_j\}$ to I, for some $I \in \binom{[n]\setminus S_j,a,c}{k-2}$. Let $M_{\mathcal{S}}(\mathbf{y})$ be some matrix obtained by evaluating the indeterminate entries of $M_{\mathcal{S}}(\mathbf{x})$ at nonzero real values y_{ij}. We may suppose $j = 1$ and that the columns of M are cyclically rotated so that b is the first column. Let $m = |[b, d] \cap I|$. For $R \subset \mathcal{S}$ and $C \subset [n]$ with $|R| = |C|$, let $M_{R,C}$ denote the determinant of the square submatrix of M with rows R and columns C. Since there is a matching from $\mathcal{S} \setminus \{S_i, S_j\}$ to I,

$$M_{\mathcal{S}\setminus S_1, I\cup a}, M_{\mathcal{S}\setminus S_1, I\cup c} \neq 0$$

at a generic evaluation of the x_{ij}. Then,

$$\operatorname{sgn}(M_{\mathcal{S},I\cup ab}) = \operatorname{sgn}(y_{1b})\operatorname{sgn}(M_{\mathcal{S}\setminus S_1,I\cup a}),$$

and

$$\operatorname{sgn}(M_{\mathcal{S},I\cup ad}) = (-1)^{m+1}\operatorname{sgn}(y_{1d})\operatorname{sgn}(M_{\mathcal{S}\setminus S_1,I\cup a}).$$

So, if $M_{\mathcal{S}}(\mathbf{y})$ represents a point in $Gr_{\geq 0}(k, n)$,

$$\operatorname{sgn}(y_{1b}) = (-1)^{m+1}\operatorname{sgn}(y_{1d}).$$

Further,

$$\operatorname{sgn}(M_{\mathcal{S},I\cup cb}) = \operatorname{sgn}(y_{1b})\operatorname{sgn}(M_{\mathcal{S}\setminus S_1,I\cup c}),$$

and

$$\operatorname{sgn}(M_{\mathcal{S},I\cup cd}) = (-1)^{m}\operatorname{sgn}(y_{1d})\operatorname{sgn}(M_{\mathcal{S}\setminus S_1,I\cup c}).$$

So, if $M_{\mathcal{S}}(\mathbf{y})$ represents a point in $Gr_{\geq 0}(k, n)$,

$$\operatorname{sgn}(y_{1a}) = (-1)^{m}\operatorname{sgn}(y_{1c}).$$

Thus, $L(\mathcal{S})$ cannot intersect the positive Grassmannian in its full dimension. If $\mathcal{B}(\mathcal{S})$ is a positroid, Theorems 3.3.1 and 2.2.20 imply

$$\dim(L(\mathcal{S}) \cap Gr_{\geq 0}(k, n)) = \dim(V_{\mathcal{B}} \cap Gr_{\geq 0}(k, n)) = \dim(V_{\mathcal{B}}).$$

So, $\mathcal{B}(\mathcal{S})$ cannot be a positroid. □

Remark 3.4.19. Beware that the asymmetry in the definition of crossing is crucial in this lemma. If a, b, c, d witness a crossing of S_i and S_j with $a, c, d \in S_i$, the computations of signs of determinants in the proof of Lemma 3.4.18 do not necessarily hold if one exchanges the roles of S_i and S_j. In fact, it is possible that there will be a matching of size $k - 2$ in $\Gamma_{\mathcal{S}}$ from $\mathcal{S} \setminus \{S_i, S_j\}$ to $[n] \setminus (S_j \cup \{a, c\})$. Consider for example the set system $\mathcal{S}$ consisting of

$$\begin{aligned} S_1 &= \{1,3,5\}, \\ S_2 &= \{2,3,4\}, \\ S_3 &= \{2,4,5\}. \end{aligned}$$

This set system may be pivoted to $\{145, 245, 345\}$, which is noncrossing. Hence $\mathcal{B}(\mathcal{S})$ is a positroid. In $\mathcal{S}$, S_2 crosses S_1 with $4, 1, 2, 3$ witnessing this crossing. Lemma 3.4.18 implies there is not a matching from S_3 to $[5] \setminus (S_1 \cup \{2,4\})$ in $\Gamma_{\mathcal{S}}$. There is however a matching from S_3 to $[5] \setminus (S_2 \cup \{1,3\})$ in $\Gamma_{\mathcal{S}}$.

Lemma 3.4.20. *Suppose that $\mathcal{S}$ is a minimal presentation of the transversal positroid $\mathcal{B}(\mathcal{S})$. Suppose that S_i crosses S_j in $\mathcal{S}$. Then, there is an exact subsystem $\mathcal{T}$ of $\mathcal{S}$ such that S_j is pivotable. Moreover, we make take $\mathcal{T}$ to contain both S_i and S_j.*

Proof. Suppose that S_i crosses S_j, and let $a, c \in S_i$ and $b, d \in S_j$ be elements witnessing this crossing. Suppose towards a contradiction that there is no exact subsystem $\mathcal{T}$ of $\mathcal{S}$ such that S_j is pivotable. Then, for all $\mathcal{T} \subseteq \mathcal{S} \setminus S_j$,

$$\left| \bigcup_{T \in \mathcal{T} \cup S_j} T \right| > |\mathcal{T}| + |S_j|.$$

So, for all $x \in [n] \setminus S_j$,

$$\left| \bigcup_{T \in \mathcal{T}} T \setminus (S_j \cup x) \right| \geq |\mathcal{T}|. \tag{3.10}$$

Since $\mathcal{B}(\mathcal{S})$ is a positroid, Lemma 3.4.18 says there cannot be a matching in $\Gamma_{\mathcal{S}}$ from $\mathcal{S} \setminus \{S_i, S_j\}$ to $[n] \setminus (S_j \cup ac)$. So, Hall's Matching Theorem guarantees some $\mathcal{T} \subseteq \mathcal{S} \setminus \{S_i, S_j\}$ such that

$$\left| \bigcup_{T \in \mathcal{T}} T \setminus (S_j \cup ac) \right| < |\mathcal{T}|.$$

From (3.10), we must have

$$\left| \bigcup_{T \in \mathcal{T}} T \setminus (S_j \cup ac) \right| = |\mathcal{T}| - 1,$$

and

$$a, c \in \bigcup_{T \in \mathcal{T}} T. \tag{3.11}$$

If $S_i \setminus S_j \subset \bigcup_{T \in \mathcal{T}} T$, then

$$\left| \bigcup_{T \in \mathcal{T} \cup S_i \cup S_j} T \right| \leq |\mathcal{T}| + |S_j| + 1.$$

Since $\mathcal{S}$ is a minimal presentation, this inequality must hold with equality, and $\mathcal{T} \cup S_i \cup S_j$ is an exact subsystem where S_j is pivotable.

Otherwise, there is some $e \in S_i \setminus S_j$ which is not in $\bigcup_{T \in \mathcal{T}} T$. Suppose without loss of generality that e is on the same side of the chord from b to d that c is on. So, a, b, e, d witnesses the fact that S_i crosses S_j as well. Again using Lemma 3.4.18 and Hall's Theorem, there is some $\mathcal{T}'$ such that

$$\left| \bigcup_{T \in \mathcal{T}'} T \setminus (S_j \cup ae) \right| < |\mathcal{T}'|.$$

As before, using (3.10), $\left|\bigcup_{T \in \mathcal{T}'} T \setminus S_j\right| = |\mathcal{T}'| + 1$ and using (3.11) $a, e \in \bigcup_{T \in \mathcal{T}'} T$.

Note that

$$\left| \left(\bigcup_{T \in \mathcal{T}} T \setminus S_j \right) \cap \left(\bigcup_{T \in \mathcal{T};} T \setminus S_j \right) \right| \geq 1,$$

since a is in this intersection. Then,

$$\left| \bigcup_{T \in \mathcal{T} \cup \mathcal{T}'} T \setminus S_j \right| \leq |\mathcal{T} \cup \mathcal{T}'| + 1.$$

Now,

$$a, c, e \in \bigcup_{T \in \mathcal{T} \cup \mathcal{T}'} T.$$

Continuing inductively in this fashion, we may take $\mathcal{T}$ to be some set such that

$$\left| \bigcup_{T \in \mathcal{T}} T \setminus S_j \right| \leq |\mathcal{T}| + 1$$

and $S_i \setminus S_j \subset \bigcup_{T\in\mathcal{T}} T$. Then, $\mathcal{T} \cup S_i \cup S_j$ is an exact subsystem where S_j is pivotable. $\square$

We call a presentation $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ of $\mathcal{B}$ *Gale minimal* if it is a minimal presentation of $\mathcal{B}$ and there is no other minimal presentation $\mathcal{S}' = \{S'_1, S'_2, \ldots, S'_k\}$ of $\mathcal{B}$ where $|S'_i| = |S_i|$ and $S'_i \leq S_i$ in Gale order for all $1 \leq i \leq k$. Suppose that the sets in $\mathcal{S}$ are indexed such that $|S_i| \leq |S_{i+1}|$ for $1 \leq i \leq k-1$. A Gale minimal presentation of $\mathcal{B}(\mathcal{S})$ may be produced algorithmically by, for each $1 \leq i \leq k$:

- Identifying the maximal (with respect to containment) exact subsystem $\mathcal{T}$ containing S_i as a set of maximal size.
- Replacing S_i with the Gale minimal basis of $\mathcal{B}^*(\mathcal{T} \setminus S_i)|_{\cup_{T\in\mathcal{T}} T}$.

Proof of Proposition 3.4.9. Let $\mathcal{S}$ be a minimal presentation of the transversal positroid $\mathcal{B}(\mathcal{S})$ and suppose that all sets in $\mathcal{S}$ have the same size. We may further suppose that $\mathcal{B}(\mathcal{S})$ is connected, since $\mathcal{B}(\mathcal{S})$ will have a noncrossing presentation if and only if the connected components of $\mathcal{B}(\mathcal{S})$ form a noncrossing partition and each connected component has a noncrossing presentation.

Suppose $\mathcal{S}$ is a Gale minimal presentation of $\mathcal{B}(\mathcal{S})$. We claim $\mathcal{S}$ is noncrossing. Suppose the sets S and S' cross each other. Then, Lemma 3.4.20 guarantees the existence of some exact subsystem $\mathcal{T} = \{T_1, T_2, \ldots, T_{|\mathcal{T}|}\}$ containing S and S'. Since $\mathcal{S}$ was a Gale minimal presentation of $\mathcal{B}(\mathcal{S})$, $\mathcal{T}$ is a Gale minimal presentation of $\mathcal{B}(\mathcal{T})$. Let

$$\{t_1, t_2, \ldots, t_{|S|+|\mathcal{T}|-1}\} = \bigcup_{T\in\mathcal{T}} T,$$

where $t_1 < t_2 < \cdots < t_{|S|+|\mathcal{T}|-1}$. Then, after possibly reindexing the sets in $\mathcal{T}$, the unique Gale minimal presentation of $\mathcal{T}$ is

$$\begin{aligned}
T_1 &= \{t_1, t_2, \ldots, t_{|S|-1}, t_{|S|}\},\\
T_2 &= \{t_1, t_2, \ldots, t_{|S|-1}, t_{|S|+1}\},\\
T_3 &= \{t_1, t_2, \ldots, t_{|S|-1}, t_{|S|+2}\},\\
&\vdots\\
T_{|\mathcal{T}|} &= \{t_1, t_2, \ldots, t_{|S|-1}, t_{|S|+|\mathcal{T}|-1}\}.
\end{aligned}$$

Observe that this presentation is noncrossing, violating that assumption that S and S' crossed. $\square$

Unfortunately, Gale minimal presentations of transversal positroids may in general feature crossings. However, we may consider presentations which are minimal in a-Gale order for some a; the cyclic shift of Gale order obtained by using $<_a$ in place of the usual order on $[n]$. So, if $I = \{i_1, i_2, \ldots, i_k\}$ and $J = \{j_1, j_2, \ldots, j_k\}$ with $i_1 <_a i_2 <_a \cdots <_a i_k$ and $j_1 <_a j_2 <_a \cdots <_a j_k$, then $I \leq_a J$ if and only if $i_\ell \leq_a j_\ell$ for each $1 \leq \ell \leq k$.

The following example illustrates a matroid $\mathcal{B}(\mathcal{S})$ whose Gale minimal presentation is crossing, but which has a noncrossing presentation which is minimal in a-Gale order, for some a. Conjecture 3.4.22 posits that this phenomena holds in general; that at least one of the a-Gale minimal presentations of a transversal positroid will always be noncrossing.

Example 3.4.21. Let $\mathcal{S}$ be the set system consisting of

$$\begin{aligned} S_1 &= \{1,2,3,4\}, \\ S_2 &= \{1,2,3,5\}, \\ S_3 &= \{4,5,6\}. \end{aligned}$$

The matroid $\mathcal{B}(\mathcal{S})$ is a positroid and this presentation is Gale minimal. However, $6,1,4,5$ witnesses a crossing of S_3 and S_2. A minimal presentation in 4-Gale order is

$$\begin{aligned} S_1' &= \{4,5,1,2\} \\ S_2' &= \{4,5,1,3\} \\ S_3' &= \{4,5,6\}. \end{aligned}$$

This presentation is noncrossing.

Conjecture 3.4.22. *Let $\mathcal{B}$ be a transversal positroid. There is a minimal presentation $\mathcal{S}$ of $\mathcal{B}$ which is noncrossing and which is minimal in a-Gale order for some a.*

This conjecture holds in the cases described in Propositions 3.4.7 and 3.4.9 and has been verified exhaustively for matroids of rank up to 4 on up to 10 elements. Additionally, it has received extensive computational verification on randomized examples of matroids of rank up to 8 on up to 14 elements.

The method of computation is to select a set system at random. If the set system satisfies the inequalities (3.3), we initialize a random integer matrix whose pattern of zero/non-zero entries is prescribed by the set system and use the matroid package in SAGE to compute the matroid represented by this matrix. If this matroid is a positroid, we compute presentations which are minimal in some cyclic shift of the Gale order until we either find a presentation which is noncrossing or have checked all cyclic shifts of the Gale order. SAGE code for these computations is available upon request. If one were inclined to further optimize the computation of a transversal matroid and to make the computation exact rather than probabilistic, the computation could be performed in the Boolean algebra, where nonzero matrix entries are set to TRUE, zero entries to FALSE, multiplication is replaced by AND, and addition is replaced by OR.

We remark that the odds that a random set system is a minimal presentation of a positroid are quite low. For a collection of 5 subsets of [10], the odds that a random set system is a minimal presentation of a positroid are roughly 0.5%, which allows a standard laptop to check one meaningful example about every 10 seconds. For rank ≥ 6, the odds of finding a minimal presentation of a positroid are vanishingly small, and this computation was performed as a large array of jobs on parallel system.

3.5 Comparison with Other Structures

We note that not all positroids are of the form $\mathcal{B}(\mathcal{S})$ for some $\mathcal{S}$.

Example 3.5.1. Consider the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

All maximal minors of this matrix are nonnegative, so the matroid $\mathcal{B}$ it represents is a positroid. However, this matroid is not a transversal matroid. Suppose $\mathcal{B} = \mathcal{B}(\{S_1, S_2\})$. Suppose that $1 \in S_1$. Then, $2 \notin S_2$, since $12 \notin \mathcal{B}$. So, $2 \in S_1$, and $1 \notin S_2$. Then, $3, 4, 5, 6 \in S_2$, since $13, 14, 15, 16 \in \mathcal{B}$. Then, since $34, 56 \notin \mathcal{B}$, $3, 4, 5, 6 \notin S_1$. Then, $35 \notin \mathcal{B}(\{S_1, S_2\})$, contradicting the fact that $\mathcal{B} = \mathcal{B}(\{S_1, S_2\})$.

The *interval rank function* is the map sending a $k \times n$ matrix M to the upper triangular $n \times n$ matrix $r(M)$ where

$$r(M)_{ij} = \text{rank(the submatrix of } M \text{ using columns } \{i, i+1, \ldots, j\}).$$

Note that the intervals appearing here are ordinary intervals, not cyclic intervals. An *open interval positroid variety* is the set of points in $Gr(k, n)$ with a fixed interval rank matrix. The matroid represented by a generic point in an interval positroid variety is an *interval positroid*. Interval positroid varieties were introduced by Knutson in [35] to study the degenerations appearing in Vakil's "geometric Littlewood-Richardson rule," [56]. Interval positroid varieties are positroid varieties. All Schubert varieties, opposite Schubert varieties, and Richardson varieties are interval positroid varieties. The matroid from Example 3.5.1 is an example of an interval positroid which is not a transversal matroid. The following example provides a transversal positroid which is not an interval positroid.

Example 3.5.2. Let $\mathcal{S} = \{1245, 23, 56\}$. The set system $\mathcal{S}$ is noncrossing, so Theorem 3.3.1 implies $\mathcal{B}(\mathcal{S})$ is a positroid. The interval rank matrix of a generic point in $L(\mathcal{S})$ is

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 \\ 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.12}$$

The interval positroid variety associated to (3.12) is the smallest interval positroid variety containing $L(\mathcal{S})$. So, if $\mathcal{B}(\mathcal{S})$ is an interval positroid, it must be the one defined by (3.12). Computing from (3.12) a bounded affine permutation as described in [35], then associating a positroid to this bounded affine permutation as described in [36], the interval positroid associated to (3.12) is

$$\{125, 126, 135, 136, 145, 146, 156, 235, 236, 245, 246, 256, 345, 346, 356\}.$$

Notably, 145 is a basis of this interval positroid, but $145 \notin \mathcal{B}(\mathcal{S})$.

Even though $MR(\mathcal{B}) \cap L(\mathcal{S})$ is dense in both $MR(\mathcal{B})$ and $L(\mathcal{S})$, neither set is in general contained in the other.

Example 3.5.3. Let $\mathcal{S} = \{134, 234\}$. Then,

$$\operatorname{span}\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \in L(\mathcal{S}).$$

This point is in $Gr_{\geq 0}(2,4)$, but represents a matroid aside from $\mathcal{B}(\mathcal{S})$. So, this point is not in $MR(\mathcal{B}(\mathcal{S}))$.

Example 3.5.4. The Marsh-Rietsch cell associated to the positroid $\mathcal{B} = \binom{[4]}{2}$ is

$$\left\{\operatorname{span}\begin{pmatrix} 1 & 0 & -a_3 & -(a_3a_4 + a_3a_2) \\ 0 & 1 & a_1 & a_1a_2 \end{pmatrix} \;:\; a_1, a_2, a_3, a_4 \in \mathbb{R}^*\right\}.$$

The set $\mathcal{S} = \{134, 234\}$ satisfies $\mathcal{B}(\mathcal{S}) = \mathcal{B}$. The subset of $MR(\mathcal{B})$ where $a_4 = -a_2$ is not contained in $L(\mathcal{S})$.

A *rank variety* is $\overline{L(\mathcal{S})}$ where each $S \in \mathcal{S}$ is an (ordinary) interval. Rank varieties were introduced by Billey and Coskun in [13] as a generalization of Richardson varieties.

Diagram varieties were introduced by Liu in [42], and studied by Pawloski in [52]. They define a diagram to be a subset D of $[k] \times [n-k]$. If $A \in M_{k,n}$, let $[A \mid I_k]$ be the $k \times n+k$ matrix obtained by appending a $k \times k$ identity matrix to the right of A. The *diagram variety* defined by D is the closure of

$$\{\text{span}[A \mid I_k] : A \in M_{k,n-k} \text{ with } A_{ij} = 0 \text{ when } (i,j) \in D\}.$$

Evidently, all diagram varieties are closures of basis shape loci, but there are basis shape loci whose closures are not diagram varieties.

3.6 Speculation: Dominoes

An amplituhedron is a projection of $Gr_{\geq 0}(k,n)$ to $Gr(k, k+m)$ for some m by a totally positive matrix. When $m = 4$, the volumes of amplituhedra conjecturally compute scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM). Toward verifying this conjecture, Arkani-Hamed and Trnka conjecture in [11] that a certain collection of positroid cells called BCFW cells project to a triangulation of the $m = 4$ amplituhedron, in the sense that their images are dense in the amplituhedron and overlap in a set of measure zero. BCFW cells, after Britto, Cachazo, Feng, and Witten, arise from the BCFW recurrence relation, which is known to compute certain amplitudes in $\mathcal{N} = 4$ SYM [18].

Karp, Williams, and Zhang provide a program for proving that the BCFW cells triangulate the amplituhedron in [33], and for finding collections of positroid cells that triangulate other amplituhedra for other even m. Roughly, their strategy is to first find a basis of a special shape, called a domino basis, for any plane in the positroid cell under consideration, then to apply sign variation techniques to these basis shapes to verify disjointness of the cells. Conjecture A.7 in [33] says that any BCFW cell admits a domino basis. Their sign variation techniques are similar to those used in [10] to give a description of amplituhedra in terms of binary codes. When $k = 1$, the conjectured triangulation of [33] is among the triangulations of cyclic polytopes described in Theorem 4.2 in [54].

Part of the impetus for this project was to serve this program of Karp, Williams, and Zhang. Currently, there is a sense of the kind of basis shapes which should be amenable to their sign variation arguments and in many cases of the cells which should appear in a triangulation of the amplituhedra, but one missing component is a formal way of connecting these two ideas. The hope is that rather than first identifying a family of positroid cells and then trying to find special basis shapes for points in these cells, one might be able to first identify the sorts of basis shapes $\mathcal{S}$ which are amenable to sign variation arguments and then work with positroid cells $L(\mathcal{S}) \cap Gr_{\geq 0}(k,n)$. Since the intersection of $L(\mathcal{S})$ and the associated positroid cell is dense in both of them, working with $L(\mathcal{S})$ is no different from working with the positroid cell from the perspective of producing a triangulation.

Briefly, we introduce the basis shapes appearing in Karp, Williams, and Zhang's program which our work is presently able to handle; this is just a subset of the basis shapes Karp, Williams, and Zhang consider. Say that a vector $\mathbf{v}$ is an *i-domino* if

$$\mathrm{supp}(\mathbf{v}) = \{i, i+1\},$$

where by convention $n+1 = 1$.[1] For $I \subset [n]$, say $\mathbf{v}$ is an *I-domino* if $\mathbf{v}$ is a sum of i-dominos with disjoint supports for all $i \in I$. For $\mathcal{I} = \{I_1, \ldots, I_k\}$, say $V \in Gr(k,n)$ admits an *$\mathcal{I}$-domino basis* if V is the span of I-dominos for $I \in \mathcal{I}$. Given such an $\mathcal{I}$, let

$$\mathcal{I}' = \{I_1', \ldots, I_k'\}, \tag{3.13}$$

where

$$I_j' = I_j \cup \{i+1 : i \in I_j\}.$$

Evidently, the set of planes admitting $\mathcal{I}$-domino bases is exactly $L(\mathcal{I}')$.

In [33], i-dominos are further required to have their adjacent entries have the same sign. We note that this requirement is actually a consequence of positivity and the shape constraint imposed on the vectors.

Proposition 3.6.1. *Suppose that $V \in Gr_{\geq 0}(k,n) \cap L(\mathcal{S})$ for some set system $\mathcal{S}$ satisfying the hypotheses of Theorem 3.1.2. Let $\mathbf{v}_{S_1}, \ldots, \mathbf{v}_{S_k}$ be a basis of shape $\mathcal{S}$ for V and let $(v_1, \ldots, v_n) = \mathbf{v}_{S_1}$. If $v_i, v_{i+1} \neq 0$, then*

$$\mathrm{sgn}(v_{i+1}) = \begin{cases} \mathrm{sgn}(v_i) & \text{if } i \neq n, \\ (-1)^{k-1}\mathrm{sgn}(v_i) & \text{if } i = n. \end{cases}$$

Proof. From point (iii) of Theorem 3.1.2, there is some

$$J = \{j_1, \ldots, j_{k-1}\} \subseteq [n] \setminus S_1$$

[1][33] treats n-dominos slightly differently, defining $\mathbf{v}$ to be an n-domino if $\mathrm{supp}(\mathbf{v}) = n$. We choose our convention since it will more natural for drawing a connection with Wilson loop cells in Chapter 5.

such that there is a matching from $S_2, \dots, S_k$ to J in $\Gamma_{\mathcal{S}}$. Let $A = (a_{\ell,m})$, where $a_{\ell,m}$ is the j_m^{th} coordinate of $\mathbf{v}_{S_{\ell+1}}$. Then,

$$\Delta_{J \cup i} = v_i \det(A),$$

and

$$\Delta_{J \cup i+1} = \begin{cases} v_{i+1} \det(A) & \text{if } i \neq n, \\ (-1)^{k-1} v_{i+1} \det(A) & \text{if } i = n. \end{cases}$$

Since $V \in Gr_{\geq 0}(k, n)$, these two Plücker coordinates have the same sign if they are both nonzero. □

These are only the simplest basis shapes appearing in Karp, Williams, and Zhang's work. There are still several gaps to be filled before this present work can be useful for describing all basis shapes appearing in their work. Notably, our results only apply in the case when the entries of the matrix $M_{\mathcal{S}}(\mathbf{x})$ are all independent. To capture the basis shapes appearing in [33], one would need to extend our results to describe cases where some of the entries of $M_{\mathcal{S}}(\mathbf{x})$ are prescribed to be equal to each other, or when they satisfy some other simple constraints. Another useful feature would be a way of translating between the set system $\mathcal{S}$ and any of the many combinatorial objects indexing positroids described in [53].

Another motivation for this present work comes from another program for computing amplitudes in $\mathcal{N} = 4$ SYM via Wilson loop diagrams (also called MHV diagrams). This program is introduced from a physical perspective in [20] and surveyed in a way more accessible to mathematicians in [4]. The geometric spaces arising in this program are basis shape loci $L(\mathcal{S})$ for a particular class of $\mathcal{S}$ defined by Wilson loop diagrams. One goal was to illustrate a connection between these shapes $\mathcal{S}$ and the domino bases from [33]. Using the perspective of basis shape loci, we obtain several corollaries about cells defined by Wilson loop diagrams in Section 5.2, and draw connections between Wilson loop cells and domino bases. Notably, Theorem 5.2.5 will show that the set of points in $Gr_{\geq 0}(k, n)$ admitting $\mathcal{I}$-domino bases where $|I| = 2$ for all $I \in \mathcal{I}$ is exactly the union of all Wilson loop cells.

Chapter 4

Deodhar Decompositions and Go-diagrams

This chapter proves several results about the Deodhar decomposition of the Grassmannian. The first three sections present results originally appearing in [45] by the author. The three main questions addressed in that paper are:

(1) Provide a set of local moves which may be used to transform any, not necessarily reduced, diagram into a Go-diagram.

(2) Describe the boundary structure of Deodhar components in the Grassmannian.

(3) Given an arbitrary filling of a Ferrers shape with black stones, white stones, and pluses, provide a test for whether this diagram is a Go-diagram.

Section 4.1 answers question (1), describing a set of *corrective flips* which may be used to transform any diagram into a Go-diagram. This should be thought of as a natural extension of Lam and Williams's ⅃-moves (introduced below) from [41] to non-reduced diagrams. However, unlike in the case of ⅃-moves on reduced diagrams, it is possible to obtain more than one Go-diagram for a fixed starting diagram via corrective flips.

Section 4.2 addresses question (2). In general, one does not expect questions of this nature to have a reasonable answer. The Deodhar decomposition is known to not even be a stratification of the flag manifold, [22]. However, we offer some evidence that there might be a reasonable answer restricted to the special case of Deodhar components within a Schubert cell. We introduce a new variety $\mathcal{V}_D$ inside a Schubert cell associated to a

$\bullet/\circ/+$-diagram D. When D is a Go-diagram, $\mathcal{V}_D$ contains and is conjecturally equal to $\overline{\mathcal{D}} \cap \mathcal{S}_\lambda$. Theorem 4.2.9 shows that $\mathcal{V}_{D'} \subset \mathcal{V}_D$ when D' and D are related by a certain diagrammatic procedure. We conjecture that the same diagrammatic procedure provides a complete charactization of when $\overline{\mathcal{D}'} \subset \overline{\mathcal{D}}$ with codimension one inside a Schubert cell, and prove this fact in the special case where D and D' are ⅃-diagrams.

Section 4.3 addresses question (3). Ideally, one would like a description of Go-diagrams in terms of forbidden subdiagrams, analogous to the description of ⅃-diagrams. We show that a reasonable description of this form cannot exist by providing an injection from the set of valid Go-diagrams to the set of "minimal forbidden subdiagrams" for the class of Go-diagrams in Theorem 4.3.1. So, the task of providing a list of forbidden subdiagrams for the class of Go-diagrams is at least as hard as providing a list of all valid Go-diagrams. In lieu of such a description, Theorem 4.3.8 provides an inductive characterization of Go-diagrams.

The final section of this chapter presents joint work with Agarwala, originally appearing in [6]. There, we describe a projection

$$\pi : Gr(k, n+1) \to Gr(k, n) \cup Gr(k-1, n),$$

and study the fibers of this map over Deodhar components in $Gr(k, n)$. We decompose the fiber over a Deodhar component into Deodhar components and describe the boundary structure of components appearing in the fiber. We further describe the fibers of the restriction of this projection to the positive Grassmannian.

4.1 Corrective Flips

The notation for $\bullet/\circ/+$-diagrams was introduced in Section 2.1. Throughout this section, stones in $\bullet/\circ/+$-diagrams will be colored black if and only if they are uncrossings. The work presented in this section originally appears in Section 3 of [45] by the author.

In [41], Lam and Williams address the problem of giving a series of local moves on $\circ/+$-diagrams which may be used to transform any reduced diagram into the ⅃-diagram corresponding to the same pair of permutations. Such moves are called ⅃-moves. They solve this problem in all cominuscule types. Their motivation was to provide a diagrammatic index set for the Deodhar components intersecting the totally nonnegative part of cominuscule Grassmannians akin to the ⅃-diagram description in type A. As in the type A case, this task is equivalent to providing a diagrammatic description of positive distinguished subexpressions in the quotient of a Weyl group by a cominuscule maximal parabolic subgroup. In type A, the only case we consider in this thesis, ⅃-moves are moves of the form given in Figure 4.1.

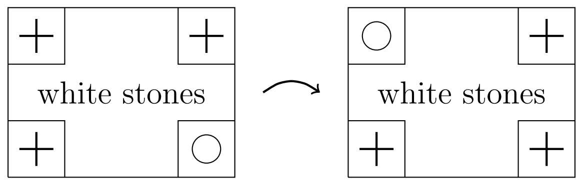


Figure 4.1: Le-move.

The following theorem collects Lemma 4.13, Proposition 4.14, and Theorem 5.3 from [41].

Theorem 4.1.1. *Let D be a reduced $\circ/+$-diagram.*

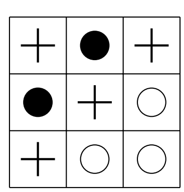
(i) If D' is obtained from D via a sequence of ⅃-moves, the permutations associated to D and D' are identical.

(ii) D is a ⅃-diagram if and only if no ⅃-moves may be applied to it.

(iii) Any sequence of ⅃-moves applied to D terminates in the unique ⅃-diagram associated to the same pair of permutations.

The goal of this section is to provide an analogous set of moves to transform any $\bullet/\circ/+$-diagram into a Go-diagram. The following example shows that ⅃-moves are not sufficient to transform any reduced diagram into a Go-diagram, so additional moves really are needed.

Example 4.1.2. Consider the following diagram.



This diagram is not a Go-diagram since the square in the top left corner violates the distinguished property. Further, there are no ⅃-moves which may be applied to this diagram.

Definition 4.1.3. Let D be a $\bullet/\circ/+$-diagram. Given a plus which violates the distinguished property, a *corrective flip*:

(i) switches the plus with either the white stone with which it violates the distinguished property or that stone's uncrossing partner if it exists, then

(ii) relabels the stones in the diagram so that a stone is black if and only if it is an uncrossing.

In the diagram from Example 4.1.2, we may perform a corrective flip, swapping the plus in the top left corner with the white stone in the bottom right corner. This corrective flip is the only corrective flip available in the diagram. After performing this flip, the diagram is a Go-diagram.

Remark 4.1.4. In [41], one of the defining features of ⅃-moves is that only two squares change filling during the ⅃-move, one from a white stone to a plus and the other from a plus to a white stone. In the case of a $\bullet/\circ/+$-diagram, the coloring of stones white or black should be thought of purely as a mnemonic for which stones correspond to crossings and uncrossings in the pipe dream. When performing a corrective flip, in the pipe dream only two tiles change: one from an elbow piece to a crossing and the other from a crossing to an elbow. The possible change in coloring of other stones in the diagram is a necessary side effect of this two square swap.

Proposition 4.1.5. *Let D be a $\bullet/\circ/+$-diagram.*

(i) D is a Go-diagram if and only if there are no available corrective flips.

(ii) Corrective flips preserve the pair of permutations associated to a diagram.

(iii) Corrective flips preserve number of black stones, white stones, and pluses in a diagram.

(iv) Suppose pipes i and j cross at the crossing tile involved in a corrective flip. Then, the only stones which change color when preforming a corrective flip are along pipes i and j on the segments between the plus and crossing tile involved in the flip.

(v) D can be transformed into a Go-diagram via corrective flips.

Proof. Points (i), (ii), and (iv) are obvious, looking at the pipe dream associated to a diagram. Point (iii) is a consequence of Proposition 2.1.9 and the fact that a corrective flip preserves the number of pluses in a diagram. For point (v), observe that if we only preform corrective flips switching pluses and white stones, the pluses only move downward. We may preform such flips until no more corrective flips are available, at which point point (i) implies the end result is a Go-diagram. $\square$

Lemma 4.1.6. *Let D be a $\bullet/\circ/+$-diagram and let D' be obtained from D by performing a corrective flip. Then, $u^D_{b^{in}} \geq u^{D'}_{b^{in}}$ for all $b \in D$.*

Proof. Let c and d be the boxes participating in the corrective flip and suppose that $c \prec d$. If $c \notin b^{in}$ or $d \in b^{in}$, then $u^D_{b^{in}} = u^{D'}_{b^{in}}$. Otherwise, if $c \in b^{in}$ and $d \notin b^{in}$, then $u^D_{b^{in}} > u^{D'}_{b^{in}}$. $\square$

In fact, a converse to Lemma 4.1.6 holds as well.

Proposition 4.1.7. *Let D and D' be $\bullet/\circ/+$-diagrams with the same associated pair of permutations and suppose that D' is obtained from D by exchanging a single elbow piece and crossing tile in the associated pipe dreams. If $u^D_{b^{in}} \geq u^{D'}_{b^{in}}$ for all $b \in D$, then D' was obtained from D by performing a corrective flip.*

Proof. If D and D' give the same pair of permutations and differ by exchanging a single elbow piece and crossing tile in their pipe dreams, the elbow piece and crossing tile exchanged must involve the same pair of pipes. Suppose the exchanged crossing tile is in box c and the elbow piece is in box d. If c contained a white stone in D, we must have $c \prec d$, otherwise $u^D_{b^{in}} < u^{D'}_{b^{in}}$ for any box b such that $d \prec b$ and $b \not\preceq c$. If there were some box e with $c \prec e \prec d$ which was an uncrossing pair with c in D, then $u^D_{b^{in}} < u^{D'}_{b^{in}}$ for any box b with $e \prec b \prec d$. So, in this case D' is obtained from D via a corrective flip. If c contained a black stone in D, we must have $d \prec c$, otherwise $u^D_{b^{in}} < u^{D'}_{b^{in}}$ for any box b such that $c \prec b$ and $b \not\preceq d$. In this case, if c had a crossing pair e with $d \prec e \prec c$, then $u^D_{b^{in}} < u^{D'}_{b^{in}}$ for any box b with $d \prec b \prec e$. $\square$

Theorem 4.1.8. *Every sequence of corrective flips terminates in a Go-diagram.*

Proof. To a diagram D, associate the tuple

$$\tau(D) = \bigoplus_{b \in D} u^D_{b^{in}} \in \bigoplus_{b \in D} \mathfrak{S}_n.$$

We endow $\bigoplus_{b \in D} \mathfrak{S}_n$ with the product partial order obtained from the Bruhat orders on each copy of $\mathfrak{S}_n$. Let D' be obtained from D by performing a corrective flip. Lemma 4.1.6 implies $\tau(D') < \tau(D)$. Since this poset is finite, any sequence of corrective flips must terminate. Then, point (ii) in Proposition 4.1.5 implies any sequence of corrective flips terminates in a Go-diagram. $\square$

Unlike point (iii) in Theorem 4.1.1, there might be more than one Go-diagram obtainable from a •/ ∘ /+-diagram via corrective flips.

Example 4.1.9. Consider the •/ ∘ /+-diagram

●	+	+
+	+	+
+	+	○

, (4.1)

which is not a Go-diagram. Using corrective flips, it may be transformed into either

+	+	+
+	●	+
+	+	○

or

●	+	+
+	○	+
+	+	+

. (4.2)

One could remove this aspect of free will from the definition of corrective flip, for instance by defining corrective flips to only switch pluses and white stones. However, we find Definition 4.1.3 is the correct choice of definition given Proposition 4.1.7 and the role corrective flips play in the boundary structure of Deodhar components, described in Section 4.2.

Remark 4.1.10. The set of corrective flips as described is not a minimal set of moves with the properties described in Proposition 4.1.5 and Theorem 4.1.8. If one wanted a smaller set of moves with these properties, they could consider only corrective flips such that the elbow and crossing pieces being switched in the pipe dream have no other elbow pieces between them involving the same pair of pipes. However, even this set of moves isn't minimal: restricted to reduced diagrams, it is a strictly larger set of moves than the set of ⅃-moves. It could be interesting to describe a set of corrective flips which is minimal and whose specialization to reduced diagrams is exactly the set of ⅃-moves.

4.2 Boundaries

The notation for •/ ∘ /+-diagrams was introduced in Section 2.1. Throughout this section, stones in •/∘/+-diagrams will be colored black if and only if they are uncrossings. The work presented in this section originally appears in Section 4 of [45] by the author.

This section introduces a new variety $\mathcal{V}_D$ inside of a Schubert cell associated to a $\bullet/\circ/+$-diagram D. This variety contains the closure of the Deodhar component $\mathcal{D}$ inside the Schubert cell $\mathcal{S}_\lambda$. We describe a combinatorial condition implying a containment $\mathcal{V}_{D'} \subset \mathcal{V}_D$. We conjecture that this same combinatorial condition implies $\overline{\mathcal{D}'}$ is a codimension one boundary of $\overline{\mathcal{D}}$ and prove this conjecture in the case where $\overline{\mathcal{D}'}$ and $\overline{\mathcal{D}}$ are positroid varieties.

Definition 4.2.1. Let D be a $\bullet/\circ/+$-diagram whose Ferrers shape λ has vertical steps $i_1 < i_2 < \cdots < i_k$ and horizontal steps $j_1 < j_2 < \cdots < j_{n-k}$. Define $I_{u_{b^{in}}}$ to be the set associated to the permutation $u_{b^{in}}$. So, $I_{u_{b^{in}}}$ is obtained diagrammatically by changing the filling of all boxes in b^{out} to white stones, computing the pipe dream associated to this diagram, then reading the labels of the pipes along the left boundary of the Ferrers shape. Define $\mathcal{V}_D$ to be the subset of $Gr(k,n)$ where:

(i) $\Delta_I = 0$ if $I \ngeq I_\lambda$,

(ii) $\Delta_{I_\lambda} \neq 0$, and

(iii) for all boxes $(i_\ell, j_m) \in D$, $\Delta_I = 0$ if $i_1, \ldots, i_{\ell-1} \in I$, $j_{n-k}, j_{n-k-1}, \ldots, j_{m+1} \notin I$ and $I \nleq I_{u_{(i_\ell,j_m)^{in}}}$,

Remark 4.2.2. When a box b contains a white stone, $I_b > I_{u_{b^{in}}}$. So, this definition captures the fact that $\Delta_{I_b} = 0$ when D is a Go-diagram and b contains a white stone.

Corollary 4.2.3. *Let D be a Go-diagram of shape λ. Then, $\overline{\mathcal{D}} \cap \mathcal{S}_\lambda \subseteq \mathcal{V}_D$.*

Proof. Theorem 2.2.25 implies that every Plücker coordinate perscribed to vanish in point (iii) of Definition 4.2.1 also vanishes uniformly on $\mathcal{D}$. $\square$

We offer two conjectures relating $\mathcal{V}_D$ to $\mathcal{D}$. Conjecture 4.2.5 is a strengthening of Conjecture 4.2.4.

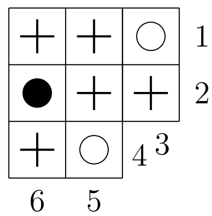
Conjecture 4.2.4. *The ideal generated by the Plücker coordinates in (i) and (iii) of Definition 4.2.1 is exactly the ideal of Plücker coordinates vanishing uniformly on $\overline{\mathcal{D}}$.*

Conjecture 4.2.5. *Let D be a Go-diagram inside the Ferrers shape λ and $\mathcal{D}$ be the associated Deodhar component. Then, $\overline{\mathcal{D}} \cap \mathcal{S}_\lambda = \mathcal{V}_D$.*

Remark 4.2.6. There is some reason to be skeptical of Conjecture 4.2.5. The variety $\overline{\mathcal{D}}$ is some matroid variety, which one does not expect to be defined by the vanishing of Plücker coordinates in general (see Counterexample 2.6 in [25]). However, given that closures of Deodhar components are considerably more restricted than general matroid varieties, we are optimistic about Conjecture 4.2.5.

Remark 4.2.7. Conjecture 4.2.5 gives an alternate, perhaps more tractable characterization of Deodroids (see Section 2.4). It says that a Deodroid is completely characterized by the smallest lattice path matroids containing each minor from Corollary 2.4.3 as a submatroid. Equivalently, a Deodhar component is determined by which Richardson cells the projection of $\mathcal{D} \cap (x_{i_1} = \cdots = x_{i_\ell} = 0)$ onto $\mathrm{span}(x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_m})$ lie in for each ℓ, m.

Example 4.2.8. This example illustrates that in general there are Plücker coordinates in the ideal of $\mathcal{V}_D$ that are not listed in points (i) or (ii) of Definition 4.2.1. Let D be the following Go-diagram.



Consider the set 236. In the Go-network $N(\mathcal{D})$, there is not a flow to 236 since there is not a path from 1 to 3. So, Δ_{236} vanishes uniformly on $\mathcal{D}$.

We show that even though 236 is not among the sets listed in Definition 4.2.1, that Δ_{236} is nontheless forced to vanish on $\mathcal{V}_D$. Evidently, $236 > 124$, so point (i) of Definition 4.2.1 does not force Δ_{236} to vanish on $\mathcal{V}_D$. Since $1 \notin 236$ and $6 \in 236$, the only constraint from (iii) which is relevant is that associated to the box $(1, 6)$. Note that $236 < I_{u_{(1,6)^{in}}} = 456$, so this constraint does not force Δ_{236} to vanish on $\mathcal{V}_D$. Consider the Plücker relation

$$\Delta_{236}\Delta_{124} = \Delta_{123}\Delta_{246} + \Delta_{234}\Delta_{126}.$$

Since $123 < 124$, Δ_{123} vanishes on $\mathcal{V}_D$. Since $234 > 134 = I_{u_{(1,3)^{in}}}$, Δ_{234} vanishes on $\mathcal{V}_D$. Then, since $\Delta_{124} \neq 0$ on $\mathcal{V}_{\mathcal{D}}$, Δ_{236} must vanish on $\mathcal{V}_D$.

Theorem 4.2.9. *Let D and D' be $\bullet/\circ/+$ diagrams. Suppose that D is obtained from D' by one of the following procedures.*

(i) Replacing the two stones in a crossing/uncrossing pair with pluses.

(ii) Replacing a white stone without an uncrossing pair with a plus.

(iii) Performing corrective flips.

Then, $\mathcal{V}_{D'} \subset \mathcal{V}_D$.

Proof. To show that $\mathcal{V}_{D'} \subset \mathcal{V}_D$, it suffices to show that $I_{u^{D'}_{b^{in}}} \leq I_{u^{D}_{b^{in}}}$ for all boxes b. We show the stronger fact the $u^{D'}_{b^{in}} \geq u^{D}_{b^{in}}$ for any box.

For case (i), let c and d form a crossing/uncrossing pair in D' where c contains a white stone and d contains a black stone and let D be the diagram obtained by replacing these two stones with pluses. Then, $u^{D'}_{b^{in}} > u^{D}_{b^{in}}$ if $c \prec b$ and $b \not\prec d$. For all other boxes $u^{D'}_{b^{in}} = u^{D}_{b^{in}}$.

For case (ii), let c contain a white stone without an uncrossing pair in D' and let D be the diagram obtained by changing c to a plus. Then, $u^{D'}_{b^{in}} > u^{D}_{b^{in}}$ if $c \prec b$. Otherwise, $u^{D'}_{b^{in}} = u^{D}_{b^{in}}$

Case (iii) is Lemma 4.1.6. □

We conjecture that the combinatorial procedure from Theorem 4.2.9 applied inductively implies a containment of closures of Deodhar components. Theorem 4.2.16 proves this conjecture for positroid varieties. Theorem 4.4.5 proves another special case of this conjecture.

Conjecture 4.2.10. *Let D and D' be Go-diagrams of the same Ferrers shape and $\mathcal{D}$ and $\mathcal{D}'$ be the associated Deodhar components. Then, $\overline{\mathcal{D}'} \subset \overline{\mathcal{D}}$ with $\dim(\mathcal{D}) = \dim(\mathcal{D}') + 1$ if and only if D is obtained from D' by:*

(i) Choosing a crossing/uncrossing pair in D',

(ii) replacing the two stones in this pair with pluses, and relabelling the other stones in the diagram such that a stone is colored black if and only if it is an uncrossing, then

(iii) performing corrective flips.

or by:

(i) Choosing a white stone without an uncrossing pair in D' such that replacing this stone with a plus decreases the length of the diagram's permutation by exactly one,

(ii) replacing this white stone with a plus, and relabelling the other stones in the diagram such that a stone is colored black if and only if it is an uncrossing, then

(iii) performing corrective flips.

Remark 4.2.11. The fact that these conditions would imply $\overline{\mathcal{D}'}$ is a codimension one boundary of $\overline{\mathcal{D}}$ would follow from Theorem 4.2.9 applied inductively together with Conjecture 4.2.5. If this is the case, the image to have in mind is that $\mathcal{V}_D = \overline{\mathcal{D}}$ when D is a Go-diagram. When, D'' is not a Go-diagram, $\mathcal{V}_{D''}$ is some possibly reducible variety. As we undo corrective flips, we shed irreducible components, eventually arriving at another Deodhar closure $\mathcal{V}_{D'} = \overline{\mathcal{D}'}$.

Remark 4.2.12. As an additional reality check for the sufficiency of this combinatorial procedure to imply a boundary structure of Deodhar closures, we have proved that if D obtained from D' by one of the procedures described, then every Plücker coordinate which is prescribed to be nonvanishing on $\mathcal{D}'$ is not prescribed to vanish on $\mathcal{V}_D$. This is a necessary, but not a priori sufficient condition for having $\overline{\mathcal{D}'} \subset \overline{\mathcal{D}}$. We can provide this proof upon request.

Remark 4.2.13. The necessity of this combinatorial condition for there to be a containment of closures of Deodhar components inside a Schubert cell is purely speculative, and there is reason to be skeptical of this half of the conjecture. In particular, Proposition 2.5 in [22] shows that the closure of a Deodhar component is not in general a union of Deodhar components by providing two Deodhar components $\mathcal{D}$ and $\mathcal{D}'$ in the same Schubert cell in the type B full flag manifold such that $\overline{\mathcal{D}} \cap \mathcal{D}'$ is a nonempty proper subset of $\mathcal{D}'$. Proposition 2.7 in [22] disproves a conjecture for determining whether $\overline{\mathcal{D}} \cap \mathcal{D}' \neq \emptyset$. While Conjecture 4.2.10 addresses a question distinct from these two issues, the general wild behavior of the Deodhar decomposition could be reason for skepticism.

Example 4.2.14. This example illustrates the first set of moves described in Theorem 4.2.10. Consider the following Go-diagram.

$$D' = \begin{array}{|c|c|c|l} \hline \bullet & \bullet & + & 1 \\ \hline + & + & \circ & 2 \\ \hline + & + & \circ & 3 \\ \hline \multicolumn{1}{c}{6} & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{4} & \end{array} \tag{4.3}$$

The white stone at $(2, 4)$ and the black stone at $(1, 5)$ form a crossing/uncrossing pair. Replacing the stones in these squares with pluses yields the diagram (4.1) from the previous section. We saw that this diagram could be transformed into either of the Go-diagrams (4.2) via corrective flips. So, $\mathcal{V}_{D'}$ is a boundary of the varieties labelled by the Go-diagrams (4.2).

Example 4.2.15. This example illustrates the second set of moves described in Theorem 4.2.10. Consider the following Go-diagram.

$D' =$

+	○	○	+	1
+	○	+	+	2
6	5	4	3	

Replacing the white stone at $(1, 5)$ with a plus decreases the length of the permutation by exactly one, so this replacement is valid. After, performing this replacement we obtain the diagram

+	+	○	+
+	○	+	+

,

which is not a Go-diagram. Performing corrective flips, which in this case are simply ⅃-moves, we arrive at the diagram

$D =$

○	+	○	+
+	+	+	+

.

So, $\mathcal{V}_{D'}$ is a boundary of $\mathcal{V}_D$. The diagrams D and D' are also ⅃-diagrams and thus index positroid cells $\mathcal{P}$ and $\mathcal{P}'$. Theorem 4.2.16 will imply that $\mathcal{P}'$ is a codimension one boundary of $\mathcal{P}$.

Theorem 4.2.16. *Let D and D' be ⅃-diagrams in the same Ferrers shape indexing positroid cells $\mathcal{P}_{u,v}$ and $\mathcal{P}_{u',v}$. Then, $\mathcal{P}_{u',v}$ is a codimension one boundary of $\mathcal{P}_{u,v}$ if and only D is obtained from D' by:*

(i) Choosing a white stone in D' such that replacing this stone with a plus decreases the length of the diagram's permutation by exactly one,

(ii) replacing this white stone with a plus, then

(iii) performing ⅃-moves.

Proof. Let D and D' be ⅃-diagrams with the same Ferrers shape indexing positroid cells $\mathcal{P}_{u,v}$ and $\mathcal{P}_{u',v}$. Combining Theorem 5.10, Theorem 5.9, and Theorem 3.16 from [36], $\overline{\mathcal{P}_{u',v}} \subset \overline{\mathcal{P}_{u,v}}$ if and only if there is a containment of Bruhat intervals $[u', v] \subset [u, v]$. In this case, the codimension of $\overline{\mathcal{P}_{u',v}}$ in $\overline{\mathcal{P}_{u,v}}$ is $\ell(u) - \ell(u')$. Let $\mathbf{u}$ be the expression obtained by omitting the identity terms in the positive distinguished expression for u in $\mathbf{v}$. Then, $\mathcal{P}_{u',v}$ is a codimension one boundary of $\mathcal{P}_{u,v}$ if and only if $\ell(u) - \ell(u') = 1$ and there is a subexpression $\mathbf{u}'$ of $\mathbf{u}$ obtained by omitting one transposition of $\mathbf{u}$. This subexpression $\mathbf{u}'$ is diagrammatically realized by replacing the white stone corresponding to the omitted transposition in the ⅃-diagram associated to u with a plus. Theorem 4.1.1 implies that the resulting diagram can be transformed into the ⅃-diagram D' indexing the positroid cell $\mathcal{P}_{u',v}$ using ⅃-moves. □

Example 4.2.17. The poset in Figure 4.2 consists of Go-diagrams labelling Deodhar components with the positroid cell $\mathcal{P}_{123456,456123} \subset Gr(3, 6)$ ordered by containment of closures of Deodhar components.

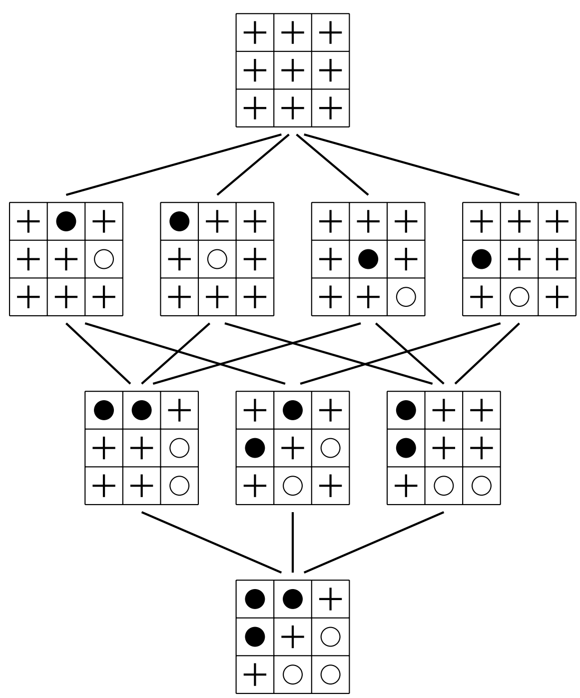


Figure 4.2: Boundary poset of Deodhar components within the positroid cell $\mathcal{P}_{123456,456123}$.

4.3 Classification of Go-diagrams

The notation for $\bullet/\circ/+$-diagrams was introduced in Section 2.1. In this section, unlike the previous two sections, we do not assume that stones in $\bullet/\circ/+$-diagrams are colored black if and only if they are uncrossings. The work presented in this section originally appears in Section 5 of [45].

The goal of this section is to give a means of verifying whether an arbitrary filling of a Ferrers shape with black stones, white stones, and pluses is a Go-diagram. For ⅃-diagrams, there is a compact description of the class of ⅃-diagrams as diagrams avoiding certain subdiagrams. Theorem 4.3.1 shows that no reasonable description of Go-diagrams in terms of forbidden subdiagrams can exist. In lieu of such a description, Theorem 4.3.8 gives an inductive characterization of the class of Go-diagrams.

We say that a rectangular diagram is a *minimal forbidden subdiagram* if the only square in the diagram which violates the distinguished property is in the top left corner, and the pair of pipes which should uncross in this square initially cross in the bottom right corner. Restricted to $\circ/+$-diagrams, minimal forbidden subdiagrams are of the form given in Figure 4.3.

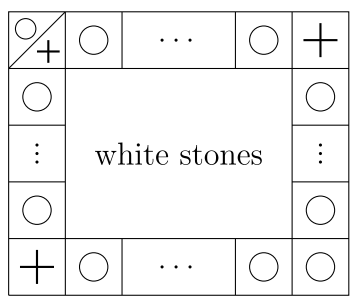


Figure 4.3: Minimal fobidden subdiagrams for the class of Le-diagrams.

It is an easy exercise to show that a $\circ/+$-diagram which violates the ⅃-condition must contain one of these minimal forbidden subdiagrams. The following theorem shows that the set of minimal violations for $\bullet/\circ/+$-diagrams is much more poorly behaved, by providing an injection from the set of valid Go-diagrams into the set of minimal forbidden subdiagrams for the class of Go-diagrams. Since every minimal forbidden subdiagram must appear on any list of forbidden subdiagrams for the class of Go-diagrams, this shows that Go-diagrams do not admit a reasonable description in terms of forbidden subdiagrams. This theorem provides a negative answer to Problem 4.9 in [39].

Theorem 4.3.1. *There is an injection from the set of valid Go-diagrams into the set of minimal forbidden subdiagrams for the class of Go-diagrams.*

Proof. Let D be a Go-diagram whose Ferrers shape λ which fits inside a $k \times (n-k)$ rectangle. Let D' be the diagram inside a $k \times (n-k)$ rectangle obtained by placing D in the top left corner, then padding out the bottom right corner with pluses. Note that D' is a Go-diagram.

Consider the 2×2 diagram

●	+
+	○

. (4.4)

Build the reflection of the shape λ over the line $y = x$ using these 2×2 blocks. Call this figure Λ. Now, build a rectangular diagram D'' which contains D' in the top left corner, Λ in the bottom right corner, and pluses padding out the rest of the squares. The dimensions of D'' do not matter as long as there is enough room so that no square in D' is adjacent to a square in Λ. Observe that D'' is a valid Go-diagram.

Finally, build a one box wide border around D'' which has:

- pluses in the top left, top right, and bottom left corners,
- a white stone in the bottom right corner,
- white stones along the bottom and right sides, and
- white or black stones along the top and left sides, as is necessary to avoid a violation of the distinguished property.

In this diagram, the top left square should be an uncrossing with the bottom right square and hence violates the distinguished property. As no other square in the diagram violates the distinguished property, this diagram is a minimal forbidden subdiagram.

Given a diagram of this form, one can recover the diagram D it came from. To do so, first delete a one square wide strip of boxes from the boundary of the diagram. Then, examine the bottom right portion of this diagram to find a Ferrers shape built out of copies of the 2×2 subdiagram (4.4). The boxes of this same Ferrers shape in the top left corner are the diagram D. Since this map is reversible, it is an injection from the set of valid Go-diagrams into the set of minimal forbidden subdiagrams for the class of Go-diagrams. □

Example 4.3.2. Figure 4.4 shows a Go-diagram and its image under the injection described in Theorem 4.3.1. The outline of the original Go-diagram and the reflection of its Ferrers shape drawn in 2×2 blocks (4.4) are highlighted in bold for clarity.

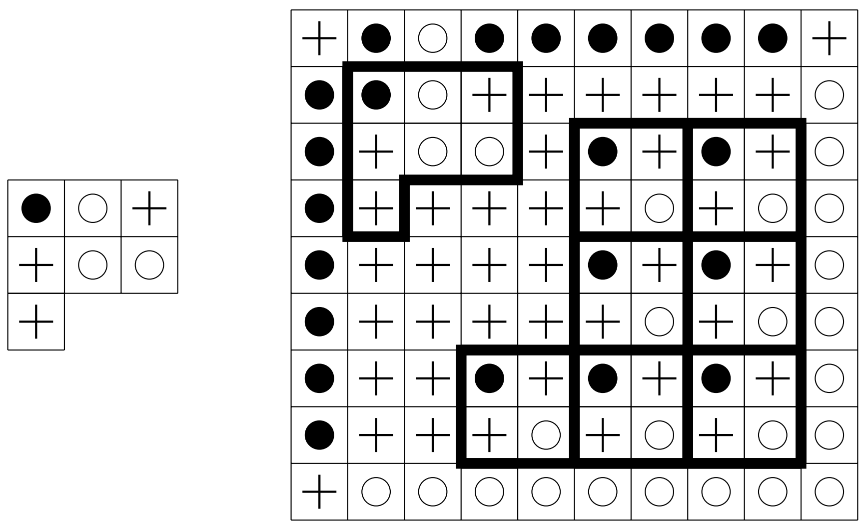


Figure 4.4: A Go-diagram and its image under the injection described in Theorem 4.3.1.

All minimal violations obtained via the injection described in Theorem 4.3.1 have the feature that, in the associated pipe dream, the pipes involved in the violation of the distinguished property take only one turn each. However, this is not necessary in general; it is possible for the pipes involved in the violation of the distinguished property to take arbitrarily many turns. For instance, one can arrange copies of the 2×2 block (4.4) in a serpentine pattern, as shown in Figure 4.5. In this figure, we've highlighted to subdiagrams (4.4) to make the pattern clearer. All boxes not drawn are filled with white stones. One may check that this example is indeed a minimal violation.

In lieu of a good description of Go-diagrams in terms of forbidden subdiagrams, we offer an algorithmic characterization of when a filling of a Ferrers shape with black stones, white stones, and pluses is a Go-diagram. Algorithm 4.3.4 provides a method of producing

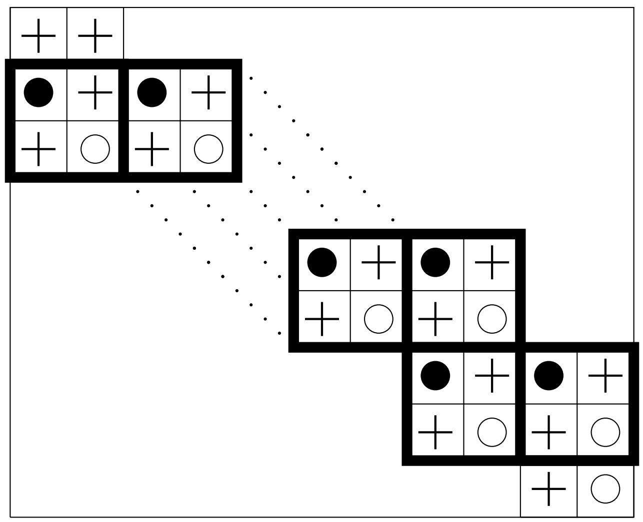


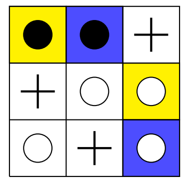
Figure 4.5: A minimal forbidden subdiagram for the class of Go-diagrams featuring a serpentine pattern.

a *partner* square to any square in the diagram. A $\bullet/\circ/+$-diagram will be a Go-diagram if and only the set of squares with partners is exactly the set of squares filled with black stones. In general, the partner of a black stone will be different than the white stone it serves as an uncrossing pair to. This notion of partner has two advantages over crossing/uncrossing pairs:

- Replacing all black stones and their partners with pluses simultaneously yields a reduced $\circ/+$-diagram for the same pair of permutations.
- For a black stone in box b, replacing all black stones in b^{in} and their partners with pluses simultaneously does not alter the location of b's partner.

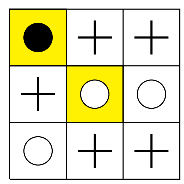
The following example shows that these properties are not enjoyed by crossing/uncrossing pairs.

Example 4.3.3. Consider the following Go-diagram, where the boxes containing one crossing/uncrossing pair have been shaded blue (dark gray in grayscale) and those containing the other have been shaded yellow (light gray).

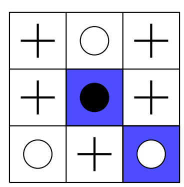


(4.5)

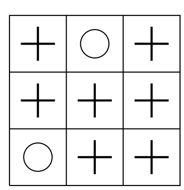
If we undo the blue crossing uncrossing pair, the diagram transforms into the following. Note that the location of the white stone which was part of the yellow pair has moved.



If we undo the yellow crossing/uncrossing pair, the diagram becomes the following. Note that in this case, the location of the other black stone has moved.



In either case, after undoing the last crossing/uncrossing pair and performing ⅃-moves if necessary, we arrive at the following ⅃-diagram.



(4.6)

Simultaneously replacing the stones the blue and yellow squares in (4.5) with pluses yields

+	+	+
+	○	+
○	+	+

,

which is not a reduced diagram for the same permutation as (4.6).

The problem of black stones moving around when undoing crossing/uncrossing pairs can be solved by undoing these crossing uncrossing pairs as black stones increase in the partial order $\prec$ on boxes. So, in Example 4.3.3 we would first undo the blue crossing/uncrossing pair, then undo the yellow one. The problem of the white stones involved in crossing/uncrossing pairs moving around is however unavoidable.

The following algorithm provides an inductive procedure to compute the *partner* of a box in a $\bullet/\circ/+$-diagram. This algorithm will be applied to all the boxes in a diagram in increasing order in the $\prec$ partial order on boxes to construct a partner for every black stone in a Go-diagram.

Algorithm 4.3.4. Let $b = (i_b, j_b)$ be a box in a $\bullet/\circ/+$-diagram.

1. If there is no black stone or plus to the right of b in row i_b or no black stone or plus below b in column j_b, then b has no partner.

2. Trace right from b row i_b until you hit a black stone or a plus in a box $c = (i_b, j_c)$ and down from b down in column j_b until you hit a black stone or plus in a box $d = (i_d, j_b)$.

3. If any of the following situations occur, b has no partner:

   3.1. The Ferrers shape does not have a box $e = (i_d, j_c)$.

   3.2. There is a plus or a black stone in a square (i, j_c) with $i_b < i < i_d$.

   3.3. There is a plus or a black stone in a square (i_d, j) with $j_c < j < j_b$.

4. If $e = (i_d, j_c)$ contains a white stone, e is b's partner. Otherwise, if e contains a plus or a black stone, construct a path P_b starting at b traveling to the right via the following procedure:

   4.1. If P_b hits a plus while traveling right, switch from traveling right to down;

4.2. If P_b hits a plus while traveling down, switch from traveling down to right;

4.3. If P_b hits a black stone while traveling right jump to that black stone's partner it exists and continue traveling down, if that black stone has no partner then b has no partner;

4.4. If P_b hits a black stone while traveling down jump to that black stone's partner and continue traveling right, if that black stone has no partner then b has no partner;

4.5. If P_b hits a white stone that was partnered with some other black stone in b^{in} while traveling right, switch from traveling right to down;

4.6. If P_b hits a white stone that was partnered with some other black stone in b^{in} while traveling down, switch from traveling down to right.

5. Construct a path Q_b starting at b and traveling down following the same rules.

6. If P_b and Q_b meet and the first square they meet in (the largest square they meet in the $\prec$ partial order) contains a white stone, that square is b's partner.

7. Otherwise, b has no partner.

Example 4.3.5. Consider the following diagram, which is a Go-diagram.

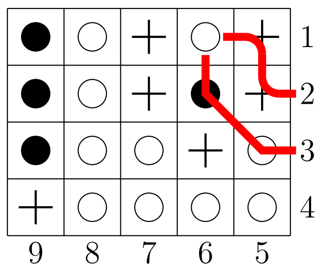
●	○	+	○	+	1
●	○	+	●	+	2
●	○	○	+	○	3
+	○	○	○	○	4
9	8	7	6	5	

None of the boxes in row 4 and in columns 5 and 8 will have partners, since none of these boxes have a plus or a black stone both below them and to their right. The boxes $(2,7), (3,6)$, and $(3,7)$ do not have partners for similar reasons.

It is straight forward to see that the black stone in box $(2,6)$ is partnered with the white stone in box $(3,5)$; that the black stone in box $(3,9)$ is partnered with the white stone in box $(4,6)$; and that the black stone in box $(2,9)$ is partnered with the white stone in box $(3,7)$.

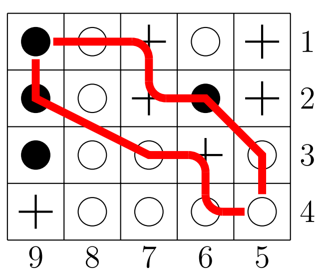
The box in $(1, 7)$ has pluses to its right and below it. However, from these pluses, if we try to trace down from $(1, 5)$ and right from $(2, 7)$ to where they meet in $(2, 5)$, we notice there is a black stone along the line from $(2, 7)$ to $(2, 5)$. So, point 3.3 in Algorithm 4.3.4 implies that $(1, 7)$ does not have a partner.

The box $(1, 6)$ has boxes with pluses or black stones below it and to its right. Tracing right from $(2, 6)$ and down from $(1, 5)$, there are no pluses or black stones before we arrive at the box $(2, 5)$. So, we proceed to point 4 of the algorithm. Since there is a plus in $(2, 5)$, we must construct paths $P_{(2,5)}$ and $Q_{(2,5)}$.



Since these paths do not meet, $(2, 6)$ does not have a partner.

Finally, the box $(1, 9)$ has boxes with pluses or black stones below it and to its right. Drawing out the paths $P_{(1,9)}$ and $Q_{(1,9)}$ as described, we obtain the following. Note that the path $Q_{(1,9)}$ takes a turn in box $(4, 6)$ because that box was partnered with the black stone in $(3, 9)$.



Since these two paths meet in a white stone at $(4, 5)$, box $(1, 9)$ is partnered to box $(4, 5)$.

Proposition 4.3.6. *Suppose $b = (i_b, j_b)$ is a box in a $\bullet/\circ/+$-diagram D which has a partner.*

(i) b's partner is in b^{in}.

(ii) *If the paths P_b and Q_b in Algorithm 4.3.4 are constructed, all boxes encountered along these paths are in b^{in}.*

(iii) *b's partner is a white stone.*

(iv) *No other box shares a partner with b.*

(v) *Let $c = (i_b, j_c)$, $d = (i_d, j_b)$, and $e = (i_d, j_c)$ be as in Algorithm 4.3.4. If there is a plus or a black stone in e, one of c or d must contain a black stone.*

(vi) *Let $a \in D$ be incomparable to b in the partial order $\prec$ on boxes. Suppose a has a partner p. Then p is not along the paths P_b or Q_b from b constructed in Algorithm 4.3.4.*

Proof. Points (i), (ii), and (iii) are immediately apparent from Algorithm 4.3.4. To prove point (iv), suppose that the boxes b and b' share a partner p. Since the construction of the paths from b and b' to p are entirely deterministic, either the paths from b to p goes through b' or the paths from b' to p goes through b. Either case creates a contradiction with point 6 of Algorithm 4.3.4.

For (v), suppose the box e contains a black stone. Then, we must construct the paths P_b and Q_b in Algorithm 4.3.4. If both c and d contain pluses, then these paths first meet at the box e. So, e is the partner of b, which contradicts point (iii) in this proposition. We remark that the case of point (v) where e contains a plus is an artifact of the distinguished property for subwords. The case where e contains a black stone is an artifact of the fact the crossings and uncrossings must alternate.

For (vi), suppose the path P_b goes through p. As remarked in the proof of (iii), the construction of these paths is reversible. So, P_b agrees with P_a or Q_a eventually and thus goes through a eventually, either before or after b. But, everything along paths from b is in b^{in} and everything along paths from a is in a^{in} by point (ii). This contradicts the incomparability of a and b. □

Observe that in Example 4.3.5 the partners $(2, 9)$ and $(3, 7)$ do not constitute a crossing/uncrossing pair in the pipe dream associated to the diagram. However, if we change the boxes $(3, 9)$ and $(4, 6)$ to pluses, the black stone in $(2, 9)$ and the white stone in $(3, 7)$ will be a crossing/noncrossing pair. In fact, for any black stone in a box b in this diagram, if we flip all of the black stones and their partners in $b^{in} \setminus b$ to pluses, b and its partner form a crossing/uncrossing pair in the new diagram. This observation generalizes.

Consider a $\bullet/\circ/+$-diagram D such that a square has a partner if and only if it is filled with a black stone. We'll see shortly that such diagrams are exactly Go-diagrams. Let b be a box in a diagram D. Let $f(D,b)$ be the diagram obtained by replacing all black stones in boxes $c \prec b$ and all white stones in the partners of these boxes with pluses. Consider the following pair of properties:

(P1) $u_{b^{in}}^{f(D,b)} = u_{b^{in}}^{D}$.

(P2) b has a partner p if and only if the boxes b and p form a crossing/uncrossing pair in $f(D,b)$.

Lemma 4.3.7. *Let D be a $\bullet/\circ/+$-diagram such that a square has a partner if and only if it is filled with a black stone. Let b be a box in D such that properties (P1) and (P2) hold for all $c \prec b$. Then, properties (P1) and (P2) hold for b.*

Proof. Let b be a box in D and suppose that properties (P1) and (P2) hold for all boxes in b^{in} aside from b. Let c be the box directly to the right of b if such a box exists. We may flip all of the black stones in c^{in} and their partners to pluses to obtain a diagram D' without changing the permutation $u_{c^{in}}$. At this point, we have flipped all black stones and their partners in b^{in} aside from those in b's column. We proceed to flip the black stones in this column and their partners starting from the bottom of the column.

Let d be the lowest box containing a black stone in the same column as b. From point (vi) in Proposition 4.3.6, the only squares along the paths P_d and Q_d from d to its partner which were flipped in passing to D' are black stones in d^{in} and partners of these stones. In the pipe dream of $f(D,d)$, follow the pipes coming out of the box d down and to the right. The pipe going to the right turns downward at the first plus it encounters; such a plus could have come from either a plus or a black in the original diagram D. Likewise, the pipe going down from d turns right at the first plus it encountered. Points 3.2 and 3.3 in Algorithm 4.3.4 and point (vi) in Proposition 4.3.6 guarantee that, after these initial turns these pipes continue without turning until they meet at some square e. If e contained a white stone in D, it still contains a white stone in $f(D,d)$. In this case, d and e were partnered in D and they form a crossing/uncrossing pair in $f(D,d)$.

If the square e contained a plus or black stone in D, it will contain a plus in $f(D,d)$. So, the pipes originating at d will continue to travel down and right according to the rules:

1. If they hit a plus that was a plus in D while traveling right, switch from traveling right to traveling down.

2. If they hit a plus that was a plus in D while traveling down, switch from traveling down to traveling right.

3. If they hit a plus that was a black stone in D, our inductive assumption tells us this plus was a crossing uncrossing pair with its partner. So, if they hit a plus that was a black stone in D while traveling right, continue to its partner, then switch to traveling downward.

4. If they hit a plus that was a black stone in D while traveling down, continue to its partner, then switch to traveling to the right.

5. If they hit a plus that was a white stone in D while traveling right, switch from traveling right to traveling down. Point (vi) of Proposition 4.3.6 guarantees such a white stone in D had to be the partner of some square in d^{in}.

6. If they hit a plus that was a white stone in D while traveling right, switch from traveling right to traveling down.

This list of rules agrees with points 4.1–4.6 in Algorithm 4.3.4. So, the pipes originating at d next share a square at the same point that the paths P_d and Q_d from Algorithm 4.3.4 meet. This square contains a white stone and is d's partner. So, d and its partner from D form a crossing/uncrossing pair in $f(D,d)$. Then, flipping d and its partner both to be pluses leaves the permutation unchanged.

Continuing in this way, we may flip all of the black stones in the same column as b and their partners to pluses without altering the permutation. So, $u_{b^{in}}^{f(D,b)} = v = u_{b^{in}}^{D}$. In the case where b contains a black stone, the same agrument as above shows that b and its partner from D form a crossing/uncrossing pair in $f(D,b)$. □

Theorem 4.3.8. *A $\bullet/\circ/+$-diagram D is a Go-diagram if and only if all boxes containing black stones have partners and all boxes with partners are filled with black stones. Changing all black stones and their partners to pluses simultaneously yields a reduced diagram for the pair of permutations determined by D.*

Proof. Let D be a Go-diagram and let b be a box containing a black stone such that no box in $b^{in} \setminus b$ contains a black stone. Since there are no black stones in $b^{in} \setminus b$, property (P1) holds for b. Consider the diagram obtained by restricting D to the subdiagram b^{in} and replacing the black stone in b with a plus. This diagram contains no black stones, and it is not a Ⅎ-diagram. So, it contains some subdiagram violating the Ⅎ-condition, of the form given in Figure 4.3. Necessarily, b is the top left corner of this subdiagram. Then, the bottom right corner of this subdiagram is b's partner in D. Evidently, these two squares also form a crossing/uncrossing pair. So, property (P2) holds for b. Then, applying Lemma 4.3.7 inductively, properties (P1) and (P2) hold for all squares in D. Notably, since (P1) and (P2) hold for the top left square, the diagram obtained by replacing all black stones and their partners with pluses is a reduced diagram for the same pair of permutations.

Let b be any square in D. We want to show that b contains a black stone if and only if it has a partner in D. From the distinguished property, b contains a black stone if and only if

$$\ell\left(u^{D}_{b^{in}} s_b\right) < \ell\left(u^{D}_{b^{in}}\right). \tag{4.7}$$

Then, (P1) says that (4.7) holds if and only if

$$\ell\left(u^{f(D,b)}_{b^{in}} s_b\right) < \ell\left(u^{f(D,b)}_{b^{in}}\right).$$

This inequality holds if and only if b forms a crossing/uncrossing pair with some box p in $f(D, b)$. Property (P2) says that b forms a crossing/uncrossing pair with p if and only if p is b's partner in D. Thus, in a Go-diagram, the set of squares filled with black stones and the set of squares with partners are identical.

Now, suppose D is a diagram such that a square contains a black stone if and only if it has a partner. Let b be a box in D such that no boxes in $b^{in} \setminus b$ contain black stones. Then, from point (v) in Proposition 4.3.6, b's partner is defined by a diagram of the following form.

b	○	$\cdots$	○	+
○				○
$\vdots$				$\vdots$
○				○
+	○	$\cdots$	○	○

,

where the interior of the diagram could be filled with anything. So, b and its partner form a crossing/uncrossing pair. Then, applying Lemma 4.3.7 inductively, properties (P1) and (P2) hold for all squares in D.

Now, let b be any box in D. To verify that D is a Go-diagram we must check that b contains a black stone if and only if (4.7) holds for b. Since (P1) holds for every box in D,

b satisfies (4.7) if and only if

$$\ell\left(u_{b^{in}}^{f(D,b)} s_b\right) < \ell\left(u_{b^{in}}^{f(D,b)}\right).$$

This condition holds if and only if b forms a crossing/uncrossing pair with some box p in $f(D,b)$. Then, Property (P2) says that b forms a crossing/uncrossing pair with p in $f(D,b)$ if and only if p is b's partner in D. From our assumption, b has a partner in D if and only b contains a black stone. □

Theorem 4.3.8 may be used to give partial lists of forbidden subdiagrams for the class of Go-diagrams. Though Theorem 4.3.1 demonstrates that there is no finite characterization of Go-diagrams in terms of forbidden subdiagrams, such tests can still be valuable as a quick reality check for whether a diagram is or is not a Go-diagram.

Corollary 4.3.9. *Any Go-diagram avoids subdiagrams given in Figure 4.6. In this figure, the boxes with slashes in them indicate that the box could be filled with the items on either side of the slash.*

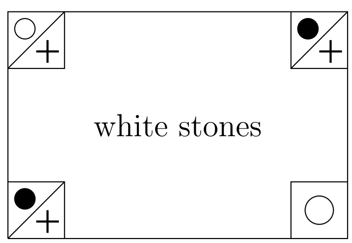
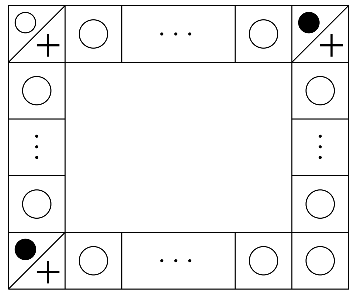


Figure 4.6: Some forbidden subdiagrams for the class of Go-diagrams.

Proof. From Theorem 4.3.8 and Point 4 in Algorithm 4.3.4, it is obvious a Go diagram must avoid subdiagrams of the form



, (4.8)

where it is not specified what the interior of the diagram is filled with. We show the existence of such a subdiagram D implies the existence of a diagram of the form given in Figure 4.6. Suppose there is a plus or a black stone in the interior of this diagram. Choose a plus or black stone in a box b in the interior of D such that there are no interior pluses or black stones to the right of b. Now, let c be the highest box in the same column as b containing a black stone or plus. Then, the square whose bottom left corner is c and whose top right corner is the top right corner of D is a subdiagram of the form given in Figure 4.6. $\square$

4.4 Fibers of the "Delete a Column" Map

This section describes a projection π from $Gr(k, n+1)$ to $Gr(k, n) \cup Gr(k-1, n)$, and describes the fibers of this map over a Deodhar component in $Gr(k, n)$. The work in this section is joint work with Agarwala, and originally appears in [6].

Choose an ordered basis, $\mathbf{b}_1, \ldots, \mathbf{b}_{n+1}$ of $\mathbb{R}^{n+1}$, so that points in $Gr(k, n+1)$ may be represented by $(n+1) \times k$ matrices. Let $\pi' : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the map which projects out the $(n+1)^{st}$ coordinate. This map yields a map of Grassmannians,

$$\begin{aligned} \pi : Gr(k, n+1) &\to Gr(k, n) \cup Gr(k-1, n) \\ \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) &\mapsto \mathrm{span}(\pi'(\mathbf{v}_1), \ldots, \pi'(\mathbf{v}_k)). \end{aligned} \tag{4.9}$$

If V is in the Schubert cell $\mathcal{S}_\lambda \subset Gr(k, n+1)$, where $n+1 \notin I_\lambda$, then $\pi(V) \in Gr(k, n)$. Otherwise, if $n+1 \in I_\lambda$, then $\pi(V) \in Gr(k-1, n)$. This section will describe the fibers of π over various subsets of $Gr(k, n)$.

The fiber over all of $Gr(k,n)$ is

$$\{V \in Gr(k,n+1) : \mathbf{b}_{n+1} \notin V\} = \bigcup_{I_\lambda \in \binom{[n+1]}{k} \,:\, n+1 \notin I_\lambda} \mathcal{S}_\lambda.$$

At the other extreme, the fiber over a single point $V = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) \in Gr(k,n)$ is

$$\{\mathrm{span}(\mathbf{v}_1 \oplus g_1, \ldots, \mathbf{v}_k \oplus g_k) : (g_1, \ldots, g_k) \in \mathbb{R}^k\},$$

which can be thought of as the set of matrices obtained by appending a column to a matrix representing V. We sometimes call this extra column vector being appended a *gauge vector*.

Proposition 4.4.1. *The fiber $\pi^{-1}(Gr(k,n))$ is a k-dimensional vector bundle over $Gr(k,n)$.*

The next several results describe the fibers of π over a Deodhar component in $Gr(k,n)$.

Theorem 4.4.2. *Let $\mathcal{D}$ be the Deodhar component in $Gr(k,n)$ labelled by the Go-diagram D. Then, $\pi^{-1}(\mathcal{D})$ is the union of all Deodhar components $\mathcal{D}'$ labelled by Go-diagrams D' obtained by adding a column of k boxes on the left of D.*

Proof. Let $V \in \pi^{-1}(\mathcal{D})$. Then, V lies in some Deodhar component in $Gr(k,n+1)$. Theorem 2.2.18 says there is some unique weighted Go-network $N(D')_V$ representing V. Corollary 2.2.24 implies that the projection $\pi(V)$ is obtained by deleting all the vertices in the leftmost column of $N(D')_V$ and all edges incident to these vertices. Since $\pi(V) \in \mathcal{D}$, this network obtained by deleting the left column of vertices must be $N(D)$.

Conversely, let $V \in \mathcal{D}'$, where $\mathcal{D}'$ Deodhar component associated to some Go-diagram D' obtained by adding a column of boxes to the left of D. Then, Theorem 2.2.18 says that V has a unique realization as a weighted Go-network $N(D')_V$. Corollary 2.2.24 says that the entries in the first n rows of a matrix representing V depend only on the part of the part of the Go-network $N(D')_V$ agreeing with D. So, $\pi(V) \in D$. □

Proposition 4.4.3. *Let $\mathcal{D}$ be a Deodhar component in $Gr(k,n)$. The fiber $\pi^{-1}(\mathcal{D})$ contains a unique top dimensional Deodhar component.*

Proof. Let $\mathcal{D}$ be the Deodhar component in $Gr(k,n)$ labelled by the Go-diagram D. Consider some Deodhar component $\mathcal{D}' \subset \pi^{-1}(\mathcal{D})$ labelled by the Go-diagram D'. Theorem 4.4.2 says that D' is obtained by adding a column of k boxes to the left of D. So,

$$\dim(\mathcal{D}') \leq \dim(\mathcal{D}) + k.$$

Theorem 2.2.12 implies that equality occurs if and only if the new column contains no white stones. There is a unique filling of the new column which does not use any white stones, which is produced by the following procedure:

- Fill the bottom box in the new column with a plus.
- Let b be a box in the new column and suppose all boxes in the new column below b have been assigned a filling.
- Check whether filling the box b with a plus causes a violation of the distinguished property.
- If it does, fill b with a black stone. Otherwise, fill b with a plus. □

In Remark 7.11 in [55], Talaska and Williams give the following algorithm for constructing a weighted Go-network from a point in a Grassmannian. Given $V \in Gr(k, n)$, one may use this algorithm to determine which Deodhar component any point V' in the fiber $\pi^{-1}(V)$ lies in.

- Let b be a box in the new column, and suppose that all boxes below b have been filled with a white stone, black stone, or plus.
- If $\ell(u_{b^{in}} s_b) < \ell(u_{b^{in}})$, fill b with a black stone.
- Otherwise, if $\ell(u_{b^{in}} s_b) > \ell(u_{b^{in}})$, compute $\Delta_{I_b}(V)$.
- If $\Delta_{I_b}(V) = 0$, fill V with a white stone. Otherwise, fill b with a plus.

We next describe the boundary structure of the Deodhar components in the fiber $\pi^{-1}(\mathcal{D})$ over the Deodhar component $\mathcal{D} \subset Gr(k, n)$. This theorem is a special case of Conjecture 4.2.10. We will need the following technical lemma.

Lemma 4.4.4. *Let D be a Go-diagram. Suppose the box $b = (i, j)$ contains a white stone or a plus and consider a box $c = (k, j)$ with $k > i$. That is, c is below b in the same column.*

(i) If the box c contains a plus and D' is the diagram obtained by replacing the plus in c with a white stone, then the box b does not violate the distinguished property in D'.

(ii) If the box c contains a black stone and D' is the diagram obtained by replacing the black stone in c with a plus, then the box b does not violate the distinguished property in D'.

Proof. We prove point (i) in the case where b contains a white stone. The proofs for all other cases are similar, and are left to the reader. Suppose that in the pipe dream associated to D, the pipes x and y cross in the square b with $x < y$. So, the pipe x is running horizontally at the square b and y is running vertically. If the pipe y does not run through the square c, replacing the plus in c with a white stone does not change the pipes crossing at b and hence b does not violate the distinguished property in D'. Suppose then that the pipes y and z appear in the square c in the pipe dream. Since c is below b in its column, the pipe y must enter c from the right and exit from the top. So, $y < z$. Then, in the pipe dream associated to D', x and z cross in the square b. Since $x < z$, the square b does not violate the distinguished property. □

Theorem 4.4.5. *Let D' and D'' be Go-diagrams indexing Deodhar components $\mathcal{D}'$ and $\mathcal{D}''$ in the fiber $\pi^{-1}(\mathcal{D})$. Then, $\mathcal{D}''$ is a codimension one boundary of $\mathcal{D}'$ if and only if D'' is obtained by changing the filling of a single box b in the leftmost column of D'' from a plus to a white stone, then reading up from b in the leftmost column, and changing black stones to pluses as is necessary to avoid a violation of the distinguished property.*

Proof. Suppose that D'' is obtained from D' in the manner described. Lemma 4.4.4 guarantees that D'' is in fact a Go-diagram. The equations defining the Deodhar components $\mathcal{D}'$ and $\mathcal{D}''$ are identical aside from the equations demanding Δ_{I_c} vanish or not, where c is a box in the leftmost column of the diagram. So, to verify $\mathcal{D}''$ is on the boundary of $\mathcal{D}'$, it suffices to verify that Δ_{I_c} vanishes on $\mathcal{D}''$, whenever c is a box in the leftmost column of the diagram containing a white stone in D'. Theorem 2.2.25 implies that the set I_c is identical in D'' and D' for every box c in the leftmost column of the diagram. Since the set of boxes containing white stones in the leftmost column of D' is a subset of the set of boxes containing white stones in D'', $\mathcal{D}''$ is on the boundary of $\mathcal{D}'$. Counting the number of pluses and black stones in D' and D'', $\dim(\mathcal{D}'') = \dim(\mathcal{D}') - 1$.

Now, suppose that $\mathcal{D}'$ and $\mathcal{D}''$ are Deodhar components in the fiber $\pi^{-1}(\mathcal{D})$ labelled by the Go-diagrams D' and D'', and that $\mathcal{D}''$ is a codimension one boundary of $\mathcal{D}'$. Since the sets I_c labelling boxes in the leftmost columns of D' and D'' are identical, the set of boxes containing white stones in the leftmost column of D' must be a subset of the set of boxes containing white stones in D''. Since $\mathcal{D}''$ has codimension one, exactly one square in the leftmost column of D' must change to a white stone in D''. In a Go-diagram a square is filled with a black stone if and only if it violates the distinguished property, so black stones in D' cannot change to white stones in D''. So, one plus in a box b must change to a white stone. Lemma 4.4.4 says that changing this plus to a white stone will not cause any of the white stones or pluses in the diagram to violate the distinguished property. However, it is possible that a black stone will no longer reduce the length of the associated subword after changing b to a white stone. Since $\mathcal{D}''$ is a codimension one boundary, any black stone which no longer violates the distinguished property must be changed to a plus. □

Example 4.4.6. Let $\mathcal{D}$ be the Deodhar component labelled by the following Go-diagram.

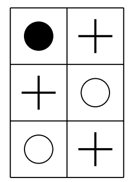


Figure 4.7 gives the boundary poset of Go-diagrams labelling Deodhar components in the fiber $\pi^{-1}(\mathcal{D})$.

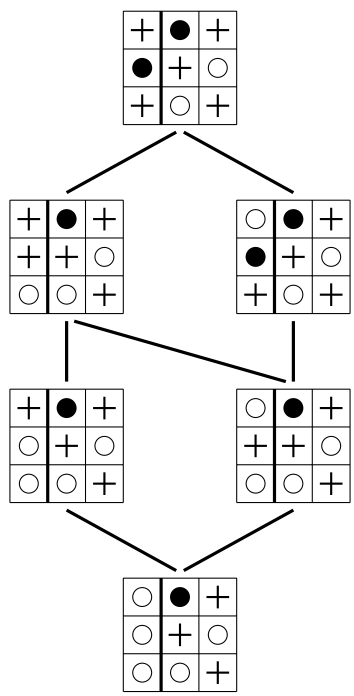


Figure 4.7: Boundary poset of Deodhar components in $\pi^{-1}(\mathcal{D})$.

We may consider the restriction of the map π to a map of totally non-negative Grassmannians,

$$\pi_{\geq 0} : Gr_{\geq 0}(k, n+1) \to Gr_{\geq 0}(k, n) \cup Gr_{\geq 0}(k-1, n).$$

Deleting a column does not alter the positivity of any minors which do not involve the deleted column, so this map is well defined. However, in this restriction, there is no longer a bundle structure, since the fibers of $\pi_{\geq 0}$ over different points are no longer equidimensional.

Example 4.4.7. Consider the point

$$V = \text{span}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in Gr_{\geq 0}(2,3).$$

The fiber of π over V is

$$\pi^{-1}(V) = \left\{ \text{span}\left(\begin{array}{ccc|c} 1 & 0 & -1 & g_1 \\ 0 & 1 & 0 & g_2 \end{array} \right) : (g_1, g_2) \in \mathbb{R}^2 \right\}.$$

A point in this fiber is positive if and only if $g_1 \leq 0$ and $g_2 = 0$. So, $\pi_{\geq 0}^{-1}(V)$ is only 1-dimensional. On the other hand, the point

$$W = \text{span}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Gr_{\geq 0}(2,3)$$

has a two dimensional fiber,

$$\pi_{\geq 0}^{-1}(W) = \left\{ \text{span}\left(\begin{array}{ccc|c} 1 & 1 & 0 & g_1 \\ 0 & 0 & 1 & g_2 \end{array} \right) : g_1 \in \mathbb{R}_{\leq 0},\ g_2 \in \mathbb{R}_{\geq 0} \right\}.$$

Let $\mathcal{P} \subset Gr(k,n)$ be a positroid cell and let $\mathcal{P}_{\geq 0} = \mathcal{P} \cap Gr_{\geq 0}(k,n)$ the positive part of this cell. Each positroid cell is a semialgebraic subset of $Gr_{\geq 0}(k,n)$, defined by setting certain Plücker coordinates to zero, demanding certain Plücker coordinates do not vanish, and demanding all other Plücker coordinates are uniformly non-negative. The fiber $\pi^{-1}(\mathcal{D})$ is the semialgebraic subset of $Gr(k, n+1)$ defined by the exact same equations that define $\mathcal{D}$, only now these equations are viewed as equations in the Plücker coordinates on $Gr(k, n+1)$. No constraints are imposed on the Plücker coordinates Δ_I when $n+1 \in I$. For $\pi_{\geq 0}^{-1}(\mathcal{P}_{\geq 0})$, we must additionally impose $\Delta_I \geq 0$ when $n+1 \in I$. The structure of $\pi_{\geq 0}^{-1}(\mathcal{P}_{\geq 0})$ is described in the following theorem.

Theorem 4.4.8. *Let $\mathcal{P}_{\geq 0} \subset Gr_{\geq 0}(k,n)$ be the positive part of the positroid cell indexed by the* Ⅎ*-diagram D. Then,*

(i) $\pi_{\geq 0}^{-1}(\mathcal{P}_{\geq 0})$ is a union of positroid cells.

(ii) $\pi_{\geq 0}^{-1}(\mathcal{P}_{\geq 0})$ *contains a unique top dimensional positroid cell* $\mathcal{P}'_{\geq 0}$.

(iii) $\dim(\mathcal{P}'_{\geq 0}) = \dim(\mathcal{P}_{\geq 0}) + k$ *if and only if* D *contains no pluses with white stones below them in their column.*

(iv) The boundary poset of all the positroid cells in $\pi_{\geq 0}^{-1}(\mathcal{P}_{\geq 0})$ *is a Boolean lattice.*

Proof. Point (i) follows from Theorems 4.4.5 and 2.2.20, which say that the intersection of a Deodhar component with $Gr_{\geq 0}(k,n)$ is either empty or the positive part of a positroid cell.

Let $\mathcal{P}_{\geq 0} \subset Gr_{\geq 0}(k,n)$ be the positive part of the positroid cell given by the Ⅎ-diagram D. Let $\mathcal{D}$ be the Deodhar component labelled by D, so $\mathcal{P}_{\geq 0} = \mathcal{D} \cap Gr_{\geq 0}(k,n)$. Then,

$$\pi_{\geq 0}^{-1}(\mathcal{P}_{\geq 0}) = \pi^{-1}(\mathcal{D}) \cap Gr_{\geq 0}(k, n+1).$$

Let $\mathcal{D}' \subset \pi^{-1}(\mathcal{D})$ be a Deodhar component and let D' be the Go-diagram labelling $\mathcal{D}'$. So, D' is obtained by adding a column of boxes to D. Theorem 2.2.20 says that $\mathcal{D}'$ intersects $Gr_{\geq 0}(k, n+1)$ if and only if D' is a Ⅎ-diagram. Let b be a box in the new column added to create D'. If there is any box containing a white stone to the right of b in its row with a plus above it, then filling b with a plus will cause a violation of the Ⅎ-property. So, such boxes must be filled with white stones, and all other boxes in the new column may be safely filled with white stones or pluses. The unique top dimensional cell is then obtained by filling all boxes which may be filled with pluses in the new column with pluses, proving points (ii) and (iii).

Now, let D' and D'' be two Ⅎ-diagrams labelling positroids $\mathcal{P}'_{\geq 0}$ and $\mathcal{P}''_{\geq 0}$ in the fiber $\pi_{\geq 0}^{-1}(\mathcal{P}_{\geq 0})$. We claim that $\mathcal{P}''_{\geq 0}$ is a codimension 1 boundary of $\mathcal{P}'_{\geq 0}$ if and only if $\mathcal{P}''_{\geq 0}$ is obtained by changing one of the pluses in the leftmost column of $\mathcal{P}'_{\geq 0}$ into a white stone. The cell $\mathcal{P}''_{\geq 0}$ is on the boundary of $\mathcal{P}'_{\geq 0}$ if and only if all Plücker coordinates vanishing on $\mathcal{P}'_{\geq 0}$ also vanish on $\mathcal{P}''_{\geq 0}$. From Theorem 2.2.25, we see that the set I_b labelling a box b in the new column does not depend on the filling of the boxes in the new column. So, $\mathcal{P}''_{\geq 0}$ is on the boundary of $\mathcal{P}'_{\geq 0}$ if and only if all boxes containing white stones in the new column in D' also contain white stones in D''. The dimension of a positroid cell is the number of pluses in its Ⅎ-diagram, proving the claim. Then, the boundary poset of the cells in the fiber $\pi_{\geq 0}^{-1}(\mathcal{P}_{\geq 0})$ is exactly the Boolean lattice of subsets of boxes containing pluses in the new column of the Ⅎ-diagram labelling the top dimensional cell of $\pi_{\geq 0}^{-1}(\mathcal{P}_{\geq 0})$. □

Chapter 5

Application: Parameterization Space of Wilson Loops

Section 5.1 introduces Wilson loop diagrams and their parameterization spaces. Section 5.2 discusses corollaries of the results of Chapter 5 to studying these geometric spaces. Section 5.3 discuses fibers of the map π from Section 4.4 over Wilson loop cells.

The results in Section 5.3, is joint work with Agarwala, originally appearing in [6]. The exposition in Sections 5.1 and 5.3 borrows heavily from this paper. The results in Section 5.2 originally appear in [46].

5.1 Wilson Loop Diagrams

Definition 5.1.1. A *Wilson loop diagram*, W, is defined by a set of unordered pairs $\mathcal{I} \subset \binom{[n]}{2}$. We denote it $W = (\mathcal{I}, n)$.

Each $p \in \mathcal{I}$ is called a *propagator*. Graphically, we represent a Wilson loop diagram by a convex polygon, whose vertices are labeled by the elements of $[n]$, respecting the cyclic ordering. Rather than label all the vertices of the boundary polygon, we draw a dot on the vertex labelled 1, and adopt the convention that the rest of the vertices are labelled counterclockwise in increasing order. For each $p = \{i_p, j_p\} \in \mathcal{I}$, draw an internal wavy line between the edges of the polygon defined by the vertices $\{i_p, i_p + 1\}$ and $\{j_p, j_p + 1\}$. For example, the Wilson loop diagram

$$W = (\{24, 46\}, 6)$$

would be represented by the arrangement of squiggly lines in a polygon in Figure 5.1.

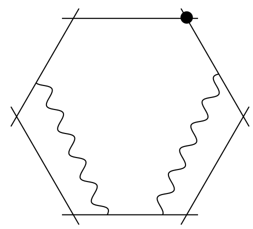


Figure 5.1: The Wilson loop diagram $(\{24, 46\}, 6)$.

Definition 5.1.2. Let $W = (\mathcal{I}, n)$ be a Wilson loop diagram. The *support* of propagator is the collection of vertices comprising the endpoints of the edges the propagator is drawn between. For $p = (i_p, j_p) \in \mathcal{I}$, the support of p is written

$$V(p) = \{i_p, i_p + 1, j_p, j_p + 1\}.$$

For $P \subset \mathcal{I}$, the support is $V(P) = \bigcup_{p \in P} V(p)$.

Definition 5.1.3. The *propagator set* of a set of a vertex v is

$$\text{Prop}(v) = \{p \in \mathcal{I} : v \in V(p)\}.$$

For $V \subset [n]$, $\text{Prop}(V) = \bigcup_{v \in V} \text{Prop}(v)$.

Given a Wilson loop diagram $W = (\mathcal{I}, n)$, consider the set system

$$\mathcal{W} = \bigcup_{p \in \mathcal{P}} \{V(p)\}.$$

We quickly recall some notation from Chapter 3. $M_{\mathcal{W}}(\mathbf{x})$ is the matrix of invertible indeterminates, where

$$M_{\mathcal{W}}(\mathbf{x})_{p,v} = \begin{cases} x_{p,v} & \text{if } v \in V(p), \\ 0 & \text{otherwise.} \end{cases}$$

$L(\mathcal{W})$ is the basis shape locus associated to the set system $\mathcal{W}$, which in this case we call the *Wilson loop cell* associated to W. So, any point in $L(\mathcal{W})$ is the row space of some matrix obtained by evaluating the entries of $M_{\mathcal{W}}(\mathbf{x})$ at nonzero real entries. We define

$$L_{\geq 0}(\mathcal{W}) = L(\mathcal{W}) \cap Gr_{\geq 0}(k, n).$$

Note that the Wilson loop cell $L(\mathcal{W})$ is exactly the set of planes admitting $\mathcal{I}$-domino bases where $|I| = 2$ for all $I \in \mathcal{I}$, described in Section 3.6. In Section 3.6, the set system $\mathcal{W}$ was denoted $\mathcal{I}'$.

5.2 Applications of Results on Basis Shape Loci

This section uses the notation of Chapter 3. So, if $\mathcal{S}$ is a set system, $\mathcal{B}(\mathcal{S})$ is the transversal matroid it defines and $L(\mathcal{S})$ is the associated basis shape locus.

Definition 5.2.1. A Wilson loop diagram $W = (\mathcal{I}, n)$ is *admissible* if the following hold:

(i) for all $P \subset \mathcal{I}$, $|P| + 3 \leq |V(P)|$, and

(ii) if $p = (i_p, j_p)$, $q = (i_q, j_q) \in \mathcal{I}$ are two propagators, then $i_p < i_q < j_q < j_p$ in the cyclic ordering of $[n]$.

The results from Chapter 3 imply several corollaries for Wilson loop cells. Let W be an admissible Wilson loop diagram and $\mathcal{W}$ be the associated set system. Theorem 3.38 in [2] says that $\mathcal{B}(\mathcal{W})$ is a positroid. One would like to be able to apply tools from the positroid literature to study Wilson loop diagrams. However, the object of interest is really the cell $L(\mathcal{W})$, not the matroid $\mathcal{B}(\mathcal{W})$. In the literature, it was understood and used, but not clear that working with the positroid cell associated to $\mathcal{B}(\mathcal{W})$ was, up to a set of measure zero, equivalent to working with the Wilson loop cell $L(\mathcal{W})$. This equivalence is a corollary of Theorems 3.3.1 and 3.4.3.

Theorem 5.2.2. *Let W be an admissible Wilson loop diagram, and $\mathcal{W}$ be the associated set system. Then,*

(i) $\dim(L(\mathcal{W})) = 3k$.

(ii) The matroid $\mathcal{B}(\mathcal{W})$ is a positroid.

(iii) $\overline{L(\mathcal{W})} = \overline{V_{\mathcal{B}(\mathcal{W})}}$.

Proof. Point (i) follows from point (i) of Theorem 3.1.2. Point (ii) follows from the fact $\mathcal{I}'$ is noncrossing and Theorem 3.4.3. The fact that the matroid represented by a generic point in $L(\mathcal{W})$ is a positroid originally appears as Theorem 3.38 in [2]. Point (iii) follows from Theorem 3.3.1. □

Wilson loop diagrams come equipped with a notion of exact subdiagrams, analogous to Definition 3.4.11.

Definition 5.2.3. Let $(\mathcal{I}, n)$ be a Wilson loop diagram satisfying point (i) of Definition 5.2.1, and let $\mathcal{W}$ be the associated set system. A subset $\mathcal{J} \subset \mathcal{I}$ is an *exact subdiagram* if its associated set system is an exact subsystem of $\mathcal{W}$ in the sense of Definition 3.4.10. That is,

$$\left| \bigcup_{j \in \mathcal{J}} V(j) \right| = |\mathcal{J}| + 3.$$

Let $(\mathcal{I}, n)$ and $(\mathcal{I}', n)$ be Wilson loop diagrams satisfying point (i) of Definition 5.2.1 and let $\mathcal{W}$ and $\mathcal{W}'$ be their associated set systems. Say that $\mathcal{I} \sim \mathcal{I}'$ if $\mathcal{B}(\mathcal{W}) = \mathcal{B}(\mathcal{W}')$. Since all sets in $\mathcal{W}$ have the same size, any set involved in an exact subsystem of $\mathcal{W}$ may be pivoted in the sense of Definition 3.4.13. Further, any exact subsystem supported on $M \subseteq [n]$ must be defined by $|M| - 3$ propagators from $\mathcal{I}$. Using these observations, one is able to show that any exact two exact subdiagrams supported on the set M may be pivoted to one another. Moreover, it is always possible to arrange $|M| - 3$ propagators supported on M vertices to obey points (i) and (ii) of Definition 5.2.1 (in fact, the number of ways to do so is a Catalan number). We record these observations.

Theorem 5.2.4 (Theorem 1.18 in [2])**.** *Let $(\mathcal{I}, n)$ and $(\mathcal{I}', n)$ be Wilson loop diagrams satisfying point (i) of Definition 5.2.1. If $\mathcal{I}$ and $\mathcal{I}'$ differ by only an exact subdiagram supported on some $M \subseteq [n]$, then $\mathcal{I} \sim \mathcal{I}'$. If $\mathcal{I}$ is exact, then $\mathcal{I} \sim \mathcal{I}''$ for some $\mathcal{I}$ satisfying both points (i) and (ii) of Definition 5.2.1.*

Theorem 5.2.5. *Let $(\mathcal{I}, n)$ be a Wilson loop diagram satisfying point (i) of Definition 5.2.1 and $\mathcal{W}$ be the associated set system. Then, $\mathcal{B}(\mathcal{W})$ is a positroid if and only if $\mathcal{I} \sim \mathcal{I}'$ for some Wilson loop diagram $(\mathcal{I}', n)$ satisfying both points (i) and (ii) of Definition 5.2.1.*

Proof. Let $(\mathcal{I}, n)$ be a Wilson loop diagram satisfying point (i) but not necessarily point (ii) of Definition 5.2.1 and let $\mathcal{W}$ its associated set systems. If $\mathcal{I} \sim \mathcal{I}'$ for some $\mathcal{I}'$ satisfying both points (i) and (ii) of Definition 5.2.1, then Theorem 3.38 in [2] or Theorem 3.4.3 above imply that $\mathcal{B}(\mathcal{W})$ is a positroid. If $\mathcal{B}(\mathcal{W})$ is a positroid, but there is some pair of crossing propagators in $\mathcal{I}$, Lemma 3.4.20 says we can find some exact subdiagram involving these crossing propagators. Then, Theorem 5.2.4 says this exact subdiagram may be replaced with any noncrossing exact subdiagram supported on the same set of vertices. Repeatedly applying this argument, $\mathcal{I} \sim \mathcal{I}'$ for some $\mathcal{I}'$ which is noncrossing. □

This theorem has a compelling rephrasing, using the language of Section 3.6.

Corollary 5.2.6. *The set of points in $Gr_{\geq 0}(k,n)$ admitting $\mathcal{J}$-domino bases where $|J| = 2$ for all $J \in \mathcal{J}$ is exactly*

$$\bigcup_{\mathcal{W}} L_{\geq 0}(\mathcal{W}),$$

where the union is across all set systems associated to admissible Wilson loop diagrams.

5.3 Fibers over Wilson Loop Cells

Let

$$\pi : Gr(k, n+1) \to Gr(k,n) \cup Gr(k-1,n)$$

be the projection map from Section 4.4. Let $\pi^{-1}(M_{\mathcal{W}}(\mathbf{x}))$ be the matrix obtained by appending the column vector $(x_{1(n+1)}, x_{2(n+1)}, \ldots, x_{k(n+1)})^T$ to $M_{\mathcal{W}}(\mathbf{x})$, where the x_{ij} are invertible variables, which are algebraically independent from the entries of $M_{\mathcal{W}}(\mathbf{x})$ and from each other.

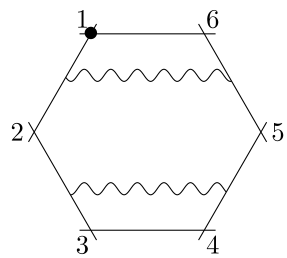
Lemma 5.3.1. *If $\mathcal{W}$ is the set system associated to an admissible Wilson loop diagram, any point in $\pi^{-1}(L_{\geq 0}(\mathcal{W}))$ is the row span of a matrix obtained by evaluating the entries of $\pi^{-1}(M_{\mathcal{W}}(\mathbf{x}))$ at nonzero real numbers.*

The following is an immediate consequence of point (i) of Theorem 5.2.2 combined with Proposition 4.4.1

Corollary 5.3.2. *Let W be an admissible Wilson loop diagram, and $\mathcal{W}$ be the associated set system. Then $\pi^{-1}(L_{\geq 0}(\mathcal{W}))$ is a $4k$-dimensional subspace of $Gr(k,n)$.*

While $L(\mathcal{W})$ always intersects the positive Grassmannian in its full dimension, this is not necessarily true $\pi^{-1}(L(\mathcal{W}))$, illustrated in the following example. This phenomenon is similar to the phenomenon observed in Example 4.4.7.

Example 5.3.3. Consider the Wilson loop diagram

$W =$ 

The Ⅎ-diagram indexing the positroid $\mathcal{B}(\mathcal{W})$ is

+	○	+	+
○	+	+	+

.

For details on how to compute this Ⅎ-diagram, see [4]. Using Proposition 4.4.3, $\pi^{-1}(\overline{L(\mathcal{W})})$ contains a unique top dimensional Deodhar component which is indexed by the Go-diagram

●	+	○	+	+
+	○	+	+	+

.

Since this diagram has a black stone, the Deodhar component it indexes does not intersect $Gr_{\geq 0}(k, n+1)$.

Alternatively, one may consider a matrix

$$\pi^{-1}(M_{\mathcal{W}}(\mathbf{y})) = \left(\begin{array}{cccccc|c} y_{11} & y_{12} & 0 & 0 & y_{15} & y_{16} & y_{17} \\ 0 & y_{22} & y_{23} & y_{24} & y_{25} & 0 & y_{27} \end{array} \right), \ y_{pq} \in \mathbb{R}^*$$

representing a point in $\pi^{-1}(L_{\geq 0}(\mathcal{W}))$. Since $\Delta_{13}, \Delta_{36} \geq 0$, we must have

$$\mathrm{sgn}(y_{11}) = (-1)\mathrm{sgn}(y_{16}).$$

Then, either Δ_{17} or Δ_{67} must be negative. So, $\dim(\pi^{-1}(L_{\geq 0}(\mathcal{W}))) < 8$.

As a third way of seeing that $\pi^{-1}(L_{\geq 0}(\mathcal{W}))$ is not a positroid, one may note that the set system $\{12567, 23457\}$ is a minimal presentation for the transversal matroid indexing this basis shape locus. Moreover, this presentation is Gale-minimal. We saw in the proof of Proposition 3.4.9 that if all the sets in a minimal presentation of a transversal positroid have the same size, then the Gale-minimal presentation of this positroid is noncrossing. However, $1, 3, 6, 7$ witness a crossing of the two sets in this presentation. So, the matroid indexing $\pi^{-1}(L_{\geq 0}(\mathcal{W}))$ is not a positroid, and thus $\pi^{-1}(L_{\geq 0}(\mathcal{W}))$ does not intersect the positive Grassmannian in its full dimension.

It would be desirable to have a way of telling when

$$\dim(\pi^{-1}(L_{\geq 0}(\mathcal{W})) \cap Gr_{\geq 0}(k, n)) = 4k,$$

or equivalently to have a way of telling when the top dimensional Deodhar component of $\pi^{-1}(\overline{L(\mathcal{W})})$ is a positroid. Using Proposition 4.4.3, this is equivalent to telling when the ꓘ-diagram indexing $\mathcal{B}(\mathcal{W})$ has no plus with a white stone below it in its column. The following conjecture has been proved for all Wilson loop diagrams with up to four propagators, and checked extensively on larger examples.

Conjecture 5.3.4. *Let $W = (\mathcal{I}, n)$ be a Wilson loop diagram with no propagator (i, n) and let D be the ꓘ-diagram labelling $\mathcal{B}(\mathcal{W})$. Then,*

$$\dim(\pi^{-1}(L_{\geq 0}(\mathcal{W})) \cap Gr_{\geq 0}(k, n)) = 4k$$

if and only if there is no pair of propagators $(i_1, j_1), (i_2, j_2)$ with $i_1 < i_2$ and $j_1 > j_2$ in the usual order on $[n]$. Moreover, if there is such a pair of propagators, then there is a plus in the row of D with vertical step labelled i_1 which has a white stone below it in its column.

Chapter 6

Further Speculation

This chapter takes the opportunity to highlight several open problems stemming from the material in this thesis. Some of these problems were already described in greater detail in Sections 2.4 and 3.6.

6.1 Basis Shape Loci

From a geometric perspective, the most interesting question left open is to describe how a basis shape variety $\overline{L(\mathcal{S})}$ compares with the matroid variety $\overline{V_{\mathcal{B}(\mathcal{S})}}$. Theorem 3.3.1 shows that $\overline{L(\mathcal{S})} = \overline{V_{\mathcal{B}(\mathcal{S})}}$ in the case where $\mathcal{B}(\mathcal{S})$ is a positroid. Corollary 3.4.8 showed that all Schubert and Richardson varieties are closures of basis shape loci. Perhaps the strongest general result we have comparing basis shape varieties to matroid varieties is Theorem 3.2.6 stating that $\operatorname{codim}(L(\mathcal{S})) = \operatorname{ec}(\mathcal{B}(\mathcal{S}))$. However, outside of the case of positroid varieties, it is not known whether $\operatorname{codim}(V_{\mathcal{B}}) = \operatorname{ec}(\mathcal{B}(\mathcal{S}))$. It seems likely that $\operatorname{ec}(\mathcal{B}(\mathcal{S}))$ is a lower bound on $\operatorname{codim}(V_{\mathcal{B}})$. However, this fact does not seem to be stated explicitly in the literature. If this were the case, then Theorem 3.2.6 would at least imply that $\overline{L(\mathcal{S})}$ is a top dimensional component of $\overline{V_{\mathcal{B}}}$.

From a combinatorial perspective, the largest question left open is to provide a characterization of when a transversal matroid is a positroid. Conjecture 3.4.5 offered a characterization in terms of a crossing condition on minimal presentations of a transversal matroid. Conjecture 3.4.22 gave a strengthened version of this conjecture. If true, these conjectures would strengthen Theorem 7.6 in [7] relating positroids and noncrossing partitions in the case of transversal positroids. Section 3.4 proved Conjecture 3.4.5 in several special cases, and developed machinery which is potentially helpful for proving this conjecture in the general case.

Conjecture 3.4.17 stated that any two minimal presentations of a transversal matroid are connected by a series of pivots of the type described in Definition 3.4.13. Remark 3.4.16 mentioned Ardila and Ruiz gave their own series of pivots and proved that any two (not necessarily minimal) presentations of a transversal matroid are related by pivots of this type. We conjecture a stronger fact, which would subsume both of these facts. Namely, we conjecture that there is a polytope associated to a transversal matroid where each face of the polytope is associated to a presentation of the matroid. The unique maximal presentation corresponds to the interior face and the minimal presentations corresponds to vertices. For any pivot replacing S by $S \setminus a \cup b$, note that $\mathcal{S} \setminus S \cup (S \cup b)$ is a (non-minimal) presentation of the same transversal matroid. We conjecture that presentations of this type are exactly the one-skeleton of the conjectured polytope. Further, pivots of the type described by Ardila and Ruiz describe when one face is a codimension one boundary of another.

In addition to these problems, Section 3.6 described a program due to Karp, Williams, and Zhang in [33] to use "domino bases" and sign variation techniques to study triangulations of amplituhedra. This program provided some motivation for the introduction of basis shape loci. Briefly, their strategy is to identify a family of positroid cells, find a basis of a particular shape for points in these cells, then apply sign variation techniques to prove the projections of these cells to the amplituhedron are disjoint. Our hope is to circumnavigate the first two steps of this process, by identifying basis shapes amenable to sign variation techniques, then taking positroid cells admitting bases of the chosen shapes. The technology of we developed is only currently able to handle the simplest basis shapes appearing in Karp, Williams, and Zhang's work. A brief description of some adaptations that need to be added on to our present machinery appears in Section 3.6. Corollary 5.2.6 provides evidence of the potential fruitfulness of the approach of defining a family of cells in terms of admittance of a particular basis shape, by proving that the previously studied family of Wilson loop cells may be defined in this way.

6.2 The Deodhar Decomposition

Section 2.4 proposed a combinatorial avenue of study, exploring the class of matroids represented by the a generic point in a Deodhar component. Conjecture 2.4.6 posits that this class is minor closed, and conjectures a specific set of excluded minors.

Section 4.2 offered several conjectures about Deodhar components and their boundary structure. Conjecture 4.2.4 stated that when D is a Go-diagram, the ideal defining $\mathcal{V}_D$ is exactly the ideal of Plücker coordinates which vanish uniformly on the Deodhar component $\mathcal{D}$. Conjecture 4.2.4 asserted that closures of Deodhar components in the Grassmannian are set theoretically defined by the vanishing of Plücker coordinates. Conjecture 4.2.10 offered a description of when one Deodhar component is on the boundary of another within a Schubert cell.

From any point in the Grassmannian, one may generate a solution to the KP-hierarchy of differential equations. In [39], Kodama and Williams prove that the shape a particular soliton system assumes asymptotic with time depends only on the Deodhar component the point used to generate the solution lies in. Algorithm 10.4 in [39] describes how to compute a graph of the shape a soliton system assumes from a Go-diagram. This procedure is similar to computing a plabic graph from a Ꟶ-diagram. Studying the interaction of this algorithm with the diagrammatic of boundaries of Deodhar components from Conjecture 4.2.10 (Theorem 4.2.16 in the case of positroids), one could obtain results describing certain phase changes in systems of solitons as one moves between different Deodhar components.

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