Exact C*-algebras and the Kirchberg-Phillips Nuclear Embedding Theorem

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

A C*-algebra A is **exact** if the functor $A \otimes_{\min} (\cdot)$ preserves short exact sequences. This is equivalent to the algebra A having a nuclear faithful *-representation. Exact C*-algebras are a class of C*-algebras which is much more broad than the class of nuclear C*-algebras; exactness passes to subalgebras. This property was studied in great detail by Kirchberg and Wassermann in the early 90's, and is still interesting today, for example in the study of exact groups. Exposition will be given to several results and characterizations related to exact C*-algebras.

We begin by examining certain C*-algebras which play a tremendous role in several characterizations of exactness in the separable case. The first being the CAR algebra, which is the UHF algebra corresponding to the infinite tensor product of M_2. The second is a family of C*-algebras originally introduced by Cuntz in [9], which are unital, simple C*-algebras generated by isometries which satisfy a certain relation. We then proceed to explore exact C*-algebras, initially through the exposition given by Brown and Ozawa in [6], and then through the work of Kirchberg and Wassermann. Kirchberg classified the separable exact C*-algebras as those that are subquotients as the CAR algebra in [18], and Wassermann gave his own proof, the one we choose to follow, in [33]. We finally examine unital, simple, purely infinite C*-algebras and their approximation properties in order to give a proof of the famous Kirchberg-Phillips nuclear embedding Theorem, as in [19].

This thesis follows work that was done in a previous USRA term (Winter 2018), supervised by Laurent Marcoux. During that term I studied nuclear C*-algebras, and the fact that they are a class of C*-algebras with a positive solution to Kadison's similarity problem, of which an exposition can be found in [25].

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Contents

1 Background

As mentioned in the Abstract, a C*-algebra A is exact if the function $A \otimes_{\min} (\cdot)$ preserved short exact sequences. This is equivalent to the algebra A having a nuclear faithful *-representation. Exact C*-algebras are class of C*-algebras which is much more broad than the class of nuclear C*-algebras, which will be defined below. For example, exactness passes to subalgebras, something which fails for nuclear C*-algebras - we will actually see that $C_r^*(\mathbb{F}_2) \subseteq \mathcal{O}_2$, where the latter is nuclear and the former is not. Exactness was studied in great detail by Kirchberg and Wassermann in the early 90's, and is still interesting today, for example in the study of exact groups.

Our main goal is to present a proof of the Kirchberg-Phillips theorem, which states that a separable C*-algebra A is exact if and only if A embeds into the Cuntz algebra $\mathcal{O}_2$.

In this chapter we *very briefly* outline some of the basic tools that will be required to study exactness of C*-algebras and the Cuntz algebra in the subsequent chapters. In order to keep this thesis reasonable length, we will not include the proofs of the results of Chapter 1, but will mostly refer the reader to the appropriate sources.

1.1 Completely Positive and Completely Bounded Maps

One of the central objects of this thesis is the class of "exact" C*-algebras, which can be defined in terms of having a nuclear, injective *-representation. These nuclear maps are defined as point-norm limits (point-wise limits) of contractive completely positive maps which factor through matrix algebras. A full exposition of the theory of these maps would take too much time and space, so we content ourselves with outlining results which will be useful later in the thesis. We refer to [22] for the proofs of the results below.

Definition 1.1.1. Let A be a C*-algebra. We call a self-adjoint, unital subspace $S \subseteq A$ an **operator system**. We say that a subspace $X \subseteq A$ is an **operator space**.

Note that operator systems have many positive elements. First, given $a \in S$, since S is a *-closed subspace, $\mathrm{Re}(a) = \frac{a+a^*}{2}, \mathrm{Im}(a) = \frac{a-a^*}{2i} \in S$, so that there are many self-adjoint elements. Now for any $a = a^* \in S \subseteq A$, an operator system, we have that $\frac{1}{2}(\|a\| \cdot 1 + a\|)$ and $\frac{1}{2}(\|a\| \cdot 1 - a)$ are both positive.

Definition 1.1.2. For operator systems S, T, we say a linear map $\phi : S \to T$ is positive, denoted $\phi \geq 0$, if $\phi(a) \geq 0$ whenever $a \geq 0$. If we have a linear map $\phi : S \to T$, there are maps $\phi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ given by $\phi^{(n)}([a_{ij}]) = [\phi(a_{ij})] \in M_n(T)$. We say that ϕ is n-positive if $\phi^{(n)} \geq 0$, and we say that ϕ is **completely positive** (c.p.) if $\phi^{(n)} \geq 0$ for all $n \in \mathbb{N}$. ϕ is **completely bounded** (c.b.) if $\|\phi\|_{cb} = \sup_n \|\phi^{(n)}\| < \infty$, and ϕ is **completely contractive** (c.c.) if $\|\phi\|_{cb} \leq 1$. Moreover, we say that ϕ is c.c.p. if it is contractive and completely positive (in particular it will be c.c. and c.p.). Note that c.b. and c.c. are analogously defined on operator spaces.

Completely positive and completely bounded maps have been well studied. We will state some results, of which the proofs can be found in [22].

Proposition 1.1.3. If A, B are unital C*-algebras, $S \subseteq A$ is an operator system, and $\phi : S \to B$ is a c.p. map, then $\|\phi\|_{cb} = \|\phi\| = \|\phi(1)\|$.

Another important fact related to this is that any unital, completely contractive map is completely positive. This is analogous to the fact that any contractive functional on a C*-algebra which is unital is in fact a state.

Theorem 1.1.4 (Stinespring dilation Theorem)**.** Let A be a unital C*-algebra, $\phi : A \to \mathcal{B}(\mathcal{H})$ be a c.p. map. Then there exists a Hilbert space $\mathcal{K}$, a unital *-representation $\pi : A \to \mathcal{B}(\mathcal{K})$, and an operator $V : \mathcal{H} \to \mathcal{K}$ with $\|V^*V\| = \|\phi(1)\|$ such that

$$\phi(\cdot) = V^*\pi(\cdot)V.$$

Theorem 1.1.5 (Arveson's extension Theorem)**.** Let A be a unital C*-algebra, $S \subseteq A$ an operator system. Then every c.c.p. map $\phi : S \to \mathcal{B}(\mathcal{H})$ extends to a c.c.p. map $\psi : A \to \mathcal{B}(\mathcal{H})$.

Theorem 1.1.6 (Wittstock's extension Theorem)**.** Let A be a unital C*-algebra, $X \subseteq A$ an operator space. Then every c.b. map $\phi : X \to \mathcal{B}(\mathcal{H})$ extends to a c.b. map $\psi : A \to \mathcal{B}(\mathcal{H})$ with $\|\psi\|_{cb} = \|\phi\|_{cb}$.

Theorem 1.1.7. Let A be a unital C*-algebra, $\phi : A \to \mathcal{B}(\mathcal{H})$ be a c.b. map. Then there exists a Hilbert space $\mathcal{K}$, a *-homomorphism $\pi : A \to \mathcal{B}(\mathcal{K})$ and bounded operators $V, W : \mathcal{H} \to \mathcal{K}$ with $\|\phi\|_{cb} = \|V\|\|W\|$ such that

$$\phi(\cdot) = V^*\pi(\cdot)W.$$

Moreover if $\|\phi\|_{cb} = 1$, V, W can be taken to be isometries.

There is also a good understanding of what happens if we consider maps from and to M_n.

Lemma 1.1.8. Let A be a C*-algebra, (e_{ij}) the matrix units for M_n. A map $\phi : M_n \to A$ is c.p. if and only if $(\phi(e_{ij}))$ is positive in $M_n(A)$. i.e., we have a bijective correspondence

$$\phi \text{ c.p. } \mapsto (\phi(e_{ij})) \in M_n(A)_+.$$

Lemma 1.1.9 ([14], Lemma 2.3)**.** Let S, T be operator systems such that $\dim(S) = n < \infty$. Then any map $\phi : S \to B$ is c.b. with

$$\|\phi\| \leq \|\phi\|_{cb} \leq n\|\phi^{(n)}\|.$$

Proposition 1.1.10. Let A, B be C*-algebras, $\phi : A \to B$ be a c.c.p. map.

1. (Schwarz Inequality) $\phi(a)^*\phi(a) \leq \phi(a^*a)$ for all $a \in A$.
2. If a is such that $\phi(a^*a) = \phi(a)^*\phi(a)$ and $\phi(aa^*) = \phi(a)\phi(a^*)$, then $\phi(ba) = \phi(b)\phi(a)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $b \in B$.
3. The subspace $A_\phi = \{a \in A \mid \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\}$ is a C*-subalgebra of A, and is called the **multiplicative domain** of ϕ. The multiplicative domain of ϕ is the largest subalgebra of A on which ϕ restricts to a *-homomorphism.

Lemma 1.1.11 ([6], B.4)**.** Let X be an operator space and $\phi : X \to M_n$ be bounded. Then

$$\|\phi\|_{cb} = \|\phi^{(n)}\|.$$

Definition 1.1.12. Let $A \subseteq B$ be C*-algebras. A map $E : B \to A$ is called a **conditional expectation** if E is a c.c.p map such that $E(a) = a$ and $E(aba') = aE(b)a'$ for all $a, a' \in A, b \in B$.

Theorem 1.1.13 (Tomiyama)**.** Let $A \subseteq B$ be C*-algebras, and $E : B \to A$ a map such that $E(a) = a$ for all $a \in A$. The following are equivalent:

1. E is a conditional expectation;
2. E is c.c.p.;
3. E is contractive.

1.2 Tensor Products of C*-Algebras

When studying algebras, tensor products are a very natural extension which allows one to "extend scalars." We will briefly outline the construction of certain tensor products of C*-algebras, and define two tensor products which exist for every pair of C*-algebras (although they may coincide) - that is, the max tensor product and the min tensor product. We will mostly be concerned with the min tensor product.

For two (algebraic) objects A, B, let $A \odot B$ denote their algebraic tensor product. For Hilbert spaces $\mathcal{H}, \mathcal{K}$, we will write $\mathcal{H} \otimes \mathcal{K}$ to denote the Hilbert space completion of $\mathcal{H} \odot \mathcal{K}$ under the inner product $\langle \sum_i x_i \otimes y_i, \sum_j u_j \otimes v_j \rangle = \sum_{i,j} \langle x_i, u_j \rangle \langle y_i, v_j \rangle$. It is easy to see that if $(e_i) \subseteq \mathcal{H}, (f_j) \subseteq \mathcal{K}$ are orthonormal bases, then $(e_i \otimes f_j) \subseteq \mathcal{H} \otimes \mathcal{K}$ is an orthonormal basis for $\mathcal{H} \otimes \mathcal{K}$.

Now if we take operators $S \in \mathcal{B}(\mathcal{H}), T \in \mathcal{B}(\mathcal{K})$, then we get a unique operator $S \otimes T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ defined on elementary tensors by $S \otimes T(v \otimes w) = Sv \otimes Tw$. Moreover $\|S \otimes T\| = \|S\|\|T\|$.

It is clear that if we have two C*-algebras A, B, then their algebraic tensor product is a *-algebra with natural operations.

Definition 1.2.1. Let A, B be C*-algebras. A **C*-norm** on $A \odot B$ is a norm $\|\cdot\|_\alpha$ such that $\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha$ and $\|x^*x\|_\alpha = \|x\|_\alpha^2$ for all $x, y \in A \odot B$. We will denote the completion of $A \odot B$ with respect to $\|\cdot\|_\alpha$ by $A \otimes_\alpha B$. This is clearly a C*-algebra.

We know that $M_n \odot A \simeq M_n(A)$ has a unique C*-norm since $M_n(A)$ is a C*-algebra. Indeed, if we take a faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$, then it has the natural operator norm on $\mathcal{B}(\mathcal{H}^{(n)})$, which restricts to a C*-norm on $M_n(A) \subseteq \mathcal{B}(\mathcal{H}^{(n)})$. Since C*-algebras have unique C*-norms, evidently this norm is unique. This will be a crucial observation once we examine nuclear C*-algebras.

Definition 1.2.2. Let A, B be C*-algebras.

1. The **maximal C*-norm** on $A \odot B$ is defined as follows: for $x \in A \odot B$,

$$\|x\|_{\max} = \sup\{\|\pi(x)\| \mid \pi : A \odot B \to \mathcal{B}(\mathcal{H}) \text{ is a *-homomorphism}\}.$$

   We let $A \otimes_{\max} B$ be the completion of $A \odot B$ with respect to $\| \cdot \|_{\max}$.

2. Let $\pi : A \to \mathcal{B}(\mathcal{H}), \sigma : B \to \mathcal{B}(\mathcal{K})$ be faithful representations. The **spatial (minimal) norm** is defined as

$$\| \sum a_i \otimes b_i \|_{\min} = \| \sum \pi(a_i) \otimes \sigma(b_i) \|_{\mathcal{B}(\mathcal{H} \otimes \mathcal{K})}$$

   for $\sum a_i \otimes b_i \in A \odot B$. We will denote the completion of $A \odot B$ with respect to $\| \cdot \|_{\min}$ by $A \otimes B$.

Both of these are C*-norms, and $\|x\|_{\min} \leq \|x\|_\alpha \leq \|x\|_{\max}$ for every C*-norm $\| \cdot \|_\alpha$ on $A \odot B$. The last inequality is clear, while the first follows from a theorem of Takesaki. This can be found in chapter 3 of [6] or chapter IV.4 of [30]. Most of the results that follow are from chapter 3 of [6] as well.

It is also true that we can extend certain tensor product maps, with the right assumptions.

Theorem 1.2.3. Let A, B, C, D be C*-algebras, $\phi : A \to C, \psi : B \to D$ be c.p. maps. Then the map

$$\phi \odot \psi : A \odot B \to C \odot D$$

extends to a c.p. map on both the minimal and maximal tensor products. Moreover, if we denote these by $\phi \otimes \psi$ and $\phi \otimes_{\max} \psi$ respectively, then

$$\|\psi \otimes_{\max} \psi\| = \|\phi \otimes \psi\| = \|\phi\|\|\psi\|.$$

This next result is quite deep, but is vital for proving that the first two characterizations of nuclearity, as in chapter 2, are actually equivalent. Because that result will be assumed, we will state this here without proof.

Theorem 1.2.4. Let $\phi : A \to M \subseteq \mathcal{B}(\mathcal{H})$ be a map from a unital C*-algebra A to a von Neumann algebra M. Then there exists c.c.p. maps $\phi_\lambda : A \to M_{k(\lambda)}, \psi_\lambda : M_{k(\lambda)} \to M$ such that $\phi(a) = \lim_\lambda \psi_\lambda \circ \phi_\lambda(a)$ (norm-convergence) for all $a \in A$ if and only if the product map $\phi \times \iota_{M'} : A \odot M' \to \mathcal{B}(\mathcal{H})$, defined by $\phi \times \iota_{M'}(a \otimes b) = \phi(a)b$, is continuous with respect to the min-norm.

The following result will be useful for exhibiting an example of a non-exact C*-algebra.

Theorem 1.2.5 (The Trick)**.** Let $A \subseteq B, C$ be C*-algebras, $\| \cdot \|_\alpha$ be a C*-norm on $B \odot C$, and $\| \cdot \|_\beta$ be the restriction of $\| \cdot \|_\alpha$ to $A \odot C$. If $\pi_A : A \to \mathcal{B}(\mathcal{H}), \pi_C : C \to \mathcal{B}(\mathcal{H})$ are representations with commuting ranges and the product $\pi_A \times \pi_C : A \odot C \to \mathcal{B}(\mathcal{H})$ is $\| \cdot \|_\beta$-continuous, then there exists a c.c.p. map $\phi : B \to \pi_C(C)'$ which extends π_A.

1.3 Crossed Products (by Discrete Groups)

Crossed products are the realization of non-commutative dynamical systems. One can consider the dynamics of a group acting by automorphism on a compact Hausdorff space, and understand many things about both the group and the topology of the space. This carries over nicely to the non-commutative setting: for example, it was proved in [15] that a discrete amenable group G is C*-simple (that is, its reduced group C*-algebra is simple) if and only if it acts freely on its Furstenburg boundary, and this is equivalent to $C(\partial_F G) \rtimes_r G$ being simple, where $\partial_F G$ is the Furstenburg boundary. Our use of crossed products will mostly be to prove that certain C*-algebras are nuclear or exact, or to preserve these properties.

Definition 1.3.1. A C*-dynamical system (A, G, α) consists of a C*-algebra A, a discrete group G, and a group homomorphism $\alpha : G \to \operatorname{Aut}(A)$. We will denote $\alpha(s)$ by α_s. Given a C*-dynamical system, a **covariant representation** is a pair (π, u) where $\pi : A \to \mathcal{B}(\mathcal{H})$ is a *-representation, and $u : G \to \mathcal{B}(\mathcal{H})$ is a unitary representation of g such that

$$u_s \pi(a) u_s^* = \pi(\alpha_s(a))$$

for all $a \in A, s \in G$, where $u_s = u(s)$.

If (A, G, α) is C*-dynamical system, we let $AG = C_c(G, A)$ be the space of finitely supported functions $G \to A$, and we write elements as finite sums $\sum_s a_s s$. We define a twisted convolution and an involution on AG as follows. For $\sum_s a_s s, \sum_t b_t t \in AG$,

$$\left(\sum_s a_s s\right)\left(\sum_t b_t t\right) = \sum_{s,t} a_s s b_t s^{-1} s t = \sum_{s,t} a_s \alpha_s(b_t) st = \sum_{s,t} a_s \alpha_s(b_{s^{-1}t}) t,$$

and

$$\left(\sum_s a_s s\right)^* = \sum_s s^* a_s^* = s^{-1} a_s^* s s^{-1} = \alpha_s^{-1}(a_s^*) s^{-1} = \sum_s \alpha_s(a_{s^{-1}}^*) s.$$

It is clear that these operations align with what we would have in a covariant representation. Note that given a *-representation $AG \to \mathcal{B}(\mathcal{H})$, its clear that we can get a covariant representation, and that given a covariant representation, we can get a *-representation.

Definition 1.3.2. Let (A, G, α) be a C*-dynamical system.

1. The **full crossed product** of (A, G, α), denoted $A \rtimes_\alpha G$, is the completion of AG with respect to the norm
$$\|x\|_u = \sup \|\pi(x)\|,$$
where the supremum is taken over all (cyclic) *-representations $\pi : AG \to \mathcal{B}(\mathcal{H})$.

2. The **reduced crossed product** of (A, G, α), denoted $A \rtimes_{r,\alpha} G$, is the closure of the following representation. Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a faithful representation , and let $\pi : A \to \mathcal{B}(\mathcal{H} \otimes \ell^2(G))$ by $\pi(a)(\xi \otimes \delta_g) = (\alpha_{g^{-1}}(a)\xi) \otimes \delta_g$. Now consider the *-representation $\pi \times (1 \otimes \lambda) : AG \to \mathcal{B}(\mathcal{H} \otimes \ell^2(G))$, where $\lambda : G \to \mathcal{B}(\ell^2(G))$ is the left regular representation. This is a covariant representation, and the C*-algebra $A \rtimes_{r,\alpha} G = C^*((\pi \times (1 \otimes \lambda))(AG))$ is independent of choice of faithful representation $A \subseteq \mathcal{B}(\mathcal{H})$.

If A is unital, then there is a copy of A in either crossed product, and a copy of G in the unitary group of either crossed product.

Theorem 1.3.3. The map $E : AG \to A$, given by $E(\sum_s a_s s) = a_e$, extends to a faithful conditional expectation from $A \rtimes_{r,\alpha} G$ onto A.

Lemma 1.3.4. Let (A, G, α) be a C*-dynamical system, $F \subseteq G$ be finite. Then there exist c.c.p. maps $\phi : A \rtimes_{\alpha,r} G \to A \otimes M_{|F|}$, $\psi : A \otimes M_{|F|} \to AG \subseteq A \rtimes_{\alpha,r} G$ such that for all $a \in A, s \in G$,

$$\psi \circ \phi(as) = \frac{|F \cap sF|}{|F|} as.$$

We require one last result, which is a Consequence of Green's imprimitivity theorem.

Definition 1.3.5. Let G be a locally compact group, X a locally compact space, and let $G \curvearrowright X$.

1. $G \curvearrowright X$ is **free** if the stabilizer of every point is trivial.
2. $G \curvearrowright X$ is **proper** if the map $\psi : G \times X \to X \times X$, given by $\psi(g, x) = (g \cdot x, x)$, satisfies $\psi^{-1}(K)$ is compact for all compact $K \subseteq X \times X$.

Theorem 1.3.6 ([34], Remark 4.16)**.** Let G be an infinite, second countable, locally compact group, and X be a second countable locally compact space. Then if $G \curvearrowright X$ is proper and free,

$$C_0(X) \rtimes G \simeq C_0(X/G) \otimes \mathcal{K}(L^2(G)),$$

where X/G denotes the quotient of X by the orbits, with the appropriate topology.

Corollary 1.3.7. Let $\tau : \mathbb{Z} \to \operatorname{Aut}(C_0(\mathbb{R}))$ be the automorphism defined by $\tau(f)(x) = f(x + 1)$. Then $C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z} \simeq C(\mathbb{T}) \otimes \mathbb{K}$.

Proof. It is clear that the action $\mathbb{Z} \curvearrowright \mathbb{R}$ given by $n \cdot x \mapsto n + x$ is both free and proper. The result follows by the above theorem. □

1.4 K-Theory

To every C*-algebra A, we associate two abelian groups $K_0(A)$ and $K_1(A)$. $K_0(\cdot)$ and $K_1(\cdot)$ are covariant functors from the category of C*-algebras to the category of abelian groups. The details of the following constructions and results can be found in [27]. These functors are of particular interest since they are an isomorphism invariant. There are certain classes of C*-algebras, where the K-theory is a complete invariant: for example, K_0 is a complete invariant for the class of approximately finite-dimensional (AF) C*-algebras.

Let $(S, +)$ be an abelian semigroup. We let $G(S)$ be the **Grothendieck group** of G. That is, $G(S) = S \times S/\sim$, where $(x_1, y_1) \sim (x_2, y_2)$ if there exists $z \in S$ such that $x_1 + y_2 + z = x_2 + y_1 + z$. We let $\gamma_S : S \to G(S)$ be the map $\gamma_S(x) = \langle x + y, y\rangle$, where $\langle a, b\rangle$ denoted the equivalence class of $(a, b) \in S \times S$, and $x, y \in S$.

$G(S)$ is an abelian group with the operation $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$, where $0 = \langle x, x\rangle$ and $-\langle x, y\rangle = \langle y, x\rangle$ for $x, y \in S$. The map γ_S above is called the **Grothendieck map** and is independent of choice of $y \in S$.

Definition 1.4.1. Let A be a unital C*-algebra. Let $P_n(A) = \{p \in M_n(A) \mid p = p^* = p^2\}$, and $P_\infty(A) = \cup_n P_n(A)$. Define an equivalence relation $\sim_\infty$ on $P_\infty(A)$ as follows. For $p \in P_n(A), q \in P_m(A)$, $p \sim_\infty q$ if there is some $m \times n$ matrix v with entries from A such that $p = v^*v$ and $q = vv^*$. Let $D(A) = P_\infty(A)/\sim$ and define a binary operation on $D(A)$ by $[p]_\infty + [q]_\infty = [p \oplus q]_\infty$. Then $(D(A), +)$ is an abelian semigroup and we let $K_0(A) = G(D(A))$. If $\gamma : D(A) \to K_0(A)$ is the Grothendieck map, we define $[\cdot]_0 : D(A) \to K_0(A)$ by $[p]_0 = \gamma([p]_\infty)$.

If A is non-unital, let $\pi : \tilde{A} \to \mathbb{C}$ be the quotient map, where $\tilde{A}$ is the unitization of A. Then define $K_0(A) = \ker K_0(\pi)$.

Definition 1.4.2. Let A be a unital C*-algebra, $U_n(A) = U(M_n(A))$, and $U_\infty(A) = \cup_n U_n(A)$. Define an equivalence relation $\sim_1$ on $U_\infty(A)$ as follows. For $u \in U_n(A), v \in U_m(A)$, $u \sim_1 v$ if there exists some $k \geq m, n$ such that $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$, where $\sim_h$ is the natural homotopy relation on the unitary group of a C*-algebra. Now for any C*-algebra, A, define $K_1(A) = U_\infty(\tilde{A})/\sim_1$, which is an abelian group with operation $[u]_1 + [v]_1 = [u \oplus v]_1$, identity $[1_k]_1$ for all k, and inverse given by $-[u]_1 = [u^*]_1$.

If A is not unital, let $K_1(A) = K_1(\tilde{A})$.

Note that if A is unital, then there is a group isomorphism $K_1(A) \simeq U_\infty(A)/\sim_1$.

Theorem 1.4.3. $K_0(\cdot)$ and $K_1(\cdot)$ are covariant functors from the category of C*-algebras to the category of abelian groups.

Remark 1.4.4. Evidently there is another way to construct $K_0(A)$ and $K_1(A)$ if A is unital. One takes the Grothendieck group of $P(\mathbb{K} \otimes A)/\sim$, where $P(\mathbb{K} \otimes A)$ is the set of projections and $\sim$ is Murray-von Neumann equivalence, and the operation is similar to the one above the one above: if $p, q \in P(\mathbb{K} \otimes A)$, find orthogonal projections p', q' such that $p \sim p', q \sim q'$, then set $[p] + [q] = [p' + q']$.

For K_1, one just takes $K_1(A) = U((\mathbb{K} \otimes A)^\sim)/U_0((\mathbb{K} \otimes A)^\sim)$.

Theorem 1.4.5. Let

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0$$

be a short exact sequence of C*-algebras. Then there is an exact sequence

$$\begin{tikzcd}
K_0(I) \arrow[r, "K_0(\iota)"] & K_0(A) \arrow[r, "K_0(\pi)"] & K_0(A/I) \arrow[d, "\delta_0"] \\
K_1(A/I) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "K_1(\pi)"] & K_1(I), \arrow[l, "K_1(\iota)"]
\end{tikzcd}$$

where δ_1 is the index map, and δ_0 is the exponential map.

It will not be necessary to know exactly what the maps δ_0, δ_1 are. All that we shall need is the exactness of the sequence. This will be used as a computational tool to compute the K-theory of the Cuntz algebras.

1.5 Excision and Glimm's Lemma

There is a useful approximation property on the states which are wk*-limits of pure states, called excision. This along with Glimm's Lemma will prove to be valuable tools when considering the structure of unital, simple, purely infinite C*-algebras. The results about excision can be found in [1], while there are proofs of Glimm's lemma in chapter 11 of [13] and chapter 1 of [6].

Definition 1.5.1. Let A be a C*-algebra, $\phi \in S(A)$ a state. A net (e_λ) of positive elements, with $\|e_\lambda\| = 1$, **excises** ϕ if $\lim_\lambda \|e_\lambda a e_\lambda - \phi(a)e_\lambda^2\| = 0$ for all $a \in A$.

Theorem 1.5.2. Let A be a unital C*-algebra. A state ϕ can be excised if and only if ϕ is a wk*-limit of pure states.

Lemma 1.5.3 (Glimm's Lemma)**.** Let $\mathcal{H}$ be separable, $\mathbb{K} = \mathcal{K}(\mathcal{H})$ be the compacts, and let $A \subseteq \mathcal{B}(\mathcal{H})$ be a C*-algebra such that $1_{\mathcal{H}} \in A$. If ϕ is a state such that $\phi|_{A\cap\mathbb{K}} = 0$, then ϕ is a wk*-limit of vector states on A. Moreover, if A is irreducible in $\mathcal{H}$, then ϕ is a wk*-limit of pure states of A.

2 Nuclear and Quasidiagonal C*-algebras

In this chapter, we outline several approximations properties for C*-algebras and groups.

2.1 Nuclear C*-Algebras

Nuclearity for C*-algebras is sometimes thought of as the non-commutative analogue of compactness, so naturally these algebras are very important. The previous statement becomes clear when one sees the proof that all abelian C*-algebras are nuclear (Proposition 2.4.2 of [6]). Most of the following can be found in chapters 2 and 3 of [6] as well.

Definition 2.1.1. A map $\theta : A \to B$ between C*-algebras is **nuclear** if there exists c.c.p. maps $\phi_\lambda : A \to M_{k(\lambda)}, \psi_\lambda : M_{k(\lambda)} \to B$ such that $\psi_\lambda \circ \phi_\lambda \to \theta$ in point-norm - that is,

$$\|\psi_\lambda \circ \phi_\lambda(a) - \theta(a)\| \to 0 \text{ for all } a \in A.$$

It is clear that this is actually quite a local property. This definition is equivalent to the condition that for every finite $\mathcal{F} \subseteq A, \varepsilon > 0$, there exists $n \in \mathbb{N}$, c.c.p. maps $\phi : A \to M_n, \psi : M_n \to B$ such that $\|\psi \circ \phi(a) - \theta(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.

This, however, is not the right notion for von Neumann algebras.

Definition 2.1.2. Let A be a C*-algebra, N a von Neumann algebra. We say a map $\theta : A \to N$ is **weakly nuclear** if there exists c.c.p. $\phi_\lambda : A \to M_n, \psi_\lambda : M_n \to N$ such that $\psi_\lambda \circ \phi_\lambda \to \theta$ point-ultraweakly - that is,

$$\|\eta(\psi_\lambda \circ \phi_\lambda(a)) - \eta(\theta(a))\| \to 0 \text{ for all } a \in A, \eta \in N_*.$$

Again, this is a local property and is equivalent to the condition that for all finite $F \subseteq A, \chi \subseteq N_*, \varepsilon > 0$, there exists c.c.p. $\phi : A \to M_n, \psi : M_n \to N$ such that $\|\eta(\psi \circ \phi(a)) - \eta(\theta(a))\| < \varepsilon$ for all $a \in F, \eta \in \chi$.

If algebras are unital, one can replace c.c.p. with u.c.p., if not, we extend nuclear maps on to their unitizations. As such, the unital and non-unital cases are not all that different. Also in the weakly nuclear case, one can ensure that the c.c.p. maps are normal.

Remark 2.1.3. Theorem 1.2.4 really says that a map $\phi : A \to M \subseteq \mathcal{B}(\mathcal{H})$ is weakly nuclear if and only the product map $\phi \times \iota_{M'} : A \odot M' \to \mathcal{B}(\mathcal{H})$ is min-continuous.

Definition 2.1.4. Let A be a C*-algebra, M a von Neumann algebra.

1. We say that A is **nuclear** if the the map $\mathrm{id}_A : A \to A$ is nuclear.
2. We say that M is **semidiscrete** if the map $\mathrm{id}_M : M \to M$ is weakly nuclear.

Theorem 2.1.5. Let A be a C*-algebra. The following are equivalent.

1. A is nuclear: i.e., $\mathrm{id}_A : A \to A$ is a nuclear map;
2. A is $\otimes$-nuclear: i.e., for any C*-algebra B, $A \otimes_{\max} B = A \otimes B$;
3. A^{**} is a semidiscrete von Neumann algebra;
4. A^{**} is an injective von Neumann algebra.

The equivalence of the first two with the last two is in chapter 9 of [6] or chapter XVI of [31]. This is a very deep result, which we will be assuming.

2.2 Amenable Groups

The concept of amenability has been around for quite some time, and it is particularly prevalent in the study of groups. The class of amenable groups has many interesting properties, one of which being that the reduced C*-algebra of a group is nuclear if and only if the group is amenable. This gives rise to many examples and non-examples of nuclear C*-algebras - and as we will see, an example of a non-nuclear but exact C*-algebra, in addition to a non-exact C*-algebra.

Recall that for $\ell^\infty(G)$, we have the left translation action given by $s \cdot f(t) = f(s^{-1}t)$ for $f \in \ell^\infty(G), s, t \in G$.

Definition 2.2.1. We say that a group G is **amenable** if there exists a state μ on $\ell^\infty(G)$ which is invariant under left translation. That is, $\mu(s \cdot f) = \mu(f)$ for all $f \in \ell^\infty(G), s \in G$.

In the discrete case, amenable groups are characterized in several ways. The following can be found in chapter 2.6 of [6]. Recall that for two sets A, B, $A\Delta B = (A \cup B) \setminus (A \cap B)$.

Theorem 2.2.2. Let G be a group. The following are equivalent.

1. G is amenable.

2. G has an **approximate invariant mean**. That is, for any finite subset $E \subseteq G$ and $\varepsilon > 0$, there exists $\mu \in \mathrm{Prob}(G)$ such that
$$\max_{s \in E} \|s \cdot \mu - \mu\|_1 < \varepsilon.$$

3. G satisfies the **Følner condition**. That is, for any finite subset $E \subseteq F$ and $\varepsilon > 0$, there exists a finite subset $F \subseteq G$ such that
$$\max_{s \in E} \frac{|sF\Delta F|}{|F|} < \varepsilon.$$
A sequence of finite sets $F_n \subseteq G$ such that
$$\frac{|sF_n\Delta F_n|}{|F_n|} \to 0$$
for every $s \in G$ is called a **Følner sequence**. It is clear that G satisfies the Føner condition if and only if there exists a Følner sequence.

4. The trivial representation τ of G is **weakly contained** in the left regular representation λ. That is, there exists unit vectors $\xi_i \in \ell^2(G)$ such that $\|\lambda_s \xi_i - \xi_i\| \to 0$ for all $s \in G$.

5. There exists a net (ϕ_i) of finitely supported positive definite functions on G such that $\phi_i \to 1$ pointwise.

6. $C^*(G) = C^*_r(G)$.

7. $C^*_r(G)$ has a character (a one-dimensional representation).

8. For any finite $E \subseteq G$, we have

$$\left\| \frac{1}{|E|} \sum_{s \in E} \lambda_s \right\| = 1.$$

9. $C_r^*(G)$ is nuclear.

10. $L(G)$ is semidiscrete.

2.3 Quasidiagonal C*-Algebras

Quasidiagonality is another very important approximation property that can be defined in many different ways, but we opt for the treatment given in chapter 7 [6]. These are algebras which "asymptotically" act in a similar fashion to a matrix algebra. These have been studied greatly by Voiculescu, and he gave work to show that quasidiagonality is preserved under homotopy. This property is equivalent to the algebra having an embedding to the quotient of a product of matrix algebras with a c.c.p lift (or u.c.p. if the algebra is unital). This will be invaluable as the cone of any C*-algebra is QD. This will allow us to construct our embeddings into $\mathcal{O}_2$.

Definition 2.3.1. A C*-algebra A is **quasidiagonal** (QD) if there exists c.c.p. maps $\phi_\lambda : A \to M_{k(\lambda)}$ such that $\|\phi_\lambda(ab) - \phi_\lambda(a)\phi_\lambda(b)\| \to 0$ and $\|a\| = \lim_\lambda \|\phi_\lambda(a)\|$ for all $a, b \in A$. This first property is called **asymptotically multiplicative** and the second is called **asymptotically isometric**.

Again, this is really a local property.

Lemma 2.3.2. A is QD if and only if for every finite set $F \subseteq A, \varepsilon > 0$, there exists c.c.p. $\phi : A \to M_n$ such that

$$\|\phi(ab) - \phi(a)\phi(b)\| < \varepsilon \text{ and } \|\phi(a)\| > \|a\| - \varepsilon$$

for all $a, b \in F$.

Lemma 2.3.3. If A is unital and QD, then there exists u.c.p. maps $\phi_\lambda : A \to M_{k(\lambda)}$ which are both asymptotically multiplicative and asymptotically isometric.

Theorem 2.3.4. A C*-algebra A is QD if and only if there exists an injective *-homomorphism

$$A \to \frac{\Pi_n M_{k(n)}}{\oplus_n M_{k(n)}}$$

which admits a c.c.p. lift $A \to \Pi_n M_{k(n)}$.

If A is unital, the above embedding and c.c.p. lift can both be taken to be unital.

Theorem 2.3.5. Let A be a C*-algebra. Let the **cone** of A be $CA = C_0((0,1]) \otimes A$, and the **suspension** of A be $SA = C_0((0,1)) \otimes A$. Then both CA and SA are QD.

2.4 Examples

Here we list several examples of objects with and without the above properties. We have examples and non-examples of each.

Example 2.4.1. First let us look at some examples of C*-algebras.

1. Abelian C*-algebras are nuclear and QD.

2. Finite dimensional C*-algebras are nuclear and QD.

3. Inductive limits of nuclear C*-algebras are nuclear. In particular, approximately finite dimensional C*-algebras are nuclear.

4. Inductive limits of QD C*-algebras are QD, as long as the connecting maps are injective.

5. The reduced C*-algebras of amenable groups are nuclear. Note that the full C*-algebra is equal to the reduced one in this case.

6. $A \otimes B$ and $A \otimes_{\max} B$ are nuclear if and only if both A, B are nuclear (in which case the two tensor products agree).

7. Nuclearity and quasidiagonality is preserved under min-tensors: that is if A, B are nuclear or QD, then so is $A \otimes B$. The converse is true as well.

8. Both finite and abelian groups are amenable. Amenability is closed under taking subgroups, quotients, extensions, and direct limits.

9. If G is an amenable group, $\alpha : G \to A$ is an action, then $A \rtimes_{r,\alpha} G = A \rtimes_\alpha G$ and A is nuclear if and only if $A \rtimes_\alpha G$ is nuclear.

10. It is well known that the free group $\mathbb{F}_2$ is not amenable. This can be seen through the existence of a paradoxical decomposition, in the sense of Banach-Tarski. Consequently $C_r^*(\mathbb{F}_2)$ is a non-nuclear C*-algebra, as is $C^*(\mathbb{F}_2)$.

11. If G is non-amenable, then $C_r^*(G)$ is not QD. In particular, $C_r^*(\mathbb{F}_2)$ is not QD, and it is not nuclear.

3 Special C*-Algebras

There will be two C*-algebras which give rise to two characterizations of exactness in the separable setting: the CAR algebra and the Cuntz algebra $\mathcal{O}_2$. We begin by studying the CAR algebra. CAR stands for canonical anticommutation relations, and the CAR algebra is a universal algebra generated by these relations, which will be defined shortly. It turns out that it is isomorphic to the UHF algebra $M_{2^\infty} = \otimes_1^\infty M_2$. We will see later on that every separable exact C*-algebra can be realized as the quotient of a subalgebra of the CAR algebra, giving us one characterization of exactness which allows us to prove that exactness is preserved under taking quotients.

The second C*-algebra is the Cuntz algebra $\mathcal{O}_2$, which is a universal C*-algebra generated by two isometries satisfying the Cuntz relations. To understand the structure of $\mathcal{O}_2$, we will study a family of algebras generated by isometries, namely the Cuntz algebras $\mathcal{O}_n$, for $n \geq 2$, and $\mathcal{O}_\infty$. The algebras $\mathcal{O}_n$ will contain canonical copies of M_{n^∞}, and $\mathcal{O}_2$ in particular will contain a copy of M_{2^∞}. M_{2^∞} will be seen to have a certain Rokhlin property, which plays a role in the proof that $\mathcal{O}_2 \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$. The algebra $\mathcal{O}_2$ will give rise to another characterization of exactness, namely a separable C*-algebra is exact if and only if it embeds into $\mathcal{O}_2$. This will require plenty of work, and we will get to it by the end.

3.1 The CAR Algebra

The UHF algebra M_{n^∞} is defined to be the direct limit of $(M_{n^k})_k$ with connecting maps $x \mapsto 1 \otimes x$. One can also view this as the infinite tensor product $\otimes_1^\infty M_n$. This is defined as the norm closure of $\cup_k A_k$ where $A_k = (\otimes_1^k M_n) \otimes 1 \otimes 1 \cdots \subseteq \mathcal{B}(\otimes_1^\infty \ell_2^2)$.

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, $\mathcal{K}$ a Hilbert space, and let $\alpha : \mathcal{H} \to \mathcal{B}(\mathcal{H})$ be a map such that for all $\xi, \eta \in \mathcal{H}$,

$$\alpha(\xi)\alpha(\eta) + \alpha(\eta)\alpha(\xi) = 0, \text{ and } \alpha(\xi)^*\alpha(\eta) + \alpha(\eta)\alpha(\xi)^* = \langle \eta, \xi \rangle I.$$

These relations are called the **canonical anticommutation relations**, abbreviated CAR. It will show that the C*-algebra generated by $\{\alpha(\xi) \mid \xi \in \mathcal{H}\}$ is independent of choice of α, and that it is isomorphic to the unique UHF algebra $M_{2^\infty} = \otimes_1^\infty M_2$. Therefore this C*-algebra will be a simple, separable, nuclear C*-algebra. The following example can be seen chapter 5.2 of [3].

Example 3.1.1 (Creation operators on the anti-symmetric Fock space)**.** Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{H}^{\wedge n}$ be the antisymmetric tensor product: that is, $\mathcal{H}^{\wedge n}$ is the subspace of $\mathcal{H}^{\otimes n}$ given by $\overline{\operatorname{span}}\{x_1 \wedge \cdots \wedge x_n \mid x_i \in \mathcal{H}\}$, where

$$x_1 \wedge \cdots \wedge x_n = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Let $\mathcal{F}_a(\mathcal{H}) = \oplus_{n=0}\mathcal{H}^{\wedge n}$, where the $n = 0$ corresponds to the summand $\mathbb{C}$. Define $\alpha : \mathcal{H} \to \mathcal{B}(\mathcal{F}_a(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H})$ by

$$\alpha(x)(x_1 \wedge \cdots \wedge x_n) = x \wedge x_1 \wedge \cdots \wedge x_n$$

and extend by linearity and continuity; clearly this map is continuous. Now it remains to show that the CAR are satisfied. First notice that $x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)} = \mathrm{sgn}(\sigma) x_1 \wedge \cdots \wedge x_n$. Thus

$$\begin{aligned} \alpha(x)\alpha(y)x_1 \wedge \cdots \wedge x_n &= x \wedge y \wedge x_1 \wedge \cdots \wedge x_n \\ &= -y \wedge x \wedge x_1 \wedge \cdots \wedge x_n \\ &= -\alpha(y)\alpha(x)x_1 \wedge \cdots \wedge x_n, \end{aligned}$$

so that the first relation holds. Now note that as in Proposition 3.8.5 of [29], we have that

$$\langle x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n \rangle = \det(\langle x_i, y_j \rangle).$$

Indeed,

$$\begin{aligned} \langle x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n \rangle &= \frac{1}{n!} \sum_{\sigma, \tau \in S_n} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \Pi_{i=1}^n \langle x_{\sigma(i)}, y_{\tau(i)} \rangle \\ &= \sum_{\sigma} \mathrm{sgn}(\sigma) \Pi_{i=1}^n \langle x_{\sigma(i)}, y_i \rangle \\ &= \det(\langle x_i, y_j \rangle). \end{aligned}$$

Therefore, letting $\hat{y_k}$ denote when we omit that part of the wedge, we have

$$\begin{aligned} \langle \alpha(x)x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_{n+1} \rangle &= \det \left(\begin{pmatrix} \langle x, y_1 \rangle & \cdots & \langle x, y_{n+1} \rangle \\ \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_{n+1} \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_{n+1} \rangle \end{pmatrix} \right) \\ &= \sum_{k=1}^{n+1} (-1)^{k+1} \langle x, y_k \rangle \langle x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge \hat{y_k} \wedge \cdots \wedge y_{n+1} \rangle \\ &= \sum_{k=1}^{n+1} (-1)^{k+1} \langle x_1 \wedge \cdots \wedge x_n, \overline{\langle x, y_k \rangle} y_1 \wedge \cdots \wedge \hat{y_k} \wedge \cdots \wedge y_{n+1} \rangle \\ &= \langle x_1 \wedge \cdots \wedge x_n, \sum_{k=1}^{n+1} (-1)^{k+1} \langle y_k, x \rangle y_1 \wedge \cdots \wedge \hat{y_k} \wedge \cdots \wedge y_{n+1} \rangle, \end{aligned}$$

which allows us to conclude that

$$\alpha(x)^*(y_1 \wedge \cdots \wedge y_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} \langle y_k, x \rangle y_1 \wedge \cdots \wedge \hat{y_k} \wedge \cdots \wedge y_{n+1}.$$

Now

$$\begin{aligned} \alpha(x)^*\alpha(y)x_1 \wedge \cdots \wedge x_n &= \alpha(x)^* y \wedge x_1 \wedge \cdots \wedge x_n \\ &= \langle y, x \rangle x_1 \wedge \cdots \wedge x_n + \sum_{k=1}^{n} (-1)^k \langle x_k, x \rangle y \wedge x_1 \wedge \cdots \wedge \hat{x_k} \wedge \cdots \wedge x_n \end{aligned}$$

and

$$\alpha(y)\alpha(x)^*x_1 \wedge \cdots \wedge x_n = \sum_{k=1}^{n}(-1)^{k+1}\langle x_k, x\rangle y \wedge x_1 \wedge \cdots \wedge \hat{x_k} \wedge \cdots \wedge x_n.$$

By summing these quantities, it is clear that we are left with $\langle y, x\rangle x_1 \wedge \cdots \wedge x_n$, hence

$$\alpha(x)^*\alpha(y) + \alpha(y)\alpha(x)^* = \langle y, x\rangle I.$$

Proposition 3.1.2 ([11], Example III.5.4)**.** Let $\alpha : \mathcal{H} \to \mathcal{B}(\mathcal{K})$ be a map satisfying CAR, and let $B = C^*(\{\alpha(\xi) \mid \xi \in \mathcal{H}\})$. Then $B \simeq M_{2^\infty}$. As such, B is independent of choice of α.

Proof. Let $\|\xi\| = 1$, then $\eta = \xi$ gives us that $\alpha(\xi)\alpha(\eta) + \alpha(\eta)\alpha(\xi) = 0$, hence $2\alpha(\xi)^2 = 0$, giving us that $\alpha(\xi)^2 = 0$. Also notice that $\alpha(\xi)^*\alpha(\xi) + \alpha(\xi)\alpha(\xi)^* = I$. Therefore

$$\begin{aligned}(\alpha(\xi)^*\alpha(\xi)) &= (\alpha(\xi)^*\alpha(\xi))\, I \\ &= (\alpha(\xi)^*\alpha(\xi))\,(\alpha(\xi)^*\alpha(\xi) + \alpha(\xi)\alpha(\xi)^*) \\ &= (\alpha(\xi)^*\alpha(\xi))^2 + \alpha(\xi)^*\alpha(\xi)^2\alpha(\xi)^* \\ &= (\alpha(\xi)^*\alpha(\xi))^2\,.\end{aligned}$$

Defining $E(\xi) = \alpha(\xi)^*\alpha(\xi) = (\alpha(\xi)^*\alpha(\xi))^2 = (\alpha(\xi)^*\alpha(\xi))^*$, this is clearly a projection and so is $\alpha(\xi)\alpha(\xi)^* = 1 - E(\xi) = E(\xi)^\perp$. So $\alpha(\xi)$ is a partial isometry, with initial projection $E(\xi)$ and range projection $E(\xi)^\perp$. We then have that $C^*(\alpha(\xi)) = \operatorname{span}\{\alpha(\xi), \alpha(\xi)^*, E(\xi), E(\xi)^\perp\} \simeq M_2$.

Letting $e_{21}^{(1)} = \alpha(\xi), e_{12}^{(1)} = \alpha(\xi)^*, e_{11}^{(1)} = E(\xi), e_{22}^{(1)} = E(\xi)^\perp$, it is clear that this is a set of matrix units for $C^*(\alpha(\xi)) \simeq M_2$.

Now for $\xi, \eta \in \mathcal{H}$ orthogonal

$$\begin{aligned}[\alpha(\eta), E(\xi)] &= \alpha(\eta)\alpha(\xi)^*\alpha(\xi) - \alpha(\xi)^*\alpha(\xi)\alpha(\eta) \\ &= \alpha(\eta)\alpha(\xi)^*\alpha(\xi) + \alpha(\xi)^*\alpha(\eta)\alpha(\xi) \\ &= (\alpha(\eta)\alpha(\xi)^* + \alpha(\xi)^*\alpha(\eta))\alpha(\xi) \\ &= \langle \eta, \xi\rangle\alpha(\xi) = 0.\end{aligned}$$

Since $\alpha(\eta)$ commutes with $E(\xi)$, it follows that $E(\eta)$ commutes with $E(\xi)$ as well. Now let $V_1 = I - 2E(\xi) = E(\xi)^\perp - E(\xi)$. Then

$$V_1\alpha(\eta)\alpha(\xi) = -V_1\alpha(\xi)\alpha(\eta) = \alpha(\xi)V_1\alpha(\eta),$$

where the first equality comes from the first anticommutation relation, and the last equality follows from the following calculation:

$$\begin{aligned}\alpha(\xi)V_1\alpha(\eta) &= \alpha(\xi)(I - 2\alpha(\xi)^*\alpha(\xi))\alpha(\eta) \\ &= \alpha(\xi)\alpha(\eta) - 2\alpha(\xi)\alpha(\xi)^*\alpha(\xi)\alpha(\eta) \\ &= -\alpha(\eta)\alpha(\xi) + 2\alpha(\xi)\alpha(\xi)^*\alpha(\eta)\alpha(\xi) \\ &= (-1 + 2(1 - E(\xi))\alpha(\eta)\alpha(\xi) \\ &= (1 - 2E(\xi))\alpha(\eta)\alpha(\xi) = V_1\alpha(\eta)\alpha(\xi).\end{aligned}$$

We also have that

$$V_1\alpha(\eta)\alpha(\xi)^* = -V_1\alpha(\xi)^*\alpha(\eta) = \alpha(\xi)^*V_1\alpha(g),$$

where the first equality comes from the second relation, and the last equality follows from the following calculation:

$$\begin{aligned}
V_1\alpha(\eta)\alpha(\xi)^* - \alpha(\xi)^*V_1\alpha(\eta) &= \alpha(\eta)\alpha(\xi)^* - 2\alpha(\xi)^*\alpha(\xi)\alpha(\eta)\alpha(\xi)^* \\
&\quad - \alpha(\xi)^*\alpha(\eta) + 2\alpha(\xi)^*\alpha(\xi)^*\alpha(\xi)\alpha(\eta) \\
&= 2\alpha(\eta)\alpha(\xi)^* - 2\alpha(\eta)\alpha(\xi)^*\alpha(\xi)\alpha(\xi)^* + 2\alpha(\xi)^*\alpha(\eta)\alpha(\xi)^*\alpha(\xi) \\
&= 2\alpha(\eta)\alpha(\xi)^* - 2\alpha(\eta)\alpha(\xi)^*\alpha(\xi)\alpha(\xi)^* - 2\alpha(\eta)\alpha(\xi)^*\alpha(\xi)^*\alpha(\xi) \\
&= 2\alpha(\eta)\alpha(\xi)^*(1 - \alpha(\xi)\alpha(\xi)^* - \alpha(\xi)^*\alpha(\xi)) \\
&= 2\alpha(\eta)\alpha(\xi)^*(1 - (1 - E(\xi)) - E(\xi)) = 0.
\end{aligned}$$

This gives us that V_1 commutes with all of $C^*(\alpha(\xi))$.

Now $C^*(V_1\alpha(\eta)) = \text{span}\{V_1\alpha(\eta), V_1\alpha(\eta)^*, E(\eta), E(\eta)^\perp\} \simeq M_2$, and this C*-algebra commutes with $C^*(\alpha(\xi))$. Letting $e_{21}^{(2)} = V_1\alpha(\eta), e_{12}^{(2)} = V_1\alpha(\eta)^*, e_{11}^{(2)} = E(\eta), e_{22}^{(2)} = E(\eta)^\perp$, these are clearly matrix units for $C^*(V_1\alpha(\eta))$. Moreover, we clearly have that $C^*(\alpha(\xi), \alpha(\eta)) \simeq M_4$ has matrix units $e_{ij}^{(1)}e_{kl}^{(2)}, 1 \leq i, j, k, l \leq 2$.

Now if $(h_n)_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, let $V_0 = I$, and $V_n = \Pi_{i=1}^{n-1}(I - 2E(h_n))$ for $n \geq 2$, and we get our V_1 just by picking V_1 with $h_1 = h, h_2 = k$ as above. Then we can always define matrix units, $e_{21}^{(n)} = \alpha(h_n), e_{12}^{(n)} = \alpha(h_n)^*, e_{11}^{(n)} = E(h_n), e_{22}^{(n)} = E(h_n)^\perp$, to get an isomorphic copy of M_2, and these copies will commute with each other. Then letting $B_n = C^*(\{\alpha(h_i) \mid 1 \leq i \leq n\})$, $B_n \simeq M_{2^n}$ with standard basis $e_{\sigma\tau} = \Pi_{i=1}^n e_{\sigma(i)\tau(i)}^{(i)}$, where $\sigma, \tau : \{1, \ldots, n\} \to \{1, 2\}$ are functions. Then $B = \overline{\cup_n B_n} = C^*(\{\alpha(\xi) \mid h \in \mathcal{H}\})$ is clearly isomorphic to the UHF algebra M_{2^∞}. □

Remark 3.1.3. In the UHF algebra M_{2^∞}, these V_n's can be seen as

$$V_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes n} \otimes 1 \otimes \cdots .$$

Moreover, we have

$$\begin{aligned}
\alpha(h_n) &= \left(\Pi_1^{n-1}(e_{11}^{(j)} - e_{22}^{(j)})\right) e_{12}^{(n)} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes(n-1)} \otimes e_{21}^{(n)} \otimes 1 \otimes \cdots .
\end{aligned}$$

We will see later that every separable exact C*-algebra is isomorphic to a subquotient of the CAR algebra. Since $\mathcal{O}_2$ will be proven to be nuclear, hence exact, and it will contain a copy of the CAR algebra, $\mathcal{O}_2$ will be isomorphic to a subquotient of a subalgebra of itself, which is certainly an intriguing property.

There is an interesting homomorphism $M_{2^\infty} \to M_{2^\infty}$ given by the Bernoulli shift. The following can be found in [4].

Lemma 3.1.4. Let $A = \otimes_1^\infty M_2 = C^*(\alpha(h) \mid h \in \mathcal{H})$ where $\alpha : \mathcal{H} \to \mathcal{B}(\mathcal{H})$ is a map satisfying the CAR. Let (h_n) be an orthonormal basis for $\mathcal{H}$. Then the maps $\sigma, \rho : A \to A$ determined by

$$\sigma(x) = 1 \otimes x, \rho(\alpha(h)) = \alpha(Sh),$$

where $S : \mathcal{H} \to \mathcal{H}$ is the unilateral forward shift with respect to (h_n), coincide on the **even algebra** $A^e = C^*(\alpha(h)\alpha(k), \alpha(h)\alpha(k)^* \mid h, k \in \mathcal{H})$.

Proof. We will show that $\sigma(\alpha(h_n)\alpha(h_m)) = \rho(\alpha(h_n)\alpha(h_m))$ for all $n < m \in \mathbb{N}$. Using the remark above,

$$\begin{aligned}
\rho(\alpha(h_n)\alpha(h_m)) &= \alpha(h_{n+1})\alpha(h_{m+1}) \\
&= \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes n} \otimes e_{21}^{(n+1)} \otimes 1 \otimes \cdots \right)\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes m} \otimes e_{21}^{(m+1)} \otimes 1 \otimes \cdots \right) \\
&= 1^{\otimes n} \otimes -e_{21}^{(n+1)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes(m-n-1)} \otimes e_{21}^{(m+1)} \otimes 1 \otimes \cdots \\
&= -1^{\otimes n} \otimes e_{21}^{(n+1)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes(m-n-1)} \otimes e_{21}^{(m+1)} \otimes 1 \otimes \cdots
\end{aligned}$$

and

$$\begin{aligned}
\sigma(\alpha(h_n)\alpha(h_m)) &= \sigma\left(\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes(n-1)} \otimes -e_{21}^{(n)} \otimes 1 \otimes \cdots \right)\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes(m-1)} \otimes 1 \otimes \cdots \right)\right) \\
&= 1 \otimes 1^{\otimes(n-1)} \otimes -e_{21}^{(n+1)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes(m-n-1)} \otimes e_{21}^{(m+1)} \otimes 1 \otimes \cdots \\
&= -1^{\otimes n} \otimes e_{21}^{(n+1)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes(m-n-1)} \otimes e_{21}^{(m+1)} \otimes 1 \otimes \cdots .
\end{aligned}$$

The case where $n > m$ and where we consider $\alpha(h_n)\alpha(h_m)^*$ are similar. The case where $n = m$ is trivial. Thus $\rho|_{A^e} = \sigma|_{A^e}$. □

Proposition 3.1.5 (The Rokhlin Property of the Bernoulli Shift)**.** Let σ be the one sided Bernoulli shift on the CAR algebra $A = M_{2^\infty} = \otimes_1^\infty M_2$: $\sigma(a_1 \otimes a_2 \otimes \cdots) = 1 \otimes a_1 \otimes a_2 \otimes \cdots$, where all but finitely many a_i are 1, and extend by linearity and continuity. Let $A_k = (\otimes_1^k M_2) \otimes 1 \cdots \simeq \otimes_1^k M_2$ be the unital subalgebra of A. For $\varepsilon > 0, r \in \mathbb{N}$, there exists $k \in \mathbb{N}$ and mutually orthogonal projections $p_0, p_1, \cdots, p_{2^r} = p_0$ in A_k such that $\sum_1^{2^r} p_j = 1$ and $\|\sigma(p_j) - p_{j+1}\| < \varepsilon$ for all $j = 1, \ldots, 2^r - 1$.

Proof. Let $\alpha : \ell^2(\mathbb{N}) = \mathcal{H} \to \mathcal{B}(\mathcal{H})$ be a map satisfying the CAR, and let $S \in \mathcal{B}(\mathcal{H})$ be the unilateral forward shift. Then the map $\beta : \mathcal{H} \to \mathcal{B}(\mathcal{H})$ given by $\beta(h) = \alpha(Sh)$ is another map satisfying the CAR. Hence there is a *-homomorphism $\rho : A \to A$ given by $\rho(\alpha(h)) = \alpha(Sh)$ for all $h \in \mathcal{H}$.

Let $\omega_k = e^{\frac{2\pi i}{2^k}}$, and choose an orthonormal basis $f_0, f_1, \ldots, f_r \in \ell^2(\mathbb{N})$ such that $\|Sf_j - \omega_j f_j\| \leq \delta$ for some $0 < \delta < \frac{\varepsilon}{4\|\alpha\|^2}$. Let $v_j = \alpha(f_j)(\alpha(f_0) - \alpha(f_0)^*)$. Since $\sigma(v_j) = \rho(v_j)$ by the above lemma, we have

$$\begin{aligned}\|\sigma(v_j) - \omega_j v_j\| &= \|\alpha(Sf_j)(\alpha(Sf_0) + \alpha(Sf_0)^*) - \omega_j\alpha(f_j)(\alpha(f_0) + \alpha(f_0)^*)\| \\ &\le \|\alpha(Sf_j)(\alpha(Sf_0) + \alpha(Sf_0)^*) - \alpha(Sf_j)(\alpha(f_0) + \alpha(f_0)^*)\| \\ &\quad + \|\alpha(Sf_j)(\alpha(f_0) + \alpha(f_0)^*) - \omega_j\alpha(f_j)(\alpha(f_0) + \alpha(f_0)^*)\| \\ &\le 2\|\alpha\|^2\delta + 2\|\alpha^2\|\delta \\ &= 4\|\alpha\|^2\delta < \varepsilon.\end{aligned}$$

Then it is not hard to see that $(v_j)_1^r$ satisfy the CAR:

$$v_j v_k + v_k v_j = 0 \text{ and } v_j v_k^* + v_k^* v_j = \delta_{j,k}1.$$

So by a similar argument to the one above, $C^*(v_j \mid 1 \le j \le r) \simeq M_{2^r}$. Now since $\|\sigma(v_j) - \omega_j v_j\| < \varepsilon$, $\|\sigma(x) - Ad(u)x\| < \varepsilon$, where $u = u_1 \otimes \cdots \otimes u_r$ and

$$u_j = \begin{pmatrix} 1 & 0 \\ 0 & \omega_k \end{pmatrix}.$$

Therefore $\sigma(u)$ is precisely the 2^rth roots of unitary, and u is unitarily equivalent to the cyclic shift v on M_{2^r}. Let $p_j = \chi_{\{\omega_j\}}(v)$, which are projections in M_{2^r} which satisfy $\sum_0^{2^r-1} p_j = 1$ and $\|\sigma(p_j) - p_{j+1}\| < \varepsilon$. $\square$

3.2 The Cuntz Algebras $\mathcal{O}_n$ and $\mathcal{O}_\infty$

Here we will define the Cuntz algebras, show that they are algebraically simple, and that they are nuclear. These initial results can be found in chapter V.4 of [11] and in Cuntz' original paper [9]. These algebras will later be seen to be unital, simple, purely infinite, so they will have a very nice structure and K-theory.

Definition 3.2.1. The **Cuntz algebra** $\mathcal{O}_n$ is the universal C*-algebra generated by n-isometries $(s_i)_1^n$ satisfying $\sum s_i s_i^* = I$. Then **Cuntz algebra** $\mathcal{O}_\infty$ is the universal C*-algebra generated by an infinite collection of isometries $(s_i)_1^\infty$ satisfying $\sum_1^n s_i s_i^* \le I$ for all $n \in \mathbb{N}$.

We mean that $\mathcal{O}_n$ is universal in the following sense. If $t_1, \dots, t_n$ are any other collection of isometries satisfying the Cuntz relation, then there is a (unique) *-homomorphism $\rho : \mathcal{O}_n \to C^*(t_1, \dots, t_n)$ such that $\rho(s_i) = t_i$.

We can construct this algebra as follows. Let (π_α) be a maximal collection of irreducible representations of the Cuntz relation, which are necessarily on a separable Hilbert space. Then form $\pi = \oplus\pi_\alpha$, so $\mathcal{O}_n = C^*(\pi(s_i) \mid 1 \le i \le n)$ has the desired property.

Remark 3.2.2. Since s_i's are isometries, $s_i s_i^*$ is a projection. Moreover $\sum s_i s_i^* = I$ implies that $(s_i s_i^*)$ are mutually orthogonal and so $s_i^* s_j = 0$ for $i \ne j$. Thus $s_i^* s_j = \delta_{ij} I$. This holds for $n \in \mathbb{N}$, or $n = \infty$.

Definition 3.2.3. For a word $\mu = (i_1, \dots, i_m) \in \{1, \dots, n\}^m$ (or $\in \mathbb{N}^m$ for $\mathcal{O}_\infty$), define

$$s_\mu = s_{i_1} \cdots s_{i_m},$$

and let $|\mu|$ denote the length of the word μ.

Lemma 3.2.4. Let μ, ν be words in $\{1, \dots, n\}$ (or $\mathbb{N}$) such that $s_\mu^* s_\nu \neq 0$. Then

1. If $|\mu| = |\nu|$, then $\mu = \nu$ and $s_\mu^* s_\nu = I$.
2. If $|\mu| > |\nu|$, then there exists a word μ' such that $\mu = \nu\mu'$ and $s_\mu^* s_\nu = s_{\mu'}^*$.
3. If $|\mu| < |\nu|$, then there exists a word ν' such that $\nu = \mu\nu'$ and $s_\mu^* s_\nu = s_{\nu'}$.

Therefore any non-zero word in $(s_i) \cup (s_i^*)$ has a unique reduced expression of the form $s_\mu s_\nu^*$. Moreover any element in the *-algebra generated by these elements can be written as a linear combination of elements of the form $s_\mu s_\nu^*$.

Definition 3.2.5. For $n \geq 2$ (including ∞), $k \in \mathbb{N}$, let

$$\mathcal{F}_k^n = \operatorname{span}\{s_\mu s_\nu^* \mid |\mu| = |\nu| = k, \mu, \nu \text{ are words in } \{1, \dots, n\}\},$$

and let

$$\mathcal{F}^n = \overline{\cup_k \mathcal{F}_k^n}.$$

When $n = \infty$, we let

$$\mathcal{F}_k^\infty = \overline{\operatorname{span}}\{s_\mu s_\nu^* \mid |\mu| = |\nu| = k, \mu, \nu \text{ are words in } \{1, 2, \dots\}\}.$$

Lemma 3.2.6. For $n \geq 2$, $\mathcal{F}_k^n \simeq M_{n^k}$ and $\mathcal{F}^n \simeq M_{n^\infty}$, the UHF algebra with supernatural number n^∞. Moreover, $\mathcal{F}_k^\infty \simeq \mathbb{K}$, the compacts, and $\mathcal{F}^\infty$ is an AF algebra.

Proof. $\mathcal{F}_k^n$ is spanned by $\{s_\mu s_\nu^* \mid |\mu| = |\nu| = k, \mu, \nu \text{ are words in } \{1, \dots, n\}\}$, which is a set of matrix units for M_{n^k} since

$$(s_\mu s_\nu^*)(s_{\mu'} s_{\nu'}^*) = s_\mu (s_\nu^* s_{mu'}) s_{\nu'}^* = \delta_{\nu\mu'} s_\mu s_{\nu'}^*.$$

If μ, ν are words with $|\mu| = |\nu| = k$, then

$$s_\mu s_\nu^* = s_\mu \left(\sum_1^n s_i^* s_i \right) s_\nu^* = \sum_1^n s_{\mu i} s_{\nu i}^*.$$

Thus the embedding $\mathcal{F}_k^n \hookrightarrow \mathcal{F}_{k+1}^n$ is unital and behaves as desired, so that $\mathcal{F}^n$ is the UHF algebra M_{n^∞}.

Seeing $\mathcal{F}_k^\infty \simeq \mathbb{K}$ is easy - we get all the matrix units in $\mathcal{B}(\oplus_1^k \ell^2(\mathbb{N}))$, and the norm closed span of this is just $\mathbb{K}$. Since $\mathcal{F}_k^n \subseteq \mathcal{F}_{k+1}^n \subseteq \mathcal{F}_{k+1}^{n+1}$ for all n, k, it follows that

$$\mathcal{F}^\infty = \overline{\cup_n \mathcal{F}_n^n},$$

which is AF. □

Theorem 3.2.7. For $n \geq 2$ (or ∞), there exists a faithful conditional expectation $\Phi : \mathcal{O}_n \to \mathcal{F}^n$.

Proof. For $\lambda \in \mathbb{T}$, notice that (λs_i) are a set of isometries satisfying the Cuntz relation. Therefore there exists a *-automorphism ρ_λ such that $\rho_\lambda(s_i) = \lambda s_i$. Hence $\rho_\lambda(s_i^*) = \lambda^{-1} s_i$, and $\rho_\lambda(s_\mu s_\nu^*) = \lambda^{|\mu|-|\nu|} s_\mu s_\nu^*$. Then the map $\lambda \mapsto \rho_\lambda(t)$ is continuous for all t in the *-algebra generated by (s_i). Since this *-algebra is dense in $\mathcal{O}_n$ and $\|\rho_\lambda\| = 1$ for all $\lambda \in \mathbb{T}$, the map $\mathbb{T} \to \mathcal{O}_n$ defined by $\lambda \mapsto \rho_\lambda(t)$ is continuous for all $t \in \mathcal{O}_n$. Define

$$\Phi_n(t) = \int_{\mathbb{T}} \rho_\lambda(t) d\lambda.$$

Then for all words μ, ν,

$$\Phi_n(s_\mu s_\nu^*) = \int_{\mathbb{T}} \lambda^{|\mu|-|\nu|} s_\mu s_\nu^* d\lambda = \begin{cases} 0 & \text{if } |\mu| \neq |\nu| \\ s_\mu s_\nu^* & \text{if } |\mu| = |\nu| \end{cases}.$$

So Φ_n actually maps to $\mathcal{F}^n$, and if $t \in \mathcal{F}_k^n$, then $\Phi_n(t) = t$, so $\Phi_n|_{\mathcal{F}^n} = \mathrm{id}$. Now since Φ_n is a contractive projection onto $\mathcal{F}^n$, it is a conditional expectation by Tomiyama (1.1.13). Now if t is positive and non-zero, then $\rho_\lambda(t)$ is positive and non-zero, hence $\Phi_n(t)$ is positive and non-zero, so Φ_n is faithful. □

Lemma 3.2.8. Let $n \in \mathbb{N}$ (or $n = \infty$). Let μ, ν be words such that $|\mu| \neq |\nu|$. Let $m \geq \max\{|\mu|, |\nu|\}$ and let $s_\gamma = s_1^m s_2$. Then $s_\gamma^*(s_\mu^* s_\nu)s_\gamma = 0$.

Proof. Since $|\mu| \neq |\nu|$, 3.2.4 gives us that if $s_\mu^* s_\nu \neq 0$, then either $s_\mu^* s_\nu = s_{\mu'}^*$, where $1 \leq |\mu'| \leq m$, or $s_\mu^* s_\mu = s_{\nu'}$ where $1 \leq |\nu'| \leq m$.

In the first case, $(s_\mu^* s_\nu)s_\gamma = s_{mu'}^* s_\gamma \neq 0$ if and only if $s_{\mu'} = s_1^{|\mu'|}$ and $|\mu'| \leq m$. However, if $s_{mu'} = s_1^{|\mu'|}$, then

$$s_\gamma^*(s_\mu^* s_\nu)s_\gamma = s_\gamma^*(s_1^*)^{|\mu'|} s_\gamma = s_2^*(s_1^*)^m s_1^{m-|\mu'|} s_2 = 0$$

since $s_1^* s_2 = 0$.

In the second case, $s_\gamma^*(s_\mu^* s_\nu) = s_\gamma^* s_{\nu'} \neq 0$ if and only if $s_{\nu'} = s_1^{|\nu'|}$ as $|\nu'| \leq m$. However, if $s_{nu'} = s_1^{|\nu'|}$, then

$$s_\gamma^*(s_\mu^* s_\nu)s_\gamma = s_\gamma^*(s_1)^{|\nu'|} s_\gamma = s_2^*(s_1^*)^{m-|\nu'|} s_1^m s_2 = 0$$

since $s_2^* s_1 = 0$. □

Theorem 3.2.9. Let $n \geq 2$. For each $m \in \mathbb{N}$, there exists an isometry $w_{nm} \in \mathcal{O}_n \cap (\mathcal{F}_m^n)'$ such that $\Phi_n(t) = w_{nm}^* t w_{nm} \in \mathcal{F}_m^n$ for all $t \in \operatorname{span}\{s_\mu s_\nu^* \mid |\mu|, |\nu| \leq m\}$.

Proof. Let $s_\gamma = s_1^m s_2$ and let $w_{nm} = \sum_{|\delta|=m} s_\delta s_\gamma s_\delta^*$. Then

$$w_{nm}^* w_{nm} = \sum_{|\epsilon|=|\delta|=m} s_\epsilon s_\gamma^* s_\epsilon^* s_\delta s_\gamma s_\delta^* = \sum_{|\delta|=m} s_\delta s_\gamma^* s_\gamma s_\delta^* = \sum_{|\delta|=m} s_\delta s_\delta^* = I,$$

so w_{nm} is an isometry.

Now if $|\mu| = m$, then

$$w_{nm}s_\mu = s_\mu s_\gamma \text{ and } s_\mu^* w_{nm} = s_\gamma s_\mu.$$

If $s_\mu s_\nu^*$ is a matrix unit for $\mathcal{F}_m^n$ (so $|\mu| = |\nu| = m$), then

$$w_{nm}s_\mu s_\nu^* = s_\mu s_\gamma s_\nu^* = s_\mu s_\nu^* w_{nm}.$$

So since w_{nm} commutes with the matrix units, it commutes with $\mathcal{F}_m^n$. Moreover since w_{nm} is an isometry, we have

$$w_{nm}^* s_\mu s_\nu w_{nm} = s_\mu s_\nu = \Phi_n(s_\mu s_\nu^*).$$

Now if $|\mu|, |\nu| \leq m$ with $|\mu| \neq |\nu|$, then

$$w_{nm}^* s_\mu s_\nu^* w_{nm} = \sum_{|\epsilon|=|\delta|=m} s_\delta s_\gamma^* s_\delta^* s_\mu s_\nu^* s_\epsilon s_\gamma s_\epsilon^* = 0 = \Phi_n(s_\mu s_\nu^*),$$

since if $s_\delta^* s_\mu s_\nu^* s_\epsilon \neq 0$, it can be written as $s_{\mu'}^* s_\nu$ with $|\mu'| = m - |\mu| \neq m - |\nu| = |\nu'|$. Thus

$$s_\gamma^* s_\delta^* s_\mu s_\nu^* s_\epsilon s_\gamma = s_\gamma^* s_{\mu'}^* s_{\nu'} s_\gamma = 0$$

by the above lemma. □

The proof of the following theorem is similar to that of Theorem 3.2.9 above.

Theorem 3.2.10. Let $n \geq 2$. For each $m \in \mathbb{N}$, there exists an isometry $w'_{nm} \in \mathcal{O}_\infty$ such that $\Phi_\infty(t) = w'^*_{nm} t w'_{nm} \in \mathcal{F}_m^n \subseteq \mathcal{O}_\infty$ for all $t \in \operatorname{span}\{s_\mu s_\nu^* \mid |\mu|, |\nu| \leq m \text{ words in } \{1, \ldots, n\}\}$.

Theorem 3.2.11. Let $n \geq 2$. If $0 \neq x \in \mathcal{O}_n$, there exist $a, b \in \mathcal{O}_n$ such that $axb = I$.

Proof. Since $x \neq 0$, $x^*x \neq 0$, and so $\Phi_n(x^*x) \neq 0$ since Φ_n is faithful. By scaling if necessary, let us assume that $\|\Phi_n(x^*x)\| = 1$. By density, there exists y in the algebraic span of $s_\mu s_\nu^*$ such that $\|x^*x - y\| < \frac{1}{4}$. By considering the real part of Y, we may assume that $y = y^*$. Thus $\|\Phi_n(x^*x) - \Phi(y)\| \leq \frac{1}{4}$, so $\|\Phi_n(y)\| \geq \frac{3}{4}$.

Since y is in the algebraic span of $s_\mu s_\nu^*$, there exists $m \in \mathbb{N}$ such that y is a linear combination of elements of the form $s_\mu s_\nu^*$ for $|\mu|, |\nu| \leq m$. Therefore by Theorem 3.2.9, there exists an isometry w_{nm} such that $\Phi_n(y) = w_{nm}^* y w_{nm} \in \mathcal{F}_m^n$. Since $\|\Phi_n(y)\| \geq \frac{3}{4}$ and $\Phi_n(Y)$ is a self-adjoint element of a matrix algebra, there exists a rank one projection $p \in \mathcal{F}_m^n$ such that

$$p\Phi_n(y) = \Phi_n(y)p = \|\Phi_n(y)\|p \geq \frac{3}{4}p.$$

Moreover since p and $s_1^m(s_1^*)^m$ are both rank one projections in $\mathcal{F}_m^n$, there exists an isometry $u \in \mathcal{F}_m^n$ such that $upu^* = s_1^m(s_1^*)^m$. Now let

$$z = \frac{1}{\|\Phi_n(y)\|^{\frac{1}{2}}}(s_1^*)upw_{nm}^* \in \mathcal{O}_n.$$

Thus

$$\|z\| \leq \frac{1}{\|\Phi_n(y)\|^{\frac{1}{2}}}\|s_1^*\|^m\|u\|\|p\|\|w_{nm}^*\| \leq \frac{1}{\|\Phi_n(y)\|^{\frac{1}{2}}}\| = \frac{1}{\sqrt{\frac{4}{3}}} = \frac{2}{\sqrt{3}},$$

and

$$zyz^* = \frac{1}{\|\Phi_n(y)\|}(s_1^*)^m upw_{nm}yw_{nm}pu^*s_1^m = (s_1^*)^m upu^*s_1^m = (s_1^*)^m s_1^m (s_1^*)^m s_1^m = I.$$

Hence

$$\|I - zx^*xz^*\| = \|z(y - x^*x)z^*\| \le \|z\|^2\|y - x^*x\| \le \frac{4}{3}\frac{1}{4} = \frac{1}{3},$$

so that zx^*xz^* is a self-adjoint invertible operator. Now let $b = z^*(zx^*xz^*)^{-\frac{1}{2}}$, so

$$(b^*x^*)xb = (zx^*xz^*)^{-\frac{1}{2}}zx^*xz^*(zx^*xz^*)^{-\frac{1}{2}} = I.$$

□

In a similar manner, one can prove the following.

Theorem 3.2.12. If $0 \neq x \in \mathcal{O}_\infty$, then there exists $a, b \in \mathcal{O}_\infty$ such that $axb = I$.

Corollary 3.2.13. $\mathcal{O}_\infty$ and $\mathcal{O}_n$ are simple for all $n \geq 2$.

Corollary 3.2.14. $\mathcal{O}_n$ is generated by any n isometries satisfying the Cuntz relation, and $\mathcal{O}_\infty$ is generated by any countable collection of isometries satisfying the (infinite) Cuntz relation.

Proposition 3.2.15. $\mathcal{O}_n \otimes \mathbb{K} \simeq (M_{n^\infty} \otimes \mathbb{K}) \rtimes \mathbb{Z}$, and $\mathcal{O}_n$ is isomorphic to a compression of $(M_{n^\infty} \otimes \mathbb{K}) \rtimes \mathbb{Z}$.

Proof. For each $j \in \mathbb{Z}$, let $\mathbb{A}_j = \otimes_{i=j}^{\infty} M_n \simeq M_{n^\infty}$, and consider the doubly infinite sequence

$$\cdots \longrightarrow \mathbb{A}_{j+1} \xrightarrow{\beta_{j+1}} \mathbb{A}_j \xrightarrow{\beta_j} \mathbb{A}_{j-1} \longrightarrow \cdots$$

where $\beta_j(x) = e_{11} \otimes x$. Since $\mathbb{K} = \varinjlim_k M_{n^k}$ under connecting maps $x \mapsto e_{11} \otimes x$, it follows that $\mathbb{A} = \varinjlim \mathbb{A}_j \simeq M_{n^\infty} \otimes \mathbb{K}$. Now let $\alpha_j : \mathbb{A}_j \to \mathbb{A}_{j+1}$ be the natural isomorphism which makes the following diagram commute:

$$\begin{tikzcd}
\mathbb{A}_j \arrow[rr, "\alpha_j"] & & \mathbb{A}_{j+1} \\
& M_{n^\infty}. \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

Then we can define an automorphism $\alpha : \mathbb{A} \to \mathbb{A}$ where $\alpha(\beta_{j,\infty}(x)) = \beta_{j+1,\infty}(\alpha_j(x))$, and extending by continuity. This α effectively shifts our sequence to the left. Indeed, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\beta_{j+2}"] & \mathbb{A}_{j+1} \arrow[r, "\beta_{j+1}"] & \mathbb{A}_j \arrow[r, "\beta_j"] & \mathbb{A}_{j-1} \arrow[r, "\beta_{j-1}"] & \cdots \arrow[r] & \mathbb{A} \\
\cdots \arrow[r, "\beta_{j+1}"] & \mathbb{A}_j \arrow[r, "\beta_j"] \arrow[u, "\alpha_j"] & \mathbb{A}_{j-1} \arrow[r, "\beta_{j-1}"] \arrow[u, "\alpha_{j-1}"] & \mathbb{A}_{j-2} \arrow[r, "\beta_{j-2}"] \arrow[u, "\alpha_{j-2}"] & \cdots \arrow[r] & \mathbb{A} \arrow[u, "\alpha"]
\end{tikzcd}$$

Now let $\mathbb{B} = \mathbb{A} \rtimes_\alpha \mathbb{Z}$ and let $u \in \mathbb{B}$ be a unitary which implements the action of α: $\alpha(x) = uxu^*$ for $x \in \mathbb{A}$. Then $\mathbb{B}$ is the closure of the algebra of operators of the form

$$a = \sum_{-N}^{N} t_i u^i,$$

for $t_i \in \mathbb{A}, N \in \mathbb{N}$. Let $\tilde{t}_i = u^{-i} t_i u^i \in \mathbb{A}$, so $\mathbb{B}$ is the closure of operators of the form

$$a = \sum_{i<0} u^i \tilde{t}_i + \sum_{i \geq 0} t_i u^i.$$

Let p_j be the unit of $\mathbb{A}_j$, which is a projection for all j. Notice that $up_j u^* = p_{j+1}$. Let $\mathbb{B}_0 = p_0 \mathbb{B} p_0$. This contains $\mathbb{A}_0 = p_0 \mathbb{A} p_0$, and $v = up_0$ since $up_0 = p_1 u = p_0 u p_0$. We claim that $\mathbb{B}_0$ is generated by $\mathbb{A}_0$ and v. Consider $p = p_0 \in \mathbb{A}_0$, and notice that for $i \geq 0$, $pu^i p = u^i p = v^i$ and so $pu^{-i} p = pu^{-i} = (v^*)^i$. Thus

$$pt_i u^i p = (pt_i p)(vp)^i \text{ for } i \geq 0,$$

$$pu^i \tilde{t}_i p = ((up)^*)^{-i} \tilde{t}_i p \text{ for } i < 0.$$

But then this gives

$$pap = \sum_{i<0} (v^*)^{-i} p \tilde{t}_i p + \sum_{i \geq 0} pt_i p v^i,$$

so that $\mathbb{B}_0$ is indeed generated by $\mathbb{A}_0$ and v.

Now identify $\mathbb{A}_0$ with $M_n(\mathbb{A}_1)$ and let (e_{ij}) be matrix units for $M_n(\mathbb{A}_1)$ such that $e_{11} = p_1$. Let $s_i = e_{i1} v$, so

$$s_i^* s_i = p_0 u^* e_{1i} e_{i1} u p_0 = p_0 u^* = e_{11} up = p_0 u^* p_1 u p_0 = p_0^3 = p_0,$$

giving that s_i is an isometry. Moreover,

$$s_i s_i^* = e_{i1} u p_0 p_0 u^* e_{1i} = e_{i1} p_1 e_{1i} = e_{i1} e_{11} e_{1i} = e_{ii},$$

hence (s_i) are n isometries satisfying the Cuntz relations, so $C^*(s_1, \ldots, s_n) \simeq \mathcal{O}_n$. We claim that $s_1, \ldots, s_n$ generate $\mathbb{B}_0$. It is sufficient to show that $\mathbb{A}_0$ is generated by $s_1, \ldots, s_n$. We can think of $\mathbb{A}_0$ as $M_{n^k}(\mathbb{A}_k)$ for all k. The matrices with scalar entries from $\mathbb{A}_k$ form a copy $\mathbb{M}_k$ of M_{n^k} in $\mathbb{A}_0$, and the union of these subalgebras is dense in the UHF algebra $\mathbb{A}_0$. Recall that elements of the form $s_\mu s_\nu^*$, where μ, ν are words in $\{1, \ldots, n\}$ of length k, generate of a copy $\mathbb{N}_k$ of M_{n^k}. To see that $\mathbb{M}_k = \mathbb{N}_k$, notice that the matrix units for $p_{k-1} \mathbb{M}_k p_{k-1}$ are just

$$\alpha^{k-1}(e_{ij}) = u^{k-1}(s_i s_j^*)(u^*)^{k-1} = (s_1^{k-1} s_i)(s_1^{k-1} s_j)^*, 1 \leq i, j \leq k.$$

The rest of the matrix units are obtained in a similar fashion. Hence $\mathbb{A}_0 \subseteq C^*(s_1, \ldots, s_n)$ and we have $\mathbb{B}_0 = p_0 \mathbb{B} p_0 \simeq \mathcal{O}_n$. Using the same method, we get that $\mathbb{B}_k = p_k \mathbb{B} p_k \simeq \mathcal{O}_n$. Now the embedding $\mathbb{B}_k \hookrightarrow \mathbb{B}_{k-1}$ is just given by $\beta_k(b) = e_{11} \otimes b$, and so

$$(M_{n^\infty} \otimes \mathbb{K}) \rtimes \mathbb{Z} \simeq \varinjlim \mathbb{B}_k \simeq \mathcal{O}_n \otimes \mathbb{K}.$$

□

Corollary 3.2.16. $\mathcal{O}_n$ is nuclear.

Proof. We saw that $\mathcal{O}_n \simeq p\mathbb{B}p = p(M_{n^\infty} \otimes \mathbb{K})p$, so it is a compression of a nuclear C*-algebra, hence nuclear. $\square$

Theorem 3.2.17. $\mathcal{O}_\infty$ is nuclear.

Proof. For each $j \in \mathbb{N} \cup \{0\}$, let $\mathbb{A}_j = s_1^j \mathcal{F}^\infty (s_1^*)^j \subseteq \mathcal{O}_\infty$. Then $\mathbb{A}_j \simeq \mathbb{A}_0 = \mathcal{F}^\infty$ for all $j \geq 0$ (via $t \mapsto (s_1^*)^j t (s_1)^j$). Moreover $\mathbb{A}_{j-1} \simeq \mathbb{C}I + (\mathbb{K} \otimes \mathbb{A}_j)$, where $\mathbb{C}I$ comes from $s_1^{j-1} I (s_1^*)^{j-1} \in \mathbb{A}_{j-1}$ and

$$s_1^{j-1}(s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{i_1})(s_1^*)^{j-1}$$

corresponds to

$$e_{i_1,j_1} \otimes (s_1^j (s_{i_2} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_2}^*)(s_1^*)^j) \in \mathbb{K} \otimes \mathbb{A}_j.$$

Now extend notation by letting $\mathbb{A}_{j-1} = \mathbb{C}I + (\mathbb{K} \otimes \mathbb{A}_j)$ for all $j \in \mathbb{Z}$. Consider the sequence of C*-algebras

$$\cdots \to \mathbb{A}_2 \to \mathbb{A}_1 \to \mathbb{A}_0 \to \mathbb{A}_{-1} \to \mathbb{A}_{-2} \to \cdots$$

where the inclusion $\mathbb{A}_j \to \mathbb{A}_{j-1}$ is given by $x \mapsto e_{11} \otimes x \in \mathbb{K} \otimes \mathbb{A}_j \subseteq \mathbb{A}_{j-1}$ (e_{ij} matrix units for $\mathbb{K}$). Let $\mathbb{A}$ be the direct limit of this chain. Since $\mathbb{A}_j$ is AF,then $\mathbb{A}$ is AF and so it is nuclear. Since $\mathbb{A}_j$ are all isomorphic, letting α be the automorphism of shifting to the left, the remainder of the proof follows as in the $\mathcal{O}_n$ case. Let $\mathbb{B} = \mathbb{A} \rtimes_\alpha \mathbb{Z}$ and continue as above. $\square$

Proposition 3.2.18. $M_2(\mathcal{O}_2) \simeq \mathcal{O}_2$.

Proof. Let $\mathcal{O}_2 = C^*(s_1, s_2)$ where s_1, s_2 are isometries satisfying the Cuntz relation. Letting

$$t_1 = \begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix}, t_2 = \begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix},$$

we have that t_1, t_2 are isometries satisfying the Cuntz relations. It is also clear that t_1, t_2 generate $M_2(\mathcal{O}_2)$, hence $M_2(\mathcal{O}_2) \simeq \mathcal{O}_2$. $\square$

Proposition 3.2.19. $M_n(\mathcal{O}_2) \simeq \mathcal{O}_2$ for all $n \in \mathbb{N}$.

Proof. The case for $n = 1, 2$ are both done. Suppose that $n = k+1$ for $k \geq 2$, $\mathcal{O}_2 = C^*(s_1, s_2)$. Let

$$t_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & s_1 & s_2 \end{pmatrix}, t_2 = \begin{pmatrix} s_1 & s_2 s_1 & s_2^2 s_1 & \cdots & s_2^{k-1} s_1 & s_2^k \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{k+1}(\mathcal{O}_2).$$

Clearly $t_1^* t_1 = 1$. Moreover, $t_1 t_1^* = \sum_{i=2}^{k+1} e_{ii}$ and $t_2 t_2^* = e_{11}$. Thus $t_1 t_1^* + t_2 t_2^* = 1$. So it suffices to show that $M_n(\mathcal{O}_2)$ is generated by t_1, t_2. Clearly $e_{11} \in C^*(t_1, t_2) =: A$. We have $e_{i+1,1} = t_1 e_{i1}$ for all $1 \leq i \leq k-1$, so we have $(e_{i1})_1^k \subseteq A$. Since this is a C*-algebra, we have that $(e_{i,j})_{i,j=1}^k \subseteq A$. Now for $1 \leq i \leq k$, $t_2 e_{i1} = s_2^{i-1} s_1 e_{11} \in A$, and so $s_1^*(s_2^*)^{i-1} e_{11} \in A$. We also have that $t_1 e_{k1} = s_1 e_{k+1,1} \in A$, so

$$(s_1 e_{k+1,1})(s_1^*(s_2^*)^{i-1} e_{11}) = s_1 s_1^* (s_2^*)^{i-1} e_{k+1,1} \in A \text{ for } 1 \leq i \leq k.$$

But

$$e_{k+1,k+1} = t_1 t_1^* - \sum_{i=2}^{k} e_{ii} \in A,$$

and so

$$e_{k+1,k+1} t_1 e_{k+1,k+1} = s_2 e_{k+1,k+1} \in A.$$

Thus

$$s_2^{i-1} s_1 s_1^* (s_2^*)^{i-1} e_{k+1,k+1} \in A \text{ for } 1 \leq i \leq k.$$

Now

$$\left(t_1 - \sum_{i=1}^{k-1} e_{i+1,i} \right) t_2^* = (s_1 s_1^* (s_2^*)^{k-1} + s_2 s_2^* (s_2^*)^{k-1}) e_{k+1,1} = (s_2^*)^{k-1} e_{k+1,1} \in A.$$

Therefore

$$(s_2^{k-1} e_{k+1,k+1})((s_2^*)^{k-1} e_{k+1,1}) = s_2^{k-1} (s_2^*)^{k-1} e_{k+1,1} \in A.$$

Now since

$$\begin{aligned}
s_2^{k-1}(s_2^*)^{k-1} + \sum_{i=1}^{k-1} s_2^{i-1} s_1 s_1^* (s_2^*)^{i-1} &= s_2^{k-1}(s_2^*)^{k-1} + \sum_{i=1}^{k-1} s_2^{i-1} s_1 s_1^* (s_2^*)^{i-1} \\
&= s_2^{k-1}(s_2^*)^{k-1} + s_2^{k-2} s_1 s_1^* (s_2^*)^{k-2} + \sum_{i=1}^{k-2} s_2^{i-1} s_1 s_1^* (s_2)^{i-1} \\
&= s_2^{k-2} s_2 s_2^* (s_2^*)^{k-2} + s_2^{k-2} s_1 s_1 (s_2^*)^{k-2} + \sum_{i=1}^{k-2} s_2^{i-1} s_1 s_1^* (s_2^*)^{i-1} \\
&= s_2^{k-2}(s_1 s_1^* + s_2 s_2^*)(s_2^*)^{k-2} + \sum_{i=1}^{k-2} s_2^{i-1} s_1 s_1^* (s_2^*)^{i-1} \\
&= s_2^{k-2}(s_2^*)^{k-2} + \sum_{i=1}^{k-2} s_2^{i-1} s_1 s_1^* (s_2^*)^{i-1} \\
&= \cdots \\
&= s_2 s_2^* + s_1 s_1^* = 1,
\end{aligned}$$

and

$$s_2^{i-1} s_1 s_1^* (s_2^*)^{i-1} e_{k+1,1} \in A,$$

we have that A contains all the matrix units. Finally since $s_1 e_{11} \in A$, and

$$e_{1,k+1}(s_2 e_{k+1,k+1}) e_{k+1,1} = s_2 e_{11} \in A,$$

$\mathcal{O}_n e_{11} \subseteq A$, so we have $M_n(\mathcal{O}_2) = A \simeq \mathcal{O}_2$. $\square$

4 Exact C*-Algebras

We finally examine exactness in this chapter. We start by proving the equivalence of the two main definitions: that A is exact if and only if it is nuclearly embeddable, and proceed to state some permanence properties: we will see that exactness is preserved under taking the min and max tensor products, subalgebras, crossed products by amenable groups, unitizations, and inductive limits. We will then look see an example of a non-nuclear exact C*-algebra, and a non-exact C*-algebra, proving that the class of exact C*-algebras is really broader class than that of nuclear C*-algebras. We will then give three other characterizations of exactness and prove that they are all equivalent in the separable setting: property C, property C', and being isomorphic to a subquotient of the CAR algebra. This will give us a proof that exact C*-algebras, in addition to the permanence properties above, will also be closed under taking quotients.

4.1 Nuclear Embeddability and ⊗-Exactness

As mentioned above, our goal here will be to explore the first two equivalent definitions of exactness. These results can found be in chapter 3 of [6] or chapter 7 of [32]. Throughout, if not explicitly stated, we will assume that A, B, C are C*-algebras, $I \lhd A, J \lhd B$ are closed two sided ideals.

Definition 4.1.1. A C*-algebra A is **exact** if there exists a nuclear faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$.

It seems as though this is a property specific to the faithful representation. However, being exact is independent of the faithful representation, as the following proposition shows.

Proposition 4.1.2. Every faithful (non-degenerate) representation of an exact C*-algebra is nuclear.

Proof. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a faithful nuclear representation and $\rho : A \to \mathcal{B}(\mathcal{K})$ be any faithful representation. Let $\phi_\lambda : A \to M_{k(\lambda)}, \psi'_\lambda : M_{k(\lambda)} \to \mathcal{B}(\mathcal{H})$ be c.c.p. maps such that $\psi_\lambda \circ \phi_\lambda \to \pi$ in point-norm. Define $\sigma : \pi(A) \to \mathcal{B}(\mathcal{K})$ by $\sigma(\pi(a)) = \rho(a)$, which is clearly a well-defined *-homomorphism (by the faithfulness of π), hence a c.c.p. map. Now by Arveson's extension Theorem (1.1.5), there exists a c.c.p. extension $\tilde{\sigma} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ of σ. Letting $\psi_\lambda = \tilde{\sigma} \circ \psi'_\lambda : M_{k(\lambda)} \to \mathcal{B}(\mathcal{K})$, these maps are clearly c.c.p. and for any $a \in A$,

$$\psi_\lambda \circ \phi_\lambda = \tilde{\sigma}(\psi'_\lambda(\phi_\lambda(a))) \to \tilde{\sigma}(\pi(a)) = \rho(a)$$

in norm. Thus ρ is nuclear as well. □

We will prove that the existence of a faithful nuclear representation for a C*-algebra A is equivalent to the property that whenever $0 \to J \to B \to B/J \to 0$ is a short exact sequence of C*-algebras, then $0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$ is short exact as well. Let us formally define this property.

Definition 4.1.3. We say that a C*-algebra A is **$\otimes$-exact** if for any short exact sequence $0 \to J \to B \to (B/J) \to 0$ of C*-algebras, we have that

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$

is exact.

The following observation always gives us the first inclusion, and it follows directly from the definition of the min-tensor product, along with the fact that it is independent of choice of faithful representations.

Lemma 4.1.4. Let $A \subseteq B, C$ be C*-algebras. Then there is an isometric inclusion $A \otimes C \subseteq B \otimes C$.

Throughout, every ideal will be norm closed. Moreover if we have a C*-algebra B and $J \lhd B$, it is clear that $0 \to J \to B \to B/J \to 0$ is exact.

Proposition 4.1.5. Let A, B be C*-algebras, $J \lhd B$. Then there is a C*-norm $\|\cdot\|_\alpha$ on $A \odot (B/J)$ such that

$$\frac{A \otimes B}{A \otimes J} \simeq A \otimes_\alpha (B/J).$$

It then follows that

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$

is exact if and only if $\|\cdot\|_\alpha = \|\cdot\|_{\min}$.

Proof. Let $\|\cdot\|_\alpha$ be the restriction of the quotient norm of $\frac{A\otimes B}{A\otimes J}$ to $A \odot (B/J) \simeq \frac{A\odot B}{A\odot J}$. Then the first condition is clear. Now the equivalence follows due to the uniqueness of a C*-norm on a C*-algebra, and the fact that this sequence is exact if and only if $\frac{A\otimes B}{A\otimes J} \simeq A\otimes(B/J)$. $\square$

Corollary 4.1.6. If there is a unique C*-norm on $A \odot (B/J)$, then

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$

is exact.

In particular, if A or B/J are nuclear C*-algebras, the above sequence is exact.

Lemma 4.1.7. If $J \lhd B$, $A \subseteq C$, then

$$(C \otimes J) \cap (A \otimes B) = A \otimes J.$$

Proof. Clearly $A \otimes J \subseteq (C \otimes J) \cap (A \otimes B)$. For the other inclusion, let $x \in (C \otimes J) \cap (A \otimes B)$. Let $(e_i) \subseteq J$ be an approximate unit. If C is unital, then $(1_C \otimes e_i) \subseteq C \otimes J$ is an approximate unit, hence $(1_C \otimes e_i)x \in A \otimes J$ since $x \in A \otimes B$. But $(1_C \otimes e_i)x \to x$ in norm, therefore $x \in A \otimes J$ since $A \otimes J$ is norm closed. The non-unital case follows from passing to the unitization and applying the same argument. $\square$

Proposition 4.1.8. If A is an exact C*-algebra, then

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$

is exact for every C*-algebra B and ideal $J \lhd B$.

Proof. Let $J \lhd B$ be an arbitrary ideal, and take our sequence to be

$$0 \longrightarrow A \otimes J \longrightarrow A \otimes B \xrightarrow{\rho} A \otimes (B/J) \longrightarrow 0$$

It is always true that $A \otimes J \subseteq \ker \rho$, so let us show the reverse. Let $x \in \ker \rho$, so we want to show that $x \in A \otimes J$. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a nuclear faithful representation and let $\phi_\lambda : A \to M_{k(\lambda)}, \psi_\lambda : M_{k(\lambda)} \to \mathcal{B}(\mathcal{H})$ be c.c.p. maps such that $\psi_\lambda \circ \phi_\lambda \to \pi$ in point-norm. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \otimes J \arrow[r] \arrow[d, "\phi_\lambda \otimes \mathrm{id}"] & A \otimes B \arrow[r, "\rho"] \arrow[d, "\phi_\lambda \otimes \mathrm{id}"] & A \otimes (B/J) \arrow[r] \arrow[d, "\phi_\lambda \otimes \mathrm{id}"] & 0 \\
0 \arrow[r] & M_{k(\lambda)} \otimes J \arrow[r] \arrow[d, "\psi_\lambda \otimes \mathrm{id}"] & M_{k(\lambda)} \otimes B \arrow[r, "\tilde{\rho}"] \arrow[d, "\psi_\lambda \otimes \mathrm{id}"] & M_{k(\lambda)} \otimes (B/J) \arrow[r] \arrow[d, "\psi_\lambda \otimes \mathrm{id}"] & 0 \\
0 \arrow[r] & \mathcal{B}(\mathcal{H}) \otimes J \arrow[r] & \mathcal{B}(\mathcal{H}) \otimes B \arrow[r] & \mathcal{B}(\mathcal{H}) \otimes (B/J) \arrow[r] & 0.
\end{tikzcd}$$

Notice that the middle row is exact since $M_{k(\lambda)}$ is nuclear. So if we start in the top row, middle column, and we know that $x \in \ker \rho$, we can follow the diagram to get that $\phi_\lambda \otimes \mathrm{id}(x) \in M_{k(\lambda)} \otimes B$. But this is in the kernel of $\tilde{\rho}$, so by the exactness of the middle row, $\phi_\lambda \otimes \mathrm{id}(x) \in M_{k(\lambda)} \otimes J$. But then $(\psi_\lambda \circ \phi_\lambda) \otimes \mathrm{id}(x) \in \mathcal{B}(\mathcal{H}) \otimes J$, and since $(\psi_\lambda \circ \phi_\lambda) \otimes \mathrm{id} \to \pi \otimes \mathrm{id}$ in point-norm, it follows that

$$\pi \otimes \mathrm{id}(x) \in (\mathcal{B}(\mathcal{H}) \otimes J) \cap (\pi(A) \otimes B)) = \pi(A) \otimes J$$

by the previous lemma. But $\pi \otimes \mathrm{id} : A \otimes J \to \pi(A) \otimes J = (\mathcal{B}(\mathcal{H}) \otimes J) \cap (\pi(A) \otimes B)$ is an isomorphism, so it follows that $x \in J \otimes B$. $\square$

So this gives us that every exact C*-algebra (nuclearly embeddable) is $\otimes$-exact. To see the other direction, it is useful to work with finite-dimensional operator systems, so we will need a notion of exactness for these. First, we will define what we mean by an operator system tensor product.

Definition 4.1.9. Let $S \subseteq \mathcal{B}(\mathcal{H}), T \subseteq \mathcal{B}(\mathcal{K})$ be (closed) operator systems. We will let

$$S \otimes T = \overline{\mathrm{span}\{x \otimes y \mid x \in S, y \in T\}}^{\mathcal{B}(\mathcal{H} \otimes \mathcal{K})}.$$

In general, one can consider abstract operator systems and there are many operator system tensor products, but we will not delve into it. One can see [16] for details.

Definition 4.1.10. Let $E \subseteq \mathcal{B}(\mathcal{H})$ be an operator system. We say that E is **$\otimes$-exact** if for every C*-algebra B and $J \lhd B$, we have

$$\frac{E \otimes B}{E \otimes J} \simeq E \otimes (B/J)$$

isometrically.

Remark 4.1.11. Note that we always have a contractive map

$$\frac{E \otimes B}{E \otimes J} \to E \otimes (B/J)$$

since the kernel of the contraction $E \otimes B \to E \otimes (B/J)$ contains $E \otimes J$. It is not always an isometry though. Indeed, in [25] Pisier defines a quantity $\mathrm{ex}(E)$, called the exactness constant, which in the finite-dimensional setting is equal to the norm of the inverse. It is shown that $ex(S') > 1$ if $S' = \mathrm{span}\{u_1, \cdots, u_n\} \subseteq C^*(\mathbb{F}_n)$, where u_j generate $\mathbb{F}_n$, for $n \geq 3$. Hence the operator system $S = \mathrm{span}\{1, u_1, u_1^*, \ldots, u_n, u_n^*\}$ is not exact. The notion of exactness presented here coincides with the notion of 1-exact in [17].

Lemma 4.1.12. Let $E \subseteq A$ be an operator system, $J \lhd B$ an ideal. Then there is an isometric inclusion

$$\frac{E \otimes B}{E \otimes J} \hookrightarrow \frac{A \otimes B}{A \otimes J}.$$

Proof. We must show that these quotient norms agree as Banach space norms. That is, for $x \in E \otimes B$, we want that

$$\inf_{y \in E \otimes J} \|x + y\| = \inf_{y \in A \otimes J} \|x + y\|.$$

But if (e_i) is an approximate unit for J, then $(1 \otimes e_i)$ is an approximate unit for $A \otimes J$, and so we know that the quotient norm is given by

$$\inf_{y \in A \otimes J} \|x + y\| = \lim_i \|x - x(1 \otimes e_i)\|.$$

But $x(1 \otimes e_i) \in E \otimes J$ since E is an operator system, so this infimum is actually achieved on $E \otimes J$. □

Proposition 4.1.13. A C*-algebra A is $\otimes$-exact if and only if all of its finite dimensional operator systems are $\otimes$-exact.

Proof. The union of $\frac{E \otimes B}{E \otimes J}$, where $E \subseteq A$ is a finite dimensional operator system, is dense in $\frac{A \otimes B}{A \otimes J}$. If all finite dimensional operator systems are exact, then $\frac{E \otimes B}{E \otimes J} \simeq E \otimes (B/J)$ for any finite dimensional $E \subseteq A$, and so for $x \in A \odot B$, we can take a finite dimensional operator system $E \subseteq A$ such that $x \in E \odot B$. Since we have our isometric inclusion by the above lemma, it follows that this quotient norm of x is equal to the min-norm, giving us that A is $\otimes$-exact.

On the other hand, we have the following diagram:

$$\begin{array}{ccc} \frac{A\otimes B}{A\otimes J} & \longrightarrow & A\otimes(B/J) \\ \Big\uparrow & & \Big\uparrow \\ \frac{E\otimes B}{E\otimes J} & \longrightarrow & E\otimes(B/J). \end{array}$$

If A is $\otimes$-exact, the top row is an isometric isomorphism, hence so is the bottom row. ☐

So $\otimes$-exactness is really a local property. As such, when dealing with $\otimes$-exactness, we will allow ourselves to assume that our C*-algebras are separable.

Definition 4.1.14. Let $(A_n)_n$ be C*-algebras. We let $\Pi_n A_n$ be the ℓ^∞ direct sum

$$\Pi_n A_n = \{(a_n)_n \in (A_n) \mid \sup_n \|a_n\| < \infty\}$$

and $\oplus_n A_n$ be the c_0 direct sum

$$\oplus_n A_n = \{(a_n)_n \in (A_n) \mid \lim_n \|a_n\| = 0\}.$$

Remark 4.1.15. The quotient norm on $\frac{\Pi_n A_n}{\oplus_n A_n}$ is $\|(a_n)_n + \oplus_n A_n\| = \limsup_n \|a_n\|$.

Lemma 4.1.16. Let $E \subseteq A$ be a finite-dimensional operator system, $(B_n)_n$ unital C*-algebras. Then there is a u.c.p. isometric isomorphism

$$E \otimes (\Pi_n B_n) \to \Pi_n(E \otimes B_n),$$

defined on elementary tensors by $e \otimes (b_n)_n \mapsto (e \otimes b_n)_n$. This map also gives the identification $E \otimes (\oplus_n B_n) \simeq \oplus_n(E \otimes B_n)$.

Proof. Let $B_n \subseteq \mathcal{B}(\mathcal{H}_n)$ and $A \subseteq \mathcal{B}(\mathcal{K})$ be faithful representations. Then we have a natural diagonal embedding $\Pi_n B_n \subseteq \mathcal{B}(\oplus_n \mathcal{H}_n)$, which induces an inclusion $A \otimes (\Pi_n B_n) \subseteq \mathcal{B}(\mathcal{K} \otimes (\oplus_n \mathcal{H}_n))$. We also have a natural diagonal embedding $\Pi_n(A \otimes B_n) \subseteq \Pi_n \mathcal{B}(\mathcal{K} \otimes \mathcal{H}_n) \subseteq \mathcal{B}(\oplus_n(\mathcal{K} \otimes \mathcal{H}_n))$. Then the canonical Hilbert space isomorphism $\mathcal{K} \otimes (\oplus_n \mathcal{H}_n) \to \oplus_n(\mathcal{K} \otimes \mathcal{H}_n)$ induces an isomorphism

$$\mathcal{B}(\mathcal{K} \otimes (\oplus_n \mathcal{H}_n)) \simeq \mathcal{B}(\oplus_n(\mathcal{K} \otimes \mathcal{H}_n)).$$

This map acts on $a \otimes (b_n)_n \in A \otimes (\Pi_n B_n) \subseteq \mathcal{B}(\mathcal{K} \otimes (\oplus_n \mathcal{H}_n))$ as follows:

$$a \otimes (b_n)_n \mapsto (a \otimes b_n)_n \in \Pi_n(A \otimes B_n) \subseteq \mathcal{B}(\oplus_n(\mathcal{K} \otimes \mathcal{H}_n)).$$

Thus we have a *-homomorphism $A \otimes (\Pi_n B_n) \to \Pi_n(A \otimes B_n)$, but it is not surjective unless A is finite-dimensional (for example: $\mathcal{K}(\mathcal{H}) \otimes \ell^\infty \to \Pi_n \mathcal{K}(\mathcal{H})$ is clearly not surjective).

So for our finite-dimensional operator system $E \subseteq A$, let $(x_1, \ldots, x_m) \subseteq E$ be a basis. Then for $(d_n)_n \in \Pi_n(E \otimes B_n)$, there is a unique representation

$$(d_n)_n = \left(\sum_j x_j \otimes b_{jn}\right) = \sum_j (x_j \otimes b_{jn}),$$

where the sequence (b_{jn}) is bounded for every j. But $(x_j \otimes b_{jn}) \in E \otimes (\Pi_n B_n)$, so this map is clearly surjective, hence we have our u.c.p. isometric isomorphism. ☐

Consequently, the above identification gives the following Lemma.

Lemma 4.1.17. If E is a finite-dimensional operator system with algebraic basis $(x_1, \dots, x_m)$ and $(B_n)_n$ are unital C*-algebras, then there is a contractive linear map

$$\frac{\Pi_n(E \otimes B_n)}{\oplus_n(E \otimes B_n)} \to E \otimes \left(\frac{\Pi_n B_n}{\oplus_n B_n}\right)$$

such that $(\sum_j x_j \otimes b_{jn}) \mapsto \sum_j x_j \otimes ((b_{jn}) + \oplus B_n)$. Moreover, if E is $\otimes$-exact, then this map is an isometric isomorphism.

Proposition 4.1.18. Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a unital, separable, $\otimes$-exact C*-algebra, represented faithfully on a separable Hilbert space $\mathcal{H}$. Let $(P_n)_n$ be a sequence of increasing finite-rank projections converging to $I_{\mathcal{H}}$ in SOT, and let $E \subseteq A$ be a finite-dimensional operator system. If $\phi_n : E \to M_n \simeq P_n\mathcal{B}(\mathcal{H})P_n$ are compressions defined by $\phi_n(x) = P_n x P_n$, then

$$\|\phi_n^{-1}|_{\phi_n(E)}\|_{cb} \to 1.$$

Note that the ϕ_n's are eventually injective due to finite-dimensionality.

Proof. Since $P_n \leq P_{n+1}$, we have that $\phi_n = \phi_n \circ \phi_{n+1} = \phi_{n+1}\phi_n$. Let $V_n = \phi_n^{-1}|_{\phi_n(E)}$, and note that $1 \leq \|V_{n+1}\|_{cb} \leq \|V_n\|_{cb}$. This is because if $\theta_n : \phi_{n+1}(E) \to \phi_n(E)$ is defined by $\theta_n(\phi_{n+1}(x)) = \phi_n(x)$, then $\|\theta_n\|_{cb} \leq 1$, and

$$\|V_{n+1}\|_{cb} \leq \|V_n\theta_n\|_{cb} \leq \|V_n\|_{cb}\|\theta_n\|_{cb} \leq \|V_n\|_{cb}.$$

Let us assume that $\mathrm{rank} P_n = n$ for all n, so that $P_n\mathcal{B}(\mathcal{H})P_n \simeq M_n$. For contradiction, suppose that $\lim_n \|V_n\|_{cb} = \beta > 1$. In particular, we must have that $1 < \|V_n\|_{cb}$ for all n large enough so that inverses exist, and so there exists a sequence $(k(n))$ of natural numbers and $(X_n)_n \in \Pi_n(E \otimes M_{k(n)})$ such that $\|X_n\| = 1$ for all n, and

$$\lim_n \|\phi_n \otimes \mathrm{id}_{k(n)}(X_n)\| = \beta^{-1} < 1.$$

Let

$$X \in \frac{\Pi_n(E \otimes M_{k(n)})}{\oplus_n(E \otimes M_{k(n)})}$$

be the image of $(X_n)_n \in \Pi_n(E \otimes M_{k(n)})$. Now let $\tilde{X}$ be the image of X under the contractive linear map

$$\frac{\Pi_n(E \otimes M_{k(n)})}{\oplus_n(E \otimes M_{k(n)})} \to E \otimes \left(\frac{\Pi_n M_{k(n)}}{\oplus_n M_{k(n)}}\right)$$

from the previous lemma. Then

$$\|\tilde{X}\| = \sup_s \|\phi_s \otimes \mathrm{id}(\tilde{X})\|_{M_s \otimes \left(\frac{\Pi_n M_{k(n)}}{\oplus_n M_{k(n)}}\right)}.$$

Since M_s is $\otimes$-exact, it follows that we can invert the isomorphism in the above lemma to get an isomorphism

$$M_s \otimes \left(\frac{\Pi_n M_{k(n)}}{\oplus_n M_{k(n)}}\right) \to \frac{\Pi_n(M_s \otimes M_{k(n)})}{\oplus_n(M_s \otimes M_{k(n)})}.$$

Now the image of $\tilde{X}$ under the composition of maps

$$E \otimes \left(\frac{\Pi_n M_{k(n)}}{\oplus_n M_{k(n)}}\right) \to M_s \otimes \left(\frac{\Pi_n M_{k(n)}}{\oplus_n M_{k(n)}}\right) \to \frac{\Pi_n (M_s \otimes M_{k(n)})}{\oplus_n (M_s \otimes M_{k(n)})}$$

has representation $(\phi_s \otimes \mathrm{id}_{k(n)}(X_n)) \in \Pi_n(M_s \otimes M_{k(n)})$. But then

$$\begin{aligned}\|\tilde{X}\| &= \sup_s \|\phi_s \otimes \mathrm{id}(\tilde{X})\| \\ &\le \sup_s \|(\phi_s \otimes \mathrm{id}_{k(n)}(X_n)) + \oplus_n(M_s \otimes M_{k(n)})\| \\ &= \sup_s \left(\limsup_n \|\phi_s \otimes \mathrm{id}_{k(n)}(X_n)\|\right).\end{aligned}$$

But since $\phi_s = \phi_s \circ \phi_n$ for all $n > s$, it follows that

$$\sup_s \left(\limsup_n \|\phi_s \otimes \mathrm{id}_{k(n)}(X_n)\|\right) \le \limsup_n \|\phi_n \otimes \mathrm{id}_{k(n)}(X_n)\|.$$

Thus

$$\|X\| > \beta^{-1} = \limsup_n \|\phi_n \otimes \mathrm{id}_{k(n)}(X_n)\| \ge \|\tilde{X}\|,$$

which contradicts the above lemma since E is $\otimes$-exact as A is. □

We will need one more lemma to prove that the two definitions of exactness are equivalent. We will state the lemma, and give the proof after the proof of the big theorem.

Lemma 4.1.19. With the assumptions of the above Proposition, there are u.c.p. maps $\psi_n : \phi_n(E) \to A$ such that $\|\psi_n - \phi_n^{-1}|_{\phi_n(E)}\| \to 0$.

Theorem 4.1.20 (Kirchberg)**.** Let A be a C*-algebra. The following are equivalent:

1. A is exact. That is, there exists a faithful *-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and c.c.p. maps $\phi_\lambda : A \to M_{k(\lambda)}, \psi_\lambda : M_{k(\lambda)} \to \mathcal{B}(\mathcal{H})$ such that $\psi_\lambda \circ \phi_\lambda \to \pi$ in point-norm.

2. A is $\otimes$-exact. That is, for any C*-algebra B and ideal $J \lhd B$, the sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$

   is exact.

Proof. We have already seen that exact C*-algebras are $\otimes$-exact. Conversely, suppose that $A \subseteq \mathcal{B}(\mathcal{H})$, and take a sequence of increasing finite-rank projections $(P_n)_n$ which converge strongly to the identity. Suppose that rank$P_n = n$ for every n. Let $E \subseteq A$ be a finite-dimensional operator system. Then there are u.c.p. maps $\phi_n : A \to M_n \simeq P_n \mathcal{B}(\mathcal{H}) P_n$ given by $\phi_n(a) = P_n a P_n$, and by finite-dimensionailty $\phi_n|_E : E \to M_n$ will be injective for sufficiently large n. Then the inverse maps $\phi_n^{-1}|_{\phi_n(E)} : \phi_n(E) \to E$ satisfy $\|\phi_n^{-1}|_{\phi_n(E)}\|_{cb} \to 1$. By the above lemma there are $\psi_n : \phi_n(E) \to A$ such that $\|\psi_n - \phi_n^{-1}|_{\phi_n(E)}\|_{cb} \to 0$. By

Arveson's extension Theorem (1.1.5), we can extend the ψ_n's to maps $\psi_n : M_n \to \mathcal{B}(\mathcal{H})$. But for every $x \in E$, we have

$$\|\psi_n(\phi_n(x)) - x\| = \|\psi_n(\phi_n(x)) - \phi_n^{-1}|_{\phi_n(E)}(x)\| \to 0.$$

Now for any finite $\mathcal{F} \subseteq A$ and $\varepsilon > 0$, we can find a finite-dimensional operator system E and maps $\phi : E \to M_n, \psi : M_n \to \mathcal{B}(\mathcal{H})$ such that $\|\psi \circ \phi(x) - x\| < \varepsilon$ for all $x \in E$. Therefore A is exact. □

So all that is left to do is some housekeeping for Lemma 4.1.19.

Lemma 4.1.21. Let $E \subseteq A$ be an operator system, $\phi : E \to \mathcal{B}(\mathcal{H})$ be a unital self-adjoint map. Then there exists u.c.p. $\psi : E \to \mathcal{B}(\mathcal{H})$ with $\|\phi - \psi\|_{cb} \leq 2(\|\phi\|_{cb} - 1)$.

Proof. By Theorem 1.1.7, there exists a *-representation $\pi : A \to \mathcal{B}(\mathcal{K})$ and isometries $V, W : \mathcal{H} \to \mathcal{K}$ such that

$$\phi(a) = \lambda V^*\pi(a)W = \lambda W^*\pi(a)V,$$

where $\lambda = \|\phi\|_{cb}$ and $a \in E$. Let $\psi : E \to \mathcal{B}(\mathcal{H})$ be the u.c.p. map defined by

$$\psi(a) = \frac{1}{2}(V^*\pi(a)V + W^*\pi(a)W).$$

Then

$$\lambda\psi(a) - \phi(a) = \frac{1}{2}\lambda(V - W)^*\pi(a)(V - W).$$

Since ϕ is unital, $\lambda V^*W = 1$, and

$$\begin{aligned}\|\phi - \psi\|_{cb} &\leq \|\phi - \lambda\psi\|_{cb} + \|\lambda\psi - \psi\|_{cb} \\ &\leq \frac{1}{2}\lambda\|(V - W)^*(V - W)\| + (\lambda - 1) \\ &= 2(\lambda - 1),\end{aligned}$$

as required. □

Lemma 4.1.22. Let E be a finite-dimensional operator system. Then there exists a basis $(x_i)_1^n$ where $x_i^* = x_i$ with $\|x_i\| = \|\hat{x}_i\| = 1$, where $(\hat{x}_i)$ consists of the dual basis.

Proof. Let (z_i) be a basis for E and consider the map $\Phi : E^n \to \mathbb{C}$ defined by

$$\Phi(y_1, \ldots, y_n) = \det(\hat{z}_i(y_j)).$$

This is clearly a multilinear function. Now let B_{sa} be the set of self-adjoint elements of the closed unit ball of E. Since B_{sa}^n is compact, there exists $(x_1, \ldots, x_n) \in B_{sa}^n$ at which Φ attains its maximum absolute value on B_{sa}^n. Then its clear that $(x_j)_1^n$ is a basis (the determinant doesn't evaluate to 0), and we have that

$$\hat{x}_i(y) = \frac{\Phi(x_1 \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)}{\Phi(x_1, \ldots, x_n)}.$$

The functionals $\hat{x}_i$ are self-adjoint , and so $\|\Phi(x_1, \ldots, x_n)\|$ being maximal gives us that $\|\hat{x}_i\| = 1$. □

Corollary 4.1.23. Let E be a finite-dimensional operator system, B a unital C*-algebra, and $\phi : E \to B$ a unital self-adjoint map. Then there exists a u.c.p. $\psi : E \to B$ with $\|\phi - \psi\|_{cb} \leq 2\dim(E)(\|\phi\|_{cb} - 1)$.

Proof. Let $B \subseteq \mathcal{B}(\mathcal{H})$, then Lemma 4.1.21 gives a u.c.p. $\psi' : E \to \mathcal{B}(\mathcal{H})$ with $\|\phi - \psi'\|_{cb} \leq 2(\|\phi\|_{cb} - 1)$. Let $\sigma = \psi' - \phi$, $n = \dim(E)$. If we can show that there is a positive linear functional $\theta \in E^*$ with $\|\theta\| \leq n\|\sigma\|$ and $\theta - \sigma$ is c.p., then $\psi = \phi + \theta$ gives our u.c.p. map. To see that ψ is c.p., note that

$$\psi = \phi + \theta = (\theta - \sigma) + \phi + \sigma = (\theta - \sigma) + \phi + \psi' - \phi = (\theta - \sigma) + \psi',$$

which is a sum of c.p. maps and thus is unital, $\theta(1) = 1$ and σ is unital, as is ψ'. Moreover it will clearly satisfy the necessary norm condition. To this end, let $(x_i)_1^n$ be a basis for E as in the above lemma, and let

$$\theta = \|\sigma\| \sum_1^n |\hat{x}_i|,$$

where for a self-adjoint functional f, we have $|f| = f_+ + f_-$, the sum of the positive and negative parts. Now for $a \geq 0$,

$$\sigma(a) = \sum_1^n \hat{x}_i(a)\sigma(x_i) \leq \sum_1^n |\hat{x}_i|(a)\|\sigma(x_i)\| \leq \|\sigma\| \sum_1^n |\hat{x}_i|(a) = \theta(a),$$

so that $\theta - \sigma$ is positive. Complete positivity follows similarly. $\square$

Now Lemma 4.1.19 follows as a special case of this the above corollary. Finally, we will end off by stating some permanence properties.

Lemma 4.1.24. Let $(A_\lambda)_\lambda, A$ be C*-algebras such that $A \subseteq \mathcal{B}(\mathcal{H})$ and there exist c.c.p. maps $\phi_\lambda : A \to A_\lambda, \psi_\lambda : A_\lambda \to \mathcal{B}(\mathcal{H})$ with $\psi_\lambda \circ \phi_\lambda \to \mathrm{id}_A$ in point-norm. If each A_λ is exact, A is.

Proof. Let $F \subseteq A$ be finite, $\varepsilon > 0$. By point-norm convergence, there exists some $B = A_{\lambda_0}$ such that $\phi = \phi_{\lambda_0}, \psi = \psi_{\lambda_0}$ are c.c.p. and satisfy $\|\psi \circ \phi(a) - a\| < \frac{\varepsilon}{2}$ for all $a \in F$. Since B is exact, there exists c.c.p. $\alpha : B \to M_n = M_{k(\lambda_0)}, \beta : M_n \to \mathcal{B}(\mathcal{H}_{\lambda_0})$, where $B \subseteq \mathcal{B}(\mathcal{H}_{\lambda_0})$, such that $\|\beta \circ \alpha \circ \phi(a) - \phi(a)\| < \frac{\varepsilon}{2}$. By Arveson's extension Theorem 1.1.5, extend $\psi : B \to A \subseteq \mathcal{B}(\mathcal{H})$ to a c.c.p. map $\Psi : \mathcal{B}(\mathcal{H}_{\lambda_0}) \to \mathcal{B}(\mathcal{H})$. But then

$$\begin{aligned} \|\Psi \circ \beta \circ \alpha \circ \phi(a) - a\| &\leq \|\Psi \circ \beta \circ \alpha \circ \phi(a) - \psi \circ \phi(a)\| + \|\psi \circ \phi(a) - a\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Clearly $\alpha \circ \phi : A \to M_n, \Psi \circ \beta : M_n \to \mathcal{B}(\mathcal{H})$ are c.c.p. maps, so the result follows. $\square$

Theorem 4.1.25 (Permanence)**.** Let A, B be C*-algebras.

1. If $A \subseteq B$ and B is exact, then A is exact.

2. If A is exact, then so is the unitization, $\tilde{A}$.

3. $A \otimes B$ is exact if and only if A and B are exact.

4. Inductive limits of exact C*-algebras are exact.

5. If $\alpha : G \to \mathrm{Aut}(A)$ is an action from an amenable group G to a C*-algebra A, then $A \rtimes_\alpha G$ is exact if and only if A is exact.

Proof. The first one is clear. (2) follows from the remarks following definition 2.1.2 about extending nuclear maps to the unitization.

For (3), if A, B are exact, one can take a nuclear embedding of both A, B, and form a net of tensors of c.c.p. maps which converges point-norm to the tensor of the nuclear embeddings. Conversely, $A \otimes B \subseteq A \otimes \tilde{B}$, of which we can see $A \simeq A \otimes 1_{\tilde{B}}$ as a subalgebra, hence A is exact by 1. B being exact follows in the same way.

(4) follows easily using nuclear embeddings.

For (5), Theorem 1.3.3. provides a faithful conditional expectation $E : A \rtimes_{r,\alpha} G = A \rtimes_\alpha G \to A$, from which it follows that A is exact if $A \rtimes_\alpha G$ is. Conversely, note that if B, C are C*-algebras, C is exact, and there exist c.c.p. maps $\phi_\lambda : B \to C, \psi_\lambda : C \to B$ such that $\psi_\lambda \circ \phi_\lambda \to \mathrm{id}$ in point-norm, then B is exact as well.

Lemma 1.3.4 provides c.c.p. maps $\phi_n : A \rtimes_\alpha G \to A \otimes M_{k(n)}$ and $\psi_n : A \otimes M_{k(n)} \to A \rtimes_\alpha G$ such that $\psi_n \circ \phi_n \to \mathrm{id}$ in point-norm, if we consider a Følner sequence $(F_n)_n$ in G, where $k(n) = |F_n|$. Since $M_{k(n)} \otimes A$ is exact if and only if A is, it follows that $A \rtimes_\alpha G$ is exact by the above lemma. □

It turns out that exact C*-algebras are also closed under taking quotients. This will be seen in chapter 4.4.

4.2 Examples

It is clear that every nuclear C*-algebra is exact, so we have a plethora of examples of exact C*-algebras already. Exactness is closed under taking subalgebras, crossed products by amenable groups, inductive limits, tensor products with other exact C*-algebras and, as we will see, quotients. Evidently this provides us with a rich supply of exact C*-algebras. We have yet to see an exact C*-algebra which is not nuclear, and a non-exact C*-algebra. As such, we will prove that $C_r^*(\mathbb{F}_2)$ is an exact, but not nuclear C*-algebra as $\mathbb{F}_2$ is not amenable, and that $C^*(\mathbb{F}_2)$ is a non-exact C*-algebra.

For the former, we follow Choi's paper [7], and the latter we look at chapter 3.7 of [6]. To show the first C*-algebra is exact, we will have a number of inclusions: $C_r^*(\mathbb{F}_2) \subseteq C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3) \subseteq \mathcal{O}_2$, and since exactness passes to subalgebras, the result will follow. For the second, we will show that $\mathbb{F}_2$ is residually finite, and that such groups G allow us to easily determine when $C^*(G)$ is exact.

Towards proving the exactness of $C_r^*(\mathbb{F}_2)$, let us start by showing $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3) \subseteq \mathcal{O}_2$.

Lemma 4.2.1. Let e be a projection of the form $e = 1 \oplus 0$. Then an operator v satisfies $v^2 = v^{-1} = v^*$ and $e + v^*ev + vev^* = 1$ if and only if

$$v = \begin{pmatrix} 0 & s_2^* \\ s_1 & s_2 s_1^* \end{pmatrix},$$

where s_1, s_2 are isometries satisfying $s_1 s_1^* + s_2 s_2^* = 1$.

Proof. If v has this matrix form with isometries s_1, s_2, then it is easy to see that $v^2 = v^{-1} = v^*$ and $e + v^*ev + vev^* = 1$. On the other hand, suppose that v satisfies the required relations and write

$$v = \begin{pmatrix} x & s_2^* \\ x_1 & y \end{pmatrix}.$$

Using our second relation, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e + v * ev + vev^* = \begin{pmatrix} 1 + x^*x + xx^* & * \\ * & s_1 s_1^* + s_2 s_2^* \end{pmatrix},$$

hence $s_1 s_1^* + s_2 s_2^* = 1$ and $x = 0$. Now since v is an order 3 unitary,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = v^*v = \begin{pmatrix} s_1^* s_1 & * \\ * & * \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = vv^* = \begin{pmatrix} s_2^* s_2 & * \\ y s_2 & * \end{pmatrix}$$

so that s_1, s_2 are isometries and $ys_2 = 0$. Finally since $v^* = v^2$, we get

$$\begin{pmatrix} 0 & s_1^* \\ s_2 & y^* \end{pmatrix} = v^* = v^2 = \begin{pmatrix} * & * \\ y s_1 & * \end{pmatrix},$$

so $s_2 = ys_1$. Thus $y = y(s_1 s_1^* + s_2 s_2^*) = y s_1 s_1^* = s_2 s_1^*$. So v has the required form. □

Theorem 4.2.2. Suppose that e is a projection and u, v are operators satisfying

1. $u = u^{-1} = u^*, e + u^*eu = 1$;
2. $v^2 = v^{-1} = v^*, e + v^*ev + vev^* = 1$.

Then $C^*(e, u, v) \simeq \mathcal{O}_2$.

Proof. Write $e = 1 \oplus 0$, $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, v = \begin{pmatrix} 0 & s_2^* \\ s_1 & s_2 s_1^* \end{pmatrix}$, where s_1, s_2 are isometries satisfying $s_1 s_1^* + s_2 s_2^* = 1$. Then we have $C^*(e, u, v) \simeq M_2(\mathcal{O}_2) \simeq \mathcal{O}_2$, where the last isomorphism comes from Proposition 3.2.18. □

Corollary 4.2.3. $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ is exact.

Proof. Using notation as above, it is clear that $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3) \simeq C^*(u, v) \subseteq C^*(e, u, v) \simeq \mathcal{O}_2$. □

Now it suffices to show that $\mathbb{F}_2$ is a subgroup of $\mathbb{Z}_2 * \mathbb{Z}_3$. To do this, we will need the ping-pong lemma, which is a tool used to prove that a given subgroup is in fact a free subgroup.

Lemma 4.2.4 (Ping-pong Lemma, [12], II.B)**.** Let G be a group acting on a set X, and $H_1, \cdots, H_n$ be non-trivial subgroups of size at least 2, where at least one has order greater than 2. Suppose that there exist pairwise disjoint non-empty subsets $X_1, \ldots, X_n \subseteq X$ such that for all $i \neq j$, and for any $h \in H_i \setminus \{e\}$, $h \cdot X_j \subseteq X_i$. Then $\langle H_1, \ldots, H_n \rangle \simeq H_1 * \cdots * H_n$, the free product of $H_1, \ldots, H_n$.

Corollary 4.2.5. $C_r^*(\mathbb{F}_2)$ is exact.

Proof. We just need to show that $\mathbb{F}_2 \leq \mathbb{Z}_2 * \mathbb{Z}_3$, in which case, both $C_r^*(\mathbb{F}_2)$ and $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ are exact but not nuclear C*-algebras as both of the groups will be non-amenable. Say $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = b^3 = 1 \rangle$. We claim that $\langle bab, ababa \rangle \simeq \mathbb{F}_2$. Indeed, consider $\mathbb{Z}_2 * \mathbb{Z}_3$ acting on itself by left concatenation. Let X_1 be the set of reduced words starting with b and X_2 be the set of reduced words starting with a. Then notice that $\langle ababa \rangle \cdot X_1 \subseteq X_2$ and $\langle bab \rangle \cdot X_2 \subseteq X_1$. Thus since $X_1 \cap X_2 = \emptyset$, the ping pong lemma implies that, $\langle bab, ababa \rangle \simeq \mathbb{Z} * \mathbb{Z} \simeq \mathbb{F}_2$. Thus $\mathbb{F}_2$ is a subgroup of $\mathbb{Z}_2 * \mathbb{Z}_3$. Consequently, since $C_r^*(\mathbb{F}_2) \subseteq C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$, and $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ is exact, $C_r^*(\mathbb{F}_2)$ is exact. $\square$

Remark 4.2.6. One can do something similar to show that $\mathbb{F}_2$ contains a copy of $\mathbb{F}_\infty$, and so all the free groups are exact. Indeed, one can check that if $\mathbb{F}_2 = \langle a, b \rangle$, then $(b^n a b^{-n})_n$ are free in $\mathbb{F}_2$.

Now we work towards the non-exactness of $C^*(\mathbb{F}_2)$.

Definition 4.2.7. A discrete group G is called **residually finite** if there exists subgroups $G \supseteq G_1 \supseteq G_2 \supseteq \cdots$ such that G_i is a finite index, normal subgroup of G and $\cap_n G_n = \{e\}$.

Remark 4.2.8. For a discrete group G, being residually finite is equivalent to the conditions that for every $g_1, \ldots, g_n \in G$, there is a homomorphism $\theta : G \to F$, where F is a finite group such that $\theta(g_1), \ldots, \theta(g_n) \in F$ are distinct. If G is finitely generated, this is equivalent to G being **maximally periodic**; that is G has a separating family of finite-dimensional representations. This can be seen in [2].

Lemma 4.2.9. If G is residually finite, then there exists a state μ on $C^*(G) \otimes C^*(G)$ such that for finite sums

$$x = \sum_s \alpha_s s, y = \sum_t \beta_t t \in C^*(G),$$

we have

$$\mu(x \otimes y) = \sum_s \alpha_s \beta_s.$$

Proof. For a group H, we can consider the product map

$$\lambda \times \rho : C^*(H) \odot C^*(H) \to \mathcal{B}(\ell^2(H))$$

induced by the commuting left and right regular representations. Taking finite sums $x = \sum_s \alpha_s s, y = \sum_t \beta_t t$, we have

$$\langle \lambda \times \rho(x \otimes y)\delta_e, \delta_e \rangle = \sum_s \alpha_s \beta_t.$$

If H is finite, then $C^*(H) \odot C^*(H) = C^*(H) \otimes C^*(H)$, and so this would be a state on the minimal tensor product satisfying the required formula.

Now since each G/G_n is finite, let μ_n be a state on $C^*(G/G_n) \otimes C^*(G/G_n)$ satisfying the formula above for G/G_n. We also have quotient maps $\pi_n : C^*(G) \to C^*(G/G_n)$ and hence tensor product *-homomorphisms

$$\pi_n \otimes \pi_n : C^*(G) \otimes C^*(G) \to C^*(G/G_n) \otimes C^*(G/G_n).$$

Since $\cap_n G_n = \{e\}$, by taking a cluster point μ of $\mu_n \circ (\pi_n \otimes \pi_n)$, it is clear that μ will satisfy the formula. $\square$

Proposition 4.2.10. If G is residually finite, then the product map

$$\lambda \times \rho : C^*(G) \odot C^*(G) \to \mathcal{B}(\ell^2(G)),$$

is min-continuous.

Proof. Let $\pi : C^*(G) \otimes C^*(G) \to \mathcal{B}(\mathcal{H})$ be the GNS representation of the state from the previous lemma. Uniqueness of GNS gives us that the algebraic representations

$$\pi|_{C^*(G)\odot C^*(G)} : C^*(G) \odot C^*(G) \to \mathcal{B}(\mathcal{H})$$

and

$$\lambda \times \rho : C^*(G) \odot C^*(G) \to \mathcal{B}(\ell^2(G))$$

are unitarily equivalent since $\delta_e \in \ell^2(G)$ is a cyclic vector for the algebra $\lambda \times \rho(C^*(G) \odot C^*(G))$ whose corresponding vector functional agrees with μ. Thus the $C^*(\lambda \times \rho(C^*(G) \odot C^*(G)))$ is a quotient of $C^*(G) \otimes C^*(G)$. $\square$

Proposition 4.2.11. Let G be residually finite. Then the following are equivalent.

1. G is amenable.
2. $C^*(G)$ is exact.
3. The sequence

$$0 \to J \otimes C^*(G) \to C^*(G) \otimes C^*(G) \to C_r^*(G) \otimes C^*(G) \to 0$$

   is exact, where J is the kernel of the quotient map $C^*(G) \to C_r^*(G)$.

Proof. Clearly (1) implies (2), and (2) implies (3) by definition, so suppose that (3) implies (1).

If the sequence is exact,

$$\frac{C^*(G) \otimes C^*(G)}{J \otimes C^*(G)} \simeq C_r^*(G) \otimes C^*(G).$$

Now since the map $\lambda \times \rho$ as above is min-continuous, it then extends to a *-homomorphism $\lambda \times_{\min} \rho$, which contains $J \otimes C^*(G)$ in the kernel. Thus it factors through the above quotient, giving us a *-homomorphism $\pi : C_r^*(G) \otimes C^*(G) \to \mathcal{B}(\ell^2(G))$ satisfying $\pi(x \otimes y) = x\rho(y)$ for $x \in C_r^*(G), y \in C^*(G)$. Now by The Trick (1.2.5), there exist a u.c.p. $\Phi : \mathcal{B}(\ell^2(G)) \to L(G) = \rho(\mathbb{C}G)'$, where ρ is the right regular representation, such that $\Phi(x) = x$ for all $x \in C_r^*(G)$. Let τ be the canonical vector state on $L(G)$, and let $\eta = \tau \circ \Phi$. Since Φ is u.c.p. and restricts to identity on $C_r^*(G)$, it follows $C_r^*(G)$ is in the multiplicative domain of Φ, hence

$$\eta(\lambda_s x \lambda_x^*) = \tau(\Phi(\lambda_s x \lambda_s^*)) = \tau(\lambda_s \Phi(x) \lambda_s^*) = \tau(\Phi(x)).$$

Now note that if $f \in \ell^\infty(G)$, then

$$\lambda_s f \lambda_s^* \delta_t = \lambda_s f \delta_{s^{-1}t} = \lambda_s f(s^{-1}t)\delta_{s^{-1}t} = f(s^{-1}t)\lambda_s \delta_{s^{-1}t} = f(s^{-1}t)\delta_t = (s \cdot f)\delta_t.$$

Thus $\lambda_s f \lambda_s^* = s \cdot f$, and

$$\eta(s \cdot f) = \nu(\lambda_s f \lambda_s^*) = \tau(\Phi(\lambda_s f \lambda_s^*)) = \tau(\lambda_s \Phi(f) \lambda_s^*) = \tau(\Phi(f)) = \eta(f).$$

So η is a left invariant mean on $\ell^\infty(G)$, so G is amenable. □

Proposition 4.2.12. Let $\mathbb{F}$ be a free group. Then $\mathbb{F}$ is residually finite.

Proof. Let $\mathbb{F}$ be generated by (g_λ) and let $h_1, \ldots, h_n \in \mathbb{F}$. There are $g_1, \ldots, g_m \in (g_\lambda)$ such that all h_j are in $\mathbb{F}' = \langle g_1, \ldots, g_m \rangle$. If $h = g_{i_1}^{n_1} \cdots g_{i_k}^{n_j} \in \mathbb{F}'$, where $i_l \neq i_{l+1}$ for $1 \leq l \leq k-1$, the length $|h|$ of h is $\sum_i |n_i|$. Let $m = \max_{1 \leq i \leq n} |h_i|$ and let $S = \{h \in \mathbb{F}' \mid |h| \leq m\}$. Then for each generator g_i of $\mathbb{F}'$, let $S_i = \{h \in S \mid g_i h \in S$. Then $S_i \subseteq S, g_i S_i \subseteq S\}$ and the map $h \mapsto g_i h$ takes s_i onto $g_i s_i$ injectively. Thus $|S_i| = |g_i S_i|, |S \setminus SS_i| = |S \setminus g_i S_i|$. Let $q_i : S \setminus S_i \to S \setminus g_i S_i$ be a bijection and define p_i by

$$p_i h = \begin{cases} g_i h, & h \in S_i \\ q_i h, & h \in S \setminus S_i. \end{cases}$$

Then we can define a homomorphism $\phi : \mathbb{F} \to \Sigma(S)$, where $\Sigma(S)$ is the permutation group on S as follows:

$$\phi(g_\lambda) = \begin{cases} p_i, & \text{if } g_\lambda = g_i, \text{ for some } 1 \leq i \leq m, \\ 1, & \text{otherwise.} \end{cases}$$

Then $\phi(h_i)$ is the permutation which sends e to h_i, and so $\phi(h_i) \neq \phi(h_j)$ for $i \neq j$. □

Corollary 4.2.13. $C^*(\mathbb{F}_n)$ is not exact for all $n \geq 2$.

4.3 Properties C and C′

Property C is a property of C*-algebras which has to do with the enveloping von Neumann algebra. Property C will be of particular interest since it will pass to quotients. Property C will be seen to imply property C', and this will be seen to be equivalent to exactness. In the next section, we will come full circle and see that exactness actually implies property C as well, giving us that exactness is preserved under quotients - a very deep result. The following can be found in chapter 9 of [6] and chapter 5 of [32].

Definition 4.3.1. If M, N, L are von Neumann algebras, a *-homomorphism $\pi : M \odot N \to L$ is said to be **bi-normal** if both of the restriction maps $\pi_M : M \otimes 1_N \to L$ and $\pi_N : 1_M \otimes N \to L$ are normal representations.

Proposition 4.3.2. Let A, B be C*-algebras. Then there is a canonical injective binormal map

$$A^{**} \odot B^{**} \hookrightarrow (A \otimes B)^{**}.$$

Proof. By considering restrictions of the universal representation of $A \otimes B$ and taking its double commutant, the inclusion $A \odot B \hookrightarrow (A \otimes B)^{**}$ comes from the commuting copies of A and B in $(A \otimes B)^{**}$. Thus the WOT closures of $A \subseteq (A \otimes B)^{**}$ and $B \subseteq (A \otimes B)^{**}$ also commute, and so there is a bi-normal map

$$A^{**} \odot B^{**} \to (A \otimes B)^{**}.$$

To see that this map is injective, note that pure tensors of functionals on A and B respectively separate the elements of $A^{**} \odot B^{**}$. But then for $\phi \in A^*, \psi \in B^*$, the functional $\phi \odot \psi$ can be extended to a functional $\phi \otimes \psi$ on $A \otimes B$ and then further extended to a normal linear functional on $(A \otimes B)^{**}$, which we will still call $\phi \otimes \psi$. Now $\phi \odot \psi$ is the same map as

$$A^{**} \odot B^{**} \longrightarrow (A \otimes B)^{**} \xrightarrow{\phi \otimes \psi} \mathbb{C},$$

which implies injectivity. □

Definition 4.3.3. Let A be a C*-algebra.

1. We say that A has **property C** if

$$A^{**} \odot B^{**} \hookrightarrow (A \otimes B)^{**}$$

is min-continuous for every C*-algebra B.

2. We say that A has **property C'** if

$$A \odot B^{**} \hookrightarrow (A \otimes B)^{**}$$

is min-continuous for every C*-algebra B.

Proposition 4.3.4. Properties C and C' pass to subalgebras.

Proof. Suppose that A has property C. Let $C \subseteq A$ be a C*-subalgebra and suppose that A has property C. Since $C^{**} \subseteq A^{**}$ and $(C \otimes B)^{**} \subseteq (A \otimes B)^{**}$, we have the following commutative diagram for every C*-algebra B

$$\begin{tikzcd} C^{**} \odot B^{**} \arrow[r] \arrow[d] & (C \otimes B)^{**} \arrow[d] \\ A^{**} \odot B^{**} \arrow[r] & (A \otimes B)^{**}. \end{tikzcd}$$

Since $C^{**} \otimes B^{**} \subseteq A^{**} \otimes B^{**}$ and the bottom is min-continuous, the top is min-continuous. Property C' is similar. □

Proposition 4.3.5. Property C passes to quotients.

Proof. Let $I \lhd A$ and suppose that A has property C. Let $\pi : A \to A/I$ be the quotient map. Then we have a canonical normal extension $\pi^{**} : A^{**} \to (A/I)^{**}$. Let $p \in A^{**}$ be a central projection such that $(A/I)^{**} = pA^{**}$. If (e_i) is an approximate unit for I, then e_i increases to $1 - p$ ultraweakly. Let θ be the *-homomorphism defined by the following diagram

$$\begin{tikzcd}
(A/I)^{**} \otimes B^{**} \arrow[rr, "\theta"] \arrow[d, "\simeq"'] & & ((A/I) \otimes B)^{**} \\
pA^{**} \otimes B^{**} \arrow[r, hook] & A^{**} \otimes B^{**} \arrow[r, "\iota"] & (A \otimes B)^{**}. \arrow[u, "(\pi\otimes \mathrm{id}_B)^{**}"']
\end{tikzcd}$$

where ι is the inclusion coming from property C. Evidently, θ is continuous. Now let us see that $\theta|_{(A/I)^{**}\odot B^{**}}$ coincides with the canonical inclusion $(A/I)^{**} \odot B^{**} \to ((A/I) \otimes B)^{**}$. For every $a \in A, b \in B$,

$$\begin{aligned}
\theta(\pi(a) \otimes b) &= (\pi \otimes \mathrm{id}_B)^{**} \circ \iota(pa \otimes b) \\
&= \lim_i (\pi \otimes \mathrm{id}_B)^{**} \circ \iota((1 - e_i)a \otimes b) \\
&= \lim_i \pi((1 - e_i)a) \otimes b \\
&= \pi(a) \otimes b.
\end{aligned}$$

□

Definition 4.3.6. If A is unital, $0 \to I \to A \to^{\pi} A/I \to 0$ is called **locally split** if for each finite-dimensional operator system $E \subseteq A/J$, there exists a u.c.p. $\sigma : E \to A$ such that $\pi \circ \sigma = \mathrm{id}_E$.

Lemma 4.3.7. Suppose that $0 \to I \to A \to^{\pi} A/I \to 0$ is locally split. Then for every B,

$$0 \to I \otimes B \to A \otimes B \to (A/I) \otimes B \to 0$$

is exact.

Proof. Let $\|\cdot\|_\alpha$ be such that

$$\frac{A \otimes B}{I \otimes B} \simeq (A/I) \otimes_\alpha B.$$

So we just need to show that $\|\cdot\|_\alpha = \|\cdot\|_{\min}$. Since $y \in (A/I) \odot B$ is a finite sum of elementary tensors, we can find a finite-dimensional operator system $E \subseteq A/I$ such that $y \in E \odot B \subseteq (A/I) \odot B$. Then there is a u.c.p. map $\theta : E \to A$ such that $\pi \circ \theta = \mathrm{id}_E$, where $\pi : A \to A/I$ is the quotient map. Thus we have a u.c.p., hence contractive map $\theta \otimes \mathrm{id}_B : E \otimes B \to A \otimes B$. Now we have the following diagram

$$\begin{tikzcd}
(A/I) \otimes B & & \\
E \otimes B \arrow[u, "\text{isometric}"] \arrow[r, "\theta\otimes\mathrm{id}_B"] & A \otimes B \arrow[r] & \frac{A\otimes B}{I\otimes B} \simeq (A/I) \otimes_\alpha B.
\end{tikzcd}$$

Both the maps on the bottom are contractions, and since θ is a lift, it follows that $y \in E \odot B \subseteq E \otimes B$ gets mapped to $y \in E \odot B \subseteq (A/J) \otimes_\alpha B$, and we have $\|y\|_\alpha \leq \|y\|_{\min}$. □

Lemma 4.3.8. Let $I \lhd A, x \in I^{**} \cap A$. Then $x \in I$.

Proof. The canonical inclusion $A^{**} \to (A/I)^{**}$ restricted to A is just the quotient map $A \to A/I$, hence the set of elements in A belonging to the kernel is equal to I. Moreover, the kernel of $A^{**} \to (A/I)^{**}$ is I^{**}, which gives the result. □

Definition 4.3.9. If A is a C*-algebra, the strong*-topology in $A^{**} \subseteq \mathcal{B}(\mathcal{H}_u)$ has convergence $x_i \to x$ strong* if and only if $x_i \to x$ and $x_i^* \to x^*$, both in SOT.

Proposition 4.3.10. A C*-algebra A is exact if and only if it has property C'.

Proof. First assume that A has property C', and let B be a C*-algebra, $J \lhd B$. Since $0 \to J^{**} \to B^{**} \to (B/J)^{**} \to 0$ splits, we have that

$$0 \to J^{**} \otimes A \to B^{**} \otimes A \to (B/J)^{**} \otimes A \to 0$$

is exact. Since A has property C', we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & J \otimes A \arrow[r] \arrow[d] & B \otimes A \arrow[r] \arrow[d] & (B/J) \otimes A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^{**} \otimes A \arrow[r] \arrow[d] & B^{**} \otimes A \arrow[r] \arrow[d] & (B/J)^{**} \otimes A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (J \otimes A)^{**} \arrow[r] & (B \otimes A)^{**} \arrow[r] & ((B/J) \otimes A)^{**} \arrow[r] & 0.
\end{tikzcd}$$

Now taking $x \in B \otimes A$ with x in the kernel of $B \otimes A \to (B/J) \otimes A$, then the exactness of the middle row implies that $x \in (J \otimes A)^{**} \cap (B \otimes A) = J \otimes A$ by the above lemma. Thus A is exact.

Conversely, suppose that A is exact, and B is any C*-algebra. For I any directed set, we let

$$B_I = \{(x_i)_i \in \Pi_I B \mid \text{strong}^* \lim_i \text{ exists in } B^{**}\}.$$

Since multiplication is jointly strong*-continuous on bounded sets, B_I is a C*-subalgebra of $\Pi_I B$. Now we have a *-homomorphism $\sigma : B_I \to B^{**}$ given by

$$(x_i)_i \mapsto \text{strong}^* \lim_i x_i \in B^{**},$$

which is surjective by Kaplansky's density theorem ([21], Theorem 4.3.3) if I is large enough (for example, the directed set of all finite subsets of B^*).

Now notice that $\mathrm{id}_A \otimes \sigma : A \odot B_I \to A \odot B^{**} \subseteq (A \otimes B)^{**}$ is min-continuous. Indeed,

$$\begin{aligned}
\left\| (\mathrm{id}_A \otimes \sigma) \left(\sum_{k=1}^n a_k \otimes (x_i^{(k)})_i \right) \right\|_{(A\otimes B)^{**}} &= \left\| \text{strong}^* \lim_i \sum_{k=1}^n a_k \otimes x_i^{(k)} \right\|_{(A\otimes B)^{**}} \\
&\le \sup_i \left\| \sum_{k=1}^n a_k \otimes x_i^{(k)} \right\|_{A\otimes B} \\
&= \left\| \sum_{k=1}^n a_k \otimes (x_i^{(k)})_i \right\|_{A\otimes B_I}
\end{aligned}$$

for all $\sum_1^n a_k \otimes (x_i^{(k)})_i \in A \odot B_I$, and where this last equality follows by Lemma 4.1.16. Now let $J \lhd B_I$ be the kernel of the map $B_I \to B^{**}$. Then $A \otimes J$ is in the kernel of $A \otimes B_I \to (A \otimes B)^{**}$, and so this *-homomorphism factors through

$$\frac{A \otimes B_I}{A \otimes J} = A \otimes (B_I/J) = A \otimes B^{**},$$

since A is exact. Now the map $A \odot B^{**} \to (A \otimes B)^{**}$ agrees with the map $A \otimes B^{**} \to (A \otimes B)^{**}$ on elementary tensors, hence A has property C'. $\square$

Proposition 4.3.11. Let A be a C*-algebra. If A^{**} is semidiscrete, then A has property C.

Proof. Suppose that $A^{**} \subseteq (A \otimes B)^{**} \subseteq \mathcal{B}(\mathcal{H})$ where B is a C*-algebra, and $(A \otimes B)^{**} \subseteq \mathcal{B}(\mathcal{H})$ is a normal representation. Since $A^{**} \to A^{**} \subseteq (A \otimes B)^{**}$ is weakly nuclear, by Theorem 1.2.4 $A^{**} \odot B^{**} \to (A \otimes B)^{**} \subseteq \mathcal{B}(\mathcal{H})$ is min-continuous. $\square$

Consequently, every nuclear C*-algebra, every subalgebra of a nuclear C*-algebra, and every subquotient of a nuclear C*-algebra has Property C. In the separable setting, we will see that property C is equivalent to exactness, since we will be able to show that every separable exact C*-algebra is a subquotient of the CAR algebra, which is nuclear.

4.4 Subquotients of the CAR Algebra

In this section, we would like to prove that every separable exact C*-algebra is a subquotient of the CAR algebra. This was originally proved in [18], but we follow Wassermann's proof which can be found in his paper [33] and chapter 9 of his book [32]. Evidently, since nuclear C*-algebras have property C, which passes to subalgebras and quotient, all separable exact C*-algebras will have property C, completing the loop.

We first need a technical lifting result. We wish to show that when L is a left ideal generated by an increasing sequence of projections and $R = L^*$, then for $x \in A/(L + R)$, viewed as a Banach space, there exists $y \in A$ such that $\pi(y) = x$ and $\|y\| \leq (1 + \varepsilon)\|x\|$, where $\pi : A \to A/(L + R)$ is the quotient map.

Lemma 4.4.1. Let A be a C*-algebra, $L, R \subseteq A$ closed left and right ideals respectively. Then the subspace $L + R \subseteq A$ is norm closed.

Proof. For $a \in L$, $\operatorname{dist}(a, R) \leq \operatorname{dist}(a, L \cap R)$. If (e_λ) is a right approximate unit for L, with $0 \leq e_\lambda \leq 1$, then for $\varepsilon > 0$, there exists λ such that $\|a - ae_\lambda\| \leq \varepsilon$. For $r \in R$,

$$\|a - r\| \geq \|ae_\lambda - re_\lambda\| \geq \|ae_\lambda - a\| + \|a - re_\lambda\| \geq \operatorname{dist}(a, L \cap R) - \varepsilon.$$

Thus $\operatorname{dist}(a, L \cap R) = \operatorname{dist}(a, R)$. If $\pi : A \to A/R$ is the quotient map, then $\pi|_L$ factors as

$$L \to L/(L \cap R) \to \overline{(L + R)}/R,$$

where the first map is the quotient map, and the second is an isometry. Thus $\pi(L)$ is closed, and so $L + R = \pi^{-1}(\pi(L))$. $\square$

Lemma 4.4.2. Let A be a unital C*-algebra, p a non-zero projection in A. If $\varepsilon > 0$ and $x \in A$ with $\|x\| \leq 1+\varepsilon$, $\|px\| \leq 1$, then there exists $y \in A$ such that $py = 0$, $\|y\| \leq \sqrt{2\varepsilon + \varepsilon^2}$, and $\|x - y\| \leq 1$.

Proof. Let $a = px, b = (1-p)x$, and

$$b' = b(1 - a^*a)^{\frac{1}{2}}((1+\varepsilon)^2 - a^*a)^{-\frac{1}{2}}.$$

Let $y = b - b'$, so $a + b' = x - y$, and y has the required properties. □

Lemma 4.4.3. Let A be a unital C*-algebra, p, q non-zero projections in A. If $0 < \varepsilon \leq 1$, $x \in A$ with $\|x\| \leq 1 + \varepsilon$, and $\|pxq\| \leq 1$, then there exist $y, z \in A$ such that $yq = pz = 0$, $\|y\|, \|z\| \leq 4\varepsilon^{\frac{1}{4}}$, and $\|x - (y+z)\| \leq 1$.

Proof. Let $a = pxq$, $b = px(1-q)$. Then

$$\|a^* + b^*\| = \|x^*p\| \leq 1 + \varepsilon,$$

$a^* = qx^*p$, and $\|a^*\| \leq 1$. By applying the above lemma to the element x^*p and projection q, there exists $y \in A$ such that $qy^* = 0, \|y\| \leq \sqrt{2\varepsilon + \varepsilon^2} \leq 4\varepsilon^{\frac{1}{4}}, \|x^*p - y^*\| \leq 1$. In particular, $yq = 0$ and $\|px - y\| \leq 1$. Replacing y with py, we can assume that $py = y$. Now let $c = px - y, d = (1-p)x$, and $x' = c + d = x - y$. Then

$$\|px'\| = \|pc\| \leq 1,$$

and

$$\|x'\| \leq 1 + \varepsilon + \sqrt{2\varepsilon + \varepsilon^2} \leq 1 + 3\sqrt{\varepsilon}.$$

Let $\delta = \varepsilon + \sqrt{2\varepsilon + \varepsilon^2} \leq 3\sqrt{\varepsilon}$. By the above lemma again, there exists $z \in A$ such that $pz = 0$, $\|x' - z\| \leq 1$, and

$$\|z\| \leq \sqrt{2\delta + \delta^2} \leq \sqrt{6\sqrt{\varepsilon} + 9\varepsilon} \leq 4\varepsilon^{\frac{1}{4}}.$$

Then $\|x - (y+z)\| = \|x' - z\| \leq 1$. □

Proposition 4.4.4. Let $(p_n)_n$ be an increasing sequence of non-zero projections in a unital C*-algebra A, and let $L = \overline{\cup_n Ap_n}$. For $x \in A$, there exists $\overline{x} \in L + L^*$ such that

$$\|x - \overline{x}\| = \mathrm{dist}(x, L + L^*) = \inf_{a \in L + L^*} \|x - a\|.$$

Proof. Let us assume that $\mathrm{dist}(x, L + L^*) \leq 1$. Choose $y_0 \in L, z_0 \in L^*$ such that $\|x - (y_0 + z_0)\| \leq 2$ and let $x' = x - (y_0 + z_0)$. We have $\mathrm{dist}(x', L + L^*) = \lim_n \|(1-p_n)x'(1-p_n)\|$, so by passing to a subsequence if necessary, we can assume that $\|(1-p_n)x'(1-p_n)\| \leq 1 + 2^{-4n}$ for all n.

We shall construct sequences $(y_n)_n, (z_n)_n$ inductively such that $y_n(1-p_n) = (1-p_n)z_n = 0$, $\|y_n\|, \|z_n\| \leq 9 \cdot \frac{1}{2^n}$, and

$$\|x' - (y_1 + \cdots + y_k + z_1 + \cdots + z_k)\| \leq 1 + \frac{1}{2^{4k}}.$$

If $y_0, \cdots, y_k, z_0, \cdots, z_k$ have been chosen, $k \geq 0$, let $x_k = x' - (y_1 + \cdots + y_k + z_1 + \cdots + z_k) = x - (y_0 + \cdots + y_k + z_0 + \cdots + z_k)$. Then

$$\|(1-p_{k+1})x_k(1-p_{k+1})\| = \|(1-p_{k+1})x'(1-p_{k+1})\| \leq 1 + \frac{1}{2^{4(k+1)}}.$$

Applying the above lemma with x, $x'_k = (\frac{1}{1+2^{4(k+1)}})x_k$, and $p = q = 1 - p_n$, since

$$\|x'_k\| \leq \frac{1+\frac{1}{2^{4k}}}{1+\frac{1}{2^{4(k+1)}}} \leq 1 + \frac{1}{2^4 k},$$

there are y'_{k+1}, z'_{k+1} such that $\|y'_{k+1}\|, \|z'_{k+1}\| \leq 6 \cdot \frac{1}{2^k}$,

$$(1-p_{k+1})y'_{k+1} = z'_{k+1}(1-p_{k+1}) = 0,$$

and $\|x'_k - (y'_{k+1} + z'_{k+1})\| \leq 1$. Now let

$$y_{k+1} = (1 + \frac{1}{2^{4(k+1)}})y'_{k+1}, z_{k+1} = (1 + \frac{1}{2^{4(k+1)}})z'_{k+1}.$$

Then y_{k+1}, z_{k+1} will have the required properties.

Now let $y = \sum_0^\infty y_n, z = \sum_0^\infty z_n$. Then $y \in L, z \in L^*$, and if $\overline{x} = y + z$,

$$\|x - \overline{x}\| = \lim_n \|x - (y_0 + \cdots + y_n + z_0 + \cdots + z_n)\| \leq 1.$$

□

Corollary 4.4.5. Let A be a unital C*-algebra, (p_n) an increasing sequence of non-zero projections, and $L = \overline{\cup_n A p_n}$. If $\rho : A \to A/(L + L^*)$ is the quotient map, then for each $x \in A/(L+L^*)$, there exists $\overline{x} \in A$ such that $x = \rho(\overline{x})$ and $\|\overline{x}\| = \|x\|$.

The following technical result will be required.

Proposition 4.4.6. Let A be a separable unital exact C*-algebra, and let $B = M_{2^\infty}$ be the CAR algebra. Then there is a closed left ideal L of B, an isometry $\iota : B/(L+L^*) \to D$, a unital C*-algebra, and a unital complete isometry $\sigma : A \to D$ such that if $\rho : B \to B/(L+L^*)$ is the quotient map,

1. $\iota \circ \rho : B \to D$ is a u.c.p. map;

2. $\sigma(A) \subseteq \iota(\rho(B))$.

Proof. Suppose that $\phi : A \to \mathcal{B}(\mathcal{H})$ is a nuclear embedding. Since A is separable, ϕ is nuclear, ϕ is the point norm limit of $\psi_n \circ \phi_n$, where $\phi_n : A \to M_{k(n)}, \psi_n : M_{k(n)} \to \mathcal{B}(\mathcal{H})$ are u.c.p. maps. But then $\phi^{(m)} : M_m(A) \to M_m(\mathcal{B}(\mathcal{H}))$ is the point-norm limit of $\psi_n^{(m)} \circ \phi_n^{(m)}$. Thus there is a sequence $A_1 \subseteq A_2 \subseteq \cdots$ of finite-dimensional operator systems in A with union dense in A. By passing to a subsequence if necessary, we can enforce that

$$\|(\phi - \psi_n \circ \phi_n)^{(m)}|_{M_m(A_j)}\| \leq \frac{1}{2^n},$$

where this norm is computed in $M_m(\mathcal{B}(\mathcal{H}))$.

Since B is UHF, there are integers $1 < s_1 < s_2 < \cdots$, subalgebras $B_1 \subseteq B_2 \subseteq \cdots$ such that $s_i | s_{i+1}$, $B_i \simeq M_{s_i}$, $B = \overline{\cup_i B_i}$. By Arveson's extension Theorem (1.1.5), $\phi_n : A \to M_{k(n)}$ extends to a u.c.p map $\phi_n' : \mathcal{B}(\mathcal{H}) \to M_{k(n)}$. Let $V_n = \phi_{n+1} \circ \psi_n : M_{k(n)} \to M_{k(n+1)}, W_n = \psi_n \circ \phi_n' : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. Now we show that there is a subsequence $0 < r_1 < r_2 < \ldots$ of (s_i) with $k(n) \leq r_n$, that there projections $p_n \in M_{k(n)}$ such that $(1-p_n)M_{r_n}(1-p_n) = M_{k(n)}$, and that if $\Phi_n : M_{r_n} \to M_{k(n)}$ denotes the compression $x \mapsto (1-p_n)x(1-p_n)$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B}(\mathcal{H}) \arrow[rr, "W_1"] \arrow[dr, "\phi_1'"] & & \mathcal{B}(\mathcal{H}) \arrow[rr, "W_2"] \arrow[dr, "\phi_2'"] & & \mathcal{B}(\mathcal{H}) \arrow[rr, "W_3"] \arrow[dr, "\phi_3'"] & & \cdots \\
& M_{k(1)} \arrow[rr, "V_1"] \arrow[ur, "\psi_1"] & & M_{k(2)} \arrow[rr, "V_2"] \arrow[ur, "\psi_2"] & & M_{k(3)} \arrow[r] \arrow[ur] & \cdots \\
& M_{r_1} \arrow[rr] \arrow[u, "\Phi_1"] & & M_{r_2} \arrow[rr] \arrow[u, "\Phi_2"] & & M_{r_3} \arrow[r] \arrow[u, "\Phi_3"] & \cdots ,
\end{tikzcd}$$

where the maps on the bottom row are the unital embeddings $M_{r_i} \to M_{r_i} \otimes M_{r_{i+1}/r_i}$ given by $x \mapsto x \otimes 1$. If we have the r_i's above, by identifying M_{r_i} with its image we have a sequence $p_1 \leq p_2 \leq \cdots$ of projections with the required properties.

To get the r_i's, let r_1 be the smallest s_i such that $k(1) \leq s_i$, let q_1 be a projection in M_{r_1} of rank $k(1)$, and let $p_1 = 1 - q_1$. Identifying $M_{k(1)}$ with $q_1 M_{r_1} q_1$, the image of the u.c.p. map $\Phi_1 : M_{r_1} \to M_{r_1}$ given by $\Phi_1(x) = q_1 x q_1$ is $M_{k(1)}$.

Now supposing that we have r_i and p_i for $i \leq n$. Then $V_n \circ \Phi_n : M_{r_n} \to M_{k(n+1)}$ is a u.c.p. map, so by Stinespring, there is a Hilbert space $\mathcal{K}$, a *-homomorphism $\pi : M_{r_n} \to \mathcal{B}(\mathcal{K})$, and an isometry $V : \mathbb{C}^{r_n} \to \mathcal{K}$ such that $V_n \circ \Phi_n(\cdot) = V^* \pi(\cdot) V$. But if $p_i = \pi(e_{11})$ (all of these are unitarily equivalent), and we let $\mathcal{L} = p\mathcal{K}$, we have $\mathcal{K} = \oplus_j \mathcal{L} \simeq \mathcal{L}^{(r_n)}$, then evidently $\pi((a_{ij})) = (a_{ij} I_{\mathcal{L}})$. Now if we let q_{k+1} be the projection in $\mathcal{K}$ onto $V(\mathbb{C}^{r_n})$, then it is clear that $V_n \circ \Phi_n(x) = q_{k+1}\pi(x)q_{k+1}$. Moreover, since this is a Stinespring dilation of a u.c.p. map from a finite-dimensional C*-algebra, we can assume $\mathcal{K}$ is finite-dimensional, and so $V_n \circ \Phi_n$ is a map between finite-dimensional C*-algebras. Since matrix algebras are simple, it follows that π is an isomorphism. Thus $\mathcal{B}(\mathcal{K}) \simeq M_{r_n} \otimes M_r$ for some $r \in \mathbb{N}$, and with this map $\pi(x) = x \otimes 1_r$. Let r_{n+1} be the smallest s_i such that $r_n r < s_i$. Then $r_n \mid r_{n+1}$, so $\mathcal{B}(\mathcal{K}) \simeq M_{r_n} \otimes M_r \subseteq M_{r_n} \otimes M_{r_{n+1}/r_n} \simeq M_{r_{n+1}}$. Identifying $M_{k(n+1)} \subseteq M_{r_{n+1}}$ as a corner, the above becomes

$$V_n(\Phi_n(x)) = (1 - p_{n+1})(x \otimes 1)(1 - p_{n+1}) =: \Phi_{n+1}(x \otimes 1),$$

where $p_i = 1 - q_i$. This gives us all of our maps, which satisfy the required commutations relations. Now identifying p_n with $p_n \otimes 1$, we see that

$$(1 - p_{n+1})p_n(1 - p_{n+1}) = V_n(\Phi_n(p_n)) = 0$$

which implies that $p_n \leq p_{n+1}$.

Now we can assume that M_{r_n} and $M_{k(n)}$ are isomorphic subalgebras of B, $1 \in M_{r_1} \subseteq M_{r_2} \subseteq \cdots$ and that $\cup_n M_{r_n}$ is dense in B. Then $p_n \in B$, so $L = \overline{\cup_n B p_n}$ is a closed left ideal of B. Letting $\tau : \ell^\infty(B) \to (B)_\infty = \ell^\infty(B)/c_0(B)$ be the quotient map, we have a map $\Psi : B \to \Pi B/\oplus B$ given by $\Psi(x) = \tau((q_n x q_n)_n)$. Since compositions of c.c.p. maps are c.c.p., it follows that Φ is c.c.p. and that $\Psi(1)$ is the projection $e = \tau((q_n)_n)$. Let $D = e(B)_\infty e$. Then $\Psi(B) \subseteq D$, and $\Psi : B \to D$ is unital.

Now if $x \in B$, since $p_1 \le p_2 \le \cdots$, $\|q_n x q_n\|$ is decreasing, and so it tends to a limit. Therefore

$$\|\Psi(x)\| = \limsup_n \|q_n x q_n\| = \lim_n \|q_n x q_n\|,$$

hence $\mathrm{dist}(x, L + L^*) \le \|\Psi(x)\|$. The reverse inequality holds since $\Phi(L) = \Psi(L^*) = 0$. Thus $\|\Psi(x)\| = \|\rho(x)\|$, where $\rho : B \to B/(L + L^*)$ is the quotient map. Consequently, we have a well-defined linear isometry $\iota : B/(L + L^*) \to D$ such that $\Psi = \iota \circ \rho$.

Now we need a unital complete isometry $\sigma : A \to D$. For $M_{k(n)} \subseteq B, y \in M_{k(n)}$, $V_n(y) = q_{n+1} y q_{n+1}$ for all n. If $x \in A_n$ is such that $\|x\| = 1$, then $\|x - W_i(x)\| \le \frac{1}{2^i}$ for $i \ge n$, and

$$\|\phi_{i+1}(x) - V_i(\psi_i(x))\| = \|\psi'_{i+1}(x - \psi_i(\phi_i(x))\| \le \|x - W_i(x)\| \le \frac{1}{2^i}.$$

Now since $q_j V_i(\phi_i(x)) q_j = q_j(\phi_i(x)) q_j$ for $j > i$, we have that $\Psi(\phi_i(x)) = \Psi(V_i(\phi_i(x)))$, hence

$$\|\Psi(\phi_{i+1}(x)) - \Psi(\phi_i(x))\| = \|\Psi(\phi_{i+1}(x)) - \Psi(V_i(\phi_i(x)))\| \le \|\phi_{i+1}(x) - (V_i(\phi_i(x)))\| \le \frac{1}{2^i}.$$

So if $x \in \cup_n A_n$, the sequence $(\Psi(\phi_n(x)))$ is Cauchy in D. Since $\cup_n A_n$ is dense in A, and $\Psi \circ \phi_n$ are all contractive, it follows that for any $x \in A$, $\sigma(x) = \lim_n \Psi(\phi_n(x))$ exists. The map $\sigma : A \to D$ is a u.c.p. map, as it is a point-norm limit of u.c.p. maps, and thus completely contractive. Moreover it is clear that $\sigma(A) \subseteq \iota(\rho(B))$.

To see that σ is completely isometric, we can show that if $m \in \mathbb{N}, x \in M_m(A), \|x\| = 1$, then $\|\sigma^{(m)}(x)\| \ge 1$. Since σ is completely contractive, it suffices to show this for $x \in M_m(\cup_n A_n)$. So for $i \in \mathbb{N}, x \in M_m(A_i)$ with $\|x\| = 1$, and $\varepsilon > 0$, we want an $N \in \mathbb{N}$ such that for $n \ge N$,

$$\|\Psi^{(m)}(\phi_n^{(m)}(x))\| = \lim_j \|q_j^{(m)} \phi_n^{(m)}(x) q_j^{(m)}\| \ge 1 - \varepsilon,$$

where $q_j^{(m)} = q_j \otimes 1_m$. For $x \in M_m(A_i)$ with $\|x\| = 1, \varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{2^N} > \frac{\varepsilon}{3}$ and $N \ge \max\{i, m\}$. Then $M_m(A_i) \subseteq M_m(A_n) \subseteq M_n(A_n)$, so that $x \in M_n(A_n)$ for $n \ge N$. Then

$$\|\phi_{l+1}^{(m)}(x) - q_{l+1}^{(m)} \phi_l^{(m)} q_{l+1}^{(m)}\| = \|(\phi'_{l+1})^{(m)}(x - \psi_l^{(m)}(\phi_l^{(m)}(x)))\| \le \frac{1}{2^l}$$

for $l \ge n$. For $j > n+1$, applying this for $l = n, \cdots, j-2$, and since $q_j \le q_{l+1}$ for these l, we have

$$\|q_j^{(m)} \phi_{l+1}^{(m)} q_j^{(m)} - q_j^{(m)} \phi_l^{(m)}(x) q_j^{(m)}\| \le \frac{1}{2^l}$$

and

$$\|\phi_j^{(m)}(x) - q_j^{(m)} \phi_{j-1}^{(m)}(x) q_j^{(m)}\| \le \frac{1}{2^{j-1}},$$

hence

$$\|\phi_j^{(m)}(x) - q_j^{(m)} \phi_n^{(m)}(x) q_j^{(m)}\| \le \frac{1}{2^{n-1}}.$$

Now

$$\|\psi_j^{(m)}(\phi_j^{(m)}(x) - q_j^{(m)}\phi_n^{(m)}q_j^{(m)})\| \leq \frac{1}{2^{n-1}},$$

and since

$$\|x - \psi_j^{(m)}(\phi_j^{(m)}(x))\| \leq \frac{1}{2^j} \leq \frac{1}{2^n},$$

we have that

$$\|q_j^{(m)}\phi_n^{(m)}(x)q_j^{(m)}\| \geq \|\psi_j^{(m)}(q_j^{(m)}\phi_n^{(m)}(x)q_j^{(m)})\| \geq \|x\| - \frac{1}{2^{n-1}} - \frac{1}{2^n} \geq 1 - \varepsilon.$$

So $\|\Psi^{(m)}(\phi_n^{(m)}(x))\| \geq 1 - \varepsilon$ for $n \geq N$, hence $\|\sigma^{(m)}(x)\| \geq 1 - \varepsilon$. Since ε was arbitrary, $\|\sigma^{(m)}(x)\| \geq 1$, so it follows that σ is completely isometric. □

Remark 4.4.7. With everything as above, once can also show that if A is nuclear, then σ, ι, L can be chosen such that $\sigma(A) = \iota(\rho(B))$.

We are finally ready to give a proof of the main theorem of this section.

Theorem 4.4.8. Let A be a separable unital C*-algebra. The following are equivalent:

1. A is exact
2. There is a unital C*-subalgebra $G \subseteq M_{2^\infty}$ of the CAR algebra, and an ideal $J \lhd G$ such that $A \simeq G/J$.

Proof. (1) implies (2). Let A be an exact C*-algebra, and let $B = M_{2^\infty}$ be the CAR algebra. So $B = \overline{\cup_n B_n}$ where $B_n \simeq M_{2^n}$ and $B_1 \subseteq B_2 \subseteq \cdots$. Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a unital faithful representation of A. By the above proposition, there exists a unital C*-algebra D, a unital complete isometry $\sigma : A \to D$, and a u.c.p. map $\iota \circ \rho : B \to D$ such that $\sigma(A) \subseteq \iota(\rho(B))$. Now $\sigma(A) \subseteq D$ is a closed operator system, and $\sigma^{-1} : \sigma(A) \to A \subseteq \mathcal{B}(\mathcal{H})$ is a unital complete isometry, so it is completely positive. By Arveson's extension Theorem (1.1.5), σ^{-1} extends to a c.p. map $\tau : D \to \mathcal{B}(\mathcal{H})$. Let $\pi = \tau \circ \iota \circ \rho$. For every $u \in U(A)$, $\sigma(u) \in \iota(\rho(B))$, so that $\iota^{-1}(\sigma(u)) \in \rho(B) = B/(L + L^*)$, and

$$\|\iota^{-1}(\sigma(u))\| = \|u\| = 1.$$

Now by Corollary 4.4.5, there exists $x \in B$ such that $\|x\| = 1$, and $\rho(x) = \iota^{-1}(\sigma(u))$; which means that $u = \pi(x)$. Since π is u.c.p. and by Choi's generalized Cauchy-Schwarz inequality (1.1.10), we have

$$1 = u^*u = \pi(x^*)\pi(x) \leq \pi(x^*x) \leq \|x\|^2\pi(1) = 1,$$

which implies that $\pi(x^*x) = \pi(x^*)\pi(x)$. Similarly $\pi(xx^*) = \pi(x)\pi(x^*)$. Thus x, x^* are in the multiplicative domain of π. Now let $X = \{x \in B \mid \|x\| = 1, \pi(x) \in U(A)\}$. Then X is self-adjoint, and closed under multiplication, so that $\mathrm{span}(X)$ is a *-subalgebra of B. Since X is in the multiplicative domain of π,$\pi|_{\mathrm{span}(X)}$ is a *-homomorphism, so if $F = \overline{\mathrm{span}}(X)$, $\pi|_F$ is a *-homomorphism by continuity. Now since $\pi(X) = U(A), \pi(F) = A$. Let $K = F \cap \ker \pi$, so that K is a closed ideal of F, and $A \simeq F/K$. Let $J = \overline{KBK} \subseteq G$ (which is hereditary). We also have $FJ \subseteq J$ and $\pi(J) = 0$. Let $G = F + J$, which is a C*-subalgebra of B, J is a closed ideal of G ,and $G/J \simeq A$. Indeed, $\overline{F+J}$ is a C*-subalgebra of B with J as a closed ideal. If $\pi' : \overline{F+J} \to \overline{(F+J)}/J$ is the quotient map, then $K = F \cap J$ and

$$\pi'(F) \simeq F/(F \cap J) = F/K \simeq A.$$

Thus $F + J = \pi'^{-1}(A)$, so that $F + J$ is closed in B.

To see (2) implies (1), notice that if $A \simeq G/J$ where $G \subseteq M_{2^\infty}$, $J \lhd G$, then since M_{2^∞} is nuclear, it has property C, which passes to the subalgebra G, and further passes to the quotient $A \simeq G/J$. Since property C implies exactness, A is exact. □

Remark 4.4.9. Another equivalent condition of exactness is that there is a unital completely isometric map $\theta : A \to M_{2^\infty}$. Moreover, A is nuclear if and only if there is a unital completely isometric map $\theta : A \to M_{2^\infty}$ such that there is a conditional expectation $M_{2^\infty} \to \theta(A)$.

Theorem 4.4.10. Let A be a separable C*-algebra. The following are equivalent:

1. A is exact,
2. A has property C,
3. A has property C'.

Proof. Clearly property C implies property C', and A being exact is equivalent to having C'. Now suppose that A is exact. Since A is separable, Theorem 4.4.8. implies that A is a subquotient of the CAR algebra, which is nuclear. Since nuclear C*-algebras have property C and property C passes to subalgebras and quotients, the result follows. □

Remark 4.4.11.

1. Another way to see that a separable C*-algebra is exact if and only if it has property C is to embed it into the Cuntz algebra $\mathcal{O}_2$. This algebra is nuclear and contains all separable exact C*-algebras, as we will see later.
2. The above theorem holds even in the non-separable setting.

Corollary 4.4.12. Quotients of exact C*-algebras are exact.

Proof. Let A be exact, and $I \lhd A$ an ideal. it suffices to show that every separable subalgebra of A/I is exact since a C*-algebra is exact if and only if all its separable subalgebras are exact. First, let us assume that A is unital, since passing to unitizations preserves exactness. Let $C \subseteq A/I$ be a separable subalgebra. Then there exists a separable subalgebra $D \subseteq A$ such that $\pi(D) = C$, where $\pi : A \to A/I$ is the quotient map. But since A is exact, D is exact, and so C is exact. Consequently A/I is exact. □

Remark 4.4.13. There is yet another property, **property C"**, defined as follows. A has property C" if $A^{**} \odot B \hookrightarrow (A \otimes B)^{**}$ is min-continuous for every C*-algebra B. This is equivalent to the notion of being **locally reflexive** - that is, for every finite-dimensional operator system $E \subseteq A^{**}$, there exists a net of c.c.p. maps $\phi_\lambda : E \to A$ which converges to id_E point-ultraweakly. Evidently, all exact C*-algebras exhibit this property. More on this can be found in [14] and chapter 9 of [6].

5 Purely Infinite C*-algebras

With a deeper understanding of exact C*-algebras, we will now move towards proving that all of the separable exact C*-algebra embed into $\mathcal{O}_2$. To do this, we study the class of unital, simple, purely infinite C*-algebras, of which $\mathcal{O}_2$ is part of. This class of C*-algebras was studied both by Cuntz and Kirchberg, and they are C*-algebras which are characterized by their projections. They have very simple K-theory, in the sense that one does not need to consider projections or unitaries in the higher matrix levels. We will compute the K-theory of the Cuntz algebras and show that the Cuntz algebras $\mathcal{O}_n$ and $\mathcal{O}_m$ can be distinguished by the natural numbers $n, m \geq 2$ (and both of these will not be isomorphic to $\mathcal{O}_\infty$). These C*-algebras are further seen to exhibit many approximations properties such as real rank zero, property weak (FU), and consequently finite exponential length - a very important fact used in the proof that $\mathcal{O}_2 \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$.

5.1 Projections and Simple C*-Algebras

As mentioned above, purely infinite C*-algebras are defined in terms of properties of their projections. We define what it means for a projection to be infinite, for a C*-algebra to be infinite, and then consider what happens when the C*-algebra is simple. In the case where an algebra is simple in infinite, it will contain a copy of $\mathcal{O}_\infty$ (unital if the C*-algebra is), and will have $\mathcal{O}_n$ as a quotient for all n. We then consider purely infinite C*-algebras, and show that unital, simple, purely infinite C*-algebras are closed under taking min tensor products, which will give us that $\mathcal{O}_2 \otimes \mathcal{O}_2$ is unital purely infinite as well.

Definition 5.1.1. We will want to work with the projections in a C*-algebra.

1. We call call two projections p, q in a C*-algebra A **equivalent (Murray-von Neumann equivalent)**, denoted $p \sim q$, if there is a partial isometry $v \in A$ such that $p = vv^*$ and $q = v^*v$.

2. A projection p is **infinite** if it is equivalent to a proper subprojection of itself (that is there exists q such that $p \sim q$ and $q \leq p$,$q \neq p$ - denoted $q < p$).

3. A projection p is **properly infinite** if there are orthogonal projections q_1, q_2 such that $p \sim q_1 \sim q_2$ such that $q_1 + q_2 \leq p$.

4. A C*-algebra A is **(properly) infinite** if it contains a (properly) infinite projection.

Example 5.1.2. It is clear that the identities in $\mathcal{O}_2, \mathcal{O}_\infty, \mathcal{B}(\mathcal{H})$ are all properly infinite projections. All projections in $M_n, \mathbb{K}$ are finite (not infinite).

The following definition, and equivalent formulations can be found in chapter 3.2 of [21].

Definition 5.1.3. We say $B \subseteq A$ is **hereditary** if it satisfies one (hence all) of the following:

1. A C*-subalgebra $B \subseteq A$ is called **hereditary** if for $0 \leq a \in A$ and $0 \leq b \in B$, $a \leq b$ implies that $a \in B$.

2. $bab' \in B$ for all $b, b' \in B$ and $a \in A$.

3. $B = L \cap L^*$ for L a left ideal of A (this is a one-to-one correspondence).

If B is separable, there exists $a \in B$ such that $B = \overline{aAa}$.

Theorem 5.1.4 ([21], Theorem 3.2.2.)**.** If $B \subseteq A$ is a hereditary C*-subalgebra, and $J \subseteq B$ is a closed ideal, then there exists a closed ideal I of A such that $J = B \cap I$.

Corollary 5.1.5. Every hereditary C*-subalgebra of a simple C*-algebra is simple.

Lemma 5.1.6. If A is simple, $q \in A$ is a projection, $a \in A$ is non-zero positive, then there are $z_i \in A$ such that $\sum_1^n z_i^* a z_i = q$.

Proof. Without loss of generality, suppose that $\|a\| = 1$. Since A is simple, q is in the closure of the algebraic ideal generated by a, so there exists x_i, y_i such that

$$\left\| q - \sum_1^n x_i a y_i \right\| < \frac{1}{2}.$$

But

$$\begin{aligned} q &\leq \sum_1^n q x_i a y_i q + \sum_1^n q y_i^* a x_i^* q \\ &\leq \sum_1^n q x_i a x_i^* q + \sum_1^n q y_i^* a y_i q =: b \leq c \cdot q, \end{aligned}$$

where $c = \sum_1^n (\|x_i\|^2 + \|y_i\|^2)$. The first inequality follows since the norm condition implies that $\|q - \mathrm{Re}(\sum_1^n x_i a y_i)\| < \frac{1}{2}$, and compression by the projection makes them commute - then it is easy since we can just think of these as real-valued functions. The second inequality follows since $x_i a y_i + y_i^* a x_i \leq x_i a x_i^* + y_i^* a y_i$. This follows because $(x_i - y_i^*)^* a (x_i^* - y_i) \geq 0$.

Now let $f \in C(\sigma(a))$ be a function such that $f(x) = x^{-\frac{1}{2}}$ on $[1, c]$. Then

$$q = f(b) b f(b) = \sum_1^n f(b) q x_i a x_i^* q f(b) + \sum_1^n f(b) q y_i^* a y_i q f(b),$$

which is a sum of the desired form. ☐

Theorem 5.1.7. Let A be a simple infinite C*-algebra. Then A contains a projection q and partial isometries (t_i) such that $t_i^* t_i = q > \sum_1^n t_i t_i^*$ for all $n \in \mathbb{N}$. In particular, A is properly infinite.

Proof. Let s be such that $p = ss^* < q = s^* s$. By working in $B = qAq$, we can view B as a unital C*-algebra with $q = 1$. Note that B is simple (its hereditary) and infinite (q is infinite). Since B is simple and $1 - p \neq 0$, there exists x_i such that $\sum_1^n x_i^* (1-p) x_i = 1$. Let

$$t_1 = \sum_1^n s^{i-1} (1-p) x_i.$$

Note that $s^i(1-p)$ are partial isometries which have pairwise orthogonal ranges for $i \geq 0$. Indeed, since $v = pv = vq = pvq$, we have$(1-p)v = (1-p)pv = 0 = s^*(1-p)$ and $(v^i(1-p))^*v^j(1-p) = (1-p)v^{j-i}(1-p) = 0$ for all $j > i$.

$$
\begin{aligned}
t_1^*t_1 &= \sum_{i=1}^n \sum_{j=1}^n x_j^*(1-p)(s^*)^{j-1}s^{i-1}(1-p)x_i \\
&= \sum_1^n x_i^*(1-p)x_i = 1.
\end{aligned}
$$

So t_1 is a partial isometry, and the range of the t_1 is clearly contained in the span of the ranges of $s^{i-1}(1-p)$, thus

$$
t_1t_1^* \leq \sum_1^n s^{i-1}(1-p)(s^*)^{i-1} = 1 - s^n(s^*)^n.
$$

Let $t_i = s^{n(i-1)}t_1$ for $i \geq 2$. Then

$$
\begin{aligned}
t_it_i^* &= s^{n(i-1)}t_1t_1^*(s^*)^{n(i-1)} \\
&\leq s^{n(i-1)}(1 - s^n(s^*)^n)(s^*)^{n(i-1)} \\
&= s^{n(i-1)}(s^*)^{n(i-1)} - s^{ni}(s^*)^{ni}.
\end{aligned}
$$

Hence $t_it_i^*$ are pairwise orthogonal projections. Since each is equivalent to the identity, B and A are properly infinite. □

Corollary 5.1.8. If A is simple and infinite, then $\mathcal{O}_\infty$ is a C*-subalgebra of A.

Lemma 5.1.9. Let $\mathcal{C}_n$ be a C*-algebra generated by n isometries (s_i) such that $\sum_1^n s_is_i^* = p < 1$. Then $\langle p^\perp \rangle \simeq \mathbb{K}$ and $\mathcal{C}_n/\mathbb{K} \simeq \mathcal{O}_n$.

Proof. Since $s_i^*p^\perp = p^\perp s_i = 0$, $\langle p^\perp \rangle$ is spanned by

$$
\{s_\mu p^\perp s_\nu^* \mid |\mu|, |\nu| < \infty\}.
$$

This follows by using that lemma about words. Moreover,

$$
(s_\mu p^\perp s_\nu^*)(s_\alpha p^\perp s_\beta^*) = \delta_{\nu\alpha}s_\mu p^\perp s_\beta^*,
$$

so these form a set of matrix units. Thus $\langle p^\perp \rangle \sim \mathbb{K}$. In the quotient, $(\tilde{s}_i)$ will be isometries such that $\sum_1^n \tilde{s}_i\tilde{s}_i{}^* = \tilde{1}$, hence $\mathcal{C}_n/\mathbb{K} \simeq \mathcal{O}_n$. □

Corollary 5.1.10. If A is a simple infinite C*-algebra, then $\mathcal{O}_n$ is a quotient of A.

Proposition 5.1.11. Let A be simple, $p, q \in A$ be projections with p infinite. Then q is a equivalent to a subprojection of p.

Proof. There exists $(x_i) \subseteq A$ such that $q = \sum_1^n x_i p x_i^*$. Moreover there exists partial isometries (v_i) such that $v_i^* v_i = p$ and $\sum_1^n v_i v_i^* < p$. Let $v = \sum_1^n x_i p v_i^*$. Then

$$vv^* = \sum_{i,j=1}^{n} x_i p v_i^* v_i p x_j^* = \sum_{1}^{n} x_i p x_i^* = q.$$

Hence v is a partial isometry, so v^*v is a projection. Moreover,

$$\sum_1^n v_i v_i^* < p,$$

$v_i^* p = v_i^*, p v_i = v_i$ and so

$$pv^*vp = \sum_{i,j=1}^{n} p v_i x_i^* x_j p v_j^* p = v^* v,$$

and so $q = vv^* \sim v^*v \leq p$. □

Definition 5.1.12. A C*-algebra A is **purely infinite** if every hereditary subalgebra contains an infinite projection.

Theorem 5.1.13. Let A be a unital, simple C*-algebra of dimension at least 2. The following are equivalent.

1. A is purely infinite.
2. For all $0 \neq a \in A$, there exists $x, y \in A$ such that $xay = 1$.
3. For all $0 \leq a \in A \setminus \{0\}$, $\varepsilon > 0$, there exists $x \in A$ such that $\|x\| < \|a\|^{-\frac{1}{2}} + \varepsilon$ and $xax^* = 1$.

Proof. If (3) holds, $a \neq 0$, then there exists $x \in A$ such that $xa^*ax^* = 1$, so $(xa^*)ax^* = 1$, giving (2).

If (2) holds, $a \neq 0$ is positive, find x, y such that $xa^{\frac{1}{2}}y = 1$. Then

$$1 = xa^{\frac{1}{2}}yy^*a^{\frac{1}{2}}x^* \leq \|y\|^2 xax^*.$$

Thus $z = xax^*$ is invertible, so $1 = (z^{-\frac{1}{2}}x)a(x^*z^{-\frac{1}{2}})$, giving 3 without the norm estimate. Now suppose that $B \subseteq A$ is hereditary and $0 \leq b \in B$ is not invertible. Then there is $x \in A$ such that $xbx^* = 1$. Let $s = b^{\frac{1}{2}}x^*$, so $s^*s = 1$ and s is not invertible, so s is a partial isometry. Moreover,

$$p = ss^* = b^{\frac{1}{2}}xx^*b^{\frac{1}{2}} \in B.$$

This is an infinite projection in B, $sp \in B$ and $(sp)^*(sp) = p$, while sps^* is a subprojection of p orthogonal to $s(1-p)s^*$. Thus B is infinite, giving (1).

Suppose that (1) holds. Let $0 \leq a \in A$ have $\|a\| = 1$, and let $0 < \varepsilon < \frac{1}{2}$. Let $c = 1 - \varepsilon$. Define

$$f(x) = \begin{cases} 0, & 0 \leq x \leq 1-\varepsilon \\ 1 - \frac{1-x}{\varepsilon}, & 1-\varepsilon \leq x \leq 1. \end{cases}$$

Let $B = \overline{f(a)Af(a)}$, which is hereditary in A. By (1), B contains an infinite projection p. Clearly $p \leq \chi_{[1-\varepsilon,1]}(a)$ (in $A \subseteq \mathcal{B}(\mathcal{H})$) and so $pap \geq (1-\varepsilon)p$. Since A is simple, the above proposition implies that the identity 1 of A is equivalent to a subprojection of p. Hence there exists a partial isometry $vv^* \leq p$. Thus $v^*p = v^*, pv = v$. Let

$$b = v^*av = (v^*p)a(pv) \geq (1-\varepsilon)v^*v = (1-\varepsilon)v^*v = (1-\varepsilon)\cdot 1.$$

So b is invertible with

$$(b^{-\frac{1}{2}}v^*)a(vb^{-\frac{1}{2}}) = 1.$$

Finally,

$$\|vb^{-\frac{1}{2}}\| \leq \|b^{-\frac{1}{2}}\| \leq (1-\varepsilon)^{-\frac{1}{2}} < 1+\varepsilon,$$

giving (3). □

Corollary 5.1.14. $\mathcal{O}_n$ is purely infinite for $n \geq 2$. $\mathcal{O}_\infty$ is also purely infinite.

Lemma 5.1.15. Let A be a unital, purely infinite C*-algebra. If $p \in A$ is a non-zero projection, then pAp is unital purely infinite. In particular, all projections in a purely infinite C*-algebra are infinite.

Proof. Clearly pAp is unital (with unit p). Suppose that $B \subseteq pAp$ is hereditary. We claim that $B \subseteq A$ is hereditary. Indeed, suppose $0 \leq a \leq b$ for $a \in A, b \in B$. Since $b \in B \subseteq pAp$ and $pAp \subseteq A$ is hereditary, it follows that $a \in pAp$. But now since $B \subseteq pAp$ is hereditary, $a \in B$. Thus $B \subseteq A$ is hereditary. Since A is purely infinite, B is infinite and so pAp is purely infinite. So pAp contains an infinite projection and thus p is infinite. □

Theorem 5.1.16. Let A be a unital, simple, purely infinite C*-algebra, $p, q \in A$ with $p \neq 0$. Then there exists a projection $q' \in A$ such that $q \sim q'$ and $q' < p$.

Proof. Since $p \neq 0$, the above implies that p is infinite. Since A is unital, simple, p is properly infinite. Thus there exists $p' \neq 0$ in A such that $p \sim p'$ and $p' < p$. The above also implies that p' is infinite, so the above proposition implies that there exists $q' \in A$ such that $q \sim q'$ and $q' \leq p' < p$. □

Tensor products have played a large role in our study. Let us see what happens if we take tensor products of purely infinite C*-algebras - in particular the unital simple ones. First let us see that the tensor product of two simple C*-algebras is simple; obviously only considering the min-tensor. The simplicity result comes from exercise 3.4.2 of [6], and the purely infinite results are from chapter 4 in [28].

Lemma 5.1.17. Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a factor, $\pi : M \odot M' \to \mathcal{B}(\mathcal{H})$ be the product map defined by $\pi(a \otimes b) = ab$. If $\pi(\sum a_i \otimes b_i) = 0$, then $\sum a_i \otimes b_i = 0$, where $\sum a_i \otimes b_i \in M \odot M'$.

Proof. Suppose that $\sum a_i b_i = \pi(\sum a_i \otimes b_i) = 0$. Let $\mathcal{H}_n \simeq \mathbb{C}^n$ with orthonormal basis $(e_i)_1^n$. Let $\mathcal{K} = \mathcal{H} \otimes \mathcal{H}_n$, and let

$$\mathcal{K}_0 = \overline{\operatorname{span}} \left\{ \sum_1^n bb_i\xi \otimes e_i \mid b \in M', \xi \in \mathcal{H} \right\}.$$

Note that $\mathcal{K}_0$ is $M' \otimes 1$ invariant by construction. Now notice that $\sum_1^n a_j^* \eta \otimes e_j \perp \mathcal{K}_0$ for all $\eta \in \mathcal{H}$. Indeed,

$$\begin{aligned}
\left\langle \sum_i bb_i \xi \otimes e_i, \sum_j a_j^* \eta \otimes e_j \right\rangle &= \sum_i \langle bb_i \xi, a_i^* \eta \rangle \\
&= \sum_i \langle a_i bb_i \xi, \eta \rangle \\
&= \left\langle \sum_i ba_i b_i \xi, \eta \right\rangle \\
&= \left\langle b \cdot \pi \left(\sum_i a_i \otimes b_i \right) \xi, \eta \right\rangle = 0.
\end{aligned}$$

Now letting p be the projection of $\mathcal{K}$ onto $\mathcal{K}_0$, since $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}_n) \simeq \mathcal{B}(\mathcal{H}^{(n)})$, we can view p as a matrix $(p_{ij}) \in \mathcal{B}(\mathcal{H})$. Then for $b \in M', \xi \in \mathcal{H}$,

$$\sum_j bb_j \xi \otimes e_j = (p_{ij}) \sum_j bb_j \xi \otimes e_j = \sum_{i,j} p_{ij} bb_j \xi \otimes e_j.$$

Thus $bb_j = \sum_i p_{ij} bb_j$ for all $b \in M'$, and in particular for $b = 1$. Thus

$$0 = p \sum_j a_j^* \eta \otimes e_j = \sum_{i,j} p_{ij} a_j^* \eta \otimes e_j$$

for all $\eta \in \mathcal{H}$, hence by the linear independence of (e_j), $\sum_j p_{ij} a_j^* = 0$, and so $\sum_j a_j p_{ji} = (\sum_j p_{ij} a_j^*)^* = 0$. Now

$$(a \otimes 1) \left(\sum_i bb_i \xi \otimes e_i \right) = \sum_i abb_i \xi \otimes e_i = \sum_i bb_i (a\xi) \otimes e_i \in \mathcal{K}_0,$$

so $\mathcal{K}_0$ is both $M \otimes 1$ and $M' \otimes 1$ invariant, and so it is $(M \cup M') \otimes 1$ invariant. Consequently $\mathcal{K}_0$ is $(M \cup M')'' \otimes 1$ invariant. But $(M \cup M')' = \mathbb{C}1$ since M is a factor, and so $\mathcal{K}_0$ is invariant under $\mathcal{B}(\mathcal{H}) \otimes 1$. Thus p must commute with $\mathcal{B}(\mathcal{H}) \otimes 1$, hence $p \in 1 \otimes \mathcal{B}(\mathcal{H}_n)$. This each p_{ij} is a scalar multiple of identity, so

$$\begin{aligned}
\sum_i a_i \otimes b_i &= \sum_i a_i \otimes \left(\sum_j p_{ij} b_j \right) \\
&= \sum_{i,j} a_i \otimes p_{ij} b_j \\
&= \sum_{i,j} p_{ij} a_i \otimes b_j \\
&= \sum_j \left(\sum_i p_{ij} a_i \right) \otimes b_j = 0.
\end{aligned}$$

$\square$

Proposition 5.1.18. Let A, B be C*-algebras. Then $A \otimes_\alpha B$ is simple if and only if $\|\cdot\| = \|\cdot\|_{\min}$ and both A, B are simple.

Proof. Clearly we can restrict ourselves to the min-tensor since any other tensor will have a surjection onto the min-tensor. With this in mind, if A is not simple, then there exists some proper ideal $I \lhd A$. Then $I \odot B$ is an algebraic ideal in $A \odot B$ and so $I \otimes B$ is non-zero in $A \otimes B$. Now there is some state $\phi \in A^*$ such that $\phi(I) = 0$, so if ψ is any state on B, then $\phi \times \psi : A \odot B \to \mathbb{C}$ extends to a state $\phi \otimes \psi : A \otimes B \to \mathbb{C}$ such that $(\phi \times \psi)(I \odot B) = 0$, and so $(\phi \times \psi)(I \otimes B) = 0$. Since $\phi \times \psi$ is non-zero, $A \otimes B$ is non-simple.

Conversely, if A, B are both simple and $A \otimes B$ is not, then there would be a non-faithful irreducible representation. Let $\pi : A \otimes B \to \mathcal{B}(\mathcal{H})$ be an irreducible representation. By taking restrictions, there are non-degenerate *-homomorphisms $\pi_A : A \to \mathcal{B}(\mathcal{H}), \pi_B : B \to \mathcal{B}(\mathcal{H})$ such that the ranges of π_A, π_B commute and $\pi = \pi_A \times \pi_B$. Since π is irreducible, $\pi(A \otimes B)' = \mathbb{C} \cdot I$. But then $\mathbb{C} \cdot I = \pi(A \otimes B)' = \pi_A(A)' \cap \pi_B(B)'$. Since $\pi_A(A) \subseteq \pi_B(B)$, $\pi_A(A)''$ is a factor, and $\pi_B(B) \subseteq \pi_A(A)'$.

Notice that if $\pi(\sum a_i \otimes b_i) = 0$, then since $\pi = \pi_A \times \pi_B$, $\sum \pi_A(a_i)\pi_B(b_i) = 0$. But by the above lemma, this implies that $\sum \pi_A(a_i) \otimes \pi_B(b_i) = 0$. But then

$$(\pi_A \otimes \pi_B)\left(\sum a_i \otimes b_i\right) = 0.$$

Since A, B are simple, both π_A, π_B are injective, and so $\pi_A \odot \pi_B$ is injective, hence $\sum a_i \otimes b_i = 0$.

We claim that this is enough. In general, if $\pi : A \otimes B \to C$ is a *-homomorphism, then π is injective if and only if $\pi|_{A \odot B}$ is. To see this, let $\|\cdot\|_\alpha$ be the C*-norm on $A \odot B \simeq \pi(A \odot B)$ which is the restriction of the norm on C. Then $A \otimes_\alpha B = C^*(\pi(A \odot B)) \subseteq C$, and so $\pi : A \otimes B \to A \otimes_\alpha B$ is continuous. Since *-homomorphisms are contractive, it follows that $\|\cdot\|_\alpha \le \|\cdot\|_{\min}$, so these norms are equal and π is isometric, hence injective. $\square$

Lemma 5.1.19. Let A be a unital C*-algebra, $0 \le a, b \in A$, $\varepsilon > \|a - b\|$. Then there exists $d \in A$ such that $dbd^* = (a - \varepsilon)_+$ and $\|d\| \le 1$, where c_+ denotes the positive part of the self-adjoint element c.

Proof. For $\delta > 1$, and let $f_\delta : [0, \infty) \to [0, \infty)$ be defined by $f_\delta(x) = \max\{x, x^\delta\}$. Then $f_\delta(b) \to b$ as $\delta \to 1_+$. Since $\|a - b\| < \varepsilon$, there exists $\delta_0 > 1$ such that $\|a - f_{\delta_0}(b)\| < \varepsilon$. Let $b_0 = f_{\delta_0}(b)$ and let $0 \le \varepsilon_1 = \|a - f_{\delta_0}(b)\| < \varepsilon$. Then $a - \varepsilon_1 \le b_0$. Moreover, we have that $b_0 \le b$ and that $b_0 \le b^{\delta_0}$. Since $\varepsilon_1 < \varepsilon$, if we assume $\varepsilon \le 1$, there is a contraction $e \in C^*(a)$ such that $e(a - \varepsilon_1 1)e = (a - \varepsilon 1)_+$, so $(a - \varepsilon 1)_+ \le eb_0e$. To see that such an e exists, if $a(t) = t$ on $[0, 1]$, then we can take

$$e(t) = \begin{cases} 0, & 0 \le t \le \varepsilon \\ \sqrt{\frac{t-\varepsilon}{t-\varepsilon_1}}, & \varepsilon \le t \le 1. \end{cases}$$

Now let $x = b_0^{\frac{1}{2}}e$, and suppose that $A \subseteq \mathcal{B}(\mathcal{H})$ is a unital embedding. Then there exists a partial isometry $v \in \mathcal{B}(\mathcal{H})$ such that $x = v|x|$. Then $v = \text{WOT-}\lim_n x(x^*x + \frac{1}{n}1)^{-\frac{1}{2}}$, and so $v \in A''$. Let $y = v((a - \varepsilon 1)_+)^{\frac{1}{2}} \in A''$. We want to show that $y \in A$. Clearly $y =$

WOT-$\lim_n x(x^*x+\frac{1}{n}1)^{-\frac{1}{2}}((a-\varepsilon)_+)^{\frac{1}{2}}$, and so if we show that $x_n = x(x^*x+\frac{1}{n})^{-\frac{1}{2}}((a-\varepsilon)_+)^{\frac{1}{2}} \in A$ is Cauchy, then y will be the norm-limit of elements in A, hence in A. Notice that

$$\begin{aligned}
&\|x_n - x_m\|^2 \\
&= \|(x_n - x_m)(x_n - x_m)^*\| \\
&= \left\| x \left((x^*x + \frac{1}{n})^{-\frac{1}{2}} - (x^*x + \frac{1}{m})^{-\frac{1}{2}} \right) (a-\varepsilon)_+ \left((x^*x + \frac{1}{n})^{-\frac{1}{2}} - (x^*x + \frac{1}{m})^{-\frac{1}{2}} \right) x^* \right\| \\
&\le \left\| x \left((x^*x + \frac{1}{n})^{-\frac{1}{2}} - (x^*x + \frac{1}{m})^{-\frac{1}{2}} \right) x^*x \left((x^*x + \frac{1}{n})^{-\frac{1}{2}} - (x^*x + \frac{1}{m})^{-\frac{1}{2}} \right) x^* \right\| \\
&= \left\| xx^*(xx^* + \frac{1}{n})^{-\frac{1}{2}} - xx^*(xx^* + \frac{1}{m})^{-\frac{1}{2}} \right\|^2.
\end{aligned}$$

But notice that $\frac{t}{\sqrt{t+\frac{1}{n}}} \to \sqrt{t}$ uniformly on $\sigma(xx^*)$, (x_n) is Cauchy, hence $y \in A$. Now we have

$$y^*y = ((a-\varepsilon)_+)^{\frac{1}{2}} v^*v((a-\varepsilon)_+)^{\frac{1}{2}} = (a-\varepsilon)_+$$

as $(a-\varepsilon)_+ \le eb_0e = x^*x$, and v^*v is the projection onto $\ker(|x|)^\perp$, which is the closure of the range of x^*x. Moreover,

$$yy^* = v(a-\varepsilon)_+v^* \le vx^*xv^* = v|x|(v|x|)^* = xx^* = b_0^{\frac{1}{2}} e^2 b_0^{\frac{1}{2}} \le \|e\| b_0 = b_0 \le b^{\delta_0}.$$

Now we will construct our desired element d by forming a Cauchy sequence of elements and letting d be its limit. Let $d_n = y^*(b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} b^{\frac{\delta_0 - 1}{2}}$. Then

$$\begin{aligned}
&\|d_n - d_m\|^2 \\
&= \|(d_n - d_m)^*(d_n - d_m)\| \\
&= \left\| b^{\frac{\delta_0-1}{2}} \left((b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} - (b^{\delta_0} + \frac{1}{m})^{-\frac{1}{2}} \right) yy^* \left((b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} - (b^{\delta_0} + \frac{1}{m})^{-\frac{1}{2}} \right) b^{\frac{\delta_0-1}{2}} \right\| \\
&\le \left\| b^{\frac{\delta_0-1}{2}} \left((b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} - (b^{\delta_0} + \frac{1}{m})^{-\frac{1}{2}} \right) b^{\delta_0} \left((b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} - (b^{\delta_0} + \frac{1}{m})^{-\frac{1}{2}} \right) b^{\frac{\delta_0-1}{2}} \right\| \\
&= \left\| b^{\frac{2\delta_0-1}{2}} (b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} - b^{\frac{2\delta_0-1}{2}} (b^{\delta_0} + \frac{1}{m})^{-\frac{1}{2}} \right\|^2.
\end{aligned}$$

Now since $\frac{t^{\frac{2\delta_0-1}{2}}}{\sqrt{t^{\frac{2\delta_0-1}{2}}}} \to \sqrt{t^{\frac{2\delta_0-1}{2}}}$ uniformly on $\sigma(b)$, (d_n) is Cauchy in A. Hence let $d = \lim_n d_n \in A$.

Now we just need to show that d satisfies the conclusions in the statement of the lemma. Firstly

$$\begin{aligned}
d_n^* d_n &= b^{\frac{\delta_0-1}{2}} (b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} yy^* (b^{\delta_0} + \frac{1}{n})^{-\frac{1}{n}} b^{\frac{\delta_0-1}{2}} \\
&\le b^{\frac{\delta_0-1}{2}} (b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} b (b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} b^{\frac{\delta_0-1}{2}} \\
&= b^{\delta_0} (b^{\delta_0} + \frac{1}{n})^{-1}
\end{aligned}$$

which implies that $\|d_n^* d_n\| \leq 1$ for all n, hence $\|d\| \leq 1$. Finally, $db^{\frac{1}{2}} = \lim_n y^*(b^{\delta_0}+\frac{1}{n})^{-\frac{1}{2}} b^{\frac{\delta_0}{2}} = y^*$ as $(b^{\delta_0} + \frac{1}{n})^{-\frac{1}{2}} b^{\frac{\delta_0}{2}} \to p$ in WOT, where p is the projection onto $\ker(b^{\delta_0})$ and $yy^* \leq b^{\delta_0}$ gives that $y^*p = y^*$. Since the norm limit exists, it must be the same as the WOT limit. Therefore

$$dbd^* = db^{\frac{1}{2}}(db^{\frac{1}{2}})^* = y^*y = (a-\varepsilon)_+.$$

□

Lemma 5.1.20 (Kirchberg's Slice Lemma)**.** Let A, B be unital, $D \subseteq A \otimes B$ hereditary. Then there exist $0 \neq z \in A$ such that $zz^* \in D$ and $z^*z = a \otimes b$ for some $0 \leq a \in A, 0 \leq b \in B$.

Proof. Let $0 \neq a \in D$ be positive and let $\phi \in A^*, \psi \in B^*$ be pure states such that $\phi \otimes \psi(a) \neq 0$. Let $b_1 = (\phi \otimes \mathrm{id}_B)(a) \in B$. Then $\psi(b_1) = \phi \otimes \psi(a) \neq 0$, so b_1 is non-zero and positive. By scaling a, we can assume that $\|b_1\| = 1$. Now we can excise ϕ (as in chapter 1.5), so there exists $a_1 \in A$ with $\|a_1\| = 1$ such that

$$\|(a_1^{\frac{1}{2}} \otimes 1)a(a_1^{\frac{1}{2}} \otimes 1) - a_1 \otimes b_1\| < \frac{1}{4}.$$

Now by the above lemma, there exists $r \in A \otimes B$ such that $r^*(a_1^{\frac{1}{2}} \otimes 1)a(a_1^{\frac{1}{2}} \otimes 1)r = ((a_1 \otimes b_1) - \frac{1}{4})_+$. Letting $\delta > 0$ such that $\frac{1}{2} < \delta < 1$ and let $a = (a_1 - \delta)_+ \in A_+, b = (b_1 - \delta)_+ \in B_+$, both of which are non-zero. We claim that there exist $s \in C^*(a_1) \otimes C^*(b_1)$ such that $s^*((a_1 \otimes b_1) - \frac{1}{4})_+ s = a \otimes b$. Then $z = a^{\frac{1}{2}}(a_1^{\frac{1}{2}} \otimes 1)rs$ will be our desired element in the statement of the result.

Notice that $((a_1 \otimes b_1) - \frac{1}{4})_+, a \otimes b$ are in the abelian C*-algebra $C^*(a_1) \otimes C^*(a_2) = C$. Moreover

$$\overline{\{\phi \in \Sigma(C) \mid \phi(a \otimes b) \neq 0\}} \subseteq \{\phi \in \Sigma(C) \mid \phi((a_1 \otimes b_1) - \frac{1}{4})_+) \neq 0\},$$

where $\Sigma(C)$ is the maximal ideal space of C. Indeed, if $\phi \in \Sigma(C)$, then $\phi = \phi_1 \otimes \phi_2$, where $\phi_1 \in \Sigma(C^*(a_1)), \phi_2 \in \Sigma(C^*(b_1))$, and $\phi(a \otimes b) \neq 0$ implies that $\phi_1(a), \phi_2(b) \neq 0$. Then

$$\phi((a_1 \otimes b_1) - \frac{1}{4}1_{A \otimes B})_+) \geq \phi_1(a_1)\phi_2(b_1) - \frac{1}{2} > \delta^2 - \frac{1}{2} > 0.$$

Now looking at $C(\Sigma(C)) \simeq C^*(a_1) \otimes C^*(b_1)$, there exists $s \in C^*(a_1) \otimes C^*(b_1)$ such that

$$s^*((a_1 \otimes b_1) - \frac{1}{4}1_{A \otimes B})_+ s = a \otimes b.$$

Now we see that

$$z^*z = s^*r^*(a_1^{\frac{1}{2}} \otimes 1)a^{\frac{1}{2}}a^{\frac{1}{2}}(a_1^{\frac{1}{2}} \otimes 1)rs = s^*((a_1 \otimes b_1) - \frac{1}{4})_+ s = a \otimes b,$$

and

$$0 \leq zz^* = a^{\frac{1}{2}}(a_1^{\frac{1}{2}} \otimes 1)rss^*r^*(a_1^{\frac{1}{2}} \otimes 1)a_1^{\frac{1}{2}} \leq \|rss^*r^*\|a,$$

which implies that $zz^* \in D$ since D is hereditary. □

Theorem 5.1.21. Let A be a unital purely infinite C*-algebra, and B a unital C*-algebra such that every hereditary C*-algebra has a non-zero projection. Then $A \otimes B$ is purely infinite.

Proof. Let $D \subseteq A \otimes B$ be hereditary. By Kirchberg's slice lemma, there exists $z \in A \otimes B$ such that $zz^* \in D$ and $z^*z = a \otimes b$ for positive $a \in A, b \in B$. Let $A \otimes B) \subseteq \mathcal{B}(\mathcal{H})$, and let $v \in (A \otimes B)''$ be the partial isometry such that $z = v|z|$. Define $\pi : \overline{z^*z(A \otimes B)z^*z} \to A \otimes B$ by $\pi(x) = vxv^*$. We will see that π is a well-defined isomorphism onto its range, which is $\overline{zz^*(A \otimes B)zz^*} \subseteq D$, which is hereditary in D. Since $vz^*zv^* = zz^*$, vv^* is the projection onto the range of z, and v^*v is the projection onto the range of z^*, π is clearly a well-defined *-homomorphism. Now if $0 \leq x \in A \otimes B$, then

$$v(z^*zxz^*z)v^* = z|z|x|z|z \leq \|x\|zz^*zz^*,$$

so that $v(z^*zxz^*z)v^* \in \overline{zz^*(A \otimes B)zz^*}$, since this algebra is hereditary in D. So π does indeed have the correct codomain. Moreover, $\pi^{-1}(y) = v^*yv$ will also be a *-homomorphism which will be the inverse, hence $\overline{z^*z(A \otimes B)z^*z} \simeq \overline{zz^*(A \otimes B)zz^*}$, the latter being hereditary in D. To show that D has an infinite projection, it suffices to show that $\overline{z^*z(A \otimes B)z^*z}$ has an infinite projection. But notice that

$$\overline{aAa} \otimes \overline{bBb} \subseteq \overline{(a \otimes b)(A \otimes B)(a \otimes b)} = \overline{z^*z(A \otimes B)z^*z}.$$

But since $\overline{aAa} \subseteq A, \overline{bBb} \subseteq B$ are hereditary, A is purely infinite, and every hereditary subalgebra of B has a non-zero projection, just take an infinite projection $p \in \overline{aAa}$, and any non-zero projection $q \in \overline{bBb}$. Then $p \otimes q$ is an infinite projection. □

Corollary 5.1.22. If A, B are unital, simple, purely infinite, then $A \otimes B$ is unital, simple, purely infinite. In particular $\mathcal{O}_2 \otimes \mathcal{O}_2$ is unital, simple, purely infinite.

5.2 K-Theory for Purely Infinite C*-Algebras

This this section, we will follow [10] to see how the K-theory of purely infinite C*-algebra behaves. In particular, we will use the 6-term exact sequence to compute the K_0 and K_1 groups of $\mathcal{O}_n$. Since K-theory provides an isomorphism invariant, we will be able to distinguish the Cuntz algebras. Moreover, we will conclude that $\mathcal{O}_2$ only has one non-trivial projection up to Murray-von Neumann equivalence, and that the unitary group is connected.

Theorem 5.2.1. Let A be a unital, simple, purely infinite C*-algebra. Then $K_0(A) = \{[p]_0 \mid 0 \neq p = p^* = p^2 \in A\}$.

Proof. First let us see why it suffices to consider projections in A. First note that $M_n(A)$ is unital, simple, purely infinite. If $p, q \in M_n(A)$, then p, q are equivalent to orthogonal projections inside $A \simeq e_{11} \otimes A \subseteq M_n(A)$. Indeed, one can just take two orthogonal projections $p', q' \in A \simeq e_{11} \otimes A$, then p will be equivalent to a subprojection of p' and q will be equivalent to a subprojection of q' since $M_n(A)$ is unital, simple, purely infinite. Let us show that we actually have an identity and inverses, so that taking the Grothendieck group will change nothing. Let $Q(A) = \{[p]_0 \mid 0 \neq p = p^* = p^2 \in A\}$.

Now let $p, q \in P_1(A)$ with $p \sim p' < p, q \sim q' < q$, which exist by Theorem 5.1.16. We can further replace q by an equivalent projection to get that $q \leq p'$. Then

$$[p - p']_0 + [q - q']_0 = [p - (p' - q + q')]_0.$$

If $w \in A$ such that $w^*w = q$ and $ww^* = q'$, then letting $v = (p' - q) + w$ gives

$$v^*v = p' \text{ and } vv^* = p' - q + q'.$$

If e, f are orthogonal projections in A such that $e, f \leq d$ and $e \sim f$, then $d - e \sim d - f$. Moreover if $e = x^*x, f = xx^*$, then $y = (d - e - f) + x$ gives

$$y^*y = d - f \text{ and } yy^* = d - e.$$

Thus if $p \sim p'' \leq p - p'$, then $p - p' \sim p - p'' \sim p - (p' - q + q')$, and so

$$[p - p']_0 = [p - p'']_0 = [p - (p' - q + q')]_0 = [p - p']_0 + [q - q']_0.$$

By a symmetric argument, $[q - q']_0 = [p - p']_0 + [q - q']_0$, hence $[p - p']_0 = [q - q']_0$. Thus $[p - p']_0$ acts as identity in $Q(A)$. Indeed, if $q \in P_1(A)$, then

$$[q]_0 + [p - p']_0 = [q]_0 + [q - q']_0 = [q - q' + q']_0 = [q]_0,$$

and if $q \sim q', q'' < q$ where $q'q'' = 0$, then

$$[q]_0 + [q - q' - q'']_0 = [q - q' - q'' + q']_0 = [q - q']_0 = [p - p']_0.$$

Thus $[q - q' - q'']_0$ is the inverse of $[q]$ in $Q(A)$. It therefore follows that $Q(A) \simeq D(A) = P_\infty(A)/\sim_\infty$ is a group, and so $K_0(A) = Q(A)$. $\square$

Lemma 5.2.2. Let A be a unital, simple, purely infinite C*-algebra, $u \in U(A)$, $\lambda_1, \ldots, \lambda_n \in \sigma(u)$ distinct. For any $\varepsilon > 0$, there exists $v \in U(A)$ and non-zero orthogonal projections $p_1, \ldots, p_n \in A$ such that $\|u - v\| < \varepsilon$, each p_k commutes with v, $p_k v p_k = \lambda_k p_k$.

Proof. Let $f_1, \ldots, f_n \in C(\sigma(u))$ be non-zero positive functions whose supports are disjoint and contained in the sets $\Omega_k = \{z \in \sigma(u) \mid |z - \lambda_k| < \varepsilon_0\}$ for ε_0 small enough so that the supports are disjoint. Let p_k be any non-zero (infinite) projection in the hereditary subalgebra $\overline{f_k(u) A f_k(u)}$. Then p_k are clearly mutually orthogonal, and $p_j u p_i = 0$ for all $i \neq j$ since u commutes with $f_k(u)$. Now let

$$v_0 = \sum_1^n \lambda_j p_j + \left(1 - \sum_1^n p_j\right) u \left(1 - \sum_1^n p_j\right) \in A,$$

so that $\|v_0 - u\| \leq \max_j \|\lambda_j p_j - p_j u p_j\| < \varepsilon$. Choosing ε_0 small enough, v_0 can be invertible, so and we can let v be the unitary part of v_0 in the polar decomposition. $\square$

Corollary 5.2.3. Let A be a unital, simple, purely infinite C*-algebra, $u \in U(A)$. Then there exists a non-trivial projection $p \in A$ and $v \in pAp$ such that $u \sim_h v + (1 - p)$.

Theorem 5.2.4. Let A be a unital, simple, purely infinite C*-algebra. Then $U(A)/U_0(A) \to K_1(A)$ given by $[u] \mapsto [u]_1$ is a group isomorphism.

Proof. This map is clearly a well-defined homomorphism, so let us show that it is injective. Suppose that $[u]_1 = 0$ in $K_1(A)$.We know that

$$\begin{pmatrix} u & 0 \\ 0 & 1_n \end{pmatrix} \sim_h 1_{n+1}$$

for some $n \in \mathbb{N}$, and consequently $u \oplus 1_k \sim_h 1_{k+1}$ for all $k \geq n$. So it is clear when we use the $(\mathbb{K} \otimes A)^\sim$ definition of K_1, then $[u]_1 = 0$ if and only if $e \otimes u + (1 - e \otimes 1) \sim_h 1$ in $(\mathbb{K} \otimes A)^\sim$ for some rank one projection $e \in \mathbb{K}$.

By the above corollary, we can assume that $u = u' + (1 - p)$ for some non-trivial projection $p \in A$, and $u' \in U(pAp)$. Since $1 - p \neq 0$, there exists $q < 1 - p$ such that $q \sim 1 - p$, and since $1 - p - q \neq 0$, there exists pairwise orthogonal projections $(r_j)_{j \geq 1}$ in A such that $r_j \leq 1 - p - q$ and $r_j \sim 1 - p - q$. Let $r_0 = p + q$. Then $r_0 \sim 1$ and r_0 is orthogonal to each r_j. Now let $f_k = \sum_0^k r_j$, so that $f_k A f_k \simeq M_{k+1}(A)$ for all k, and so $(\mathbb{K} \otimes A)^\sim \simeq C^*(\cup_k f_k A f_k, 1) \subseteq A$. Associating r_0 with a rank one projection e in the copy of $\mathbb{K}$ in $(\mathbb{K} \otimes A)^\sim$, we have that $e \otimes u + (1 - e \otimes 1)$ gets sent to u under this isomorphism. Since $e \otimes u + (1 - e \otimes 1) \sim_h 1_{\mathbb{K} \otimes A}$ and *-homomorphisms send unitaries to unitaries and are continuous, it follows that $u \sim_h 1$.

For surjectivity, since A is unital, simple, and infinite, Theorem 5.1.7 and its proof give us isometries $(s_i)_1^\infty$ such that $\sum_1^n s_i s_i^* < 1$ for all n. Now let $u \in U(M_n(A))$. We claim that there exists $v \in U(A)$ such that $[u]_1 = [v]_1$.

First note that if $w \in U(A)$ and $s \in A$ is an isometry, then $[w]_1 = [sws^* + (1 - ss^*)]_1$. Indeed, $sws^* + 1 - ss^*$ is clearly unitary and

$$v(w \oplus 1)v^* = (sws^* + (1 - ss^*)) \oplus 1,$$

where

$$v = \begin{pmatrix} s & 1 - ss^* \\ 0 & s^* \end{pmatrix}.$$

Thus since v is clearly an isometry, $[w]_1 = [w \oplus 1]_1 = [(sws^* + 1 - ss^*) \oplus 1]_1 = [sws^* + 1 - ss^*]_1$.

This argument carries through further: if $w_1, \ldots, w_n \in U(A)$ and $t_1, \ldots, t_n \in A$ are isometries with mutually orthogonal ranges, then $w = \sum_1^n t_i w_i t_i^* + 1 - \sum_1^n t_i t_i^* \in U(A)$ is a unitary since $w = \Pi_1^n (t_i w_i t_i^* + 1 - t_i t_i^*)$, and we have $[w]_1 = \sum_1^n [t_i w_i t_i^* + 1 - t_i t_i^*]_1 = \sum_1^n [w_i]_1$. Now let

$$t = \begin{pmatrix} s_1 & \cdots & s_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_n(A).$$

Clearly t is an isometry and $tut^* = v \oplus 0_{n-1}$ for some $v \in A$. But since $tt^* = 0 \oplus 1_{n-1}$, $tut^* + 1 - tt^* = v \oplus 1_{n-1}$ and $tut^* + 1 - tt^*$ is unitary, so v is unitary as well. But clearly $[u]_1 = [v]_1$, hence the map is indeed surjective. $\square$

Proposition 5.2.5. Let $\mathrm{Hom}(\mathcal{O}_n)$ denote the space of unital *-homomorphisms $\mathcal{O}_n \to \mathcal{O}_n$ with the point-norm topology. Then the map $U(\mathcal{O}_n) \to \mathrm{Hom}(\mathcal{O}_n)$, given by $u \mapsto \phi_u$ where $\phi_u(s_i) = us_i$ (which exists by the universal property of the Cuntz algebra), is a homeomorphism.

Proof. First notice that every unital *-homomorphism does indeed arise as ϕ_u for some $u \in U(\mathcal{O}_n)$. Clearly by the universal property of $\mathcal{O}_n$, each ϕ_u is a homomorphism. Now if $\phi : \mathcal{O}_n \to \mathcal{O}_n$ is a unital *-homomorphism, let $u = \sum_1^n \phi(s_i)s_i^*$. Then

$$u^*u = \sum_{i,j} s_i\phi(s_i)^*\phi(s_j)s_j^* = \sum_1^n s_is_i^* = 1,$$

$$uu^* = \sum_{i,j} \phi(s_i)s_i^*s_j\phi(s_j)^* = \sum_1^n \phi(s_i)\phi(s_i)^* = 1,$$

so that u is unitary. Moreover,

$$us_j = \sum_1^n \phi(s_i)s_i^*s_j = \phi(s_j).$$

Now to see why this is a homeomorphism, it is clear that if $\phi_{u_\lambda} \to \phi_u$, then $u_\lambda \to u$ in $U(A)$. For the converse, if $u_\lambda \to u$ in $U(A)$, then clearly $\phi_{u_\lambda}(s_i) = u_\lambda s_i \to us_i = \phi_u(s_i)$ for all i. But this clearly implies that $\phi_{u_\lambda}(a) \to \phi_u(a)$ for a in the *-algebra generated by $(s_i)_1^n$ since the *-algebra operations are continuous. We just need to show that this passes to the closure. Suppose that (a_n) has the property that $\phi_{u_\lambda}(a_n) \to \phi_u(a_n)$ for all n and that $a_n \to a$. Then

$$\begin{aligned}
\|\phi_{u_\lambda}(a) - \phi_u(a)\| &\leq \|\phi_{u_\lambda}(a) - \phi_{u_\lambda}(a_n)\| + \|\phi_{u_\lambda}(a_n) - \phi_u(a_n)\| + \|\phi_u(a_n) - \phi_u(a)\| \\
&\leq \|a - a_n\| + \|\phi_{u_\lambda}(a_n) - \phi_u(a_n)\| + \|a - a_n\| \\
&= 2\|a - a_n\| + \|\phi_{u_\lambda}(a_n) - \phi_u(a_n)\|
\end{aligned}$$

since *-homomorphisms are contractive. Now $a_n \to a$ and $\phi_{u_\lambda}(a_n) \to \phi_u(a_n)$, so $\|\phi_{u_\lambda}(a) - \phi_u(a)\| \to 0$. ☐

Proposition 5.2.6. The map $\lambda : \mathcal{O}_n \to \mathcal{O}_n$ given by $\lambda(a) = \sum_1^n s_ias_i^*$ is homotopic to $\mathrm{id}_{\mathcal{O}_n}$.

Proof. Let $\lambda = \phi_u$, where $u \in U(\mathcal{O}_n)$. Then notice that

$$u = \sum_1^n \lambda(s_i)s_i^* = \sum_{i,j} s_js_is_j^*s_i^* = \left(\sum_{i,j} s_js_is_j^*s_i\right)^* = u^*.$$

So u is a self-adjoint unitary, hence $\sigma(u) \subseteq \mathbb{T} \cap \mathbb{R} = \{-1, 1\}$. But this implies that $u \sim_h 1$ since we can take a logarithm. Consequently, $\mathrm{id} \sim_h \phi_u = \lambda$. ☐

Corollary 5.2.7. Let B be a unital C*-algebra. Then $(n-1)g = 0$ for all $g \in K_0(B \otimes \mathcal{O}_n)$ and $(n-1)h = 0$ for all $h \in K_1(B \otimes \mathcal{O}_n)$.

Proof. If A is a C*-algebra, $p \in P_n(A)$, $s \in M_n(A)$ is an isometry, then $[sps^*]_0 = [p]_0$. Indeed, the partial isometry $v = ps^*$ gives the equivalence. Therefore for $p \in P_\infty(\mathcal{O}_n)$,

$$K_0(\lambda)([p]_0) = \sum_1^n [p]_0 = n \cdot [p]_0.$$

It then follows that

$$K_0(\mathrm{id} \otimes \lambda)([p]_0) = n \cdot [p]_0$$

for all $p \in P_\infty(B \otimes \mathcal{O}_n)$. The same argument applies with $K_1(\cdot)$. □

Corollary 5.2.8. $K_0(\mathcal{O}_2) = K_1(\mathcal{O}_2) = K_0(\mathcal{O}_2 \otimes \mathcal{O}_2) = K_1(\mathcal{O}_2 \otimes \mathcal{O}_2) = 0$.

Corollary 5.2.9. The unitary groups $U(\mathcal{O}_2)$ and $U(\mathcal{O}_2 \otimes \mathcal{O}_2)$ are connected.

Although the above corollaries are all the K-theory we require to prove the nuclear embedding theorem, we will finish this section off by using the 6-term exact sequence to compute the K-theory of all the Cuntz algebras.

Recall as in lemma 5.1.9, that if $\mathcal{C}_n$ is a C*-algebra with n isometries $(s_i)_1^n$ such that $\sum_1^n s_i s_i^* < 1$, then $\langle 1 - \sum_1^n s_i s_i^* \rangle \simeq \mathbb{K}$ and $\mathcal{C}_n/\mathbb{K} \simeq \mathcal{O}_n$. Let $\mathcal{E}_n = C^*(s_1, \ldots, s_n) \subseteq C^*(s_1, \ldots, s_{n+1}) = \mathcal{O}_{n+1}$. Let us denote the ideal $\mathbb{K} \simeq \langle 1 - \sum_1^n s_i s_i^* \rangle \lhd \mathcal{E}_n$ by $\mathcal{J}_n$.

Lemma 5.2.10. Let $v_1, \ldots, v_n \in \mathcal{B}(\mathcal{H})$, with $\mathcal{H}$ separable, be isometries such that $\sum_1^n v_i v_i^* < 1$. Then the map $v_i \mapsto s_i$ extends to an isomorphism $C^*(v_1, \ldots, v_n) \simeq \mathcal{E}_n$.

Proof. Let $p = 1 - \sum_1^n v_i v_i^*$, and let $\mathcal{K} = p\mathcal{H} \subseteq \mathcal{H}$. By replacing v_i with $v_i \otimes 1 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \simeq \mathcal{B}(\mathcal{H})$ if necessary, we can assume that $\dim \mathcal{K} = \dim \mathcal{H}$. Let $v_{n+1} : \mathcal{H} \to \mathcal{K}$ be a Hilbert space isomorphism. Then there there is an isomorphism $C^*(v_1, \ldots, v_{n+1}) \simeq \mathcal{O}_{n+1}$ which takes $v_i \mapsto s_i$, and so this map will take $C^*(v_1, \ldots, v_n)$ to $\mathcal{E}_n$. □

Now let $\mathcal{E}_n'$ be the smallest C*-subalgebra of $\mathcal{O}_{n+1}$ which is invariant under λ_{n+1}, given by $\lambda_{n+1}(x) = \sum_1^{n+1} s_i x s_i^*$, and $\mathcal{E}_n \subseteq \mathcal{E}_n'$. Evidently, $\mathcal{E}_n'$ is generated by $\cup_{k \geq 0} \lambda_{n+1}^k(\mathcal{E}_n)$. We also have that for every $x \in \mathcal{E}_n'$, $s_{n+1} x s_{n+1}^* = \lambda_{n+1}(x) - \sum_1^n s_i x s_i^* \in \mathcal{E}_n'$. Let $\mathcal{J}_n' \lhd \mathcal{E}_n'$ be the closed ideal generated by $s_{n+1} \mathcal{E}_n' s_{n+1}^*$.

Proposition 5.2.11. $\mathcal{J}_n' \lhd \mathcal{E}_n'$ is a proper ideal with $\mathcal{J}_n' \simeq \mathbb{K} \otimes \mathcal{E}_n'$ and $\mathcal{E}_n'/\mathcal{J}_n' \simeq \mathcal{O}_n$. Moreover if $q : \mathcal{E}_n \to \mathcal{O}_n, q' : \mathcal{E}_n' \to \mathcal{O}_n$ are the quotient maps, the following diagram commutes:

$$\begin{tikzcd} \mathcal{E}_n \arrow[r, "q"] \arrow[d, hook] & \mathcal{O}_n \arrow[d, "\phi"] \\ \mathcal{E}_n' \arrow[r, "q'"'] & \mathcal{O}_n, \end{tikzcd}$$

where ϕ is the isomorphism of $\mathcal{O}_n \simeq \mathcal{O}_n$ which takes $q(s_i)$ to $q'(s_i)$.

Proof. Since $\mathcal{E}_n'$ is the smallest algebra which contains $\mathcal{E}_n$ and is invariant under $x \mapsto s_{n+1} x s_{n+1}^*$, it is clear that $\mathcal{E}_n$ is generated by $s_1, \ldots, s_n$ together with $s_{n+1} \mathcal{E}_n' s_{n+1}^*$. Just as in the proof of Lemma 5.1.9, it is clear that if a is a word in $s_1, \ldots, s_n, s_1^*, \cdots, s_n^*$, $x \in s_{n+1} \mathcal{E}_n' s_{n+1}^*$, then $ax = 0$ or $a = s_\mu$ for some word μ in $\{1, \ldots, n\}$. Thus the span of $s_\mu x s_\nu^*$ for μ, ν words and $x \in s_{n+1}\mathcal{E}_n' s_{n+1}^*$ is dense in $\mathcal{J}_n'$. Moreover these are matrix units. Indeed, $(s_\mu x s_\nu^*)(s_\alpha y s_\beta^*) = \delta_{\nu\alpha} s_\mu x y s_\beta^*$. Thus there is an isomorphism of $\mathcal{J}_n'$ onto $\mathcal{K} \otimes \mathcal{E}_n'$ which sends $s_\mu x s_\nu'$ to $e_{\mu\nu} \otimes x$. In particular $\mathcal{J}_n'$ is not unital, so it is proper. The existence of ϕ is clear from the universal property of the Cuntz algebra. $\square$

Proposition 5.2.12. Let $\iota : \mathcal{J}_n \to \mathcal{E}_n$ be the inclusion map, and $q : \mathcal{E}_n \to \mathcal{O}_n$ be the quotient map. Then the following hold.

1. $K_0(q) : K_0(\mathcal{E}_n) \to K_0(\mathcal{O}_n)$ is surjective;
2. $K_1(q) : K_1(\mathcal{E}_n) \to K_1(\mathcal{O}_n)$ is an isomorphism;
3. $K_0(\iota) : K_0(\mathcal{J}_n) \to K_0(\mathcal{E}_n)$ is injective.

Proof. We have the short exact sequence $0 \to \mathcal{J}_n \to \mathcal{E}_n \to \mathcal{O}_n \to 0$, and so we have the 6-term exact sequence

$$\begin{tikzcd} \mathbb{Z} = K_0(\mathcal{J}_n) \arrow[r, "K_0(\iota)"] & K_0(\mathcal{E}_n) \arrow[r, "K_0(q)"] & K_0(\mathcal{O}_n) \arrow[d, "\delta_0"] \\ K_1(\mathcal{O}_n) \arrow[u, "\delta_1"] & K_1(\mathcal{E}_n) \arrow[l, "K_1(q)"] & K_1(\mathcal{J}_n) = 0. \arrow[l, "K_1(\iota)"] \end{tikzcd}$$

Since $\mathcal{J}_n \simeq \mathbb{K}$, $K_0(\mathcal{J}_n) = \mathbb{Z}$ and $K_1(\mathcal{J}_n) = 0$. By exactness, since $K_1(\mathcal{J}_n) = 0$, $\mathrm{Im} K_0(q) = \ker \delta_0 = K_0(\mathcal{O}_n)$, and so $K_0(q)$ is surjective. Moreover, $K_1(\mathcal{O}_n)$ is torsion by Corollary 5.2.7, hence $\delta_1 = 0$, and so $\ker K_0(\iota) = \mathrm{Im}\delta_1 = 0$, giving that $K_0(\iota)$ is injective. To see that $K_1(\mathcal{E}_n) \simeq K_1(\mathcal{O}_n)$, notice that $\ker K_1(q) = \mathrm{Im} K_1(\iota) = 0$, so that this homomorphism is injective. We further get that $\mathrm{Im} K_1(q) = \ker \delta_1 = K_1(\mathcal{O}_n)$ since $\delta_1 = 0$, so we have surjectivity and injectivity, hence $K_1(q)$ is a group isomorphism. $\square$

Proposition 5.2.13.

1. The homomorphisms $K_0(q') : K_0(\mathcal{E}_n') \to K_0(\mathcal{O}_n)$ and $K_1(q') : K_1(\mathcal{E}_n') \to K_1(\mathcal{O}_n)$ are surjective;
2. the homomorphisms $K_0(\iota) : K_0(\mathcal{J}_n') \to K_0(\mathcal{E}_n')$ and $K_1(\iota) : K_1(\mathcal{J}_n') \to K_1(\mathcal{E}_n')$ are injective.

Proof. This follows by the above result combined with the the fact that the diagram

$$\begin{tikzcd} \mathcal{E}_n \arrow[r, "q"] \arrow[d, hook] & \mathcal{O}_n \arrow[d, "\phi"] \\ \mathcal{E}_n' \arrow[r, "q'"'] & \mathcal{O}_n, \end{tikzcd}$$

commutes. $\square$

Lemma 5.2.14. The *-homomorphisms $\mathrm{id}_{\mathcal{E}_n}, \lambda_{n+1}|_{\mathcal{E}_n} : \mathcal{E}_n \to \mathcal{E}_n'$ are homotopic.

Proof. By the same argument in Proposition 5.2.5, $\lambda_{n+1} = \phi_u$ for the self-adjoint unitary $u = \sum_{i,j} s_i s_j s_i^* s_j^* \in \mathcal{O}_{n+1}$, hence $\lambda_{n+1}(s_i) = us_i$ for all i. Since $s_{n+1}xs_{n+1}^* \in \mathcal{E}_n'$ for all $x \in \mathcal{E}_n'$, $s_i(s_j s_i^* s_j^*) \in \mathcal{E}_n'$ for $i \neq n+1$ and, $(s_i s_j s_i^*)s_j^* \in \mathcal{E}_n'$ for $j \neq n+1$. Also $s_{n+1}^2(s_{n+1}^*)^2 = s_{n+1}(1 - \sum_1^n s_i s_i^*)s_{n+1}^* \in \mathcal{E}_n'$, so $s_i s_j s_i^* s_j^* \in \mathcal{E}_n'$ for all $1 \leq i, j \leq n+1$. Now let $\alpha : [0,1] \to \mathrm{Hom}(\mathcal{E}_n, \mathcal{E}_n')$, the space of unital *-homomorphisms $\mathcal{E}_n \to \mathcal{E}_n'$, be given by $\alpha(t)(s_i) = v_t s_i$, where the map $t \mapsto v_t$ is a homotopy such that $v_0 = 1, v_1 = u$, which exists since u is self-adjoint. It is clear that α provides a homotopy for $\mathrm{id}_{\mathcal{E}_n}$ and $\lambda_{n+1}|_{\mathcal{E}_n}$. □

Let $\rho : \mathcal{E}_n' \to \mathcal{E}_n'$ be the map $\rho(x) = s_{n+1}xs_{n+1}^*$. Then the homomorphisms $K_0(\rho)$ and $K_1(\rho)$ are injective. Indeed, we have $K_i(\mathcal{E}_n') \to K_i(\mathcal{J}_n') \to K_i(\mathcal{E}_n')$, where the first homomorphism is induced from the map $\rho : \mathcal{E}_n' \to \mathcal{J}_n'$ is given by $\rho(x) = s_{n+1}xs_{n+1}^*$ is an isomorphism since $K_i(\iota) : K_i(\mathcal{J}_n') \to K_i(\mathcal{E}_n')$ are injective by Proposition 5.2.13, and the following diagram commutes

$$\begin{tikzcd} \mathcal{E}_n' \arrow[r, "\rho"] \arrow[d, "\mathrm{id}"'] & \mathcal{J}_n' \arrow[d, "\simeq"] \\ \mathcal{E}_n' \arrow[r, hook] & \mathbb{K} \otimes \mathcal{E}_n'. \end{tikzcd}$$

Proposition 5.2.15.

1. $[p]_0 = n[p]_0 + K_0(\rho)([p]_0)$ in $K_0(\mathcal{E}_n')$ for all projections $p \in \mathcal{E}_n \subseteq \mathcal{E}_n'$.

2. $[p]_0 \neq n[p]_0$ in $K_0(\mathcal{E}_n)$ for all $p \in \mathcal{E}_n$ such that $K_0(\rho)([p]_0) \neq 0$.

Proof. Notice that $[\lambda_{n+1}(p)]_0 = n[p]_0 + K_0(\rho)([p]_0)$ in $K_0(\mathcal{E}_n')$ for all projections $p \in \mathcal{E}_n'$. If $p \in \mathcal{E}_n$, then $[p]_0 = [\lambda_{n+1}(p)]_0$ in $K_0(\mathcal{E}_n')$ and so (1) follows.

Now let $K_0(q), K_0(q')$ and $K_0(j)$ be the maps induced from the quotient maps q, q' and the inclusion map $j : \mathcal{E}_n \to \mathcal{E}_n'$ respectively. Then since $q = q' \circ j$, $K_0(q) = K_0(q') \circ K_0(j)$. Thus if $K_0(q)([p]_0) \neq 0$, then $K_0(j)([p]_0) \neq 0$, so

$$K_0(j)([p]_0) = nK_0(j)([p]_0) + K_0(\rho) \circ K_0(j)([p]_0) \neq nK_0(j)([p]_0)$$

in $K_0(\mathcal{E}_n')$ by (1) and the fact that ρ is injective. □

Theorem 5.2.16. $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ for $2 \leq n < \infty$.

Proof. Since $\mathcal{O}_n$ is unital, simple, purely infinite, we only need to consider equivalence classes of projections in $\mathcal{O}_n$ itself by Theorem 5.2.1. Let $r = [1 - \sum_1^n s_i s_i^*]_0 \in K_0(\mathcal{E}_n)$. Then the kernel of $K_0(q) : K_0(\mathcal{E}_n) \to K_0(\mathcal{O}_n)$ is just $\mathbb{Z}r$ since r generates $K_0(\iota)(K_0(\mathcal{J}_n)) = \ker K_0(q)$. Thus for every $p \in K_0(\mathcal{E}_n)$, $np = p + kr$ for some $k \in \mathbb{Z}$ since every element has order $n-1$. Now since $n[1]_0 = [1]_0 - r$, we have

$$n(p + k[1]_0) = p + kr + [1]_0 - kr = p + k[1]_0.$$

But then we have $K_0(q)(p) = -kK_0(q)([1]_0) = -k[1]_0$ in $K_0(\mathcal{O}_n)$ by part (2) of the above proposition. Thus $K_0(\mathcal{O}_n) = \mathbb{Z}[1]_0$, so it is cyclic. Now let us see that $k[1]_0 = 0$ in $K_0(\mathcal{O}_n)$ if and only if $k \equiv 0 \bmod (n-1)$.

If $k[1]_0 = 0$ in $K_0(\mathcal{O}_n)$ for $k \in \mathbb{Z}$, then $k[1]_0 = jr$ in $K_0(\mathcal{E}_n)$ for some $j \in \mathbb{Z}$. Multiplying both sides by n yields

$$k([1]_0) - r = njr$$

since $n[1]_0 = [1]_0 - r$ in $K_0(\mathcal{E}_n)$. As $k[1]_0 = jr$,

$$kr = -(n-1)jr.$$

Since $lr \neq 0$ for all $0 \neq l \in \mathbb{Z}$, it follows that $k \equiv 0 \bmod (n-1)$. □

Corollary 5.2.17. $\mathcal{O}_n \not\simeq \mathcal{O}_m$ for $n \neq m$.

Theorem 5.2.18. $K_1(\mathcal{O}_n) = 0$ for all $2 \leq n < \infty$.

Proof. Let $K_1(j) : K_1(\mathcal{E}_n) \to K_1(\mathcal{E}'_n)$ be the map induced from the inclusion $j : \mathcal{E}_n \to \mathcal{E}'_n$. It suffices to show that the restriction of $K_1(\rho) : K_1(\mathcal{E}'_n) \to K_1(\mathcal{E}'_n)$ to $K_1(\mathcal{E}_n)$ is trivial. Indeed, since $K_1(\rho)$ is injective, this will imply that $K_1(\rho)(K_1(\mathcal{E}_n)) = 0$, and the result will follow from the surjectivity of $K_1(q) : K_1(\mathcal{E}_n) \to K_1(\mathcal{O}_n)$.

Now $u = nu + K_1(\rho)(u)$ for all $u \in K_1(j)(K_1(\mathcal{E}_n))$ since $\mathrm{id}_{\mathcal{E}_n} \sim_h \lambda_{n+1}|_{\mathcal{E}_n}$ by Lemma 5.2.14. But we also have that $u = nu$ for all $u \in K_1(j)(K_1(\mathcal{E}_n))$ by Corollary 5.2.7 and the fact that $K_1(\mathcal{E}_n) \simeq K_1(\mathcal{O}_n)$ by part (2) of Proposition 5.2.12. But this implies that $K_1(\rho)(u) = 0$ for all $u \in K_1(j)(K_1(\mathcal{E}_n))$. □

Finally, let us end off by working toward the K-theory of $\mathcal{O}_\infty$.

Proposition 5.2.19. $K_0(\mathcal{E}_n) = \mathbb{Z}$ and $K_1(\mathcal{E}_n) = 0$ for all $2 \leq n < \infty$.

Proof. Since $K_0(\mathcal{J}_n) \to K_0(\mathcal{E}_n) \to K_0(\mathcal{O}_n)$ is exact and $K_0(\mathcal{O}_n) \simeq \mathbb{Z}_{n-1}$, it follows that $K_0(\mathcal{E}_n)$ is generated by $K_0(\iota)(K_0(\mathcal{J}_n))$ and $[1]_0$. But $K_0(\iota)(K_0(\mathcal{J}_n)) = \mathbb{Z}r$ with $r = [1 - \sum_1^n s_i s_i^*]_0$ as above, and $r = -(n-1)[1]_0$. Thus $K_0(\mathcal{E}_n) = \mathbb{Z}[1]_0$. Since $K_0(\iota)$ is injective, $\mathbb{Z} \simeq \iota(K_0(\mathcal{J}_n)) \subseteq \mathbb{Z}[1]_0$, and since $\mathbb{Z}[1]_0$ is torsion free, it is isomorphic to $\mathbb{Z}$.

$K_1(\mathcal{E}_n) = 0$ just follows from the fact that $K_1(\mathcal{E}_n) \simeq K_1(\mathcal{O}_n)$ by part (2) of Proposition 5.2.12. □

Theorem 5.2.20. $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$.

Proof. Let $(s_i)_1^\infty$ be a sequence of isometries with pairwise orthogonal ranges such that $\mathcal{O}_\infty = C^*(s_1, \dots)$. Then $\mathcal{O}_\infty$ is the inductive limit of the subalgebras $C^*(s_1, \dots, s_n)$ which are isomorphic to $\mathcal{E}_n$. Thus for $K_i(\mathcal{O}_\infty)$ is the inductive limit of

$$K_i(\mathcal{E}_2) \xrightarrow{K_i(j)} K_i(\mathcal{E}_3) \xrightarrow{K_0(j)} \cdots,$$

where j is the inclusion $C^*(s_1, \dots, s_n) \to C^*(s_1, \dots, s_{n+1})$. Thus it follows that $K_1(\mathcal{O}_\infty) = 0$. Moreover $K_0(\mathcal{E}_n) = \mathbb{Z}[1]_0$ and $K_0(j)([1_n]_0) = [1_{n+1}]_0$ where 1_n is the unit for $\mathcal{E}_n$. This $K_0(j)$ is an isomorphism for all n. □

Corollary 5.2.21. $\mathcal{O}_n \not\simeq \mathcal{O}_\infty$ for all $n \geq 2$.

Corollary 5.2.22. For all $2 \leq n \leq \infty$, the unitary group of $\mathcal{O}_n$ is connected. Moreover if $2 \leq n < \infty$, then every projection in $\mathcal{O}_n$ is equivalent to one of the form $\sum_1^k s_i s_i^*$ for $1 \leq k < n$. In $\mathcal{O}_\infty$, every projection is equivalent to a projection of the form $\sum_1^k s_i s_i^*$ or $1 - \sum_1^k s_i s_i^*$ for $1 \leq k < \infty$.

5.3 Real Rank Zero, Exponential Rank, and Exponential Length

We will now be concerned with certain approximation properties of purely infinite C*-algebras. We first study algebras of real rank zero, of which many of the results can be found in [5] and chapter V.7 of [11]. We then proceed to study the finite unitary property (FU) and the weak finite unitary property weak (FU), which were studied by Phillips in [23] and [24]. This paves way for the study of exponential rank and exponential length.

Definition 5.3.1. A unital C*-algebra A has **real rank zero** if the invertible self-adjoint elements A_{sa}^{-1} are dense in the self-adjoint elements A_{sa}. If A is non-unital, we say it has real rank zero if its unitization does.

Example 5.3.2.

1. $C(X)$ has real rank zero if and only if X is totally disconnected.

2. Finite-dimensional C*-algebras real rank zero.

3. von Neumann algebras are real rank zero. This is because we have the L^∞ functional calculus, and we know that simple functions are dense.

4. Inductive limits of real rank zero algebras are real rank zero. In particular, AF algebras are real rank zero.

Proposition 5.3.3. Let A be a C*-algebra. The following are equivalent.

(RR0) A has real rank zero.

(FS) The self-adjoint elements with finite spectrum are dense in the self-adjoint elements.

(HP) Every hereditary subalgebra of A has an approximate unit of projections (which is not necessarily increasing).

Proof. Suppose that (RR0) holds, let $a \in A_{sa}$ with $\|a\| = 1$, and fix $\varepsilon > 0$. Let $-1 = t_1, \cdots, tn = 1$ be an increasing subset of non-zero points in $[-1,1]$. Then, by (RR0), there exists $a_1 \in A_{sa}$ such that $a_1 - t_1 1$ is invertible and $\|a - a_1\| = \|(a - t_1 1) - (a_1 - t_1 1)\| < e_1 = \frac{\varepsilon}{4}$. Now let $\varepsilon_2 < \frac{\varepsilon}{8}$ such that $[t_1 - \varepsilon_2, t_1 + \varepsilon_2] \cap \sigma(a_1) = \emptyset$. Then by (RR0), let $a_2 \in A_{sa}$ be such that $a_2 - t_2 1$ is invertible and $\|(a_2 - t_2 1) - (a_1 - t_2 1)\| = \|a_2 - a_1\| < \varepsilon_2$. Then $t_1, t_2 \in \sigma(a_2)$. Inductively do this to get $a_1, \ldots, a_n \in S_{sa}$ such that $t_1, \ldots, t_n \in \sigma(a_n)$ and

$$\|a - a_n\| < \sum_1^n \varepsilon_i < \sum_1^n \frac{1}{2^{i+1}} < \frac{\varepsilon}{2}.$$

Since $t_i \notin \sigma(a_n)$ for all i,

$$b = -\chi_{(-1-\frac{\varepsilon}{2},-1]}(a_n) + \sum_2^n t_i \chi_{(t_{i-1},t_i]}(a_n) + \chi_{(1,1+\frac{\varepsilon}{2})}(a_n) \in A,$$

where $\chi_S(x)$ is the spectral projection of x onto $S \subseteq \mathbb{C}$. Then b has finite spectrum and

$$\|b - a\| \leq \|b - a_n\| + \|a_n - a\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now assume that (FS) holds and suppose that $B \subseteq A$ is hereditary. To see that B has an approximate identity of projections, it suffices to show that for any$b_1, \ldots, b_n \in B$ and $0 < \varepsilon < 1$, there exists $p = p^* = p^2 \in B$ such that $\|b_i - b_i p\| < \varepsilon$. If $b = \sum_1^n b_i b_i^*$, then

$$\|b_i - b_i p\|^2 = \|b_i(1-p)\|^2 = \|(1-p)b_i^* b_i(1-p)\| \leq \|b(1-p)\| = \|b - bp\|.$$

So it suffices to show this for $\|b - bp\| < \varepsilon$. Without loss of generality, suppose that $\|b\| = 1$. Let $\delta > 0$ such that $\delta < \frac{\varepsilon - \varepsilon^2}{6}$, and let $n \in \mathbb{N}$ be such that $\delta^{\frac{2}{n}} > 1 - \delta$. Then there exists $0 \leq c \in A$ with finite spectrum such that $\|b^{\frac{1}{n}} - c\| < \frac{\delta}{n}$ with $\|c\| \leq 1$. Thus $a = c^n$ satisfies

$$\|a - b\| = \|\sum_0^{n-1} c^{n-1-i}(c - b^{\frac{1}{n}})b^{\frac{i}{n}}\| \leq \sum_0^{n-1} \|c^{n-1-i}\|\|c - b^{\frac{1}{n}}\|\|b^{\frac{1}{n}}\|^{n-i-1} \leq n\|b^{\frac{1}{n}} - c\| < \delta.$$

Since c has finite spectrum, a has finite spectrum by the spectral theorem, and so $\chi_{[\delta,1]}$ is continuous on $\sigma(a)$, hence letting $q = \chi_{[\delta,1]}(a)$, we have that $\|a - aq\| < \delta$ and

$$\|a^{\frac{1}{n}} q a^{\frac{1}{n}} - q\| \leq 1 - \delta^{\frac{2}{n}} < \delta.$$

Now since B is hereditary, $x = b^{\frac{1}{n}} q b^{\frac{1}{n}} \in B$. Thus

$$\|x - q\| \leq 2\|b^{\frac{1}{n}} - a^{\frac{1}{n}}\| + \|a^{\frac{1}{n}} q a^{\frac{1}{n}} - q\| < 3\delta.$$

Now

$$\|x - x^2\| = \|(1-q)(x-q) - (x-q)x\| \leq 6\delta < \varepsilon - \varepsilon^2.$$

Hence $\sigma(x) \subseteq [0, \varepsilon] \cup [1-\varepsilon, 1]$. Thus $p = \chi_{[1-\varepsilon,1]}(x) \in B$ and $\|p - x\| \leq \varepsilon$. Finally,

$$\|b - bp\| \leq \|p - q\| + \|a - b\| + \|a - aq\| \leq \varepsilon + 5\delta < 2\varepsilon.$$

Now assume that (HP) holds and let $a = a^* \in A$ with $\|a\| = 1$. Decompose a into its positive an negative parts, $a = a_+ - a_-$. Let $B = \overline{a_+ A a_+}$, so by (HP) there exists $p \in B$ such that $\|a_+ - a_+ p\| < \varepsilon$. Since $a_- a_+ = 0$, we also have that $a_- p = 0$. Let

$$\begin{aligned} b &= pap + 2\varepsilon p + (1-p)a(1-p) - 2\varepsilon(1-p) \\ &= a - pa(1-p) + (1-p)ap + 2\varepsilon(p - (1-p)). \end{aligned}$$

Then

$$\|a - b\| \leq \|pa(1-p) + (1-p)ap\| + 2\varepsilon\|p - (1-p)\| \leq \varepsilon + 2\varepsilon = 2\varepsilon.$$

But

$$\begin{aligned} pbp &= (1-p)a_-(1-p + (1-p)a_+(1-p) - 2\varepsilon(1-p) \\ &\leq \varepsilon(1-p) - 2\varepsilon(1-p) \\ &= -\varepsilon(1-p). \end{aligned}$$

Since b and p commute, b must be invertible. It follows that A has (RR0). $\square$

Theorem 5.3.4. Let A be unital, simple, purely infinite. Then A has real rank zero.

Proof. Suppose that $a \in A_{sa}$ and $\varepsilon > 0$. Let

$$f_\varepsilon(t) = \begin{cases} 0, & \text{if } |t| \leq \varepsilon, \\ t-\varepsilon, & \text{if } t \geq \varepsilon \\ t+\varepsilon, & \text{if } t \leq -\varepsilon, \end{cases}$$

and let $g_\varepsilon(t) = \max\{\varepsilon - |t|, 0\}$. Let $B = \overline{g_\varepsilon(a)Ag_\varepsilon(a)} \subseteq A$, which is hereditary, and so there exists an infinite projection $p \in B$ since A is purely infinite. By Theorem 5.1.16, $1-p \sim q \leq p$. So let $v \in A$ be a partial isometry such that $v^*v = 1-p$ and $vv^* = q \leq p$. Since $f_\varepsilon(t)g_\varepsilon(t) = 0$, we have that $f_\varepsilon(a) = (1-p)f_\varepsilon(a)(1-p)$. Now let

$$b = f_\varepsilon(a) + \varepsilon(v+v^*) + \varepsilon(p-q) \simeq \begin{pmatrix} f_\varepsilon(a) & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix},$$

where this matrix comes from taking a unital faithful representation $A \subseteq \mathcal{B}(\mathcal{H})$, and decomposing $\mathcal{H} = (1-p)\mathcal{H} \oplus q\mathcal{H} \oplus (p-q)\mathcal{H}$ with matrix unit $e_{21} = v : (1-p)\mathcal{H} \to q\mathcal{H}$. This matrix has inverse

$$\begin{pmatrix} 0 & \frac{1}{\varepsilon} & 0 \\ \frac{1}{\varepsilon} & -\frac{1}{\varepsilon^2}f_\varepsilon(a) & 0 \\ 0 & 0 & \frac{1}{\varepsilon} \end{pmatrix}.$$

Moreover b is self-adjoint, and

$$\|b-a\| \leq \|f_\varepsilon(a) - a\| + \varepsilon\|v + v^* + (p-q)\| \leq \varepsilon + 3\varepsilon = 4\varepsilon.$$

$\square$

Corollary 5.3.5. The Cuntz algebras $\mathcal{O}_n, \mathcal{O}_\infty$, and the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathbb{K}$ have real rank zero.

We now follow [23] and [24] to prove that the unitary groups of the Cuntz algebras have desirable approximation properties.

Definition 5.3.6. Let A be a unital C*-algebra. We say that A has the **finite unitary property**, abbreviated (FU), if the elements of $U(A)$ with finite spectrum are dense in $U(A)$. We say that A has the **weak finite unitary property**, abbreviated weak (FU), if the element of $U_0(A)$ with finite spectrum are dense in $U_0(A)$. If A is non-unital, we say that A has these respective properties if $\tilde{A}$ does.

Property (FU) is clearly stronger than weak (FU).

Example 5.3.7.

1. Finite dimensional C*-algebras clearly have property (FU) since every unitary has finite spectrum.

2. von Neumann algebras have property (FU) since unitaries are normal and simple functions are dense in the bounded Borel functions.

Definition 5.3.8. Let A be a unital C*-algebra. The **exponential rank** of A, written $\mathrm{cer}(A)$, is the largest element of the set of symbols $\{1, 1+\varepsilon, 2, 2+\varepsilon, \ldots, \infty\}$, with obvious order, such that

1. $\mathrm{cer}(A) \leq n$ if every $u \in U_0(A)$, $u = \Pi_1^n e^{ia_j}$ for $a_j = a_j^* \in A$.
2. $\mathrm{cer}(A) \leq n + \varepsilon$ if every $u \in U_0(A)$ is a norm limit of products as in (1).

If A is non-unital, we let $\mathrm{cer}(A) = \mathrm{cer}(\tilde{A})$.

Proposition 5.3.9. Let A be a unital C*-algebra. Then A has weak (FU) if and only if A has real rank 0 and $\mathrm{cer}(A) \leq 1 + \varepsilon$. Also A has (FU) if and only if A has real rank 0, $\mathrm{cer}(A) \leq 1 + \varepsilon$ and $U(A)$ is connected.

Proof. Suppose that A has weak (FU). Then $\mathrm{cer}(A) \leq 1 + \varepsilon$ since every unitary with finite spectrum is an exponential. To see that A has (RR0), let $a \in A_{sa}$, and without loss of generality, suppose that $\|a\| < \pi$. Let $e^{ia} = \lim_n u_n$ where $u_n \in U(A)$ have finite spectrum. Let f be a branch of the logarithm. Then $-if(u_n)$ is self-adjoint, has finite spectrum, and $-if(u_n) \to a$.

Conversely, suppose that A has (RR0), $\mathrm{cer}(A) \leq 1 + \varepsilon$, and let $u \in U_0(A)$. Write $u = \lim_n e^{ia_n}$ for $a_n \in A_{sa}$. Since A has (RR0), let $b_n \in A_{sa}$ have finite spectrum such that $\|a_n - b_n\| < \frac{1}{n}$. Then $u_n = e^{ib_n}$ is unitary with finite spectrum and $u_n \to u$.

Finally, it is clear that A has (FU) if and only if A has weak (FU) and $U(A)$ is connected. $\square$

Remark 5.3.10. If A is unital, $\mathrm{cer}(A \oplus C) = \mathrm{cer}(A)$, and so $\mathrm{cer}(\tilde{A}) = \mathrm{cer}(A)$ for any C*-algebra.

Proposition 5.3.11. Let $A = \lim\limits_{\rightarrow} A_\alpha$ is a direct limit of C*-algebras (our index set being directed) and that $\mathrm{cer}(A_\alpha) \leq n + \varepsilon$ for all α. Then $\mathrm{cer}(A) \leq n + \varepsilon$.

Proof. Without loss of generality, we can assume that the A and the A_α's are unital by unitizing everything. Let $\beta_\alpha : A_\alpha \to A$ be the inclusion maps. Let $u \in U_0(A)$ and let $u = \Pi_1^N e^{ia_j}$ for some N and $a_j \in A_{sa}$. Note that we can assume that $N \geq n$ since otherwise it is trivial. Since $\cup_\alpha \beta_\alpha(A_\alpha)$ is dense in A, and the index set is directed, there exists $\alpha(k), a_j^{(k)} \in (A_{\alpha(k)})_{sa}$ such that $\beta_{\alpha(k)}(a_j^{(k)}) \to a_j$ for all $j = 1, \ldots, N$. Let $v_k = \Pi_1^N e^{ia_j^{(k)}} \in U_0(A_{\alpha(k)})$, so $\beta_{\alpha(k)}(v_k) \to u$. But for each k there is $u_k \in U_0(A_{\alpha(k)})$ which is a product of n exponentials with $\|u_k - v_k\| < \frac{1}{k}$. Then $\beta_{\alpha(k)}(u_k)$ is a product of n exponentials which converges to u. $\square$

Lemma 5.3.12. Let $\phi : A \to B$ be a surjective, unital *-homomorphism. Then $\phi(U_0(A)) = U_0(B)$.

Proof. A unital *-homomorphism is continuous and maps unitaries to unitaries, so $\phi(U_0(A)) \subseteq U_0(B)$. Conversely, say $u = \Pi_1^n e^{ib_j} \in U_0(B)$ for some $b_j \in B_{sa}$. Since ϕ is surjective, there exists $a_j \in A$ such that $\phi(a_j) = b_j$. Sine ϕ is a *-homomorphism and clearly $\phi(\frac{a_j + a_j^*}{2}) = b_j$, we can assume that $a_j \in A_{sa}$. Now let $v = \Pi_1^n e^{ia_j} \in U_0(A)$, so that $\phi(v) = u$. $\square$

Corollary 5.3.13. If $\phi : A \to B$ is a surjective unital *-homomorphism, then $\mathrm{cer}(B) \leq \mathrm{cer}(A)$.

Example 5.3.14.

1. Finite-dimensional C*-algebras have exponential rank 1. Indeed, since we can take a logarithm f, we can write a unitary as $u = e^{f(u)}$ where $f(u)$ is self-adjoint.

2. von Neumann algebras have exponential rank 1 by the same argument, except the logarithm comes from the L^∞-functional calculus.

3. The Calkin algebra has exponential rank 1 since $\mathcal{B}(\mathcal{H})$ does and the quotient map is surjective.

4. Commutative C*-algebras have exponential rank 1. This is because for self-adjoints $a, b \in A$, a commutative C*-algebra, we have $ab = ba$ and so $e^{ia}e^{ib} = e^{i(a+b)}$.

5. AF algebras have exponential rank 1 or $1 + \varepsilon$. This follows from Proposition 5.3.11 and the fact that finite-dimensional C*-algebras have exponential rank 1.

6. Let $D \subseteq \mathbb{T}$ be the dyadic rational mod $\mathbb{Z}$ and let $G = D \rtimes (\mathbb{Z}/2\mathbb{Z})$ with the action of $\mathbb{Z}/2\mathbb{Z}$ on D being inversion. Let $D \curvearrowright \mathbb{T}$ by rotation and $\mathbb{Z}/2\mathbb{Z}$ act by $z \mapsto -z$. Thus $G \curvearrowright \mathbb{T}$ with these actions. Let $A = C(\mathbb{T}) \rtimes D$ be the crossed product. One can see in [20] that A is AF, and we will see that and $\mathrm{cer}(A) = 1 + \varepsilon$, so that proposition 5.3.11 does not hold if we replace $n + \varepsilon$ by n.

   To see that $\mathrm{cer}(A) = 1 + \varepsilon$, let $f(z) = z$. If $f = e^{ia}$ for some $a = a^* \in A$, then a commutes with f and f^*, so $a \in C(\mathbb{T})$. Since $f \notin U_0(C(\mathbb{T}))$, this is a contradiction. Hence $\mathrm{cer}(A) \neq 1$.

Our goal will be to show that $\mathrm{cer}(A) \leq 1 + \varepsilon$ for any unital, simple, purely infinite C*-algebra. Consequently, since such an algebra has real rank 0, it will follow that A has weak (FU). Since all of the Cuntz algebras have connected unitary groups, it will further follow that they have (FU).

Lemma 5.3.15. Let A be unital, $\alpha : [0,1] \to U(A)$, $t \mapsto u_t$, by be a piecewise C^1 path in $U(A)$ such that $\alpha(0) = 1$. Let L be length of the path. Then $\sigma(\alpha(1)) \subseteq \{e^{i\theta} \mid -L \leq \theta \leq L\}$.

Proof. If $u, v \in U(A)$ and $\lambda \in \sigma(v)$, then there exists $\mu \in \sigma(u)$ such that $|\lambda - \mu| \leq \|v - u\|$. Indeed, if $|\lambda - \mu| > \|u - v\|$ for all $\mu \in \sigma(u)$, then

$$\|1 - (\lambda 1 - u)\| = \|(\lambda 1 - v)^{-1}(\lambda 1 - v) - (\lambda 1 - u)\| \leq \|(\lambda 1 - v)^{-1}\|\|u - v\| < 1$$

since $\|(\lambda 1 - v)^{-1}\| \leq \mathrm{dist}(\lambda, \sigma(v))^{-1} < \|u - v\|^{-1}$. But then $\lambda 1 - v \in A^{-1}$, which is a contradiction.

Now let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of $[0,1]$. Let $\lambda \in \sigma(\alpha(1))$ and let $\mu_k \in \sigma(\alpha(t_k))$ be such that $\mu_n = \lambda$ and $|\mu_k - \mu_{k-1}| \leq \|\alpha(t_k) - \alpha(t_{k-1})\|$. Then $\mu_0 = 1$ and

$$\sum_1^n |\mu_k - \mu_{k-1}| \leq \sum_1^n \|\alpha(t_k) - \alpha(t_{k-1})\|.$$

Now taking a limit as $\max_k |t_k - t_{k-1}| \to 0$, the right hand side approaches L. Since $\max_k |\mu_k - \mu_{k-1}| \le \max_k \|\alpha(t_k) - \alpha(t_{k-1})\|$, which approaches 0, the lim inf of the left hand side of the inequality above is at most the length of the path around the unit circle from 1 to λ. Thus $\lambda = e^{i\theta}$ for some $\theta \in [-L, L]$. $\square$

Corollary 5.3.16. Let A be a unital C*-algebra, $u \in U(A), \varepsilon > 0$. Then there exists $a \in M_2(A)_{sa}$ such that $\|u \oplus u^* - e^{ia}\| < \varepsilon$.

Proof. Let

$$\alpha(t) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

for $t \in [0, \frac{\pi}{2}]$. Then $\alpha(0) = 1 \oplus 1$ and $\alpha(\frac{\pi}{2}) = u \oplus u^*$. Differentiating α by using the product rule gives a sum of two unitaries, hence $\|\alpha'(t)\| \le 2$ for all t. Thus the length of the path $\alpha|_{[0,\lambda]}$ is less that π for any $\lambda < \frac{\pi}{2}$. Hence the above lemma implies that $-1 \notin \sigma(\alpha(t))$ for all $t < \frac{\pi}{2}$. In particular, each $\alpha(t)$ is of the form e^{ia} for some $a = a^* \in A$, since we are able to take a logarithm on $\sigma(\alpha(t))$. $\square$

Lemma 5.3.17. Let A be a unital, simple, purely infinite C*-algebra, e_1, e_2, e_3, e_4 be non-zero orthogonal projections such that $e_1 + e_2 + e_3 + e_4 = 1$, and let $s \in A$ be a partial isometry such that $s^*s = e_2, ss^* = e_3$. Let $u \in U(e_1Ae_1)$ satisfy $\sigma(u) = \mathbb{T}$, and let $v \in U(e_2Ae_2)$. Then for any $\varepsilon > 0$, there are unitaries $z \in U(A), w \in U(e_4Ae_4)$, where w has finite spectrum, and

$$(*)\ \|z^*(u + 1 - e_1)z - (u + v + sv^*s^* + w)\| < \varepsilon.$$

Proof. Since we have 4 orthogonal projections which add to identity, we can think of elements of A as 4x4 matrices with the the ijth entry in e_iAe_j, and we can identify e_2Ae_2 and e_3Ae_3 via the partial isometry v

Define $\phi : M_2(e_2Ae_2) \to (e_2 + e_3)A(e_2 + e_3)$ by

$$\phi \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11} + x_{12}s^* + sx_{21} + sx_{22}s^*.$$

Then this is clearly an isomorphism since s has initial projection e_2 and range projection e_3. Then $v + sv^*s^* = \phi(v \oplus v^*)$, and so the above corollary implies that it suffices to prove that $(*)$ holds if we replace $v + sv^*s^*$ by e^{ia} for some $a \in ((e_2 + e_3)A(e_2 + e_3))_{sa}$. Clearly $(e_2 + e_3)A(e_2 + e_3) \subseteq A$ has unit $e_2 + e_3$, and this is hereditary in A, so that it is both simple and purely infinite. Thus by Theorem 5.3.4, $(e_2 + e_3)A(e_2 + e_3)$ has real rank zero. Thus we can further assume that $\sigma(a)$ is finite. So it suffices to show that $(*)$ holds when we replace $v + sv^*s^*$ by $\sum_1^n \lambda_k q_k$ where $\lambda_k \in \mathbb{T}$ and q_k are non-zero mutually orthogonal projections which sum to $e_2 + e_3$. By Lemma 5.2.2, we can further assume that $u = u_0 + \sum_1^n \lambda_k p_k$ where p_j are mutually orthogonal projections in e_1Ae_1, and with $p = e_1 - \sum_1^n p_j$, we have that $u_0 \in U(pAp)$.

Now to find z, w such that $z^*(u + 1 - e_1)z = u + \sum_1^n \lambda_k q_k + w$, choose partial isometries c_k such that $c_k^*c_k = p_k, c_kc_k^* < p_k$, which exist by Theorem 5.1.16. Then $c = p + \sum_1^n c_k$ is a partial isometry with

$$c^*c = e_1;\ cc^* = e_1 - \sum_1^n (p_k - c_kc_k^*);\ cuc^* = u_0 + \sum_1^n \lambda_k(p_k - c_kc_k^*).$$

Now let d_k be partial isometries such that $d_k^*d_k = q_k$ and $d_kd_k^* \leq p_k - c_kc_k^*$, which again exist by Theorem 5.1.16. Let $d = \sum_1^n d_k$, which is a partial isometry such that

$$d^*d = e_2 + e_3;\ dd^* \leq \sum_1^n (p_k - c_kc_k^*);\ d\left(\sum_1^n \lambda_k q_k\right) d^* = \sum_1^n \lambda_k d_k d_k^*.$$

Next, let b be a partial isometry such that

$$b^*b < e_4, bb^* = \sum_1^n (p_k - c_kc_k^* - d_kd_k^*),$$

and let

$$w_0 = \sum_1^n \lambda_k b^*(p_k - c_kc_k^* - d_kd_k^*)b,$$

which is unitary in $(b^*b)A(b^*b)$ with finite spectrum. Consequently, $z_0 = b + c + d$ is a partial isometry such that

$$z_0^*z_0 = e_1 + e_2 + e_4 + b^*b;\ z_0z_0^* = e_1;\ z_0\left(u + \sum_1^n \lambda_k q_k + w_0\right) z_0^* = u.$$

Thus $[e_1]_0 = [e_1 + e_2 + e_4 + b^*b]_0$ in $K_0(A)$, hence $[1 - e_1]_0 = [e_4 - b^*b]_0$. Therefore by Theorem 5.2.1 there exists a partial isometry $y \in A$ such that $yy^* = 1 - e_1$ and $y^*y = e_4 - b^*b$. Let $z = z_0 + y$ and $w = w_0 + e_4 - b^*b$. Then it isn't difficult to see that $z^*z = 1 = zz^*$ and that $w \in e_4Ae_4$ with finite spectrum. Now

$$\begin{aligned}
z^*(u + 1 - e_1)z &= (z_0^* + y^*)(u + 1 - e_1)(z_0 + y) \\
&= z_0^*uz_0 + y^*(1 - e_1)y \\
&= z_0^*\left(z_0\left(u + \sum_1^n \lambda_k q_k + w_0\right) z_0^*\right) z_0 + (e_4 - b^*b) \\
&= (e_1 + e_2 + e_3 + b^*b)\left(u + \sum_1^n \lambda_k q_k + w_0\right)(e_1 + e_2 + e_3 + b^*b) + (e_4 - b^*b) \\
&= u + \sum_1^n \lambda_k q_k + w_0 + (e_4 - b^*b) \\
&= u + \sum_1^n \lambda_j q_j + w.
\end{aligned}$$

This completes the proof, since we can approximate $u + v + svs^* + w$ by the above element. $\square$

Remark 5.3.18. If we let everything be as in the above lemma, then this is saying that

$$\left\| z^* \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} z - \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v^* & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \right\| < \varepsilon.$$

Theorem 5.3.19. Let A be a unital simple purely infinite C*-algebra. Then $\operatorname{cer}(A) \leq 1+\varepsilon$.

Proof. Let $u \in U_0(A)$ and let $\varepsilon > 0$. If $\sigma(u) \neq \mathbb{T}$, then u has a logarithm and we are done, so suppose that $\sigma(u) = \mathbb{T}$. Using Lemma 5.2.3, we can approximate u within $\frac{\varepsilon}{4}$ by $u_0 + p$, where p is a non-zero projection and $u_0 \in U((1-p)A(1-p))$. Let x be a partial isometry such that $x^*x = p$ and $xx^* < p$. If $\alpha : [0,1] \to U(A)$ is a continuous path connecting u to 1, then $t \mapsto (1-p+x)\alpha(t)(1-p+x)^*$ is a continuous path connected $u_0 + xx^*$ to the identity in $(1-p+xx^*)A(1-p+xx^*)$. By replacing p with $p - xx^*$ and u_0 with $u_0 + xx^*$, we can assume that $u_0 \in U_0((1-p)A(1-p))$.

Now since $u_0 \in U_0((1-p)A(1-p))$, let $u_1, \dots, u_N \in U((1-p)A(1-p))$ be such that $\|u_j - u_{j+1}\| < \frac{\varepsilon}{4}$ for $j = 0, \dots, N-1$ and $u_N = 1-p$. Let c_j, d_j be partial isometries such that $c_j^*c_j = d_j^*d_j = 1-p$ and $p_j = c_jc_j^*, q_j = d_jd_j^*$ are all mutually orthogonal and satisfy

$$\sum_1^N p_j + \sum_1^N q_j < p.$$

Let $v = \sum_1^N c_j u_j^* c_j^*$ and $s = \sum_1^N d_j c_j^*$. Then $svs^* = \sum_1^N d_j u_j^* d_j^*$. Applying the above lemma with $e_1 = 1-p, e_2 = \sum_1^N c_jc_j^*, e_3 = \sum_1^N d_jd_j^*, e_4 = p - e_2 - e_3$, there exists a unitary $u \in U(e_4Ae_4)$ with finite spectrum and $z \in U(A)$ such that

$$\|z^*(u_0+p)z - (u_0 + v + sv^*s^* + w)\| < \frac{\varepsilon}{4}.$$

Let $d_0 = 1-p, b = \sum_1^N d_{j-1}c_j^*$. Then

$$b^*b = \sum_1^N p_j;\ bb^* = 1 - p + \sum_1^{N-1} q_j.$$

Moreover,

$$\begin{aligned}\|(u_0 + (sv^*s^* - q_N) + v) - (bv * b^* + v)\| &= \left\|\sum_1^N d_{j-1}(u_{j-1} - u_j)d_{j-1}^*\right\| \\ &= \max_j \|u_{j-1} - u_j\| < \frac{\varepsilon}{4}.\end{aligned}$$

Now let

$$f = \left(1 - p + \sum_1^{N-1} q_j\right) + \sum_1^N p_j = 1 - e_4 - qN.$$

Using the above corollary like before, there exists $a_0 = a_0^* \in fAf$ such that $\|e^{ia_0} - (bv^*b^* + v)\| < \frac{\varepsilon}{4}$. Now since w has finite spectrum, it has a logarithm $ia_1 \in e_4Ae_4$. Letting $a = a_0 + a_1$, $e^{ia} = e^{ia_0} + q_N + w$ since the summand corresponding to q_N is 0. Thus

$$\begin{aligned}\|e^{ia} - z^*uz\| \leq{}& \|e^{ia_0} - (bv^*b^* + v)\| + \|(bv^*b^* + v) - (u_0 + sv^*s^* - q_N + v)\| \\ &+ \|(u_0 + v + sv^*s^* + w) - z^*(u_0+p)z\| + \|(u_0 + p) - u\| < \varepsilon.\end{aligned}$$

Thus $\|e^{i(zaz^*)} - u\| < \varepsilon$, giving us the result. ∎

Corollary 5.3.20. If A is a unital simple purely infinite C*-algebra, then A has property weak (FU). Consequently since the Cuntz algebras have connected unitary groups, it follows that $\mathcal{O}_n$ has property (FU) for all n.

Proof. A has real rank zero, so the corollary follows from Proposition 5.3.9. □

Definition 5.3.21. Let A be a unital C*-algebra. We say that A has **exponential length** L if every $u \in U_0(A)$ can be written as $u = \Pi_1^n e^{ia_j}$ where $a_j \in A_{sa}$ are such that $\sum_1^n \|a_j\| \leq L$. We say that $\mathrm{cel}(A) \leq L$.

Proposition 5.3.22. Let A be a unital C*-algebra with property weak (FU). Then $\mathrm{cel}(A) \leq 4$.

Proof. Let $u \in U_0(A)$. Then there exists $v \in U_0(A)$ with finite spectrum such that $\|u - v\| < \frac{2}{\pi}(4-\pi)$. But then $\|uv^* - 1\| < 1$, so that $-1 \notin \sigma(uv^*)$, hence $uv^* = e^{-ia_1}$ for some $a_1 \in A_{sa}$ with $\|a_1\| \leq 4 - \pi$. Moreover since v has finite spectrum, $v = e^{ia_2}$ for some $a_2 \in A_{sa}$ with $\|a_2\| \leq \pi$. Thus $u = e^{ia_1}e^{ia_2}$ with $a_1, a_2 \in A_{sa}$ and $\|a_1\| + \|a_2\| \leq 4 - \pi + \pi = 4$. □

Corollary 5.3.23. A unital simple purely infinite C*-algebra A has $\mathrm{cel}(A) = \pi$.

Proof. By Theorem 5.3.19, $\mathrm{cer}(A) \leq 1 + \varepsilon$. Firstly, $-1 = e^{ia}$ some self-adjoint a, and so $\mathrm{cel}(A) \geq \pi$.

Conversely, since A has property weak (FU), it suffices to consider a dense set of unitaries in $U_0(A)$, which we can take to be unitaries with finite spectrum. Thus $\mathrm{cel}(A) \leq \pi$ as well since if v has finite spectrum, $v = e^{ia}$ for some a such that $\|a\| \leq \pi$. □

6 The Nuclear Embedding Theorem

In this chapter, we aim to finally prove the Kirchberg-Phillips nuclear embedding theorem, the main result of this thesis. We will start by using the approximation properties of unital, simple, purely infinite C*-algebras to prove that $\mathcal{O}_2 \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$ by proving that the unitary group of a unital, simple, purely infinite C*-algebra A has a certain stability property. We will use this fact to prove that any two unital injective *-homomorphisms $\mathcal{O}_2 \to A$ are approximately unitarily equivalent. Letting $A = \mathcal{O}_2 \otimes \mathcal{O}_2$, the isomorphism result will follow. We then prove the Effros-Haagerup lifting theorem, which will be useful for proving the embedding theorem in full generality. In Section 6.3, we prove that any two unital injective *-homomorphism $A \to \mathcal{O}_2$ from a unital, separable, exact C*-algebra are approximately unitarily equivalent. We then use this, along with the fact that the cone of a C*-algebra is QD, to finally prove the theorem.

6.1 $\mathcal{O}_2 \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$

We first wish to prove a stability condition for unital, simple purely infinite C*-algebras. This is the condition that if γ is defined as in the following paragraph, then if u is a unitary, there is always another unitary such that $\|v\gamma(v)^* - u\|$ is small. We follow Section 4 of [26], and then Chapter 5 of [28].

Let $\mathcal{O}_2 = C^*(s_1, s_2)$ where s_1, s_1 are isometries satisfying the Cuntz relations. Let $\lambda : \mathcal{O}_2 \to \mathcal{O}_2$ be defined by $\lambda(a) = s_1 a s_1^* + s_2 a s_2^*$. Let A be a unital, simple, purely infinite C*-algebra two isometries t_1, t_2 satisfying the Cuntz relations, $u \in U(A)$, and let $\phi, \psi : \mathcal{O}_2 \to A$ be defined by $\phi(s_i) = t_i, \psi(s_i) = ut_i$. Then it is clear that $u = \psi(s_1)\phi(s_1)^* + \psi(s_2)\phi(s_2)^*$. Let $\gamma : A \to A$ be the map $\gamma(a) = t_1 a t_1^* + t_2 a t_2^*$.

Note that since A is unital, simple, purely infinite $U(A)/U_0(A) \simeq K_1(A)$, so in the following lemmas one assumes that $u \in U_0(A)$. More generally, one can work with unitaries u such that $[u]_1 = 0$ in $K_1(A)$.

Lemma 6.1.1. For $k \in \mathbb{N}$, $\mathrm{Im}\gamma^k = \phi(\mathcal{F}_k^2)' \cap A$.

Proof. We have $\gamma^k(a)\phi(s_\mu) = \phi(s_\mu)a$ and $\phi(s_\mu)^*\gamma^k(a) = a\phi(s_\mu)^*$ for all $k \in \mathbb{N}, a \in A, \mu$ word with with $|\mu| = k$. Thus $\phi(s_\mu s_\nu^*)\gamma^k(a) = \phi(s_\mu)a\phi(s_\nu^*) = \gamma^k(a)\phi(s_\mu s_\nu^*)$ for $|\mu| = |\nu| = k$, and so $\mathrm{Im}\gamma^k \subseteq \phi(\mathcal{F}_k^2)' \cap A$.

Conversely, let $b \in \phi(\mathcal{F}_k^2)' \cap A$ and let μ, ν be words with $|\mu| = |\nu| = k$. Then $\phi(s_\mu s_\nu^*)b = b\phi(s_\mu s_\nu^*)$, so multiplying on the right by $\phi(s_\mu^*)$ and on the left by $\phi(s_\nu)$ gives us

$\phi(s_\nu^*)b\phi(s_\nu) = \phi(s_\mu^*)b\phi(s_\mu)$. Now let $a = \phi(s_\nu^*)b\phi(s_\nu)$, so

$$\begin{aligned}
\gamma^k(a) &= \sum_{|\mu|=k} \phi(s_\mu)a\phi(s_\mu^*) \\
&= \sum_{|\mu|=k} \phi(s_\mu)\phi(s_\nu^*)b\phi(s_\nu)\phi(s_\mu)^* \\
&= \sum_{|\mu|=k} \phi(s_\mu)\phi(s_\mu^*)b\phi(s_\mu)\phi(s_\mu^*) \\
&= \sum_{|\mu|=k} b\phi(s_\mu)\phi(s_\mu)^*\phi(s_\mu)\phi(s_\mu)^* \\
&= \sum_{|\mu|=k} b\phi(s_\mu)\phi(s_\mu^*) = b.
\end{aligned}$$

□

Let $u_k = \sum_{|\mu|=k} \psi(s_\mu)\phi(s_\mu)^*$. Then u_k are all unitary, $u_1 = u$, and $\psi(s_\mu) = u_k\phi(s_\mu)$ for all words μ with $|\mu| = k$.

Lemma 6.1.2. For $k \in \mathbb{N}, u = u_k\gamma^k(u)\gamma(u_k)^*$.

Proof. Let us prove this by induction on k. $k = 1$ follows since $u_1 = u$. Before we proceed by induction, notice that

$$\gamma(u_k) = \sum_1^2 \phi(s_j)u_k\phi(s_j)^* = u^* \sum_1^2 \psi(s_j)u_k\phi(s_j)^* = u^*u_{k+1}.$$

Now assume that $u = u_k\gamma^k(u)\gamma(u_k)^*$ for some $k \geq 1$. Then

$$\gamma^{k+1}(u) = \gamma(u_k^* u\gamma(u_k)) = \gamma(u_k^* uu^*u_{k+1}) = \gamma(u_k^* u_{k+1}).$$

Therefore

$$u_{k+1}\gamma^{k+1}(u)\gamma(u_{k+1})^* = u_{k+1}\gamma(u_k)^* = u_{k+1}u_{k+1}^* u = u.$$

□

Lemma 6.1.3. Let A be a C*-algebra. Then for every unitary $u \in U_0(A)$ and $m \in \mathbb{N}$, if $u = \Pi_1^n e^{ia_j}$ such that $a_j \in A_{sa}$ and $\sum \|a_j\| \leq C$, then there exists unitary $v_1, \ldots, v_m \in U(A)$ such that $u = v_1 \cdots v_m$ and $\|v_j - 1\| \leq \frac{C}{m}$.

Proof. We will prove this by induction on m. For $m = 1$, if $u \in U_0(A)$, then $u = \Pi_1^n e^{ia_j}$ where $a_j \in A_{sa}$ and $\sum_j \|a_j\| \leq C$. So letting $v_1 = u$, we have

$$\|v_1 - 1\| = \|\Pi_1^n e^{ia_j} - 1\| \leq \sum_j \|a_j\| \leq C.$$

Now suppose that for some $m \geq 1$, any unitary $w = \Pi_1^l e^{ib_j} \in U_0(A)$ such that $\sum \|b_j\| \leq C$ can be written as $w = w_1 \cdots w_m$, where $\|w_j - 1\| \leq \frac{C}{m}$.

For $u = \Pi_1^n e^{ia_j}$ with $\sum_1^n \|a_j\| \le C$, if for some $1 \le k \le n$, $\sum_1^k \|a_j\| \le \frac{C}{m+1}$, let p be the largest such k. Otherwise, let $p = 0$. If $p = 0$, we will adopt the notion that $\sum_1^p \|a_j\| = 0$. Let $t \in (0, 1]$ be such that

$$\sum_1^p \|a_j\| + t\|a_{p+1}\| = \frac{C}{m+1}.$$

Let $v_1 = e^{ia_1} \cdots e^{ia_p} e^{ita_{p+1}}$, where if $p = 0$, then the first terms are all just 1. But then $v_1^* u = e^{i(1-t)a_{p+1}} \cdots e^{ia_n}$ satisfies

$$(1-t)\|a_{p+1}\| + \cdots + \|a_n\| \le C - \frac{C}{m+1} = \frac{mC}{m+1}.$$

By the induction hypothesis, $v_1^* u = v_2 \cdots v_{m+1}$ where $\|v_j - 1\| \le \frac{1}{m}(\frac{mC}{m+1}) = \frac{C}{m+1}$. Thus $u = v_1 \cdots v_{m+1}$ has the required form. □

Note that since we are assuming that A is unital simple purely infinite, A has finite exponential length $L = \pi$. The above lemma works well in this situation, since we can take $C = L$ for every unitary in the connected component.

Lemma 6.1.4. Let $k, m \in \mathbb{N}, l = k+m-1$. Let $u \in U_0(A)$. Then there are $w_0, w_1, \ldots, w_{m-1} \in A \cap \phi(\mathcal{F}_k^2)'$ such that

$$\Pi_0^{m-1} \gamma^j(w_j) = 1$$

and $\|\gamma^l(u) - w_j\| \le \frac{L}{m}$ for all $j = 0, 1, \ldots, m-1$, where L is the finite exponential length.

Proof. Let $x_j = \gamma^l(u)\gamma^{l+1}(u) \cdots \gamma^{l+j}(u)$ for $j = 0, 1, \ldots, m-1$, and let $v = u\gamma(u) \cdots \gamma^{m-1}(u)$. Then $x_{m-1} = \gamma^l(v)$ and $v \in U_0(A)$. Then the above lemma lets us write v as a product of m unitaries which are $\frac{L}{m}$ of 1, and applying γ^l to these unitaries gives us that there are $y_0, \ldots, y_{m-1} \in \phi(\mathcal{F}_l^2)'$ such that $x_{m-1} = y_{m-1} y_{m-2} \cdots y_1 y_0$ with $\|y_j - 1\| \le \frac{L}{m}$ for $j = 0, 1, \ldots, m-1$ by Lemma 6.1.1.

Now since $x_j^* y_j^* x_j \in \phi(\mathcal{F}_k^2)' \cap A \subseteq \gamma^j(D \cap \phi(\mathcal{F}_k^2)')$ for $j = 0, \ldots, m-1$, there are unitaries $z_0, z_1, \ldots, z_{m-1} \in \phi(\mathcal{F}_k^2)' \cap A$ such that $\gamma(z_j) = x_j^* y_j^* x_j$. Now let $w_j = \gamma^l(u) z_j$. Then since $x_{j-1}^* x_i = \gamma^{l+j}(u)$,

$$\begin{aligned} w_0 \gamma(w_1) \cdots \gamma^{m-1}(w_{m-1}) &= \gamma^l(u) z_0 \gamma^{l+1}(u) \gamma(z_1) \cdots \gamma^{l+m-1}(u) \gamma^{m-1}(z_{m-1}) \\ &= (x_0 z_0 x_0^*)(x_1 \gamma(z_1) x_1^*) \cdots (x_{m-1} \gamma^{m-1}(z_{m-1}) x_{m-1}^*) x_{m-1} \\ &= y_0^* y_1^* \cdots y_{m-1}^* x_{m-1} = 1. \end{aligned}$$

Now since γ is isometric,

$$\|\gamma^l(u) - w_j\| = \|1 - z_j\| = \|1 - \gamma^j(z_j)\| = \|1 - y_j^*\| \le \frac{L}{m}.$$

□

Lemma 6.1.5. For $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $u \in U_0(A)$, there exists $v \in D$ such that $\|\gamma^k(u) - v\gamma(v)^*\| < \varepsilon$.

Proof. Let m be a power of 2 satisfying $\frac{L}{m} < \frac{\varepsilon}{2}$. By the Rokhlin property (3.1.5), there are $l \in \mathbb{N}$, m mutually orthogonal projections $e_0, \ldots, e_m = 0$ in $\phi(\mathcal{F}_l^2)$ such that $1 = \sum_1^m e_j$ and $\|\gamma(e_j) - e_{j-1}\| < \frac{\varepsilon}{2m}$ for $j = 1, \ldots, m$. Let $k = l + m - 1$. Let $u \in U(A)$ be such that $[u]_1 = [0]_1$ in $K_1(A)$. Then by the previous lemma, there are unitaries $w_0, \ldots, w_{m-1} \in \phi(\mathcal{F}_l^2)' \cap A$ with $\|\gamma^k(u) - w_j\| \leq \frac{L}{m} < \frac{\varepsilon}{2}$ and $w_0\gamma(w_1)\cdots\gamma^{m-1}(w_{m_1}) = 1$. Let $v_m = 1$ and $v_j = w_j\gamma(w_{j+1})\cdots\gamma^{m-j-1}(w_{m-1})$ for $j = 0, 1, \ldots, m-1$. Then $v_j \in \phi(\mathcal{F}_l^2)' \cap A$ and $v_0 = v_m = 1$. Moreover $w_j = v_j\gamma(v_{j+1})^*$ for all j. Let

$$v = \sum_1^m v_j e_j.$$

Then $v \in U(D)$ since the v_j's commute with the e_j's. Let

$$\Delta = v\sum_1^m (\gamma(e_j) - e_{j-1})\gamma(v_j)^*.$$

Then $\|\Delta\| < \frac{\varepsilon}{2}$ and

$$v\gamma(v)^* = v\sum_j^m e_{j-1}\gamma(v_j)^* + \Delta = \sum_0^{m-1} v_j\gamma(v_{j+1})^* e_j + \Delta = \sum_0^{m-1} w_j e_j + \Delta.$$

Therefore

$$\|\gamma^k(u) - v\gamma(v)^*\| \leq \left\|\sum_1^m (\gamma^k(u) - w_j)e_j\right\| + \|\Delta\| = \max_j \|\gamma^k(u) - w_j\| + \Delta\| < \varepsilon.$$

$\square$

Lemma 6.1.6 (Stability)**.** Let $u \in U(A)$. Then there exists $w \in U(A)$ such that $\|u - w\gamma(w)^*\| < \varepsilon$.

Proof. First let us assume that $u \in U_0(A)$. Let k, v be as in Lemma 6.1.5. Since $u = u_k\gamma^k(u)\gamma(u_k)^*$ by Lemma 6.1.2, we have

$$\begin{aligned}
\|u - (u_k v)\gamma(u_k v)^*\| &= \|u_k\gamma^k(u)\gamma(u_k)^* - u_k v\gamma(v)^*\gamma(u_k)^*\| \\
&\leq \|u_k\|\|\gamma^k(u) - v\gamma(v)^*\|\|\gamma(u_k)^*\| \\
&= \|\gamma^k(u) - v\gamma(v)^*\| < \varepsilon.
\end{aligned}$$

Thus the required unitary is $w = u_k v$.

Now if $u \in U(A)$, then $[u^2\gamma(u)^*]_1 = 0$ in $K_1(A)$ since $K_1(\gamma) = 2\mathrm{id}_{K_1(A)}$, and so $u^2\gamma(u)^* \in U_0(A)$ since A is K_1-injective. Then find $v \in A$ such that $\|v\gamma(v)^* - u^2\gamma(u)^*\| < \varepsilon$, so that $\|w\gamma(w)^* - u\| < \varepsilon$, where $w = u^*v$. $\square$

Definition 6.1.7. Let A, B be unital, separable C*-algebras, $\phi, \psi : A \to B$ unital *-homomorphisms. We say that ϕ is **approximately unitarily equivalent** to ψ if there exists a sequence (v_n) of unitaries in B such that $v_n\phi(a)v_n^* \to \psi(a)$ for all $a \in A$.

Theorem 6.1.8. Let A be a unital, simple, purely infinite C*-algebra. Then any two unital *-homomorphisms $\phi, \psi : \mathcal{O}_2 \to A$ are approximately unitarily equivalent.

Proof. Let $u = \psi(s_1)\phi(s_1)^* + \psi(s_2)\phi(s_2)^*$, where $\mathcal{O}_2 = C^*(s_1, s_2)$ with $s_1, s_2 \in \mathcal{O}_2$ being isometries satisfying the Cuntz relations. Let $t_j = \phi(s_j)$. By stability, there exists unitaries $(v_n) \subseteq U(A)$ such that $v_n\gamma(v_n)^* \to u$, where $\gamma(a) = t_1at_1^* + t_2at_2^*$. But then

$$v_n\phi(s_j)v_n^* = v_nt_jv_n^* = v_n\gamma(v_n)^*t_j \to ut_j = \psi(s_j).$$

□

Lemma 6.1.9. Let A, B be separable C*-algebras, where B is unital, and let $\pi : A \to B$ be an injective *-homomorphism. If there exists $(u_n) \subseteq U(B)$ such that

$$\|u_n\pi(a) - \pi(a)u_n\| \to 0 \text{ and } \operatorname{dist}(u_n^*bu_n, \pi(A)) \to 0$$

for all $a \in A, b \in B$, then there exists a *-isomorphism $\sigma : A \to B$ which is approximately unitarily equivalent to π.

Proof. Let $(a_n), (b_n)$ be countable dense sets for A, b respectively. Then we can inductively find unitaries $(v_n) \subseteq U(B)$ and elements $(a_{j,n})_{j=1}^n$ such that

1. $\|v_n^* \cdots v_1^*b_jv_1 \cdots v_n - \pi(a_{jn})\| \leq \frac{1}{n}$ for $1 \leq j \leq n$;
2. $\|v_n\pi(a_j) - \pi(a_j)v_n\| \leq \frac{1}{2^n}$ for $1 \leq j \leq n$;
3. $\|v_n\pi(a_{jm}) - \pi(a_{jm})v_n\| \leq \frac{1}{2^n}$ for $1 \leq m \leq n-1, 1 \leq j \leq m$.

By the first condition, $(v_1 \cdots v_n\pi(a_j)v_n^* \cdots v_1^*)_n \subseteq B$ is Cauchy for all j. Thus by the density of (a_j), $(v_1 \cdots v_n\pi(a)v_n^* \cdots v_1^*)$ is Cauchy in B for all $a \in A$. Thus we can define an isometric *-homomorphism $\sigma(a) = \lim_n v_1 \cdots v_n\pi(a)v_n^* \cdots v_1^*$ since v_i are unitaries.

Now notice that for $j \leq n$,

$$\|\sigma(a_{jm}) - v_1 \cdots v_n\pi(a_{jn})v_n^* \cdots v_1^*\| \leq \sum_{m=n+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^n}.$$

Thus

$$\|b_j - \sigma(a_{jn})\| \leq \frac{1}{2^n} + \|b_j - v_1 \cdots v_n\pi(a_{jn})v_n^* \cdots v_1^*\| \leq \frac{1}{2^n} + \frac{1}{n}.$$

Since $\sigma(A)$ is closed, $b_j \in \sigma(A)$ for all j, and so $\sigma(A) = B$ by density. □

Definition 6.1.10. Let A, B be C*-algebras. We say the sequence $(b_n) \subseteq B$ is **asymptotically central** if $\lim_n \|bb_n - b_nb\| = 0$ for all $b \in B$. We say that a sequence (π_n) of *-homomorphisms $\pi_n : A \to B$ is **asymptotically central** if $(\pi_n(a))$ if asymptotically central in B for all $a \in A$.

Lemma 6.1.11. There is an asymptotically central sequence (ρ_n) of unital *-homomorphisms $\rho_n : \mathcal{O}_2 \to \mathcal{O}_2$.

Proof. Let $\lambda(a) = s_1 a s_1^* + s_2 a s_2^*$, where $\mathcal{O}_2 = C^*(s_1, s_2)$. Then $(a_n) \subseteq \mathcal{O}_2$ is asymptotically central if and only if $\|\lambda(a_n) - a_n\| \to 0$. If $(v_n) \subseteq U(\mathcal{O}_2)$ with $\rho_n = \phi_{v_n}$ where $\phi_{v_n}(s_i) = v_n s_i$, then (ρ_n) is asymptotically central if and only if $\|\lambda(\rho_n(s_j)) - \rho_n(s_j)\| \to 0$ and

$$\|\lambda(\rho_n(s_j)) - \rho_n(s_j)\| = \|\lambda(v_n s_j) - v_n s_j\| = \|\lambda(s_j) - \lambda(v_n)^* v_n s_j\| = \|u s_j - \lambda(v_n)^* v_n s_j\|,$$

so it suffices to construct unitaries (v_n) such that $\lambda(v_n)^* v_n \to u = \lambda(s_1)s_1^* + \lambda(s_2)s_2^* = \sum_{i,j=1}^2 s_i s_j s_i^* s_j^*$. Now since $U(\mathcal{O}_2)/U_0(\mathcal{O}_2) \simeq K_1(\mathcal{O}_2) = 0$, stability ensures that there exists $v_n \in U(\mathcal{O}_2)$ such that $v_n \lambda(v_n)^* \to u$. Since u is self-adjoint, by replacing v_n with v_n^*, we get that $\lambda(v_n)^* v_n \to u$. □

Theorem 6.1.12. $\mathcal{O}_2 \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$.

Proof. Let $\mathcal{O}_2 = C^*(s_1, s_2)$, $\phi : \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ be defined by $\phi(x) = x \otimes 1$, which is clearly an injective *-homomorphism. It suffices to show that for all $\varepsilon > 0$, there exists $v \in U(\mathcal{O}_2 \otimes \mathcal{O}_2)$ such that

$$\|v(s_j \otimes 1) - (s_j \otimes 1)v\| < \varepsilon$$
$$\operatorname{dist}(v^*(1 \otimes s_j)v, \mathcal{O}_2 \otimes 1) < \varepsilon,$$

from which it will follow that there is a *-isomorphism approximately unitarily equivalent to ϕ by Lemma 6.1.9. So let $\varepsilon > 0$. Since $\mathcal{O}_2 \otimes \mathcal{O}_2$ is unital, simple, purely infinite, Theorem 6.1.8 implies that the maps $\mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ given by $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$ are approximately unitarily equivalent, so there exists a unitary $w \in U(\mathcal{O}_2 \otimes \mathcal{O}_2)$ such that

$$\|w(s_j \otimes 1)w^* - 1 \otimes s_j\| < \varepsilon$$

for $j = 1, 2$. Now let (ρ_n) be a sequence of asymptotically central *-homomorphisms from $\mathcal{O}_2$ to $\mathcal{O}_2$, and let $\psi_n = \rho_n \otimes \mathrm{id} : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$. Let $w_n = \psi_n(w)$, so that

$$\|w_n(s_j \otimes 1) - (s_j \otimes 1)w_n\| \to 0$$

and

$$\begin{aligned} \operatorname{dist}(w_n^*(1 \otimes s_j)w_n, \mathcal{O}_2 \otimes 1) &\le \|w_n^*(1 \otimes s_j)w_n - \rho_n(s_j) \otimes 1\| \\ &= \|\psi_n(w^*(1 \otimes s_j)w_n - s_j \otimes 1)\| < \varepsilon. \end{aligned}$$

Thus let $v = w_n$ for $n \in \mathbb{N}$ large enough so that the the quantities above are less than ε. □

Remark 6.1.13. It clearly follows that any finite tensor of $\mathcal{O}_2$ with itself is isomorphic to $\mathcal{O}_2$. It is also true that the infinite tensor product (the inductive limit of the finite ones with connecting maps $x \mapsto x \otimes 1$) is isomorphic to $\mathcal{O}_2$. One must study approximate intertwinings to get the result. Chapter 2 of [28] covers these, and the result about the infinite tensor product is Corollary 5.1.5 of [28].

6.2 Lifting Theorems

To prove the nuclear embedding theorem to its fullest, we will need to lift certain *-homomorphisms into $(\mathcal{O}_2)_\infty = \ell^\infty(\mathcal{O}_2)/c_0(\mathcal{O}_2)$ to u.c.p. maps to $\ell^\infty(\mathcal{O}_2)$. The Effros-Haagerup lifting theorem will be crucial.

Definition 6.2.1. Let B be a C*-algebra, $J \lhd B$, and E be an operator system. A c.c.p. map $\phi : E \to B/J$ is **liftable** if there exists a c.c.p. map $\psi : E \to B$ such that $\pi \circ \psi = \phi$, where $\pi : B \to B/J$ is the quotient map. It is **locally liftable** if for every finite-dimensional operator system $F \subseteq E$, the c.c.p. map $\phi|_F$ is liftable.

Lemma 6.2.2. Let $J \lhd B$, E a separable operator system. Then the set of liftable c.c.p. maps $E \to B/J$ is closed in the point-norm topology.

Proof. Let $\phi : E \to B/J$ be c.c.p. and let $\psi'_n : E \to B$ be c.c.p. maps such that $\pi \circ \psi'_n \to \phi$ in point-norm. Let $(x_k)_k$ be a dense sequence in E. Passing to a subsequence if necessary, we can assume that $\|\pi \circ \psi'_n(x_k) - \phi(x_k)\| < \frac{1}{2^n}$ for $k \leq n$. We claim that there exists c.c.p. maps $\psi'_n : E \to B$ such that $\|\pi \circ \psi_n(x_k) - \phi(x_k)\| < \frac{1}{2^n}$ and $\|\psi_{n+1}(x_k) - \psi_n(x_k)\| < \frac{1}{2^{n-1}}$ for $k \leq n$. We prove this by inductions. Let $\psi_1 = \psi'_1$. Now suppose that we have constructed $\psi_1, \dots, \psi_n$ with the desired property. Let (e_λ) be a quasicentral approximate unit for J in B. Then for $k \leq n$,

$$\lim_\lambda \|(1 - e_\lambda)^{\frac{1}{2}} \psi_n(x_k)(1 - e_\lambda)^{\frac{1}{2}} + e_\lambda^{\frac{1}{2}} \psi_n(x_k) e_\lambda^{\frac{1}{2}} - \psi_n(x_k)\| = 0$$

and for $b_k = \psi'_{n+1}(x_k) - \psi_n(x_k)$ with $k \neq n$, we have

$$\lim_\lambda \|(1 - e_\lambda)^{\frac{1}{2}} b_k (1 - e_\lambda)^{\frac{1}{2}}\| = \|\pi(b_k)\| < \frac{3}{2^{n+1}}.$$

So let $e = e_\lambda \in J$ be such that for every $k \leq n$ we have

$$\|(1 - e)^{\frac{1}{2}} \psi_n(x_k)(1 - e)^{\frac{1}{2}} + e^{\frac{1}{2}} \psi_n(x_k) e^{\frac{1}{2}} - \psi_n(x_k)\| < \frac{1}{2^{n+1}}$$

and

$$\|(1 - e)^{\frac{1}{2}} b_k (1 - e)^{\frac{1}{2}}\| < \frac{3}{2^{n+1}}.$$

Then the map $E \to B$ given by

$$\psi_{n+1}(x) = (1 - e)^{\frac{1}{2}} \psi'_{n+1}(x)(1 - e)^{\frac{1}{2}} + e^{\frac{1}{2}} \psi_n(x) e^{\frac{1}{2}}$$

is c.c.p. and satisfies the desired property.

Now since (ψ_n) is a sequence of c.c.p. maps which converges point-norm on a dense set (x_k), it converges everywhere to a c.c.p. map $\psi : E \to B$ which is a lift of ϕ. □

Theorem 6.2.3 (Choi-Effros lifting Theorem)**.** Every nuclear c.c.p. map from a separable C*-algebra A into a quotient B/J is liftable. In particular, every c.c.p. map from a separable nuclear C*-algebra is liftable.

Proof. Since the liftable c.c.p. maps are point-norm closed and nuclear maps factor through c.c.p. maps to and from matrix algebras, it suffices to show that every c.c.p. map $\phi : M_n \to B/J$ is liftable. Let $0 \le a = (\phi(e_{ij})) \in M_n(B/J)$. Since $\pi_n : M_n(B) \to M_n(B/J)$ is a surjective *-homomorphism, the positive element a lifts to a positive element $b = (b_{ij}) \in M_n(B)$, and the corresponding $\psi' : M_n \to B$ coming from Lemma 1.1.8 is a c.p. lift. □

Theorem 6.2.4 (Effros-Haagerup lifting Theorem)**.** Let $J \lhd B, \pi : B \to B/J$ be the quotient map. The following are equivalent.

1. For any C*-algebra A, the sequence
$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$
is exact;

2. The sequence
$$0 \to \mathcal{B}(\mathcal{H}) \otimes J \to \mathcal{B}(\mathcal{H}) \otimes B \to \mathcal{B}(\mathcal{H}) \otimes (B/J) \to 0$$
is exact, where $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space;

3. For any finite-dimensional operator system $E \subseteq B/J$, the inclusion $E \hookrightarrow B/J$ is liftable.

Proof. Clearly (1) implies (2). To see that (2) implies (3), suppose $E \subseteq B/J$ is a finite-dimensional operator system. By operator space duality ([22], chapter 14), $E \subseteq B/J$ corresponds to an element $z \in E^* \otimes (B/J)$ with $\|z\| = 1$. Assuming that $E^* \subseteq \mathcal{B}(\mathcal{H})$, we have
$$\frac{E^* \otimes B}{E^* \otimes J} \subseteq \frac{\mathcal{B}(\mathcal{H}) \otimes B}{\mathcal{B}(\mathcal{H}) \otimes J} = \mathcal{B}(\mathcal{H}) \otimes (B/J)$$
isometrically by Lemma 4.1.12, and so
$$E^* \otimes (B/J) = \frac{E^* \otimes B}{E^* \otimes J}$$
isometrically. So for any $\varepsilon > 0$, one can lift z to an element $\tilde{z} \in E^* \otimes B$ with $\|\tilde{z}\| < 1 + \varepsilon$. Then the map $\psi' : E \to B$ corresponding to $\tilde{z}$ is a lift of ϕ with $\|\psi'\|_{cb} < 1 + \varepsilon$. Now we can assume that ψ' is self-adjoint since $\psi'(1) - 1 \in J$, so there exists $0 \le e \le 1$ in J such that
$$\|(1-e)^{\frac{1}{2}}(\psi'(1) - 1)(1-e)^{\frac{1}{2}}\| < \varepsilon.$$
Now let $\theta \in E^*$ be a unital positive linear functional and let $\psi'' : E \to B$ be defined by
$$\begin{aligned}\psi''(x) &= (1-e)^{\frac{1}{2}} \left(\psi'(x) - \theta(x)(\psi'(x) - 1)\right)(1-e)^{\frac{1}{2}} + \theta(x)e \\ &= \begin{pmatrix}(1-e)^{\frac{1}{2}} & e^{\frac{1}{2}}\end{pmatrix} \begin{pmatrix}\psi'(x) & 0 \\ 0 & \theta(x)\end{pmatrix} \begin{pmatrix}(1-e)^{\frac{1}{2}} \\ e^{\frac{1}{2}}\end{pmatrix} + \theta(x)(1-e)^{\frac{1}{2}}(\psi'(1) - 1)(1-e)^{\frac{1}{2}}.\end{aligned}$$
Then ψ'' is a lift of ϕ with $\psi''(1) = 1$ and $\|\psi''|_{cb} < 1 + 2\varepsilon$. By Corollary 4.1.23, there exists a u.c.p. map $\psi : E \to B$ such that $\|psi - \psi''\|_{cb} < 2\dim(E)(1 + 2\varepsilon - 1) = 4\dim(E)\varepsilon$. Since $\varepsilon > 0$ was arbitrary, and the space of liftable u.c.p. maps is closed in point-norm, the result follows. □

6.3 Homomorphisms into $\mathcal{O}_2$

We will now work towards showing that if A is unital, separable, exact, then any two unital injective *-homomorphisms $A \to \mathcal{O}_2$ are approximately unitarily equivalent. To this end, it will be necessary to understand the structure of the state space, as well as c.p. maps to and from unital, simple, purely infinite C*-algebras.

We now follow Dixmier's book [13] to show that the wk*-closure of the pure states in a unital, simple, purely infinite C*-algebra is the whole state space. Consequently, due to real rank zero, we will be able to excise nets by projections. We start by looking at antiliminal C*-algebras, of which unital, simple, purely infinite C*-algebras are a subclass.

Definition 6.3.1. Let A be a C*-algebra. For a representation $\pi : A \to \mathcal{B}(\mathcal{H})$, let $K_\pi = \{x \in A \mid \pi(x) \in \mathcal{K}(\mathcal{H})\} = \pi^{-1}(\mathcal{K}(\mathcal{H}))$ which is a closed ideal of A. We say that A is **liminal** if for every irreducible representation π, $K_\pi = A$. We say that A is **antiliminal** if the zero ideal is its only liminal closed two-sided ideal.

Example 6.3.2. The following examples can be found in Chapter 5.6 of [21].

1. Abelian C*-algebras are liminal since every irreducible representation is 1-dimensional.
2. Finite-dimensional C*-algebras are liminal since every irreducible representation is finite-dimensional.
3. $\mathcal{K}(\mathcal{H})$ is liminal since every non-zero irreducible representation is unitarily equivalent to the identity representation of $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$.
4. Purely infinite C*-algebras are antiliminal.

   Proof. Let A be a purely infinite C*-algebra, and suppose that $I \lhd A$ is liminal. Since I is liminal, for every irreducible representation π, $K_\pi = I$. Now $I \subseteq A$ is hereditary, so I is also purely infinite by Lemma 5.1.15. Thus there exists a partial isometry $v \in A$ such that $q = vv^* < v^*v = p$, where p, q are infinite projections. So $\pi(p) = \pi(v^*v) = \pi(v)^*\pi(v)$ and $\pi(q) = \pi(v)\pi(v)^*$ for all irreducible representations π. In particular, since the image of a projection is a projection, $\pi(p) \sim \pi(q)$ for every irreducible representation π. Now since $\pi(p)$ and $\pi(q)$ are compact, hence finite rank, $\pi(q) \leq \pi(p)$ implies that $\pi(p) = \pi(q)$ for all irreducible representations π. But since irreducible representations separate the points of I, this is a contradiction since $p \neq q$. $\square$

The following is a standard result, which is a consequence of the Hahn-Banach Separation Theorem ([8], Theorem IV.3.7).

Theorem 6.3.3 ([21], Theorem 5.1.14)**.** Let A be a unital C*-algebra, $S \subseteq S(A)$ be a subset of states such that if $a \in A_{sa}$ satisfies $f(a) \geq 0$ for all $f \in S$, then $a \geq 0$. Then the $\overline{\mathrm{conv}}^{\mathrm{wk}*}(S) = S(A)$, and $\overline{S}^{\mathrm{wk}*} \supseteq PS(A)$.

Lemma 6.3.4. Let A be a unital antiliminal C*-algebra, and let $S'(A)$ be the set of states which vanish on at least one K_π, for π an irreducible representation. Then $\overline{PS(A)}^{\mathrm{wk}*} = \overline{S'(A)}^{\mathrm{wk}*}$.

Proof. Let $a = a^* \in A$ be such that $f(a) \geq 0$ for every $f \in S'(A)$. Let π be an irreducible representation and let ρ_π be a representation with $\ker \rho_\pi = K_\pi$. Then every state associated to ρ_π corresponds to an element of $S'(A)$, and so every states is ≥ 0 at x. Thus since $\oplus_\pi \rho_\pi$ is isometric, it follows that $x \geq 0$. By the above theorem, this implies that the wk*-closure of $S'(A)$ contains $PS(A)$.

Conversely, suppose that $f \in S'(A)$ and that there exists an irreducible representation π such that $f(K_\pi) = 0$. Then f defines a state g of $\pi(A)$ which vanishes on $\pi(A) \cap \mathcal{K}(\mathcal{H}_\pi)$. Then by Glimm's Lemma (Lemma 1.5.3), g is a wk*-limit of pure states on $\pi(A)$. Consequently, f is a wk*-limit of pure states on A. Thus the wk*-closure of $PS(A)$ contains $S'(A)$. □

Proposition 6.3.5. Let A be a unital, simple, antiliminal C*-algebra. Then $\overline{PS(A)}^{\mathrm{wk}*} = S(A)$.

Proof. Since $\cap_\pi K_\pi = 0$, where the intersection is taken over all unitary equivalence classes of irreducible representations, it follows that $K_\pi = 0$ for some irreducible representation by simplicity. In particular, every state vanishes on K_π, so the above lemma implies that $\overline{PS(A)}^{\mathrm{wk}*} = S(A)$. □

Corollary 6.3.6. Let A be a unital, simple, purely infinite C*-algebra. Then $\overline{PS(A)}^{\mathrm{wk}*} = S(A)$.

We will now prove several lemmas which will be required for the proof of the nuclear embedding theorem. We will now be following Kirchberg's and Phillips' paper [19].

Lemma 6.3.7. Let A be a unital simple purely infinite C*-algebra, ϕ a state on A. Then for every $\varepsilon > 0$ and every finite subset $F \subseteq A$, there exists a non-zero projection $p \in A$ such that $\|pap - \phi(a)p\| < \varepsilon$ for all $a \in F$.

Proof. Since the wk*-closure of the pure states is the whole state space, ϕ can be excised by a net of positive elements (e_λ) with $\|e_\lambda\| = 1$. That is, $\lim_\lambda \|e_\lambda a e_\lambda - \phi(a)e_\lambda^2\| = 0$. Now let λ be such that $\|e_\lambda a e_\lambda - \phi(a)e_\lambda^2\| < \frac{\varepsilon}{2}$ for all $a \in F$. Since A has real rank zero, $e_\lambda \geq 0$, there is a positive element $e \in A$ with finite spectrum such that $\|e\| = 1$ and $\|eae - \phi(a)e^2\| < \varepsilon$ for all $a \in F$. Since e has finite spectrum, it is positive, and $\|e\| = 1$, $p = \chi_{\{1\}}(e)$ is a non-zero projection in A. Moreover,

$$\|pap - \phi(a)p\| = \|peaep - p\phi(a)e^2p\| \leq \|eae - \phi(a)e^2\| < \varepsilon.$$

□

Lemma 6.3.8. Let A be a C*-algebra, $p \in A$ a projection, and suppose $a \in A$ is such that $ap = a$ and $\|a^*x - a\| < 1$. Then the partial isometry v in the polar decomposition of a in pAp satisfies $v^*v = p$ and

$$\|v - a\| \leq 1 - (1 - \|a^*a - p\|)^{\frac{1}{2}} \leq \|a^*a - p\|.$$

Proof. Notice that $pa^*ap = (ap)^*(ap) = a^*a$ since $ap = a$, and so $\|a^*a - p\| < 1$, so that a^*a is invertible in the hereditary subalgebra pAp. In particular since $pa^*ap = a^*a$, $v = ab$ where $b = |a|^{-1}$ in pAp. So $b = (pa^*ap)^{-\frac{1}{2}}$ in pAp, and

$$v^*v = ba^*ab = vpa^*apb = (pa^*ap)^{-\frac{1}{2}}pa^*ap(pa^*ap)^{-\frac{1}{2}} = p.$$

Now let $\delta = \|a^*a - p\|$. Then using the fact that $(a^*a)^{\frac{1}{2}} \in pAp$, we have

$$\begin{aligned}(v-a)^*(v-a) &= v^*v - v^*a - a^*v + a^*a \\ &= p - (pa^*ap)^{-\frac{1}{2}}a - a^*(pa^*ap)^{-\frac{1}{2}}) + a^*a \\ &= a^*a - (a^*a)^{-\frac{1}{2}}a^*a - a^*a(a^*a)^{-\frac{1}{2}} + p \\ &= a^*a - 2(a^*a)^{\frac{1}{2}} + p \\ &= \left((a^*a)^{\frac{1}{2}} - p\right)^2.\end{aligned}$$

And so

$$\|v - a\| = \|(a^*a)^{\frac{1}{2}} - p\| \leq \sup\{|\sqrt{t} - 1| \mid |t-1| \leq \delta\} = 1 - (1-\delta)^{\frac{1}{2}},$$

which gives the first inequality. For the second, $(1-\delta)^{\frac{1}{2}} \geq 1-\delta$, and so $-\delta \leq (1-\delta)^{\frac{1}{2}} - 1$. $\square$

Lemma 6.3.9. Let A be a unital simple purely infinite C*-algebra, $T : A \to M_n$ be a u.c.p. map, and let $\phi : M_n \to A$ be a *-homomorphism. Then for every $\varepsilon > 0$ and finite $F \subseteq A$, there exists a partial isometry $s \in A$ such that $s^*s = \phi(1)$ and $\|s^*as - \phi(T(a))\| < \varepsilon$ for all $a \in F$.

Proof. Without loss of generality, suppose that $1 \in F$ and that $\|a\| \leq 1$ for all $a \in F$. Let $0 < \delta < \min\{\frac{1}{n^3}, \frac{\varepsilon}{4n^3}\}$.

Let $(e_1, \dots, e_n)$ be the standard orthonormal basis for $\mathbb{C}^n$, and let (e_{ij}) be the matrix units of M_n. Define

$$\tau((a_{ij})) = \tau\left(\sum_{i,j} e_{ij} \otimes a_{ij}\right) = \frac{1}{n}\sum_{k,l=1}^{n} \langle T(a_{kl}e_l, e_k\rangle,$$

which is a state on $M_n(A)$. Then by the correspondence between $CP(M_n, A)$ and $(M_n(A)^*)_+$ (1.1.8), we have

$$T(a) = n\sum_{k,l=1}^{n} \tau(e_{ij} \otimes a)e_{ij}$$

for all $a \in A$. Since $M_n(A)$ is unital, simple, purely infinite, Lemma 6.3.7 gives a non-zero projection $p_0 \in M_n(A)$ such that

$$\|p_0(e_{ij} \otimes a)p_0 - \tau(e_{ij} \otimes a)p_0\| < \delta$$

for all $a \in F$ and $1 \leq i, j \leq n$. But because A is simple and purely infinite, there exists $p \leq p_0 \in M_n(A)$ and a partial isometry $s_1 \in M_n(A)$ such that $s_1s_1^* = p$ and $s_1^*s_1 = e_{11} \otimes \phi(e_{11})$. For $2 \leq j \leq n$, let $s_j \in M_n(A)$ be the partial isometry defined by

$$s_j = s_1(e_{11} \otimes \phi(e_{1j})).$$

Then
$$\|s_i^*(e_{kl} \otimes a)s_j - \tau(e_{kl} \otimes a)(e_{11} \otimes \phi(e_{ij}))\| < \delta$$
for all $a \in F$ and $1 \leq i, j, k, l \leq n$. Let
$$c = \sum_k (e_{1k} \otimes 1)s_k \in M_n(A).$$

Then for $a \in F$, we have
$$\begin{aligned} c^*(e_{11} \otimes a)c &= \left(\sum_k s_k^*(e_{k1} \otimes 1)\right)(e_{11} \otimes a)\left(\sum_l (e_{1l} \otimes 1)s_l\right) \\ &= \sum_{k,l} s_k^*(e_{kl} \otimes a)s_l. \end{aligned}$$

Therefore
$$\|nc^*(e_{11} \otimes a)c - e_{11} \otimes \phi(T(a))\| \leq n \sum_{k,l} \|s_k^*(e_{kl} \otimes a)s_l - \tau(e_{kl} \otimes a)(e_{11} \otimes \phi(e_{kl}))\| < n^3\delta$$

for $a \in F$. In particular, letting $a = 1$, we have that $\|nc^*(e_{11} \otimes 1)c - e_{11} \otimes \phi(1)\| < n^3\delta$. Let
$$d = \sqrt{n}(e_{11} \otimes 1)c(e_{11} \otimes \phi(1)) \in (e_{11} \otimes 1)M_n(A)(e_{11} \otimes 1).$$

Then $\|d^*d - e_{11} \otimes \phi(1)\| < n^3\delta$, hence the above lemma gives us that $d(d^*d)^{-\frac{1}{2}}$ (in $(e_{11} \otimes \phi(1))M_n(A)(e_{11} \otimes \phi(1))$ is a partial isometry in $(e_{11} \otimes 1)M_n(A)(e_{11} \otimes 1)$ such that
$$\left(d(d^*d)^{-\frac{1}{2}}\right)^*\left(d(d^*d)^{-\frac{1}{2}}\right) = e_{11} \otimes \phi(1)$$

and
$$\|d(d^*d)^{-\frac{1}{2}} - d\| \leq \|d^*d - (e_{11} \otimes \phi(1))\| < n^3\delta.$$
Now let $s \in A$ be the partial isometry such that $d(d^*d)^{\frac{1}{2}} = e_{11} \otimes s$. Then clearly $s^*s = \phi(1)$ and
$$\|nc^*(e_{11} \otimes a)c - e_{11} \otimes \phi(T(a))\| < n^3\delta,$$
hence
$$\|d^*(e_{11} \otimes a)d - e_{11} \otimes \phi(T(a))\| < n^3\delta.$$
Moreover, $\|e_{11} \otimes s - d\| = \|d(d^*d)^{-\frac{1}{2}} - d\| \leq n^3\delta < 1$, and so $\|d\| = 2$. Thus
$$\begin{aligned} \|s^*as - \phi(T(a))\| &= \|e_{11} \otimes s)^*(e_{11} \otimes a)(e_{11} \otimes s) - e_{11} \otimes \phi(T(a))\| \\ &\leq n^3\delta + 2n^3\delta + \|d^*(e_{11} \otimes a)d - e_{11} - \phi(T(a))\| \\ &< 3n^3\delta + n^3\delta = 4n^3\delta < \varepsilon. \end{aligned}$$

$\square$

Lemma 6.3.10. Let A be a unital C*-algebra, $T : M_n \to A$ be a u.c.p. map. Then there exists a partial isometry $t \in M_n \otimes M_n \otimes A$ such that

$$t^*t = e_{11} \otimes e_{11} \otimes 1 \text{ and } t^*(b \otimes 1 \otimes 1)t = e_{11} \otimes e_{11} \otimes T(b)$$

for all $b \in M_n$.

Proof. Let $x = \sum_{i,j} e_{ij} \otimes e_{ij} \in M_n \otimes M_n$, and note that $\frac{1}{n}x$ is a projection, so that $x \geq 0$. Therefore

$$y = (\mathrm{id} \otimes T)(x) = \sum_{i,j} e_{ij} \otimes T(e_{ij}) \in M_n \otimes A$$

is positive as well. Letting $y^{\frac{1}{2}} = \sum_{i,j} e_{ij} \otimes a_{ij}$, $y^{\frac{1}{2}}$ is self-adjoint and squares to y, so $a_{ik}^* = a_{ki}$ and $\sum_{j=1}^n a_{ij}a_{jk} = T(e_{ik})$ for $1 \leq i, k \leq n$. Let $t = \sum_{i,j} e_{i1} \otimes e_{j1} \otimes a_{ji}$. Then

$$\begin{aligned} t^*(e_{ik} \otimes 1 \otimes 1)t &= \left(\sum_{i,j} e_{1i} \otimes e_{1j} \otimes a_{ij}\right)\left(\sum_{k,l} e_{k1} \otimes e_{l1} \otimes a_{lk}\right) \\ &= \sum_{j=1}^n e_{11} \otimes e_{11} \otimes a_{ij}a_{jk} \\ &= e_{11} \otimes e_{11} \otimes T(e_{ik}). \end{aligned}$$

It then follows that $t^*(b \otimes 1 \otimes 1)t = e_{11} \otimes e_{11} \otimes T(b)$ for all $b \in M_n$. In particular,

$$t^*t = e_{11} \otimes e_{11} \otimes T(1) = e_{11} \otimes e_{11} \otimes 1.$$

□

Proposition 6.3.11. Let A be a unital, simple, purely infinite C*-algebra, and let $\rho : A \to A$ be a unital nuclear map. Then for every $\varepsilon > 0$, finite $F \subseteq A$, there exists a non-unitary isometry $s \in A$ such that $\|s^*as - \rho(a)\| < \varepsilon$ for all $a \in F$.

Proof. Since ρ is nuclear and unital, there exists $n \in \mathbb{N}$ and u.c.p. maps $\phi : A \to M_n, \psi : M_n \to A$ such that $\|\psi \circ \phi(a) - \rho(a)\| < \varepsilon$ for all $a \in F$. Since A is unital, simple, purely infinite, so is $M_n \otimes M_n \otimes A$, and so there is an isometry $t_1 \in M_n \otimes M_n \otimes A$ such that $t_1^*t_1 = 1$ and $t_1t_1^* < e_{11} \otimes e_{11} \otimes 1$. Applying the previous lemma to ψ, we have a partial isometry t_2 such that $t_2^*t_2 = e_{11} \otimes e_{11} \otimes 1$ and $t_2^*(b \otimes 1 \otimes 1)t = e_{11} \otimes e_{11} \otimes \psi(b)$ for all $b \in M_n$. Define a *-homomorphism $\pi_0 : M_n \to A$ by identifying A with $A_0 = (e_{11} \otimes e_{11} \otimes 1)(M_n \otimes M_n \otimes A)(e_{11} \otimes e_{11} \otimes 4)$, and letting

$$\pi_0(b) = t_1(b \otimes 1 \otimes 1)t_1^*.$$

Let $t = t_1t_2$, which is an isometry in A_0, and identify it with an element of A, still call it t. Using the previous lemma, we have that $\psi(b) = t^*\pi_0(b)t$ for all $b \in M_n$.

Now let $p = t_1t_1^*$, which we can regard as a projection in A. Since A is purely infinite and $1 - p \neq 0$, there is a non-zero homomorphism $\pi_1 : M_n \to (1 - p)A(1 - p)$ defined by $\pi_1(a) = (1 - p)\pi_0(a)(1 - p)$. Now let $\pi(b) = \pi_0(b) + \pi_1(b)$, so that $\rho(b) = t^*\pi(b)t$ for all $b \in M_n$.

Now using Lemma 6.3.9, there is a partial isometry $s_0 \in A$ such that $s_0^* s_0 = \pi(1)$ and $\|s_0^* a s_0 - \pi(\phi(a))\| < \varepsilon$ for all $a \in F$. Let $s = s_0 t$, so that for all $a \in F$,

$$\|s^* a s - \psi \circ \phi(a)\| = t^* s_0^* a s_0 t - t^* \pi(\psi(a)) t\| < \varepsilon.$$

Moreover s is an isometry since $s^* s = t^* s_0^* s_0 t = t^* \pi(1) t = \rho(1) = 1$. We also have $s_0^* s_0 \geq t_1 t_1^* \geq t t^*$ and $s_0^* s_0 < t_1 t_1^*$, so if it was unitary then

$$s_0^* s_0 = s_0^* s s^* s_0^* \leq s_0^* s_0 t_1 t_1^* s_0 = \pi(1) t_1 t_1^* \pi(1) = t_1 t_1^*,$$

giving us that $s_0^* s_0 < s_0^* s_0$, a contradiction. $\square$

Lemma 6.3.12. Let A be a unital C*-algebra, $a_1, \ldots, a_m \in A$ linearly independent such that $E = \operatorname{span}\{a_1, \ldots, a_m\} \subseteq A$ is an operator system. Let

$$M = \sup \left\{ \max_l |\alpha_k| \mid \left\| \sum \alpha_l a_l \right\| \leq 1 \right\}.$$

Then for $b_1, \ldots, b_m \in A$, the map $W : E \to \operatorname{span}\{b_1, \ldots, b_m\}$ given by $W(a_l) = b_l$, satisfies

$$\|W\|_{cb} \leq 1 + mM \sum_l \|a_l - b_l\|$$

and if $mM \sum_l \|a_l - b_l\| < 1$, then

$$\|W^{-1}\|_{cb} \leq \left(1 - mM \sum_l \|a_l - b_l\|\right)^{-1}.$$

Proof. Consider the space $X = \ell_m^\infty$, which is the m-dimensional space with norm $\|(\alpha_1, \ldots, \alpha_m)\|_\infty = \max_i |\alpha_i|$. Define $Q : E \to X$ by $a_i \to e_i$, where $(e_i)_1^m$ is the standard basis for ℓ_m^∞. Define $R : \ell_m^\infty \to A$ by $R(e_i) = b_i - a_i$. Then $\|Q\| = M$ and $\|R\| \leq \sum_i \|a_i - b_i\|$. Then by Lemma 1.1.9,

$$\|R \circ Q\|_{cb} \leq m \|R \circ Q\| \leq mM \sum_i \|a_i - b_i\|.$$

Since $W(a) = a + R(Q(a))$, we clearly have $\|W\|_{cb} \leq 1 + mM \sum_i \|a_i - b_i\|$. Now for any $n, a \in M_n(E)$,

$$\|W^{(n)}(a)\| \geq \|a\| - \|(R \circ Q)^{(n)}(a)\| \geq \|a\|(1 - \|R \circ Q\|_{cb}).$$

Therefore

$$\|(W^{(n)})^{-1}\| \leq \left(1 - mM \sum_i \|a_i - b_i\|\right)^{-1}$$

for all n. $\square$

Lemma 6.3.13. Let A be a unital C*-algebra, $E \subseteq A$ an operator system, and let $\phi : E \to \mathcal{B}(\mathcal{H})$ be a unital self-adjoint c.b. map. Then there exists a u.c.p. map $\psi : A \to \mathcal{B}(\mathcal{H})$ such that $\|\psi|_E - \phi\|_{cb} \leq 2(\|\phi\|_{cb} - 1)$.

Proof. By Wittstock's extension Theorem (1.1.6), there exists a c.b. map $\phi_0 : A \to \mathcal{B}(\mathcal{H})$ such that $\phi_0|_E = \phi$ and $\|\phi_0\|_{cb} = \|\phi\|_{cb}$. Now by Lemma 4.1.21, there exists a u.c.p. map $\psi : A \to \mathcal{B}(\mathcal{H})$ such that $\|\phi_0 - \psi\|_{cb} \leq 2(\|\phi_0\|_{cb} - 1) = 2(\|\phi\|_{cb} - 1)$. Then it is also true that $\|\phi - \psi|_E\|_{cb} \leq 2(\|\phi\|_{cb} - 1)$. □

Lemma 6.3.14. Let A be a unital, separable, exact C*-algebra, $E \subseteq A$ a finite-dimensional operator system, and $\varepsilon > 0$. Then for every $0 < \delta < \frac{\varepsilon}{2}$, there exists $n \in \mathbb{N}$ such that if B_1, B_2 are unital, separable C*-algebras and $\phi : E \to B_1, \psi : E \to B_2$ are u.c.p. maps such that

1. ϕ is injective;
2. $\|\phi^{-1}\|_n \leq 1 + \delta$, where $\phi^{-1} : \phi(E) \to E$;
3. B_2 is nuclear,

then there is a u.c.p. map $\eta : B_1 \to B_2$ such that $\|\eta \circ \phi - \psi\| < \varepsilon$.

Proof. Let $\rho = \frac{\varepsilon - 2\delta}{4(1+\delta)}$. Since A is exact, there is a nuclear embedding $A \subseteq \mathcal{B}(\mathcal{H})$. Let $\{a_1, \dots, a_m\}$ be a basis for E such that $a_1 = 1$ and let $\mu > 0$ be small enough such that if $b_1, \dots, b_m \in \mathcal{B}(\mathcal{H})$ and $\|a_j - b_j\| < \mu$ for all $j = 1, \dots, m$, then the map $T : E \to \operatorname{span}\{b_1, \dots, b_m\}$ defined by $T(a_j) = b_j$ satisfies $\|T^{-1}\|_{cb} < 1 + \rho$ by Lemma 6.3.13.

Now since the inclusion $A \subseteq \mathcal{B}(\mathcal{H})$ is nuclear, there exists $n \in \mathbb{N}$ and u.c.p. maps $S_1 : E \to M_n, \tilde{S}_2 : M_n \to \mathcal{B}(\mathcal{H})$ such that $b_j = \tilde{S}_2(S_1(a_j))$ satisfy $\|a_j - b_j\| < \mu$ for all $j = 1, \dots, m$. Let T be as above for these b_j's, and let $E_0 = S_1(E)$, which is an operator system in M_n. Define $S_2 : E_0 \to E$ by $S_2 = T^{-1} \circ \tilde{S}_2$. Thus S_2 is unital, $S_2 \circ S_1 = \mathrm{id}_E$, and $\|S_2\|_{cb} < 1 + \rho$. Moreover, S_1 is u.c.p., hence self-adjoint, so therefore S_2 is as well. Indeed, we have $S_2(S_1(a_j^*)) = a_j^* = S_2(S_1(a_j))^*$ for all $j = 1, \dots, m$. It is also clear that S_2 is unital.

Now since B_2 is nuclear, there exists $r \in \mathbb{N}$ and u.c.p. maps $W_1 : E \to M_r, W_2 : M_r \to B_2$ such that $\|W_2 \circ W_1 - \psi\| < \rho$. Now since $E_0 \subseteq M_n$ is an operator system, the above lemma provides a u.c.p. map $Q : M_n \to M_r$ such that $\|Q|_{E_0} - W_1 \circ S_2\| < 2(\|W_1 \circ S_2\|_{cb} - 1) = 2\rho$.

Consider $S_1 \circ \phi^{-1} : \phi(E) \to E_0 \subseteq M_n$. Since S_1 is u.c.p.,

$$\|(S_1 \circ V^{-1})^{(n)}\| \leq \|S_1\|_{cb} \|\phi^{-1}\|_n \leq 1 + \delta.$$

Now by Lemma 1.1.10, $\|\phi^{-1}\|_n = \|\phi^{-1}\|_{cb}$, hence $\|S_1 \circ \phi^{-1}\|_{cb} \leq 1 + \delta$. Now Lemma 6.3.13 again provides a u.c.p. map $R : B_1 \to M_n$ such that $\|R|_{\phi(E)} - S_1 \circ \phi^{-1}\| \leq 2\delta$. Now let $\eta = W_2 \circ Q \circ R : B_1 \to B_2$. Then η is u.c.p. and

$$\begin{aligned}
\|\eta|_{\phi(E)} - \psi \circ \phi^{-1}\| &\leq \|\psi - W_2 \circ W_1\| \|\phi^{-1}\| + \|W_2\| \|Q \circ R|_{\phi(E)} - W_1 \circ S_2 \circ S_1 \circ \phi^{-1}\| \\
&\leq \rho(1 + \delta) + \|R|_{\phi(E)} - S_1 \circ \phi^{-1}\| + \|Q|_{\phi(E)} - W_1 \circ S_2\| \|S_1 \circ \phi^{-1}\| \\
&< \rho(1 + \delta) + 2\rho(1 + \delta) + 2\delta \\
&= 3\rho(1 + \delta) + 2\delta = \frac{3}{4}\varepsilon - \frac{3}{2}\delta + 2\delta \\
&= \frac{3}{4}\varepsilon + \frac{1}{2}\delta < \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

□

Lemma 6.3.15. Let A be a unital C*-algebra, $u \in U(A), s \in A$ an isometry with range projection $e = ss^*$. Then

$$\|u - (eue + (1-e)u(1-e))\| \leq \inf_{v \in U(A)} \sqrt{2\|s^*us - v\|}.$$

Proof. Notice that if v is unitary, then svs^* is unitary in eAe, and

$$\|eue - svs^*\| = \|ss^*uss^* - svs^*\| = \|s^*us - v\|.$$

Thus $\|(eue)^*(eue) - e\| \leq 2\|s^*us - v\|$. Furthermore, we have

$$e = eu^*ue = (eue)^*(eue) + ((1-e)ue)^*((1-e)ue),$$

and so

$$\|((1-e)ue)^*((1-e)ue)\| \leq 2\|s^*us - v\|.$$

Therefore $\|(1-e)ue\| \leq \sqrt{2\|s^*us - v\|}$. Similarly, since $uu^* = 1$, we get that $\|eu(1-e)\| \leq \sqrt{2\|s^*us - v\|}$. Since e and $1-e$ are orthogonal, it follows that

$$\|u - (eue + (1-e)u(1-e))\| = \|(1-e)ue + eu(1-e)\| \leq \sqrt{2\|s^*us - v\|}.$$

□

Lemma 6.3.16. Let A be a unital C*-algebra, $s, t \in A$ isometries in A. Let $D \subseteq A$ be a subalgebra which is isomorphic to $\mathcal{O}_2$ such that every element of D commutes with s and t. Then there exists a unitary $z \in A$ such that whenever $u, v \in U(A)$ commute with every element of D, then

$$\|z^*uz - v\| \leq 11\sqrt{\max\{\|s^*us - v\|, \|t^*vt - u\|\}}.$$

Proof. Let $B = D' \cap A$ be the relative commutant. Since $\mathcal{O}_2$ is nuclear, the *-homomorphism $\rho \times \iota : \mathcal{O}_2 \odot B \to A$, where $\rho : \mathcal{O}_2 \to D$ is an isomorphism and ι is the inclusion map $B \hookrightarrow A$, extends to a unique *-homomorphism $\pi : \mathcal{O}_2 \otimes B \to A$ such that $\pi(1 \otimes b) = b$ for all $b \in B$ and $\pi|_{\mathcal{O}_2 \otimes 1}$ is an isomorphism between $\mathcal{O}_2$ are D. Now we will show that there exists $z \in U(\mathcal{O}_2 \otimes B)$ such that whenever $u, v \in U(B)$,

$$\|z(1 \otimes u)z^* - 1 \otimes v\| \leq 11\sqrt{\max\{\|s^*us - v\|, \|t^*vt - u\|\}}.$$

We will take certain compressions of $1 \otimes u$ and rearrange them. To this end, let $e_1 = ss^*$, $f_1 = tt^*$, and

$$e_2 = sf_1s^* \leq e_1; \quad f_2 = te_1t^* \leq f_1; \quad f_3 = te_2t^* \leq f_2.$$

Let

$$p_1 = 1 - e_1; \quad p_2 = e_1 - e_2; \quad p_3 = e_2,$$

which are mutually orthogonal projections which sum to 1, as are

$$q_1 = 1 - f_1; \quad q_2 = f_1 - f_2; \quad q_3 = f_2 - f_3; \quad q_4 = f_3.$$

Let

$$c_1 = p_2 s q_1; \quad c_2 = p_1 t^* q_2; \quad c_3 = p_2 t^* q_3; \quad c_4 = p_3 t^* q_4,$$

which are partial isometries such that $c_1 = s - sf_1$, so

$$c_1^* c_1 = (s - sf_1)^*(s - sf_1) = s^*s - s^*sf_1 - f_1 s^*s + f_1 s^* s f_1 = 1 - f_1 = q_1$$

and

$$c_1 c_1^* = (s - sf_1)(s - sf_1)^* = ss^* - sf_1 s^* - sf_1 s^* + sf_1 s^* = e_1 - sf_1 s^* = e_1 - e_2 = p_2.$$

Moreover by similar computations, once can see that

$$c_j^* c_j = q_j; \quad c_j c_j^* = p_{j-1}$$

for $j = 2, 3, 4$. Now let $\mathcal{O}_2 = C^*(s_1, s_2)$, where s_1, s_2 are isometries satisfying the Cuntz relation. Define

$$z = s_1 \otimes c_1 + 1 \otimes c_2 + s_2 \otimes c_3 + 1 \otimes c_4 \in \mathcal{O}_2 \otimes B.$$

Then since $c_1^* c_2 = c_1^* c_4 = c_3^* c_2 = c_3^* c_4 = c_j c_k^* = 0$ for all $j \neq k$, we have that

$$\begin{aligned} z^* z &= s_1^* s_1 \otimes c_1^* c_1 + 1 \otimes c_2^* c_2 + s_2^* s_2 \otimes c_3^* c_3 + 1 \otimes c_4^* c_4 + s_1^* s_2 \otimes c_1^* c_3 + s_2^* s_1 \otimes c_3^* c_1 \\ &= 1 \otimes (q_1 + q_2 + q_3 + q_4) = 1 \otimes 1, \end{aligned}$$

and

$$\begin{aligned} zz^* &= s_1 s_1^* \otimes p_2 + 1 \otimes p_1 + s_2 s_2^* \otimes p_2 + 1 \otimes p_3 \\ &= 1 \otimes (p_1 + p_2 + p_3) = 1 \otimes 1. \end{aligned}$$

Now

$$\begin{aligned} z(1 \otimes q_1) z^* &= s_1 s_1^* \otimes c_1 q_1 c_1^* = s_1 s_1^* \otimes c_1 c_1^* = s_1 s_1^* \otimes p_2, \\ z(1 \otimes q_2) z^* &= 1 \otimes c_2 q_2 c_2^* = 1 \otimes c_2 c_2^* = 1 \otimes p_1, \\ z(1 \otimes q_3) z^* &= s_2 s_2^* \otimes c_3 q_3 c_3^* = 1 \otimes c_3 c_3^* = s_2 s_2^* \otimes p_2, \\ z(1 \otimes q_4) z^* &= 1 \otimes c_4 q_4 c_4^* = 1 \otimes c_4 c_4^* = 1 \otimes p_3. \end{aligned}$$

Now let $u, v \in B$ be unitaries and let $\delta = \max\{\|s^* u s - v\|, \|t^* v t - u\|\}$. Since $sq_1 = p_2 s$ and $p_2 \leq ss^*$, we get

$$\begin{aligned} \|c_1(q_1 v q_1) c_1^* - p_2 u p_2\| &= \|p_2 s v s^* p_2 - p_2 u p_2\| \\ &= \|p_2 s v s^* p_2 - p_2 s s^* u s s^* p_2\| \\ &\leq \|v - s^* u s\| \leq \delta. \end{aligned}$$

Similarly, one obtains

$$\|c_j (q_j v q_j) c_j^* - p_{j-1} u p_{j-1}\| \leq \|t v t^* - u\| \leq \delta$$

for $j = 2,3,4$. But now

$$\begin{aligned}&z^*(1 \otimes (q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4))z^* \\ &= s_1s_1^* \otimes c_1(q_1vq_1)c_1^* + 1 \otimes c_2(q_2vq_2)c_2^* + s_2s_2^* \otimes c_3(q_3vq_3)c_3^* + 1 \otimes c_4(q_4vq_4)c_4^*,\end{aligned}$$

so that

$$\begin{aligned}&\|z(1 \otimes v)z^* - 1 \otimes u\| \\ &\leq \delta + \|q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4 - v\| + \|p_1up_1 + p_2up_2 + p_3up_3 - u\|\end{aligned}$$

Therefore since $e_1 = ss^*$,

$$\|u - (e_1ue_1) + (1-e_1)u(1-e_1))\| \leq \sqrt{2\|s^*us - v\|} \leq \sqrt{2\delta}$$

by the above lemma. Since $e_2 = stt^*s^*$,

$$\|e_2ue_2 - stut^*s^*\| \leq \|t^*s^*ust - u\| \leq \|s^*us - v\| + \|t^*vt - u\| \leq 2\delta.$$

Consequently,

$$\|u - (e_2ue_2 + (1-e_2)u(1-e_2))\| \leq \sqrt{4\delta}$$

by the above lemma again. Now since $e_1 \geq e_2$, we can compress by e_1 to get

$$\|e_1ue_1 - (e_2ue_2 + (e_1 - e_2)u(e_1 - e_2))\|\sqrt{4\delta}.$$

It then follows that

$$\|p_1up_1 + p_2up_2 + p_3up_3 - u\| \leq (\sqrt{2} + 2)\sqrt{\delta}.$$

Similarly since $f_1 = tt^*, f_2 = tss^*t^*, f_3 = (tst)(tst)^*$, we repeat the above process and keep applying the above lemma to get

$$\begin{aligned}&\|v - (f_1vf_1 + (1-f_1)v(1-f_1))\| \leq \sqrt{2\delta}, \\ &\|f_1vf_1 - (f_2vf_2 + (1-f_2)v(1-f_2))\| \leq \sqrt{4\delta}, \\ &\|f_2vf_2 - (f_3vf_3 + (1-f_3)v(1-f_3))\| \leq \sqrt{6\delta},\end{aligned}$$

so that

$$q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4 - v\| \leq (\sqrt{2} + \sqrt{4} + \sqrt{6})\sqrt{\delta}.$$

Now since $\delta \leq 2$, $\delta \leq \sqrt{2\delta}$, and so

$$\begin{aligned}\|z^*(1 \otimes u)z - 1 \otimes v\| &= \|z(1 \otimes v)z^* - 1 \otimes u\| \\ &\leq \delta + \|q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4 - v\| + \|p_1up_1 + p_2up_2 + p_3up_3 - u\| \\ &\leq \sqrt{2\delta} + (\sqrt{2} + \sqrt{4})\sqrt{\delta} + (\sqrt{2} + \sqrt{4} + \sqrt{6})\sqrt{\delta} \\ &= (4 + 3\sqrt{2} + \sqrt{6})\sqrt{\delta} \leq 11\sqrt{\delta}.\end{aligned}$$

□

Lemma 6.3.17. Let A be a unital, separable, exact C*-algebra, and let B be a separable nuclear, unital, simple, purely infinite C*-algebra. Let $\phi, \psi : A \to B$ be two injective unital *-homomorphisms. Then the homomorphisms from A to $\mathcal{O}_2 \otimes B$ given by $a \mapsto 1 \otimes \phi(a)$ and $a \mapsto 1 \otimes \psi(a)$ are approximately unitarily equivalent.

Proof. Let $u_1, \ldots, u_n \in U(A)$ and $\varepsilon > 0$. We will find a unitary $z \in \mathcal{O}_2 \otimes B$ such that

$$\|z^*(1 \otimes \phi(u_j))z^* - 1 \otimes \psi(u_j)\| < \varepsilon$$

for all $1 \leq j \leq n$. Let $E = \text{span}\{1, u_1, u_1^*, \ldots, u_n, u_n^*\}$. Then applying Lemma 6.3.14 to $\phi|_E, \psi|_E$, there are u.c.p. map $S, T : B \to B$ such that

$$\|S \circ \phi(u_j) - \psi(u_j)\| < \frac{1}{2}\left(\frac{\varepsilon}{11}\right)^2; \quad \|T \circ \psi(u_j) - \phi(u_j)\| < \frac{1}{2}\left(\frac{\varepsilon}{11}\right)^2$$

for all $j = 1, \ldots, n$. Then Lemma 6.3.11 gives isometries $s, t \in B$ such that

$$\|s^*\phi(u_j)s - \psi(u_j)\| < \left(\frac{\varepsilon}{11}\right)^2; \quad \|t^*\psi(u_h)t - \phi(u_j)\| < \left(\frac{\varepsilon}{11}\right)^2$$

for all $j = 1, \ldots, n$. Then applying the above lemma, we get the desired z. $\square$

Theorem 6.3.18. Let A be a unital, separable, exact C*-algebra. Then any two injective unital *-homomorphisms from A to $\mathcal{O}_2$ are approximately unitarily equivalent.

Proof. Let $\phi, \psi : A \to \mathcal{O}_2$ be unital, injective *-homomorphisms. Let $\mu : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism, and let $\beta : \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ be the map $\beta(a) = 1 \otimes a$. The $\mu \circ \beta : \mathcal{O}_2 \to \mathcal{O}_2$ is approximately unitarily equivalent to $\beta \circ \psi$ by the above lemma. Hence ϕ is approximately unitarily equivalent to $\mu \circ \beta \circ \phi$, which is approximately unitarily equivalent to $\mu \circ \beta \circ \psi$, which is approximately unitarily equivalent to ψ. $\square$

6.4 Embedding Separable Exact C*-Algebras into $\mathcal{O}_2$

We now follow E. Kirchberg and C. Phillips original paper [19]. First, it is shown that every unital separable exact quasidiagonal C*-algebra embeds into $\mathcal{O}_2$.

Lemma 6.4.1. Let A be a unital, separable, exact C*-algebra such that there is an injective unital *-homomorphism $\phi : A \to (\mathcal{O}_2)_\infty = \Pi_n \mathcal{O}_2 / \oplus_n \mathcal{O}_2 = \ell^\infty(\mathcal{O}_2)/c_0(\mathcal{O}_2)$ with a u.c.p. lift $\rho : A \to \ell^\infty(\mathcal{O}_2)$. Then there is an injective unital *-homomorphism $A \to \mathcal{O}_2$.

Proof. Let (u_n) be a sequence of unitaries in A which have dense span. Let $E_n \subseteq A$ be the operator system defined by $E_n = \text{span}\{1, u_1, u_1^*, \ldots, u_n, u_n^*\} \subseteq A$. Then $\mathbb{C} \cdot 1 \subseteq E_0 \subseteq E_1 \subseteq \cdots A$ and $\overline{\cup_n E_n} = A$. We will show that there is an injective *-homomorphism $\psi : A \to (\mathcal{O}_2)_\infty$ with a c.p. lift $a \mapsto V(a) = (V_n(a))_n$ such that for all n and sufficiently large m, $V_m|_{E_n}$ is injective, and its inverse defined on $V_m(E_n)$ satisfies $\lim_m \|(V_m|_{E_n}^{-1})^{(k)}\| = 1$ for all $k \in \mathbb{N}$.

Let $a \mapsto Q(a) = (Q_n(a))_n$ be a lift of ϕ to a u.c.p. map from A to $\ell^\infty(\mathcal{O}_2)$. Since ϕ is injective, for ever $N \in \mathbb{N}$, the map $a \mapsto \phi_N(a) = \pi_\infty(Q_{N+1}(a), Q_{N+2}(a), \ldots)$ is again injective. Thus for every $N, k \in \mathbb{N}$, $a \in M_k(A)$, we have

$$\lim_m \|(Q_{N+1}^{(k)}(a), \ldots, Q_{N+m}^{(k)}(a))\| = \|\phi_n^{(k)}(a)\| = \|a\|.$$

Now since each E_n is finite-dimensional, we can inductively construct a sequence $0 = N_1 < N_2 < \cdots$ of integers such that

$$\|(Q^{(k)}_{(N_m)+1}(a), \dots, Q^{(k)}_{N_{m+1}}(a))\| \geq (1 - \frac{1}{2^m})\|a\|$$

for all $k \leq m$ and $a \in M_k(E_m)$. Let $\sigma_m : \mathcal{O}_2^{N_{m+1}-N_m} \to \mathcal{O}_2$ be any unital *-homomorphism. Let $V_m : A \to \mathcal{O}_2$ be defined by

$$V_m(a) = \sigma_m((Q_{(N_m)+1}(a), \dots, Q_{N_{m+1}}(a)),$$

which is u.c.p. since the Q_j's are, and so

$$V(a) = (V_1(a), V_2(a), \dots)$$

defined a u.c.p. map $A \to \ell^\infty(\mathcal{O}_2)$. Moreover we have that $\lim_m \|((V_m|_{E_n})^{-1})^{(k)}\| = 1$ for every fixed $k, n \in \mathbb{N}$. Letting $k = 1$, $\psi = \pi_\infty \circ V$ is isometric, hence injective. Since $\lim_j(Q_j(ab) - Q_j(a)Q_j(b)) = 0$ for $a, b \in A$, we also have that $\lim_m(V_m(ab) - V_m(a)V_m(b)) = 0$ for $a, b \in A$. Thus ψ is a homomorphism with lift V which satisfies the desired properties.

Let $\delta_m > 0$ be such that $\delta_0 \geq \delta_1 \geq \cdots$ and $2\delta_m + 11\sqrt{5\delta_m} < \frac{1}{2^m}$. By Lemma 6.3.14, there exists $k(m)$ with $k(0) \leq k(1) \leq \cdots$ such that if $V, W : E_m \to \mathcal{O}_2$ are u.c.p. with V injective, and

$$\|(V^{-1})^{(k(m))}\| \leq 1 + \delta_m,$$

then there is a u.c.p. map $T : \mathcal{O}_2 \to \mathcal{O}_2$ such that $\|T \circ V - W\| < 2\delta_m$. With our V above, we can therefore pass to a subsequence of m such that $V_m|_{E_n}$ is injective for $n \leq m$, and

$$\|((V_m|_{E_n})^{-1})^{(k(m))}\| \leq 1 + \delta_m; \; \|V_m(u_n)^* V_m(u_n) - 1\| < \delta_m; \; \|V_m(u_n)V_m(u_n)^* - 1\| < \delta_m$$

for all m and $n \leq m$. Using Lemma 6.3.14 again with the above approximations, we find u.c.p. maps $S_m, T_m : \mathcal{O}_2 \to \mathcal{O}_2$ such that

$$\|T_m \circ V_m|_{E_m} - V_{m+1}|_{E_m}\| \leq 2\delta_m; \; \|S_m \circ V_{m+1}|_{E_m} - V_m|_{E_m}\| \leq 2\delta_m.$$

For $1 \leq j \leq m$, define unitaries $x^{(j)}_m = V_m(u_j)(V_m(u_j)^*(V_m(u_j))^{-\frac{1}{2}}$. Then

$$\|x^{(j)}_m - V_m(u_j)\| \leq \delta_m$$

by Lemma 6.3.8. Therefore

$$\|T_m(x^{(j)}_m) - x^{(j)}_{m+1}\| \leq 4\delta_m; \; \|S_m(x^{(j)}_{m+1}) - s^{(j)}_m\| \leq 4\delta_m.$$

Then Proposition 6.3.11 gives isometries $s_m, t_m \in \mathcal{O}_2$ such that

$$\|s_m^* x^{(j)}_m s_m - x^{(j)}_{m+1}\| \leq 5\delta_m; \; \|t_m^* x^{(j)}_{m+1} t_m - x^{(j)}_m\| \leq 5\delta_m$$

for $1 \leq j \leq m$. Lemma 6.3.16 gives unitaries $z_m \in \mathcal{O}_2 \otimes \mathcal{O}_2$ such that

$$\|z_m(1 \otimes x^{(j)}_m)z_m^* - 1 \otimes V_{m+1}(u_j)\| \leq 2\delta_m + 11\sqrt{5\delta_m} < \frac{1}{2^m}$$

for all $1 \leq j \leq m$.

Define $y_n = z_1^* \cdots z_{n-1}^*$, so that y_n are unitaries such that

$$\lim_n y_n(1 \otimes V_n(u_j))y_n^*$$

exists for all j. It follows that $\psi_0(a) = \lim_n y_n(1 \otimes V_n(a))y_n^*$ exists for all $a \in \cup_n E_n$. Furthermore, for all n, m, $V_m|_{E_n}$ is a u.c.p. map. Thus $\|\psi_0|_{E_n}\| \leq 1$, and so we can extend ψ_0 to a u.c.p. map $\psi : A \to \mathcal{O}_2 \otimes \mathcal{O}_2$. Since

$$\lim_m (V_m(ab) - V_m(a)V_m(b)) = 0$$

for all $a \in \cup_n E_n$, it follows that ψ is a homomorphism. Finally $\|\psi(a)\| = \lim_m \|V_m(a)\| = \|a\|$ for all $a \in \cup_n E_n$, so ψ is isometric, hence injective. Thus $\psi : A \to \mathcal{O}_2 \otimes \mathcal{O}_2$ is an injective unital *-homomorphism, and so it follows that there exists an injective unital *-homomorphism $A \to \mathcal{O}_2$ since $\mathcal{O}_2 \otimes \mathcal{O}_2$. $\square$

Corollary 6.4.2. Every separable quasidiagonal unital exact C*-algebra A has a unital embedding $A \to \mathcal{O}_2$.

Proof. Since A is separable quasidiagonal unital, we have the following commutative diagram

$$\begin{tikzcd}
 & \Pi_n M_{k(n)} \arrow[r, "\iota"] \arrow[d, "\pi"] & \ell^\infty(\mathcal{O}_2) \arrow[d, "\pi_\infty"] \\
A \arrow[ur, "\rho"] \arrow[r, "\phi"'] & \Pi_n M_{k(n)}/\oplus_n M_{k(n)} \arrow[r, "\tilde{\iota}"'] & (\mathcal{O}_2)_\infty,
\end{tikzcd}$$

where $(M_{k(n)})_n$ is some sequence of matrix algebras, ϕ is a unital embedding, ρ is a u.c.p. lift of ϕ, and $\pi_\infty : \ell^\infty(\mathcal{O}_2) \to (\mathcal{O}_2)_\infty$ is the quotient map.. Note that the map ι exists since $\mathcal{O}_2$ contains a unital copy of M_k for all k, and so one can construct a unital injective homomorphism $\Pi_n M_{k(n)} \to \ell^\infty(\mathcal{O}_2)$. If A is exact, then the above lemma implies that A has a unital embedding into $\mathcal{O}_2$. $\square$

The following definitions comes from [14].

Definition 6.4.3. We say that a C*-algebra A is **approximately injective** if given finite dimensional operator systems $E_1 \subseteq E_2 \subseteq \mathcal{B}(\mathcal{H})$, a c.p. map $\phi_1 : E_1 \to A, \varepsilon > 0$, there exists a c.p. map $\phi_2 : E_2 \to A$ such that

$$\|\phi_2|_{E_1} - \phi_1\| < \varepsilon.$$

Remark 6.4.4. Let B is a C*-algebra, $J \triangleleft B$ be approximately injective. If

$$0 \to \mathcal{B}(\mathcal{H}) \otimes J \to \mathcal{B}(\mathcal{H}) \otimes B \to \mathcal{B}(\mathcal{H}) \otimes (B/J) \to 0$$

is exact, then the Effros-Haagerup lifting theorem provides a lift for any finite-dimensional operator system. If B is separable, we can take an increasing union of finite-dimensional operator systems, and the approximately injective property will give us a lift since the liftable maps are closed in point-norm.

Lemma 6.4.5. Let A, B be unital C*-algebras where A is separable, $J \lhd B$ is an ideal which is approximately injective, and let $\phi : A \to B/J$ be an injective homomorphism. Let H be a separable infinite dimensional Hilbert space, and suppose that the induced map

$$A \odot \mathcal{B}(\mathcal{H}) \to \frac{B \odot \mathcal{B}(\mathcal{H})}{J \odot \mathcal{B}(\mathcal{H})}$$

extends continuously to an injective homomorphism

$$\bar{\phi} : A \otimes \mathcal{B}(\mathcal{H}) \to \frac{B \otimes \mathcal{B}(\mathcal{H})}{J \otimes \mathcal{B}(\mathcal{H})}.$$

Then there is a u.c.p. map $T : A \to B$ which lifts ϕ.

Proof. Let $\rho : B \to B/J$ be the quotient map. Let $B_0 = \rho^{-1}(\phi(A)) \subseteq B$, and let $\rho_0 = \rho|_{B_0}$. Since the min-tensor preserves inclusions,

$$J \otimes \mathcal{B}(\mathcal{H}) \subseteq B_0 \otimes \mathcal{B}(\mathcal{H}) \subseteq B \otimes \mathcal{B}(\mathcal{H}).$$

So the hypothesis of the lemma still holds if we replace B by B_0. Hence we can assume that $A = B/J$ and $\phi = \mathrm{id}_{B/J}$. By the Effros-Haagerup lifting theorem, since J is approximately injective, it suffices to show that the sequence

$$0 \to \mathcal{B}(\mathcal{H}) \otimes J \to \mathcal{B}(\mathcal{H}) \otimes B \to \mathcal{B}(\mathcal{H}) \otimes (B/J) \to 0$$

is exact. But this is clear from hypothesis since the map

$$\bar{\phi} : A \otimes \mathcal{B}(\mathcal{H}) = (B/J) \otimes \mathcal{B}(\mathcal{H}) \to \frac{B \otimes \mathcal{B}(\mathcal{H})}{J \otimes \mathcal{B}(\mathcal{H})}$$

is injective (and is clearly surjective). $\square$

Lemma 6.4.6. Let G be a discrete amenable group, $\alpha : G \to \mathrm{Aut}(A)$ be an action of G on a unital C*-algebra A. Let (ϕ, u) be a covariant representation of (A, G, α) in a unital C*-algebra B, where ϕ is injective. Then there is an injective homomorphism $\psi : A \rtimes_{r,\alpha} G = A \rtimes_\alpha G \to C^*_r(G) \otimes B$ determined by $\psi(a) = 1 \otimes \phi(a)$ for $a \in A$, and $\psi(g) = g \otimes u_g$ for $g \in G$.

Proof. Since G is amenable, the full crossed product equals the reduced one, and so the existence and uniqueness of ψ follows from the universal property of the full crossed product. To show injectivity,, let $\pi_0 : B \to \mathcal{B}(\mathcal{K})$ be a faithful representation of B on $\mathcal{K}$, and let $\lambda : C^*_r(G) \to \mathcal{B}(\ell^2(G))$ be the left regular representation. Then $\sigma = (\pi_0 \otimes \lambda) \circ \psi$ is a representation of $A \rtimes_\alpha G$ on $\mathcal{H} = \mathcal{K} \otimes \ell^2(G)$. If we can show that this representation is unitarily equivalent to the canonical regular representation given in (2) of Definition 1.3.2, with $\pi = \pi_0 \circ \phi : B \to \mathcal{B}(\mathcal{K})$, then we will be done. Note

$$\pi(a)(\xi \otimes g) = (\pi_0 \circ \phi \circ \alpha_{g^{-1}}(a)\xi) \otimes \delta_g \text{ and } \sigma(a)(\xi \otimes \delta_g) = ((\pi_0 \circ \psi(a))\xi) \otimes \delta_g,$$

and

$$\pi(g) = 1 \otimes \lambda_g \text{ and } \sigma(g) = \pi_0(u_g) \otimes \lambda_g,$$

since $\pi_0 \otimes \lambda(\psi(g)) = \pi_0 \otimes \lambda(g \otimes u_g)$. Define $V \in \mathcal{B}(\mathcal{K} \otimes \ell^2(G))$ by $V(\xi \otimes \delta_g) = \pi_0(u_g)\xi \otimes \delta_g$ and extend by linearity and continuity. This is clearly a unitary operator. Then

$$\begin{aligned} V^*\sigma(a)V(\xi \otimes \delta_g) &= V^*\sigma(a)((\pi_0(u_g)\xi) \otimes \delta_g) \\ &= V^*(\pi_0 \circ \phi(a)\pi_0(u_g)\xi) \otimes \delta_g \\ &= (\pi_0(u_g^*)\pi_0 \circ \phi(a)\pi_0(u_g)\xi) \otimes \delta_g \\ &= \pi_0 \circ \phi(\alpha_g^{-1}(a))\xi \otimes \delta_g \\ &= \pi(a)(\xi \otimes \delta_g), \end{aligned}$$

so that $V^*\sigma(a)V = \pi(a)$. Moreover by a similar computation, $V^*\sigma(g)V = 1 \otimes \lambda_g$. Hence $\pi_0 \circ \phi$ is unitarily equivalent to the canonical regular covariant representation, and so this is injective. Thus ψ must be injective. $\square$

Lemma 6.4.7. Let B be a unital C*-algebra, $A \subseteq B$ which contains the identity, and let $\sigma \in \mathrm{Aut}(A)$. Suppose that σ is **approximately inner** in B, that is, there is a sequence $v_1, v_2, \ldots$ of unitaries in B such that $\lim_n v_n a v_n^* = \sigma(a)$ for all $a \in A$. Let z be the standard generator for $C(\mathbb{T})$, and let u be the canonical unitary in $A \rtimes_\sigma \mathbb{Z}$ which implements σ. Then

$$a \mapsto 1 \otimes \pi^B(a, a, \ldots);\ u \mapsto z \otimes \pi^B(v_1, v_2, \ldots)$$

defines an injective homomorphism $\phi : A \rtimes_\alpha \mathbb{Z} \to C_r^*(\mathbb{Z}) \otimes (B)_\infty \simeq C(\mathbb{T}) \otimes (B)_\infty$, where $\pi^B : \ell^\infty(B) \to (B)_\infty$ is the quotient map. Moreover for any unital C*-algebra C, this homomorphism extends continuously to an injective homomorphism

$$(A \rtimes_\sigma \mathbb{Z}) \otimes C \to C(\mathbb{T}) \otimes \left(\frac{\ell^\infty(B) \otimes C}{c_0(B) \otimes C}\right).$$

Proof. We clearly have that

$$(v_1, v_2, \ldots) \cdot (a, a, \ldots) \cdot (v_1, v_2, \ldots)^* - (\sigma(a), \sigma(a), \ldots) \in c_0(B)$$

for all $a \in A$. Thus

$$a \mapsto \pi^B(a, a, \ldots) \text{ and } u \mapsto \pi^B(v_1, v_2, \ldots)$$

define a homomorphism from $A \rtimes_\sigma \mathbb{Z} \to \ell^\infty(B)/c_0(B) = (B)_\infty$. Moreover since the first map is injective,

$$a \mapsto 1 \otimes \pi^B(a, a, \ldots) \text{ and } u \mapsto z \otimes \pi^B(v_1, v_2, \ldots)$$

defines an injective *-homomorphisms from $\phi : A \rtimes_\sigma \mathbb{Z} \to C(\mathbb{T}) \otimes (B)_\infty$.

For the latter part of the lemma, first note that $(A \rtimes_\sigma \mathbb{Z}) \otimes C = (A \otimes C) \rtimes_{\sigma \otimes \mathrm{id}_C} \mathbb{Z}$ for any C*-algebra C, and we clearly have

$$\lim_n (v_n \otimes 1)x(v_n \otimes 1)^* = (\sigma \otimes \mathrm{id}_C)(x)$$

for all $x \in A \otimes C$, as this holds for all $x \in A \odot C$. Thus ϕ extends continuously to an injective *-homomorphism

$$\bar{\phi} : (A \rtimes_\sigma \mathbb{Z}) \otimes C \to C(\mathbb{T}) \otimes (B \otimes C)_\infty.$$

Now there is a natural inclusion $\ell^\infty(B) \otimes C \subseteq \ell^\infty(B \otimes C)$. Since

$$c_0(B) \otimes C = c_0 \otimes B \otimes C = c_0(B \otimes C),$$

the inclusion of $\ell^\infty(B) \otimes C$ into $\ell^\infty(B \otimes C)$ gives an injective homomorphism

$$\frac{\ell^\infty(B) \otimes C}{c_0(B) \otimes C} \to \frac{\ell^\infty(B \otimes C)}{c_0(B \otimes C)}.$$

Then the image of $\bar{\phi}$ is clearly contained in

$$C(\mathbb{T}) \otimes \left(\frac{\ell^\infty(B) \otimes C}{c_0(B) \otimes C} \right),$$

which gives the desired extension. □

Lemma 6.4.8. Let B be a separable nuclear unital C*-algebra, let $A \subseteq B$ be a subalgebra containing the identity, and let $\sigma \in \mathrm{Aut}(A)$ be approximately inner in B. Then the *-homomorphism $A \rtimes_\sigma \mathbb{Z} \to C(\mathbb{T}) \otimes (B)_\infty$ from the previous lemma has a lift to a u.c.p. map $A \rtimes_\sigma \mathbb{Z} \to C(\mathbb{T}) \otimes \ell^\infty(B)$.

Proof. Let $A' = A \rtimes_\sigma \mathbb{Z}$, $B' = C(\mathbb{T}) \otimes \ell^\infty(B)$ and $J' = C(\mathbb{T}) \otimes c_0(B)$. Then the ideal J' is approximately inner since it is nuclear, and the map

$$A' \odot \mathcal{B}(\mathcal{H}) \to \frac{B' \odot \mathcal{B}(\mathcal{H})}{J' \odot \mathcal{B}(\mathcal{H})}$$

extends continuously to an injective homomorphism

$$\bar{\phi} : A' \otimes \mathcal{B}(\mathcal{H}) \to \frac{B' \otimes \mathcal{B}(\mathcal{H})}{J' \otimes \mathcal{B}(\mathcal{H})}$$

by the last lemma. Now applying Lemma 6.4.5 gives a u.c.p. lift of the *-homomorphism $A \rtimes_\sigma \mathbb{Z} \to C(\mathbb{T}) \otimes (B)_\infty$. □

Theorem 6.4.9 (Kirchberg-Phillips Nuclear Embedding Theorem)**.** A separable C*-algebra A is exact if and only if A embeds into $\mathcal{O}_2$. This embedding can be unital if A is.

Proof. Clearly if a C*-algebra is isomorphic to a subalgebra of $\mathcal{O}_2$, it is exact. Conversely, notice that the cone $C_0([0,1)) \otimes A$ is quasidiagonal. The unitization $B_0 = (C_0(\mathbb{R}) \otimes A)^\sim$ is the unitization of a subalgebra of $C_0([0,1)) \otimes A$, hence it is also quasidiagonal. It will also still be exact, as exactness passes to unitizations, A is exact, abelian C*-algebras are exact, and exactness is preserved by the min-tensor. By Corollary 6.4.2, there is a unital embedding $\phi : B_0 \to \mathcal{O}_2$. Let $\tau \in \mathrm{Aut} B_0$ be defined by $\tau = \tau_1 \otimes \mathrm{id}$, where $\tau_1(f)(x) = f(x+1)$, and extending to the unitization. Let $B = B_0 \rtimes_\tau \mathbb{Z}$, which is unital, separable exact. Let $\psi = \phi \circ \tau : B_0 \to \mathcal{O}_2$, which is another unital, injective *-homomorphism. Since B_0 is unital, separable, exact, ϕ and ψ are approximately unitarily equivalent by Theorem 6.3.18.

Now using the embedding $\phi : B_0 \to \mathcal{O}_2$, the automorphism τ is approximately inner in $\mathcal{O}_2$, and so Lemma 6.4.8 provides an injective *-homomorphism $B \to C(\mathbb{T}) \otimes (\mathcal{O}_2)_\infty$, which has a lifting to a u.c.p. map $B \to C(\mathbb{T}) \otimes \ell^\infty(\mathcal{O}_2)$. Now since $C(\mathbb{T})$ is unital, separable, quasidiagonal, and exact, Corollary 6.4.2 gives a unital embedding $C(\mathbb{T}) \to \mathcal{O}_2$. Thus we have embeddings

$$B \to C(\mathbb{T}) \otimes (\mathcal{O}_2)_\infty \to \mathcal{O}_2 \otimes (\mathcal{O}_2)_\infty \to (\mathcal{O}_2 \otimes \mathcal{O}_2)_\infty \to (\mathcal{O}_2)_\infty,$$

with a u.c.p. lift given by

$$B \to C(\mathbb{T}) \otimes \ell^\infty(\mathcal{O}_2) \to \mathcal{O}_2 \otimes \ell^\infty(\mathcal{O}_2) \to \ell^\infty(\mathcal{O}_2 \otimes \mathcal{O}_2) \to \ell^\infty(\mathcal{O}_2).$$

Now B is still exact, so there is an injective unital *-homomorphism $\gamma : B \to \mathcal{O}_2$ by Lemma 6.4.1. But B contains $(C_0(\mathbb{R}) \otimes A) \rtimes_\tau \mathbb{Z} \simeq (C_0(\mathbb{R}) \rtimes_{\tau_1} \mathbb{Z}) \otimes A \simeq C(\mathbb{T}) \otimes \mathbb{K} \otimes A$, this last isomorphism coming from Corollary 1.3.7. This algebra clearly contains an isomorphic copy A_0 of A as $C(\mathbb{T}) \otimes \mathbb{K}$ has projections. Let $p \in B$ be the identity of A_0, so

$$\gamma|_{A_0} : A_0 \to \gamma(p)\mathcal{O}_2\gamma(p)$$

is a unital embedding of A in $\gamma(p)\mathcal{O}_2\gamma(p)$. Now since $K_0(\mathcal{O}_2) = 0$, and $\mathcal{O}_2$ is unital, simple, purely infinite,$K_0(\mathcal{O}_2)$ is just the Murray von-Neumann classes of non-zero projections by Theorem 5.2.1, it follows that $[\gamma(p)]_0 = [1]_0 = 0$. Thus there exists an isometry $v \in \mathcal{O}_2$ such that $vv^* = \gamma(p)$. Therefore if $\mathcal{O}_2 = C^*(s_1, s_2)$, where s_1, s_2 are isometries satisfying the Cuntz relations, then vs_1v^* and vs_2v^* are isometries in $\gamma(p)\mathcal{O}_2\gamma(p)$ satisfying the Cuntz relations, so $\gamma(p)\mathcal{O}_2\gamma(p) \simeq \mathcal{O}_2$. So we have

$$A \xrightarrow{\simeq} A_0 \hookrightarrow C(\mathbb{T}) \otimes \mathbb{K} \otimes A \hookrightarrow B \xhookrightarrow{\gamma} \mathcal{O}_2,$$

giving the inclusion

$$A \xrightarrow{\simeq} A_0 \xhookrightarrow{\gamma|_{A_0}} \gamma(p)\mathcal{O}_2\gamma(p) \xrightarrow{\simeq} \mathcal{O}_2.$$

□

To summarize, exact C*-algebras can be characterized in several different ways.

Corollary 6.4.10. Let A be a separable C*-algebra. Then the following are equivalent.

1. There exists a nuclear embedding $\pi : A \to \mathcal{B}(\mathcal{H})$.
2. For any C*-algebra $B, J \lhd B$, we have that
$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$
is exact.
3. A has Property C.
4. A has Property C'.
5. There exists a C*-subalgebra $G \subseteq M_{2^\infty}, J \lhd G$ such that $A \simeq G/J$.
6. There exists an embedding $\phi : A \to \mathcal{O}_2$.

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