Spectral Properties of Structured Kronecker Products and Their Applications

by

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Statement of contributions

The bulk of this thesis was authored by me alone. Material in Chapter 5 is related to the manuscript [75] that is a joint work with my supervisor Levent Tunçel.

Abstract

We study certain spectral properties of some fundamental matrix functions of pairs of symmetric matrices. Our study includes eigenvalue inequalities and various interlacing properties of eigenvalues. We also discuss the role of interlacing in inverse eigenvalue problems for structured matrices.

Interlacing is the main ingredient of many fundamental eigenvalue inequalities. This thesis also recounts a historical development of the eigenvalue inequalities relating the sum of two matrices to its summands with some recent findings motivated by problems arising in compressed sensing.

One of the fundamental matrix functions on pairs of matrices is the Kronecker product. It arises in many fields such as image processing, signal processing, quantum information theory, differential equations and semidefinite optimization. Kronecker products enjoy useful algebraic properties that have proven to be useful in applications. The less-studied symmetric Kronecker product and skew-symmetric Kronecker product (a contribution of this thesis) arise in semidefinite optimization. This thesis focuses on certain interlacing and eigenvalue inequalities of structured Kronecker products in the context of semidefinite optimization.

A popular method used in semidefinite optimization is the primal-dual interior point path following algorithms. In this framework, the Jordan-Kronecker products arise naturally in the computation of Newton search direction. This product also appears in many linear matrix equations, especially in control theory. We study the properties of this product and present some nice algebraic relations. Then, we revisit the symmetric Kronecker product and present its counterpart the skew-symmetric Kronecker product with its basic properties. We settle the conjectures posed by Tunçel and Wolkowicz, in 2003, on interlacing properties of eigenvalues of the Jordan-Kronecker product and inequalities relating the extreme eigenvalues of the Jordan-Kronecker product. We disprove these conjectures in general, but we also identify large classes of matrices for which the interlacing properties hold. Furthermore, we present techniques to generate classes of matrices for which these conjectures fail. In addition, we present a generalization of the Jordan-Kronecker product (by replacing the transpose operator with an arbitrary symmetric involution operator). We study its spectral structure in terms of eigenvalues and eigenvectors and show that the generalization enjoys similar properties of the Jordan-Kronecker product. Lastly, we propose a related structure, namely Lie-Kronecker products and characterize their eigenvectors.

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Chapter 1

Introduction

The main themes of this thesis are eigenvalue inequalities, interlacing and Kronecker products of some fundamental matrix functions of pairs of symmetric matrices. These concepts arise in many research and application areas in mathematical sciences and engineering in general, and in combinatorics and optimization in particular.

Eigenvalue inequalities on sums of Hermitian matrices have a long history and have been studied extensively in theory [85, 139, 66]. These inequalities play a major role in perturbation theory. Furthermore, these eigenvalue inequalities have been important in applications in statistics and engineering problems [39, 34], and in particular in signal processing [107, 143]. Horn established a certain set of inequalities relating the eigenvalues of the sum of two Hermitian matrices to its summands, and conjectured that they describe exactly the attainable set of eigenvalues of the summands and the sum [66]. The eigenvalue inequalities of sums of Hermitian matrices have gained an interest in the last two decades by the settlement of the Horn's conjecture by Klyachko [77], and Knutson and Tao [78]. A comprehensive survey of these results can be found in [49]. In addition, many such eigenvalue inequalities such as Lidskii inequalities [13] and Horn inequalities [55] can be extended for the characteristic roots of hyperbolic polynomials.

Interlacing is the main ingredient in many eigenvalue inequalities. One of the well-known results in perturbation theory states that the eigenvalues of a rank one Hermitian perturbation of a Hermitian matrix interlace the eigenvalues of the unperturbed matrix. Eigenvalue inequalities and interlacing has many applications in algebraic graph theory. In particular, they are useful in obtaining inequalities on the eigenvalues of the adjacency matrix or the Laplacian matrix of the graph. These turn out to be useful in bounding some parameters of a graph such as the size of a maximal independent set, the chromatic number, the diameter and the bandwidth, the edge connectivity and the isoperimetric number. For a more detailed account of this topic, see Haemers' survey paper [56] and also [1]. Recently, interlacing has gained more popularity by the work of Markus, Spielman and Srivastava, in which they developed the method of interlacing polynomials and proved the existence of infinite families of regular bipartite Ramanujan graphs of every degree greater than 2 [89]. Also, they solved the long standing Kadison-Singer problem [90].

One of the main concepts studied in this thesis is one of the fundamental matrix functions on pairs of matrices: the Kronecker product. It arises in many fields such as image processing, signal processing, quantum information theory, differential equations and semidefinite optimization. Kronecker products enjoy useful algebraic properties that have proven to be useful in applications. The less-studied symmmetric Kronecker product and skew-symmetric Kronecker product (a contribution of this thesis) arise in semidefinite optimization. This thesis focuses on certain interlacing and eigenvalue inequalities of structured Kronecker products in the context of semidefinite optimization.

In the remainder of this chapter, we present some background material on convex sets and functions, matrix theory, interlacing and the notation used throughout this thesis. We put some elementary definitions and notations on the vector spaces and normed linear spaces to Appendix A.1-A.2. We finish this section by providing the outline and the contribution of this thesis in Section 1.4.

1.1 Convex sets and functions

The definitions and the concepts covered here can also be found in [64].

Definition 1.1.1 (Convex set)**.** A set $C \subseteq \mathbb{R}^n$ is *convex* if $\alpha\mathbf{x} + (1-\alpha)\mathbf{y}$ is in C whenever $\mathbf{x}, \mathbf{y} \in C$ and $\alpha \in (0,1)$.

Definition 1.1.2 (Convex hull)**.** Given a set $C \subseteq \mathbb{R}^n$, the *convex hull* of C is defined as

$$\operatorname{conv}(C) := \left\{ \sum_{i=0}^{n} \alpha_i \mathbf{x}_i : \mathbf{x}_i \in C, \alpha_i \geq 0, \sum_{i=0}^{n} \alpha_i = 1 \right\},$$

which is the set of all convex combinations of the points of C.

Theorem 1.1.1. *For every $C \subseteq \mathbb{R}^n$, $\operatorname{conv}(C)$ is the intersection of all convex sets containing C.*

Definition 1.1.3 (Extreme point)**.** Given a convex set $C \subseteq \mathbb{R}^n$, a point $\mathbf{x} \in C$ is an *extreme point of* C if there are no two distinct points $\mathbf{x}_1, \mathbf{x}_2 \in C$ and $\alpha \in (0,1)$ such that $\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$.

Definition 1.1.4 (Cone)**.** A set $K \subseteq \mathbb{R}^n$ is said to be a *cone* if $0 \in K$ and $\lambda\mathbf{x} \in K$ for every $\mathbf{x} \in K$ and $\lambda \geq 0$.

Definition 1.1.5 (Polar cone)**.** Let K be a convex cone. The *polar cone* of K is defined as

$$K^\circ := \{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{y}, \mathbf{x} \rangle \leq 0 \text{ for every } \mathbf{x} \in K\}.$$

Definition 1.1.6 (Tangent cone)**.** Given a nonempty set $S \subseteq \mathbb{R}^n$, $\mathbf{d} \in \mathbb{R}^n$ is a direction *tangent* to S at $\mathbf{x} \in S$ if there exists sequences $\{\mathbf{x}_k\} \subset S$ and $\{t_k\}$ such that $t_k > 0$ and when $k \to \infty$

$$\mathbf{x}_k \to \mathbf{x},\ t_k \to 0,\ \frac{\mathbf{x}_k - \mathbf{x}}{t_k} \to \mathbf{d}.$$

for every $\mathbf{x} \in K$. The set of all such directions is called the *tangent cone* to S at $\mathbf{x}$ (or the contingent cone), and is denoted by $T_S(\mathbf{x})$.

Proposition 1.1.2. *Given a closed convex set $C \subset \mathbb{R}^n$, the tangent cone of C at $\mathbf{x} \in C$ is the closure of the cone generated by $C \setminus \{\mathbf{x}\}$:*

$$T_K(\mathbf{x}) = cl\{\alpha(\mathbf{y} - \mathbf{x}) : \alpha \geq 0, \mathbf{y} \in C\}.$$

Definition 1.1.7 (Convex function)**.** Let $C \subseteq \mathbb{R}^n$ be a convex set. A function $f : C \to \mathbb{R}$ is said to be *convex* on C if for every $\mathbf{x}, \mathbf{y} \in C$ and $\alpha \in (0, 1)$,

$$f(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1 - \alpha)f(\mathbf{y}).$$

Definition 1.1.8 (Gauge)**.** Given a closed, convex set $C \subset \mathbb{R}^n$ containing the origin, the *gauge of* C (or *Minkowski functional* of C) is defined as

$$\gamma_C(\mathbf{x}) := \inf\{t > 0 : \mathbf{x} \in tC\},$$

where $\gamma_C(\mathbf{x}) := +\infty$ if $\mathbf{x} \in tC$ for no $t > 0$.

Theorem 1.1.3. *Let $C \subseteq \mathbb{R}^n$ be a closed, convex set containing the origin. Then the gauge of C, γ_C is*

(i) nonnegative and positively homogenous,

(ii) subadditive*, i.e., for every $\mathbf{x}, \mathbf{y} \in C$, $\gamma_C(\mathbf{x} + \mathbf{y}) \leq \gamma_C(\mathbf{x}) + \gamma_C(\mathbf{y})$.*

1.2 Matrices

We provide basic definitions and notations from linear algebra here. Most of these definitions and results can be found in [67].

Throughout this thesis, we reserved bold lower-case letters to denote vectors and bold upper-case letters to denote matrices. All vectors are column vectors. $\mathbb{R}^n$ denotes the n-dimensional vector space over real numbers. $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote nonnegative and positive real numbers, respectively. The set of all m-by-n matrices over $\mathbb{R}$ is denoted by $\mathbb{R}^{m \times n}$. $(\cdot)^\top$ and $(\cdot)^H$ denote transpose and conjugate transpose, respectively.

Definition 1.2.1 (Trace)**.** Given an n-by-n matrix $\mathbf{A} := [a_{ij}]_{i,j=1}^n$, the *trace* of $\mathbf{A}$ is denoted by $\operatorname{tr}(\mathbf{A})$ and is defined as the sum of its diagonal elements

$$\operatorname{tr}(\mathbf{A}) := \sum_{i=1}^{n} a_{ii}.$$

The vector space $\mathbb{R}^{m \times n}$ is an inner product space (see Appendix A.2) with the inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij} = \operatorname{tr}\left(\mathbf{A}^\top \mathbf{B}\right).$$

For every matrix $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$

- $\mathrm{tr}\left(\mathbf{A}^\top\right) = \mathrm{tr}\left(\mathbf{A}\right)$ (given $\mathbf{A}$ is a square matrix),
- $\mathrm{tr}\left(\mathbf{A}^\top\mathbf{B}\right) = \mathrm{tr}\left(\mathbf{B}\mathbf{A}^\top\right)$,

Definition 1.2.2 (Eigenvalue, eigenvector)**.** We say that λ is the eigenvalue of a matrix $\mathbf{A}$ corresponding to the eigenvector $\mathbf{x} \neq 0$ if

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

For a given matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we define $|\mathbf{A}| := (\mathbf{A}^\top\mathbf{A})^{1/2}$. The eigenvalues of $|\mathbf{A}|$, $\sigma_1(\mathbf{A}) \geq \sigma_2(\mathbf{A}) \geq \cdots \geq \sigma_{\min\{m,n\}}(\mathbf{A})$ are called as the *singular values* of $\mathbf{A}$. Whenever convenient we may write σ_i to represent the corresponding singular value. $\mathbf{A}_{[1:k]}$ denotes the matrix formed by the first k terms of the singular value decomposition of $\mathbf{A}$. $\mathbf{A}(i:j;p:q)$ denotes the submatrix of $\mathbf{A}$ formed from the intersection of rows i to j and columns p to q, or $\mathbf{A}(I;J)$ denotes the submatrix of $\mathbf{A}$ formed from the intersection of rows indexed by I and columns indexed by J. For a vector $\mathbf{x} := \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$, x_i denotes the ith entry of $\mathbf{x}$, unless defined otherwise.

Definition 1.2.3 (vec)**.** For an m-by-n matrix $\mathbf{X} := [x_{ij}]$, we define the linear transformation $\mathrm{vec} : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$

$$\mathrm{vec}(\mathbf{X}) := \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{m1} & x_{12} & x_{22} & \cdots & x_{m2} & \cdots & x_{mn} \end{bmatrix}^\top,$$

where $\mathrm{vec}(\mathbf{X})$ is an mn-by-1 vector formed by stacking the columns of $\mathbf{X}$ consecutively.

Definition 1.2.4 (Mat)**.** We define $\mathrm{Mat} : \mathbb{R}^{n^2} \to \mathbb{R}^{n\times n}$ as the linear map which reshapes n^2-dimensional vector and maps it into an n-by-n matrix, by assigning the first n entry of the vector $\mathbf{x}$ as the first column of the matrix $\mathbf{X} := \mathrm{Mat}(\mathbf{x})$, the second n entries of $\mathbf{x}$ as its second column and so on.

1.2.1 Matrix Norms

Definition 1.2.5 (Matrix norm)**.** A matrix norm $\|\cdot\|$ is function from the set of all m-by-n matrices to $\mathbb{R}$ that satisfies the following properties:

(i) $\|\mathbf{A}\| \geq 0$,

(ii) $\|\mathbf{A}\| = 0 \Leftrightarrow \mathbf{A} = \mathbf{0}$,

(iii) $\|\alpha\mathbf{A}\| = |\alpha|\|\mathbf{A}\|$,

(iv) $\|\mathbf{A}+\mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$, (where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$),

(v) $\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$, if $m = n$.

We list the definitions of commonly used matrix norms.

(i) **(Frobenius norm)** The *Frobenius norm* of $\mathbf{A} := [a_{ij}] \in \mathbb{R}^{m\times n}$ is defined as

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

Note that $\|\mathbf{A}\|_F = \|\operatorname{vec}(\mathbf{A})\|_2$.

(ii) **(Spectral norm)** The *spectral norm* of $\mathbf{A} := [a_{ij}] \in \mathbb{R}^{m\times n}$ is defined as

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \sigma_1(\mathbf{A}),$$

where $\sigma_1(A)$ is the largest singular value of $\mathbf{A}$.

(iii) **(Schatten-p norm)** The *Schatten-p norm* of $\mathbf{X}$ is defined as

$$|||\mathbf{X}|||_p := \|\sigma(\mathbf{X})\|_p, \quad p \geq 1, \tag{1.1}$$

where $\sigma : \mathbb{R}^{m\times n} \to \mathbb{R}^{\min\{m,n\}}$is defined by

$$\sigma(\mathbf{X}) := \begin{bmatrix}\sigma_1(\mathbf{X}) & \cdots & \sigma_{\min\{m,n\}}(\mathbf{X})\end{bmatrix}^\top.$$

When $p = 1$, this norm is called the *nuclear norm* or the *trace norm*. When $p = 2$, $|||\cdot|||_p$ corresponds to the Frobenius norm.

In the case of $p \in (0, 1)$, (1.1) does not define a norm; it defines a quasi-norm, and called the *Schatten-p quasi-norm* [67].

Definition 1.2.6 (Unitarily invariant matrix norm)**.** A matrix norm $|||\cdot|||$ on $\mathbb{R}^{m\times n}$ is *unitarily invariant* if $|||\mathbf{UAV}||| = |||\mathbf{A}|||$, for all unitary matrices $\mathbf{U} \in \mathbb{R}^{m\times m}$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$, and for all $\mathbf{A} \in \mathbb{R}^{m\times n}$.

Schatten-p norm is one of the typical examples of unitarily invariant norms.

Definition 1.2.7 (Symmetric gauge function)**.** A function $\phi : \mathbb{R}^n \to \mathbb{R}_+$ is a *symmetric gauge function* if it satisfies the following:

(i) $\phi(\cdot)$ is a norm on $\mathbb{R}^n$,

(ii) $\phi(\mathbf{x}) = \phi(|\mathbf{x}|)$, $\forall \mathbf{x} \in \mathbb{R}^n$ (where $|\cdot|$ is taken elementwise), and

(iii) $\phi(\mathbf{x}) = \phi(\mathbf{P}\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^n$, for all permutation matrices $\mathbf{P} \in \{0, 1\}^{n\times n}$.

Let $\mathbf{A}$ have the following singular value decomposition $\mathbf{A} = \mathbf{U\Sigma V}^\top$. Let $|||\cdot|||$ be a unitarily invariant norm. Since $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices,

$$|||\mathbf{A}||| = |||\mathbf{\Sigma}|||.$$

This shows that every unitarily invariant norm is a symmetric gauge function of the singular values of its argument.

1.2.2 Some Special Structured Matrices

In control theory, signal processing, machine learning and differential equations, one often makes certain assumptions when solving certain systems of linear equations. These include stationarity, time-invariance, homogeneity and sparsity. Such assumptions lead to certain structured matrices. Solving linear systems of structured matrices has more advantages, as they accommodate more possibilities to exploit their structure and various fast algorithms have been studied for structured matrices [74]. A recent work on the fast algorithms for structured matrices can be found in [142], which in particular studies the design of fast algorithms in terms of the *bilinear complexity* for various bilinear operations.

In this section, we define some matrices of special structure, as we refer to them in the subsequent chapters.

Definition 1.2.8 (Diagonal matrices)**.** The matrix $\mathbf{D} := [d_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$ is called a *diagonal matrix* if for every $i \neq j \in \{1, 2, \ldots, n\}$, $d_{ij} = 0$. We will use the notation $\mathrm{Diag}(d_1, d_2, \ldots, d_n)$ to denote the n-by-n diagonal matrix $\mathbf{D}$ with diagonal entries $d_{ii} := d_i$, $i \in \{1, 2, \ldots, n\}$. The *identity matrix* $\mathbb{I}_n \in \mathbb{R}^{n\times n}$ is an example of a diagonal matrix whose diagonal entries are equal to one. We will omit the subscript and use $\mathbb{I}$ to denote the identity matrix whenever its dimension can be understood from the context.

Definition 1.2.9 (Tridiagonal matrices)**.** The matrix $\mathbf{A} := [a_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$ is called a *tridiagonal matrix* if for every $i, j \in \{1, 2, \ldots, n\}$

$$a_{ij} = 0, \text{ whenever } |i - j| > 1.$$

Definition 1.2.10 (Permutation matrices)**.** A *permutation matrix* is a square matrix each of whose row and column contains exactly one element equal to 1 and 0s elsewhere.

Definition 1.2.11 (Doubly stochastic matrices)**.** A *doubly stochastic matrix* is a square matrix with non-negative entries such that each row and column sums to one.

Definition 1.2.12 (Hankel matrices)**.** The matrix $\mathbf{A} := [a_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$ is called a *Hankel matrix* if for every $i, j \in \{1, 2, \ldots, n\}$

$$a_{ij} = a_{|i-j|},$$

for some numbers $a_0, \ldots, a_{n-2}, a_{n-1} \in \mathbb{R}$.

Definition 1.2.13 (Toeplitz matrices)**.** The matrix $\mathbf{A} := [a_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$ is a *Toeplitz matrix* if the entries are constant on the diagonals, i.e., for every $i, j \in \{1, 2, \ldots, n\}$

$$a_{ij} = a_{j-i},$$

for some $(2n - 1)$ numbers $a_{-(n-1)}, a_{-(n-2)}, \ldots, a_0, \ldots, a_{n-2}, a_{n-1} \in \mathbb{R}$.

Toeplitz matrices are ubiquitous and arise in many applications in areas such as signal processing, compressed sensing and differential equations. In addition, Toeplitz matrices are computationally desirable as one can solve a Toeplitz system of linear equations in $O(n \log^2 n)$

time [15]. The inversion, determinant computation and LU and QR decompositions can be computed in $O(n^2)$ time compared to arbitrary matrices which usually take $O(n^3)$ time. Even though the multiplication of two Toeplitz matrices are not necessarily Toeplitz, still the multiplication of two Toeplitz matrices harbor some nice properties as they have low displacement rank which reduces the computational complexity of solving a system of linear equations [73]. Furthermore, a recent study showed that every n-by-n matrix is a product of at most $2n + 5$ Toeplitz matrices [141].

Theorem 1.2.1. *[51] Two Toeplitz matrices commute if and only if one of the matrices is the value of a linear function at the other one, or both matrices belong to the same algebra of generalized circulant matrices.*

Definition 1.2.14 (Circulant matrices)**.** The matrix $\mathbf{A} := [a_{ij}]_{i,j=1}^n \in \mathbb{R}^{n\times n}$ is a *circulant matrix* if for every $i, j \in \{1, 2, \ldots, n\}$

$$a_{ij} = a_{((j-i) \mod n)},$$

for some numbers $a_0, a_1 \ldots, a_{n-1} \in \mathbb{R}$.

Definition 1.2.15 (Centrosymmetric matrices)**.** The matrix $\mathbf{A} := [a_{ij}]_{i,j=1}^n \in \mathbb{R}^{n\times n}$ is a *centrosymmetric matrix* if

$$\mathbf{A} = \mathbf{JAJ},$$

where $\mathbf{J}$ is a special permutation matrix with ones on the secondary diagonal (from bottom left to the upper right corner) and zeros elsewhere. Centrosymmetric matrices are symmetric about their diagonal and secondary diagonal, e.g.,

$$\mathbf{A} = \begin{bmatrix} a & b & c & d & e \\ b & f & g & f & d \\ c & g & i & g & c \\ d & f & g & f & b \\ e & d & c & b & a \end{bmatrix}$$

1.3 Interlacing

Definition 1.3.1 (Interlace)**.** Given two sequences of real numbers $a_1 \geq a_2 \geq \cdots \geq a_s$ and $b_1 \geq b_2 \geq \cdots \geq b_t$, where $t \leq s$. The sequence $\{b_i\}_{i=1}^t$ is said to *interlace* $\{a_i\}_{i=1}^s$, if for every $i \in \{1, 2, \ldots, t\}$,

$$a_i \geq b_i \geq a_{i+s-t}.$$

Interlacing has been studied extensively in the literature and recently has gained more popularity when Markus, Spielman and Srivastava developed the method of interlacing polynomials and proved the existence of infinite families of regular bipartite Ramanujan graphs of every degree greater than 2 [89]. It has many applications in matrix theory, real stable polynomials and spectral graph theory [56, 1].

The most well-known interlacing result follows from Rolle's theorem. If $p : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree n with all real roots $r_1 \geq r_2 \geq \cdots \geq r_n$ then each interval (r_i, r_{i+1}), $i \in \{1, \ldots, n-1\}$ contains at least one root of its derivative, p'. This result was generalized by Gauss.

Theorem 1.3.1 (Gauss-Lucas)**.** *Given a polynomial $p \in \mathbb{C}[x]$ (with complex coefficients), the roots of p' are in the convex hull of the roots of p.*

The following well-known theorems provide a characterization of the interlacing polynomials.

Theorem 1.3.2 (Theorem 6.3.4, [110])**.** *Let $p, q \in \mathbb{R}[x]$ be non-constant polynomials such that their degrees are equal or differ by one. p and q have strictly interlacing zeros if and only if the polynomial $r := p + iq$ has all its roots either in the upper half plane $\mathcal{H}_U := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ or in the lower half plane $\mathcal{H}_L := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$.*

Theorem 1.3.3 (Theorem 6.3.8, [110])**.** *Let $p, q \in \mathbb{R}[x]$ be non-constant polynomials such that their degrees are equal or differ by one. p and q have strictly interlacing roots if and only if for every $\lambda, \mu \in \mathbb{R}$ such that $\lambda^2 + \mu^2 \neq 0$, the polynomial $r := \lambda p + \mu q$ has distinct and real roots.*

Definition 1.3.2 (Bezout matrix)**.** Let p, q be polynomials, each of degree at most n. The *Bezout matrix* of p and q is the symmetric matrix $\mathbf{B} := [b_{ij}]$ such that

$$\mathcal{B}(p, q; x, y) := \frac{p(x)q(y) - p(y)q(x)}{x - y} = \sum_{i,j=1}^{n} b_{ij} x^{i-1} y^{j-1}, \tag{1.2}$$

where $\mathcal{B}(p, q; x, y)$, the polynomial in x and y, is called the *Bezoutiant*.

The following result provides the relation between positive definiteness of the Bezout matrix of two polynomials and the interlacement of the roots of these polynomials. The proof can be also found in [105].

Theorem 1.3.4. *Let p, q be two real polynomials of the same degree that have no common roots. If the Bezout matrix defined by (1.2) of p, q is positive definite then the roots of p and q are distinct, real numbers. Furthermore, the roots of q and p interlace.*

If a symmetric (or Hermitian) matrix $\mathbf{A}$ is perturbed by a rank one positive semidefinite matrix $\mathbf{B}$, then the eigenvalues of the matrix $\mathbf{A}$ and the perturbed matrix $\mathbf{C} := \mathbf{A} + \mathbf{B}$ alternate, the largest belonging to $\mathbf{C}$. There is a close relationship between this result and the interlacement of the eigenvalues of a real symmetric (or Hermitian) matrix or oder n and its leading principal matrix of order $n - 1$ in the sense that one can be derived from the other. The latter result is generally known as *Cauchy's interlacing theorem* [40]. The following well-known interlacing theorem, a generalization of Cauchy's interlacing theorem can be found in [56, Theorem 2.1].

Theorem 1.3.5 (Cauchy's interlacing theorem)**.** *Let $\mathbf{A} \in \mathbb{S}^n$ be a symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, and let $\mathbf{Q} \in \mathbb{R}^{n\times m}$ be such that $\mathbf{Q}^\top\mathbf{Q} = \mathbb{I}$. If $\mathbf{B} := \mathbf{Q}^\top A\mathbf{Q}$ with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$, then*

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$$

for all $i \in \{1, \dots, m\}$.

The basic inequalities between singular values and eigenvalues of real square matrices are consequences of the interlacing property given in Cauchy's theorem. Some well-known examples are Thompson-Freede inequalities [127], Lidskii inequalities and Mirsky's inequality.

Interlacing theorems also shed light on many inequalities and regularity results concerning the structure of graphs in terms of eigenvalues of the adjacency matrix. For example, Cauchy's interlacing theorem can be employed to find a bound on the size of the largest independent set of G as well as the bounds on the chromatic number, the diameter and the bandwidth of graphs [56].

Our interest in interlacing is based on the work of Tunçel and Wolkowicz. In [131], Tunçel and Wolkowicz conjectured interesting interlacing relations on the roots of the characteristic polynomials of certain structured matrices arising from the Jordan-Kronecker products of real symmetric matrices, namely $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$. We restate the conjectures below for the convenience of the reader.

Let $\mathbb{S}^n$ denote *the set of n-by-n real symmetric matrices* and $\mathbb{K}^n$ denote *the set of n-by-n real skew-symmetric matrices*. Define a *transpose operator* $\mathcal{T} : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ such that $\mathrm{Mat}(\mathcal{T}\mathbf{x}) = (\mathrm{Mat}(\mathbf{x}))^\top$ (see Definition 1.2.4). Let $\mathbf{T}$ be the n^2-by-n^2 matrix representation of the transpose operator $\mathcal{T}$. This matrix will be defined in more detail later in Chapter 4.4.5.

Conjecture 1.3.6. [131] *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. Then*

$$\min_{\mathbf{Tu}=\mathbf{u}} \frac{\mathbf{u}^\top(\mathbf{A}\otimes\mathbf{B})\mathbf{u}}{\mathbf{u}^\top\mathbf{u}} \leq \min_{\mathbf{Tw}=-\mathbf{w}} \frac{\mathbf{w}^\top(\mathbf{A}\otimes\mathbf{B})\mathbf{w}}{\mathbf{w}^\top\mathbf{w}}, \tag{1.3}$$

$$\max_{\mathbf{Tw}=-\mathbf{w}} \frac{\mathbf{w}^\top(\mathbf{A}\otimes\mathbf{B})\mathbf{w}}{\mathbf{w}^\top\mathbf{w}} \leq \max_{\mathbf{Tu}=\mathbf{u}} \frac{\mathbf{u}^\top(\mathbf{A}\otimes\mathbf{B})\mathbf{u}}{\mathbf{u}^\top\mathbf{u}}. \tag{1.4}$$

Equivalently,

$$\min_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \mathrm{tr}\left(\mathbf{AUBU}\right) \leq \min_{\mathbf{W}\in\mathbb{K}^n, \|\mathbf{W}\|_F=1} \mathrm{tr}\left(\mathbf{AWBW}^\top\right), \tag{1.5}$$

$$\max_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \mathrm{tr}\left(\mathbf{AUBU}\right) \geq \max_{\mathbf{W}\in\mathbb{K}^n, \|\mathbf{W}\|_F=1} \mathrm{tr}\left(\mathbf{AWBW}^\top\right). \tag{1.6}$$

Conjecture 1.3.7. [131] *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. Also, let $\mathbf{w} \in \mathbb{R}^{n^2}$ such that $\mathbf{Tw} = -\mathbf{w}$ and $\mathbf{w}$ is the eigenvector of $(\mathbf{A}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{A})$ corresponding to its kth largest eigenvalue. Then λ_{k-1} and λ_{k+1} of the matrix are well-defined and they are determined by some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{n^2}$ such that $\mathbf{Tu} = \mathbf{u}$ and $\mathbf{Tv} = \mathbf{v}$.*

1.4 Outline and contributions of the thesis

- In Chapter 2, we present a survey of convex optimization in compressed sensing with a focus on matrix and eigenvalue inequalities.
- In Chapter 3, we give a historical exposition of the eigenvalue inequalities relating the sum of matrices to its summands with some recent developments on Mirsky type inequalities which have applications in compressed sensing.
- In Chapter 4, we cover the ubiquitous Kronecker products in general. We also introduce skew-symmetric Kronecker products and their properties. We also discuss the role of interlacing in various inverse eigenvalue problems for structured matrices.
- In Chapter 5, we settle the conjectures posed by Tunçel and Wolkowicz. We introduce Jordan-Kronecker products and provide some algebraic results and inequalities. We disprove these conjectures in general, but we also identify large classes of matrices for which the interlacing properties hold. Furthermore, we present techniques to generate classes of matrices for which these conjectures fail. In addition, we present a generalization of the Jordan-Kronecker product (by replacing the transpose operator with an arbitrary symmetric involution operator). We study its spectral structure in terms of eigenvalues and eigenvectors and show that the generalization enjoys similar properties of the Jordan-Kronecker product. Lastly, we propose a related structure, namely Lie-Kronecker products and characterize their eigenvectors.
- In Chapter 6, we provide a brief discussion on the areas where Jordan-Kronecker products appear and we conclude the thesis with open problems.

As a general rule, throughout the thesis we labelled new results as "(new)". Whenever the result is not strong or the contribution is negligible, we tried to avoid putting the label and presented without any reference.

Chapter 2

Compressed Sensing

The classical *Nyquist-Shannon Theorem* states that a band-limited signal (i.e., a signal whose Fourier transform is zero outside of some bounded frequency interval) can be reconstructed exactly from its samples taken at points separated by twice its maximum frequency component [119]. Although Nyquist-Shannon condition guarantees the exact recovery, it is not a necessary condition for it. It is now well-known that some sparse signals can be recovered from fewer samples than required by the Nyquist-Shannon theorem. In particular, an unknown signal from $\mathbb{R}^n$ with support $S \subseteq [n]$ can be recovered by solving a linear programming problem from almost every set of frequencies of size $O(|S| \log(n))$ with high probability [21].

In signal processing, it is common to model the output $\mathbf{b} \in \mathbb{R}^m$ of a system as a linear function of the input signal $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{b} = \mathbf{A}\mathbf{x}.$$

Recovering the signal $\mathbf{x}$ from $\mathbf{b}$ is an inverse linear problem. In general, the dimension of the output signal, or the number of measurements must be larger than or equal the dimension of the input signal to recover it. However, if the unknown signal can be represented with a linear combination of a few basis vectors from some basis then the recovery of the unknown signal is possible even if the number of measurements are far fewer than unknowns. *Compressed sensing* is a method of reconstructing signals using as few measurements as possible. The main idea is to find the sparsest representation of the signal via solving an underdetermined linear system.

In this chapter, we introduce the basic mathematical tools of compressed sensing, the notion of sparsity, null-space property, coherence, restricted isometry property and their relation to sparse recovery of signals from structured matrices.

Definition 2.0.1 (s-sparse)**.** A signal $\mathbf{x} \in \mathbb{R}^n$ is called *s-sparse* if the number of nonzero entries of $\mathbf{x}$ is at most s.

Definition 2.0.2 (ℓ_0-"norm")**.** Let $\|\cdot\|_0 : \mathbb{R}^n \to \mathbb{N}$ be such that for every $\mathbf{x} \in \mathbb{R}^n$

$$\|\mathbf{x}\|_0 := |\{i : x_i \neq 0\}|,$$

is the number of non-zero entries of $\mathbf{x}$.

Note that $\|\cdot\|_0$ is neither a norm nor a quasi-norm (see Definition A.2.4) as $\|\alpha\mathbf{x}\|_0 \neq |\alpha|\|\mathbf{x}\|_0$ for some $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

Consider an underdetermined system of linear equations

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{w},$$

where $\mathbf{b} \in \mathbb{R}^m$ is a measurement vector, $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a measurement matrix with $m < n$, $\mathbf{x}$ is an unknown vector and $\mathbf{w}$ is a noise vector. This system has infinitely many solutions provided there exists at least one solution. Among the infinitely many solutions, a common preference is the sparsest one for several reasons. First of all, a sparse signal enables fast computation and easier analysis especially when it has a certain structure. Many signals used in practice such as audio and image signals have sparse representations with respect to bases such as Fourier and Wavelet [88]. Furthermore, there are many efficient algorithms available based on convex optimization to compute such representations [35, 22, 21]. In this line of work, the central interest has been on the necessary and sufficient conditions for the existence and uniqueness of sparse solutions that can be computed with efficient (polynomial-time) algorithms.

Finding the sparsest signal satisfying an underdetermined system of linear equations can be formulated by minimizing the number of non-zero entries over all the feasible points:

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} && \|\mathbf{x}\|_0 && (P_0) \\ &\text{subject to} && \mathbf{A}\mathbf{x} = \mathbf{b}. \end{aligned} \tag{2.1}$$

A more general formulation for a given $\epsilon \geq 0$ is

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} && \|\mathbf{x}\|_0 && (P_{(0,\epsilon)}) \\ &\text{subject to} && \|\mathbf{A}\mathbf{x} - \mathbf{b}\| \leq \epsilon. \end{aligned} \tag{2.2}$$

The optimization problems in (2.1) and (2.2) are NP hard, as the instance of the exact cover by 3-sets problem can be reduced in polynomial time to (2.2) [102]. Although there are some approximation algorithms and heuristics to attack this problem, we will focus on approaches considering the "convex relaxations" of such problems [21].

The ℓ_1-norm unit ball $B_1 := \{\mathbf{x} : \|\mathbf{x}\|_1 \leq 1\}$ is the convex hull of $B_0 := \{\mathbf{x} : \|\mathbf{x}\|_0 \leq 1\}$, i.e., the smallest convex set containing B_0. See Figure 2.1 on page 14 for the illustration on $\mathbb{R}^2$. In addition, for every $\mathbf{x} \in \mathbb{R}^n$

$$\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_\infty \|\mathbf{x}\|_0.$$

Assuming the minimizer of (P_0) is contained in $B_\infty(R) = \{\mathbf{x} : \|\mathbf{x}\|_\infty \leq R\}$ where $R > 0$ is a large number, instead of (2.1) one can solve

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} && \|\mathbf{x}\|_0 \\ &\text{subject to} && \mathbf{x} \in C \cap B_\infty(R), \end{aligned} \tag{2.3}$$

where $C := \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$. Since $\|\mathbf{x}\|_0 \geq \frac{1}{R}\|\mathbf{x}\|_1$ for every $x \in C \cap B_\infty(R)$, $\frac{1}{R}\|\cdot\|_1$ is a convex relaxation of $\|\cdot\|_0$ over the set $C \cap B_\infty(R)$. Multiplying the objective function by a positive constant does not change the set of minimizers, therefore, a convex relaxation of the cardinality minimization problem in (2.3) can be written as ℓ_1-norm minimization problem:

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} && \|\mathbf{x}\|_1 \\ &\text{subject to} && \mathbf{x} \in C \cap B_\infty(R). \end{aligned} \tag{2.4}$$

In fact, $\|\cdot\|_1$ is the *lower semi-continuous convex envelope* of $\|\cdot\|_0$ over the unit ℓ_∞-ball.

Definition 2.0.3 (Convex envelope)**.** The *convex envelope* of a function $f : \Omega \to \mathbb{R}$ is the supremum of all possible convex functions $g : \Omega \to \mathbb{R}$ such that $g(\mathbf{x}) \leq f(\mathbf{x})$ for every $\mathbf{x} \in \Omega$.

Given a convex feasible region Ω, the ℓ_1-norm minimization problem $\min\{\|\mathbf{x}\|_1 : \mathbf{x} \in \Omega\}$ finds a minimizer (or minimizers) by expanding the ℓ_1-ball of radius r, i.e., $B_1(r) := \{\mathbf{x} : \sum_{i=1}^n |x_i| = r\}$ from radius 0 until $B_1(r)$ hits a point in the feasible region Ω for the first time. Because the ℓ_1-norm unit ball has a lot of extreme points in $\mathbb{R}^n$, in practice it often hits a point (or points) with a lot of coordinates being zero, especially when the feasible region is a polyhedron. However in some cases, this point (or points) may not be the sparsest one. In fact, since the ℓ_1-norm is not strictly convex, the solution may not even be unique. Therefore, (2.4) is not necessarily the best convex relaxation for (2.3) in the sense that it may not always provide the sparsest solution. The following illustrates this discussion.

Example 2.0.1. Consider

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} && \|\mathbf{x}\|_0 \\ &\text{subject to} && x_1 + 5x_2 = 2 \\ & && 2x_1 - 5x_3 = 4. \end{aligned}$$

The ℓ_1-norm relaxation of this problem yields the solution $(x_1^*, x_2^*, x_3^*) := (0, 0.4, -0.8)$. Note that this is the first point that $B_1(r)$ hits the feasible region when expanding from radius $r = 0$. However, this is clearly not the sparsest solution as $(2, 0, 0)$ is also a feasible solution.

In the ℓ_1-norm minimization, the coefficients of the unknown vector is penalized equally. This results in penalization of large coefficients more than smaller coefficients. To address this issue and to promote more sparsity, in her thesis [43], Fazel suggested to replace the ℓ_1 norm in the objective by the following concave function: $\sum_{i=1}^n \log(|x_i| + \epsilon)$, where $\epsilon > 0$ is a small number. As $\log(\cdot)$ is a concave function, a common approach is to minimize its linearization [43]. This led to the development of weighted ℓ_1-norm minimization algorithms [43, 86, 23]. The weighted ℓ_1-norm minimization is an iterative algorithm where the solution of the weighted linear program is used to calculate the weights which are fed back to the weighted linear program (where the initial weights are all equal). We refer the reader to [145] for a summary of different weights used in iterative reweighted ℓ_1-norm minimization approaches.

Although the ℓ_1-norm relaxation of the objective may not always give the sparsest solution(s), a body of research shows that under certain conditions, it can give the exact solution

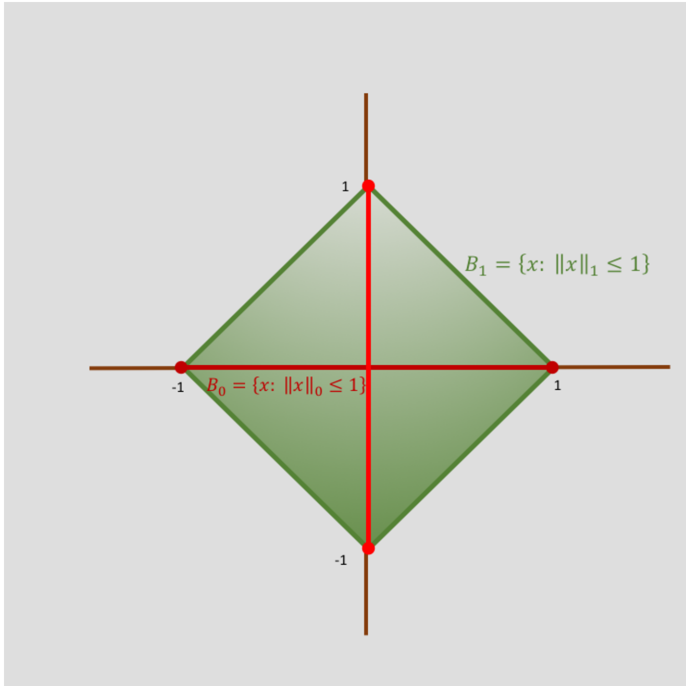

Figure 2.1: Convex hull of $B_0 = \{\mathbf{x} : \|\mathbf{x}\|_0 \le 1\}$ is $B_1 := \{\mathbf{x} : \|\mathbf{x}\|_1 \le 1\}$.

or the approximation to the solution(s) of the optimization problem in (2.1). We formulate a relaxation of (2.1) in which the objective is replaced by the ℓ_1-norm of $\mathbf{x}$ as follows:

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad \|\mathbf{x}\|_1 \qquad (P_1) \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}. \end{aligned} \tag{2.5}$$

Since (P_1) can be formulated as a linear programming problem, (2.5) can be solved using efficient methods such as the simplex method or interior point methods [16, 104].

It is now well known that an unknown signal from $\mathbb{R}^n$ with support $S \subseteq [n]$ can be recovered from its discrete Fourier transform samples chosen uniformly at random of size $O(|S|\log(n))$ with high probability by solving a linear programming problem. Even if the unknown signal is not s-sparse, under certain conditions depending on the restricted isometry property of the measurement matrix (which will be discussed later in this chapter), the stable recovery, i.e.,

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_q^q \le c \min_{\mathbf{x}'} \left\{ \|\mathbf{x} - \mathbf{x}'\|_q^q : \mathbf{x}' \text{ is } s\text{-sparse} \right\},$$

via ℓ_q-minimization ($0 < q \le 1$) is possible [46]. Here $\hat{\mathbf{x}}$ is the estimated vector via ℓ_q-minimization and c is a positive constant depending on the restricted isometry property of the measurement matrix. An interesting observation made in [46] states that the number of minimum measurements required to allow a stable recovery (independent of the algorithm recovering the unknown vector) is $O(s\log(n))$. In particular, for a given vector $\mathbf{x}$ if an algorithm outputs $\hat{\mathbf{x}} \in \mathbb{R}^n$ from the observations $\mathbf{b} = \mathbf{A}\mathbf{x}$ such that

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_q^q \le c \min_{\mathbf{x}'} \left\{ \|\mathbf{x} - \mathbf{x}'\|_q^q : \mathbf{x}' \text{ is an } s\text{-sparse vector} \right\},$$

where $c > 0$ and $0 < q \leq 1$, then the number of the rows of $\mathbf{A}$ must satisfy

$$m \geq D(c)(s\log(n/s)),$$

where $D(c) > 0$ is a constant number that depends on c [46].

Although the ℓ_1-norm minimization approach works in many situations to recover the sparsest signals, it does not give a general recipe to solve similar natured minimization problems containing different feasible sets and different non-convex objectives which are hard to solve. This problem is targeted in [25] by Chandrasekaran, Recht, Parrilo, and Willsky. The authors propose a new framework in which the cardinality function in the objective is relaxed by the notion of *atomic norm*. Minimizing this function over the solutions to the set of underdetermined system of linear equations allows generalizing many results on well-known sparse and low rank recovery problems.

Definition 2.0.4 (Atomic norm)**.** Let $\mathcal{F} \subseteq \mathbb{R}^n$ be a compact set of *atoms* which constitute simple building blocks of a signal. Assume that $\mathcal{F}$ is *centrally symmetric* about the origin (a set $\mathcal{F}$ is called centrally symmetric if for every $f \in \mathcal{F} \Leftrightarrow -f \in \mathcal{F}$) and the elements of $\mathcal{F}$ are the extreme points of $\operatorname{conv}(\mathcal{F})$. The *atomic norm* induced by $\mathcal{F}$ is defined as

$$\|\mathbf{x}\|_{\mathcal{F}} := \inf\{t > 0 : \mathbf{x} \in t\operatorname{conv}(\mathcal{F})\}.$$

Since $\mathcal{F}$ is centrally symmetric, $\|\cdot\|_{\mathcal{F}}$ is a norm. If not, then $\|\cdot\|_{\mathcal{F}}$ is a semi-norm. Atomic norm of a point x induced by the set $\mathcal{F}$ is basically the Minkowski function of $\operatorname{conv}(\mathcal{F})$ at x.

For example, if the set of atoms is $\mathcal{E} = \{\pm e_1, \ldots, \pm e_n\}$ then the atomic norm induced by $\mathcal{E}$ is the ℓ_1-norm. If $\mathcal{M} = \{\mathbf{M} \in \mathbb{R}^{m\times n} : \operatorname{rank}(\mathbf{M}) = 1, \|\mathbf{M}\|_F = 1\}$ then the atomic norm induced by $\mathcal{M}$ is the nuclear norm.

In [25], the authors assume that the unknown vector (signal) is of the form

$$\mathbf{x} = \sum_{i=1}^{s} \alpha_i \mathbf{v}_i,\ \mathbf{v}_i \in \mathcal{F}, \alpha_i \geq 0,$$

where s is relatively small. Although the model may seem restricted, it encompasses many structured mathematical objects encountered frequently in compressed sensing.

Lemma 2.0.1. *If C is a closed convex, centrally symmetric set in $\mathbb{R}^n$ with $0 \in int(C)$, then the gauge (see Definition 1.1.8) of C, γ_C, defines a norm. Furthermore, C is the unit ball for this norm.*

Proof. We refer the reader to [116, Theorem 1.35] to show γ_C is a seminorm. Since $0 \in \operatorname{int}(C)$, it follows that γ_C is a norm. The proof of the second part can be found in [64, p.129-130]. □

The atomic norm minimization problem is formulated as

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad && \|\mathbf{x}\|_{\mathcal{F}} \qquad (P_{\mathcal{F}}) \\ &\text{subject to} && \mathbf{A}\mathbf{x} = \mathbf{b}. \end{aligned} \tag{2.6}$$

In the case of noisy measurements $\mathbf{Ax} + \mathbf{w} = \mathbf{b}$, the linear constraint can be replaced by the relaxed constraint $\|\mathbf{b} - \mathbf{Ax}\| \leq \varepsilon$, where ε is an upper bound on the norm of the noise $\mathbf{w}$:

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad \|\mathbf{x}\|_{\mathcal{F}} \qquad\qquad (P_{F_n}) \\ &\text{subject to} \quad \|\mathbf{b} - \mathbf{Ax}\| \leq \delta. \end{aligned} \tag{2.7}$$

The recovery of sparse signals have been studied immensely so it is impossible to include all the references (see [19, 22, 23, 46, 25] and the references therein for a start). The majority of the focus has been on the necessary and sufficient conditions to recover the signal exactly or robustly (i.e., when the observation contains noise/error, i.r., $\mathbf{b} = \mathbf{Ax} + \mathbf{n}$, $\mathbf{n}$ is the noise). For this, many approaches exist such as the null-space property, restricted isometry property and coherence.

2.0.1 Null Space Property

In this section, we collect some necessary and sufficient conditions for the recovery of sparse signals using convex optimization techniques which are based on the null-space of the measurement matrix $\mathbf{A}$.

Definition 2.0.5 (Null-space property of order s)**.** A matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ has the *null-space property of order s* (with respect to the ℓ_1-norm) if for every subset $S \subseteq [n]$ with $|S| \leq s$

$$2\sum_{i\in S} |x_i| < \sum_{j\in[n]} |x_j| \quad \text{for every } \mathbf{x} \in \text{Null}(\mathbf{A}) \setminus \{0\}.$$

The null-space property is a necessary and sufficient condition for the exact recovery of s-sparse vectors through the ℓ_1-norm minimization problem [36, 35].

Theorem 2.0.2. *Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Then an s-sparse vector $\mathbf{x}_0 \in \mathbb{R}^n$ with $\mathbf{b} = \mathbf{Ax}_0$ is the unique solution of the ℓ_1-norm minimization problem*

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\textit{minimize}} \quad \|\mathbf{x}\|_1 \qquad (P_1) \\ &\textit{subject to} \quad \mathbf{Ax} = \mathbf{b} \end{aligned} \tag{2.8}$$

if and only if $\mathbf{A}$ satisfies the null space property of order s.

It is also well known that when the objective is a weighted ℓ_1-norm, i.e., $\sum_{i=1}^n w_i|x_i|$ for some positive weight vector $\mathbf{w}$, then

$$2\sum_{i\in S} w_i|x_i| < \sum_{j\in[n]} w_j|x_j| \quad \text{for every } \mathbf{x} \in \text{Null}(\mathbf{A}) \setminus \{0\}$$

is a necessary and sufficient condition for the recovery through the weighted ℓ_1-norm minimization [76].

The following optimization problem is equivalent to (2.8)

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \|\mathbf{x}_0+\mathbf{d}\|_1 \qquad (P_1') \\ \text{subject to} \quad & \mathbf{d}\in \text{Null}(\mathbf{A}), \end{aligned}$$

where $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$. Here, $\mathbf{x}_0$ is the unique optimal solution if and only if $\mathbf{0}$ is the unique solution of (P_1'). Note that in (P_1'), the set of feasible points can be restricted to the points from null-space of $\mathbf{A}$ that decreases the objective, i.e., to the set $\{\mathbf{d} : \mathbf{d} \in \text{Null}(\mathbf{A}) \text{ and } \|\mathbf{x}_0+\mathbf{d}\|_1 < \|\mathbf{x}_0\|_1\}$.

The condition in Theorem 2.0.2 can be written in terms of atomic norms. Before introducing the general condition, we define the following cone

$$T_{\mathcal{F}}(\mathbf{x}_0) := \text{cone}\{\mathbf{x}-\mathbf{x}_0 : \|\mathbf{x}\|_{\mathcal{F}} \le \|\mathbf{x}_0\|_{\mathcal{F}}, x\in C\}, \tag{2.9}$$

where $C := \|\mathbf{x}_0\|_{\mathcal{F}} \,\text{conv}(\mathcal{F})$. Note that for every $\mathbf{x}\in C$, $\|\mathbf{x}\|_F \le \|\mathbf{x}_0\|$ by the definition of C. In particular, $T_{\mathcal{F}}(\mathbf{x}_0)$ is the cone of descent directions with respect to $\|\cdot\|_{\mathcal{F}}$ at a point $\mathbf{x}_0$ over the set C. We would like to emphasize that this is, in general, not equal to the tangent cone to $\|\mathbf{x}_0\|_{\mathcal{F}} \,\text{conv}(\mathcal{F})$ at $\mathbf{x}_0$. For example, take $\mathcal{F}$ as the points on the Euclidean unit ball. Then the tangent cone at $\mathbf{x}_0$ to the scaled unit ball $\|\mathbf{x}_0\|_{\mathcal{F}} \,\text{conv}(\mathcal{F})$ is not equal to (2.9). By the definition of the tangent cone for closed convex sets one needs to take the closure of the cone consisting of the directions $\mathbf{x}-\mathbf{x}_0$ where $\mathbf{x}\in\|\mathbf{x}_0\|_{\mathcal{F}} \,\text{conv}(\mathcal{F})$ (which is equal to the closure of $T_{\mathcal{F}}(\mathbf{x}_0)$).

The following is now well known in convex optimization, giving a necessary and sufficient condition for the uniqueness of the solution to the general problem (2.6).

Theorem 2.0.3. *[25] Let $\mathbf{A}\in\mathbb{R}^{m\times n}$ and $\mathbf{b} = \mathbf{A}\mathbf{x}_0$. Then $\hat{\mathbf{x}} = \mathbf{x}_0$ is the unique solution to (2.6) if and only if $\text{Null}(\mathbf{A}) \cap T_{\mathcal{F}}(\mathbf{x}_0) = \{\mathbf{0}\}$, where $T_{\mathcal{F}}(\mathbf{x}_0)$ is defined as in (2.9).*

The result in Theorem 2.0.3 gives a characterization of exact recovery based on the condition that the null space of linear measurement matrix $\mathbf{A}$ has zero intersection with the cone $T_{\mathcal{F}}(\mathbf{x}_0)$. Observe that

$$\text{Null}(\mathbf{A}) \cap T_{\mathcal{F}}(\mathbf{x}_0) = \{\mathbf{0}\}$$

$$\text{if and only if}$$

$$\exists \varepsilon > 0 \text{ such that } \ \|\mathbf{A}\mathbf{z}\|_2 \ge \varepsilon, \ \ \forall \mathbf{z}\in T_{\mathcal{F}}(\mathbf{x}_0)\cap\mathcal{S}^{n-1}.$$

In some sense, assuming $\mathbf{A}$ is a *random matrx* (i.e., a matrix whose entries are random variables), we are interested in bounding its restricted spectral norm.

For a random measurement matrix $\mathbf{A}$ with $\mathbf{b} = \mathbf{A}\mathbf{x}_0$, the uniqueness of the solution $\mathbf{x}_0$ can be characterized from

$$P\left[\min_{\mathbf{z}\in\Omega} \|\mathbf{A}\mathbf{z}\|_2 \ge \varepsilon\right], \tag{2.10}$$

where $\Omega := T_{\mathcal{F}}(\mathbf{x}_0)\cap\mathcal{S}^{n-1}$ and P denotes a probability measure. Using concentration inequalities it is possible to find a lower bound for (2.10).

Lemma 2.0.4. *[84, Section 1.1] Let $f : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz continuous on $\mathbb{R}^N$ with Lipschitz constant L and let $\mathbf{w} \in \mathbb{R}^N$ be a Gaussian vector with zero mean and identity variance. Then, for every $t > 0$,*

$$P\left[f(\mathbf{w}) > \mathbb{E}\left[f(\mathbf{w})\right] - t\right] \geq 1 - \exp\left(-\frac{t^2}{2L^2}\right). \tag{2.11}$$

A critical step in applying (2.11) is to find a good lower bound for $\mathbb{E}[f(\mathbf{w})]$. Before introducing a lower bound, we find it useful to provide a notation for the expected norm of a Gaussian vector. For a given Gaussian vector $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbb{I})$ of dimension m, we denote

$$a_m := \mathbb{E}\left[\left(\sum_{i=1}^{m} w_i^2\right)^{\frac{1}{2}}\right] = \frac{\sqrt{2}\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)},$$

where $\Gamma(\cdot)$ is the gamma function. Then we have [53]

$$\frac{m}{\sqrt{m+1}} \leq a_m \leq \sqrt{m}. \tag{2.12}$$

Lemma 2.0.5. *([53, Corollary 1.2]) Let $\Omega \subseteq \mathcal{S}^{n-1}$ be a closed set. Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a map with independent and identically distributed (i.i.d.) zero-mean, unit variance Gaussian entries and $\mathbf{g} \in \mathbb{R}^n$ be a vector with i.i.d. zero-mean, unit variance Gaussian entries. Then*

$$\mathbb{E}\left[\min_{\mathbf{z}\in\Omega} \|\mathbf{A}\mathbf{z}\|_2\right] \geq a_m - \mathbb{E}\left[\max_{\mathbf{z}\in\Omega} \mathbf{g}^\top \mathbf{z}\right],$$

where $a_m := \mathbb{E}\left[\|\mathbf{w}\|_2\right]$ is the expected value of the norm of the Gaussian vector $\mathbf{w} \in \mathbb{R}^m$.

Definition 2.0.6 (Gaussian width)**.** The *Gaussian width* of a set $C \subseteq \mathbb{R}^n$ is

$$w(C) := \mathbb{E}\left[\max_{\mathbf{x}\in C} \mathbf{g}^\top \mathbf{x}\right],$$

where $\mathbf{g} \sim \mathcal{N}(\mathbf{0}, \mathbb{I})$ is a Gaussian vector with independent identically distributed coordinates that has zero mean and unit variance.

The following result and its proof from [25] provide a recipe to find the number of measurements required to recover sparse objects from linear measurements with high probability in the atomic norm framework.

Proposition 2.0.6. *[25, Corollary 3.3] Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a random linear map with i.i.d. zero mean Gaussian entries having variance $1/m$. Let $\Omega := T_{\mathcal{F}}(\mathbf{x}_0) \cap \mathcal{S}^{n-1}$. Suppose $\mathbf{b} = \mathbf{A}\mathbf{x}_0$. Then, $\mathbf{x}_0$ is the unique solution of (P_1) in (2.5) with probability at least*

$$1 - \exp\left(-\frac{1}{2}(a_m - w(\Omega) - \sqrt{m}\varepsilon)^2\right),$$

provided $m \geq \dfrac{w(\Omega)^2 + 3/2}{(1-\varepsilon)^2}$. Here, a_m is the expected norm of a Gaussian vector ($\sim \mathcal{N}(0, \mathbb{I})$) of dimension m.

Proof. For any closed subset $\Omega \subseteq \mathcal{S}^{n-1}$, the unit sphere in n dimension, the function

$$\mathbf{A} \mapsto \min_{\mathbf{z}\in\Omega} \|\mathbf{A}\mathbf{z}\|_2$$

is Lipschitz continuous with constant 1 (with respect to the Frobenius norm). Since each entry of $\mathbf{A}$ has variance $1/m$, by Lemma 2.0.5 we have

$$\mathbb{E}\left[\min_{\mathbf{z}\in\Omega} \|\sqrt{m}\mathbf{A}\mathbf{z}\|_2\right] \geq a_m - \mathbb{E}\left[\max_{\mathbf{z}\in\Omega} \mathbf{g}^\top \mathbf{z}\right].$$

Also,

$$P\left[\min_{\mathbf{z}\in\Omega} \|\mathbf{A}\mathbf{z}\|_2 \geq \varepsilon\right] = P\left[\min_{\mathbf{z}\in\Omega} \left\|\sqrt{m}\mathbf{A}\mathbf{z}\right\|_2 \geq \sqrt{m}\varepsilon\right].$$

Then, by Lemma 2.0.4 we get

$$P\left[\min_{\mathbf{z}\in\Omega} \left\|\sqrt{m}\mathbf{A}\mathbf{z}\right\|_2 \geq \sqrt{m}\varepsilon\right] \geq 1 - \exp\left(-\frac{1}{2}\left(a_m - w(\Omega) - \sqrt{m}\varepsilon\right)^2\right)$$

provided that $a_m - w(\Omega) - \sqrt{m}\varepsilon \geq 0$. We show that this condition indeed holds. By the assumption on m, we have

$$\begin{aligned}
\frac{w(\Omega)^2 + 3/2}{(1-\varepsilon)^2} \leq m \implies w(\Omega)^2 + 1 &\leq m(1-\varepsilon)^2 - 1/2 \\
&\leq m(1-\varepsilon)^2 - 2\varepsilon(1-\varepsilon) + \varepsilon^2/m \\
&\leq \left(\sqrt{m}(1-\varepsilon-\varepsilon/\sqrt{m}\right)^2
\end{aligned}$$

the last inequality follows since $\varepsilon(1-\varepsilon) \leq 1/4$ for every $\varepsilon \in (0,1)$. By (2.12) and the above result, we have

$$a_m - \sqrt{m}\varepsilon \geq \frac{m}{\sqrt{m+1}} - \sqrt{m+1}\varepsilon \geq \frac{m-(m+1)\varepsilon}{\sqrt{m+1}} \geq w(\Omega).$$

Hence, the result follows. □

In [53], Gordon provided a sufficient and necessary condition for the existence of a k-dimensional subspace having a zero intersection with a subset of $\mathbb{R}^n$. Using a result from [53], the authors [115, 25] provided upper bounds on the Gaussian width of some convex sets.

Proposition 2.0.7. *[25, Proposition 3.6] Let $\mathcal{C}$ be a nonempty convex cone in $\mathbb{R}^n$ and let $\mathbf{g} \sim \mathcal{N}(\mathbf{0}, \mathbb{I})$ be a random Gaussian vector. Then,*

$$w(\mathcal{C} \cap S^{n-1}) \leq \mathbb{E}\left[\operatorname{dist}(\mathbf{g}, C^\circ)\right],$$

where C° is the polar cone of C, $S^{n-1} \subset \mathbb{R}^n$ is the unit norm sphere, and dist *denotes the Euclidean distance between a point and a set.*

The recovery of an s-sparse signal $\mathbf{x}_0$ from its measurements $\mathbf{A}\mathbf{x}_0$ has been studied extensively, and the exact reconstruction from these measurements is possible if the number of measurements is of order $O(s\log(n/s))$ [115]. In a previous work [115], the method used to recover the s-sparse signal also exploits the concept of Gaussian width and follows similar lines to the result in [25].

Theorem 2.0.8. *[25, Proposition 3.10] Let $\hat{\mathbf{x}} \in \mathbb{R}^n$ be an s-sparse vector. Let $\mathcal{F} := \{\pm e_1, \ldots, \pm e_n\}$. Then $\|\cdot\|_{\mathcal{F}}$ is the ℓ_1-norm and*

$$w(T_{\mathcal{F}}(\hat{\mathbf{x}}) \cap \mathbb{S}^{n-1})^2 \leq 2s\log(\frac{n}{s}) + \frac{11}{8}s.$$

Thus, $2s\log(n/s) + 11s/8$ random Gaussian measurements suffice to recover $\hat{\mathbf{x}}$ by ℓ_1-norm minimization with high probability.

Note that in [25], it is stated that $2s\log(n/s) + 5s/4$ random measurements suffice, however the second term contains an insignificant calculation error. Some other results in [25] using the bounds on the Gaussian width and Proposition 2.0.6 include low rank recovery via nuclear norm minimization, recovery of orthogonal matrices via spectral norm minimization and recovery of permutation matrices via the norm induced by the Birkhoff polytope of doubly stochastic matrices.

Null-space based recovery with quasi-norm minimization

Definition 2.0.7 (ℓ_q quasi-norm)**.** Let $\mathbf{x} \in \mathbb{R}^n$ and $q \in (0,1)$. The ℓ_q quasi-norm of $\mathbf{x}$ is defined as

$$\|\mathbf{x}\|_q := \left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q}.$$

Consider the unit balls for ℓ_q-quasi norm and and ℓ_0-"norm". As $q \to 0$, the unit ball for ℓ_q quasi-norm approaches to that of ℓ_0-"norm". Motivated by this ℓ_q quasi-norm has been used as a surrogate function for the cardinality function or added as a penalty function to a convex objective to induce more sparsity even though it is a concave function. There is a similar null-space property for recovering s-sparse signals with ℓ_q quasi-norm minimization. In fact, a generalization of this holds [54].

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad \|\mathbf{x}\|_q \qquad (P_q) \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned} \tag{2.13}$$

where $0 < q < 1$ has a unique s-sparse solution. It turns out that under an analogous nullspace property condition uniqueness can be guaranteed for a class of functions including ℓ_q quasi-norm with $q \in (0,1]$.

Theorem 2.0.9. *[54, Theorem 2] Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *and let* f *be a nondecreasing function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *, not identically zero, with* $f(0) = 0$ *and such that* $x \mapsto f(x)/x$ *is non-increasing on* $\mathbb{R}_{++}$*. Then every* s*-sparse vector* $\mathbf{x}$ *can be uniquely recovered by solving*

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{i=1}^{n} f(\|x_i\|) \quad (P_f) \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned}$$

if and only if for every subset $S \subseteq [n]$ *with* $|S| \le s$

$$\sum_{i\in S} f(|x_i|) < \sum_{j\in[n]\setminus S} f(|x_j|) \quad \text{for every } \mathbf{x} \in \mathrm{Null}(\mathbf{A}) \setminus \{0\}.$$

Note that any concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ is nondecreasing. In addition, the condition $f(0) = 0$ implies $f(\alpha x) \ge \alpha f(x)$ for every $\alpha \in (0, 1)$ which in turn implies that $\frac{1}{\alpha x} f(\alpha x) \ge \frac{1}{x} f(x)$. This shows that Theorem 2.0.9 applies to an important class of functions including $f(x) := |x|^q$, where $q \in (0, 1]$.

Corollary 2.0.10. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *and* $q \in (0, 1]$*. Then every* s*-sparse vector* $\mathbf{x}$ *can be uniquely recovered by solving* (P_q) *in* (2.13) *if and only if for every subset* $S \subseteq [n]$ *with* $|S| \le s$

$$\sum_{i\in S} |x_i|^q < \sum_{j\in[n]\setminus S} |x_j|^q \quad \text{for every } \mathbf{x} \in \mathrm{Null}(\mathbf{A}) \setminus \{0\}.$$

Furthermore it follows from [54] that the exact recovery by ℓ_r-minimization implies exact recovery by ℓ_q-minimization whenever $0 < q < r \le 1$.

Theorem 2.0.11. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$*, and* $0 < q < r \le 1$*. If every* s*-sparse vector* $\mathbf{x}$ *can be uniquely recovered from the measurements* $\mathbf{b} = \mathbf{A}\mathbf{x}$ *by solving* ℓ_r*-quasi-norm minimization problem as in* (2.13)*, then every* s*-sparse vector* $\mathbf{x}$ *can be uniquely recovered from the same measurements* $\mathbf{b} = \mathbf{A}\mathbf{x}$ *by solving* ℓ_q*-quasi-norm minimization problem* (2.13)*.*

This theorem is important in the sense that it also gives a sufficient condition for a linear programming problem to solve an instance of NP-hard nonconvex programs.

Next, we provide the definition of coherence in the context of compressed sensing.

Definition 2.0.8 (Coherence)**.** Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the columns of $\mathbf{A}$. The coherence of $\mathbf{A}$ is the largest absolute value of the inner product between the different normalized columns of $\mathbf{A}$

$$\mu(\mathbf{A}) := \max_{i\ne j} \frac{|\mathbf{a}_i^\top \mathbf{a}_j|}{\|\mathbf{a}_i\|_2 \|\mathbf{a}_j\|_2}.$$

2.0.2 Restricted Isometry Property

The restricted isometry property is another fundamental concept and one of the most popular tools for studying the recovery of sparse signals using linear programming [22]. The restricted isometry property of a matrix requires the distance between some sparse vectors to be "approximately" preserved. Our aim here is just to give a brief overview of the main results rather than opening Pandora's box and list all of the works in this field. In particular, we are interested in the results related to structured matrices.

Definition 2.0.9 (Restricted Isometry Property (RIP))**.** Let $\mathbf{A}$ be an $m \times n$ matrix with $m \leq n$. For every integer s where $1 \leq s \leq n$; we define the *s-restricted isometry constant* δ_s to be the smallest real number in $(0, 1)$ such that

$$(1 - \delta_s)\|\mathbf{v}\|_2^2 \leq \|\mathbf{A}\mathbf{v}\|_2^2 \leq (1 + \delta_s)\|\mathbf{v}\|_2^2 \tag{2.14}$$

for all $\mathbf{v} \in \mathbb{R}^n$ with at most s nonzero entries.

Suppose that we want to recover an s-sparse signal $\mathbf{x}$ from linear measurements $\mathbf{y} = \mathbf{A}\mathbf{x}$ and also suppose δ_{2s} is sufficiently smaller than 1. Then RIP implies that

$$(1 - \delta_{2s})\|\mathbf{x}_1 - \mathbf{x}_2\|_2^2 \leq \|\mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2)\|_2^2 \leq (1 + \delta_{2s})\|\mathbf{x}_1 - \mathbf{x}_2\|_2^2 \tag{2.15}$$

pairwise distances between s-sparse signals must be well preserved. Furthermore, (2.15) implies that two s-sparse signals have the same image under $\mathbf{A}$ i.e., satisfy $\mathbf{A}\mathbf{x}_1 = \mathbf{A}\mathbf{x}_2$ if and only if $\mathbf{x}_1 = \mathbf{x}_2$ as $0 < \delta_{2s} < 1$.

Let $S \subseteq [n]$ and $\mathbf{A}_S$ be the matrix formed from the columns of $\mathbf{A}$ indexed by S, then the RIP property is equivalent to

$$\left\|\mathbf{A}_S^\top \mathbf{A}_S - \mathbb{I}\right\|_2 \leq \delta_s,$$

for each $S \subseteq [n]$ with $|S| \leq s$. The number δ_s measures how close the columns of $\mathbf{A}$ are to behave like an orthonormal system when restricted to the linear combination of at most s many vectors.

A lot of work in compressed sensing that analyzed RIP has focused on the equivalence condition between ℓ_1-norm minimization and ℓ_0-"norm". In [19], it was shown that $\delta_s + \delta_{2s} + \delta_{3s} < 1$ is a sufficient condition for the recovery of s-sparse signals using (P_1). Later in [20], Candes gave a sufficient condition for robust recovery as well as the equivalence of (P_0) and (P_1) in the sense that both programs have the same unique optimal solution whenever there is an s-sparse solution to the linear system.

Theorem 2.0.12 (Theorem 1.2, [20])**.** *If $\delta_{2s} < \sqrt{2} - 1$, then there exists a positive number C such that every solution $\mathbf{x}^*$ of (P_1) satisfies*

$$\|\mathbf{x} - \mathbf{x}^*\|_1 \leq C\|\mathbf{x} - \mathbf{x}^{(s)}\|_1$$

and

$$\|\mathbf{x} - \mathbf{x}^*\|_2 \leq Cs^{-1/2}\|\mathbf{x} - \mathbf{x}^{(s)}\|_1,$$

where $\mathbf{x}^{(s)} \in \mathbb{R}^n$ is the vector of all zeros except the first s entries are composed of the absolute wise s largest entries of $\mathbf{x}$. Furthermore, if there exists an s-sparse $\mathbf{x}$ solution to (P_1) then the recovery is exact.

This result has been extended to the case where the observations are corrupted by some noise. Suppose $\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{w}$ and consider the following

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{x}\|_1 \quad \text{subject to } \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 \leq \varepsilon. \quad (\tilde{P}_1) \tag{2.16}$$

Theorem 2.0.13 (Theorem 1.3, [20])**.** *If $\delta_{2s} < \sqrt{2} - 1$ and $\|\mathbf{w}\| \leq \varepsilon$, then there exist positive numbers $\tilde{C}$ and $\tilde{C}_1$ such that every solution $\mathbf{x}^*$ of $(\tilde{P}_1)$ satisfies*

$$\|\mathbf{x} - \mathbf{x}^*\|_2 \leq \tilde{C} s^{-1/2} \|\mathbf{x} - \mathbf{x}^{(s)}\|_1 + \tilde{C}_1 \varepsilon,$$

where $\mathbf{x}^{(s)} \in \mathbb{R}^n$ is the vector of all zeros except the first s entries are composed of the absolute wise s largest entries of $\mathbf{x}$.

In addition, Foucart and Lai showed that $\delta_{2s} < 2(3 - \sqrt{2})/7$ is sufficient for the recovery of s-sparse signals by ℓ_1-minimization [45]. Other sufficient conditions for the signal recovery include $\delta_{2s} < 0.4931$ in [97], $\delta_s < 0.307$ in [17]. In addition, the recovery of certain s-sparse signals are impossible when δ_{2s} is arbitrarily close to $\frac{1}{\sqrt{2}}$ [31] or when $\delta_s = \frac{s-1}{2s-1} < 0.5$ [17]. To the best of our knowledge, the most recent result about RIP related to the recovery of s-sparse signals using (P_1) is provided by Cai and Zhang [18]. The authors showed that the sufficient condition $\delta_s < 1/3$ for the recovery of s-sparse signals using ℓ_1-norm minimization is sharp, i.e., in general it is not possible to recover every s-sparse signal if $\delta_s \geq 1/3$.

2.1 Compressed Sensing in Structured Matrices

In signal processing and machine learning, structured matrices are ubiquitous and arise in many areas. Therefore, in this section we find it useful to provide a small survey of recent works on the recovery of structured matrices.

Toeplitz and Hankel matrices are one of the most commonly used structure in signal processing as they arise naturally from the convolutional structure inherent to linear time invariant systems.

In [58], a restricted isometry property has been established for m-by-n Toeplitz matrices with bounded or Gaussian i.i.d entries. Using the Gersgorin's Theorem and some variants of Hoeffding's theorem, they established that the number of measurements m required for RIP property with constant δ_s to be satisfied with probability $1 - \epsilon$ is of order $O(Cs^2 \log(n^2/\epsilon))$, where C is a constant that is inversely proportional to δ_s^2. Later, in [58], using the coherence for Toeplitz matrices with entries that are Rademacher random variables (i.e., each random variable takes value $+1$ with probability 0.5, and -1 with probability 0.5), the number of necessary measurements were established as $O(Cs^2 \log^2(n^2/\epsilon))$.

In [111], Rauhut studied the recovery of partial random circulant and Toeplitz matrices in the context of ℓ_1-minization. Utilizing a new version of the non-commutative Khintchine inequality, it is shown that the necessary number of observations to guarantee sparse reconstruction by ℓ_1-minimization is linearly proportional to the sparsity up to a log-factor in the input size which is an improvement over previous recovery results for such matrices [58]. Using a recovery theorem for ℓ_1-minimization which is based on KKT complementary slackness conditions and a new version of Khintchine inequalities, the following results are established [111].

Theorem 2.1.1. *[111, Theorem 2.1] Let* $\mathbf{A}$ *be an* $m \times n$ *partial circulant matrix or a Toeplitz matrix with its entries being* ± 1 *Bernoulli variables. Assume that the sign of the nonzero entries of the unknown* s*-sparse vector are Bernoulli random variables. Then the exact recovery the unknown vector with* ℓ_1 *minimization* (P_1) *with probability at least* $1-\varepsilon$ *requires at least*

$$Cs\log^3(n/\varepsilon),$$

many measurements.

Proposition 2.1.2. *[111, Proposition 3.2] Let* μ *be the coherence of the random partial circulant matrix or Toeplitz matrix* $\frac{1}{\sqrt{n}}\mathbf{A}$*, where* $\mathbf{A}$ *is an* $n \times N$ *partial circulant matrix with its entries being* ± 1 *Bernoulli variables. Then with with probability at least* $1-\varepsilon$*, the coherence satisfies*

$$\mu \leq 4\frac{\log(2N^2/\varepsilon)}{\sqrt{n}}$$

Although the method followed in atomic norm exploits a min-max inequality due to Gordon [53, Corollary 1.2] in which the underlying matrix is a Gaussian matrix, we would like to know if this inequality can be modified to serve a purpose for the Gaussian Toeplitz matrices and reproduce or obtain better results in terms of number of measurements to recover the unknown sparse signal with high probability.

It is still an open question whether it is possible for these structured matrices to satisfy RIP with constant δ_s when the number of measurements is $O(s\log(n))$.

Partial circulant matrices are also commonly studied in compressed sensing. Romberg showed that the restricted isometry property holds for the partial random circulant matrices with random sampling sets and random generators when the number of observations is $O(Cs\log^6 n)$ [114]. On the other hand, if the condition for random sampling is replaced by a fixed sampling then $O(s\log^2 s\log^2 n)$ many measurements is sufficient for the RIP to hold. [80]. In addition, it is possible to recover the s-sparse signal in a given orthonormal representation from $O(s\log n)$ samples from its convolution with a pulse whose Fourier transform has unit magnitude and random phase at all frequencies [114].

Furthermore, a recent work established that if $s \leq D\sqrt{n/\log(n)}$, then $O(Cs\log(en/s))$ measurements are sufficient for the robust recovery of s-sparse vector from $\mathbf{b} = \mathbf{Ax} + \mathbf{e}$, where $\mathbf{A}$ is a Gaussian partial circulant matrix and $\mathbf{e}$ is a noise vector [95]. If s is larger than $D\sqrt{n/\log(n)}$ then $m \geq Cs\log^2(s)\log(\log(S))\log(n)$ measurements are sufficient for the robust recovery.

In 2011, Oymak et al. studied the relations between sparse signal recovery and low rank matrix recovery. They proposed a simple and transparent way to extend the recovery conditions given for vectors to matrices [107]. In the next section, we analyze the recovery of low rank matrices.

2.2 Recovery of Low Rank Matrices

A famous problem known as the *Netflix problem* was to suggest movie ratings to users based on their previous ratings. In 2009, Netflix provided a training data set of $100,480,507$ ratings that $480,189$ users gave to $17,770$ movies. The challenge was to predict $2,817,131$ entries of the matrix whose columns where indexed by movies and rows indexed by users. In this problem, one may assume that there is a high correlation between certain columns. For example, if a user rates the movie "The Matrix" with grade 1, she will probably rate the movie "The Matrix Reloaded" with grade 1 as well. Such possible linear dependence between the columns of the users versus movie ratings matrix suggests that the big matrix is rank deficient. When the underlying measurement matrix has a low rank, it is often possible to exploit its structure and recover it using computationally efficient algorithms. Therefore, one can pose the prediction of incomplete entries as a rank minimization problem subject to given linear measurements. Such problems also arise in machine learning, signal processing, image recognition and in finance.

The problem of low rank matrix recovery subject to linear constraints can be formulated as

$$\begin{aligned} &\underset{\mathbf{X}\in\mathbb{R}^{n\times n}}{\text{minimize}} \quad \text{rank}(\mathbf{X}) \qquad (P) \\ &\text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{y}. \end{aligned} \tag{2.17}$$

Although one can find the best rank-k approximation of a matrix with respect to Frobenius norm by singular value decomposition, the rank minimization problem (2.17) is NP-hard. The heuristic approaches have been used to attack the problem [43]. In her seminal work [43], Fazel proposed the nuclear norm of the matrix and log det of the regularized matrix as surrogate functions for the rank of a matrix. The former can be posed as an SDP problem [136, 43] and the second can be approximated by using some linearization and solved as an SDP problem efficiently in polynomial time. Note that in the rank minimization problem above, if the rank one matrices with unit Frobenius norm is defined as the atomic set, then the convex hull of this atomic set is the nuclear norm ball. Relaxing the objective of (2.17) by the nuclear norm which is its largest lower convex envelope aligns with the general setting proposed in [25].

The null-space property of order s can be generalized for the nuclear norm minimization and it is sufficient and necessary condition for the unique recovery of rank-k matrices.

Proposition 2.2.1. *[107] Let $\mathcal{A}:\mathbb{R}^{m\times n}\to\mathbb{R}^{\ell}$ be a linear mapping. All matrices $\mathbf{X}\in\mathbb{R}^{m\times n}$ with $\operatorname{rank}(\mathbf{X})\le k$ can be recovered from*

$$\begin{aligned} &\underset{\mathbf{X}\in\mathbb{R}^{m\times n}}{\textit{minimize}} \quad \min\|\mathbf{X}\|_* \qquad (P_*) \\ &\textit{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{y}. \end{aligned} \tag{2.18}$$

if and only if for every $W\in\text{Null}(\mathcal{A})$

$$2\sum_{i=1}^{k}\sigma_i(W) < \sum_{j=k+1}^{n}\sigma_j(W).$$

A similar statement (when replacing the nuclear norm with positively weighted nuclear norm) also holds for the low rank recovery of the weighted nuclear norm minimization.

In [107], Oymak et. al. showed how to extend several known results for sparse vector recovery to matrices, including the restricted isometry and null-space conditions to low rank matrix recovery.

In [108], the authors also established a list of matrix robustness conditions for the nuclear norm and Frobenius norm. Here, we state some generalizations of these matrix noise robustness results. We generalize the necessary and sufficient conditions and replace the nuclear norm with unitarily invariant norm and the ℓ_2-norm for measuring the error by a general norm whenever it is possible.

Here, we consider the problem of recovering a low rank matrix $\mathbf{X}_0 \in \mathbb{R}^{m\times n}$, with $m \leq n$ from the corrupted measurements $y = \mathcal{A}(\mathbf{X}_0) + \mathbf{z}$, with $\|\mathbf{z}\| \leq \varepsilon$, where ε denotes the noise level, and $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^{\ell}$ is a linear operator.

Lemma 2.2.2. *Let $\mathbf{X}_0 \in \mathbb{R}^{m\times n}$, $m \leq n$, such that* $\operatorname{rank}(\mathbf{X}_0) \leq k$ *and* $\|\mathcal{A}(\mathbf{X}_0) - \mathbf{y}\| \leq \varepsilon$. *Let $\hat{\mathbf{X}} \in \mathbb{R}^{m\times n}$ be such that $\|\hat{\mathbf{X}}\|_* \leq \|\mathbf{X}_0\|_*$ and $\|\mathcal{A}(\hat{\mathbf{X}}) - \mathbf{y}\| \leq \varepsilon$. Then for every $C > 0$,*

$$\left|\left|\left|\mathbf{X}_0 - \hat{\mathbf{X}}\right|\right|\right| < C\varepsilon \tag{2.19}$$

if and only if for all $\mathbf{W}$ with $\sum_{i=1}^{k} \sigma_i(\mathbf{W}) \geq \sum_{i=k+1}^{n} \sigma_i(\mathbf{W})$,

$$|||\mathbf{W}||| < \frac{C}{2}\|\mathcal{A}(\mathbf{W})\|. \tag{2.20}$$

Lemma 2.2.2 differs from the result in [107]. We used unitarily invariant norm, $|||\cdot|||$, for matrices and a general norm, $\|\cdot\|$ for vectors in conditions (2.19)-(2.20).

Proof. Assume that for every $\mathbf{W}$ with $\sum_{i=1}^{k} \sigma_i(\mathbf{W}) \geq \sum_{i=k+1}^{n} \sigma_i(\mathbf{W})$,

$$|||\mathbf{W}||| < \frac{C}{2}\|\mathcal{A}(\mathbf{W})\|.$$

Let $\mathbf{X}_0 \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(\mathbf{X}_0) \leq k$ and $\|\hat{\mathbf{X}}\|_* \leq \|\mathbf{X}_0\|_*$ and $\|\mathcal{A}(\hat{\mathbf{X}}) - \mathbf{y}\| \leq \varepsilon$. Then

$$\begin{align}
\sum_{i=1}^{k} \sigma_i(\mathbf{X}_0 - \hat{\mathbf{X}}) &\geq \sum_{i=1}^{k} \sigma_i(\mathbf{X}_0) - \sum_{i=1}^{k} \sigma_i(\hat{\mathbf{X}}) \tag{2.21}\\
&\geq \sum_{i=1}^{n} \sigma_i(\hat{\mathbf{X}}) - \sum_{i=1}^{k} \sigma_i(\hat{\mathbf{X}}) \tag{2.22}\\
&\geq \sum_{i=k+1}^{n} \sigma_i(\hat{\mathbf{X}}) - \sum_{i=k+1}^{n} \sigma_i(\mathbf{X}_0) \tag{2.23}\\
&\geq \sum_{i=k+1}^{n} \sigma_i(\hat{\mathbf{X}} - \mathbf{X}_0). \tag{2.24}
\end{align}$$

Here, (2.21) and (2.22) follow by Mirsky's singular value inequalities (which will be discussed in Chapter 3 in more detail, but we give below for completeness), i.e., for every $k \in \{1, 2, \ldots, m\}$

$$\sum_{i=1}^{k} \sigma_i(\mathbf{X}_0 - \hat{\mathbf{X}}) \geq \sum_{i=1}^{k} \left| \sigma_i(\mathbf{X}_0) - \sigma_i(\hat{\mathbf{X}}) \right|,$$

and by the assumption on $\mathbf{X}_0$ and $\hat{\mathbf{X}}$, respectively. In addition, (2.24) follows from the triangle inequality (given below) combined with the trace equality. Note that by triangle inequality

$$\sum_{i=1}^{k} \sigma_i(\hat{\mathbf{X}}) \leq \sum_{i=1}^{k} \sigma_i(\hat{\mathbf{X}} - \mathbf{X}_0) + \sum_{i=1}^{k} \sigma_i(\mathbf{X}_0).$$

Also, by the trace equality

$$\sum_{i=1}^{n} \sigma_i(\hat{\mathbf{X}}) = \sum_{i=1}^{n} \sigma_i(\hat{\mathbf{X}} - \mathbf{X}_0) + \sum_{i=1}^{n} \sigma_i(\mathbf{X}_0).$$

Multiplying the trace inequality by -1 and adding to the triangle inequality above gives

$$-\sum_{i=k+1}^{n} \sigma_i(\hat{\mathbf{X}}) \leq -\sum_{i=k+1}^{n} \sigma_i(\hat{\mathbf{X}} - \mathbf{X}_0) - \sum_{i=k+1}^{n} \sigma_i(\mathbf{X}_0).$$

Hence,

$$\sum_{i=k+1}^{n} \sigma_i(\hat{\mathbf{X}}) \geq \sum_{i=k+1}^{n} \sigma_i(\hat{\mathbf{X}} - \mathbf{X}_0) + \sum_{i=k+1}^{n} \sigma_i(\mathbf{X}_0).$$

Then (2.20) holds for $\mathbf{W} = \mathbf{X}_0 - \hat{\mathbf{X}}$, i.e.,

$$\left|\left|\left| \mathbf{X}_0 - \hat{\mathbf{X}} \right|\right|\right| < \frac{C}{2} \|\mathcal{A}(\mathbf{X}_0 - \hat{\mathbf{X}})\|. \tag{2.25}$$

Since $\|\mathcal{A}(\mathbf{X}_0) - \mathbf{y}\| \leq \varepsilon$ and $\|\mathcal{A}(\hat{\mathbf{X}}) - \mathbf{y}\| \leq \varepsilon$, by triangle inequality we have $\|\mathcal{A}(\mathbf{X}_0) - \mathcal{A}(\mathbf{X}^*)\| \leq 2\varepsilon$. This and (2.25) imply

$$\frac{2}{C} \|\mathbf{X}_0 - \hat{\mathbf{X}}\| < \|\mathcal{A}(\mathbf{X}_0 - \hat{\mathbf{X}})\| = \|\mathcal{A}(\mathbf{X}_0) - \mathcal{A}(\hat{\mathbf{X}})\| \leq 2\varepsilon, \tag{2.26}$$

where the equality is due to the linearity of the $\mathcal{A}$.

To prove the converse, suppose that (2.20) does not hold. Then there exists $\mathbf{W}$ such that $\|\mathbf{W}_{[1:k]}\|_* \geq \|\mathbf{W} - \mathbf{W}_{[1:k]}\|_*$, but

$$|||\mathbf{W}||| \geq \frac{C}{2} \|\mathcal{A}(\mathbf{W})\|. \tag{2.27}$$

Choose $\mathbf{X}_0 := 2\frac{\varepsilon}{\eta}\mathbf{W}_{[1:k]}$, and $\hat{\mathbf{X}} := -2\frac{\varepsilon}{\eta}\big(\mathbf{W} - \mathbf{W}_{[1:k]}\big)$, where $\eta := \|\mathcal{A}\,(\mathbf{W})\|$, and $\mathbf{y} := \dfrac{\mathcal{A}(\mathbf{X}_0) + \mathcal{A}(\hat{\mathbf{X}})}{2}$. Then, $\|\mathcal{A}(\mathbf{X}_0) - \mathbf{y}\| = \|\mathcal{A}(\hat{\mathbf{X}}) - \mathbf{y}\| \leq \varepsilon$. However,

$$\begin{aligned}\left\|\left|\mathbf{X}_0 - \hat{\mathbf{X}}\right|\right\| &= \frac{2\varepsilon}{\eta}\|\|\mathbf{W}\|\| \\ &\geq \frac{2\varepsilon}{\eta}\frac{C}{2}\|\mathcal{A}(\mathbf{W})\| = C\varepsilon.\end{aligned}$$

This completes the proof. $\square$

Lemma 2.2.3. *Let $\mathbf{X}_0, \hat{\mathbf{X}} \in \mathbb{R}^{m\times n}$ such that $\|\hat{\mathbf{X}}\|_* \leq \|\mathbf{X}_0\|_*$ and $\mathcal{A}(\hat{\mathbf{X}}) = \mathcal{A}(\mathbf{X}_0)$. Then, for $C > 0$ large enough,*

$$\left\|\left|\mathbf{X}_0 - \hat{\mathbf{X}}\right|\right\| < \frac{C}{\sqrt{k}}\|\mathbf{X}_0 - \mathbf{X}_{0[1:k]}\|_* \tag{2.28}$$

if and only if for all $\mathbf{W} \in \mathrm{Null}(\mathcal{A})$,

$$\|\mathbf{W} - \mathbf{W}_{[1:k]}\|_* - \|\mathbf{W}_{[1:k]}\|_* > \frac{2\sqrt{k}}{C}\|\|\mathbf{W}\|\|. \tag{2.29}$$

This lemma differs from [107] as the Frobenius norm is replaced by unitarily invariant norm in (2.28) and (2.29).

Proof. Suppose that for every $\mathbf{W} \in \mathrm{Null}(\mathcal{A})$, (2.29) holds.
Let $\mathbf{X}_0, \hat{\mathbf{X}}$ be as in the assumption and let $\mathbf{W} = \mathbf{X}_0 - \hat{\mathbf{X}}$. Then $\mathbf{W} \in \mathrm{Null}(\mathcal{A})$. We have

$$\begin{aligned}\|\mathbf{X}_0 - \mathbf{X}_{0[1:k]}\|_* &= \|\mathbf{X}_0 - \mathbf{W} + \mathbf{W} - \mathbf{X}_{0[1:k]}\|_* \\ &\geq \|\mathbf{W} - \mathbf{X}_{0[1:k]}\|_* - \|\mathbf{X}_0 - \mathbf{W}\|_* \\ &\geq \sum_{\ell=1}^{m}\big|\sigma_\ell(\mathbf{W}) - \sigma_\ell(\mathbf{X}_{0[1:k]})\big| - \|\mathbf{X}_0 - \mathbf{W}\|_* \\ &= \sum_{\ell=1}^{k}\big|\sigma_\ell(\mathbf{W}) - \sigma_\ell(\mathbf{X}_{0[1:k]})\big| + \|\mathbf{W}_{[k+1:n]}\|_* - \|\hat{\mathbf{X}}\|_* \\ &\geq \|\mathbf{X}_{0[1:k]}\|_* - \|\mathbf{W}_{[1:k]}\|_* + \|\mathbf{W}\|_* - \|\mathbf{W}_{[1:k]}\|_* - \|\hat{\mathbf{X}}\| \\ &= \|\mathbf{X}_{0[1:k]}\|_* - \|\mathbf{W}_{[1:k]}\|_* + \|\mathbf{W} - \mathbf{W}_{[1:k]}\|_* - \|\hat{\mathbf{X}}\| \\ &\geq \|\mathbf{X}_{0[1:k]}\|_* - \|\mathbf{X}_0\|_* + \|\mathbf{W} - \mathbf{W}_{[1:k]}\|_* - \|\mathbf{W}_{[1:k]}\|_* \\ &> \|\mathbf{X}_{0[1:k]}\|_* - \|\mathbf{X}_0\|_* + \frac{2\sqrt{k}}{C}\left\|\left|\mathbf{X}_0 - \hat{\mathbf{X}}\right|\right\| \\ &\geq -\|\mathbf{X}_{0[1:k]} - \mathbf{X}_0\|_* + \frac{2\sqrt{k}}{C}\left\|\left|\mathbf{X}_0 - \hat{\mathbf{X}}\right|\right\|.\end{aligned}$$

This implies $\left\|\left|\mathbf{X}_0 - \hat{\mathbf{X}}\right|\right\| < \dfrac{C}{\sqrt{k}}\|\mathbf{X}_0 - \mathbf{X}_{0[1:k]}\|_*$.

To prove the converse, suppose (2.29) does not hold. After some algebraic manipulations and using the equivalence of norms, we get

$$(C-\alpha)\|\mathbf{W}-\mathbf{W}_{[1:k]}\|_* \le (C+\alpha)\|\mathbf{W}_{[1:k]}\|_*,$$

for some $\alpha > 0$. Here $\alpha := 2\sqrt{k}\alpha'$, where α' is a fixed number depending on the equivalence of $|||\cdot|||$ and $\|\cdot\|_*$. We choose C large enough so that $C-\alpha > 0$.

Let $\mathbf{X}_0 := -(C+\alpha)\mathbf{W}_{[1:k]}$ and $\hat{\mathbf{X}} := (C-\alpha)\mathbf{W} + \mathbf{X}_0$. Then $\|\hat{\mathbf{X}}\|_* \le (C-\alpha)\|\mathbf{W} - \mathbf{W}_{[1:k]}\|_* - 2\alpha\|\mathbf{W}_{[1:k]}\|_* \le (C+\alpha)\|\mathbf{W}_{[1:k]}\|_* = \|\mathbf{X}_0\|_*$. So, $\|\hat{\mathbf{X}}\|_* \le \|\mathbf{X}_0\|_*$ and $\|\mathcal{A}(\hat{\mathbf{X}}) - \mathbf{y}\| \le \varepsilon$, but $\left|\left|\left|\mathbf{X}_0 - \hat{\mathbf{X}}\right|\right|\right| \ge \dfrac{C}{\sqrt{k}}\|\mathbf{X}_0 - \mathbf{X}_{0[1:k]}\|_*$. □

Lemma 2.2.4. *[107, Lemma 1] Let $C > 1$ be a constant. Let $\mathbf{X}_0, \hat{\mathbf{X}} \in \mathbb{R}^{m\times n}$ such that $\|\hat{\mathbf{X}}\|_* \le \|\mathbf{X}_0\|_*$ and $\mathcal{A}(\mathbf{X}_0) = \mathcal{A}(\hat{\mathbf{X}})$. Then*

$$\|\mathbf{X}_0 - \hat{\mathbf{X}}\|_* < 2C\|\mathbf{X}_0 - \mathbf{X}_{0[1:k]}\|_* \tag{2.30}$$

if and only if for all $\mathbf{W} \in \mathrm{Null}(\mathcal{A})$,

$$\|\mathbf{W}_{[1:k]}\|_* < \frac{C-1}{C+1}\|\mathbf{W} - \mathbf{W}_{[1:k]}\|_* \tag{2.31}$$

In the following remark, we make a minor observation about the above lemma.

Remark 2.2.5. *If the nuclear norm in the above lemma is replaced by $\|.\|_p$, where $p \ge 2$, $p \in \mathbb{N}$, then the above lemma does not hold. We show this by giving a set of counterexamples. Suppose $\mathrm{Null}(\mathcal{A}) = \{\alpha\,\mathrm{Diag}(\mathbf{e}) : \alpha \in \mathbb{R}\}$. Let $p \in \mathbb{N}, p \ge 2$, $n \in \mathbb{N}$, $n \ge 2$, $C = (1 + 1/(n-1)^{1/p})/(1 - 1/(n-1)^{1/p}) + 0.02$, and $k = 1$. For all $\mathbf{W} \in \mathrm{Null}(\mathcal{A})$ (2.31) holds. However for $\mathbf{X}_0 = \mathrm{Diag}[p\ \ (1/2^p) \times ones(1, n-1)]$ and $\hat{\mathbf{X}} = [(p-1)\ \ (-1 + (1/2^p)) \times ones(1, n-1)]$, (2.30) does not hold.*

Proof. To see this, let $p \ge 2$ and $n \ge 2$ be arbitrarily chosen. Note that $\|\mathbf{X}_0 - \hat{\mathbf{X}}\|_p = n^{\frac{1}{p}}$ and $\|\mathbf{X}_0 - \mathbf{X}_0^k\|_p = (\dfrac{n-1}{2^{p^2}})^{1/p}$.

Let $C^* := 2^{\frac{p^2-p}{p}}\left(\dfrac{n^{1/2}}{n-1}\right)^{\frac{1}{p}}$. Since $2^{\frac{p^2-p}{p}}\left(\dfrac{n^{1/2}}{n-1}\right)^{\frac{1}{p}} < 2^{\frac{p^2-p}{p}}\left(\dfrac{n}{n-1}\right)^{\frac{1}{p}}$.

As $(2C^*)^p\|\mathbf{X}_0 - \mathbf{X}_0^k\|_p^p = (2C^*)^p\left(\dfrac{n-1}{2^{p^2}}\right) = n^{1/2} < n = \|\mathbf{X}_0 - \hat{\mathbf{X}}\|_p^p$ for $n \ge 2$, we conclude (2.30) does not hold. □

2.2.1 Null-space Based Recovery Results with Schatten-q Quasi Norm Minimization

Definition 2.2.1 (Schatten-q Quasi Norm)**.** Let $\mathbf{x} \in \mathbb{R}^n$ and $q \in (0, 1)$. The ℓ_q quasi-norm of $\mathbf{x}$ is defined as

$$\|\mathbf{x}\|_q := \left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q}.$$

Define $\sigma : \mathbb{R}^{m\times n} \to \mathbb{R}^{\min\{m,n\}}$ by

$$\sigma(\mathbf{X}) := \begin{bmatrix}\sigma_1(\mathbf{X}) & \cdots & \sigma_{\min\{m,n\}}(\mathbf{X})\end{bmatrix}^\top.$$

Then, for every $\mathbf{X} \in \mathbb{R}^{m\times n}$, ℓ_q quasi-norm is defined as

$$\|\mathbf{X}\|_q := \|\sigma(\mathbf{X})\|_q.$$

Consider the following minimization problems:

$$\begin{aligned} &\min \ ||\mathbf{X}||_* \\ &\text{s. t}\ \mathcal{A}(\mathbf{X}) = \mathbf{y} \qquad (P_*), \end{aligned}$$

and

$$\begin{aligned} &\min \ ||\mathbf{X}||_q \\ &\text{s. t}\ \mathcal{A}(\mathbf{X}) = \mathbf{y}, \qquad (P_q) \end{aligned}$$

where $q \in (0, 1)$.

ℓ_q quasi-norm has been considered as a sparseness measure and been used to recover sparse vectors [54]. Although ℓ_q norm is concave, many numerical algorithms exist to approximately solve (P_q) (see for example [45]). Experimentally, it has been shown that ℓ_q quasi-norm based minimization algorithms may give better recovery performance compared to some ℓ_1-norm based minimization algorithms [45].

Below we provide a similar matrix robustness result for Schatten-q quasi norm.

Lemma 2.2.6. *Let $C > 1$ and $q \in (0, 1)$. For every pair $\mathbf{X}_0, \hat{\mathbf{X}} \in \mathbb{R}^{m\times n}$ with $\|\hat{\mathbf{X}}\|_q \le \|\mathbf{X}_0\|_q$ such that $\mathcal{A}(\hat{\mathbf{X}}) = \mathcal{A}(\mathbf{X}_0)$,*

$$\|\mathbf{X}_0 - \hat{\mathbf{X}}\|_q < (2C)^{1/q}\|\mathbf{X}_0 - \mathbf{X}_{0[1:k]}\|_q \tag{2.32}$$

if for every $\mathbf{W} \in \mathrm{Null}(\mathcal{A})$

$$\|\mathbf{W}_{[1:k]}\|_q < (\frac{C-1}{2C})^{1/q}\|\mathbf{W}\|_q. \tag{2.33}$$

Proof. Suppose that for all $\mathbf{W} \in \mathrm{Null}(\mathcal{A})$, (2.33) holds. Let $\mathbf{X}_0, \hat{\mathbf{X}}$ be as in the assumption, and let $\mathbf{W} = \mathbf{X}_0 - \hat{\mathbf{X}}$. Then $\mathbf{W} \in \mathrm{Null}(\mathcal{A})$.

$$\begin{aligned}
\|\mathbf{X}_0 - \mathbf{X}_{0[1:k]}\|_q^q &\ge \|\mathbf{X}_{0[1:k]} - \mathbf{W}\|_q^q - \|\mathbf{X}_0 - \mathbf{W}\|_q^q && (2.34)\\
&\ge \sum_{i=1}^n |\sigma_i^q(\mathbf{X}_{0[1:k]}) - \sigma_i^q(\mathbf{W})| - \|\mathbf{X}_*\|_q^q && (2.35)\\
&\ge \sum_{i=1}^k (\sigma_i^q(\mathbf{X}_{0[1:k]}) - \sigma_i^q(\mathbf{W})) + \sum_{i=k+1}^n \sigma_i^q(\mathbf{W}) - \|\mathbf{X}_0\|_q^q && (2.36)\\
&\ge \|\mathbf{X}_{0[1:k]}\|_q^q - \|\mathbf{X}_0\|_q^q + \sum_{i=1}^n \sigma_i^q(\mathbf{W}) - 2\sum_{i=1}^k \sigma_i^q(\mathbf{W}) && (2.37)\\
&\ge -\|\mathbf{X}_0 - \mathbf{X}_{0[1:k]}\|_q^q + \|\mathbf{W}\|_q^q - \frac{1}{C}\|\mathbf{W}\|_q^q + \frac{1}{C}\|\mathbf{W}\|_q^q - 2\|\mathbf{W}_{[1:k]}\|_q^q && (2.38)\\
&\ge -\|\mathbf{X}_0 - \mathbf{X}_{0[1:k]}\|_q^q + \frac{1}{C}\|\mathbf{W}\|_q^q && (2.39)
\end{aligned}$$

Here, (2.34) follows since $(\cdot)^q$, where $q \in (0,1)$ is subadditive on $\mathbb{R}_+$. (2.35) follows by the generalization of Mirsky's inequality for $f := (\cdot)^q$, which will be discussed later in Theorem 2.2.8. The last inequality (2.38) holds since $\mathbf{W}$ satisfies (2.33). The last inequality implies (2.32).

$\square$

In addition, it follows immediately from Theorem 2.0.11 that if every k-rank matrix can be uniquely recovered from (P_r), then every k-rank matrix can be uniquely recovered from (P_q) whenever $0 < q < r \leq 1$.

In [107], Oymak et al. gave a necessary condition for the exact recovery of low rank matrices by solving (P_q) [107]. Their result attracted a lot of interest because the sufficiency of the condition depended on a well-known conjecture in linear algebra [9, Conjecture 6].

Theorem 2.2.7 (Theorem 3, [107])**.** *Let* $\operatorname{rank}(\mathbf{X}_0) = k$ *and let* $\bar{\mathbf{X}}$ *denote the global minimizer of* (P_q)*. For every* $\mathbf{W} \in \operatorname{Null}(\mathbf{A})$,

$$\sum_{i=1}^{k} \sigma_i^q(\mathbf{W}) < \sum_{i=k+1}^{m} \sigma_i^q(\mathbf{W}), \tag{2.40}$$

if every matrix $\mathbf{X}_0$ *with* $\operatorname{rank}(\mathbf{X}_0) \leq k$ *can be exactly recovered by solving* (P_q)*.*

Note that (2.40) with $p = 1$ implies (2.40) for $p \in (0,1)$. In fact the unique minimizer of (P_*) is the rank k solution of $\mathcal{A}(\mathbf{X}) = \mathbf{y}$, which is also the unique minimizer of (P_q).

The authors conjectured that this condition is also sufficient [107]. The proof of their conjecture (now a theorem) depends on the validity of a special version of the conjecture which was posed by W. Miao and appears in [9, Conjecture 6]. For the convenience of the reader, we restate this below.

Theorem 2.2.8. *([9, Conjecture 6],[8]) Let* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$*, and let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be a concave function satisfying* $f(0) = 0$*. Also, let* $\{i_1, \ldots, i_k\}$ *be an increasing subsequence of* $\{1, \ldots, \min\{m,n\}\}$*. Then*

$$\sum_{k=1}^{\min\{m,n\}} |f(\sigma_{i_k}(\mathbf{A})) - f(\sigma_{i_k}(\mathbf{B}))| \leq \sum_{k=1}^{\min\{m,n\}} f(\sigma_i(\mathbf{A} - \mathbf{B})). \tag{2.41}$$

The inequality (2.41) corresponds to the well-known *Mirsky's inequalities* on singular values [96, Theorem 5] if we choose $f(x) = |x|$. A version of this inequality without the absolute values on the left hand side was proved in [132, Corollary 4.5]. The generalization of the Mirsky's signular value inequalities (2.41) was proved by Audenaert in 2014 [8] (although the result has not been published yet) utilizing the Thompson-Freede inequalites [127].

The perturbation inequalities such as (2.41) arise in many areas such as perturbation theory and compressed sensing. In the next chapter, we study such eigenvalue based inequalities including the generalized Mirsky's inequality.

Chapter 3

Eigenvalue Inequalities

In this chapter, we study inequalities relating the eigenvalues of a sum of two Hermitian matrices to the eigenvalues of its summands. In addition, we look into some singular value inequalities which has applications to compressed sensing.

3.1 Eigenvalue Inequalities for a Sum of Two Hermitian Matrices

Throughout this chapter we will assume that the eigenvalues of Hermitian matrices are indexed in nonincreasing order. For example, if $\lambda_1(\mathbf{A}), \ldots, \lambda_n(\mathbf{A})$ are the eigenvalues of an n-by-n Hermitian matrix $\mathbf{A}$ then

$$\lambda_1(\mathbf{A}) \geq \cdots \geq \lambda_n(\mathbf{A}).$$

Whenever the dependence is obvious we will only use λ_i to denote the ith eigenvalue. Also, we will denote the vector formed by these eigenvalues as

$$\boldsymbol{\lambda}(\mathbf{A}) := \begin{bmatrix} \lambda(\mathbf{A})_1 & \cdots & \lambda_n(\mathbf{A}) \end{bmatrix}^\top \quad \text{so that} \quad \boldsymbol{\lambda} : \mathbb{H}^n \to \mathbb{R}^n.$$

Characterization of the eigenvalues of a Hermitian matrix sum is an old problem: Given two Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$ what can be said about the spectrum of their sum $\mathbf{A}+\mathbf{B}$? If the second matrix has very small entries then the problem can also be viewed as finding the spectrum of the perturbation of $\mathbf{A}$ in terms of $\mathbf{A}$ and the perturbation.

This problem has been studied by many well-known mathematicians [85, 139, 14, 66]. In 1950, Lidskiĭ who was a student of Gelfand showed that the vector formed from the eigenvalues of the sum of two Hermitian matrices say $\boldsymbol{\lambda}(\mathbf{A}+\mathbf{B})$ is in the convex hull of $\boldsymbol{\lambda}(\mathbf{A}) + \mathbf{P}\boldsymbol{\lambda}(\mathbf{B})$ where $\mathbf{P}$ varies over all n-by-n permutation matrices [85]. As pointed out by Fulton [48], in 1956 in [14], Berezin and Gelfand, also proved the same result. As opposed to Lidskiĭ's proof which uses more elementary techniques [85], the proof in [14] is more advanced and is based on the techniques from the representation theory of Lie groups and Lie algebras.

Theorem 3.1.1 (Lidskiĭ)**.** *Let* $\mathbf{A}, \mathbf{B}$ *be n-by-n Hermitian matrices. Then the vector*

$$\boldsymbol{\lambda}(\mathbf{A}+\mathbf{B}) := \begin{bmatrix}\lambda(\mathbf{A}+\mathbf{B})_1 & \cdots & \lambda_n(\mathbf{A}+\mathbf{B})\end{bmatrix}^\top$$

is in the convex hull of

$$\{\boldsymbol{\lambda}(\mathbf{A}) + \boldsymbol{\lambda}(\mathbf{B})\mathbf{P} : \mathbf{P} \text{ is an } n\text{-by-}n \text{ permutation matrix}\}.$$

Later in 1955, Wielandt showed that this geometric condition given in Theorem 3.1.1 is equivalent to the following set of inequalities

$$\sum_{i\in J} \lambda_i(\mathbf{A}+\mathbf{B}) \leq \sum_{i\in J} \lambda_i(\mathbf{A}) + \sum_{r=1}^{k} \lambda_r(\mathbf{B}),$$

for every subset J of $\{1, \dots, n\}$ with cardinality r [139, Theorem 2]. Wielandt's proof is based on a minmax principle.

Theorem 3.1.2 (Wielandt, [139])**.** *Let* $\mathcal{R}_k$ *denote a k-dimensional inner product space over the field of complex numbers, and let* $1 \leq i_1 < \cdots < i_k \leq n$*. Then*

$$\lambda_{i_1}(\mathbf{A}) + \cdots + \lambda_{i_k}(\mathbf{A}) = \max_{\substack{\mathcal{R}_{i_1} \subseteq \cdots \subseteq \mathcal{R}_{i_k} \\ \dim(\mathcal{R}_{i_j}) = i_j}} \min_{\substack{\mathbf{x}_j \in \mathcal{R}_{i_j}, \\ \mathbf{x}_j^H \mathbf{x}_\ell = \delta_{j\ell}}} \sum_{j=1}^{k} \mathbf{x}_j^H \mathbf{A} \mathbf{x}_j,$$

where $\delta_{j\ell} := \begin{cases} 1, & \text{if } j = \ell, \\ 0, & \text{otherwise} \end{cases}$.

Theorem 3.1.3 (Wielandt)**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be n-by-n Hermitian matrices. If* $1 \leq i_1 < \cdots < i_k \leq n$*, then*

$$\sum_{r=1}^{k} \lambda_{i_r}(\mathbf{A}+\mathbf{B}) \leq \sum_{r=1}^{k} \lambda_{i_r}(\mathbf{A}) + \sum_{r=1}^{k} \lambda_r(\mathbf{B}). \tag{3.1}$$

Thompson and Freede generalized *Wielandt-Lidskiii inequalities* (3.1) [127], which will be discussed in the next section in more detail.

In his famous paper [66], Horn raised the following question: For which nonincreasing sequences $\{\alpha_i\}_{i=1}^n$,$\{\beta_i\}_{i=1}^n$ and $\{\gamma_i\}_{i=1}^n$ do there exist Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$ such that and $\mathbf{A}, \mathbf{B}$ and $\mathbf{A}+\mathbf{B}$ have $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ and $\{\gamma_i\}_{i=1}^n$ as their respective eigenvalues? He conjectured the following.

Theorem 3.1.4 (Horn's Conjecture)**.** *Sequences* $\{\alpha_i\}_{i=1}^n$,$\{\beta_i\}_{i=1}^n$ *and* $\{\gamma_i\}_{i=1}^n$ *occur as eigenvalues of Hermitian n-by-n matrices* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{A}+\mathbf{B}$ *respectively, if and only if*

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i,$$

and the inequalities

$$\sum_{k\in K}\gamma_k \le \sum_{i\in I}\alpha_i + \sum_{j\in J}\beta_j$$

hold for every $(I,J,K)\in T_r^n$ *and for every* $r<n$*, where* $I := \{i_1 < \cdots < i_r\}$*,* $J := \{j_1 < \cdots < j_r\}$ *and* $K := \{k_1 < \cdots < k_r\}$*. The set* T_r^n *is defined by the following inductive procedure. Define*

$$U_r^n := \left\{(I,J,K) : \sum_{i\in I} i + \sum_{j\in J} j = \frac{r(r+1)}{2} + \sum_{k\in K} k\right\}.$$

For $r=1$ *set* $T_1^n = U_1^n$*. If* $n\ge 2$*,*

$$T_r^n := \left\{(I,J,K)\in U_r^n : \text{ for all } p<r \text{ and } (F,G,H)\in T_p^r,\ \sum_{f\in F} i_f + \sum_{g\in G} j_g \le \sum_{h\in H} k_h + \frac{p(p+1)}{2}\right\}.$$

Horn proved his conjecture for $n\in\{3,4\}$ [66]. For general n, Horn's conjecture was proved by Knutson and Tao in [78] following the result of Klyachko [77].

Example 3.1.1. Let $\alpha_1\ge\alpha_2\ge\alpha_3$, $\beta_1\ge\beta_2\ge\beta_3$ and $\gamma_1\ge\gamma_2\ge\gamma_3$ be the eigenvalues of Hermitian 3-by-3 matrices $\mathbf{A},\mathbf{B}$ and $\mathbf{C} := \mathbf{A}+\mathbf{B}$, respectively. Then Horn's inequalities consist of 13 inequalities:

$$\gamma_1 \le \alpha_1+\beta_1 \tag{3.2}$$
$$\gamma_2 \le \alpha_1+\beta_2 \tag{3.3}$$
$$\gamma_2 \le \alpha_2+\beta_1 \tag{3.4}$$
$$\gamma_3 \le \alpha_1+\beta_3 \tag{3.5}$$
$$\gamma_3 \le \alpha_3+\beta_1 \tag{3.6}$$
$$\gamma_3 \le \alpha_2+\beta_2 \tag{3.7}$$
$$\gamma_1+\gamma_2 \le \alpha_1+\alpha_2+\beta_1+\beta_2 \tag{3.8}$$
$$\gamma_1+\gamma_3 \le \alpha_1+\alpha_2+\beta_1+\beta_3 \tag{3.9}$$
$$\gamma_1+\gamma_3 \le \alpha_1+\alpha_3+\beta_1+\beta_2 \tag{3.10}$$
$$\gamma_2+\gamma_3 \le \alpha_1+\alpha_2+\beta_2+\beta_3 \tag{3.11}$$
$$\gamma_2+\gamma_3 \le \alpha_1+\alpha_3+\beta_1+\beta_3 \tag{3.12}$$
$$\gamma_2+\gamma_3 \le \alpha_2+\alpha_3+\beta_1+\beta_2 \tag{3.13}$$
$$\sum_{i=1}^{3}\gamma_i = \sum_{i=1}^{3}\alpha_i + \sum_{i=1}^{3}\beta_i \tag{3.14}$$

When the eigenvalues of $\mathbf{B}$ is a permutation of the eigenvalues of $\mathbf{A}$ then $\beta_i=\alpha_i$ for every i, and since (3.4), (3.6), (3.10) and (3.13) are redundant, the Horn inequalities reduce to the following system:

$$\begin{aligned}
\gamma_1 &\leq 2\alpha_1 \\
\gamma_2 &\leq \alpha_1 + \alpha_2 \\
\gamma_3 &\leq \alpha_1 + \alpha_3 \\
\gamma_3 &\leq 2\alpha_2 \\
\gamma_1 + \gamma_2 &\leq 2\alpha_1 + 2\alpha_2 \\
\gamma_1 + \gamma_3 &\leq 2\alpha_1 + \alpha_2 + \alpha_3 \\
\gamma_2 + \gamma_3 &\leq \alpha_1 + 2\alpha_2 + \alpha_3 \\
\gamma_2 + \gamma_3 &\leq 2\alpha_1 + 2\alpha_3 \\
\sum_{i=1}^{3} \gamma_i &= 2\sum_{i=1}^{3} \alpha_i.
\end{aligned}$$

One way to check whether one of the above inequalities, LHS $\leq$ RHS, is redundant is to solve the linear program where the objective is to maximize LHS $-$ RHS subject to the rest of the inequalities. If the optimum value is nonpositive, this implies that the inequality is redundant, otherwise it is not. However, checking each inequality one by one with this approach may not be efficient when n is large.

It is well-known that for large $n \geq 6$, Horn inequalities are not minimal and as n increases so does the number of redundant inequalities. In some problems, the eigenvalues of $\mathbf{A}$ are a permutation of the eigenvalues of $\mathbf{B}$. An interesting direction here is to investigate the set of minimal Horn inequalities and understand how the number of redundant Horn inequalities change as n increases.

3.2 Mirsky's Inequality and Its Generalizations

The purpose of this section is to study Mirsky's inequalities and some of its recent generalizations. First, we present some basic concepts and results. Some of the classical eigenvalue inequalities like Weilandt and Mirsky can be explained more elegantly through majorization. We refer the reader to the book [93] by Marshall, Olkin and Arnold for a comprehensive reference on majorization and its applications.

Definition 3.2.1 (Majorization)**.** For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we say that $\mathbf{x}$ is *majorized* by $\mathbf{y}$ (or $\mathbf{y}$ *majorizes* $\mathbf{x}$), and denote by $\mathbf{x} \prec \mathbf{y}$, if

$$\begin{aligned}
\sum_{i=1}^{k} x_{[i]} &\leq \sum_{i=1}^{k} y_{[i]}, \quad k \in \{1, \ldots, n-1\}, \\
\sum_{i=1}^{n} x_{[i]} &= \sum_{i=1}^{n} y_{[i]}.
\end{aligned}$$

Theorem 3.2.1 (Hardy-Littlewood-Polya)**.** *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ *satisfy* $x_1 \geq \cdots \geq x_m$*, and* $y_1 \geq \cdots \geq y_m$*. Then, there exists a doubly stochastic matrix* $\mathbf{S}$ *with* $\mathbf{y} = \mathbf{S}\mathbf{x}$ *if and only if* $\mathbf{y} \prec \mathbf{x}$*.*

One of the well-known results on majorization is due to Schur. In 1923, Schur showed that the eigenvalues of a Hermitian matrix $\mathbf{A} \in \mathbb{H}^n$, say $\boldsymbol{\lambda}$, majorizes its diagonal elements $\mathbf{d} := \begin{bmatrix} a_{11} & \cdots & a_{nn} \end{bmatrix}^\top$, i.e., $\mathbf{d} \prec \boldsymbol{\lambda}$. By the spectral decomposition of $\mathbf{A} := \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H$, one can show that $a_{ii} = \sum_{j=1}^n \lambda_j |u_{ij}|^2$, where u_{ij} is the ijth entry of the unitary matrix $\mathbf{U}$. Then, the result follows from Hardy-Littlewood-Polya Theorem.

Definition 3.2.2 (Symmetric gauge function)**.** A function $\phi : \mathbb{R}^n \to \mathbb{R}_+$ is a *symmetric gauge function* if

(i) ϕ is a norm on $\mathbb{R}^n$,

(ii) $\phi(|x_1|, \ldots, |x_n|) = \phi(x_1, \ldots, x_n)$.

(iii) $\phi(\mathbf{x}) = \phi(\mathbf{P}\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^n$, for all permutation matrices $\mathbf{P} \in \{0,1\}^{n\times n}$.

Definition 3.2.3 (Schur-convexity)**.** A real valued function f on $\mathbb{R}^n$ is called *Schur-convex* or *S-convex*, if $\mathbf{x} \prec \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$.

Lemma 3.2.2 (Fan's Lemma, [122])**.** *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$ *satisfy*

$$x_1 \geq \cdots \geq x_n,$$

and

$$y_1 \geq \cdots \geq y_n.$$

Then

$$\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i, \ \forall k \in \{1, \ldots, n\}$$

if and only if

$$\phi(\mathbf{x}) \geq \phi(\mathbf{y})$$

for all symmetric gauge functions ϕ *on* $\mathbb{R}^n$*.*

By Fan's Lemma one can observe that every symmetric gauge function is also Schur-convex. In 1953, Wielandt and Hoffman showed that for n-by-n normal matrices

$$\sum_{i=1} |\lambda_i(\mathbf{A}) - \lambda_i(\mathbf{B})|^2 \leq \|\mathbf{A} - \mathbf{B}\|_F^2.$$

Later, in 1960, Mirsky generalized this to unitarily invariant norms for Hermitian matrices, i.e., he showed

$$|||\mathrm{Diag}(\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{B}), \ldots, \lambda_n(\mathbf{A}) - \lambda_n(\mathbf{B})||| \leq |||\mathbf{A} - \mathbf{B}|||.$$

He also showed the following for the singular values of complex matrices. Here, we provide it for real valued matrices, however the technique used below is the same provided for complex matrices in [96].

Theorem 3.2.3 (Mirsky, [96]). *Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ be matrices with singular values $\sigma_1(\mathbf{A}) \geq \cdots \geq \sigma_r(\mathbf{A})$ and $\sigma_1(\mathbf{B}) \geq \cdots \geq \sigma_r(\mathbf{B})$, respectively, where $r := \min\{m, n\}$. Then for every unitarily invariant norm $|||\cdot|||$,*

$$|||\Sigma_{\mathbf{A}} - \Sigma_{\mathbf{B}}||| \leq |||\mathbf{A} - \mathbf{B}|||.$$

As the unitarily invariant norms play an important role in Mirsky's theorem, it is important to understand which matrix norms are unitarily invariant. A famous result due to von Neumann characterizes all unitarily invariant norms as symmetric gauge functions of singular values. Therefore, Mirsky's theorem can be equivalently stated in terms of a symmetric gauge function.

Theorem 3.2.4. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ be matrices with singular values $\sigma_1(\mathbf{A}) \geq \sigma_2(\mathbf{A}) \geq \cdots \geq \sigma_r(\mathbf{A})$ and $\sigma_1(\mathbf{B}) \geq \sigma_2(\mathbf{B}) \geq \cdots \geq \sigma_r(\mathbf{B})$, respectively, where $r := \min\{m, n\}$. Then for every symmetric gauge function ϕ, we have*

$$\phi(|\sigma(\mathbf{B}) - \sigma(\mathbf{A})|) \leq \phi(\sigma(\mathbf{B} - \mathbf{A})). \tag{3.15}$$

Proof. The proof below is based on techniques from a paper by L. Mirsky [96] which borrows ideas from [139].

It is well known that the eigenvalues of $\tilde{\mathbf{A}} := \begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{A}^\top & 0 \end{bmatrix}$ and $\tilde{\mathbf{B}} := \begin{bmatrix} 0 & \mathbf{B} \\ \mathbf{B}^\top & 0 \end{bmatrix}$ are

$$\begin{aligned} &\sigma_1(\mathbf{A}) \geq \cdots \geq \sigma_r(\mathbf{A}) \geq -\sigma_r(\mathbf{A}) \geq \cdots \geq -\sigma_1(\mathbf{A}), \\ &\sigma_1(\mathbf{B}) \geq \cdots \geq \sigma_r(\mathbf{B}) \geq -\sigma_r(\mathbf{B}) \geq \cdots \geq -\sigma_1(\mathbf{B}) \end{aligned}$$

respectively. This observation is originally due to Camille Jordan (1838-1922) [67, p. 135].

Let $\mathcal{R}_{j_i}$ be the space spanned by the eigenvectors of $\tilde{\mathbf{B}}$ corresponding to the eigenvalues $\lambda_{j_1}(\tilde{\mathbf{B}}), \ldots, \lambda_{j_i}(\tilde{\mathbf{B}})$. Then for every unit norm vector $\mathbf{x} \in \mathcal{R}_{j_i}$, $\mathbf{x}^\top \mathbf{A}\mathbf{x} \geq \lambda_{j_i}(\mathbf{A})$. This implies that for every pairwise orthonormal vectors $\{\mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_k}\}$ such that $x_{j_i} \in \mathcal{R}_{j_i}$, $\lambda_{j_i}(\tilde{\mathbf{B}}) \leq \mathbf{x}_{j_i}^\top \tilde{\mathbf{B}} \mathbf{x}_{j_i}$. Hence,

$$\sum_{i=1}^{k} \lambda_{j_i}(\tilde{\mathbf{B}}) \leq \sum_{i=1}^{k} \mathbf{x}_{j_i}^\top \tilde{\mathbf{B}} \mathbf{x}_{j_i}$$

In addition, by Theorem 3.1.2, for the given subspaces $\mathcal{R}_{j_1} \subseteq \cdots \subseteq \mathcal{R}_{j_k}$, there exists an orthonormal matrix $\mathbf{Q} := \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_k \end{bmatrix}$ such that $\mathbf{q}_i \in \mathcal{R}_{j_i}$ and

$$\sum_{i=1}^{k} \mathbf{q}_i^\top \tilde{\mathbf{A}} \mathbf{x}_{\mathbf{q}_i} = \operatorname{tr}\left(\mathbf{Q}^\top \tilde{\mathbf{A}} \mathbf{Q}\right) \leq \sum_{i=1}^{k} \lambda_{j_i}(\tilde{\mathbf{A}}).$$

Then

$$\sum_{i=1}^{k} \lambda_{j_i}(\tilde{\mathbf{B}}) \leq \operatorname{tr}\left(\mathbf{Q}^\top \tilde{\mathbf{B}} \mathbf{Q}\right) \tag{3.16}$$

$$= \operatorname{tr}\left(\mathbf{Q}^\top \tilde{\mathbf{A}} \mathbf{Q}\right) + \operatorname{tr}\left(\mathbf{Q}^\top (\tilde{\mathbf{B}} - \tilde{\mathbf{A}}) \mathbf{Q}\right) \tag{3.17}$$

$$\leq \sum_{i=1}^{k} \lambda_{j_i}(\tilde{\mathbf{A}}) + \operatorname{tr}\left(\mathbf{Q}^\top (\tilde{\mathbf{B}} - \tilde{\mathbf{A}}) \mathbf{Q}\right) \tag{3.18}$$

$$\leq \sum_{i=1}^{k} \lambda_{j_i}(\tilde{\mathbf{A}}) + \sum_{i=1}^{k} \sigma_i(\mathbf{B} - \mathbf{A}) \tag{3.19}$$

Here, (3.19) follows from Cauchy's interlacing theorem (see e.g., Theorem 9.1.1 in[52]). The inequality in (3.19) holds for every subsequence $1 \leq j_1 < \cdots < j_k \leq m$. Therefore, we can choose $j_i := i$ whenever $\sigma_i(\mathbf{B}) \geq \sigma_i(\mathbf{A})$ and $j_i := i + r$ whenever $\sigma_i(\mathbf{B}) < \sigma_i(\mathbf{A})$. Then (3.19) reduces to

$$\sum_{i=1}^{k} |\sigma_i(\mathbf{B}) - \sigma_i(\mathbf{A})| \leq \sum_{i=1}^{k} \sigma_i(\mathbf{B} - \mathbf{A}).$$

The result then follows by Fan's Lemma (see Lemma 3.2.2). □

Corollary 3.2.5. *Theorem 3.2.3 (Mirsky's Theorem) follows from Theorem 3.2.4.*

Proof. Let $|||\cdot|||$ be a given unitarily invariant norm on $\mathbb{R}^{m\times n}$. Define $\phi(\mathbf{x}) := |||\mathbf{X}|||$, where $\mathbf{X}_{ii} := x_i$, and $\mathbf{X}_{ij} := 0$, if $i \neq j$. Clearly $\phi(\cdot)$ is a norm on $\mathbb{R}^m$. Since $|||\cdot|||$ is unitarily invariant,

$$\phi(\mathbf{x}) = |||\mathbf{X}||| = |||-\mathbf{X}||| = \phi(|\mathbf{x}|), \ \forall x \in \mathbb{R}^m, \ \text{(here absolute norm is taken elementwise)}$$

and

$$\phi(\mathbf{x}) = |||\mathbf{X}||| = |||\mathbf{P}\mathbf{X}||| = \phi(\mathbf{P}\mathbf{x}),$$

for every permutation matrix $\mathbf{P}$. Therefore, for every unitarily invariant norm one can find a symmetric gauge function ϕ such that $|||\mathbf{A}||| = \phi(\sigma_1(\mathbf{A}), \ldots, \sigma_m(\mathbf{A}))$. Since Theorem 3.2.3 holds for all symmetric gauge functions, and since there is a symmetric gauge function for every unitarily invariant norm, the result follows. □

The following theorem due to Lidskii can be proved using Theorem 3.1.2.

Theorem 3.2.6 (Lidskii's Inequality)**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be n-by-n real symmetric matrices. If* $1 \leq i_1 < i_2 < \cdots < i_m \leq n$*, then*

$$\sum_{k=1}^{m} \lambda_{i_k}(\mathbf{A} + \mathbf{B}) \leq \sum_{k=1}^{m} \lambda_{i_k}(\mathbf{A}) + \sum_{k=1}^{m} \lambda_k(\mathbf{B}).$$

Note that one can prove Mirsky's Theorem (Theorem 3.2.3) using Lidskii's Inequality (Theorem 3.2.6). We will state this as a corollary to Theorem 3.2.6.

Corollary 3.2.7. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ *be matrices with singular values* $\sigma_1(\mathbf{A}) \geq \sigma_2(\mathbf{A}) \geq \cdots \geq \sigma_r(\mathbf{A})$ *and* $\sigma_1(\mathbf{B}) \geq \sigma_2(\mathbf{B}) \geq \cdots \geq \sigma_r(\mathbf{B})$*, respectively, where* $r := \min\{m, n\}$ *and* $m \leq n$*. Then, for every* $\ell \in \{1, 2, \ldots, r\}$*,*

$$\sum_{i=1}^{\ell} |\sigma_i(\mathbf{A}) - \sigma_i(\mathbf{B})| \leq \sum_{i=1}^{\ell} |\sigma_i(\mathbf{A} - \mathbf{B})|. \tag{3.20}$$

Proof. Define $\tilde{\mathbf{A}} := \begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{A}^\top & 0 \end{bmatrix}$ and $\tilde{\mathbf{B}} := \begin{bmatrix} 0 & \mathbf{B} \\ \mathbf{B}^\top & 0 \end{bmatrix}$. The eigenvalues of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are $\pm\sigma_1(\mathbf{A}), \ldots, \pm\sigma_r(\mathbf{A})$ and $\pm\sigma_1(\mathbf{B}), \ldots, \pm\sigma_r(\mathbf{B})$, respectively. Furthermore, the eigenvalues of $(\tilde{\mathbf{A}} - \tilde{\mathbf{B}})$ are $\pm\sigma_1(\mathbf{A} - \mathbf{B}), \ldots, \pm\sigma_r(\mathbf{A} - \mathbf{B})$. If $\boldsymbol{\lambda}(\mathbf{A})$ denotes the vector whose entries are eigenvalues of $\mathbf{A}$ sorted in nonincreasing order, then the entries of $\boldsymbol{\lambda}(\tilde{\mathbf{A}}) - \boldsymbol{\lambda}(\tilde{\mathbf{B}})$ are $\pm(\sigma_1(\mathbf{A}) - \sigma_1(\mathbf{B})), \ldots, \pm(\sigma_r(\mathbf{A}) - \sigma_r(\mathbf{B}))$. Let

$$i_k := \begin{cases} k, & \text{if } \sigma_k(\mathbf{A}) \geq \sigma_k(\mathbf{B}), \\ k + r, & \text{otherwise.} \end{cases} \tag{3.21}$$

By Lidskii's Inequality, we have

$$\sum_{k=1}^{\ell} \lambda_{i_k}(\mathbf{A}) \leq \sum_{k=1}^{\ell} \lambda_{i_k}(\mathbf{B}) + \sum_{k=1}^{\ell} \lambda_k(\mathbf{A} - \mathbf{B}). \tag{3.22}$$

For all $k \leq \ell$, $\lambda_k(\mathbf{A} - \mathbf{B}) = \sigma_k(\mathbf{A} - \mathbf{B})$. Moreover, $\lambda_{i_k}(\mathbf{A}) - \lambda_{i_k}(\mathbf{B}) = |\sigma_k(\mathbf{A}) - \sigma_k(\mathbf{B})|$ by the choice of i_k. Substitution of these into (3.22) yields the result. □

Next, we introduce *Amir-Moéz inequalities*. Let $i_1 \leq \cdots \leq i_k \leq n$, and $j_1 \leq \cdots \leq j_k \leq n$ be sequences of positive integers, where $i_\ell + j_\ell \leq n - k + \ell + 1$, $\ell \in \{1, \ldots, k\}$ and $k \leq n$. Let $\mathbf{A}, \mathbf{B}$ be n-by-n Hermitian matrices, $\lambda_1 \geq \cdots \geq \lambda_n$, $\beta_1 \geq \cdots \geq \beta_n$ and $\gamma_1 \geq \cdots \geq \gamma_n$ be the eigenvalues of $\mathbf{A}, \mathbf{B}$ and $\mathbf{A} + \mathbf{B}$, respectively. Then the following inequalities, which are called as *Amir-Moéz inequalities*, hold:

$$\lambda_{i'_1} + \cdots + \lambda_{i'_k} + \beta_{j'_1} + \cdots + \beta_{j'_k} \geq \gamma_{i'_1 + j'_1 - 1} + \cdots + \gamma_{i'_k + j'_k - k}, \tag{3.23}$$

where $\{i'_1, \ldots, i'_k\}$ and and $\{j'_1, \ldots, j'_k\}$ are strictly increasing subsequences of $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_k\}$, respectively [4].

Theorem 3.2.8 (Thomspon - Freede Inequalities, [127])**.** *Let* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C} := \mathbf{A} + \mathbf{B}$ *be* n*-by-*n *Hermitian matrices. Then*

$$\sum_{k=1}^{m} \lambda_{i_k + j_k - k}(\mathbf{C}) \leq \sum_{k=1}^{m} \lambda_{i_k}(\mathbf{A}) + \sum_{k=1}^{m} \lambda_{j_k}(\mathbf{B}) \tag{3.24}$$

whenever $1 \leq i_1 < i_2 < \cdots < i_m \leq n$*,* $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ *and* $i_m + j_m \leq m + n$*.*

Thompson-Freede inequalities are a family of eigenvalue inequalities that are a generalization of the Lidskii inequalities. This can be easily observed from Theorem 3.2.8 by substituting $j_k = k$. These inequalities relating the eigenvalues of $\mathbf{A}, \mathbf{B}$, and $\mathbf{A} + \mathbf{B}$ (see Theorem 3.2.8) also hold when the eigenvalues of the respective matrices are replaced by their the singular values [127, Theorem 3]. Furthermore, Thompson-Freede inequalities are at least as strong as the *Amir-Moez inequalities* [127].

In the following, we present *a generalization of Mirsky's theorem*. As discussed in Chapter 2, this theorem was originally posed as a conjecture by W. Miao and appears in [9, Conjecture 6].

Theorem 3.2.9. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave, non-decreasing function such that $f(0) = 0$. Consider*

$$\sum_{i=1}^{k} |f(\sigma_i(\mathbf{A})) - f(\sigma_i(\mathbf{B}))| \leq \sum_{i=1}^{k} f(\sigma_i(\mathbf{A} - \mathbf{B})), \quad \forall k \in \{1, \ldots, \min\{m, n\}\}. \tag{3.25}$$

Note that replacing $f(x) := x$ reduces (3.25) to the well known Mirsky inequalities.

The above inequality (3.25) with $k := \min\{m, n\}$ and $f(x) = x^q$, where $q \in (0, 1)$ was of particular interest in a low-rank recovery problem [107]. The confirmation of this case, under the nullspace condition give in (2.40) implies the exact recovery of every matrix of rank at most k from linear measurements $\mathcal{A}\mathbf{X} = \mathbf{y}$ by minimizing $\|\mathbf{X}\|_q$, as in (P_q) in (2.13). A special case was reproved in [82], for positive semidefinite matrices when $f(x) = x^q$, $q \in (0, 1)$, which was previously proved in the work of Ando [5]. In [143], the authors attempted to prove the inequality (3.25) when $k = \min\{m, n\}$ to remove of the gap between the sufficient and necessary conditions for the recovery of low rank matrices by the Schatten-q quasi norm heuristic [107], however the proof is erroneous, see the discussion in [44, p. 1]. The generalization of (3.25) was proved by Audenaert in [8] where the core of the proof relied on Thompson-Freede inequalities [127]. Audenart's idea originates from the fact that every non-decreasing concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ can be represented as a finite or infinite sum of positive linear combinations of "hook" functions $h : \mathbb{R} \to \mathbb{R}$, $h_t(x) := \min\{t, x\}$ for some $t > 0$. Then the author reduces the problem to showing (3.25) for $f := h_t$. The rest of the argument follows from application of the Thompson-Freede inequalities [127] for singular values. Recently, Foucart presented a shorter proof for $k := \min\{m, n\}$, following the idea of Audenaert. The key result that shortened his proof was to show the sufficiency of proving the Thompson-Freede inequalities for $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} - \mathbf{B}$ is of rank one [44].

In the following, we state a generalized version of Thompson-Freede inequalities.

Theorem 3.2.10. *[8, Theorem 2] Let $\{\alpha(i)\}_{i=1}^n$, $\{\beta(i)\}_{i=1}^n$ and $\{\gamma(i)\}_{i=1}^n$ be nonnegative sequences sorted in non-increasing order and satisfy Thompson-Freede inequalities. If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is concave, non-decreasing, and $f(0) = 0$, then for every $1 \leq m \leq n$*

$$\sum_{k=1}^{m} f(\gamma(i_k + j_k - k)) \leq \sum_{k=1}^{m} f(\alpha(i_k)) + \sum_{k=1}^{m} f(\beta(j_k)), \tag{3.26}$$

whenever $i_1 < \cdots < i_m$, $j_1 < \cdots < j_m$ and $i_m + j_m \leq m + n$.

The ideas of this proof are useful in the proof of generalized Mirsky's inequality for singular values. Therefore we include it here.

Proof. The proof is due to Audenaert [8]. Any function f that is concave, non-decreasing and nonnegative on $\mathbb{R}_+$ can be expressed as a finite or infinite positive linear combination of functions $h_t(x) := \min\{t, x\}$ i.e., $f(x) = \int_0^\infty h_t(x)dt$. By linearity of the integral and since multiplying an inequality by a positive number does not change the sign of the inequality, it suffices to prove the inequality (3.26) when $f(x) := h(x) = \min\{1, x\}$. In the rest we show that for a given m, $i_1 < \cdots < i_m$, $j_1 < \cdots < j_m$ and $i_m + j_m \leq m + n$

$$\sum_{k=1}^{m} h(\gamma(i_k + j_k - k)) \leq \sum_{k=1}^{m} h(\alpha(i_k)) + \sum_{k=1}^{m} h(\beta(j_k)). \tag{3.27}$$

Suppose $\alpha(i_k) \geq 1$ for every k, then the RHS of (3.27) becomes $m + \sum_{k=1}^m h(\beta(j_k))$. Since $\sum_{k=1}^m h(\gamma(i_k + j_k - k)) \leq \sum_{k=1}^m 1 = m$ and $\sum_{k=1}^m h(\beta(j_k)) \geq 0$, LHS $\leq$ RHS. So without loss of generality we may assume both sequences $\{\alpha(i_k)\}$ and $\{\beta(j_k)\}$ have entries less than 1. If we suppose that all entries of $\alpha(i_k) < 1$ and $\beta(j_k) < 1$ for every k then the result follows, since by assumption $\{\alpha(i)\}_{i=1}^n$, $\{\beta(i)\}_{i=1}^n$, $\{\gamma(i)\}_{i=1}^n$ satisfy the Thompson-Freede inequalities and by the definition, $h(\gamma(i_k + j_k - k) \leq \gamma(i_k + j_k - k)$.

Now, assume $\alpha(i_s) < 1 \leq \alpha(i_s)$ and $\beta(j_{t+1}) < 1 \leq \beta(j_t)$ for some indices s and t. By the definition of $h(x) = 1$ if $x \geq 1$ and $h(x) = x$, otherwise. Then

$$\text{RHS} := \sum_{k=1}^{m} h(\alpha(i_k)) + \sum_{k=1}^{m} h(\beta(j_k)) = (s-1) + \sum_{k=s}^{m} \alpha(i_k) + (t-1) + \sum_{k=t}^{m} \beta(j_k).$$

Since $h(x) \leq 1$ for every x, we also have

$$\sum_{k=1}^{s+t-2} h(\gamma(i_k + j_k - k) \leq s + t - 2. \tag{3.28}$$

By assumption $\{\alpha(i)\}_{i=1}^n$, $\{\beta(i)\}_{i=1}^n$ and $\{\gamma(i)\}_{i=1}^n$ satisfies the Thompson-Freede inequalities. Then we have

$$\sum_{k=1}^{m-(s-1)+(t-1)} \gamma(i_{k+s-1} + j_{k+t-1} - k) \leq \sum_{k=1}^{m-(s-1)+(t-1)} \alpha(i_{k+s-1}) + \sum_{k=1}^{m-(s-1)+(t-1)} \beta(j_{k+t-1}),$$

where we assume $s + t - 1 \leq m$. If $s + t - 1 > m$, the above inequality holds trivially.

Furthermore, by assumption the index subsequences are strictly increasing. Therefore

$$\begin{aligned} i_{k+s-1} &\geq i_{k+s-1+(t-1)} - (t-1) \\ j_{k+t-1} &\geq j_{k+t-1+(s-1)} - (s-1). \end{aligned}$$

Summing up the above inequalities up and adding $-k$ to the both sides yield $i_{k+s-1}+j_{k+t-1}-k \geq i_{k+s-1+(t-1)} + j_{k+t-1+(s-1)} - (s-1) - (t-1)$. Therefore

$$\begin{aligned}\sum_{k=1}^{m-(s-1)+(t-1)} \gamma(i_{k+s-1} + j_{k+t-1} - k) &\geq \gamma(i_{k+s+t-2} + j_{k+s+t-2} - (k+s+t-2)) \\ &= \sum_{k=s+t-1}^{m} \gamma(i_k + j_k - k),\end{aligned}$$

Then by combining the two inequalities above we get

$$\begin{aligned}\sum_{k=s+t-1}^{m} \gamma(i_k + j_k - k) &\leq \sum_{k=1}^{m-(a-1)+(b-1)} \alpha(i_{k+s-1}) + \sum_{k=1}^{m-(a-1)+(b-1)} \beta(i_{k+t-1}) \\ &= \sum_{k=s}^{m-t+1} \alpha(i_k) + \sum_{k=t}^{m-s+1} \beta(j_k). \end{aligned} \tag{3.29}$$

This yields

$$\begin{aligned}\sum_{k=s+t-1^m} h(\gamma(i_k + j_k - k)) \leq \gamma(i_k + j_k - k) \\ &\leq \sum_{k=s}^{m-t+1} \alpha(i_k) + \sum_{k=t}^{m-s+1} \beta(j_k) \\ &\leq \sum_{k=s}^{m} \alpha(i_k) + \sum_{k=t}^{m} \beta(j_k).\end{aligned}$$

Here, the first inequality follows since $h(x) \leq x$ for every x, the second inequality is by (3.29) and the last inequality follows since both $\alpha(i_k)$ and $\beta(j_k)$ are nonnegative. Adding the last inequality and (3.28) gives the Thompson-Freede inequality for h as desired.

If $\alpha(i_s) < 1 \leq \alpha(i_s)$ for some s and $\beta(j_k) < 1$ for every k. Then the proof given above still applies by replacing $t = 1$. □

The following is a useful lemma that will be used in the proof of the generalization of Mirsky's inequality.

Lemma 3.2.11. *Let* $\mathbf{A}, \mathbf{B}$ *be* n*-by-*n *matrices such that* $\operatorname{rank}(\mathbf{A}-\mathbf{B}) = 1$*. Then*

$$\sum_{i=1}^{n} |\min\{1, \sigma_i(\mathbf{A})\} - \min\{1, \sigma_i(\mathbf{B})\}| \leq 1.$$

Proof. The proof follows similar lines with the one in [44]. Because $\operatorname{rank}(\mathbf{A}-\mathbf{B}) = 1$ by [126, Theorem 1] it follows that

$$\sigma_1(\mathbf{A}) \geq \sigma_2(\mathbf{B}) \geq \sigma_3(\mathbf{A}) \geq \cdots \geq \sigma_{n-1}(\mathbf{B}) \geq \sigma_n(\mathbf{A}) \tag{3.30}$$

$$\sigma_1(\mathbf{B}) \geq \sigma_2(\mathbf{A}) \geq \sigma_3(\mathbf{B}) \geq \cdots \geq \sigma_{n-1}(\mathbf{A}) \geq \sigma_n(\mathbf{B}). \tag{3.31}$$

Suppose $\sigma_{i^*}(\mathbf{A}) < 1 \le \sigma_{i^*-1}(\mathbf{A})$ for some i^* and $\sigma_{j^*}(\mathbf{B}) < 1 \le \sigma_{j^*-1}(\mathbf{B})$ for some j^*. Without loss of generality we may assume $i^* \le j^*$, otherwise the roles of $\mathbf{A}$ and $\mathbf{B}$ can be swapped. Then $\sigma_{i^*}(\mathbf{A}) < 1 \le \sigma_{i^*-1}(\mathbf{A}) \le \sigma_{i^*-2}(\mathbf{B})$. In addition by (3.30)-(3.31), for every $1 \le i \le n$ (assigning $\sigma_{n+1}(\mathbf{A}) = 0$)

$$\sigma_i(\mathbf{A}) - \sigma_i(\mathbf{B}) \ge \sigma_i(\mathbf{A}) - \sigma_{i+1}(\mathbf{A}).$$

As a result, we get

$$\begin{aligned}\sum_{i=1}^{n}|\min\{1,\sigma_i(\mathbf{A})\} - \min\{1,\sigma_i(\mathbf{B})\}| &= 1 - \min\{1,\sigma_{i^*-1}(\mathbf{B})\} + \sum_{i=i^*}^{n}|\sigma_i(\mathbf{A}) - \sigma_i(\mathbf{B})| \\ &\le (1 - \min\{1,\sigma_{i^*-1}(\mathbf{B})\}) + \sum_{i=i}^{n}(\sigma_i(\mathbf{A}) - \sigma_{i+1}(\mathbf{A})) \\ &= \sigma_{i^*}(\mathbf{A}) + 1 - \min\{1,\sigma_{i^*-1}(\mathbf{B})\} \\ &\le 1.\end{aligned}$$

Note that if $\min\{1,\sigma_{i^*-1}(\mathbf{B})\} = 1$, the result follows since $\sigma_{i^*}(\mathbf{A}) \le 1$, if not then the result follows because $\sigma_{i^*-1}(\mathbf{B}) \ge \sigma_{i^*}(\mathbf{A})$. □

Finally, we present a sketch of the proof of the generalization of Mirsky's theorem (3.25) following the proof presented in [44].

Proof. As in the proof of Theorem 3.2.10, it suffices to prove the inequality (3.25) when f is the hook function. In particular, we show

$$\sum_{i=1}^{m}|h(\sigma_i(\mathbf{A})) - h(\sigma_i(\mathbf{B}))| \le \sum_{i=1}^{m} h(\sigma_i(\mathbf{A} - \mathbf{B})). \tag{3.32}$$

- Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ where $m \le n$. Since the singular values of $\begin{bmatrix}\mathbf{A} & 0\end{bmatrix} \in \mathbb{R}^{n\times n}$ is the same as of $\mathbf{A}$, one may assume that the matrices are n-by-n.
- It suffices to prove (3.32) when $\mathbf{A}$ is a rank one perturbation of $\mathbf{B}$, i.e.,under the assumption $\text{rank}(\mathbf{A} - \mathbf{B}) = 1$ it suffices to prove

$$\sum_{i=1}^{n}|h(\sigma_i(\mathbf{A})) - h(\sigma_i(\mathbf{A}))| \le h(\sigma_1(\mathbf{A} - \mathbf{B})). \tag{3.33}$$

- Since the hook function is Lipschitz continuous with constant 1 and by Mirsky's theorem, it follows that

$$\sum_{i=1}^{n}|h(\sigma_i(\mathbf{A})) - h(\sigma_i(\mathbf{A}))| \le \sum_{i=1}^{n}|\sigma_i(\mathbf{A}) - \sigma_i(\mathbf{B})| \le \sum_{i=1}^{n}|\sigma_i(\mathbf{A} - \mathbf{B})|.$$

  This together with Lemma 3.2.11 completes the proof of (3.33).

□

Chapter 4

Kronecker Products, Schur Products and Interlacing

4.1 The Kronecker Product

Given two matrices of arbitrary dimensions $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{f} := \mathbf{Ax}$ and $\mathbf{g} := \mathbf{By}$, how can the product $f_i g_j$ be represented? The answer to this question entails the concept of Kronecker products.

Definition 4.1.1 (Kronecker product)**.** Given an m-by-n matrix $\mathbf{A}$ and a p-by-q matrix $\mathbf{B}$, the *Kronecker product* of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \otimes \mathbf{B}$ and is the mp-by-nq matrix

$$\mathbf{A} \otimes \mathbf{B} := \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}.$$

According to [60], the first result on the Kronecker product of two matrices was published in 1858 by Johann Georg Zehfuss. Zehfuss showed that

$$\det(\mathbf{A} \otimes \mathbf{B}) = \det(\mathbf{A})^m \det(\mathbf{B})^n,$$

whenever $\mathbf{A}$ is an m-by-m and $\mathbf{B}$ is an n-by-n matrix. The operation $\otimes$ was originally named after Leopold Kronecker by his student Kurt Hensel, however in some references this product is named as Zehfuss matrix [60]. Later, Hurwitz denotes the Kronecker product by $\times$ and uses the terminology *Producttransformationen* (product transformation) of matrices [69].

Kronecker products have many applications in signal processing [112], semi-definite programming [3, 123], and quantum computing. They have also been used extensively in the theory and applications of linear matrix equations such as Sylvester equations and Lyapunov problem [133], in some compressed sensing applications using sparsification [137], in constructing convex relaxations of non-convex sets [7]. Tensor product preconditioners used in conjugate gradient method [134] and image restoration [113], low rank tensor approximations arising in the context of certain quantum chemistry systems [135], quantum many-body systems [81], and high dimensional partial differential equations [10, 11] are among many other applications which utilize the rich properties of tensor products that can transfer the structure of the individual elements to the product itself.

There are several operations that we will use in the context of Kronecker products. One of them is the vector valued function $\mathrm{vec} : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ which takes an m-by-n matrix and returns an mn dimensional vector which is formed by stacking the columns of the matrix by taking the columns in order from first to last.

Definition 4.1.2 (vec)**.** For an m-by-n matrix $\mathbf{X} := [x_{ij}]$, we define the linear transformation $\mathrm{vec} : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ as

$$\mathrm{vec}(\mathbf{X}) := \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{m1} & x_{12} & x_{22} & \cdots & x_{m2} & \cdots & x_{mn} \end{bmatrix}^\top,$$

where $\mathrm{vec}(\mathbf{X})$ is an mn-by-1 vector formed by stacking the columns of the matrix $\mathbf{X}$ consecutively.

For matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{X}$ of appropriate dimensions, the following identity holds

$$\mathrm{vec}(\mathbf{B}\mathbf{X}\mathbf{A}^\top) = (\mathbf{A} \otimes \mathbf{B})\,\mathrm{vec}(\mathbf{X}). \tag{4.1}$$

As pointed out in [60], the study of equation $\mathbf{B}\mathbf{X}\mathbf{A}^\top$ appears in the *Lectures on the theory of determinants* of Kronecker in 1903 which is edited by his student Hensel. However, Kronecker only studies this for 2-by-2 matrices and then derives the determinant of the product as the product of the determinants of the constituent matrices.

Definition 4.1.3 (Mat)**.** For a vector $\mathbf{v} \in \mathbb{R}^{n^2}$, $\mathrm{Mat} : \mathbb{R}^{n^2} \to \mathbb{R}^{n\times n}$ is a matrix valued function mapping an n^2-by-1 vector into an n-by-n matrix whose ith column is $\begin{bmatrix} v_{(i-1)\cdot n+1} & \cdots & v_{i\cdot n} \end{bmatrix}^\top$.

Definition 4.1.4 (Kronecker sum)**.** Given an n-by-n matrix $\mathbf{A}$ and an m-by-m matrix $\mathbf{B}$, their *Kronecker sum*, denoted by $\mathbf{A} \oplus \mathbf{B}$, is defined as

$$\mathbf{A} \oplus \mathbf{B} := \mathbf{A} \otimes \mathbb{I}_m + \mathbb{I}_n \otimes \mathbf{B}.$$

4.1.1 Properties of the Kronecker Product

Some important properties that Kronecker product satisfies are given below.

(a) If $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{p\times q}$, then

$$(\mathbf{A} \otimes \mathbf{B})^\top = \mathbf{A}^\top \otimes \mathbf{B}^\top.$$

(b) If $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{p\times q}$ and $\mathbf{C} \in \mathbb{R}^{s\times t}$ then

$$\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}.$$

(c) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{p\times q}$, $\mathbf{X} \in \mathbb{R}^{n\times r}$ and $\mathbf{Y} \in \mathbb{R}^{q\times s}$ be real matrices. Then

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{X} \otimes \mathbf{Y}) = (\mathbf{AX}) \otimes (\mathbf{BY}).$$

(d) If $\mathbf{A}, \mathbf{B}$ are square invertible matrices, then

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}.$$

The above three properties were initially established by Hurwitz in 1894 [69, Section 6].

(d) $\operatorname{tr}(\mathbf{A} \otimes \mathbf{B}) = \operatorname{tr}(\mathbf{A}) \operatorname{tr}(\mathbf{B})$, when $m = n$ and $p = q$

(e) Let $\{\lambda_i\}$ be the eigenvalues of $\mathbf{A}$ corresponding to the eigenvectors $\{\mathbf{u}_i\}$ and let $\{\mu_j\}$ be the eigenvalues of $\mathbf{B}$ corresponding to the eigenvectors $\{\mathbf{v}_j\}$. Then $\{\lambda_i\mu_j\}$ are the eigenvalues of $\mathbf{A}\otimes\mathbf{B}$ corresponding to the eigenvectors $\{\mathbf{u}_i \otimes \mathbf{v}_j\}$. According to [60], this result was first formulated by Cyparissos Stephanos in 1898 [60], although it can also be attributed to Hurwitz due to the property (c). In the same work, Stephanos also states the determinant result (originally due to Zehfuss) as a corollary of his proposition [121].

Proof. Since

$$\begin{aligned}(\mathbf{A} \otimes \mathbf{B})(\mathbf{u}_i \otimes \mathbf{v}_j) &= (\mathbf{A}\mathbf{u}_i) \otimes (\mathbf{B}\mathbf{v}_j) \\ &= (\lambda_i \mathbf{u}_i) \otimes (\mu_j \mathbf{v}_j) \\ &= \lambda_i \mu_j (\mathbf{u}_i \otimes \mathbf{v}_j),\end{aligned}$$

the result follows. □

(f) Let $\{\lambda_i\}$ be the eigenvalues of $\mathbf{A}$ corresponding to the eigenvectors $\{\mathbf{u}_i\}$ and let $\{\mu_j\}$ be the eigenvalues of $\mathbf{B}$ corresponding to the eigenvectors $\{\mathbf{v}_j\}$. Then $\{\lambda_i + \mu_j\}$ are the eigenvalues of $\mathbf{A} \oplus \mathbf{B}$ corresponding to the eigenvectors $\{\mathbf{u}_i \otimes \mathbf{v}_j\}$.

Proof. Since

$$\begin{aligned}(\mathbf{A} \oplus \mathbf{B})(\mathbf{u}_i \otimes \mathbf{v}_j) &= (\mathbf{A} \otimes \mathbb{I}_m + \mathbb{I}_n \otimes \mathbf{B})\,(\mathbf{u}_i \otimes \mathbf{v}_j) \\ &= (\mathbf{A}\mathbf{u}_i \otimes \mathbb{I}_m\, \mathbf{v}_j) + (\mathbb{I}_n\, \mathbf{u}_i \otimes \mathbf{B}\mathbf{v}_j) \\ &= (\lambda_i \mathbf{u}_i) \otimes \mathbf{v}_j + \mathbf{u}_i \otimes (\mu_j \mathbf{v}_j) \\ &= (\lambda_i + \mu_j)(\mathbf{u}_i \otimes \mathbf{v}_j),\end{aligned}$$

the result follows. □

For a more detailed account of the history of the Kronecker products, we refer the reader to [60].

The Kronecker product of two structured matrices usually maintains the structure [133].

Theorem 4.1.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be given matrices.*

$$\text{If } \mathbf{A} \text{ and } \mathbf{B} \text{ are } \begin{cases} \text{symmetric} \\ \text{diagonal} \\ \text{unitary} \\ \text{positive definite} \\ \text{permutation} \\ \text{Toeplitz} \\ \text{doubly stochastic} \end{cases}, \text{ then } \mathbf{A} \otimes \mathbf{B} \text{ is a } \begin{cases} \text{symmetric} \\ \text{diagonal} \\ \text{unitary} \\ \text{positive definite} \\ \text{permutation} \\ \text{block Toeplitz} \\ \text{doubly stochastic} \end{cases}.$$

Furthermore, some well-known compressed sensing properties such as coherence (see Definition 2.0.8), restricted isometry property (see Definition 2.0.9) of the Kronecker product of two matrices can be expressed in terms of individual matrices.

Theorem 4.1.2. *[71, Theorem 3.5] Given real matrices* $\mathbf{A}_1, \dots, \mathbf{A}_m$ *of arbitrary dimensions, the coherence of the Kronecker product of these matrices*

$$\mu(\mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_m) = \max_{i \in \{1,\dots,m\}} \mu(\mathbf{A}_i).$$

Theorem 4.1.3. *[71, Theorem 3.7] Let* $\mathbf{A}_i \in \mathbb{R}^{k_i \times \ell_i}$, $i \in \{1, \dots, m\}$ *and let* $\mathbf{C} := \mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_m$*, then the restricted isometry constant of* $\mathbf{C}$ *denoted by* $\delta_s(\mathbf{C})$ *satisfies*

$$\delta_s(\mathbf{C}) \geq \max_{i \in \{1,\dots,m\}} \delta_s(\mathbf{A}_i).$$

Some results on the sums of Kronecker products are also established in [71]. The following is useful in building a bound for the coherence of the sums of Kronecker products.

Theorem 4.1.4. *[71, Theorem 4.4] Let* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$ *have normalized columns and let* $\mathbf{a}_i, \mathbf{b}_j$ *denote the* i*th and* j*th column of* $\mathbf{A}$ *and* $\mathbf{B}$ *respectively. Then the following holds for the coherence of the sum of* $\mathbf{A}$ *and* $\mathbf{B}$*:*

$$\mu(\mathbf{A} + \mathbf{B}) \leq \frac{\mu(\mathbf{A}) + \mu(\mathbf{B}) + \max_{i \neq j} |\langle \mathbf{a}_i, \mathbf{b}_j \rangle|}{2 - 2 \max_j |\langle \mathbf{a}_j, \mathbf{b}_j \rangle|},$$

provided $\max_j |\langle \mathbf{a}_j, \mathbf{b}_j \rangle| \neq 1$.

The authors showed that the bound given in Theorem 4.1.4 is sharp, meaning it attains the upper bound for some matrices. This result can also be applied to the sums of Kronecker products. Application of this bound for a common sum of Kronecker products is given below.

Example 4.1.1. Let $\mathbf{X} \in \mathbb{R}^{n \times n}$, $\mathbf{Y} \in \mathbb{R}^{n \times n}$, $\mathbf{A} := \mathbf{X} \otimes \mathbf{Y}$ and $\mathbf{B} := \mathbf{Y} \otimes \mathbf{X}$, where the columns of $\mathbf{X}$ and $\mathbf{Y}$ are normalized. Since

$$\begin{aligned} \max_k |\langle \mathbf{a}_k, \mathbf{b}_k \rangle| : &= \max_{i,j \in [n]} |\langle \mathbf{x}_i \otimes \mathbf{y}_j, \mathbf{y}_i \otimes \mathbf{x}_j \rangle| \\ &= \max_{i,j \in [n]} |\langle \mathbf{x}_i, \mathbf{y}_i \rangle \langle \mathbf{y}_j, \mathbf{x}_j \rangle| \\ &= \max_{j \in [n]} |\langle \mathbf{x}_j, \mathbf{y}_j \rangle|^2, \end{aligned}$$

$$\begin{aligned}\max_{r\neq s}|\langle \mathbf{a}_r,\mathbf{b}_s\rangle| &:= \max\left\{|\langle \mathbf{x}_i\otimes\mathbf{y}_j,\mathbf{y}_k\otimes\mathbf{x}_\ell\rangle|:\ i,j,k,l\in[n]\text{ s.t. } i=k\Rightarrow j\neq\ell \text{ or } j=\ell\Rightarrow i\neq k\right\}\\ &=\max\left\{|\langle \mathbf{x}_i,\mathbf{y}_k\rangle\langle \mathbf{x}_\ell,\mathbf{y}_j\rangle|:\ i,j,k,l\in[n]\text{ s.t. } i=k\Rightarrow j\neq\ell \text{ or } j=\ell\Rightarrow i\neq k\right\}\\ &=\max_{r\neq s}\left\{|c_r\cdot c_s|:\ c_r,c_s\in\{\langle \mathbf{x}_i,\mathbf{y}_j\rangle : i,j\in[n]\}\right\},\end{aligned}$$

by Theorem 4.1.2 and Theorem 4.1.4 we have

$$\begin{aligned}\mu(\mathbf{A}+\mathbf{B}) &\leq \frac{2\mu(\mathbf{X}\otimes\mathbf{Y})+\max_{i\neq j}|\langle \mathbf{a}_i,\mathbf{b}_j\rangle|}{2-2\max_j|\langle \mathbf{a}_j,\mathbf{b}_j\rangle|}\\ &=\frac{2\max\{\mu(\mathbf{X}),\mu(\mathbf{Y})\}+\max_{r\neq s}\{|c_r\cdot c_s|:\ c_r,c_s\in\{\langle \mathbf{x}_i,\mathbf{y}_j\rangle : i,j\in[n]\}\}}{2-2\max_{j\in[n]}|\langle \mathbf{x}_j,\mathbf{y}_j\rangle|^2},\end{aligned}$$

provided $\max_{j\in[n]}|\langle \mathbf{x}_j,\mathbf{y}_j\rangle|\neq 1$.

For the null-space property, it can be easily verified that whenever one of the matrices $\mathbf{A},\mathbf{B}$ has null-space property of order s with respect to ℓ_1-norm and the other has full rank, then $\mathbf{A}\otimes\mathbf{B}$ also has the null-space property of order s. This naturally raises the following question. Given both $\mathbf{A}$ and $\mathbf{B}$ satisfy null-space property order s, does $\mathbf{A}\otimes\mathbf{B}$ have the null-space property order s? We do not have an answer to this question. However, in [47], the authors proposed a method which involves a series of ℓ_1 minimization to uniquely recover s-sparse vectors $\mathbf{x}$ from $\mathbf{y}=(\mathbf{A}\otimes\mathbf{B})\mathbf{x}$, when both $\mathbf{A}$ and $\mathbf{B}$ satisfy the null-space property of order s.

4.2 Sylvester Equations

Definition 4.2.1. The *Sylvester equation*, named after the English mathematician James Joseph Sylvester (1814-1897), is the linear matrix equation

$$\mathbf{AX}-\mathbf{XB}=\mathbf{C}, \tag{4.2}$$

where $\mathbf{A}\in\mathbb{R}^{n\times n}$, $\mathbf{B}\in\mathbb{R}^{m\times m}$, $\mathbf{C}\in\mathbb{R}^{n\times m}$ are given, and $\mathbf{X}$ is an unknown $n\times m$ matrix.

As pointed out in Higham's article [63] on Sylvester's contribution on applied mathematics, Sylvester considered the homogeneous version of (4.2), i.e., when $\mathbf{C}=0$. He showed that the homogeneous Sylvester equation has a unique solution if $\mathbf{A}$ and $\mathbf{B}$ have no common eigenvalues. To show this, we write the Sylvester equation equivalently as the Kronecker sum of $\mathbf{B}^\top$ and $\mathbf{A}$. Since

$$\mathbf{AX}-\mathbf{XB}=\mathbf{C}\Leftrightarrow(\mathbb{I}_m\otimes\mathbf{A}-\mathbf{B}^\top\otimes\mathbb{I}_n)\,\mathrm{vec}(\mathbf{X})=\mathrm{vec}(\mathbf{C})\quad\Leftrightarrow\big((-\mathbf{B})^\top\oplus\mathbf{A}\big)\,\mathrm{vec}(\mathbf{X})=\mathrm{vec}(\mathbf{C}),$$

$\mathbf{AX}-\mathbf{XB}=\mathbf{C}$ has a unique solution if and only if $\big((-\mathbf{B})^\top\oplus\mathbf{A}\big)$ has all nonzero eigenvalues, i.e., if $\{\lambda_i\}$ and $\{\mu_j\}$ are the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ respectively, then $\lambda_i-\mu_j\neq 0$ for every i,j.

Sylvester equation arises in various areas such as control systems, discretization of partial differential equations, and linear matrix equations. Consider the function $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $F(\mathbf{X}) := \mathbf{X}^2$. The Fréchet derivative of the quadratic function F at $\mathbf{X}$ in the direction $\mathbf{U}$ is

$$\mathbf{XU} + \mathbf{UX},$$

where $\mathbf{U} \in \mathbb{R}^{n\times n}$, since

$$F(\mathbf{X} + \mathbf{U}) = F(\mathbf{X}) + \mathbf{XU} + \mathbf{UX} + o(\|\mathbf{U}\|).$$

A special case of Sylvester equation is *Lyapunov equation* in which $\mathbf{B} = -\mathbf{A}^\top$ and $\mathbf{C} = \mathbf{C}^\top$ in (4.2). Lyapunov type equations arise in control theory, in particular in the study of the stability of solutions of systems of linear differential equations [124]. Lyapunov showed that the linear system of ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$$

is asymptotically stable if and only if for every symmetric positive-definite matrix $\mathbf{S}$ the solution of

$$\mathbf{A}^\top\mathbf{X} + \mathbf{XA} = -\mathbf{S}$$

is positive definite.

Several numerical algorithms based on Galerkin methods are developed for computing low-rank approximate solutions to the Sylvester type equations [68] and Lyapunov equations [120, 70, 37]. In Chapter 6, we provide a discussion on the generalizations of these equations as well.

4.3 The Schur Product

Definition 4.3.1 (Schur product)**.** Given $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$, the *Schur product* of $\mathbf{A}$ and $\mathbf{B}$ is the m-by-n matrix which is defined by

$$\mathbf{A} \circ \mathbf{B} := \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{m1} & a_{m2}b_{m2} & \cdots & a_{mn}b_{mn} \end{bmatrix}.$$

The Schur product is named after Issai Schur. It is also called the *Hadamard product*. The Schur product $\mathbf{A} \circ \mathbf{B}$ is a principal submatrix of the Kronecker product $\mathbf{A} \otimes \mathbf{B}$.

Lemma 4.3.1. *Given* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$*,*

$$\mathbf{A} \circ \mathbf{B} = (\mathbf{A} \otimes \mathbf{B})(I, J),$$

where $I := \{1, m + 2, 2m + 3, \ldots, m^2\}$ *and* $J := \{1, n + 2, 2n + 3, \ldots, n^2\}$*.*

This lemma basically states that $\mathbf{A} \circ \mathbf{B} = \mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{Q}$, where $\mathbf{Q}$ is an n^2-by-n matrix defined by

$$\mathbf{Q}(i,j) := \begin{cases} 1, & \text{if } i \in \{1, n+2, 2n+3, \dots, n^2\} \\ 0, & \text{otherwise,} \end{cases}$$

where $j \in \{1, 2, \dots, n\}$. Note that $\mathbf{Q}^\top \mathbf{Q} = \mathbb{I}$. An immediate conclusion of this result with Cauchy's interlacing theorem (see Theorem 1.3.5) states that the eigenvalues of the Schur product $\mathbf{A} \circ \mathbf{B}$ interlace the eigenvalues of $\mathbf{A} \otimes \mathbf{B}$, whenever $\mathbf{A}, \mathbf{B}$ are n-by-n Hermitian matrices.

The following is a well-known result.

Theorem 4.3.2 (Schur Product Theorem)**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ be positive semidefinite, then so is $\mathbf{A} \circ \mathbf{B}$. In addition, if $\mathbf{B}$ is positive definite and $\mathbf{A}$ has no diagonal entry equal to* 0*, then $\mathbf{A} \circ \mathbf{B}$ is positive definite. In particular, if both $\mathbf{A}$ and $\mathbf{B}$ are positive definite, then so is $\mathbf{A} \circ \mathbf{B}$.*

Lemma 4.3.1 enables derivation of new inequalities on the Schur product by applying some inequalities on matrices to the Kronecker products and utilizing the properties of the Kronecker product. A well-known inequality on matrix convex function f is

$$f(\mathbf{Q}^H \mathbf{A} \mathbf{Q}) \preceq \mathbf{Q}^H f(\mathbf{A}) \mathbf{Q}. \tag{4.3}$$

Lemma 4.3.1 and (4.3) gives

$$(\mathbf{A} \circ \mathbf{B})^2 = (\mathbf{J}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{J})^2 \preceq (\mathbf{J}^\top(\mathbf{A} \otimes \mathbf{B})^2\mathbf{J}) = (\mathbf{J}^\top(\mathbf{A}^2 \otimes \mathbf{B}^2)\mathbf{J}) = \mathbf{A}^2 \circ \mathbf{B}^2,$$

where $\mathbf{J}$ is n^2-by-n and $\mathbf{J}^\top\mathbf{J} = \mathbb{I}$. Similarly, assuming $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices, one can show

$$(\mathbf{A} \circ \mathbf{B})^{-1} \preceq \mathbf{A}^{-1} \circ \mathbf{B}^{-1},$$

by taking $f(t)$ as t^2 and t^{-1}, respectively [100]. Mond and Pečarić used a number of inequalities on positive definite matrices [92, 98, 99] to derive similar inequalities on the Schur product, by using the properties of Kronecker product as described in the above example. We summarize their results in the following theorem.

Theorem 4.3.3. *[100] Let $\mathbf{A}$ and $\mathbf{B}$ be n-by-n positive definite Hermitian matrices. Then the following hold.*

1. *Let $r, s \in \mathbb{R}$, $s > r$ such that M is the largest eigenvalue and m is the smallest eigenvalue of $\mathbf{A} \otimes \mathbf{B}$. Then,*

$$r(\mathbf{A}^r \circ \mathbf{B}^r - \alpha \mathbf{A}^s \circ \mathbf{B}^s - \beta\, \mathbb{I}) \succeq 0,$$

where $\alpha := (M^r - m^r)/(M^s - m^s)$, and $\beta := (M^s m^r - M^r m^s)/(M^s - m^s)$.

2. *Let $r, s \in \mathbb{R} \setminus (-1, 1)$ such that $s > r$. Then,*

$$(\mathbf{A}^s \circ \mathbf{B}^s)^{1/s} \succeq (\mathbf{A}^r \circ \mathbf{B}^r)^{1/r}.$$

3. *Let* $r, s \in \mathbb{R} \setminus \{0\}$ *such that* $s > r$ *and*

$$\gamma := \max_{\alpha\in[0,1]} (\alpha M^s + (1-\alpha)m^s)^{1/s} - (\alpha M^r + (1-\alpha)m^r)^{1/r}.$$

Then, $(\mathbf{A}^s \circ \mathbf{B}^s)^{1/s} - (\mathbf{A}^r \circ \mathbf{B}^r)^{1/r} \preceq \gamma\,\mathbb{I}$.

4. *Let* $r, s \in \mathbb{R} \setminus (-1, 1)$ *such that* $s > r$ *and*

$$\gamma' := \frac{r^{1/s}((M/m)^s - (M/m)^r)}{(s-r)^{1/s}((M/s)^r - 1)^{1/s}} \frac{(r-s)^{1/r}((M/s)^s - 1)^{1/r}}{s^{1/r}((M/m)^r - (M/m)^s)^{1/r}}.$$

Then,

$$\gamma'(\mathbf{A}^r \circ \mathbf{B}^r)^{1/r} - (\mathbf{A}^s \circ \mathbf{B}^s)^{1/s} \succeq 0.$$

The following are some results from Schur's famous paper [118].

Theorem 4.3.4. *Let* $\mathbf{A} := [a_{ij}], \mathbf{B} \in \mathbb{R}^{n\times n}$.

1. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semidefinite, then*

$$\min_{1\le i\le n} a_{ii}\lambda_{min}(\mathbf{B}) \le \lambda_{min}(\mathbf{A}\circ\mathbf{B}) \le \lambda_{max}(\mathbf{A}\circ\mathbf{B}) \le \max_{1\le i\le n} a_{ii}\lambda_{max}(\mathbf{B}).$$

2. *If* $\mathbf{A}$ *is positive semidefinite, then*

$$\sigma_1(\mathbf{A}\circ\mathbf{B}) \le \max_{1\le i\le n} a_{ii}\sigma_1(\mathbf{B}).$$

3.
$$\sigma_1(\mathbf{A}\circ\mathbf{B}) \le \sigma_1(\mathbf{A})\sigma_1(\mathbf{B}).$$

Some other well-known results on the Schur product are given below.

Lemma 4.3.5. *[12] Let* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{x} \in \mathbb{R}^n$. *Then,*

$$\operatorname{diag}(\mathbf{A}\operatorname{Diag}(\mathbf{x})\mathbf{B}) = (\mathbf{A}\circ\mathbf{B}^\top)\mathbf{x}.$$

In particular, the vector formed by the diagonal entries of $\mathbf{AB}$) *is the vector of row sums of* $\mathbf{A}\circ\mathbf{B}^\top$.

The next result can be proved by applying the above lemma and Hardy-Littlewood-Polya Theorem (see Theorem 3.2.1).

Proposition 4.3.6. *[12] Let* $\mathbf{A}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m \in \mathbb{R}^{n\times n}$ *such that* $\mathbf{A}$ *is symmetric and*

$$\sum_{i=1}^m \mathbf{S}_i^\top \mathbf{S}_i = \sum_{i=1}^m \mathbf{S}_i \mathbf{S}_i^\top = \mathbb{I}\,.$$

If $\mathbf{X} = \sum_{i=1}^m \mathbf{S}_i\mathbf{A}\mathbf{S}_i^\top$, *then eigenvalues of* $\mathbf{A}$ *majorize eigenvalues of* $\mathbf{X}$, *i.e.,* $\lambda(\mathbf{X}) \prec \lambda(\mathbf{A})$.

In [106], Oppenheim established the following inequality relating the determinant of the Schur product of two matrices to the determinant of individual ones.

Theorem 4.3.7 (Oppenheim inequality)**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ be positive semidefinite. Then*

$$\det(\mathbf{A} \circ \mathbf{B}) \geq (\det \mathbf{A})(\det \mathbf{B}).$$

Some other results on the Schur product are given below.

Theorem 4.3.8 (Oppenheim-Schur inequalities)**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ be positive semidefinite. Then*

$$\max\left\{ (\prod_{i=1}^{n} a_{ii}) \det(\mathbf{B}), (\prod_{i=1}^{n} b_{ii}) \det(\mathbf{A}) \right\} \leq \det(\mathbf{A} \circ \mathbf{B})$$

and

$$(\prod_{i=1}^{n} a_{ii}) \det(\mathbf{B}) + (\prod_{i=1}^{n} b_{ii}) \det(\mathbf{A}) \leq \det(\mathbf{A} \circ \mathbf{B}) + \det(\mathbf{AB}).$$

Theorem 4.3.9. *[5, 138] Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ be positive semidefinite, $r \in [0, 1]$ and $k \in \{1, 2, \ldots, n\}$. Then*

$$\prod_{i=k}^{n} \lambda_i(\mathbf{A} \circ \mathbf{B}) \geq \prod_{i=k}^{n} (\lambda_i(\mathbf{A}^r \circ \mathbf{B}^r))^{1/r} \geq \prod_{i=k}^{n} (\lambda_i(\mathbf{A}^r \mathbf{B}^r))^{1/r} \geq \prod_{i=k}^{n} \lambda_i(\mathbf{AB}).$$

In [91], it was shown that

$$\operatorname{per}(\mathbf{A}) \geq \prod_{i=1}^{n} a_{ii}.$$

Later in [26], Chollet posed a conjecture (which is still open) asking if an analog of Oppenheim's inequality can be proven for permanent.

Conjecture 4.3.10. *[26] Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ be positive semidefinite. Then*

$$\operatorname{per}(\mathbf{A} \circ \mathbf{B}) \leq (\operatorname{per} \mathbf{A})(\operatorname{per} \mathbf{B}).$$

4.4 Structured Eigenvectors and Interlacing

4.4.1 Even and Odd Factorization of Eigenvectors

In many eigenvalue problems, special structure and symmetry give rise to structure in eigenfunctions. In such problems, often the characteristic polynomial factorizes into two polynomials one whose roots that corresponds "even" eigenvectors and the other corresponds to "odd" eigenvectors. In many applications, the largest or the smallest eigenvalue or the eigenvector corresponding to these eigenvalues are required [72]. When the eigenvectors decompose in such manners, it may suffice to find the roots of a polynomial whose degree is half of the original polynomial. In this section, we investigate such cases where eigenvectors decompose into symmetric and skew-symmetric vectors.

Definition 4.4.1 (Symmetric vector, Skew-symmetric vector)**.** Let $\mathbf{P} \in \mathbb{R}^{n\times n}$ be a real symmetric, involutory matrix (i.e., $\mathbf{P}^2 = \mathbb{I}$). A vector $\mathbf{x} \in \mathbb{R}^n$ is a *symmetric vector* if

$$\mathbf{Px} = \mathbf{x},$$

and a *skew-symmetric vector* if

$$\mathbf{Px} = -\mathbf{x}.$$

In some literature, symmetric and skew-symmetric vectors are named as reciprocal and anti-reciprocal vectors [33]. In special cases of $\mathbf{P}$ e.g., when $\mathbf{P} := \mathbf{J}$ (a permutation matrix with ones on the secondary diagonal), as described in [6, 24], the symmetric centrosymmetric matrices of order n can be decomposed such that the eigenvectors can be classified into $n - \lfloor \frac{n}{2} \rfloor$ even and $\lfloor \frac{n}{2} \rfloor$ odd vectors. Under some circumstances, the eigenvalues corresponding to even and odd eigenvectors interlace. This interlacing has implications for the inverse eigenvalue problem for symmetric Toeplitz matrices [33, 28, 83]. For a detailed overview of the results on the inverse eigenvalue problems for structured matrices, we refer the reader to [29].

Given some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, an *inverse eigenvalue problem* aims to find a matrix that belongs to a set of special structured matrices with the spectrum consisting of these real numbers. The structural constraint is usually enforced to satisfy requirements of the underlying physical system.

4.4.2 Centrosymmetric Matrices and Related Matrices

An n-by-n real matrix $\mathbf{A} = [a_{i,j}]$ is *centrosymmetric* if $\mathbf{A} = \mathbf{JAJ}$, i.e.,

$$a_{i,j} = a_{n-i+1,n-j+1} \text{ for } 1 \le i, j \le n.$$

A well known class of matrices that are subsets of centrosymmetric matrices are Toeplitz matrices. Centrosymmetric matrices arise in LTI (linear time invariant) systems as a representing matrix for convolution operation, as a transition matrix of a Markov process and it appears in numerical solution of certain differential equations.

A subclass of centrosymmetric matrices are symmetric centrosymmetric matrices. These matrices possess nice algebraic structure. It is well known that symmetric centrosymmetric matrices form an Abelian group under addition and the nonsingular ones form a non-Abelian group under multiplication. Due to their structure, every such matrix can be characterized by certain partitions based on whether the size of the matrix is even or odd. This is demonstrated in [24] and we state it here for completeness.

Lemma 4.4.1. *[24, Lemma 2, Theorem 2] Every symmetric n-by-n centrosymmetric matrix can be represented as*

$$\mathbf{C}_{even} := \begin{bmatrix} \mathbf{A} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{JAJ} \end{bmatrix} \tag{4.4}$$

if n is even,

$$\mathbf{C}_{odd} := \begin{bmatrix} \mathbf{A} & \mathbf{x} & \mathbf{B}^\top \\ \mathbf{x}^\top & \mathbf{q} & \mathbf{x}^\top\mathbf{J} \\ \mathbf{B} & \mathbf{J}\mathbf{x} & \mathbf{J}\mathbf{A}\mathbf{J} \end{bmatrix}$$

if n is odd, where $\mathbf{A}, \mathbf{B}$ are $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$ matrices with $\mathbf{A}^\top = \mathbf{A}$, $\mathbf{B}^\top = \mathbf{J}\mathbf{B}\mathbf{J}$. If n is even, the $n/2$ skew symmetric orthonormal eigenvectors $\mathbf{v}_i$ corresponding to eigenvalues λ_i of $\mathbf{C}_{even}$ can be determined from the solution of the equation

$$(\mathbf{A} - \mathbf{J}\mathbf{B})\mathbf{u}_i = \tilde{\lambda}_i \mathbf{u}_i,$$

where $i \in I(n/2)$, $\mathbf{u}_i$ form an orthonormal set and $\mathbf{v}_i = (1/\sqrt{2}) \begin{bmatrix} \mathbf{u}_i & -\mathbf{J}\mathbf{u}_i \end{bmatrix}^\top$. Also, the $n/2$ symmetric orthonormal eigenvectors $\mathbf{w}_i$ corresponding to eigenvalues $\tilde{\lambda}_i$ of $\mathbf{C}_{even}$ can be determined from the solution of the equation

$$(\mathbf{A} + \mathbf{J}\mathbf{B})\mathbf{y}_i = \lambda_i \mathbf{y}_i,$$

where $i \in I(n/2)$, $\mathbf{w}_i = (1/\sqrt{2}) \begin{bmatrix} \mathbf{y}_i & \mathbf{J}\mathbf{y}_i \end{bmatrix}^\top$ form an orthonormal set and $\mathbf{w}_i = (1/\sqrt{2}) \begin{bmatrix} \mathbf{y}_i & \mathbf{J}\mathbf{y}_i \end{bmatrix}^\top$. Moreover, the set of skew symmetric and symmetric eigenvectors of $\mathbf{C}_{even}$ forms an orthonormal set which therefore spans the eigenspace of $\mathbf{C}_{even}$.

If n is odd, the skew symmetric orthonormal eigenvectors $\mathbf{v}_i$ corresponding to eigenvalues λ_i of $\mathbf{C}_{odd}$ can be determined from the solution of the equation

$$(\mathbf{A} - \mathbf{J}\mathbf{B})\mathbf{u}_i = \lambda_i \mathbf{u}_i,$$

where $i \in I(n/2)$, $\mathbf{u}_i$ form an orthonormal set and $\mathbf{v}_i = (1/\sqrt{2}) \begin{bmatrix} \mathbf{u}_i & 0 & -\mathbf{J}\mathbf{u}_i \end{bmatrix}^\top$. Also, the symmetric orthonormal eigenvectors $\mathbf{w}_i$ corresponding to eigenvalues $\tilde{\lambda}_i$ of $\mathbf{C}_{odd}$ can be determined from the solution of the equation

$$\begin{bmatrix} \mathbf{A} + \mathbf{J}\mathbf{C} & \sqrt{2}\mathbf{x} \\ \sqrt{2}\mathbf{x}^\top & \mathbf{q} \end{bmatrix} \begin{bmatrix} \mathbf{y}_i \\ \alpha_i \end{bmatrix} = \tilde{\lambda}_i \begin{bmatrix} \mathbf{y}_i \\ \alpha_i \end{bmatrix}$$

where $i \in \{1, 2, \ldots, \frac{n}{2}\}$, $\begin{bmatrix} \mathbf{y}_i \\ \alpha_i \end{bmatrix}$ form an orthonormal set and $\mathbf{w}_i = (1/\sqrt{2}) \begin{bmatrix} \mathbf{y}_i & 2\alpha_i & \mathbf{J}\mathbf{y}_i \end{bmatrix}^\top$. Moreover, the set of skew symmetric and symmetric eigenvectors of $\mathbf{C}$ form an orthonormal set which therefore spans the eigenspace of $\mathbf{C}_{odd}$.

The symmetric centrosymmetric matrices exhibit certain interlacing properties under certain structure restrictions.

Theorem 4.4.2. *[24, Theorem 5] Let n be an odd number and let $\mathbf{C} \in \mathbb{R}^{n \times n}$ be such that*

$$\mathbf{C} := \begin{bmatrix} \mathbf{A} & \mathbf{x} & \mathbf{0} \\ \mathbf{x}^\top & q & \mathbf{x}^\top\mathbf{J} \\ \mathbf{0} & \mathbf{J}\mathbf{x} & \mathbf{J}\mathbf{A}\mathbf{J} \end{bmatrix}.$$

If the eigenvalues of $\mathbf{C}$ are distinct and sorted in ascending order, then the corresponding eigenvectors are alternately symmetric and skew symmetric, starting with symmetric.

In the same work [24], Cantoni also established that when n is even and $\mathbf{B}$ in (4.4) is of rank one, then the even and odd eigenvalues of the symmetric centrosymmetric matrices alternate [24]. Furthermore, the largest eigenvalue corresponds to a symmetric eigenvector if $\mathbf{B}$ is positive semidefinite, and skew symmetric eigenvector, otherwise.

4.4.3 Real Symmetric Toeplitz Matrices

The set of real symmetric Toeplitz matrices is a subset of symmetric centrosymmetric matrices. The even and odd spectra of Toeplitz matrices can be exploited for efficient computation of eigenvectors and its extreme eigenvalues[6, 94].

An interest in the interlacement of the odd and even eigenvalues of real symmetric Toeplitz matrices arise from the inverse eigenvalue problem. The goal here is to reconstruct a real symmetric Toeplitz matrix from a prescribed spectrum. In [33], it is showed that for every $n \leq 4$, the inverse eigenvalue problem always has a solution, if its prescribed eigenvalues when sorted in ascending order alternate in parity, i.e., if the even and odd spectra are interlaced with the largest being even. In addition, if the two prescribed spectra are not interlaced, then the inverse eigenvalue problem fails to have a solution for some choices of prescribed eigenvalues. The next result gives a sufficient condition for the interlacement of the eigenvalues of certain Toeplitz matrices.

Theorem 4.4.3. *[129, Theorem 4] Let f be nonincreasing and $f(0) = M > m = f(\pi)$. Let $\mathbf{T}_{r,s} = t_{r-s}$, $1 \leq r, s \leq n$ be a Toeplitz matrix such that $t_k = \frac{1}{\pi}\int_0^\pi f(\theta)\cos\theta d\theta$. Then for every n the matrix $\mathbf{T}_n$ has n distinct eigenvalues in (m, M), its even and odd spectra are interlaced, and its eigenvector corresponding to its largest eigenvalue is even.*

For a general n, the existence of an n-by-n real symmetric Toeplitz matrix with prescribed eigenvalues was solved by Landau [83]. He showed that for the set of symmetric Toeplitz matrices that are *regular* i.e., for which every principal submatrix has distinct eigenvalues whose eigenvalues alternate parity with the largest determined by an even eigenvector, the inverse eigenvalue problem is always solvable.

4.4.4 Tridiagonal Matrices

One of the most studied problems in the area of inverse eigenvalue problem stems from Sturm-Liouville problem. For numerical solutions, this problem was discretized. Let $\mathbf{T}_0$ be a tridiagonal matrix whose main diagonal entries are equal to 2 and super and subdiagonal entries are equal to -1. Given $h > 0$, and a set of negative numbers $\lambda_1, \ldots, \lambda_n$, the *discretized Sturm-Liouville problem* seeks to find a positive diagonal matrix $\mathbf{Q}$ such that $-\frac{1}{h}\mathbf{T}_0 + \mathbf{Q}$ has the prescribed eigenvalues. Although the Sturm-Liouville problem has infinitely many eigenvalues, the eigenvalues of the discretized problem imitate the first few smallest eigenvalues of the (continuous) Sturm-Liouville problem [29].

Definition 4.4.2 (Jacobi matrix)**.** A square matrix with real entries is a *Jacobi matrix* if it is a symmetric tridiagonal matrix with positive super and subdiagonal entries.

This structure arises in many applications in physics especially in Sturm-Liouville problem. The eigenvalues of a Jacobi matrix are real and distinct. In [65], Hochstadt showed that a real symmetric Jacobi matrix with positive subdiagonal elements is uniquely determined by its eigenvalues and those of $(n-1)$-by-$(n-1)$ leading principal submatrix. Later, in [57], Hald showed that given

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{n-1} < \lambda_n,$$

a unique n-by-n Jacobi matrix exists with the prescribed eigenvalues $\{\lambda_i\}$, where $\{\mu_i\}$ are the eigenvalues of the leading principal submatrix of order $(n-1)$.

For a comprehensive survey on the structured inverse eigenvalue problems, including the Jacobi matrices, we refer the reader to [28].

4.4.5 Symmetric Perfect Shuffle Invariant Matrices

Define a *transpose operator* $\mathcal{T} : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$, such that $\mathrm{Mat}(\mathcal{T}\mathbf{x}) = \mathrm{Mat}(\mathbf{x})^\top$. Let $\mathbf{T}$ be the n^2-by-n^2 matrix representation of the transpose operator $\mathcal{T}$ which we call as the *transposition matrix*. More specifically, $\mathbf{T}$ is an n^2-by-n^2 permutation matrix such that if the columns of $\mathbf{T}$ are in the order of $(1,1),\ldots,(1,n)$, $(2,1),\ldots,(2,n)$,...,$(n,1),\ldots,(n,n)$, then the (i,j)th column of T is

$$T_{(i,j)} := e_j \otimes e_i, \quad \text{for } i,j \in \{1,2,\ldots,n\}. \tag{4.5}$$

Note that $\mathbf{T}$ is a symmetric involutory matrix. In some literature, the permutation matrix $\mathbf{T}$ is named as the commutation matrix [87].

Definition 4.4.3 (Perfect shuffle invariant)**.** A matrix $\mathbf{A} \in \mathbb{R}^{n^2 \times n^2}$ is *perfect shuffle invariant* if

$$\mathbf{A} = \mathbf{TAT}.$$

Definition 4.4.4 (Perfect shuffle symmetric matrix (PS-symmetric))**.** A matrix $\mathbf{A} \in \mathbb{R}^{n^2 \times n^2}$ is *perfect shuffle symmetric* (PS-symmetric) if it is both perfect shuffle invariant and symmetric.

Define an n^2-by-$\frac{n(n+1)}{2}$ orthogonal matrix $\mathbf{Q}$ whose columns form a basis for n^2-by-n^2 symmetric matrices. If we label the columns of $\mathbf{Q}$ in the order $(1,1),(1,2),\ldots,(1,n)$, $(2,2),(2,3),\ldots,(2,n),\ldots,$ $(n-1,n-1),(n-1,n),(n,n)$, then

$$\mathbf{Q}_{(i,j)} := \begin{cases} \dfrac{1}{\sqrt{2}}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), & \text{if } 1 \le i < j \le n, \\ \mathbf{e}_i \otimes \mathbf{e}_i, & \text{if } 1 \le i = j \le n, \end{cases} \tag{4.6}$$

where $\mathbf{e}_i$ is an n dimensional vector with all components equal to zero, except ith component which is equal to one. Note that $\mathbf{Q}^\top\mathbf{Q} = \mathbb{I}$, where $\mathbb{I}$ is an identity matrix of appropriate dimension and $\mathbf{Q}\mathbf{Q}^\top$ is the orthogonal projector mapping every point in $\mathbb{R}^{n^2}$ onto the set of symmetric vectors in $\mathbb{R}^{n^2}$ [131].

Similarly, we define an orthogonal matrix $\tilde{\mathbf{Q}} \in \mathbb{R}^{n^2 \times \frac{n(n-1)}{2}}$ whose columns form a basis for n^2-by-n^2 symmetric matrices. If we label the columns of $\tilde{\mathbf{Q}}$ in the order $(1,2),\ldots,(1,n)$, $(2,3),\ldots,(2,n),\ldots,$ $(n-1,n)$, then

$$\tilde{\mathbf{Q}}_{(i,j)} := \frac{1}{\sqrt{2}}(\mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i), \quad \text{for } 1 \le i < j \le n. \tag{4.7}$$

Note that $\tilde{\mathbf{Q}}^\top \tilde{\mathbf{Q}} = \mathbb{I}$ and $\tilde{\mathbf{Q}}\tilde{\mathbf{Q}}^\top$ is the orthogonal projector, mapping $\mathbb{R}^{n^2}$ onto the set of skew-symmetric vectors in $\mathbb{R}^{n^2}$.

Proposition 4.4.4. *[135, Theorem 2.1] Let $\mathbf{A} \in \mathbb{R}^{n^2 \times n^2}$ be a PS-symmetric matrix and $\mathbf{Q}_{nn} := \begin{bmatrix}\mathbf{Q} & \tilde{\mathbf{Q}}\end{bmatrix}$. Then, $\mathbf{Q}_{nn}$ is orthogonal and block diagonalizes $\mathbf{A}$ as*

$$\mathbf{Q}_{nn}^\top \mathbf{A}\mathbf{Q}_{nn} = \begin{bmatrix}\mathbf{A}^{sym} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{skew}\end{bmatrix},$$

where $\mathbf{A}^{sym}$ is $\frac{n(n+1)}{2}$-by-$\frac{n(n+1)}{2}$ and $\mathbf{A}^{skew}$ is $\frac{n(n-1)}{2}$-by-$\frac{n(n-1)}{2}$.

Proof. The proof is taken from [135].
Observe that

$$\mathbf{Q}_{nn}^\top \mathbf{A}\mathbf{Q}_{nn} = \begin{bmatrix}\mathbf{Q}^\top\mathbf{A}\mathbf{Q} & \mathbf{Q}^\top\mathbf{A}\tilde{\mathbf{Q}} \\ \tilde{\mathbf{Q}}^\top\mathbf{A}\mathbf{Q} & \tilde{\mathbf{Q}}^\top\mathbf{A}\tilde{\mathbf{Q}}\end{bmatrix}. \tag{4.8}$$

By the definition of $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$, we have $\mathbf{T}\mathbf{Q} = \mathbf{Q}$ and $\mathbf{T}\tilde{\mathbf{Q}} = -\tilde{\mathbf{Q}}$. This implies that $\mathbf{Q}^\top\mathbf{A}\tilde{\mathbf{Q}} = 0$, as

$$\mathbf{Q}^\top\mathbf{A}\tilde{\mathbf{Q}} = \mathbf{Q}\mathbf{T}\mathbf{A}\mathbf{T}\tilde{\mathbf{Q}} = -\mathbf{Q}^\top\mathbf{A}\tilde{\mathbf{Q}} = \mathbf{0}.$$

Assigning $\mathbf{A}^{sym} := \mathbf{Q}^\top\mathbf{A}\mathbf{Q}$ and $\mathbf{A}^{skew} := \tilde{\mathbf{Q}}^\top\mathbf{A}\tilde{\mathbf{Q}}$ yields the result. □

Using the above result one can show that the eigenvectors of PS-symmetric matrix can be decomposed into symmetric and skew-symmetric vectors [135]. Let $\mathbf{A} \in \mathbb{R}^{n^2 \times n^2}$ be a PS-symmetric matrix. Let

$$\begin{aligned}\mathbf{A}^{sym} &= \mathbf{U}\mathbf{D}\mathbf{U}^\top, \text{ and} \\ \mathbf{A}^{skew} &= \mathbf{W}\tilde{\mathbf{D}}\mathbf{W}^\top\end{aligned}$$

be the spectral decompositions of the diagonal blocks as given in Proposition 4.4.4. Then

$$\mathbf{R} := \mathbf{Q}_{nn}\begin{bmatrix}\mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}\end{bmatrix} = \begin{bmatrix}\mathbf{Q}\mathbf{U} & \tilde{\mathbf{Q}}\mathbf{V}\end{bmatrix}$$

diagonalizes $\mathbf{A}$ as follows:

$$\mathbf{A} = \mathbf{R}\begin{bmatrix}\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{D}}\end{bmatrix}\mathbf{R}^\top.$$

By the definition of $\mathbf{Q}_{nn}$, the eigenvectors corresponding to $\mathbf{D}$ are the columns of $\mathbf{Q}\mathbf{U}$. Since $\mathbf{T}\mathbf{Q}\mathbf{U} = \mathbf{Q}\mathbf{U}$ as $\mathbf{T}\mathbf{Q} = \mathbf{Q}$, each column of $\mathbf{Q}\mathbf{U}$ is a symmetric vector, i.e., when reshaped to n-by-n matrix it is a symmetric matrix. In addition, the eigenvectors corresponding to $\tilde{\mathbf{D}}$ are the columns of $\tilde{\mathbf{Q}}\mathbf{V}$ are skew-symmetric.

Remark 4.4.5. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$. Then*

$$\mathbf{T}(\mathbf{A}\otimes\mathbf{B})\mathbf{T} = \mathbf{B}\otimes\mathbf{A}.$$

This implies that $\mathbf{T}(\mathbf{A}\otimes\mathbf{B}+\mathbf{B}\otimes\mathbf{A})\mathbf{T} = (\mathbf{A}\otimes\mathbf{B}+\mathbf{B}\otimes\mathbf{A})$. Therefore, it is a perfect shuffle matrix.

Although one can always create a perfect shuffle matrix with a prescribed set of scalars as its eigenvalues (see Proposition 4.4.4), we raise the following inverse eigenvalue problem for a subclass of perfect shuffle matrices: Given a prescribed set of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n^2}$, under what conditions do there exist n-by-n real symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ such that these eigenvalues belong to $\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A}$?

Chapter 5

Jordan-Kronecker Products and Interlacing

This chapter contains the main contributions of this thesis. We study *Jordan-Kronecker products* which have many applications in semidefinite programming, control theory, and solutions of differential matrix equations. First, we introduce special Kronecker products that are essential ingredients and connect to the structure of Jordan-Kronecker products. We study basic properties of these special Kronecker products. Then we study the spectral structure of Jordan-Kronecker products. We also provide a generalization of Jordan-Kronecker product based on replacing the transpose matrix by an arbitrary symmetric involutary matrix. We study its spectral structure in terms of eigenvalues and eigenvectors and show that the generalization enjoys similar properties of the Jordan-Kronecker product. We then study the conjectures posed by Tunçel and Wolkowicz in [131] on the interlacing properties and the extreme eigenvalues of the Jordan-Kronecker product. We disprove these conjectures in general, but we also provide a number of structured matrices and classes of matrices for which these conjectures hold. Furthermore, we present techniques to generate classes of matrices for which these conjectures fail. Lastly, we propose a related structure, namely Lie-Kronecker products and characterize their eigenstructure.

5.1 The Symmetric Kronecker Product

In primal-dual interior-point path following methods, a sequence of primal and dual variables, converging to the primal and dual solutions of SDP, are generated to approximate the central path. The points representing the central path satisfy the primal and dual feasibility conditions and the centering condition. The iteration is usually carried out by Newton's method. However, solving these equations in the space of square matrices, in general does not produce symmetric primal variables. Alizadeh, Haeberly and Overton proposed the $\mathbf{XZ} + \mathbf{ZX}$ *method* in [3] to deal with this issue. The symmetric Kronecker product (for symmetric matrices) was introduced in this work to represent the centering condition for the $\mathbf{XZ} + \mathbf{ZX}$ method [3]. We discuss this concept in more detail in Chapter 6.

Recall $\mathbb{S}^n$ denotes *the set of n-by-n real symmetric matrices* and let $\mathrm{sym}(n) := n(n+1)/2$ denote its dimension. Define the mapping $\mathrm{s2vec} : \mathbb{S}^n \to \mathbb{R}^{\mathrm{sym}(n)}$ as the isometry between $\mathbb{S}^n$ and $\mathbb{R}^{\mathrm{sym}(n)}$ such that for every n-by-n real symmetric matrix $\mathbf{X} = [x_{ij}]$,

$$\mathrm{s2vec}(\mathbf{X}) := \begin{bmatrix} x_{11} & \sqrt{2}x_{21} & \cdots & \sqrt{2}x_{n1} & x_{22} & \sqrt{2}x_{32} & \cdots & \sqrt{2}x_{n2} & \cdots & x_{nn} \end{bmatrix}^\top.$$

The usual trace inner-product in $\mathbb{S}^n$ can be expressed as

$$\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{s2vec}(\mathbf{A})^\top \mathrm{s2vec}(\mathbf{B}),$$

for every $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$.

Definition 5.1.1 (Symmetric Kronecker product)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be two arbitrary n-by-n real matrices. The *symmetric Kronecker product* of $\mathbf{A}$ and $\mathbf{B}$ is defined by its action on an n-by-n symmetric matrix $\mathbf{X}$ as

$$(\mathbf{A} \overset{s}{\otimes} \mathbf{B})\, \mathrm{s2vec}(\mathbf{X}) := \frac{1}{2} \mathrm{s2vec}(\mathbf{B}\mathbf{X}\mathbf{A}^\top + \mathbf{A}\mathbf{X}\mathbf{B}^\top).$$

Let $\mathbf{Q}$ be the n^2-by-$\frac{n(n+1)}{2}$ orthogonal matrix as defined in 4.4.5 (see (4.6)). For the convenience of the reader, we rewrite its definition and some others here.

Definition 5.1.2. Let the columns be labelled in the order $(1,1), (1,2), \ldots, (1,n)$, $(2,2)$, $(2,3), \ldots, (2,n), \ldots,$ $(n-1,n-1), (n-1,n), (n,n)$, and define

$$\mathbf{Q}_{(i,j)} := \begin{cases} \frac{1}{\sqrt{2}}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), & \text{if } 1 \le i < j \le n, \\ \mathbf{e}_i \otimes \mathbf{e}_i, & \text{if } 1 \le i = j \le n, \end{cases} \tag{5.1}$$

where $\mathbf{e}_i$ is an n dimensional vector with all components equal to zero, except ith component which is equal to one.

As can be observed $\mathbf{Q}$ is a sparse matrix, it has only n^2 nonzero entries out of $n^3(n+1)/2$ entries. Since the columns of $\mathbf{Q}$ form a basis (when matricized to n-by-n matrix) for n-by-n real symmetric matrices, $\mathbf{Q}^\top \mathbf{Q} = \mathbb{I}$. Furthermore, $\mathbf{Q}\mathbf{Q}^\top$ is the orthogonal projector mapping every point in $\mathbb{R}^{n^2}$ onto the set of symmetric vectors in $\mathbb{R}^{n^2}$ [131].

Define a *transpose operator* $\mathcal{T} : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ such that $\mathrm{Mat}(\mathcal{T}\mathbf{x}) = (\mathrm{Mat}(\mathbf{x}))^\top$. Let $\mathbf{T}$ be the n^2-by-n^2 matrix representation of the transpose operator $\mathcal{T}$ (see (4.5)). More specifically, $\mathbf{T}$ is an n^2-by-n^2 permutation matrix with kth column $\mathbf{T}_k$, where $k \in \{1, 2, \ldots, n^2\}$ such that

$$\mathbf{T}_{i+(j-1)n} := \mathbf{e}_i \otimes \mathbf{e}_j, \quad \text{for } i, j \in \{1, 2, \ldots, n\}. \tag{5.2}$$

Definition 5.1.3 (Symmetric/skew-symmetric vector)**.** We call a vector $\mathbf{x} \in \mathbb{R}^{n^2}$ a *symmetric vector* if $\mathbf{T}\mathbf{x} = \mathbf{x}$. It is called a *skew-symmetric vector* if $\mathbf{T}\mathbf{x} = -\mathbf{x}$.

We note here that this definition aligns with Definition 4.4.1 as $\mathbf{T}$ is also a symmetric, involutory matrix.

For a symmetric vector $\mathbf{v} \in \mathbb{R}^{n^2}$, the following identities hold:

$$\mathbf{Q}^\top \mathbf{v} = \mathrm{s2vec}(\mathrm{Mat}(\mathbf{v})), \text{ and } \mathbf{Q}\, \mathrm{s2vec}(\mathrm{Mat}(\mathbf{v})) = \mathbf{v}.$$

Theorem 5.1.1. *Let* $\mathbf{Q}$ *be the orthogonal* $\mathrm{sym}(n)$*-by-*n *matrix defined as in* (5.1). *For every* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$,

$$(\mathbf{A} \overset{s}{\otimes} \mathbf{B}) = \frac{1}{2}\mathbf{Q}^\top(\mathbf{A}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{A})\mathbf{Q} = \mathbf{Q}^\top(\mathbf{A}\otimes\mathbf{B})\mathbf{Q}.$$

Proof. The proof can be found in [32, Appendix E]. □

Although the eigenstructure of Kronecker products is well known, the eigenstructure of symmetric Kronecker products seems much more complicated and so far, much less understood.

5.1.1 Properties of Symmetric Kronecker Product

In this section, we list some properties of symmetric Kronecker products.

Theorem 5.1.2. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{p\times q}$, $\mathbf{C} \in \mathbb{R}^{n\times r}$ *and* $\mathbf{D} \in \mathbb{R}^{q\times s}$. *Then*

(i) $(\mathbf{A} \overset{s}{\otimes} \mathbf{B})^\top = (\mathbf{A}^\top \overset{s}{\otimes} \mathbf{B}^\top)$,

(ii) $(\mathbf{A} \overset{s}{\otimes} \mathbf{B}) = (\mathbf{B} \overset{s}{\otimes} \mathbf{A})$,

(iii) $(\mathbf{A} \overset{s}{\otimes} \mathbf{B})(\mathbf{C} \overset{s}{\otimes} \mathbf{D}) = \frac{1}{2}\Big((\mathbf{AC} \overset{s}{\otimes} \mathbf{BD}) + (\mathbf{AD} \overset{s}{\otimes} \mathbf{BC})\Big)$

(iv)

$$\begin{aligned}(\mathbf{A} \overset{s}{\otimes} \mathbf{B}) \succeq 0 &\iff (\mathbf{A}\otimes\mathbf{B}) \succeq 0\\ (\mathbf{A} \overset{s}{\otimes} \mathbf{B}) \succ 0 &\iff (\mathbf{A}\otimes\mathbf{B}) \succ 0.\end{aligned}$$

(v) Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ *be positive definite and let* $p, q \in \mathbb{R}\setminus(-1,1)$ *such that* $q > p$. *Then,*

$$(\mathbf{A}^q \overset{s}{\otimes} \mathbf{B}^q)^{1/q} \succeq (\mathbf{A}^p \overset{s}{\otimes} \mathbf{B}^p)^{1/p}.$$

(vi) Let Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ *be positive definite. Then* $(\mathbf{A} \overset{s}{\otimes} \mathbf{B})^{-1} \preceq \mathbf{A}^{-1} \overset{s}{\otimes} \mathbf{B}^{-1}$.

(vii) Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ *be positive definite. Then* $(\mathbf{A} \overset{s}{\otimes} \mathbf{B})^2 \preceq \mathbf{A}^2 \overset{s}{\otimes} \mathbf{B}^2$.

Proof. See [32, Appendix E] for the proof of Theorem 5.1.2.(i)-(iii), and [131, Theorem 2.8] for the proof of Theorem 5.1.2.(iv).

(*v*) The proof follows from similar arguments given in Chapter 4.3. For the sake of completeness, we provide it here. In [98], Mond and Pečarić proved the following. Given a positive definite matrix $\mathbf{A}$,

$$(\mathbf{G}^H\mathbf{A}^q\mathbf{G})^{1/q} \succeq (\mathbf{G}^H\mathbf{A}^p\mathbf{G})^{1/p}, \tag{5.3}$$

where $\mathbf{G}$ is an n-by-ℓ matrix such that $\mathbf{G}^H\mathbf{G} = \mathbb{I}$, $p, q \in \mathbb{R} \setminus (-1, 1)$ such that $q > p$. Since $\mathbf{Q}$ given in (5.1) satisfies $\mathbf{Q}^\top\mathbf{Q} = \mathbb{I}$, then applying (5.3) to $\mathbf{A} \otimes \mathbf{B}$, gives

$$(\mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})^q\mathbf{Q})^{1/q} \succeq (\mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})^p\mathbf{Q})^{1/p}.$$

Since $(\mathbf{A} \otimes \mathbf{B})^p = \mathbf{A}^p \otimes \mathbf{B}^p$, and $\mathbf{Q}^\top(\mathbf{A}^p \otimes \mathbf{B}^p)\mathbf{Q} = \mathbf{A}^p \overset{s}{\otimes} \mathbf{B}^p$, the result follows.

$(vi) - (vii)$ Note that (vi) is a special case of (v). Also, as mentioned in Chapter 4.3, for a given convex function f, and a positive definite matrix $\mathbf{A}$

$$f(\mathbf{Q}^H\mathbf{A}\mathbf{V}) \preceq \mathbf{Q}^H f(\mathbf{A})\mathbf{Q}. \tag{5.4}$$

Applying this for $f(t) = t^{-1}$ and $f(t) = t^2$ and using the definition of symmetric Kronecker product yield the results.

□

The symmetric Kronecker product differs in a number of properties compared to the Kronecker product. For example, unlike the Kronecker product, the symmetric Kronecker product is commutative. Also, although the Kronecker product satisfies the associativity property, the symmetric Kronecker product does not satisfy it in general. In addition if $\mathbf{A}$ and $\mathbf{B}$ are arbitrary nonsingular matrices, $(\mathbf{A} \overset{s}{\otimes} \mathbf{B})^{-1}$ does not in general equal to $\mathbf{A}^{-1} \overset{s}{\otimes} \mathbf{B}^{-1}$. For an example of this, we refer the reader to [117, p. 22].

5.2 The Skew-Symmetric Kronecker Product

Similar to the symmetric Kronecker product, we define a *skew-symmetric Kronecker product*. Let $\mathbb{K}^n$ denote the set of n-by-n real skew-symmetric matrices, following the notation from [3]. Then the dimension of this space is $\mathrm{skew}(n) := n(n-1)/2$. Note that $\mathbb{K}^n$ is a linear subspace in $\mathbb{R}^{n^2}$. In this subspace, $\mathbb{S}^n$ and $\mathbb{K}^n$ are orthogonal complements of each other. Define the mapping $\mathrm{kvec} : \mathbb{K}^n \to \mathbb{R}^{\mathrm{skew}(n)}$ as the isometry between $\mathbb{K}^n$ and $\mathbb{R}^{\mathrm{skew}(n)}$ such that for every $\mathbf{X} \in \mathbb{K}^n$,

$$\mathrm{kvec}(\mathbf{X}) := \begin{bmatrix} \sqrt{2}x_{21} & \cdots & \sqrt{2}x_{n1} & \sqrt{2}x_{32} & \sqrt{2}x_{n2} & \cdots & \sqrt{2}x_{n(n-1)} \end{bmatrix}^\top.$$

$\mathrm{kvec}(\mathbf{X})$ is a $\mathrm{sym}(n)$ dimensional vector formed by stacking the columns of the lower triangular part of $\mathbf{X}$ and by multiplying the off-diagonal elements by $\sqrt{2}$ (to preserve the inner product).

Definition 5.2.1. Let $\tilde{\mathbf{Q}} \in \mathbb{R}^{n^2 \times \mathrm{skew}(n)}$ the columns of $\tilde{\mathbf{Q}}$ form a basis for n^2-by-n^2 skew-symmetric matrices and defined by

$$\tilde{\mathbf{Q}}_{(i,j)} := \frac{1}{\sqrt{2}}(\mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i), \quad \text{for } 1 \le i < j \le n, \tag{5.5}$$

where the columns of $\tilde{\mathbf{Q}}$ are labeled in the order $(1,2), \ldots, (1,n)$, $(2,3), \ldots, (2,n), \ldots$, $(n-1, n)$.

By the definition, $\tilde{\mathbf{Q}}^\top \tilde{\mathbf{Q}} = \mathbb{I}$ and $\tilde{\mathbf{Q}}\tilde{\mathbf{Q}}^\top$ is the orthogonal projector, mapping $\mathbb{R}^{n^2}$ onto the set of skew-symmetric vectors in $\mathbb{R}^{n^2}$. Then for a skew-symmetric vector $\mathbf{w} \in \mathbb{R}^{n^2}$,

$$\tilde{\mathbf{Q}}^\top \mathbf{w} = \operatorname{kvec}(\operatorname{Mat}(\mathbf{w})), \text{ and } \tilde{\mathbf{Q}} \operatorname{kvec}(\operatorname{Mat}(\mathbf{w})) = \mathbf{w}.$$

Definition 5.2.2 (Skew-symmetric Kronecker product)**.** The *symmetric Kronecker product* of any two n-by-n matrices $\mathbf{A}$ And $\mathbf{B}$ is defined implicitly by its action on an n-by-n skew-symmetric matrix $\mathbf{W}$ as

$$(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B}) \operatorname{kvec}(\mathbf{W}) := \frac{1}{2} \operatorname{kvec}(\mathbf{B}\mathbf{W}\mathbf{A}^\top + \mathbf{A}\mathbf{W}\mathbf{B}^\top).$$

Theorem 5.2.1. *Let $\tilde{\mathbf{Q}}$ be the orthogonal* skew(n)*-by-n matrix defined as in Definition 5.5. For every $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$*

$$(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B}) = \frac{1}{2}\tilde{\mathbf{Q}}^\top (\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}^\top (\mathbf{A} \otimes \mathbf{B})\tilde{\mathbf{Q}}.$$

Analogous to the symmetric Kronecker product, the following properties hold for the skew-symmetric product. We skip the proofs as they are elementary.

Theorem 5.2.2 (new)**.** *Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{p\times q}$, $\mathbf{C} \in \mathbb{R}^{n\times r}$ and $\mathbf{D} \in \mathbb{R}^{q\times s}$. Then*

(i) $(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B})^\top = (\mathbf{A}^\top \overset{\tilde{s}}{\otimes} \mathbf{B}^\top)$,

(ii) $(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B}) = (\mathbf{B} \overset{\tilde{s}}{\otimes} \mathbf{A})$,

(iii) $(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B})(\mathbf{C} \overset{\tilde{s}}{\otimes} \mathbf{D}) = \frac{1}{2}\Big((\mathbf{AC} \overset{\tilde{s}}{\otimes} \mathbf{BD}) + (\mathbf{AD} \overset{\tilde{s}}{\otimes} \mathbf{BC})\Big)$

(iv)

$$(\mathbf{A}\otimes\mathbf{B}) \succeq 0 \implies (\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B}) \succeq 0,$$
$$(\mathbf{A}\otimes\mathbf{B}) \succ 0 \implies (\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B}) \succ 0.$$

(v) Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ be positive definite and let $p, q \in \mathbb{R} \setminus (-1, 1)$ such that $q > p$. Then,

$$(\mathbf{A}^q \overset{\tilde{s}}{\otimes} \mathbf{B}^q)^{1/q} \succeq (\mathbf{A}^p \overset{\tilde{s}}{\otimes} \mathbf{B}^p)^{1/p}.$$

(vi) Let Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ be positive definite. Then $(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B})^{-1} \preceq \mathbf{A}^{-1} \overset{\tilde{s}}{\otimes} \mathbf{B}^{-1}$.

(vii) Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ be positive definite. Then $(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B})^2 \preceq \mathbf{A}^2 \overset{\tilde{s}}{\otimes} \mathbf{B}^2$.

These properties differ in the item (iv) compared to the properties of the symmetric Kronecker product. Unlike in Theorem 5.1.2-(iv), $(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B}) \succ 0$ does not necessarily imply that $(\mathbf{A}\otimes\mathbf{B}) \succ 0$. To see this, consider the following example.

Example 5.2.1. Let

$$\mathbf{A} := \begin{bmatrix} 44 & 2 & -1 \\ 2 & 31 & -6 \\ -1 & -6 & 7 \end{bmatrix} \text{ and } \mathbf{B} := \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & 0 \\ -2 & 0 & 0 \end{bmatrix}.$$

Although

$$(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B}) = \begin{bmatrix} 108.5 & -6 & -26.5 \\ -6 & 8.5 & 1 \\ 26.5 & 1 & 10.5 \end{bmatrix} \succ 0,$$

the matrix $\mathbf{A} \otimes \mathbf{B}$ has negative eigenvalues, taking

$$\mathbf{v} := \begin{bmatrix} -5 & 2 & -8 & -1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}^\top,$$

gives $\mathbf{v}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{v} = -5063 < 0$.

In the vast research area of semidefinite programming, there has been an increased amount of interest in systems of linear equations involving symmetric Kronecker products which arise when computing search directions in primal-dual interior-point methods [128, 131]. Given the applications of these equations, seeking a better understanding of the eigenvector/eigenvalue structure of the generalized Lyapunov operators constituting these equations seems valuable.

5.3 Jordan-Kronecker Product

Definition 5.3.1 (Jordan-Kronecker Product)**.** Given matrices $\mathbf{A}$, $\mathbf{B}$, the *Jordan-Kronecker product* of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A}).$$

Indeed, this is the Jordan product of $\mathbf{A}$ and $\mathbf{B}$ [41], where the matrix multiplication is replaced by the Kronecker product. This is also related to the notion of *Jordan triple product* [42] since

$$(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})\,\mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{AXB}) + \mathrm{vec}(\mathbf{BXA}), \quad \forall \mathbf{A}, \mathbf{B} \in \mathbb{S}^n.$$

Consider the eigenvalue/eigenvector structure of the Jordan-Kronecker product of $\mathbf{A}$ and $\mathbf{B}$. A nice characterization for the eigenstructure of the Jordan-Kronecker product of n-by-n matrices $\mathbf{A}$ and $\mathbf{B}$ is provided in [131, Section 2] which shows that the eigenvectors of $(\mathbf{A}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{A})$ can be chosen so that they form an orthonormal basis where each eigenvector is either a symmetric vector or a skew-symmetric vector. We restate it below.

Proposition 5.3.1 (Theorem 2.9 in [131])**.** *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. *For every* $\mathbf{u} \in \mathbb{R}^{\frac{1}{2}n(n+1)}$, *we have the eigenpair relationship* $(\mathbf{A} \overset{s}{\otimes} \mathbf{B})\mathbf{u} = \lambda\mathbf{u} \Rightarrow \frac{1}{2}(\mathbf{A}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{A})(\mathbf{Q}^\top\mathbf{u}) = \lambda(\mathbf{Q}^\top\mathbf{u})$.

Henceforth, we say that an eigenvalue of $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ belongs to its *odd spectrum* if it corresponds to a skew-symmetric eigenvector and *even spectrum* if it corresponds to a symmetric eigenvector. Furthermore, we call an eigenvalue of $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ an *odd eigenvalue* if it belongs to its odd spectrum and an *even eigenvalue* if it belongs to its even spectrum.

The characterization of structured matrices by a symmetry property of their eigenvectors is not a new concept. We give a number of such examples in Chapter 4.4. There are various structured matrices other than Jordan-Kronecker products such that for a special involutory matrix $\mathbf{J}$, an eigenvector is called even if $\mathbf{J}\mathbf{x} = \mathbf{x}$ and called odd if $\mathbf{J}\mathbf{x} = -\mathbf{x}$. This terminology has been used for centrosymmetric matrices [6] in which $\mathbf{J}$ is the matrix with ones on the secondary diagonal (from the top right corner to the bottom left corner), and zeros elsewhere and also has been used with a general involutary matrix in [140]. Utilization of this structure leads to efficient solution for the eigenvalue problem (with complexity $\frac{1}{4}$ of the original one) and helps understanding of the properties of the solution preserved through some standard methods to solve Sturm-Liouville problem [6]. Some sufficient conditions on the interlacement of odd and even eigenvalues are provided in [24, Theorem 5, 6] for centrosymmetric matrices, and in [129] for some real symmetric matrices. The interlacement property, alone, is algebraically interesting itself but it also plays an important role in real symmetric Toeplitz matrices in answering the inverse eigenvalue problem [129, 83]. Such structured eigenvectors also arise in *perfect shuffle symmetric matrices* (e.g. matrices $A \in \mathbb{S}^{n^2}$ such that $\mathbf{TAT} = \mathbf{A}$, see for instance [30]) which are used in certain quantum chemistry applications [135]. As we remarked in Chapter 4.4.5, the Jordan-Kronecker product of two matrices is an example of a perfect shuffle invariant matrix, since $\mathbf{T}(\mathbf{A} \otimes \mathbf{B})\mathbf{T} = \mathbf{B} \otimes \mathbf{A}$ and $\mathbf{T}(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})\mathbf{T} = \mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A}$.

In [131], Tunçel and Wolkowicz conjectured interesting interlacing relations on the roots of the characteristic polynomials of certain structured matrices arising from the Jordan-Kronecker products of real symmetric matrices. In this chapter, we investigate these interlacing relationships.

We define a number of interlacing properties which will be used throughout this chapter.

The following property is defined to indicate when the odd spectrum of the Jordan-Kronecker product interlaces its even spectrum in the sense of Definition 1.3.1.

Definition 5.3.2 (Interlacing Property)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be both n-by-n symmetric matrices (or skew-symmetric matrices). Denote the even eigenvalues and the odd eigenvalues of $\mathbf{C} := (\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ in non-increasing order by $\lambda_1 \geq \cdots \geq \lambda_s$ and $\beta_1 \geq \cdots \geq \beta_t$, respectively, where $s := \mathrm{sym}(n)$ and $t := \mathrm{skew}(n)$. We say that the *odd eigenvalues of* $\mathbf{C}$ *interlace its even eigenvalues* if for an eigenvalue β_i belonging to the odd spectrum of $\mathbf{C}$, there are even eigenvalues of $\mathbf{C}$ such that

$$\lambda_{s-t+i} \leq \beta_i \leq \lambda_i, \text{ for } i \in \{1, \ldots, t\}.$$

Definition 5.3.3 (Weak-Interlacing Property)**.** Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. We say that $(\mathbf{A}\otimes\mathbf{B}+\mathbf{B}\otimes\mathbf{A})$ satisfies *weak-interlacing* if

$$\min_{\mathbf{Tu}=\mathbf{u}} \frac{\mathbf{u}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{u}}{\mathbf{u}^\top\mathbf{u}} \leq \min_{\mathbf{Tw}=-\mathbf{w}} \frac{\mathbf{w}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{w}}{\mathbf{w}^\top\mathbf{w}}, \tag{5.6}$$

$$\max_{\mathbf{Tw}=-\mathbf{w}} \frac{\mathbf{w}^\top(\mathbf{A}\otimes\mathbf{B})\mathbf{w}}{\mathbf{w}^\top\mathbf{w}} \le \max_{\mathbf{Tu}=\mathbf{u}} \frac{\mathbf{u}^\top(\mathbf{A}\otimes\mathbf{B})\mathbf{u}}{\mathbf{u}^\top\mathbf{u}}. \tag{5.7}$$

or equivalently,

$$\min_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}(\mathbf{AUBU}) \le \min_{\mathbf{W}\in\mathbb{K}^n, \|\mathbf{W}\|_F=1} \operatorname{tr}\left(\mathbf{AWBW}^\top\right), \tag{5.8}$$

$$\max_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}(\mathbf{AUBU}) \ge \max_{\mathbf{W}\in\mathbb{K}^n, \|\mathbf{W}\|_F=1} \operatorname{tr}\left(\mathbf{AWBW}^\top\right). \tag{5.9}$$

The weak-interlacing property is introduced to avoid repeating the claim of Conjecture 1.3.6. Instead of saying $\mathbf{A},\mathbf{B}\in\mathbb{S}^n$ satisfy the Conjecture 1.3.6, we will say they satisfy the weak interlacing property.

Definition 5.3.4 (Strong Interlacing Property)**.** Let $\mathbf{A},\mathbf{B}\in\mathbb{S}^n$. We say that there is *strong interlacing* between the odd and even eigenvalues of $(\mathbf{A}\otimes\mathbf{B}+\mathbf{B}\otimes\mathbf{A})$ if for each kth odd eigenvalue, where $k\in\{2,\dots,n-1\}$, the $(k+1)$th and $(k-1)$th eigenvalues are associated with even eigenvectors.

Similarly, instead of saying $\mathbf{A},\mathbf{B}\in\mathbb{S}^n$ satisfy the Conjecture 1.3.7, we will say they satisfy the strong interlacing property. Note that this property implies the weak-interlacing property.

Proposition 5.3.2. *Let $\mathbf{A},\mathbf{B}\in\mathbb{S}^n$. Regarding the conjectures, without loss of generality, we may assume $\mathbf{B}$ is a diagonal matrix with diagonal entries sorted in descending order.*

Proof. Let $\mathbf{B}\in\mathbb{S}^n$ have the spectral decomposition $\mathbf{B}=\mathbf{VDV}^\top$, where $\mathbf{D}\in\mathbb{S}^n$ is the diagonal matrix whose diagonal entries are the eigenvalues of $\mathbf{B}$ sorted in descending order, and $\mathbf{V}^\top\mathbf{V}=\mathbf{VV}^\top=\mathbb{I}$. For a given $\mathbf{X}\in\mathbb{S}^n$ with $\|\mathbf{X}\|_F=1$, we define $\mathbf{U}:=\mathbf{V}^\top\mathbf{XV}$ and $\bar{\mathbf{A}}:=\mathbf{V}^\top\mathbf{AV}$. Then $\mathbf{U}^\top=\mathbf{U}$ and $\|\mathbf{U}\|_F^2=1$. Using the commutativity of the trace operator and the orthogonality of $\mathbf{V}$, we get

$$\begin{aligned}\operatorname{tr}(\mathbf{AXBX}) &= \operatorname{tr}\left(\mathbf{AVV}^\top\mathbf{XVV}^\top\mathbf{BVV}^\top\mathbf{XVV}^\top\right)\\ &= \operatorname{tr}\left(\left(\mathbf{V}^\top\mathbf{AV}\right)\left(\mathbf{V}^\top\mathbf{XV}\right)\left(\mathbf{V}^\top\mathbf{BV}\right)\left(\mathbf{V}^\top\mathbf{XV}\right)\right)\\ &= \operatorname{tr}\left(\bar{\mathbf{A}}\mathbf{UDU}\right).\end{aligned}$$

Therefore, instead of solving

$$\min_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}(\mathbf{AUDU})$$

one may equivalently solve

$$\min_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}\left(\bar{\mathbf{A}}\mathbf{UDU}\right)$$

The proofs for the "max" case and for the case when $\mathbf{U}$ is a skew-symmetric matrix follow along similar lines and are omitted.

The transformation $\mathbf{D} \mapsto \mathbf{PDP}^\top$ permutes the rows and columns of $\mathbf{D}$ in the same order. Assume $\mathbf{P}$ sorts the diagonal entries of $\mathbf{D}$ so that $\mathbf{PDP}^\top$ is sorted in descending order. Since

$$\begin{aligned}
\operatorname{tr}\left(\bar{\mathbf{A}}\mathbf{UDU}\right) &= \operatorname{tr}\left(\mathbf{P}^\top\mathbf{P}\bar{\mathbf{A}}\mathbf{P}^\top\mathbf{PUP}^\top\mathbf{PDP}^\top\mathbf{PU}\right)\\
&= \operatorname{tr}\left(\left(\mathbf{P}\bar{\mathbf{A}}\mathbf{P}^\top\right)\left(\mathbf{PUP}^\top\right)\left(\mathbf{PDP}^\top\right)\left(\mathbf{PUP}^\top\right)\right)\\
&= \operatorname{tr}\left(\mathbf{A}'\mathbf{U}'\mathbf{D}'\mathbf{U}\right),
\end{aligned}$$

where $\mathbf{A}' := \mathbf{P}\bar{\mathbf{A}}\mathbf{P}^\top$, $\mathbf{U}' := \mathbf{PUP}^\top$ and $\mathbf{D} = \mathbf{PDP}^\top$ whose diagonal entries are sorted in descending order.

Hence, in investigating the conjectures, without loss of generality we may assume $\mathbf{B}$ is a diagonal matrix with diagonal entries sorted in descending order. □

5.3.1 Preliminaries

In this section, we provide some algebraic results related to the Jordan-Kronecker product and the conjectures.

Lemma 5.3.3. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$*. If every eigenvalue of* $\mathbf{A} \otimes \mathbf{B}$ *has multiplicity one, then any vector satisfying* $\mathbf{Tw} = -\mathbf{w}$ *cannot be an eigenvector of* $\mathbf{A} \otimes \mathbf{B}$*.*

Proof. For the sake of contradiction, suppose $\mathbf{w} \in \mathbb{R}^{n^2} - \{0\}$ with $\|\mathbf{w}\|_2 = 1$ is an eigenvector of $\mathbf{A} \otimes \mathbf{B}$ such that $\mathbf{Tw} = -\mathbf{w}$. By the assumption on the multiplicity of the eigenvalues, $\mathbf{w}$ has the following structure

$$\mathbf{w} := \begin{bmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_1\beta_n \\ \vdots \\ \alpha_n\beta_1 \\ \vdots \\ \alpha_n\beta_n \end{bmatrix},$$

for some $\alpha_i, \beta_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$. However, $\mathbf{Tw} = -\mathbf{w}$ implies $\alpha_i\beta_i = 0, \forall i \in \{1, \ldots, n\}$, as $\mathbf{w}$ is skew symmetric. If $\alpha_i\beta_i = 0$, either $\alpha_i = 0$ or $\beta_i = 0$. Suppose $\alpha_i = 0$. Then $\alpha_i\beta_j = 0$ for all $j \in \{1, 2, \ldots, n\}$. But this implies that

$$\begin{aligned}
0 =& \alpha_i\beta_1 = -\alpha_1\beta_i,\\
0 =& \alpha_i\beta_2 = -\alpha_2\beta_i,\\
&\vdots\\
0 =& \alpha_i\beta_n = -\alpha_n\beta_i.
\end{aligned}$$

From the above equalities, we get that either $\beta_i = 0$ or $\alpha_1 = \cdots = \alpha_n = 0$. The latter implies $\mathbf{w} = 0$, so assume $\beta_i = 0$ must hold. Then $\alpha_j\beta_i = 0$ for all $j \in \{1, 2, \ldots, n\}$. Similarly this implies that

$$\begin{aligned}
0 =& \beta_i\alpha_1 = -\alpha_i\beta_1,\\
0 =& \beta_i\alpha_2 = -\alpha_i\beta_2,\\
&\vdots\\
0 =& \beta_i\alpha_n = -\alpha_i\beta_n.
\end{aligned}$$

Then either $\beta_1 = \beta_2 = \ldots = \beta_n = 0$ or $\alpha_i = 0$. The first implies that $\mathbf{w} = 0$, which is a contradiction, so as a result we get that $\alpha_i = 0 = \beta_i$. Since this holds for each $i \in \{1, 2, \ldots, n\}$, we get $\mathbf{w} = 0$, which is a contradiction. Therefore, we conclude that an eigenvector of $\mathbf{A} \otimes \mathbf{B}$ cannot be skew-symmetric. □

The following are some interesting algebraic results on the projection of the extreme eigenvectors of $\mathbf{A} \otimes \mathbf{B}$ onto symmetric vectors and skew symmetric vectors in $\mathbb{R}^{n^2}$.

Proposition 5.3.4 (new)**.** *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$*. Assume every eigenvalue of* $\mathbf{A}\otimes\mathbf{B}$ *has multiplicity one and* $\mathbf{v}$ *be an eigenvector of* $\mathbf{A} \otimes \mathbf{B}$*. If*

$$\bar{\mathbf{v}} := \arg\min_{\mathbf{Tu}=\mathbf{u}} \|\mathbf{u} - \mathbf{v}\|_2 \quad \textit{and} \quad \tilde{\mathbf{v}} := \arg\min_{\mathbf{Tw}=-\mathbf{w}} \|\mathbf{w} - \mathbf{v}\|_2,$$

for every $i, j \in \{1, 2, \ldots, n\}$

$$\begin{aligned}
\bar{v}_{(i-1)n+j} = \bar{v}_{(j-1)n+i} &= \frac{v_{(i-1)n+j} + v_{(j-1)n+i}}{2},\\
\tilde{v}_{(i-1)n+j} = -\tilde{v}_{(j-1)n+i} &= \frac{v_{(i-1)n+j} - v_{(j-1)n+i}}{2}.
\end{aligned}$$

Furthermore, $||\bar{\mathbf{v}}||_2 \geq ||\tilde{\mathbf{v}}||_2$*.*

Proof. Let $\mathbf{v} := \begin{bmatrix} v_1 & v_2 & \cdots & v_{n^2}\end{bmatrix}^\top$ be an eigenvector of $\mathbf{A} \otimes \mathbf{B}$. It is easy to show that for every $i, j \in \{1, 2, \ldots, n\}$

$$\begin{aligned}
\bar{v}_{(i-1)n+j} = \bar{v}_{(j-1)n+i} &= \frac{v_{(i-1)n+j} + v_{(j-1)n+i}}{2},\\
\tilde{v}_{(i-1)n+j} = -\tilde{v}_{(j-1)n+i} &= \frac{v_{(i-1)n+j} - v_{(j-1)n+i}}{2}.
\end{aligned}$$

Since $\mathbf{v}$ is an eigenvector of $\mathbf{A} \otimes \mathbf{B}$ corresponding to an eigenvalue of multiplicity one, it has the following form

$$\mathbf{v} = \begin{bmatrix}\alpha_1\beta_1 & \alpha_1\beta_2 & \cdots & \alpha_1\beta_n & \cdots & \alpha_n\beta_n\end{bmatrix}^\top,$$

where $\boldsymbol{\alpha} := \begin{bmatrix}\alpha_1 \cdots \alpha_n\end{bmatrix}^\top$ is an eigenvector of $\mathbf{A}$ and $\boldsymbol{\beta} := \begin{bmatrix}\beta_1 \cdots \beta_n\end{bmatrix}^\top$ is an eigenvector of $\mathbf{B}$, and $\alpha_i, \beta_j \in \mathbb{R}$, $i, j \in \{1, \dots, n\}$. Then

$$\begin{aligned}
\|\bar{\mathbf{v}}\|_2^2 - \|\tilde{\mathbf{v}}\|_2^2 &= \left(\frac{\mathbf{Tv}+\mathbf{v}}{2}\right)^\top \left(\frac{\mathbf{Tv}+\mathbf{v}}{2}\right) - \left(\frac{\mathbf{Tv}-\mathbf{v}}{2}\right)^\top \left(\frac{\mathbf{Tv}+\mathbf{v}}{2}\right) \\
&= \mathbf{v}^\top \mathbf{Tv} \\
&= \sum_{i=1}^n v_i^2 + 2\sum_{i<j} v_i v_j \\
&= (\sum_{i=1}^n \alpha_i \beta_i)^2 \\
&= \|\boldsymbol{\alpha} \odot \boldsymbol{\beta}\|_2^2 \geq 0.
\end{aligned}$$

□

Proposition 5.3.5 (new)**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. Assume the largest and the smallest eigenvalues have multiplicity one and $\mathbf{v}_1$ and $\mathbf{v}_n$ be the unit-norm eigenvectors corresponding to the largest and smallest eigenvalues of $\mathbf{A} \otimes \mathbf{B}$, respectively. Let*

$$\mathbf{u}_i := \arg\min_{\mathbf{Tu}=\mathbf{u}} \|\mathbf{u} - \mathbf{v}_i\|_2, \quad \mathbf{w}_i := \arg\min_{\mathbf{Tw}=-\mathbf{w}} \|\mathbf{w} - \mathbf{v}_i\|_2,$$

and let

$$\rho_{\mathbf{u}_i} := \frac{\mathbf{u}_i^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{u}_i}{\mathbf{u}_i^\top \mathbf{u}_i}, \quad \rho_{\mathbf{w}_i} := \frac{\mathbf{w}_i^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{w}_i}{\mathbf{w}_i^\top \mathbf{w}_i},$$

where $i \in \{1, n\}$. Then $\rho_{\mathbf{u}_1} \geq \rho_{\mathbf{w}_1}$ and $\rho_{\mathbf{u}_n} \leq \rho_{\mathbf{w}_n}$.

Proof. First, we note that $\mathbf{u}_i = (\mathbf{v}_i + \mathbf{Tv}_i)/2$ and $\mathbf{w}_i = (\mathbf{v}_i - \mathbf{Tv}_i)/2$ by Proposition 5.3.4. Substituting these expressions into corresponding Rayleigh quotients, we get the following for $i = 1$:

$$\begin{aligned}
\rho_{\mathbf{u}_1} &= \frac{(\mathbf{v}_1 + \mathbf{Tv}_1)^\top (\mathbf{A} \otimes \mathbf{B})(\mathbf{v}_1 + \mathbf{Tv}_1)}{(\mathbf{v}_1 + \mathbf{Tv}_1)^\top (\mathbf{v}_1 + \mathbf{Tv}_1)} \\
&= \frac{\lambda_1 + 2\lambda_1 \mathbf{v}_1^\top \mathbf{Tv}_1 + \mathbf{v}_1^\top \mathbf{T}^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{Tv}_1}{2 + 2\mathbf{v}_1^\top \mathbf{Tv}_1} \\
&= \lambda_1 - \frac{\lambda_1 - \mathbf{v}_1^\top \mathbf{T}^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{Tv}_1}{2 + 2\mathbf{v}_1^\top \mathbf{Tv}_1},
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
\rho_{\mathbf{w}_1} &= \frac{(\mathbf{v}_1 - \mathbf{Tv}_1)^\top (\mathbf{A} \otimes \mathbf{B})(\mathbf{v}_1 - \mathbf{Tv}_1)}{(\mathbf{v}_1 - \mathbf{Tv}_1)^\top (\mathbf{v}_1 - \mathbf{Tv}_1)} \\
&= \lambda_1 - \frac{\lambda_1 - \mathbf{v}_1^\top \mathbf{T}^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{Tv}_1}{2 - 2\mathbf{v}_1^\top \mathbf{Tv}_1}.
\end{aligned} \tag{5.11}$$

Note that $\lambda_1 - \mathbf{v}_1^\top \mathbf{T}^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{Tv}_1 \geq 0$ since $\|\mathbf{Tv}_1\|_2 = 1$ and λ_1 is the largest eigenvalue of $\mathbf{A} \otimes \mathbf{B}$. Due to the assumption on the multiplicity of the largest eigenvalue, $\mathbf{v}_1^\top \mathbf{Tv}_1 = \|\boldsymbol{\alpha} \odot \boldsymbol{\beta}\|_2^2 \geq 0$, for some $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$. Then $2 + 2\mathbf{v}_1^\top \mathbf{Tv}_1 > 2 - 2\mathbf{v}_1^\top \mathbf{Tv}_1$. Hence, from (5.10) and (5.11), we conclude that $\rho_{\mathbf{u}_1} \geq \rho_{\mathbf{w}_1}$.
The proof of $\rho_{\mathbf{u}_n} \leq \rho_{\mathbf{w}_n}$ follows similar lines to the above, therefore it is omitted. □

Under certain multiplicity conditions, Proposition 5.3.4 states that the distance (with respect to ℓ_2-norm) between the eigenvectors of $\mathbf{A} \otimes \mathbf{B}$ and the set of symmetric vectors is shorter than their distance to the skew-symmetric vectors. In addition, Proposition 5.3.5 gives orderings between the Rayleigh quotients of $\mathbf{A} \otimes \mathbf{B}$ for the projected eigenvectors (to the set of symmetric vectors and skew-symmetric vectors). We emphasize that Proposition 5.3.5 does not directly give an answer to the Conjecture 1.3.6.

Lemma 5.3.6. *Let* $\mathbf{A} \in \mathbb{S}^n$*. Then*

$$\max_{\mathbf{U} \in \mathbb{S}^n, \|\mathbf{U}\|_F = 1} \operatorname{tr}(\mathbf{UAU}) = \lambda_1(\mathbf{A})$$

and

$$\min_{\mathbf{U} \in \mathbb{S}^n, \|\mathbf{U}\|_F = 1} \operatorname{tr}(\mathbf{UAU}) = \lambda_n(\mathbf{A}), \tag{5.12}$$

where $\lambda_1(\mathbf{A})$ *is the largest eigenvalue and* $\lambda_n(\mathbf{A})$ *is the smallest eigenvalue of* $\mathbf{A}$*.*

Proof. We have

$$\begin{aligned}
\max_{\substack{\mathbf{U} \in \mathbb{S}^n, \\ \|\mathbf{U}\|_F = 1}} \operatorname{tr}(\mathbf{UAU}) &= \max_{\mathbf{U} \in \mathbb{S}^n} \frac{\operatorname{vec}(\mathbf{U})^\top (\mathbb{I} \otimes \mathbf{A}) \operatorname{vec}(\mathbf{U})}{\operatorname{vec}(\mathbf{U})^\top \operatorname{vec}(\mathbf{U})} \\
&\leq \lambda_1(\mathbb{I} \otimes \mathbf{A}) \\
&= \lambda_1(\mathbf{A}).
\end{aligned}$$

The first line follows from $\operatorname{tr}(\mathbf{UAU}) = \operatorname{vec}(\mathbf{U}^\top)^\top \operatorname{vec}(\mathbf{AU}\,\mathbb{I})$ and the identity (4.1). The second line follows since the maximization of the Rayleigh quotient of $(\mathbb{I} \otimes \mathbf{A})$ is carried over a subset of $\mathbb{R}^{n^2}$. If $\hat{\mathbf{U}} := \mathbf{v}_1 \mathbf{v}_1^\top$, where $\mathbf{v}_1$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_1(A)$, then $\hat{\mathbf{U}} \in \mathbb{S}^n$ with $\|\hat{\mathbf{U}}\|_F = \mathbf{v}_1^\top \mathbf{v}_1 = 1$ and

$$\operatorname{tr}\left(\hat{\mathbf{U}} \mathbf{A} \hat{\mathbf{U}}\right) = \lambda_1(\mathbf{A}).$$

Since the upper bound of $\max_{\mathbf{U} \in \mathbb{S}^n, \|\mathbf{U}\|_F = 1} \operatorname{tr}(\mathbf{UAU})$ is achieved by $\hat{\mathbf{U}}$, the first result follows. The proof of the second result (5.12) is similar to the first one and is omitted here. □

Next, we give a useful result that sheds light on the interlacing relation between the "odd" and "even" eigenvalues of $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ (the ones corresponding to the odd and even eigenvectors).

Proposition 5.3.7. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be both* n*-by-*n *symmetric (or skew-symmetric) matrices. If* $\mathbf{G}(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B})\mathbf{G}^\top$ *is a principal submatrix of* $\mathbf{A} \overset{s}{\otimes} \mathbf{B}$ *for some* $\operatorname{skew}(n)$*-by-*$\operatorname{skew}(n)$ *orthogonal matrix* $\mathbf{G}$*, then the odd eigenvalues of* $\mathbf{C} := (\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ *interlace its even eigenvalues.*

Proof. This proof is based on the factorization of the symmetric perfect shuffle invariant matrices provided in [135, Theorem 2.1]. Let $\mathbf{Q}' := \begin{bmatrix} \mathbf{Q} & \tilde{\mathbf{Q}} \end{bmatrix}$, where $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$ are defined in

Definition 5.1.2 and Definition 5.5. Then

$$\begin{aligned}\mathbf{Q}'^{\top}\mathbf{C}\mathbf{Q}' &= \begin{bmatrix}\mathbf{Q}^{\top}\mathbf{C}\mathbf{Q} & \mathbf{Q}^{\top}\mathbf{C}\tilde{\mathbf{Q}}\\ \tilde{\mathbf{Q}}^{\top}\mathbf{C}\mathbf{Q} & \tilde{\mathbf{Q}}^{\top}\mathbf{C}\tilde{\mathbf{Q}}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{Q}^{\top}\mathbf{C}\mathbf{Q} & 0\\ 0 & \tilde{\mathbf{Q}}^{\top}\mathbf{C}\tilde{\mathbf{Q}}\end{bmatrix} = \begin{bmatrix}2(\mathbf{A}\overset{s}{\otimes}\mathbf{B}) & 0\\ 0 & 2(\mathbf{A}\overset{\tilde{s}}{\otimes}\mathbf{B})\end{bmatrix}.\end{aligned}$$

The off diagonal blocks are zero since

$$\mathbf{Q}^{\top}\mathbf{C}\tilde{\mathbf{Q}} = \mathbf{Q}^{\top}\mathbf{T}\mathbf{C}\mathbf{T}\tilde{\mathbf{Q}} = -\mathbf{Q}^{\top}\mathbf{T}\mathbf{C}\mathbf{T}\tilde{\mathbf{Q}} = -\mathbf{Q}^{\top}\mathbf{C}\tilde{\mathbf{Q}}.$$

Let $(\mathbf{A}\overset{s}{\otimes}\mathbf{B}) = \mathbf{U}\boldsymbol{\Lambda}_e\mathbf{U}^{\top}$ and $(\mathbf{A}\overset{\tilde{s}}{\otimes}\mathbf{B}) = \mathbf{V}\boldsymbol{\Lambda}_o\mathbf{V}^{\top}$ be the spectral decomposition of $(\mathbf{A}\overset{s}{\otimes}\mathbf{B})$ and $(\mathbf{A}\overset{\tilde{s}}{\otimes}\mathbf{B})$, respectively. Then

$$\mathbf{C} = \begin{bmatrix}\mathbf{Q}\mathbf{U} & \tilde{\mathbf{Q}}\mathbf{V}\end{bmatrix}\begin{bmatrix}2\boldsymbol{\Lambda}_e & 0\\ 0 & 2\boldsymbol{\Lambda}_o\end{bmatrix}\begin{bmatrix}\mathbf{Q}\mathbf{U} & \tilde{\mathbf{Q}}\mathbf{V}\end{bmatrix}^{\top}.$$

By the definition of $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$, the columns of $\mathbf{QU}$ are symmetric and the columns of $\mathbf{QV}$ are skew-symmetric. Therefore, the even spectrum of $\mathbf{C}$ consists of the eigenvalues of $2(\mathbf{A}\overset{s}{\otimes}\mathbf{B})$ and the odd spectrum of $\mathbf{C}$ consists of the eigenvalues of $2(\mathbf{A}\overset{\tilde{s}}{\otimes}\mathbf{B})$. Since $\mathbf{G}$ is an orthogonal matrix, the eigenvalues of $\mathbf{G}(\mathbf{A}\overset{\tilde{s}}{\otimes}\mathbf{B})\mathbf{G}^{\top}$ is the same as the eigenvalues of $(\mathbf{A}\overset{\tilde{s}}{\otimes}B)$. Then the result follows by Theorem 1.3.5.

□

Proposition 5.3.8. *Let* $\mathbf{A},\mathbf{B}\in\mathbb{S}^3$ *where* $\mathbf{A} := [a_{ij}]$ *and* $\mathbf{B} := \operatorname{Diag}(b_1,b_2,b_3)$. *Then, for every* $\mathbf{W}\in\mathbb{K}^3$, *there exists* $\mathbf{U}\in\mathbb{S}^3$ *such that* $\operatorname{Tr}(\mathbf{AUBU}) \le \operatorname{Tr}(A\mathbf{WBW}^{\top})$ *if* $b_1b_2b_3a_{21}a_{31}a_{32}\le 0$.

5.3.2 Cases when interlacing properties hold

We show that the odd spectrum of the Jordan-Kronecker product interlaces its even spectrum for a number of structured matrices.

Theorem 5.3.9 (new)**.** *Let* $\mathbf{A},\mathbf{B}\in\mathbb{S}^n$ *such that* $\min\{\operatorname{rank}(\mathbf{A}),\operatorname{rank}(\mathbf{B})\}\le 2$. *Then the odd eigenvalues of* $(\mathbf{A}\otimes\mathbf{B}+\mathbf{B}\otimes\mathbf{A})$ *interlace its even eigenvalues.*

Proof. Assume $\mathbf{A},\mathbf{B}\in\mathbb{S}^n$. Without loss of generality, we may assume $\operatorname{rank}(\mathbf{B})\le 2$. So, we let $\mathbf{A} := \sum_{i=1}^n \alpha_i\mathbf{a}_i\mathbf{a}_i^{\top}$ and $\mathbf{B} := \beta_1\mathbf{e}_1\mathbf{e}_1^{\top} + \beta_2\mathbf{e}_2\mathbf{e}_2^{\top}$, where $\mathbf{e}_i$ is a vector of all zeros except its ith term is 1. (We used Proposition 5.3.2.)

Denote the last $n-2$ entries of $\mathbf{a}_i$ by $\underline{\mathbf{a}}_i := \begin{bmatrix}a_{i3} & a_{i4} & \cdots & a_{in}\end{bmatrix}^{\top}$ and the jth entry of $\mathbf{a}_i$ by a_{ij}. Let $\mathbf{W} := [w_{ij}]\in\mathbb{K}^n$ with $\|\mathbf{W}\|_F = 1$. Define $\mathbf{w} := \operatorname{vec}(\mathbf{W})$ and $\underline{\mathbf{w}}_i :=$

$\begin{bmatrix} w_{3i} & w_{4i} & \cdots & w_{ni} \end{bmatrix}^\top$, which consists of the last $n-2$ entries of $\mathbf{W}(:,i)$, where $i \in \{1,2\}$. Then

$$\begin{aligned}
\mathbf{w}^\top(\mathbf{B}\otimes\mathbf{A})\mathbf{w} &= \beta_1\left(\alpha_1\left(W(:,1)^\top\mathbf{a}_1\right)^2+\cdots+\alpha_n\left(\mathbf{W}(:,1)^\top\mathbf{a}_n\right)^2\right)\\
&+\beta_2\left(\alpha_1\left(\mathbf{W}(:,2)^\top\mathbf{a}_1\right)^2+\cdots+\alpha_n\left(\mathbf{W}(:,2)^\top\mathbf{a}_n\right)^2\right)\\
&=\beta_1\sum_{i=1}^n\alpha_i\left(w_{21}^2a_{i2}^2+(\underline{\mathbf{w}}_1^\top\underline{\mathbf{a}}_i)^2+2w_{21}a_{i2}\left(\underline{\mathbf{w}}_1^\top\underline{\mathbf{a}}_i\right)\right)\\
&+\beta_2\sum_{i=1}^n\alpha_i\left(w_{21}^2a_{i1}^2+(\underline{\mathbf{w}}_2^\top\underline{\mathbf{a}}_i)^2-2w_{21}a_{i1}\left(\underline{\mathbf{w}}_2^\top\underline{\mathbf{a}}_i\right)\right).
\end{aligned}$$

Let $\mathbf{U} := [u_{ij}] \in \mathbb{S}^n$ such that all of its diagonal elements are zeros. Define $\mathbf{u} := \mathrm{vec}(\mathbf{U})$ and $\underline{\mathbf{u}}_i := \begin{bmatrix} u_{3i} & u_{4i} & \cdots & u_{ni} \end{bmatrix}^\top$, for $i \in \{1,2\}$. Choosing $u_{21} := -w_{21}$, $\underline{\mathbf{u}}_1 := -\underline{\mathbf{w}}_1$ and assigning the upper triangular part of $\mathbf{W}(2:n,2:n)$ to the upper triangular part of $\mathbf{U}(2:n,2:n)$ gives $\|\mathbf{U}\|_F = \|\mathbf{W}\|_F = 1$. Then

$$\begin{aligned}
\mathbf{u}^\top(\mathbf{B}\otimes\mathbf{A})\mathbf{u} &= \beta_1\left(\alpha_1\left(\mathbf{U}(:,1)^\top\mathbf{a}_1\right)^2+\cdots+\alpha_n\left(\mathbf{U}(:,1)^\top\mathbf{a}_n\right)^2\right)\\
&+\beta_2\left(\alpha_1\left(\mathbf{U}(:,2)^\top\mathbf{a}_1\right)^2+\cdots+\alpha_n\left(\mathbf{U}(:,2)^\top\mathbf{a}_n\right)^2\right)\\
&=\beta_1\sum_{i=1}^n\alpha_i\left(u_{21}^2a_{i2}^2+(\underline{\mathbf{u}}_1^\top\underline{\mathbf{a}}_i)^2+2u_{21}a_{i2}\left(\underline{\mathbf{u}}_1^\top\underline{\mathbf{a}}_i\right)\right)\\
&+\beta_2\sum_{i=1}^n\alpha_i\left(u_{21}^2a_{i1}^2+(\underline{\mathbf{u}}_2^\top\underline{\mathbf{a}}_i)^2+2u_{21}a_{i1}\left(\underline{\mathbf{u}}_2^\top\underline{\mathbf{a}}_i\right)\right)\\
&=\mathbf{w}^\top(\mathbf{B}\otimes\mathbf{A})\mathbf{w}.
\end{aligned}$$

This shows that for a given $\mathbf{W} \in \mathbb{K}^n$ with $\|\mathbf{W}\|_F = 1$, one can find $\mathbf{U} \in \mathbb{S}^n$ with $\|\mathbf{U}\|_F = 1$ such that

$$\mathrm{tr}\left(\mathbf{AWBW}^\top\right) = \mathrm{tr}\left(\mathbf{AUBU}\right).$$

Therefore, the weak interlacing property holds. Note that the claim of this theorem is stronger than the weak interlacing. Define the diagonal matrix $\boldsymbol{\Phi} := [\phi_{ij}] \in \mathbb{S}^{\mathrm{skew}(n)}$ by

$$\phi_{kk} := \begin{cases} -1, & \text{if } k \in \{1,2,\ldots,n-1\},\\ 1, & \text{otherwise.}\end{cases}$$

Then $\boldsymbol{\Phi}(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B})\boldsymbol{\Phi}$ is a principal submatrix of $\mathbf{A} \overset{s}{\otimes} \mathbf{B}$.

To see this, observe that if we define $\mathbf{u} := \mathrm{vec}(\mathbf{U})$, where $\mathbf{U} := [u_{ij}] \in \mathbb{S}^n$, then $\mathbf{QQ}^\top\mathbf{u} = \mathbf{u}$. Then

$$\mathbf{u}^\top(\mathbf{B}\otimes\mathbf{A})\mathbf{u} = \mathbf{u}^\top\mathbf{Q}\left(\mathbf{Q}^\top(\mathbf{B}\otimes\mathbf{A})\mathbf{Q}\right)\mathbf{Q}^\top\mathbf{u} = \mathbf{u}^\top\mathbf{Q}\left(\mathbf{A}\overset{s}{\otimes}\mathbf{B}\right)\mathbf{Q}^\top\mathbf{u}.$$

Let $\mathbf{W} := [b_{ij}] \in \mathbb{K}^n$ be such that the upper triangular part of $\mathbf{W}$ is the same as $\mathbf{U}$. Let $\bar{\mathbf{w}} = \mathrm{vec}(\bar{\mathbf{W}})$, where $\bar{\mathbf{W}} := [\bar{w}_{ij}] \in \mathbb{K}^n$ be such that its first row (exlcuding the first entry) is the negative of the first row exlcuding the first entry) of $\mathbf{U}$. Let the rest of the upper triangular part of $\bar{\mathbf{W}}$ be the same as of $\mathbf{U}$. Note that if one removes the entries of $\text{s2vec}(\mathbf{U}) := \mathbf{Q}^\top \mathbf{u}$ corresponding to the diagonal entries then we get exactly $\text{kvec}(\mathbf{W}) = \tilde{\mathbf{Q}}^\top \mathbf{w}$. Let the diagonals of $\mathbf{U}$ be all zero. Then,

$$
\begin{aligned}
\mathbf{u}^\top \mathbf{Q}(\mathbf{A} \overset{s}{\otimes} \mathbf{B})\mathbf{Q}^\top \mathbf{u} &= \mathbf{u}^\top \mathbf{Q} \left(\mathbf{Q}^\top (\mathbf{B} \otimes \mathbf{A})\mathbf{Q}\right) \mathbf{Q}^\top \mathbf{u} \\
&= \bar{\mathbf{w}}^\top \tilde{\mathbf{Q}} \left(\tilde{\mathbf{Q}}^\top (\mathbf{B} \otimes \mathbf{A})\tilde{\mathbf{Q}}\right) \tilde{\mathbf{Q}}^\top \bar{\mathbf{w}} \\
&= \mathbf{w}^\top \tilde{\mathbf{Q}} \boldsymbol{\Phi} \left(\tilde{\mathbf{Q}}^\top (\mathbf{B} \otimes \mathbf{A})\tilde{\mathbf{Q}}\right) \boldsymbol{\Phi} \tilde{\mathbf{Q}}^\top \mathbf{w} \\
&= \mathbf{w}^\top \tilde{\mathbf{Q}} \boldsymbol{\Phi} (\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B}) \boldsymbol{\Phi} \tilde{\mathbf{Q}}^\top \mathbf{w}.
\end{aligned}
$$

Since this holds for every $\mathbf{U} \in \mathbb{S}^n$ with zero diagonals, $\boldsymbol{\Phi}(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B})\boldsymbol{\Phi}$ is a principal submatrix of $\mathbf{A} \overset{s}{\otimes} \mathbf{B}$.

Therefore, by Theorem 1.3.5 and Proposition 5.3.7, the odd eigenvalues of $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ interlace its even eigenvalues, i.e., the interlacing property holds. $\square$

Corollary 5.3.10. *If $\mathbf{A}, \mathbf{B} \in \mathbb{S}^2$, then the interlacing property and the strong interlacing property hold.*

Although the weak interlacing property [131, Conjecture 2.10] is stated for real symmetric matrices, we show that it holds for certain real skew-symmetric matrices as well. Real skew-symmetric matrices have purely imaginary eigenvalues; on the other hand, the Jordan-Kronecker product of two skew-symmetric matrices are symmetric and therefore have real eigenvalues. In the following, we show that the odd eigenvalues of the Jordan-Kronecker product of two skew-symmetric matrices (for which one of the matrices has rank at most two) interlace its even eigenvalues.

Theorem 5.3.11 (new)**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{K}^n$ such that $\min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\} \leq 2$. Then the odd eigenvalues of $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ interlace its even eigenvalues.*

Proof. Without loss of generality, we may assume $\text{rank}(\mathbf{B}) \leq 2$. By the block diagonalization of skew-symmetric matrices, we may write B as

$$
\mathbf{B} := \begin{bmatrix} 0 & \lambda_1 & 0 & \cdots & 0 \\ -\lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix},
$$

where $\lambda_1 \in \mathbb{R}$. Let $\bar{\mathbf{W}} \in \mathbb{K}^n$ be given. Then

$$
\text{tr}\left(\mathbf{A}\bar{\mathbf{W}}\mathbf{B}\bar{\mathbf{W}}^\top\right) = \lambda_1 \bar{\mathbf{W}}(:,2)^\top \mathbf{A}\bar{\mathbf{W}}(:,1) - \lambda_1 \bar{\mathbf{W}}(:,1)^\top \mathbf{A}\bar{\mathbf{W}}(:,2).
$$

We construct a symmetric matrix $\bar{\mathbf{U}} \in \mathbb{S}^n$ as follows. Here $\mathbf{U}(i,j)$ means the ijth entry of the matrix $\mathbf{U}$.
$\bar{\mathbf{U}}(1,1) = \bar{\mathbf{W}}(2,1)$ $\bar{\mathbf{U}}(2,2) = \bar{\mathbf{W}}(2,1), \bar{\mathbf{U}}(2,1) = 0, \bar{\mathbf{U}}(3:n,1) = \bar{\mathbf{W}}(3:n,2)$, $\bar{\mathbf{U}}(3:n,2) = -\bar{\mathbf{W}}(3:n,1)$, and we assign the upper triangular part of
$\bar{\mathbf{W}}(3:n,3:n)$ (including its main diagonal) to the upper triangular part of $\bar{\mathbf{U}}(3:n,3:n)$ which results in $\|\bar{\mathbf{U}}\|_F = \|\bar{\mathbf{W}}\|_F = 1$. Then

$$\operatorname{tr}\left(\mathbf{A}\bar{\mathbf{W}}\mathbf{B}\bar{\mathbf{W}}^\top\right) = \operatorname{tr}\left(\mathbf{A}\bar{\mathbf{U}}\mathbf{B}\bar{\mathbf{U}}\right).$$

Hence, Conjecture 1.3.6 holds. The proof of interlacing is very similar to the proof in Theorem 5.3.9. Therefore, it is omitted. □

Corollary 5.3.12. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{K}^2$. Then the interlacing property and the strong interlacing property hold.*

Consider mutually diagonalizable matrices. Some examples of this class are diagonal matrices and circulant matrices which are used extensively in signal processing and statistics. A result from [131] shows that the strong interlacing (which implies the weak interlacing) property holds for pair of mutually diagonalizable matrices. We restate it below.

Lemma 5.3.13. [131, Corollary 2.5] *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ be commuting matrices. Let λ_i, μ_j denote the eigenvalues and $\mathbf{v}_i, \mathbf{v}_j$ be the corresponding eigenvectors of $\mathbf{A}$ and $\mathbf{B}$, respectively. Then, for $1 \le i \le j \le n$, we get $\frac{1}{2}(\lambda_i\mu_j + \lambda_j\mu_i)$ as the eigenvalues and $\operatorname{s2vec}(\mathbf{v}_i\mathbf{v}_j^\top + \mathbf{v}_j\mathbf{v}_i^\top)$ as the corresponding eigenvectors of $\mathbf{A} \overset{s}{\otimes} \mathbf{B}$.*

Lemma 5.3.13 also applies to a symmetrized similarity operator [144] as follows. For every nonsingular matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$, [144] defines $H_P : \mathbb{R}^{n\times n} \to \mathbb{S}^n$ by

$$H_P(\mathbf{X}) := \mathbf{P}\mathbf{X}\mathbf{P}^{-1} + \mathbf{P}^{-\top}\mathbf{X}^\top\mathbf{P}^\top.$$

Let us restrict the domain of H_P to $\mathbb{S}^n$ and restrict $\mathbf{P}$ to symmetric matrices. Then the resulting operator H_P is representable by a Jordan-Kronecker product of symmetric matrices $\mathbf{P}$ and $\mathbf{P}^{-1}$. Since $\mathbf{P}, \mathbf{P}^{-1}$ commute, Lemma 5.3.13 applies. We discuss the role of this operator in Chapter 6.3.

Given the result for mutually diagonalizable matrices, an interesting direction to explore is to determine how one can perturb $\mathbf{A}$ or $\mathbf{B}$ so that the weak (or the strong) interlacing property will still be preserved. We provide some results which are based on perturbing one of the matrices.

The next two propositions can be proved using the proof technique given for Theorem 5.3.9. So, proofs of Propositions 5.3.14 and 5.3.15 are omitted.

Proposition 5.3.14 (new)**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ be diagonal matrices, where $\mathbf{A}_{kk} = a_k$, for every $k \in \{1, 2, \dots, n\}$. Let*

$$\tilde{\mathbf{A}} := \mathbf{A} + \sum_{i=1}^{\lfloor k/2 \rfloor} \alpha_i^{(k)}(\mathbf{E}_{i(k-i+1)} + \mathbf{E}_{i(k-i+1)}^\top),$$

i.e.,

$$\tilde{\mathbf{A}} := \begin{bmatrix} a_1 & 0 & \cdots & \cdots & \cdots & 0 & \alpha_1^{(k)} & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 & \alpha_2^{(k)} & 0 & \vdots & \cdots & 0 \\ \vdots & 0 & \ddots & & ⋰ & 0 & \vdots & & \cdots & 0 \\ \vdots & \vdots & \vdots & ⋰ & ⋰ & \vdots & \vdots & \vdots & & \vdots \\ \vdots & 0 & ⋰ & ⋰ & \ddots & 0 & \vdots & & & \\ 0 & \alpha_2^{(k)} & 0 & \cdots & 0 & a_{k-1} & 0 & \vdots & & \vdots \\ \alpha_1^{(k)} & 0 & \vdots & & \vdots & 0 & a_k & 0 & & \\ 0 & \vdots & \vdots & & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix},$$

where $\mathbf{E}_{ik}$ *is an* n*-by-*n *matrix with all entries zero except* $\mathbf{E}(i,k) = 1$*, and* $\alpha_i^{(k)} \in \mathbb{R}$*,* $i \in \{1, 2, \ldots, \lfloor k/2 \rfloor\}$*. Then, the weak interlacing property holds for the pair* $\tilde{\mathbf{A}}, \mathbf{B}$*.*

Proposition 5.3.15. *Let* $\mathbf{A}$ *be an* n*-by-*n *symmetric tridiagonal matrix and* $\mathbf{B}$ *be an* n*-by-*n *diagonal matrix. Then the odd eigenvalues of* $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ *interlace its even eigenvalues. Furthermore, define* $\bar{\mathbf{A}} \in \mathbb{S}^n$ *such that* $\bar{\mathbf{A}}_{ij} = \mathbf{A}_{ij}$ *for all* $i, j \in \{1, \ldots, n\}$ *except* $\mathbf{A}_{rs} \neq 0$*,* $(r > s+1)$*, and* $\mathbf{A}_{(s+1)s} = 0$ *for some* r, s*. Then the odd eigenvalues of* $\left(\bar{\mathbf{A}} \otimes \mathbf{B} + \mathbf{B} \otimes \bar{\mathbf{A}}\right)$ *interlace its even eigenvalues.*

Proposition 5.3.15 gives another sufficient condition for the weak interlacing property. For every pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ where $\mathbf{B}$ is diagonal and $\mathbf{A}$ is formed by perturbing a diagonal matrix where each row of the upper triangular matrix of $\mathbf{A}$ has at most one nonzero entry, the odd eigenvalues of $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ interlace its even eigenvalues.

Some minor results for $\mathbf{A}, \mathbf{B} \in \mathbb{S}^3$ are given below.

Proposition 5.3.16. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}^3$*, where* $\mathbf{A} := [a_{ij}]$ *and* $\mathbf{B} := \mathrm{Diag}(b_1, b_2, b_3)$ *is a diagonal matrix. Then, the odd eigenvalues of* $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ *interlace the even eigenvalues when* $b_1 b_2 b_3 a_{21} a_{31} a_{32} = 0$*.*

Proposition 5.3.17. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}^3$*, where* $\mathbf{B}$ *is a diagonal unimodular matrix. (Recall, a* unimodular matrix *is a square matrix whose determinant is either* $+1$ *or* -1*.) Then the interlacing property holds for* $\mathbf{A}, \mathbf{B}$ *if one of the followings holds:*

1. $a_{11} = a_{22}$,
2. $a_{31} = \pm a_{32}$,
3. *one of the non-diagonal entries of* $\mathbf{A}$ *is zero.*

Proof. Without loss of generality we may assume $\mathbf{B} := \mathrm{Diag}(1, 1, -1)$. Let $\mathbf{U} \in \mathbb{S}^3$ and $\mathbf{W} \in \mathbb{K}^3$. Since

$$\begin{aligned}\operatorname{tr}(\mathbf{AUBU}) = {} & u_{21}^2(b_1a_{22} + b_2a_{11}) + u_{31}^2(b_1a_{33} + b_3a_{11}) + u_{32}^2(b_2a_{33} + b_3a_{22}) \\ & + u_{11}^2(b_1a_{11}) + u_{22}^2(b_2a_{22}) + u_{33}^2(b_3a_{33}) \\ & + 2a_{21}(b_1u_{11}u_{21} + b_2u_{21}u_{22} + b_3u_{31}u_{32}) \\ & + 2a_{31}(b_1u_{11}u_{31} + b_3u_{31}u_{33} + b_2u_{21}u_{32}) \\ & + 2a_{32}(b_2u_{22}u_{32} + b_3u_{32}u_{33} + b_1u_{21}u_{31})\end{aligned}$$

the eigenvalues corresponding to the symmetric eigenvectors of $\mathbf{C}$ are the eigenvalues of

$$\bar{\mathbf{C}}_{sym} := \begin{bmatrix} a_{11} & 0 & 0 & \frac{a_{21}}{\sqrt{2}} & \frac{a_{31}}{\sqrt{2}} & 0 \\ 0 & a_{22} & 0 & \frac{a_{21}}{\sqrt{2}} & 0 & \frac{a_{32}}{\sqrt{2}} \\ 0 & 0 & a_{33} & 0 & -\frac{a_{31}}{\sqrt{2}} & -\frac{a_{32}}{\sqrt{2}} \\ \frac{a_{21}}{\sqrt{2}} & \frac{a_{21}}{\sqrt{2}} & 0 & \frac{a_{11}+a_{22}}{2} & \frac{a_{32}}{2} & \frac{a_{31}}{2} \\ \frac{a_{31}}{\sqrt{2}} & 0 & -\frac{a_{31}}{\sqrt{2}} & \frac{a_{32}}{2} & \frac{a_{33}-a_{11}}{2} & -\frac{a_{21}}{2} \\ 0 & \frac{a_{32}}{\sqrt{2}} & -\frac{a_{32}}{\sqrt{2}} & \frac{a_{31}}{2} & -\frac{a_{21}}{2} & \frac{a_{33}-a_{22}}{2} \end{bmatrix}.$$

Also,

$$\begin{aligned}\operatorname{tr}\left(\mathbf{AWBW}^\top\right) = {} & (a_{11} + a_{22})w_{21}^2 + (a_{33} - a_{11})w_{31}^2 + (a_{33} - a_{22})w_{32}^2 \\ & + 2(w_{21}w_{31}a_{32} - w_{21}w_{32}a_{31} - w_{31}w_{32}a_{21}).\end{aligned}$$

Therefore, the eigenvalues corresponding to the skew-symmetric eigenvectors of $\mathbf{C} := \mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A}$ are the eigenvalues of

$$\bar{\mathbf{C}}_{skew} := \frac{1}{2}\begin{bmatrix} (a_{11} + a_{22}) & a_{32} & a_{31} \\ a_{32} & (a_{33} - a_{11}) & a_{21} \\ a_{31} & a_{21} & (a_{33} - a_{22}) \end{bmatrix}.$$

1. First, we assume $a_{11} = a_{22}$. Choosing

$$\mathbf{Q}_1 := \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

   we see that $\bar{\mathbf{C}}_{skew} = \mathbf{Q}_1\bar{\mathbf{C}}_{sym}\mathbf{Q}_1^\top$ since $a_{11} = a_{22}$. Therefore, by Interlacing Theorem (Theorem 1.3.5), the eigenvalues of C corresponding to the symmetric eigenvectors interlace its eigenvalues corresponding to the skew symmetric eigenvectors.

2. Next, we assume $a_{31} = sa_{32}$, where $s \in \{+1, -1\}$. Choosing

$$\mathbf{Q}_s := \begin{bmatrix} s\frac{1}{\sqrt{2}} & -s\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

   gives $\bar{\mathbf{C}}_{skew} = \mathbf{Q}_s\bar{\mathbf{C}}_{sym}\mathbf{Q}_s^\top$ when $a_{31} = sa_{32}$.

3. Finally, we assume that one of the non-diagonal entries of $\mathbf{A}$ is zero.
If $a_{21} = 0$, choosing
$$\mathbf{Q}_2 := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
gives $\bar{\mathbf{C}}_{skew} = \mathbf{Q}_2 \bar{\mathbf{C}}_{sym} \mathbf{Q}_2^\top$.
If $a_{31} = 0$, choosing
$$\mathbf{Q}_3 := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$
gives $\bar{\mathbf{C}}_{skew} = \mathbf{Q}_3 \bar{\mathbf{C}}_{sym} \mathbf{Q}_3^\top$.
If $a_{32} = 0$, choosing
$$\mathbf{Q}_4 := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
gives $\bar{\mathbf{C}}_{skew} = \mathbf{Q}_4 \bar{\mathbf{C}}_{sym} \mathbf{Q}_4^\top$. Therefore, the result follows by Theorem 1.3.5.

□

Although we provided some results regarding the interlacing properties of $\mathbf{A}, \mathbf{B} \in \mathbb{S}^3$, we still do not have any counterexamples or a proof whether the interlacing properties hold in general.

5.3.3 Cases when interlacing conjectures fail

Proposition 5.3.18 (new)**.** *For every integer $k \geq 3$ and for every integer $n \geq \max\{4, k\}$, there exist symmetric matrices $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ such that $\min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\} = k$, $\max\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\} = n$ and the weak interlacing property fails for the pair $\mathbf{A}, \mathbf{B}$.*

Proof. We prove this using Theorem 5.3.9.

- Consider the following 4-by-4 symmetric matrices
$$\mathbf{A}_0 := \begin{bmatrix} -2 & -1 & -4 & 2 \\ -1 & 1 & -4 & -3 \\ -4 & -4 & 1 & 0 \\ 2 & -3 & 0 & 2 \end{bmatrix} \text{ and } \mathbf{B}_0 := \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \tag{5.13}$$
Here, $\min\{\operatorname{rank}(\mathbf{A}_0), \operatorname{rank}(\mathbf{B}_0)\} = 3$. For the following skew-symmetric matrix
$$\mathbf{W}_0 := \begin{bmatrix} 0 & 9 & -6 & -10 \\ -9 & 0 & 4 & -5 \\ 6 & -4 & 0 & -5 \\ 10 & 5 & 5 & 0 \end{bmatrix},$$

define $\mathbf{w}_0 := \mathrm{vec}(W_0)$. Then

$$\rho_{\mathbf{w}_0}(\mathbf{A}_0, \mathbf{B}_0) := \frac{\mathbf{w}_0{}^\top (\mathbf{A}_0 \otimes \mathbf{B}_0)\mathbf{w}_0}{\mathbf{w}_0{}^\top \mathbf{w}_0} = \frac{6311}{566} \geq 11.$$

If we show that for every symmetric vector $\mathbf{u}$,

$$\mathit{\Delta}_{\mathbf{u}} := \mathbf{u}^\top (\mathbf{A}_0 \otimes \mathbf{B}_0)\mathbf{u} - 11\mathbf{u}^\top \mathbf{u} < 0,$$

then that will imply the maximum eigenvalue of $C_0 := (\mathbf{A}_0 \otimes \mathbf{B}_0 + \mathbf{B}_0 \otimes \mathbf{A}_0)$ corresponds to a skew-symmetric vector. Note that we can get a lower dimensional quadratic representation of $\mathit{\Delta}_{\mathbf{u}}$ by gathering the terms for each distinct entry u_{ij}.

$$\begin{aligned}
\mathit{\Delta}_{\mathbf{u}} &= \mathbf{u}^\top (\mathbf{A}_0 \otimes \mathbf{B}_0)\mathbf{u} - 11\mathbf{u}^\top \mathbf{u} \\
&= \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \\ u_{22} \\ u_{32} \\ u_{42} \\ u_{33} \\ u_{43} \\ u_{44} \end{bmatrix}^\top
\begin{bmatrix}
-17 & -3 & -12 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -23 & -12 & -9 & -2 & -8 & 4 & 0 & 0 & 0 \\
-12 & -12 & -19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & -9 & 0 & -12 & 0 & 0 & 2 & 0 & 8 & -4 \\
0 & -2 & 0 & 0 & -9 & -8 & -6 & 0 & 0 & 0 \\
0 & -8 & 0 & 0 & -8 & -20 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 2 & -6 & 0 & -20 & 0 & 8 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -11 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 8 & 0 & -24 & 0 \\
0 & 0 & 0 & -4 & 0 & 0 & 6 & 0 & 0 & -15
\end{bmatrix}
\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \\ u_{22} \\ u_{32} \\ u_{42} \\ u_{33} \\ u_{43} \\ u_{44} \end{bmatrix}.
\end{aligned}$$

It suffices to show that the above 10-by-10 matrix, which we denote by $\mathbf{C}_s$, is negative definite, or its negative is positive definite. We show that $-\mathbf{C}_s$ is positive definite.

We observe that it is sufficient to show the matrix $\overline{\mathbf{C}}_s$ obtained by removing the 8th row and column of $-\mathbf{C}_s$ is positive definite, as the 8th row and column of $-\mathbf{C}_s$ has only a positive diagonal entry and the other entries are zero. Furthermore,

$$\begin{bmatrix}
20 & 0 & 0 & 0 \\
0 & 20 & -8 & -6 \\
0 & -8 & 24 & 0 \\
0 & -6 & 0 & 15
\end{bmatrix}$$

is positive definite, since it is a symmetric strictly diagonally-dominant matrix. In order to show

$$\begin{bmatrix}
9 & 8 & 6 & 0 & 0 \\
8 & 20 & 0 & 0 & 0 \\
6 & 0 & 20 & -8 & -6 \\
0 & 0 & -8 & 24 & 0 \\
0 & 0 & -6 & 0 & 15
\end{bmatrix}$$

is positive definite, we use Schur Complement Lemma (see, for instance, [130]), and compute

$$
\begin{aligned}
& 9-\begin{bmatrix}8 & 6 & 0 & 0\end{bmatrix}\begin{bmatrix}20 & 0 & 0 & 0\\ 0 & 20 & -8 & -6\\ 0 & -8 & 24 & 0\\ 0 & -6 & 0 & 15\end{bmatrix}^{-1}\begin{bmatrix}8\\6\\0\\0\end{bmatrix}\\
&= 9-\frac{1}{840}\begin{bmatrix}4 & 3 & 0 & 0\end{bmatrix}\begin{bmatrix}168 & 0 & 0 & 0\\ 0 & 225 & 75 & 90\\ 0 & 75 & 165 & 30\\ 0 & 90 & 30 & 260\end{bmatrix}\begin{bmatrix}4\\3\\0\\0\end{bmatrix}\\
&= \frac{949}{280} > 0.
\end{aligned}
$$

Since $\overline{\mathbf{C}}_s(4:9, 4:9)$ is diagonally dominant, it is positive semidefinite. Also, the columns are linearly independent, therefore it is positive definite.

Let $\mathbf{W} := \begin{bmatrix}17 & 3 & 12\\ 3 & 23 & 12\\ 12 & 12 & 19\end{bmatrix}$, $Y := \begin{bmatrix}12 & 0 & 0 & -2 & -8 & -4\\ 0 & 9 & 8 & 6 & 0 & 0\\ 0 & 8 & 20 & 0 & 0 & 0\\ -2 & 6 & 0 & 20 & -8 & -6\\ -8 & 0 & 0 & -8 & 24 & 0\\ -4 & 0 & 0 & -6 & 0 & 15\end{bmatrix}$ and $\mathbb{S} := \begin{bmatrix}-6 & 0 & 0 & 0 & 0 & 0\\ 9 & 2 & 8 & -4 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$.

To show that $\overline{\mathbf{C}}_s$ is positive definite, we show that

$$
\mathbf{M} := \mathbf{W} - \mathbf{S}\mathbf{Y}^{-1}\mathbf{S}^{\top}
$$

is positive definite. (In the previous part, we have already shown $\mathbf{Y}$ is positive definite). Here,

$$
\mathbf{Y}^{-1} := \frac{1}{163736}\begin{bmatrix}22776 & -8280 & 3312 & 8004 & 10260, & -2872\\ -8280 & 51320 & -20528 & -22320 & -10200 & -6720\\ 3312 & -20528 & 16398 & 8928 & 4080 & 2688\\ 8004 & -22320 & 8928 & 21576 & 9860 & 6496\\ 10260 & -10200 & 4080 & 9860 & 13529 & 1208\\ -2872 & -6720 & 2688 & 6496 & 1208 & 14280\end{bmatrix}.
$$

Then $\mathbf{M} = \dfrac{1}{20467}\begin{bmatrix}245447 & 198579 & 245604\\ 198579 & 198714 & 245604\\ 245604 & 245604 & 388873\end{bmatrix}$.

We show that $20467\mathbf{M}$ is positive definite by using Schur Complement Lemma. For this, we compute

$$\begin{bmatrix} 245447 & 198579 \\ 198579 & 198714 \end{bmatrix} - \frac{245604^2}{388873} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \frac{1}{19} \begin{bmatrix} 1716245 & 825753 \\ 825753 & 828318 \end{bmatrix}.$$

Even though the numbers are large in the resulting 2-by-2 matrix, it is not hard to observe that the above matrix is symmetric row diagonally-dominant with positive diagonal entries. Hence it is positive definite. Therefore $-\mathbf{C}_s$ is positive definite. We have shown that

$$\operatorname{tr}(\mathbf{A}_0\mathbf{U}\mathbf{B}_0\mathbf{U}) - \rho_{\mathbf{w}_0}(\mathbf{A}_0, \mathbf{B}_0) < 0 \text{ for every } \mathbf{U} \in \mathbb{S}^n, \text{ with } \|\mathbf{U}\|_F = 1.$$

This completes the proof for $k = 3$.

- Using the counterexample for $k = 3$, given in (5.13), we show that the weak interlacing property fails when $n = 4, k = 4$. Construct $\mathbf{A}_1 := \mathbf{A}_0$, $\mathbf{B}_1 := \mathbf{B}_0 + \varepsilon\,\mathbb{I}$, where ε is a very small number such that $\operatorname{rank}(\mathbf{B}_1) = 4$. Since $\operatorname{rank}(A_1) = \operatorname{rank}(B_1) = 4$, $k = 4$ for this pair. Let $\mathbf{W}_1 := \mathbf{W}_0/\|\mathbf{W}_0\|_F$. Then

$$\begin{aligned}
\operatorname{tr}(\mathbf{A}_1\mathbf{U}\mathbf{B}_1\mathbf{U}) - \rho_{\mathbf{w}_1}(\mathbf{A}_1, \mathbf{B}_1) &:= \operatorname{tr}(\mathbf{A}_1\mathbf{U}\mathbf{B}_1\mathbf{U}) - \operatorname{tr}\left(\mathbf{A}_1\mathbf{W}_1\mathbf{B}_1\mathbf{W}_1^\top\right) \\
&= \operatorname{Tr}(\mathbf{A}_0\mathbf{U}\mathbf{B}_0\mathbf{U}) - \operatorname{Tr}(\mathbf{A}_0\mathbf{W}_1\mathbf{B}_0\mathbf{W}_1^\top) \\
&\quad + \varepsilon\left(\operatorname{tr}(\mathbf{A}_0\mathbf{U}\mathbf{U}) - \operatorname{tr}\left(\mathbf{A}_0\mathbf{W}_1\mathbf{W}_1^\top\right)\right) \\
&= \operatorname{tr}(\mathbf{A}_0\mathbf{U}\mathbf{B}_0\mathbf{U}) - \rho_{\mathbf{w}_0}(\mathbf{A}_0, \mathbf{B}_0) \\
&\quad + \varepsilon\left(\operatorname{tr}(\mathbf{A}_0\mathbf{U}\mathbf{U}) - \operatorname{tr}\left(\mathbf{A}_0\mathbf{W}_1\mathbf{W}_1^\top\right)\right).
\end{aligned}$$

  Since $\operatorname{tr}(\mathbf{A}_0\mathbf{U}\mathbf{B}_0\mathbf{U}) - \rho_{\mathbf{w}_0}(\mathbf{A}_0, \mathbf{B}_0) < 0$, choosing ε small enough, we get $\operatorname{tr}(\mathbf{A}_1\mathbf{U}\mathbf{B}_1\mathbf{U}) - \rho_{\mathbf{w}_1}(\mathbf{A}_1, \mathbf{B}_1) < 0$ for every $\mathbf{U} \in \mathbb{S}^n$ with $\|\mathbf{U}\|_F = 1$. This implies that for the pair $\mathbf{A}_1, \mathbf{B}_1$ the weak interlacing property fails.

- Now, we show that the weak interlacing property fails for $k = 4$ and arbitrarily chosen $n > 4$. Let

$$\mathbf{A}_2 := \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \varepsilon\,\mathbb{I} \end{bmatrix} \in \mathbb{S}^n \text{ and } \mathbf{B}_2 := \begin{bmatrix} \mathbf{B}_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}^n,$$

  where ε is a very small number. Let $\mathbf{W}_2 := \begin{bmatrix} \mathbf{W}_1 & 0 \\ 0 & 0 \end{bmatrix} \in \tilde{\mathbb{S}}^n$, then $\|\mathbf{W}_2\|_F = 1$. Suppose $\mathbf{U}_2 := \arg\max_{\mathbf{U}_2 \in \mathbb{S}^n, \|\mathbf{U}\|_F = 1} \operatorname{tr}(\mathbf{A}_2\mathbf{U}\mathbf{B}_2\mathbf{U})$, and $\mathbf{U}_2 = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{21} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix}$, such that the size of $\mathbf{U}_{11}$ is the same as the size of $\mathbf{W}_1$. Then

$$\begin{aligned}
\operatorname{tr}(\mathbf{A}_2\mathbf{U}_2\mathbf{B}_2\mathbf{U}_2) - \rho_{\mathbf{w}_2}(\mathbf{A}_2, \mathbf{B}_2) &:= \operatorname{tr}(\mathbf{A}_2\mathbf{U}_2\mathbf{B}_2\mathbf{U}_2) - \operatorname{tr}\left(\mathbf{A}_2\mathbf{W}_2\mathbf{B}_2\mathbf{W}_2^\top\right) \\
&= \operatorname{tr}(\mathbf{A}_1\mathbf{U}_{11}\mathbf{B}_1\mathbf{U}_{11}) + \varepsilon\operatorname{tr}(\mathbf{U}_{21}\mathbf{B}_1\mathbf{U}_{21}) \\
&\quad - \operatorname{tr}\left(\mathbf{A}_1\mathbf{W}_1\mathbf{B}_1\mathbf{W}_1^\top\right) \\
&= \|U_{11}\|_F^2 \operatorname{tr}\left(\mathbf{A}_1\frac{\mathbf{U}_{11}}{\|\mathbf{U}_{11}\|_F}\mathbf{B}_1\frac{\mathbf{U}_{11}}{\|\mathbf{U}_{11}\|_F}\right) \\
&\quad + \varepsilon\operatorname{tr}(\mathbf{U}_{21}\mathbf{B}_1\mathbf{U}_{21}) - \rho_{\mathbf{w}_1}(\mathbf{A}_1, \mathbf{B}_1).
\end{aligned}$$

For every every unit norm matrix $\mathbf{U} \in \mathbb{S}^n$ (by the counterexample constructed for the proof for $n = 4, k = 4$),

$$\operatorname{tr}(\mathbf{A}_1\mathbf{U}\mathbf{B}_1\mathbf{U}) < \rho_{\mathbf{w}_1}(\mathbf{A}_1, \mathbf{B}_1). \tag{5.14}$$

Recall that

$$\begin{aligned}\rho_{\mathbf{w}_1}(\mathbf{A}_1, \mathbf{B}_1) &= \operatorname{tr}\left(\mathbf{A}_0\mathbf{W}_1(\mathbf{B}_0 + \varepsilon\,\mathbb{I})\mathbf{W}_1^\top\right)\\ &= \rho_{\mathbf{w}_0}(\mathbf{A}_0, \mathbf{B}_0) + \varepsilon \operatorname{tr}\left(\mathbf{A}_0\mathbf{W}_1\mathbf{W}_1^\top\right).\end{aligned}$$

Without loss of generality, we may assume that $\rho_{\mathbf{w}_1}(\mathbf{A}_1, \mathbf{B}_1) > 0$ as $\rho_{\mathbf{w}_0}(\mathbf{A}_0, \mathbf{B}_0) > 0$ and ε is chosen to be very small number. Note that if $\|\mathbf{U}_{11}\|_F = 0$ the result follows, so assume $\|\mathbf{U}_{11}\|_F \neq 0$. Hence scaling the left hand side of (5.14) by $\|\mathbf{U}_{11}\|_F^2 < 1$, gives

$$\|\mathbf{U}_{11}\|_F^2 \operatorname{tr}\left(\mathbf{A}_1 \frac{\mathbf{U}_{11}}{\|\mathbf{U}_{11}\|_F}\mathbf{B}_1\frac{\mathbf{U}_{11}}{\|\mathbf{U}_{11}\|_F}\right) < \rho_{\mathbf{w}_1}(\mathbf{A}_1, \mathbf{B}_1).$$

By choosing ε small enough, we get $\operatorname{tr}(\mathbf{A}_2\mathbf{U}_2\mathbf{B}_2\mathbf{U}_2) - \rho_{\mathbf{w}_2}(\mathbf{A}_2, \mathbf{B}_2) < 0$, which completes the proof for $k = 4$ and arbitrary $n > 4$.

- The proof for arbitrary k follows along similar lines as the proof of the case $n = k = 4$ (i.e., one can choose $\mathbf{A}_3 := \mathbf{A}_2$ and $\mathbf{B}_3 := \mathbf{B}_2 + \varepsilon\,\mathbb{I}$, where ε is a small number), and is omitted here.

□

By Perron-Frobenius theorem, for nonnegative symmetric matrices $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$, (1.4) always holds; however, we constructed examples of such nonnegative symmetric matrices where (1.3) fails. In addition, we constructed examples where for a pair of full rank 6-by-6 real skew-symmetric matrices for which the weak interlacing property fails.

5.3.4 Asymptotic behavior

Lastly, we consider a number of different perturbations to arbitrary pairs of symmetric matrices $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ where the perturbed pair is guaranteed to satisfy the weak interlacing property. The following theorem provides a set of perturbations which allows constructing nontrivial pairs of matrices satisfying the weak interlacing property. Furthermore, it helps improve our understanding of the spectral properties of $(\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A})$ in terms of the spectral properties of $\mathbf{A}$ and $\mathbf{B}$.

Theorem 5.3.19 (new)**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. Then*

1. $(\mathbf{A} + \mu\,\mathbb{I}, \mathbf{B})$ *satisfies the weak interlacing property for $\mu > 0$ large enough, if $\mathbf{B}$ is indefinite and the multiplicity of the smallest and largest eigenvalues of $\mathbf{B}$ is* 1.

2. $(\mathbf{A} + \beta\mu\,\mathbb{I}, \mathbf{B} + \mu\,\mathbb{I})$ *satisfies the weak interlacing property for $\mu > 0$ large enough where $\beta > 0$ is a constant, if $\mathbf{A} + \beta\mathbf{B}$ is indefinite and the geometric multiplicity of the smallest and largest eigenvalues of $\mathbf{A} + \beta\mathbf{B}$ is* 1.

3. $(\mathbf{A}+\beta\mathbf{B}, \mathbf{B}+\alpha\mathbf{A})$ *satisfies the weak interlacing property for* $\alpha > 0$ *large enough and* $\beta > 0$ *small enough such that* $\alpha\beta$ *is constant, if* $\mathbf{A}$ *is indefinite and the geometric multiplicity of the smallest and largest eigenvalues of* $\mathbf{A}$ *is* 1.

4. $(\mathbf{A}+\mu\mathbf{D}, \mathbf{B})$ *satisfies the weak interlacing property for* $\mu > 0$ *large enough where* $\mathbf{B}$ *and* $\mathbf{D}$ *are diagonal matrices, if* $\mathbf{B} \otimes \mathbf{D}$ *is indefinite and the geometric multiplicity of the smallest and largest eigenvalues of* $\mathbf{B} \otimes \mathbf{D}$ *is* 1.

5. $(\mathbf{A}+\mu\mathbf{D}, \mathbf{B}+\mu\mathbb{I})$ *satisfies the weak interlacing property for* $\mu > 0$ *large enough, if* $\mathbf{D}$ *is indefinite and the geometric multiplicity of the smallest and largest eigenvalues of* $\mathbf{D}$ *is* 1.

6. $(\mathbf{A}+\mu\mathbf{D}_1, \mathbf{B}+\mu\mathbf{D}_2)$ *satisfies the weak interlacing property for* $\mu > 0$ *large enough, if* $\mathbf{D}_1 \otimes \mathbf{D}_2$ *is indefinite and the geometric multiplicity of the smallest and the largest eigenvalues of* $\mathbf{D}_1 \otimes \mathbf{D}_2$ *is* 1.

Proof. 1. Let $\mathbf{B}$ be an indefinite symmetric matrix and $\mu > 0$. Since

$$\operatorname{tr}\left((\mathbf{A}+\mu\,\mathbb{I})\mathbf{UBU}\right) = \operatorname{Tr}(\mathbf{AUBU}) + \mu \operatorname{tr}(\mathbf{UBU})$$
$$\operatorname{tr}\left((\mathbf{A}+\mu\,\mathbb{I})\mathbf{WBW}^\top\right) = \operatorname{tr}\left(\mathbf{AWBW}^\top\right) + \mu \operatorname{tr}\left(\mathbf{WBW}^\top\right)$$

By Lemma 5.3.6, we have $\max_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}(\mathbf{UBU}) = \lambda_1(\mathbf{B})$ and $\min_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}(\mathbf{UBU}) = \lambda_n(\mathbf{B})$. If the eigenspaces of the largest and the smallest eigenvalues of $\mathbf{B}$ both have dimension 1, then

$$\max_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}(\mathbf{UBU}) > \max_{\mathbf{W}\in\mathbb{K}^n, \|\mathbf{W}\|_F=1} \operatorname{tr}\left(\mathbf{WBW}^\top\right),$$

and

$$\min_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}(\mathbf{UBU}) < \min_{\mathbf{W}\in\mathbb{K}^n, \|\mathbf{W}\|_F=1} \operatorname{tr}\left(\mathbf{WBW}^\top\right).$$

Then, for μ large enough,

$$\max_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}\left((\mathbf{A}+\mu\,\mathbb{I})\,\mathbf{UBU}\right) \geq \max_{\mathbf{W}\in\mathbb{K}^n, \|\mathbf{W}\|_F=1} \operatorname{tr}\left((\mathbf{A}+\mu\,\mathbb{I})\,\mathbf{WBW}^\top\right),$$

and

$$\min_{\mathbf{U}\in\mathbb{S}^n, \|\mathbf{U}\|_F=1} \operatorname{tr}\left((\mathbf{A}+\mu\,\mathbb{I})\,\mathbf{UBU}\right) \leq \min_{\mathbf{W}\in\mathbb{K}^n, \|\mathbf{W}\|_F=1} \operatorname{tr}\left((\mathbf{A}+\mu\,\mathbb{I})\,\mathbf{WBW}^\top\right).$$

The proofs for parts $2-6$ of Theorem 5.3.19 are along similar lines with the proof of part 1 above and are omitted. □

Using a similar construction given as in Theorem 5.3.19, it is possible to generate infinitely many pairs of matrix pencils formed by perturbing $\mathbf{A}$ and $\mathbf{B}$ for which the weak interlacing property fails.

Theorem 5.3.20 (new)**.** *Let $\bar{\mathbf{A}}, \bar{\mathbf{B}} \in \mathbb{S}^n$, (where $n \geq 4$) such that the weak interlacing property does not hold. Then for every $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$, the weak interlacing property fails for the pairs*

1. $(\mathbf{A}+\mu\bar{\mathbf{A}}, \bar{\mathbf{B}})$ *and* $(\mathbf{A}+\beta\mu\bar{\mathbf{A}}, \mathbf{B}+\mu\bar{\mathbf{B}})$ *for $\mu > 0$ large enough, where $\beta > 0$ is a constant, and*

2. $(\mathbf{A} + \beta\bar{\mathbf{B}}, \mathbf{B} + \alpha\bar{\mathbf{A}})$ *for $\alpha, \beta > 0$ large enough.*

Proof of Theorem 5.3.20 is elementary (similar to the above proof of Theorem 5.3.19) and is omitted.

5.4 A Generalization of Jordan-Kronecker Product

We construct a generalization[1] of the Jordan-Kronecker product with similar eigenvalue, eigenvector structure to the Jordan-Kronecker product. Let $\mathbf{T}$ be an n^2-by-n^2 symmetric involutory matrix (i.e., $\mathbf{T}^\top = \mathbf{T}$ and $\mathbf{T}^2 = \mathbb{I}$). Given $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, we define the *generalized Jordan-Kronecker product* as

$$\mathbf{C} := \mathbf{A} \otimes \mathbf{B} + \mathbf{T}(\mathbf{A} \otimes \mathbf{B})\mathbf{T}.$$

Recall, that in the Jordan-Kronecker product the symmetric involutory matrix is the n^2-by-n^2 commutation matrix.

Similar to the Jordan-Kronecker product, $\mathbf{TCT} = \mathbf{C}$. Since $\mathbf{T}^2 = \mathbb{I}$,

$$\begin{aligned}
\mathbf{TCT} &= \mathbf{T}(\mathbf{A} \otimes \mathbf{B})\mathbf{T} + \mathbf{T}^2(\mathbf{A} \otimes \mathbf{B})\mathbf{T}^2 \\
&= \mathbf{T}(\mathbf{A} \otimes \mathbf{B})\mathbf{T} + (\mathbf{A} \otimes \mathbf{B}) \\
&= \mathbf{C}.
\end{aligned}$$

Define the spaces

$$\begin{aligned}
\mathcal{R}_{sym} &:= \{\mathbf{x} \in \mathbb{R}^{n^2} : \mathbf{Tx} = \mathbf{x}\}, \\
\mathcal{R}_{skew} &:= \{\mathbf{x} \in \mathbb{R}^{n^2} : \mathbf{Tx} = -\mathbf{x}\}.
\end{aligned}$$

Let s denote the dimension of $\mathcal{R}_{sym}$ and t denote the dimension of $\mathcal{R}_{skew}$. As it is the convention, we call vectors in $\mathcal{R}_{sym}$ symmetric and the ones in $\mathcal{R}_{skew}$ as skew-symmetric. Another similar observation is that both $\mathcal{R}_{sym}$ and $\mathcal{R}_{skew}$ are linear subspaces of $\mathbb{R}^{n^2}$ and $\mathbb{R}^{n^2} = \mathcal{R}_{sym} \oplus \mathcal{R}_{skew}$. In addition $\mathcal{R}_{sym}$ is the orthogonal complement of $\mathcal{R}_{skew}$ in $\mathbb{R}^{n^2}$. These observations can be justified by simple algebra. Every $\mathbf{x} \in \mathbb{R}^{n^2}$ can be written as

$$\mathbf{x} = \left(\frac{\mathbf{x} + \mathbf{Tx}}{2}\right) + \left(\frac{\mathbf{x} - \mathbf{Tx}}{2}\right).$$

[1] We thank Chris Godsil for suggesting this generalization.

In addition, for every $\mathbf{x} \in \mathcal{R}_{sym} \setminus \{0\}$ and $\mathbf{y} \in \mathcal{R}_{skew} \setminus \{0\}$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{T}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{T}^\top \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{T}\mathbf{y} \rangle = -\langle \mathbf{x}, \mathbf{y} \rangle.$$

Then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. This shows that $\mathcal{R}_{sym} \perp \mathcal{R}_{skew}$.

Now, let $\mathbf{Q}$ be an n^2-by-s orthogonal matrix such that $\mathbf{TQ} = \mathbf{Q}$ and let $\tilde{\mathbf{Q}}$ be a n^2-by-t matrix such that $\mathbf{T}\tilde{\mathbf{Q}} = -\tilde{\mathbf{Q}}$. The following gives a characterization of the eigenvalue/eigenvector structure of the generalized Jordan-Kronecker product.

Proposition 5.4.1. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ (or both in $\mathbb{K}^n$). Let $\mathbf{T} \in \mathbb{R}^{n^2 \times n^2}$ be an arbitrary symmetric involutory matrix and $\mathbf{C} := \mathbf{A} \otimes \mathbf{B} + \mathbf{T}(\mathbf{A} \otimes \mathbf{B})\mathbf{T}$. Then, the eigenvectors of $\mathbf{C}$ can be decomposed into symmetric and skew-symmetric vectors. Furthermore, the eigenvalues corresponding to the symmetric eigenvectors are the eigenvalues of $2\mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{Q}$ and the ones corresponding to the skew-symmetric eigenvectors are the eigenvalues of $2\tilde{\mathbf{Q}}^\top(\mathbf{A} \otimes \mathbf{B})\tilde{\mathbf{Q}}$.*

Proof. The proof is very similar to the proof of Theorem 5.3.7. We provide it here just for the sake of completeness.

Let $\mathbf{Q}' := \begin{bmatrix} \mathbf{Q} & \tilde{\mathbf{Q}} \end{bmatrix}$, where $\mathbf{Q}$ is the n^2-by-s orthogonal matrix such that $\mathbf{TQ} = \mathbf{Q}$ and $\tilde{\mathbf{Q}}$ is the n^2-by-t matrix such that $\mathbf{T}\tilde{\mathbf{Q}} = -\tilde{\mathbf{Q}}$, as described above. Then

$$\begin{aligned} \mathbf{Q}'^\top \mathbf{C} \mathbf{Q}' &= \begin{bmatrix} \mathbf{Q}^\top \mathbf{C} \mathbf{Q} & \mathbf{Q}^\top \mathbf{C} \tilde{\mathbf{Q}} \\ \tilde{\mathbf{Q}}^\top \mathbf{C} \mathbf{Q} & \tilde{\mathbf{Q}}^\top \mathbf{C} \tilde{\mathbf{Q}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}^\top \mathbf{C} \mathbf{Q} & 0 \\ 0 & \tilde{\mathbf{Q}}^\top \mathbf{C} \tilde{\mathbf{Q}} \end{bmatrix}. \end{aligned}$$

The off diagonal blocks are zero since

$$\mathbf{Q}^\top \mathbf{C} \tilde{\mathbf{Q}} = \mathbf{Q}^\top \mathbf{TCT} \tilde{\mathbf{Q}} = -\mathbf{Q}^\top \mathbf{C} \tilde{\mathbf{Q}}.$$

Note that $\mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{Q}$ is symmetric, since

$$(\mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{Q})^\top = \mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})^\top \mathbf{Q} = \mathbf{Q}^\top(\mathbf{A}^\top \otimes \mathbf{B}^\top)\mathbf{Q} = \mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{Q}.$$

Similarly, one can show that $\tilde{\mathbf{Q}}^\top(\mathbf{A} \otimes \mathbf{B})\tilde{\mathbf{Q}}$ is symmetric. Let $\mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{Q} = \mathbf{U}\boldsymbol{\Lambda}_e\mathbf{U}^\top$ and $\tilde{\mathbf{Q}}^\top(\mathbf{A} \otimes \mathbf{B})\tilde{\mathbf{Q}} = \mathbf{V}\boldsymbol{\Lambda}_o\mathbf{V}^\top$ be the spectral decomposition of $\mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{Q}$ and $\tilde{\mathbf{Q}}^\top(\mathbf{A} \otimes \mathbf{B})\tilde{\mathbf{Q}}$, respectively. Then,

$$\mathbf{C} = \begin{bmatrix} \mathbf{QU} & \tilde{\mathbf{Q}}\mathbf{V} \end{bmatrix} \begin{bmatrix} 2\boldsymbol{\Lambda}_e & 0 \\ 0 & 2\boldsymbol{\Lambda}_o \end{bmatrix} \begin{bmatrix} \mathbf{QU} & \tilde{\mathbf{Q}}\mathbf{V} \end{bmatrix}^\top.$$

By the definition of $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$, the columns of $\mathbf{QU}$ are symmetric and the columns of $\mathbf{QV}$ are skew-symmetric. Therefore, the even spectrum of $\mathbf{C}$ consists of the eigenvalues of $2(\mathbf{Q}^\top(\mathbf{A} \otimes \mathbf{B})\mathbf{Q})$ and the odd spectrum of $\mathbf{C}$ consists of the eigenvalues of $2(\tilde{\mathbf{Q}}^\top(\mathbf{A} \otimes \mathbf{B})\tilde{\mathbf{Q}})$. ☐

Finally, we remark that a sufficient condition for the interlacement of skew-symmetric and symmetric eigenvectors of the generalized Jordan-Kronecker product can be stated in the same spirit of Theorem 5.3.7.

5.5 Lie-Kronecker Product

We exposed some nice characteristics of eigenspaces of Jordan-Kronecker products of pairs of symmetric matrices and skew-symmetric matrices. One may also wonder if similar characterizations can be established for matrices of the form $(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})$. For $\mathbf{A},\mathbf{B}\in\mathbb{R}^{n\times n}$, we define $(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})$ as the *Lie-Kronecker product* of $\mathbf{A}$ and $\mathbf{B}$. Note that for every pair of symmetric matrices $\mathbf{A},\mathbf{B}$ (or skew-symmetric matrices $\mathbf{A},\mathbf{B}$), $\mathbf{A}\otimes\mathbf{B}$ is symmetric.

The following proposition characterizes the eigenvector/eigenvalue structure of the Lie-Kronecker product of symmetric matrices and skew-symmetric matrices.

Proposition 5.5.1 (new)**.** *Let $\mathbf{A},\mathbf{B}\in\mathbb{S}^n$ (or both in $\mathbb{K}^n$). Then the following properties hold.*

1. *If $\lambda\neq 0$ is an eigenvalue of $(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})$ corresponding to the eigenvector $\mathbf{v}$, then $-\lambda$ is also an eigenvalue corresponding to the eigenvector $\mathbf{Tv}$.*

2. $\operatorname{rank}(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})\leq n^2-n$.

3. *Let $t:=n(n-1)/2$. Then, the eigenvectors of $(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})$ can be chosen in the following form*

$$\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_t,\mathbf{Tv}_1,\mathbf{Tv}_2,\ldots,\mathbf{Tv}_t,\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\},$$

such that $\mathbf{Tu}_i=\mathbf{u}_i$ for every $i\in\{1,2,\ldots,n\}$, where $\mathbf{T}$ is the commutation matrix (see Definition 4.5). Furthermore, the symmetric eigenvectors $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}$ belong to the null space of $(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})$.

Proof. 1. Suppose that λ is an eigenvalue of $(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})$ corresponding to the eigenvector $\mathbf{v}$. Then

$$(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})\,\mathbf{v}=\lambda\mathbf{v}.$$

Note that

$$\mathbf{T}\,(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})\,\mathbf{Tv}=-\,(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})\,\mathbf{v}=-\lambda\mathbf{v}$$

implies

$$\mathbf{T}\,(\mathbf{T}\,(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})\,\mathbf{Tv}))=-\lambda\mathbf{Tv}\implies(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})\,\mathbf{Tv}=-\lambda\mathbf{Tv}.$$

2. From the previous part we know that if $\mathbf{v}$ is an eigenvector corresponding to a nonzero eigenvalue λ then $\mathbf{Tv}$ is an eigenvector corresponding to $-\lambda$. Since the nonzero eigenvectors of $(\mathbf{A}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{A})$ of are orthonormal, the cardinality of the set

$$\{(\mathbf{v}_i,\mathbf{Tv}_i):i=1,\ldots,\ell\}$$

may not exceed $t:=\frac{n^2-n}{2}$. If it does then the vector $\mathbf{v}_{t+1}$ must be a symmetric vector in which case $(\mathbf{v}_{t+1},\mathbf{Tv}_{t+1})$ is not an an orthogonal pair of vectors.

3. Suppose that we have orthonormal vectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\ell-1}, \mathbf{T}\mathbf{v}_1, \mathbf{T}\mathbf{v}_2, \ldots, \mathbf{T}\mathbf{v}_{\ell-1}\},$$

where $\ell - 1 < t$ which are eigenvectors of $(\mathbf{A} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{A})$ corresponding to the nonzero eigenvalues. Denote $E := \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\ell-1}, \mathbf{T}\mathbf{v}_1, \mathbf{T}\mathbf{v}_2, \ldots, \mathbf{T}\mathbf{v}_{\ell-1}\}$. Let $E^{\perp}$ be the orthogonal complement of E.

Note that if $\mathbf{v} \in E$, then $\mathbf{v} = \sum_{i=1}^{\ell-1}(\alpha_i \mathbf{v}_i + \beta \mathbf{T}\mathbf{v}_i)$ for some α_i, β_i's. Then $\mathbf{T}\mathbf{v} = \sum_{i=1}^{\ell-1}(\alpha_i \mathbf{T}\mathbf{v}_i + \beta_i \mathbf{v}_i) \in E$.

Also, if $\mathbf{w} \in E^{\perp}$, then $\mathbf{T}\mathbf{w} \in E^{\perp}$. Suppose that $\mathbf{w} \in E^{\perp}$, then $\mathbf{w} = \sum_{i=\ell}^{n} \alpha_i \mathbf{w}_i$ for some orthonormal basis vectors $\mathbf{w}_i$ in $S^{\perp}$. Then $\mathbf{T}\mathbf{w} = \sum_{i=\ell}^{n} \alpha_i \mathbf{T}\mathbf{w}_i$. Then

$$\langle \mathbf{v}, \mathbf{T}\mathbf{w} \rangle = \langle \sum_{k=1}^{\ell-1} (\alpha_i \mathbf{T}\mathbf{v}_i + \beta_i \mathbf{v}_i), \sum_{i=\ell}^{n} \alpha_i \mathbf{w}_i \rangle = 0.$$

Therefore $\mathbf{T}\mathbf{w} \in E^{\perp}$.

Define

$$M_{skew} := \{\tilde{\mathbf{m}}_i := \mathbf{v}_i - \mathbf{T}\mathbf{v}_i : i = 1, \ldots, \ell - 1\}.$$

The elements of M_{skew} are linearly independent, orthogonal and skew-symmetric. Since $|M_{skew}| < t$, one can complete the set $\{\tilde{\mathbf{m}}_1, \tilde{\mathbf{m}}_2, \ldots, \tilde{\mathbf{m}}_{\ell-1}\}$ such that they form an orthogonal basis for the set of skew vectors of dimension n^2. Suppose that $\tilde{\mathbf{m}}_\ell$ is formed in this way and $\tilde{\mathbf{m}}_\ell \perp \{\tilde{\mathbf{m}}_1, \tilde{\mathbf{m}}_2, \ldots, \tilde{\mathbf{m}}_{\ell-1}\}$.

One can write $\tilde{\mathbf{m}}_\ell = \tilde{\mathbf{m}}_u - \tilde{\mathbf{m}}_u^{\top}$ such that $\text{Mat}(\tilde{\mathbf{m}}_u)$ is upper triangular matrix with zero main diagonal.

Suppose $\tilde{\mathbf{m}}_u = \mathbf{v} + \mathbf{w}$ where $\mathbf{v} \in E, \mathbf{w} \in E^{\perp}$. Then

$$\begin{aligned}
\langle (\tilde{\mathbf{m}}_u - \mathbf{T}\tilde{\mathbf{m}}_u), \tilde{\mathbf{m}}_i \rangle &= \langle (\mathbf{v} + \mathbf{w} - (\mathbf{T}\mathbf{v} + \mathbf{T}\mathbf{w})), \tilde{\mathbf{m}}_i \rangle \\
&= \langle (\mathbf{w} - \mathbf{T}\mathbf{w}), \tilde{\mathbf{m}}_i \rangle + \langle (\mathbf{v} - \mathbf{T}\mathbf{v}), \tilde{\mathbf{m}}_i \rangle \\
&= \langle (\mathbf{v} - \mathbf{T}\mathbf{v}), \tilde{\mathbf{m}}_i \rangle
\end{aligned}$$

For some i, $\langle (\mathbf{v} - \mathbf{T}\mathbf{v}), \tilde{\mathbf{m}}_i \rangle \neq 0$ since $\mathbf{v} \in E$. Therefore if $\mathbf{v} \neq 0$ then $\tilde{\mathbf{m}}_\ell$ is not orthogonal to $\{\tilde{\mathbf{m}}_1, \tilde{\mathbf{m}}_2, \ldots, \tilde{\mathbf{m}}_{\ell-1}\}$. Hence we conclude that $\tilde{\mathbf{m}}_u \in E$. By the definition of $\tilde{\mathbf{m}}_u$, it is easy to see that $\langle \tilde{\mathbf{m}}_u, \mathbf{T}\tilde{\mathbf{m}}_u \rangle = 0$. Using this construction we can construct $\mathbf{v}_k := \tilde{\mathbf{m}}_u$, for $k \in \{\ell, \ell + 1, \ldots, t\}$ such that $\langle \mathbf{v}_k, \mathbf{T}\mathbf{v}_k \rangle = 0$. Since the set of all skew symmetric vectors has rank t, any other vector orthogonal to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t, \mathbf{T}\mathbf{v}_1, \mathbf{T}\mathbf{v}_2, \ldots, \mathbf{T}\mathbf{v}_t\}$ will be symmetric. Therefore the remaining n vectors belonging to the null space can be decomposed into symmetric orthogonal vectors.

$\square$

Chapter 6

Applications of Jordan-Kronecker Products and Open Problems

Jordan-Kronecker products arise mainly in linear matrix equations which have vast applications in control theory, differential equations, and optimization theory. In this section, we briefly mention some of these areas along with the role of the structure in this field.

6.1 Matrix Differentiation

The Jordan-Kronecker product may arise in many equations as a result of "differentiation".

The *Frechet derivative* of a matrix function $f : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is a linear mapping $\mathbf{U} \mapsto L_f(\mathbf{X}, \mathbf{U})$ such that for every $\mathbf{U} \in \mathbf{C}^{n\times n}$,

$$f(\mathbf{X}+\mathbf{U}) - f(\mathbf{X}) - L_f(\mathbf{X}, \mathbf{U}) = o(\|\mathbf{U}\|).$$

If the Frechet derivative exists, it is unique [62]. Since L_f is linear in $\mathbf{U}$

$$\operatorname{vec}(L_f(\mathbf{X}, \mathbf{U})) = K(\mathbf{X}) \operatorname{vec}(\mathbf{U}),$$

where $K(\mathbf{X})$ is referred as the Kronecker form of Frechet derivative.

For example, if $p(x) := \sum_{k=0}^{m} a_k x^k$, then

$$L_p(\mathbf{X}, \mathbf{U}) = \sum_{k=1}^{m} a_k \sum_{j=1}^{k} \mathbf{X}^{j-1}\mathbf{U}\mathbf{X}^{k-j},$$

and therefore

$$K(\mathbf{X}) = \sum_{k=1}^{m} a_k \sum_{j=1}^{k} (\mathbf{X}^{j-1})^{\top} \otimes \mathbf{X}^{k-j}.$$

For the derivation, we refer the reader to [62, Section 3.2]. Note that at a real symmetric matrix $\mathbf{X}$ the above expression is the summation of Jordan-Kronecker products of the powers of $\mathbf{X}$.

In [103], Neudecker studied the partial derivatives of the elements of a matrix function with respect to the elements of the argument matrix. He introduced some definitions for partial derivatives of different matrix functions (the ones in the form of Kronecker product and ordinary product). One form of matrix differentiation defined by Neudecker in [103] is $d\mathbf{X} := [dx_{ij}]$. Then, for matrices $\mathbf{X}, \mathbf{Y}$ of appropriate dimensions the following holds for the differential of the product of two matrices [103]

$$d(\mathbf{X}\mathbf{Y}) = (d\mathbf{X})\mathbf{Y} + \mathbf{X}(d\mathbf{Y}).$$

Some examples of the instance the form of Jordan-Kronecker product appear in the matrix differentiation are listed below. For n-by-n square matrices $\mathbf{A}, \mathbf{X}, \mathbf{B}$, the followings hold [103].

- If $f(\mathbf{X}) := \mathbf{X} \otimes \mathbf{X}$, then
$$df(\mathbf{X}) = d\mathbf{X} \otimes \mathbf{X} + \mathbf{X} \otimes d\mathbf{X}.$$

- Let $d^2 f := \operatorname{tr}\left(\mathbf{A} d\mathbf{X}^\top \mathbf{B} d\mathbf{X}\right) + \operatorname{tr}\left(\mathbf{B} d\mathbf{X}^\top \mathbf{A} d\mathbf{X}\right)$, where $\mathbf{A}$ and $\mathbf{B}$ are real matrices of appropriate dimensions. By definition $d^2 f = \operatorname{vec}(d\mathbf{X})^\top (\nabla^2_{\operatorname{vec}(\mathbf{X})} f) \operatorname{vec}(d\mathbf{X})$. Since
$$d^2 f = \frac{1}{2} \operatorname{vec}(d\mathbf{X})^\top \left(\mathbf{A}^\top \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{B}^\top + \mathbf{B}^\top \otimes \mathbf{A} + \mathbf{B} \otimes \mathbf{A}^\top\right) \operatorname{vec}(d\mathbf{X}),$$
the Hessian of the scalar function f with respect to $\operatorname{vec}(\mathbf{X})$ is
$$\nabla^2_{\operatorname{vec}(\mathbf{X})} f = \frac{1}{2} \left(\mathbf{A}^\top \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{B}^\top + \mathbf{B}^\top \otimes \mathbf{A} + \mathbf{B} \otimes \mathbf{A}^\top\right).$$
When $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$, the Hessian reduces to the Jordan-Kronecker product of $\mathbf{A}$ and $\mathbf{B}$.

6.2 Generalized Lyapunov Matrix Equations

Let $\mathbf{A}, \mathbf{B}, \mathbf{Y} \in \mathbb{C}^{n\times n}$. The generalized continuous-time algebraic Lyapunov equations (GCALE) are defined as

$$\mathbf{A}^H \mathbf{X} \mathbf{B} + \mathbf{B}^H \mathbf{X} \mathbf{A} = -\mathbf{Y}, \tag{6.1}$$

where $\mathbf{X} \in \mathbb{C}^{n\times n}$ is an unknown matrix. When $\mathbf{B} = \mathbb{I}$, it reduces to the well-known Lyapunov equation. These equations arise in the stability analysis of differential equations and control theory. As discussed in Chapter 4, Lyapunov equations are special cases of Sylvester equations. Similarly, the GCALE equations are the special case of the generalized Sylvester equation

$$\mathbf{A}\mathbf{X}\mathbf{B} - \mathbf{C}\mathbf{X}\mathbf{D} = -\mathbf{Y}. \tag{6.2}$$

By using Kronecker product, (6.2) can be equivalently written as

$$(\mathbf{B}^\top \otimes \mathbf{A} - \mathbf{D}^\top \otimes \mathbf{C}) \operatorname{vec}(\mathbf{X}) = -\operatorname{vec}(\mathbf{Y}).$$

The uniqueness of the solution depends on the nonsingularity of $(\mathbf{B}^\top \otimes \mathbf{A} - \mathbf{D}^\top \otimes \mathbf{C})$. In 1987, Chu proved that necessary and sufficient conditions for the existence of a unique solution of

(6.2) are (i) the pencils $\mathbf{A} - \alpha\mathbf{C}$ and $\mathbf{D} - \alpha\mathbf{B}$ are regular (a matrix pencil $\alpha\mathbf{A} - \beta\mathbf{B}$ is called *regular* if $\mathbf{A}$ and $\mathbf{B}$ are square, and $\det(\alpha\mathbf{A} - \beta\mathbf{B}) \neq 0$ for some $\alpha, \beta \in \mathbb{C}$) and ($ii$) these pencils have no common eigenvalues. For the GCALE, this result was restated as a special case of Chu's result in [109]. Later, Stykel generalized the condition for the stability of the GCALE in [124].

Theorem 6.2.1. *[124] If $\alpha\mathbf{B} - \mathbf{A}$ is a regular pencil and all eigenvalues of $\alpha\mathbf{B} - \mathbf{A}$ are finite and lie in the open left half-plane, then for every Hermitian, positive (semi)definite matrix $\mathbf{Y}$, the (6.1) has a unique Hermitian, positive (semi)definite solution $\mathbf{X}$. Conversely, if there exist Hermitian, positive definite matrices $\mathbf{X}$ and $\mathbf{Y}$ satisfying (6.1), then all eigenvalues of the pencil $\alpha\mathbf{B} - \mathbf{A}$ are finite and lie in the open left half-plane.*

There is a vast literature on the solutions and numerical algorithms for the solutions of generalized Sylvester equations and the GCALE, see [27, 61, 50, 68] and [109, 120, 37, 125] and the references therein.

6.3 Interior-Point Methods

Consider a semidefinite programming problem (SDP)

$$\begin{aligned} \min \quad & \langle \mathbf{C}, \mathbf{X} \rangle & (P) \\ \text{subject to} \quad & \langle \mathbf{A}_i, \mathbf{X} \rangle = b_i, \quad i = 1, \dots, m \\ & \mathbf{X} \succeq 0, \end{aligned}$$

and its dual

$$\begin{aligned} \max \quad & \mathbf{b}^\top \mathbf{y} & (D) \\ \text{subject to} \quad & \sum_{i=1}^m y_i \mathbf{A}_i + \mathbf{Z} = \mathbf{C} \\ & \mathbf{Z} \succeq 0, \end{aligned}$$

where $\mathbf{C} \in \mathbb{S}^n$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{A}_i \in \mathbb{S}^n$ for every $i \in \{1, \dots, m\}$ are all data. The set of strictly feasible solutions of (P) and (D) are

$$\mathcal{F}_o(P) := \{\mathbf{X} \in \mathbb{S}^n : \langle \mathbf{A}_i, \mathbf{X} \rangle = b_i \; i = 1, \dots, m, \; \mathbf{X} \succ 0\},$$

$$\mathcal{F}_o(D) := \{(\mathbf{Z}, \mathbf{y}) \in \mathbb{S}^n \times \mathbb{R}^m : \sum_{i=1}^m y_i \mathbf{A}_i + \mathbf{Z} = \mathbf{C}, \; \mathbf{Z} \succ 0\}.$$

Under strict feasibility conditions, i.e., when $\mathcal{F}_o(P) \neq \emptyset$ and $\mathcal{F}_o(D) \neq \emptyset$, both (P) and (D) attain their optimal values (i.e., there exist solutions $\mathbf{X}^*$ and $(\mathbf{Z}^*, \mathbf{y}^*)$) and the optimal values are equal, i.e., $\langle \mathbf{C}, \mathbf{X}^* \rangle = \mathbf{b}^\top \mathbf{y}^*$.

Barrier method for semidefinite programming approximates (P) by replacing the objective with $f(X) := \langle \mathbf{C}, \mathbf{X} \rangle + \mu F(\mathbf{X})$ and removing the $\mathbf{X} \succeq 0$ constraint, where $\mu \geq 0$ is a paramater, $F : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}$,

$$F(\mathbf{X}) := \begin{cases} -\log\det(\mathbf{X}), & \text{if } \mathbf{X} \succ 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

For each $\mu \geq 0$, let us denote the resulting convex optimization problem by (P_μ). Under the strict feasibility conditions for both (P) and (D), $(\mathbf{X}_\mu, \mathbf{y}_\mu, \mathbf{Z}_\mu)$ are primal and dual optimal solutions of (P_μ) if and only if they solve the following system of equations:

$$\langle \mathbf{A}_i, \mathbf{X} \rangle = b_i \quad i = 1, \dots, m \tag{6.3}$$

$$\sum_{i=1}^{m} y_i \mathbf{A}_i + \mathbf{Z} = \mathbf{C} \tag{6.4}$$

$$\mathbf{X}\mathbf{Z} = \mu\, \mathbb{I} \tag{6.5}$$

$$\mathbf{X} \succeq 0, \mathbf{Z} \succeq 0. \tag{6.6}$$

The family of solutions $\{(\mathbf{X}_\mu, \mathbf{y}_\mu, \mathbf{Z}_\mu) : \mu \geq 0\}$ is called the *central path*. Maintaining the last two conditions (6.5)-(6.6) is difficult. Different linearization techniques which deal with the nonlinearity of (6.5) lead to different search directions.

In [59], the following linearization was proposed

$$\mathbf{X}\mathbf{Z} + (\varDelta\mathbf{X})\mathbf{Z} + \mathbf{X}(\varDelta\mathbf{Z}) = \mu\, \mathbb{I}\,. \tag{6.7}$$

However, this does not provide a symmetric solution for $\varDelta\mathbf{X}$. A lot of different approaches were considered to overcome this issue.

Recall the symmetrized similarity operator, the linear map $H_P : \mathbb{R}^{n\times n} \to \mathbb{S}^n$ such that

$$H_P(\mathbf{X}) = \mathbf{P}\mathbf{X}\mathbf{P}^{-1} + \mathbf{P}^{-\top}\mathbf{X}^\top\mathbf{P}^\top,$$

where $\mathbf{P}$ is an n-by-n nonsingular matrix.

In his paper [144], Zhang points out that if $(\varDelta\mathbf{X}, \varDelta\mathbf{Z})$ satisfies

$$H_P((\mathbf{X}\mathbf{Z} + (\varDelta\mathbf{X})\mathbf{Z} + \mathbf{X}(\varDelta\mathbf{Z})) = \mu\, \mathbb{I},$$

then it also satisfies (6.7). This operator also serves useful in unifying the other approaches proposed to deal with the symmetry issue.

Restricting the domain of H_P to $\mathbb{S}^n$ enables H_P to represented by a Jordan-Kronecker product:

$$\operatorname{vec}(H_P(\mathbf{W})) = \big((\mathbf{P} \otimes \mathbf{P}^{-\top}) + (\mathbf{P}^{-\top} \otimes \mathbf{P})\big)\operatorname{vec}(\mathbf{W}). \tag{6.8}$$

If $\bar{\mathbf{W}} := \mathbf{X}(\varDelta\mathbf{Z}) + \mathbf{Z}(\varDelta\mathbf{X})$, then

$$\operatorname{vec}(H_P(\bar{\mathbf{W}})) = \big((\mathbf{P} \otimes \mathbf{P}^{-\top}) + (\mathbf{P}^{-\top} \otimes \mathbf{P})\big)\left((\mathbb{I} \otimes \mathbf{X})\operatorname{vec}(\varDelta\mathbf{Z}) + (\mathbf{Z} \otimes \mathbb{I})\operatorname{vec}(\varDelta\mathbf{X})\right).$$

A generic primal dual interior point method solves the following system of equations to find a new direction:

$$\langle \mathbf{A}_i, \Delta\mathbf{X}\rangle = 0 \quad i = 1, \dots, m, \tag{6.9}$$

$$\sum_{i=1}^{m} \Delta y_i \mathbf{A}_i + \Delta\mathbf{Z} = 0, \tag{6.10}$$

$$H_P\left(\mathbf{X}(\Delta\mathbf{Z}) + (\Delta\mathbf{X})\mathbf{Z}\right) = H_P\left(\sigma\mu\,\mathbb{I} - \mathbf{X}\mathbf{Z}\right) := \mathbf{R}, \tag{6.11}$$

where $\sigma \in (0, 1]$ is a centrality parameter and $\mu := \frac{1}{n}\langle \mathbf{X}, \mathbf{Z}\rangle > 0$ is the barrier parameter. If $\mathbf{P} = \mathbb{I}$, then the direction is called Alizadeh-Haeberly-Overton (AHO) search direction [2], if $\mathbf{P} = \mathbf{Z}^{1/2}$ this gives HKM direction [59], [79], [101].

Let $\bar{\mathbf{W}} := \mathbf{X}(\Delta\mathbf{Z}) + \mathbf{Z}(\Delta\mathbf{X})$. Multiplying (6.11) from left and right by $\mathbf{Z}^{-1/2}$ gives (assuming $\mathbf{Z} \in \mathcal{F}_o(D)$)

$$\mathbf{Z}^{-1/2}\mathbf{R}\mathbf{Z}^{-1/2} := \mathbf{Z}^{-1/2}\left(\mathbf{P}\bar{\mathbf{W}}\mathbf{P}^{-1} + \mathbf{P}^{-\top}\bar{\mathbf{W}}^{\top}\mathbf{P}^{\top}\right)\mathbf{Z}^{-1/2}$$

If we substitute $\mathbf{P} = \mathbf{Z}^{1/2}$ (which gives the HKM direction), then the above equation reduces to

$$\begin{aligned}\mathbf{Z}^{-1/2}\mathbf{R}\mathbf{Z}^{-1/2} &= \bar{\mathbf{W}}\mathbf{Z}^{-1} + \mathbf{Z}^{-1}\bar{\mathbf{W}} \\ &= 2\Delta\mathbf{X} + \mathbf{X}(\Delta\mathbf{Z})\mathbf{Z}^{-1} + \mathbf{Z}^{-1}(\Delta\mathbf{Z})\mathbf{X},\end{aligned}$$

This can be written equivalently as

$$\operatorname{vec}(\mathbf{Z}^{-1/2}\mathbf{R}\mathbf{Z}^{-1/2}) = \left(\mathbf{X} \otimes \mathbf{Z}^{-1} + \mathbf{Z}^{-1} \otimes \mathbf{X}\right)\operatorname{vec}(\Delta\mathbf{Z}). \tag{6.12}$$

Denote $\mathbf{M} := \mathbf{X}\otimes\mathbf{Z}^{-1} + \mathbf{Z}^{-1}\otimes\mathbf{X}$, $\Delta\mathbf{x} := \operatorname{vec}(\Delta\mathbf{X})$, $\Delta\mathbf{z} := \operatorname{vec}(\Delta\mathbf{Z})$, $\mathbf{h} := \operatorname{vec}(\mathbf{Z}^{-1/2}\mathbf{R}\mathbf{Z}^{-1/2})$ and $\mathbf{A} := \begin{bmatrix}\operatorname{vec}(\mathbf{A}_1)^\top \\ \vdots \\ \operatorname{vec}(\mathbf{A}_m)^\top\end{bmatrix}$. Then the system of equations (6.9)-(6.11) are equivalent to

$$\mathbf{A}\Delta\mathbf{x} = 0 \tag{6.13}$$

$$\mathbf{A}^\top\Delta\mathbf{y} + \Delta\mathbf{z} = 0 \tag{6.14}$$

$$\mathbf{M}\Delta\mathbf{z} + 2\Delta\mathbf{x} = \mathbf{h}. \tag{6.15}$$

The solution of (6.13)- (6.15) requires solving the normal equations (implicitly or explicitly):

$$\mathbf{A}\mathbf{M}\mathbf{A}^\top\Delta\mathbf{y} = \mathbf{h},$$

which is the main computational work in solving SDPs with this type of interior point methods [123]. A common approach is to compute the Cholesky factorization of $\mathbf{A}\mathbf{M}\mathbf{A}^\top$. Given the fact that $\mathbf{M}$ is the Jordan-Kronecker product of $\mathbf{X}$ and $\mathbf{Z}^{-1}$, can we find more efficient factorization techniques to solve this system of linear equations? Also, spectral information on $\mathbf{M}$ should be helpful in the design and analysis of underlying algorithms.

6.4 Open Problems

In this section, we list some open problems and conjectures arising from the work of this thesis.

In Chapter 2, we presented a number convex optimization problems in compressed sensing with a focus on matrix and eigenvalue inequalities. In particular, we looked into the recovery problems when the measurement matrix is structured, such as a Toeplitz matrix or a Hankel matrix. It is still an open question whether it is possible for these structured random matrices to satisfy (with high probability) the restricted isometry property with constant δ_s when the number of measurements is $O(s \log(n))$.

Conjecture 6.4.1. *The weak interlacing property holds for every* $\mathbf{A}, \mathbf{B} \in \mathbb{S}^3$.

Conjecture 6.4.2. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$*, where* $\mathbf{A} := \mathbf{P} + \mathbf{P}^\top$ *and* $\mathbf{B} := \mathbf{P}' + \mathbf{P}'^\top$*,* $\mathbf{P}, \mathbf{P}'$ *are permutation matrices of order* n*. Then*

$$\min_{\mathbf{Tu}=\mathbf{u}} \frac{\mathbf{u}^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{u}}{\mathbf{u}^\top \mathbf{u}} \leq \min_{\mathbf{Tw}=-\mathbf{w}} \frac{\mathbf{w}^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{w}}{\mathbf{w}^\top \mathbf{w}}. \tag{6.16}$$

We remark here that by Perron-Frobenius theorem, for nonnegative symmetric matrices $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$, (1.4) always holds.

The eigenvalues of the Jordan-Kronecker product corresponding to the even eigenvectors are the same as the eigenvalues of $\mathbf{A} \overset{s}{\otimes} \mathbf{B}$. From Theorem 5.1.2.($iv$) (due to [131]), one can easily obtain the following

$$(\mathbf{A} \overset{s}{\otimes} \mathbf{B}) \succeq 0 \quad \Longleftrightarrow \quad (\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A}) \succeq 0,$$

In addition, as discussed in Example 5.2.1, $(\mathbf{A} \overset{\tilde{s}}{\otimes} \mathbf{B}) \succeq 0$ does not necessarily imply $(\mathbf{A} \otimes \mathbf{B}) \succeq 0$. Furthermore, based on a lot of numerical experiments, we have a strong belief that the minimum eigenvalue of the Jordan-Kronecker product of positive semidefinite matrices $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$ corresponds to a symmetric eigenvector. We state this formally below.

Conjecture 6.4.3. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n_+$*. Then*

$$\min_{\mathbf{Tu}=\mathbf{u}} \frac{\mathbf{u}^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{u}}{\mathbf{u}^\top \mathbf{u}} \leq \min_{\mathbf{Tw}=-\mathbf{w}} \frac{\mathbf{w}^\top (\mathbf{A} \otimes \mathbf{B}) \mathbf{w}}{\mathbf{w}^\top \mathbf{w}}. \tag{6.17}$$

If the minimum eigenvalue of a real symmetric (or Hermitian) matrix is nonnegative then this matrix is positive semidefinite. In primal-dual interior-point methods for semidefinite optimization, the new iterates are computed by adding the search directions to the current primal and dual points using a line search to make sure the new iterates are feasible. Usually the feasibility condition requires positive semidefiniteness of certain matrices, which is determined by the minimum eigenvalue of those matrices. In this respect, we believe the study of this conjecture may be useful.

It would also be interesting to find classes of structured matrices other than commutative matrices, and 2-by-2 matrices for which the strong interlacing property holds.

As discussed in Chapter 4, one can always create a perfect shuffle matrix with a prescribed set of scalars as its eigenvalues (see Proposition 4.4.4), we raise the following inverse eigenvalue problem for a subclass of perfect shuffle matrices: Given a prescribed set of eigenvalues $\lambda_1, \ldots, \lambda_{n^2}$, under what conditions do there exist n-by-n real symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ such that these eigenvalues belong to $\mathbf{A} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{A}$?

Another potential application of the Jordan-Kronecker products can be the estimation of a quantum channel (if it exists) which sends a given set of quantum states to another set of quantum states, see [38] and the references therein. This problem can be stated as follows. Given some positive semidefinite Hermitian matrices $\{\mathbf{A}_1, \ldots, \mathbf{A}_\ell\}$ of order n and $\{\mathbf{B}_1, \ldots, \mathbf{B}_\ell\}$ of order m, with trace one, find a positive linear map $\mathcal{A}$:

$$\mathcal{A}(\mathbf{X}) := \sum_{k=1}^{r} \mathbf{F}_k \mathbf{X} \mathbf{F}_k^H,$$

such that $\sum_{k=1}^{r} \mathbf{F}_k^H \mathbf{F}_k = \mathbb{I}$ and $\mathcal{A}(\mathbf{A}_i) = \mathbf{B}_i$ for every $i \in \{1, 2, \ldots, \ell\}$.

Finally, as a future work we would like to investigate how much Horn's eigenvalue inequalities simplify for general n, when the summands are $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$, where $\mathbf{A}$ and $\mathbf{B}$ are n-by-n real symmetric matrices.

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Appendix A

APPENDICES

A.1 Vector Spaces and Inner Product Spaces

In this section, we provide some basic notations and elementary concepts as a reference.

Definition A.1.1 (Vector Spaces)**.** A *vector space* over a field $\mathbb{F}$ is a nonempty set V equipped with two binary operations: vector addition on $V \times V \to V$ defined as $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u}+\mathbf{v}$ and scalar multiplication $\mathbb{F} \times V \to V$ defined as $(\alpha, \mathbf{v}) \to \alpha\mathbf{v}$, satisfying

1. (Associativity of vector addition) For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$,
2. (Commutativity of vector addition) For every $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$,
3. (talk element of vector addition) There exists a zero vector $\mathbf{0} \in V$ such that for every $\mathbf{u} \in V$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$,
4. (Inverse elements of vector addition) For every $\mathbf{u} \in V$ there exists an inverse element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$,
5. (Associativity of scalar multiplication) For every $\alpha, \beta \in \mathbb{F}$ and $\mathbf{u} \in V$, $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$,
6. (Identity element of scalar multiplication) For every $\mathbf{u} \in V$, $1\mathbf{u} = \mathbf{u}$, where $1 \in \mathbb{F}$ is the identity element of $\mathbb{F}$,
7. (Distributivity of scalar multiplication with respect to vector addition) For every $\alpha \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$, $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$,
8. (Distributivity of scalar multiplication with respect to field addition) For every $\alpha, \beta \in \mathbb{F}$ and $\mathbf{u} \in V$, $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

For a given field $\mathbb{F}$ and positive integer n, the set $\mathbb{F}^n$ of n-tuples with components from $\mathbb{F}$ forms a vector space over $\mathbb{F}$ under the (componentwise) addition and scalar multiplication.

Definition A.1.2. A *subspace* U of a vector space W is a subset of W that is a vector space over the same field.

Definition A.1.3 (Inner product space)**.** An *inner product space* is a vector space V with an inner product defined on V, with an inner product on V is a mapping of $V \times V$ into the scalar field $\mathbb{F}$ of V; that is, with every pair of vectors u, v there is associated a scalar which is written

$$\langle \mathbf{u}, \mathbf{v} \rangle,$$

and is called the *inner product* of u and v, such that for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{F}$

1. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$,
2. $\langle \alpha\mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$,
3. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$, (here $*$ denotes the complex conjugation)
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Definition A.1.4 (Minkowski sum)**.** Given two subsets U, V of a vector space W, the *Minkowski sum* of U and V is defined as

$$U + V := \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}.$$

Definition A.1.5 (Internal direct sum)**.** Let U, V be two subspaces of a vector space W with $U \cap V = \{\mathbf{0}\}$. We say $U \boxplus V$ is the *(internal) direct sum* of U and V if every vector $\mathbf{x} \in (U \boxplus V)$ has a unique representation of the form $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in U$ and $\mathbf{v} \in V$.

The internal direct sum requires uniqueness in the representation compared to the Minkowski sum. Also, note that Minkowski sum applies to every pair of sets in W; on the other hand, $\boxplus$ is only given for subspaces.

Definition A.1.6 ((External) direct sum)**.** Let U, V be two vector spaces over the filed F. We define the *(external) direct sum* of U and V as

$$U \oplus V := \{(\mathbf{u}, \mathbf{v}) : \mathbf{u} \in U, \mathbf{v} \in V\},$$

with addition given by

$$(\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{u}_2, \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2)$$

and scalar multiplication given by

$$\alpha(\mathbf{u}, \mathbf{v}) = (\alpha\mathbf{u}, \alpha\mathbf{v}).$$

Note that $U \oplus V$ is also a vector space. If U, V are subspaces of a vector space W with $U \cap V = \{0\}$ then $U \oplus V$ is isomorphic to $U \boxplus V$.

A.2 Normed Linear Spaces

Definition A.2.1 (Seminorm and norm)**.** A *seminorm* on a real vector space V is a real-valued function on V whose value at a $\mathbf{v} \in V$ is denoted by $\|\mathbf{v}\|$ has the properties

(i) $\|\mathbf{v}\| \geq 0$,

(ii) $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|$,

(iii) $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$,

where $\mathbf{v}, \mathbf{w}$ are arbitrary vectors in V and α is a scalar. A *norm* on V is a seminorm which also satisfies

(vi) $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$,

An inner product on V defines a norm on V given by

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

Definition A.2.2 (Normed linear space)**.** A *normed linear space* V is a vector space with a norm defined on it.

Definition A.2.3 (Equivalent norms)**.** A norm $\|\cdot\|_a$ on a vector space V is said to be *equivalent to a norm* $\|\cdot\|_b$ on V if there are positive numbers $\beta > \alpha > 0$ such that for every $\mathbf{v} \in V$ we have

$$\alpha \|\mathbf{v}\|_b \leq \|\mathbf{v}\|_a \leq \beta \|\mathbf{v}\|_b.$$

Theorem A.2.1. *Given two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a finite dimensional vector space V, $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$.*

Some Commonly Used Norms

In this section, we list a number of norms that are of primary interest to us. Let $\mathbf{x} := \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top$ be a vector in n-dimensional Euclidean space $\mathbb{R}^n$.

(i) **(Euclidean norm)** The *Euclidean norm* on $\mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

(ii) **(ℓ_p-norm)** The *ℓ_p-norm* on $\mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

(iii) **(ℓ_∞-norm)** The ℓ_∞-*norm (or max norm)* on $\mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Definition A.2.4 (Quasi-norm)**.** A *quasi-norm* on a real vector space V is a real-valued function on V whose value at a $\mathbf{v} \in V$ is denoted by $\|\mathbf{v}\|$ has the properties

(i) $\|\mathbf{v}\| \geq 0$,

(ii) $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$,

(iii) $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$,

(iv) there is a constant C such that

$$\|\mathbf{v} + \mathbf{w}\| \leq C\left(\|\mathbf{v}\| + \|\mathbf{w}\|\right). \tag{A.1}$$

where $\mathbf{v}, \mathbf{w}$ are arbitrary vectors in V and α is a scalar.

Note that $\|\cdot\|_p$ is not a norm when $p \in (0, 1)$, as the triangle inequality fails. However, when it defines a quasi-norm and the constant satisfying (A.1) is $C = 2^{\frac{1}{p}-1}$.

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