On the Excluded Minors for Dyadic Matroids

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The study of the class of dyadic matroids, the matroids representable over both $GF(3)$ and $GF(5)$, is a natural step to finding the excluded minors for $GF(5)$-representability. In this thesis we characterize the ternary matroids M that are excluded minors for dyadic matroids and contains a 3-separation. We will show that one side of the separation has size at most four, and that M is obtained by adding at most four elements to another excluded minor M'. This reduces the problem of finding the excluded minors for dyadic matroids to the problem of finding the vertically 4-connected excluded minors for dyadic matroids.

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Chapter 1

Introduction

The main result in this thesis reduces the problem of finding the excluded minors for the class of dyadic matroids to the problem of finding the vertically 4-connected excluded minors.

1.1 Background

In his introductory paper on matroids, Whitney [23] posed the problem of characterizing representable matroids. The class $\mathcal{M}$ of matroids that are representable over a field $\mathbb{F}$ is closed under taking minors, so it is natural to characterize $\mathcal{M}$ by its set of excluded minors; these are the minor minimal-matroids not in $\mathcal{M}$. When $\mathbb{F}$ is infinite, the excluded minors were shown to be as least as wild as the class itself; see Mayhew, Newman, and Whittle [12]. However, there are very natural questions to be considered for finite fields. For every finite field $\mathbb{F}$, is this set of excluded minors finite? And if so, what are they? The former question is known as Rota's Conjecture, which was one of the major unsolved problems in matroid theory until a proof was announced in 2014 by Geelen, Gerards, and Whittle [8]. For the latter question, the cases for $GF(2), GF(3), GF(4)$ have been solved over the past 60 years.

In 1958, Tutte proved that a matroid M is binary if and only if M contains no minor isomorphic to $U_{2,4}$, the 4-point line [20]. This gives a strikingly simple connection between the algebraic property of being representable over $GF(2)$, and the combinatorial property of containing a certain minor. Then in 1979, Bixby [1] and Seymour [18] indepedently proved that the excluded minors for ternary matroids are $U_{2,5}, U_{3,5}, F_7$, and F_7^*. Bixby attributes this characterization to unpublished work of Reid. These proofs relied on the fact that matroids in these two classes are uniquely representable over their respective field, a technique that would be used in the proofs for larger fields. In 1988, Kahn showed that a quarternary matroid is uniquely representable if and only if it is not the 2-sum of two non-binary matroids [11]. Twelve years later, this was followed by a proof for the excluded minors of $GF(4)$-representable matroids, published by Geelen, Gerards, and Kapoor [6] in 2000, using the result of Kahn. They showed that there are seven excluded minors for $GF(4)$-respresentability, and they are $U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*, P_8$, and P_8''.

Logically, the next goal is to study $GF(5)$, which is the field that we are interested in for this thesis. It is known that there are at least 564 excluded minors for $GF(5)$-representabilty, found via a database of all matroids with up to nine elements that was compiled computationally by Mayhew and Royle [13]. A first step to understanding the excluded minors for $GF(5)$-representability is to study the class of dyadic matroids, which are the ones representable over both $GF(5)$ and $GF(3)$. The motivation for this stems backs to the problem of limiting the number of inequivalent representations, which proved fruitful for $GF(2), GF(3)$, and $GF(4)$. While quinary matroids, those that are $GF(5)$-representable, may have up to six inequivalent $GF(5)$-representations, the story is different if the matroid is also ternary. Whittle proved that dyadic matroids have at most three inequivalent representations over $GF(5)$ [24]. A strengthening of this result was proved by Pendavingh and van Zwam in [17], where they prove that a ternary matroid may have exactly zero, one, or three inequivalent $GF(5)$-representations, and describe the conditions for each case to occur. In particular, if a dyadic matroid has three inequivalent $GF(5)$-representations, then M is near-regular but not regular. This is useful since the list of excluded for regular matroids (Tutte [20]) and near-regular matroids (Hall, Mayhew, and van Zwam [10]) are known.

Most of the difficulties of the proof of the excluded minors for $GF(4)$-representability disappear when considering 6-connected excluded minors. These difficulties gradually return as we consider the 5-connected, 4-connected, and, finally the 3-connected cases. Here we are proposing an alternative approach of applying connectivity reductions to determine the excluded minors with low connectivity.

Our contribution for this thesis is a characterization of the case when an excluded minor for dyadic matroids is has an exact 3-separation. Our theorem would imply that, given a list of all vertically 4-connected excluded minors for dyadic matroids, one can generate the full list of dyadic excluded minors after some case analysis.

1.2 Main Theorem

Observe that any excluded minor for dyadic matroids that is not ternary contains one of $U_{2,5}, U_{3,5}, F_7$, or F_7^* as a minor. Hence we look at the excluded minors for dyadic matroids that are ternary, and they are precisely the ternary matroids that are excluded minors for $GF(5)$-representability, as all of their proper minors are dyadic.

Let M be a ternary matroid with an exact 3-separation (X, Y). We construct a ternary matroid $M^+_{X,Y}$ by extending M by a set L of four elements added into the span of both X and Y such that L is independent. As ternary matroids are uniquely representable (Brylawski 1976: [2]), this is well-defined. We call L the *boundary line* of (X, Y). Note that, since $M^+_{X,Y}$ need not be simple, technically L may not be a line of $M^+_{X,Y}$, however we will call it as such for convenience. We then let $M_X = M^+_{X,Y}|(X \cup L)$ and $M_Y = M^+_{X,Y}|(Y \cup L)$.

Our main theorem is the following:

Theorem 1.2.1. *Let M be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation (X, Y) such that $|X|, |Y| \geq 4$. Then either:*

- *M is isomorphic to T_8,*
- *$|X| = 4$ and for each $e \in Y$, both $M_Y \backslash e$ and M_Y/e are $GF(5)$-representable, or*
- *$|Y| = 4$ and for each $e \in X$, both $M_X \backslash e$ and M_X/e are $GF(5)$-representable.*

The matroid T_8 is a well-known excluded minor for dyadic matroids (see page 649 of [16]). From the diagram of a geometric representation of T_8 in Figure 1.1 it is clear that the two planes drawn induce an exact 3-separation of T_8.

It follows from Theorem 1.2.1 that an excluded minor containing an exact 3-separation must be a minor of a matroid with at most four extra elements added to a previously-known excluded minor. So we can deduce the list of ternary excluded minors for dyadic matroids from the set $\mathcal{M}$ of vertically 4-connected ternary excluded minors for dyadic matroids: For $M \in \mathcal{M}$, consider the set of matroids obtained by a sequence of four ternary extensions or coextensions of M, check if each of them contain an excluded minor, adding any new excluded minors to $\mathcal{M}$, and repeating this procedure until all excluded minors have been found. There is no inherent reason that this procedure does not run forever; its termination is a consequence of Rota's Conjecture.

Note that the result that we prove in Chapter 3 also characterizes the specific matroids M_X, M_Y that correspond to the side of size four.

An informal outline of our proof strategy will be given in the next section.

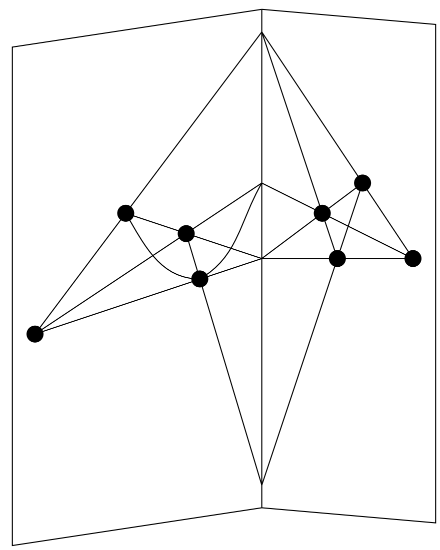

Figure 1.1: The matroid T_8

1.3 Proof Outline

Preliminary results and definitions can be found in Chapter 2 and the text of Oxley [16]. The proof of our main result follows in Chapter 3.

The main strategy that we will use is to decompose the excluded minor M along the separation (X, Y), analyze each side, then find the minimal obstructions that prevent us from reassembling the two sides into a quinary matroid. We start off in Section 3.1 by constructing the two matroids corresponding to each side, $M_X = M^+_{X,Y}|(X \cup L)$ and $M_Y = M^+_{X,Y}|(Y \cup L)$.

Now we need to consider the relationship between the non-$GF(5)$-represtability $M^+_{X,Y}$ with M_X and M_Y, and between $M^+_{X,Y}$ and M. For the former question, the tool that we will use is the generalized parallel connection introduced by Brylawski [4]. Generalized parallel connections gives us a way to glue two matroids together along a common flat if that flat is modular on at least one side, and we begin by showing that $M^+_{X,Y}$ can be constructed by a generalized parallel connection of M_X and M_Y. This is important for representablility since if a matroid N is a generalized parallel connection of two $\mathbb{F}$-representable matroids N_1 and N_2, and they have representations that agree on $E(N_1) \cap E(N_2)$, then the N itself is $\mathbb{F}$-repersentable; see Lemma 2.6.2.

Since M is an excluded minor for quinary matroids, $M^+_{X,Y}$ must be non-quinary, and so it must not be the case that M_X and M_Y are quinary and have a representation that match along L. Over $GF(5)$, a 4-point line L can have three inequivalent representations, which are given by the matrices of the form $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{bmatrix}$ for $x \in \{2, 3, 4\}$. The value x is the *cross-ratio* of the representation of L with respect to that ordering of its elements.

Now that we have constructed $M^+_{X,Y}$, we want to find the conditions for when adding the elements of L to M does not change the representability. While adding elements on a line is well-defined for ternary matroids, this isn't the case for quinary matroids. The classic example is a pair of representations of $U_{3,6}$ with respect to a given line (Figure 1.2).

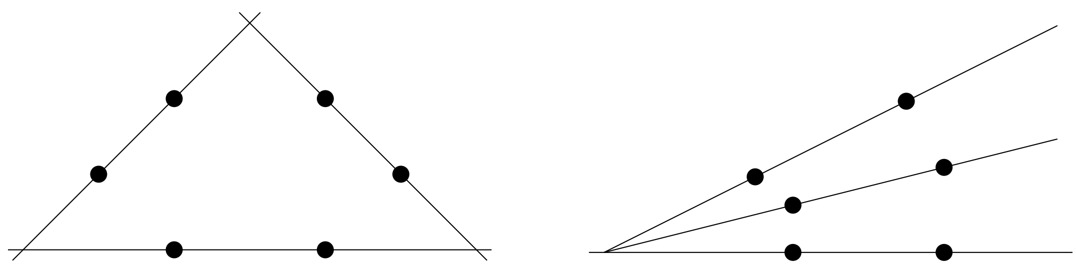

Figure 1.2: Two representations of the matroid $U_{3,6}$

However, in Section 3.2 we show that this is well-defined if both M_X and M_Y "distinguish" the element. Let N be a subset of the groundset of a matroid M, if there is a set $S \subseteq E(M) \setminus \{e\}$ that spans e but not N, we call S a strand for N. If $M|N$ is a line, we say that $e \in N$ is *distinguished*. An element e in a boundary line L is *pinned* if both M_X and M_Y distinguish e.

Lemma 1.3.1 (Lemma 3.2.1)**.** *Let M be a ternary matroid with an exact 3-separation (X, Y). Let M' be a ternary matroid obtained from M via an extension by a non-loop element e such that $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. If there exists strands $S_X \subseteq X$ for Y and $S_Y \subseteq Y$ for X such that $e \in \mathrm{cl}(S_X) \cap \mathrm{cl}(S_Y)$, then M is $\mathbb{F}$-representable if and only if M' is $\mathbb{F}$-representable.*

So we want to show that all four elements of L are pinned in $M^+_{X,Y}$; our next lemma shows what would happen if this were not the case:

Lemma 1.3.2 (Lemma 3.2.3)**.** *Let M be a 3-connected ternary matroid whose proper minors are all quinary and let (X, Y) be an exact 3-separation of M with $|X| \geq 4$. If M_X does not distinguish all four elements in the boundary line, then M is quinary.*

The proof strategy is as follows: If $e \in L$ is not distinguished, then $M_X \backslash e$ has a modular 3-point line and $M_Y \backslash e$ is a minor of M so it is quinary. A theorem of Seymour [19] implies that $M_X \backslash e$ is binary. Since matroids that are both binary and ternary are in fact regular, $M_X \backslash e$ is quinary. It is then easy to glue the $M_X \backslash e$ and $M_Y \backslash e$ together, since we can use row operations to find representations that agree along the modular 3-point line of $M_X \backslash e$. Then applying Lemma 2.6.2 implies that M is quinary.

Our excluded minor M is not quinary, thus all four elements of L are pinned if X and Y have size at least four.

Lemma 1.3.3 (Lemma 3.2.4)**.** *Let M be a ternary excluded minor for $GF(5)$-representability and let (X, Y) be an exact 3-separation of M. If $|X|, |Y| \geq 4$, then $M^+_{X,Y}$ pins all elements of the boundary line.*

So we have established that a ternary matroid M is quinary if and only if $M^+_{X,Y}$ is, which in turn is quinary if and only if there exists quinary representations of M_X and M_Y that match along L. Next we consider whether M_X and M_Y are quinary, first dealing with the case that we have M_Y non-quinary. In Section 3.4, we prove the following lemma:

Lemma 1.3.4 (Lemma 3.4.1)**.** *Let M be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation (X, Y). If M_Y is non-quinary, then there is no proper minor of M'_X of M_X such that $M|L = M'|L$ and all four elements of L are distinguished in M'_X.*

The proof of this lemma relies on a simple observation: if a minor M'_X of M_X keeps L and keeps all four elements of L distinguished, then the generalized parallel connection of M'_X and M_Y results in a non-quinary matroid that is a minor of $M^+_{X,Y}$, which we can then use to construct a non-quinary minor of M.

So then our goal is to characterize such minor-minimal matroids. Our next lemma will show that they must have at most four elements apart from the line. As a result, if one side of the separation is non-quinary, then the other side has size at most four.

Lemma 1.3.5 (Lemma 3.4.2)**.** *Let L be a modular set in a ternary matroid M such that $M|L$ is isomorphic to $U_{2,4}$. If all four elements of L are distinguished, then M contains an 8-element minor N such that $M|L = N|L$ and all elements of L are distinguished in N.*

By Lemma 3.4.2, if M_Y is not quinary, then $|X| = 4$. By case analysis, M_X is either the unique matroid with a modular 4-point line L and an independent coline, or one of the four matroids in Figure 1.3.

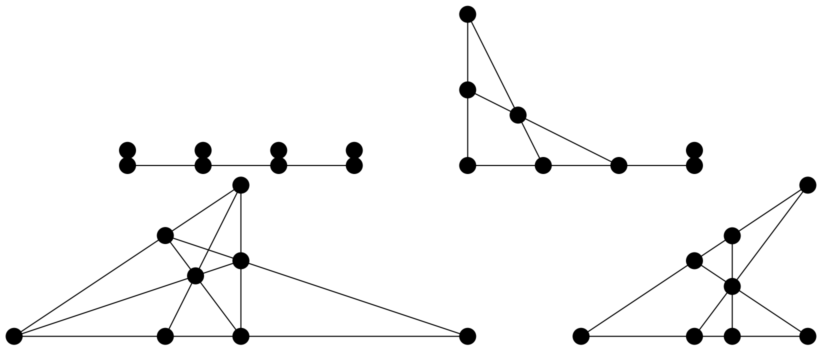

Figure 1.3: Four of the matroids that satisfy the conditions of Lemma 3.4.2

In the remaining case we have that both M_X and M_Y are quinary, so we want to know whether they have representations that match along the boundary line L. In Section 3.5, we focus on the next question: how many $GF(5)$-representations do M_X and M_Y have? A result from Whittle [24] states that $U_{2,4}$ is a stabilizer for ternary matroids. In other words, if a 3-connected ternary matroid N has a $U_{2,4}$-minor, then every $GF(q)$-representation of N is uniquely determined by the $GF(q)$-representation of that $U_{2,4}$-minor. Since M_X contains the $U_{2,4}$-restriction L, the number of quinary representations of L that extend to a representation of M_X is equal to the number of quinary representations of M_X itself, and similary for M_Y.

Pendavingh and van Zwam proved that that 3-connected dyadic matroids have either one or three inequivalent representations over $GF(5)$ [17], and characterized exactly when each case happens:

Theorem 1.3.6. *For a 3-connected dyadic matroid M, either*

(i) M is regular, and M is uniquely representable over $GF(5)$.

(ii) M is near-regular but not regular, and M has exactly three representations over $GF(5)$.

(iii) M is dyadic but not near-regular, and M is uniquely representable over $GF(5)$.

We will use this theorem to show that neither M_X nor M_Y can be near-regular.

Lemma 1.3.7 (Corollary 3.5.3)**.** *Let M be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation (X, Y). If both M_X and M_Y are quinary, then neither M_X nor M_Y are near-regular.*

Suppose, towards a contradiction, that M_X is near-regular. Note that M_X is not binary as it contains a $U_{2,4}$-restriction, so M_X is near-regular but not regular. Since M_X is ternary, it does not contain $U_{2,5}$ or $U_{3,5}$ as a minor, so we can apply Theorem 1.3.6 to M_X. It follows that M_X has three inequivalent representations over $GF(5)$, and in particular all three possible $GF(5)$-representations of L extend to a representation of M_X. Now given any quinary representation of M_Y, we can take its representation of L and find a $GF(5)$-representaton of M_X that extends this. We thus have quinary representations of M_X and M_Y that match along L, and we can apply Lemma 2.6.2 to obtain a contradiction.

Since we have proven that M_X and M_Y are not near-regular, the final part of the proof is concerned with finding the minor-minimal dyadic matroids that have a $U_{2,4}$-restriction and are not near-regular. If a matroid is both dyadic and not near-regular, it is not $GF(4)$-representable. Then we can use the characterization of quarternary matroids of Geelen et al. in [7].

Theorem 1.3.8. *If M is a 3-connected non-$GF(4)$ representable matroid, then either*

(i) M has a $U_{2,6}$-, $U_{4,6}$-, P_6-, F_7^--, or $(F_7^-)^*$-minor,

(ii) M is isomorphic to P_8'', or

(iii) M is isomoprhic to a minor of $S(5,6,12)$ with rank and corank at least 4.

By eliminating the cases of Theorem 1.3.8 that are not compatible with a matroid that is dyadic but not near-regular, we deduce that M_X and M_Y has an F_7^-- or $(F_7^-)^*$-minor, whose elements we will call F. We further prove that they have no elements other than the ones in $F \cup L$ and by case analysis we will show that they are in fact isomorphic to the matroid that we will refer to as O_8 (see Figure 1.4).

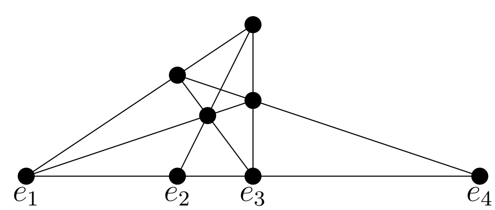

Figure 1.4: The matroid O_8

It is a single element extension of the matroid O_7 and is the unique matroid with a modular 4-point line along with a 4-element *circuit-cocircuit*, a set that is both a circuit and a cocircuit. Observe that T_8 can be obtained by a generalized parallel connection (see Section 2.6) of two copies of O_8 along their respective 4-point lines where (e_1, e_2, e_3, e_4) is matched with (e_1, e_2, e_4, e_3).

Lemma 1.3.9 (Lemma 3.6.1)**.** *Let L be a $U_{2,4}$-restriction in a 3-connected matroid M that is dyadic but not near-regular. Then M has a minor N isomorphic to O_8 such that L is a $U_{2,4}$-restriction of N.*

Thus, the only excluded minors of this form are constructed from taking of these non-Fano extensions and gluing them along their 4-point line in a way that the cross ratios do not match up. We will show that the only such matroid that arises this way is T_8.

Chapter 2

Preliminaries

We will assume the reader has familiarity with the definitions of a matroid, its rank function, flats, uniform matroids, parallel elements, simplication, cosimplicaton, representability, duality, minors, connectivity, 3-connectivity, k-separation, extensions, and coextensions; see Oxley's textbook [16].

2.1 Representable Matroids/Projective Geometries

A matroid M is *uniquely representable* over $\mathbb{F}$ if all $\mathbb{F}$-representations of M are equivalent under row operations, column scaling, and automorphisms of $\mathbb{F}$. A matroid is *binary*, *ternary*, *quaternary*, or *quinary* if it is $GF(2)$-, $GF(3)$-, $GF(4)$-, or $GF(5)$-representable, respectively. Additionally, a matroid is *regular* if it is representable over all fields, *dyadic* if it is representable over $GF(3)$ and $GF(5)$, and *near-regular* if it is representable over $GF(3)$, $GF(4)$, and $GF(5)$.

To prove that a matroid M is regular, it suffices to show that M is both binary and ternary (Theorem 6.6.3 of [16]). This was proved by Tutte in [21].

Theorem 2.1.1. *The following statements are equivalent for a matroid M.*

- *M is regular.*
- *M is representable over $GF(2)$ and $GF(3)$.*

Let $GF(q)$ be the finite field of order q. Consider the maximum number of elements in a simple $GF(q)$-representable matroid with rank n. Observe that there are $q^n - 1$ non-zero vectors in $GF(q)^n$, and $\frac{q^n-1}{q-1}$ subspaces of rank one. Taking a representative vector from each rank one subspace, the associated vector matroid has $\frac{q^n-1}{q-1}$ elements, and we denote this simple rank-n $GF(q)$-representable matroid by $PG(n-1, q)$.

By convention, for a matrix representation A of a matroid M we may also label the row i of A by an element b of M if b labels the elementary vector e_i.

2.2 Duality

The following proposition (Proposition 2.1.6 of [16]) characterizes some elementary properties of the dual matroid:

Proposition 2.2.1. *Let M be a matroid and let $X \subseteq E(M)$. Then*

- *X is independent in M^* if and only if $E(M) \setminus X$ is spanning in M, and*
- *X is a hyperplane in M if and only if $E(M) \setminus X$ is a circuit in M^*.*

For an $\mathbb{F}$-representable matroid M, the following proposition gives a $\mathbb{F}$-representation of its dual.

Proposition 2.2.2. *Let M be a matroid M with n elements and rank r such that $M = M[A]$ where A is the matrix $[I_r|B]$. Let $A^* = [I_{n-r}| - B^T]$. Then $M^* = M[A^*]$.*

A directly corollary is that a matroid M is $\mathbb{F}$-representable if and only if M^* is $\mathbb{F}$-representable.

A *cyclic flat* F of a matroid M is a flat that is the union of circuits. It is routine to show that F is a cyclic flat of M if and only if $E(M) \setminus F$ is a cyclic flat of M^*.

2.3 Minors

For minors of matroids representable over a field $\mathbb{F}$, observe that deleting an element in a matroid clearly corresponds to deleting the column representing that element. Since contraction is defined as $M/e = ((M^*)\backslash e)^*$, and both deletion and duality preserve $\mathbb{F}$-representability, so contraction also preserves $\mathbb{F}$-representability. Thus the class of $\mathbb{F}$-representable matroids is minor-closed.

Let N be a minor of M. Since the deletion and contraction operations commute with each other, it is often useful to consider N as obtained from M by contracting a set of elements C then deleting a set of elements of D. The Scum Theorem (Theorem 3.3.1 of [16]) from Crapo and Rota [5] says that we may assume that C is indepedent and D is coindependent.

Theorem 2.3.1 (Scum Theorem)**.** *Let N be a minor of M. Then there exists an independent set C and a coindependent set D such that $N = M/C\backslash D$.*

2.4 Connectivity

The connectivity function of a matroid M is defined as $\lambda_M(X) = r(X)+r(E(M)\backslash X)-r(M)$ for $X \subseteq E(M)$. A subset X of the groundset of a matroid M is *k-separating* if $\lambda_M(X) < k$. A k-separation (X, Y) is *exact* if neither X nor Y are $k-1$-separating.

We say that two separations (X, Y) and (X', Y') *cross* if $X \nsubseteq X'$, $X \nsubseteq Y'$, $X' \nsubseteq X$, and $X' \nsubseteq Y$. When k-separations in a k-connected matroid cross, we will often want $X \cap X'$ to be a separation, and the following lemma from Oxley, Semple, and Whittle [15] proves that this is the case, provided that each side has sufficiently many elements. The proof of this lemma, referred to as the Uncrossing Lemma, relies only on the submodularity of the connectivity function.

Lemma 2.4.1 (Uncrossing Lemma)**.** *Let X and X' be k-separating sets of a k-connected matroid M.*

(i) If $|X \cap X'| \geq k-1$, then $X \cup X'$ is k-separating.

(ii) If $|E(M) - (X \cup X')| \geq k-1$, then $X \cap X'$ is k-separating.

An element e is in the *guts* of (X, Y) if $e \in cl_M(X \setminus e)$ and $e \in cl_M(Y \setminus e)$. Dually e is in the *coguts* of (X, Y) if $e \in cl^*_M(X \setminus e)$ and $e \in cl^*_M(Y \setminus e)$.

Local connectivity is a related notion that captures the connectivity between two subsets in a matroid. For $X, Y \subseteq E(M)$, the *local connectivity* $\sqcap_M(X, Y)$ is defined as

$$\sqcap_M(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

For a subset N of $E(M)$, we can consider how the rest of the matroid connects to it. We call a set $S \subseteq E(M) \setminus N$ a *strand* for N if it is a minimal set such that $\sqcap(S, N) = 1$. We say that a strand S *distinguishes an element* e in N if $e \in \text{cl}(S)$, and a matroid M *distinguishes* e if $E(M) \setminus N$ contains a strand that distinguishes e. Note that if $M|N$ is a simple line, then e is distinguished if and only if it is a fixed element of M, since we can extend freely on the line if and only if e is not distinguished.

Another related function, κ_M describes the connectivity between two sets in the context of the matroid as a whole. For disjoint subsets $X, Y \subseteq E(M)$, we define:

$$\kappa_M(X, Y) = \min\{\lambda_M(S) : X \subseteq S \subseteq E(M) \setminus Y\}.$$

Tutte [22] proved the following matroid analogue of Menger's theorem:

Theorem 2.4.2 (Tutte's Linking Theorem)**.** *Let X, Y be disjoint subsets of elements in a matroid M. Then $\kappa_M(X, Y)$ is the maximum of $\kappa_N(X, Y)$ for all minors N of M with groundset $X \cup Y$.*

Tutte's Linking Theorem can be strengthened to finding a minor N that preserves not only $\kappa_M(X, Y)$, but the structure of $M|X$ and $M|Y$ as well (Theorem 8.5.7 of [16]).

Theorem 2.4.3. *Let X, Y be disjoint subsets of elements in a matroid M. Then there exists a minor N of M such that $E(N) = X \cup Y$, $\kappa_N(X, Y) = \kappa_M(X, Y)$, $N|X = M|X$, and $N|Y = M|Y$.*

When we delete or contract an element from a 3-connected matroid, we often need to consider the case when this results in a matroid M that is not 3-connected. Let (X, Y) be a 2-separation of M. Consider the matroid M_X^+ by contracting an independent set $S \subseteq Y$ of rank $r(Y) - 1$ such that $\sqcap_M(S, X) = 0$ and simplifying Y to an element p. It is routine to show that M_X^+ does not depend on the choice of S, so it is well-defined. Let M_Y^+ be the analguous matroid obtained by contracting from X. Then M is the 2-sum of M_X and M_Y with basepoint p. For the 2-separating set Y, we call M_X^+ the matroid obtained by *contracting Y down to its basepoint*. We first show that this operation preserves 3-connected minors that have at most one element in Y.

Proposition 2.4.4. *Let X, Y be an exact 2-separation of a matroid M. If N is a 3-connected minor with $|E(N) \cap X| \geq |E(N) - 1|$, then M_X^+ contains an N minor.*

Proof. Let $E(N) \cap Y = \{e\}$. By the Scum Theorem, there exists an independent set C and coindependent set D such that $N = M/C \backslash D$. First, $C \cap (cl(X) \cap \mathrm{cl}(Y))$ is non-empty, otherwise N is disconnected with $\{e\}$ as a component. We claim that $|C \cap Y| = r(Y) - 1$. Clearly $|C \cap Y| < r(Y)$ as e is not a loop in N. Suppose $|C \cap Y| < r(Y) - 1$. Then $\mathrm{si}((M|Y)/(C \cap Y)$ is a 2-separating set in N, a contradiction. Thus $M/(C \cap Y) = M_X^+$ and so M_X^+ contains an N-minor. □

Contracting Y down to its basepoint also preserves strands for sets of X.

Proposition 2.4.5. *Let (X, Y) be an exact 2-separation (X, Y) in a matroid M, and let $N \subseteq X$. If an element $e \in N$ is distinguished in M, then e is distinguished in M_X^+.*

Proof. Let S be a strand for N in M that distinguishes e. If $S \subseteq X$ then we are done, so assume that $S_Y = S \cap Y$ is non-empty and that $S_Y \not\subseteq \mathrm{cl}(X)$ (otherwise $S_Y \in \mathrm{cl}(p)$, where p is the basepoint of the 2-separation). Suppose $|S_Y| = 1$, then $(S \setminus S_Y) \cup \{e\}$ is a subset of X that spans S_Y, contradicting that $S_Y \not\subseteq \mathrm{cl}(X)$. So $|S_Y| > 1$, and $|S \setminus S_Y| \geq 1$ since $e \notin \mathrm{cl}(p)$. However, this implies that $S \cup \{e\}$ is a circuit and S_Y is 2-separating in $S \cup \{e\}$, contradicting that a circuit is 3-connected. □

By applying Proposition 2.4.4, we can determine the structure of 2-separations in a matroid M relative to a 3-connected minor N.

Lemma 2.4.6. *Let N be a 3-connected minor of a 2-connected matroid M such that $|E(N)| \geq 4$. Then there exists a partition $B_1, \ldots, B_k$ of $E(M)$ such that:*

(i) $\lambda_M(B_i) = 1$ *for* $i \in \{1, \ldots, k\}$,

(ii) $|E(N) \cap B_i| \leq 1$ *for* $i \in \{1, \ldots, k\}$, *and*

(iii) if M'' is the matroid obtained by contracting all B_i down to their respective basepoints e_i, then M'' is 3-connected with groundset $\{e_1, \ldots, e_k\}$ with an N-minor.

Proof. Let $\mathcal{T}$ be the collection of 2-separating sets B of M such that $|E(N) \cap B| \leq 1$. We define the following binary relation $(E(M), \sim)$: for elements $e, f \in E(M)$, we let $e \sim f$ if there exists a 2-separation (A, B) of M such that $e, f \in B$ and $B \in \mathcal{T}$. We claim that $(E(M), \sim)$ is an equivalence relation. Clearly $(E(M), \sim)$ is reflexive and symmetric, so it remains to show that it is transitive. Let $e, f, g \in E(M)$ such that (A, B) and (A', B') are 2-separations of $E(M)$ such that $e, f \in B$, $f, g \in B'$, and $B, B' \in \mathcal{T}$. Now, by the uncrossing lemma, $B \cup B'$ is 2-separating. Since $|E(N)| \geq 4$, there are at least two elements of N in $A \cap A'$. So $(A \cap A', B \cup B')$ is a 2-separation of M. Then as N is 3-connected, $B \cup B' \in \mathcal{T}$ since there can not be at least two elements of N in both sides of a 2-separation. Thus $e \sim g$ as desired.

Let $B_1, \ldots, B_k$ be the partition of $E(M)$ formed by the equivalence classes of $(E(M), \sim)$. We want to show that this partition satisfies the properties (i), (ii), and (iii). Suppose $\lambda_M(B_i) > 1$ for some $i \in \{1, \ldots, k\}$, then $|B_i| \geq 2$. Let B_i' be the union of all 2-separating sets in $\mathcal{T}$ that have non-empty intersection with B_i. Then $B_i \subsetneq B_i'$ since B_i is not 2-separating but B_i' is 2-separating by the uncrossing lemma. Let $g \in B_i' \setminus B_i$. Then $(E(M) \setminus B_i', B_i')$ is a 2-separation of M such that $e, g \in B_i'$ and $B_i' \in \mathcal{T}$, contradicting that $g \notin B_i$. This proves (i).

To prove (ii), suppose that $B_i \notin \mathcal{T}$, so let $n, n' \in E(N) \cap B_i$. Then there exists a 2-separation (A, B) of M such that $n, n' \in B$ and $B \in \mathcal{T}$, which is a contradiction as $|E(N) \cap B| \geq 2$. Finally, condition (iii) follows from applying Proposition 2.4.4 on each B_i. $\square$

The following lemma describes the structure of a matroid with a chain of 2-separations that separate two elements e, f.

Lemma 2.4.7. *Let M be a 2-connected matroid with a set S of k elements and $e, f \in E(M) \setminus S$. If M/s has two components where e, f are in distinct components for all $s \in S$, then there exists an ordering $(s_1, \ldots, s_k)$ of S and a partition $(A_0, \ldots, A_k)$ of $E(M) \setminus S \setminus \{e, f\}$ such that, for all $i \in \{1, \ldots, k\}$, the sets $\{e, s_1, \ldots, s_{i-1}\} \cup A_0 \cup \ldots \cup A_{i-1}$ and $\{s_{i+1}, \ldots, s_k, f\} \cup A_i \cup \ldots \cup A_k$ are the components of M/s_i.*

Proof. We proceed by induction on k. This is clear for $k = 1$, so let $k > 1$ and for some element $s \in S$, by applying the inductive hypothesis on $S \setminus \{s\}$ there exists an ordering $(s_1, \ldots, s_{k-1})$ and a partition $(A_0, \ldots, A_{k-1})$ satisfying the connectivity properties of the lemma. Now s is an element of some A_i. Since $\{e, s_1, \ldots, s_{i-1}\} \cup A_0 \cup \ldots \cup A_{i-1}$ is a component in M/s_i and does not contain s, it is connected in M/s and its component contains s_i since it spans s_i in M. Similarly $\{s_{i+1}, s_{i+2}, \ldots, s_k, f\} \cup A_{i+1} \cup \ldots \cup A_k$ is connected in M/s. Since M/s has 2 components, let A_i' be the elements of A_i in the component containing $\{e, s_1, \ldots, s_{i-1}\} \cup A_0 \cup \ldots \cup A_{i-1}$ and A_i'' be the elements of A_i in the component containing $\{s_{i+1}, s_{i+2}, \ldots, s_k, f\} \cup A_{i+1} \cup \ldots \cup A_k$ (these components are distinct since e, f are in different components of M/s).

Then $(s_1, \ldots, s_i, s, s_{i+1}, \ldots, s_k)$ and $(A_0, \ldots, A_{i-1}, A_i', A_i'', A_{i+1}, A_k)$ satisfies the desired conditions. $\square$

2.5 Modularity/Lines

A set X is *modular* if, for all flats F in M, the following equation holds:

$$\sqcap(X, F) = r(F \cap X).$$

The proposition below (6.9.2 of [16]) gives a useful characterization of when a flat in a matroid is modular.

Proposition 2.5.1. *A flat X in a matroid M is a modular flat if and only if $r(X)+r(Y) = r(X \cup Y)$ for all flats Y such that $X \cap Y = \emptyset$.*

In particular, if a flat F is modular, then every strand for F distinguishes an element in F.

Seymour proved the following theorem [19], which implies that any matroid with a modular 3-point line is binary:

Theorem 2.5.2. *Let M be a 3-connected non-binary matroid. Then for any two elements $e_1, e_2 \in E(M)$, M contains a $U_{2,4}$-minor N such that $e_1, e_2 \in E(N)$.*

Corollary 2.5.3. *Let T be a 3-point line in a 3-connected matroid M. If T is modular, then M is binary.*

Proof. Suppose, for a contradiction, that M is not binary. Let e_1 and e_2 be elements of T. By Theorem 2.5.2, M contains a $U_{2,4}$-minor N such that $e_1, e_2 \in E(N)$. Let $e_4 \in E(N) \setminus \{e_1, e_2, e_3\}$. By Theorem 2.3.1, there exists an indepedent set C and a coindepedent set D such that $N = M/C\backslash D$. Let $F = cl_M(C \cup \{e_4\})$. Then $\sqcap_M(F, T) = 1$, but $r_M(F, T) = 0$, contradicting that T is modular. $\square$

In a 3-connected matroid M with a modular line L, if L is neither spanning nor cospanning, then we can find a minor M' of M such that there are 3 elements in a triangle T, each parallel to an element of L. We will prove a short proposition to aid the proof.

Proposition 2.5.4. *Let M be a 3-connected matroid on groundset $S \cup L$ such that S is a circuit and L is a line. If $e \in S \setminus \mathrm{cl}(L)$, then M/e is 3-connected.*

Proof. Let $M' = M/e$, and $S' = S \setminus \{e\}$. Suppose, for a contradiction, that (A, B) is a 2-separation of M'. Since $e \notin L$, L is a line in M' and we may assume that $L \subseteq A$. Since contraction inside of a circuit preserves the circuit, S' is a circuit in M', at least $|S'| - 1$ elements of S' are in either A or B. Since $|B| \geq 2$, we have $|S' \cap B| \geq |S'| - 1$.

Then $\lambda_M(A) = r_M(A) + r_M(B \cup \{e\}) - r(M) = r_{M'}(A) + (r_{M'}(B) + 1) - (r(M') + 1) = r_{M'}(A) + r_{M'}(B) - r(M') = 1$. Since $|A| \geq 2, |B| \geq 2$, we have that $(A, B \cup \{e\})$ is a 2-separation in M, a contradiction. □

Lemma 2.5.5. *Let M be a 3-connected matroid with a modular line L. If $E(M) \setminus L$ is non-empty and not a coline, then there exists sets $C \subset E(M) \setminus L$ and $\{x_1, x_2, x_3\} \subseteq E(M) \setminus L$ such that*

- *$\{x_1, x_2, x_3\}$ is a triangle in M/C, and*
- *each of x_1, x_2, x_3 is parallel to an element of L in M/C.*

Proof. First, $E(M) \setminus L$ contains a circuit. Supposing otherwise, then $E(M) \setminus L$ is independent so L is cospanning, which implies that $E(M) \setminus L$ is a coline.

Let S be a circuit of $E(M) \setminus L$, and since M is 3-connected, we have $\kappa_M(L, S) = 2$. By Theorem 2.4.3 (Tutte's Linking Theorem) there exists a minor N of M with groundset $L \cup S$, such that $\kappa_N(L, S) = 2$, $M|L = N|L$, and $M|S = N|S$. By Theorem 2.3.1 (Scum Theorem), we can write $N = M/C' \setminus D$ where C' is independent.

Suppose $S = \{s_1, s_2, s_3\}$. Then S is a triangle and all elements of S lie on L as $\kappa_N(L, S) = 2$. Then $C = C'$ satisfies the conditions. Otherwise, consider the sequence of matroids $N_1, N_2, \ldots N_{|S|-2}$ obtained as follows: set $N_1 = N$, then $N_i = N_{i-1}/f$ where f_i is an element of S that is not on L. By Proposition 2.5.4, each N_i is 3-connected. Such elements exist since when $|S| \geq 3$, at most two elements of S lie on L. Now let $S' = S \setminus \{f_1, \ldots, f_{|S|-3}\} = \{s_1, s_2, s_3\}$ which is a circuit (and thus a triangle) in $N_{|S|-2}$, with the three elements of S' on L by the 3-connectivity of $N_{|S|-2}$. Since L is modular, each element of S' is parallel to an element of L. Then $C = C' \cup \{s_i\} \cup \{f_1, \ldots, f_{|S|-3}\}$ satisfies the conditions, as desired. □

Let e_1, e_2, e_3 be the elements of L that are parallel to x_1, x_2, x_3 in M/C of the previous lemma. Then $C \cap \{x_i, e_i\}$ is a circuit for $i \in \{1, 2, 3\}$ in M, and so a consequence is that M distinguishes e_i for $i \in \{1, 2, 3\}$.

Corollary 2.5.6. *Let L be a line in a 3-connected matroid M such that $E(M) \neq E(L)$. If L is modular, then at least 3 elements of L are distinguished.*

The following lemma describes the structure of a matroid with a line that has exactly two elements distinguished.

Lemma 2.5.7. *Let M be a matroid with a modular line L. If exactly two elements e_1, e_2 of L are distinguished, then there exists a partition (C, D) of $E(M) \setminus L$ such that C and D are 2-separating in M with basepoints e_1 and e_2, respectively.*

Proof. To partition $E(M) \setminus L$, we define the following binary relation $(E(M) \setminus L, \sim)$: for elements $e, f \in E(M) \setminus L$, we let $e \sim f$ if there exists a 2-separation (A, B) of M such that $e, f \in B$ and $L \subseteq A$. We claim that $(E(M) \setminus L, \sim)$ is an equivalence relation. Clearly it is reflexive and symmetric, so it remains to show that $(E(M) \setminus L, \sim)$ is transitive. Let $e, f, g \in E(M)$ such that (A, B) and (A', B') are 2-separations of $E(M)$ such that $e, f \in B$, $f, g \in B'$, $L \subseteq A$, and $L \subseteq A'$. Now, by the uncrossing lemma, $B \cup B'$ is 2-separating. So $(A \cap A', B \cup B')$ is a 2-separation of M with $e, g \in B \cup B'$ and $L \subseteq A \cap A'$. Thus $e \sim g$ as desired.

Let $B_1, \ldots, B_k$ be the partition of $E(M) \setminus L$ formed by the equivalence classes under $(E(M) \setminus L, \sim)$. First we will show that B_i is 2-separating for $i \in \{1, \ldots, k\}$. Suppose $\lambda_M(B_i) > 1$, then $|B_i| \geq 2$. Let B_i' be the union of all 2-separating sets B such that B has non-empty intersection with B_i and B is disjoint from L. Then $B_i \subsetneq B_i'$ since B_i is not 2-separating but B_i' is 2-separating by the uncrossing lemma. Let $g \in B_i' \setminus B_i$. Then $(E(M) \setminus B_i', B_i')$ is a 2-separation of M such that $e, g \in B_i'$ and $L \subseteq E(M) \setminus B_i'$, contradicting that $g \notin B_i$.

Now let M' be the matroid obtained from M by contracting each B_i down to its basepoint b_i for $i \in \{1, \ldots, k\}$. Then $M \backslash L$ is simple and $\mathrm{si}(M')$ is 3-connected. If there exists an element $b_i \in E(M')$ such that $b_i \notin \mathrm{cl}(L)$, then by Corollary 2.5.6, $\mathrm{si}(M')$ distinguishes at least three elements of L, contradicting that M distinguishes exactly two elements of L. Hence $b_i \in \mathrm{cl}(L)$ for all $i \in \{1, \ldots, k\}$.

We claim that $k = 2$. Since M distinguishes only e_1 and e_2, each b_i is parallel to e_1 or e_2 as L is modular. Suppose b_1, b_2 are both parallel to e_1. Then let $e \in B_1$ and $f \in B_2$. Then $(E(M) \setminus B_1 \setminus B_2, B_1 \cup B_2)$ is a 2-separation of M where $e, f \in B_1 \cup B_2$ and $L \subset E(M) \setminus B_1 \setminus B_2$. So $e \sim f$, contradicting that e, f are in different parts of the partition. Thus $k \leq 2$, and $k > 1$ otherwise only one element of L is distinguished, establishing our claim.

Finally, we may assume that $\{b_1, e_1\}$ and $\{b_2, e_2\}$ are parallel pairs in M'. Then $C = B_1$ and $D = B_2$ satisfies the desired conditions. □

2.6 Generalized Parallel Connection

Let M_1 and M_2 be matroids such that $E(M_1) \cap E(M_2) = T$, $M_1|T = M_2|T = N$, and $\mathrm{si}(M_2|T)$ is a modular flat of $\mathrm{si}(M_2)$. Then the *generalized parallel connection*, $P_N(M_1, M_2)$, is the matroid on groundset $E(M_1) \cup E(M_2)$ with its set of flats being the subsets F such that $F \cap E_1$ and $F \cap E_2$ are flats of M_1 and M_2, respectively.

The following proposition (Proposition 11.4.15 of [16]) gives us a way to decompose a matroid along a k-separation.

Proposition 2.6.1. *Let M be a simple matroid with a subset of elements T such that M/T is the direct sum of M_1 and M_2. If T is a modular flat of $M\backslash E(M_2)$, then $M = P_{M|T}(M\backslash E(M_1), M\backslash E(M_2))$.*

Our next lemma will show that, if there exist quinary representations for both sides of our ternary matroid with a k-separation, and they match along the guts, then our ternary matroid is also quinary.

Lemma 2.6.2. *Let (X, Y) be an exact k-separation in a simple matroid M, such that $T \subseteq \mathrm{cl}(X) \cap \mathrm{cl}(Y)$ has rank $k-1$ and T is a modular flat of $M|(Y \cup T)$. Let B_T be a basis of T. Let A_X and A_Y be $\mathbb{F}$-representations of $M|(X \cup T)$ and $M|(Y \cup T)$ (respectively) of the form:*

$$A_X = \begin{array}{c c} & \begin{array}{cc} \phantom{A_{X'}} & T \end{array} \\ \begin{array}{c} B_X \\ B_T \end{array} & \left[\begin{array}{c|c} A_{X'} & \mathit{0} \\ \cline{2-2} & A_T \end{array}\right] \end{array}, \quad A_Y = \begin{array}{c c} & \begin{array}{cc} T & \phantom{A_{Y'}} \end{array} \\ \begin{array}{c} B_T \\ B_Y \end{array} & \left[\begin{array}{c|c} A_T & A_{Y'} \\ \cline{1-1} \mathit{0} & \end{array}\right] \end{array}.$$

Then

$$A = \begin{array}{c c} & \begin{array}{ccc} \phantom{A_{X'}} & T & \phantom{A_{Y'}} \end{array} \\ \begin{array}{c} B_X \\ B_T \\ B_Y \end{array} & \left[\begin{array}{c|c|c} A_{X'} & \mathit{0} & \mathit{0} \\ \cline{2-3} & A_T & A_{Y'} \\ \cline{1-2} \mathit{0} & \mathit{0} & \end{array}\right] \end{array}$$

is an $\mathbb{F}$-representation of M.

Proof. First we will show that $M[A] = P_{M|T}(X \cup T, Y \cup T)$ by showing their set of flats are equal. To ease notation, let $M' = P_{M|T}(X \cup T, Y \cup T)$, $F_X = F \cap (X \cup T)$, and $F_Y = F \cap (Y \cup T)$. Observe that $M'|(X \cup T) = M[A]|(X \cup T)$ and $M'|(Y \cup T) = M[A]|(Y \cup T)$.

Let F be a flat of $M[A]$. As taking a restriction preserves flats, F_X is a flat of $M[A]|(X \cup T)$, and F_Y is a flat of $M[A]|(Y \cup T)$. Then F_X, F_Y are flats of $'|(X \cup T)$ and $M'|(Y \cup T)$, respectively. Hence F is a flat of M' by the definition of generalized parallel connection.

Conversely, let F be a flat of M' and suppose for a contradiction that F is not a flat of $M[A]$. Then there exists $e \in cl_{M[A]}(F)$ such that $e \notin F$. Let C be a circuit in $M[A]$ such that $C \setminus F = \{e\}$. Observe that $C \cap (X \setminus T)$ and $C \cap (Y \setminus T)$ are both non-empty, otherwise $C \subseteq X \cup T$ or $C \subseteq X \cup T$. Then $C \setminus \{e\} \subseteq F_Y$ or $C \setminus \{e\} \subseteq F_X$. Both cases imply that $e \in F$ as F_Y is a flat of $M[A]|(Y \cup T)$ and F_X is a flat of $M[A]|(X \cup T)$, a contradiction. Let $\vec{v_1}, \dots, \vec{v_i}, \vec{v_e}$ denote the vectors corresponding to C with $\vec{v_e}$ the vector labelled by e. Then $\vec{v_e}$ can be written as the linear combination $a_1\vec{v_1} + \dots + a_i\vec{v_i} = \vec{v_e}$ where $a_1, \dots, a_i \in GF(q_2) \setminus \{0\}$.

- Case 1: $e \in X \cup T$. By the above observation $C \cap (Y \setminus T) \neq \emptyset$. We may assume that $\vec{v_1}, \dots \vec{v_j}$ are labelled by elements of $C \cap (Y \setminus T)$, and let $\vec{v_t} = a_1\vec{v_1} + \dots + a_i\vec{v_i}$. Since $\vec{v_{j+1}}, \dots, \vec{v_i}, \vec{v_e}$ all have zero entries in the rows labelled by B_Y, $\vec{v_t}$ has only zero entries in those rows. Then the only non-zero entries of $\vec{v_t}$ are in the rows labelled by B_T. Since T is a modular flat of $M|(Y \cup T) = M[A]|(Y \cup T)$, there exists an element $t \in T$ such that $\vec{v_t}$ is a scalar multiple of the column corresponding to t. So $C \cap (Y \setminus T)$ spans t, which implies that $t \in F$. As $t \in F$ and $e \notin F$, $t \neq e$. Then $a_t\vec{v_t} + a_{j+1}\vec{v_{j+1}} + \dots + a_i\vec{v_i} = \vec{v_e}$, so $(C \setminus (Y \setminus T)) \cup \{t\}$ is a subset of $X \cup T$ and contains a circuit involving e, contradicting our observation.

- Case 2: $e \in Y \setminus T$. Similar to above, we may assume that $\vec{v_1}, \dots \vec{v_j}$ are labelled by elements of $C \cap (Y \setminus T)$. Then $a_1\vec{v_1} + \dots + a_j\vec{v_j} = \vec{v_t}$ for some vector v_t that is in the span of the vectors of T. First suppose that there exists $t \in T$ such that $\vec{v_t}$ is a scalar multiple of the column corresponding to t. Then $C \setminus (X \setminus T) \cup \{t\}$ is a subset of $X \cup T$ and contains a circuit involving e, a contradiction. Next, suppose that there does not exist such an element t. Then as $a_e\vec{v_e} - a_{j+1}\vec{v_{j+1}} - \dots - a_i\vec{v_i} = \vec{v_t}$, we have a strand of $Y \cup T$ that does not distinguish an element of T, contradicting that T is modular.

As T is the guts of a separation, M/T is disconnected where M/T is the direct sum of X and Y. Applying Proposition 2.6.1 yields that $M = M'$, and thus $M = M[A]$. □

2.7 Generalized Δ-Y Exchange

The generalized $\Delta - Y$ exchange is an operation introduced by Oxley, Semple, and Vertigan [14] that transforms a line L in a matroid into a coline in M'. The name is inspired from the $\Delta - Y$ exchange which replaces a triangle with a triad. This operation is described as follows: fix an integer $k \geq 2$ and let Θ_k be the unique matroid on groundset $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ such that $\{a_1, \ldots, a_k\}$ is a modular line and $\{b_1, \ldots, b_k\}$ is an independent coline, and $\{b_1, \ldots, b_k\} \triangle \{a_i, b_i\}$ is a circuit for all $i \in \{1, \ldots, k\}$. Now let M be a matroid with a $U_{2,k}$-restriction $A = \{a_1, \ldots, a_k\}$ such that A is coindepedent (this last condition is imposed for duality to work). Then the generalized $\Delta - Y$ exchange on A of size k, denoted $\Delta_A(M)$, is defined to be the generalized parallel connection $P_A(\Theta_k, M) \setminus A$. To preserve the groundset of M in $\Delta_A(M)$, relabel b_i with a_i for all $i \in \{1, \ldots, k\}$. Geometrically, the $\Delta - Y$ exchange replaces a line in a matroid by a coline, as shown in the following lemma (Lemma 2.5 of [14]).

Lemma 2.7.1. *For all $k \geq 2$, the restriction of $(P_A(\Theta_k, M) \setminus A)^*$ to B is isomorphic to $U_{2,k}$ if and only if A is coindepdent in M.*

This result yields a dual operation, the $Y - \Delta$ exchange. Let M be a matroid with an independent set A such that $M^*|A$ is isomorphic to $U_{2,k}$. Then the $Y - \Delta$ exchange on A is defined as $\nabla_A(M) = [P_A(\Theta_k, M^*) \setminus A]^*$, and Lemma 2.11 and Corollary 2.12 of [14] proves that it is the inverse of the $\Delta - Y$ exchange.

Lemma 2.7.2. *Let A be a coindependent set in a matroid M where every 3 element subset of A is a circuit. Then $\nabla_A(\Delta_A(M))$ is well-defined and $\nabla_A(\Delta_A(M)) = M$.*

Lemma 2.7.3. *Let A be an independent set in a matroid M where every 3 element subset of A is a cocircuit. Then $\Delta_A(\nabla_A(M))$ is well-defined and $\Delta_A(\nabla_A(M)) = M$.*

Lemma 2.7.4. *Suppose that $\nabla_A(M)$ is defined. If $x \in A$ and $|A| \geq 3$, then $\nabla_{A\setminus\{x\}}(M/x)$ is also defined and $\nabla_A(M)\backslash x = \nabla_{A\setminus\{x\}}(M/x)$*

The $\Delta - Y$ exchange also preserves representability over any field $\mathbb{F}$, as Corollary 3.7 of [14] shows. Note that the original lemma states the result for partial fields, as every field is a partial field.

Lemma 2.7.5. *Let M be a matroid with a coindependent $U_{2,k}$-restriction A. Then M is $\mathbb{F}$-representable if and only if $\Delta_A(M)$ is $\mathbb{F}$-representable.*

Chapter 3

Main Result

In this chapter we will prove our main theorem. Recall that O_8 is the following matroid:

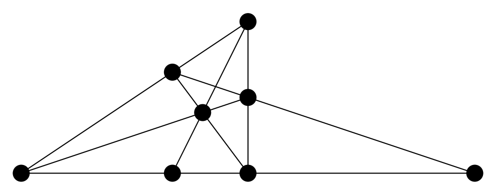

Figure 3.1: The matroid O_8

Theorem 3.0.1 (Theorem 3.8.1)**.** *Let M be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation (X, Y) such that $|X|, |Y| \geq 4$. Then either:*

- *M is isomorphic to T_8,*
- *M_X is isomorphic to O_8 and M_Y contains an excluded minor for $GF(5)$-representability with up to two fewer elements than M_Y,*
- *M_Y is isomorphic to O_8 and M_X contains an excluded minor for $GF(5)$-representability with up to two fewer elements than M_X,*
- *$|X| = 4$, M_X is not isomorphic to O_8, and M_Y is an excluded minor for $GF(5)$-representability, or*
- *$|Y| = 4$, M_Y is not isomorphic to O_8, and M_X is an excluded minor for $GF(5)$-representability.*

3.1 Adding guts

Let M be a ternary matroid with an exact 3-separation (X, Y). In order to decompose M into two matroids corresponding to each side of the 3-separation, we will construct an auxiliary matroid $M^+_{X,Y}$ where X and Y span a common 4-point line. As mentioned previously, adding elements arbitrarily on a line of a matroid is not well-defined. However, we can take advantage of the unique representability of ternary matroids to do this for M.

Let M have rank n and let A be a $GF(3)$-representation of M (by removing redundant rows, we may assume A has n rows). We can view M as a restriction of $PG(n-1, 3)$ by taking the columns of the $GF(3)$-representation of $PG(n-1, 3)$ which match the columns of A. This is well-defined since projective geometries are uniquely representable (Fundamental Theorem of Projective Geometry) combined with the fact that ternary matroids are uniquely representable [2].

Hence let $G = PG(n-1, 3)$ and $E(M)$ is a spanning subset of $E(G)$ such that $M = G|E(M)$. Now consider the boundary $B = cl_G(X) \cap cl_G(Y)$, which has rank two. As G is a projective geometry, $G|B \cong PG(1, 3) \cong U_{2,4}$. Let G^+_B be the matroid obtained from G by taking four extensions $\{e_1, e_2, e_3, e_4\}$ parallel to the four elements of B.

We then define the matroid $M^+_{X,Y} = G^+_B|(E(M) \cup \{e_1, e_2, e_3, e_4\})$. We call $L = \{e_1, e_2, e_3, e_4\}$ the *boundary line* of (X, Y). Note while L may not be a line in $M^+_{X,Y}$ if $M^+_{X,Y}$ is not simple, we nevertheless refer to it as a line for convenience. We also define the matroids corresponding to X and Y as $M_X = M^+_{X,Y}|(X \cup L)$ and $M_Y = M^+_{X,Y}|(Y \cup L)$, respectively. It is clear that $M^+_{X,Y} = P_L(M_X, M_Y)$, and that $M^+_{X,Y} \backslash L, M_X \backslash L, M_Y \backslash L$ are simple. Moreover, each of $\mathrm{si}(M^+_{X,Y}), \mathrm{si}(M_X), \mathrm{si}(M_Y)$ are 3-connected.

3.2 Pinning the points

Let (X, Y) be a 3-separation in a representable matroid, and let B be the set of guts elements of (X, Y). We say that an element $e \in B$ is *pinned* with respect to B if M_X and M_Y distinguish e. In this section we will prove that, in our excluded minor, all four elements of L are pinned with respect to L. The importance of this is highlighted by the next lemma, which shows that adding a pinned element of the line implies that does not change its representability over $GF(5)$.

Lemma 3.2.1. *Let M be a ternary matroid with an exact 3-separation (X, Y). Let M' be a ternary matroid obtained from M via an extension by a non-loop element e in the* *guts of (X, Y). If there exists strands $S_X \subseteq X$ for Y and $S_Y \subseteq Y$ for X such that $e \in \mathrm{cl}(S_X) \cap \mathrm{cl}(S_Y)$, then M is $\mathbb{F}$-representable if and only if M' is $\mathbb{F}$-representable.*

Proof. The "if" direction is clear, as M is a restriction of M'. Conversely, suppose that A is an $\mathbb{F}$-representation of M. We may assume that A is of the form

$$A = \left[\begin{array}{c|c} A_X & 0 \\ \hline 0 & A_Y \end{array}\right]$$

where A_X is a represenation of $A|X$ and A_Y is a representation of $A|Y$, and there are exactly two rows that intersect A_X and A_Y. Let r_1, r_2 denote these two rows.

We have $\sqcap_M(S_X, S_Y) = 1$, so the subspaces spanned by the columns of S_X and S_Y in A intersect at a point. Let $\vec{v}$ be the vector representing this point. Since both S_X and S_Y span $\vec{v}$, its only non-zero entries are in r_1 and r_2. Clearly $\vec{v}$ must have at least one non-zero entry.

Now let A' be the matrix where $\vec{v}$ is added to A. We claim that $M[A'] = M'$. Suppose, for a contradiction, that $M[A'] \neq M$ and let $\{M_1, M_2\} = \{M[A'], M'\}$ such that C is a circuit in M_1 and independent in M_2. Since $M_1 \backslash e = M_2 \backslash e$, we have $e \in C$ and let $C^- = C \setminus \{e\}$. Let L be the boundary line of (X, Y) in $M^+_{X,Y}$. Observe that $\sqcap_M(C^-, L) \neq 2$, otherwise C^- spans e in M_2, contradicting that C is independent in M_2. Similarly $\sqcap_M(C^-, L) \neq 0$, otherwise C^- does not span e in M_1. Hence $\sqcap_M(C^-, L) = 1$, and we claim that either $C^- \cap X$ or $C^- \cap Y$ is empty. Suppose otherwise, and we have that one of $\sqcap_M(C^- \cap X) = 0$ or $\sqcap_M(C^- \cap Y) = 0$ as $\sqcap_M(C^-, L) \neq 2$ and C is independent in M_2. We may assume that $\sqcap_M(C^- \cap X) = 0$ and so $(C^- \setminus X) \cup \{e\}$ is a circuit in M_1. Thus $C^- \cap X = \emptyset$ as $C^- \cup \{e\}$ is a circuit in M_1. So C^- is a strand of X that spans e in M_1, so $\sqcap_M(C^-, S_X) = 1$. Then e is in the span of C^- in both M' and $M[A']$, contradicting that C is independent in M_2. $\square$

By applying Lemma 3.2.1 to each element of L in $M^+_{X,Y}$, we get that M is quinary if and only if $M^+_{X,Y}$ is quinary.

Corollary 3.2.2. *Let M be a ternary matroid with an exact 3-separation (X, Y). Let L be the boundary line of (X, Y) in $M^+_{X,Y}$. If all elements of L are pinned, then M is $GF(q)$-representable if and only if $M^+_{X,Y}$ is $GF(q)$-representable.*

Proof. Let $e \in L$. Since e is pinned, M_X and M_Y distinguish X. Then X contains a strand S_X for L in $M^+_{X,Y}$ that distinguishes e, which is also a strand for Y. Similarly Y contains a strand S_Y for X that distinguishes e. Then applying Lemma 3.2.1 to M and e yields that M is $GF(q)$-representable if and only if $M^+_{X,Y} \backslash (L \setminus \{e\})$ is $GF(q)$-representable. Applying this argument (along with Lemma 3.2.1) three more times for the other three elements of L yields the result. $\square$

So we want to see whether all the elements of L are pinned in $M^+_{X,Y}$. If this is not the case, and if neither X, Y is a line nor a coline, our main result of this section shows that M is in fact quinary.

Lemma 3.2.3. *Let M be a 3-connected ternary matroid whose proper minors are all quinary and let (X, Y) be an exact 3-separation of M with $|X| \geq 4$. If M_X does not distinguish all four elements in the boundary line, then M is quinary.*

Proof. Suppose that $L = \{e_1, e_2, e_3, e_4\}$ is the boundary line where e_4 is not distinguished in M_X. Let $M'_X = M_X \backslash e_4$ and $M'_Y = M_Y \backslash e_4$.

Observe that $\{e_1, e_2, e_3\}$ is a modular line in $\mathrm{si}(M'_X)$. By applying Corollary 2.5.3, we get that $\mathrm{si}(M'_X)$ is binary. Then M'_X is binary and ternary, so by Theorem 2.1.1 we have that M'_X is regular. Thus M'_X is quinary, and we want to show that M_Y is quinary.

Since $|X| \geq 4$ and M_X does not distinguish all elements of L, X is not a coline. Further, $\mathrm{si}(M'_X)$ is 3-connected with the modular line $\{e_1, e_2, e_3\}$, so we can apply Lemma 2.5.5 to obtain the sets $C \subseteq X$ and $\{x_1, x_2, x_3\} \subseteq X$ such that in M'_X/C we have that $\{x_1, x_2, x_3\}$ is a triangle where x_i is parallel to e_i for $i \in \{1, 2, 3\}$. Observe that M/C is quinary since it is a minor of M, and $\mathrm{si}(M/C) = \mathrm{si}((M^+_{X,Y} \backslash e_4)/C) = \mathrm{si}(M'_Y)$ so M'_Y is quinary.

So both M'_X and M'_Y are quinary, and by applying row operations there exists $GF(5)$ matrices A_X and A_Y where e_1, e_2 label the last two rows of A_X and the first two rows of A_Y. Further by column scaling we may assume that the column representing e_3 has entry 1 in the rows corresponding to e_1, e_2 and zeroes elsewhere.

$$A_X = \begin{array}{cc} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \left[\begin{array}{c} \\ \ast \\ \\ \end{array} \right. & \left| \begin{array}{ccc} & 0 & \\ \hline 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right] \end{array}, \qquad A_Y = \begin{array}{cc} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} & \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline & 0 & \end{array} \right| & \left. \begin{array}{c} \\ \ast \\ \\ \end{array} \right] \end{array}$$

Then A_X, A_Y match along the set $T = \{e_1, e_2, e_3\}$ which is a modular flat in X. Applying Lemma 2.6.2 yields a $GF(5)$-representation of $P_T(M'_X, M'_Y)$, which has M as a restriction by deleting $\{e_1, e_2, e_3\}$, thus M is quinary as desired. □

It immediately follows that $M^+_{X,Y}$ must pin all four elements of its boundary line.

Corollary 3.2.4. *Let M be a ternary excluded minor for $GF(5)$-representability and let (X, Y) be an exact 3-separation of M. If $|X|, |Y| \geq 4$, then $M^+_{X,Y}$ pins all elements of the boundary line.*

3.3 Duality with Generalized $\Delta - Y$ Exchange

Since duality preserves representability and connectivity, if M is a ternary excluded minor for $GF(5)$-representability with an exact 3-separation (X, Y), then so is M^*. We will establish the relationship between M_X and M^*_X in this section, using the Generalized $\Delta - Y$ Exchange in Section 2.7.

Lemma 3.3.1. *Let M be a 3-connected ternary matroid with an exact 3-separation (X, Y), and let L label the boundary lines of $M^+_{X,Y}$ and $(M^*)^+_{X,Y}$. Then $(M^*)_X = \nabla_L((M_X)^*)$.*

Proof. Observe that both matroids have the groundset $X \cup L$. Clearly L is isomorphic to $U_{2,4}$ in both matroids. Hence, to establish the equality of the two matroids, we must show first that they agree on the rank of every subset A of X, then show the equality of local connnectivity from A to L, and finally show that if $\sqcap(A, L) = 1$ then A spans $e \in L$ in $(M^*)_X$ if and only if A spans e in $\nabla_L((M_X)^*)$.

Let $A \subseteq X$ and $E = E(M)$. Since neither deleting outside A nor extensions change the rank of A,

$$r_{(M^*)_X}(A) = r_{(M^*)|X}(A) = r_{M^*}(A)$$

By the dual rank function, we have

$$r_{(M^*)_X}(A) = r_M(E \setminus A) + |A| - r(M).$$

Since $Y \subseteq E \setminus A$ and Y spans L (as M is 3-connected), $r_M(E \setminus A) = r_{M^+_{X,Y}}(E \cup L \setminus A)$ and $r(M) = r(M^+_{X,Y})$. Clearly $|A|$ is not changed, which implies

$$r_{(M^*)_X}(A) = r_{M^+_{X,Y}}((E \cup L) \setminus A) + |A| - r(M^+_{X,Y}) = r_{(M^+_{X,Y})^*}(A).$$

Observe that we obtain M_X from $M^+_{X,Y}$ by contracting elements in Y and then simplifying. Both operations keep $r_{M^+_{X,Y}}(E(L) \setminus A) - r(M^+_{X,Y})$ invariant, so

$$r_{(M^*)_X}(A) = r_{(M_X)^*}(A).$$

It is clear that performing a $Y - \Delta$ exchange does not change the rank of a set disjoint from L, thus

$$r_{(M^*)_X}(A) = r_{\nabla_L(M_X)^*)}(A)$$

as desired.

Next we show that $\sqcap_{(M^*)_X}(A, L) = \sqcap_{(M_X)^*}(A, L)$. Since deleting outside of $A \cup L$ does not change the local connectivity, we have

$$\begin{aligned}\sqcap_{(M^*)_X}(A, L) &= \sqcap_{(M^*)^+_{X,Y}}(A, L) \\ &= \sqcap_{M^*}(A, Y).\end{aligned}$$

Now by applying the definition of local connectivity,

$$\begin{aligned}\sqcap_{(M^*)_X}(A, L) &= r_{M^*}(A) - r_{M^*/Y}(A) \\ &= r_{M^*}(A) - r_{((M^+_{X,Y})^*/L)/Y}(A) \\ &= r_{M^*}(A) - r_{(M^+_{X,Y})^*/Y/L}(A).\end{aligned}$$

Since $M_X = M^+_{X,Y} \setminus Y$, $(M_X)^* = (M^+_{X,Y})^*/Y$ and so

$$\sqcap_{(M^*)_X}(A, L) = r_{M^*}(A) - r_{(M_X)^*/L}(A).$$

We proved above that $r_{M^*}(A) = r_{(M_X)^*}(A)$, thus

$$\sqcap_{(M^*)_X}(A, L) = r_{(M_X)^*}(A) - r_{(M_X)^*/L}(A)$$
$$= \sqcap_{(M_X)^*}(A, L).$$

It remains to show that for all A such that $\sqcap(A, L) = 1$ we have that A spans $e \in L$ in $(M^*)_X$ if and only if A spans e in $\nabla_L((M_X)^*)$. Observe that we may assume that the elements $\{e_1, e_2, e_3, e_4\}$ of L are labelled such that if there exists a set A such that $\sqcap(A, L) = 1$ and $\sqcap_{(M_X)^*}(A, L \setminus \{e_i\}) = 1$ then A distinguishes e_i in $(M^*)_X$. Then equality follows by the construction of $\nabla_L((M_X)^*)$. □

So this lemma allows us to dualize and construct the matroids for the two sides of the dual. We want the properties for M_X to also hold for $\nabla_L((M^*)_X)$, specifically that this perserves representability and distinguishing the points of L. That representability is preserved follows directly from the previous lemma and Lemma 2.7.5.

Lemma 3.3.2. *Let (X, Y) be an exact 3-separation in a 3-connected ternary matroid M. For any field $\mathbb{F}$, $(M^*)_X$ is $\mathbb{F}$-representable if and only if M_X is $\mathbb{F}$-representable.*

Proof. Let L be the boundary line of (X, Y). As duality preserves representability, M_X is $\mathbb{F}$-representable if and only if $(M_X)^*$ is $\mathbb{F}$-representable. Now L is an independent coline in $co((M_X)^*)$ since it is a coindepdent line in $\mathrm{si}(M_X)$. Then by the dual of Lemma 2.7.5, $co((M_X)^*)$ is $\mathbb{F}$-representable if and only if $co(\nabla_L((M_X)^*))$ is $\mathbb{F}$-representable. Since cosimplication preserves representability, applying Lemma 3.3.1 gives the result. □

Lemma 3.3.3. *Let (X, Y) be an exact 3-separation in a 3-connected ternary matroid M. Then all four elements of the boundary line is pinned in $M^+_{X,Y}$ if and only if all four elements of the boundary line is pinned in $(M^*)^+_{X,Y}$.*

Proof. Let $L = \{a_1, a_2, a_3, a_4\}, L'$ be the boundary lines of $M^+_{X,Y}$ and $(M^*)^+_{X,Y}$, respectively. Suppose that all four elements of L are pinned in $M^+_{X,Y}$. We want to show that if M_X distinguishes a_i then $(M^*)_X$ distinguishes a_i for $i \in \{1, \ldots, 4\}$.

Recall that $\{a_1, a_2, a_3, a_4\}$ is a 4-point line in Θ_4 and consider $M^+_X = P_{\{a_1,a_2,a_3,a_4\}}(\Theta_4, M_X)$, and observe that $\Delta_L(M_X) = M^+_X \backslash L$. Since M_X distinguishes a_1, there exists a circuit C with $\mathrm{cl}(C) \cap \mathrm{cl}(L) = \{a_1\}$. Then $\mathrm{cl}(C)$ is a cyclic flat of M_X, and let $F = \mathrm{cl}(C) \cup \{b_2, b_3, b_4\}$. F is also a cyclic flat of M^+_X since $\{a_1, b_2, b_3, b_4\}$ is a circuit of M_X and adding $\{b_2, b_3, b_4\}$ to $\mathrm{cl}(C)$ does not introduce any of b_1, a_2, a_3, a_4 into its span. Now since $C \cup \{b_2, b_3, b_4\}$ is a circuit, $F \setminus \{a_1\}$ is a cyclic flat of $M_X^+ \backslash a_1$. Since deleting outside a cyclic flat preserves the cyclic flat, $F \setminus \{a_1\}$ is a cyclic flat of $M_X^+ \backslash a_1, a_2, a_3, a_4$. Observe that by relabelling b_i to a_i for $i \in \{1, \dots, 4\}$ in $M_X^+ \backslash a_1, a_2, a_3, a_4$, we have $M_X^+ \backslash a_1, a_2, a_3, a_4 = \Delta_L(M_X)$.

Now $F^* = \Delta_L(M_X) \setminus F$ is a cyclic flat in $(\Delta_L(M_X))^*$ and observe that F^* intersects the 4-point line $\{a_1, a_2, a_3, a_4\}$ at exactly $\{a_1\}$. Hence there exists a circuit C^* in $(\Delta_L(M_X))^*$ that intersects $\{a_1, a_2, a_3, a_4\}$ at exactly $\{a_1\}$, so a_1 is distinguished in $(\Delta_L(M_X))^*$. Now $(\Delta_L(M_X))^* = \nabla_L((M_X)^*)$, and $\nabla_L((M_X)^*) = (M^*)_X$ by Lemma 3.3.1 as desired. Applying the same argument for a_2, a_3, a_4 yields the result. □

So whenever we dualize, we may assume that $(M^*)_X$ distinguishes all points of L^*, and $(M^*)_X$ is representable over the same fields that M_X is.

3.4 Non-quinary side

In this section, we will show that, in an excluded minor, if M_Y is not quinary, then $|X| \leq 4$. We begin by showing that if one side is not quinary, then the other side must have size at most 4.

Lemma 3.4.1. *Let M be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation (X, Y). If M_Y is non-quinary, then there is no proper minor of M'_X of M_X such that $M|L = M'|L$ and all four elements of L are distinguished in M'_X.*

Proof. Suppose there exists a proper minor M'_X of M_X such that L is a 4-point line with all elements distinguished in M'_X. Let $N^+ = P_L(M'_X, M_Y)$. Then N^+ is not quinary since it contains M_Y is a restriction. Since M'_X distinguishes all four elements of L in N^+, by Lemma 3.2.1 we have that $N^+ \backslash L$ is not quinary. However, $N^+ \backslash L$ is a proper minor of M, so this contradicts that M is an excluded minor. □

Hence our problem becomes characterizing the 3-connected matroids M with a 4-point line L where all elements of L are distinguished, but every proper minor that keeps L does not distinguish L. Moreover, by dualizing with the Generalized Delta-Y, $\nabla_L(M^*)$ has the same properties, so we may switch between M and $\nabla_L(M^*)$.

Lemma 3.4.2. *Let L be a modular set in a ternary matroid M such that $M|L$ is isomorphic to $U_{2,4}$. If all four elements of L are distinguished, then M contains an 8-element minor N such that $M|L = N|L$ and all elements of L are distinguished in N.*

Proof. Suppose that M is a minor-minimal counterexample. Then $|E(M)| \geq 9$. We prove the result via a series of claims.

(1) *M is 3-connected.*

First we show that $M \backslash L$ is simple. Suppose $\{e, e'\}$ is a parallel pair in $M \backslash L$. So if S is a strand that contains e and distinguishes $e_1 \in L$, then $S \triangle \{e, e'\}$ is a strand distinguishing e_1. Hence $M \backslash e$ distinguishes all four elements of L, contradicting minimality of M.

Next we will show that $\mathrm{si}(M)$ is 3-connected. Let $\mathrm{si}(M) = M \backslash P$ where $P \cap L = \emptyset$, and let (A, B) be a 2-separation of $\mathrm{si}(M)$ such that $L \subseteq B$. We may assume that this separation is exact, otherwise $L \setminus A$ contradicts minimality. Then by Proposition 2.4.5, M_B^+ is a minor of $\mathrm{si}(M)$ and preserves strands for L. By Theorem 2.3.1, $M_B^+ = \mathrm{si}(M)/C \backslash D = M/C \backslash D \backslash P$ for some $C, D \subseteq E(M)$ where at least one of C, D is non-empty. Then $M/C\ D$ distinguishes all four elements of L and contradicts minimality.

To prove that M is 3-connected, it remains to show that M is simple. Let e be an element parallel to $e_4 \in L$, and let $L \setminus e_4 = \{e_1, e_2, e_3\}$. Then by minimality of M, $M \backslash e$ distinguishes e_1, e_2, e_3 but not e_4. If $E(M) \setminus L \setminus e$ is a line or coline that distinguishes e_1, e_2, e_3, then the result is immediate. Since $\mathrm{si}(M \backslash e)$ is 3-connected and $E(M) \setminus \mathrm{cl}(L)$ is not a coline, by Lemma 2.5.5 we have that there is minor M' of $\mathrm{si}(M)$ such that M' contains a triangle $\{x_1, x_2, x_3\}$ parallel to $\{e_1, e_2, e_3\}$. Then adding e to M' is a parallel extension of M' at e_4 which distinguishes all elements of L, contradicting minimality of M and proving (1).

Note that by Lemma 3.3.3, $\nabla_L(M^*)$ is also a minimal counterexample. Then by duality, (1) implies that there are no elements in the coguts of $(L, E(M) \setminus L)$, as $\nabla_L(M^*)$ would not be cosimple.

(2) *M/L and $M \backslash L$ is connected.*

Let C be a component of M/L. Suppose for a contradiction that $C \neq M/L$. Consider $M' = M|(C \cup L)$. Since C is a component of M/L, $\lambda_M(E(C)) = 2$ and C spans L, so $\mathrm{si}(M')$ is 3-connected. By Corollary 2.5.6, we may assume that M' distinguishes e_1, e_2, e_3 of L. Note that by minimality of M, M' does not distinguish e_4. Since e_4 is distinguished in M, there exists some component C' of M/L containing a strand S that distinguishes e_4. Then we can find a set of $|S| - 1$ elements of S to contract to obtain a minor of M'' of M with an element e parallel to e_4. Then $N = M''|(C \cup L \cup \{e\})$ is a parallel extension of M' at e_4, contradicting the minimality of M, so M/L is connected.

Now by duality, $\nabla_L(M^*)/L$ is connected. Since dualizing preserves connectivity, $\Delta_L(M)\backslash L$ is connected, which implies that $M\ L$ is connected, proving (2).

(3) *For every $e \notin L$, the line L has exactly three distinguished elements in $M\backslash e$.*

Consider $M\backslash e$. Since $\mathrm{si}(M)\backslash e$ is 2-connected, at least one element of L is distinguished. First note $M\backslash e$ distinguishes more than one element of L, as otherwise e is in the coguts of $(L, E(M) \setminus L)$. So suppose that exactly two elements of L are distinguished in $M\backslash e$. Applying Lemma 2.5.7, there exists a partition (C, D) of $E(M\backslash e) \setminus L$ such that C and D are 2-separating in $M\backslash e$ with basepoints e_1 and e_2. Then $(M\backslash e)/L$ is disconnected. Applying the dual argument to $\nabla_L(M^*)\backslash e$, we get that $(\nabla_L(M^*)\backslash e)/L$ is disconnected. As duality preserves connectivity, $\Delta_L(M)\backslash L/e$ is disconnected. Then $M/L/e$ and $M/L\backslash e$ are disconnected, a contradiction and establishing (3).

Consider $M' = M\backslash e$. We may assume that M' not distinguish some $e_4 \in L$, as otherwise we contradict the minimality of M. In particular, this implies that every set S of M such that $\mathrm{cl}(L) \cap \mathrm{cl}(S) = e_4$ must contain e. Note that $e \in S$. Let $S \subseteq E(M) \setminus L$ be a maximal set whose closure contains e_4 but not L. Note that $S \cup L$ is spanning, as any element outside of $\mathrm{cl}(S \cup L)$ can be added to S.

(4) *There exist distinct elements $e_1, e_2 \in L\backslash\{e_4\}$ such that for every element $f \in (S\backslash\{e\})$, there exists a partition (C_f, D_f) of $E(M'/f) \setminus L$ such that C_f, D_f are 2-separating in M'/f with basepoints e_1, e_2 (respectively).*

Let $f \in S \setminus \{e\}$. First we want to show that M'/f distinguishes exactly two elements of L. Suppose M'/f distinguishes all three elements of $L \setminus \{e_4\}$. Then M/f also distinguishes $L \setminus \{e_4\}$, but since $S \setminus \{f\}$ is a strand of L that distinguishes e_4 in M/f, this contradicts the minimality of M. Next, suppose that M'/f distinguishes only one element of $L \setminus \{e_4\}$. Then $(L, E(M) \setminus L)$ is a 2-separation of M'/f and since L is a line in M', we have that $(L, E(M) \setminus L)$ is a 2-separation of M', contradicting (3). Then applying Lemma 2.5.7 gives us the desired partition (C_f, D_f), and it remains to show that the basepoints are invariant over the choices of f. Let e_3 be the element of $L \setminus \{e_4\}$ that is not distinguished in M'/f.

Suppose, for a contradiction, that there exists $f' \in (S \setminus \{e\})$ such that M'/f' distinguishes e_3. We may assume that $f' \in C_f$. Then $e_1 \in cl_{M'}(C_f \cup \{f\}$, so M'/f' distinguishes e_1. Since $\sqcap_{M'}(L, D_f \cup \{f\}) = 1$ and $e_2 \in cl_{M'}(D_f \cup \{f\})$ but e_2 is not distinguished in M'/f', so $\sqcap_{M'/f'}(L, D_f \cup \{f\}) = 2$. Hence $f' \in cl_{M'}(D_f \cup \{f\} \cup L) = cl_{M'}(D_f \cup \{f, e_1\})$. Since $f' \in C_f$ and C_f is 3-separating in M' with f, e_1 in the guts of $(C_f, E \setminus C_f)$, it follows that f' is also in the guts of $(C_f, E \setminus C_f)$. So f, e_1, f' is a triangle in M'. However, this contradicts that $\sqcap_M(S, L) = 1$ since $e_1, e_4 \in \mathrm{cl}(S)$. This proves (4).

(5) *For every element $f \in (S \setminus \{e\})$, $(C_f \cup \{e_1\}, D_f \cup \{e_2\})$ is a 2-separation of $M'\backslash e_3, e_4$ with f in the guts.*

Let $f \in S \setminus \{e\}$, and $C = C_f \cup \{e_1\}$, $D = D_f \cup \{e_2\}$. Since C_f and D_f are 2-separating sets in M'/f by (4), C and D are 3-separating in M' with $\{e_1, f\}$ and $\{e_2, f\}$ in the respective guts. Since C is 3-separating in M', $r(C) + r(D \cup \{e_3, e_4\}) - r(M') = 2$. As e_1, e_2 spans both e_3 and e_4, $r(M') = r(M'\backslash e_3, e_4)$. Since D contains e_2 but spans neither e_3 nor e_4, $r(D) = r(D \cup \{e_3, e_4\}) - 1$. Hence $r(C) + r(D) - r(M'\backslash e_3, e_4) = 1$, and clearly $|C|, |D| \geq 2$, proving (5).

Applying Lemma 2.4.7 to $M'\backslash e_3, e_4$, $S \setminus \{e\}$ and the special elements e_1 and e_2, we get an ordering $(s_1, \ldots, s_k)$ of $S \setminus \{e\}$ and a partition $(A_0, \ldots, A_k)$ of $E(M') \setminus S \setminus L$ satisfying the conditions of Lemma 2.4.7.

(6) $|S \setminus \{e\}| = 1$.

Suppose $k > 1$. Then $M'/s_1, s_2$ is disconnected with A_1 as a component (A_1 is non-empty, otherwise M' only distinguishes e_1 and e_2). Since $S \cup L$ spans M, A_1 is in the span s_1, s_2. Let $a \in A_1$, and consider $M'' = M/a\backslash s_2$. Clearly M'' distinguishes e_1 and e_2, and e_3 is distinguished as e_1, e_2 are in the same component in $M'/a\backslash e_3, e_4$. Finally e_4 is distinguished since $\sqcap_{M/a}(S, L) = 1$ as $a \in \mathrm{cl}(S)$ and s_2 is parallel to s_1 in M/a, so $S \setminus \{s_2\}$ distinguishes e_4. This contradicts minimality of M, proving (6).

Now let $S = \{e, f\}$, then $r(M) = 3$ since $\{e, f\} \cup L$ spans M. Let g be an element of M that is in neither of the span of S nor L, and g exists since M does not only distinguish e_4. Then by modularity of L, we may assume that $\{e, g\}$ is a strand for e_1 and $\{f, g\}$ is a strand for e_2. Since M distinguishes e_3, there exists an element h that is not parallel to e, f, g and not spanned by L.

If one of $\{e, h\}, \{f, h\}, \{g, h\}$ is a strand for e_3, then $E(M) = \{e, f, g, h\} \cup L$ and we are done. First suppose that h is spanned by one of the lines $\{e, f\}, \{e, g\}, \{f, g\}$. By symmetry we may assume that $h \in \mathrm{cl}(\{e, f\})$. Then $\{h, g\}$ does not span e_1 since $\{e, g\}$ spans e_1, it does not span e_2 since $\{f, g\}$ spans e_2, and it does not span e_4 since $\{e, f, h\}$ spans e_4. Hence as L is modular, $\{h, g\}$ spans e_3 as desired. So h is not in any of the spans of the lines $\{e, f\}, \{e, g\}, \{f, g\}$. A similar check shows that one of $\{e, h\}$, $\{f, h\}$, or $\{g, h\}$ spans e_3, as desired. $\square$

By case analysis, we can deduce all possible 8-element matroids M with a line L such that all elements of L are distinguished. Let M_1, M_2, M_3, M_4 be the matroids in Figure 3.2

(given by their geometric representations), where the elements of L are denoted by filled circles, and let M_5 be the unique matroid with a modular 4-point line L and an indepedent coline.

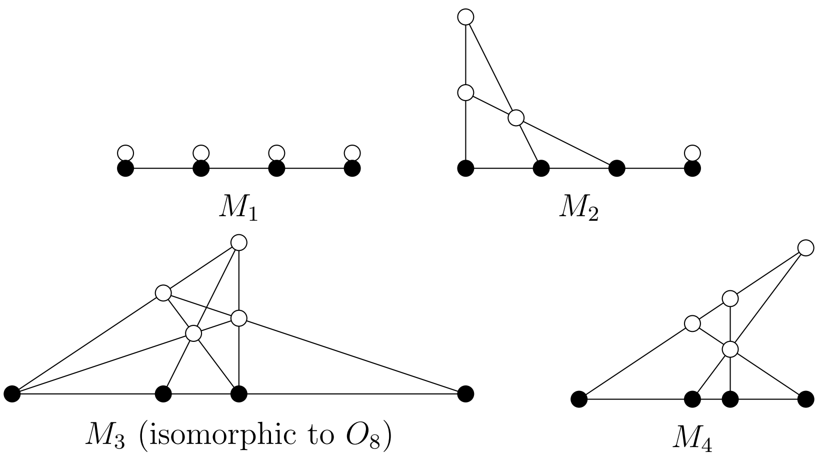

Figure 3.2: Four of the matroids that satisfy the conditions of Lemma 3.4.2

Lemma 3.4.3. *Let $L = \{e_1, e_2, e_3, e_4\}$ be a modular $U_{2,4}$-restriction in an 8-element matroid M. If all elements of L are distinguished in M, then M is isomorphic to one of M_1, M_2, M_3, M_4, or M_5.*

Proof. Let $N = E(M) \setminus L = \{e_5, e_6, e_7, e_8\}$, let n be the number of elements of N that in the span of L. Clearly M contains neither loops nor coloops, and $M|N$ is simple. If $n = 4$, then $M|N$ is isomorphic to $U_{2,4}$, and M is isomorphic to M_1. Clearly $n \neq 2$ and $n \neq 3$, otherwise M distinguishes at most three elements of L. Suppose $n = 1$. Then we may assume that $\{e_4, e_8\}$ is a parallel pair and e_5, e_6, e_7 distinguishes e_1, e_2, e_3. It follows that $\{e_5, e_6, e_7\}$ is independent and $\mathrm{cl}(\{e_5, e_6, e_7\})$ spans L (and thus also spans M). Then $L \cup \{e_8\}$ is a hyperplane of M, so $\{e_5, e_6, e_7\}$ is a triad, and M is isomorphic to M_2. Finally suppose that $n = 0$. Then $\lambda_M(L) = \sqcap_M(L, N) = 2$ otherwise M distinguishes at most one element of L. It follows that N spans M. If $T = \{e_5, e_6, e_7\}$ is a triangle, then L is on the plane spanned by N. Since L is modular, we may assume that $e_1 \in \mathrm{cl}(T)$ and the sets $\{e_2, e_5, e_8\}$, $\{e_3, e_6, e_8\}$, $\{e_4, e_7, e_8\}$ are triangles. In this case, M is isomorphic to M_3. If N is a circuit, so $r(N) = |N| - 1 = 3$ and N is a circuit-cocircuit, and hence M is isomorphic O_8. In the case where N is not a circuit and does not contain a 3-point line, then N is independent and spans L, so M is isomorphic to M_5. $\square$

3.5 Near-regular

Now that we have taken care of the case where one of M_X and M_Y are non-quinary, we turn our attention our focus to whether M_X and M_Y have representations that match along L. As mentioned in Chapter 1, there are three inequivalent representations of L, which are given by $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{bmatrix}$ for $x \in \{2, 3, 4\}$. Each of these representations of L may or may not extend to a representation of M_X or M_Y. However we do know that any representation of L extends to at most one representation of M_X or M_Y, by the following theorem from Whittle [24].

Theorem 3.5.1. *Let M be a 3-connected ternary matroid with a $U_{2,4}$-minor N. Then a representation of M over a field $\mathbb{F}$ is uniquely determined by an $\mathbb{F}$-representation of N.*

So the number of inequivalent $GF(5)$-representations for M_X is equal to the number of inequivalent representations for L that extend to a represention of M_X. Hence we want to know the number of inequivalent $GF(5)$-representations M_X may have. A result from Pendavingh and van Zwam [17] shows that a dyadic matroid has either one or three inequivalent $GF(5)$-representations, and characterizes exactly when each case is realized; see Theorem 1.3.6. Since M_X is not binary, case (i) of Theorem 1.3.6 does not apply. Our next lemma shows that, in fact, case (iii) must apply to M_X (and analoguously for M_Y).

Lemma 3.5.2. *Let (X, Y) be an exact 3-separation in a matroid M where M is a ternary excluded minor for $GF(5)$-representability. If M_X and M_Y are both quinary. Then M_X is uniquely representable over $GF(5)$.*

Proof. Suppose that there exists three inequivalent $GF(5)$-representations $A_{X,1}, A_{X,2}, A_{X,3}$ of M_X. Let L be the line common to M_X and M_Y. By performing row operations, we may assume that the representations are of the form:

$$A_{X,r} = \left[\begin{array}{c|cccc} & & & \mathbf{0} & \\ \cline{2-5} * & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & r+1 \end{array}\right]$$

for $r \in \{1, 2, 3\}$, where L labels the last four columns of $A_{X,r}$.

Now let A_Y be a $GF(5)$-representations of M_Y, and by row operations we may assume it is of the form:

$$A_Y = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \\ 0 & 1 & 1 & r_Y & * \\ \hline & \multicolumn{2}{c}{\mathbf{0}} & & \end{array}\right]$$

for some $r_Y \in \{1, 2, 3\}$, where L labels the first four columns of $A_{X,r}$.

Then applying Lemma 2.6.2 on the representations A_{X,r_Y-1}, A_Y for M_X and M_Y (respectively) gives us that $M^+_{X,Y}$ is $GF(5)$-representable. By Lemma 3.2.4, all elements of L are pinned in $M^+_{X,Y}$. Hence applying Lemma 3.2.1 gives us that M is $GF(5)$-representable, a contradiction. $\square$

Putting together the above lemma and Theorem 1.3.6, it follows that neither M_X nor M_Y are near-regular.

Corollary 3.5.3. *Let M be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation (X, Y). If both M_X and M_Y are quinary, then neither M_X nor M_Y are near-regular.*

3.6 Reducing to O_8

In this section, we will study the structure of M_X and M_Y given that they are quinary but not near-regular. If M is an excluded minor, then M_X and M_Y are minor-minimal with respect to keeping the boundary line and being not near-regular. The main result of this section shows that M_X and M_Y are both isomorphic to O_8.

Lemma 3.6.1. *Let L be a $U_{2,4}$-restriction in a 3-connected matroid M that is dyadic but not near-regular. Then M has a minor N isomorphic to O_8 such that L is a $U_{2,4}$-restriction of N.*

In order to prove that our minor-minimal dyadic matroid is isomorphic to O_8, we only need to show that if it is not O_8 then we can find a minor preserving the line L and a 4-element circuit-cocircuit.

Since M is dyadic but not near-regular, M is not $GF(4)$-representable. We will prove that M must contain a non-Fano or its dual as a minor, using an excluded minor theorem for 3-connected non-$GF(4)$-representable matroids from Theorem 1.3.8.

In our proof we will often want to dualize our matroid M. The next proposition shows that we can take the dual of M and use a generalized $\Delta - Y$ exchange to preserve all the desired properties.

Proposition 3.6.2. *Let M be a 3-connected, dyadic, non-near-regular matroid with a $U_{2,4}$-restriction L. Then $\nabla_L(M^*)$ is also 3-connected, dyadic, non-near-regular with a $U_{2,4}$-restriction.*

Proof. Since representability and connectivity are preserved by duality, M^* is 3-connected, dyadic, and not near-regular. Note that L is an coline in M^* (as L is a coindepdent line in M by 3-connectivity), so $\nabla_L(M^*)$ is well-defined and L is a $U_{2,4}$-restriction in $\nabla_L(M^*)$. $\square$

Lemma 3.6.3. *Let M be a 3-connected, dyadic, non-near-regular matroid with a $U_{2,4}$-restriction. Then M has an F_7^-- or $(F_7^-)^*$-minor.*

Proof. Since M is not $GF(4)$-representable, M must satisfy one of the three cases of Theorem 1.3.8. M does not satisfy case (ii) as P_8'' is not ternary. Next, it can be easily verified that contracting at most two elements of $S(5, 6, 12)$ does not produce a $U_{2,4}$-restriction, hence M does not satisfy case (iii). Thus M satisfies case (i). As neither $U_{2,6}$, $U_{4,6}$, nor P_6 is ternary, it follows that M contains an F_7^- or $(F_7^-)^*$ as a minor. $\square$

Lemma 3.6.4. *Let L be a subset of elements of a 3-connected, dyadic matroid M such that $M|L$ is isomorphic to $U_{2,4}$ and M contains an F_7^--minor N. If all proper minors M' of M that satisfy $M|L = M'|L$ has the property that M' does not contain an F_7^--minor, then $E(M) = E(N) \cup L$.*

Proof. Suppose M contains an element e such that $e \notin L$ and $e \notin E(N)$. Then either $M \backslash e$ or M/e preserves both L as a $U_{2,4}$-restriction and N as an F_7^--minor. Let $M' \in \{M\backslash e, \nabla_L(M^*)\backslash e\}$ such that N or N^* is an F_7^--minor of M'. Since M is minor-minimal, M' is not 3-connected.

By applying Lemma 2.4.6 on M', we get that a partition $B_1, \dots, B_k$ of $E(M')$ such that $\lambda_M(B_i) \leq 1$ and $|E(N) \cap B_i| \leq 1$ for $i \in \{1, \dots, k\}$, and the matroid M'' obtained by contracting all B_i down to their respective basepoints e_i is 3-connected with groundset $\{e_1, \dots, e_k\}$ with an N-minor.

Note that L is contained in some B_i, otherwise M'' contradicts the minimality of M. Let B_L be the side of the 2-separation $(B_L, E(M') \setminus B_L)$ of M' that L is contained in, so $(B_L \cup \{e\}, E(M) \setminus (B_L \cup \{e\}))$ is a 3-separation of M. We may assume that we have chosen e such that $|B_L|$ is minimal.

Our next claim is that B_L contains no other elements. Suppose there exists $e' \in B_L$ such that $e \notin L$, then there exists a 3-separation $(B_L' \cup \{e'\}, E(M) \setminus B_L')$ of M such that L is contained in B_L'. Now by Lemma 2.4.1 we get that $B_L \cap B_L'$ is 3-separating for M, and as $e \notin B_L'$, this is strictly smaller than B_L, contradicting our choice of e.

Finally we want to show that M/e contradicts minimality. Since $(L, E(M) \setminus L)$ is not a 2-separation in M and $r_M(L) = r_{M/e}(L)$, L is not 2-separating in M/e, so $M' = M\backslash e$. Now M/e is dyadic and preserves L as $e \notin L$. Further, M/e is 3-connected as . Also we may assume that $cl_M(L) \cap cl_M(E(M) \setminus L) = \ell_1$, otherwise M contains a $U_{2,5}$-minor, contradicting that M is ternary. Then $\{e, \ell_2, \ell_3, \ell_4\}$ is a cocircuit of M. As $M/\ell_4\backslash e$ contains an F_7^--minor by claim (1), $M|((E(M) \setminus (L \cup e)) \cup \ell_1)$ is a hyperplane of M that does not contain e and contains an F_7^--minor. So M/e contains an F_7^--minor, contradicting the minimality of M. □

Now that we have proved that M has no other elements apart from the line and the non-Fano minor, we can do case analysis to show that M is in fact O_8. To ease notation, we will assign canonical labels for the elements of F_7^- and $(F_7^-)^*$, and use these labels for the remainder of the chapter.

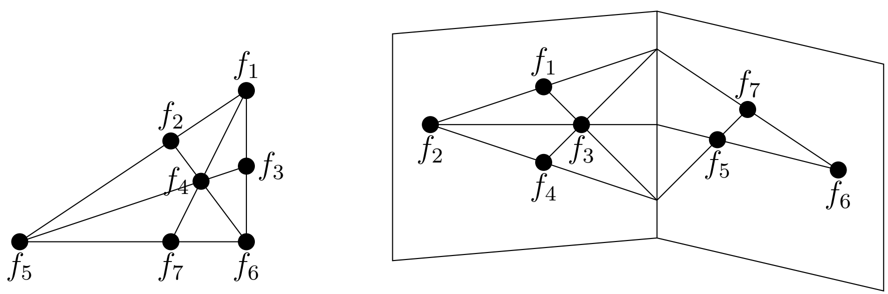

Figure 3.3: A labelling of F_7^- and $(F_7^-)^*$

First we will prove a proposition regarding the non-Fano matroid.

Proposition 3.6.5. *Let M be a ternary extension of F_7^- by a non-loop element e. Then M contains an F_7^--restriction using e.*

Proof. Observe that F_7^- is the unique rank-3, 7-element ternary matroid such that $\{e_1, e_2, e_5\}$, $\{e_1, e_3, e_6\}$, $\{e_1, e_4, e_7\}$, $\{e_5, e_6, e_7\}$ are 3-point lines but $\{e_2, e_3, e_4\}$ is not.

We may assume that e is a simple extension, otherwise the result is immediate. Let $L = \{f_5, f_6, f_7\}$. First suppose that e is not in the span of any 3-point lines. Then the line spanned by $\{f_1, e\}$ does not contain any element of L. Let f_8 be a ternary extension of M in the spans of $\{f_1, e\}$ and L. Then the lines spanned by $\{e, f_i\}$ for $i \in \{2, 3, 4\}$ must not contain f_8 (otherwise $\{f_1, f_i\}$ spans both f_8 and f_{i+3}, a contradiction as f_1 is not on L), so they must span distinct elements of L. Since $\{f_2, f_3, f_4\}$ is not a 3-point line, by our observation above, $M\backslash f_1 \cong F_7^-$.

Now suppose e is in the span of a 3-point line, which we may assume to be L. Then $\{f_2, f_3\}$ spans e, otherwise M/f_2 contains a $U_{2,5}$-restriction, contradicting that M is ternary. Then $\{f_2, f_3, e\}$, $\{f_2, f_1, f_5\}$, $\{f_2, f_4, f_7\}$, and $\{e, f_5, f_7\}$ are all 3-point lines and $\{f_1, f_3, f_4\}$ is not. By our observation, $M\backslash f_6 \cong F_7^-$, as desired. □

We are now ready to prove Lemma 3.6.1.

Proof of Lemma 3.6.1. By Lemma 3.6.3 and Lemma 3.6.4, M contains a $U_{2,4}$-restriction L and an F_7^--minor N and no other elements. Since we obtain N by a sequence of at most two contractions, $r(M) \leq 5$. Clearly we also have $r(M) \geq 3$. We then split our analysis on the rank of M. Let $L = \{e_1, e_2, e_3, e_4\}$ and $N = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$, noting that e_i and f_j may not be distinct.

Case 1: $r(M) = 3$.

In this case, N is in fact an F_7^--restriction, and $N \setminus L$ is a cocircuit. Observe that deleting any line of N preserves a 4-element circuit, hence $N \setminus L$ contains a 4-element circuit-cocircuit, so $M\backslash L$ contains a 4-element circuit. Then $M|(L \cup C)$ is isomorphic to O_8.

Case 2: $r(M) = 4$.

By switching labels of elements of L, we may assume that M/e_1 has a F_7^--restriction, so the projection of M from e_1 onto a hyperplane $H \subseteq (E(M) \setminus \{e_1\})$ contains a non-Fano restriction. Since L is modular, we may assume that e_4 is in the span of H. By Proposition 3.6.5, M/e_1 contains an F_7^--restriction using e_4, so we may assume that e_4 is an element of N. As the automorphism group of F_7^- has the two orbits (f_1, f_4, f_5, f_6) and (f_2, f_3, f_7), we consider the two subcases where e_4 is an element of a representative from each orbit:

Subcase 2.1: $e_4 = f_5$.

Then M/f_6 preserves L. Since $L \cup \{f_6, f_7\}$ is a plane, $\{f_1, f_2, f_3, f_4\}$ is a cocircuit in M/f_6. If $\{f_1, f_2, f_3, f_4\}$ is a circuit in M/f_6 then we have found our circuit-cocircuit. So in M, there exists a plane P containing f_6 and at least three of $\{f_1, f_2, f_3, f_4\}$. By symmetry we may assume that $f_2, f_4, f_6 \in P$ and $e_1 \notin P$ as there are no lines more than three elements in F_7^-. So the plane containing $\{e_1, f_2, f_4, f_6\}$ meets P along a line. Hence $\{f_2, f_4, f_6\}$ is a triangle in M. By applying the above analysis on M/f_1, we get that $\{f_1, f_4, f_7\}$ is a triangle M. As these two triangle have a common element f_4, they lie on a plane P'. Since

M is 3-connected, f_3 is not in the span of P'. Then M/f_3 contains L and $\{f_1, f_2, f_6, f_7\}$ as a circuit-cocircuit, as desired.

Subcase 2.2: $e_4 = f_2$.

In this case consider M/f_3, which preserves L and has $\{f_1, f_4, f_5, f_6, f_7\}$ as a cocircuit. We claim that at least one 4-element subset of $\{f_1, f_4, f_5, f_6, f_7\}$ is a circuit in M/f_3. Suppose otherwise, then in M every 4-element subset of $\{f_1, f_4, f_5, f_6, f_7\}$ lies on a plane with e_3, and hence the entire set lies on a plane with e_3 in M. However, this contradicts 3-connectivity of M.

Case 3: $r(M) = 5$.

In this case $E(N)$ and L are disjoint, so M has 11 elements, and we may assume that $M/e_1, e_2, e_3, e_4 \cong F_7^-$. Now consider M^*, which has rank six and $M^* \backslash e_1, e_2, e_3, e_4 \cong (F_7^-)^*$, so deleting the coline L preserves the $(F_7^-)^*$-minor. Hence applying the generalized $Y - \Delta$ exchange on L in M^* preserves the $(F_7^-)^*$-minor. Let $M' = \nabla_L(M^*)$. Then M' contains a 4-point line, an $(F_7^-)^*$-minor, and $\mathrm{si}(M')$ is 3-connected. Since a $Y - \Delta$ exchange on a 4-element coline drops the rank by two, M' has rank four and the $(F_7^-)^*$-minor is in fact an $(F_7^-)^*$-restriction.

Next we want to show that M' contains rank-3 minor that has L as a 4-point line along with a 4-element circuit-cocircuit. Let $e \in E(M')$ such that e is not in the span of L. Let $P_1, \ldots, P_k$ be the planes that contain L in M'. We have $k \geq 3$ as $\mathrm{si}(M')$ is 3-connected, and $|P_i \cap E(N)| < 5$ since $(F_7^-)^*$ has no plane with five elements. We split our analysis on the size of $\mathrm{cl}(L) \cap E(N)$, noting that $|cl_{M'}(L) \cap E(N)| < 3$ since $(F_7^-)^*$ has no 3-point line. Let $E(N) = \{n_1, \ldots, n_7\}$.

Subcase 3.1: $|cl_{M'}(L) \cap E(N)| = 0$.

Then there are seven elements of N and least three planes, we may assume $|P_1 \cap E(N)| \leq 2$. Let $n_1 \in P_1$. Then $E(N) \setminus P_1$ has size ≥ 5, so we may assume $\{n_2, n_3, n_4, n_5, n_6\} \subseteq E(N) \setminus P_1$. Since $\{n_2, n_3, n_4, n_5, n_6\}$ is not in the span of L in M'/n_1, so it is a cocircuit in M'/n_1. We want to show that this also contains a circuit C in M'/n_1. Suppose otherwise, then we may assume that $\{n_1, n_2, n_3, n_4\}$ is a plane in M'. There is also a plane involving n_5, n_6, otherwise $C = \{n_3, n_4, n_5, n_6\}$. Then we may assume that $\{n_1, n_4, n_5, n_6\}$ is a plane in M'. Then $C = \{n_2, n_3, n_5, n_6\}$ as desired.

Subcase 3.2: $|cl_{M'}(L) \cap E(N)| = 1$.

Suppose that some P_1 contains only two elements of N and let $n \in P_1 \setminus cl_{M'}(L)$. Then $E(N) \setminus P_1$ has size ≥ 5, so we can apply the analysis from Subcase 3.1 on $E(N) \setminus P_1$ to obtain a 4-element circuit-cocircuit C in M'/n. Hence each P_i contains exactly three elements of N and hence $k = 3$. Since the automorphism group is preserved by duality, the automorphism group of $(F_7^-)^*$ also has the two orbits (f_1, f_4, f_5, f_6) and (f_2, f_3, f_7). Hence we may assume that either $f_1 \in cl_{M'}(L)$ (referring to Figure 3.3) or $f_2 \in cl_{M'}(L)$.

Suppose $f_1 \in cl_{M'}(L)$. Let $f_2 \in P_1$, then as $|P_1| = 3$ we have that $P_1 = \{f_1, f_2, f_5\}$, since each of $\{f_3, f_4, f_6, f_7\}$ lie on a 4-element plane with f_1, f_2. Then M'/f_2 preserves L and $\{f_3, f_4, f_6, f_7\}$ is a 4-element plane in M' that does not span f_2 and is disjoint from $cl_{M'}(L \cup \{f_2\})$, so it is a circuit-cocircuit in M'/f_2 as desired. Now suppose $f_2 \in cl_{M'}(L)$, then let $f_1 \in P_1$. Then as above, $P_1 = \{f_1, f_2, f_5\}$ and $\{f_3, f_4, f_6, f_7\}$ is a circuit-cocircuit in M'/f_1.

Subcase 3.3: $|cl_{M'}(L) \cap E(N)| = 2$.

Let $n_1, n_2 \in cl_{M'}(L)$. Observe that at least one plane contains four elements of N since every line in $(F_7^-)^*$ is contained in a 4-element plane, we may assume that $P_1 \cap E(N) = \{n_1, n_2, n_3, n_4\}$. Then there are three remaining elements of $E(N) \setminus P_1$ and least two remaining planes, so we may assume that $|P_2 \cap (E(N) \setminus L)| = 1$. Let $n_5 \in (P_2 \setminus L)$ and consider M'/n_5. Then $\{n_3, n_4, n_6, n_7\}$ has size four and is not in the span of L in M'/n_5, so it is a cocircuit in M'/n_5. If it is also a circuit, then we are done, so suppose otherwise. Hence there exists a plane P' of M' that contains f and exactly three elements of $\{n_3, n_4, n_6, n_7\}$ (it cannot contain more than three as F_7^- has no plane with five elements).

Suppose both of n_3, n_4 are in P'. Then we may assume that $n_6 \notin P'$. Then $\{n_3, n_4, n_5, n_7\}$ is a circuit-cocircuit in M'/n_6, and we want that no element is in $cl_{M'/n_6}(L)$. Since $n_6 \notin P_1$ and $n_6 \notin P_2$, the plane P'' spanned by L and n_6 does not contain any of $\{n_3, n_4, n_5\}$, so $n_7 \in P''$. Then $\{n_1, n_2, n_7\}$ is a cocircuit of $(F_7^-)^*$ (via P'), and $\{n_5, n_3, n_4\}$ is a cocircuit of $(F_7^-)^*$ (via P''), however this is a contradiction as F_7^- contains no two disjoint 3-element lines.

Next suppose $n_3 \notin P'$ but $n_4 \in P'$. Then $\{n_1, n_2, n_3\}$ is a cocircuit of $(F_7^-)^*$ (via P'), and $\{n_5, n_6, n_7\}$ is a cocircuit of $(F_7^-)^*$ (via P_1), a contradiction as above and proving Subcase 3.3.

So in all subcases, we find a minor of M'' of M' that contains L and has a 4-element circuit-cocircuit disjoint from L. Then $\nabla((M'')^*)$ is a minor of M, and as circuit-cocircuits are obviously self-dual, M contains L and a circuit-cocircuit disjoint from L, as desired. □

3.7 Matching Cross Ratios

As we have proven in the previous section that both sides of our excluded minor must be isomorphic to O_8, our final task is to determine the excluded minors themselves. To ease notation in this section, we will use the labelling of the elements of O_8 given in Figure 3.4.

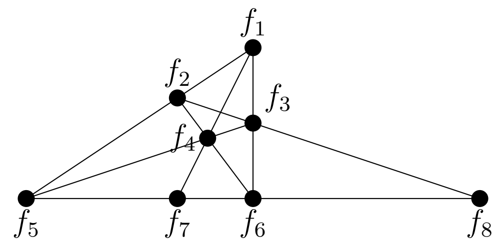

Figure 3.4: A labelling of the elements of O_8

It is easy to check that there is an automorphism of O_8 that swaps (f_5, f_6) while f_7, f_8 are fixed elements (and vice versa), and another that swaps both pairs (f_5, f_6) and (f_7, f_8). However, there is no automorphism that sends either f_5 or f_6 to f_7 or f_8 (since f_5 and f_6 are contained in three 3-point lines, while f_7 and f_8 are contained in two). Hence we consider f_5, f_6 as "red" elements and f_7, f_8 as "blue" elements.

Lemma 3.7.1. *Let M be a ternary excluded minor with an exact 3-separation (X, Y), such that M_X and M_Y are both quinary. Then M is isomorphic to T_8.*

Proof. By Corollary 3.5.3 and Lemma 3.6.1, M_X and M_Y are both isomorphic to O_8. Let $L = \{e_1, e_2, e_3, e_4\}$ be the boundary line of (X, Y) in $M^+_{X,Y}$. We colour the elements of L in M_X and M_Y with red and blue as above. By relabeling, we may assume that e_1 and e_2 are red elements in M_Y, and e_3, e_4 are blue. Furthermore, if exactly one of e_1, e_2 is red in M_X, we may assume that e_1 is red, by swapping the labels of e_1, e_2 in M_Y. Similarly e_3 is red in M_X if one of e_3, e_4 is red.

Given an ordering of elements $(\ell_1, \ell_2, \ell_3, \ell_4)$ of a $U_{2,4}$-minor in a uniquely $\mathbb{F}$-representable matroid M, we denote its cross ratio by $cr_M(\ell_1, \ell_2, \ell_3, \ell_4)$. Let $x = cr_{M_X}(e_1, e_2, e_3, e_4)$, and $y = cr_{M_Y}(e_1, e_2, e_3, e_4)$. First, we compute y. Let A be a $GF(5)$-representation of M_Y. By row operations, we may assume that the columns of A corresponding to L have the form

$$\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & y \end{bmatrix}.$$

Observe that this is row equivalent to

$$\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{1}{y} & 1 \end{bmatrix}.$$

Applying an automorphism on M_Y that swaps e_3, e_4 and fixes e_1, e_2, we get a representation A' of M_Y where the columns corresponding to L are

$$\begin{array}{cccc} e_1 & e_2 & e_4 & e_3 \\ \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & y \end{bmatrix}.$$

Since O_8 is uniquely representable over $GF(5)$, $y = \frac{1}{y}$ so $y = 3$. Now that we have the cross ratio for the Y side, we consider the possible orderings of L in M_X, splitting our analysis by the number of red elements in $\{e_1, e_2\}$ in M_X.

Case 1: e_1, e_2 are both red in M_X.

Then e_3, e_4 are both blue. Since the elements of L match in M_Y and M_X, clearly their cross ratios match as well, so $x = 3$. Hence $M^+_{X,Y}$ is quinary, a contradiction.

Case 2: e_1, e_2 are both blue in M_X.

Then e_3, e_4 are both red. Similarly to our argument that $y = 3$, we have that

$$\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \end{array} \quad \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \frac{1}{x} \end{bmatrix}$$

are both $GF(5)$-representations of L that extend to M_X, by applying an automorphism that swaps e_3, e_4 while fixing e_1, e_2. Hence $x = 3$, which implies that $M^+_{X,Y}$ is quinary, a contradiction.

Case 3: e_1 is red, e_2 is blue in M_X.

Then e_3 is red and e_4 is blue. It is routine to check that M is isomorphic to T_8, and T_8 is indeed an excluded minor for dyadic matroids, see page 649 of [16]. □

3.8 Main Theorem

We conclude by applying the results from the previous sections to prove our main theorem.

Theorem 3.8.1. *Let M be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation (X, Y) such that $|X|, |Y| \geq 4$. Then either:*

(i) M is isomorphic to T_8,

(ii) X is a circuit-cocircuit and M_Y contains an excluded minor for $GF(5)$-representability with up to two fewer elements than M_Y,

(iii) Y is a circuit-cocircuit and M_X contains an excluded minor for $GF(5)$-representability with up to two fewer elements than M_X,

(iv) $|X| = 4$, M_X is not isomorphic to O_8, and M_Y is an excluded minor for $GF(5)$-representability, or

(v) $|Y| = 4$, M_Y is not isomorphic to O_8, and M_X is an excluded minor for $GF(5)$-representability.

Proof. Let $L = \{e_1, e_2, e_3, e_4\}$ be the boundary line of (X, Y) in $M^+_{X,Y}$. Since $|X|, |Y| \geq 4$, M pins all four elements of L by Lemma 3.2.4. If both M_X and M_Y are quinary, then by Lemma 3.7.1, M satisfies (i). By symmetry we may assume that M_Y is not quinary. Then by Lemma 3.4.1 and Lemma 3.4.2, M_X has eight elements. Since M_X distinguishes all four elements of L, M_X is one of the five matroids M_1, M_2, M_3, M_4, M_5 given by Lemma 3.4.3. Let x_1, x_2, x_3, x_4 label the elements of $M_X \backslash L$. Note that if M_X is isomorphic to M_5, then M_Y is isomorphic to $\nabla_X(M)$.

It remains to show that, if M_X is not isomorphic to M_3, then M_Y is an excluded minor for $GF(5)$-representability. Since M_Y is not quinary, it suffices to show that every deletion or contraction of M_Y results in a quinary matroid.

Let M_Y^- be a minor of M_Y obtained by contracting $e_1 \in L$. Observe that in all five possible matroids for M_X, we can contract a (possibly empty) set S such that L is a spanning line in M_X/S with e_1 parallel to some $e \in X$. Then $\mathrm{si}(M_Y^-/e_1)$ is isomorphic to $\mathrm{si}(M/S/e)$, and since simplification does not change representability, M_Y^- is quinary. Next let M_Y^- be obtained by deleting $e_1 \in L$. If M_X is isomorphic to M_1, M_2 or M_4 then clearly we can contract a set S such that L is a spanning line in M_X/S with e_2, e_3, e_4 each parallel to some $x_2, x_3, x_4 \in X$. Then $M_Y \backslash e_1$ is a restriction of M/S so M_Y^- is quinary. If M_X is isomorphic to M_5, let $\{x_2, x_3, x_4, e_1\}$ be the unique 4-element circuit of M_X containing e_1. Then M_X/x_1 contains a triad $\{x_2, x_3, x_4\}$ with e_2, e_3, e_4 distinguished. Then $\nabla_{\{x_2,x_3,x_4\}}(M/x_1)$ is isomorphic to $M_Y\backslash e_1$, so M_Y^- is quinary.

Let M_Y^- be a minor of M_Y obtained by deleting or contracting an element $e \in Y$. If $M_Y^- = M_Y/e$ and e is parallel to an element e_1 of L, then clearly this is equivalent to contracting e_1, which was proved above. So M_Y^- contains L as a $U_{2,4}$-restriction. Now $M' = P_L(M_X, M_Y^-)\backslash L$ is quinary as it is a minor of M, and M' has the 3-separation $(X, Y \setminus \{e\})$, so it has a quinary representation of the form

$$A = \left[\begin{array}{c|c} A_X & 0 \\ \cline{2-2} & A_{Y-e} \\ \cline{1-1} 0 & \end{array}\right]$$

where A_X and A_{Y-e} intersect at most two rows. Let A_L be a $GF(5)$-matrix such that the only non-zero entries are in the last two rows and $[A_X|A_L]$ is a $GF(5)$-representation of M_X. Then

$$A = \left[\begin{array}{c|c} A_L & 0 \\ \cline{2-2} & A_{Y-e} \\ \cline{1-1} 0 & \end{array}\right]$$

is a $GF(5)$-representation of M_Y^- since an element on L is not pinned in M' if and only if M_Y^- does not distinguish the element, so M satisfies (iv). Analogously, if M_X is not quinary, then M satisfies (v).

Now suppose M_X is isomorphic to M_3, so M_X is isomorphic to O_8 and X is a 4-element circuit-cocircuit. We colour the elements of L in M_X with red and blue as in Section 3.7. If M_Y^- is a minor of M_Y obtained by contracting $e_1 \in L$, deleting $e \in Y$, or contracting $e \in Y$, then M_Y^- is quinary by an argument similar to above cases. If M_Y^- is a minor of M_Y obtained by deleting $e_1 \in L$, then M_Y^- is a minor of M if and only if e_1 is a blue element in M_X. We may assume that e_1, e_2 are blue in M_X. Then one of $M_Y, M_Y\backslash e_1, M_Y\backslash e_2, M_Y\backslash\{e_1, e_2\}$ is an excluded minor for $GF(5)$-representability, satisfying (ii). Analogously, if M_X is not quinary and M_Y isomorphic to O_8, then M satisfies (iii). $\square$

From Theorem 1.2.1, if we are given the set $\mathcal{M}$ of vertically 4-connected ternary excluded minors for dyadic matroids, then we can deduce the list of ternary excluded minors for dyadic matroids as follows: For $M \in \mathcal{M}$, let L be a line of M. If $|L| = 4$, then $P_L(M, M_2)$, $P_L(M, M_4)$, and $P_L(M, M_5)$ are all excluded minors for dyadic matroids (for all possible orderings of L), and $P_L(M, M_3)$ either is an excluded minor or contains a T_8-minor. If $|L| = 3$, then $\nabla_L(M)$ is an excluded minor. Moreover, let M^+ be the ternary extension of M such that L spans a 4-point line L^+. Then $P_{L^+}(M^+, M_3)$, where $L^+ \setminus L$ is matched with a blue element of M_3, contains an excluded minor. Finally, if $|L| = 2$, let M^+ be the ternary extension of M such that L spans a 4-point line L^+. Then $P_{L^+}(M^+, M_3)$, where $L^+ \setminus L$ is matched with blue elements of M_3, contains an excluded minor. Now add any new excluded minors that were found to the set $\mathcal{M}$, and repeat until all excluded minors are found. Assuming Rota's Conjecture, this procedure terminates. It follows from Theorem 1.2.1 that the matroids found by this construction are the only ternary dyadic excluded minors with a 3-separation such that one side of the separation has rank at least three.

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