On the Excluded Minors for Dyadic Matroids

by

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Author’s Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The study of the class of dyadic matroids, the matroids representable over both $GF(3)$ and $GF(5)$, is a natural step to finding the excluded minors for $GF(5)$-representability. In this thesis we characterize the ternary matroids $M$ that are excluded minors for dyadic matroids and contains a 3-separation. We will show that one side of the separation has size at most four, and that $M$ is obtained by adding at most four elements to another excluded minor $M'$. This reduces the problem of finding the excluded minors for dyadic matroids to the problem of finding the vertically 4-connected excluded minors for dyadic matroids.
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Chapter 1

Introduction

The main result in this thesis reduces the problem of finding the excluded minors for the class of dyadic matroids to the problem of finding the vertically 4-connected excluded minors.

1.1 Background

In his introductory paper on matroids, Whitney [23] posed the problem of characterizing representable matroids. The class $M$ of matroids that are representable over a field $F$ is closed under taking minors, so it is natural to characterize $M$ by its set of excluded minors; these are the minor minimal-matroids not in $M$. When $F$ is infinite, the excluded minors were shown to be as least as wild as the class itself; see Mayhew, Newman, and Whittle [12]. However, there are very natural questions to be considered for finite fields. For every finite field $F$, is this set of excluded minors finite? And if so, what are they? The former question is known as Rota’s Conjecture, which was one of the major unsolved problems in matroid theory until a proof was announced in 2014 by Geelen, Gerards, and Whittle [8]. For the latter question, the cases for $GF(2), GF(3), GF(4)$ have been solved over the past 60 years.

In 1958, Tutte proved that a matroid $M$ is binary if and only if $M$ contains no minor isomorphic to $U_{2,4}$, the 4-point line [20]. This gives a strikingly simple connection between the algebraic property of being representable over $GF(2)$, and the combinatorial property of containing a certain minor. Then in 1979, Bixby [1] and Seymour [18] independently proved that the excluded minors for ternary matroids are $U_{2,5}, U_{3,5}, F_7$, and $F_7^*$. Bixby
attributes this characterization to unpublished work of Reid. These proofs relied on the fact that matroids in these two classes are uniquely representable over their respective field, a technique that would be used in the proofs for larger fields. In 1988, Kahn showed that a quarternary matroid is uniquely representable if and only if it is not the 2-sum of two non-binary matroids [11]. Twelve years later, this was followed by a proof for the excluded minors of $GF(4)$-representable matroids, published by Geelen, Gerards, and Kapoor [6] in 2000, using the result of Kahn. They showed that there are seven excluded minors for $GF(4)$-representability, and they are $U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*, P_8,$ and $P_8''$.

Logically, the next goal is to study $GF(5)$, which is the field that we are interested in for this thesis. It is known that there are at least 564 excluded minors for $GF(5)$-representability, found via a database of all matroids with up to nine elements that was compiled computationally by Mayhew and Royle [13]. A first step to understanding the excluded minors for $GF(5)$-representability is to study the class of dyadic matroids, which are the ones representable over both $GF(5)$ and $GF(3)$. The motivation for this stems back to the problem of limiting the number of inequivalent representations, which proved fruitful for $GF(2), GF(3),$ and $GF(4)$. While quinary matroids, those that are $GF(5)$-representable, may have up to six inequivalent $GF(5)$-representations, the story is different if the matroid is also ternary. Whittle proved that dyadic matroids have at most three inequivalent representations over $GF(5)$ [24]. A strengthening of this result was proved by Pendavingh and van Zwam in [17], where they prove that a ternary matroid may have exactly zero, one, or three inequivalent $GF(5)$-representations, and describe the conditions for each case to occur. In particular, if a dyadic matroid has three inequivalent $GF(5)$-representations, then $M$ is near-regular but not regular. This is useful since the list of excluded for regular matroids (Tutte [20]) and near-regular matroids (Hall, Mayhew, and van Zwam [10]) are known.

Most of the difficulties of the proof of the excluded minors for $GF(4)$-representability disappear when considering 6-connected excluded minors. These difficulties gradually return as we consider the 5-connected, 4-connected, and, finally the 3-connected cases. Here we are proposing an alternative approach of applying connectivity reductions to determine the excluded minors with low connectivity.

Our contribution for this thesis is a characterization of the case when an excluded minor for dyadic matroids is has an exact 3-separation. Our theorem would imply that, given a list of all vertically 4-connected excluded minors for dyadic matroids, one can generate the full list of dyadic excluded minors after some case analysis.
1.2 Main Theorem

Observe that any excluded minor for dyadic matroids that is not ternary contains one of $U_{2,5}, U_{3,5}, F_7,$ or $F_7^*$ as a minor. Hence we look at the excluded minors for dyadic matroids that are ternary, and they are precisely the ternary matroids that are excluded minors for $GF(5)$-representability, as all of their proper minors are dyadic.

Let $M$ be a ternary matroid with an exact 3-separation $(X, Y)$. We construct a ternary matroid $M^+_{X,Y}$ by extending $M$ by a set $L$ of four elements added into the span of both $X$ and $Y$ such that $L$ is independent. As ternary matroids are uniquely representable (Brylawski 1976; [2]), this is well-defined. We call $L$ the \textit{boundary line} of $(X, Y)$. Note that, since $M^+_{X,Y}$ need not be simple, technically $L$ may not be a line of $M^+_{X,Y}$, however we will call it as such for convenience. We then let $M_X = M^+_{X,Y} | (X \cup L)$ and $M_Y = M^+_{X,Y} | (Y \cup L)$.

Our main theorem is the following:

\textbf{Theorem 1.2.1.} Let $M$ be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation $(X, Y)$ such that $|X|, |Y| \geq 4$. Then either:

- $M$ is isomorphic to $T_8$,
- $|X| = 4$ and for each $e \in Y$, both $M_Y \backslash e$ and $M_Y / e$ are $GF(5)$-representable, or
- $|Y| = 4$ and for each $e \in X$, both $M_X \backslash e$ and $M_X / e$ are $GF(5)$-representable.

The matroid $T_8$ is a well-known excluded minor for dyadic matroids (see page 649 of [16]). From the diagram of a geometric representation of $T_8$ in Figure 1.1 it is clear that the two planes drawn induce an exact 3-separation of $T_8$.

It follows from Theorem 1.2.1 that an excluded minor containing an exact 3-separation must be a minor of a matroid with at most four extra elements added to a previously-known excluded minor. So we can deduce the list of ternary excluded minors for dyadic matroids from the set $\mathcal{M}$ of vertically 4-connected ternary excluded minors for dyadic matroids: For $M \in \mathcal{M}$, consider the set of matroids obtained by a sequence of four ternary extensions or coextensions of $M$, check if each of them contain an excluded minor, adding any new excluded minors to $\mathcal{M}$, and repeating this procedure until all excluded minors have been found. There is no inherent reason that this procedure does not run forever; its termination is a consequence of Rota’s Conjecture.

Note that the result that we prove in Chapter 3 also characterizes the specific matroids $M_X, M_Y$ that correspond to the side of size four.

An informal outline of our proof strategy will be given in the next section.


1.3 Proof Outline

Preliminary results and definitions can be found in Chapter 2 and the text of Oxley [16]. The proof of our main result follows in Chapter 3.

The main strategy that we will use is to decompose the excluded minor $M$ along the separation $(X,Y)$, analyze each side, then find the minimal obstructions that prevent us from reassembling the two sides into a quinary matroid. We start off in Section 3.1 by constructing the two matroids corresponding to each side, $M_X = M_{X,Y}^+(X \cup L)$ and $M_Y = M_{X,Y}^+(Y \cup L)$.

Now we need to consider the relationship between the non-$GF(5)$-represtability $M_{X,Y}^+$ with $M_X$ and $M_Y$, and between $M_{X,Y}^+$ and $M$. For the former question, the tool that we will use is the generalized parallel connection introduced by Brylawski [4]. Generalized parallel connections gives us a way to glue two matroids together along a common flat if that flat is modular on at least one side, and we begin by showing that $M_{X,Y}^+$ can be constructed by a generalized parallel connection of $M_X$ and $M_Y$. This is important for representability since if a matroid $N$ is a generalized parallel connection of two $F$-representable matroids $N_1$ and $N_2$, and they have representations that agree on $E(N_1) \cap E(N_2)$, then the $N$ itself is $F$-repersentable; see Lemma 2.6.2.
Since $M$ is an excluded minor for quinary matroids, $M_{X,Y}^+$ must be non-quinary, and so it must not be the case that $M_X$ and $M_Y$ are quinary and have a representation that match along $L$. Over $GF(5)$, a 4-point line $L$ can have three inequivalent representations, which are given by the matrices of the form $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{bmatrix}$ for $x \in \{2, 3, 4\}$. The value $x$ is the cross-ratio of the representation of $L$ with respect to that ordering of its elements.

Now that we have constructed $M_{X,Y}^+$, we want to find the conditions for when adding the elements of $L$ to $M$ does not change the representability. While adding elements on a line is well-defined for ternary matroids, this isn’t the case for quinary matroids. The classic example is a pair of representations of $U_{3,6}$ with respect to a given line (Figure 1.2).

![Figure 1.2: Two representations of the matroid $U_{3,6}$](image)

However, in Section 3.2 we show that this is well-defined if both $M_X$ and $M_Y$ “distinguish” the element. Let $N$ be a subset of the groundset of a matroid $M$, if there is a set $S \subseteq E(M) \setminus \{e\}$ that spans $e$ but not $N$, we call $S$ a strand for $N$. If $M|N$ is a line, we say that $e \in N$ is distinguished. An element $e$ in a boundary line $L$ is pinned if both $M_X$ and $M_Y$ distinguish $e$.

**Lemma 1.3.1** (Lemma 3.2.1). Let $M$ be a ternary matroid with an exact 3-separation $(X,Y)$. Let $M'$ be a ternary matroid obtained from $M$ via an extension by a non-loop element $e$ such that $e \in \text{cl}(X) \cap \text{cl}(Y)$. If there exists strands $S_X \subseteq X$ for $Y$ and $S_Y \subseteq Y$ for $X$ such that $e \in \text{cl}(S_X) \cap \text{cl}(S_Y)$, then $M$ is $F$-representable if and only if $M'$ is $F$-representable.

So we want to show that all four elements of $L$ are pinned in $M_{X,Y}^+$; our next lemma shows what would happen if this were not the case:

**Lemma 1.3.2** (Lemma 3.2.3). Let $M$ be a 3-connected ternary matroid whose proper minors are all quinary and let $(X,Y)$ be an exact 3-separation of $M$ with $|X| \geq 4$. If $M_X$ does not distinguish all four elements in the boundary line, then $M$ is quinary.
The proof strategy is as follows: If \( e \in L \) is not distinguished, then \( M_X \setminus e \) has a modular 3-point line and \( M_Y \setminus e \) is a minor of \( M \) so it is quinary. A theorem of Seymour [19] implies that \( M_X \setminus e \) is binary. Since matroids that are both binary and ternary are in fact regular, \( M_X \setminus e \) is quinary. It is then easy to glue the \( M_X \setminus e \) and \( M_Y \setminus e \) together, since we can use row operations to find representations that agree along the modular 3-point line of \( M_X \setminus e \). Then applying Lemma 2.6.2 implies that \( M \) is quinary.

Our excluded minor \( M \) is not quinary, thus all four elements of \( L \) are pinned if \( X \) and \( Y \) have size at least four.

**Lemma 1.3.3** (Lemma 3.2.4). Let \( M \) be a ternary excluded minor for \( GF(5) \)-representability and let \((X,Y)\) be an exact 3-separation of \( M \). If \(|X|,|Y| \geq 4\), then \( M_{X,Y}^+ \) pins all elements of the boundary line.

So we have established that a ternary matroid \( M \) is quinary if and only if \( M_{X,Y}^+ \) is, which in turn is quinary if and only if there exists quinary representations of \( M_X \) and \( M_Y \) that match along \( L \). Next we consider whether \( M_X \) and \( M_Y \) are quinary, first dealing with the case that we have \( M_Y \) non-quinary. In Section 3.4, we prove the following lemma:

**Lemma 1.3.4** (Lemma 3.4.1). Let \( M \) be a ternary excluded minor for \( GF(5) \)-representability with an exact 3-separation \((X,Y)\). If \( M_Y \) is non-quinary, then there is no proper minor of \( M_X' \) of \( M_X \) such that \( M|L = M'|L \) and all four elements of \( L \) are distinguished in \( M_X' \).

The proof of this lemma relies on a simple observation: if a minor \( M_X' \) of \( M_X \) keeps \( L \) and keeps all four elements of \( L \) distinguished, then the generalized parallel connection of \( M_X' \) and \( M_Y \) results in a non-quinary matroid that is a minor of \( M_{X,Y}^+ \), which we can then use to construct a non-quinary minor of \( M \).

So then our goal is to characterize such minor-minimal matroids. Our next lemma will show that they must have at most four elements apart from the line. As a result, if one side of the separation is non-quinary, then the other side has size at most four.

**Lemma 1.3.5** (Lemma 3.4.2). Let \( L \) be a modular set in a ternary matroid \( M \) such that \( M|L \) is isomorphic to \( U_{2,4} \). If all four elements of \( L \) are distinguished, then \( M \) contains an 8-element minor \( N \) such that \( M|L = N|L \) and all elements of \( L \) are distinguished in \( N \).

By Lemma 3.4.2, if \( M_Y \) is not quinary, then \(|X| = 4\). By case analysis, \( M_X \) is either the unique matroid with a modular 4-point line \( L \) and an independent coline, or one of the four matroids in Figure 1.3.
In the remaining case we have that both $M_X$ and $M_Y$ are quinary, so we want to know whether they have representations that match along the boundary line $L$. In Section 3.5, we focus on the next question: how many $GF(5)$-representations do $M_X$ and $M_Y$ have? A result from Whittle [24] states that $U_{2,4}$ is a stabilizer for ternary matroids. In other words, if a 3-connected ternary matroid $N$ has a $U_{2,4}$-minor, then every $GF(q)$-representation of $N$ is uniquely determined by the $GF(q)$-representation of that $U_{2,4}$-minor. Since $M_X$ contains the $U_{2,4}$-restriction $L$, the number of quinary representations of $L$ that extend to a representation of $M_X$ is equal to the number of quinary representations of $M_X$ itself, and similarly for $M_Y$.

Pendavingh and van Zwam proved that that 3-connected dyadic matroids have either one or three inequivalent representations over $GF(5)$ [17], and characterized exactly when each case happens:

**Theorem 1.3.6.** For a 3-connected dyadic matroid $M$, either

(i) $M$ is regular, and $M$ is uniquely representable over $GF(5)$.

(ii) $M$ is near-regular but not regular, and $M$ has exactly three representations over $GF(5)$.

(iii) $M$ is dyadic but not near-regular, and $M$ is uniquely representable over $GF(5)$.

We will use this theorem to show that neither $M_X$ nor $M_Y$ can be near-regular.

**Lemma 1.3.7** (Corollary 3.5.3). Let $M$ be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation $(X,Y)$. If both $M_X$ and $M_Y$ are quinary, then neither $M_X$ nor $M_Y$ are near-regular.
Suppose, towards a contradiction, that $M_X$ is near-regular. Note that $M_X$ is not binary as it contains a $U_{2,4}$-restriction, so $M_X$ is near-regular but not regular. Since $M_X$ is ternary, it does not contain $U_{2,5}$ or $U_{3,5}$ as a minor, so we can apply Theorem 1.3.6 to $M_X$. It follows that $M_X$ has three inequivalent representations over $GF(5)$, and in particular all three possible $GF(5)$-representations of $L$ extend to a representation of $M_X$. Now given any quinary representation of $M_Y$, we can take its representation of $L$ and find a $GF(5)$-representation of $M_X$ that extends this. We thus have quinary representations of $M_X$ and $M_Y$ that match along $L$, and we can apply Lemma 2.6.2 to obtain a contradiction.

Since we have proven that $M_X$ and $M_Y$ are not near-regular, the final part of the proof is concerned with finding the minor-minimal dyadic matroids that have a $U_{2,4}$-restriction and are not near-regular. If a matroid is both dyadic and not near-regular, it is not $GF(4)$-representable. Then we can use the characterization of quaternary matroids of Geelen et al. in [7].

**Theorem 1.3.8.** If $M$ is a 3-connected non-$GF(4)$ representable matroid, then either

(i) $M$ has a $U_{2,6}$-, $U_{4,6}$-, $P_6^-$, $F_7^-$, or $(F_7^-)^*$-minor,

(ii) $M$ is isomorphic to $P_8''$, or

(iii) $M$ is isomorphic to a minor of $S(5, 6, 12)$ with rank and corank at least 4.

By eliminating the cases of Theorem 1.3.8 that are not compatible with a matroid that is dyadic but not near-regular, we deduce that $M_X$ and $M_Y$ has an $F_7^-$ or $(F_7^-)^*$-minor, whose elements we will call $F$. We further prove that they have no elements other than the ones in $F \cup L$ and by case analysis we will show that they are in fact isomorphic to the matroid that we will refer to as $O_8$ (see Figure 1.4).

![Figure 1.4: The matroid $O_8$](image)

It is a single element extension of the matroid $O_7$ and is the unique matroid with a modular 4-point line along with a 4-element circuit-cocircuit, a set that is both a circuit...
and a cocircuit. Observe that $T_8$ can be obtained by a generalized parallel connection (see Section 2.6) of two copies of $O_8$ along their respective 4-point lines where $(e_1, e_2, e_3, e_4)$ is matched with $(e_1, e_2, e_4, e_3)$.

**Lemma 1.3.9** (Lemma 3.6.1). Let $L$ be a $U_{2,4}$-restriction in a 3-connected matroid $M$ that is dyadic but not near-regular. Then $M$ has a minor $N$ isomorphic to $O_8$ such that $L$ is a $U_{2,4}$-restriction of $N$.

Thus, the only excluded minors of this form are constructed from taking of these non-
Fano extensions and gluing them along their 4-point line in a way that the cross ratios do not match up. We will show that the only such matroid that arises this way is $T_8$. 
Chapter 2

Preliminaries

We will assume the reader has familiarity with the definitions of a matroid, its rank function, flats, uniform matroids, parallel elements, simplication, cosimplicaton, representability, duality, minors, connectivity, 3-connectivity, \( k \)-separation, extensions, and coextensions; see Oxley’s textbook [16].

2.1 Representable Matroids/Projective Geometries

A matroid \( M \) is uniquely representable over \( \mathbb{F} \) if all \( \mathbb{F} \)-representations of \( M \) are equivalent under row operations, column scaling, and automorphisms of \( \mathbb{F} \). A matroid is binary, ternary, quaternary, or quinary if it is \( GF(2) \)-, \( GF(3) \)-, \( GF(4) \)-, or \( GF(5) \)-representable, respectively. Additionally, a matroid is regular if it is representable over all fields, dyadic if it is representable over \( GF(3) \) and \( GF(5) \), and near-regular if it is representable over \( GF(3) \), \( GF(4) \), and \( GF(5) \).

To prove that a matroid \( M \) is regular, it suffices to show that \( M \) is both binary and ternary (Theorem 6.6.3 of [16]). This was proved by Tutte in [21].

Theorem 2.1.1. The following statements are equivalent for a matroid \( M \).

- \( M \) is regular.
- \( M \) is representable over \( GF(2) \) and \( GF(3) \).
Let $GF(q)$ be the finite field of order $q$. Consider the maximum number of elements in a simple $GF(q)$-representable matroid with rank $n$. Observe that there are $q^n - 1$ non-zero vectors in $GF(q)^n$, and $\frac{q^n - 1}{q - 1}$ subspaces of rank one. Taking a representative vector from each rank one subspace, the associated vector matroid has $\frac{q^n - 1}{q - 1}$ elements, and we denote this simple rank-$n$ $GF(q)$-representable matroid by $PG(n - 1, q)$.

By convention, for a matrix representation $A$ of a matroid $M$ we may also label the row $i$ of $A$ by an element $b$ of $M$ if $b$ labels the elementary vector $e_i$.

### 2.2 Duality

The following proposition (Proposition 2.1.6 of [16]) characterizes some elementary properties of the dual matroid:

**Proposition 2.2.1.** Let $M$ be a matroid and let $X \subseteq E(M)$. Then

- $X$ is independent in $M^*$ if and only if $E(M) \setminus X$ is spanning in $M$, and
- $X$ is a hyperplane in $M$ if and only if $E(M) \setminus X$ is a circuit in $M^*$.

For an $F$-representable matroid $M$, the following proposition gives a $F$-representation of its dual.

**Proposition 2.2.2.** Let $M$ be a matroid $M$ with $n$ elements and rank $r$ such that $M = M[A]$ where $A$ is the matrix $[I_r|B]$. Let $A^* = [I_{n-r}|-B^T]$. Then $M^* = M[A^*]$.

A directly corollary is that a matroid $M$ is $F$-representable if and only if $M^*$ is $F$-representable.

A cyclic flat $F$ of a matroid $M$ is a flat that is the union of circuits. It is routine to show that $F$ is a cyclic flat of $M$ if and only if $E(M) \setminus F$ is a cyclic flat of $M^*$.

### 2.3 Minors

For minors of matroids representable over a field $F$, observe that deleting an element in a matroid clearly corresponds to deleting the column representing that element. Since
contraction is defined as $M/e = ((M^*)\setminus e)^*$, and both deletion and duality preserve $\mathbb{F}$-representability, so contraction also preserves $\mathbb{F}$-representability. Thus the class of $\mathbb{F}$-representable matroids is minor-closed.

Let $N$ be a minor of $M$. Since the deletion and contraction operations commute with each other, it is often useful to consider $N$ as obtained from $M$ by contracting a set of elements $C$ then deleting a set of elements of $D$. The Scum Theorem (Theorem 3.3.1 of [16]) from Crapo and Rota [5] says that we may assume that $C$ is independent and $D$ is coindependent.

**Theorem 2.3.1 (Scum Theorem).** Let $N$ be a minor of $M$. Then there exists an independent set $C$ and a coindependent set $D$ such that $N = M/C\setminus D$.

### 2.4 Connectivity

The connectivity function of a matroid $M$ is defined as $\lambda_M(X) = r(X) + r(E(M) \setminus X) - r(M)$ for $X \subseteq E(M)$. A subset $X$ of the groundset of a matroid $M$ is $k$-separating if $\lambda_M(X) < k$. A $k$-separation $(X, Y)$ is exact if neither $X$ nor $Y$ are $k - 1$-separating.

We say that two separations $(X, Y)$ and $(X', Y')$ cross if $X \nsubseteq X'$, $X \nsubseteq Y'$, $X' \nsubseteq X$, and $X' \nsubseteq Y$. When $k$-separations in a $k$-connected matroid cross, we will often want $X \cap X'$ to be a separation, and the following lemma from Oxley, Semple, and Whittle [15] proves that this is the case, provided that each side has sufficiently many elements. The proof of this lemma, referred to as the Uncrossing Lemma, relies only on the submodularity of the connectivity function.

**Lemma 2.4.1 (Uncrossing Lemma).** Let $X$ and $X'$ be $k$-separating sets of a $k$-connected matroid $M$.

(i) If $|X \cap X'| \geq k - 1$, then $X \cup X'$ is $k$-separating.

(ii) If $|E(M) - (X \cup X')| \geq k - 1$, then $X \cap X'$ is $k$-separating.

An element $e$ is in the guts of $(X, Y)$ if $e \in cl_M(X \setminus e)$ and $e \in cl_M(Y \setminus e)$. Dually $e$ is in the coguts of $(X, Y)$ if $e \in cl^*_M(X \setminus e)$ and $e \in cl^*_M(Y \setminus e)$.

Local connectivity is a related notion that captures the connectivity between two subsets in a matroid. For $X, Y \subseteq E(M)$, the local connectivity $\cap_M(X, Y)$ is defined as
\[ \cap_M(X, Y) = r(X) + r(Y) - r(X \cup Y). \]

For a subset \( N \) of \( E(M) \), we can consider how the rest of the matroid connects to it. We call a set \( S \subseteq E(M) \setminus N \) a strand for \( N \) if it is a minimal set such that \( \cap(S, N) = 1 \). We say that a strand \( S \) distinguishes an element \( e \) in \( N \) if \( e \in \text{cl}(S) \), and a matroid \( M \) distinguishes \( e \) if \( E(M) \setminus N \) contains a strand that distinguishes \( e \). Note that if \( M|N \) is a simple line, then \( e \) is distinguished if and only if it is a fixed element of \( M \), since we can extend freely on the line if and only if \( e \) is not distinguished.

Another related function, \( \kappa_M \) describes the connectivity between two sets in the context of the matroid as a whole. For disjoint subsets \( X, Y \subseteq E(M) \), we define:

\[
\kappa_M(X, Y) = \min\{\lambda_M(S) : X \subseteq S \subseteq E(M) \setminus Y\}.
\]

Tutte [22] proved the following matroid analogue of Menger’s theorem:

**Theorem 2.4.2** (Tutte’s Linking Theorem). Let \( X, Y \) be disjoint subsets of elements in a matroid \( M \). Then \( \kappa_M(X, Y) \) is the maximum of \( \kappa_N(X, Y) \) for all minors \( N \) of \( M \) with groundset \( X \cup Y \).

Tutte’s Linking Theorem can be strengthened to finding a minor \( N \) that preserves not only \( \kappa_M(X, Y) \), but the structure of \( M|X \) and \( M|Y \) as well (Theorem 8.5.7 of [16]).

**Theorem 2.4.3.** Let \( X, Y \) be disjoint subsets of elements in a matroid \( M \). Then there exists a minor \( N \) of \( M \) such that \( E(N) = X \cup Y \), \( \kappa_N(X, Y) = \kappa_M(X, Y) \), \( N|X = M|X \), and \( N|Y = M|Y \).

When we delete or contract an element from a 3-connected matroid, we often need to consider the case when this results in a matroid \( M \) that is not 3-connected. Let \((X, Y)\) be a 2-separation of \( M \). Consider the matroid \( M_X^+ \) by contracting an independent set \( S \subseteq Y \) of rank \( r(Y) - 1 \) such that \( \cap_M(S, X) = 0 \) and simplifying \( Y \) to an element \( p \). It is routine to show that \( M_X^+ \) does not depend on the choice of \( S \), so it is well-defined. Let \( M_Y^+ \) be the analogous matroid obtained by contracting from \( X \). Then \( M \) is the 2-sum of \( M_X \) and \( M_Y \) with basepoint \( p \). For the 2-separating set \( Y \), we call \( M_X^+ \) the matroid obtained by **contracting \( Y \) down to its basepoint**. We first show that this operation preserves 3-connected minors that have at most one element in \( Y \).

**Proposition 2.4.4.** Let \( X, Y \) be an exact 2-separation of a matroid \( M \). If \( N \) is a 3-connected minor with \( |E(N) \cap X| \geq |E(N) - 1| \), then \( M_X^+ \) contains an \( N \) minor.
Proof. Let $E(N) \cap Y = \{e\}$. By the Scum Theorem, there exists an independent set $C$ and coidependent set $D$ such that $N = M/C \setminus D$. First, $C \cap (cl(X) \cap cl(Y))$ is non-empty, otherwise $N$ is disconnected with $\{e\}$ as a component. We claim that $|C \cap Y| = r(Y) - 1$. Clearly $|C \cap Y| < r(Y)$ as $e$ is not a loop in $N$. Suppose $|C \cap Y| < r(Y) - 1$. Then $\text{si}((M|Y)/(C \cap Y)$ is a 2-separating set in $N$, a contradiction. Thus $M/(C \cap Y) = M_X^+$ and so $M_X^+$ contains an $N$-minor.

Contracting $Y$ down to its basepoint also preserves strands for sets of $X$.

**Proposition 2.4.5.** Let $(X, Y)$ be an exact 2-separation $(X, Y)$ in a matroid $M$, and let $N \subseteq X$. If an element $e \in N$ is distinguished in $M$, then $e$ is distinguished in $M_X^+$.

**Proof.** Let $S$ be a strand for $N$ in $M$ that distinguishes $e$. If $S \subseteq X$ then we are done, so assume that $S_Y = S \cap Y$ is non-empty and that $S_Y \not\subseteq \text{cl}(X)$ (otherwise $S_Y \in \text{cl}(p)$, where $p$ is the basepoint of the 2-separation). Suppose $|S_Y| = 1$, then $(S \setminus S_Y) \cup \{e\}$ is a subset of $X$ that spans $S_Y$, contradicting that $S_Y \not\subseteq \text{cl}(X)$. So $|S_Y| > 1$, and $|S \setminus S_Y| \geq 1$ since $e \not\in \text{cl}(p)$. However, this implies that $S \cup \{e\}$ is a circuit and $S_Y$ is 2-separating in $S \cup \{e\}$, contradicting that a circuit is 3-connected.

By applying Proposition 2.4.4, we can determine the structure of 2-separations in a matroid $M$ relative to a 3-connected minor $N$.

**Lemma 2.4.6.** Let $N$ be a 3-connected minor of a 2-connected matroid $M$ such that $|E(N)| \geq 4$. Then there exists a partition $B_1, \ldots, B_k$ of $E(M)$ such that:

(i) $\lambda_M(B_i) = 1$ for $i \in \{1, \ldots, k\}$,

(ii) $|E(N) \cap B_i| \leq 1$ for $i \in \{1, \ldots, k\}$, and

(iii) if $M''$ is the matroid obtained by contracting all $B_i$ down to their respective basepoints $e_i$, then $M''$ is 3-connected with groundset $\{e_1, \ldots, e_k\}$ with an $N$-minor.

**Proof.** Let $\mathcal{T}$ be the collection of 2-separating sets $B$ of $M$ such that $|E(N) \cap B| \leq 1$. We define the following binary relation $(E(M), \sim)$: for elements $e, f \in E(M)$, we let $e \sim f$ if there exists a 2-separation $(A, B)$ of $M$ such that $e, f \in B$ and $B \in \mathcal{T}$. We claim that $(E(M), \sim)$ is an equivalence relation. Clearly $(E(M), \sim)$ is reflexive and symmetric, so it remains to show that it is transitive. Let $e, f, g \in E(M)$ such that $(A, B)$ and $(A', B')$ are 2-separations of $E(M)$ such that $e, f \in B$, $f, g \in B'$, and $B, B' \in \mathcal{T}$. Now, by the uncrossing lemma, $B \cup B'$ is 2-separating. Since $|E(N)| \geq 4$, there are at least two elements
of $N$ in $A \cap A'$. So $(A \cap A', B \cup B')$ is a 2-separation of $M$. Then as $N$ is 3-connected, $B \cup B' \in \mathcal{T}$ since there can not be at least two elements of $N$ in both sides of a 2-separation. Thus $e \sim g$ as desired.

Let $B_1, \ldots, B_k$ be the partition of $E(M)$ formed by the equivalence classes of $(E(M), \sim)$. We want to show that this partition satisfies the properties (i), (ii), and (iii). Suppose $\lambda_M(B_i) > 1$ for some $i \in \{1, \ldots, k\}$, then $|B_i| \geq 2$. Let $B_i'$ be the union of all 2-separating sets in $\mathcal{T}$ that have non-empty intersection with $B_i$. Then $B_i \subseteq B_i'$ since $B_i$ is not 2-separating but $B_i'$ is 2-separating by the uncrossing lemma. Let $g \in B_i' \setminus B_i$. Then $(E(M) \setminus B_i', B_i')$ is a 2-separation of $M$ such that $e, g \in B_i'$ and $B_i' \in \mathcal{T}$, contradicting that $g \not \in B_i$. This proves (i).

To prove (ii), suppose that $B_i \not \in \mathcal{T}$, so let $n, n' \in E(N) \cap B_i$. Then there exists a 2-separation $(A, B)$ of $M$ such that $n, n' \in B$ and $B \in \mathcal{T}$, which is a contradiction as $|E(N) \cap B| \geq 2$. Finally, condition (iii) follows from applying Proposition 2.4.4 on each $B_i$. \hfill \Box

The following lemma describes the structure of a matroid with a chain of 2-separations that separate two elements $e, f$.

**Lemma 2.4.7.** Let $M$ be a 2-connected matroid with a set $S$ of $k$ elements and $e, f \in E(M) \setminus S$. If $M/s$ has two components where $e, f$ are in distinct components for all $s \in S$, then there exists an ordering $(s_1, \ldots, s_k)$ of $S$ and a partition $(A_0, \ldots, A_k)$ of $E(M) \setminus \{e, f\}$ such that, for all $i \in \{1, \ldots, k\}$, the sets $\{e, s_1, \ldots, s_i\} \cup A_0 \cup \ldots \cup A_{i-1}$ and $\{s_{i+1}, \ldots, s_k, f\} \cup A_i \cup \ldots \cup A_k$ are the components of $M/s$.

**Proof.** We proceed by induction on $k$. This is clear for $k = 1$, so let $k > 1$ and for some element $s \in S$, by applying the inductive hypothesis on $S \setminus \{s\}$ there exists an ordering $(s_1, \ldots, s_{i-1})$ and a partition $(A_0, \ldots, A_{i-1})$ satisfying the connectivity properties of the lemma. Now $s$ is an element of some $A_i$. Since $\{e, s_1, \ldots, s_{i-1}\} \cup A_0 \cup \ldots \cup A_{i-1}$ is a component in $M/s$ and does not contain $s$, it is connected in $M/s$ and its component contains $s_i$ since it spans $s_i$ in $M$. Similarly $\{s_{i+1}, s_{i+2}, \ldots, s_k, f\} \cup A_{i+1} \cup \ldots \cup A_k$ is connected in $M/s$. Since $M/s$ has 2 components, let $A_i'$ be the elements of $A_i$ in the component containing $\{e, s_1, \ldots, s_{i-1}\} \cup A_0 \cup \ldots \cup A_{i-1}$ and $A_i''$ be the elements of $A_i$ in the component containing $\{s_{i+1}, s_{i+2}, \ldots, s_k, f\} \cup A_{i+1} \cup \ldots \cup A_k$ (these components are distinct since $e, f$ are in different components of $M/s$).

Then $(s_1, \ldots, s_i, s, s_{i+1}, \ldots, s_k)$ and $(A_0, \ldots, A_{i-1}, A_i', A_i'' \cup A_{i+1}, A_k)$ satisfies the desired conditions. \hfill \Box

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2.5 Modularity/Lines

A set $X$ is modular if, for all flats $F$ in $M$, the following equation holds:

$$\cap (X, F) = r(F \cap X).$$

The proposition below (6.9.2 of [16]) gives a useful characterization of when a flat in a matroid is modular.

**Proposition 2.5.1.** A flat $X$ in a matroid $M$ is a modular flat if and only if $r(X) + r(Y) = r(X \cup Y)$ for all flats $Y$ such that $X \cap Y = \emptyset$.

In particular, if a flat $F$ is modular, then every strand for $F$ distinguishes an element in $F$.

Seymour proved the following theorem [19], which implies that any matroid with a modular 3-point line is binary:

**Theorem 2.5.2.** Let $M$ be a 3-connected non-binary matroid. Then for any two elements $e_1, e_2 \in E(M)$, $M$ contains a $U_{2,4}$-minor $N$ such that $e_1, e_2 \in E(N)$.

**Corollary 2.5.3.** Let $T$ be a 3-point line in a 3-connected matroid $M$. If $T$ is modular, then $M$ is binary.

**Proof.** Suppose, for a contradiction, that $M$ is not binary. Let $e_1$ and $e_2$ be elements of $T$. By Theorem 2.5.2, $M$ contains a $U_{2,4}$-minor $N$ such that $e_1, e_2 \in E(N)$. Let $e_4 \in E(N) \setminus \{e_1, e_2, e_3\}$. By Theorem 2.3.1, there exists an independent set $C$ and a coindependent set $D$ such that $N = M/C \setminus D$. Let $F = \text{cl}_M(C \cup \{e_4\})$. Then $\cap_M(F, T) = 1$, but $r_M(F, T) = 0$, contradicting that $T$ is modular. $\square$

In a 3-connected matroid $M$ with a modular line $L$, if $L$ is neither spanning nor cospanning, then we can find a minor $M'$ of $M$ such that there are 3 elements in a triangle $T$, each parallel to an element of $L$. We will prove a short proposition to aid the proof.

**Proposition 2.5.4.** Let $M$ be a 3-connected matroid on groundset $S \cup L$ such that $S$ is a circuit and $L$ is a line. If $e \in S \setminus \text{cl}(L)$, then $M/e$ is 3-connected.
Proof. Let $M' = M/e$, and $S' = S \setminus \{e\}$. Suppose, for a contradiction, that $(A,B)$ is a 2-separation of $M'$. Since $e \not\in L$, $L$ is a line in $M'$ and we may assume that $L \subseteq A$. Since contraction inside of a circuit preserves the circuit, $S'$ is a circuit in $M'$, at least $|S'|-1$ elements of $S'$ are in either $A$ or $B$. Since $|B| \geq 2$, we have $|S' \cap B| \geq |S'|-1$.

Then $\lambda_M(A) = r_M(A) + r_M(B \cup \{e\}) - r(M) = r_{M'}(A) + (r_{M'}(B) + 1) - (r(M') + 1) = r_{M'}(A) + r_{M'}(B) - r(M') = 1$. Since $|A| \geq 2$, $|B| \geq 2$, we have that $(A, B \cup \{e\})$ is a 2-separation in $M$, a contradiction.

**Lemma 2.5.5.** Let $M$ be a 3-connected matroid with a modular line $L$. If $E(M) \setminus L$ is non-empty and not a coline, then there exists sets $C \subset E(M) \setminus L$ and $\{x_1, x_2, x_3\} \subset E(M) \setminus L$ such that

- $\{x_1, x_2, x_3\}$ is a triangle in $M/C$, and
- each of $x_1, x_2, x_3$ is parallel to an element of $L$ in $M/C$.

**Proof.** First, $E(M) \setminus L$ contains a circuit. Supposing otherwise, then $E(M) \setminus L$ is independent so $L$ is cospanning, which implies that $E(M) \setminus L$ is a coline.

Let $S$ be a circuit of $E(M) \setminus L$, and since $M$ is 3-connected, we have $\kappa_M(L,S) = 2$. By Theorem 2.4.3 (Tutte’s Linking Theorem) there exists a minor $N$ of $M$ with groundset $L \cup S$, such that $\kappa_N(L,S) = 2$, $M|L = N|L$, and $M|S = N|S$. By Theorem 2.3.1 (Scum Theorem), we can write $N = M/C'\setminus D$ where $C'$ is independent.

Suppose $S = \{s_1, s_2, s_3\}$. Then $S$ is a triangle and all elements of $S$ lie on $L$ as $\kappa_N(L,S) = 2$. Then $C = C'$ satisfies the conditions. Otherwise, consider the sequence of matroids $N_1, N_2, \ldots N_{|S|-2}$ obtained as follows: set $N_1 = N$, then $N_i = N_{i-1}/f$ where $f_j$ is an element of $S$ that is not on $L$. By Proposition 2.5.4, each $N_i$ is 3-connected. Such elements exist since when $|S| \geq 3$, at most two elements of $S$ lie on $L$. Now let $S' = S \setminus \{f_1, \ldots, f_{|S|-3}\} = \{s_1, s_2, s_3\}$ which is a circuit (and thus a triangle) in $N_{|S|-2}$, with the three elements of $S'$ on $L$ by the 3-connectivity of $N_{|S|-2}$. Since $L$ is modular, each element of $S'$ is parallel to an element of $L$. Then $C = C' \cup \{s_1\} \cup \{f_1, \ldots, f_{|S|-3}\}$ satisfies the conditions, as desired.

Let $e_1, e_2, e_3$ be the elements of $L$ that are parallel to $x_1, x_2, x_3$ in $M/C$ of the previous lemma. Then $C \cap \{x_i, e_i\}$ is a circuit for $i \in \{1, 2, 3\}$ in $M$, and so a consequence is that $M$ distinguishes $e_i$ for $i \in \{1, 2, 3\}$.

**Corollary 2.5.6.** Let $L$ be a line in a 3-connected matroid $M$ such that $E(M) \neq E(L)$. If $L$ is modular, then at least 3 elements of $L$ are distinguished.
The following lemma describes the structure of a matroid with a line that has exactly two elements distinguished.

**Lemma 2.5.7.** Let $M$ be a matroid with a modular line $L$. If exactly two elements $e_1, e_2$ of $L$ are distinguished, then there exists a partition $(C, D)$ of $E(M) \setminus L$ such that $C$ and $D$ are 2-separating in $M$ with basepoints $e_1$ and $e_2$, respectively.

**Proof.** To partition $E(M) \setminus L$, we define the following binary relation $(E(M) \setminus L, \sim)$: for elements $e, f \in E(M) \setminus L$, we let $e \sim f$ if there exists a 2-separation $(A, B)$ of $M$ such that $e, f \in B$ and $L \subseteq A$. We claim that $(E(M) \setminus L, \sim)$ is an equivalence relation. Clearly it is reflexive and symmetric, so it remains to show that $(E(M) \setminus L, \sim)$ is transitive. Let $e, f, g \in E(M)$ such that $(A, B)$ and $(A', B')$ are 2-separations of $E(M)$ such that $e, f \in B$, $f, g \in B'$, $L \subseteq A$, and $L \subseteq A'$. Now, by the uncrossing lemma, $B \cup B'$ is 2-separating. So $(A \cap A', B \cup B')$ is a 2-separation of $M$ with $e, g \in B \cup B'$ and $L \subseteq A \cap A'$. Thus $e \sim g$ as desired.

Let $B_1, \ldots, B_k$ be the partition of $E(M) \setminus L$ formed by the equivalence classes under $(E(M) \setminus L, \sim)$. First we will show that $B_i$ is 2-separating for $i \in \{1, \ldots, k\}$. Suppose $\lambda_M(B_i) > 1$, then $|B_i| \geq 2$. Let $B_i'$ be the union of all 2-separating sets $B$ such that $B$ has non-empty intersection with $B_i$ and $B$ is disjoint from $L$. Then $B_i \subseteq B_i'$ since $B_i$ is not 2-separating but $B_i'$ is 2-separating by the uncrossing lemma. Let $g \in B_i' \setminus B_i$. Then $(E(M) \setminus B_i', B_i')$ is a 2-separation of $M$ such that $e, g \in B_i'$ and $L \subseteq E(M) \setminus B_i'$, contradicting that $g \not\in B_i$.

Now let $M'$ be the matroid obtained from $M$ by contracting each $B_i$ down to its base-point $b_i$ for $i \in \{1, \ldots, k\}$. Then $M \setminus L$ is simple and $\text{si}(M')$ is 3-connected. If there exists an element $b_i \in E(M')$ such that $b_i \not\in \text{cl}(L)$, then by Corollary 2.5.6, $\text{si}(M')$ distinguishes at least three elements of $L$, contradicting that $M$ distinguishes exactly two elements of $L$. Hence $b_i \in \text{cl}(L)$ for all $i \in \{1, \ldots, k\}$.

We claim that $k = 2$. Since $M$ distinguishes only $e_1$ and $e_2$, each $b_i$ is parallel to $e_1$ or $e_2$ as $L$ is modular. Suppose $b_1, b_2$ are both parallel to $e_1$. Then let $e \in B_1$ and $f \in B_2$. Then $(E(M) \setminus B_1 \setminus B_2, B_1 \cup B_2)$ is a 2-separation of $M$ where $e, f \in B_1 \cup B_2$ and $L \subseteq E(M) \setminus B_1 \setminus B_2$. So $e \sim f$, contradicting that $e, f$ are in different parts of the partition. Thus $k \leq 2$, and $k > 1$ otherwise only one element of $L$ is distinguished, establishing our claim.

Finally, we may assume that $\{b_1, e_1\}$ and $\{b_2, e_2\}$ are parallel pairs in $M'$. Then $C = B_1$ and $D = B_2$ satisfies the desired conditions. \qed
2.6 Generalized Parallel Connection

Let $M_1$ and $M_2$ be matroids such that $E(M_1) \cap E(M_2) = T$, $M_1|T = M_2|T = N$, and \( \text{si}(M_2|T) \) is a modular flat of \( \text{si}(M_2) \). Then the generalized parallel connection, \( P_N(M_1, M_2) \), is the matroid on groundset $E(M_1) \cup E(M_2)$ with its set of flats being the subsets $F$ such that $F \cap E_1$ and $F \cap E_2$ are flats of $M_1$ and $M_2$, respectively.

The following proposition (Proposition 11.4.15 of [16]) gives us a way to decompose a matroid along a $k$-separation.

**Proposition 2.6.1.** Let $M$ be a simple matroid with a subset of elements $T$ such that $M/T$ is the direct sum of $M_1$ and $M_2$. If $T$ is a modular flat of $M\setminus E(M_2)$, then $M = P_{M\setminus T}(M\setminus E(M_1), M\setminus E(M_2))$.

Our next lemma will show that, if there exist quinary representations for both sides of our ternary matroid with a $k$-separation, and they match along the guts, then our ternary matroid is also quinary.

**Lemma 2.6.2.** Let $(X,Y)$ be an exact $k$-separation in a simple matroid $M$, such that $T \subseteq \text{cl}(X) \cap \text{cl}(Y)$ has rank $k - 1$ and $T$ is a modular flat of $M \mid (Y \cup T)$. Let $B_T$ be a basis of $T$. Let $A_X$ and $A_Y$ be $\mathbb{F}$-representations of $M \mid (X \cup T)$ and $M \mid (Y \cup T)$ (respectively) of the form:

\[
A_X = B_X \begin{bmatrix} T \\ A_X' \\ 0 \\ 0 \\ A_T \\ B_T \end{bmatrix},
A_Y = B_Y \begin{bmatrix} T \\ A_T \\ 0 \\ 0 \\ A_Y' \\ B_Y \end{bmatrix}.
\]

Then

\[
A = B_X \begin{bmatrix} T \\ A_X' \\ 0 \\ 0 \\ A_T \\ B_T \\ 0 \\ 0 \\ A_Y' \end{bmatrix}
\]

is an $\mathbb{F}$-representation of $M$. 19
Proof. First we will show that $M[A] = P_{M\mid T}(X \cup T, Y \cup T)$ by showing their set of flats are equal. To ease notation, let $M' = P_{M\mid T}(X \cup T, Y \cup T)$, $F_X = F \cap (X \cup T)$, and $F_Y = F \cap (Y \cup T)$. Observe that $M'((X \cup T) = M[A]((X \cup T)$ and $M'((Y \cup T) = M[A]((Y \cup T)$.

Let $F$ be a flat of $M[A]$. As taking a restriction preserves flats, $F_X$ is a flat of $M[A]((X \cup T)$, and $F_Y$ is a flat of $M[A]((Y \cup T)$. Then $F_X, F_Y$ are flats of $M(((X \cup T)$ and $M'((Y \cup T)$, respectively. Hence $F$ is a flat of $M'$ by the definition of generalized parallel connection.

Conversely, let $F$ be a flat of $M'$ and suppose for a contradiction that $F$ is not a flat of $M[A]$. Then there exists $e \in cl_{M[A]}(F)$ such that $e \notin F$. Let $C$ be a circuit in $M[A]$ such that $C \setminus F = \{e\}$. Observe that $C \cap (X \setminus T)$ and $C \cap (Y \setminus T)$ are both non-empty, otherwise $C \subseteq X \cup T$ or $C \subseteq X \cup T$. Then $C \setminus \{e\} \subseteq F_Y$ or $C \setminus \{e\} \subseteq F_X$. Both cases imply that $e \in F$ as $F_Y$ is a flat of $M[A]((Y \cup T)$ and $F_X$ is a flat of $M[A]((X \cup T)$, a contradiction. Let $\vec{v}_1, \ldots, \vec{v}_i, \vec{v}_e$ denote the vectors corresponding to $C$ with $\vec{v}_e$ the vector labelled by $e$. Then $\vec{v}_e$ can be written as the linear combination $a_1 \vec{v}_1 + \ldots + a_i \vec{v}_i = \vec{v}_e$ where $a_1, \ldots, a_i \in GF(q_2) \setminus \{0\}$.

- **Case 1**: $e \in X \cup T$. By the above observation $C \cap (Y \setminus T) \neq \emptyset$. We may assume that $\vec{v}_1, \ldots, \vec{v}_j$ are labelled by elements of $C \cap (Y \setminus T)$, and let $\vec{v}_i = a_1 \vec{v}_1 + \ldots + a_i \vec{v}_i$. Since $\vec{v}_j, \ldots, \vec{v}_i, \vec{v}_e$ all have zero entries in the rows labelled by $B_Y$, $\vec{v}_e$ has only zero entries in those rows. Then the only non-zero entries of $\vec{v}_i$ are in the rows labelled by $B_T$. Since $T$ is a modular flat of $M((Y \cup T) = M[A]((Y \cup T)$, there exists an element $t \in T$ such that $\vec{v}_i$ is a scalar multiple of the column corresponding to $t$. So $C \cap (Y \setminus T)$ spans $t$, which implies that $t \in F$. As $t \in F$ and $e \notin F$, $t \neq e$. Then $a_1 \vec{v}_1 + a_j \vec{v}_j + a_{j+1} \vec{v}_{j+1} + \ldots + a_i \vec{v}_i = a_t \vec{v}_e$, so $(C \setminus (Y \setminus T)) \cup \{t\}$ is a subset of $X \cup T$ and contains a circuit involving $e$, contradicting our observation.

- **Case 2**: $e \in Y \setminus T$. Similar to above, we may assume that $\vec{v}_1, \ldots, \vec{v}_j$ are labelled by elements of $C \cap (Y \setminus T)$. Then $a_1 \vec{v}_1 + \ldots + a_j \vec{v}_j = \vec{v}_i$ for some vector $v_i$ that is in the span of the vectors of $T$. First suppose that there exists $t \in T$ such that $\vec{v}_i$ is a scalar multiple of the column corresponding to $t$. Then $C \setminus (X \setminus T) \cup \{t\}$ is a subset of $X \cup T$ and contains a circuit involving $e$, a contradiction. Next, suppose that there does not exist such an element $t$. Then as $a_e \vec{v}_e - a_{j+1} \vec{v}_{j+1} - \ldots - a_i \vec{v}_i = \vec{v}_e$, we have a strand of $Y \cup T$ that does not distinguish an element of $T$, contradicting that $T$ is modular.

As $T$ is the guts of a separation, $M/T$ is disconnected where $M/T$ is the direct sum of $X$ and $Y$. Applying Proposition 2.6.1 yields that $M = M'$, and thus $M = M[A]$.


2.7 Generalized $\Delta$-Y Exchange

The generalized $\Delta - Y$ exchange is an operation introduced by Oxley, Semple, and Vertigan [14] that transforms a line $L$ in a matroid into a coline in $M'$. The name is inspired from the $\Delta - Y$ exchange which replaces a triangle with a triad. This operation is described as follows: fix an integer $k \geq 2$ and let $\Theta_k$ be the unique matroid on groundset $\{a_1,\ldots,a_k,b_1,\ldots,b_k\}$ such that $\{a_1,\ldots,a_k\}$ is a modular line and $\{b_1,\ldots,b_k\}$ is an independent coline, and $\{b_1,\ldots,b_k\} \Delta \{a_i,b_i\}$ is a circuit for all $i \in \{1,\ldots,k\}$. Now let $M$ be a matroid with a $U_{2,k}$-restriction $A = \{a_1,\ldots,a_k\}$ such that $A$ is coindependent (this last condition is imposed for duality to work). Then the generalized $\Delta - Y$ exchange on $A$ of size $k$, denoted $\Delta A(M)$, is defined to be the generalized parallel connection $P_A(\Theta_k,M) \setminus A$.

To preserve the groundset of $M$ in $\Delta A(M)$, relabel $b_i$ with $a_i$ for all $i \in \{1,\ldots,k\}$. Geometrically, the $\Delta - Y$ exchange replaces a line in a matroid by a coline, as shown in the following lemma (Lemma 2.5 of [14]).

**Lemma 2.7.1.** For all $k \geq 2$, the restriction of $(P_A(\Theta_k,M) \setminus A)^*$ to $B$ is isomorphic to $U_{2,k}$ if and only if $A$ is coindependent in $M$.

This result yields a dual operation, the $Y - \Delta$ exchange. Let $M$ be a matroid with an independent set $A$ such that $M^*|A$ is isomorphic to $U_{2,k}$. Then the $Y - \Delta$ exchange on $A$ is defined as $\nabla A(M) = [P_A(\Theta_k,M^*) \setminus A]^*$, and Lemma 2.11 and Corollary 2.12 of [14] proves that it is the inverse of the $\Delta - Y$ exchange.

**Lemma 2.7.2.** Let $A$ be a coindependent set in a matroid $M$ where every 3 element subset of $A$ is a circuit. Then $\nabla A(\Delta A(M))$ is well-defined and $\nabla A(\Delta A(M)) = M$.

**Lemma 2.7.3.** Let $A$ be an independent set in a matroid $M$ where every 3 element subset of $A$ is a cocircuit. Then $\Delta A(\nabla A(M))$ is well-defined and $\Delta A(\nabla A(M)) = M$.

**Lemma 2.7.4.** Suppose that $\nabla A(M)$ is defined. If $x \in A$ and $|A| \geq 3$, then $\nabla A\setminus\{x\}(M/x)$ is also defined and $\nabla A(M)\setminus x = \nabla A\setminus\{x\}(M/x)$

The $\Delta - Y$ exchange also preserves representability over any field $\mathbb{F}$, as Corollary 3.7 of [14] shows. Note that the original lemma states the result for partial fields, as every field is a partial field.

**Lemma 2.7.5.** Let $M$ be a matroid with a coindependent $U_{2,k}$-restriction $A$. Then $M$ is $\mathbb{F}$-representable if and only if $\Delta A(M)$ is $\mathbb{F}$-representable.
Chapter 3

Main Result

In this chapter we will prove our main theorem. Recall that $O_8$ is the following matroid:

![Diagram of O_8](image)

Figure 3.1: The matroid $O_8$

**Theorem 3.0.1** (Theorem 3.8.1). Let $M$ be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation $(X, Y)$ such that $|X|, |Y| \geq 4$. Then either:

- $M$ is isomorphic to $T_8$,
- $M_X$ is isomorphic to $O_8$ and $M_Y$ contains an excluded minor for $GF(5)$-representability with up to two fewer elements than $M_Y$,
- $M_Y$ is isomorphic to $O_8$ and $M_X$ contains an excluded minor for $GF(5)$-representability with up to two fewer elements than $M_X$,
- $|X| = 4$, $M_X$ is not isomorphic to $O_8$, and $M_Y$ is an excluded minor for $GF(5)$-representability, or
- $|Y| = 4$, $M_Y$ is not isomorphic to $O_8$, and $M_X$ is an excluded minor for $GF(5)$-representability.
3.1 Adding guts

Let $M$ be a ternary matroid with an exact 3-separation $(X,Y)$. In order to decompose $M$ into two matroids corresponding to each side of the 3-separation, we will construct an auxiliary matroid $M'_{X,Y}$ where $X$ and $Y$ span a common 4-point line. As mentioned previously, adding elements arbitrarily on a line of a matroid is not well-defined. However, we can take advantage of the unique representability of ternary matroids to do this for $M$.

Let $M$ have rank $n$ and let $A$ be a $GF(3)$-representation of $M$ (by removing redundant rows, we may assume $A$ has $n$ rows). We can view $M$ as a restriction of $PG(\binom{n-1}{3},3)$ by taking the columns of the $GF(3)$-representation of $PG(n-1,3)$ which match the columns of $A$. This is well-defined since projective geometries are uniquely representable (Fundamental Theorem of Projective Geometry) combined with the fact that ternary matroids are uniquely representable [2].

Hence let $G = PG(n-1,3)$ and $E(M)$ is a spanning subset of $E(G)$ such that $M = G|E(M)$. Now consider the boundary $B = cl_G(X) \cap cl_G(Y)$, which has rank two. As $G$ is a projective geometry, $G|B \cong PG(1,3) \cong U_{2,4}$. Let $G'_{B}$ be the matroid obtained from $G$ by taking four extensions $\{e_1,e_2,e_3,e_4\}$ parallel to the four elements of $B$.

We then define the matroid $M'_{X,Y} = G'_{B}|(E(M) \cup \{e_1,e_2,e_3,e_4\})$. We call $L = \{e_1,e_2,e_3,e_4\}$ the boundary line of $(X,Y)$. Note while $L$ may not be a line in $M'_{X,Y}$ if $M'_{X,Y}$ is not simple, we nevertheless refer to it as a line for convenience. We also define the matroids corresponding to $X$ and $Y$ as $M_X = M'_{X,Y}|(X \cup L)$ and $M_Y = M'_{X,Y}|(Y \cup L)$, respectively. It is clear that $M'_{X,Y} = P_L(M_X,M_Y)$, and that $M'_{X,Y}\backslash L,M_X\backslash L,M_Y\backslash L$ are simple. Moreover, each of $si(M'_{X,Y}),si(M_X),si(M_Y)$ are 3-connected.

3.2 Pinning the points

Let $(X,Y)$ be a 3-separation in a representable matroid, and let $B$ be the set of guts elements of $(X,Y)$. We say that an element $e \in B$ is pinned with respect to $B$ if $M_X$ and $M_Y$ distinguish $e$. In this section we will prove that, in our excluded minor, all four elements of $L$ are pinned with respect to $L$. The importance of this is highlighted by the next lemma, which shows that adding a pinned element of the line implies that does not change its representability over $GF(5)$.

**Lemma 3.2.1.** Let $M$ be a ternary matroid with an exact 3-separation $(X,Y)$. Let $M'$ be a ternary matroid obtained from $M$ via an extension by a non-loop element $e$ in the
guts of \((X,Y)\). If there exists strands \(S_X \subseteq X\) for \(Y\) and \(S_Y \subseteq Y\) for \(X\) such that \(e \in \cl(S_X) \cap \cl(S_Y)\), then \(M\) is \(\mathbb{F}\)-representable if and only if \(M'\) is \(\mathbb{F}\)-representable.

**Proof.** The “if” direction is clear, as \(M\) is a restriction of \(M'\). Conversely, suppose that \(A\) is an \(\mathbb{F}\)-representation of \(M\). We may assume that \(A\) is of the form

\[
A = \begin{bmatrix}
A_X & 0 \\
0 & A_Y
\end{bmatrix}
\]

where \(A_X\) is a representation of \(A|X\) and \(A_Y\) is a representation of \(A|Y\), and there are exactly two rows that intersect \(A_X\) and \(A_Y\). Let \(r_1, r_2\) denote these two rows.

We have \(\cap_M(S_X, S_Y) = 1\), so the subspaces spanned by the columns of \(S_X\) and \(S_Y\) in \(A\) intersect at a point. Let \(\vec{v}\) be the vector representing this point. Since both \(S_X\) and \(S_Y\) span \(\vec{v}\), its only non-zero entries are in \(r_1\) and \(r_2\). Clearly \(\vec{v}\) must have at least one non-zero entry.

Now let \(A'\) be the matrix where \(\vec{v}\) is added to \(A\). We claim that \(M[A'] = M'\). Suppose, for a contradiction, that \(M[A'] \neq M\) and let \(\{M_1, M_2\} = \{M[A'], M'\}\) such that \(C\) is a circuit in \(M_1\) and independent in \(M_2\). Since \(M_1\setminus e = M_2\setminus e\), we have \(e \in C\) and let \(C^- = C \setminus \{e\}\). Let \(L\) be the boundary line of \((X,Y)\) in \(M_{X,Y}^+\). Observe that \(\cap_M(C^-, L) \neq 2\), otherwise \(C^-\) spans \(e\) in \(M_2\), contradicting that \(C\) is independent in \(M_2\). Similarly \(\cap_M(C^-, L) \neq 0\), otherwise \(C^-\) does not span \(e\) in \(M_1\). Hence \(\cap_M(C^-, L) = 1\), and we claim that either \(C^- \cap X\) or \(C^- \cap Y\) is empty. Suppose otherwise, and we have that one of \(\cap_M(C^- \cap X) = 0\) or \(\cap_M(C^- \cap Y) = 0\) as \(\cap_M(C^-, L) \neq 2\) and \(C\) is independent in \(M_2\). We may assume that \(\cap_M(C^- \cap X) = 0\) and so \((C^- \cap X) \cup \{e\}\) is a circuit in \(M_1\). Thus \(C^- \cap X = \emptyset\) as \(C^- \cup \{e\}\) is a circuit in \(M_1\). So \(C^-\) is a strand of \(X\) that spans \(e\) in \(M_1\), so \(\cap_M(C^-, S_X) = 1\). Then \(e\) is in the span of \(C^-\) in both \(M'\) and \(M[A']\), contradicting that \(C\) is independent in \(M_2\).

By applying Lemma 3.2.1 to each element of \(L\) in \(M_{X,Y}^+\), we get that \(M\) is quinary if and only if \(M_{X,Y}^+\) is quinary.

**Corollary 3.2.2.** Let \(M\) be a ternary matroid with an exact 3-separation \((X,Y)\). Let \(L\) be the boundary line of \((X,Y)\) in \(M_{X,Y}^+\). If all elements of \(L\) are pinned, then \(M\) is \(GF(q)\)-representable if and only if \(M_{X,Y}^+\) is \(GF(q)\)-representable.
Proof. Let $e \in L$. Since $e$ is pinned, $M_X$ and $M_Y$ distinguish $X$. Then $X$ contains a strand $S_X$ for $L$ in $M^+_{X,Y}$ that distinguishes $e$, which is also a strand for $Y$. Similarly $Y$ contains a strand $S_Y$ for $X$ that distinguishes $e$. Then applying Lemma 3.2.1 to $M$ and $e$ yields that $M$ is $GF(q)$-representable if and only if $M^+_{X,Y} \setminus (L \setminus \{e\})$ is $GF(q)$-representable. Applying this argument (along with Lemma 3.2.1) three more times for the other three elements of $L$ yields the result.

So we want to see whether all the elements of $L$ are pinned in $M^+_{X,Y}$. If this is not the case, and if neither $X,Y$ is a line nor a coline, our main result of this section shows that $M$ is in fact quinary.

Lemma 3.2.3. Let $M$ be a 3-connected ternary matroid whose proper minors are all quinary and let $(X,Y)$ be an exact 3-separation of $M$ with $|X| \geq 4$. If $M_X$ does not distinguish all four elements in the boundary line, then $M$ is quinary.

Proof. Suppose that $L = \{e_1, e_2, e_3, e_4\}$ is the boundary line where $e_4$ is not distinguished in $M_X$. Let $M'_X = M_X \setminus e_4$ and $M'_Y = M_Y \setminus e_4$.

Observe that $\{e_1, e_2, e_3\}$ is a modular line in $si(M'_X)$. By applying Corollary 2.5.3, we get that $si(M'_X)$ is binary. Then $M'_X$ is binary and ternary, so by Theorem 2.1.1 we have that $M'_X$ is regular. Thus $M'_X$ is quinary, and we want to show that $M_Y$ is quinary.

Since $|X| \geq 4$ and $M_X$ does not distinguish all elements of $L$, $X$ is not a coline. Further, $si(M'_X)$ is 3-connected with the modular line $\{e_1, e_2, e_3\}$, so we can apply Lemma 2.5.5 to obtain the sets $C \subseteq X$ and $\{x_1, x_2, x_3\} \subseteq X$ such that in $M'_X / C$ we have that $\{x_1, x_2, x_3\}$ is a triangle where $x_i$ is parallel to $e_i$ for $i \in \{1, 2, 3\}$. Observe that $M / C$ is quinary since it is a minor of $M$, and $si(M / C) = si((M^+_{X,Y} \setminus e_4) / C) = si(M'_Y)$ so $M'_Y$ is quinary.

So both $M'_X$ and $M'_Y$ are quinary, and by applying row operations there exists $GF(5)$ matrices $A_X$ and $A_Y$ where $e_1, e_2$ label the last two rows of $A_X$ and the first two rows of $A_Y$. Further by column scaling we may assume that the column representing $e_3$ has entry 1 in the rows corresponding to $e_1, e_2$ and zeroes elsewhere.

$$A_X = \begin{bmatrix} * & e_1 & e_2 & e_3 \\ 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_Y = \begin{bmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & * \end{bmatrix}$$
Then $A_X, A_Y$ match along the set $T = \{e_1, e_2, e_3\}$ which is a modular flat in $X$. Applying Lemma 2.6.2 yields a $GF(5)$-representation of $P_T(M'_X, M'_Y)$, which has $M$ as a restriction by deleting $\{e_1, e_2, e_3\}$, thus $M$ is quinary as desired.

It immediately follows that $M^+_X$ must pin all four elements of its boundary line.

**Corollary 3.2.4.** Let $M$ be a ternary excluded minor for $GF(5)$-representability and let $(X, Y)$ be an exact 3-separation of $M$. If $|X|, |Y| \geq 4$, then $M^+_X$ pins all elements of the boundary line.

### 3.3 Duality with Generalized $\Delta - Y$ Exchange

Since duality preserves representability and connectivity, if $M$ is a ternary excluded minor for $GF(5)$-representability with an exact 3-separation $(X, Y)$, then so is $M^*$. We will establish the relationship between $M_X$ and $M^*_X$ in this section, using the Generalized $\Delta - Y$ Exchange in Section 2.7.

**Lemma 3.3.1.** Let $M$ be a 3-connected ternary matroid with an exact 3-separation $(X, Y)$, and let $L$ label the boundary lines of $M^+_X Y$ and $(M^*)_X Y$. Then $(M^*)_X = \nabla_L((M_X)^*)$.

**Proof.** Observe that both matroids have the groundset $X \cup L$. Clearly $L$ is isomorphic to $U_{2,4}$ in both matroids. Hence, to establish the equality of the two matroids, we must show first that they agree on the rank of every subset $A$ of $X$, then show the equality of local connectivity from $A$ to $L$, and finally show that if $\cap(A, L) = 1$ then $A$ spans $e \in L$ in $(M^*)_X$ if and only if $A$ spans $e$ in $\nabla_L((M_X)^*)$.

Let $A \subseteq X$ and $E = \mathcal{E}(M)$. Since neither deleting outside $A$ nor extensions change the rank of $A$,

$$r_{(M^*)_X}(A) = r_{(M^*)_X}(A) = r_{M^*}(A)$$

By the dual rank function, we have

$$r_{(M^*)_X}(A) = r_M(E \setminus A) + |A| - r(M).$$
Since $Y \subseteq E \setminus A$ and $Y$ spans $L$ (as $M$ is 3-connected), $r_M(E \setminus A) = r_{M^+_{X,Y}}(E \cup L \setminus A)$ and $r(M) = r(M^+_{X,Y})$. Clearly $|A|$ is not changed, which implies

$$r_{(M^*)_X}(A) = r_{M^+_{X,Y}}((E \cup L) \setminus A) + |A| - r(M^+_{X,Y}) = r_{(M^*)_{X,Y}}(A).$$

Observe that we obtain $M_X$ from $M^+_{X,Y}$ by contracting elements in $Y$ and then simplifying. Both operations keep $r_{M^+_{X,Y}}(E(L) \setminus A) - r(M^+_{X,Y})$ invariant, so

$$r_{(M^*)_X}(A) = r_{(M_X)^*}(A).$$

It is clear that performing a $Y - \Delta$ exchange does not change the rank of a set disjoint from $L$, thus

$$r_{(M^*)_X}(A) = r_{\nabla_L(M_X)^*}(A)$$

as desired.

Next we show that $\nabla_{(M^*)_X}(A,L) = \nabla_{(M_X)^*}(A,L)$. Since deleting outside of $A \cup L$ does not change the local connectivity, we have

$$\nabla_{(M^*)_X}(A,L) = \nabla_{(M^*)_{X,Y}}(A,L)$$

$$= \nabla_{M^*}(A,Y).$$

Now by applying the definition of local connectivity,

$$\nabla_{(M^*)_X}(A,L) = r_{M^*}(A) - r_{M^*/Y}(A)$$

$$= r_{M^*}(A) - r_{((M^*_{X,Y})^*/L)/Y}(A)$$

$$= r_{M^*}(A) - r_{((M^*_{X,Y})^*/Y/L}(A).$$

Since $M_X = M^+_{X,Y} \setminus Y$, $(M_X)^* = (M^+_{X,Y})^*/Y$ and so

$$\nabla_{(M^*)_X}(A,L) = r_{M^*}(A) - r_{(M_X)^*/L}(A).$$
We proved above that \( r_{M^*}(A) = r_{(M_X)^*}(A) \), thus

\[
\cap_{(M^*)_X}(A, L) = r_{(M_X)^*}(A) - r_{(M_X)^*/L}(A) = \cap_{(M_X)^*}(A, L).
\]

It remains to show that for all \( A \) such that \( \cap(A, L) = 1 \) we have that \( A \) spans \( e \in L \) in \( (M^*)_X \) if and only if \( A \) spans \( e \) in \( \nabla_L((M_X)^*) \). Observe that we may assume that the elements \( \{e_1, e_2, e_3, e_4\} \) of \( L \) are labelled such that if there exists a set \( A \) such that \( \cap(A, L) = 1 \) and \( \cap_{(M_X)^*}(A, L \setminus \{e_i\}) = 1 \) then \( A \) distinguishes \( e_i \) in \( (M^*)_X \). Then equality follows by the construction of \( \nabla_L((M_X)^*) \).

So this lemma allows us to dualize and construct the matroids for the two sides of the dual. We want the properties for \( M_X \) to also hold for \( \nabla_L((M^*)_X) \), specifically that this preserves representability and distinguishing the points of \( L \). That representability is preserved follows directly from the previous lemma and Lemma 2.7.5.

**Lemma 3.3.2.** Let \( (X, Y) \) be an exact 3-separation in a 3-connected ternary matroid \( M \). For any field \( \mathbb{F} \), \( (M^*)_X \) is \( \mathbb{F} \)-representable if and only if \( M_X \) is \( \mathbb{F} \)-representable.

**Proof.** Let \( L \) be the boundary line of \((X, Y)\). As duality preserves representability, \( M_X \) is \( \mathbb{F} \)-representable if and only if \( (M_X)^* \) is \( \mathbb{F} \)-representable. Now \( L \) is an independent co-line in \( co((M_X)^*) \) since it is a coineendent line in \( si(M_X) \). Then by the dual of Lemma 2.7.5, \( co((M_X)^*) \) is \( \mathbb{F} \)-representable if and only if \( co(\nabla_L((M_X)^*)) \) is \( \mathbb{F} \)-representable. Since cosimplication preserves representability, applying Lemma 3.3.1 gives the result.

**Lemma 3.3.3.** Let \( (X, Y) \) be an exact 3-separation in a 3-connected ternary matroid \( M \). Then all four elements of the boundary line is pinned in \( M_{X,Y}^+ \) if and only if all four elements of the boundary line is pinned in \( (M^*)_{X,Y}^+ \).

**Proof.** Let \( L = \{a_1, a_2, a_3, a_4\} \), \( L' \) be the boundary lines of \( M_{X,Y}^+ \) and \( (M^*)_{X,Y}^+ \), respectively. Suppose that all four elements of \( L \) are pinned in \( M_{X,Y}^+ \). We want to show that if \( M_X \) distinguishes \( a_i \) then \( (M^*)_X \) distinguishes \( a_i \) for \( i \in \{1, \ldots, 4\} \).

Recall that \( \{a_1, a_2, a_3, a_4\} \) is a 4-point line in \( \Theta_4 \) and consider \( M_X^+ = P_{\{a_1, a_2, a_3, a_4\}}(\Theta_4, M_X) \), and observe that \( \Delta_L(M_X) = M_X^+ \setminus L \). Since \( M_X \) distinguishes \( a_1 \), there exists a circuit \( C \) with \( cl(C) \cap cl(L) = \{a_1\} \). Then \( cl(C) \) is a cyclic flat of \( M_X \), and let \( F = cl(C) \cup \{b_2, b_3, b_4\} \). \( F \) is also a cyclic flat of \( M_X^+ \) since \( \{a_1, b_2, b_3, b_4\} \) is a circuit of \( M_X \) and adding \( \{b_2, b_3, b_4\} \) to \( cl(C) \) does not introduce any of \( b_1, a_2, a_3, a_4 \) into its span. Now since \( C \cup \{b_2, b_3, b_4\} \) is
a circuit, $F \setminus \{a_1\}$ is a cyclic flat of $M_X^\lor \setminus a_1$. Since deleting outside a cyclic flat preserves the cyclic flat, $F \setminus \{a_1\}$ is a cyclic flat of $M_X^\lor \setminus a_1, a_2, a_3, a_4$. Observe that by relabelling $b_i$ to $a_i$ for $i \in \{1, \ldots, 4\}$ in $M_X^\lor \setminus a_1, a_2, a_3, a_4$, we have $M_X^\lor \setminus a_1, a_2, a_3, a_4 = \Delta_L(M_X)$.

Now $F^* = \Delta_L(M_X) \setminus F$ is a cyclic flat in $(\Delta_L(M_X))^*$ and observe that $F^*$ intersects the 4-point line $\{a_1, a_2, a_3, a_4\}$ at exactly $\{a_1\}$. Hence there exists a circuit $C^*$ in $(\Delta_L(M_X))^*$ that intersects $\{a_1, a_2, a_3, a_4\}$ at exactly $\{a_1\}$, so $a_1$ is distinguished in $(\Delta_L(M_X))^*$. Now $(\Delta_L(M_X))^* = \nabla_L((M_X)^*)$, and $\nabla_L((M_X)^*) = (M^*)_X$ by Lemma 3.3.1 as desired. Applying the same argument for $a_2, a_3, a_4$ yields the result. 

So whenever we dualize, we may assume that $(M^*)_X$ distinguishes all points of $L^*$, and $(M^*)_X$ is representable over the same fields that $M_X$ is.

### 3.4 Non-quinary side

In this section, we will show that, in an excluded minor, if $M_Y$ is not quinary, then $|X| \leq 4$. We begin by showing that if one side is not quinary, then the other side must have size at most 4.

**Lemma 3.4.1.** Let $M$ be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation $(X, Y)$. If $M_Y$ is non-quinary, then there is no proper minor of $M_X'$ of $M_X$ such that $M|_L = M'|_L$ and all four elements of $L$ are distinguished in $M_X'$.

**Proof.** Suppose there exists a proper minor $M_X'$ of $M_X$ such that $L$ is a 4-point line with all elements distinguished in $M_X'$. Let $N^+ = P_L(M_X', M_Y)$. Then $N^+$ is not quinary since it contains $M_Y$ is a restriction. Since $M_X'$ distinguishes all four elements of $L$ in $N^+$, by Lemma 3.2.1 we have that $N^+ \setminus L$ is not quinary. However, $N^+ \setminus L$ is a proper minor of $M$, so this contradicts that $M$ is an excluded minor. 

Hence our problem becomes characterizing the 3-connected matroids $M$ with a 4-point line $L$ where all elements of $L$ are distinguished, but every proper minor that keeps $L$ does not distinguish $L$. Moreover, by dualizing with the Generalized Delta-Y, $\nabla_L(M^*)$ has the same properties, so we may switch between $M$ and $\nabla_L(M^*)$.

**Lemma 3.4.2.** Let $L$ be a modular set in a ternary matroid $M$ such that $M|_L$ is isomorphic to $U_{2,4}$. If all four elements of $L$ are distinguished, then $M$ contains an 8-element minor $N$ such that $M|_L = N|_L$ and all elements of $L$ are distinguished in $N$. 

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Proof. Suppose that $M$ is a minor-minimal counterexample. Then $|E(M)| \geq 9$. We prove the result via a series of claims.

(1) $M$ is 3-connected.

First we show that $M \setminus L$ is simple. Suppose $\{e, e'\}$ is a parallel pair in $M \setminus L$. So if $S$ is a strand that contains $e$ and distinguishes $e_1 \in L$, then $S \Delta \{e, e'\}$ is a strand distinguishing $e_1$. Hence $M \setminus e$ distinguishes all four elements of $L$, contradicting minimality of $M$.

Next we will show that $\text{si}(M)$ is 3-connected. Let $\text{si}(M) = M \setminus P$ where $P \cap L = \emptyset$, and let $(A, B)$ be a 2-separation of $\text{si}(M)$ such that $L \subseteq B$. We may assume that this separation is exact, otherwise $L \setminus A$ contradicts minimality. Then by Proposition 2.4.5, $M_B^+$ is a minor of $\text{si}(M)$ and preserves strands for $L$. By Theorem 2.3.1, $M_B^+ = \text{si}(M)/C \setminus D = M/C \setminus D^P$ for some $C, D \subseteq E(M)$ where at least one of $C, D$ is non-empty. Then $M/C D$ distinguishes all four elements of $L$ and contradicts minimality.

To prove that $M$ is 3-connected, it remains to show that $M$ is simple. Let $e$ be an element parallel to $e_4 \in L$, and let $L \setminus e_4 = \{e_1, e_2, e_3\}$. Then by minimality of $M$, $M \setminus e$ distinguishes $e_1, e_2, e_3$ but not $e_4$. If $E(M) \setminus L \setminus e$ is a line or coline that distinguishes $e_1, e_2, e_3$, then the result is immediate. Since $\text{si}(M \setminus e)$ is 3-connected and $E(M) \setminus \text{cl}(L)$ is not a coline, by Lemma 2.5.5 we have that there is minor $M'$ of $\text{si}(M)$ such that $M'$ contains a triangle $\{x_1, x_2, x_3\}$ parallel to $\{e_1, e_2, e_3\}$. Then adding $e$ to $M'$ is a parallel extension of $M'$ at $e_4$ which distinguishes all elements of $L$, contradicting minimality of $M$ and proving (1).

Note that by Lemma 3.3.3, $\nabla_L(M^*)$ is also a minimal counterexample. Then by duality, (1) implies that there are no elements in the coguts of $(L, E(M) \setminus L)$, as $\nabla_L(M^*)$ would not be cosimple.

(2) $M/L$ and $M \setminus L$ is connected.

Let $C$ be a component of $M/L$. Suppose for a contradiction that $C \neq M/L$. Consider $M' = M|(C \cup L)$. Since $C$ is a component of $M/L$, $\lambda_M(E(C)) = 2$ and $C$ spans $L$, so $\text{si}(M')$ is 3-connected. By Corollary 2.5.6, we may assume that $M'$ distinguishes $e_1, e_2, e_3$ of $L$. Note that by minimality of $M$, $M'$ does not distinguish $e_4$. Since $e_4$ is distinguished in $M$, there exists some component $C'$ of $M/L$ containing a strand $S$ that distinguishes $e_4$. Then we can find a set of $|S| - 1$ elements of $S$ to contract to obtain a minor of $M''$ of $M$ with an element $e$ parallel to $e_4$. Then $N = M''|(C \cup L \cup \{e\})$ is a parallel extension of $M'$ at $e_4$, contradicting the minimality of $M$, so $M/L$ is connected.
Now by duality, $\nabla_L(M^*)/L$ is connected. Since dualizing preserves connectivity, $\Delta_L(M)\setminus L$ is connected, which implies that $M \setminus L$ is connected, proving (2).

(3) For every $e \notin L$, the line $L$ has exactly three distinguished elements in $M\setminus e$.

Consider $M\setminus e$. Since $\si(M)\setminus e$ is 2-connected, at least one element of $L$ is distinguished. First note $M\setminus e$ distinguishes more than one element of $L$, as otherwise $e$ is in the coguts of $(L, E(M) \setminus L)$. So suppose that exactly two elements of $L$ are distinguished in $M\setminus e$. Applying Lemma 2.5.7, there exists a partition $(C, D)$ of $E(M\setminus e) \setminus L$ such that $C$ and $D$ are 2-separating in $M\setminus e$ with basepoints $e_1$ and $e_2$. Then $(M\setminus e)/L$ is disconnected. Applying the dual argument to $\nabla_L(M^*)\setminus e$, we get that $(\nabla_L(M^*)\setminus e)/L$ is disconnected. As duality preserves connectivity, $\Delta_L(M)\setminus L/e$ is disconnected. Then $M/L/e$ and $M/L\setminus e$ are disconnected, a contradiction and establishing (3).

Consider $M' = M\setminus e$. We may assume that $M'$ not distinguish some $e_4 \in L$, as otherwise we contradict the minimality of $M$. In particular, this implies that every set $S$ of $M$ such that $\cl(L) \cap \cl(S) = e_4$ must contain $e$. Note that $e \in S$. Let $S \subseteq E(M) \setminus L$ be a maximal set whose closure contains $e_4$ but not $L$. Note that $S \cup L$ is spanning, as any element outside of $\cl(S \cup L)$ can be added to $S$.

(4) There exist distinct elements $e_1, e_2 \in L\setminus\{e_4\}$ such that for every element $f \in (S\setminus \{e\})$, there exists a partition $(C_f, D_f)$ of $E(M'\setminus f) \setminus L$ such that $C_f, D_f$ are 2-separating in $M'\setminus f$ with basepoints $e_1, e_2$ (respectively).

Let $f \in S\setminus \{e\}$. First we want to show that $M'\setminus f$ distinguishes exactly two elements of $L$. Suppose $M'\setminus f$ distinguishes all three elements of $L\setminus \{e_4\}$. Then $M'\setminus f$ also distinguishes $L \setminus \{e_4\}$, but since $S \setminus \{f\}$ is a strand of $L$ that distinguishes $e_4$ in $M'\setminus f$, this contradicts the minimality of $M$. Next, suppose that $M'\setminus f$ distinguishes only one element of $L \setminus \{e_4\}$. Then $(L, E(M) \setminus L)$ is a 2-separation of $M'\setminus f$ and since $L$ is a line in $M'$, we have that $(L, E(M) \setminus L)$ is a 2-separation of $M'$, contradicting (3). Then applying Lemma 2.5.7 gives us the desired partition $(C_f, D_f)$, and it remains to show that the basepoints are invariant over the choices of $f$. Let $e_3$ be the element of $L \setminus \{e_4\}$ that is not distinguished in $M'\setminus f$.

Suppose, for a contradiction, that there exists $f' \in (S \setminus \{e\})$ such that $M'\setminus f'$ distinguishes $e_3$. We may assume that $f' \in C_f$. Then $e_1 \in \cl_{M'}(C_f \cup \{f\})$, so $M'\setminus f'$ distinguishes $e_1$. Since $\cap_{M'}(L, D_f \cup \{f\}) = 1$ and $e_2 \in \cl_{M'}(D_f \cup \{f\})$ but $e_2$ is not distinguished in $M'\setminus f'$, so $\cap_{M'\setminus f'}(L, D_f \cup \{f\}) = 2$. Hence $f' \in \cl_{M'}(D_f \cup \{f\} \cup L) = \cl_{M'}(D_f \cup \{f, e_1\})$. Since $f' \subset C_f$ and $C_f$ is 3-separating in $M'$ with $f, e_1$ in the guts of $(C_f, E \setminus C_f)$, it follows that $f'$ also in the guts of $(C_f, E \setminus C_f)$. So $f, e_1, f'$ is a triangle in $M'$. However, this contradicts that $\cap_M(S, L) = 1$ since $e_1, e_4 \in \cl(S)$. This proves (4).
(5) For every element $f \in (S \setminus \{e\})$, $(C_f \cup \{e_1\}, D_f \cup \{e_2\})$ is a 2-separation of $M' \setminus e_3, e_4$ with $f$ in the guts.

Let $f \in S \setminus \{e\}$, and $C = C_f \cup \{e_1\}$, $D = D_f \cup \{e_2\}$. Since $C_f$ and $D_f$ are 2-separating sets in $M'/f$ by (4), $C$ and $D$ are 3-separating in $M'$ with $\{e_1, f\}$ and $\{e_2, f\}$ in the respective guts. Since $C$ is 3-separating in $M'$, $r(C) + r(D \cup \{e_3, e_4\}) - r(M') = 2$. As $e_1, e_2$ spans both $e_3$ and $e_4$, $r(M') = r(M' \setminus e_3, e_4)$. Since $D$ contains $e_2$ but spans neither $e_3$ nor $e_4$, $r(D) = r(D \cup \{e_3, e_4\}) - 1$. Hence $r(C) + r(D) - r(M' \setminus e_3, e_4) = 1$, and clearly $|C|, |D| \geq 2$, proving (5).

Applying Lemma 2.4.7 to $M' \setminus e_3, e_4, S \setminus \{e\}$ and the special elements $e_1$ and $e_2$, we get an ordering $(s_1, \ldots, s_k)$ of $S \setminus \{e\}$ and a partition $(A_0, \ldots, A_k)$ of $E(M') \setminus S \setminus L$ satisfying the conditions of Lemma 2.4.7.

(6) $|S \setminus \{e\}| = 1$.

Suppose $k > 1$. Then $M'/s_1, s_2$ is disconnected with $A_1$ as a component ($A_1$ is non-empty, otherwise $M'$ only distinguishes $e_1$ and $e_2$). Since $S \cup L$ spans $M$, $A_1$ is in the span $s_1, s_2$. Let $a \in A_1$, and consider $M'' = M/a \setminus s_2$. Clearly $M''$ distinguishes $e_1$ and $e_2$, and $e_3$ is distinguished as $e_1, e_2$ are in the same component in $M'/a \setminus e_3, e_4$. Finally $e_4$ is distinguished since $\cap_{M/a}(S, L) = 1$ as $a \in \cl(S)$ and $s_2$ is parallel to $s_1$ in $M/a$, so $S \setminus \{s_2\}$ distinguishes $e_4$. This contradicts minimality of $M$, proving (6).

Now let $S = \{e, f\}$, then $r(M) = 3$ since $\{e, f\} \cup L$ spans $M$. Let $g$ be an element of $M$ that is in neither of the span of $S$ nor $L$, and $g$ exists since $M$ does not only distinguish $e_4$. Then by modularity of $L$, we may assume that $\{e, g\}$ is a strand for $e_1$ and $\{f, g\}$ is a strand for $e_2$. Since $M$ distinguishes $e_3$, there exists an element $h$ that is not parallel to $e, f, g$ and not spanned by $L$.

If one of $\{e, h\}, \{f, h\}, \{g, h\}$ is a strand for $e_3$, then $E(M) = \{e, f, g, h\} \cup L$ and we are done. First suppose that $h$ is spanned by one of the lines $\{e, f\}, \{e, g\}, \{f, g\}$. By symmetry we may assume that $h \in \cl(\{e, f\})$. Then $\{h, g\}$ does not span $e_1$ since $\{e, g\}$ spans $e_1$, it does not span $e_2$ since $\{f, g\}$ spans $e_2$, and it does not span $e_4$ since $\{e, f, h\}$ spans $e_4$. Hence as $L$ is modular, $\{h, g\}$ spans $e_3$ as desired. So $h$ is not in any of the spans of the lines $\{e, f\}, \{e, g\}, \{f, g\}$. A similar check shows that one of $\{e, h\}, \{f, h\}, or \{g, h\}$ spans $e_3$, as desired.

By case analysis, we can deduce all possible 8-element matroids $M$ with a line $L$ such that all elements of $L$ are distinguished. Let $M_1, M_2, M_3, M_4$ be the matroids in Figure 3.2
(given by their geometric representations), where the elements of $L$ are denoted by filled circles, and let $M_5$ be the unique matroid with a modular 4-point line $L$ and an independent coline.

Figure 3.2: Four of the matroids that satisfy the conditions of Lemma 3.4.2

**Lemma 3.4.3.** Let $L = \{e_1, e_2, e_3, e_4\}$ be a modular $U_{2,4}$-restriction in an 8-element matroid $M$. If all elements of $L$ are distinguished in $M$, then $M$ is isomorphic to one of $M_1, M_2, M_3, M_4,$ or $M_5$.

**Proof.** Let $N = E(M) \setminus L = \{e_5, e_6, e_7, e_8\}$, let $n$ be the number of elements of $N$ that in the span of $L$. Clearly $M$ contains neither loops nor coloops, and $M|N$ is simple. If $n = 4$, then $M|N$ is isomorphic to $U_{2,4}$, and $M$ is isomorphic to $M_1$. Clearly $n \neq 2$ and $n \neq 3$, otherwise $M$ distinguishes at most three elements of $L$. Suppose $n = 1$. Then we may assume that $\{e_4, e_8\}$ is a parallel pair and $e_5, e_6, e_7$ distinguishes $e_1, e_2, e_3$. It follows that $\{e_5, e_6, e_7\}$ is independent and $\text{cl}(\{e_5, e_6, e_7\})$ spans $L$ (and thus also spans $M$). Then $L \cup \{e_8\}$ is a hyperplane of $M$, so $\{e_5, e_6, e_7\}$ is a triad, and $M$ is isomorphic to $M_2$. Finally suppose that $n = 0$. Then $\lambda_M(L) = \text{cl}_M(L, N) = 2$ otherwise $M$ distinguishes at most one element of $L$. It follows that $N$ spans $M$. If $T = \{e_5, e_6, e_7\}$ is a triangle, then $L$ is on the plane spanned by $N$. Since $L$ is modular, we may assume that $e_1 \in \text{cl}(T)$ and the sets $\{e_2, e_5, e_8\}, \{e_3, e_6, e_8\}, \{e_4, e_7, e_8\}$ are triangles. In this case, $M$ is isomorphic to $M_3$. If $N$ is a circuit, so $r(N) = |N| - 1 = 3$ and $N$ is a circuit-cocircuit, and hence $M$ is isomorphic $O_8$. In the case where $N$ is not a circuit and does not contain a 3-point line, then $N$ is independent and spans $L$, so $M$ is isomorphic to $M_5$. \hfill $\square$
3.5 Near-regular

Now that we have taken care of the case where one of $M_X$ and $M_Y$ are non-quinary, we turn our attention to whether $M_X$ and $M_Y$ have representations that match along $L$. As mentioned in Chapter 1, there are three inequivalent representations of $L$, which are given by

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{bmatrix}$$

for $x \in \{2, 3, 4\}$. Each of these representations of $L$ may or may not extend to a representation of $M_X$ or $M_Y$. However we do know that any representation of $L$ extends to at most one representation of $M_X$ or $M_Y$, by the following theorem from Whittle [24].

**Theorem 3.5.1.** Let $M$ be a 3-connected ternary matroid with a $U_{2,4}$-minor $N$. Then a representation of $M$ over a field $\mathbb{F}$ is uniquely determined by an $\mathbb{F}$-representation of $N$.

So the number of inequivalent $GF(5)$-representations for $M_X$ is equal to the number of inequivalent representations for $L$ that extend to a representation of $M_X$. Hence we want to know the number of inequivalent $GF(5)$-representations $M_X$ may have. A result from Pendavingh and van Zwam [17] shows that a dyadic matroid has either one or three inequivalent $GF(5)$-representations, and characterizes exactly when each case is realized; see Theorem 1.3.6. Since $M_X$ is not binary, case (i) of Theorem 1.3.6 does not apply. Our next lemma shows that, in fact, case (iii) must apply to $M_X$ (and analoguously for $M_Y$).

**Lemma 3.5.2.** Let $(X, Y)$ be an exact 3-separation in a matroid $M$ where $M$ is a ternary excluded minor for $GF(5)$-representability. If $M_X$ and $M_Y$ are both quinary. Then $M_X$ is uniquely representable over $GF(5)$.

**Proof.** Suppose that there exists three inequivalent $GF(5)$-representations $A_{X,1}, A_{X,2}, A_{X,3}$ of $M_X$. Let $L$ be the line common to $M_X$ and $M_Y$. By performing row operations, we may assume that the representations are of the form:

$$A_{X,r} = \begin{bmatrix} * & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & r + 1 \end{bmatrix}$$

for $r \in \{1, 2, 3\}$, where $L$ labels the last four columns of $A_{X,r}$.

Now let $A_Y$ be a $GF(5)$-representations of $M_Y$, and by row operations we may assume it is of the form:
for some \( r_Y \in \{1, 2, 3\} \), where \( L \) labels the first four columns of \( A_{X,r} \).

Then applying Lemma 2.6.2 on the representations \( A_{X,r_Y-1}, A_Y \) for \( M_X \) and \( M_Y \) (respectively) gives us that \( M^+_{X,Y} \) is \( GF(5) \)-representable. By Lemma 3.2.4, all elements of \( L \) are pinned in \( M^+_{X,Y} \). Hence applying Lemma 3.2.1 gives us that \( M \) is \( GF(5) \)-representable, a contradiction. \( \square \)

Putting together the above lemma and Theorem 1.3.6, it follows that neither \( M_X \) nor \( M_Y \) are near-regular.

**Corollary 3.5.3.** Let \( M \) be a ternary excluded minor for \( GF(5) \)-representability with an exact 3-separation \((X,Y)\). If both \( M_X \) and \( M_Y \) are quinary, then neither \( M_X \) nor \( M_Y \) are near-regular.

### 3.6 Reducing to \( O_8 \)

In this section, we will study the structure of \( M_X \) and \( M_Y \) given that they are quinary but not near-regular. If \( M \) is an excluded minor, then \( M_X \) and \( M_Y \) are minor-minimal with respect to keeping the boundary line and being not near-regular. The main result of this section shows that \( M_X \) and \( M_Y \) are both isomorphic to \( O_8 \).

**Lemma 3.6.1.** Let \( L \) be a \( U_{2,4} \)-restriction in a 3-connected matroid \( M \) that is dyadic but not near-regular. Then \( M \) has a minor \( N \) isomorphic to \( O_8 \) such that \( L \) is a \( U_{2,4} \)-restriction of \( N \).

In order to prove that our minor-minimal dyadic matroid is isomorphic to \( O_8 \), we only need to show that if it is not \( O_8 \) then we can find a minor preserving the line \( L \) and a 4-element circuit-cocircuit.

Since \( M \) is dyadic but not near-regular, \( M \) is not \( GF(4) \)-representable. We will prove that \( M \) must contain a non-Fano or its dual as a minor, using an excluded minor theorem for 3-connected non-\( GF(4) \)-representable matroids from Theorem 1.3.8.

In our proof we will often want to dualize our matroid \( M \). The next proposition shows that we can take the dual of \( M \) and use a generalized \( \Delta - Y \) exchange to preserve all the desired properties.
Proposition 3.6.2. Let $M$ be a 3-connected, dyadic, non-near-regular matroid with a $U_{2,4}$-restriction $L$. Then $\nabla_L(M^*)$ is also 3-connected, dyadic, non-near-regular with a $U_{2,4}$-restriction.

Proof. Since representability and connectivity are preserved by duality, $M^*$ is 3-connected, dyadic, and not near-regular. Note that $L$ is an coline in $M^*$ (as $L$ is a coincident line in $M$ by 3-connectivity), so $\nabla_L(M^*)$ is well-defined and $L$ is a $U_{2,4}$-restriction in $\nabla_L(M^*)$. □

Lemma 3.6.3. Let $M$ be a 3-connected, dyadic, non-near-regular matroid with a $U_{2,4}$-restriction. Then $M$ has an $F_7^-$ or $(F_7^-)^*$-minor.

Proof. Since $M$ is not $GF(4)$-representable, $M$ must satisfy one of the three cases of Theorem 1.3.8. $M$ does not satisfy case (ii) as $P_8''$ is not ternary. Next, it can be easily verified that contracting at most two elements of $S(5,6,12)$ does not produce a $U_{2,4}$-restriction, hence $M$ does not satisfy case (iii). Thus $M$ satisfies case (i). As neither $U_{2,6}$, $U_{4,6}$, nor $P_6$ is ternary, it follows that $M$ contains an $F_7^-$ or $(F_7^-)^*$ as a minor. □

Lemma 3.6.4. Let $L$ be a subset of elements of a 3-connected, dyadic matroid $M$ such that $M|L$ is isomorphic to $U_{2,4}$ and $M$ contains an $F_7^-$-minor $N$. If all proper minors $M'$ of $M$ that satisfy $M|L = M'|L$ has the property that $M'$ does not contain an $F_7^-$-minor, then $E(M) = E(N) \cup L$.

Proof. Suppose $M$ contains an element $e$ such that $e \not\in L$ and $e \not\in E(N)$. Then either $M\setminus e$ or $M/e$ preserves both $L$ as a $U_{2,4}$-restriction and $N$ as an $F_7^-$-minor. Let $M' \in \{M\setminus e, \nabla_L(M^*)\setminus e\}$ such that $N$ or $N^*$ is an $F_7^-$-minor of $M'$. Since $M$ is minor-minimal, $M'$ is not 3-connected.

By applying Lemma 2.4.6 on $M'$, we get that a partition $B_1, \ldots, B_k$ of $E(M')$ such that $\lambda_M(B_i) \leq 1$ and $|E(N) \cap B_i| \leq 1$ for $i \in \{1, \ldots, k\}$, and the matroid $M''$ obtained by contracting all $B_i$ down to their respective basepoints $e_i$ is 3-connected with groundset $\{e_1, \ldots, e_k\}$ with an $N$-minor.

Note that $L$ is contained in some $B_i$, otherwise $M''$ contradicts the minimality of $M$. Let $B_L$ be the side of the 2-separation $(B_L, E(M') \setminus B_L)$ of $M'$ that $L$ is contained in, so $(B_L \cup \{e\}, E(M) \setminus (B_L \cup \{e\}))$ is a 3-separation of $M$. We may assume that we have chosen $e$ such that $|B_L|$ is minimal.

Our next claim is that $B_L$ contains no other elements. Suppose there exists $e' \in B_L$ such that $e' \not\in L$, then there exists a 3-separation $(B_L' \cup \{e'\}, E(M) \setminus B_L')$ of $M$ such that $L$ is contained in $B_L'$. Now by Lemma 2.4.1 we get that $B_L \cap B_L'$ is 3-separating for $M$, and as $e \not\in B_L'$, this is strictly smaller than $B_L$, contradicting our choice of $e$. 

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Finally we want to show that $M/e$ contradicts minimality. Since $(L,E(M) \setminus L)$ is not a 2-separation in $M$ and $r_M(L) = r_{M/e}(L)$, $L$ is not 2-separating in $M/e$, so $M' = M \setminus e$. Now $M/e$ is dyadic and preserves $L$ as $e \not\in L$. Further, $M/e$ is 3-connected as . Also we may assume that $cl_M(L) \cap cl_{M}(E(M) \setminus L) = \ell_1$, otherwise $M$ contains a $U_{2,5}$-minor, contradicting that $M$ is ternary. Then $\{e,\ell_2,\ell_3,\ell_4\}$ is a cocircuit of $M$. As $M/\ell_4 \setminus e$ contains an $F_7^-$-minor by claim (1), $M|((E(M) \setminus (L \cup e)) \cup \ell_1)$ is a hyperplane of $M$ that does not contain $e$ and contains an $F_7^-$-minor. So $M/e$ contains an $F_7^-$-minor, contradicting the minimality of $M$.

Now that we have proved that $M$ has no other elements apart from the line and the non-Fano minor, we can do case analysis to show that $M$ is in fact $O_8$. To ease notation, we will assign canonical labels for the elements of $F_7^-$ and $(F_7^-)^*$, and use these labels for the remainder of the chapter.

![Figure 3.3: A labelling of $F_7^-$ and $(F_7^-)^*$](image)

First we will prove a proposition regarding the non-Fano matroid.

**Proposition 3.6.5.** Let $M$ be a ternary extension of $F_7^-$ by a non-loop element $e$. Then $M$ contains an $F_7^-$-restriction using $e$.

**Proof.** Observe that $F_7^-$ is the unique rank-3, 7-element ternary matroid such that $\{e_1, e_2, e_5\}$, $\{e_1, e_3, e_6\}$, $\{e_1, e_4, e_7\}$, $\{e_5, e_6, e_7\}$ are 3-point lines but $\{e_2, e_3, e_4\}$ is not.

We may assume that $e$ is a simple extension, otherwise the result is immediate. Let $L = \{f_5, f_6, f_7\}$. First suppose that $e$ is not in the span of any 3-point lines. Then the line spanned by $\{f_1, e\}$ does not contain any element of $L$. Let $f_8$ be a ternary extension of $M$ in the spans of $\{f_1, e\}$ and $L$. Then the lines spanned by $\{e, f_i\}$ for $i \in \{2, 3, 4\}$ must not contain $f_8$ (otherwise $\{f_1, f_i\}$ spans both $f_8$ and $f_{i+3}$, a contradiction as $f_1$ is not on $L$),

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so they must span distinct elements of \( L \). Since \( \{ f_2, f_3, f_4 \} \) is not a 3-point line, by our observation above, \( M \setminus f_1 \cong F_7^- \).

Now suppose \( e \) is in the span of a 3-point line, which we may assume to be \( L \). Then \( \{ f_2, f_3 \} \) spans \( e \), otherwise \( M/f_2 \) contains a \( U_{2,5} \)-restriction, contradicting that \( M \) is ternary. Then \( \{ f_2, f_3, e \}, \{ f_2, f_1, f_3 \}, \{ f_2, f_4, f_7 \} \), and \( \{ e, f_5, f_7 \} \) are all 3-point lines and \( \{ f_1, f_3, f_4 \} \) is not. By our observation, \( M \setminus f_6 \cong F_7^- \), as desired.

We are now ready to prove Lemma 3.6.1.

**Proof of Lemma 3.6.1.** By Lemma 3.6.3 and Lemma 3.6.4, \( M \) contains a \( U_{2,4} \)-restriction \( L \) and an \( F_7^- \)-minor \( N \) and no other elements. Since we obtain \( N \) by a sequence of at most two contractions, \( r(M) \leq 5 \). Clearly we also have \( r(M) \geq 3 \). We then split our analysis on the rank of \( M \). Let \( L = \{ e_1, e_2, e_3, e_4 \} \) and \( N = \{ f_1, f_2, f_3, f_4, f_5, f_6, f_7 \} \), noting that \( e_i \) and \( f_j \) may not be distinct.

**Case 1:** \( r(M) = 3 \).

In this case, \( N \) is in fact an \( F_7^- \)-restriction, and \( N \setminus L \) is a cocircuit. Observe that deleting any line of \( N \) preserves a 4-element circuit, hence \( N \setminus L \) contains a 4-element circuit-cocircuit, so \( M \setminus L \) contains a 4-element circuit. Then \( M \vert (L \cup C) \) is isomorphic to \( O_8 \).

**Case 2:** \( r(M) = 4 \).

By switching labels of elements of \( L \), we may assume that \( M/e_1 \) has a \( F_7^- \)-restriction, so the projection of \( M \) from \( e_1 \) onto a hyperplane \( H \subseteq (E(M) \setminus \{ e_1 \}) \) contains a non-Fano restriction. Since \( L \) is modular, we may assume that \( e_4 \) is in the span of \( H \). By Proposition 3.6.5, \( M/e_1 \) contains an \( F_7^- \)-restriction using \( e_4 \), so we may assume that \( e_4 \) is an element of \( N \). As the automorphism group of \( F_7^- \) has the two orbits \( \{ f_1, f_4, f_5, f_6 \} \) and \( \{ f_2, f_3, f_7 \} \), we consider the two subcases where \( e_4 \) is an element of a representative from each orbit:

**Subcase 2.1:** \( e_4 = f_5 \).

Then \( M/f_6 \) preserves \( L \). Since \( L \cup \{ f_6, f_7 \} \) is a plane, \( \{ f_1, f_2, f_3, f_4 \} \) is a cocircuit in \( M/f_6 \). If \( \{ f_1, f_2, f_3, f_4 \} \) is a circuit in \( M/f_6 \) then we have found our circuit-cocircuit. So in \( M \), there exists a plane \( P \) containing \( f_6 \) and at least three of \( \{ f_1, f_2, f_3, f_4 \} \). By symmetry we may assume that \( f_2, f_4, f_6 \in P \) and \( e_1 \notin P \) as there are no lines more than three elements in \( F_7^- \). So the plane containing \( \{ e_1, f_2, f_4, f_6 \} \) meets \( P \) along a line. Hence \( \{ f_2, f_4, f_6 \} \) is a triangle in \( M \). By applying the above analysis on \( M/f_1 \), we get that \( \{ f_1, f_4, f_7 \} \) is a triangle \( M \). As these two triangle have a common element \( f_4 \), they lie on a plane \( P' \). Since
$M$ is 3-connected, $f_3$ is not in the span of $P'$. Then $M/f_3$ contains $L$ and \{$f_1, f_2, f_6, f_7$\} as a circuit-cocircuit, as desired.

**Subcase 2.2: $e_4 = f_2$.**

In this case consider $M/f_3$, which preserves $L$ and has \{$f_1, f_4, f_5, f_6, f_7$\} as a cocircuit. We claim that at least one 4-element subset of \{$f_1, f_4, f_5, f_6, f_7$\} is a circuit in $M/f_3$. Suppose otherwise, then in $M/e$ every 4-element subset of \{$f_1, f_4, f_5, f_6, f_7$\} lies on a plane with $e_3$, and hence the entire set lies on a plane with $e_3$ in $M$. However, this contradicts 3-connectivity of $M$.

**Case 3: $r(M) = 5$.**

In this case $E(N)$ and $L$ are disjoint, so $M$ has 11 elements, and we may assume that $M/e_1, e_2, e_3, e_4 \cong F_7^-$. Now consider $M^*$, which has rank six and $M^* \setminus e_1, e_2, e_3, e_4 \cong (F_7^-)^*$, so deleting the coline $L$ preserves the $(F_7^-)^*$-minor. Hence applying the generalized $Y - \Delta$ exchange on $L$ in $M^*$ preserves the $(F_7^-)^*$-minor. Let $M' = \nabla_L(M^*)$. Then $M'$ contains a 4-point line, an $(F_7^-)^*$-minor, and $\si(M')$ is 3-connected. Since a $Y - \Delta$ exchange on a 4-element coline drops the rank by two, $M'$ has rank four and the $(F_7^-)^*$-minor is in fact an $(F_7^-)^*$-restriction.

Next we want to show that $M'$ contains rank-3 minor that has $L$ as a 4-point line along with a 4-element circuit-cocircuit. Let $e \in E(M')$ such that $e$ is not in the span of $L$. Let $P_1, \ldots, P_k$ be the planes that contain $L$ in $M'$. We have $k \geq 3$ as $\si(M')$ is 3-connected, and $|P_i \cap E(N)| < 5$ since $(F_7^-)^*$ has no plane with five elements. We split our analysis on the size of $\cl(L) \cap E(N)$, noting that $|\cl_{M'}(L) \cap E(N)| < 3$ since $(F_7^-)^*$ has no 3-point line. Let $E(N) = \{n_1, \ldots, n_7\}$.

**Subcase 3.1: $|\cl_{M'}(L) \cap E(N)| = 0$.**

Then there are seven elements of $N$ and least three planes, we may assume $|P_i \cap E(N)| \leq 2$. Let $n_1 \in P_1$. Then $E(N) \setminus P_1$ has size $\geq 5$, so we may assume $\{n_2, n_3, n_4, n_5, n_6\} \subseteq E(N) \setminus P_1$. Since $\{n_2, n_3, n_4, n_5, n_6\}$ is not in the span of $L$ in $M'/n_1$, so it is a cocircuit in $M'/n_1$. We want to show that this also contains a circuit $C$ in $M'/n_1$. Suppose otherwise, then we may assume that $\{n_1, n_2, n_3, n_4\}$ is a plane in $M'$. There is also a plane involving $n_5, n_6$, otherwise $C = \{n_3, n_4, n_5, n_6\}$. Then we may assume that $\{n_1, n_4, n_5, n_6\}$ is a plane in $M'$. Then $C = \{n_2, n_3, n_5, n_6\}$ as desired.

**Subcase 3.2: $|\cl_{M'}(L) \cap E(N)| = 1$.**

Suppose that some $P_1$ contains only two elements of $N$ and let $n \in P_1 \setminus \cl_{M'}(L)$. Then $E(N) \setminus P_1$ has size $\geq 5$, so we can apply the analysis from Subcase 3.1 on $E(N) \setminus P_1$ to obtain a 4-element circuit-cocircuit $C$ in $M'/n$. Hence each $P_i$ contains exactly three
elements of $N$ and hence $k = 3$. Since the automorphism group is preserved by duality, the automorphism group of $(F_7^-)^*$ also has the two orbits $(f_1, f_4, f_5, f_6)$ and $(f_2, f_3, f_7)$. Hence we may assume that either $f_1 \in cl_{M'}(L)$ (referring to Figure 3.3) or $f_2 \in cl_{M'}(L)$.

Suppose $f_1 \in cl_{M'}(L)$. Let $f_2 \in P_1$, then as $|P_1| = 3$ we have that $P_1 = \{f_1, f_2, f_3\}$, since each of $\{f_3, f_4, f_5, f_7\}$ lie on a 4-element plane with $f_1, f_2$. Then $M'/f_2$ preserves $L$ and $\{f_3, f_4, f_6, f_7\}$ is a 4-element plane in $M'$ that does not span $f_2$ and is disjoint from $cl_{M'}(L \cup \{f_2\})$, so it is a circuit-cocircuit in $M'/f_2$ as desired. Now suppose $f_2 \in cl_{M'}(L)$, then let $f_1 \in P_1$. Then as above, $P_1 = \{f_1, f_2, f_3\}$ and $\{f_3, f_4, f_6, f_7\}$ is a circuit-cocircuit in $M'/f_1$.

**Subcase 3.3: $|cl_{M'}(L) \cap E(N)| = 2$.**

Let $n_1, n_2 \in cl_{M'}(L)$. Observe that at least one plane contains four elements of $N$ since every line in $(F_7^-)^*$ is contained in a 4-element plane, we may assume that $P_1 \cap E(N) = \{n_1, n_2, n_3, n_4\}$. Then there are three remaining elements of $E(N) \setminus P_1$ and least two remaining planes, so we may assume that $|P_2 \cap (E(N) \setminus L)| = 1$. Let $n_5 \in (P_2 \setminus L)$ and consider $M'/n_5$. Then $\{n_3, n_4, n_6, n_7\}$ has size four and is not in the span of $L$ in $M'/n_5$, so it is a cocircuit in $M'/n_5$. If it is also a circuit, then we are done, so suppose otherwise. Hence there exists a plane $P'$ of $M'$ that contains $f$ and exactly three elements of $\{n_3, n_4, n_6, n_7\}$ (it cannot contain more than three as $F_7^-$ has no plane with five elements).

Suppose both of $n_3, n_4$ are in $P'$. Then we may assume that $n_6 \notin P'$. Then $\{n_3, n_4, n_5, n_7\}$ is a circuit-cocircuit in $M'/n_6$, and we want that no element is in $cl_{M'/n_6}(L)$. Since $n_6 \notin P_1$ and $n_6 \notin P_2$, the plane $P''$ spanned by $L$ and $n_6$ does not contain any of $\{n_3, n_4, n_5\}$, so $n_7 \in P''$. Then $\{n_1, n_2, n_7\}$ is a cocircuit of $(F_7^-)^*$ (via $P''$), and $\{n_5, n_3, n_4\}$ is a cocircuit of $(F_7^-)^*$ (via $P''$), however this is a contradiction as $F_7^-$ contains no two disjoint 3-element lines.

Next suppose $n_3 \notin P'$ but $n_4 \in P'$. Then $\{n_1, n_2, n_3\}$ is a cocircuit of $(F_7^-)^*$ (via $P'$), and $\{n_5, n_6, n_7\}$ is a cocircuit of $(F_7^-)^*$ (via $P_1$), a contradiction as above and proving Subcase 3.3.

So in all subcases, we find a minor of $M''$ of $M'$ that contains $L$ and has a 4-element circuit-cocircuit disjoint from $L$. Then $\nabla((M'')^*)$ is a minor of $M$, and as circuit-cocircuits are obviously self-dual, $M$ contains $L$ and a circuit-cocircuit disjoint from $L$, as desired. \qed

### 3.7 Matching Cross Ratios

As we have proven in the previous section that both sides of our excluded minor must be isomorphic to $O_8$, our final task is to determine the excluded minors themselves. To ease
notation in this section, we will use the labelling of the elements of \(O_8\) given in Figure 3.4.

![Figure 3.4: A labelling of the elements of \(O_8\)](image)

It is easy to check that there is an automorphism of \(O_8\) that swaps \((f_5, f_6)\) while \(f_7, f_8\) are fixed elements (and vice versa), and another that swaps both pairs \((f_5, f_6)\) and \((f_7, f_8)\). However, there is no automorphism that sends either \(f_5\) or \(f_6\) to \(f_7\) or \(f_8\) (since \(f_5\) and \(f_6\) are contained in three 3-point lines, while \(f_7\) and \(f_8\) are contained in two). Hence we consider \(f_5, f_6\) as “red” elements and \(f_7, f_8\) as “blue” elements.

**Lemma 3.7.1.** Let \(M\) be a ternary excluded minor with an exact 3-separation \((X, Y)\), such that \(M_X\) and \(M_Y\) are both quinary. Then \(M\) is isomorphic to \(T_8\).

**Proof.** By Corollary 3.5.3 and Lemma 3.6.1, \(M_X\) and \(M_Y\) are both isomorphic to \(O_8\). Let \(L = \{e_1, e_2, e_3, e_4\}\) be the boundary line of \((X, Y)\) in \(M_{X,Y}^+\). We colour the elements of \(L\) in \(M_X\) and \(M_Y\) with red and blue as above. By relabeling, we may assume that \(e_1\) and \(e_2\) are red elements in \(M_Y\), and \(e_3, e_4\) are blue. Furthermore, if exactly one of \(e_1, e_2\) is red in \(M_X\), we may assume that \(e_1\) is red, by swapping the labels of \(e_1, e_2\) in \(M_Y\). Similarly \(e_3\) is red in \(M_X\) if one of \(e_3, e_4\) is red.

Given an ordering of elements \((\ell_1, \ell_2, \ell_3, \ell_4)\) of a \(U_{2,4}\)-minor in a uniquely \(\mathbb{F}\)-representable matroid \(M\), we denote its cross ratio by \(cr_M(\ell_1, \ell_2, \ell_3, \ell_4)\). Let \(x = cr_{M_X}(e_1, e_2, e_3, e_4)\), and \(y = cr_{M_Y}(e_1, e_2, e_3, e_4)\). First, we compute \(y\). Let \(A\) be a \(GF(5)\)-representation of \(M_Y\). By row operations, we may assume that the columns of \(A\) corresponding to \(L\) have the form

\[
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & y
\end{bmatrix}.
\]
Observe that this is row equivalent to

\[
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & \frac{1}{y} & 1
\end{bmatrix}.
\]

Applying an automorphism on \( M_Y \) that swaps \( e_3, e_4 \) and fixes \( e_1, e_2 \), we get a representation \( A' \) of \( M_Y \) where the columns corresponding to \( L \) are

\[
\begin{bmatrix}
e_1 & e_2 & e_4 & e_3 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & y
\end{bmatrix}.
\]

Since \( O_8 \) is uniquely representable over \( GF(5) \), \( y = \frac{1}{y} \) so \( y = 3 \). Now that we have the cross ratio for the \( Y \) side, we consider the possible orderings of \( L \) in \( M_X \), splitting our analysis by the number of red elements in \( \{ e_1, e_2 \} \) in \( M_X \).

**Case 1:** \( e_1, e_2 \) are both red in \( M_X \).

Then \( e_3, e_4 \) are both blue. Since the elements of \( L \) match in \( M_Y \) and \( M_X \), clearly their cross ratios match as well, so \( x = 3 \). Hence \( M_{X,Y}^+ \) is quinary, a contradiction.

**Case 2:** \( e_1, e_2 \) are both blue in \( M_X \).

Then \( e_3, e_4 \) are both red. Similarly to our argument that \( y = 3 \), we have that

\[
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & x
\end{bmatrix}, \quad \begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & \frac{1}{x}
\end{bmatrix}
\]

are both \( GF(5) \)-representations of \( L \) that extend to \( M_X \), by applying an automorphism that swaps \( e_3, e_4 \) while fixing \( e_1, e_2 \). Hence \( x = 3 \), which implies that \( M_{X,Y}^+ \) is quinary, a contradiction.

**Case 3:** \( e_1 \) is red, \( e_2 \) is blue in \( M_X \).

Then \( e_3 \) is red and \( e_4 \) is blue. It is routine to check that \( M \) is isomorphic to \( T_8 \), and \( T_8 \) is indeed an excluded minor for dyadic matroids, see page 649 of [16].

\( \square \)
3.8 Main Theorem

We conclude by applying the results from the previous sections to prove our main theorem.

**Theorem 3.8.1.** Let $M$ be a ternary excluded minor for $GF(5)$-representability with an exact 3-separation $(X,Y)$ such that $|X|, |Y| \geq 4$. Then either:

(i) $M$ is isomorphic to $T_8$,

(ii) $X$ is a circuit-cocircuit and $M_Y$ contains an excluded minor for $GF(5)$-representability with up to two fewer elements than $M_Y$,

(iii) $Y$ is a circuit-cocircuit and $M_X$ contains an excluded minor for $GF(5)$-representability with up to two fewer elements than $M_X$,

(iv) $|X| = 4$, $M_X$ is not isomorphic to $O_8$, and $M_Y$ is an excluded minor for $GF(5)$-representability, or

(v) $|Y| = 4$, $M_Y$ is not isomorphic to $O_8$, and $M_X$ is an excluded minor for $GF(5)$-representability.

**Proof.** Let $L = \{e_1, e_2, e_3, e_4\}$ be the boundary line of $(X,Y)$ in $M_X^{+Y}$. Since $|X|, |Y| \geq 4$, $M$ pins all four elements of $L$ by Lemma 3.2.4. If both $M_X$ and $M_Y$ are quinary, then by Lemma 3.7.1, $M$ satisfies (i). By symmetry we may assume that $M_Y$ is not quinary. Then by Lemma 3.4.1 and Lemma 3.4.2, $M_X$ has eight elements. Since $M_X$ distinguishes all four elements of $L$, $M_X$ is one of the five matroids $M_1, M_2, M_3, M_4, M_5$ given by Lemma 3.4.3. Let $x_1, x_2, x_3, x_4$ label the elements of $M_X \setminus L$. Note that if $M_X$ is isomorphic to $M_5$, then $M_Y$ is isomorphic to $\nabla_X(M)$.

It remains to show that, if $M_X$ is not isomorphic to $M_3$, then $M_Y$ is an excluded minor for $GF(5)$-representability. Since $M_Y$ is not quinary, it suffices to show that every deletion or contraction of $M_Y$ results in a quinary matroid.

Let $M_Y^{-}$ be a minor of $M_Y$ obtained by contracting $e_1 \in L$. Observe that in all five possible matroids for $M_X$, we can contract a (possibly empty) set $S$ such that $L$ is a spanning line in $M_X/S$ with $e_1$ parallel to some $e \in X$. Then $si(M_Y^{-}/e_1)$ is isomorphic to $si(M/S/e)$, and since simplification does not change representability, $M_Y^{-}$ is quinary. Next let $M_Y^{-}$ be obtained by deleting $e_1 \in L$. If $M_X$ is isomorphic to $M_1, M_2$ or $M_4$ then clearly we can contract a set $S$ such that $L$ is a spanning line in $M_X/S$ with $e_2, e_3, e_4$ each parallel to some $x_2, x_3, x_4 \in X$. Then $M_Y\setminus e_1$ is a restriction of $M/S$ so $M_Y^{-}$ is
quinary. If $M_X$ is isomorphic to $M_5$, let $\{x_2, x_3, x_4, e_1\}$ be the unique 4-element circuit of $M_X$ containing $e_1$. Then $M_X/x_1$ contains a triad $\{x_2, x_3, x_4\}$ with $e_2, e_3, e_4$ distinguished. Then $\nabla_{\{x_2, x_3, x_4\}}(M/x_1)$ is isomorphic to $M_Y\setminus e_1$, so $M_Y$ is quinary.

Let $M_Y^-$ be a minor of $M_Y$ obtained by deleting or contracting an element $e \in Y$. If $M_Y^- = M_Y/e$ and $e$ is parallel to an element $e_1$ of $L$, then clearly this is equivalent to contracting $e_1$, which was proved above. So $M_Y^-$ contains $L$ as a $U_{2,4}$-restriction. Now $M' = P_L(M_X, M_Y^-) \setminus L$ is quinary as it is a minor of $M$, and $M'$ has the 3-separation $(X, Y \setminus \{e\})$, so it has a quinary representation of the form

$$A = \begin{bmatrix} A_X & 0 \\ 0 & A_{Y-e} \end{bmatrix}$$

where $A_X$ and $A_{Y-e}$ intersect at most two rows. Let $A_L$ be a $GF(5)$-matrix such that the only non-zero entries are in the last two rows and $[A_X | A_L]$ is a $GF(5)$-representation of $M_X$. Then

$$A = \begin{bmatrix} A_L & 0 \\ 0 & A_{Y-e} \end{bmatrix}$$

is a $GF(5)$-representation of $M_Y^-$ since an element on $L$ is not pinned in $M'$ if and only if $M_Y^-$ does not distinguish the element, so $M$ satisfies (iv). Analogously, if $M_X$ is not quinary, then $M$ satisfies (v).

Now suppose $M_X$ is isomorphic to $M_3$, so $M_X$ is isomorphic to $O_8$ and $X$ is a 4-element circuit-cocircuit. We colour the elements of $L$ in $M_X$ with red and blue as in Section 3.7. If $M_Y^-$ is a minor of $M_Y$ obtained by contracting $e_1 \in L$, deleting $e \in Y$, or contracting $e \in Y$, then $M_Y^-$ is quinary by an argument similar to above cases. If $M_Y^-$ is a minor of $M_Y$ obtained by deleting $e_1 \in L$, then $M_Y^-$ is a minor of $M$ if and only if $e_1$ is a blue element in $M_X$. We may assume that $e_1, e_2$ are blue in $M_X$. Then one of $M_Y, M_Y\setminus e_1, M_Y\setminus e_2, M_Y\setminus \{e_1, e_2\}$ is an excluded minor for $GF(5)$-representability, satisfying (ii). Analogously, if $M_X$ is not quinary and $M_Y$ isomorphic to $O_8$, then $M$ satisfies (iii).

From Theorem 1.2.1, if we are given the set $M$ of vertically 4-connected ternary excluded minors for dyadic matroids, then we can deduce the list of ternary excluded minors for dyadic matroids as follows: For $M \in M$, let $L$ be a line of $M$. If $|L| = 4$, then $P_L(M, M_2), P_L(M, M_4)$, and $P_L(M, M_5)$ are all excluded minors for dyadic matroids (for
all possible orderings of \( L \), and \( P_L(M, M_3) \) either is an excluded minor or contains a \( T_8 \)-minor. If \( |L| = 3 \), then \( \nabla_L(M) \) is an excluded minor. Moreover, let \( M^+ \) be the ternary extension of \( M \) such that \( L \) spans a 4-point line \( L^+ \). Then \( P_{L^+}(M^+, M_3) \), where \( L^+ \setminus L \) is matched with a blue element of \( M_3 \), contains an excluded minor. Finally, if \( |L| = 2 \), let \( M^+ \) be the ternary extension of \( M \) such that \( L \) spans a 4-point line \( L^+ \). Then \( P_{L^+}(M^+, M_3) \), where \( L^+ \setminus L \) is matched with blue elements of \( M_3 \), contains an excluded minor. Now add any new excluded minors that were found to the set \( \mathcal{M} \), and repeat until all excluded minors are found. Assuming Rota’s Conjecture, this procedure terminates. It follows from Theorem 1.2.1 that the matroids found by this construction are the only ternary dyadic excluded minors with a 3-separation such that one side of the separation has rank at least three.
References


