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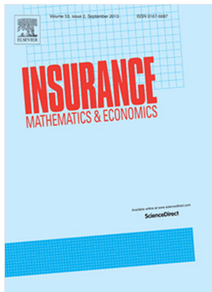

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Optimal investment-reinsurance strategies with state dependent risk aversion and VaR constraints in correlated markets

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Abstract

In this paper, we investigate the optimal time-consistent investment-reinsurance strategies for an insurer with state dependent risk aversion and Value-at-Risk (VaR) constraints. The insurer can purchase proportional reinsurance to reduce its insurance risks and invest its wealth in a financial market consisting of one risk-free asset and one risky asset, whose price process follows a geometric Brownian motion. The surplus process of the insurer is approximated by a Brownian motion with drift. The two Brownian motions in the insurer's surplus process and the risky asset's price process are correlated, which describe the correlation or dependence between the insurance market and the financial market. We introduce the VaR control levels for the insurer to control its loss in investment-reinsurance strategies, which also represent the requirement of regulators on the insurer's investment behavior. Under the mean-variance criterion, we formulate the optimal investment-reinsurance problem within a game theoretic framework. By using the technique of stochastic control theory and solving the corresponding extended Hamilton-Jacobi-Bellman (HJB) system of equations, we derive the closed-form expressions of the optimal investment-reinsurance strategies. In addition, we illustrate the optimal investment-reinsurance strategies by numerical examples and discuss the impact of the risk aversion, the correlation between the insurance market and the financial market, and the VaR control levels on the optimal strategies.

Keywords: Optimization Techniques; VaR constraint; Equilibrium investment-reinsurance strategy; Stochastic control; Extended HJB system of equations; Mean-variance criterion.

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1 Introduction

An insurer can manage its assets by investment and reduce its insurance risks by reinsurance. Optimal investment and reinsurance problems for an insurer have attracted much attention in the actuarial literature. The optimization criteria commonly used in these optimal investment and reinsurance problems include maximizing the expected utility of the terminal wealth of an insurer and minimizing the ruin probability of an insurer. Some recent works related to these criteria can be found in Browne (1995), Schmidli (2002), Liu and Yang (2004), Promislow and Young (2005), Yang and Zhang (2005), Liang et al. (2011), Bi and Guo (2013), and so on. In an optimal investment problem, there is a trade-off between the expected return of the investment and the risk of the investment over a fixed time horizon. In the fundamental work of Markowitz (1952), the risk of a portfolio is measured by the variance of its return in a single-period model, and the mean-variance criterion is used to seek the best allocation of wealth among a variety of securities so as to achieve the optimal trade-off between the expected return of the investment and its risk over a fixed time horizon. The mean-variance criterion has become one of the important criteria used in optimal investment problems. This criterion is also useful in insurance/reinsuarnce decision problems, as pointed out in Bäuerle (2005). Some recent applications of the mean-variance criterion in insurance/reinsurance problems can be found in Bai and Zhang (2008), Zeng and Li (2011), Bi and Guo (2013), Zeng et al. (2013), Li and Li (2013), Wu and Zeng (2015), Zhang and Liang (2017), and references therein.

It is well known that an optimal investment-reinsurance problem under the mean-variance criterion in a multi-period or continuous time framework lacks of the iterated-expectation property, which leads to time-inconsistent investment-reinsurance problems in the sense that the Bellman optimality principle does not hold for such optimal control problems. One of the important ways to deal with the time inconsistency in the optimization problem is to study the optimization problem within a game theoretic framework, in which a decision-maker's preferences change in a temporally inconsistent way as time goes by, and the mean-variance optimization problem is viewed as a game, where the players are the future incarnations of the decision-maker's own preferences. The decision-maker looks for a subgame perfect Nash equilibrium point for this game. The first paper to treat the time inconsistency in more general frameworks by the game theoretic approach was Björk and Murgoci (2010), in which they considered a general class of time-inconsistent objective functions and a general controlled Markov process and derived an extension of the standard dynamic programming equation in the form of a system of equations. Since then, the game theoretic approach has been applied in different optimization problems. For instance, Zeng et al. (2013) considered the optimal investment-reinsurance problem with one risky asset followed by a geometric Brownian motion (GBM) in a compound Poisson risk model. Björk et al. (2014) studied the mean-variance problems with a state dependent risk aversion and assumed that the risk aversion depends dynamically on current wealth. Li and Li (2013) extended the work of Björk et al. (2014) to the optimal investment-reinsurance problem, in which the surplus process is approximated by a diffusion process. Under the same criterion, Zhang and Liang (2016) discussed the optimal portfolio selection problem with one risk-free asset and two jump-diffusion risky assets, where the two risky asset price processes are correlated through a common shock. Further work about time-inconsistent problem was discussed in Zeng and Li (2011) and Wu and Zeng (2015).

When we consider a continuous time mean-variance investment problem, the wealth of an investor over any time period in the investment horizon and the terminal wealth may occur huge loss. To prevent investors from extremely dangerous positions in the market, it is helpful if we can use risk measures to limit the risk exposures to the market. The risk measure of Value-at-Risk (VaR) is often used to describe the market risk of a trading portfolio. Generally speaking, the VaR of a portfolio is the maximum possible loss of the portfolio at a given confidence level. Indeed, in practice, in order to fulfill the regulation requirements, an insurance company or a financial institution has to control the VaR of its portfolio. Hence, it is an interesting topic if we consider an investment-reinsurance problem with VaR constraints. Recently, Chen et al. (2010) and Ye and Li (2012) have investigated the optimal investment-reinsurance problems for an insurance company with VaR constraints under the criterion of minimizing the probability of ruin and the mean-variance criterion, respectively. Other works about optimal investment or optimal reinsurance problems with VaR constraints can be found in Yiu (2004), Zhang et al. (2016), Chen et al. (2018) and the references therein.

Optimal time-inconsistent investment-reinsurance problems have been extensively studied in the literature. However, very few of these contributions deal with the problems under VaR constraints. In this paper we are going to study the optimal time-inconsistent investment-reinsurance problem under VaR constraints with state dependent risk aversion for an insurer. The insurer's surplus process is approximated by a Brownian motion with drift. The risky asset's process follows a geometric Brownian motion. This paper extends the work of Li and Li (2013) in two ways. On the one hand, the two Brownian motions in the insurer's surplus process and the risky asset's price process are correlated with a correlation coefficient. This makes our model more flexible but makes the extended HJB system of equations in our paper more complicated. On the other hand, the VaR constraints on the future net loss over any time period with a fixed time length is incorporated in the model. This provides us an opportunity to observe the effect of VaR constraints on the optimal investment-reinsurance strategies. To the best of our knowledge, this paper is the first one to study optimal time-consistent strategies with dependent insurance and investment risks as well as VaR constraints.

This paper is organized as follows. In Section 2, we give the model settings consisting of the insurance risk process, the price processes of the risk-free asset and the risky asset, as well as the corresponding wealth process with investment and reinsurance. In Section 3, we formulate the optimization problem within a game theoretic framework without VaR constraints. By solving an extended HJB system of equations, the closed-form expressions of the equilibrium investment-reinsurance strategies and the corresponding equilibrium value function for the problem are derived. In Section 4, we consider the optimization problem with VaR constraints and solve the optimization problem using the results derived in Section 3. In Section 5, we illustrate our results by numerical examples. Finally, Section 6 concludes our results.

2 Model settings and problem formulations

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions, i.e., $\{\mathcal{F}_t\}_{t\in[0,T]}$ is right continuous and $\mathcal{P}$ complete, and containing the information of the market available up to time t. $T > 0$ is a fixed time horizon. In addition, we assume that there is no consumption, no income, no transaction cost and no tax in the financial market or the insurance market, and trading takes place continuously.

2.1 Reserve process of an insurer and the financial market

The dynamic of the reserve process $\{\bar{R}(t)\}_{t\geq 0}$ of an insurer is modeled by

$$d\bar{R}(t) = cdt - d\sum_{i=1}^{N(t)} Y_i, \tag{2.1}$$

where the constant $c > 0$ is the premium rate, $\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ representing the number of claims occurring in time interval $[0, t]$, and Y_i is the size of the ith claim.

In addition, $\{Y_i, i \geq 1\}$ is assumed to be an i.i.d sequence of random variable and be independent of $\{N(t)\}_{t\geq 0}$. The compound Poisson process $\sum_{i=1}^{N(t)} Y_i$ represents the cumulative amount of claims in time interval $[0,t]$. Let Y be a generic random variable which has the same distribution as $Y_i, i \geq 1$. Let $F_Y(\cdot)$ denote the cumulative distribution function of Y. Denote the expectation and the second moment of Y by $\mathbb{E}(Y) = \mu_1 > 0$ and $\mathbb{E}(Y^2) = \mu_2 > 0$, respectively. Assume that the insurance premium rate at time t is calculated by the expected value principle, that is, $c = (1+\eta)\lambda\mu_1$, where $\eta > 0$ is the safety loading.

Due to the jumps in the reserve process $\{\bar{R}(t)\}_{t\geq 0}$, it is not feasible to solve the mean-variance optimal investment-reinsurance problem directly under the reserve process $\{\bar{R}(t)\}_{t\geq 0}$. As most studies on the mean-variance optimal investment-reinsurance problem (see, for example, Browne (1995), Bai and Zhang (2008), Liang and Yuen (2016), and so on), we can consider the problem under the diffusion approximation of the reserve process $\{\bar{R}(t)\}_{t\geq 0}$. According to Grandell (1991) (pages 15-17), the diffusion approximation $\{\hat{R}(t)\}_{t\geq 0}$ of the reserve process $\{\bar{R}(t)\}_{t\geq 0}$ is given by

$$d\hat{R}(t) = cdt - \lambda\mu_1 dt + \sqrt{\lambda\mu_2}\, dW_1(t),$$

where $W_1(t)$ is a standard Brownian motion.

Now, suppose that the insurer with an initial wealth $X_0 > 0$ is able to invest its wealth in a financial market consisting of one risk-free asset and one risky asset, which are traded continuously on a finite time horizon $[0,T]$. The price process of the risk-free asset is given by

$$\begin{cases} dP_0(t) = r_0 P_0(t)dt, \ t \in [0,T], \\ P_0(0) = p_0, \end{cases}$$

where $r_0 (> 0)$ is the interest rate of the risk-free asset.

The price of the risky asset is modeled by the following stochastic differential equation (SDE)

$$\begin{cases} dP_1(t) = P_1(t)\left[r_1 dt + \sigma dW_2(t)\right], \ t \in [0,T], \\ P_1(0) = p_1, \end{cases}$$

where $r_1 (> r_0)$ is the appreciation rate, σ is the volatility coefficient, and $W_2(t)$ is a standard Brownian motion. The Brownian motion $W_1(t)$ in the approximated reserve process $\{\hat{R}(t)\}_{t\geq 0}$ and the Brownian motion $W_2(t)$ in the risky asset are possibly correlated with correlation coefficient $\rho \in [-1,1]$, which represents the dependence between the stock market and the insurance market. This kind of dependence may be due to an extreme event (such as a natural disaster) which has the common impact on both the financial and insurance markets.

Let X_t denote the insurer's total wealth at time t and $u(t)$ denote the total market value of the insurer's wealth in the risky asset at time t. Then $X_t - u(t)$ is the value of the insurer's wealth in the risk-free asset. Assume $u(t) \geq 0$, i.e., the short-selling of the stock is prohibited. Let $q(t)(\geq 0)$ represent the retention level of new business (reinsurance) acquired at time t, which means that the insurer pays $q(t)Y$ of a claim occurring at time t and the new businessman (reinsurer) pays $(1-q(t))Y$. Suppose that the reinsurance premium is also calculated by the expected value principle. For this business, the reinsurance premium is paid at rate $(1-q(t))(1+\theta)\lambda\mu_1$, where $\theta(>\eta)$ is the safety loading of the reinsurer and the condition of $\theta > \eta$ is required for avoiding the insurer's arbitrage. Note that for the insurance company, $q(t) \in [0,1]$ corresponds to a reinsurance cover and $q(t) > 1$ would mean that the company can take an extra insurance business from other companies (i.e., act as a reinsurer for other cedents).

A strategy $\pi(t) = (q(t), u(t))$ is said to be admissible if $q(t), u(t)$ are $\mathcal{F}_t$-predictable processes, and satisfy $q(t) \geq 0$, $u(t) \geq 0$, $\mathbb{E}[\int_0^t q^2(s)ds] < \infty$ and $\mathbb{E}[\int_0^t u^2(s)ds] < \infty$ for all $t \geq 0$. We denote the set of all admissible strategies by Π. Let X_t^π denote the insurer's total wealth at time t under the strategy $\pi(t) = (q(t), u(t))$. Then, the dynamics of X_t^π is given by

$$dX_t^\pi = \{r_0 X_t^\pi + \lambda\mu_1\theta q(t) + (r_1 - r_0)u(t) + \lambda\mu_1(\eta - \theta)\}\, dt + q(t)\sqrt{\lambda\mu_2}dW_1(t) + u(t)\sigma dW_2(t). \quad (2.2)$$

Note that due to the diffusion approximation, the wealth process X_t^π satisfying (2.2) is not always positive, which is a quite common situation when a compound Poisson risk process is approximated by a diffusion process. In our model, the total amount invested in the risky asset at time t satisfies $u(t) \geq 0$ or short selling is prohibited. As pointed out at the end of Section 2 of the celebrated paper of Browne (1995), the situation that $X_t^\pi < 0$ (or in general $u(t) > X_t^\pi$ in this paper) means that the investor/company is borrowing money to invest long in the risky asset. In fact, Section 3 of Browne (1995) has studied the negative wealth case and derived the optimal investment strategy that maximizes the expected utility of the investor/company at a terminal time when a wealth process is allowed to be negative. In practice, if the wealth process is negative or the company is in deficit, the company may need inject capital to keep the wealth process positive. This is an interesting question, but is not considered in this paper. In this paper, from the perspective of risk management, besides maximizing the expected mean-variance utility of the terminal wealth at the terminal time, we also want to control the VaR of the loss of the company over any time period prior to the terminal time. That is the novel point of our paper. The studies of the negative wealth cases with capital injections in the context of optimal investments, optimal portfolio sections, optimal dividend payments, and optimal reinsurances can be found in Zhou and Yuen (2012, 2015), Zhu and Yang (2016), Zhao, Chen and Yang (2017), Zhao, Jin and Wei (2018), and the references therein.

2.2 Value-at-Risk constraints for the investment-reinsurance strategy

Under the investment-reinsurance strategy $\pi(t) = (q(t), u(t))$, the insurer's wealth process $\{X_t^\pi, t \geq 0\}$ is a risk process. As discussed before in Section 1, the insurer may or has to use the risk measure of VaR to control its wealth for avoiding huge loss. For time interval $[t, t+h]$ with a small time step $h > 0$, assume that the investment-reinsurance strategy does not change over this short time period, i.e., $\pi(l) = \pi(t),\ l \in [t, t+h]$. This assumption is reasonable because in practice the insurer usually adjust its investment-reinsurance policy on a monthly (quarterly, yearly) basis. Thus, the loss of the insurer in time interval $[t, t+h]$ can be expressed as $\Delta X_{t,h}^\pi := X_t^\pi e^{r_0 h} - X_{t+h}^\pi$. According to the Itô's formula, the SDE (2.2) admits a solution

$$
\begin{aligned}
X_s^\pi = \quad & X_t^\pi e^{r_0(s-t)} + \int_t^s e^{r_0(s-z)} \left[\lambda\mu_1\theta q(z) + (r_1 - r_0)u(z) + \lambda\mu_1(\eta - \theta)\right] dz \\
& + \int_t^s e^{r_0(s-z)} \left[q(z)\sqrt{\lambda\mu_2}\, dW_1(z) + u(z)\sigma dW_2(z)\right]. \qquad (2.3)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta X_{t,h}^\pi \quad = \quad & -\frac{e^{r_0 h} - 1}{r_0} \left[\lambda\mu_1\theta q(t) + (r_1 - r_0)u(t) + \lambda\mu_1(\eta - \theta)\right] \\
& - \int_t^{t+h} e^{r_0(t+h-z)} \left[q(t)\sqrt{\lambda\mu_2}\, dW_1(z) + u(t)\sigma dW_2(z)\right]. \qquad (2.4)
\end{aligned}
$$

One feasible way for the insurer to control its wealth risk is to control the VaR of $\Delta X_{t,h}^\pi$ for any $t \in [0, T]$ with a small fixed time step h, say $h = 1/365$ (any day), $h = 1/12$ (any month), $h = 1/4$ (any quarter), and $h = 1$ (any year).

For a given risk level $p \in (0, 1)$ and a time step h, we denote the conditional VaR of $\Delta X_{t,h}^\pi$ conditioning on $\mathcal{F}_t$ by $\mathrm{VaR}_t^{p,h,\pi}$, namely,

$$
\mathrm{VaR}_t^{p,h,\pi} := \inf\{L \in \mathbb{R};\, \mathbb{P}(\Delta X_{t,h}^\pi \geq L \,|\, \mathcal{F}_t) \leq p\}. \qquad (2.5)
$$

In other words, $\mathrm{VaR}_t^{p,h,\pi}$ is the maximum possible loss over the next time period of length h at the confidence level $1 - p$. We point out that $\mathbb{P}(\Delta X_{t,h}^\pi \geq L | \mathcal{F}_t)$ in (2.5) is the conditional expectation of $\mathbb{E}\left[\mathbf{1}_{\{\Delta X_{t,h}^\pi \geq L\}} | \mathcal{F}_t\right]$ which is a random variable. However, as we see from (6.1) in the proof of Lemma 2.1 in Appendix A that given $\mathcal{F}_t$, the conditional probability $\mathbb{P}(\Delta X_{t,h}^\pi \geq L | \mathcal{F}_t)$ is almost surely equal to the normal distribution function (6.1). This is due to the well-known fact that given $\mathcal{F}_t$, the stochastic integral (6.2) has a normal distribution with mean zero and variance (6.3). Hence, $\mathbb{P}(\Delta X_{t,h}^{\pi} \geq L|\mathcal{F}_t)$ is almost surely a deterministic function (6.1). Thus, $\mathrm{VaR}_t^{p,h,\pi}$ defined by (2.5) is almost surely a deterministic function, which is given in Lemma 2.1.

In this paper, we will derive the optimal strategy π under the constraint that the investor wants to limit the VaR of its loss over any time period of length h at a constant $\overline{\mathrm{VaR}}$, that is to say that at any time $t \in [0, T]$, the strategy $\pi(t)$ should satisfy

$$\mathrm{VaR}_t^{p,h,\pi} \leq \overline{\mathrm{VaR}}. \tag{2.6}$$

To derive the optimal strategy in Section 4, we first give the expression of $\mathrm{VaR}_t^{p,h,\pi}$.

Lemma 2.1. *Given risk level $p \in (0, 1)$ and time length $h > 0$, we have*

$$\begin{aligned} \mathrm{VaR}_t^{p,h,\pi} \quad = \quad & -\frac{e^{r_0 h} - 1}{r_0} \left[\lambda\mu_1\theta q(t) + (r_1 - r_0)u(t) + \lambda\mu_1(\eta - \theta)\right] \\ & -\Phi^{-1}(p)\sqrt{\frac{e^{2r_0 h} - 1}{2r_0}\left[\lambda\mu_2 q^2(t) + \sigma^2 u^2(t) + 2\rho\sqrt{\lambda\mu_2}\sigma q(t)u(t)\right]}, \end{aligned} \tag{2.7}$$

where $\Phi^{-1}(\cdot) = \inf\{x \in \mathbb{R} : \Phi(x) \geq p\}$ is the inverse function of the cumulative standard normal distribution function $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$.

Proof. See Appendix A. □

2.3 Problem formulation

In this subsection, we will formulate the problem within a game theoretic framework, which is developed by Björk and Murgoci (2010). We consider an optimization problem for the insurer to maximize the expected mean-variance utility of its terminal wealth, i.e., the objective function which we want to maximize is given by

$$J(t, x, \pi) = \mathbb{E}_{t,x}[X_T^{\pi}] - \frac{\gamma(x)}{2}\mathbb{V}ar_{t,x}[X_T^{\pi}], \tag{2.8}$$

where x is the initial capital of the investor at the initial time t, $\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot|X_t^{\pi} = x]$, $\mathbb{V}ar_{t,x}[\cdot] = \mathbb{V}ar_{t,x}[\cdot|X_t^{\pi} = x]$. Furthermore, we let $\gamma(x) = \frac{\gamma}{x}$. It is known that $\gamma(x) = \frac{\gamma}{x}$ is a suitable choice of the state dependent risk aversion function. It was suggested by Björk et al. (2014) and has been studied by Li and Li (2013), Zhang and Liang (2017), and so on. The detailed discussion for this choice of the state dependent risk aversion function $\gamma(x)$ was given in Björk et al. (2014). We added these comments after (2.8).

Due to the fact that this objective functional (2.8) involves with a non-linear function $\mathbb{V}ar_{t,x}[\cdot]$ and the current wealth x at current time t, the optimization problem is time-inconsistent. We solve this time-inconsistent problem within a game theoretic framework and look for Nash subgame perfect equilibrium solutions.

For convenience, we rewrite the function (2.8) as

$$J(t,x,\pi) = \mathbb{E}_{t,x}\left[X_T^\pi - \frac{\gamma}{2x}(X_T^\pi)^2\right] + \frac{\gamma}{2x}\left[\mathbb{E}_{t,x}(X_T^\pi)\right]^2 = \mathbb{E}_{t,x}[F(x,X_T^\pi)] + G(x,\mathbb{E}_{t,x}[X_T^\pi])$$

with $F(x,y) = y - \frac{\gamma}{2x}y^2$ and $G(x,y) = \frac{\gamma}{2x}y^2$.

Now we recall the following definition of an equilibrium control and equilibrium value function, which is from Björk and Murgoci (2010).

Definition 2.1. *Given a control law π^*, which can be informally viewed as a candidate equilibrium law. Choose a fixed $\pi \in \Pi$, a fixed real number $l > 0$ and a fixed arbitrarily chosen initial point $(t,y) \in [0,T] \times \mathbb{R}$. Construct a control law π_l by*

$$\pi_l(s,y) = \begin{cases} \pi(s,y), & t \le s < t+l,\ y \in \mathbb{R}, \\ \pi^*(s,y), & t+l \le s \le T,\ y \in \mathbb{R}. \end{cases}$$

If

$$\liminf_{l\to 0} \frac{J(t,x,\pi^*) - J(t,x,\pi_l)}{l} \ge 0$$

for all $\pi \in \Pi$ and $(t,x) \in [0,T] \times \mathbb{R}$, we say that π^* is an equilibrium control law. The equilibrium value function is defined by

$$W(t,x) = J(t,x,\pi^*).$$

Based on the definition above, the equilibrium strategy is time-consistent, the equilibrium strategy is thus the optimal time-consistent strategy. Our goal is to find an equilibrium strategy π^* and the corresponding equilibrium value function.

Before giving the extended HJB system of equations and the verification theorem, we define a infinitesimal generator. Let $C^{1,2}([0,T] \times \mathbb{R})$ denote the space of the bivariate functions $\phi(t,x)$ such that $\phi(t,x)$ and its derivatives $\phi_t(t,x)$, $\phi_x(t,x)$, $\phi_{xx}(t,x)$ are continuous on $[0,T] \times \mathbb{R}$. For any function $\phi(t,x) \in C^{1,2}([0,T] \times \mathbb{R})$ and any fixed $\pi \in \Pi$, the usual infinitesimal generator $\mathcal{A}$ for process (2.2) is defined by

$$\mathcal{A}^\pi \phi(t,x) = \phi_t + [r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)]\,\phi_x + \frac{1}{2}\left[\lambda\mu_2 q^2 + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2}qu\right]\phi_{xx}. \tag{2.9}$$

Theorem 2.1. *(Verification Theorem). For the Nash equilibrium problem, if there exist functions $V(t,x)$, $f(t,x,y)$ and $g(t,x)$ satisfying the following conditions: $\forall (t,x) \in [0,T]\times\mathbb{R}$ and $y \in \mathbb{R}$,*

$$\begin{cases} \sup\limits_{\pi\in\Pi}\left\{\mathcal{A}^\pi V(t,x) - \mathcal{A}^\pi f(t,x,x) + \mathcal{A}^\pi f^x(t,x) - \mathcal{A}^\pi (G\diamond g)(t,x) + \mathcal{H}^\pi g(t,x)\right\} = 0,\ 0\le t\le T,\\ \mathcal{A}^{\pi^*} f^y(t,x) = 0,\ 0\le t\le T,\\ \mathcal{A}^{\pi^*} g(t,x) = 0,\ 0\le t\le T,\\ V(T,x) = F(x,x) + G(x,x),\\ f(T,x,y) = F(y,x),\\ g(T,x) = x, \end{cases} \tag{2.10}$$

and

$$\pi^* = \arg\sup_{\pi\in\Pi}\left\{\mathcal{A}^\pi V(t,x) - \mathcal{A}^\pi f(t,x,x) + \mathcal{A}^\pi f^x(t,x) - \mathcal{A}^\pi (G\diamond g)(t,x) + \mathcal{H}^\pi g(t,x)\right\},$$

then $W(t,x) = V(t,x)$, i.e., $V(t,x)$ is the equilibrium value function, π^* is the equilibrium reinsurance-investment strategy and

$$\begin{cases} f(t,x,y) = \mathbb{E}_{t,x}\left[F(y, X_T^{\pi^*})\right] = \mathbb{E}_{t,x}\left[X_T^{\pi^*} - \frac{\gamma}{2y}(X_T^{\pi^*})^2\right],\\ g(t,x) = \mathbb{E}_{t,x}\left[X_T^{\pi^*}\right], \end{cases} \tag{2.11}$$

where the operators f^y, $G\diamond g$ as well as $\mathcal{H}^\pi g$ are defined as follows:

$$\begin{cases} f^y(t,x) = f(t,x,y),\\ G\diamond g(t,x) = G(x, g(t,x)),\\ \mathcal{H}^\pi g(t,x) = G_y(x,g(t,x)) \times \mathcal{A}^\pi g(t,x),\\ G_y(x,y) = \dfrac{\partial G}{\partial y}(x,y). \end{cases}$$

Equation (2.10) is also called the extended HJB system of equations.

Proof. The derivation of the extended HJB system of equations (2.10) and the proof of the verification theorem can be obtained by using the standard arguments similar to those used in Section 4 of Björk and Murgoci (2010). We just give a sketch of the derivation of the extended HJB system of equations (2.10) and the proof of Theorem 2.1 and omit the detailed proof here.

The derivation of the extended HJB system of equations (2.10) can be derived in the following way: First, we discretize the continuous time problem and obtain a discretized recursion for π^* by using the results of Björk and Murgoci (2010) for discrete time control theory. Then, letting the time step tend to zero, we obtain the continuous time extension of the HJB system of equations (2.10). The proof of Theorem 2.1 consists of two steps: First, using the martingale approach, it can be proved that $V(t,x)$ is the value function corresponding to π^* and that the functions $f(t,x,y)$ and $g(t,x)$ have the probabilistic interpretations (2.11). Second, applying the discretization method, it can be proved that π^* is indeed an equilibrium control law. □

3 Solution to the optimization problem without VaR constraints

In this section, we first solve the optimal investment-reinsurance problem under the mean-variance criterion for state dependent risk aversion without VaR constraints. Note that by (2.11), we have

$$\begin{aligned} V(t,x) &= J(t,x,\pi^*) = \mathbb{E}_{t,x}[X_T^{\pi^*}] - \frac{\gamma}{2x}\,\mathbb{V}ar_{t,x}[X_T^{\pi^*}] \\ &= \mathbb{E}_{t,x}\left[X_T^{\pi} - \frac{\gamma}{2x}(X_T^{\pi^*})^2\right] + \frac{\gamma}{2x}\left[\mathbb{E}_{t,x}(X_T^{\pi^*})\right]^2 \\ &= f(t,x,x) + \frac{\gamma}{2x}\,g^2(t,x). \end{aligned} \tag{3.1}$$

First, after detailed calculations, we obtain the following result about the extended HJB system of equations and the equilibrium strategy.

Proposition 3.1. *The extended HJB system of equations (2.10) is reduced to the following system* *of equations:*

$$
\begin{cases}
f_t + \frac{\gamma}{x} g g_t + \sup\limits_{(q,u)\in\Pi} \Big\{ [r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)] \times \Big[f_x + \frac{\gamma}{x} g g_x \Big] \\
\qquad + \frac{1}{2}\Big[\lambda\mu_2 q^2 + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2} q u \Big] \times \Big[f_{xx} + \frac{\gamma}{x} g g_{xx} \Big] \Big\} = 0, \\
f_t(t,x,y) + [r_0 x + \lambda\mu_1\theta q^* + (r_1 - r_0)u^* + \lambda\mu_1(\eta-\theta)]\, f_x(t,x,y) \\
\qquad + \frac{1}{2}\Big[\lambda\mu_2 (q^*)^2 + \sigma^2 (u^*)^2 + 2\rho\sigma\sqrt{\lambda\mu_2} q^* u^* \Big] f_{xx}(t,x,y) = 0, \\
g_t(t,x) + [r_0 x + \lambda\mu_1\theta q^* + (r_1 - r_0)u^* + \lambda\mu_1(\eta-\theta)]\, g_x(t,x) \\
\qquad + \frac{1}{2}\Big[\lambda\mu_2 (q^*)^2 + \sigma^2 (u^*)^2 + 2\rho\sigma\sqrt{\lambda\mu_2} q^* u^* \Big] g_{xx}(t,x) = 0,
\end{cases}
$$

and the equilibrium strategy is given by

$$
\begin{cases}
q^* = \left\{ \frac{\rho\sigma\sqrt{\lambda\mu_2}(r_1 - r_0) - \sigma^2\lambda\mu_1\theta}{\sigma^2\lambda\mu_2(1-\rho^2)} \times \frac{f_x + \frac{\gamma}{x} g g_x}{f_{xx} + \frac{\gamma}{x} g g_{xx}} \right\} \vee 0, \\
u^* = \left\{ \frac{\rho\sigma\sqrt{\lambda\mu_2}\lambda\mu_1\theta - \lambda\mu_2(r_1 - r_0)}{\sigma^2\lambda\mu_2(1-\rho^2)} \times \frac{f_x + \frac{\gamma}{x} g g_x}{f_{xx} + \frac{\gamma}{x} g g_{xx}} \right\} \vee 0.
\end{cases}
$$

Here $f_t = \frac{\partial f(t,x,y)}{\partial t}$, $f_x = \frac{\partial f(t,x,y)}{\partial x}$, $f_{xx} = \frac{\partial^2 f(t,x,y)}{\partial x^2}$, $g_t = \frac{\partial g(t,x)}{\partial t}$, $g_x = \frac{\partial g(t,x)}{\partial x}$ *and* $g_{xx} = \frac{\partial^2 g(t,x)}{\partial x^2}$ *are the partial derivatives of* $f(t,x,y)$ *and* $g(t,x)$*.*

Proof. See Appendix B. □

Next, we give the explicit solution of the equilibrium strategy in the following theorem.

Theorem 3.1. *The equilibrium strategy (optimal time-consistent strategy) of the extended HJB system of equations (2.10) is given by*

$$
\begin{cases}
q^*(t) = [c_1(t)x + k_1(t)] \vee 0, \\
u^*(t) = [c_2(t)x + k_2(t)] \vee 0,
\end{cases}
\tag{3.2}
$$

where

$$
\begin{cases}
c_1(t) = \bar{G} \times \left[-\frac{1}{\gamma}\, e^{-\int_t^T \bar{F}_s\, ds} + 1 - e^{\int_t^T (\bar{A}_s - \bar{F}_s) ds} \right], \\
c_2(t) = \bar{H} \times \left[-\frac{1}{\gamma}\, e^{-\int_t^T \bar{F}_s\, ds} + 1 - e^{\int_t^T (\bar{A}_s - \bar{F}_s)\, ds} \right], \\
k_1(t) = \bar{G} \times \left[-e^{-\int_t^T \bar{F}_s\, ds} \times \int_t^T e^{\int_s^T \bar{A}_z\, dz}\, \bar{E}_s\, ds + \int_t^T e^{-\int_t^s \bar{F}_z\, dz}\, \bar{E}_s\, ds \right], \\
k_2(t) = \bar{H} \times \left[-e^{-\int_t^T \bar{F}_s\, ds} \times \int_t^T e^{\int_s^T \bar{A}_z\, dz}\, \bar{E}_s\, ds + \int_t^T e^{-\int_t^s \bar{F}_z\, dz}\, \bar{E}_s\, ds \right],
\end{cases}
\tag{3.3}
$$

and

$$\begin{aligned}
\bar{A}_t &= r_0 + \lambda\mu_1\theta c_1(t) + (r_1 - r_0)c_2(t); && (3.4)\\
\bar{E}_t &= \lambda\mu_1\theta k_1(t) + (r_1 - r_0)k_2(t) + \lambda\mu_1(\eta - \theta) - \lambda\mu_2 c_1(t)k_1(t)\\
&\quad -\sigma^2 c_2(t)k_2(t) - \rho\sigma\sqrt{\lambda\mu_2}c_1(t)k_2(t) - \rho\sigma\sqrt{\lambda\mu_2}c_2(t)k_1(t), && (3.5)\\
\bar{F}_t &= r_0 + \lambda\mu_1\theta c_1(t) + (r_1 - r_0)c_2(t) + \lambda\mu_2 c_1^2(t) + \sigma^2 c_2^2(t) + 2\rho\sqrt{\lambda\mu_2}\sigma c_1(t)c_2(t); && (3.6)\\
\bar{G} &= \frac{\rho\sqrt{\lambda\mu_2}(r_1 - r_0) - \sigma\lambda\mu_1\theta}{\sigma\lambda\mu_2(1-\rho^2)}; && (3.7)\\
\bar{H} &= \frac{\rho\sigma\sqrt{\lambda\mu_2}\lambda\mu_1\theta - \lambda\mu_2(r_1 - r_0)}{\sigma^2\lambda\mu_2(1-\rho^2)}. && (3.8)
\end{aligned}$$

Proof. See Appendix C. □

Remark 3.1. *We point out that when $\rho = 0$, the results in Theorem 3.1 recover the results of Li and Li (2013). So, in this section, we extend the research of Li and Li (2013).*

Next we consider the equilibrium value function. Because of the constraints of $q(\cdot) \geq 0$, $u(\cdot) \geq 0$, we need to discuss the following four cases:

$$\begin{cases}
\textbf{Case A: } \bar{G} < 0, \ \bar{H} < 0,\\
\textbf{Case B: } \bar{G} < 0, \ \bar{H} \geq 0,\\
\textbf{Case C: } \bar{G} \geq 0, \ \bar{H} < 0,\\
\textbf{Case D: } \bar{G} \geq 0, \ \bar{H} \geq 0.
\end{cases}$$

We only give the detail discussion for **Case A** in the following theorem. The results in other cases can be derived similarly.

Theorem 3.2. *For **Case A** if the initial reserve x at the initial time t satisfies*

$$\begin{aligned}
&x \times \left[-\frac{1}{\gamma}e^{-\int_t^T \bar{F}_s ds} + 1 - e^{\int_t^T \bar{A}_s ds}e^{-\int_t^T \bar{F}_s ds}\right]\\
&+\left[-e^{-\int_t^T \bar{F}_s ds} \times \int_t^T e^{\int_s^T \bar{A}_z dz}\bar{E}_s ds + \int_t^T e^{-\int_t^s \bar{F}_z dz}\bar{E}_s ds\right] < 0, \qquad (3.9)
\end{aligned}$$

the equilibrium value function of the extended HJB system of equations (2.10) is given by

$$\begin{aligned}
V(t,x) &= x\left[P_1(t) + \frac{\gamma}{2}P_1^2(t) - \frac{\gamma}{2}P_2(t)\right]\\
&\quad +Q_1(t) - \frac{\gamma}{2}Q_2(t) + \gamma P_1(t)Q_1(t) + \frac{\gamma}{2x}\left[Q_1^2(t) - R(t)\right], \qquad (3.10)
\end{aligned}$$

where $P_1(t)$, $Q_1(t)$, $P_2(t)$, $Q_2(t)$ and $R(t)$ are given in (6.10), (6.11), (6.12), (6.13) and (6.14) respectively.

Otherwise, if the initial reserve x at the initial time t satisfies

$$\begin{aligned} &x \times \left[-\frac{1}{\gamma} e^{-\int_t^T \bar{F}_s ds} + 1 - e^{\int_t^T \bar{A}_s ds} e^{-\int_t^T \bar{F}_s ds}\right] \\ &+ \left[-e^{-\int_t^T \bar{F}_s ds} \times \int_t^T e^{\int_s^T \bar{A}(z)dz} \bar{E}_s ds + \int_t^T e^{-\int_t^s \bar{F}_z dz} \bar{E}_s ds\right] \geq 0, \end{aligned} \tag{3.11}$$

the equilibrium value function of the extended HJB system of equations (2.10) is given by

$$V(t,x) = e^{r_0(T-t)} x + \frac{\lambda \mu_1(\eta - \theta)}{r_0} \left[e^{r_0(T-t)} - 1\right]. \tag{3.12}$$

Proof. See Appendix D. □

In the following theorem, we show that the system of integral equations (3.3) has a unique global solution.

Theorem 3.3. *The system of integral equations (3.3) admits a unique solution $c_1(t)$, $c_2(t)$, $k_1(t)$, $k_2(t) \in C[0,T]$, where $C[0,T]$ is the space of continuous functions defined on $[0,T]$.*

Proof. The theorem can be obtained easily by arguments similar to those used in Li and Li (2013) (or Björk and Murgoci 2010, Björk et al. 2014 and Zhang and Liang 2016). Thus, we omit its proof. □

4 The equilibrium strategies under VaR constraints

In this section, we will use the results in Section 3 to solve the optimal investment-reinsurance problem with VaR constraints (2.6). To do so, we make the following assumption.

Assumption 4.1. *We assume that*

$$\overline{\text{VaR}} \geq \frac{e^{r_0 h} - 1}{r_0} \lambda \mu_1(\theta - \eta) \tag{4.1}$$

and $\Phi^{-1}(p) < 0$ or equivalently $p < 1/2$.

We point out that the conditions of Assumption 4.1 are mild ones. To see that, note that the VaR control level $\overline{\text{VaR}}$ is a given constant and usually is a large value. With small time step h, the right hand side of (4.1) is small, in fact, the right hand side of (4.1) converges to zero as $h \to 0$. So with a big $\overline{\text{VaR}}$ or a small h, (4.1) can hold easily. In addition, in practice, the risk level for VaR is a small value such as $p = 0.01, 0.05$. Hence the condition $p < 1/2$ can be also satisfied easily.

In the following proposition, we give an equivalent expression for the VaR constraints (2.6).

Proposition 4.1. *Under Assumption 4.1, the VaR constraints (2.6) or* $\text{VaR}_t^{p,h,\pi} \leq \overline{\text{VaR}}$ *is equivalent to*

$$\tilde{A}q^2(t) + \tilde{B}u^2(t) + \tilde{C}q(t)u(t) + \tilde{D}q(t) + \tilde{E}u(t) + \tilde{F} \leq 0, \tag{4.2}$$

where

$$\begin{aligned}
\tilde{A} &= (\Phi^{-1}(p))^2 \frac{e^{2r_0h}-1}{2r_0}\lambda\mu_2 - \left(\frac{e^{r_0h}-1}{r_0}\lambda\mu_1\theta\right)^2; \\
\tilde{B} &= (\Phi^{-1}(p))^2 \frac{e^{2r_0h}-1}{2r_0}\sigma^2 - \left[\frac{e^{r_0h}-1}{r_0}(r_1-r_0)\right]^2; \\
\tilde{C} &= (\Phi^{-1}(p))^2 \frac{e^{2r_0h}-1}{r_0}\rho\sigma\sqrt{\lambda\mu_2} - 2\left(\frac{e^{r_0h}-1}{r_0}\right)^2 \lambda\mu_1\theta(r_1-r_0); \\
\tilde{D} &= -2\frac{e^{r_0h}-1}{r_0}\lambda\mu_1\theta\left[\frac{e^{r_0h}-1}{r_0}\lambda\mu_1(\eta-\theta)+\overline{\text{VaR}}\right]; \\
\tilde{E} &= -2\frac{e^{r_0h}-1}{r_0}(r_1-r_0)\left[\frac{e^{r_0h}-1}{r_0}\lambda\mu_1(\eta-\theta)+\overline{\text{VaR}}\right]; \\
\tilde{F} &= -\left[\frac{e^{r_0h}-1}{r_0}\lambda\mu_1(\eta-\theta)+\overline{\text{VaR}}\right]^2.
\end{aligned} \tag{4.3}$$

Proof. By (2.7), we see that

$$\begin{aligned}
& \text{VaR}_t^{p,h,\pi} \leq \overline{\text{VaR}} \\
\Longleftrightarrow\quad & -\Phi^{-1}(p)\sqrt{\frac{e^{2r_0h}-1}{2r_0}\left[\lambda\mu_2q^2(t)+\sigma^2u^2(t)+2\rho\sqrt{\lambda\mu_2}\sigma q(t)u(t)\right]} \leq \overline{\text{VaR}} \\
& +\frac{e^{r_0h}-1}{r_0}[\lambda\mu_1\theta q(t)+(r_1-r_0)u(t)+\lambda\mu_1(\eta-\theta)] \\
\Longleftrightarrow\quad & (\Phi^{-1}(p))^2\frac{e^{2r_0h}-1}{2r_0}\left[\lambda\mu_2q^2(t)+\sigma^2u^2(t)+2\rho\sqrt{\lambda\mu_2}\sigma q(t)u(t)\right] \leq \\
& \left\{\overline{\text{VaR}}+\frac{e^{r_0h}-1}{r_0}[\lambda\mu_1\theta q(t)+(r_1-r_0)u(t)+\lambda\mu_1(\eta-\theta)]\right\}^2 \\
\Longleftrightarrow\quad & \tilde{A}q^2(t)+\tilde{B}u^2(t)+\tilde{C}q(t)u(t)+\tilde{D}q(t)+\tilde{E}u(t)+\tilde{F} \leq 0.
\end{aligned}$$

In the above calculation, we use Assumption 4.1. □

Remark 4.1 *Note that* $\tilde{F} \leq 0$*, so there exists at least one strategy* $(q(t), u(t)) \equiv (0, 0)$ *that satisfies (4.2). Then the control space defined in Proposition 4.1 is not empty.*

Remark 4.2. *After a simple calculation, we see that if*

$$\tilde{C}^2 - 4\tilde{A}\tilde{B} < 0,$$

which is equivalent to

$$\left[\Phi^{-1}(p)\right]^2 > \frac{\left(\frac{e^{r_0 h}-1}{r_0}\right)^2 \left\{[\lambda\mu_2(r_1-r_0) - \sigma\lambda\mu_1\theta]^2 + 2(1-\rho)\sigma\sqrt{\lambda\mu_2}\lambda\mu_1\theta(r_1-r_0)\right\}}{\frac{e^{2r_0 h}-1}{2r_0}\sigma^2\lambda\mu_2(1-\rho^2)}, \tag{4.4}$$

then, the control space of the strategy $(q(\cdot), u(\cdot))$ is the first quadrant of an ellipse; otherwise, it is the first quadrant of a parabolic.

Note that $\Phi^{-1}(p) \to -\infty$ as $p \to 0$. Hence, $[\Phi^{-1}(p)]^2 \to \infty$ as $p \to 0$. Thus, the condition (4.4) will hold for a small value of p and the control space of the strategy $(q(\cdot), u(\cdot))$ is the first quadrant of an ellipse for the small p.

Now, we can solve the optimization problem (2.8) subjected to (2.2) and (2.6), as well as $q(t) \geq 0$, $u(t) \geq 0$. We denote the optimal solution of this problem by π^*_{VaR} if it exists.

According to Theorem 3.1 and Proposition 4.1, the equilibrium strategy under VaR constraints should satisfy

$$\begin{cases} q(t) = c_1(t)x + k_1(t), \\ u(t) = c_2(t)x + k_2(t), \\ q(t) \geq 0, \\ u(t) \geq 0, \\ \tilde{A}[q(t)]^2 + \tilde{B}[u(t)]^2 + \tilde{C}q(t)u(t) + \tilde{D}q(t) + \tilde{E}u(t) + \tilde{F} \leq 0, \end{cases} \tag{4.5}$$

where $c_1(t)$, $c_2(t)$, $k_1(t)$, $k_2(t)$ are given in (3.3), and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$, $\tilde{F}$ are given in (4.3). The first two equations in (4.5) are the equilibrium strategy without any constraint. The third and fourth inequalities in (4.5) present the nonnegative constraints on the strategy, and the fifth inequality in (4.5) denotes the VaR constraint. The last three inequalities in (4.5) constitute the control space, which is the first quadrant of an ellipse or a parabolic (see Remark 4.2).

If the strategy $(q(t), u(t))$ defined in the first and second equations in (4.5) or the equilibrium strategy without any constraint satisfies the last three inequalities in (4.5) or locates in the control space for any $t \in [0, T]$, then the strategy $(q(t), u(t))$ is also a solution with the constraints, namely $(q(t), u(t)) = \pi^*_{\text{VaR}}$. Otherwise, if the strategy $(q(t), u(t))$ defined in the first and second equations in (4.5) or the equilibrium strategy without any constraint does not satisfy the last three inequalities in (4.5) or does not locate in the control space for some $t \in [0, T]$, we consider the following two situations. First, if the strategy $(q(t), u(t))$ defined in the first and second equations in (4.5) is outside the control space at the initial time $t = 0$, then the equilibrium investment-reinsurance strategy is just the boundary of the control space. Second, if the strategy $(q(t), u(t))$ defined in the first and second equations in (4.5) is inside the control space at the initial time $t = 0$, but it leaves the control space at sometime before T, we define the first exit time $\bar{t}$ of the control space as

$$\begin{aligned}\bar{t} := \inf\Big\{ & t > 0 : \tilde{A}[c_1(t)x + k_1(t)]^2 + \tilde{B}[c_2(t)x + k_2(t)]^2 + \tilde{C}[c_1(t)x + k_1(t)] \times \\ & [c_2(t)x + k_2(t)] + \tilde{D}[c_1(t)x + k_1(t)] + \tilde{E}[c_2(t)x + k_2(t)] + \tilde{F} > 0, \\ & \text{or } c_1(t)x + k_1(t) < 0, \text{ or } c_2(t)x + k_2(t) < 0 \Big\}.\end{aligned}$$

Then, under the VaR constraints, the optimal time-consistent strategy (equilibrium strategy) is $\pi^*_{\text{VaR}}(t) = (q^*_{\text{VaR}}(t), u^*_{\text{VaR}}(t))$ with

$$q^*_{\text{VaR}}(t) = \begin{cases} c_1(t)x + k_1(t), & t \in [0, \bar{t} \wedge T], \\ c_1(\bar{t})x + k_1(\bar{t}), & t \in (\bar{t} \wedge T, T], \end{cases} \tag{4.6}$$

and

$$u^*_{\text{VaR}}(t) = \begin{cases} c_2(t)x + k_2(t), & t \in [0, \bar{t} \wedge T], \\ c_2(\bar{t})x + k_2(\bar{t}), & t \in (\bar{t} \wedge T, T]. \end{cases} \tag{4.7}$$

Remark 4.3. *We can extend our model and results to the financial market model with multiple risky assets which are correlated. Assume that there are m risky assets (stocks), and their price processes $P_i(t), i = 1, 2, \cdots, m$, satisfy the following SDEs*

$$\begin{cases} dP_i(t) = P_i(t)\left[r_{1i}dt + \sum_{j=1}^{m} \sigma_{ij} dW_{2j}(t)\right], & t \in [0, T], \\ P_i(0) = p_i, & i = 1, 2, \cdots, m, \end{cases}$$

where $r_1 := (r_{11}, r_{12}, \cdots, r_{1m})^\top$, $r_{1i} > r_0,\ i = 1, 2, \cdots, m$, is the appreciation rate, $\sigma := (\sigma_{ij})_{m\times m}$ is the volatility coefficient, $W_2(t) := (W_{21}(t), W_{22}(t), \cdots, W_{2m}(t))^\top$ is a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted m-dimensional Brownian motion, with the superscript $\top$ means the transpose of a matrix or a vector. The Brownian motion $W_1(t)$ in the approximated reserve process of the insurer and the m-dimensional Brownian motion $W_2(t)$ in the risky asset are possibly correlated with correlation *coefficient* $\rho := (\rho_1, \rho_2, \cdots, \rho_m)^\top$, $\rho_i \in [-1, 1]$, $i = 1, 2, \cdots, m$, *which represents the dependence between the stock market and the insurance market. Let* $u_i(t)$, $i = 1, 2, \cdots, m$, *denote the total market value of the insurer's wealth in the i-th risky asset at time t.* $u(t) := (u_1(t), u_2(t), \cdots, u_m(t))^\top$ *is the investment strategy.*

All the main results about the equilibrium strategies presented in Proposition 3.1, Theorem 3.1, Theorem 3.2, and (4.5) can be obtained by the same arguments used in this paper for the financial market model with multiple risky assets. To show the results can be extended the financial market model with multiple risky assets, we give the equilibrium strategy for the financial market model with multiple risky assets as follows:

$$\begin{cases} q^* = \left\{ \dfrac{|\Sigma_1|}{|\Sigma|} \times \dfrac{f_x + \frac{\gamma}{x} g g_x}{f_{xx} + \frac{\gamma}{x} g g_{xx}} \right\} \vee 0, \\ u^* = \left\{ \dfrac{|\Sigma_2|}{|\Sigma|} \times \dfrac{f_x + \frac{\gamma}{x} g g_x}{f_{xx} + \frac{\gamma}{x} g g_{xx}} \right\} \vee 0, \end{cases} \tag{4.8}$$

where

$$\Sigma := \begin{pmatrix} \lambda\mu_2 & \sqrt{\lambda\mu_2}\rho^\top\sigma \\ \sqrt{\lambda\mu_2}\sigma^\top\rho & \sigma^\top\sigma \end{pmatrix}$$

$$\Sigma_1 := \begin{pmatrix} -\lambda\mu_1\theta & \sqrt{\lambda\mu_2}\rho^\top\sigma \\ -(r_1 - r_0\mathbf{1}) & \sigma^\top\sigma \end{pmatrix}$$

$$\Sigma_2 := \begin{pmatrix} \lambda\mu_2 & -\lambda\mu_1\theta \\ \sqrt{\lambda\mu_2}\sigma^\top\rho & -(r_1 - r_0\mathbf{1}) \end{pmatrix}$$

are $(m+1) \times (m+1)$ *matrices,* $|\Sigma|$ *means the determinant of matrix* Σ*, and* $\mathbf{1} := (1, 1, \cdots, 1)^\top$ *is a m-dimensional vector. The strategy (4.8) extends the equilibrium strategy of Proposition 3.1. We omit the proof of (4.8) since the proof is similar to that for Proposition 3.1.*

We will study the impact of the parameters on the control space and the equilibrium investment-reinsurance strategies under VaR constraints in the following section through some numerical examples.

5 Numerical examples

In this section, we illustrate the results obtained in Sections 3 and 4 by numerical examples.

5.1 The equilibrium investment-reinsurance strategies without VaR constraints

In this subsection, we numerically show the impact of the risk aversion and the correlation between the financial market and the insurance market on the equilibrium strategies without VaR constraints, which have been derived explicitly in (3.2) and (3.3) in Theorem 3.1. In doing so, we set the model parameters of the insurer's reserve process and the financial market in Table 1 with $x = 0.6$ and $T = 2$ (years).

λ	μ_1	μ_2	r_0	r_1	σ	η	θ
1	0.1	0.2	0.1	0.2	0.6	0.3	0.5

Table 1: Parameters of the insurance market and the financial market.

- First, we keep the correlation coefficient $\rho = 0.3$ and calculate the equilibrium strategies $(q^*(t), u^*(t))$ by using (3.2) and (3.3) for different risk aversion parameters of $\gamma = 1,\ 2,\ 3$. The equilibrium strategies are shown in Figure 1. From Figure 1, we see that for a given risk aversion γ, both the optimal reinsurance retention levels of $q^*(t)$ and the optimal investment amounts of $u^*(t)$ to the risky asset increase as t increases, which means that the insurer should retain more and more insurance risks and invest more and more money into the risky asset if it has no VaR constraints. Moreover, at a given time t, both the optimal reinsurance retention levels and the optimal investment amounts to the risky asset are decreasing while γ increases, which are reasonable because a large value of γ means that the insurer is more risk averse. Such an insurer (more risk averse) would like to retain less proportion of the insurance risk and to invest less money into the risky asset.

- Second, we keep the risk aversion $\gamma = 1$ and calculate the equilibrium investment-reinsurance strategies $(q^*(t), u^*(t))$ by using (3.2) and (3.3) for different correlation coefficients of $\rho = 0,\ 0.15,\ 0.3$. The equilibrium investment-reinsurance strategies $(q^*(t), u^*(t))$ are shown in Figure 2. From Figure 2, we can see that at a given time t, both $q^*(t)$ and $u^*(t)$ are decreasing while ρ increases. These findings are also reasonable because a large value of ρ means both the financial market and the insurance market are more risky, so the insurer will retain less proportion of the insurance risks and invest less money into the risky asset. In addition, for a given correlation coefficient ρ, both the optimal reinsurance retention levels of $q^*(t)$ and the optimal investment amounts of $u^*(t)$ to the risky asset increase as t increase, which again

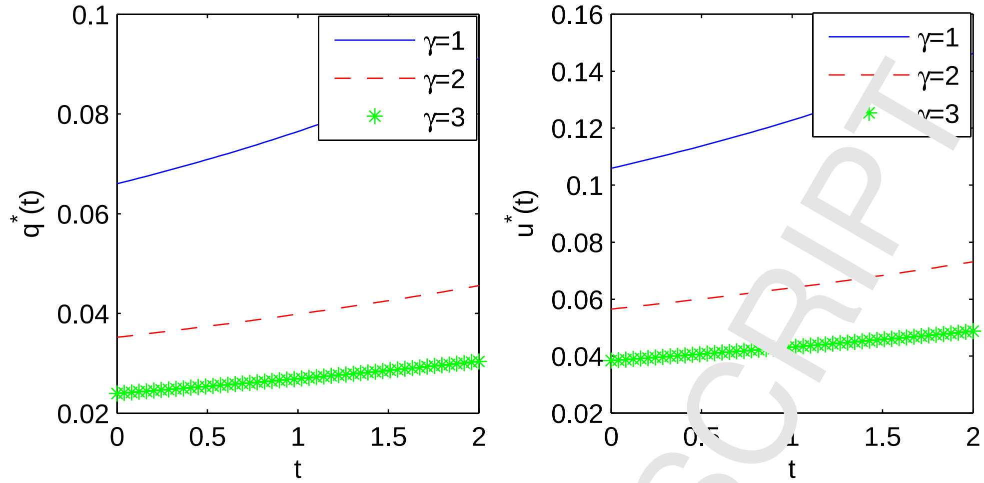

Figure 1: $q^*(t)$ and $u^*(t)$ for $x = 0.6$, $\rho = 0.3$; $\gamma = 1, 2, 3$ without VaR constraint.

means that the insurer should retain more and more insurance risks and invest more and more money into the risky asset if it has no VaR constraints.

5.2 The equilibrium investment-reinsurance strategies with VaR constraints

In this subsection, we numerically illustrate the influence of VaR constraints in the equilibrium investment-reinsurance strategies with VaR constraints, which have been presented explicitly in Section 4.

We use the model parameters of the insurer's reserve process and the financial market as in Table 1 with $\rho = 0.3$ and consider the VaR control levels $\overline{\text{VaR}}$ as well as the risk levels p for three different cases/combinations as in Table 2, where the time interval h is equal to $\frac{1}{12}$ (one month), and the VaR control levels of $\overline{\text{VaR}}$ are set so that (4.1) holds.

- First, we show the effects of VaR constraints on the control space. By using (4.2) of Proposition 4.1, we present the numerical solutions of the control space with the VaR constraints under the model setting in Figure 3. From Figure 3 (Cases 1-2), we see that for fixed p, a bigger value of $\overline{\text{VaR}}$ (or a relaxed requirement on VaR control level) means a bigger control area. For fixed $\overline{\text{VaR}}$, a higher risk level p (or a lower confidence level) means a bigger control area, see Cases 2-3 in Figure 3.

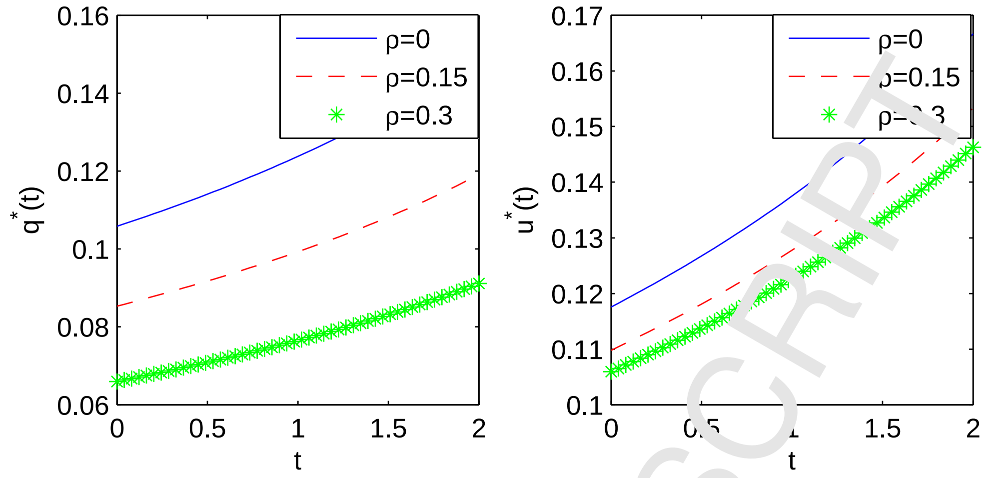

Figure 2: $q^*(t)$ and $u^*(t)$ for $x = 0.6$, $\gamma = 1$; $\rho = 0,\ 0.15,\ 0.3$ without VaR constraint.

	Case 1	Case 2	Case 3
$\overline{\text{VaR}}$	0.05	0.1	0.1
p	0.01	0.01	0.05

Table 2: VaR control levels and risk levels with time interval $h = 1/12$.

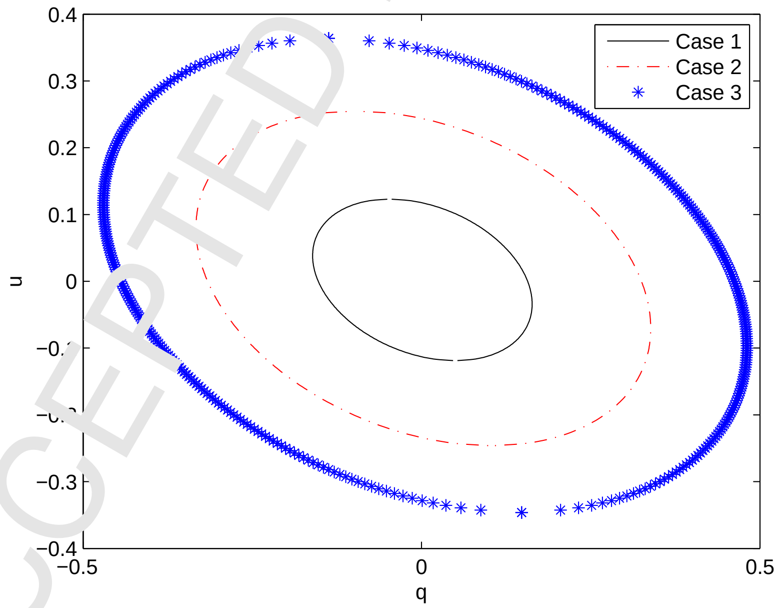

Figure 3: The control space under the VaR constraints in Cases 1-4.

- Second, for the VaR control level $\overline{\text{VaR}}$ and risk level p given in Case 1 of Table 2, time interval $h = \frac{1}{12}$ and the model parameter values given in Table 1 with $x = 0.6$, and $T = 2$, we illustrate the impact of the VaR constraints on the equilibrium investment-reinsurance strategies by calculating the equilibrium investment-reinsurance strategies, respectively, for different risk aversion parameters and different correlation coefficients by using (4.6) and (4.7).

  We keep the correlation coefficient $\rho = 0.3$, and obtain the equilibrium investment-reinsurance strategies for different risk aversion parameters of $\gamma = 1,\ 2,\ 3$, which are presented in Figure 4. From Proposition 4.1, we know that the control space does not depend on γ. The equilibrium reinsurance strategy $q^*_{\text{VaR}}(t)$ and the equilibrium investment strategy $u^*_{\text{VaR}}(t)$ have the upper bounds 0.0609 and 0.0978, respectively, for different values of γ, over the investment period. For $\gamma = 2$ or 3, the equilibrium reinsurance strategy $q^*_{\text{VaR}}(t)$ and the equilibrium investment strategy $u^*_{\text{VaR}}(t)$ with VaR constraint are the same as $q^*(t)$ and $u^*(t)$ without VaR constraint, because they do not exceed the upper bounds 0.0609 and 0.0978 over the whole investment period. For $\gamma = 1$, because $q^*(t)$ and $u^*(t)$ without VaR constraint exceed the upper bounds 0.0609 and 0.0978 at the initial time, the equilibrium reinsurance strategy $q^*_{\text{VaR}}(t)$ and the equilibrium investment strategy $u^*_{\text{VaR}}(t)$ with VaR constraint are 0.0609 and 0.0978, respectively, over the whole investment period. The result can be seen in Figure 1 and Figure 4.

  Moreover, we keep the risk aversion parameter $\gamma = 1$, and obtain the equilibrium investment-reinsurance strategies for different correlation coefficients of $\rho = 0,\ 0.15,\ 0.3$, which are presented in Figure 5. From Proposition 4.1, we know that the value of ρ influences $\tilde{C}$, so the control space depends on ρ. The equilibrium reinsurance strategy $q^*_{\text{VaR}}(t)$ and the equilibrium investment strategy $u^*_{\text{VaR}}(t)$ have the different upper bounds for different values of ρ. When $\rho = 0$, the upper bounds of the equilibrium reinsurance strategy and the equilibrium investment strategy are 0.0888 and 0.0987, respectively. When $\rho = 0.15$, the upper bounds of the equilibrium reinsurance strategy and the equilibrium investment strategy are 0.0748 and 0.0982, respectively. When $\rho = 0.3$, the upper bounds of the equilibrium reinsurance strategy and the equilibrium investment strategy are 0.0609 and 0.0978, respectively. For $\rho = 0,\ 0.15,\ 0.3$, because $q^*(t)$ and $u^*(t)$ without VaR constraint exceed the upper bounds at the initial time, so the equilibrium reinsurance strategy $q^*_{\text{VaR}}(t)$ and the equilibrium investment strategy $u^*_{\text{VaR}}(t)$ equal to the upper bounds over the whole investment period. The result can

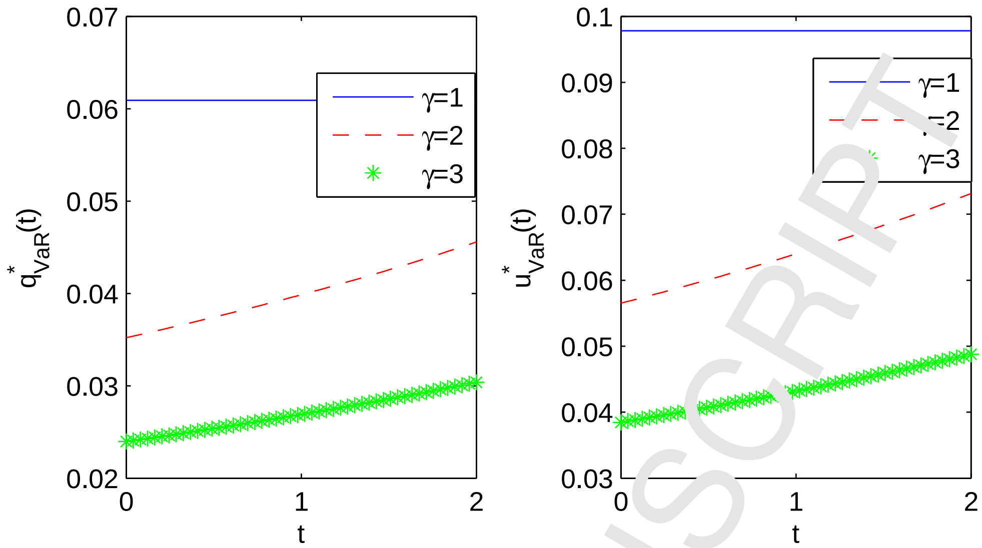

Figure 4: The equilibrium strategies under the VaR constraints for different γ (Case 1).

be seen in Figure 2 and Figure 5.

- Third, we illustrate the impact of the VaR constraints on the equilibrium investment-reinsurance strategies by calculating the equilibrium investment-reinsurance strategies in Cases 1 and 2 of Table 2, respectively, for different initial capitals of x by using (4.6) and (4.7).

  Case 1 of Table 2: For the VaR control level $\overline{\text{VaR}}$ and risk level p given in Case 1 of Table 2, time interval $h = \frac{1}{12}$ and the model parameter values given in Table 1 with $\rho = 0.3$, $\gamma = 1$ and $T = 2$, by using (4.6) and (4.7), we obtain the equilibrium investment-reinsurance strategies for different initial capitals of $x = 0.3,\ 0.4,\ 0.5,\ 0.6$, which are presented in Figure 6. Under the VaR constraints, at a given time t, both the equilibrium reinsurance strategy $q^*_{\text{VaR}}(t)$ and the equilibrium investment strategy $u^*_{\text{VaR}}(t)$ are increasing while the initial capital x increases. It is a reasonable result because when the insurer has a bigger initial wealth x, the insurer would like to retain a bigger proportion of its insurance risks and to invest more money into the risky asset. Moreover, the equilibrium reinsurance strategy $q^*_{\text{VaR}}(t)$ and the equilibrium investment strategy $u^*_{\text{VaR}}(t)$ have the upper bounds 0.0609 and 0.0978, respectively for different values of x, over the investment period. The upper bounds represent the effects of the VaR constraints on the equilibrium strategies, which implies that to limit the loss of the insurer at the VaR control level $\overline{\text{VaR}}$, the insurer has to limit both the amounts

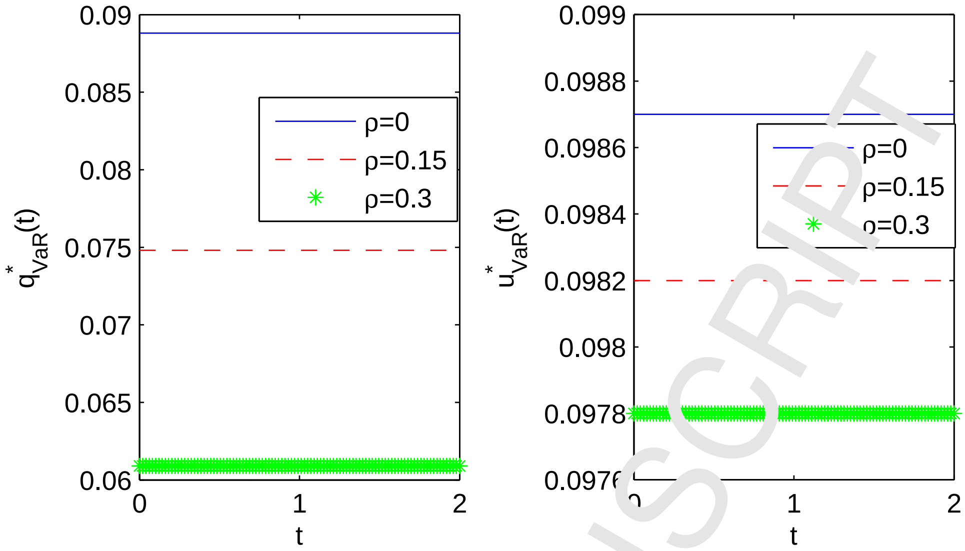

Figure 5: The equilibrium strategies under the VaR constraints for different ρ (Case 1).

invested into the risky asset and the retained insurance risks.

Case 2 of Table 2: For the VaR control level $\overline{\mathrm{VaR}}$ and risk level p given in Case 2 of Table 2, the time interval $h = \frac{1}{12}$ and the model parameter values given in Table 1 with $\rho = 0.3$, $\gamma = 1$ and $T = 2$, by using (4.6) and (4.7), we obtain the equilibrium investment-reinsurance strategies for different initial capitals of $x = 0.3,\ 0.4,\ 0.5,\ 0.6,\ 1.0$, which are presented in Figure 7. In this case, the equilibrium reinsurance strategy $q^*_{\mathrm{VaR}}(t)$ and the equilibrium investment strategy $u^*_{\mathrm{VaR}}(t)$ have the upper bounds 0.1293 and 0.2076, respectively, for different values of x. By comparing Figure 6 with Figure 7, we see that the upper bounds of the equilibrium reinsurance strategy $q^*_{\mathrm{VaR}}(t)$ and the equilibrium investment strategy $u^*_{\mathrm{VaR}}(t)$ in Case 2 are bigger than those in Case 1, which means that the insurer can invest more money into the risky asset and retain more insurance risks in Case 2 than in Case 1. This finding is consistent with the fact that the insurer has a tougher VaR control level of 0.05 in Case 1 than that the VaR control level 0.1 in Case 2.

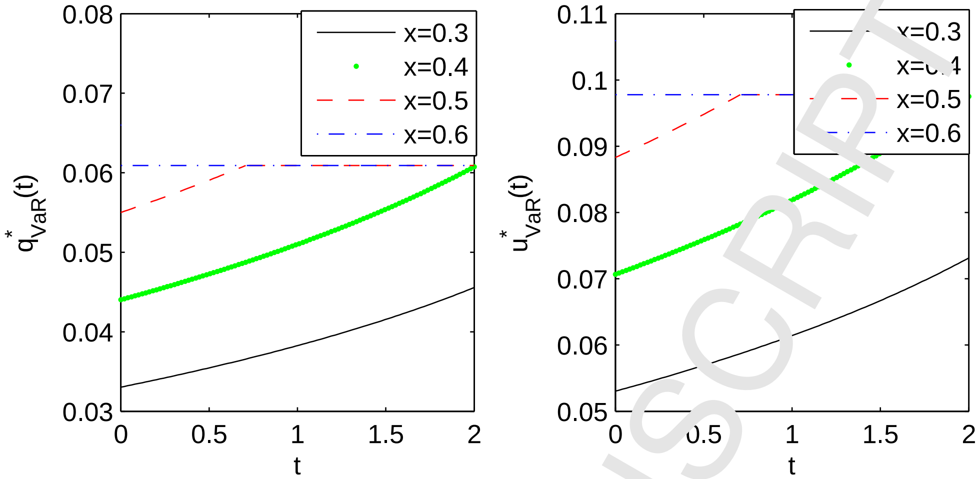

Figure 6: The equilibrium strategies under the VaR constraints for different x (Case 1).

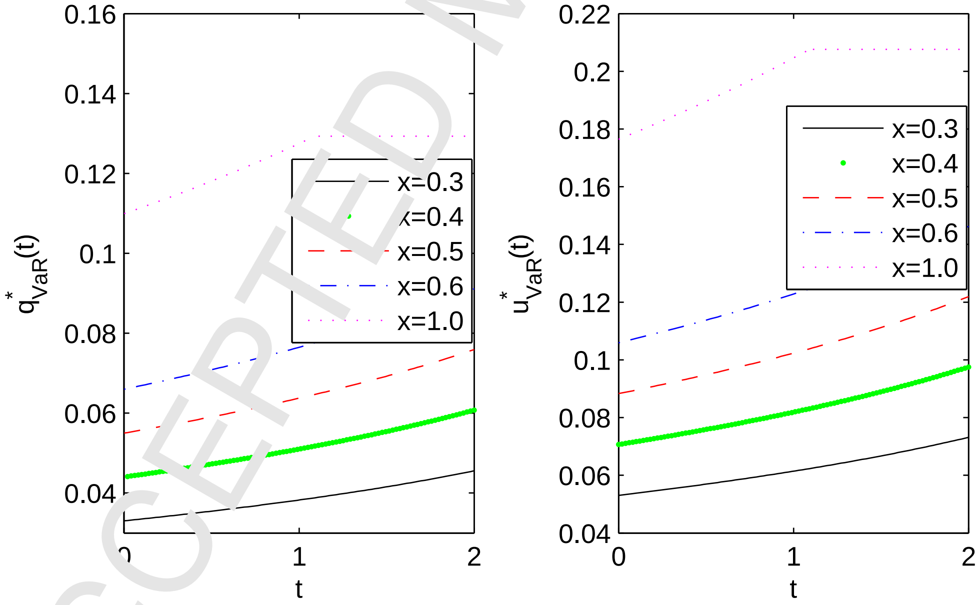

Figure 7: The equilibrium strategies under the VaR constraints for different x (Case 2).

6 Concluding remarks

This paper studies the insurer's optimal time-consistent investment-reinsurance strategies (equilibrium strategies) under the mean-variance criterion with state dependent risk aversion and VaR constraints and discuss the impact of the risk aversion, the correlation between the insurance market and the financial market, the risk level, and the VaR control level on the investment-reinsurance strategies for an insurer. The results suggest that the more risk aversion an insurer has, the less insurance risks it will retain and the less money it will invest into any risky asset, and that if there is a stronger correlation between the insurance market and the financial market, the insurer should retain less insurance risks and invest less money into risky assets. If there is a VaR constraint on the loss of the insurer, it has to limit both the retained insurance risk and the amounts invested into the risky assets. These results and findings are consistent with the practices of an insurer in investment-reinsurance decisions.

Appendix A: The proof of Lemma 2.1

Proof. We have

$$\begin{aligned}
&\mathbb{P}(\Delta X_{t,h}^{\pi} \geq L|\mathcal{F}_t) = \mathbb{P}(X_t^{\pi} e^{r_0 h} - X_{t+h}^{\pi} \geq L \,\big|\, \mathcal{F}_t) \\
= \;& \mathbb{P}\Big(\int_t^{t+h} e^{r_0(t+h-z)} \left[q(t)\sqrt{\lambda\mu_2}dW_1(z) + u(t)\sigma dW_2(z)\right] \\
&\quad \leq -L - \frac{e^{r_0 h}-1}{r_0}\left[\lambda\mu_1\theta q(t) + (r_1-r_0)u(t) + \lambda\mu_1(\eta-\theta)\right] \,\Big|\, \mathcal{F}_t \Big) \\
= \;& \mathbb{P}\Bigg(\frac{\int_t^{t+h} e^{r_0(t+h-z)} \left[q(t)\sqrt{\lambda\mu_2}dW_1(z) + u(t)\sigma dW_2(z)\right]}{\sqrt{\frac{e^{2r_0 h}-1}{2r_0}\left[\lambda\mu_2 q^2(t) + \sigma^2 u^2(t) + 2\rho\sqrt{\lambda\mu_2}\sigma q(t)u(t)\right]}} \\
&\quad \leq \frac{-L - \frac{e^{r_0 h}-1}{r_0}\left[\lambda\mu_1\theta q(t) + (r_1-r_0)u(t) + \lambda\mu_1(\eta-\theta)\right]}{\sqrt{\frac{e^{2r_0 h}-1}{2r_0}\left[\lambda\mu_2 q^2(t) + \sigma^2 u^2(t) + 2\rho\sqrt{\lambda\mu_2}\sigma q(t)u(t)\right]}} \,\Bigg|\, \mathcal{F}_t \Bigg) \\
= \;& \Phi\Bigg(\frac{-L - \frac{e^{r_0 h}-1}{r_0}\left[\lambda\mu_1\theta q(t) + (r_1-r_0)u(t) + \lambda\mu_1(\eta-\theta)\right]}{\sqrt{\frac{e^{2r_0 h}-1}{2r_0}\left[\lambda\mu_2 q^2(t) + \sigma^2 u^2(t) + 2\rho\sqrt{\lambda\mu_2}\sigma q(t)u(t)\right]}} \Bigg),
\end{aligned} \tag{6.1}$$

where the last equality follows from the fact that the random variable

$$\int_t^{t+h} e^{r_0(t+h-z)} \left[q(t)\sqrt{\lambda\mu_2}dW_1(z) + u(t)\sigma dW_2(z)\right] \tag{6.2}$$

conditionally on the filtration $\mathcal{F}_t$, is normally distributed with mean zero and variance

$$\frac{e^{2r_0 h}-1}{2r_0}\left[\lambda\mu_2 q^2(t) + \sigma^2 u^2(t) + 2\rho\sqrt{\lambda\mu_2}\sigma q(t)u(t)\right]. \tag{6.3}$$

Thus

$$
\begin{aligned}
& \mathbb{P}(X_t^\pi e^{r_0 h} - X_{t+h} \geq L|\mathcal{F}_t) < p \\
\iff \quad & \Phi\left(\frac{-L - \frac{e^{r_0 h}-1}{r_0}\left[\lambda\mu_1\theta q(t) + (r_1 - r_0)u(t) + \lambda\mu_1(\eta-\theta)\right]}{\sqrt{\frac{e^{2r_0 h}-1}{2r_0}\left[\lambda\mu_2 q^2(t) + \sigma^2 u^2(t) + 2\rho\sqrt{\lambda\mu_2}\sigma q(t)u(t)\right]}}\right) < p \\
\iff \quad & L > -\frac{e^{r_0 h}-1}{r_0}\left[\lambda\mu_1\theta q(t) + (r_1 - r_0)u(t) + \lambda\mu_1(\eta-\theta)\right] \\
& -\Phi^{-1}(p)\sqrt{\frac{e^{2r_0 h}-1}{2r_0}\left[\lambda\mu_2 q^2(t) + \sigma^2 u^2(t) + 2\rho\sqrt{\lambda\mu_2}\sigma q(t)u(t)\right]}.
\end{aligned}
$$

According to the definition (2.5), we obtain (2.7). □

Appendix B: The proof of Proposition 3.1

Proof. Recall the wealth process and the infinitesimal generator given in (2.2) and (2.9), respectively, we have

$$
\begin{aligned}
\mathcal{A}^\pi V(t,x) &= V_t + \left[r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)\right] V_x \\
&\quad + \frac{1}{2}\left[\lambda\mu_2 q^2 + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2}qu\right] V_{xx}; \\
\mathcal{A}^\pi f(t,x,x) &= f_t(t,x,x) + \left[r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)\right] \times \left[f_x(t,x,x) + f_y(t,x,x)\right] \\
&\quad + \frac{1}{2}\left[\lambda\mu_2 q^2 + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2}qu\right] \times \left[f_{xx}(t,x,x) + f_{yy}(t,x,x) + 2f_{xy}(t,x,x)\right]; \\
\mathcal{A}^\pi f^x(t,x) &= f_t(t,x,x) + \left[r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)\right] f_x(t,x,x) \\
&\quad + \frac{1}{2}\left[\lambda\mu_2 q^2 + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2}qu\right] f_{xx}(t,x,x); \\
\mathcal{A}^\pi(G\diamond g)(t,x) &= \mathcal{A}^\pi G(x, g(t,x)) = G_y g_t + \left[r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)\right] \times (G_x + G_y g_x) \\
&\quad + \frac{1}{2}\left[\lambda\mu_2 q^2 + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2}qu\right] \times \left[G_{xx} + G_{yy}g_x^2 + G_y g_{xx} + 2G_{xy}g_x\right]; \\
\mathcal{H}^\pi g(t,x) &= G_y(x, g(t,x)) \times \mathcal{A}^\pi g(t,x) \\
&= G_y\Big\{g_t + \left[r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)\right] g_x \\
&\quad + \frac{1}{2}\left[\lambda\mu_2 q^2 + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2}qu\right] g_{xx}\Big\};
\end{aligned}
$$

where G is evaluated at $G(x, g(t,x))$ and g is evaluated at $g(t,x)$.

Thus the extended HJB system of equations (2.10) can be rewritten as the following system of equations:

$$
\begin{cases}
V_t + \sup_{\pi\in\Pi} \Big\{ [r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)] \times \left[V_x - f_y - \frac{\gamma'(x)}{2} g^2\right] + \frac{1}{2}[\lambda\mu_2 q^2 \\
\qquad + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2} q u] \times \left[V_{xx} - f_{yy} - 2f_{xy} - \frac{\gamma''(x)}{2} g^2 - 2\gamma'(x) g g_x - \gamma(x) g_x^2\right] \Big\} = 0; \\
f_t(t,x,y) + [r_0 x + \lambda\mu_1\theta q^* + (r_1 - r_0)u^* + \lambda\mu_1(\eta-\theta)]\, f_x(t,x,y) \\
\qquad + \frac{1}{2}\left[\lambda\mu_2 (q^*)^2 + \sigma^2 (u^*)^2 + 2\rho\sigma\sqrt{\lambda\mu_2} q^* u^*\right] f_{xx}(t,x,y) = 0; \\
g_t(t,x) + [r_0 x + \lambda\mu_1\theta q^* + (r_1 - r_0)u^* + \lambda\mu_1(\eta-\theta)]\, g_x(t,x) \\
\qquad + \frac{1}{2}\left[\lambda\mu_2 (q^*)^2 + \sigma^2 (u^*)^2 + 2\rho\sigma\sqrt{\lambda\mu_2} q^* u^*\right] g_{xx}(t,x) = 0.
\end{cases}
\tag{6.4}
$$

Note that $\gamma(x) = \frac{\gamma}{x}$ with $\gamma'(x) = -\frac{\gamma}{x^2}$, $\gamma''(x) = \frac{2\gamma}{x^3}$ and

$$V(t,x) = f(t,x,x) + \frac{\gamma}{2x} g^2(t,x).$$

Thus we have

$$
\begin{aligned}
V_t &= f_t + \frac{\gamma}{x} g g_t, \\
V_x &= f_x + f_y - \frac{\gamma}{2x^2} g^2 + \frac{\gamma}{x} g g_x, \\
V_{xx} &= f_{xx} + f_{yy} + 2f_{xy} + \frac{\gamma}{x^3} g^2 - \frac{\gamma}{x^2} g g_x - \frac{\gamma}{x^2} g g_x + \frac{\gamma}{x} g_x^2 + \frac{\gamma}{x} g g_{xx},
\end{aligned}
$$

where f and its derivatives are evaluated at (t,x,x), while g and its derivatives are evaluated at (t,x). Using these expressions, the first equation of the system (6.4) becomes

$$
\begin{aligned}
f_t + \frac{\gamma}{x} g g_t + \sup_{q,u} \Big\{ & [r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)] \times \left[f_x + \frac{\gamma}{x} g g_x\right] \\
& + \frac{1}{2}\left[\lambda\mu_2 q^2 + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2} q u\right] \times \left[f_{xx} + \frac{\gamma}{x} g g_{xx}\right] \Big\} = 0.
\end{aligned}
\tag{6.5}
$$

Let

$$
\begin{aligned}
H(q,u) = {} & [r_0 x + \lambda\mu_1\theta q + (r_1 - r_0)u + \lambda\mu_1(\eta-\theta)] \times \left[f_x + \frac{\gamma}{x} g g_x\right] \\
& + \frac{1}{2}\left[\lambda\mu_2 q^2 + \sigma^2 u^2 + 2\rho\sigma\sqrt{\lambda\mu_2} q u\right] \times \left[f_{xx} + \frac{\gamma}{x} g g_{xx}\right].
\end{aligned}
$$

Differentiating the function $H(q,u)$ with respect to q and u respectively, we obtain

$$\begin{cases} \dfrac{\partial H(q,u)}{\partial q} = \left(\lambda\mu_2 q + \rho\sigma\sqrt{\lambda\mu_2}u\right)\left[f_{xx} + \dfrac{\gamma}{x}gg_{xx}\right] + \lambda\mu_1\theta\left[f_x + \dfrac{\gamma}{x}gg_x\right]; \\ \dfrac{\partial H(q,u)}{\partial u} = \left(\rho\sigma\sqrt{\lambda\mu_2}q + \sigma^2 u\right)\left[f_{xx} + \dfrac{\gamma}{x}gg_{xx}\right] + (r_1 - r_0)\left[f_x + \dfrac{\gamma}{x}gg_x\right]; \\ \dfrac{\partial H^2(q,u)}{\partial q^2} = \lambda\mu_2\left[f_{xx} + \dfrac{\gamma}{x}gg_{xx}\right]; \\ \dfrac{\partial H^2(q,u)}{\partial u^2} = \sigma^2\left[f_{xx} + \dfrac{\gamma}{x}gg_{xx}\right]; \\ \dfrac{\partial H^2(q,u)}{\partial q\partial u} = \rho\sigma\sqrt{\lambda\mu_2}\left[f_{xx} + \dfrac{\gamma}{x}gg_{xx}\right]. \end{cases}$$

The Hessian matrix is

$$H = \begin{pmatrix} \frac{\partial H^2(q,u)}{\partial q^2} & \frac{\partial H^2(q,u)}{\partial q\partial u} \\ \frac{\partial H^2(q,u)}{\partial q\partial u} & \frac{\partial H^2(q,u)}{\partial u^2} \end{pmatrix} = \begin{pmatrix} \lambda\mu_2 & \rho\sigma\sqrt{\lambda\mu_2} \\ \rho\sigma\sqrt{\lambda\mu_2} & \sigma^2 \end{pmatrix}\left[f_{xx} + \frac{\gamma}{x}gg_{xx}\right].$$

Because of

$$|H| = \sigma^2\lambda\mu_2(1-\rho^2)\left(f_{xx} + \frac{\gamma}{x}gg_{xx}\right)^2 \geq 0,$$

it is easy to see that the maximizer $(\hat{q},\hat{u})$ of (6.5) is the solution of the equations

$$\begin{cases} \left(\lambda\mu_2 q + \rho\sigma\sqrt{\lambda\mu_2}u\right)\left[f_{xx} + \dfrac{\gamma}{x}gg_{xx}\right] + \lambda\mu_1\theta\left[f_x + \dfrac{\gamma}{x}gg_x\right] = 0 \\ \left(\rho\sigma\sqrt{\lambda\mu_2}q + \sigma^2 u\right)\left[f_{xx} + \dfrac{\gamma}{x}gg_{xx}\right] + (r_1 - r_0)\left[f_x + \dfrac{\gamma}{x}gg_x\right] = 0. \end{cases}$$

That is,

$$\begin{cases} \hat{q} = \dfrac{\rho\sqrt{\lambda\mu_2}(r_1 - r_0) - \sigma\lambda\mu_1\theta}{\sigma\lambda\mu_2(1-\rho^2)} \times \dfrac{f_x + \frac{\gamma}{x}gg_x}{f_{xx} + \frac{\gamma}{x}gg_{xx}}, \\ \hat{u} = \dfrac{\rho\sqrt{\lambda\mu_2}\lambda\mu_1\theta - \lambda\mu_2(r_1 - r_0)}{\sigma^2\lambda\mu_2(1-\rho^2)} \times \dfrac{f_x + \frac{\gamma}{x}gg_x}{f_{xx} + \frac{\gamma}{x}gg_{xx}}. \end{cases} \tag{6.6}$$

This completes the proof. □

Appendix C: The proof of Theorem 3.1

Proof. From the form of $\hat{\pi} = (\hat{q},\hat{u})$ in (6.6), we conjecture that $\hat{q}$ and $\hat{u}$ are affine form of x. So we guess that

$$\begin{cases} \hat{q} = c_1(t)x + k_1(t); \\ \hat{u} = c_2(t)x + k_2(t); \end{cases} \tag{6.7}$$

for some deterministic functions $c_1,\ c_2,\ k_1$ and k_2. In this case, the wealth process (2.2) becomes

$$\begin{aligned} dX_t^{\hat{\pi}} &= \Big\{[r_0 + \lambda\mu_1\theta c_1(t) + (r_1 - r_0)c_2(t)]\, X_t^{\hat{\pi}} + [\lambda\mu_1\theta k_1(t) + (r_1 - r_0)k_2(t) + \lambda\mu_1(\eta - \theta)]\Big\}\, dt \\ &\quad + \sqrt{\lambda\mu_2}\left[c_1(t)X_t^{\hat{\pi}} + k_1(t)\right] dW_1(t) + \sigma\left[c_2(t)X_t^{\hat{\pi}} + k_2(t)\right] dW_2(t) \\ &= \left(\bar{A}_t X_t^{\hat{\pi}} + \bar{B}_t\right) dt + \left(\bar{C}_{1t}X_t^{\hat{\pi}} + \bar{D}_{1t}\right) dW_1(t) + \left(\bar{C}_{2t}X_t^{\hat{\pi}} + \bar{D}_{2t}\right) dW_2(t), \end{aligned}$$

with $\bar{A}_t$ given in (3.4) and

$$\begin{aligned} \bar{B}_t &= \lambda\mu_1\theta k_1(t) + (r_1 - r_0)k_2(t) + \lambda\mu_1(\eta - \theta), \\ \bar{C}_{1t} &= \sqrt{\lambda\mu_2}\, c_1(t), \\ \bar{D}_{1t} &= \sqrt{\lambda\mu_2}\, k_1(t), \\ \bar{C}_{2t} &= \sigma c_2(t), \\ \bar{D}_{2t} &= \sigma k_2(t). \end{aligned}$$

Next we calculate $\mathbb{E}_{t,x}[X_T^{\hat{\pi}}]$ and $\mathbb{E}_{t,x}[(X_T^{\hat{\pi}})^2]$. To do so, we construct the following exponential martingale:

$$d\bar{\rho}_t = \bar{\rho}_t\left[\left(-\bar{A}_t + \bar{C}_{1t}^{\,2} + \bar{C}_{2t}^{\,2} + 2\rho\bar{C}_{1t}\bar{C}_{2t}\right) dt - \bar{C}_{1t}dW_1(t) - \bar{C}_{2t}dW_2(t)\right],$$

or equivalently,

$$\bar{\rho}_t = \bar{\rho}_0 \exp\left\{\int_0^t \left[\left(-\bar{A}_s + \frac{1}{2}\bar{C}_{1s}^{\,2} + \frac{1}{2}\bar{C}_{2s}^{\,2} + \rho\bar{C}_{1s}\bar{C}_{2s}\right) ds - \bar{C}_{1s}dW_1(s) - \bar{C}_{2s}dW_2(s)\right]\right\},$$

and then

$$\frac{\bar{\rho}_t}{\bar{\rho}_T} = \exp\left\{\int_t^T \left[\left(\bar{A}_s - \frac{1}{2}\bar{C}_{1s}^{\,2} - \frac{1}{2}\bar{C}_{2s}^{\,2} - \rho\bar{C}_{1s}\bar{C}_{2s}\right) ds + \bar{C}_{1s}dW_1(s) + \bar{C}_{2s}dW_2(s)\right]\right\}. \tag{6.8}$$

Applying the generalized Itô's formula to $\bar{\rho}_t X_t^{\hat{\pi}}$ yields

$$\begin{aligned} & d\left(\bar{\rho}_t X_t^{\hat{\pi}}\right) \\ =\ & X_t^{\hat{\pi}} d\bar{\rho}_t + \bar{\rho}_t dX_t^{\hat{\pi}} + < X_t^{\hat{\pi}}, \bar{\rho}_t > \\ =\ & X_t^{\hat{\pi}}\bar{\rho}_t\left[\left(-\bar{A}_t + \bar{C}_{1t}^{\,2} + \bar{C}_{2t}^{\,2} + 2\rho\bar{C}_{1t}\bar{C}_{2t}\right) dt - \bar{C}_{1t}dW_1(t) - \bar{C}_{2t}dW_2(t)\right] \\ & + \bar{\rho}_t\left(\bar{A}_t X_t^{\hat{\pi}} + \bar{B}_t\right) dt + \bar{\rho}_t\left(\bar{C}_{1t}X_t^{\hat{\pi}} + \bar{D}_{1t}\right) dW_1(t) + \bar{\rho}_t\left(\bar{C}_{2t}X_t^{\hat{\pi}} + \bar{D}_{2t}\right) dW_2(t) \\ & - \Big[\bar{C}_{1t}\left(\bar{C}_{1t}X_t^{\hat{\pi}} + \bar{D}_{1t}\right) + \rho\bar{C}_{2t}\left(\bar{C}_{1t}X_t^{\hat{\pi}} + \bar{D}_{1t}\right) + \rho\bar{C}_{1t}\left(\bar{C}_{2t}X_t^{\hat{\pi}} + \bar{D}_{2t}\right) \\ & + \bar{C}_{2t}\left(\bar{C}_{2t}X_t^{\hat{\pi}} + \bar{D}_{2t}\right)\Big]\bar{\rho}_t dt \\ =\ & \bar{\rho}_t\left(\bar{B}_t - \bar{C}_{1t}\bar{D}_{1t} - \bar{C}_{2t}\bar{D}_{2t} - \rho\bar{C}_{1t}\bar{D}_{2t} - \rho\bar{C}_{2t}\bar{D}_{1t}\right) dt + \bar{\rho}_t\left[\bar{D}_{1t}dW_1(t) + \bar{D}_{2t}dW_2(t)\right]. \end{aligned}$$

Integrating from t to T on the above equation and rearranging it, we have

$$X_T^{\hat{\pi}} = \frac{\bar{\rho}_t}{\bar{\rho}_T} x + \int_t^T \frac{\bar{\rho}_s}{\bar{\rho}_T} \left[\bar{E}_s ds + \bar{D}_{1s} dW_1(s) + \bar{D}_{2s} dW_2(s) \right] \tag{6.9}$$

with $x = X_t^{\hat{\pi}}$ and

$$\bar{E}_t = \bar{B}_t - \bar{C}_{1t}\bar{D}_{1t} - \bar{C}_{2t}\bar{D}_{2t} - \rho\, \bar{C}_{1t}\bar{D}_{2t} - \rho\, \bar{C}_{2t} D_{1t}.$$

Note that $\mathbb{E}\left(\frac{\bar{\rho}_t}{\bar{\rho}_T}\right) = e^{\int_t^T \bar{A}_s ds}$, then we have $\mathbb{E}_{t,x}[X_T^{\hat{\pi}}] = P_1(t)x + Q_1(t)$ with

$$P_1(t) = e^{\int_t^T \bar{A}_s ds} \tag{6.10}$$

and

$$Q_1(t) = \int_t^T e^{\int_s^T \bar{A}_z dz} \bar{E}_s ds \tag{6.11}$$

By (6.9), we have

$$\begin{aligned} (X_T^{\hat{\pi}})^2 \quad = \quad & \left(\frac{\bar{\rho}_t}{\bar{\rho}_T}\right)^2 x^2 + 2x \frac{\bar{\rho}_t}{\bar{\rho}_T} \int_t^T \frac{\bar{\rho}_s}{\bar{\rho}_T} \left[\bar{E}_s ds + \bar{D}_{1s} dW_1(s) + \bar{D}_{2s} dW_2(s) \right] \\ & + \left\{ \int_t^T \frac{\bar{\rho}_s}{\bar{\rho}_T} \left[\bar{E}_s ds + \bar{D}_{1s} dW_1(s) + \bar{D}_{2s} dW_2(s) \right] \right\}^2, \end{aligned}$$

which implies $\mathbb{E}_{t,x}[(X_T^{\hat{\pi}})^2] = P_2(t)x^2 + Q_2(t)x + R(t)$ with

$$P_2(t) = e^{\int_t^T (\bar{A}_s + \bar{F}_s) ds}; \tag{6.12}$$

$$Q_2(t) = 2 \int_t^T e^{\int_t^s \bar{A}_z dz} e^{\int_s^T (\bar{A}_z + \bar{F}_z) dz} \bar{E}_s ds; \tag{6.13}$$

$$R(t) = \mathbb{E}\left\{ \left[\int_t^T \frac{\bar{\rho}_s}{\bar{\rho}_T} \left(\bar{E}_s ds + \bar{D}_{1s} dW_1(s) + \bar{D}_{2s} dW_2(s) \right) \right]^2 \right\}, \tag{6.14}$$

where

$$\bar{F}_t = \bar{A}_t + \bar{C}_{1t}^2 + \bar{C}_{2t}^2 + 2\rho \bar{C}_{1t}\bar{C}_{2t}.$$

We recall that

$$f(t,x,y) = \mathbb{E}_{t,x}\left[X_T^{\pi^*}\right] - \frac{\gamma}{2y} \mathbb{E}_{t,x}\left[(X_T^{\pi^*})^2\right] = P_1(t)x + Q_1(t) - \frac{\gamma}{2y}\left[P_2(t)x^2 + Q_2(t)x + R(t)\right], \tag{6.15}$$

$$g(t,x) = \mathbb{E}_{t,x}\left[X_T^{\pi^*}\right] = P_1(t)x + Q_1(t). \tag{6.16}$$

Then we have

$$\begin{aligned} f_t(t,x,y) &= \dot{P}_1(t)x + \dot{Q}_1(t) - \frac{\gamma}{2y}\left[\dot{P}_2(t)x^2 + \dot{Q}_2(t)x + \dot{R}(t)\right]; \\ f_x(t,x,y) &= P_1(t) - \frac{\gamma}{2y}\left[2P_2(t)x + Q_2(t)\right]; \\ f_{xx}(t,x,y) &= -\frac{\gamma}{y}P_2(t); \\ g_t(t,x) &= \dot{P}_1(t)x + \dot{Q}_1(t); \\ g_x(t,x) &= P_1(t); \\ g_{xx}(t,x) &= 0. \end{aligned}$$

Thus

$$\begin{aligned} \frac{f_x + \frac{\gamma}{x}gg_x}{f_{xx} + \frac{\gamma}{x}gg_{xx}} &= \frac{P_1(t) - \frac{\gamma}{2x}\left[2P_2(t)x + Q_2(t)\right] + \frac{\gamma}{x}[P_1(t)x + Q_1(t)]P_1(t)}{-\frac{\gamma}{x}P_2(t)} \\ &= \frac{x\left[P_1(t) - \gamma P_2(t) + \gamma P_1^2(t)\right] + \gamma P_1(t)Q_1(t) - \frac{\gamma}{2}Q_2(t)}{-\gamma P_2(t)}, \end{aligned}$$

where f and its derivatives are evaluated at (t,x,x) while g and its derivatives are evaluated at (t,x).

Comparing $(\hat{q},\hat{u})$ in (6.6) and (6.7), we have

$$\begin{cases} c_1(t) = \dfrac{\rho\sqrt{\lambda\mu_2}(r_1-r_0) - \sigma\lambda\mu_1\theta}{\sigma\lambda\mu_2(1-\rho^2)} \times \dfrac{P_1(t) - \gamma P_2(t) + \gamma P_1^2(t)}{-\gamma P_2(t)}, \\ c_2(t) = \dfrac{\rho\sigma\sqrt{\lambda\mu_2}\lambda\mu_1\theta - \lambda\mu_2(r_1-r_0)}{\sigma^2\lambda\mu_2(1-\rho^2)} \times \dfrac{P_1(t) - \gamma P_2(t) + \gamma P_1^2(t)}{-\gamma P_2(t)}, \\ k_1(t) = \dfrac{\rho\sqrt{\lambda\mu_2}(r_1-r_0) - \sigma\lambda\mu_1\theta}{\sigma\lambda\mu_2(1-\rho^2)} \times \dfrac{P_1(t)Q_1(t) - \frac{1}{2}Q_2(t)}{-P_2(t)}, \\ k_2(t) = \dfrac{\rho\sigma\sqrt{\lambda\mu_2}\lambda\mu_1\theta - \lambda\mu_2(r_1-r_0)}{\sigma^2\lambda\mu_2(1-\rho^2)} \times \dfrac{P_1(t)Q_1(t) - \frac{1}{2}Q_2(t)}{-P_2(t)}. \end{cases}$$

Note that

$$\begin{aligned} \frac{P_1(t) - \gamma P_2(t) + \gamma P_1^2(t)}{-\gamma P_2(t)} &= -\frac{1}{\gamma}\frac{P_1(t)}{P_2(t)} + 1 - \frac{P_1^2(t)}{P_2(t)} \\ &= -\frac{1}{\gamma}e^{-\int_t^T \bar{F}_s ds} + 1 - e^{\int_t^T \bar{A}_s ds}e^{-\int_t^T \bar{F}_s ds}, \end{aligned}$$

and

$$\begin{aligned} \frac{P_1(t)Q_1(t) - \frac{1}{2}Q_2(t)}{-P_2(t)} &= -\frac{e^{\int_t^T \bar{A}_s ds}}{e^{\int_t^T(\bar{A}_s + \bar{F}_s)ds}}\int_t^T e^{\int_s^T \bar{A}_z dz}\bar{E}_s ds + \frac{\int_t^T e^{\int_t^s \bar{A}_z dz}e^{\int_s^T(\bar{A}_z+\bar{F}_z)dz}\bar{E}_s ds}{e^{\int_t^T(\bar{A}_s+\bar{F}_s)ds}} \\ &= -e^{-\int_t^T \bar{F}_s ds} \times \int_t^T e^{\int_s^T \bar{A}_z dz}\bar{E}_s ds + \int_t^T e^{-\int_t^s \bar{F}_z dz}\bar{E}_s ds. \end{aligned}$$

Thus, we obtain (3.3) and finish the proof. □

Appendix D: The proof of Theorem 3.2

Proof. If (3.9) is satisfied, we have $c_1(t)x + k_1(t) \geq 0$; $c_2(t)x + k_2(t) \geq 0$. Inserting (6.15) and (6.16) into (3.1), we get the equilibrium value function

$$\begin{aligned} V(t,x) &= f(t,x,x) + \frac{\gamma}{2x} g^2(t,x) \\ &= P_1(t)x + Q_1(t) - \frac{\gamma}{2x}\left[P_2(t)x^2 + Q_2(t)x + R(t)\right] + \frac{\gamma}{2x}[P_1(t)x + Q_1(t)]^2. \end{aligned}$$

Then we obtain (3.10).

Otherwise, if (3.11) holds, we have $c_1(t)x + k_1(t) < 0$ and $c_2(t)x + k_2(t) < 0$. The equilibrium strategy is $q^*(\cdot) = 0$, $u^*(\cdot) = 0$, the wealth process is

$$dX_t = [r_0 X_t + \lambda\mu_1(\eta - \theta)]dt,$$

then we have

$$X_T = e^{r_0(T-t)} X_t + \frac{\lambda\mu_1(\eta-\theta)}{r_0}\left[e^{r_0(T-t)} - 1\right],$$

which implies that $\mathbb{E}[X_T] = e^{r_0(T-t)}x + \frac{\lambda\mu_1(\eta-\theta)}{r_0}\left[e^{r_0(T-t)} - 1\right]$ and $\mathbb{V}ar[X_T] = 0$, hence,

$$V(t,x) = \mathbb{E}[X_T] = e^{r_0(T-t)}x + \frac{\lambda\mu_1(\eta-\theta)}{r_0}\left[e^{r_0(T-t)} - 1\right].$$

The proof is finished. □

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Highlights:

- The optimal time-consistent investment-reinsurance strategies for an insurer with state dependent risk aversion are considered.
- The Value-at-Risk control levels for the insurer are introduced to control its loss in investment-reinsurance strategies.
- The optimal investment-reinsurance problem is formulated within a game theoretic framework.
- An extended Hamilton-Jacobi-Bellman system of equations is solved.
- The closed-form expressions of the optimal investment-reinsurance strategies are derived.