Results on Chromatic Polynomials Inspired by a Correlation Inequality of G.E. Farr

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In [9] Graham Farr gave a proof of a correlation inequality involving colourings of graphs. His work eventually led to a conjecture that number of colourings of a graph with certain properties gave a log-concave sequence. We restate Farr's work in terms of the bivariate chromatic polynomial of Dohmen, Poenitz, Tittman [7] and give a simple, self-contained proof of Farr's inequality using a basic combinatorial approach. We attempt to prove Farr's conjecture through methods in stable polynomials and computational verification, ultimately leading to a stronger conjecture.

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Chapter 1

Introduction

We let G denote an arbitrary undirected graph on n vertices. When there is no confusion, we will use V to denote $V(G)$, and E to denote $E(G)$. A subset W of V is said to be *independent* (or *stable*) if its vertices are pairwise non-adjacent, and is said to be *dependent* otherwise. In particular, a vertex with a loop cannot be in any independent set. We denote by $\mathcal{I}(G)$ the set of all independent sets of G.

Fix a non-negative integer k and a probability p such that $kp \leq 1$. We will write $[k]$ to denote the set $\{1, \ldots, k\}$. We will often think of the elements of $[k]$ as colours. Independently for each $v \in V$ and each $i \in [k]$, v is assigned colour i with probability p. Hence a vertex is assigned no colour with probability $1 - kp$. At times it may be convenient to think of uncoloured vertices as having been assigned a $(k+1)$-th colour. For each $i \in [k]$, denote by $\Gamma_i^{(p)}$ (or Γ_i when p is clear) the set of vertices assigned colour i. Thus $\Gamma_1^{(p)}, \ldots, \Gamma_k^{(p)}$ are k mutually dependent random variables, each depending only on p.

For all $\ell \in \{0, \ldots, k\}$, we call the probabilities of the form

$$\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G))$$

the *joint independence probabilities* of G. In [9] Graham Farr proved that the joint independence probabilities are negatively correlated. Specifically, he showed:

Theorem (Theorem 2 of [9])**.** *Let G be a graph, k a non-negative integer, and p a probability, such that $kp \leq 1$. For every $\ell \in [k]$ we have*

$$\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G)) \leq \Pr(\Gamma_1, \ldots, \Gamma_{\ell-1} \in \mathcal{I}(G)) \Pr(\Gamma_\ell \in \mathcal{I}(G)).$$

We will refer to this result as *Farr's correlation inequality*. Farr's original proof, as well as other proofs by McDiarmid [17] and Dubhashi and Ranjan [8], have relied on arguments grounded in probabilistic methods. In Section 4 we will give a proof using a bare-bones combinatorial approach.

Farr also defined a sequence $(a_\ell^{(k)})_{\ell=0}^k$ where the ℓ-th term is

$$a_\ell^{(k)} = \Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G)) \Pr(\Gamma_1, \ldots, \Gamma_{k-\ell} \in \mathcal{I}(G))$$

for ℓ ranging from 0 to k. Farr's correlation inequality and other results led Farr to conjecture that this sequence is *log-concave*. That is to say, for each $i \in \{1, \ldots, k-1\}$

$$(a_i^{(k)})^2 \geq a_{i-1}^{(k)} a_{i+1}^{(k)}.$$

It seems that little to no progress has been made toward proving or disproving this conjecture. Following our proof of Farr's Correlation inequality we will attempt to prove that the sequence $(a_\ell^{(k)})_{\ell=0}^k$ is log-concave.

Chapter 2 will review what is currently known about the joint independence probabilities. This will include relations with the independence and chromatic polynomials of a graph, a proof of Farr's correlation inequality using basic probability theory, and what is currently known about log-concavity of the sequence $(a_\ell^{(k)})_{\ell=0}^k$.

In Chapter 3, we will focus on finding a better description for joint independence probabilities. We show that it is not an evaluation of the Tutte polynomial. An exact expression is then given in terms of a generalisation of the chromatic polynomial. This description has implications for joint independent probabilities and the sequence $(a_\ell^{(k)})_{\ell=0}^k$.

Chapter 4 will give a proof of an extension of Farr's correlation inequality using a simple combinatorial approach.

Chapter 5 will introduce a new graph invariant, $B_k(G; y)$, which guarantees the log-concavity of the $a_\ell^{(k)}$'s when it has only real roots. Delving into known theory of real-rooted polynomials, we show that $B_k(G; y)$ possesses many attractive, and sometimes peculiar, properties with regards to real-rootedness. We also present a loosely related, yet interesting, result which came up during this investigation.

The final chapter will list computational results and further work that they motivate. These results include exhaustive verifications of log-concavity and real-rootedness of $B_k(G; y)$.

Chapter 2

Background

We begin by considering only a single Γ_i. Fix an arbitrary set $W \subseteq V$. For each vertex $v \in V$, $v \in \Gamma_i$ with probability p, independently of any other vertex being in Γ_i. Therefore the probability that Γ_i is exactly W is $p^{|W|}(1-p)^{n-|W|}$. It follows that the probability of Γ_i being an independent set is given by the polynomial

$$A(G;p) = \sum_{W \in \mathcal{I}(G)} p^{|W|}(1-p)^{n-|W|}.$$

Farr calls this polynomial the *stability polynomial* of G.

Using this idea, for any given k we can express $\Pr(\Gamma_1, \ldots, \Gamma_k \in \mathcal{I}(G))$ as a polynomial in p. Fix disjoint sets $W_1, \ldots, W_k \subseteq V$. For each $v \in V$ and $i \in [k]$, $v \in W_i$ with probability p, and therefore

$$\Pr(\Gamma_1 = W_1, \ldots, \Gamma_k = W_k) = p^{|W_1|+\cdots+|W_k|}(1-kp)^{n-|W_1|-\cdots-|W_k|}.$$

Thus we have

$$\Pr(\Gamma_1^{(p)}, \ldots, \Gamma_k^{(p)} \in \mathcal{I}(G)) = \sum_{\substack{W_1,\ldots,W_k \in \mathcal{I}(G) \\ \text{disjoint}}} p^{|W_1|+\cdots+|W_k|}(1-kp)^{n-|W_1|-\cdots-|W_k|}. \qquad (2.1)$$

Hence these probabilities are all given by polynomials in p. The forms given above are not particularly enlightening. However, it has been shown that these polynomials, especially $A(G;p)$, are related to other graph polynomials [9] [19]. We will focus on the relations to the independence polynomial and the chromatic polynomial.

2.1 Relation to Polynomials of Graphs

The *independence polynomial* was first studied by Gutman and Harary [11]. Denoted by $I(G; x)$, it is defined to be the generating series for independent sets with respect to cardinality, that is

$$I(G; x) = \sum_{W \in \mathcal{I}(G)} x^{|W|}.$$

A detailed overview of the independence polynomial can be found in the survey [16] of Levit and Mandrescu.

It is immediately clear that the stability polynomial and the independence polynomial are related by a simple transformation

$$A(G; p) = \sum_{W \in \mathcal{I}(G)} p^{|W|} (1-p)^{n-|W|} = (1-p)^n I(G; p/(1-p)). \tag{2.2}$$

For example, consider the complete graph on n vertices, K_n. The only independent sets of K_n are the empty set and singleton. Therefore its independence polynomial is $I(K_n; x) = 1 + nx$. It follows from (2.2) and a bit of rearranging that

$$A(K_n; p) = (1-p)^{n-1}(1 - p(n-1)).$$

Unfortunately, computing the independence polynomial for a general graph is a hard problem (specifically, it is #P-hard [27]). However there are many recurrence results that can be used to compute $I(G; x)$ if it is known for smaller graphs.

Before stating a few such results, we establish some notation. For $U \subseteq V(G)$, $G - U$ denotes the graph obtained by deleting the vertices in U from G. If v is a vertex of G, we write $G - v$ to mean $G - \{v\}$. For an edge e of G, we let $G \setminus e$ and G / e denote, respectively, the graphs obtained by deleting and contracting the edge e. If the edge e is a loop, then $G \setminus e = G / e$. We also define a third edge-removal operation we call *edge extraction*, denoted by $G - e$, in which the ends of the edge are removed.

Theorem 2.1.1. *If G_1 and G_2 are disjoint graphs, then*

$$I(G_1 \cup G_2; x) = I(G_1; x)I(G_2; x).$$

Proof. Since G_1 and G_2 are disjoint, the independent sets of $G_1 \cup G_2$ are unions of an independent set of G_1 with an independent set of G_2. The result now follows from the definition of $I(G; x)$. $\square$

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A graph is the disjoint union of its connected components. Denote by $\kappa(G)$ the number of connected components of G. An immediate consequence of Theorem 2.1.1 is that if $G_1, \ldots, G_\kappa$ are the connected components of G, then

$$I(G; x) = \prod_{i=1}^{\kappa} I(G_i; x).$$

The analogous result also holds for $A(G; p)$.

Theorem 2.1.2. *Let G be a graph and e be an edge of G. The independence polynomial of G satisfies the following recurrence relation:*

$$I(G; x) = I(G \setminus e; x) - x(I(G / e; x) - I(G - e; x)).$$

This recurrence is usually given as $I(G; x) = I(G \setminus e; x) - x^2 I(G - N(u) - N(v); x)$ where u, v are the ends of e and $N(u)$ is the set containing u and all vertices adjacent to u in G. For our purposes, the form we gave will be more relevant.

Proof. Let u, v be the ends of e. Independent sets of $G \setminus e$ can be divided into two classes, those that contain both u and v, and those that contain at most one. The latter are in a one-to-one correspondence with the independent sets of G.

Let w be the image of u and v under the contraction of e. The independent sets of G / e can also be divided into two classes, those that contain w and those that do not. The former are in bijection with the independent sets of $G \setminus e$ which contain both u and w. The latter are in bijection with the independent sets of $(G / e) - w$ which is exactly $G - e$.

Since the independent sets of $G \setminus e$ containing both u and v contain one more vertex than the corresponding independent sets of G / e which contain w, we must multiply the independence polynomial of the latter by x. $\qquad\square$

Since every $W \subseteq V$ is independent in an edgeless graph, its independence polynomial is clearly

$$\sum_{W \subseteq V} x^{|W|} = \sum_{i=0}^{n} \binom{n}{i} x^i = (1 + x)^n.$$

Thus, one can always compute the independence polynomial of a graph by repeatedly applying Theorem 2.1.2. However, this quickly becomes infeasible as the number of polynomials that must be computed triples with every edge removed.

We have shown $\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G))$ is related to the independence polynomial when $\ell = 1$. The other main relation with a graph invariant comes when $\ell = k$ and $kp = 1$. In this case, $\Gamma_1, \ldots, \Gamma_k$ form a colouring of G.

A *colouring* of G is any function $c : V \to K$ where K is a finite set. The elements of K are referred to as *colours* and the preimage, $c^{-1}(i)$, of any colour $i \in K$ is called the *i-th colour-class of c*. If K is a set of k colours, we say that c is a *k-colouring*. We do not require that all colours have a non-empty preimage, thus a $(k-1)$-colouring is also a k-colouring. A colouring whose colour-classes are all independent sets is said to be *proper*. In particular, a graph with a loop cannot have a proper colouring. Unless we specify otherwise, we will always use the set $K = [k]$ as the colours of our k-colourings.

A colouring can be viewed as a partition of the vertex set along with a choice of colour for each part of the partition. A proper colouring's corresponding partition will contain only independent sets. This mapping is a bijection, thus we can express the number of k-colourings of a graph in terms of the number of ways its vertices can be partitioned into independent sets.

For each $j \in \{0, \ldots, n\}$, define $\lambda_j(G)$ to be the number of partitions of V into exactly j non-empty independent sets. Given k colours and a partition with j parts, there are $(k)_j$ ways of assigning a different colour to each part, where $(k)_j$ is the falling factorial $k(k-1)\cdots(k-j+1)$. Therefore, the number of k-colourings of G is given by

$$\sum_{j=0}^{n} \lambda_j(G)(k)_j. \tag{2.3}$$

This shows that there is a polynomial in k which gives the number of k-colourings of G for every non-negative integer k. We call the polynomial (2.3) the *chromatic polynomial of G* and denote it by $P(G; k)$. Notice that $\lambda_n = 1$ since we can always partition V into singletons. Therefore, the chromatic polynomial is always a monic polynomial of degree n.

For example, the chromatic polynomial of $\overline{K_n}$, the edgeless graph on n vertices, is k^n since every colouring is proper. On the other hand, the only independent sets of vertices in the complete graph K_n are singletons, thus $P(K_n; k) = (x)_n$.

The chromatic polynomial was first studied for graphs in the plane by Birkhoff [3] in 1912 in an effort to prove the Four-Colour Map Theorem. It was generalised to all graphs in 1932 by Whitney [32, 33]. For detailed surveys of the chromatic polynomial see [2, 22].

If $kp = 1$, then every vertex is assigned to one of $\Gamma_1, \ldots, \Gamma_k$. In this case we obtain a k-colouring of G whose i-th colour-class is Γ_i. Since the probability that $v \in V$ receives

colour $i \in [k]$ is independent of v and i, each k-colouring is equally likely to occur. In total there are k^n k-colourings, therefore

$$\Pr(\Gamma_1, \ldots, \Gamma_k \in \mathcal{I}(G)) = \frac{P(G; k)}{k^n}. \tag{2.4}$$

The chromatic polynomial also provides a crude lower bound on the stability polynomial. Since the colour-classes of a proper colouring are independent sets, choosing a proper colouring uniformly at random among all colourings is more restrictive than choosing an independent set among all subsets of V. Hence we have

$$\frac{P(G; k)}{k^n} \leq A(G; 1/k). \tag{2.5}$$

Unfortunately, as was the case for the independence polynomial, determining the chromatic polynomial is #P-hard [13]. The simplest way of computing the chromatic polynomial is to use recursive results similar to those we listed for the independence polynomial. In proving these we make use of the fact that two polynomials which agree at infinitely many points are identical.

Theorem 2.1.3. *Let G_1 and G_2 be graphs such that $V(G_1) \cap V(G_2)$ induces a complete graph on r vertices in each of G_1 and G_2 (r is 0 if G_1 and G_2 are disjoint). Then*

$$P(G_1 \cup G_2; k) = \frac{P(G_1; k) P(G_2; k)}{(k)_r}$$

where $(k)_0 = 1$.

Proof. Let $C = V(G_1) \cap V(G_2)$. A proper k-colouring of $G_1 \cup G_2$ corresponds to the union of a proper k-colouring of G_1 with a proper k-colouring of G_2 such that both colour C in the same way. Since C induces a complete subgraph in both G_1 and G_2, its vertices must be assigned distinct colours. We know that proper colourings are partitions into independent sets along with a choice of colour for each part. Since the colours for the vertices of C are determined by the colourings of G_1, the colourings of G_2 are restricted by a factor of $(k)_r$. Thus after choosing a colouring for G_1, we lose r colour choices. Hence the number of colourings of G_2 is reduced by a factor of $(k)_r$. $\qquad \square$

In particular, $P(G_1 \cup G_2; k) = P(G_1; k) P(G_2; k)$ when G_1 and G_2 are disjoint. Hence, as was true for $I(G; x)$, if G has connected components $G_1, \ldots, G_m$, then

$$P(G; k) = \prod_{i=1}^{m} P(G_i; k).$$

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Another useful observation is that if T is a tree, $v, w \in V(T)$ are adjacent, and v is not adjacent to any other vertex of T, then $T = (T - v) \cup K_2$ where the vertex set of K_2 is $\{v, w\}$. These graphs have one vertex in common hence

$$P(T; k) = \frac{(k)_2 P(T - v; k)}{(k)_1} = (k - 1)P(T - v; k).$$

Since $P(K_1; k) = k$ it follows by induction on n that the chromatic polynomial of any tree on n vertices is $k(k - 1)^{n-1}$. This means that the number of proper k-colourings of a tree is entirely determined by the number of vertices.

The next result is similar to Theorem 2.1.2 and is one of the best known properties of the chromatic polynomial.

Theorem 2.1.4 (Deletion-Contraction Recurrence). *Let G be a graph and e be an edge of G. The chromatic polynomial of G satisfies*

$$P(G; k) = P(G \setminus e; k) - P(G \,/\, e; k).$$

Proof. Let e be any edge in G and consider a proper k-colouring c of $G \setminus e$. If c colours the ends of e with different colours, then c is a proper k-colouring of G, otherwise it corresponds to a proper k-colouring of $G \,/\, e$ (after identifying the two ends of e). This mapping is a bijection between the proper k-colourings of $G \setminus e$ and the union of those of G and $G \,/\, e$. Thus we have the deletion-contraction recurrence. $\square$

Since the chromatic polynomial of an edgeless graph on n vertices is k^n, this recurrence provides a second way of showing that $P(G; k)$ is a polynomial of degree n with integer coefficients.

2.2 Farr's Correlation Inequality

Suppose that k and p are such that $kp = 1$. The stability polynomial gives the probability that a single Γ_i is independent. Since the Γ_i are symmetric, it follows that

$$A(G; 1/k)^k = \prod_{i=1}^{k} \Pr(\Gamma_i \in \mathcal{I}(G)). \tag{2.6}$$

On the other hand (2.4) states that

$$p^n P(G; 1/p) = \Pr(\Gamma_1, \ldots, \Gamma_k \in \mathcal{I}(G)).$$

The left-hand side of (2.4) requires that $\Gamma_1, \ldots, \Gamma_k$, in addition to being independent, form a partition of V. In Farr's words [9, p. 15], a comparison of these two equalities suggests that the events $(\Gamma_i \in \mathcal{I}(G))$ are negatively correlated. That is

$$\Pr(\Gamma_1, \ldots, \Gamma_k \in \mathcal{I}(G)) \leq \prod_{i=1}^{k} \Pr(\Gamma_i \in \mathcal{I}(G)). \tag{2.7}$$

We recall Farr's correlation inequality:

Theorem (Theorem 2 of [9])**.** *Let G be a graph, k a non-negative integer, and p a probability, such that $kp \leq 1$. For every $\ell \in [k]$ we have*

$$\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G)) \leq \Pr(\Gamma_1, \ldots, \Gamma_{\ell-1} \in \mathcal{I}(G)) \Pr(\Gamma_\ell \in \mathcal{I}(G)).$$

By repeatedly applying this result, one easily obtains (2.7). Furthermore, when $kp = 1$ we obtain the following inequality involving the chromatic polynomial and the stability polynomial:

$$\frac{P(G;k)}{k^n} \leq A(G;1/k)^k.$$

It is important to note that this bound is a big improvement over (2.5). Of course, the right-hand side can also be expressed in terms of the independence polynomial as

$$(k+1/k)^{nk} I(G;1/k-1)^k.$$

Farr's original proof is by no means basic as it uses the Ahlswede-Daykin Inequality (also known as the Four Functions Theorem), a very powerful correlation inequality result. Farr even mentions [9, p. 15]:

> I found this surprisingly hard to prove, and indeed the proof given uses the considerable power of the Ahlswede-Daykin Theorem.

Soon after, an alternate proof was provided by McDiarmid [17] which relied on Harris' Inequality, a weaker, but not elementary, correlation inequality result implied by the Ahlswede-Daykin Inequality. McDiarmid's proof was stated as a consequence of a lemma stronger than Farr's result. In [8, Section 4.4] Dubhashi and Ranjan recast Farr and Mc-Diarmid's results in terms of their balls and bins experiment framework. Their proof uses only basic probability theory and their result is a very strong extension of Farr's correlation inequality. To state their result, we define a more general way of assigning colours to

vertices at random. Let k be a non-negative integer and let $\boldsymbol{q} = (q_{i,v} : i \in [k], v \in V)$ be a vector of probabilities such that $\sum_{i \in [k]} q_{i,v} = 1$ for all $v \in V$. Independently, for each vertex $v \in V$, v is assigned colour i with probability $q_{i,v}$ for each $i \in [k]$. Denote by $\Gamma_i^{(\boldsymbol{q})}$ (or Γ_i when $\boldsymbol{q}$ is clear) the set of vertices assigned colour i.

Theorem 2.2.1. *Let k be a non-negative integer and $\boldsymbol{q}$ be a vector of probabilities such that $\sum_{i \in [k]} q_{i,v} = 1$. For any $j, \ell \in [k]$ such that $j < \ell$ we have*

$$\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G)) \leq \Pr(\Gamma_1, \ldots, \Gamma_j \in \mathcal{I}(G)) \Pr(\Gamma_{j+1}, \ldots, \Gamma_\ell \in \mathcal{I}(G)).$$

It is easy to recover Farr's result from Theorem 2.2.1. Consider $k + 1$ colours, for all $v \in V$ set $q_{i,v} = p$ for all $i \in [k]$, and set $q_{k+1,v} = 1 - kp$. Then with $j = \ell - 1$ we obtain Farr's correlation inequality.

We include the following proof of Theorem 2.2.1, adapted from [8], as an example of the probabilistic approach to Farr's correlation inequality. This will contrast with Chapter 4 in which we provide a proof using an entirely combinatorial approach. Unfortunately, our result won't be quite as strong as Theorem 2.2.1.

When $\boldsymbol{X}$ is a vector with entries indexed by a set A, and B is a subset of A, then by $\boldsymbol{X}_B$ we mean the vector $(X_i : i \in B)$. A vector $\boldsymbol{X} = (X_1, \ldots, X_k)$ of random variables is *negatively associated* if it satisfies the following condition: for any disjoint sets $I, J \subseteq [k]$ and any functions $f : \mathbb{R}^I \to \mathbb{R}$ and $g : \mathbb{R}^J \to \mathbb{R}$ that are both non-decreasing or both non-increasing we have

$$\mathrm{E}[f(\boldsymbol{X}_I)g(\boldsymbol{X}_J)] \leq \mathrm{E}[f(\boldsymbol{X}_I)]\mathrm{E}[g(\boldsymbol{X}_J)].$$

We note that the inequality above holds for f and g if and only if it holds for $-f$ and $-g$. Therefore, to show $\boldsymbol{X}$ is negatively associated it suffices to show the inequality holds for non-decreasing f and g.

Lemma 2.2.2 (Lemma 9 of [8]). *If $\boldsymbol{X} = (X_1, \ldots, X_m)$ are binary random variables satisfying $\sum_{i \in [m]} X_i = 1$, then $\boldsymbol{X}$ is negatively associated.*

The following elementary proof is due to McDiarmid [8, Remark 11].

Proof. Let I, J be disjoint subsets of $[m]$ and let $f : \{0,1\}^I \to \mathbb{R}$ and $g : \{0,1\}^J \to \mathbb{R}$ be non-decreasing functions. Define f^* and g^* by

$$f^*(\boldsymbol{x}_I) := f(\boldsymbol{x}_I) - f(\boldsymbol{0})$$
$$g^*(\boldsymbol{x}_J) := g(\boldsymbol{x}_J) - g(\boldsymbol{0}).$$

Since f and g are non-decreasing, f^* and g^* are non-negative functions and

$$f(\mathbf{0}) = 0 = g(\mathbf{0}).$$

However, we know $\sum_i X_i = 1$, hence f^* and g^* cannot be non-zero at the same time. Thus we have

$$\mathrm{E}[f^*(\boldsymbol{X}_I)g^*(\boldsymbol{X}_J)] = 0 \leq \mathrm{E}[f^*(\boldsymbol{X}_I)]\mathrm{E}[g^*(\boldsymbol{X}_J)].$$

It follows from the linearity of expectation that

$$\begin{aligned}
\mathrm{E}[f(\boldsymbol{X}_I)g(\boldsymbol{X}_J)] &= \mathrm{E}[f^*(\boldsymbol{X}_I)g^*(\boldsymbol{X}_J)] + f(\mathbf{0})\mathrm{E}[g^*(\boldsymbol{X}_J)] + g(\mathbf{0})\mathrm{E}[f^*(\boldsymbol{X}_I)] + f(\mathbf{0})g(\mathbf{0}) \\
&\leq \mathrm{E}[f^*(\boldsymbol{X}_I)]\mathrm{E}[g^*(\boldsymbol{X}_J)] + f(\mathbf{0})\mathrm{E}[g^*(\boldsymbol{X}_J)] + g(\mathbf{0})\mathrm{E}[f^*(\boldsymbol{X}_I)] + f(\mathbf{0})g(\mathbf{0}) \\
&= \mathrm{E}[f(\boldsymbol{X}_I)]\mathrm{E}[g(\boldsymbol{X}_J)].
\end{aligned}$$

Thus $\boldsymbol{X}$ is negatively associated. $\qquad\square$

Next we require a tool to show that a union of negatively associated vectors of variables is also negatively associated. The following result appears as part 1 of Proposition 8 in [8]. but the proof we give is due to Joag-Dev and Proschan [14, Property P_7].

Lemma 2.2.3. *Let $\boldsymbol{X} = (X_1, \ldots, X_m)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be vectors of negatively associated random variables such that $\boldsymbol{X}$ is independent from $\boldsymbol{Y}$. Then their union $(\boldsymbol{X}, \boldsymbol{Y})$ is also negatively associated.*

Proof. Let $(\boldsymbol{X}_f, \boldsymbol{X}_g)$ and $(\boldsymbol{Y}_f, \boldsymbol{Y}_g)$ be partitions of $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $f : \mathbb{R}^{|\boldsymbol{X}_f|+|\boldsymbol{Y}_f|} \to \mathbb{R}$ and $g : \mathbb{R}^{|\boldsymbol{X}_g|+|\boldsymbol{Y}_g|} \to \mathbb{R}$ be non-decreasing functions.

Use $h_f(\boldsymbol{Y}_f)$ to denote $\mathrm{E}[f(\boldsymbol{X}_f, \boldsymbol{Y}_f) \mid \boldsymbol{Y}_f]$. Since $h_f(\boldsymbol{Y}_f)$ is a measurable function we have

$$\mathrm{E}[f(\boldsymbol{X}_f, \boldsymbol{Y}_f) \mid \boldsymbol{Y}_f] = \mathrm{E}[f(\boldsymbol{X}_f, \boldsymbol{Y}_f) \mid \boldsymbol{Y}_f, \boldsymbol{Y}_g]$$

almost surely. A similar result holds for $h_g(\boldsymbol{Y}_g) = \mathrm{E}[g(\boldsymbol{X}_g, \boldsymbol{Y}_g) \mid \boldsymbol{Y}_g]$.

We know that h_f and h_g are non-decreasing as f and g are non-decreasing. Since $\boldsymbol{X}$ is independent of $\boldsymbol{Y}$, the negative association of $\boldsymbol{X}$ is preserved when the expectation is conditional on $\boldsymbol{Y}$. Hence we have

$$\mathrm{E}[f(\boldsymbol{X}_f, \boldsymbol{Y}_f)g(\boldsymbol{X}_g, \boldsymbol{Y}_g) \mid \boldsymbol{Y}_f, \boldsymbol{Y}_g] \leq h_f(\boldsymbol{Y}_f)h_g(\boldsymbol{Y}_g).$$

It then follows that

$$
\begin{aligned}
\mathrm{E}[f(\boldsymbol{X}_f, \boldsymbol{Y}_f)g(\boldsymbol{X}_f, \boldsymbol{Y}_f)] &= \mathrm{E}[\mathrm{E}[f(\boldsymbol{X}_f, \boldsymbol{Y}_f)g(\boldsymbol{X}_g, \boldsymbol{Y}_g) \mid \boldsymbol{Y}_f, \boldsymbol{Y}_g]] \\
&\leq \mathrm{E}[h_f(\boldsymbol{Y}_f)h_g(\boldsymbol{Y}_g)] \\
&\leq \mathrm{E}[h_f(\boldsymbol{Y}_f)]\mathrm{E}[h_g(\boldsymbol{Y}_g)] \\
&= \mathrm{E}[f(\boldsymbol{X}_f, \boldsymbol{Y}_f)]\mathrm{E}[f(\boldsymbol{X}_g, \boldsymbol{Y}_g)]
\end{aligned}
$$

where the second inequality is due to $\boldsymbol{Y}$ being negatively associated. We conclude that the union of $\boldsymbol{X}$ and $\boldsymbol{Y}$ is negatively associated. $\qquad\square$

We translate our random colouring into binary random variables with k variables per vertex. For each $i \in [k]$ and $v \in V$ let the distribution of $X_{i,v}$ be such that $X_{i,v} = 1$ if $v \in \Gamma_i$ and $X_{i,v} = 0$ otherwise. Thus, for a fixed $v \in$ we have $\sum_i X_{i,v} = 1$. The idea is to show that $(X_{i,v} : i \in [k], v \in V)$ is negatively associated and then produce non-increasing functions which indicate when certain colour-classes are independent sets.

Proof of Theorem 2.2.1. Since each vertex is coloured independently in our random colour assignment, the vectors $(X_{i,v} : i \in [k])$ are all pairwise independent. Since each of these vectors is negatively associated, it follows from Lemma 2.2.3 that the entire vector $(X_{i,v} : i \in [k], v \in V)$ is negatively associated.

Now we create non-increasing functions which indicate when certain colour-classes are independent sets. For each $i \in [k]$ we write $\boldsymbol{X}_i$ to mean the vector $(X_{i,v} : v \in V)$ and we define $f_i : \{0,1\}^V \to \mathbb{R}$ by

$$
f_i(\boldsymbol{X}_i) = \begin{cases} 1, & \text{if } \Gamma_i \text{ is an independent set} \\ 0, & \text{otherwise} \end{cases}
$$

where $\Gamma_i = \{v \in V : X_{i,v} = 1\}$.

Each f_i is non-increasing as we cannot make a dependent set of vertices independent by adding vertices. Since these functions are non-negative, the property of being non-increasing is preserved by products. Therefore

$$
\mathrm{E}\left[\prod_{i=1}^k f_i(\boldsymbol{X}_i)\right] \leq \mathrm{E}\left[\prod_{i=1}^j f_i(\boldsymbol{X}_i)\right]\mathrm{E}\left[\prod_{i=j+1}^k f_i(\boldsymbol{X}_i)\right].
$$

Note that the expectation of a binary random variable is equal to the probability that this variable is 1. Thus we conclude that

$$
\Pr(\Gamma_1, \ldots, \Gamma_k \in \mathcal{I}(G)) \leq \Pr(\Gamma_1, \ldots, \Gamma_j \in \mathcal{I}(G))\Pr(\Gamma_{j+1}, \ldots, \Gamma_n \in \mathcal{I}(G)). \qquad\square
$$

While Theorem 2.2.1 holds for probabilities that depend on both i and v, Farr's work, and consequently the majority of this thesis, is only concerned with the case where the probability is fixed for all $i \in [k]$ and $v \in V$. Thus, with the exception of Chapter 4, we will always assume that there is a single p being used to define $\Gamma_1^{(p)}, \ldots, \Gamma_k^{(p)}$.

2.3 The Sequence $a^{(k)}$

By the symmetry between the colour-classes, Farr's correlation inequality can be written as

$$\Pr(\Gamma_1, \ldots, \Gamma_k \in \mathcal{I}(G)) \leq \Pr(\Gamma_1, \ldots, \Gamma_{k-1} \in \mathcal{I}(G)) \Pr(\Gamma_1 \in \mathcal{I}(G)).$$

This led Farr to study the following sequence: for $j = 0, \ldots, k$ we define

$$a_j^{(k)}(G; p) := \Pr(\Gamma_1, \ldots, \Gamma_j \in \mathcal{I}(G)) \Pr(\Gamma_1, \ldots, \Gamma_{k-j} \in \mathcal{I}(G)).$$

We refer to the entire sequence as $a^{(k)}(G; p)$. When there is no ambiguity, we will omit p and G.

This sequence is symmetric, that is to say $a_j^{(k)} = a_{k-j}^{(k)}$ for all $j = 0, \ldots, k$. Farr's correlation inequality states that $a_0^{(k)} \leq a_1^{(k)}$ and, by symmetry, that $a_{k-1}^{(k)} \geq a_k^{(k)}$. In Chapter 6, we will give results of verification by computer which shows that the pattern continues with terms increasing as they approach the centre of the sequence.

We say that a sequence of real numbers $\alpha_0, \ldots, \alpha_n$ is *unimodal* if it has a single peak. That is, there exists some $i \in \{0, \ldots, n\}$ such that

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{i-1} \leq \alpha_i \geq \alpha_{i+1} \geq \cdots \geq \alpha_{n-1} \geq \alpha_n.$$

Since $a(G; k)$ is symmetric, it being unimodal would mean the single peak is in the centre of the sequence. A stronger condition on a sequence is logarithmic concavity.

Let $\alpha_0, \ldots, \alpha_n$ be a sequence of non-negative real numbers. For $i \in \{1, \ldots, n-1\}$, α_i is said to be an *internal zero* of the sequence if $\alpha_i = 0$ and there exist $i_< \in \{0, \ldots, i-1\}$ and $i_> \in \{i+1, \ldots, n\}$ such that $\alpha_{i_<}$ and $\alpha_{i_>}$ are non-zero. The sequence is said to be *logarithmically concave* (abbreviated to *log-concave*) if for all $i \in \{1, \ldots, n-1\}$ we have $\alpha_i^2 \geq \alpha_{i-1}\alpha_{i+1}$. For a detailed survey of log-concave and unimodal sequences see [23].

The prototypical example of a log-concave sequence is a row of Pascal's triangle. Indeed, one can easily verify that

$$\binom{k}{\ell}^2 \binom{k}{\ell-1}^{-1} \binom{k}{\ell+1}^{-1} = \frac{(\ell+1)(k-\ell+1)}{\ell(k-\ell)} > 1.$$

As mentioned earlier, log-concavity is a stronger condition on a sequence than unimodality.

Theorem 2.3.1. *If $\alpha_0, \ldots, \alpha_n$ is a log-concave sequence with no internal zeroes, then $\alpha_0, \ldots, \alpha_n$ is unimodal.*

Proof. As there are no internal zeroes, we lose no generality in assuming that $\alpha_1, \ldots, \alpha_{n-1}$ are all non-zero. Let $i \in \{1, \ldots, n-1\}$ be arbitrary. If $\alpha_{i-1} \geq \alpha_i$, then it follows from $\alpha_i^2 \geq \alpha_{i-1}\alpha_{i+1}$ that $\alpha_i \geq \alpha_{i+1}$. Similarly, $\alpha_i \leq \alpha_{i+1}$ will imply $\alpha_{i-1} \leq \alpha_i$. Thus there can only be one peak in the sequence. $\square$

While the requirements for log-concavity are much stronger than those for unimodality, Graham Farr was able to show that, under moderate conditions, unimodality and log-concavity are equivalent for the sequence a_k. In fact, his result is slightly stronger and shows that a peak in the centre of a_k is enough to guarantee log-concavity (and unimodality).

Theorem 2.3.2 (Private communication from G. E. Farr). *If $2kp \leq 1$ and $a^{(m)}$ has a local central maximum for all even $m < 2k$, then $a^{(k)}$ is log-concave.*

Proof. Let j be a positive integer such that $j \leq \frac{k}{2}$. Since $a^{(2j)}$ and $a^{(2k-2j)}$ have a central peak we have
$$a_j^{(2j)} a_{k-j}^{(2k-2j)} \geq a_{j-1}^{(2j)} a_{k-j-1}^{(2k-2j)}.$$
We can rewrite this as

$$\Pr(\Gamma_1, \ldots, \Gamma_j \in \mathcal{I}(G))^2 \Pr(\Gamma_1, \ldots, \Gamma_{k-j} \in \mathcal{I}(G))^2$$
$$\geq \Pr(\Gamma_1, \ldots, \Gamma_{j-1} \in \mathcal{I}(G)) \Pr(\Gamma_1, \ldots, \Gamma_{j+1} \in \mathcal{I}(G))$$
$$\Pr(\Gamma_1, \ldots, \Gamma_{k-(j+1)} \in \mathcal{I}(G)) \Pr(\Gamma_1, \ldots, \Gamma_{k-(j-1)} \in \mathcal{I}(G))$$

which is easily seen to be
$$(a_j^{(k)})^2 \geq a_{j-1}^{(k)} a_{j+1}^{(k)}.$$
Thus $a^{(k)}$ is log-concave. $\square$

While this result is an interesting partial converse to Proposition 2.3.1, it requires information about many sequences and puts a big restriction on k. This restriction is particularly unfortunate as it means that a vertex receives no colour at least half the time. Thus the result tells us nothing when $p = 1/k$, which the most interesting case to consider. With this in mind, and having undoubtedly verified many examples, Farr made the following conjecture:

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Conjecture 2.3.3 (Private communication from G. E. Farr). *Let G be a graph. For any non-negative integer k and any probability p such that $kp \leq 1$, the sequence $a^{(k)}(G; p)$ is log-concave.*

Chapter 3

The Joint Independence Probabilities as a Polynomial

We wish to better describe $\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G))$ as a polynomial. We know that for each ℓ, the probability is a polynomial in p. This polynomial can be written in terms of the independence polynomial when $\ell = 1$ and in terms of the chromatic polynomial when $\ell p = 1$. These polynomials have some similar properties such as being multiplicative over the connected components of G and satisfying some sort of deletion-contraction recurrence. It stands to reason that $\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G))$ might be a polynomial in both k and p which possesses these properties.

We begin by exploring possible connections to the Tutte polynomial. This polynomial is an important tool for studying graph invariants which have these kinds of properties. Unfortunately, we will see that the Tutte polynomial cannot describe the joint independence probabilities in general. We will turn to $P(G; k, \ell)$, a two-variable generalisation of the chromatic polynomial due to Dohmen, Poenitz, and Tittmann [7]. Thanks to its combinatorial interpretation, we will see that this polynomial is equivalent to $\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G))$. We explore the implications of this relation, especially in regards to the sequence $a^{(k)}$, and conclude with some important properties of $P(G; k, \ell)$.

3.1 The Tutte Polynomial

The Tutte polynomial is one of the most important and well-studied graph invariants. Via Theorem 3.1.3 it generalises graph invariants which possess a multiplicative property and satisfy a deletion-contraction recurrence.

The *rank of* G, denoted $r(G)$, is defined to be the number of edges in a maximal acyclic subgraph of G. Since a tree on n vertices has $n-1$ edges, $r(G) = n(G) - \kappa(G)$. For $F \subseteq E$, we identify F with the spanning subgraph of G with edge set F. The Tutte polynomial, $T(G; x, y)$, is defined as follows:

$$T(G; x, y) = \sum_{F \subseteq E} (x-1)^{r(G)-r(F)} (y-1)^{|F|-r(F)}. \tag{3.1}$$

The Tutte polynomial was first studied by Tutte in [26], however the coefficients of the equivalent rank polynomial, $R(G; x, y) = T(G; x-1, y-1)$, were studied earlier by Whitney [32, 33]. A great account of the history of the Tutte polynomial can be found in Section 3 of [10].

We denote the graph with a single vertex and a single loop by $\mathcal{L}$. The Tutte polynomial can also be defined recursively in the following way.

Theorem 3.1.1. *$T(G; x, y)$ is the unique polynomial which satisfies all of the following:*

(i) $T(K_1; x, y) = 1$, $T(K_2; x, y) = x$, *and* $T(\mathcal{L}; x, y) = y$;

(ii) *if G_1 and G_2 are graphs with at most one vertex in common, then*

$$T(G_1 \cup G_2; x, y) = T(G_1; x, y) T(G_2; x, y);$$

(iii) *for every edge $e \in E$ that is neither a cut-edge, nor a loop*

$$T(G; x, y) = T(G \setminus e; x, y) + T(G \,/\, e; x, y).$$

The following proof is adapted from [20].

Proof. It is easy to verify (i) holds.

If G_1 and G_2 have at most one vertex in common, then a maximal acyclic subgraph of G is the union of maximal acyclic subgraphs of G_1 and G_2. Thus

$$r(G_1 \cup G_2) = r(G_1) + r(G_2).$$

Since G_1 and G_2 have no edges in common, (ii) follows from (3.1).

Fix an edge e that is neither a cut-edge nor a loop. Observe that $r(G \setminus e) = r(G)$ and $r(G \,/\, e) = r(G) - 1$. If $F \subseteq E$ does not contain e, then

$$(x-1)^{r(G)-r(F)} (y-1)^{|F|-r(F)} = (x-1)^{r(G \setminus e)-r(F)} (y-1)^{|F|-r(F)}.$$

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On the other hand, if F contains e, then

$$(x-1)^{r(G)-r(F)}(y-1)^{|F|-r(F)} = (x-1)^{r(G/e)-r(F/e)}(y-1)^{|F/e|-r(F/e)}.$$

We conclude from these observations that (iii) holds.

Now suppose that $f(G; x, y)$ is a graph invariant satisfying (i), (ii), and (iii). We will show that T and f are the same by induction on m, the number of edges of G. The basis of induction is handled by (i). We assume that for some $m \geq 2$, T and f agree for all graphs with fewer than m edges. Suppose G has m edges. If G has an edge e that is neither a cut-edge nor a loop, then $f(G; x, y) = T(G; x, y)$ since they both satisfy (iii). Otherwise, every edge of G is either a cut-edge or a loop. We will show that $G = G_1 \cup G_2$ where G_1 and G_2 have at most one vertex in common and both have fewer than m edges. It will then follow that

$$T(G; x, y) = T(G_1; x, y)T(G_2; x, y) = f(G_1; x, y)f(G_2; x, y) = f(G; x, y).$$

If G is not connected, let G_1 be some connected component of G and let $G_2 = G - V(G_1)$. If G is connected, then G must have a vertex v incident with at least two distinct edges. In this case G is the union of two graphs G_1 and G_2, each having at least one edge, whose only common vertex is v. In both cases, G_1 and G_2 have fewer edges than G, completing the proof. $\square$

Remark 3.1.2. Using (i), (ii) and straightforward induction on the number of edges, one can easily show that $T(G; x, y) = x^i y^j$ when G is has i cut-edges, j loops, and no other edges.

As mentioned earlier, the Tutte polynomial generalises all graph invariants which satisfy a deletion-contraction recurrence and a multiplicativity property.

Theorem 3.1.3 (Theorem 1 in Section 2 of [20]). *Suppose that f is a graph invariant satisfying all of the following:*

(i) $f(K_1) = 1$;

(ii) $f(G_1 \cup G_2) = f(G_1)f(G_2)$ *whenever G_1 and G_2 have at most one vertex in common;*

(iii) *there exist a, b such that $f(G) = af(G \setminus e) + bf(G / e)$ whenever $e \in E$ is neither a cut-edge or a loop.*

Then, for any graph G, we have

$$f(G) = a^{|E(G)|-r(G)}b^{r(G)}T(G; b^{-1}f(K_2), a^{-1}f(L)). \tag{3.2}$$

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Proof. Let $f'(G)$ denote the right-hand side of (3.2). We already argued that $r(G) = r(G_1) + r(G_2)$ when G_1 and G_2 have at most one vertex in common. Hence $f'(G)$ satisfies (ii) since $T(G; x, y)$ satisfies it. If $e \in E$ is neither a loop nor a cut-edge, then $r(G \backslash e) = r(G)$ and $r(G / e) = r(G) - 1$. Thus, since $T(G; x, y)$ satisfies a deletion-contraction recurrence, f' satisfies the following:

$$f'(G) = a^{|E(G)|-r(G)}b^{r(G)}T(G; b^{-1}f(K_2), a^{-1}f(L))$$
$$= a^{|E(G)|-r(G)}b^{r(G)}[T(G \backslash e; b^{-1}f(K_2), a^{-1}f(L)) + T(G / e; b^{-1}f(K_2), a^{-1}f(L))]$$
$$= af'(G \backslash e) + bf'(G / e).$$

That $f'(G) = f(G)$ now follows in the same way we showed $T(G; x, y)$ was unique in the proof of Theorem 3.1.1. $\square$

The chromatic polynomial does not quite satisfy the conditions of Theorem 3.1.3. In particular, $P(K_1; k) = k$, and if G_1 and G_2 have exactly one vertex in common, then $P(G_1 \cup G_2; k) = k^{-1}P(G_1; k)P(G_2; k)$. To address this, we define the polynomial

$$P'(G; k) = k^{-\kappa(G)}P(G; k).$$

It is clear that $P'(K_1; k) = 1$. Moreover, if G_1 and G_2 have at most one vertex in common, it is easy to show $P'(G_1 \cup G_2; k) = P'(G_1; k)P'(G_2; k)$ using Theorem 2.1.3 and the following observation:

$$\kappa(G_1 \cup G_2) = \begin{cases} \kappa(G_1) + \kappa(G_2), & \text{if } |V(G_1) \cap V(G_2)| = 0 \\ \kappa(G_1) + \kappa(G_2) - 1, & \text{if } |V(G_1) \cap V(G_2)| = 1. \end{cases}$$

Finally, when e is not a cut-edge we are guaranteed that deleting e will not change the number of connected components. Hence P' satisfies the deletion-contraction recurrence just as $P(G; k)$ does

$$P'(G; k) = P'(G \backslash e; k) - P'(G / e; k).$$

It now follows from Theorem 3.1.3 that the chromatic polynomial is the following evaluation of the Tutte polynomial:

$$P(G; k) = (-1)^{r(G)}k^{\kappa(G)}T(G; 1 - k, 0).$$

Unfortunately, the independence polynomial cannot be expressed as an evaluation of the Tutte polynomial. It follows from Remark 3.1.2 that the Tutte polynomial of a tree on n vertices is x^{n-1}. On the other hand, $I(G; x)$ differs for the two trees on four vertices:

$$I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3$$
$$I(P_4; x) = 1 + 4x + 3x^2.$$

Thus, while the joint independent probabilities may be expressed in terms of the Tutte polynomial when $kp = 1$, this does not hold for general k. In particular, the probability that a subset Γ_i of vertices is independent is not an evaluation of the Tutte polynomial.

3.2   The Bivariate Chromatic Polynomial

Let K be a finite set of colours and let L be a subset of K. A (K, L)-*colouring* of G is a colouring $c : V \to K$ such that $c^{-1}(i)$ is an independent set whenever $i \in L$. We call the colours in L the *proper colours* of K and the colours in $K \setminus L$ the *improper colours* of K.

For convenience, if k and ℓ are non-negative integers such that $\ell \leq k$, then by a (k, ℓ)-colouring we will mean a $([k], [\ell])$-colouring of G. Similarly, if $L \subseteq [k]$, then we may use (k, L)-colouring to mean a $([k], L)$-colouring.
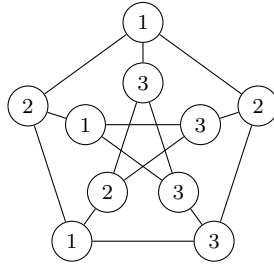
Figure 3.2.1: A $(3, 2)$-colouring of the Petersen graph

When none of the colours are proper we simply have a colouring and when all the colours are proper we have a proper colouring. Hence (k, ℓ)-colourings sit in between k-colourings and proper k-colourings. We also note that a vertex with a loop-edge can only be coloured with an improper colour.

For $U \subseteq V$, we denote by $G[U]$ the graph induced by the vertices in U. Just as the number of proper k-colourings is given by a polynomial in k, the number of (k, ℓ)-colourings is given by a polynomial in k and ℓ.

Theorem 3.2.2 (Theorem 1 of [7])**.** *For any non-negative integers k and ℓ such that $\ell \leq k$, the number of (k, ℓ)-colourings of G is given by*

$$\sum_{W \subseteq V} (k - \ell)^{n - |W|} P(G[W]; \ell). \tag{3.3}$$

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Proof. We consider all possible ways of partitioning the vertices into a set of properly coloured vertices and a set of improperly coloured vertices. For each $W \subseteq V$ there are $P(G[W]; \ell)$ colourings of U with the proper colours, and $(k - \ell)^{|W|}$ ways of colouring the remaining vertices with the improper colours. □

We call (3.3) the *bivariate chromatic polynomial* of G and denote it by $P(G; k, \ell)$. As we mentioned earlier, when all colours are proper we recover the notion of proper colourings, hence $P(G; k, k) = P(G; k)$. On the other hand, in the absence of proper colours we recover the notion of arbitrary colourings, hence $P(G; k, 0) = k^n$. This generalisation of the chromatic polynomial was first introduced in 2003 by Dohmen, Poenitz, and Tittmann [7].

A graph has a proper 1-colouring if and only if it has no edge. Hence, it follows from Theorem 3.2.2 that

$$P(G; x + 1, 1) = \sum_{W \subseteq V} x^{n-|W|} P(G[W]; 1) = \sum_{W \in \mathcal{I}(G)} x^{n-|W|}.$$

Thus, the independence polynomial is the following evaluation of $P(G; k, \ell)$:

$$I(G; x) = x^n P(G; (x + 1)/x, 1).$$

Using (2.2) we deduce a similar expression for the stability polynomial in terms of the bivariate chromatic polynomial

$$A(G; p) = p^n P(G; 1/p, 1).$$

Recall also (2.4) which gives a similar relation expressing the joint independence probabilities in terms of $P(G; \ell)$ when $\ell = 1/p$:

$$\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G)) = \frac{P(G; \ell)}{\ell^n} = \frac{P(G; \ell, \ell)}{\ell^n}.$$

These relations for specific values of ℓ are generalised in the following result which shows that $\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G))$ and $P(G; k, \ell)$ coincide.

Theorem 3.2.3. *Let G be a graph and let ℓ be a non-negative integer. Then*

$$\Pr(\Gamma_1^{(p)}, \ldots, \Gamma_\ell^{(p)} \in \mathcal{I}(G)) = p^n P(G; 1/p, \ell)$$

as polynomials in p.

Proof. Recall (2.1) which stated

$$\Pr(\Gamma_1^{(p)}, \ldots, \Gamma_\ell^{(p)} \in \mathcal{I}(G)) = \sum_{\substack{W_1, \ldots, W_k \in \mathcal{I}(G) \\ \text{disjoint}}} p^{|W_1| + \cdots + |W_k|}(1 - kp)^{n - |W_1| - \cdots - |W_k|}.$$

Since proper ℓ-colourings are in one-to-one correspondence with partitions of the vertices into ℓ independent sets (some of which may be empty), we can rewrite (2.1) as a sum over $W = W_1 \cup \cdots \cup W_\ell$

$$\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G)) = \sum_{W \subseteq V} p^{|W|}(1 - \ell p)^{n - |W|} P(G[W]; \ell).$$

Using Theorem 3.2.2 we easily rewrite $p^n P(G; 1/p, \ell)$ as

$$\sum_{W \subseteq V} p^{|W|}(1 - p\ell)^{n - |W|} P(G[W]; \ell).$$

Since both identities hold for all real numbers p between 0 and $1/\ell$, we conclude that $p^n P(G; 1/p, \ell)$ and $\Pr(\Gamma_1^{(p)}, \ldots, \Gamma_\ell^{(p)} \in \mathcal{I}(G))$ agree as polynomials in p. $\qquad\square$

Theorem 3.2.3 has some important consequences for the bivariate chromatic polynomial. It gives meaning to $P(G; k, \ell)$ when k is a positive non-integer number. It also implies a result analogous to Farr's correlation inequality holds for the bivariate chromatic polynomial. As for the joint independence probabilities, Theorem 3.2.3 tells us that it can be studied under the lens of (k, ℓ)-colourings. Moreover, many of its properties can be deduced from properties of $P(G; k, \ell)$. For example, computing $\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G))$ for K_n and $K_{m,n}$ can easily be done by counting (k, ℓ)-colourings.

Consider the complete graph on n vertices. To count the number of (k, ℓ)-colourings it suffices to count all possible ways of choosing a set of vertices to be properly coloured, and colouring the vertices accordingly. For each i, there are $\binom{n}{i}$ ways of choosing i vertices and $(\ell)_i$ ways of colouring them with proper colours. The remaining vertices are coloured however we choose, thus

$$P(K_n; k, \ell) = \sum_{i=0}^{n} \binom{n}{i} (k - \ell)^i (\ell)_{n-i}.$$

For the complete bipartite graph $K_{a,b}$ with bipartition (A, B) where $|A| = a$ and $|B| = b$, we count the ways of choosing i vertices from A and colour them with j colours. Let $\left\{ {i \atop j} \right\}$

denote *Stirling numbers of the second kind* which count the number of ways in which an i-element set can be partitioned into j non-empty subsets. Then there are $\left\{{i \atop j}\right\}(\ell)_j$ ways of colouring the vertices chosen from A, $(k-\ell)^{a-i}$ ways of colouring the remainder of A with improper colours, and $(k-j)^b$ ways of colouring the vertices of B. Therefore

$$P(K_{a,b}; k, \ell) = \sum_{i=0}^{a} \binom{a}{i}(k-\ell)^{a-i} \sum_{j=0}^{i} \left\{{i \atop j}\right\}(k-j)^b (\ell)_j.$$

We conclude that

$$\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G)K_n) = p^n \sum_{i=0}^{n} \binom{n}{i}(1/p - \ell)^i (\ell)_{n-i},$$

$$\Pr(\Gamma_1, \ldots, \Gamma_\ell \in \mathcal{I}(G)K_{a,b}) = p^{a+b} \sum_{i=0}^{a} \binom{a}{i}(1/p - \ell)^{a-i} \sum_{j=0}^{i} \left\{{i \atop j}\right\}(1/p - j)^b (\ell)_j.$$

An additional consequence of Theorem 3.2.3 is that, for any positive integer k,

$$a_k^{(p)}(G; \ell) = p^{2n} P(G; 1/p, \ell) P(G; 1/p, k - \ell).$$

Thus if the sequence $(P(G; 1/p, \ell))_{\ell=0}^k$ is log-concave, then it is immediate that $a_k^{(p)}(G)$ is as well. This equality also suggests that the sequence of evaluations of $P(G; k, \ell)$ may be a more basic object to study, and that $a_k^{(p)}(G)$ is simply inheriting its log-concavity. For example, Farr's correlation inequality can equivalently be stated as saying

$$P(G; k, \ell) P(G; k, 0) \leq P(G; k, \ell - 1) P(G; k, 1).$$

In particular, when $\ell = 2$,

$$P(G; k, 2) P(G; k, 0) \leq P(G; k, 1)^2$$

hence the first three terms of $(P(G; 1/p, \ell))_{\ell=0}^k$ satisfy the definition of log-concavity. To take advantage of the combinatorial interpretation of $P(G; k, \ell)$ we will focus on the case where $p = 1/k$. Since increasing the number of proper colours adds restrictions on the colouring it follows that the sequence is decreasing

$$p^{-n} = P(G; 1/p, 0) \geq P(G; 1/p, 1) \geq \cdots \geq P(G; 1/p,) = P(G; k).$$

In fact, the sequence is strictly decreasing when G has an edge since the edge will restrict the assignment of any proper colours. In Section 6.1 we give computational verification that $(P(G; k, \ell))_{\ell=0}^k$ is log-concave for all G and k where $n(G) \leq 10$ and $k \leq n(G)+1$. The results of these computations motivated a shift from studying $a^{(k)}$ to studying $(P(G; 1/p, \ell))_{\ell=0}^k$.

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3.3 Properties of $P(G; k, \ell)$

We will make frequent use of the following result to prove identities involving the bivariate chromatic polynomial.

Lemma 3.3.1. *Let $p \in \mathbb{R}[x, y]$ be arbitrary. Then p is uniquely determined by the values $p(k, \ell)$ for all non-negative integers k, ℓ such that $\ell \leq k$.*

Proof. Let N be a non-negative integer such that each of x and y appear in p at most to the N-th power. Let $S = \{N, \ldots, 2N + 1\} \times \{N, \ldots, 2N + 1\}$. For $s = (i, j) \in S$ let m_s be the monomial $(x - N)_i (y - N)_j$. Define M as the matrix

$$M = (m_i(j))_{i,j \in S}.$$

The set $\{(x)_a : a \in \{0, \ldots, N\}\}$ is a basis for polynomials in x of degree at most N. This is easily seen by induction on N and the observation that $(x)_N$ is the only falling factorial of degree N. It follows that $\{m_s : s \in S\}$ is a basis for $\mathbb{R}[x, y]$ of degree at most N for each variable individually. Hence $p(x, y)$ is in the column space of M. Thus, if M is non-singular, $p(x, y)$ is uniquely determined by $\{p(s) : s \in S\}$. Notice that $(x)_i (y)_j = 0$ when $x \in \{0, \ldots, i - 1\}$ or $y \in \{0, \ldots, j - 1\}$. Thus, with an appropriate ordering of the rows and columns, M is a triangular matrix with non-zero entries on its diagonal. We conclude that M is non-singular. $\qquad\square$

It is a consequence of Lemma 3.3.1 that identities involving $P(G; x, y)$ need only be shown to hold for non-negative integers ℓ, k such that $\ell \leq k$. We will use this fact many times without explicit mention.

By considering (3.3) it is clear that the term corresponding to $W = \emptyset$ is the only one in which k appears to the n-th power. This shows that the degree of $P(G; k, \ell)$ in terms of k is n. Moreover, the coefficient of k^n is 1. In order to say something similar in regards to ℓ we will express $P(G; k, \ell)$ in terms of falling factorials of ℓ. To do so, we will make use of a standard technique in enumerative combinatorics, the Inclusion-Exclusion Principle.

Inclusion-Exclusion Principle. *Let S and $\mathcal{A}$ be finite sets. Suppose that we have a set $A_s \subseteq \mathcal{A}$ for each $s \in S$. For every $T \subseteq S$, denote by A_T the set $\bigcap_{t \in T} A_t$, where $A_\emptyset = \mathcal{A}$. Then the number of elements of $\mathcal{A}$ that are not contained in any A_s is*

$$\left| \mathcal{A} \setminus \bigcup_{s \in S} A_s \right| = \sum_{T \subseteq S} (-1)^{|T|} |A_T|.$$

A proof of the Inclusion-Exclusion Principle can be found in Section 2.1 of [24].

Theorem 3.3.2. *Fix a graph G. For every non-negative integer i and every $W \subseteq V$, define $\mu_i(W)$ to be the number of partitions of W into exactly i dependent sets. Then*

$$P(G; k, \ell) = \sum_{i \geq 0} (-1)^i (\ell)_i \sum_{W \subseteq V} \mu_i(W)(k-i)^{n-|W|}.$$

Proof. Fix arbitrary non-negative integers k, ℓ such that $\ell \leq k$. For each $i \in [\ell]$, define A_i to be the set of k-colourings for which the i-th colour-class is dependent. Then $P(G; k, \ell)$ counts the number of k-colourings which are not in any A_i. We let $\mathcal{A}$ be the set of all k-colourings of G and use the notation given in the statement of the Inclusion-Exclusion Principle. It follows that we have

$$P(G; k, \ell) = \sum_{I \subseteq [\ell]} (-1)^{|I|} |A_I|.$$

For $I \subseteq [\ell]$, A_I is the set of k-colourings c of G which have the property that $c^{-1}(j)$ is dependent for each $j \in I$. To count such colourings we consider partitioning the vertices into two sets: those which are coloured with colours in I, and those which are not. Let $i = |I|$. For $W \subseteq V$, there are $\mu_i(W) \cdot i!$ ways of colouring the vertices of W with the colours of I and $(k-i)^{n-|W|}$ ways of colourings the other vertices with the remaining colours. Therefore

$$|A_I| = \sum_{W \subseteq V} \mu_i(W)(k-i)^{n-|W|} i!.$$

Since there are $(\ell)_i / i!$ subsets of $[\ell]$ of cardinality i, the result follows. $\square$

If π is a partition of W into i dependent sets, then each part of π contains two vertices which are joined by an edge, hence $|W| \geq 2i$. It follows from Theorem 3.3.2 that the coefficient of $k^j \ell^i$ in $P(G; k, \ell)$ can only be non-zero when $j + 2i \leq n$. In particular, the total degree of $P(G; k, \ell)$ is n. It also means that π must contain a set of vertices matched by a matching of size i. We denote by $\nu(G)$ the maximum size of a matching of G. It follows that $\mu_i(W)$ is zero for every W when $i > \nu(G)$. Furthermore, for every $i \leq \nu(G)$ there is a $W \subseteq V$ such that $\mu_i(W)$ is non-zero. We deduce that the degree of $P(G; k, \ell)$ in terms of ℓ is $\nu(G)$.

If $W \subseteq V$ is such that $|W| = 2i$ for some $i \in \{0, \ldots, \nu(G)\}$, then $\mu_i(W)$ is the number of matchings of G for which the set of matched vertices is exactly W. By fixing i, and summing $\mu_i(W)$ over all $W \subseteq V$ satisfying $|W| = 2i$, we obtain the number of matchings

of G of cardinality i. Call this last quantity $m_i(G)$. Then the coefficient of the terms of total degree n are

$$[k^{n-2i}\ell^i]P(G; k, \ell) = (-1)^i m_i(G).$$

It is interesting to note that $(-1)^i m_i(G)$ is the coefficient of the i-th term in the matching polynomial of G.

The bivariate chromatic polynomial also possesses a multiplicativity property, and satisfies a deletion-contraction recurrence, like the independence and chromatic polynomials.

Theorem 3.3.3. *If G_1 and G_2 are disjoint graphs, then*

$$P(G_1 \cup G_2; k, \ell) = P(G_1; k, \ell)P(G_2; k, \ell).$$

Proof. Since the assignment of colours in one connected component cannot place restrictions on the (k, ℓ)-colouring of another component, the (k, ℓ)-colourings of $G_1 \cup G_2$ are in one-to-one correspondence with pairs of (k, ℓ)-colourings for G_1 and G_2. $\square$

Once again, this means that $P(G; k, \ell)$ multiplies over connected components. However, it also has an important implication for the sequences $(P(G; k, \ell))_{\ell=0}^k$ and $a^{(k)}(G; p)$. Since the term-wise product of two log-concave sequences is also log-concave, Theorem 3.2.2 tells us that in order to show $(P(G; k, \ell))_{\ell=0}^k$ is log-concave, it suffices to check this result for the connected components of G.

Unfortunately, when G_1 and G_2 have vertices in common we cannot hope for a result like Theorem 2.1.3. Such a result would have to account for all possible ways of partitioning the common vertices between the proper and improper colours. In particular, we would require an expression for the number of (k, ℓ)-colourings where some vertices must receive proper colours.

The deletion-contraction recurrence that $P(G; k, \ell)$ satisfies is similar to the expression given for the independence polynomial in Theorem 2.1.2. Since $P(G; k)$ and $I(G; x)$ are evaluations of $P(G; k, \ell)$, this recurrence generalises Theorem 2.1.2 and Theorem 2.1.4.

Theorem 3.3.4 (Lemma 12 of [1]). *If e is an edge in a graph G, then*

$$P(G; k, \ell) = P(G \setminus e; k, \ell) - P(G / e; k, \ell) + (k - \ell)P(G - e; k, \ell). \qquad (3.4)$$

Proof. As in the proof of Theorem 2.1.4, $P(G \setminus e; k, \ell) - P(G / e; k, \ell)$ corresponds to the number of (k, ℓ)-colourings of G where the ends of e get different colours. The remaining (k, ℓ)-colourings of G must assign the same improper colour to both ends of e. The number of such colourings is given by $(x - y)P(G - e; k, \ell)$. $\square$

Given Theorems 3.3.3 and 3.3.4 it is natural to seek a polynomial which would be to $P(G; k, \ell)$, what the Tutte polynomial is to $P(G; k)$. One such object has been studied by Averbouch, Godlin, and Makowsky [1]. The polynomial in question is a three-variable generalisation of the Tutte polynomial which specialises to all graph invariants satisfying Theorems 3.3.3 and 3.3.4.

Table 3.3.5 lists the bivariate chromatic polynomial for the complete graphs, the complete bipartite graphs, the path graphs, and the cycle graphs. Details and justification for the cycle and path graphs can be found in Section 5 of [7].

G	$P(G; k, \ell)$
K_n	$\displaystyle\sum_{i=0}^{n} \binom{n}{i} (k-\ell)^i (\ell)_{n-i}$
$K_{m,n}$	$\displaystyle\sum_{i=0}^{m} \binom{m}{i} (k-\ell)^{m-i} \sum_{j=0}^{i} \left\{ {i \atop j} \right\} (k-j)^n k(\ell)_j$
P_n	$\displaystyle\sum_{0<i+2j\leq n} (-1)^{n-i-j} \binom{i+j}{i} \binom{n-i-j-1}{n-i-2j} k^i \ell^j$
C_n	$\displaystyle (-1)^n \ell + n \sum_{0<i+2j\leq n} \frac{(-1)^{n-i-j}}{i+j} \binom{i+j}{i} \binom{n-i-j-1}{n-i-2j} k^i \ell^j$

Table 3.3.5: $P(G; k, \ell)$ for some common families of graphs

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Chapter 4

A Combinatorial Proof of Farr's Correlation Inequality

Recall Farr's correlation inequality which states that for a number of colours k, a probability p, and $\ell \in [k]$, we have

$$\Pr(\Gamma_1^{(p)}, \ldots, \Gamma_\ell^{(p)} \in \mathcal{I}(G)) \leq \Pr(\Gamma_1^{(p)}, \ldots, \Gamma_{\ell-1}^{(p)} \in \mathcal{I}(G)) \Pr(\Gamma_\ell^{(p)} \in \mathcal{I}(G)). \qquad (4.1)$$

In this chapter we will provide a combinatorial proof of this inequality in the more general setup used for Theorem 2.2.1. Specifically, $\boldsymbol{q} = (q_{i,v} : i \in [k], v \in V)$ is a vector of probabilities, for each each $v \in V$, $\sum_{i \in [k]} q_{i,v} = 1$ and $v \in \Gamma_i^{(\boldsymbol{q})}$ with probability $q_{i,v}$. We aim to prove the following theorem:

Theorem 4.0.1. *Let k be a non-negative integer, and $\boldsymbol{q}$ be a vector of probabilities such that $\sum_{i \in [k]} q_{i,v} = 1$ for each $v \in V$. Then, for each $\ell \in [k]$, we have*

$$\Pr(\Gamma_1^{(\boldsymbol{q})}, \ldots, \Gamma_\ell^{(\boldsymbol{q})} \in \mathcal{I}(G)) \leq \Pr(\Gamma_1^{(\boldsymbol{q})}, \ldots, \Gamma_{\ell-1}^{(\boldsymbol{q})} \in \mathcal{I}(G)) \Pr(\Gamma_\ell^{(\boldsymbol{q})} \in \mathcal{I}(G)).$$

Fix a graph G. We let $\boldsymbol{\Gamma}$ denote the k-colouring whose i-th colour-class is Γ_i for each $i \in [k]$. For convenience, if $L \subseteq [k]$, we will denote by $\mathcal{C}_k(L)$ the set of (k, L)-colourings of G. Since $\mathcal{C}_k(\emptyset)$ is the set of all k-colourings, we can rewrite the inequality of Theorem 4.0.1 as follows:

$$\Pr(\boldsymbol{\Gamma} \in \mathcal{C}_k([\ell])) \cdot \Pr(\boldsymbol{\Gamma} \in \mathcal{C}_k(\emptyset)) \leq \Pr(\boldsymbol{\Gamma} \in \mathcal{C}_k([\ell-1])) \cdot \Pr(\boldsymbol{\Gamma} \in \mathcal{C}_k(\{\ell\})).$$

The key ingredient of the proof is an injective function, $\Psi_{k,\ell}$, which will map $\mathcal{C}_k([\ell]) \times \mathcal{C}(\emptyset)$ into $\mathcal{C}_k([\ell-1]) \times \mathcal{C}_k(\{\ell\})$. Section 4.1 will focus on defining $\Psi_{k,\ell}$ and Section 4.2 will contain the remainder of the proofs of Theorem 4.0.1 and Farr's correlation inequality.

4.1 An Injective Function on Pairs of Colourings

Fix a graph G. For $L \subseteq [k]$, let $\mathcal{C}_k(L)$ will be the set of (k, L)-colourings of G. We will define an injective function $\Psi_{k,\ell}$ mapping $\mathcal{C}_k([\ell]) \times \mathcal{C}_k(\emptyset)$ into $\mathcal{C}_k([\ell - 1]) \times \mathcal{C}_k(\{\ell\})$. We begin with an example showing the result of applying $\Psi_{k,\ell}$ to a pair of 3-colourings of the Petersen graph. Afterwards, we will give a formal treatment of $\Psi_{k,\ell}$.

For our example ℓ will be 3. We consider the colourings $c_1 \in \mathcal{C}_3([3])$ and $c_2 \in \mathcal{C}_3(\emptyset)$ which are depicted in Figure 4.1.1; c_1 is shown on the left and c_2 on the right.
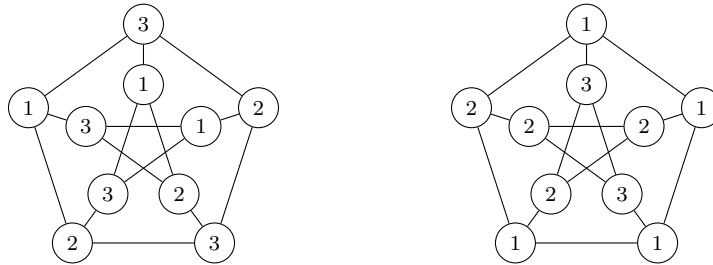
Figure 4.1.1: The colourings c_1 and c_2

For every vertex $v \in V$, if either $c_1(v)$ or $c_2(v)$ is ℓ (3 in this case), then we swap the colour of $c_1(v)$ and $c_2(v)$. Doing so yields the colourings shown in Figure 4.1.2.

Figure 4.1.2: The colourings after the first swap

In Figure 4.1.2 the colouring on the right is a $(3, \{3\})$-colouring, but the colouring on the left is not a $(3, \{1, 2\})$-colouring. Call the left colouring c_1'. For each colour $i \in \{1, 2\}$ we define the set A_i to be the set of vertices $v \in V$ which satisfy the following:

- v was originally assigned colour ℓ by c_1,

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- v is assigned colour i by c_1', and
- v is adjacent to a vertex that is also assigned colour i by c_1'.

In our example, A_1 and A_2 each contain a single vertex. These vertices are indicated in Figure 4.1.3.

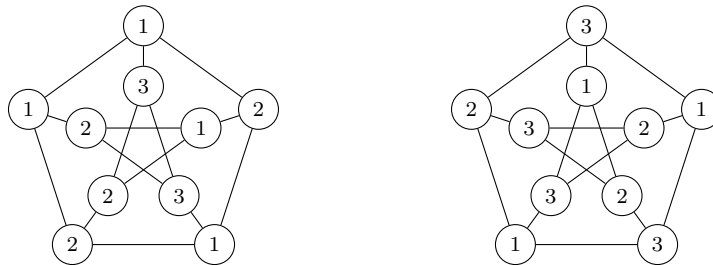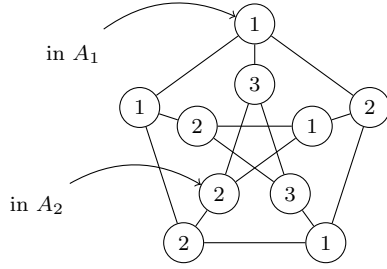Figure 4.1.3: The sets A_j for the left colouring

For every colour $i \in \{1, 2\}$ and every $v \in A_i$, we swap the colours assigned to v in the left and right colouring of Figure 4.1.2. Each of these vertices will now be assigned the same colour they were originally assigned by c_1 and c_2. The first and second colour-classes of the left colouring will now be independent sets. The resulting colourings are shown in Figure 4.1.4.
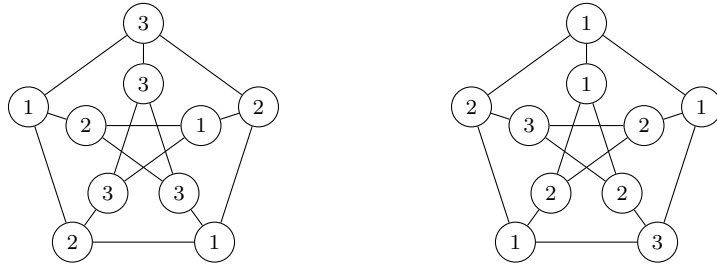
Figure 4.1.4: Final colourings produced by $\Psi_{3,3}$

From left to right denote the colourings of Figure 4.1.4 by d_1 and d_2 respectively. We observe that d_1 is a $(k, \{1, 2\})$-colouring of G and that d_2 is a $(k, \{3\})$-colouring of G. The pair (d_1, d_2) is the image of (c_1, c_2) under $\Psi_{k,\ell}$.

We now formally define $\Psi_{k,\ell}$. The reader is encouraged to verify that the example above conforms with the following definition. Fix k, ℓ and let $(c_1, c_2) \in \mathcal{C}_{k,\ell} \times \mathcal{C}_{k,0}$. If no such

(c_1, c_2) exist for this choice of k, ℓ, then we are done. For all $i, j \in [k]$, we define

$$C_{i,j} = c_1^{-1}(i) \cap c_2^{-1}(j).$$

The $C_{i,j}$'s partition V into k^2 sets, some of which may be empty, such that

$$\bigcup_{j \in [k]} C_{i,j} = c_1^{-1}(i)$$

is an independent set for each $i \in [\ell]$. Let $u \sim v$ denote adjacency of u and v in G. For $j \in [\ell - 1]$ we define A_j to be the set

$$\{u \in C_{1,j} : u \sim v, \text{ for some } v \in C_{j,i} \text{ and some } i \neq \ell\}.$$

To make notation easier, we will set $A_j = \emptyset$ for all other j. We now swap the vertices in $C_{1,j} \setminus A_j$ with those in $C_{j,1}$, for all $j \in [k]$. Specifically, we define $D_{i,j}$, where $i, j \in [k]$, as follows:

$$D_{i,j} = \begin{cases} C_{\ell,i} \setminus A_i, & \text{if } j = \ell \\ C_{j,\ell} \cup A_j, & \text{if } i = \ell \\ C_{i,j}, & \text{otherwise.} \end{cases}$$

We note that since $A_\ell = \emptyset$, $D_{\ell,\ell} = C_{\ell,\ell}$ in both of the first two conditions. It is quickly verified that the $D_{i,j}$'s form a partition of V. We define d_1 and d_2 to be the unique k-colourings of G which satisfy, for all $i, j \in [k]$,

$$d_1^{-1}(i) \cap d_2^{-1}(j) = D_{i,j}.$$

Claim 4.1.5. *The k-colourings d_1 and d_2 defined above are, respectively, a $(k, [\ell - 1])$-colouring and a $(k, \{\ell\})$-colouring of G.*

Proof. Let $i \in [\ell - 1]$ be arbitrary. We begin by showing the i-th colour-class of d_1 is an independent set. It is clear that

$$\bigcup_{j \neq \ell} D_{i,j} = \bigcup_{j \neq \ell} C_{i,j} \subseteq c_1^{-1}(i),$$

hence the union on the left is an independent set of G. As for $D_{i,\ell}$, it is defined to be $C_{\ell,i} \setminus A_i$. Since the ℓ-th colour-class of c_1 is an independent set, so is $C_{\ell,i}$. Our choice of A_i ensures there is no vertex in $D_{i,\ell}$ adjacent to a vertex in any $D_{i,j}$ where $j \neq \ell$. Thus, $d_1^{-1}(i)$

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is an independent set. As i was arbitrary, we conclude that d_1 is a $(k, [\ell - 1])$-colouring of G. On the other hand, we have

$$d_2^{-1}(\ell) = \bigcup_{i=1}^{k} D_{i,\ell} = \bigcup_{i=1}^{k} (C_{i,\ell} \setminus A_i).$$

In particular, the ℓ-th colour-class of d_2 is a subset of the ℓ-th colour-class of c_1. This colour-class is independent, and so d_2 is a $(k, \{\ell\})$-colouring of G. $\quad\square$

Claim 4.1.6. *The mapping* $(c_1, c_2) \mapsto (d_1, d_2)$ *given above is injective.*

Proof. Just as d_1, d_2 are uniquely determined from the $D_{i,j}$'s, c_1, c_2 can be uniquely determined from the $C_{i,j}$'s. Hence, it suffices to show that the $C_{i,j}$'s can be uniquely determined using only the $D_{i,j}$'s. We begin by defining A_i in terms of $D_{i,j}$ A_i.

We know $A_i = \emptyset$ when $i \notin [\ell - 1]$, hence we fix $i \in [\ell - 1]$. From the definition, $D_{\ell,i} = C_{i,\ell} \cup A_i$. Since $c_1^{-1}(i)$ is independent, $C_{i,\ell}$ has no neighbours in $C_{i,j}$ for all $j \neq \ell$. But $C_{i,j} = D_{i,j}$ when neither i nor j is ℓ. Thus $C_{i,\ell}$ has no neighbours in $D_{i,j}$ for all $j \neq \ell$. The vertices in A_i were chosen because they were adjacent to vertices in some $C_{i,j}$ for some $j \neq \ell$. This means that

$$A_i = \{u \in D_{1,i} : u \sim v, \text{ for some } v \in D_{i,j} \text{ and some } j \neq \ell\}.$$

Now, using the definition of $D_{i,j}$, we have

$$C_{i,j} = \begin{cases} D_{\ell,i} \cup A_i, & \text{if } j = \ell \\ D_{j,\ell} \setminus A_j, & \text{if } i = \ell \\ D_{i,j}, & \text{otherwise.} \end{cases}$$

This shows that c_1, c_2 can be uniquely determined from d_1, d_2, thus the mapping is injective. $\quad\square$

We define $\Psi_{k,\ell}$ to be the injective function described above which takes $(c_1, c_2) \in \mathcal{C}_k([\ell]) \times \mathcal{C}_k(\emptyset)$ to (d_1, d_2) in $\mathcal{C}_k([\ell - 1]) \times \mathcal{C}_k(\{\ell\})$.

4.2 Proof of Farr's Correlation Inequality

Fix a graph G. Let $\mathcal{T}_k$ be the set of all n-tuples, indexed by V, of unordered pairs $\{a, b\}$ where $a, b \in [k]$. The idea is to use $\boldsymbol{t} \in \mathcal{T}_k$ to prescribe the possible colours that can be

assigned to each vertex by a pair of k-colourings. In particular, for a pair of k-colourings (c_1, c_2) and $\boldsymbol{t} \in \mathcal{T}_k$, we wish to have $\{c_1(v), c_2(v)\} = t_v$ for each $v \in V$. For $L_1, L_2 \subseteq [k]$, we define $\mathcal{K}_k(\boldsymbol{t}; L_1, L_2)$ to be the set

$$\{(c_1, c_2) \in \mathcal{C}_k(L_1) \times \mathcal{C}_k(L_2) : \{c_1(v), c_2(v)\} = t_v, \ \forall v \in V\}.$$

The key to the proof is that $\Psi_{k,\ell}$ is an injective function which preserves $\{c_1(v), c_2(v)\}$ for each $v \in V$. This is captured by the following lemma:

Lemma 4.2.1. *Let $\boldsymbol{t} \in \mathcal{T}_k$ and $\ell \in [k]$ be arbitrary. Then we have*

$$|\mathcal{K}_k(\boldsymbol{t}; [\ell], \emptyset)| \leq |\Gamma_k(\boldsymbol{t}; [\ell - 1], \{\ell\})|.$$

Proof. Let $(c_1, c_2) \in \mathcal{K}_k(\boldsymbol{t}; [\ell], \emptyset)$ be arbitrary and let $(d_1, d_2) = \Psi_{k,\ell}(c_1, c_2)$. Since $\Psi_{k,\ell}$ is an injective function and (c_1, c_2) is arbitrary, it will be sufficient to show that $(d_1, d_2) \in \mathcal{K}_k(\boldsymbol{t}; [\ell - 1], \{\ell\})$.

Thanks to Claim 4.1.5, we only need to show that for every $v \in V$

$$\{c_1(v), c_2(v)\} = \{d_1(v), d_2(v)\}. \tag{4.2}$$

We recall the definitions of $C_{i,j}$ and $D_{i,j}$ from the previous section. To verify (4.2) holds we can equivalently show that for all $i, j \in [k]$

$$C_{i,j} \cup C_{j,i} = D_{i,j} \cup D_{j,i}. \tag{4.3}$$

If $i = \ell$, then $D_{\ell,j} = C_{j,\ell} \cup A_j$ and $D_{j,\ell} = C_{\ell,j} \setminus A_j$. The case for $j = \ell$ is similar. Otherwise $D_{i,j} = C_{i,j}$ and $D_{j,i} = C_{j,i}$. In all three cases (4.3) holds. $\qquad \square$

Proof of Theorem 4.0.1. For any $L_1, L_2 \subseteq [k]$, we can break up the probabilities

$$Pr(\boldsymbol{\Gamma} \in \mathcal{C}_k(L_1) \Pr(\boldsymbol{\Gamma} \in \mathcal{C}_k(L_2))$$

as a sum over $\boldsymbol{t} \in \mathcal{T}_k$ to obtain

$$\sum_{\boldsymbol{t} \in \mathcal{T}_k} \sum_{(c_1, c_2) \in \mathcal{K}_k(\boldsymbol{t}; L_1, L_2)} \Pr(\boldsymbol{\Gamma} = c_1) \Pr(\boldsymbol{\Gamma} = c_2).$$

The probability that $v \in V$ get colour $i \in [k]$ is $q_{i,v}$, hence for a k-colouring c, we have

$$\Pr(\boldsymbol{\Gamma} = c) = \prod_{v \in V} q_{c(v), v}.$$

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In particular, if (c_1, c_2) are such that $\{c_1(v), c_2(v)\} = t_v$ for each $v \in V$, then

$$\Pr(\mathbf{\Gamma} = c_1) \Pr(\mathbf{\Gamma} = c_2) = \prod_{\substack{v \in V \\ \{a,b\} = t_v}} q_{a,v} q_{b,v}.$$

The right-hand side depends only on $\boldsymbol{t}$; for convenience we will denote this product by $\boldsymbol{t}(\boldsymbol{q})$. Thus we have

$$Pr(\mathbf{\Gamma} \in \mathcal{C}_k(L_1) \Pr(\mathbf{\Gamma} \in \mathcal{C}_k(L_2)) = \sum_{\boldsymbol{t} \in \mathcal{T}_k(V)} \boldsymbol{t}(\boldsymbol{q}) \cdot |\mathcal{K}_k(\boldsymbol{t}; L_1, L_2)|. \tag{4.4}$$

We set $L_1 = [\ell]$ and $L_2 = \emptyset$ in (4.4). Applying Lemma 4.2.1 yields

$$Pr(\mathbf{\Gamma} \in \mathcal{C}_k(\{1, \ldots, \ell\}) \Pr(\mathbf{\Gamma} \in \mathcal{C}_k(\emptyset))$$
$$= \sum_{\boldsymbol{t} \in \mathcal{T}_k(V)} \boldsymbol{t}(\boldsymbol{q}) \cdot |\mathcal{K}_k(\boldsymbol{t}; [\ell], \emptyset)|$$
$$\leq \sum_{\boldsymbol{t} \in \mathcal{T}_k(V)} \boldsymbol{t}(\boldsymbol{q}) \cdot |\mathcal{K}_k(\boldsymbol{t}; [\ell-1], \{\ell\})|$$
$$= \Pr(\mathbf{\Gamma} \in \mathcal{C}_k([\ell-1]) \Pr(\mathbf{\Gamma} \in \mathcal{C}_k(\{\ell\}))$$

This concludes the proof of Theorem 4.0.1. $\qquad\square$

Farr's correlation inequality is a simple corollary of Theorem 4.0.1.

Corollary 4.2.2 (Farr's correlation inequality). *Let k be a non-negative integer and p be a probability such that $kp \leq 1$. Then for any $\ell \in [k]$ we have*

$$\Pr(\mathbf{\Gamma} \in \mathcal{C}_k([\ell])) \cdot \Pr(\mathbf{\Gamma} \in \mathcal{C}_k(\emptyset)) \leq \Pr(\mathbf{\Gamma} \in \mathcal{C}_k(\{1, \ldots, \ell-1\})) \cdot \Pr(\mathbf{\Gamma} \in \mathcal{C}_k(\{\ell\})).$$

Proof. Let $k' = k + 1$. For each $v \in V$ set $q_{i,v} = p$ for all $i \in [k]$, and set $q_{k+1,v} = 1 - kp$. It follows from Theorem 4.0.1 that

$$\Pr(\mathbf{\Gamma} \in \mathcal{C}_k([\ell])) \cdot \Pr(\mathbf{\Gamma} \in \mathcal{C}_k(\emptyset)) \leq \Pr(\mathbf{\Gamma} \in \mathcal{C}_k([\ell-1])) \cdot \Pr(\mathbf{\Gamma} \in \mathcal{C}_k(\{\ell\})).$$

This concludes the proof of Farr's correlation inequality using a basic combinatorial approach. $\qquad\square$

We have tried without success to extend our approach to prove Theorem 2.2.1 or to prove log-concavity of $(P(G; k, \ell))_{\ell=0}^{k}$ beyond the first three terms. Our attempts seem to suggest that our approach relies heavily on the ability to freely change the colours assigned to vertices by c_2 without having to worry about keeping some colour-classes independent. A different approach will likely be needed to provide combinatorial proofs of these stronger results.

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Chapter 5

The Polynomial $B_k(G; y)$

While testing various approaches to log-concavity of the sequence $(P(G; k, \ell))_{\ell=0}^k$, we came across a peculiar polynomial, which we will call $B_k(G; y)$, whose roots seem to always be real. For this particular polynomial, real-rootedness implies the log-concavity of $(P(G; k, \ell))_{\ell=0}^k$, motivating us to study $B_k(G; y)$. For any positive integer k, the $B_k(G; y)$ is defined as follows:

$$B_k(G; y) = \sum_{\ell=0}^{k} \binom{k}{\ell} P(G; k, \ell) y^\ell.$$

The focus of this chapter is the study of $B_k(G; y)$ and its peculiar properties.

Section 5.1 will focus on the motivation for studying $B_k(G; y)$. This includes the result due to Newton which inspired the definition of $B_k(G; y)$ and some basic examples. In Section 5.2 we introduce some basic results in the theory of stable polynomials, a more general notion of real-rootedness, which we will use in the remainder of the chapter. Section 5.3 will discuss what we know about the coefficients and roots of $B_k(G; y)$. Section 5.4 gives a proof that the real-rootedness of $B_k(G; y)$ depends on the roots of the polynomial for the connected components of G. An analogue of the deletion-contraction recurrence for $B_k(G; y)$ is explored in Section 5.5. Section 5.6 will discuss a related transformation on polynomials that may preserve real-rootedness. Finally, Section 5.7 will give a related result which came up during our investigation.

5.1 Motivation to Study $B_k(G; y)$

The inspiration for defining $B_k(G; y)$ is the following result due to Newton, commonly referred to as Newton's Inequalities.

Newton's Inequalities. *Let*

$$\phi(x) = \sum_{\ell=0}^{m} \binom{m}{\ell} \alpha_\ell x^\ell$$

be a polynomial of degree m with real coefficients. If $\phi(x)$ has only real roots, then the sequence $(\alpha_\ell)_{\ell=0}^{m}$ is log-concave.

For simplicity, we will write D_x to mean $\partial/\partial x$ and similarly for other variables. The following proof is adapted from Stanley [23].

Proof. First, we show that if $p \in \mathbb{R}[x]$ has only real roots, then so does $D_x p$. Let $\theta_1 < \cdots < \theta_r$ be the distinct roots of p with multiplicities $m_1, \ldots, m_r$. Then we have

$$p(x) = \prod_{i=1}^{n} (x - \theta_i)^{m_i}$$

Using the product and chain rules for differentiation, it is straightforward to show that there exists some $q \in \mathbb{R}[x]$ of degree $r - 1$ such that

$$D_x p = q(x) \prod_{i=1}^{n} (x - \theta_i)^{m_i - 1}$$

To show the roots of $q(x)$ are real, we recall Rolle's Theorem which states that if f is a continuous real-valued function on an interval $[a, b]$, which is differentiable on (a, b), and $f(a) = f(b)$, then $D_x f(c) = 0$ for some $c \in (a, b)$. Applying Rolle's Theorem with $a = \theta_i$ and $b = \theta_{i+1}$ for each $i \in [r-1]$ we can show that the $r-1$ roots of q are all real. We conclude that $D_x p$ has only real roots.

Next, we fix $j \in \{1, \ldots, m-1\}$ and let $\phi_1(x)$ be the polynomial

$$\phi_1(x) = D_x^{j-1} \phi(x) = \sum_{\ell=j-1}^{m} \binom{m}{\ell} (\ell)_{j-1} \alpha_\ell x^{\ell-j+1}.$$

Our argument above guarantees that $\phi_1(x)$ has only real roots. Define $\phi_2(x)$ to be

$$\phi_2(x) = x^{m-j+1}\phi_1(x^{-1}) = \sum_{\ell=j-1}^{m} \binom{m}{\ell}(\ell)_{j-1}\alpha_\ell x^{m-\ell}.$$

Notice that the roots of $\phi_2(x)$ are the inverses of the non-zero roots of $\phi_1(x)$, hence $\phi_2(x)$ has only real roots. Finally, define $\phi_3(x)$ to be

$$\phi_3(x) = \mathrm{D}_x^{m-j-1}\phi_2(x) = \sum_{\ell=j-1}^{j+1} \binom{m}{\ell}(\ell)_{j-1}(m-\ell)_{m-j-1}\alpha_\ell x^{j+1-\ell}.$$

As with $\phi_1(x)$, all the roots of $\phi_3(x)$ are real. It is straightforward to simplify the expression for $\phi_3(x)$ given above to

$$\phi_3(x) = \frac{m!}{2}(\alpha_{j-1}x^2 + 2\alpha_j x + \alpha_{j+1}).$$

This quadratic has real roots exactly when $\alpha_j^2 \leq \alpha_{j-1}\alpha_{j+1}$. Since $j \in \{1, \ldots, m-1\}$ was arbitrary, we conclude that $(\alpha_j)_{j=0}^m$ is log-concave. $\qquad\square$

In particular, it follows from Newton's Inequalities that the sequence $(P(G;k,\ell))_{\ell=0}^k$ is log-concave when $B_k(G;y)$ has only real roots. Since $\binom{k}{0}, \ldots, \binom{k}{k}$ is a log-concave sequence, log-concavity of the evaluations of $P(G;k,\ell)$ also follows from the real-rootedness of the following polynomial

$$\sum_{\ell=0}^{k} P(G;k,\ell)y^\ell.$$

However, it is easy to check that

$$\sum_{\ell=0}^{k} P(K_1;k,\ell)y^\ell = k\sum_{\ell=0}^{k} y^\ell$$

has non-real roots when $k = 2$. The polynomial $B_k(G;y)$, on the other hand, is not known to have non-real roots for any G and k. We discuss an exhaustive search for such a pair in Section 6.1.

Table 5.1.1 lists some of the simpler examples of $B_k(G;y)$. Each can be deduced from the definition of $B_k(G;y)$ and the forms for $P(G;k,\ell)$ given in Table 3.3.5. Unfortunately, even for simple families of graphs, expressions for $B_k(G;y)$ tend to be lengthy. This can be seen in the progression from $B(K_{1,m};y)$ to $B_k(K_{2,m};y)$.

G	$B_k(G; y)$
$\overline{K_n}$	$k^n(1+y)^n$
$K_{1,m}$	$k((k-1)^m y + k^m)(y+1)^{k-1}$
$K_{2,m}$	$k(y+1)^{k-2}[k^m(y+k) + (k-1)^m y(2k-1+y) + (k-2)^m y(ky+1)]$

Table 5.1.1: Some simple examples of $B_k(G; y)$

It is immediately clear that, for every positive integer k, $B_k(\overline{K_n}; y)$ and $B_k(K_{1,m}; y)$ have only real roots for all k. Real-rootedness of $B_k(K_{2,m}; y)$ is equivalent to showing that the following quadratic has real roots

$$[(k-1)^m + k(k-2)^m]y^2 + [k^m + (2k-1)(k-1)^m + (k-2)^m]y + k^{m+1}.$$

Showing this quadratic has real roots for arbitrary k and m does not appear to be straightforward.

5.2 Basic Theory of Stable Polynomials

For $z \in \mathbb{C}$, let $\Im(z)$ denote the imaginary part of z. Define $\mathcal{H}$ to be the half-plane $\{z \in \mathbb{C} : \Im(z) > 0\}$. Let $\boldsymbol{z}$ be a sequence of indeterminates $(z_1, \ldots, z_\eta)$ for some η. A polynomial $f \in \mathbb{C}[\boldsymbol{z}]$ is said to be *stable* if one of the following is true:

- f is identically zero, or
- for all $\boldsymbol{z} \in \mathcal{H}^\eta$, we have $f(\boldsymbol{z}) \neq 0$.

A stable polynomial in $\mathbb{R}[\boldsymbol{z}]$ is called *real-stable*. Thus for $B_k(G; y)$ to have only real roots is equivalent to $B_k(G; y)$ being real-stable. We denote the set of stable polynomials in $\boldsymbol{z}$ by $\mathfrak{S}[\boldsymbol{z}]$, and the set of real-stable polynomials in $\boldsymbol{z}$ by $\mathfrak{S}_{\mathbb{R}}[\boldsymbol{z}]$. It should be noted that the notion of stability for polynomials is not related to that of stability for sets of vertices. For a detailed survey of the theory of stable polynomials in multiple variables we direct the reader to [30].

The following theorem is a basic result used throughout the theory of stable polynomials:

Hurwitz's Theorem (Theorem 1.3.8 or [21])**.** *Let $S \subseteq \mathbb{C}^\eta$ be a non-empty connected open set, and let $(f_n)_{n=1}^\infty$ be a sequence of functions, each analytic and non-vanishing on*

S, which converges to a limit *f* uniformly on compact subsets of *S*. Then either $f(\boldsymbol{z}) \neq 0$ for all $z \in S$, or *f* is identically zero.

The set $\mathcal{H}^\eta$ is both connected and open. Hence, if a sequence of polynomials in $\mathfrak{S}[\boldsymbol{z}]$ converges uniformly to a limit *f* on compact subsets of *S*, then either *f* is identically zero, or $f(\boldsymbol{z}) \neq 0$ for all $\boldsymbol{z} \in \mathcal{H}^\eta$. In particular, if *f* is a polynomial, then *f* is stable.

Lemma 5.2.1 (Lemma 2.4 of [30]). *For any $f \in \mathfrak{S}[\boldsymbol{z}]$, the following operations preserve stability:*

(a) *permutation:* $f \mapsto f(z_{\tau(1)}, \ldots, z_{\tau(\eta)})$ *for any permutation τ of $1, \ldots, \eta$,*
(b) *scaling:* $f \mapsto cf(a_1 z_1, \ldots, a_\eta z_\eta)$ *where $c \in \mathbb{C}$ and $a_1, \ldots, a_\eta$ are positive real numbers,*
(c) *diagonalisation: for any $j \in \{2, \ldots, \eta\}$, $f \mapsto f(\boldsymbol{z})|_{z_1 = z_j}$,*
(d) *specialisation:* $f \mapsto f(a, z_2, \ldots, z_\eta)$ *for any $a \in (\mathcal{H} \cup \mathbb{R})$,*
(e) *inversion:* $f \mapsto z_1^d f(-z_1^{-1}, z_2, \ldots, z_\eta)$ *where d is the degree of z_1 in f,*
(f) *differentiation:* $f \mapsto D_{z_1} f$.

By permuting the variables parts (c) through (f) apply to all indeterminates.

Proof. It is clear that parts (a), (b), (c) hold. For part (d), the result is clear when $\Im(a) > 0$. When $\Im(a) = 0$, we consider the sequence $(a_m)_{m=1}^\infty$, where $a_m = a + i2^{-m}$. Since $\Im(a_m) > 0$, we know $f(a_m, z_2, \ldots, z_\eta)$ is stable for all $m \geq 1$. The f_m's converge uniformly to $f(a, z_2, \ldots, z_\eta)$ which must be stable by Hurwitz's Theorem. Part (e) follows from the fact that $-(a + ib)^{-1} = (-a + ib)/(a^2 + b^2)$.

To prove part (f) holds, we let *d* be the degree of *f* with respect to z_1 and consider the sequence $f_j(\boldsymbol{z}) = j^{-d} f(jz_1, z_2, \ldots, z_\eta)$ for $j \geq 1$. By part (b), all f_j are stable. The sequence converges to a polynomial, which is stable by Hurwitz's Theorem. Our choice of *d* ensures that this limit is not identically zero. So for any choice of $w_2, \ldots, w_\eta \in \mathcal{H}$, $g(x) = f(x, w_2, \ldots, w_\eta)$ is a stable polynomial of degree *d*. Let $\xi_1, \ldots, \xi_\eta$ be the roots of $g(x)$ and write

$$g(x) = c \prod_{j=1}^{d} (x - \xi_j)$$

for some $c \in \mathbb{C}$. Since *g* is stable, $\xi_j \notin \mathcal{H}$ for every $j \in [d]$. Then we have

$$\frac{D_x g(x)}{g(x)} = D_x \log(g(x)) = \sum_{j=1}^{d} (x - \xi_j)^{-1}.$$

Note that if $\Im(x) > 0$, then $\Im((x - \xi_j)^{-1}) < 0$ for all $j \in [m]$ and therefore $D_x g(x) \neq 0$. Thus, if $\boldsymbol{z} \in \mathcal{H}^\eta$, then $D_{z_1} f(\boldsymbol{z}) \neq 0$. We conclude that this polynomial is stable. □

We denote by $\overline{\mathfrak{S}[\boldsymbol{z}]}$ (resp. $\overline{\mathfrak{S}_{\mathbb{R}}[\boldsymbol{z}]}$) the set of all power series in $\mathbb{C}[[\boldsymbol{z}]]$ that arise as the limit of a sequence of polynomials in $\mathfrak{S}[\boldsymbol{z}]$ (resp. $\mathfrak{S}_{\mathbb{R}}[\boldsymbol{z}]$) which converge uniformly on compact sets.

For example, it is well-known that

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n \to \infty} (1 + zn^{-1})^n. \tag{5.1}$$

It is easy to show that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges uniformly on any compact set. Furthermore, the only root of $(1 + zn^{-1})$ is $z = -n$. Therefore $(1 + zn^{-1})^n \in \mathfrak{S}[z]$ and we conclude that $\exp(z) \in \overline{\mathfrak{S}[z]}$. We now give a few results which allow us to produce new objects in $\overline{\mathfrak{S}[z]}$ from existing ones.

Lemma 5.2.2. *If $f(z) \in \mathbb{C}[z]$, then $f(-wz) \in \mathfrak{S}[w, z]$ if and only if $f(z)$ has only real non-positive roots.*

Proof. Observe that $\mathcal{H}$ is exactly the set of non-zero $z \in \mathbb{C}$ whose argument lies in the open interval $(0, \pi)$. Hence

$$\{xy : x, y \in \mathcal{H}\} = \mathbb{C} \setminus \{r \in \mathbb{R} : r \geq 0\}$$

and the result follows. $\square$

Lemma 5.2.3. *Suppose $A(z) \in \overline{\mathfrak{S}_{\mathbb{R}}[z]}$. Then $A(-z) \in \overline{\mathfrak{S}_{\mathbb{R}}[z]}$. Moreover, if $A(z)$ arises a limit of real-stable polynomials whose roots are all non-positive, then $A(-xy) \in \overline{\mathfrak{S}[x, y]}$.*

Proof. Let $(f_j)_{j=1}^{\infty}$ be a sequence of polynomials in $\mathfrak{S}_{\mathbb{R}}[z]$ converging uniformly to $A(z)$ on compact sets. Since $f_j(z)$ is a real polynomial, $f_j(-z) \in \mathfrak{S}_{\mathbb{R}}[z]$ as well. It follows that $A(-z)$, which arises as the limit of $f_j(-z)$ as $j \to \infty$, is in $\overline{\mathfrak{S}_{\mathbb{R}}[z]}$. Next, assume that each f_j has only non-positive roots. Notice that if a set K is compact, then the set $\{-xy : x, y \in K\}$ is compact as well. Thus it follows from Lemma 5.2.2 that $f_j(-xy) \in \mathfrak{S}[x, y]$ for every $j \geq 1$, and hence $A(-xy) \in \overline{\mathfrak{S}[x, y]}$. $\square$

Since the only root of $(1 + zn^{-1})^n$ is $z = -n$, it is immediate from Lemma 5.2.3 that $\exp(-z) \in \overline{\mathfrak{S}_{\mathbb{R}}[z]}$ and $\exp(-wz) \in \overline{\mathfrak{S}[w, z]}$.

Lemma 5.2.4. *Suppose $A(\boldsymbol{z}) \in \overline{\mathfrak{S}[\boldsymbol{z}]}$ and $p(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}]$. Then $p(\boldsymbol{z})A(\boldsymbol{z}) \in \overline{\mathfrak{S}[\boldsymbol{z}]}$ if and only if $p(\boldsymbol{z}) \in \mathfrak{S}[\boldsymbol{z}]$.*

Proof. By definition, there exists a sequence of polynomials $(f_i)_{i=1}^\infty$, each in $\mathfrak{S}[\boldsymbol{z}]$, which converge uniformly on compact sets to $A(\boldsymbol{z})$. If $p(\boldsymbol{z}) \in \mathfrak{S}[\boldsymbol{z}]$, then $(pf_i)_{i=1}^\infty$ is a sequence of stable polynomials which converge uniformly on compact sets to $p(\boldsymbol{z})A(\boldsymbol{z})$ (note that $|p(\boldsymbol{z})|$ is bounded on any given compact set). Hence $p(\boldsymbol{z})A(\boldsymbol{z}) \in \overline{\mathfrak{S}[\boldsymbol{z}]}$ when $p(\boldsymbol{z}) \in \mathfrak{S}[\boldsymbol{z}]$.

Conversely, suppose $p(\boldsymbol{z})A(\boldsymbol{z})$ arises as the limit of a sequence of polynomials $(g_j)_{j=1}^\infty$, each in $\mathfrak{S}\boldsymbol{z}$. By Hurwitz's theorem $p(\boldsymbol{z})A(\boldsymbol{z})$ cannot vanish on $\mathcal{H}^\eta$. In particular, $p(\boldsymbol{z})$ cannot vanish on $\mathcal{H}^\eta$, thus it is stable. $\qquad\square$

In particular, if $p(z) \in \mathbb{C}[z]$, then $p(-wz)\exp(-wz) \in \overline{\mathfrak{S}[w,z]}$ if and only if $p(z)$ has only real non-positive roots.

5.3 Coefficients and Roots of $B_k(G; y)$

Some basic results regarding the roots of $B_k(G; y)$ can be obtained by understanding its coefficients, $\binom{k}{\ell}P(G; k, \ell)$. The binomial coefficient $\binom{k}{\ell}$ can be thought of as the number of ways of choosing ℓ proper colours out of a set of k colours. Hence we may think of the coefficient of y^ℓ in $B_k(G; y)$ as counting all (k, Y)-colourings of G over all sets $Y \subseteq [k]$ of size ℓ. Note that this counts a colouring more than once if it is a valid (k, Y)-colouring for multiple Y.

Unfortunately, this means that if G has no k-colouring, then the coefficient of y^k will be zero. However, a graph must always have a $(k, k-1)$-colouring as we can simply colour all the vertices with the single improper colour. Therefore, the degree of $B_k(G; y)$ is k when G has a k-colouring and $k-1$ otherwise. To remedy this, we can consider replacing $B_k(G; y)$ with the polynomial

$$\hat{B}_k(G; y) = y^k B_k(G; y^{-1}) = \sum_{j=0}^{k} \binom{k}{j} P(G; k, k-j)y^j$$

to obtain a polynomial of degree k. It follows from Lemma 5.2.1 that the stability of $\hat{B}_k(G; y)$ is equivalent to the stability of $B_k(G; y)$. Most of the properties of $B_k(G; y)$ also hold for $\hat{B}_k(G; y)$. Furthermore, since $\binom{k}{\ell} = \binom{k}{k-\ell}$, the coefficients have a combinatorial interpretation similar to the one described for $B_k(G; y)$. For our purposes the degree of $B_k(G; y)$ is not important so we will stick to $B_k(G; y)$.

Since all the coefficients of $B_k(G; y)$ are non-negative, Descartes' Rule of Signs guarantees that all the real roots of $B_k(G; y)$ are non-positive. For a proof of Descartes' Rule of Signs we refer the reader to [31].

Descartes' Rule of Signs. *Let $\phi(x) = \sum_{j=0}^{n} \alpha_j x^j$ be a polynomial with real coefficients. The number of sign changes (ignoring zeroes) in the sequence $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n$ is an upper bound on the number of positive real roots of ϕ (counted with their multiplicity).*

It isn't immediately clear that $B_k(G; y)$ has any real roots. However, a glance at some examples quickly reveals that -1 is often, if not always, a root with high multiplicity. The following result confirms this and gives the multiplicity of -1 as a root of $B_k(G; y)$.

Lemma 5.3.1. *For any $W \subseteq V$, define $\rho(W)$ to be 1 if W is an independent set and 0 otherwise. Then, for any graph G and any non-negative integer k, we have*

$$B_k(G; y) = \sum_c \prod_{i=1}^{k} (1+y)^{\rho(c^{-1}(i))} \tag{5.2}$$

where the sum is over all k-colourings c.

Proof. We can think of the coefficient of y^ℓ in $B_k(G; y)$, as counting the number of pairs (L, c) where $L \subseteq [k]$, $|L| = \ell$, and c is a (k, L)-colouring of G.

A colouring c with exactly m independent colour-classes can contribute to the coefficient of y^ℓ in exactly $\binom{m}{\ell}$ ways, once for each choice of ℓ proper colours. Note that when $m < \ell$, the colouring c cannot contribute to the coefficient of y^ℓ and $\binom{m}{\ell} = 0$ accordingly. We conclude that the contribution of the colouring c to $B_k(G; y)$ is

$$\sum_{\ell=0}^{m} \binom{m}{\ell} y^\ell = (1+y)^m.$$

This can be rewritten as

$$\prod_{i=1}^{k} (1+y)^{\rho(c^{-1}(i))}.$$

Summing over all possible k-colourings completes the proof. $\square$

Corollary 5.3.2. *Let G be any graph and k be any positive integer. The following are true:*

- *-1 is a root of $B_k(G; y)$ with multiplicity $k - \nu(G)$; and*
- *if θ is a real root of $B_k(G; y)$, then $\theta \leq -1$.*

Proof. Every colouring has at most $\nu(G)$ dependent colour-classes since any dependent set contains two adjacent vertices. This means that each k-colouring contributes a factor $(1+y)$ at least $k - \nu(G)$ times. Moreover, there exists at least one k-colouring with exactly $k - \nu(G)$ independent colour-classes, hence -1 must be a root of $B_k(G; y)$ with exactly this multiplicity.

A closer look at the proof of Lemma 5.3.1 reveals that the coefficient of y^ℓ in

$$B_k(G; y - 1) = \sum_c \prod_{i=1}^{k} y^{\rho(c^{-1}(i))}$$

is the number of k-colourings of G with exactly ℓ independent colour-classes. It follows from Descartes' Rule of Signs that the real roots of $B_k(G; y - 1)$ are non-positive as well, thus the real roots of $B_k(G; y)$ are at most -1. We summarize these results in the following corollary. $\qquad\square$

In particular, it follows from Corollary 5.3.2 that if $\nu(G) \leq 1$, then $B_k(G; y)$ has at most one root that is not -1. Since the non-real roots of $B_k(G; y)$ must come in pairs, it follows that, for such a graph, $B_k(G; y)$ is always stable. The graphs with $\nu(G) = 1$ are exactly $K_{1,m}$ for some m, hence we have a second proof that $B_k(K_{1,m}; y)$ is always stable.

Results similar to those of Corollary 5.3.2 also exist for $\hat{B}_k(G; y)$. As we mentioned earlier, the inverses of the non-zero roots of $\hat{B}_k(G; y)$ are the roots of $B_k(G; y)$. Thus we have the following results:

- 0 is a root of $\hat{B}_k(G; y)$ with multiplicity 1 if and only if G has no proper k-colouring;
- -1 is a root of $\hat{B}_k(G; y)$ with multiplicity $k - \nu(G)$; and
- if θ is a real root of $\hat{B}_k(G; y)$, then $0 \leq \theta \leq 1$.

5.4 $\quad B_k(G; y)$ and Connected Components of G

Recall that to show $(P(G; k, \ell))_{\ell=0}^{k}$ was log-concave, it sufficed to show it for the connected components of G. In general, most nice invariants of a graph can be determined from the connected components of the graph. Ideally, the stability of B_k for every connected component of G would be enough to determine if $B_k(G; y)$ is stable. However, it is far from obvious that this will work for $B_k(G; y)$. We will make use of powerful results from the theory of stable polynomials to show that this is indeed the case.

For $\ell \in \{0, \dots, \eta\}$, the ℓ-th *elementary symmetric function* in $\boldsymbol{z}$ is defined to be

$$e_\ell(\boldsymbol{z}) = \sum_{1 \le i_1 < \cdots < i_\ell \le \eta} z_{i_1} \cdots z_{i_\ell}.$$

The function e_ℓ is called *symmetric* since it is invariant under permutation of $z_1, \dots, z_\eta$. We observe that e_ℓ has $\binom{\eta}{\ell}$ terms, each of total degree ℓ. It follows that

$$e_\ell(z, \dots, z) = \binom{\eta}{\ell} z^\ell.$$

If $f(z) \in \mathbb{C}[z]$ is a polynomial in a single variable, let $\mathrm{Pol}_\eta f(\boldsymbol{z})$ denote the polynomial in $\mathbb{C}[\boldsymbol{z}]$ obtained by replacing occurrences of z^ℓ in f with $\binom{\eta}{\ell}^{-1} e_\ell(\boldsymbol{z})$. Thus, if $f(z) = \sum_{\ell=0}^m \alpha_\ell z^\ell$, then

$$\mathrm{Pol}_\eta f(\boldsymbol{z}) = \sum_{\ell=0}^m \alpha^\ell \binom{\eta}{\ell}^{-1} e_\ell(\boldsymbol{z}).$$

This is called the *η-th polarisation of f*. It follows from our remarks above that $\mathrm{Pol}_\eta f$ is symmetric and $\mathrm{Pol}_\eta f(z, \dots, z) = f(z)$.

A *circular region* of $\mathbb{C}$ is a non-empty subset A of $\mathbb{C}$ that is open or closed, and is bounded by a circle or a straight line. The following theorem due to Grace, Walsh, and Szegö tells us that in a circular region the polarisation of f can only attain values attainable by f itself.

Theorem 5.4.1 (Theorem 3.4.1b of [21]). *Suppose $f \in \mathbb{C}[z]$ has degree at most η and $A \subseteq \mathbb{C}$ is a circular region. If either f has degree exactly η, or A is convex, then for every $\boldsymbol{w} \in A^\eta$ there exists $w \in A$ such that $\mathrm{Pol}_\eta f(\boldsymbol{w}) = f(w)$.*

Corollary 5.4.2. *If $f \in \mathbb{C}[z]$ has degree at most d, then $\mathrm{Pol}_d f$ is stable if and only if f is stable.*

Proof. Recall that $\mathrm{Pol}_d f(z, \dots, z) = f(z)$, thus it follows from Lemma 5.2.1 that $\mathrm{Pol}_d f$ cannot be stable when f is not stable. For the converse implication, observe that $\mathcal{H}$ is a convex circular region. Therefore, if there exists $\boldsymbol{w} \in \mathcal{H}^d$ such that $\mathrm{Pol}_d f(\boldsymbol{w}) = 0$, then Theorem 5.4.1 guarantees the existence of $w \in \mathcal{H}$ such that $f(w) = 0$. In other words, if f cannot be stable when $\mathrm{Pol}_\eta f$ is not stable. $\qquad\square$

For univariate polynomials $f(z) = \sum_{\ell=0}^k \binom{k}{\ell} \alpha_\ell z^\ell$ and $g(z) = \sum_{\ell=0}^k \binom{k}{\ell} \beta_\ell z^\ell$, we define the *Schur-Szegö composition* of f and g as follows:

$$f * g = \sum_{\ell=0}^k \binom{k}{\ell} \alpha_\ell \beta_\ell z^\ell.$$

Notice that it follows from Theorem 3.3.3 that if G has connected components $G_1, \ldots, G_m$, then

$$B_k(G; y) = B_k(G_1; y) * B_k(G_2; y) * \cdots * B_k(G_m; y).$$

We are now ready to show that stability of every $B_k(G_i; y)$ guarantees the stability of $B_k(G; y)$. The following is a special case of a theorem of Schur and Szegö. The proof given is adapted from that of part (a) of Proposition 2.4 in [28].

Theorem 5.4.3. *If $f, g \in \mathbb{C}[z]$ are such that f is stable and g has only real non-positive roots, then $f * g$ is stable.*

Proof. If $f * g$ is identically zero there is nothing to show. Assume otherwise and let η be a positive integer such that $\eta = \max\{\deg(f), \deg(g)\}$. For $j \in \{0, \ldots, \eta\}$ let $\beta_j \in \mathbb{C}$ be such that

$$g(z) = \sum_{j=0}^{\eta} \binom{\eta}{j} \beta_j z^j.$$

We begin by showing the result holds when 0 is not a root of g. In this case, there exist $c \in \mathbb{C}$ and real positive $\theta_1, \ldots, \theta_\eta$ such that

$$g(z) = c \prod_{i=1}^{\eta} (1 + \theta_i z). \tag{5.3}$$

The coefficient of z^j in (5.3) must be $ce_j(\theta_1, \ldots, \theta_\eta)$ since e_j gives the sum over all j-subsets of the indices. Thus, it must be that

$$c \binom{\eta}{j}^{-1} e_j(\theta_1, \ldots, \theta_\eta) = \beta_j$$

for all $j \in \{0, \ldots, \eta\}$. It is now clear that

$$f * g = c \cdot \mathrm{Pol}_\eta f(\theta_1 z, \ldots, \theta_\eta z).$$

It follows from Corollary 5.4.2 that $\mathrm{Pol}_\eta f$ is stable. By Lemma 5.2.1, so is $f * g$.

Now we handle the case where 0 is a root of g with multiplicity $d > 0$. Using θ_i's as above, we write

$$g(z) = cz^d \prod_{i=1}^{\eta-d} (1 + \theta_i z).$$

For each positive integer m, define

$$g_m(z) = cm^{-d}(1 + mz)^d \prod_{i=1}^{\eta-d}(1 + \theta_i z).$$

The sequence $f * g_m$ converges uniformly to $f * g$. We know $f * g_m$ is stable for each m and so Hurwitz's Theorem implies that the limit, $f * g$, is stable as well. $\qquad\square$

Since the real roots of $B_k(G; y)$ are at most -1, it follows immediately that $B_k(G; y)$ is stable when B_k is stable for all connected components of G. This is also true of $\hat{B}_k(G; y)$ since its roots are non-positive as well.

5.5   A Deletion-Contraction Recurrence for $B_k(G; y)$

Since $B_k(G; y)$ is so closely related to $P(G; x, y)$, the deletion-contraction recurrence for $P(G; x, y)$ induces a similar recurrence for $B_k(G; y)$. It is straightforward to show that applying $y\mathrm{D}_y$ to a polynomial multiplies the coefficient of y^j by j. Thus the following identity follows from (3.4):

$$B_k(G; y) = B_k(G \setminus e; y) - B_k(G / e; y) + (k - y\mathrm{D}_y)B_k(G - e; y). \qquad (5.4)$$

We can expand the term $(k - y\mathrm{D}_y)B_k(G; y)$ as

$$\sum_{\ell=0}^{k}(k - \ell)\binom{k}{\ell}P(G; k, \ell)y^\ell.$$

We note that the coefficient of y^k must be zero, but that the coefficient of y^{k-1} is non-zero. This means that, unlike $B_k(G; y)$ which can be of degree k or $k - 1$, $(k - y\mathrm{D}_y)B_k(G; y)$ is always polynomial of degree $k - 1$. Furthermore, we can think of $(k - \ell)$ as being the number of ways of assigning a colour to a vertex with a loop. Thus if $\mathcal{L}$ is a graph consisting of a vertex not in G and a loop, then

$$(k - y\mathrm{D}_y)B_k(G; y) = B_k(G \cup \mathcal{L}; y).$$

In general, it can be quite difficult to determine if the sum of two polynomials is stable. In the case of real-stable polynomials, we can guarantee the sum is stable when the roots of the polynomials satisfy an interlacing condition. Let f and g be real-stable polynomials,

let $\theta_1 \le \cdots \le \theta_\ell$ be the roots of f, and let $\xi_1 \le \cdots \le \xi_m$ be the roots of g. We say the roots of f and g are *interlaced* if

$$\theta_1 \le \xi_1 \le \theta_2 \le \xi_2 \le \cdots .$$

Theorem 5.5.1 (Hermite, Kakeya, Obreschkoff, Theorem 6.3.8 of [21])**.** *If $f, g \in \mathbb{R}[z]$, then the following are equivalent:*

- $af + bg$ *is stable for all $a, b \in \mathbb{R}$,*
- *f and g are stable and their roots are interlaced.*

Proposition 5.5.2. *Let G be a graph and k be a positive integer. If $B_k(G; y)$ is stable, then so is $(k - y\mathrm{D}_y)B_k(G; y)$.*

Proof. Rolle's Theorem guarantees that $\mathrm{D}_y B_k(G; y)$ has a root in the open interval between any two distinct roots of $B_k(G; y)$. Furthermore, if θ is a root of $B_k(G; y)$ with multiplicity r, then it is a root of $\mathrm{D}_y B_k(G; y)$ with multiplicity $r - 1$ (write $B_k(G; y)$ as a product over its roots and use the product rule to differentiate). Thus, if $\theta_1 \le \cdots \le \theta_d$ are the roots of $B_k(G; y)$ and $\xi_1 \le \cdots \le \xi_{d-1}$ are the roots of $\mathrm{D}_y B_k(G; y)$, then we have

$$\theta_1 \le \xi_1 \le \theta_2 \le \cdots \le \xi_{d-1} \le \theta_d.$$

Multiplying $\mathrm{D}_y B_k(G; y)$ by y adds a root at 0. Corollary 5.3.2 showed that $\theta_d = -1$, thus the roots of $B_k(G; y)$ and $y\mathrm{D}_y B_k(G; y)$ are interlaced. It follows from Theorem 5.5.1 that $(k - y\mathrm{D}_y)B_k(G; y)$ is stable. $\qquad\square$

Unfortunately, we cannot hope for all the terms of (5.4) to have their roots interlaced with one another. As an example, we consider the cycle and path graphs on four vertices and the edges e_1, e_2 as shown below.
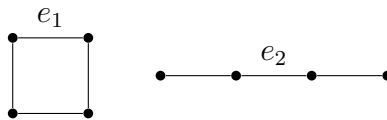

Table 5.5.3 lists some of the B_k polynomials for these graphs where $k = 4$. We see that the roots of $B_4(C_4; y)$ and $B_4(C_4 / e_1; y)$ are not interlaced. Interlacing of roots for all the other possible pairs are ruled out by P_4 and its subgraphs, with the exception of $B_k(G / e; y)$ and $(k - y\mathrm{D}_y)B_k(G - e; y)$. There are many graphs with more than four vertices which exhibit similar non-interlacing properties. However, for all these graphs the roots are interlaced in the exceptional case of $B_k(G / e; y)$ and $(k - y\mathrm{D}_y)B_k(G - e; y)$. We give more details of this in Section 6.4.

polynomial	approximate roots
$B_4(C_4; y)$	-2.58, -1.18, -1, -1
$B_4(C_4 \, / \, e_1; y)$	-2.67, -1, -1, -1
$B_4(P_4; y)$	-2.16, -1.10, -1, -1
$B_4(P_4 \setminus e_2; y)$	-1.54, -1.15, -1, -1
$B_4(P_4 \, / \, e_2; y)$	-1.78, -1, -1, -1
$(4 - y\mathrm{D}_y)B_4(P_4 - e_2; y)$	-1, -1, -1

Table 5.5.3: Roots of $B_4(G; y)$ for graphs related to C_4 and P_4

5.6 The Operator $P(G; k, y\mathrm{D}_y)$

It is well-known that

$$(1 + y)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} y^\ell.$$

With this fact in mind, we can think of $B_k(G; y)$ as being the result of applying a linear transformation T_k to $(1 + y)^\ell$ where T_k replaces y^ℓ with $P(G; k, \ell)y^\ell$ for all $\ell \geq 0$. Since $(1 + y)^k$ is stable, we wish for T_k to preserve the stability of $(1 + y)^k$. Our first insight into T_k is the following lemma:

Lemma 5.6.1. *Let $f(x)$ be a polynomial and define T to be the linear transformation mapping x^m to $f(m)x^m$ for every non-negative integer m. Then $T = f(x\mathrm{D}_x)$ where multiplication is replaced by composition.*

Proof. It suffices to show $T(x^m) = f(x\mathrm{D}_x)(x^m)$ for all non-negative integers m. We write $f(x)$ as $\sum_{i=0}^{m} \alpha_i x^i$. For all non-negative integers j, we have

$$T(x^j) = f(j)x^j = \sum_{i=0}^{m} \alpha_i j^i x^j = \sum_{i=0}^{m} \alpha_i (x\mathrm{D}_x)^i x^j = f(x\mathrm{D}_x)x^j,$$

thus $T = f(x\mathrm{D}_x)$. $\qquad\square$

It is immediate from Lemma 5.6.1 that

$$B_k(G; y) = \sum_{j=0}^{k} \binom{k}{j} P(G; k, j)y^j = P(G; k, y\mathrm{D}_y)(1 + y)^k \tag{5.5}$$

In this section we will study the operator $P(G; k, y\mathrm{D}_y)$ and its surprising connections to $B_k(G; y)$. We will make use of Theorems 5.6.2 and 5.6.4, two theorems due to Julius Borcea and Petter Brändén [4] which characterise stability-preserving operators. Proofs of these theorems are quite long and can be found in [4] as well as in [30].

This first theorem characterises operators which preserve stability of polynomials whose degree is bounded. For a positive integer d, denote by $\mathbb{C}[z]^{\leq d}$ the set of polynomials in $\mathbb{C}[z]$ of degree at most d.

Theorem 5.6.2 (Univariate case of Theorem 1.1 of [4]). *Suppose $T : \mathbb{C}[z]^{\leq d} \to \mathbb{C}[z]$ is a linear transformation. Then T maps $\mathfrak{S}[z]^{\leq d}$ into $\mathfrak{S}[z]$ if and only if either of the following holds:*

- *$T(f) = \eta(f) \cdot g$ for some linear functional $\eta : \mathbb{C}[z]^{\leq d} \to \mathbb{C}$ and $g \in \mathfrak{S}[z]$, or*
- *the polynomial*

$$T[(w + z)^d] = \sum_{j=0}^{d} \binom{d}{j} T(z^j) w^{d-j}$$

is in $\mathfrak{S}[w, z]$.

A useful fact is that $T(w + z)^d \in \mathfrak{S}[w, z]$ if and only if $T(1 - wz)^d \in \mathfrak{S}[w, z]$. One can easily verify this using Lemma 5.2.1 and the following equality:

$$T[(1 - zw)^d] = \sum_{j=0}^{d} \binom{d}{j} (-1)^j T(z^j) w^j = (-1)^d w^d T[(-w^{-1} + z)^d]. \qquad (5.6)$$

Theorem 5.6.3. *Let G be a graph and k be non-negative integer. Then $B_k(G; y) \in \mathfrak{S}[y]$ if and only if $P(G; k, y\mathrm{D}_y)$ maps $\mathfrak{S}[y]^{\leq k}$ into $\mathfrak{S}[y]$.*

Proof. For simplicity we write $T_G^{(k)}$ to mean $P(G; k, y\mathrm{D}_y)$. It follows from Theorem 5.6.2 and (5.6) that $T_G^{(k)}$ maps $\mathfrak{S}[z]^{\leq k}$ into $\mathfrak{S}[z]$ if and only if $T_G^{(k)}[(1 - yz)^k] \in \mathfrak{S}[y, z]$. Using (5.5) and the fact that $y\mathrm{D}_y(-yz) = -yz$, it is easy to see that

$$T_G^{(k)}[(1 - yz)^k] = B_k(G; -yz).$$

Lemma 5.2.2 tells us this is equivalent to $B_k(G; y)$ having only real, non-positive roots, which is equivalent to $B_k(G; y)$ being stable thanks to Corollary 5.3.2. $\qquad \square$

Hence the operator $P(G; k, y\mathrm{D}_y)$ has the peculiar property that it suffices to check its effect on $(1 + y)^k$ to determine if it maps $\mathfrak{S}[y]^{\leq k}$ in $\mathfrak{S}[y]$.

We now turn to the second characterisation theorem of Borcea and Brändén to determine if a strengthening of Theorem 5.6.3 exists.

Theorem 5.6.4 (Univariate case of Theorem 1.3 in [4]). *Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear transformation. Then T maps $\mathfrak{S}[z]$ into $\mathfrak{S}[z]$ if and only if either of the following holds:*

- $T(f) = \eta(f) \cdot p$ *for some linear functional $\eta : \mathbb{C}[z] \to \mathbb{C}$ and $p \in \mathfrak{S}[z]$, or*
- *the power series*

$$T(\exp(-zw)) = \sum_{j \geq 0} T(z^j) \frac{(-w)^j}{j!}$$

is in $\overline{\mathfrak{S}[z, w]}$.

Using induction on j and the product rule for differentiation, one can easily show that the following useful identity holds for all non-negative integers j.

$$(z\mathrm{D}_z)_j = z^j \mathrm{D}_z^j \tag{5.7}$$

Using (5.7) and Theorem 5.6.4 we prove the following two results.

Theorem 5.6.5. *Suppose $f(z) = \sum_{j=0}^m a_j z^j \in \mathbb{R}[z]$ be arbitrary and let $g(z)$ to be the polynomial $\sum_{j=0}^m a_j(z)_j$. Then $g(z\mathrm{D}_z)$ is an operator which maps $\mathfrak{S}[z]$ into itself if and only if $f(z)$ has only real non-positive roots.*

Proof. Using (5.7) we see that $g(z\mathrm{D}_z)$ maps

$$g(z\mathrm{D}_z) = \sum_{j=0}^m a_j z^j (\mathrm{D}_z)^j.$$

Hence we have

$$g(z\mathrm{D}_z)[(\exp(-zw)] = \sum_{j=0}^n a_j z^j (\mathrm{D}_z)^j \exp(-zw)$$

$$= \sum_{j=0}^n a_j (-zw)^j \exp(-zw)$$

$$= f(-zw)e^{-zw}.$$

It follows from Theorem 5.6.4 that $g(z\mathrm{D}_z)$ preserves stability if and only if $g(z\mathrm{D}_z)\exp(-zw)$ is in $\overline{\mathfrak{S}[w, z]}$. From Lemma 5.2.3 we conclude that $g(z\mathrm{D}_z)$ preserves stability if and only if $f(z)$ has only real, non-positive roots. $\square$

Lemma 5.6.6. *Suppose $f(z) \in \mathbb{R}[z]$ is not identically zero, then $f(wz)\exp(wz) \notin \overline{\mathfrak{S}[w,z]}$.*

Proof. Define g from f as in the statement of Theorem 5.6.5. Let T be the operator which maps z^j to $g(j)(-z)^j$. From Lemma 5.6.1 we conclude that T maps $(-z)^j$ to $f(z\mathrm{D}_z)z^j$. Using our work in the proof of Theorem 5.6.5 we have

$$T(\exp(-wz)) = g(z\mathrm{D}_z)\exp(wz) = f(wz)\exp(wz).$$

It is clear that the image of T is not one-dimensional (consider $T(z^a)$ and $T(z^b)$ where a, b are non-negative integers such that $g(a) \neq g(b)$). Thus it follows from Theorem 5.6.4 that $f(wz)\exp(wz) \in \overline{\mathfrak{S}[w,z]}$ if and only if T is a stability-preserving operator. We will show that T will never preserve stability.

Since f is not identically zero, neither is g. So there exists a non-negative integer j such that $g(j)$ and $g(j+1)$ are both non-zero and of the same sign. Now consider $cz^j + z^{j+1}$ for some $c \in \mathcal{H}$. Notice that this polynomial is stable as its only non-zero root is $-c$. On the other hand, we have

$$T(cz^j + z^{j+1}) = (-z)^j[cg(j) - g(j+1)z].$$

Since $g(j)$ and $g(j+1)$ have the same sign, $cg(j)/g(j+1) \in \mathcal{H}$ and hence $T(cz^j + z^{j+1})$ is not stable. We have shown T does not preserve stability, hence $f(wz)\exp(wz) \notin \overline{\mathfrak{S}[w,z]}$. $\square$

In particular, choosing $f(z) = 1$ above shows that $\exp(wz)$ does not arise as a limit of stable polynomials.

We recall Theorem 3.3.2 which showed

$$P(G; k, \ell) = \sum_{i=0}^{\nu(G)} (-1)^i \sum_{W \subseteq V} \mu_i(W)(k-i)^{n-|W|}(\ell)_i,$$

where $\mu_i(W)$ is the number of ways W can be partitioned into i dependent sets. We define

$$Q(G; k, y) = \sum_{i=0}^{\nu(G)} (-1)^i y^i \sum_{W \subseteq V} \mu_i(W)(k-i)^{n-|W|}.$$

For $i \in \{0, \ldots, \nu(G)\}$, there is always some $W \subseteq V$ for which $\mu_i(W)$ is non-zero (just take the set of matched vertices of a matching of cardinality i). Thus, the coefficients of $Q(G; k, y)$ are non-zero and alternate in sign. It follows from Descartes' Rule of Signs that

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any real root of $Q(G; k, y)$ is strictly positive. The strictness is due to the coefficient of y^0 being non-zero.

For a fixed k, it follows from Theorem 5.6.5 that $P(G; k, y\mathrm{D}_y)$ preserves stability if and only if $Q(G; k, y)$ has only real non-positive roots. When $Q(G; k, y)$ is constant, G is an edgeless graph and $P(G; k, y\mathrm{D}_y)$ trivially preserves stability. However, when the degree of $Q(G; k, y)$ is at least 1, we conclude that $P(G; k, y\mathrm{D}_y)$ cannot preserve stability. Hence, there must exist some b for which $P(G; k, y\mathrm{D}_y)$ does not map $\mathfrak{S}[y]^{\leq b}$ into $\mathfrak{S}[y]$. It is immediate from Theorem 5.6.3 that $B_k(G; y)$ is stable if and only if any such b must be greater than k. It is unclear how one could go about determining, for a given graph G, if a pair of k and b exist such that $P(G; k, y\mathrm{D}_y)$ does not map $\mathfrak{S}[y]^{\leq b}$ into $\mathfrak{S}[y]$.

Remark 5.6.7. Theorem 1.4 of [4] is a slightly different version of Theorem 5.6.4 for operators T which map $\mathbb{R}[z]$ into $\mathbb{R}[z]$. In particular, $P(G; k, y\mathrm{D}_y)$ maps $\mathfrak{S}_{\mathbb{R}}[z]$ into itself if and only if it maps at least one of $\exp(-xy)$ and $\exp(xy)$ into $\overline{\mathfrak{S}_{\mathbb{R}}[x, y]}$. Proceeding as in the proof of Theorem 5.6.5 we see that

$$P(G; k, y\mathrm{D}_y) \exp(xy) = Q(G; k, xy) \exp(xy).$$

We already showed that $Q(G; k, -xy) \exp(-xy) \notin \overline{\mathfrak{S}[x, y]}$. As for $Q(G; k, xy) \exp(xy)$, it follows from Lemma 5.6.6 that it is not in $\overline{\mathfrak{S}[x, y]}$. Since stable polynomials encompass the real-stable ones we conclude that $P(G; k, y\mathrm{D}_y)$ does not preserve real-stability either.

While the locations of the real roots of $Q(G; k, y)$ guarantee that $P(G; k, y\mathrm{D}_y)$ does not always preserve stability, stability of $Q(G; k, y)$ implies the stability of $B_k(G; y)$. To show this we require another powerful theorem of Borcea and Brändén:

Theorem 5.6.8 (Special case of Theorem 5.1 of [4])**.** *Suppose that $F(w, z) \in \mathbb{C}[w][[z]]$ and that $p_a(w) \in \mathbb{C}[w]$ for each non-negative integer a, such that $F(w, z) = \sum_{a \geq 0} z^a p_a(w)$. Then $F(w, z)$ is in $\overline{\mathfrak{S}[w, z]}$ if and only if*

$$\sum_{a=0}^{b} (b)_a z^a p_a(w)$$

is in $\mathfrak{S}[w, z]$ for every non-negative integer n.

Theorem 5.6.9. *Let G be a graph and k be a non-negative integer. If $Q(G; k, y) \in \mathfrak{S}[y]$, then so is $B_k(G; y)$.*

Proof. By Lemma 5.2.4, if $Q(G; k, y) \in \mathfrak{S}[y]$, then $Q(G; k, y) \exp(y) \in \overline{\mathfrak{S}[y]}$. This power series has the following form:

$$\sum_{a \geq 0} P(G; k, a) \frac{y^a}{a!}.$$

Stability is not violated by adding variables to the ambient space, hence this power series is also in $\overline{\mathfrak{S}[x, y]}$. By Theorem 5.6.8 this is equivalent to

$$\sum_{a=0}^{b} \binom{b}{a} P(G; k, a) y^a$$

being stable for all non-negative integers b. In particular, when $b = k$ it implies that

$$\sum_{a=0}^{k} \binom{k}{a} P(G; k, a) y^a = B_k(G; y)$$

is stable. $\qquad\qquad\square$

This result is important as it translates a problem about the $B_k(G; y)$'s, a family of univariate polynomials with varying degrees, into one about $Q(G; k, y)$, a single polynomial in two variables. Section 6.5 uses this to guarantee stability of $B_k(G; y)$ for some graphs.

5.7 A Power Series and an Interesting Result

Motivated by Theorem 5.6.8, we now attempt to show $B_k(G; y)$ is stable for all k, using a well-chosen power series. Unfortunately, these attempts will fail. Fortuitously, in doing so we happen upon an interesting, but only loosely related result.

The idea is to define a power series such as

$$C(G; x, y) = \sum_{k \geq 0} \frac{x^k}{k!} B_k(G; y).$$

If $F(x, y)$ in Theorem 5.6.8 is $C(G; x, y)$, then the appropriate $p_a(y)$ would be $B_a(G; y)/a!$. Assume that

$$\sum_{a=0}^{b} \frac{(b)_a x^a}{a!} B_a(G; y)$$

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is in $\mathfrak{S}[x, y]$ for some non-negative integer b. It follows from Lemma 5.2.1 that, for each $k \le b$, the following is stable

$$B_k(G; y) = \frac{1}{(b)_k} \left[(\mathrm{D}_x)^k \sum_{a=0}^{b} \frac{(b)_a x^a}{a!} B_a(G; y) \right]_{x=0}.$$

Thus, $C(G; x, y) \in \overline{\mathfrak{S}[x, y]}$ implies that $B_k(G; y) \in \mathfrak{S}[y]$ for every non-negative integer k. Unfortunately, this fails spectacularly. Using Lemma 5.6.1 for y and then for x, we can show

$$C(G; x, y) = P(G; x\mathrm{D}_x, y\mathrm{D}_y) \exp(x(y + 1)).$$

Since differentiating $\exp(x(y + 1))$ is equivalent to multiplying it by a polynomial, there must exist some polynomial $\varphi(G; x, y)$ such that

$$C(G; x, y) = P(G; x\mathrm{D}_x, y\mathrm{D}_y) \exp(x(y + 1)) = \varphi(G; x, y) \exp(x(y + 1)).$$

For example, for the edgeless graph on n vertices, $P(\overline{K_n}; k, \ell) = k^n$ so $\varphi(\overline{K_n}; x, y) = (x(y+1))^n$. Since $\mathcal{H}$ is invariant under the map $y \mapsto y+1$, it follows from Lemma 5.6.6 that $C(G; x, y)$ is not even in $\overline{\mathfrak{S}[x, y]}$ for the edgeless graphs. Instead, we consider $C(G; -x, y) = P(G; x\mathrm{D}_x, y\mathrm{D}_y) \exp(-x(y + 1))$. In this case

$$C(\overline{K_n}; -x, y) = (x\mathrm{D}_x)^n \exp(-x(y + 1)) = x^n (y + 1)^n \exp(-x(y + 1))$$

which we know to be in $\overline{\mathfrak{S}[x, y]}$ thanks to Lemma 5.2.4. While this may seem promising, these are likely to be the only graphs for which this is true. Indeed $P(K_2; k, j) = k^2 - j$, and so

$$C(K_2; -x, y) = (x^2 y^2 + 2x^2 y + x^2 - x) \exp(-x(y + 1)).$$

The polynomial $x^2 y^2 + 2x^2 y + x^2 - x$ is not stable; we can verify that it vanishes when $x = (-1 + \mathrm{i}\sqrt{3})/2$ and $y = (-3 + \mathrm{i}\sqrt{3})/2$. Thus $C(K_2; x, y)$ cannot arise as a limit of stable polynomials. In fact, we have checked the partial sums from Theorem 5.6.8 using a computer, and found that $C(G; -x, y) \notin \overline{\mathfrak{S}[x, y]}$ for all connected graphs with at most five vertices.

However, in our investigation of $C(G; -x, y)$ we did stumble across an interesting result. We recall (3.3) which stated

$$P(G; k, \ell) = \sum_{W \subseteq V} (k - \ell)^{n - |W|} P(G[W]; \ell).$$

With some manipulation we have

$$C(G; -x, y) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} P(G; k, \ell) y^\ell$$

$$= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} y^\ell \sum_{W \subseteq V} (k - \ell)^{n - |W|} P(G[W]; \ell)$$

$$= \sum_{W \subseteq V} \sum_{\ell=0}^{\infty} \frac{(-xy)^\ell}{\ell!} P(G[W]; \ell) \sum_{k \geq \ell} \frac{(-x)^{k-\ell}}{(k-\ell)!} (k - \ell)^{n - |W|}.$$

Using our usual tricks the rightmost sum is easily seen to be $(x D_x)^{n - |W|} \exp(-x)$. We define

$$R(G; z) = \sum_{k \geq 0} P(G; k) \frac{z^k}{k!}.$$

Then we have

$$C(G; -x, y) = \sum_{W \subseteq V} (-x D_x)^{n - |W|} \exp(-x) R(G[W]; -xy).$$

This expression is interesting as $(-x D_x)^{n - |W|} \exp(-x)$ is entirely independent of G, depending only on $n - |W|$, and $R(G; z)$ is the generating function for evaluations of the chromatic polynomial. We thought it good to ask: when is $R(G; z)$ or $R(G; -xy)$ the limit of a sequence of stable polynomials?

To answer this question, we use Lemma 5.6.1 to obtain

$$R(G; z) = \sum_{k \geq 0} P(G; k) \frac{z^k}{k!} = P(G; z D_z) \exp(z).$$

As usual, we are interested in the polynomial obtained from $P(G; k)$ by replacing $(z)_j$ with z^j. Luckily, this polynomial and its roots have already received much attention. Recall (2.3) in which we defined the chromatic polynomial to be

$$P(G; k) = \sum_{j=0}^{n} \lambda_j(G)(k)_j,$$

where $\lambda_j(G)$ was the number of partitions of V into exactly j, possibly empty, independent sets. First studied by Korfhage [15], the *sigma polynomial of G*, denoted $\sigma(G; z)$, is defined

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to be the polynomial

$$\sum_{j=0}^{n} \lambda_j(G)z^j.$$

We say that a graph is σ-real if $\sigma(G; z)$ has only real roots.

Theorem 5.7.1. *For all graphs G, $D(G; z) = \sigma(G; z)\exp(z)$. Furthermore, the following are equivalent:*

(i) $P(G; z\mathrm{D}_z)$ maps $\mathfrak{S}[z]$ into itself,
(ii) $D(G; z) \in \overline{\mathfrak{S}[z]}$,
(iii) G is σ-real.

Proof. It is easy to deduce from (5.7) that

$$D(G; z) = P(G; z\mathrm{D}_z)\exp(z) = \sigma(G; z)\exp(z) \tag{5.8}$$

Proceeding as in the proof of Theorem 5.6.5 we also see that

$$P(G; z\mathrm{D}_z)\exp(-wz) = \sigma(-wz)\exp(-wz) = D(G; -wz).$$

The coefficients of $\sigma(G; z)$ are non-negative integers, hence we deduce from Descartes' Rule of Signs that real roots of $\sigma(G; z)$ are non-positive. The equivalence of (i) and (ii) is immediate from Theorem 5.6.4 and Lemmas 5.2.2 and 5.2.3. The equivalence of (ii) and (iii) follows from (5.8) and Lemma 5.2.4. $\qquad\square$

The property of being σ-real is not particularly rare among graphs. There are many families of graphs that have been shown to be σ-real. The following is a list of some of the most important families of graphs known to be σ-real:

Theorem 5.7.2 (Theorem 2.2 of [6]). *Let G be a graph. If G satisfies any of the following conditions, then G is σ-real.*

- *G has 7 or fewer vertices (Section 3 of [5]),*
- *G has no $(n-4)$-colouring (Theorem 3.7 of [6]),*
- *G is a* chordal graph, *that is it has no induced cycle of length greater than 3 (Theorem 3.5 of [29]),*
- *G has no independent set of cardinality 3 (in this case $\sigma(G; x)$ is the matching polynomial of G which was famously shown to be real-stable by Heilmann and Lieb in Lemma 4.1 of [12]),*
- *G is an* incomparability graph, *that is there exists a partial order $\prec$ on V such that uv is an edge of $\overline{G}$ whenever u and v are incomparable. (Theorem 2.5 of [29])*

Chapter 6

Computational Results and Open Problems

This final chapter focuses on computational results and the further work they motivate. Specifically, we give computer evidence in support of Conjecture 2.3.3 as well as log-concavity of other sequences and give grounds for conjecturing that $B_k(G; y)$ is stable for all graphs G and all positive integers k.

6.1 Log-Concavity of $(P(G; k, \ell))_{\ell=0}^{k}$

Recall that Farr's Correlation Inequality implies that the first three terms of sequence $(P(G; k, \ell))_{\ell=0}^{k}$ satisfy the definition of log-concavity. Following our proof in Chapter 4 we mentioned that our attempts to extend this result to show log-concavity of further terms in the sequence were unsuccessful. Using a computer we have verified that this sequence is log-concave for all graphs G with $n(G) \leq 10$ and all $k \leq n(G) + 1$.

Since the bivariate chromatic polynomial is multiplicative over connected components, it sufficed to check only connected graphs. For each n up to 10, we generated a list of all connected graphs on n vertices, up to isomorphism, using the the program Nauty, created by Brendan McKay and Adolfo Piperno [18]. Throughout this chapter, these lists will be used when checking all connected graphs with a given number of vertices.

A program written in C checked log-concavity of the sequence for each graph G and k. To do so, this program created a list of all partitions of an n-element set. For each graph, it ran through all partitions and recorded, for all i and j, the number of partitions which

have exactly j independent parts out of a total of i parts. Since a colouring corresponds to a partition of the vertices and a choice of colour for each part, it was then easy to compute the sequence $(P(G;k,\ell))_{\ell=0}^k$ and check if it is log-concave. Numerical computations were performed using the GNU Multiple Precision Arithmetic Library to avoid problems with integer overflow. A few computations were compared against the output of a similar program written separately in Racket (a Scheme-derived language) to verify correctness. The code for the C program can be found in Appendix A.

As mentioned above, we found that $(P(G;k,\ell))_{\ell=0}^k$ is log-concave for connected graphs G with 10 or fewer vertices and all non-negative integers k such that $k \leq n(G) + 1$. We chose to not go beyond 10 vertices as the computations became too lengthy. Recall that log-concavity of this sequence implies log-concavity of $a^{(k)}(G; 1/k)$, hence these results lend credence to Conjecture 2.3.3 (Farr undoubtedly performed similar computations as well). Our findings motivate the following strengthening of Farr's conjecture:

Conjecture 6.1.1. *For any graph G and any non-negative integer k, the sequence*

$$P(G;k,0), P(G;k,1), \ldots, P(G;k,k)$$

is log-concave.

6.2   Terms in the Deletion-Contraction Recurrence

Recall (3.4), the generalised deletion-contraction recurrence for $P(G;k,\ell)$, which states that
$$P(G;k,\ell) = P(G \setminus e;k,\ell) - P(G \,/\, e;k,\ell) + (k-\ell)P(G-e;k,\ell).$$
In Section 6.1, we found that the sequence $(P(G;k,\ell))_{\ell=0}^k$ is log-concave for many G and k. This turns out to also be the case for the following sequences:

$$(P(G \setminus e;k,\ell) - P(G \,/\, e;k,\ell))_{\ell=0}^k,$$
$$(P(G \setminus e;k,\ell) + (k-\ell)P(G-e;k,\ell))_{\ell=0}^k,$$
$$(P(G \,/\, e;k,\ell) - (k-\ell)P(G-e;k,\ell))_{\ell=0}^k.$$

These three sequences correspond to the three possible pairs of terms on the right-hand side of (3.4). We verified their log-concavity using SageMath and the same lists of connected graphs generated by Nauty. The SageMath script used can be found in Appendix B.

For each graph G in the lists, and each edge e of G, we computed the bivariate-chromatic polynomial of the required graphs using (3.3). The computations of the chromatic polynomial, as well as all vertex deletions, edge deletions, and edge contractions were performed using SageMath's built-in graph library. For each $k \leq n(G) + 1$ we evaluated the polynomials at every value of ℓ, and tested log-concavity of all three sequences.

We checked all connected graphs G such that $n(G) \leq 8$, going no further due to time constraints. We found that for all such graphs, all edges e, and all $k \leq n(G) + 1$, the three sequences were log-concave. It is possible that these sequences and the generalised deletion-contraction recurrence can be exploited to prove Conjecture 6.1.1. However it unlikely that this can be achieved without a much better understanding of the behaviour of these sequences.

6.3   Stability of $B_k(G; y)$

The only families of graphs for which we know $B_k(G; y)$ is stable are the edgeless graphs and star graphs ($K_{1,m}$ for some positive integer m) have the property that $B_k(G; y)$ is stable. To better understand how common this property is, we checked the roots of $B_k(G; y)$ for many graphs. To do so, we made use of the following result:

Proposition 6.3.1. *Let $\boldsymbol{t}$ be a vector $(t_v : v \in V)$ of variables. For $W \subseteq V$, we will write $\boldsymbol{t}^W$ to mean the monomial $\prod_{v \in W} t_v$.*

Define the power series $S(G; y, \boldsymbol{t})$ as follows:

$$S(G; y, \boldsymbol{t}) = \exp\left(\sum_{v \in V} t_v\right) + y \sum_{W \subseteq \mathcal{I}(G)} \boldsymbol{t}^W.$$

Then for any non-negative integer k, we have

$$B_k(G; y) = [\boldsymbol{t}^V z^k](1 - zS(G; y, \boldsymbol{t}))^{-1}.$$

Proof. Recall Lemma 5.3.1 which states that

$$B_k(G; y) = \sum_c \prod_{j=1}^{k} (1 + y)^{\rho(c^{-1}(j))},$$

where the sum is over all k-colourings c. In this expression a k-colouring can be viewed as an ordered partition of V into k (possibly empty) parts, where independent parts contribute a factor of $(1 + y)$ and dependent parts contribute 1.

Observe that for each $W \subseteq V$ the coefficient of $\boldsymbol{t}^W$ in $\exp(\sum_{v \in V} t_v)$ must be 1. Furthermore, these are the only monomials in which every indeterminate occurs at most to the first power. Adding $y \sum_{W \subseteq \mathcal{I}(G)} \boldsymbol{t}^W$ ensures that the coefficient of t^W is $(1 + y)$ for independent W, while remaining 1 otherwise. We think of an ordered partition of V into k, possibly empty, parts as a choice of k monomials in $S(G; y, \boldsymbol{t})$ such that for each $v \in V$, t_v occurs exactly once among all the monomials. The product of the coefficients of all the monomials will be the contribution of this partition to $B_k(G; y)$. Thus

$$B_k(G; y) = [t^V](S(G; y, \boldsymbol{t}))^k.$$

It follows that

$$B_k(G; y) = [t^V z^k] \sum_{k \geq 0} z^k (S(G; y, \boldsymbol{t}))^k = [t^v z^k](1 - zS(G; y, \boldsymbol{t}))^{-1}. \qquad \square$$

We found that using Proposition 6.3.1 was the quickest way to compute $B_k(G; y)$ in Maple for the graphs we were interested in. For each connected graph G, a Maple script computed $\mathcal{I}(G)$, $(1 - zS(G; y, \boldsymbol{t}))^{-1}$, and then $B_k(G; y)$ for all desired k. We used Maple's `fsolve` function to find the roots numerically as complex numbers. Maple represents complex numbers as two floating-point numbers, hence the usual uncertainties involving floating-point numbers are present. For this reason, this is an imperfect approach. In the future it would be better to use Sturm's Theorem [25] and exact integer arithmetic. Issues regarding approximations aside, we checked all connected graphs G with $n(G) \leq 9$ and $k \leq n(G) + 1$ without `fsolve` returning a root θ with $|\Im(\theta)| > 10^{-7}$. We chose 10^{-7} arbitrarily, but also examined a small sample of the roots produced by Maple and found that any imaginary parts were much smaller than this bound. It is very surprising that none of the $B_k(G; y)$ examined had a root with significant imaginary part. For this reason, we find it plausible that the $B_k(G; y)$ we tested were all real-stable. The Maple code used can be found in Appendix C.

6.4 Interlacing of Roots

In Section 5.5 we gave recurrence (5.4) for $B_k(G; y)$ which stated that for all G and all non-negative integers k

$$B_k(G; y) = B_k(G \setminus e; y) - B_k(G \mathbin{/} e; y) + (k - yD_y)B_k(G - e; y).$$

Using Table 5.5.3 we showed that the polynomials appearing in this equation do not interlace each other's roots, with the exception of $B_k(G \mathbin{/} e; y)$ and $(k - yD_y)B_k(G - e; y)$.

We used Maple to compute the roots of both polynomials approximately and check if they interlace. The relevant Maple code can be found in Appendix C. We allowed for roots to have imaginary part less than 10^{-7} in magnitude and we considered two roots equal if they differed by less than this quantity. We also ignored the small imaginary parts when checking for interlacing. The edge contractions and vertex deletions were done using Maple's `GraphTheory` package. We performed these computations for all connected graphs G with $n(G) \leq 8$, all edges e of G, and all $k \leq n(G) + 1$. In all cases we found that the roots were interlaced. This is a bit mysterious and perhaps points to $(k-yD_y)B_k(G-e; y) - B_k(G/e; y)$ being a bit special.

6.5 Roots of $Q(G; k, y)$

Our last computational results are related to Theorem 5.6.9 which stated that if $Q(G; k, y)$ is stable, then so is $B_k(G; y)$. The upshot of this result is that the analysis of $Q(G; k, y)$, which is a single polynomial in two variables, is at times simpler than that of infinitely many different $B_k(G; y)$'s.

For a real polynomial of small degree, we can use its discriminant to determine stability. We denote by $\delta(G; k)$ the discriminant of $Q(G; k, y)$ with respect to y. We know the degree of $Q(G; k, y)$ in terms of y is $\nu(G)$, hence when G has fewer than 8 vertices, $Q(G; k, y)$ is, at worst, a cubic in y. We obtain $Q(G; k, y)$ from $P(G; k, \ell)$ using the following result adapted from Section 1.9 of [24]. Recall that $\left\{ {i \atop j} \right\}$ counts the number of partitions of an i-element set with exactly j non-empty parts.

Proposition 6.5.1. *Let $f(x) = \sum_{i=0}^{m} a_i(x)_i$ be an arbitrary polynomial and define $g(x) = \sum_{i=0}^{m} a_i x^i$. Further define*

$$s_i(x) = \sum_{j=0}^{i} \left\{ {i \atop j} \right\} x^j$$

for each $i \in \{0, \ldots, m\}$. If $b_0, \ldots, b_m$ are such that $f(x) = \sum_{i=0}^{m} b_i x^i$, then

$$g(x) = \sum_{i=0}^{m} b_i s_i(x).$$

Proof. We will begin by showing

$$x^i = \sum_{j=0}^{i} \left\{ {i \atop j} \right\} (x)_j \tag{6.1}$$

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for all non-negative integers i. Suppose x is a positive integer. The left-hand side counts the number of functions from $[i]$ to $[x]$. Every such function f is surjective onto a set Y whose cardinality is at most m. The pre-images $f^{-1}(y)$ for all $y \in Y$ form a partition of $[i]$ into $|Y|$ parts. For each $j \in \{0, \dots, m\}$, there are $\left\{{i \atop j}\right\}$ partitions of $[i]$ into j parts and $(x)_j$ possible ways of mapping the parts to a subset of $[x]$ with j elements. One can easily verify that this is a bijection. Therefore (6.1) holds for all positive integers x. It follows that the two sides must agree as polynomials.

We know $\{x^i : i \geq 0\}$ and $\{(x)_i : i \geq 0\}$ are bases for the space of polynomials in x. Thus, if we are replacing $(x)_i$ with x^i to obtain $g(x)$ from $f(x)$, then we are replacing x^i with $s_i(x)$. $\qquad \square$

For each connected graph we had Maple compute $P(G; k, \ell)$ using (3.3). We used Proposition 6.5.1 to compute $Q(G; k, y)$, and then computed its discriminant $\delta(G; k)$ as a polynomial in k. By considering only graphs with 7 or fewer vertices, we ensured the degree of $Q(G; k, y)$ in terms of y was at most 3. It is well-known that the discriminant of a polynomial $ay^3 + by^2 + cy + d$ is

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

We used `fsolve` to find the largest real root θ of $\delta(G; k)$. Again, we allowed for $|\Im(\theta)|$ to be at most 10^{-7} (in this case this is quite generous). In the case that the value of θ, as returned by `fsolve`, was slightly non-real, we threw away the imaginary part. For quadratics and cubics, the polynomial has only real roots when the discriminant is non-negative. Thus we had Maple evaluate $\delta(G; k)$ at $k = \theta + 1$ and check that it is positive. If we found the discriminant to be positive beyond its largest real root, then we checked stability of $B_k(G; y)$ for $k \leq \theta + 1$ in the way described in Section 6.3. The Maple code used can, once again, be found in Appendix C.
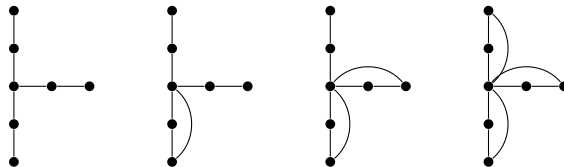

Figure 6.5.2: Four graphs for which $\delta(G; k)$ is negative as $k \to \infty$

These computations were performed on all connected graphs G with $n(G) \leq 7$. Only four graphs were found for which the $\delta(G; k)$ is not positive beyond its largest real root.

These graphs are shown in Figure 6.5.2. For each of these four graphs, $Q(G; k, y)$ has two non-real roots for all positive integers k. However, for each of these graphs we used the methods described in Section 6.3 to check the roots of $B_k(G; y)$ for all $k \leq 21$. Since the largest real root of $\delta(G; k)$ was less than 13 for all four graphs, we are confident that their discriminant being negative is not an indicator of non-stability for $B_k(G; y)$. We also performed these computations on some of the connected graphs with 8 vertices but found many graphs for which the determinant was negative beyond its largest real root.

Overall, we have good evidence that $B_k(G; y)$ is stable in many cases. We summarise the cases in which we have proven, or have strong computational evidence, that $B_k(G; y)$ is stable below:

- $\nu(G) \leq 1$ and k is arbitrary.
- G is a connected graph with $n(G) \leq 9$ and $k \leq n + 1$.
- G is a connected graph with $n(G) \leq 7$, G is not one of the four graphs in Figure 6.5.2, and k is arbitrary.
- G is a graph and k is a non-negative integer such that for each connected component H of G, H and k satisfy one of the conditions above.

These results gives us the confidence to make the following conjecture:

Conjecture 6.5.3. *For all graphs G and all non-negative integers k $B_k(G; y)$ is real-stable.*

References

[1] I. Averbouch, B. Godlin, and J.A. Makowsky, *An extension of the bivariate chromatic polynomial*, European Journal of Combinatorics **31** (2010), no. 1, 1–17.

[2] G.D. Birkhoff and D.C. Lewis, *Chromatic polynomials*, Trans. Amer. Math. Soc. **60** (1946), 355–451.

[3] George D. Birkhoff, *A determinant formula for the number of ways of coloring a map*, Annals of Mathematics **14** (1912), no. 1/4, 42–46.

[4] J. Borcea and P. Brändén, *The Lee-Yang and Pólya-Schur programs. I. linear operators preserving stability*, Inventiones mathematicae **177** (2009mar), no. 3, 541–569.

[5] F. Brenti, G.F. Royle, and D.G. Wagner, *Location of zeros of chromatic and related polynomials of graphs*, Canad. J. Math. (1994), 55–80.

[6] J. Brown and A. Erey, *On the roots of σ-polynomials*, Journal of Graph Theory **82** (2015), no. 1, 90–102.

[7] K. Dohmen, A. Poenitz, and P. Tittmann", *A new two-variable generalization of the chromatic polynomial*, Discrete Mathematics and Theoretical Computer Science, DMTCS **6** (2003), no. 1, 69–90.

[8] D. Dubhashi and D. Ranjan, *Balls and bins: A study in negative dependence*, Random Structures & Algorithms **13** (1998), no. 2, 99–124.

[9] G.E. Farr, *A correlation inequality involving stable set and chromatic polynomials*, Journal of Combinatorial Theory, Series B **58** (1993), no. 1, 14–21.

[10] _____, *Tutte-Whitney polynomials: Some history and generalizations* (2007), 28–52.

[11] I. Gutman and F. Harary, *Generalizations of the matching polynomial*, Utilitas Mathematica **24** (1983), 97–106.

[12] O.J. Heilmann and E.H. Lieb, *Theory of monomer-dimer systems*, Comm. Math. Phys. **25** (1972), no. 3, 190–232.

[13] F. Jaeger, D.L. Vertigan, and D.J.A. Welsh, *On the computational complexity of the Jones and Tutte polynomials*, Mathematical Proceedings of the Cambridge Philosophical Society **108** (1990), no. 1, 35–53.

[14] K. Joag-Dev and F. Proschan, *Negative association of random variables with applications*, The Annals of Statistics **11** (1983), no. 3, 286–295.

[15] R.R. Korfhage, *σ-polynomials and graph coloring*, Journal of Combinatorial Theory, Series B **24** (1978), no. 2, 137–153.

[16] V.E. Levit and E. Mandrescu, *The independence polynomial of a graph - a survey*, Proceedings of the 1st international conference on algebraic informatics, 2005, pp. 233–254.

[17] C. McDiarmid, *On a correlation inequality of Farr*, Combinatorics, Probability and Computing **1** (1992), 157–160.

[18] B. D. McKay and A. Piperno, *Practical graph isomorphism, II*, Journal of Symbolic Computation **60** (2014), no. 0, 94–112.

[19] R. Mo, *Certificates for properties of stability polynomials of graphs*, Master's Thesis, 2012.

[20] J.G. Oxley and D.J.A. Welsh, *The Tutte polynomial and percolation, in: J.A. Bondy, U.S.R. Murty (Eds.), Graph theory and related topics*, Academic Press, London, 1988.

[21] Q.I. Rahman and G. Schmeisser, *Analytic theory of polynomials* **26** (2002).

[22] R.C. Read and W.T. Tutte, *Chromatic polynomials, in: L.W. Beineke, R.J. Wilson (Eds.), Selected topics in graph theory*, Vol. 3, Academic Press, London, 1988.

[23] R.P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Annals of the New York Academy of Sciences **576** (1989), 500–535.

[24] _____, *Enumerative combinatorics: Volume 1*, 2nd ed., Cambridge University Press, New York, 2011.

[25] J.M. Thomas, *Sturm's theorem for multiple roots*, National Mathematics Magazine **15** (1941), no. 8, 391–394.

[26] W.T. Tutte, *A ring in graph theory*, Mathematical Proceedings of the Cambridge Philosophical Society **43** (1947), no. 1, 26–40.

[27] L. Valiant, *The complexity of enumeration and reliability problems*, SIAM Journal on Computing **8** (1979), no. 3, 410–421.

[28] David G. Wagner, *Weighted enumeration of spanning subgraphs with degree constraints*, Journal of Combinatorial Theory, Series B **99** (2009), no. 2, 347–357.

[29] D.G. Wagner, *The partition polynomial of a finite set system*, Journal of Combinatorial Theory, Series A **56** (1991), no. 1, 138–159.

[30] _____, *Multivariate stable polynomials: Theory and applications*, Bulletin of the Amer. Math. Soc. **48** (2011), 53–84.

[31] X. Wang, *A simple proof of Descartes' rule of signs*, The American Mathematical Monthly **111** (2004), no. 6, 525–526.

[32] H. Whitney, *The coloring of graphs*, Annals of Mathematics **33** (1932), no. 4, 688–718.

[33] _____, *A logical expansion in mathematics*, Bull. Amer. Math. Soc. **38** (1932), no. 8, 572–579.

APPENDICES

Appendix A

C Program Used to Check Log-Concavity of $(P(G; k, \ell))_{\ell=0}^{k}$

```c
#include <stdio.h>
#include <stdlib.h>
#include <unistd.h>
#include "gmp-6.1.0/gmp.h"

// Assumes all graphs read have same number of vertices
#define N // Set to number of vertices
#define ADJ_MAT_SIZE ((N * (N - 1)) / 2)
#define R_ROWS ((N / 2) + 1)
#define R_COLS (N + 1)
#define RESULTS_SIZE (R_ROWS * R_COLS)
#define MAX_K (N + 1) --max number of colours
typedef unsigned int uint;
struct _ILst { // List if integers
  int first;
  struct _ILst *rest;
};
struct _LLst { // List of lists
  ILst *first;
  struct _LLst *rest;
};
struct _LLLst { // List of lists of lists
```

```c
  LLst *first;
  struct _LLLst *rest;
};
struct _Graph { // Graph
  int nVerts;
  char *adjMat;
};


// Returns the number of elements in lst
int lengthLLst (LLst *lst) {
  int len = 0;
  while (lst != NULL) { len += 1; lst = lst->rest; }
  return len;
}


// Append elem to the front of lst
ILst *consILst (int elem, ILst *lst) {
  ILst *newCell = (ILst*)malloc(sizeof(ILst));
  newCell->first = elem; newCell->rest = lst;
  return newCell;
}
LLst *consLLst (ILst *elem, LLst *lst) {
  LLst *newCell = (LLst*)malloc(sizeof(LLst));
  newCell->first = elem; newCell->rest = lst;
  return newCell;
}
LLLst *consLLLst (LLst *elem, LLLst *lst) {
  LLLst *newCell = (LLLst*)malloc(sizeof(LLLst));
  newCell->first = elem; newCell->rest = lst;
  return newCell;
}


// Appends elem to the front of every element of lst
LLst *consAllILst (int elem, LLst *lst) {
    LLst *pos = lst;
    while (pos != NULL) {
        pos->first = consILst(elem, pos->first);
        pos = pos->rest;
```

```
    }
    return lst;
}
LLLst *consAllLLst (ILst *elem, LLLst *lst) {
    LLLst *pos = lst;
    while (pos != NULL) {
        pos->first = consLLst(elem, pos->first);
        pos = pos->rest;
    }
    return lst;
}

// Appends lst1 to the front of lst2
LLLst *appendLLLst (LLLst *lst1, LLLst *lst2) {
  if (lst1 == NULL) return lst2;
  if (lst2 == NULL) return lst1;
  LLLst *newLst = consLLLst(lst1->first, NULL);
  LLLst *pos = newLst;
  lst1 = lst1->rest;
  while (lst1 != NULL) {
    pos->rest = consLLLst(lst1->first, NULL);
    pos = pos->rest;
    lst1 = lst1->rest;
  }
  pos->rest = lst2;
  return newLst;
}

// Add elem tor prtn in all possible ways
LLLst *addToPrtn (int elem, LLst *prtn) {
  if (prtn == NULL) {
    return consLLLst(consLLst(consILst(elem, NULL), NULL), NULL);
  }
  return consLLLst(consLLst(consILst(elem, prtn->first), prtn->rest),
   consAllLLst(prtn->first, addToPrtn(elem, prtn->rest)));
}

// Add elem to each partition in prtns in all ways possible
```

```
LLLst *addToAllPrtns (int elem, LLLst *prtns) {
  LLLst *result = NULL;
  while (prtns != NULL) {
    result = appendLLLst(addToPrtn(elem, prtns->first), result);
    prtns = prtns->rest;
  }
  return result;
}

// Produces a list of all set partitions of the set {0,...,n-1}
LLLst *partitionsList (int n) {
  if (n == 0) {
    return consLLLst(NULL, NULL);
  } else {
    return addToAllPrtns(n-1, partitionsList(n-1));
} }

// Read a graph (in g6 format) from the file descriptor 'in'
// Assumes the graph being read has no more than 62 vertices
int ReadGraph (int in, struct Graph *g) {
  int c = 0;
  if (read(in, &c, sizeof(char)) == 0) return 0; // Reached EOF
  g->nVerts = c - 63;
  int pos = 0;
  while (read(in, &c, sizeof(char)) != 0) {
    if (c == '\n') break;
    c = c - 63;
    for (int i = 5; i >= 0 && pos < ADJ_MAT_SIZE; i -= 1) {
      g->adjMat[pos++] = (c >> i) & 1;
  } }
  return 1;
}

// Returns 1 if the set is stable in G, 0 otherwise
int IsStable(struct ILst *set, struct Graph *G) {
  while(set != NULL) {
    int u = set->first;
    set = set->rest;
```

```c
    int colStartIndx = (u * (u - 1)) / 2;
    for (ILst *pos = set; pos != NULL; pos = pos->rest) {
      if (G->adjMat[colStartIndx + pos->first]) {
        return 0;
  } } }
  return 1;
}


// Returns the number of sets in the given list which are stable
int numStableSets (LLst *sets, Graph *g) {
  int n = 0;
  while (sets != NULL) {
    n += isStable(sets->first, g);
    sets = sets->rest;
  }
  return n;
}


// Returns the number of partitions (provided as ps) of V(G)
// with exactly i stable parts as the i-th entry of 'results'
// Assumes 'results' is initialised to zero
void CountPartitions (struct LLLst *ps, struct Graph *G, int *results) {
  while (ps != NULL) {
    int stbl = numStableSets(ps->first, G);
    results[((N + 1) * (lengthLLst(ps->first) - stbl)) + stbl] += 1;
    ps = ps->rest;
  }
}


// Compute the falling factorial (n)_k, z is return value
void FallingFact (mpz_t z, uint n, uint k) {
  if (k > n) {
    mpz_set_ui(z, 0);
  } else {
    mpz_set_ui(z,1);
    for (uint i = n; i > n - k; i -= 1) {
      mpz_mul_ui (z, z, i);
} } }
```

```
// Compute the number of (k,i)-colourings for each i in [k]
// returned as i-th entry of b
// 'results' is assumed to be the vector produced by CountPartitions
void ComputeBs (int k, int *results, mpz_t *b) {
  mpz_t z1, z2; mpz_init(z1); mpz_init(z2);
  for (uint i = 0; i < MAX_COLOURS; i += 1) {
    mpz_set_ui(b[i], 0);
    if (i > k) continue;
    int resPos = 0;
    for (uint ns = 0; ns < min((uint) R_ROWS, k - i + 1); ns += 1) {
      for (uint s = 0; s < R_COLS; s += 1) {
        if (ns + s <= k) {
          fallingFact(z1, k, ns + s);
          mpz_addmul_ui(b[i], z1, (uint) results[resPos]);
        }
        resPos += 1;
} } } }

// Check if b's are log-concave (k indicates the length of b)
// Prints graph, k, and i for which sequence is not log-concave
void CheckLogConc(int line, int k, mpz_t *b) {
  mpz_t bb, ac;
  mpz_init(bb);
  mpz_init(ac);
  for (int x = 0, y = 1, z = 2; z < MAX_K; x += 1, y += 1, z += 1) {
    mpz_mul(ac, b[x], b[z]);
    mpz_mul(bb, b[y], b[y]);
    if (mpz_cmp(bb,ac) < 0) {
      printf("line = %d -- k = %d -- i = %d\n", line, k, y);
} } }

int main() {
  // Initialize graph structure G, vector of mpz_t b,
  // open file created by Nauty, create list of partitions
  int line = 0;
  while (readGraph(in, &G) != 0) {
    for (int i = 0; i < RESULTS_SIZE; i += 1) results[i] = 0;
```

```
      CountPartitions(partitions, &G, results);
      for (int k = 0; k < MAX_K; k += 1) {
        ComputeBs(k, results, b);
        CheckLogConc(line, k, b);
      }
      line += 1;
  } }
```

Appendix B

Sage Script Used to Check Log-Concavitiy of Related Sequences

```
def AddSeqs(s1,s2):
  return [s1[j] + s2[j] for j in range(len(s1))]

def SubSeqs(s1,s2):
  return [s1[j] - s2[j] for j in range(len(s1))]

def IsLC(seq):
  for j in range(1,len(seq)-1):
    if seq[j]^2 < seq[j-1]*seq[j+1]: return false
  return true

def TestGraph(g6str,kmax):
  G = Graph(g6str)
  P = G.bivariate_chrom_poly()
  for e in G.edges():
    Gdel = G.copy(); Gcon = G.copy(); Gext = G.copy()
    Gdel.delete_edge(e)
    Gcon.contract_edge(e)
    Gext.delete_vertices([e[0],e[1]])
    Pdel = Gdel.bivariate_chrom_poly()
    Pcon = Gcon.bivariate_chrom_poly()
    Pext = Gext.bivariate_chrom_poly()
```

```
  for k in range(1,kmax):
    S = [P(x=k,y=j) for j in range(k+1)]
    Sdel = [Pdel(x=k,y=j) for j in range(k+1)]
    Scon = [Pcon(x=k,y=j) for j in range(k+1)]
    Sext = [(k-j)*Pext(x=k,y=j) for j in range(k+1)]
    if not (IsLC(SubSeqs(Sdel,Scon)) and
      IsLC(SubSeqs(Sext, Scon)) and
      IsLC(AddSeqs(Sdel,Sext))): return false
return true
```

Appendix C

Maple Code Used to Check Properties of $B_k(G; y)$

```
# Returns S(G;x,t)
SS := proc(V::set, I::set, $)
  return exp(add(t[v], v in V)) + x*add(mul(t[v], v in S), I in I);
end proc:

# Returns true if no element of E is a subset of W
IsIndep := proc(W::set, E::set(set), $)
  return not ormap((e) -> e subset W, E);
end proc:

# Initialize a graph G for future use of the Bkpoly function
_VSet[0] := {}:    # List of vertex sets
_VPowSet[0] := {}: # List of powersets of vertex sets
InitGraph := proc(G::Graph, $)
  global _VSet, _VPowSet;
  local n, StableSets;
  n := NumberOfVertices(G);
  if not type(_VSet[n], set) then
    _VSet[n] := {seq(i, i=1..n)};
    _VPowSet[n] := powerset(_VSet[n]);
  end if;
  Indeps := select(IsIndep, _VPowSet[n], Edges(G));
```

```
    SetGraphAttribute(G, "Sgeo"=1 / (1 - y*SS(_VSet[n], Indeps)));
end proc:


# Returns the coefficient of the the monomial prod_{(t,i) in S} t^i
coeffList := proc(expr, S::set(list), $)
  return foldl((Z, x) -> coeftayl(Z, x[1] = 0, x[2]), expr, op(S));
end proc:


# Returns B_k(G;x) (assumes InitGraph has been called on G)
Bkpoly := proc(G::Graph, k::nonnegint, $)
  local Z;
  Z := GetGraphAttribute(G, "Sgeo");
  return simplify(coeffList(coeftayl(Z, y = 0, k),
    {seq([t[v], 1], v in _vSet[NumberOfVertices(G)])}));
end proc:


# Returns false if some bk has non-real roots, true otherwise
AreAllBkStable := proc(kmax::nonnegint, path::string)
  local NextGraph, k, G;
  NextGraph = ImportGraph(path, "graph6", output=iterator);
  do:
    G := NextGraph();
    if (G = FAIL) then return true; end if;
    InitGraph(G);
    for k from 1 to kmax do
      if (not andmap(IsReal, [fsolve(Bkpoly(G,k)=1,complex,x)])) then
        return false;
      end if;
    end do;
  end do;
end proc:


# Version of the 'Contract' function that works with Bkpoly
SafeContract := proc(G::Graph, e, $)
    local n,H;
    n := NumberOfVertices(G) - 1 ;
    H := RelabelVertices(Contract(G,e), [seq(i, i=1..n)]);
    InitGraph(H);
```

```
    return H;
end proc;


# Version of the 'DeleteVertex' function that works with Bkpoly
SafeDeleteVertex := proc(G::Graph, v, $)
    local n,H;
    n := NumberOfVertices(G) - nops(v);
    if (n = 0) then
        H := Graph();
    else
        H := RelabelVertices(DeleteVertex(G,v), [seq(i, i=1..n)]);
    end if;
    InitGraph(H);
    return H;
end proc;


# "Approximately equal" function to deal with numerical inacuracies
ApproxEq := proc(x,y)
    return (abs(x-y) < 0.0000001);
end proc:


# Returns true if the elements of L1 and L2 interlace each other
Interlace := proc(L1::list, L2::list, $)
  local i,L1max,L2max;
  i := 1;
  L1max := nops(L1);
  L2max := nops(L2);
  while(i <= L1max and i <= L2max and ApproxEq(L1[i],L2[i])) do
    i := i + 1;
  end do;
  if (i > L1max) then return i >= L2max; end if;
  if (i > L2max) then return i >= L1max; end if;
  if (L1[i] > L2[i]) then return Interlace(L2, L1); end if;
  while (i < L1max) do
    if (i > L2max) then return false; end if;
    if (not (ApproxEq(L1[i], L2[i]) or ApproxEq(L2[i],L1[i+1])))
        and (L1[i] > L2[i] or L2[i] > L1[i+1]) then
      return false;
```

```
      end if;
      i := i + 1;
    end do;
   if (i < L2max) then return false;
   if (i = L2max) then return ApproxEq(L1[i],L2[i]) or L1[i]<=L2[i]; fi;
   return true;
end proc:

# Returns false if no interlacing is found, true otherwise
CheckInterlacing := proc(kmax::nonnegint, path::string)
  local NextGraph, k;
  NextGraph := ImportGraph(path, "graph6", output=iterator);
  do:
    G := NextGraph();
    if (G = FAIL) then return true; end if;
    InitGraph(G);
    for e in Edges(G) do
      Gcon := SafeContract(G, e);
      Gext := SafeDeleteVertex(G, convert(e, list));
      for k from 1 to kmax do
        Bkcon := Bkpoly(Gcon, k);
        Bkext := Bkpoly(Gext, k);
        A := k*Bkext - x*diff(Bkext, x);
        Brs := map((z) -> Re(z), [fsolve(Bkcon=0, complex)]);
        Ars := map((z) -> Re(z), [fsolve(A=0, complex)]);
        if (not Interlace(Brs,Ars)) then return false; end if;
      end do;
    end do;
  end do;
end proc:

# Compute the Bivariate Chromatic Polynomial of G
BivChromPoly := proc(G::Graph, x::name, y::name, $)
    global _vSet, _vPowSet;
    local n;
    n := NumberOfVertices(G);
    if (n = 0) then return 1; elif (n = 1) then return x; end if;
    if not type(_vSet[n], set) then
```

```
        _vSet[n] := {seq(i, i=1..n)};
        _vPowSet[n] := powerset(_vSet[n]);
    end if;
    return simplify(add((x-y)^(n - nops(W))*ChromaticPolynomial(
        InducedSubgraph(G,W),y), W in _vPowSet[n]);)
end proc:


# Precompute the stirling polynomials.
MAX_STIRLING := 12: # Max degree we will compute
strlY[0] := 1:
for k from 1 to MAX_STIRLING do:
  strlY[k] := sort(expand(y*strlY[k-1] + y*diff(strlY[k-1],y))):
end do:


# Maps sum_{k=0}^n a_k*(t)_k to sum_{k=0}^n a_k*y^k
FallingFactsToYPowers := proc(p::polynom(integer), t::name, $)
    local P,d;
    P := expand(p): d := degree(p,x);
    return simplify(add(coeff(P,t,k)*strlY[k], k=0..d));
end proc:


IsReal := proc(z::complex)
    return evalb(abs(Im(z)) < 0.0000001);
end proc:


# Discriminant of quadratic
QuadDiscr := proc(p::polynom(integer), y::name, x::name, $)
    local A,B,C;
    A := coeff(p,y,2); B := coeff(p,y,1); C := coeff(p,y,0);
    return B^2 - 4*A*C;
end proc:


# Discriminant of cubic
CubicDiscr := proc(p::polynom(integer), y::name, x::name, $)
    local A,B,C,D;
    A := coeff(p,y,3); B := coeff(p,y,2);
    C := coeff(p,y,1); D := coeff(p,y,0);
    return B^2*C^2 - 4*A*C^3 - 4*B^3*D - 27*A^2*D^2 + 18*A*B*C*D;
```

```
end proc:

# Check if discriminant
CheckDiscr := proc(p::polynom(integer), y::name, x::name, $)
  local d,discr,m;
  d := degree(p,y);
  if (d <= 1) then return -infinity;
  elif (d = 2) then discr := QuadDiscr(p,y,x);
  elif (d = 3) then discr := CubicDiscr(p,y,x);
  else return FAIL; end if; # Should not occur
  m := Re(max(select(IsReal, [fsolve(disc=0, complex)])));
  if (eval(disc,x=m+1) > 0) then return m; else return FAIL; end if;
end proc:

# Check if Q(G;k,y) has positive discriminant as we tend to infinity
CheckQpolys := proc(path::string)
  local NextGraph, maxM, m;
  m := -infinity;
  NextGraph := ImportGraph(path, "graph6", output=iterator);
  do:
    G := NextGraph()
    if (G = FAIL) then return true; end if;
    InitGraph(G)
    P := BivariateChromaticPoly(G,x,y);
    sig := FallingFactsToYPowers(P,y);
    m := CheckDiscr(sig,y,x);
    if (m = FAIL) then return false; end if;
    if (m > maxM) then maxM := m; end if;
  end do;
  print("maxM =", maxM);
  return true;
end proc:
```