Supplementary Information Appendix: Competition between injunctive social norms and conservation priorities give rise to complex dynamics in a model of forest growth and opinion dynamics

Appendix A.

(a) Stability Analysis–Main Model:

For the set $x' = f(x)$ of n variables x_1,x_2, $\ldots, x_n$ written explicitly as

$$\begin{aligned} x'_1 &= f_1(x_1, ..x_n) \\ &\vdots \\ x'_n &= f_n(x_1, ..x_n). \end{aligned} \tag{1}$$

The Jacobian matrix (J) of the above system is the $n \times n$ matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Now, the Jacobian matrix for our system is 2×2 matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial F} & \frac{\partial f_1}{\partial x} \\ \\ \frac{\partial f_2}{\partial F} & \frac{\partial f_2}{\partial x} \end{bmatrix},$$

where

$$f_1 = RF(1-F) - h(1-x)F$$

$$f_2 = \kappa x(1-x)\left(c - F + \delta(2x-1)\right).$$

That is,

$$J = \begin{bmatrix} R(1-2F) - h(1-x) & hF \\ \\ -\kappa x(1-x) & \kappa(1-2x)(c-F) + \kappa\delta(-1+6x-6x^2) \end{bmatrix}.$$

Now the Jacobian matrix at the steady state $A_1 = (0,0)$ is

$$J_{A_1} = \begin{bmatrix} R-h & 0 \\ \\ 0 & \kappa(c-\delta) \end{bmatrix},$$

and it has the eigenvalues $\lambda_1 = R - h < 0$, if $R < h$ and $\lambda_2 = \kappa(c-\delta) < 0$, if $c < \delta$. Therefore for $\boxed{1 < \frac{h}{R}}$ and $\boxed{1 < \frac{\delta}{c}}$, $A_1 = (0,0)$ is locally asymptotically stable(LAS).

$$J_{A_2} = \begin{bmatrix} -R+h & \left(1-\frac{h}{R}\right)h \\ \\ 0 & \kappa\left(c-1-\delta+\frac{h}{R}\right) \end{bmatrix},$$

has the eigenvalues $\lambda_1 = -R+h < 0$ if $\frac{h}{R} < 1$ and $\lambda_2 = \kappa\left(c-1-\delta+\frac{h}{R}\right) < 0$ if $\frac{h}{R} < 1+\delta-c$. Therefore for $\boxed{\frac{h}{R} < 1+\delta-c < 1 \text{ or } \frac{h}{R} < 1 < 1+\delta-c}$, $A_2 = \left(1-\frac{h}{R}, 0\right)$ is LAS. We note that if A_1 is LAS then A_2 is biologically meaningless and if A_2 is stable then A_1 is unstable.

$$J_{A_3} = \begin{bmatrix} R & 0 \\ \\ 0 & -\kappa(c+\delta) \end{bmatrix},$$

has the eigenvalues $\lambda_1 = R > 0$, and $\lambda_2 = -\kappa(c+\delta) < 0$. Therefore, $A_3 = (0,1)$ is unstable.

$$J_{A_4} = \begin{bmatrix} -R & h \\ & \\ 0 & \kappa(1-c-\delta) \end{bmatrix},$$

has the eigenvalues $\lambda_1 = -R < 0$ and $\lambda_2 = \kappa(1-c-\delta) < 0$ if $1 < c+\delta$. Therefore for $\boxed{c+\delta > 1}$, $A_4 = (1,1)$ is LAS.

$$J_{A_5} = \begin{bmatrix} R - \frac{h}{2\delta}(\delta + c) & 0 \\ & \\ \frac{\kappa(c^2-\delta^2)}{4\delta^2} & -\frac{\kappa c^2}{2\delta} + \frac{1}{2}\kappa\delta \end{bmatrix},$$

has one of the eigenvalue $\lambda_1 = -\frac{\kappa c^2}{2\delta} + \frac{1}{2}\kappa\delta > 0$ for $c < \delta$ ($c < \delta$ is must for A_5 to be biologically meaningful). Therefore $A_5 = \left(0, \frac{\delta - c}{2\delta}\right)$ is unstable.

Finally

$$J_{A_6} = \begin{bmatrix} -R(1+hS) & h(1+hS) \\ & \\ \kappa RS(1+RS) & -2\kappa\delta RS(1+RS) \end{bmatrix} = \begin{bmatrix} -RF^* & hF^* \\ & \\ \kappa x^*(x^*-1) & 2\kappa\delta(1-x^*)x^* \end{bmatrix},$$

where $F^* = 1 + hS$ and $x^* = 1 + RS$. If $2\kappa\delta(1-x^*)x^* < RF^*$, then trace $(J_{A_6}) = 2\kappa\delta(1-x^*)x^* - RF^* < 0$. And $det.(J_{A_6}) = 2\kappa\delta RF^*(1-x^*)x^* - \kappa h x^* F^*(x^*-1) = \kappa x^*(1-x^*)F^*(h - 2\delta R)$.

A_6 is biologically meaningful if $1 > 1 + RS > 0$ and $1 > 1 + hS > 0$ and it implies, $S = \frac{1-c-\delta}{2R\delta - h}$ is negative. This leads the following possibilities:

(1) $1 - c - \delta < 0$ & $2R\delta - h > 0$

(2) $1 - c - \delta > 0$ & $2R\delta - h < 0$.

Now under case (1), det$.(J_{A_6}) < 0$. Hence A_6 is unstable. And under case (2), $0 > RS > -1$ and $0 > hS > -1$ implies $1+\delta-c < \frac{h}{R}$ or $\frac{2\delta}{\delta+c} < \frac{h}{R}$ and det$.(J_{A_6}) > 0$. Hence $A_6 = (1+hS, 1+RS)$ is LAS if $\boxed{1+\delta-c < \frac{h}{R} \text{ or } \frac{2\delta}{\delta+c} < \frac{h}{R}}$, $\boxed{\delta+c<1}$ and $\boxed{0 < 2\kappa\delta(1-x^*)x^* < RF^*}$ are satisfied.

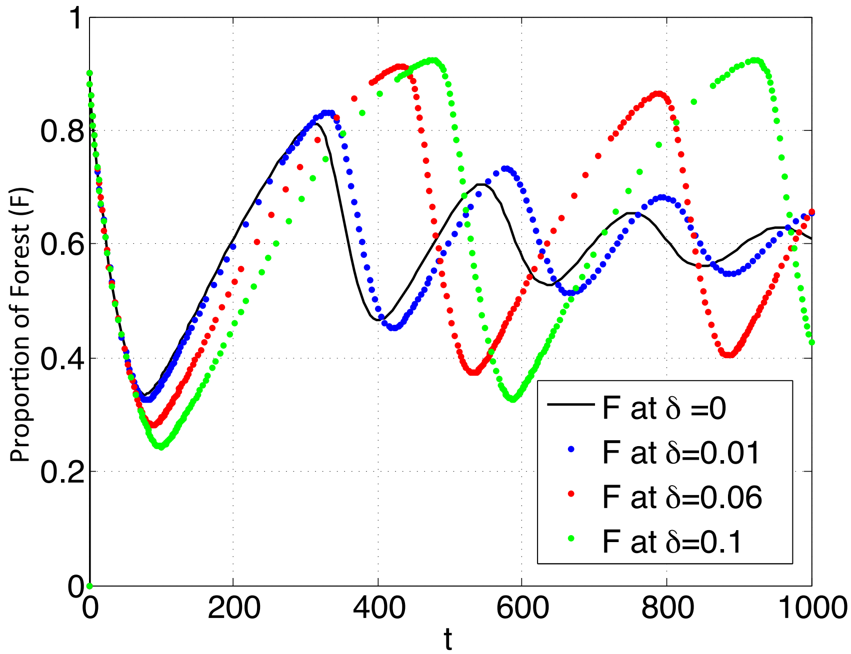

FIGURE S1. For the initial condition $(F_0, x_0) = (0.9, 0.1)$, increasing injunctive social norms bring larger decrease and increase but reduce the oscillation. All other parameters except those being varied appear in the table 1 of the main text.

(b) Non-existence of the Periodic Orbit:

From the direction field plot we notice that there may exist the periodic orbit for particular parameter values. For other set of parameter values, we can ruled out the periodic orbit under certain conditions.

$x = 0$ implies $x' = 0$, $F = 0$ implies $F' = 0$ and $x = 1$ implies $x' = 0$. Which means that, the lines $x = 0$, $x = 1$ and $F = 0$ are invariant. Moreover, $F = 1$ implies $F' = -h(1-x) \le 0$ implies solutions can not increase through the line $F = 1$. Therefore there does not exist the periodic orbit including the lines $x = 0$, $x = 1$, $F = 0$ and $F = 1$. So, we are interested in $D = \{(F, x) : 0 < F < 1, 0 < x < 1\}$, the interior of a unit square with boundary $x = 0$, $x = 1$, $F = 0$, $F = 1$.

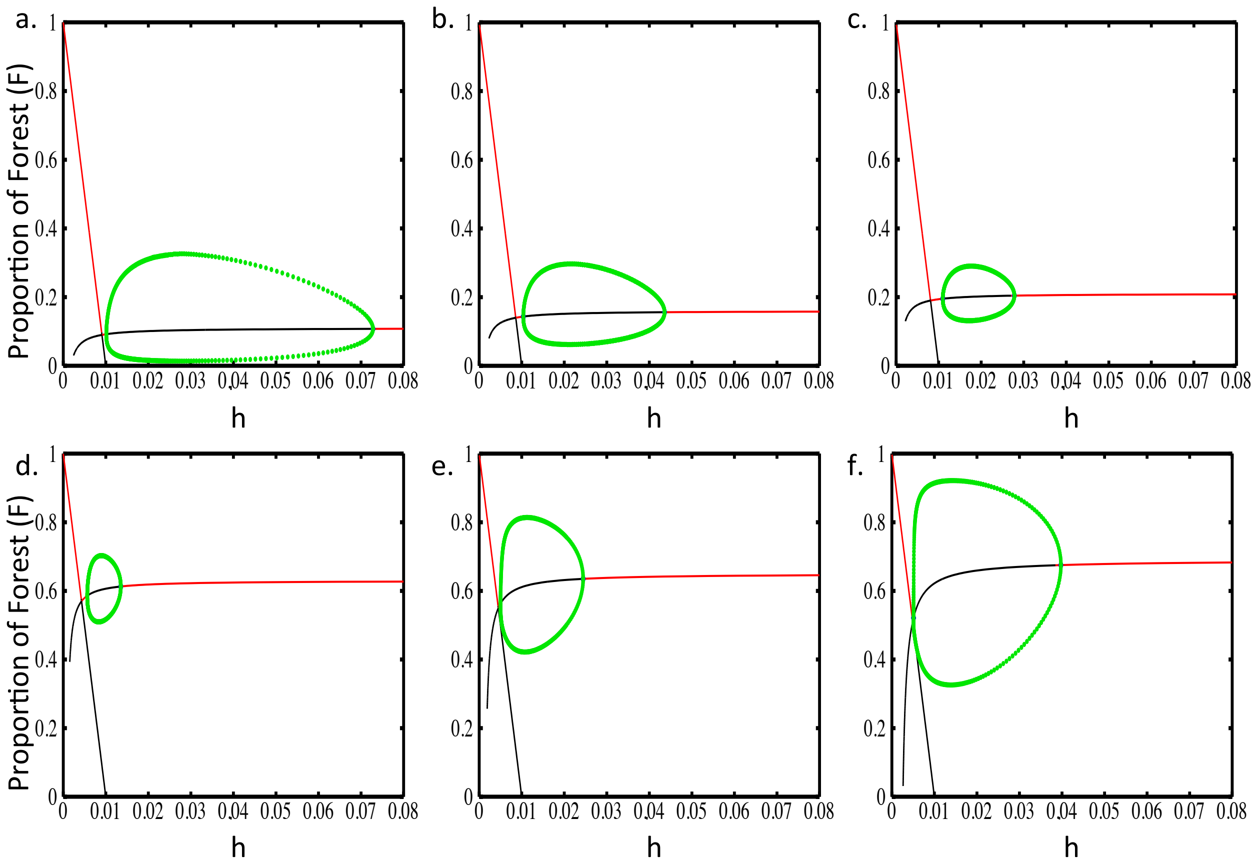

Figure S2. Conservation value and the social norms has opposite impact in system stability because the increase in one increase in oscillation but increase in other decrease the oscillation. Increasing strength of conservation value (a) $c = 0.1$ (b) $c = 0.15$ (c) $c = 0.2$, helps to reduce the oscillation and increasing strength of injunctive social norms (d) $d = 0.03$ (e) $d = 0.05$ (f) $d = 0.09$, increase confusion. All other parameters except those being varied appear in the table 1 of the main text.

Consider the vector field

$$\underline{\dot{x}} = \underline{f}(x) = \begin{pmatrix} \dot{x} \\ \dot{F} \end{pmatrix} = \begin{pmatrix} kx(1-x)(c - F + \delta(2x-1)) \\ RF(1-F) - h(1-x)F \end{pmatrix} = \begin{pmatrix} \lambda(F, x) \\ \nu(F, x) \end{pmatrix}.$$

At the boundary $x = 1$ and $0 \le F \le 1$ of D, $\lambda(F,x) = 0$, $\nu(F,x) = RF(1-F) \ge 0$, the vector field is the upward tangent. At the boundary $F = 0$ and $0 \le x \le 1$ of D, $\lambda(F,x) = kcx(1-x)[c+\delta(2x-1)] \ge$ or ≤ 0, $\nu(F,x) = 0$, the vector field is the forward or backward tangent. At the boundary $x = 0$ and $0 \le F \le 1$ of D, $\lambda(F,x) = 0$, $\nu(F,x) = RF(1-F) - hF \le$ or ≥ 0, the vector field is the upward or downward tangent. At the boundary $F = 1$ and $0 \le x \le 1$ of D, $\lambda(F,x) = kx(1-x)[c-1+\delta(2x-1)] \le$ or ≥ 0, $\nu(F,x) = -h(1-x) \le 0$, the vector field is inward to D. Hence D is the trapping reason.

For the Dulac function $\alpha = \frac{1}{x(1-x)F^2}$ in the region D,

$$\begin{aligned} div.(\alpha \underline{f}) &= \frac{\partial}{\partial F}\left(\alpha\nu(F,x)\right) + \frac{\partial}{\partial x}\left(\alpha\lambda(F,x)\right) \\ &= \frac{-R + h(1-x) + 2\delta k x(1-x)}{x(1-x)F^2} \\ &= \frac{(-R+h) - hx + \frac{k\delta}{2} - 2k\delta(x-0.5)^2}{x(1-x)F^2} \quad < 0 \end{aligned}$$

for $R > h$ and sufficiently small value of $\kappa\delta$. Hence for $R > h$ and sufficiently small value of $\kappa\delta$, using Dulac's criterion (Hale and Koçak, 2012, p.373), we conclude that there does not exist the periodic orbit. Thus using poincaré–Bendixson theorem (Hale and Koçak, 2012, p.366), $A_6 = (1+hS, 1+RS)$ is globally asymptotically stable in D.

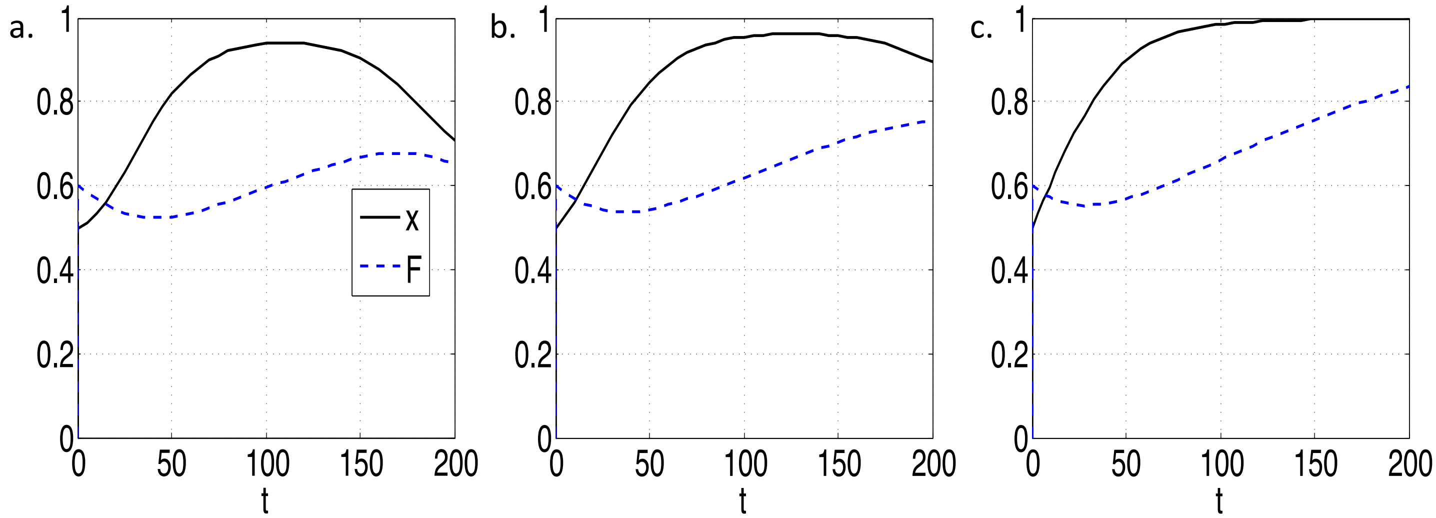

FIGURE S3. When, $(F_0, x_0) = (0.6, 0.5)$, increasing strength of injunctive social norms do not bring much changes into the system, (a) $\delta = 0.05$ (b) $\delta = 0.2$ (c) $\delta = 0.4$. All other parameters except those being varied appear in the table 1 of the main text.

APPENDIX B.

(a) Stability Analysis–Model with $\delta = 0$:

As in the main model, we have

$$J = \begin{bmatrix} R(1-2F) - h(1-x) & hF \\ \\ -\kappa m x(1-x) & \kappa(1-2x)(c-F) \end{bmatrix}.$$

Now,

$$J_{B_1} = \begin{bmatrix} R-h & 0 \\ 0 & \kappa c \end{bmatrix},$$

has the eigenvalues $\lambda_1 = R - h < 0$ if $R < h$ and $\lambda_2 = \kappa c > 0$. Therefore, $B_1 = (0,0)$ is unstable.

$$J_{B_2} = \begin{bmatrix} -R+h & \left(1-\frac{h}{R}\right)h \\ & \\ 0 & \kappa\left(c-1+\frac{h}{R}\right) \end{bmatrix},$$

has the eigenvalues $\lambda_1 = -R+h < 0$ if $\frac{h}{R} < 1$ and $\lambda_2 = \kappa\left(c-1+\frac{h}{R}\right) < 0$ if $\frac{h}{R} < 1-c$. Therefore for $\boxed{\frac{h}{R} < 1-c}$, $B_2 = \left(1-\frac{h}{R}, 0\right)$ is LAS.

$$J_{B_3} = \begin{bmatrix} R & 0 \\ 0 & -\kappa c \end{bmatrix},$$

has the eigenvalues $\lambda_1 = R > 0$ and $\lambda_2 = -\kappa c < 0$. Therefore, $B_3 = (0, 1)$ is unstable.

$$J_{B_4} = \begin{bmatrix} -R & h \\ 0 & \kappa(1-c) \end{bmatrix},$$

has the eigenvalues $\lambda_1 = -R < 0$ and $\lambda_2 = \kappa(1-c) > 0$. Therefore, $B_4 = (1, 1)$ is unstable.

$$J_{B_5} = \begin{bmatrix} -cR & hc \\ & \\ -\kappa\frac{R}{h}(1-c)\{1-\frac{R}{h}(1-c)\} & 0 \end{bmatrix},$$

trace $(J_{B_5}) = -cR < 0$ and $det.\,(J_{B_5}) = \kappa Rc(1-c)\{1-\frac{R}{h}(1-c)\} > 0$, if $\frac{h}{R} > 1-c$. Therefore for $\boxed{\frac{h}{R} > 1-c}$, $B_5 = \left(c, 1-\frac{R}{h}(1-c)\right)$ is LSA.

(b) Non-existence of the Periodic Orbit:

Since, our solutions lies on and inside the lines $x = 0$, $x = 1$, $F = 0$ and $F = 1$, we focus our study in this region. The lines $x = 0$, $x = 1$ and $F = 0$ are invariant as $x = 0$ implies $x' = 0$; $F = 0$ implies $F' = 0$ and $x = 1$ implies $x' = 0$. Also, the solution does not increases through the line $F = 1$ as $F = 1$ implies $F' = -h(1-x) \leq 0$. Therefore there does not exist the periodic orbit including the lines $x = 0$, $x = 1$, $F = 0$ and $F = 1$. So, we are interested in $D = \{(F, x) : 0 < F < 1, 0 < x < 1\}$, the interior of a unit square with boundary $x = 0$, $x = 1$, $F = 0$, $F = 1$.

Now for the vector field

$$\underline{\dot{x}} = \underline{f}(x) = \begin{pmatrix} \dot{x} \\ \\ \dot{F} \end{pmatrix} = \begin{pmatrix} kx(1-x)(c-F) \\ \\ RF(1-F) - h(1-x)F \end{pmatrix} = \begin{pmatrix} \gamma(F, x) \\ \\ \mu(F, x) \end{pmatrix},$$

if we follow the similar process to appendix A, we can show that the simply connected region D is the trapping reason. For the Dulac function $\alpha' = \frac{1}{x(1-x)F}$ in the region D,

$$\begin{aligned} div.(\alpha' f) &= \frac{\partial}{\partial F}\left(\alpha' \mu(F, x)\right) + \frac{\partial}{\partial x}\left(\alpha' \gamma(F, x)\right) \\ &= \frac{1}{x(1-x)} \frac{\partial}{\partial F}[R(1-F) - h(1-x)] + \frac{\partial}{\partial x}\left(\kappa\left(\frac{c}{F} - 1\right)\right) \\ &= -\frac{R}{x(1-x)} \\ &< 0. \end{aligned}$$

Hence the Dulac's criterion (Hale and Koçak, 2012, p.373) guaranteed the non-existence of periodic orbit in the simply connected region D. Thus using poincaré–Bendixson theorem (Hale and Koçak, 2012, p.366), the interior steady state $B_5 = \left(c, 1 - \frac{R}{h}(1-c)\right)$ is globally asymptotically stable in D.

References

Hale, J. K. and Koçak, H. (2012). *Dynamics and bifurcations*, volume 3. Springer Science & Business Media.