# Induction Relations in the Symmetric Groups and Jucys-Murphy Elements

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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## Abstract

Transitive factorizations faithfully encode many interesting objects. The well-known ones include ramified coverings of the sphere and hypermaps. Enumeration of specific classes of such objects have been known for quite some time now. Hurwitz numbers, monotone Hurwitz numbers and hypermaps numbers were discovered using different techniques. Recently, Carrell and Goulden found a unified algebraic approach to count these objects in genus 0. Jucys-Murphy elements and centrality play important roles in establishing induction relations. Such a method is interesting in its own right. Its corresponding combinatorial decomposition is however intriguingly mysterious. Towards a understanding of direct combinatorial analysis of multiplication of arbitrary permutations, we consider methods, especially operators on symmetric functions, and related problems in symmetric groups.

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| | |
|---|---|
| $\alpha \vdash n$ | An integer partitions of $n$ |
| $m_i(\lambda)$ | Multiplicity of $i$ in a partition $\lambda$ |
| $\ell(\lambda)$ | Number of parts in a partition $\lambda$ |
| $H_\lambda$ | Hook product of a partition $\lambda$ |
| $c(\square)$ | Content of a cell |
| $\mathcal{S}_n$ | Symmetric group on $\{1, \ldots, n\}$ |
| $\mathrm{cyc}(\sigma)$ | Cycle type of the permutation $\sigma$ |
| $\mathbb{C}[\mathcal{S}_n]$ | The group algebra of $\mathcal{S}_n$ |
| $\chi^\lambda$ | Irreducible character of $\mathcal{S}_n$ indexed by $\lambda$ |
| $Z(\mathbb{C}[\mathcal{S}_n]), \mathcal{Z}_n$ | The centre of group algebra of $\mathcal{S}_n$ |
| $J_i$ | The $i$-th Jucys-Murphy element |
| $\mathcal{C}_\alpha$ | Conjugacy class |
| $\mathcal{F}^\lambda$ | Central orthogonal idempotent |
| $\theta^\lambda$ | Central (irreducible) character indexed by $\lambda$ |
| ch | Frobenius characteristic map |
| $\mathrm{Sym}_n$ | The algebra of symmetric polynomials in $n$ variable |
| Sym | The algebra of symmetric functions |
| $\mathrm{Sym}^d$ | The space of homogeneous symmetric functions of degree $d$ |
| $e_k$ | Elementary symmetric function |
| $h_k$ | Complete symmetric function |
| $p_i$ | Power sum symmetric function |
| $s_\lambda$ | Schur function |
| $f^\perp$ | Adjoint to multiplication by $f$ |
| $\Delta$ | Join-Cut operator |
| $\mathcal{L}_k$ | Lassalle's operator |
| $\mathcal{L}_k(t)$ | Lassalle's operator (generalized) |
| $\mathcal{U}_k$ | Up operator |
| $\mathcal{U}_k^{(h)}$ | Up operator (graded) |
| $\mathcal{U}(x)$ | Generating series of $\mathcal{U}$ operators |

| | |
|---|---|
| $\varphi_{\alpha}^{f}$ | Class expansion coefficient of central element $\zeta$ |
| $\Phi^{f}$ | Generating series of central element described by $f$ |
| $\Phi^{H}, \Phi^{(1-x)^{-1}}$ | Generating series of monotone transposition factorizations |
| $\Phi^{f(x)}$ | Content series |
| $\Psi^{f(x)}$ | Content series (connected) |
| $\Psi_{h}^{f(x)}$ | Content series (connected, genus $g$ portion) |

# Chapter 1

# Introduction

This thesis is motivated by counting problems of a family of objects that come up in many different areas of mathematics [22]. Grothendieck calls them Dessins d'Enfants — children's drawings. Algebraic geometers see them coming from moduli spaces of complex algebraic curves. Mathematical physicists often use them to model objects arising from quantum mechanics. Topologists know them as branched coverings of the Riemann sphere. Bijective combinatorialists prefer to think of them as (combinatorial) maps — 2-cell embeddings of graphs on orientable surfaces. Conveniently, all of the above can be encoded as factorizations of permutations.

Not surprisingly, enumerative problems of permutation factorizations are posed and studied in many different ways. One often counts factorizations with respect to a genus parameter due to connections to branched covering of the Riemann sphere. For example, the earliest result dates back to 1886 when Hurwitz sketched the proof for the number of genus 0 branched coverings where all but 1 branching points are simple [18]. The geometry of Hurwitz Numbers remains an active research area [21]. On the other hand, physical interpretations of factorizations lead to interest in enumerating maps equipped with some additional objects, e.g. spanning trees. Such problems are of course inherently more difficult. Many are open problems [2].

There are many beautiful results in permutation factorizations that promote the subject from within combinatorics. A classic one is the enumeration of minimum factorizations of a full $n$-cycle into transpositions. The answer is the tree number $n^{n-2}$. Elegant results like this often excite combinatorialists for they tend to reveal interesting decompositions. For example, Bousquet-Mélou and Schaeffer discovered a fruitful bijection between factorizations whose corresponding combinatorial maps are connected and certain families of decorated trees [3]. This bijection has been extended in many different directions to count various classes of combinatorial maps with or without additional structures [1, 9]. For a survey on the planar combinatorial maps, see [28].

The symmetric group algebra is a suitable setting for permutation factorization problems. Its centre, consisting of elements that commute with every element, can be identified with symmetric functions. This correspondence opens doors to many effective algebraic enumerative techniques. Goulden and Jackson were the first to write down a proof for the aforementioned Hurwitz number using symmetric functions and combinatorial analysis on multiplication of transpositions [16]. Moreover, this correspondence is often a 2-way street. Many interesting questions in symmetric functions arise from working with permutation factorizations. A well-known one is the so called $b$-conjecture regarding coefficients of Jack symmetric functions [14].

Many methods have been used to attack enumeration of permutation factorizations. Although it is not possible to list them all, we nonetheless give references to some relevant works here. Tutte enumerated various classes of combinatorial maps using the so called quadratic method [17, Section 2.9]. Representation theory and symmetric functions are used by Goulden and Jackson [16], Goulden, Guay-Paquet, and Novak [12, 13], and Carrell and Goulden [4]. Integrable hierarchy is used by Goulden and Jackson [15], Carrell and Chapuy [5], and Carrell [6]. Bijective approaches have been used by Bousquet-Mélou and Schaeffer [3].

One of the recent results is a unified algebraic proof of three special classes of permutation factorizations discovered using very different tools [4]. An interesting induction technique is used in the centre of group algebra. The purpose of this thesis is to collect methods related to this approach to obtain induction relations in the symmetric groups.

## 1.1 Overview

One of the contributions of this thesis is to bring together results related to operators on symmetric functions that model an induction behaviour in symmetric groups involving multiplication by Jucys-Murphy elements. We survey works by Lassalle [24], Féray [10] and Carrell and Goulden Carrell and Goulden. Another contribution of this thesis is a generalization of Lassalle's group specific operators to operators that describes the same behaviour simultaneously for all symmetric groups, thus providing a more general context for extension. Our generalization allows systematic and possibly computation for such operators as oppose to Lassalle's ad-hoc approach.

Section 1.2 is a brief discussion on related counting problems on permutation factorizations. We consider permutation factorizations in connected and not necessarily connected cases.

Chapter 2 is a short essay where we set up combinatorial and algebraic frameworks. We bring in relevant facts from representations of finite groups and symmetric functions. We explain how each of these tools enters the big picture.

Chapter 3 describes a family of differential operators $\mathcal{L}$ which is one of the tools for finding induction relations. As an application of $\mathcal{L}$ operators, we obtain some linear relations in central characters.

Chapter 4 considers a problem generalized from permutation factorizations in the not necessarily connected case. Using linear relations in central characters, we obtain induction relations for their coefficients. We also discuss the combinatorics in a special case which leads to a generating series involving contents defined in Chapter 6.

Chapter 5 picks apart the $\mathcal{L}$ operators. The method used in Chapter 4 is extended to obtain another family of operators $\mathcal{U}$ first considered by Carrell and Goulden. The combinatorics of these operators involves an add-a-vertex operation and multiplication by powers of Jucys-Murphy elements. As we will see, the $\mathcal{L}$ operators are group specific. We present a new result generalizing that to a family called $\mathcal{L}(t)$ operators that does not depend on a particular symmetric group. In other words, the $\mathcal{L}(t)$ operators describe the same multiplication by Jucys-Murphy elements behaviour simultaneously for all symmetric groups. We also reveal a natural parameter $h$ and obtain a generating series with respect to $h$ using some tools from mathematical physics.

Chapter 6 is our final chapter. Generalizing from 3 classes of permutation factorizations, we define a generating series $\Phi^{f(x)}(z, y, \mathbf{p})$ involving contents of partitions. The $\mathcal{U}$ operators and its combinatorial interpretation are used to obtain a partial differential equation for which $\Phi^{f(x)}(z, y, \mathbf{p})$ is the unique solution satisfying some initial condition. We will also see that the $h$ parameter can be interpreted as the genus of $\mathcal{U}$ operators. As a final application, we obtain a partial differential equations for the genus 0 portion of $\Phi^{f(x)}(z, y, \mathbf{p})$.

## 1.2 Two Counting Problems

Let $\sigma_0 \in \mathcal{S}_n$ be an arbitrary permutation. An *(unrestricted and ordered) $m$-factorization* of $\sigma_0$ is a tuple $(\sigma_1, \ldots, \sigma_m)$ such that $\sigma_1, \ldots, \sigma_m \in \mathcal{S}_n$ and $\sigma_1 \cdots \sigma_m = \sigma_0$. For example, $(id, id)$ and $((12), (12))$ are all the ordered 2-factorizations of the identity permutation $id \in \mathcal{S}_2$. For each $m \geq 1$, one can ask for the number of $m$-factorizations of $\sigma_0$. The answer is fairly straightforward: If $m = 1$, then the factorization is unique. For $m \geq 2$, we can first freely choose $m-1$ permutations $\sigma_1, \ldots, \sigma_{m-1}$ from $\mathcal{S}_n$. The last factor $\sigma_m = \sigma_{m-1}^{-1} \cdots \sigma_1^{-1} \sigma_0$ is then completely determined. Therefore the answer is $n!^{m-1}$.

The problem is not much more difficult when we require some but not all factors to be chosen from non-empty strict subsets of $\mathcal{S}_n$. Let $S_1, \ldots, S_n$ be non-empty subsets of $\mathcal{S}_n$ with some $S_i = \mathcal{S}_n$. We can freely choose $\sigma_k \in S_n$ for $k \in \{1, \ldots, i-1, i+1, \ldots, m\}$. Then $\sigma_i = \sigma_{i-1}^{-1} \cdots \sigma_1^{-1} \sigma_0 \sigma_m^{-1} \cdots \sigma_{i+1}^{-1}$ is completely determined. Hence, the answer is $|S_1| \cdots |S_m|/n!$. If all restriction sets $S_1, \ldots, S_m$ are non-empty strict subsets of $\mathcal{S}_n$, then knowledge about the last choice $\sigma_i$ becomes crucial. In general, this is a challenging problem. Some structure on the restrictions is required to make it tractable.

This motivates the *class expansion* problem: Given conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_m$ of $\mathcal{S}_n$, not necessarily distinct, what is the cycle type of the product $\sigma_1 \cdots \sigma_m$ if $\sigma_i$ must be chosen from $\mathcal{C}_i$ for $i = 1, \ldots, m$?

The class expansion problem has a nice formulation in the group algebra $\mathbb{C}[\mathcal{S}_n]$. We explain these notions and properties in the next chapter. It turns out that conjugacy classes are linearly independent in $\mathbb{C}[\mathcal{S}_n]$ and products of conjugacy classes are linear combinations of conjugacy classes. So the class expansion problem can be restated as follows: *Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be conjugacy classes in $\mathbb{C}[\mathcal{S}_n]$, not necessarily distinct. Let $\mathcal{C}_\alpha \in \mathcal{S}_n$ be another conjugacy class. What is the coefficient of $\mathcal{C}_\alpha$ in $\mathcal{C}_1 \cdots \mathcal{C}_m$?* This is equivalent to counting $m$-factorizations with restrictions $\mathcal{C}_1, \ldots, \mathcal{C}_m$ and $\sigma_0 \in \mathcal{C}_\alpha$.

Historically speaking, class expansion with a connectivity condition attracted more attention than the vanilla version thanks to various bijections. An $m$-factorization $\sigma_1 \cdots \sigma_m = \sigma_0$ in $\mathcal{S}_n$ is *transitive* if the subgroup $\langle \sigma_1, \ldots, \sigma_m \rangle$ acts transitively on $\{1, \ldots, n\}$. Equivalently, the diagram obtained by superimposing functional diagrams of individual factors $\sigma_1, \cdots, \sigma_m$ is connected.

There is a well-known bijection between transitive $m$-factorizations and *branched coverings* of the Riemann sphere. The transitivity condition translates to topological connectedness of covering spaces. Furthermore, $m$-factorizations also faithfully encode *(combinatorial) hypermaps* — certain 2-cell embedding of graphs on orientable surfaces. The transitivity condition again translates to connectedness of surfaces.

More interestingly, the genera of corresponding branched covering and the hypermap for a given transitive $m$-factorization agree. Hence, transitive factorizations are often enumerated with respect to a genus parameter defined using the Riemann-Hurwitz formula: If an $m$-factorization $\sigma_1 \cdots \sigma_m = \sigma_0$ in $\mathcal{S}_n$ is transitive with $\sigma_0 \in \mathcal{C}_\alpha$, then the *genus* $h$ of its corresponding branched covering is given by

$$n - \ell(\alpha) + \sum_{i=1}^{m} \Big( n - \ell(\operatorname{cyc}(\sigma_i)) \Big) = 2n - 2 + 2h,$$

where $\operatorname{cyc}(\sigma_i)$ is the cycle type of $\sigma_i$ and $\ell(\alpha)$ is the number of parts in $\alpha$. *Minimum* transitive factorizations are the ones with genus 0. We now give a historical account of related results.

Enumeration of minimum transitive factorizations into transpositions is the first result of its kind. The genus condition together with cycle type restriction completely determines the number of factors. The number of minimum transitive factorizations of $\sigma_0 \in \mathcal{C}_\alpha$ in $\mathcal{S}_n$ into transpositions is

$$(n + \ell(\alpha) - 2)! n^{\ell(\alpha)-3} \prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{(\alpha_i - 1)!}.$$

Hurwitz [18] wrote down a proof sketch in 1886. Fast forward a century, Goulden and Jackson [16] rediscovered this result in 1997 independently by analyzing the combinatorics of joins and cuts of transpositions acting on arbitrary permutations. This is usually called the *Hurwitz number.*

In 2000, Bousquet-Mélou and Schaeffer [3] enumerated minimum transitive factorizations with no restriction on conjugacy classes. In this case, the number of factors is no longer determined by genus. The number of minimum transitive $m$-factorizations of $\sigma_0 \in \mathcal{C}_\alpha$ in $\mathcal{S}_n$ is

$$m\frac{((m-1)n-1)!}{((m-1)n-\ell(\alpha)+2)!}\prod_{i=1}^{\ell(\alpha)}\binom{m\alpha_j-1}{\alpha_j}.$$

Their method involves a bijection to constellations. This is usually called the *m-hypermap number*.

In 2013, Goulden, Guay-Paquet, and Novak [12] considered Hurwitz Numbers with a twist and obtained a similar formula. A list of transpositions $(a_1b_1), \ldots, (a_mb_m)$ with $a_i < b_i$ is *monotone* if $b_1 \le b_2 \le \cdots \le b_m$. Similar to Hurwitz numbers, the number of factors is determined by genus. The number of minimum transitive factorizations of $\sigma_0 \in \mathcal{C}_\alpha$ in $\mathcal{S}_n$ into monotone transpositions is

$$\frac{(2n+\ell(\alpha)-3)!}{(2n)!}\prod_{j=1}^{\ell(\alpha)}\binom{2\alpha_j}{\alpha_j}.$$

Their proof uses sophisticated algebraic tools that capture the finer details of join and cut actions on the level of generating series. This is called the *monotone Hurwitz number*. Similar problems in higher genera have also been studied by the same group of authors [13].

Proofs of the above formulas took different approaches. A common feature is the use of induction on the number of factors. Carrell and Goulden took a different point of view. A *unified* algebraic method is obtained when induction is applied on the order of the group $\mathcal{S}_n$ [4]. Their work will be described in Chapter 6.

A similar technique was used in an earlier paper by Lassalle [24] who studied a generalized factorization problem (without the transitivity condition). The *generalized class expansion* problem in $\mathcal{S}_n$ is the determination of the coefficients

$$[\mathcal{C}_\alpha]\left(\sum_{\alpha\vdash n} f_\alpha^{(1)}\mathcal{C}_\alpha\right)\cdots\left(\sum_{\alpha\vdash n} f_\alpha^{(m)}\mathcal{C}_\alpha\right) = [\mathcal{C}_\alpha]\sum_{\alpha^{(1)},\cdots,\alpha^{(k)}\vdash n} f^{(1)}_{\alpha^{(1)}}\cdots f^{(m)}_{\alpha^{(m)}}\mathcal{C}_{\alpha^{(1)}}\cdots\mathcal{C}_{\alpha^{(m)}},$$

where $f_\alpha^{(i)}$ are scalars for $i = 1, \ldots, m$ and $\alpha \vdash n$.

Linear combinations of conjugacy classes are central elements of $\mathbb{C}[\mathcal{S}_n]$. A celebrated fact is that such elements can be written as symmetric polynomials evaluated at Jucys-Murphy elements. Let $J_1, \ldots, J_n$ denote Jucys-Murphy elements in $\mathbb{C}[\mathcal{S}_n]$. The first result of this kind is due to Jucys [20]: For $k = 1, \ldots, n$

$$e_k(J_1,\ldots,J_n) = \sum_{\substack{\alpha\vdash n\\ \ell(\alpha)=n-k}} \mathcal{C}_\alpha,$$

where $e_k$ is the $k$-th elementary symmetric functions.

Motivated by connections to mathematical physics, Lascoux and Thibon [23] considered the expansion of power sum symmetric functions and obtained a solution as the constant term of the following generating series

$$\frac{1}{(q-1)(1-q^{-1})}(V(z;q)-1)p_1^n,$$

where $V$ is a vertex operator

$$V(z;q) = \exp\left(\sum_{k\geq 1}(q^k-1)p_k\frac{z^k}{k}\right)\left(\sum_{\ell\geq 1}(1-q^{-\ell})p_\ell^\perp\frac{z^{-\ell}}{\ell}\right)$$

and $p_i$ and $p_i^\perp$ are the power sum symmetric functions and their adjoints respectively.

Lassalle [24] engineered a family of differential operators in the underlying variables of symmetric polynomials. These operators provide a *unified* approach to obtaining recurrences for the expansion of some fundamental families of symmetric functions. This method recovers the results of Jucys and Lascoux-Thibon. New results include recurrences of expansion of complete symmetric functions and 1-row Hall-Littlewood symmetric functions.

Lassalle's work and a follow up by Féray will be described in Chapter 3, 4 and 5.

# Chapter 2

# Background

## 2.1 Partitions

A *partition* of a positive integer $n$, denoted as $\lambda \vdash n$, is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 + \cdots + \lambda_k = n$. The set of all partitions is denoted by $\mathcal{P} = \{\alpha \vdash n : n \geq 1\}$. The $\lambda_i$'s are called *parts* of $\lambda$. The sum of parts is called the *weight* of $\lambda$, denoted as $|\lambda|$. The number of parts in $\lambda$ is called the *length* or *degree* of $\lambda$, denoted as $\ell(\lambda)$. For each $i \in \mathbb{N}$, the number of times $i$ appears in $\lambda$ is called the *multiplicity* of $i$ in $\lambda$, denoted as $m_i(\lambda)$. The vector $(m_1(\lambda), m_2(\lambda), \ldots)$ is known as the *multiplicity vector* of $\lambda$. In writing a partition, we often use a shorthand $\lambda = (n^{m_n(\lambda)}, (n-1)^{m_{n-1}(\lambda)}, \cdots, 1^{m(1)(\lambda)})$. The commas are omitted sometimes. For example, the partition $\lambda = (5, 3, 3, 2, 1, 1) \vdash 15$ is written as $\lambda = (53^2 21^2)$.

Partitions have useful geometric representations. The *Ferrers diagram* of $\lambda \vdash n$ is an array of left-justified boxes such that there are $\lambda_i$ number of boxes in row $i$. Row indices grow from top to bottom and column indices grow from left to right. The top-left box has coordinate $(1, 1)$. The *conjugate* of a partition $\lambda \vdash n$ is obtained by reflecting its Ferrers diagram along the diagonal. It is denoted as $\lambda'$. For example, the Ferrers diagrams corresponding to $\lambda = (53^2 21^2)$ and its conjugate $\lambda' = (6431^2)$ are drawn below.

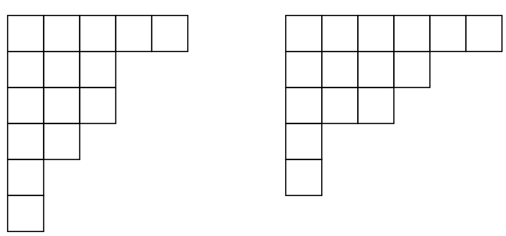


Figure 2.1: Ferrers Diagrams of $\lambda = (53^2 21^2)$ and $\lambda' = (6431^2)$.

We introduce some notations for manipulating partitions. Let $\alpha, \beta$ be partitions not necessarily of the same weight. We say $\beta$ is *contained* in $\alpha$ as a subshape, denoted as $\beta \subseteq \alpha$. More precisely, $\beta \subseteq \alpha$ if $m_i(\alpha) \geq m_i(\beta)$ for all $i \geq 1$. For any partition $\nu$, define $\alpha \setminus \beta \cup \nu$ the partition obtained by removing parts of $\beta$ from $\alpha$ then adding parts of $\nu$ if $\beta \subseteq \alpha$ and $\alpha \setminus \beta \cup \nu = 0$ otherwise. We can write this in terms of their multiplicity vectors

$$\alpha \setminus \beta \cup \nu = (1^{m_1(\alpha)-m_1(\beta)+m_1(\nu)}, 2^{m_2(\alpha)-m_2(\beta)+m_2(\nu)}, \dots),$$

if $\alpha \subseteq \beta$. Note the above operations do not commute in general. For example, if $\lambda = (4331)$, then $\lambda \setminus (3) \cup (2) = (4321)$ but $\lambda \setminus (2) \cup (3) = 0$ since $\lambda$ does not contain a row of length 2.

We will encounter the following operations quite often. The operation $\alpha \setminus (i) \cup (i+1)$ means we replace a part $i$ with $i+1$ in $\alpha$. If $\alpha$ does not have a part $i$ then $\alpha \setminus (i) \cup (i+1) = 0$. The operation $\alpha \setminus (i,j) \cup (i+j)$ says we *join* an $i$-part and a $j$-part to make an $(i+j)$-part. The operation $\alpha \setminus (i+j) \cup (i,j)$ says we *cut* an $(i+j)$-part into an $i$-part and a $j$-part.

For convenience, we denote for any $\alpha \in \mathcal{P}$

$$z(\alpha) = \frac{|\mathcal{C}_\alpha|}{n!} = \prod_{i \geq 1} i^{m_i(\alpha)} m_i(\alpha)!.$$

## 2.2 The Symmetric Groups

The *symmetric group* $\mathcal{S}_X$ is the group of permutations on some ground set $X$. When $X = \{1, \dots, n\}$, we denote $\mathcal{S}_n = \mathcal{S}_{\{1,\dots,n\}}$. The *cycle type* $\text{cyc}(\sigma)$ of a permutation $\sigma \in \mathcal{S}_n$ is the multiset of lengths of cycles in $\sigma$ when represented as a product of disjoint cycles. Hence, $\text{cyc}(\sigma)$ is a partition of $n$. A conjugacy class of $\mathcal{S}_n$ contains all permutations of the same cycle type. We denote a conjugacy class by $\mathcal{C}_\alpha$ where $\alpha \vdash n$.

The *group algebra* $\mathbb{C}[\mathcal{S}_n]$ is the vector space over $\mathbb{C}$ spanned by $\mathcal{S}_n$ treated as formal symbols. Hence, an element $v \in \mathbb{C}[\mathcal{S}_n]$ has the form

$$v = \sum_{\sigma \in \mathcal{S}_n} v_\sigma \sigma, \quad v_\sigma \in \mathbb{C}.$$

Additions in $\mathbb{C}[\mathcal{S}_n]$ are performed pointwise and multiplication as convolutions, i.e.,

$$\left( \sum_{\sigma \in \mathcal{S}_n} u_\sigma \sigma \right) + \left( \sum_{\pi \in \mathcal{S}_n} v_\pi \pi \right) = \sum_{\sigma \in \mathcal{S}_n} (u_\sigma + v_\sigma) \sigma$$

and

$$\left(\sum_{\sigma \in \mathcal{S}_n} u_\sigma \sigma\right)\left(\sum_{\pi \in \mathcal{S}_n} v_\pi \pi\right) = \sum_{\sigma, \pi \in \mathcal{S}_n} u_\sigma v_\pi \sigma\pi,$$

where the multiplication $\sigma\pi$ is carried out in $\mathcal{S}_n$.

The group algebras are useful in combinatorics since convolution is faithful to Cartesian products of $\mathcal{S}_n$. They are also instances of representations. The next section outlines the general theory of representations of finite groups with a focus on the group algebras.

### 2.2.1 Representations of Finite Groups

We follow the textbook by Sagan [27] in this section. Unless otherwise stated, all groups in this section are finite. Let $G$ be a finite group. A *matrix representation* of $G$ is a set of $d \times d$ invertible matrices $\{X(g) : g \in G\}$, not necessarily distinct, with complex entries such that

$$X(g)X(h) = X(gh)$$

where the multiplication $gh$ is carried out in $G$. Note $X$ is a homomorphism from $G$ to $GL(d)$ the group of invertible complex $d \times d$ matrices. The *degree* of $X$ is $d$ which is also is also the rank of the matrix $X(id)$. We can also phrase representations in a coordinate free setting. The group $GL(d)$ is isomorphic to the group $GL(V)$ of invertible linear transformations of some $d$-dimensional vector space $V$. If we replace $GL(d)$ with $GL(V)$ in the above definition, we get an equivalent definition in terms of modules. A *$G$-module* is a vector space $V$ together with a homomorphism $G \to GL(V)$. We can easily convert a $G$-module to a matrix representation. Given a $G$-module, we simply take the matrices representing its action on some basis in the usual way. Conversely, given a matrix representation, the vector space $V = \text{Span}\{e_1, \ldots, e_d\}$ makes $GL(d) = GL(V)$. A *representation* of $G$ can mean either a matrix representation or a $G$-module whichever is convenient.

It is possible to define representations over other fields, but we confine ourselves to $\mathbb{C}$ in this thesis.

It is clear from the definition that a $G$-module $V$ carries a group action on bases of $V$. The converse is also possible. If $G$ acts on some set $S$ and $\mathbb{C}S$ is the vector space spanned by linear combinations of $S$, then this action extended linearly to all of $\mathbb{C}S$ is a $G$-module. The group algebra $\mathbb{C}[\mathcal{S}_n]$ was defined in this way. Note not all representations induced from group action have a natural multiplication structure. For example, the $G$-module $G\{1, \ldots, n\}$ induced by permutation action does not have a natural multiplication. However, we can always obtain an algebra when $G$ acts on itself by multiplication. This is called the *group algebra* of $G$, denoted as $\mathbb{C}[G]$.

Superficial differences between modules are captured by isomorphism. Let $U, V$ be $G$-modules. A *$G$-homomorphism* is a vector space homomorphism $\varphi : U \to V$ such that

$$g\varphi(u) = \varphi(gu)$$

for all $g \in G$ and $u \in U$. If $\varphi$ is also a vector space isomorphism, then we say $\varphi$ is a $G$-isomorphism. If there exists a $G$-isomorphism between $U$ and $V$, then we say $U$ and $V$ are *equivalent* as $G$-modules, or simply equivalent, denoted by $U \cong V$. Otherwise, they are inequivalent. In terms of matrix representations $X$ and $Y$ of the same degree, they are equivalent if and only if there exists an invertible matrix $T$ of the same degree such that

$$TX(g)T^{-1} = Y(g), \quad \text{for all } g \in G.$$

Let $V$ be a $G$-module. The $G$-action on a basis of $V$ partitions them into orbits and $G$-action is closed under each orbit. If $\mathcal{O}$ is one of such orbits, then the subspace $\mathrm{Span}_{\mathbb{C}}\, \mathcal{O}$ is once again a $G$-module. In general, a subspace $W \subseteq V$ is a *submodule* if $W$ itself is a $G$-module. A $G$-module is said to be *irreducible* if the only submodules are the trivial ones — $\{0\}$ and itself. Otherwise, it is said to be *reducible*. Let $U$ be another $G$-modules. The *multiplicity* of $U$ in $V$ is the number of distinct submodules in $V$ that are equivalent to $U$.

For example, any 1-dimensional $G$-module is automatically irreducible. The permutation representation $\mathbb{C}\{1, \ldots, n\}$ of a permutation group $G \leq \mathcal{S}_n$ is reducible because it contains a 1-dimensional submodule $W = \mathrm{Span}\{1 + \cdots + n\}$.

When a $G$-module $V$ contains a submodule $W$, its compliment $U$, defined by $V = W \oplus U$, is necessarily another $G$-module. An important reducibility result in representation theory is due to Masche. It states that every $G$-module $V$ can be decomposed into irreducible modules. This is known as *complete reducibility* of representation of finite groups.

**Theorem 2.1** (Masche)**.** *Let $V^{(1)}, \ldots, V^{(k)}$ be a complete list of pairwise inequivalent irreducible $G$-modules. Then any $G$-modules $V$ can be decomposed into*

$$V \cong \bigoplus_{i=1}^{k} m_i V^{(i)},$$

*for some non-negative integers $m_1, \ldots, m_k$.*

The multiplicity coefficients $m_i$'s are related to an algebra associated with representations. Note the kernel and the image of a $G$-homomorphism are also $G$-modules. In particular, *Schur's Lemma* states that non-zero $G$-homomorphisms between irreducible $G$-modules must be $G$-isomorphisms.

Using the notation from Masche's Theorem, denote $\operatorname{Hom}(V^{(i)}, V)$ the space of $G$-homomorphisms from $V^{(i)}$ to $V$. Write Masche's Theorem in terms of precise decompositions

$$V = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{i} \pi_{i,j} V^{(i)}$$

where $\pi_{i,j}$'s are linearly independent $G$-homomorphisms from $V$ to isomorphic copies of $V^{(i)}$ in $V$. We can show $m_i = \dim \operatorname{Hom}(V^{(i)}, V)$ by thinking of them as projections. Hence one is led to study the matrix version of $\operatorname{Hom}(V, V)$ because it is easier. If $X$ is a matrix representation of $G$, then its *commutant algebra* is

$$\operatorname{Com}(X) = \{T \in \operatorname{Mat}(d) : TX(g) = X(g)T, g \in G\} \cong \operatorname{Hom}(V, V).$$

It is fairly straightforward to show that the only matricies commuting with $X(g)$ for all $g \in G$ are scalar multiples of the identity matrix. With some work, we can get the following decomposition

$$\operatorname{Com}(X) = \left\{ \bigoplus_{i=1}^{k} (M_{m_i} \otimes I_{d_i}) : M_{m_i} \in \operatorname{Mat}(m_i), i = 1, \ldots, k \right\} \quad \text{and} \quad \dim Z(\operatorname{Com}(X)) = k,$$

where $Z(\operatorname{Com}(X))$ is the centre of the commutant algebra.

A group algebra and its commutant algebra are isomorphic. To see this, we can check that the right multiplication maps $\varphi_g(h) = hg$ are all the elements of $\operatorname{Hom}(\mathbb{C}[G], \mathbb{C}[G])$. It follows that the number of inequivalent irreducible representation of $\mathbb{C}[G]$ is $k = \dim Z(\mathbb{C}[G])$. A central element $\zeta \in Z(\mathbb{C}[G])$ satisfies $v = \zeta v \zeta^{-1}$ for all $v \in \mathbb{C}[G]$. By considering conjugation action on basis elements, we immediately see that if $\sigma, \tau \in G$ are in the same conjugacy class, then their coefficients in $v$ must be the same. In other words, the centre $Z(\mathbb{C}[G])$ is spanned by conjugacy classes (as 1-dimensional sums of their elements).

Centres of group algebras and a particular basis which turn out to be projections onto irreducible representations of $\mathbb{S}_n$ and will play a central role in this thesis. For now, we continue with general theory.

Early representation theory was developed using only a simple statistics called group characters. This turns out to be an extremely useful tool. Not only will it help us to find decompositions of group algebras but it also provide connections to other areas of mathematics.

Let $X$ be a matrix representation of a group $G$. The *character* of a representing matrix $X(g)$ is

$$\chi(g) = \operatorname{tr} X(g).$$

If $X$ and $Y$ are equivalent representations, then there exists an invertible matrix $T$ such that $TX(g)T^{-1} = Y(g)$. Since similar matrices have the same trace, we have

$$\operatorname{tr} X(g) = \operatorname{tr} TX(g)T^{-1} = \operatorname{tr} Y(g).$$

In other words, characters are invariant for representations.

Broadly speaking, invariants are useful objects in mathematics. This is also the case for characters. An elementary consequence of conjugation points us in the right direction: If $g, g' \in G$ live in the same conjugacy class, then for some $h \in G$ we have $\chi(g) = \chi(hgh^{-1}) = \chi(g')$.

So characters live in the space $\mathcal{K}(G)$ of class functions — the algebra of functions $f : G \to \mathbb{C}$ such that $f(g) = f(h)$ if $g, h$ belong to the same conjugacy class. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be a complete list of conjugacy classes of $G$. A natural basis for $\mathcal{K}(G)$ is the set of indicator functions $\delta_1, \ldots, \delta_k$ where $\delta_i(g) = 1$ if $g \in \mathcal{C}_i$ otherwise $\delta_i(g) = 0$. Addition on $\mathcal{K}(G)$ is performed pointwise and multiplication is performed as convolution: If $\varphi, \psi \in \mathcal{K}(G)$, then

$$(\varphi\psi)(g) = \sum_{h \in G} \varphi(g)\psi(gh^{-1}).$$

The space $\mathcal{K}(G)$ has a nice inner product. Define

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\psi(g^{-1}),$$

for all $\varphi, \psi \in \mathcal{K}(G)$. The indicator functions are pairwise orthogonal, i.e., $\langle \delta_i, \delta_j \rangle = \delta_{ij}|\mathcal{C}_i|/|G|$. A pleasant surprise is that the irreducible characters are orthonormal.

**Theorem 2.2.** *Let $\chi^{(1)}, \ldots, \chi^{(k)}$ be a complete list of irreducible characters of a representation of $G$. Then*

$$\langle \chi^{(i)}, \chi^{(j)} \rangle = \delta_{ij}.$$

This theorem has an important consequence. Let $V$ be $G$-module with character $\chi$ and decomposition into pairwise inequivalent irreducible representations $V = m_1 V^{(1)} \oplus \cdots \oplus m_k V^{(k)}$. For each $i = 1, \ldots, k$, let $\chi^{(i)}$ be the character of the irreducible representation $V^{(i)}$. Then the above theorem implies $\langle \chi, \chi^{(i)} \rangle = m_i$ for each $i = 1, \ldots, k$. If $W$ is another $G$-module with the same character $\chi$, then it follows immediately that $V = W$ by simply comparing multiplicity of irreducible representations. Hence, characters are not just invariants of representations. They in fact completely determine representations.

Now we can decompose group algebra $\mathbb{C}[G]$. We need to answer 2 questions:

1. How many irreducible representations are there?

2. What are their multiplicities?

We deal with the second question first. Let $\chi$ be the character of $\mathbb{C}[G]$ with representing matrix $X$. Note $X(g)$ is a $\{0,1\}$-matrix with exactly one 1 in each row and exactly one 1 in each column. Let $g \in G$. If $X(g)$ has a 1 on its diagonal, then there exists some $h \in G$ such that $gh = h$. But that implies $g = id$. Hence, $\chi(g) = 0$ unless $g = id$. Then

$$\langle \chi, \chi^{(i)} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi^{(i)}(g^{-1}) = \chi^{(i)}(id).$$

Hence each irreducible appears in $\mathbb{C}[G]$ with multiplicity being its dimension.

To answer the first question, we use the fact that the number of inequivalent irreducible representations in $\mathbb{C}[G]$ is the dimension of its centre $Z(\mathbb{C}[G])$. Let $K_1, \ldots, K_k$ be the conjugacy classes of $G$. A routine computation shows $Z(\mathbb{C}[G])$ is spanned by conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_k$ where

$$\mathcal{C}_i = \sum_{g \in K_i} g.$$

Orthonormality of irreducible characters implies they are linearly independent in $\mathcal{K}(G)$. Together with their correspondence with conjugacy class, we get that they in fact form a basis of $\mathcal{K}(G)$. Furthermore, by identifying conjugacy the class basis of $Z(\mathbb{C}[G])$ with the indicator function basis of $\mathcal{K}(G)$, we see that they are isomorphic as algebras.

We now turn our attention to the case $G = \mathcal{S}_n$.

**Corollary 2.3.** *There is exactly one irreducible representation of $\mathcal{S}_n$ for each conjugacy class of $\mathcal{S}_n$. We have*

$$\mathbb{C}[\mathcal{S}_n] = \bigoplus_{\lambda \vdash n} (\dim V^\lambda) V^\lambda,$$

*where $V^\lambda$ is the irreducible representation corresponding to the conjugacy class $\mathcal{C}_\lambda$.*

The construction of irreducible representations of $\mathcal{S}_n$ will be treated in the next section. We conclude the general theory by introducing a pair of important tools.

Let $H \leq G$ be subgroups. Let $X$ be a matrix representation of $G$ with character $\chi$. The *restriction* of $X$ to $H$ is simply $X\downarrow_H^G = \{X(h) : h \in H\}$. We use similar notation $\chi\downarrow_H^G$ for its character. Now suppose $Y$ is a degree $d$ matrix representation of $H$ with character $\psi$. The *induction* of $X$ to $G$ is the block matrix

$$Y\uparrow_G^H(g) = \left[ Y(g_i^{-1} g g_j) \right]_{i,j=1,\ldots,k}$$

where $g_1 H, \ldots, g_k H$ is a complete list of distinct $H$-cosets and $Y(x) = [0]_{d \times d}$ if $x \notin H$. The notation for its character is $\psi\uparrow_H^G$. An elementary property of an induced representation is that the choice of coset representatives is irrelevant. The following property provides a hint to finding irreducible representations of $\mathcal{S}_n$.

**Proposition 2.4.** *Let $H \leq G$ be groups with a complete list of distinct $H$-cosets $\mathcal{H} = \{g_1 H, \ldots, g_k H\}$. Let $\mathbb{C}\mathcal{H}$ be the representation extended from $G$ acting on $\mathcal{H}$ by (left) multiplication. Then*

$$\mathbb{C}\mathcal{H} = 1\!\uparrow_H^G.$$

### 2.2.2 Specht Modules

We now construct irreducible representations of $\mathcal{S}_n$ known as *Specht* modules, denoted by $S^\lambda, \lambda \vdash n$. Masche's theorem tells us that every representation is the direct sum of some irreducible representations. The clever trick is to find a total ordering $\lambda^{(1)} < \cdots < \lambda^{(k)}$ of partitions of $n$ and construct a sequence of representations $M^{\lambda^{(1)}}, \cdots, M^{\lambda^{(k)}}$ such that $S^{\lambda^{(1)}} = M^{\lambda^{(1)}}$ is irreducible and $M^{\lambda^{(i)}}$ contains only copies of $S^{\lambda^{(1)}}, \ldots, S^{\lambda^{(i)}}$ with multiplicity of $S^{\lambda^{(i)}}$ being 1.

Let $\lambda = (\lambda_1, \ldots, \lambda_i)$ and $\mu = (\mu_1, \ldots, \mu_j)$ be partitions of $n$. In *lexicographic ordering*, $\lambda < \mu$ if there exists some $m \geq 1$ such that $\lambda_m < \mu_m$ but $\lambda_k = \mu_k$ for $k = 1, \ldots, m-1$. It is easily verified that this is a total order. The order we will use is the *reverse lexicographic order*. For example, partitions of 5 in reverse lexicographic order are

$$(5) < (41) < (32) < (311) < (221) < (2111) < (1^5).$$

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ be a partition. A *tableau* $t$ of shape $\lambda$ is Ferrers diagram of $\lambda$ labelled with $1, \ldots, n$, one for each box. We denote the set of tableaux of shape $\lambda$ by $\mathrm{Tab}(\lambda)$. The symmetric group $\mathcal{S}_n$ acts on $t$ by permuting its labels. Let $\bar{t}_i$ be the set of labels in row $i$ of $t$. The *row stabilizer* subgroup for $t$ is defined as $R_t = \mathcal{S}_{\bar{t}_1} \times \cdots \times \mathcal{S}_{\bar{t}_k}$. Since row labels are are pairwise disjoint, we can write $\rho = \rho_1 \cdots \rho_k$ where we view $\rho_i$ as an element of $\mathcal{S}_n$ by adding enough fixed points. The *column stabilizer* subgroup of $t$ is $C_t = R_{t'}$ where $t'$ is the conjugate (with labels) of $t$. Similarly, we can write $\pi = (\pi_1, \ldots, \pi_{\lambda_1}) \in C_t$ as $\pi = \pi_1 \cdots \pi_{\lambda_1}$ by viewing each $\pi_i \in \mathcal{S}_n$. The *tabloid* associated to $t$ is

$$\bar{t} = \sum_{\rho \in R_t} \rho t.$$

Permutation action on tabloids is defined as $\sigma \bar{t} = \overline{\sigma t}$ for all $\sigma \in \mathcal{S}_n$. Define $\kappa_t = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\pi$. The *polytabloid* associated to $t$ is defined as

$$e_t = \kappa_t \bar{t} = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\pi\bar{t}.$$

It is straightforward to check that these objects are well-defined and $\sigma e_t = e_{\sigma t}$ for all $\sigma \in \mathcal{S}_n$.

For example, let $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ be a tableau. Its stabilizer subgroups are $R_t = id + (12)$ and $C_t = id + (13)$. Then $\kappa_t = id - (13)$. The tabloid and the polytabloid associated to $t$ are

$$\bar{t} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad e_t = \left( \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \right) - \left( \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \right). \tag{2.1}$$

Let $\lambda \vdash n$. Let $S^\lambda$ denote the vector space spanned by $\{e_t : t \in \text{Tab}(\lambda)\}$. This is naturally a subspace of $M^\lambda = \text{Span}_\mathbb{C}\{\bar{t} : t \in \text{Tab}(\lambda)\}$. The $\mathcal{S}_n$ action on tabloids implies $S^\lambda$ and $M^\lambda$ are both $\mathcal{S}_n$-modules. The $S^\lambda, \lambda \vdash n$ are called the *Specht* modules of $\mathcal{S}_n$.

The submodule theorem is used to show that $S^\lambda$ are irreducible. Proofs for the following results can be found in [27].

**Theorem 2.5.** *If $U$ is a submodule of $M^\lambda$, then either $S^\lambda \subseteq U$ or $U \subseteq (S^\lambda)^\perp$.*

Let $\text{STY}(\lambda)$ denote the set of standard Young tableaux of shape $\lambda$. It turns out that $e_T, T \in \text{STY}(\lambda)$ are linearly independent. One can then use a straightening algorithm involving Garnir elements to show that every polytabloid can be written as a linearly combinations of these basis.

**Theorem 2.6.** *A basis of the Specht module $S^\lambda$ is $\{e_T : T \in \text{STY}(\lambda)\}$.*

In group algebra $\mathcal{S}_n$, there exists central elements $\mathcal{F}^\lambda = \chi^\lambda / H_\lambda$ one for each partitions $\lambda \vdash n$ such that $\mathcal{F}^\lambda \mathcal{F}^\mu = \delta_{\lambda\mu}\mathcal{F}^\lambda$ for all $\mu \vdash n$. These are projection operators from $\mathbb{C}[\mathcal{S}_n]$ to $S^\lambda$ and they can be used to construct Specht modules algebraically. Since they are clearly linearly independent, they are also a basis of $\mathcal{Z}_n$. The following basis change formula is most useful in this thesis.

**Lemma 2.7.** *The central elements $\mathcal{F}^\lambda = \frac{\chi^\lambda}{H_\lambda}, \lambda \vdash n$ are orthogonal idempotents. Furthermore, we have*

$$\mathcal{F}^\lambda = \frac{\chi^\lambda_{(1^n)}}{n!} \sum_{\alpha \vdash n} \chi^\lambda_\alpha \mathcal{C}_\alpha \quad \textit{and} \quad \mathcal{C}_\alpha = |\mathcal{C}_\alpha| \sum_{\lambda \vdash n} \frac{\chi^\lambda_\alpha}{\chi^\lambda_{(1^n)}} \mathcal{F}^\lambda.$$

The $\mathcal{F}^\lambda$ are in fact minimum central projections onto irreducible representations contained in $\mathbb{C}[\mathcal{S}_n]$. See [8, Section 1.2] for details. By group algebra decomposition, we immediately have

$$\sum_{\lambda \vdash n} \mathcal{F}^\lambda = 1.$$

## 2.3 Jucys-Murphy Elements

In group algebra $\mathbb{C}[\mathcal{S}_n]$, *Jucys-Murphy* elements are the sums of transpositions

$$J_k = (1, k) + (2, k) + \cdots + (k-1, k), \quad k = 2, \ldots, n.$$

For cosmetic reasons, we define $J_1 = 0$. They were first studied by Jucys [20] and later independently by Murphy [26]. We list several properties that are useful in this thesis. For detailed descriptions of these results and related theorems, especially a construction of irreducible representations of the symmetric groups starting with Jucys-Murphy elements, please see [8, 30].

First, note $J_2, \ldots, J_n$ pairwise commute. They generate the symmetric group in the following sense.

**Theorem 2.8.** *Let $t$ be an indeterminate. Then*

$$\prod_{k=1}^{n}(t+J_k) = \sum_{\sigma\in\mathcal{S}_n} \sigma t^{\ell(\mathrm{cyc}(\sigma))}.$$

Note that the left-hand side of the above lemma is the generating series $E(t)$ for elementary symmetric polynomials. By comparing coefficients of $t$, we get the following property.

**Corollary 2.9.** *Let $e_k(x_1,\ldots,x_n)$ denote the elementary symmetric polynomial of degree $k$. Then*

$$e_k(J_1,\ldots,J_n) = \sum_{\substack{\alpha\vdash n \\ \ell(\alpha)=n-k}} \mathcal{C}_\alpha.$$

Consider $\mathcal{S}_3$ for example. We have

$$\begin{aligned}
e_0(J_1,J_2,J_3) &= 1 = \mathcal{C}_{(1^3)} \\
e_1(J_1,J_2,J_3) &= J_1 + J_2 + J_3 \\
&= (12) + (13) + (23) = \mathcal{C}_{(21)} \\
e_2(J_1,J_2,J_3) &= J_1J_2 + J_1J_3 + J_2J_3 \\
&= (12)(13) + (12)(23) = \mathcal{C}_{(3)} \\
e_3(J_1,J_2,J_3) &= J_1J_2J_3 = 0.
\end{aligned}$$

**Theorem 2.10.** *The centre $\mathcal{Z}_n$ is generated by symmetric polynomials in $\mathrm{Sym}_n$ evaluated at Jucys-Murphy elements.*

Jucys-Murphy elements are simultaneous eigenoperators for Young's basis.

**Theorem 2.11.** *Let $\lambda \vdash n$. Then for $k = 1,\ldots,n$ and for each standard Young tableau $T$ of shape $\lambda$ we have*

$$J_k v_T = c(T^{-1}(k))v_T,$$

*where $T^{-1}(k)$ is the cell in tableau $T$ with label $k$ and $c(T^{-1}(k))$ is its content in $\lambda$.*

To state the next theorem, we introduce a notation. For a partition $\lambda \vdash n$, we denote the set of contents by $\mathbf{c}_\lambda = \{c(\square) : \square \in \lambda\}$. Symmetric polynomials evaluated at contents of a partition is denoted as $f(\mathbf{c}_\lambda) = f(c(\square) : \square \in \lambda)$. If $\mu \vdash k$ with $k < n$, then we pad $\mathbf{c}_\mu$ with enough 0's so that $f(\mathbf{c}_\mu)$ is well-defined.

**Theorem 2.12.** *Let $f \in \mathrm{Sym}_n$ be a symmetric polynomial. Then for each $\lambda \vdash n$ we have*

$$f(J_1,\ldots,J_n)\chi^\lambda = f(\mathbf{c}_\lambda)\chi^\lambda.$$

## 2.4 Symmetric Functions

Our algebraic tools of choice are symmetric function due to their useful correspondence to the centre of the group algebras. In this section, we recall fundamental facts of the algebra Sym.

### 2.4.1 Fundamental Bases

Let $x_1, \ldots, x_n$ be algebraically independent indeterminates. A permutation $\sigma \in \mathcal{S}_n$ acts on a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ by permuting its variables, i.e., $(\sigma f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. A polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is *symmetric* if $\sigma f = f$ for all $\sigma \in \mathcal{S}_n$. Let $\mathrm{Sym}_n \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ denote the set of symmetric polynomials in $x_1, \ldots, x_n$. It is straightforward to check that $\mathrm{Sym}_n$ is closed under addition and multiplication. So $\mathrm{Sym}_n$ is a commutative ring with unity.

A *monomial symmetric polynomial* $m_\lambda(x_1, \ldots, x_n)$ indexed by a partition $\lambda \vdash d$ is a homogeneous symmetric polynomial of degree $d$ obtained by the minimum symmetrization under $\mathcal{S}_n$ of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ where we set $\lambda_i = 0$ if $i > \ell(\lambda)$. Note $m_\lambda(x_1, \ldots, x_n), \lambda \vdash d$ are linearly independent. Hence, the vector space over $\mathbb{Z}$ spanned by $\{m_\lambda(x_1, \ldots, x_n) : \lambda \vdash d\}$ is the set of homogeneous symmetric polynomials having degree $d$, denoted by $\mathrm{Sym}_n^d$. We set $\mathrm{Sym}_n^0 = \mathbb{Z}$. So $\mathrm{Sym}_n$ is a vector space with a compatible multiplication, i.e., an algebra. Furthermore, if $f \in \mathrm{Sym}_n^d$ and $g \in \mathrm{Sym}_n^{d'}$, then $fg \in \mathrm{Sym}_n^{d+d'}$. This means $\mathrm{Sym}_n$ is a graded algebra (by degree) and we write

$$\mathrm{Sym}_n = \bigoplus_{d \geq 0} \mathrm{Sym}_n^d.$$

It is advantageous to think in countably many variables. A permutation $\sigma \in \mathcal{S}_n$ acts on a formal power series in $x_1, x_2, \ldots$ by permuting the first $n$ variables. A *symmetric function* is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ such that $\sigma f = f$ for any permutation $\sigma \in \mathcal{S}_n$ and any $n \geq 1$. The argument list of a symmetric function is preferred for a cleaner presentation. If $f \in \mathrm{Sym}$, then $f(x_1, \ldots, x_n, 0, 0, \ldots) \in \mathrm{Sym}_n$.

A *monomial symmetric function* $m_\lambda$ with $\lambda \vdash d$ is a homogeneous symmetric function of degree $d$ obtained by the minimum symmetrization of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ under all symmetric groups where we set $\lambda_i = 0$ if $i > \ell(\lambda)$. Similar to that of the polynomial case, $\{m_\lambda : \lambda \vdash d\}$ is linearly independent over $\mathbb{Z}$ and forms a basis for the vector space of homogeneous symmetric functions of degree $d$, denoted as $\mathrm{Sym}^d$. The algebra Sym is also graded by degree, i.e.,

$$\mathrm{Sym} = \bigoplus_{d \geq 0} \mathrm{Sym}^d.$$

An *elementary symmetric functions* is $e_k = m_{(1^k)}$ where $(1^k)$ is the partition having $k$ number of 1's. Define $e_0 = 1$. For example,

$$e_2 = x_1x_2 + x_1x_3 + \cdots + x_2x_3 + x_2x_4 + \cdots + x_3x_4 + x_3x_5 + \cdots .$$

A monomial in $e_k$ contains $k$ distinct variables each having exponent 1. So its generating series $E(t)$ marked by degree has a simple form

$$E(t) = \sum_{k\geq 0} e_k t^k = \prod_{i\geq 1}(1 + x_i t).$$

The *Fundamental Theorem of Symmetric Functions* says each symmetric function can be written uniquely as a polynomial in elementary symmetric functions. In other words, we have an algebra isomorphism

$$\mathrm{Sym} \cong \mathbb{Z}[e_1, e_2, \ldots].$$

Then the vector space $\mathrm{Sym}^d$ has another basis $\{e_\lambda : \lambda \vdash d\}$ where $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$.

Another intimately related family is the family of *complete symmetric functions* defined by

$$h_k = \sum_{\lambda\vdash k} m_\lambda, \quad k \geq 1.$$

Define $h_0 = 1$. Each monomial of degree $k$ appears in $h_k$ exactly once. Hence, its generating series marked by degree also have a nice form

$$H(t) = \sum_{k\geq 0} h_k t^k = \prod_{i\geq 1} \frac{1}{1 - x_i t}.$$

The two families are related through an algebra homomorphism $\omega : \mathrm{Sym} \to \mathrm{Sym}$ defined by $\omega : e_k \mapsto h_k$ for all $k \geq 1$ and extended to all of Sym. The relation $E(t)H(-t) = 1$ together with the fact $H(t)^{-1}$ exists means $\omega$ is an involution. So $\omega$ is an algebra isomorphism, known as the *fundamental involution*. It follows that the complete symmetric functions are also algebraically independent so we have another isomorphism $\mathrm{Sym} \cong \mathbb{Z}[h_1, h_2, \ldots]$. The vector space $\mathrm{Sym}^d$ has a third basis $\{h_\lambda : \lambda \vdash d\}$ where $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$.

A *power sum symmetric function*, or simply just *power sum*, is $p_k = m_{(k)}$ where $(k)$ is an one-part partition. They have simple expressions, e.g., $p_k = x_1^k + x_2^k + \cdots$. On the generating series level,

$$\log H(t) = \sum_{i\geq 1}\sum_{k\geq 1} x_i^k \frac{t^k}{k} = \sum_{k\geq 1} p_k \frac{t^k}{k}.$$

We take this as the definition for its generating series $P(t)$. This relation also produces a pair of recurrence relations

$$h_k = \frac{1}{k}\sum_{i=1}^{k} h_{k-i} p_i \quad \text{and} \quad p_k = k h_k - \sum_{i=1}^{k-1} h_{k-i} p_i, \quad k \geq 1.$$

This triangularity implies the power sums are algebraically independent over $\mathbb{Q}$. Hence, $\{p_\alpha : \alpha \vdash k\}$ forms a vector space basis for $\mathrm{Sym}_{\mathbb{Q}}^d$ consisting of homogeneous symmetric functions having degree $d$ with coefficients in $\mathbb{Q}$. We also have an isomorphism $\mathrm{Sym}_{\mathbb{Q}} = \oplus_{d\geq 1}\mathrm{Sym}_{\mathbb{Q}}^d \cong \mathbb{Q}[p_1, p_2, \ldots]$.

Now we introduce a more interesting family of symmetric function which provides many connections into the world of combinatorics. The *Schur polynomial* $s_\lambda(x_1, \ldots, x_n)$ with $\lambda \vdash k$ is defined as

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1,\ldots,n}}{\det\left(x_i^{n-j}\right)_{i,j=1,\ldots,n}}.$$

This is a homogeneous symmetric polynomial of degree $k$. Schur polynomials $\{s_\lambda : \lambda \vdash d\}$ forms a vector space basis for $\mathrm{Sym}_n^d$. The proof of this fact can be found in standard symmetric functions textbooks, for example [27]. Another standard fact is the *Jacobi-Trudi* identity which says

$$s_\lambda(x_1, \ldots, x_n) = \det\left(h_{\lambda_j-j+i}(x_1, \ldots, x_n)\right)_{i,j=1,\ldots,n}.$$

The $h$'s can be interpreted as generating series for certain families of lattice paths. One can apply Gessel-Viennot methodology to show that $s_\lambda(x_1, \ldots, x_n)$ is the generating series for all non-crossing families. The non-crossing families are in one-to-one correspondence with semi-standard Young tableaux.

Recall a tableau is a filling of some Ferrers diagram. A *semi-standard Young tableau* of shape $\lambda$ is a filling with symbols $1, \ldots, n$ of its Ferrers diagram where symbols are allowed to repeat with the condition that labels must increase strictly down the columns but increase only weakly along the rows. The weight of a tableau $T$ is the vector $(m_1(T), m_2(T), \ldots, m_n(T))$ where $m_i(T)$ is the number of times the symbol $i$ appears in $T$. The combinatorial definition of Schur polynomial is

$$s_\lambda(x_1, \ldots, x_n) = \sum_T x_1^{m_1(T)} \cdots x_n^{m_n(T)},$$

ranging over all semi-standard Young tableaux of shape $\lambda$. We extend our vocabulary to say $x_1^{m_1(T)} \cdots x_n^{m_n(T)}$ is the weight of $T$.

The combinatorial definition extends easily to countable many variables. The *Schur function* indexed by a partition $\lambda$ is

$$s_\lambda = \sum_T x_1^{m_1(T)} x_2^{m_2(T)} \cdots ,$$

ranging over semi-standard Young tableaux filled with symbols $1, 2, \ldots$. Note Jacobi-Trudi identity continues to hold in this setting. Not surprisingly, the fourth $\mathbb{Z}$ basis of $\mathrm{Sym}^k$ is $\{s_\lambda : \lambda \vdash k\}$

$$s_{(21)} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} + \cdots$$

Figure 2.2: Tableaux Definition of $s_{(21)}$.

The tableau definition is a hint that Schur functions are closely related to the group algebras. Let $T$ be a tableau of shape $\lambda$ with weight $x_1 \cdots x_n$. This means $T$ contains exactly one of each symbol $1, \ldots, n$. This uniqueness forces row-weakness to become row-strictness. Hence $T$ is in fact a standard Young tableaux. So the coefficient of $x_1 \cdots x_n$ in $s_\lambda$ is the number of Young tableaux of shape $\lambda$. From Section 2.2.2, we know this is the $\mathcal{S}_n$ irreducible character $\chi^\lambda_{(1^n)}$.

The full detail will be explained in Section 2.5. For now, we continue with the structure of Sym.

### 2.4.2 Orthogonality and Adjoint Operators

The space Sym has a nice inner product. Define

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

for all partitions $\lambda, \mu$ and extend bi-linearly to all of Sym. We summarize its elementary properties of as follows. For all $f, g \in \mathrm{Sym}$, we have

(i) $\langle f, g \rangle = \langle g, f \rangle$, i.e., $\langle \cdot, \cdot \rangle$ is symmetric, and

(ii) $\langle f, g \rangle \geq 0$ and $\langle f, g \rangle = 0$ if and only if $f = g$, i.e., $\langle \cdot, \cdot \rangle$ is positive definite, and

(iii) $\langle \omega(f), \omega(g) \rangle = \langle f, g \rangle$, i.e., the fundamental involution $\omega$ is an isometry.

The power sums are orthogonal with $\langle p_\alpha, p_\beta \rangle = z(\alpha)\delta_{\alpha\beta}$ for all partitions $\alpha, \beta$. There is a general criterion for when two basis are orthogonal.

**Theorem 2.13** (Cauchy)**.** *If $\{u_\lambda\}$ and $\{v_\mu\}$ are two bases for* Sym*. Then $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$ for all partitions $\lambda, \mu$ if and only if*

$$\sum_\lambda u_\lambda(x_1, x_2, \ldots) v_\mu(y_1, y_2, \ldots) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}.$$

A pair of bases that satisfy Cauchy's formula are called *dual bases*. By definition, the complete symmetric functions are dual to the monomial ones. Another pair is $\{p_\alpha\}$ and $\{p_\alpha / z(\alpha)\}$. More interestingly, Schur functions are self-dual.

**Theorem 2.14.**

$$\sum_{\lambda} s_\lambda(x_1, x_2, \dots) s_\lambda(y_1, y_2, \dots) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}.$$

It turns out treating $f$ as an operator $f : g \mapsto fg$ for all $g \in \mathrm{Sym}$ is useful for our purpose. The operator $f^\perp$ *adjoint* to $f$ as a multiplication operator is defined as

$$\langle f^\perp g, g' \rangle = \langle g, f g' \rangle, \quad g, g' \in \mathrm{Sym}.$$

Adjoint operators are also called *skewing* operators. Let $\lambda, \mu$ be partitions such that $\lambda$ is contained in $\lambda$. A *skew shape* $\lambda/\mu$ is obtained by removing boxes belonging to $\lambda$ from the Ferrers diagram of $\mu$. A semi-standard *skew tableau* of shape $\mu/\lambda$ is a filling of the skew shape $\mu/\lambda$ with strictly increasing columns and weakly increasing rows. Here is an example. The *weight* of a skew tableaux $T$ is the vector $(m_1(T), m_2(T), \dots)$.

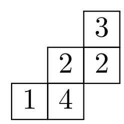



Figure 2.3: A shew tableau of shape $\mu/\lambda$ with $\mu = (3, 3, 2)$ and $\lambda = (2, 1)$ of weight $(1, 2, 1, 1)$

We can extend Schur functions to include skew shapes. A skew Schur function is defined as

$$s_{\mu/\lambda} = \sum_{T} \prod_{i \geq 1} x_i^{m_i(T)}.$$

where $T$ sums over all semi-standard skew tableaux of shape $\mu/\lambda$ and $m_i(T)$ is the number of $i$'s in $T$.

**Theorem 2.15** (Littlewood-Richardson)**.** *Let $\lambda, \mu, \nu$ be partitions. Then*

$$\langle s_\nu, s_\lambda s_\mu \rangle = \langle s_{\nu/\lambda}, s_\mu \rangle.$$

*Remark.* For completeness, we note that $\langle s_{\nu/\lambda}, s_\mu \rangle$ is called the Littlewood-Richardson coefficient of skew-shape $\nu/\lambda$ with type $\mu$. These coefficients are widely studied due to its importance in representation theory.

Now consider the adjoint operator $h_k^\perp$. Let $\lambda, \mu$ be partitions. The duality between $\{h_\lambda\}$ and $\{m_\mu\}$ implies $\langle h_\lambda, f \rangle = [m_\lambda] f$. Then

$$\langle h_k^\perp m_\mu, h_\lambda \rangle = \langle m_\mu, h_k h_\lambda \rangle = \langle m_\mu, h_{\lambda \cup (k)} \rangle = \delta_{\mu, \lambda \cup (k)}.$$

That means when $h_k^{\perp} m_\mu$ is expanded in the monomial basis, the only non-zero coefficient is indexed by a partition obtained by removing a part of size $k$ from $\mu$. Hence, $h_k^{\perp} m_\mu = m_{\mu\setminus(k)}$ where $m_{\mu\setminus(k)} = 0$ if $\mu$ does not contain a part of length $k$. It follows that for any $f \in \mathrm{Sym}_{n+1}$, we have

$$(h_k^{\perp} f)(x_1, \cdots, x_n, 0) = [x_{n+1}^k] f(x_1, \ldots, x_n, x_{n+1}).$$

This implies

$$f(x_1, \ldots, x_n, x_{n+1}) = f(x_1, \ldots, x_n, 0) + \sum_{k=1}^{n+1} x_{n+1}^k (h_k^{\perp} f)(x_1, \ldots, x_n, 0) \tag{2.2}$$

when $f$ is thought of as a formal power series in $x_{n+1}$. In the case $f$ belongs to a family with nice enough structure and the summation reduces to something of smaller order in the same family, we get recursive definitions. For example,

$$e_k(x_1, \ldots, x_n, x_{n+1}) = e_k(x_1, \ldots, x_n, 0) + x_{n+1} e_{k-1}(x_1, \ldots, x_n, 0).$$

Recursive families in Sym, Theorem 2.10, and results from the next section are the basic ingredients behind finding induction relations in the symmetric groups. Section 4.1 discusses this idea in more detail.

Lastly we briefly consider adjoints of power sums. Recall as multiplication operators acting on power sums they simply insert rows into the target, i.e., $p_k p_\alpha = p_{\alpha\cup(k)}$. Their adjoints "undo" the add-a-row operation. It is straightforward to verify that $p_k^{\perp} = i \frac{\partial}{\partial p_k}$ and

$$p_k^{\perp} p_\alpha = i m_i(\alpha) p_{\alpha\setminus(k)},$$

where $m_i(\alpha)$ is the number of $i$'s in $\alpha$. The $p_i$'s and $p_i^{\perp}$'s turn out to be a valuable tool. Add/remove-a-row operation will be used extensively in Section 5.5. The reason this operation is so useful is because of correspondence between $p_\alpha$'s and conjugacy classes in centre of group algebras which we now describe.

## 2.5 The Characteristic Map

The centre of group algebra $\mathcal{Z}_n$ is intimately connected to symmetric polynomials. We have already discussed the connection through Jucys-Murphy elements in Theorem 2.10. Now we describe a natural correspondence through characters.

Let $d \geq 1$. Define a map between vector spaces $\mathrm{ch}^d : \mathcal{Z}_d \to \mathrm{Sym}^d$

$$\mathrm{ch}^d : \chi \mapsto \sum_{\alpha \vdash d} \chi_\alpha \frac{p_\alpha}{z_\alpha},$$

where $\chi$ is a (not necessarily irreducible) character of $\mathcal{S}_d$. A fundamental result is that $\mathrm{ch}^d$ is an inner product preserving isomorphism.

**Theorem 2.16.** *If $\lambda \vdash d$ and $\chi^\lambda$ is an irreducible character of $\mathcal{S}_d$, then*

$$\mathrm{ch}^d(\chi^\lambda) = s_\lambda.$$

A proof can be found in [27].

Let $\mathcal{Z} = \oplus_{d\geq 1}\mathcal{Z}_d$ and define $\mathrm{ch} = \oplus_{d\geq 1}\mathrm{ch}^d$. The above result can be extended to an algebra isomorphism $\mathrm{ch} : \mathcal{Z} \to \mathrm{Sym}$. Furthermore, we can take advantage of the basis change formula in $\mathcal{Z}_d$ and $\mathcal{F}^\lambda = \chi^\lambda / H_\lambda$ to get

$$\mathrm{ch}^d(\mathcal{C}_\alpha) = \frac{p_\alpha}{z_\alpha}, \quad \alpha \vdash d.$$

We can formulate factorization problems using the characteristic map. We say an element $\zeta \in \mathcal{Z}_n$ is *set-like* if it can be written as

$$\zeta = \sum_{\alpha\vdash n} \varphi_\alpha^\zeta \mathcal{C}_\alpha,$$

where $\varphi_\alpha^\zeta \in \{0,1\}$. Clearly, if $\zeta \in \mathcal{Z}_n$ is set-like then we can identify it with a set $S_\zeta(n) = \cup\{\mathcal{C}_\alpha : \alpha \vdash n, \varphi_\alpha^\zeta = 1\}$. Let $\zeta_1, \ldots, \zeta_m \in \mathcal{Z}_n$ be set-like elements and let $\zeta = \zeta_1 \cdots \zeta_m$. Then $\varphi_\alpha^\zeta = [\mathcal{C}_\alpha]\zeta$ counts the number of $m$-factorizations of a permutation in $\mathcal{C}_\alpha$ with restrictions $S_{\zeta_1}, \ldots, S_{\zeta_m}$. Then we have a generating series

$$\Phi_n^\zeta(\mathbf{p}) = \mathrm{ch}\,\zeta = \frac{1}{n!}\sum_{\alpha\vdash n} \varphi_\alpha^\zeta |\mathcal{C}_\alpha| p_\alpha.$$

This tool is particularly useful when restriction sets can be described simultaneously for all symmetric groups using Theorem 2.10. We say a symmetric function $f \in \mathrm{Sym}$ is *set-like* if for each $n \geq 0$ its corresponding central element $f(J_1, \ldots, J_n)$ is set-like. Let $f_1, \ldots, f_m \in \mathrm{Sym}$ be set-like and let $f = f_1 \cdots f_m$. For each $n \geq 0$ and $\alpha \vdash n$

$$\varphi_\alpha^f = [\mathcal{C}_\alpha] f(J_1, \ldots, J_n) = [\mathcal{C}_\alpha] f_1(J_1, \ldots, J_n) \cdots f_m(J_1, \ldots, J_n)$$

counts the number of $m$-factorizations of a permutation in $\mathcal{C}_\alpha$ with restrictions described by $f_1, \ldots, f_m$ as subsets of $\mathcal{S}_n$. Let

$$\Phi_n^f(\mathbf{p}) = \mathrm{ch}\, f(J_1, \ldots, J_n) = \frac{1}{n!}\sum_{\alpha\vdash n} \varphi_\alpha^f |\mathcal{C}_\alpha| p_\alpha.$$

Sum over all $n \geq 0$ and substitute $zp_i$ for $p_i$. Then

$$\Phi^f(z, \mathbf{p}) = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\alpha\vdash n} \varphi_\alpha^f |\mathcal{C}_\alpha| p_\alpha = \sum_{n\geq 0} z^n \Phi_n^f(\mathbf{p}). \tag{2.3}$$

is an exponential generating series in $z$ for the $m$-factorizations problem with restrictions described by $f_1, \ldots, f_m$. Furthermore, $p_\alpha$ marks a product whose cycle type is $\alpha$. By standard theory of generating series [17], the logarithm of an exponential generating series is an exponential generating series for its connected objects. Recall in Section 1.2, connected factorizations are transitive factorizations. Hence,

$$\Psi^f(z, \mathbf{p}) = \log \Phi^f(z, \mathbf{p})$$

is the generating series for transitive $m$-factorizations with restriction described by $f_1, \ldots, f_m$.

We now use this method to recover a generating series first proved by Goulden and Jackson [15].

**Example 2.17.** Let

$$E = E(x_1, \ldots, x_n; t)\big|_{t=1} = \prod_{i\geq 1}(1 + x_i).$$

By Theorem 2.8, we have $E(J_1, \ldots, J_n) = \mathcal{S}_n$. Then $\Psi^{E^m}(z, \mathbf{p})$ is the generating series for transitive $m$-factorizations with no restrictions. These factorizations are also known as *$m$-hypermaps*.

Recall $\sum_{\lambda\vdash n} \mathcal{F}^\lambda = 1$. By Theorem 2.12, we have

$$\begin{aligned}
\Phi^f(z, \mathbf{p}) &= \sum_{n\geq 0} z^n \operatorname{ch} E^m(J_1, \ldots, J_n) \sum_{\lambda\vdash n} \mathcal{F}^n \\
&= \sum_{n\geq 0} z^n \sum_{\lambda\vdash n} \operatorname{ch} E^m(\mathbf{c}_\lambda) \frac{\chi^\lambda}{H_\lambda} \\
&= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \frac{n!}{H_\lambda} E^m(\mathbf{c}_\lambda) \operatorname{ch} \chi^\lambda \\
&= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \chi^\lambda_{(1^n)} E^m(\mathbf{c}_\lambda) s_\lambda.
\end{aligned}$$

But $E^m(\mathbf{c}_\lambda) = \prod_{\Box\in\lambda}(1 + c(\Box))$. Hence, the $m$-hypermap generating series can be written as

$$\Psi^{E^m}(z, \mathbf{p}) = \log\left(\sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \chi^\lambda_{(1^n)} s_\lambda \prod_{\Box\in\lambda}(1 + c(\Box))\right). \tag{2.4}$$

Chapter 6 is devoted to study generating series of this form in more detail.

Character map allows one to define another multiplication on Sym. Let $f, g \in \mathrm{Sym}^d$, define

$$f \times g = \operatorname{ch}(ch^{-1}(f) \operatorname{ch}^{-1}(g)),$$

where the multiplication $\operatorname{ch}^{-1}(f) \operatorname{ch}^{-1}(g)$ is carried out in $\mathcal{Z}^d$. For example

$$s_\lambda \times s_\mu = \operatorname{ch}(\chi^\lambda \chi^\mu) = \frac{1}{\chi^\lambda_{(1^n)}} s_\lambda.$$

We extend this to all of Sym by setting $F \times G = 0$ if both are homogeneous but $\deg F \neq \deg G$. This definition was first written down by Lascoux and Thibon [23].

This is a useful tool for describing multiplication behaviours in group algebra as operators in symmetric functions. If we think of $\omega \in \mathbb{C}[\mathcal{S}_n]$ as a multiplication operator, then its corresponding element $\operatorname{ch}\omega$ as a multiplication operator on symmetric functions faithfully describes the action of $\omega$ in group algebra.

The interesting case is the existence of operators in symmetric functions that describes the behaviours of multiplication in group algebra simultaneously in all symmetric groups. The first such operator is the join-cut operator $\Delta$ describing the behaviour of multiplication by transpositions. It will be described in Section 5.1. We then use $\Delta$ to derive a family of operators that describe a lifting behaviour in the symmetric groups.

# Chapter 3

# A Family of Differential Operators

Characters arise in a natural way in the pursue of counting problems introduced in Section 1.2. Consider multiplication of conjugacy classes from central orthogonal idempotent point of view using Lemma 2.7. Multiplication of conjugacy classes turns into convolution of central orthogonal idempotents. This is rather easy to write down since multiplication of idempotents are trivial. If $\alpha, \beta \vdash n$, then

$$\mathcal{C}_\alpha \mathcal{C}_\beta = |\mathcal{C}_\alpha||\mathcal{C}_\beta| \sum_{\lambda \vdash n} \frac{\chi^\lambda_\alpha \chi^\lambda_\beta}{\left(\chi^\lambda_{(1^n)}\right)^2} \mathcal{F}^\lambda.$$

Then after expanding $\mathcal{F}^\lambda$'s in terms of $\mathcal{C}_\gamma$'s we have

$$[\mathcal{C}_\gamma]\mathcal{C}_\alpha \mathcal{C}_\beta = \frac{|\mathcal{C}_\alpha||\mathcal{C}_\beta|}{n!} \sum_{\lambda \vdash n} \frac{\chi^\lambda_\alpha \chi^\lambda_\beta \chi^\lambda_\gamma}{\chi^\lambda_{(1^n)}}.$$

One is then hopeful that obtaining relations in characters would be a useful first step in getting recurrence relations for above coefficients. This is the approach taken by Lassalle.

Since characters correspond to Schur functions under the characteristic map, we turn to operations on symmetric functions to find such relations in characters. Jack symmetric functions enter the picture because they are generalizations of Schur functions and possesses a uniqueness property. Macdonald found a differential operator $\mathcal{D}(\zeta)$ on symmetric polynomials for which Jack functions are simultaneous eigenfunctions. Lassalle [24] turned its specialization on Schur functions into a family of operators $\mathcal{L}_k$ to muscle out linear relations in central characters.

The goal of this chapter is to describe this development.

## 3.1 Macdonald's Operator

We give a brief historical account of a family of operators to be introduced in Section 3.2.

In the interest of finding relations on characters, we turn to generalizations of Schur functions. Note Schur functions are characterized by (i) orthogonality and (ii) triangularity with respect to monomial symmetric functions. It turns out many useful symmetric functions also share this property. Let $\mathrm{Sym}\otimes\mathbb{Q}(\zeta)$ denote the ring of symmetric functions over the field of rational functions $\mathbb{Q}(\zeta)$. It is a well-known fact that this space has a nice orthogonal basis.

**Theorem 3.1.** *Define an inner product* $\langle p_\alpha, p_\beta\rangle_\zeta = \delta_{\alpha\beta} z_\alpha \zeta^{\ell(\alpha)}$ *on* $\mathrm{Sym}\otimes\mathbb{Q}(\zeta)$*. There are unique symmetric functions* $J_\lambda = J_\lambda(\mathbf{x};\zeta) \in \mathrm{Sym}\otimes\mathbb{Q}(\zeta)$ *indexed by partitions such that*

*(i)* $\langle J_\lambda, J_\mu\rangle_\zeta = 0$ *if* $\lambda \neq \mu$*, and*

*(ii)* $[m_\mu]J_\lambda = 0$ *if* $\mu > \lambda$ *in dominance order, and*

*(iii)* $[m_{(1^n)}]J_\lambda = n!$ *if* $|\lambda| = n$*.*

These symmetric functions $J_\lambda$ are called *Jack symmetric functions* and are very interesting in general. Their polynomial specialization $J_\lambda(x_1,\ldots,x_n;\zeta) = J_\lambda(x_1,\ldots,x_n,0,0,\ldots;\zeta)$ are homogeneous of degree $|\lambda|$ and $J_\lambda(x_1,\ldots,x_n;\zeta) = 0$ if $\ell(\lambda) > n$. They form a basis for $\mathrm{Sym}\otimes\mathbb{Q}(\zeta)$ and they generalize Schur functions

$$J(\mathbf{x};1) = H_\lambda s_\lambda(\mathbf{x}) = H_\lambda s_\lambda.$$

Theorem 3.1 was obtained by Macdonald [25, Chapter VI] in a more general setting of symmetric algebra $\mathrm{Sym}\otimes\mathbb{Q}(q,t)$ where $\mathbb{Q}(q,t)$ is the field of rational functions in $q$ and $t$ over $\mathbb{Q}$. Macdonald shows that there exists symmetric functions $P_\lambda(\mathbf{x};q,t) \in \mathrm{Sym}\otimes\mathbb{Q}(q,t)$ and differential operators $D_n^r$ such that the polynomial specialization $F_\lambda(x_1,\ldots,x_n;q,t)$ are simultaneous eigenfunctions with distinct eigenvalues.

A specialization of $D_n^r$ operators is the following

$$\boldsymbol{D}(\zeta) = \frac{\zeta}{2}\sum_{k=1}^{n} x_k^2 \frac{\partial^2}{\partial x_k^2} + \sum_{\substack{1\le k,k'\le n\\ k\neq k'}} \frac{x_k^2}{x_k - x_{k'}}\frac{\partial}{\partial x_k}. \tag{3.1}$$

We first need to check self-adjointness. We compute its action on power sums. The process is mechanical but tedious. Let $\alpha$ be a partition. By product rule, for $k \geq 1$

$$\frac{\partial}{\partial x_k} p_\alpha = \sum_{i=1}^{\ell(\alpha)} \left(\frac{\partial}{\partial x_k} p_{\alpha_i}\right) p_{\alpha\setminus(\alpha_i)} = \sum_{i=1}^{\ell(\alpha)} \alpha_i x_k^{\alpha_i - 1} p_{\alpha\setminus(\alpha_i)}. \tag{3.2}$$

Then

$$\sum_{k=1}^{n} x_k^2 \frac{\partial^2}{\partial x_k^2} p_\alpha = \sum_{i=1}^{\ell(\alpha)} \alpha_i(\alpha_i - 1) p_\alpha + \sum_{\substack{1\le i,j\le \ell(\alpha)\\ i\neq j}} \alpha_i \alpha_j p_{\alpha\setminus(\alpha_i,\alpha_j)\cup(\alpha_i+\alpha_j)}. \tag{3.3}$$

Furthermore, for any $b \ge 1$

$$\begin{aligned} 2\sum_{\substack{1\le k,k'\le n\\ k\neq k'}} \frac{x_k^{b+1}}{x_k - x_{k'}} &= \sum_{\substack{1\le k,k'\le n\\ k\neq k'}} \frac{x_k^{b+1} - x_{k'}^{b+1}}{x_k - x_{k'}} = \sum_{a=0}^{b} \sum_{\substack{1\le k,k'\le n\\ k\neq k'}} x_k^{b-a} x_{k'}^{a} \\ &= 2(n-1)p_b + \sum_{a=1}^{b-1}\sum_{k=1}^{n} x_k^{b-a}(p_a - x_k^a) \\ &= (2n-b-1)p_b + \sum_{a=1}^{b-1} p_{b-a}p_a. \end{aligned} \tag{3.4}$$

Therefore, we have

$$\begin{aligned} 2\sum_{\substack{1\le k,k'\le n\\ k\neq k'}} \frac{x_k^2}{x_k - x_{k'}} \frac{\partial}{\partial x_k} p_\alpha &= \sum_{i=1}^{\ell(\alpha)} 2 \sum_{\substack{1\le k,k'\le n\\ k\neq k'}} \frac{x_k^{\alpha_i+1}}{x_k - x_{k'}} \alpha_i p_{\alpha\setminus(\alpha_i)} \\ &= \sum_{i=1}^{\ell(\alpha)} \left( \alpha_i(2n - \alpha_i - 1)p_\alpha + \sum_{a=1}^{\alpha_i - 1} \alpha_i p_{\alpha\setminus(\alpha_i)\cup(a,\alpha_i - a)} \right). \end{aligned}$$

Putting the two together, we get

$$\begin{aligned} \boldsymbol{D}(\zeta)p_\alpha = \frac{\zeta}{2} &\left( \sum_{i=1}^{\ell(\alpha)} \alpha_i(\alpha_i - 1)p_\alpha + \sum_{\substack{1\le i,j\le \ell(\alpha)\\ i\neq j}} \alpha_i\alpha_j p_{\alpha\setminus(\alpha_i,\alpha_j)\cup(\alpha_i+\alpha_j)} \right) \\ &+ \frac{1}{2} \sum_{i=1}^{\ell(\alpha)} \left( \alpha_i(2n - \alpha_i - 1)p_\alpha + \sum_{a=1}^{\alpha_i - 1} \alpha_i p_{\alpha\setminus(\alpha_i)\cup(a,\alpha_k - a)} \right). \end{aligned} \tag{3.5}$$

Then we can read off the self-adjointness.

**Lemma 3.2.** *The operator $\boldsymbol{D}(\zeta)$ is self-adjoint, i.e.,*

$$\langle \boldsymbol{D}(\zeta) f, g\rangle_\zeta = \langle f, \boldsymbol{D}(\zeta) g\rangle_\zeta,$$

*for any symmetric polynomials $f, g \in \mathrm{Sym}_n \otimes \mathbb{Q}(\zeta)$.*

The $P_\lambda(\mathbf{x}; q, t)$ functions are commonly known as the *Macdonald* symmetric functions. These are generalizations of Hall-Littlewood functions, Zonal polynomials and Jack symmetric functions, and hence Schur functions. The proof of Theorem 3.1, when specialized by replacing the dominance order with reverse lexicographical order, can be used to obtain the following result due to Stanley [29].

**Theorem 3.3.** *The Jack symmetric polynomials $J_\lambda$ with $\ell(\lambda) \leq n$ form a full set of eigenfunctions of operator $\boldsymbol{D}(\zeta)$ with eigenvalues*

$$b_\lambda = \zeta b(\lambda') - b(\lambda) + |\lambda|(n-1),$$

*where $b(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i = \sum_{i=1}^{\ell(\lambda)} \binom{\lambda_i'}{2}$.*

*Proof.* We want to take advantage of the uniqueness of Jack symmetric functions described in Theorem 3.1. Let $\leq$ be the reverse lexicographical order on partitions $\{\lambda \vdash n\}$. Compute $\mathbf{D}(\zeta)$ with respect to monomial basis. We find a nice triangularity relation

$$\mathbf{D}(\zeta) m_\lambda = \sum_{\mu \leq \lambda} b_{\lambda\mu} m_\mu, \quad \lambda \vdash n,$$

with $b_{\lambda\lambda} = b_\lambda \neq 0$.

We want to obtain a set of orthogonal eigenvectors $\{E_\lambda : \lambda \vdash n\}$ of $\mathbf{D}(\zeta)$. Note $\lambda = (1^n)$ is the smallest partition in reverse lexicographical order. Triangularity says $\mathbf{D}(\zeta) m_\lambda = b_{\lambda\lambda} m_\lambda = b_\lambda m_\lambda$. Set $E_{(1^n)} = m_{(1^n)}$. If $\mu$ is the second smallest partition of $n$ in reverse lexicographical order, then triangularity implies some linear combination of $m_\lambda$ and $m_\mu$ is an eigenvector of $\mathbf{D}(\zeta)$. Apply Gram-Schmidt to obtain $E_\mu$. It follows that $E_\mu$ is also a linear combination of $m_\lambda, m_\mu$. Repeat this process until we obtain a full set of eigenvectors of $\mathbf{D}(\zeta)$. Note we have triangularity by design, i.e., for each $\lambda \vdash n$, we have

$$E_\lambda = \sum_{\mu \leq \lambda} e_{\lambda\mu} m_{\lambda\mu}.$$

Since $E_\lambda$ are orthogonal by Gram-Schmidt and lower triangular with respect to the monomials, it follows from Theorem 3.1 that each $E_\lambda$ differs from $J_\lambda$ by a scalar constant. So the Jack symmetric functions are also eigenfunctions of $\mathbf{D}(\zeta)$. The eigenvalues follows from the values of $b_{\lambda\lambda}$. □

## 3.2 Lassalle's Differential Operators

The character map faithfully translates conjugacy class $\mathcal{C}_\alpha$ to power sum $p_\alpha z(\alpha)^{-1}$. Hence, the combinatorics of $\mathbf{D}(\zeta)$ is best understood through its action on power sums. As multiplication operators they simply add parts. Symmetrized differentiation with respect to $x_i$ removes parts.

Now we define a family of operators in $\mathrm{Sym}_n$ due to Lassalle [24]. Define $\mathcal{L}_0 = p_1$ as multiplication operator and

$$\mathcal{L}_k = [\boldsymbol{D}(1), \mathcal{L}_{k-1}], \quad \text{for each } k \geq 1. \tag{3.6}$$

We will refer to this family of operators collectively as $\mathcal{L}$ operators. Their actions on Schur functions are easy to understand.

**Lemma 3.4.** *For each partition* $\lambda$

$$\mathcal{L}_k s_\lambda(x_1, \ldots, x_n) = \sum_{\mu=\lambda+\square} (c(\square) + n - 1)^k \, s_\mu(x_1, \ldots, x_n), \quad \textit{for each } k \geq 0.$$

*Proof.* Note that $b(\lambda)$ in Theorem 3.3 can be written as a summation over cells, i.e.,

$$b(\lambda) = \sum_{i=1}^{\ell} \sum_{j=1}^{\lambda_i} i - 1 = \sum_{(i,j)\in\lambda} (i-1) \quad \text{and} \quad b(\lambda') = \sum_{(i,j)\in\lambda} (j-1).$$

Hence

$$b(\lambda') - b(\lambda) = \sum_{(i,j)\in\lambda} (j-i) = \sum_{\square\in\lambda} c(\square).$$

Therefore, $\boldsymbol{D}(1)s_\lambda = (p_1(\mathbf{c}_\lambda) + |\lambda|(n-1))s_\lambda$ by Theorem 3.3. Furthermore,

$$\begin{aligned}
\mathcal{L}_1 s_\lambda(x_1, \ldots, x_n) &= (\boldsymbol{D}(1)p_1 - p_1\boldsymbol{D}(1))s_\lambda(x_1, \ldots, x_n) \\
&= \sum_{\mu=\lambda+\square} \left( \sum_{\square'\in\mu} c(\square') - \sum_{\square''\in\lambda} c(\square'') + (|\mu| - |\lambda|)(n-1) \right) s_\mu(x_1, \ldots, x_n) \\
&= \sum_{\mu=\lambda+\square} (c(\square) + n - 1) \, s_\mu(x_1, \ldots, x_n).
\end{aligned}$$

The result follows by a straightforward induction. $\square$

The actions of $\mathcal{L}$ operators on power sums are not easy to write down due to cancellations in nested commutators. The first few operators can be computed by brute force. Lassalle does so by generalizing $\boldsymbol{D}(1)$. Define for each $a \geq 1$ a pair of differential operators on $\mathrm{Sym}_n$

$$E_a = \sum_{k=1}^{n} x_k^a \frac{\partial}{\partial x_k} \quad \text{and} \quad D_a = \frac{1}{2} \sum_{k=1}^{n} x_k^a \frac{\partial^2}{\partial x_k^2} + \sum_{\substack{1\leq k,k'\leq n \\ k\neq k'}} \frac{x_k^a}{x_k - x_{k'}} \frac{\partial}{\partial x_k}.$$

Note $D_2 = \boldsymbol{D}(1)$. Equation (3.2), (3.3), and (3.4) imply the following pair of formulas.

**Lemma 3.5.** *For any partition $\alpha \in \mathcal{P}$ and any $a \geq 2$*

$$E_a p_\alpha = \sum_{i=1}^{\ell(\alpha)} \alpha_i p_{\alpha \setminus (\alpha_i) \cup (\alpha_i + a - 1)},$$

*and*

$$D_a p_\alpha = \sum_{i=1}^{\ell(\alpha)} \sum_{\substack{r,s \geq 1 \\ r+s=a+k-2}} \alpha_i p_{\alpha \setminus (\alpha_i) \cup (r,s)} + \sum_{\substack{1 \leq i,j \leq \ell(\alpha) \\ i \neq j}} \alpha_i \alpha_j p_{\alpha \setminus (\alpha_i, \alpha_j) \cup (\alpha_i + \alpha_j + a - 2)}$$
$$+ \sum_{i=1}^{\ell(\alpha)} (2n - a) p_{\alpha \setminus (\alpha_i) \cup (\alpha_i + a - 2)}$$

We omit the proof of the following result as we will later obtain a generalized version.

**Lemma 3.6.** *We have*

$$\begin{aligned}
\mathcal{L}_0 &= p_1, \\
\mathcal{L}_1 &= E_2 + (n-1)p_1, \\
\mathcal{L}_2 &= 2D_3 + E_2 + (n-1)^2 p_1.
\end{aligned}$$

## 3.3 Linear Relations in Characters

For $\lambda, \alpha \vdash n$, define the *central character* indexed by $\lambda$ evaluated at $\alpha$ by

$$\theta_\alpha^\lambda = \frac{H_\lambda}{z(\alpha)} \chi_\alpha^\lambda.$$

Since $|\mathcal{C}_\alpha|/n! = z(\alpha)^{-1}$, we have

$$J_\lambda(1) = H_\lambda s_\lambda = \sum_{\alpha \vdash n} H_\lambda \frac{|\mathcal{C}_\alpha|}{n!} \chi_\alpha^\lambda p_\alpha = \sum_{\alpha \vdash n} \theta_\alpha^\lambda p_\alpha. \tag{3.7}$$

The following result is due to Lassalle [24, Theorem 4.1] and the third relation is stated in a slightly different form here.

We need a notation for summing over all "add-a-box" operation. Let $\lambda \vdash n$ be a partition. An *outer corner* of $\lambda$ is a cell $\square$ such that the shape obtained by adding $\square$ to $\lambda$, denoted as $\lambda + \square$, is still a partition.

**Theorem 3.7.** *Let $\lambda \vdash n$ and $\beta \vdash (n+1)$. Then*

$$\sum_{\mu=\lambda+\square} \frac{H_\lambda}{H_\mu} \theta^\mu_\beta = \theta^\lambda_{\beta\setminus(1)}, \tag{3.8}$$

$$\sum_{\mu=\lambda+\square} \frac{H_\lambda}{H_\mu} c(\square) \theta^\mu_\beta = \sum_{i\geq 1} i(m_i(\beta)+1)\theta^\lambda_{\beta\setminus(i+1)\cup(i)} \tag{3.9}$$

$$\sum_{\mu=\lambda+\square} \frac{H_\lambda}{H_\mu} c(\square)^2 \theta^\mu_\beta = \sum_{i,j\geq 1} \Big( ij(m_i(\beta)+1)(m_j(\beta)+\delta_{ij}+1)\theta^\lambda_{\beta\setminus(i+j+1)\cup(i,j)} + (i+j-1)(m_{i+j-1}(\beta)+1)\theta^\lambda_{\beta\setminus(i,j)\cup(i+j-1)}\Big), \tag{3.10}$$

*where sums run over outer boxes of $\lambda$.*

The quantities $H_\lambda/H_\mu$ and the summation $\sum_{\mu=\lambda+\square} \frac{H_\lambda}{H_\mu} c(\square)^k$ will be explained in the Section 3.4. We also postpone the full proof until Section 5.3 where cleaner expressions of $\mathcal{L}$ operators are obtained. For now, we outline Lassalle's original approach.

*Proof Sketch.* Consider applying $\mathcal{L}_2$ to the polynomial version of Equation (3.7). On one hand, we have

$$\mathcal{L}_2 H_\lambda s_\lambda(x_1,\ldots,x_n) = \sum_{\mu=\lambda+\square} H_\lambda (c(\square)+n-1)^2 s_\mu(x_1,\ldots,x_n).$$

In the summation, multiply by $H_\mu/H_\mu$ and apply Equation (3.7) again to $H_\mu s_\mu(x_1,\ldots,x_n)$.

$$= \sum_{\beta\vdash(n+1)} \sum_{\mu=\lambda+\square} \frac{H_\lambda}{H_\mu} (c(\square)+n-1)^2 \theta^\mu_\beta p_\beta(x_1,\ldots,x_n),$$

$$= \sum_{\beta\vdash(n+1)} \sum_{\mu=\lambda+\square} \frac{H_\lambda}{H_\mu} \left(c(\square)^2 + 2(n-1)c(\square) + (n-1)^2\right) \theta^\mu_\beta p_\beta(x_1,\ldots,x_n).$$

Note that this is an expression in $n-1$ with coefficients in $\mathrm{Sym}_n$.

On the other hand, use the expression for $\mathcal{L}_2$ in Lemma 3.6 and apply Lemma 3.5. By linearity of $\mathcal{L}$ operators, we get an expression

$$\sum_{\alpha\vdash n} \theta^\lambda_\alpha \mathcal{L}_2 p_\alpha = \sum_{\alpha\vdash n} \theta^\lambda_\alpha \left( L^{(0)}_\alpha(\mathbf{p}) + L^{(1)}_\alpha(\mathbf{p})(n-1) + L^{(2)}_\alpha(\mathbf{p})(n-1)^2 \right), \tag{3.11}$$

where $L^{(i)}_\alpha(\mathbf{p})$ are expressions in $\mathrm{Sym}_n$ not involving $n$.

To conclude the proof, we simply identify coefficients of $n-1$. □

## 3.4 Transition Measures

To take advantage of $\mathcal{L}$ operators, we collect some results involving the quantity $H_\lambda/H_\mu$ appearing in Theorem 3.7. Proofs can be found in [24].

The (discrete) *transition measure* of $\lambda$ with respect to an outer corner $\square$ is

$$\gamma_\lambda(\square) = \frac{H_\lambda}{H_\mu}.$$

For $k \geq 0$, the $k$-th *moment* of the transition measure of $\lambda$ is

$$\Gamma_k(\lambda) = \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square)^k.$$

where the sum goes over outer corner cells of $\lambda$. The following result is due to Lassalle [24] using the theory of shifted symmetric functions.

**Theorem 3.8.** *Let $\lambda \vdash n$ be a partition. Then*

$$\Gamma_0(\lambda) = 1, \quad \Gamma_1(\lambda) = 0, \quad \Gamma_2(\lambda) = n \quad \textit{and} \quad \Gamma_3(\lambda) = 2p_1(\mathbf{c}_\lambda).$$

# Chapter 4

# Generalized Class Expansion

Let $f \in \mathrm{Sym}_n$ be a symmetric function. Theorem 2.10 states that $f(J_1, \ldots, J_n)$ is an element in $\mathcal{Z}_n$. It is natural to ask for an expression in the basis consisting of conjugacy classes. The *expansion coefficients* $\varphi^f_\alpha$ of $f$ indexed by partitions $\alpha \vdash n$ are defined as

$$f(J_1, \ldots, J_n) = \sum_{\alpha \vdash n} \varphi^f_\alpha \mathcal{C}_\alpha.$$

In Section 1.2 we introduced this problem as *generalized class expansion*. Expansion coefficients are obviously useful in understanding the centre of the group algebra. When $f$ encodes interesting factorizations in $\mathcal{S}_n$, expansion coefficients are counting coefficients for not necessarily transitive factorizations. Hence useful techniques on class expansion problems could be tweaked and adapted into tools on transitive factorization problems. This is our motivation to understand Lassalle's work.

In this chapter, we aim to apply linear relations to obtain induction relations in expansion coefficients whenever a reduction strategy is applicable. We pay special attention to complete symmetric functions. Féray noticed that the expansion of complete symmetric functions affords a nice combinatorial point of view where multiplication by transpositions enters the picture. This argument paves way to a unified algebraic method for counting transitive factorizations which we explain in Chapter 6.

## 4.1 General Strategy

First, we translate generalized class expansion to a problem in central characters. Note multiplication of central orthogonal idempotents is trivial by Lemma 2.7. If $\alpha, \lambda \vdash n$, then

$$\begin{aligned}\mathcal{C}_\alpha \mathcal{F}^\lambda &= |\mathcal{C}_\alpha| \sum_{\mu \vdash n} \frac{|\mathcal{C}_\alpha|}{\chi^\mu_{(1^n)}} \chi^\mu_\alpha \mathcal{F}^\mu \mathcal{F}^\lambda \\ &= \frac{H_\lambda}{z(\alpha)} \chi^\lambda_\alpha \mathcal{F}^\lambda \\ &= \theta^\lambda_\alpha \mathcal{F}^\lambda.\end{aligned}$$

Let $f \in \mathrm{Sym}_n$ with expansion coefficients $\varphi^f_\alpha$ where $\alpha \vdash n$. Then by Theorem 2.12 we have for each $\lambda \vdash n$

$$f(J_1, \ldots, J_n)\mathcal{F}^\lambda = \sum_{\alpha \vdash n} \varphi^f_\alpha \mathcal{C}_\alpha \mathcal{F}^\lambda = \sum_{\alpha \vdash n} \varphi^f_\alpha \theta^\lambda_\alpha \mathcal{F}^\lambda.$$

By Theorem 2.12 $f(J_1, \ldots, J_n)\mathcal{F}^\lambda = f(\mathbf{c}_\lambda)\mathcal{F}_\lambda$. So we have

$$f(\mathbf{c}_\lambda)\mathcal{F}^\lambda = \sum_{\alpha \vdash n} \varphi^f_\alpha \theta^\lambda_\alpha \mathcal{F}^\lambda. \tag{4.1}$$

The problem is now amenable to the linear relations in central characters developed in Section 3.3. The idea behind this method is best first illustrated through an example. The simplest one is the expansion of $e_k(J_1, \ldots, J_n)$. The general strategy will be discussed in full detail at the end of the section.

Let $\lambda \vdash n$. Our goal is to apply Theorem 3.7. Recall

$$e_k(x_1, \ldots, x_n, x_{n+1}) = e_k(x_1, \ldots, x_n, 0) + x_{n+1} e_{k-1}(x_1, \ldots, x_n, 0).$$

If $\mu = \lambda + \square$, then

$$e_k(\mathbf{c}_\mu) = e_k(\mathbf{c}_\lambda) + c(\square) e_{k-1}(\mathbf{c}_\lambda).$$

Multiply the above by $\gamma_\lambda(\square)c(\square)^i$ and sum over all $\mu = \lambda + \square$ where $\square$ is an outer corner $\lambda$. We obtain 2 equations to which we apply Theorem 3.8:

$$\begin{aligned}\sum_{\mu = \lambda + \square} \gamma_\lambda(\square) e_k(\mathbf{c}_\mu) &= \Gamma_0(\lambda) e_k(\mathbf{c}_\lambda) + \Gamma_1(\lambda) e_{k-1}(\mathbf{c}_\lambda) \\ &= e_k(\mathbf{c}_\lambda), \end{aligned}\tag{4.2}$$

$$\begin{aligned}\sum_{\mu = \lambda + \square} \gamma_\lambda(\square) c(\square) e_k(\mathbf{c}_\mu) &= \Gamma_1(\lambda) e_k(\mathbf{c}_\lambda) + \Gamma_2(\lambda) e_{k-1}(\mathbf{c}_\lambda) \\ &= n e_{k-1}(\mathbf{c}_\lambda). \end{aligned}\tag{4.3}$$

Now we apply Theorem 3.7 to the left-hand side of each of these equations after writing both sides in terms of central characters. For Equation (4.2), we have

$$\begin{aligned}\sum_{\alpha\vdash n}\varphi_\alpha^{e_k}\theta_\alpha^\lambda &= e_k(\mathbf{c}_\lambda)\\ &= \sum_{\beta\vdash(n+1)}\varphi_\beta^{e_k}\sum_{\mu=\lambda+\square}\gamma_i(\square)\theta_\beta^\mu\\ &= \sum_{\beta\vdash(n+1)}\varphi_\beta^{e_k}\theta_{\beta\setminus(1)}^\lambda.\end{aligned}$$

For Equation (4.3), we have

$$\begin{aligned}n\sum_{\alpha\vdash n}\varphi_\alpha^{e_{k-1}}\theta_\alpha^\lambda &= ne_{k-1}(\mathbf{c}_\lambda)\\ &= \sum_{\beta\vdash(n+1)}\varphi_\beta^{e_{k-1}}\sum_{\mu=\lambda+\square}\gamma_i(\square)c(\square)\theta_\beta^\mu\\ &= \sum_{\beta\vdash(n+1)}\varphi_\beta^{e_{k-1}}\sum_{i\geq 1}i(m_i(\beta)+1)\theta_{\beta\setminus(i+1)\cup(i)}^\lambda.\end{aligned}$$

Since central characters are non-zero multiples of irreducible characters, they are linearly independent. Compare the coefficients of central characters to get a pair of relations

$$\varphi_\alpha^{e_k} = \varphi_{\alpha\cup(1)}^{e_k} \tag{4.4}$$

$$n\varphi_\alpha^{e_{k-1}} = \sum_{i\geq 1} im_i(\alpha)\varphi_{\alpha\setminus(i)\cup(i+1)}^{e_k}. \tag{4.5}$$

**Lemma 4.1.** *Equations (4.4) and (4.5) determines $\varphi_\alpha^{e_k}$ for all $k\geq 0$ and all partitions $\alpha$.*

*Proof.* We proceed by a triple induction, first on $k$, on $n$, then on $\min(\alpha)$. Note for $k=0$

$$\varphi_\alpha^{e_0} = \delta_{(1^n),\alpha},\quad \text{for all } \alpha\vdash n, n\geq 1.$$

Suppose for some $k\geq 1$ the coefficient $\varphi_\alpha^{e_{k-1}}$ has been determined for all $\alpha\vdash n$ and $n\geq 1$. The initial condition on the inner induction is $n=1$ and $\varphi_{(1)}^{e_k}=0$. Now suppose for some $n\geq 1$, the coefficient $\varphi_\alpha^{e_k}$ has been determined for all $\alpha\vdash n$. We need to determine $\varphi_\beta^{(k)}$ for all $\beta\vdash n+1$.

Note if $\min(\beta)=1$, then Equation (4.4) implies

$$\varphi_\beta^{e_k} = \varphi_{\beta\setminus(1)}^{e_k},$$

which has been determined by hypothesis. Suppose further for some $b \geq 1$ the coefficient $\varphi^{e_k}_{\beta^*}$ has been determined for all $\beta^*$ with $\min(\beta^*) \leq b$. Let $\beta \vdash (n+1)$ with $\min(\beta) = b+1$. Remove a box from the smallest part of $\beta$, i.e., choose $\alpha = \beta \setminus (b+1) \cup (b)$. Note $m_b(\alpha) = 1$. Then Equation (4.5) can be written as

$$b\varphi^{e_k}_{\beta} = -n\varphi^{e_{k-1}}_{\alpha} + \sum_{\substack{i\geq 1\\ i\neq b}} im_i(\alpha)\varphi^{e_k}_{\alpha\setminus(i)\cup(i+1)}.$$

But every non-zero summand has a part $j$. Their coefficients have already been determined by hypothesis. Finally, $\varphi^{e_{k-1}}_{\alpha}$ has already been determined by the 2 outer induction hypotheses. □

**Corollary 4.2.** *Equations (4.4) and (4.5) imply for each $k \geq 0$ and $n \geq 1$*

$$e_k(J_1, \ldots, J_n) = \sum_{\substack{\alpha\vdash n\\ \ell(\alpha)=n-k}} \mathcal{C}_\alpha.$$

*Proof Sketch.* Replace the inner most induction hypothesis by the following: Suppose for some $j \geq 1$

$$\varphi^{e_k}_{\beta^*} = \begin{cases} 1, & \ell(\beta^*) = n-k \\ 0, & \text{otherwise}, \end{cases}$$

for all $\min(\beta^*) \leq j$. □

Now we describe the general strategy. If $\lambda \vdash n$ and $\mu = \lambda + \square$ for some outer corner $\square$ of $\lambda$, then by Equation (2.2) we have a recurrence

$$f(\mathbf{c}_\mu) = f(\mathbf{c}_\lambda) + \sum_{k=1}^{\deg f} c(\square)^k (h_k^\perp f)(\mathbf{c}_\lambda).$$

Let $g_k = h_k^\perp f$ and let $i \geq 0$. Multiply above by $\gamma_\lambda(\square)c(\square)^i$ and sum over all outer boxes $\square$ of $\lambda$. We get

$$\sum_{\mu=\lambda+\square} \gamma_\lambda(\square)c(\square)^i f(\mathbf{c}_\mu) = \Gamma_i(\lambda)f(\mathbf{c}_\lambda) + \sum_{k=1}^{\deg f} \Gamma_{i+k}(\lambda)g_k(\mathbf{c}_\lambda).$$

We look for symmetric functions $F, G$ with known class expansion coefficients. The pair should satisfy

$$F(\mathbf{c}_\lambda) = \Gamma_i(\lambda)f(\mathbf{c}_\lambda)$$

and

$$\sum_{\mu=\lambda+\square} \gamma_\lambda(\square)c(\square)^j G(\mathbf{c}_\mu) = \sum_{k=1}^{\deg f} \Gamma_{i+k}(\lambda) g_k(\mathbf{c}_\lambda),$$

for some $i \geq 0$. Expand $f, F, G$ in terms of central characters. Then expansion coefficients of $f, F, G$ are related by

$$\sum_{\beta\vdash(n+1)} \varphi_\beta^f \left( \sum_{\mu=\lambda+\square} \gamma_\lambda(\square)c(\square)^i \theta_\beta^\mu \right) = \sum_{\alpha\vdash n} \varphi_\alpha^F \theta_\alpha^\lambda + + \sum_{\beta\vdash(n+1)} \varphi_\beta^G \left( \sum_{\mu=\lambda+\square} \gamma_\lambda(\square)c(\square)^j \theta_\beta^\mu \right),$$

for some $j \geq 0$.

If $i$ and $j$ are sufficiently small, then we can apply Theorem 3.7 to reduce weights of all partitions appearing in brackets by 1. We obtain relations in $\varphi$ by comparing coefficients of $\theta_\alpha^\lambda$.

Typically we take $f$ from a family of recursively defined symmetric functions, say $f_k, k \geq 1$ so that Equation (2.2) simplifies to

$$f_k(x_1, \ldots, x_{n+1}) = f_k(x_1, \ldots, x_n, 0) + x^a f_b(x_1, \ldots, x_{n+1})$$

for some $a \leq k$ and $b < k$. In such case, the path of finding $G$ completely bypasses the non-trivial task of evaluating $\Gamma(\lambda)$. Proceed by a double induction to assume $f_k(J_1, \ldots, J_n, 0)$ and $f_b(J_1, \ldots, J_{n+1})$ both have known expansion coefficients. Apply the general strategy with $i = 0$. Then we take advantage of $\Gamma_0(\lambda) = 1$ and simply let $F = f_k$ and $G = f_b$. Depending on the values of $a, b$, we may have to apply the general strategy more than once with increasing values of $i$ to obtain desirable relations.

In the above demonstration of class expansion of $e_k$, we applied the general strategy with $i = 0, 1$. When $i = 0$, we used $F = e_k$ and $G = 0$. When $i = 1$, we used $F = ne_{k-1}$ and $G = 0$.

## 4.2 Expansion of Complete Symmetric Polynomials

Now we apply the above strategy to obtain a recurrence for the coefficients of the generalized class expansion of central elements corresponding to complete symmetric polynomials.

Fix $n \geq 1$. Consider $h_k \in \mathrm{Sym}_{n+1}$ as a power series in $x_{n+1}$. Then we have

$$\begin{aligned} h_k(x_1, \ldots, x_n, x_{n+1}) &= h_k(x_1, \ldots, x_n, 0) + \sum_{j=1}^{k} x_{n+1}^j h_{k-j}(x_1, \ldots, x_n, 0) \\ &= h_k(x_1, \ldots, x_n, 0) + x_{n+1} h_{k-1}(x_1, \ldots, x_n, x_{n+1}). \end{aligned}$$

Fix some $\lambda \vdash n$. By Equation (4.1), the expansion of the corresponding central element has a recurrence for each $\mu = \lambda + \square$

$$h_k(\mathbf{c}_\mu) = h_k(\mathbf{c}_\lambda) + c(\square) h_{k-1}(\mathbf{c}_\mu).$$

We get

$$\begin{aligned}
\sum_{\mu=\lambda+\square} \gamma_\lambda(\square) h_k(\mathbf{c}_\mu) &= \Gamma_0 h_k(\mathbf{c}_\lambda) + \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square) h_{k-1}(\mathbf{c}_\mu) \\
&= h_k(\mathbf{c}_\lambda) + \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square) h_{k-1}(\mathbf{c}_\mu) \qquad (4.6) \\
\sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square) h_k(\mathbf{c}_\mu) &= \Gamma_1 h_k(\mathbf{c}_\lambda) + \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square)^2 h_{k-1}(\mathbf{c}_\mu) \\
&= \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square)^2 h_{k-1}(\mathbf{c}_\mu). \qquad (4.7)
\end{aligned}$$

Expand each $h_k$ into central characters. Equation (4.6) becomes

$$\sum_{\beta \vdash (n+1)} \varphi_\beta^{h_k} \left( \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) \theta_\beta^\mu \right) = \sum_{\alpha \vdash n} \varphi_\alpha^{h_k} \theta_\alpha^\lambda + \sum_{\beta \vdash (n+1)} \varphi_\beta^{h_{k-1}} \left( \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square) \theta_\beta^\mu \right).$$

Apply Theorem 3.7 to summations in brackets. We get

$$\sum_{\beta \vdash (n+1)} \varphi_\beta^{h_k} \theta_{\beta\setminus(1)}^\lambda = \sum_{\alpha \vdash n} \varphi_\alpha^{h_k} \theta_\alpha^\lambda + \sum_{\beta \vdash (n+1)} \sum_{i \geq 1} \varphi_\beta^{h_k} i(m_i(\beta)+1) \theta_{\beta\setminus(i+1)\cup(i)}^\lambda.$$

Identify the coefficients of central characters on both sides. For a fixed $\alpha \vdash n$, we have

$$\varphi_{\alpha\cup(1)}^{h_k} = \varphi_\alpha^{h_k} + \sum_{i \geq 1} i m_i(\alpha) \varphi_{\alpha\setminus(i)\cup(i+1)}^{h_{k-1}}. \qquad (4.8)$$

After expansion, Equation (4.7) becomes

$$\sum_{\beta \vdash (n+1)} \varphi_\beta^{h_k} \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square) h_k(\mathbf{c}_\mu) = \sum_{\beta \vdash (n+1)} \varphi_\beta^{h_{k-1}} \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square)^2 \theta_\beta^\mu.$$

Apply Theorem 3.7 to both sides to get

$$\begin{aligned}
\sum_{\mu=\lambda+\square} \sum_{i \geq 1} i(m_i(\beta)+1) \varphi_\beta^{h_k} \theta_{\beta\setminus(i+1)\cup(i)}^\lambda j = \sum_{\mu=\lambda+\square} \sum_{i,j \geq 1} \Big( & ij(m_i(\beta)+1)(m_j(\beta)+\delta_{ij}+1) \theta_{\beta\setminus(i+j+1)\cup(i,j)}^\lambda \\
&+ (i+j-1)(m_{i+j-1}(\beta)+1) \theta_{\beta\setminus(i,j)\cup(i+j-1)}^\lambda \Big).
\end{aligned}$$

Identify coefficients of central characters. We get the second recurrence relation. For each $\alpha \vdash n$ and $i \geq 1$

$$\sum_{i\geq 1} i m_i(\alpha)\varphi^{h_k}_{\alpha\setminus(i)\cup(i+1)} = \sum_{i,j\geq 1}\Big(ijm_i(\alpha)(m_j(\alpha)-\delta_{ij})\varphi^{h_{k-1}}_{\alpha\setminus(i,j)\cup(i+j+1)} + (i+j-1)m_{i+j-1}(\alpha)\varphi^{h_{k-1}}_{\alpha\setminus(i+j-1)\cup(i,j)}\Big). \tag{4.9}$$

**Corollary 4.3.** *Equations (4.8) and (4.9) together determine expansion coefficients $\varphi^{h_k}_{\alpha}$ for all $k \geq 0$ and $\alpha \in \mathcal{P}$.*

*Proof.* We use a triple induction first on $k$, on $n$, then on $\min(\alpha)$. All steps are very similar to the ones in the proof of Lemma 4.1. For $k = 0$, we have $h_0 = 1$. So

$$\varphi^{h_0}_{\alpha} = \begin{cases} 1, & \alpha = () \\ 0, & \alpha \in \mathcal{P}. \end{cases}$$

Suppose for some $k \geq 0$ the coefficient $\varphi^{h_{k-1}}_{\alpha}$ has been determined for all $n \geq 1$ and $\alpha \vdash n$. The initial condition on the inner induction is $n = 1$ and $\varphi^{h_k}_{(1)} = 0$ since $h_k(J_1) = 0$ for all $k \geq 1$. Now suppose for some $n \geq 1$, the coefficient $\varphi^{h_k}_{\alpha}$ has been determined for all $\alpha \vdash n$. We need to determine $\varphi^{h_k}_{\beta}$ for all $\beta \vdash (n+1)$.

In the case of $\min(\beta) = 1$, we let $\beta = \alpha \cup (1)$. Equation (4.8) implies

$$\varphi^{h_k}_{\beta} = \varphi^{h_k}_{\alpha} + \sum_{i\geq 1} i m_i(\alpha)\varphi^{h_{k-1}}_{\alpha\setminus(i)\cup(i+1)}.$$

All coefficients in the RHS have been determined by hypothesis. Suppose further for some $b \geq 1$ the coefficient $\varphi^{h_k}_{\beta^*}$ has been determined for all $\beta^*$ with $\min(\beta^*) \leq b$. Let $\beta \vdash (n+1)$ with $\min(\beta) = b+1$. Remove a box from the smallest part of $\beta$, i.e., choose $\alpha = \beta \setminus (b+1) \cup (b)$. Note $m_b(\alpha) = 1$. Then Equation (4.9) can be written as

$$b\varphi^{h_k}_{\beta} = -\sum_{\substack{i\geq 1\\ i\neq b}} i m_i(\alpha)\varphi^{h_k}_{\alpha\setminus(i)\cup(i+1)} + \sum_{i,j\geq 1}\Big(ijm_i(\alpha)(m_j(\alpha)+\delta_{ij})\varphi^{h_{k-1}}_{\alpha\setminus(i,j)\cup(i+j+1)} + (i+j-1)m_{i+j-1}(\alpha)\varphi^{h_{k-1}}_{\alpha\setminus(i+j-1)\cup(i,j)}\Big).$$

All coefficients on the RHS has been determined by hypothesis. ∎

| $\alpha$ \ $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| (11) | | 1 | | 1 | | 1 | |
| (2) | 1 | | 1 | | 1 | | 1 |
| (111) | | 3 | | 11 | | 43 | |
| (21) | 1 | | 5 | | 21 | | 85 |
| (3) | | 2 | | 10 | | 42 | |
| (1111) | | 6 | | 41 | | 316 | |
| (211) | 1 | | 10 | | 91 | | 820 |
| (22) | | 1 | | 20 | | 231 | |
| (31) | | 2 | | 25 | | 252 | |
| (4) | | | 5 | | 70 | | 735 |
| (11111) | | 10 | | 105 | | 1240 | |
| (2111) | 1 | | 16 | | 231 | | 3382 |
| (221) | | 1 | | 28 | | 567 | |
| (311) | | 2 | | 42 | | 714 | |
| (32) | | | 2 | | 84 | | 1974 |
| (41) | | | 5 | | 126 | | 2415 |
| (5) | | | | 14 | | 420 | |

Table 4.1: Coefficients of Low Order $h_k(J_1, \ldots, J_n)$.

## 4.3 A Partial Differential Equation

Consider expressing the above recurrence as a relation on the generating series level. Define

$$\Phi^H(t, z, \mathbf{p}) = \sum_{k \geq 1} t^k \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\alpha \vdash n} \varphi_\alpha^{h_k} |\mathcal{C}_\alpha| p_\alpha.$$

Note $\Phi^H = \Phi^H(t, z, \mathbf{p})$ is ordinary in $t$ but exponential in $z$. Since $z(\alpha) = |\mathcal{C}_\alpha|/n!$, we can rewrite it as

$$\Phi^H = \sum_{k \geq 1} t^k \sum_{n \geq 1} z^n \sum_{\alpha \vdash n} \varphi_\alpha^{h_k} \frac{p_\alpha}{z(\alpha)}.$$

Let $\alpha \in \mathcal{P}$. Then $p_1^\perp p_{\alpha \cup (1)} z(\alpha \cup (1))^{-1} = p_\alpha z(\alpha)^{-1}$. Furthermore if $i \geq 1$ and $\beta \in \mathcal{P}$, then

$$i m_i(\alpha) \frac{p_\alpha}{z(\alpha)} = p_i \frac{p_{\alpha \setminus (i)}}{z(\alpha \setminus (i))} \quad \text{and} \quad p_{i+1}^\perp \frac{p_{\beta \cup (i+1)}}{z(\beta \cup (i+1))} = \frac{p_\beta}{z(\beta)}. \tag{4.10}$$

In particular, if $\beta = \alpha \setminus (i)$ then

$$im_i \frac{p_\alpha}{z(\alpha)} = p_i p_{i+1}^\perp p_{\alpha\setminus(i)\cup(i+1)}.$$

Multiply Equation (4.8) by $p_\alpha z(\alpha)^{-1} z^{|\alpha|+1} t^k$ and sum over all $\alpha \in \mathcal{P}$ and $k \geq 1$. Then

$$\begin{aligned}
left-handside &= \sum_{k\geq 1}\sum_{n\geq 1}\sum_{\alpha\vdash n} \varphi^{h_k}_{\alpha\cup(1)} \frac{p_\alpha}{z(\alpha)} z^{n+1} t^k \\
&= \sum_{k\geq 1} t^k \sum_{n\geq 1} z^{n+1} \sum_{\alpha\vdash n} \varphi^{h_k}_{\alpha\cup(1)} p_1^\perp \frac{p_{\alpha\cup(1)}}{z(\alpha\cup(1))} \\
&= p_1^\perp \sum_{k\geq 1} t^k \sum_{n\geq 1} z^{n+1} \sum_{\substack{\beta\vdash(n+1)\\ 1\in\beta}} \varphi^{h_k}_\beta \frac{p_\beta}{z(\beta)}.
\end{aligned}$$

Note if $\beta \in \mathcal{P}$ does not contain a part of size 1, then $p_1^\perp p_\beta = 0$. Together with the fact $\varphi^{h_k}_{(1)} = 0$ for all $k \geq 1$, we can free the restriction in the inner summation and sum over $n \geq 0$. Then the left-hand side is simply $p_1^\perp \Phi^H$. Now consider the summation in the RHS. We have

$$\begin{aligned}
RHS &= \sum_{i\geq 1}\sum_{k\geq 1}\sum_{n\geq 1}\sum_{\alpha\vdash n} \varphi^{h_{k-1}}_{\alpha\setminus(i)\cup(i+1)} im_i(\alpha) \frac{p_\alpha}{z(\alpha)} z^{n+1} t^k \\
&= t \sum_{i\geq 1}\sum_{k\geq 1} t^{k-1} \sum_{n\geq 1} z^{n+1} \sum_{\alpha\vdash n} \varphi^{h_{k-1}}_{\alpha\setminus(i)\cup(i+1)} p_i p_{i+1}^\perp \frac{p_{\alpha\setminus(i)\cup(i+1)}}{z(\alpha\setminus(i)\cup(i+1))} \\
&= t \sum_{i\geq 1} p_i p_{i+1}^\perp \sum_{k\geq 1} t^{k-1} \sum_{n\geq 1} z^{n+1} \sum_{\substack{\beta\vdash(n+1)\\ i+1\in\beta}} \varphi^{h_{k-1}}_\beta \frac{p_\beta}{z(\beta)}.
\end{aligned}$$

Similar to the previous case, if $\beta \in \mathcal{P}$ does not contain a part of size $i+1$, then $p_{i+1}^\perp p_\beta = 0$. In particular, $p_{i+1}^\perp p_{(1)} = 0$ for all $i \geq 1$. Hence we can free the restriction in the inner-most sum and sum over $n \geq 0$. Equation (4.8) can be written as a PDE

$$p_1^\perp \Phi^H = z\Phi^H + t\sum_{i\geq 1} p_i p_{i+1}^\perp \Phi^H. \tag{4.11}$$

Similarly, multiply Equation (4.9) by $p_\alpha z(\alpha)^{-1} z^{|\alpha|+1} t^k$ and sum over all $\alpha \in \mathcal{P}$ and $k \geq 1$. Apply relation (4.10) to the summand in the RHS. For any partition $\alpha \in \mathcal{P}$ and $i, j \geq 1$ we have

$$\begin{aligned}
ijm_i(\alpha)(m_j(\alpha) - \delta_{ij}) \frac{p_\alpha}{z(\alpha)} &= p_i p_j p_{i+j+1}^\perp \frac{p_{\alpha\setminus(i,j)\cup(i+j+1)}}{z(\alpha\setminus(i,j)\cup(i+j+1))}, \quad \text{using } \beta = \alpha\setminus(i,j), \\
(i+j-1)m_{i+j-1}(\alpha)\frac{p_\alpha}{z(\alpha)} &= p_{i+j-1} p_i^\perp p_j^\perp \frac{p_{\alpha\setminus(i+j-1)\cup(i,j)}}{z(\alpha\setminus(i+j-1)\cup(i,j))}, \quad \text{using } \beta = \alpha\setminus(i+j-1).
\end{aligned}$$

Note the left-hand side of Equation (4.9) is similar to the RHS of Equation (4.8). Hence, we can express Equation (4.9) as a PDE

$$\sum_{i\geq 1} p_i p_{i+1}^{\perp} \Phi^H = t \left( \sum_{i,j\geq 1} p_i p_j p_{i+j+1}^{\perp} + p_{i+j-1} p_i^{\perp} p_j^{\perp} \right) \Phi^H. \tag{4.12}$$

Combining Equation (4.11) and (4.12), we get the following result.

**Proposition 4.4.** *The series $\Phi^H$ is the unique solution to the partial differential equation*

$$p_1^{\perp} \Phi^H = z\Phi^H + t^2 \left( \sum_{i,j\geq 1} p_i p_j p_{i+j+1}^{\perp} + p_{i+j-1} p_i^{\perp} p_j^{\perp} \right) \Phi^H, \tag{4.13}$$

*with initial condition $\Phi^H(t, 0, \mathbf{p}) = 1$.*

Equation (4.13) reveals an interesting combinatorial property of $\mathcal{L}$ operators. From the derivation above, the differential operator $p_1^{\perp}$ kills off all $p_\alpha z(\alpha)^{-1}$ where $\alpha$ does not contain a part of size 1 and $p_1^{\perp} p_\alpha z(\alpha)^{-1} = p_{\alpha\setminus(1)} z(\alpha \setminus (1))^{-1}$. Since $p_\alpha z(\alpha)^{-1}$ marks a conjugacy class, we can interpret $p_1^{\perp}$ as a projection operator $\mathcal{S}_{n+1} \to \mathcal{S}_{n+1}$ defined

$$p_1^{\perp} : \sigma' \mapsto \begin{cases} \sigma', & \sigma'(n+1) = n+1 \\ 0, & \text{otherwise,} \end{cases} \qquad \sigma' \in \mathcal{S}_{n+1}.$$

Clearly, the range of the above projection can be identified with the canonical embedding of $\mathcal{S}_n$ in $\mathcal{S}_{n+1}$. Féray took advantage of it and provided a combinatorial proof of Equations (4.8) and (4.9).

## 4.4 Féray's Combinatorial Arguments

We prove Equations (4.8) and (4.9) using a combinatorial argument due to Féray [10]. Recall

$$\varphi_{\alpha\cup(1)}^{h_k} = \varphi_{\alpha}^{h_k} + \sum_{i\geq 1} i m_i(\alpha) \varphi_{\alpha}^{h_{k-1}}, \tag{4.8}$$

$$\sum_{i\geq 1} i m_i(\alpha) \varphi_{\alpha\setminus(i)\cup(i+1)}^{h_k} = \sum_{i,j\geq 1} \Big( ijm_i(\alpha)(m_j(\alpha) + \delta_{ij}) \varphi_{\alpha\setminus(i,j)\cup(i+j+1)}^{h_k} + (i+j-1) m_{i+j-1}(\alpha) \varphi_{\alpha\setminus(i+j-1)\cup(i,j)}^{h_k} \Big). \tag{4.9}$$

With mild abuse of notation, let $p_1^\perp : \mathbb{C}[\mathcal{S}_{n+1}] \to \mathbb{C}[\mathcal{S}_n]$ denote the extension of above projection operator with its range restricted to $\mathcal{S}_n$, i.e., define

$$p_1^\perp \sigma' = \begin{cases} \sigma, & \sigma'(n+1) = n+1 \\ 0, & \text{otherwise,} \end{cases} \qquad \sigma' \in \mathcal{S}_n.$$

where $\sigma$ is obtained from $\sigma'$ by removing the fixed point $n+1$. Extend linearly to all of $\mathbb{C}[\mathcal{S}_{n+1}]$. We will use $\sigma'$ to denote adding $n+1$ as a fixed point to $\sigma \in \mathcal{S}_n$.

Recall Equations (4.8) and (4.9) are derived from the group algebra recurrence

$$h_k(J_1, \dots, J_{n+1}) = h_k(J_1, \dots, J_n, 0) + J_{n+1} h_{k-1}(J_1, \dots, J_{n+1}).$$

Let $\sigma \in \mathcal{S}_n$ with cycle type $\alpha \vdash n$. Consider its coefficient after applying $p_1^\perp$ to the above equation. A surviving permutation $\tau$ in $p_1^\perp h_k(J_1, \dots, J_{n+1})$ must contain $n+1$ as a fixed point by definition. Hence, the cycle type of $\tau$ can be expressed as $\beta \cup (1)$ for some unique $\beta \vdash n$. Hence,

$$[\sigma] p_1^\perp h_k(J_1, \dots, J_{n+1}) = \varphi^{h_k}_{\alpha \cup (1)}.$$

Since every permutation in $h_k(J_1, \dots, J_n, 0)$ has $n+1$ as a fixed point, we get

$$[\sigma] p_1^\perp h_k(J_1, \dots, J_n, 0) = [\sigma] h_k(J_1, \dots, J_n) = \varphi^k_\alpha.$$

For the rightmost term, we note as operator $[\sigma] p_1^\perp = [\sigma']$. Then

$$\begin{aligned} [\sigma] p_1^\perp J_{n+1} h_{k-1}(J_1, \dots, J_{n+1}) &= [\sigma'] J_{n+1} h_{k-1}(J_1, \dots, J_{n+1}) \\ &= [\sigma' J_{n+1}] h_{k-1}(J_1, \dots, J_{n+1}) \\ &= \sum_{v=1}^{n} [(v, n+1)\sigma'] h_{k-1}(J_1, \dots, J_{n+1}). \end{aligned}$$

For each $v \in \{1, \dots, n\}$, the cycle structure of $(v, n+1) \cdot \sigma'$ differs from that of $\sigma'$ by only the cycle containing $v$ and $n+1$. Note

$$(v, n+1) \cdot (v, \sigma(v), \dots, \sigma^{\alpha_v - 1}(vjs)) = (v, \sigma(v), \dots, \sigma^{\alpha_v - 1}(v), n+1),$$

where $\alpha_v$ is the order of the cycle containing $v$. Hence the cycle type of $(v, n+1) \cdot \sigma'$ is $\alpha \setminus (\alpha_v) \cup (\alpha_v + 1)$. It follows that

$$[\sigma] p_1^\perp J_{n+1} h_{k-1}(J_1, \dots, J_{n+1}) = \sum_{i \geq 1} i m_i(\alpha) \varphi^{h_{k-1}}_{\alpha \setminus (i) \cup (i+1)}.$$

We get Equation (4.8) by putting the above together.

To get the second relation, we apply $p_1^\perp$ after multiplying the group algebra recurrence by $J_{n+1}$. Note

$$J_{n+1}h_k(J_1,\dots,J_{n+1}) = J_{n+1}h_k(J_1,\dots,J_n,0) + J_{n+1}^2 h_{k-1}(J_1,\dots,J_{n+1}).$$

Again we consider the coefficient of $\sigma$ after applying $p_1^\perp$ to the above. Its coefficient from the left-hand side is

$$[\sigma]p_1^\perp J_{n+1}h_k(J_1,\dots,J_{n+1}) = \varphi^{h_k}_{\alpha\setminus(i)\cup(i+1)}.$$

Since $\tau \in h_k(J_1,\dots,J_n,0)$ contains $n+1$ as a fixed point, then every monomial in $J_{n+1}\tau$ does not contain $n+1$ as a fixed point. Hence, $[\sigma]p_1^\perp J_{n+1}h_k(J_1,\dots,J_n,0) = 0$. For the last expression, we want to evaluate

$$[\sigma]p_1^\perp J_{n+1}^2 h_{k-1}(J_1,\dots,J_{n+1}) = \sum_{\substack{u,v=1\\ \text{distinct}}}^{n} [(u,n+1)(v,n+1)\sigma']h_{k-1}(J_1,\dots,J_{n+1}).$$

We need to figure out the cycle type of $(u,n+1)(v,n+1)\sigma'$. Note the cycle structure of $(u,n+1)(v,n+1)\sigma'$ differ from that of $\sigma'$ only on cycles containing $u$ and $v$. There are 2 cases.

(i) Suppose $u,v$ both appear in a cycle of length $\alpha_i$ with $\sigma^a(u) = v$ and $\sigma^b(v) = u$. Then

$$\begin{aligned}&(u,n+1)\cdot(v,n+1)\cdot(v,\sigma(v),\dots,\sigma^{b-1}(v),u,\sigma(u)\dots,\sigma^{a-1}(u))\\ =&(u,n+1)\cdot(v,\sigma(v),\dots,\sigma^{b}(v),u,\sigma(u)\dots,\sigma^{a-1}(u),n+1)\\ =&(v,\sigma(v),\dots,\sigma^{b-1}(v),n+1)\cdot(u,\sigma(u),\dots,\sigma^{a-1}(u)).\end{aligned}$$

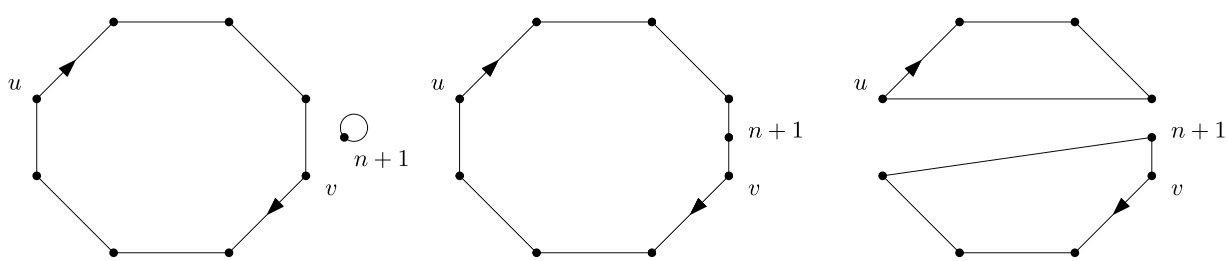



Figure 4.1: $u$ and $v$ appear in the same cycle.

Hence the cycle type of $(u,n+1)\cdot(v,n+1)\cdot\sigma'$ is $\alpha\setminus(\alpha_i)\cup(a,b+1)$. Note there are $\alpha_i$

choices for $v$. The corresponding coefficient is

$$\sum_{i=1}^{\ell(\alpha)} \sum_{\substack{a,b\geq 1\\ a+b=\alpha_i}} \alpha_i \varphi^{h_{k-1}}_{\alpha\setminus(\alpha_i)\cup(a,b+1)} = \sum_{i,j\geq 1} (i+j-1) m_{i+j-1}(\alpha) \varphi^{h_k}_{\alpha\setminus(i+j-1)\cup(i,j)}.$$

(ii) Suppose $u, v$ appear in different cycles of lengths $\alpha_u$ and $\alpha_v$ respectively. Then

$$\begin{aligned}
&(u, n+1)\cdot(v, n+1)\cdot(v, \sigma(v), \ldots, \sigma^{\alpha_v-1}(v))\cdot(u, \sigma(u), \ldots, \sigma^{\alpha_u-1}(u))\\
=&(u, n+1)\cdot(v, \sigma(v), \ldots, \sigma^{\alpha_v-1}(v), n+1)\cdot(u, \sigma(u), \ldots, \sigma^{\alpha_u-1}(u))\\
=&(v, \sigma(v), \ldots, \sigma^{\alpha_v-1}(v), u, \sigma(u), \ldots, \sigma^{\alpha_u-1}(u), n+1).
\end{aligned}$$

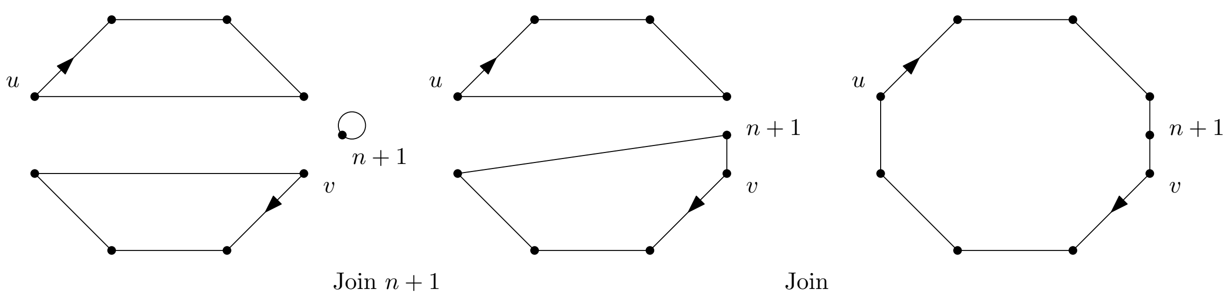



Figure 4.2: $u$ and $v$ appear in different cycles.

Hence the cycle type of $(u, n+1)\cdot(v, n+1)\cdot\sigma'$ is $\alpha\setminus(\alpha_u, \alpha_v)\cup(\alpha_u+\alpha_v+1)$. For each pair of distinct $i, j = 1, \ldots, \ell(\alpha)$, there are $\alpha_i$ choices for $u$ and $\alpha_j$ choices for $v$. The corresponding coefficient is

$$\sum_{\substack{i,j=1\\ \text{distinct}}}^{\ell(\alpha)} \alpha_i\alpha_j \varphi^{h_{k-1}}_{\alpha\setminus(\alpha_i,\alpha_j)\cup(\alpha_i+\alpha_j+1)} = \sum_{i,j\geq 1} ij m_i(\alpha)(m_j(\alpha)-\delta_{ij})\varphi^{h_{k-1}}_{\alpha\setminus(i,j)\cup(i+j-1)}.$$

Note the term $\delta_{ij}$ deals with the case when we choose 2 distinct $i$-cycles.

Putting the 2 cases together, we have Equation (4.9). This concludes the proof.

This method can be extended to prove new induction relations. The relation

$$p_b^{\perp}\frac{p_{\alpha\cup(b)}}{z(\alpha\cup(b))} = \frac{p_\alpha}{z(\alpha)}, \quad b\geq 1$$

can be used to interpret $p_b^\perp$ as a projection operator. For each $n \geq 1$, define $p_b^\perp : \mathbb{C}[\mathcal{S}_n] \to \mathbb{C}[\mathcal{S}_n]$ as

$$p_b^\perp \sigma = \begin{cases} \sigma\big|_{\{1,\dots,n-k\}}, & (n-k+1,\dots,n) \text{ is a cycle in } \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

Note the cycle $(n-k+1,\dots,n)$ can be replaced by any cycle on the ground set $\{n-k+1,\dots,n\}$. The operator $p_b^\perp$ simply removes some pre-determined $b$-cycle containing $n+1$ so that the remaining permutation lives in $\mathcal{S}_{n-k}$. We achieve the desired effect

$$p_b^\perp \mathcal{C}_{\beta\cup(b)} = \mathcal{C}_\beta.$$

**Theorem 4.5.** *The relation*

$$\varphi^{h_k}_{\beta\cup(b)} = \delta_{1,b}\varphi^{h_k}_\beta + \sum_{j\geq 1} \beta_j \varphi^{h_{k-1}}_{\beta\setminus(\beta_j)\cup(\beta_j+b)} + \sum_{\substack{r,s\geq 1\\ r+s=b}} \varphi^{h_{k-1}}_{\beta\cup(r,s)}, \quad b\geq 1 \tag{4.14}$$

*completely determines expansion coefficients $\varphi^{h_k}_\beta$ for all $k \geq 1$ and $\beta \in \mathcal{P}$.*

We first recover Equations (4.8) and (4.9). Let $\alpha \in \mathcal{P}$ be chosen arbitrarily. Set $b = 1$ and $\beta = \alpha$. We get

$$\varphi^{h_k}_{\alpha\cup(1)} = \varphi^{h_k}_\alpha + \sum_{i\geq 1} i m_i(\alpha) \varphi^{h_{k-1}}_{\alpha\setminus(i)\cup(i+1)}.$$

If $i \geq 1$, specializing $\beta = \alpha \setminus (\alpha_i)$ and $b = \alpha_i + 1$ yields

$$\varphi^{h_k}_{\alpha\setminus(\alpha_i)\cup(\alpha_i+1)} = \sum_{\substack{j\geq 1\\ j\neq i}} \alpha_j \varphi^{h_{k-1}}_{\alpha\setminus(\alpha_i,\alpha_j)\cup(\alpha_i+\alpha_j+1)} + \sum_{\substack{r,s\geq 1\\ r+s=\alpha_i+1}} \varphi^{h_{k-1}}_{\alpha\setminus(\alpha_i)\cup(r,s)}.$$

Multiply through by $\alpha_i$ and sum over all $i \geq 1$. We get Equation (4.9) after re-indexing. The term $\delta_{i,j}$ in Equation (4.9) is accounted for by the condition $j \neq i$.

*Proof of Theorem 4.5.* Let $b \geq 1$ and let $\sigma \in \mathcal{S}_{n+1-b}$ be a permutation with cycle type $\alpha \vdash (n+1-b)$. We apply $p_b^\perp$ to the group algebra recurrence

$$h_k(J_1,\dots,J_{n+1}) = h_k(J_1,\dots,J_n,0) + J_{n+1}h_{k-1}(J_1,\dots,J_{n+1}).$$

Note

$$p_b^\perp h_k(J_1,\dots,J_{n+1}) = \sum_{\beta\vdash(n+1)} \varphi^{h_k}_\beta p_b^\perp \mathcal{C}_\beta = \sum_{\alpha\vdash(n+1-k)} \varphi^{h_k}_{\alpha\cup(b)} \mathcal{C}_\alpha.$$

In other words $[\sigma]p_b^\perp h_k(J_1,\dots,J_{n+1}) = [\mathcal{C}_\alpha]p_b^\perp h_k(J_1,\dots,J_{n+1}) = \varphi^{h_k}_{\alpha\cup(b)}$.

Since $h_k(J_1, \dots, J_n, 0)$ is the canonical embedding of $\mathcal{S}_n$ in $\mathcal{S}_{n+1}$ with $n+1$ being a fixed point, then $[\sigma]p_b^\perp h_k(J_1, \dots, J_n, 0) = \delta_{1,b}\varphi_\alpha^{h_k}$. Now let $\sigma^* \in \mathcal{S}_{n+1}$ denote the permutation obtained from $\sigma$ by adding the cycle $(n+1, \dots, n+k)$. Then

$$\begin{aligned}
[\sigma]p_k^\perp J_{n+1}h_k(J_1, \dots, J_{n+1}) &= [\sigma^*]J_{n+1}h_k(J_1, \dots, J_{n+1}) \\
&= [J_{n+1}\sigma^*]h_k(J_1, \dots, J_{n+1}) \\
&= \sum_{v=1}^{n}[(v, n+1)\sigma^*]h_k(J_1, \dots, J_{n+1}).
\end{aligned}$$

Once again, there are 2 cases.

- If $v$ is in the ground set of $\sigma$, then $v$ and $n+1$ appear in 2 different cycles in $\sigma^*$ by definition. Hence, the cycle type of $(v, n+1) \cdot \sigma^*$ is $\alpha \setminus (\alpha_i) \cup (\alpha_i + b)$ for some $\alpha_i \in \alpha$. It follows that

$$\sum_{v=1}^{n+1-k}[(v, n+1)\sigma^*]h_k(J_1, \dots, J_{n+1}) = \sum_{i=1}^{\ell(\alpha)} \alpha_i \varphi_{\alpha\setminus(\alpha_i)\cup(\alpha_i+b)}^{h_{k-1}}.$$

- If $v$ is not on the ground set of $\sigma$, then $v$ and $n+1$ both appear in the cycle $(n-k+1, \dots, n+1)$. The cycle $(n+1, \dots, n+k)$ is cut into 2 smaller cycles by $(v, n+1)$. Hence, the cycle type of $(v, n+1) \cdot \sigma^*$ is $\alpha \cup (r, s)$ for some $r, s \geq 1$ such that $r + s = b$. It follows that

$$\sum_{v=n+1-k+1}^{n+1}[(v, n+1)\sigma^*]h_k(J_1, \dots, J_{n+1}) = \sum_{\substack{r,s=b \\ r,s\geq 1}}^{\ell(\alpha)} \varphi_{\alpha(r,s)}^{h_{k-1}}.$$

The proof of Corollary 4.3 can be repeated to prove that this is a recurrence relation. □

We remark that since $p_b^\perp$ is a differential operator on Sym, its action can be interpreted as marking a canonical cycle in $\mathcal{S}_n$. In the context of the above proof, the canonical cycle is $(n-k+2, \dots, n+1)$.

Finally, consider turning induction relation (4.14) into a differential equation in terms of $\Phi^H$. Note

$$\varphi_{\beta\cup(b)}^{h_k}\frac{p_\beta}{z(\beta)} = p_b^\perp \varphi_{\beta\cup(b)}^{h_k}\frac{p_{\beta\cup(b)}}{z(\beta\cup(b))}.$$

If we sum over $\beta \in \mathcal{P}$, then the summand can no longer be collected into $\Phi^H$. The problem is uniqueness. If $\alpha \in \mathcal{P}$, then there are non-unique choices $\beta \in \mathcal{P}$ such that $\beta \cup (b) = \alpha$. We would have to sum over $\beta$ of the form $\alpha \setminus (\alpha_i)$ for some $i$ and then sum over $i \geq 1$. As explained in the paragraph following the statement of Theorem 4.5, we recover Lassalle's result. Hence, the PDE is Equation (4.13). This is not surprising since Lassalle's relations and Féray's relation are both obtained from the same group algebra recurrence.

## 4.5 Related Results

For completeness, we include some known results on expansions of other symmetric functions. We omit the proofs since they are fairly similar to the derivation of Equations (4.8) and (4.9). These results are due to Lassalle.

Using Theorem 3.8, we can expand the product $e_{(k1)}$ [24, Proposition 5.2]. For $n \geq k \geq 1$

$$e_{(k1)}(J_1, \dots, J_n) = \sum_{\substack{\alpha \vdash n \\ \ell(\alpha) = n-k-1}} \sum_{i \geq 2} m_i(\alpha) \binom{i}{2} \mathcal{C}_\alpha + \sum_{\substack{\alpha \vdash n \\ \ell(\alpha) = n-k+1}} \left( \binom{n}{2} - \sum_{i \geq 2} m_i(\alpha) \binom{i}{2} \right) \mathcal{C}_\alpha.$$

The power sums have an obvious 2 term recursive definition: $p_k(x_1, \dots, x_{n+1}) = p_{k-1}(x_1, \dots, x_n, 0) + x_k^{n+1}$. Apply the general strategy we obtain [24, Equation 6.1]

$$\begin{aligned} \varphi_{\alpha \cup (1)}^{p_k} &= \varphi_\alpha^{p_k} + \sum_{i \geq 1} i m_i(\alpha) \varphi_{\alpha \setminus (i) \cup (i+1)}^{p_{k-1}}, \\ \sum_{i \geq 1} i m_i(\alpha) \varphi_{\alpha \setminus (i) \cup (i+1)}^{p_k} &= -\varphi_\alpha^{p_{k-1}} \\ &\quad + \sum_{i,j \geq 1} \Big( ij m_i(\alpha)(m_j(\alpha) - \delta_{ij}) \varphi_{\alpha \setminus (i,j) \cup (i+j+1)}^{p_{k-1}} \\ &\qquad + (i+j-1) m_{i+j-1}(\alpha) \varphi_{\alpha \setminus (i+j-1) \cup (i,j)}^{p_{k-1}} \Big). \end{aligned}$$

Note that the power sum expansion coefficients recurrence are very similar to the ones for complete symmetric functions. Lassalle considered Hall-Littlewood symmetric functions $P_\alpha = P_\alpha(z)$. These are generalizations of well-known symmetric functions. In the one-row case, we recover $P_k(0) = h_k$ and $P_k(1) = p_k$. Lassalle worked out the following recurrences [24, Equations (8.1) and (8.2)]

$$\begin{aligned} \varphi^{P_k} &= \varphi_\alpha^{P_k} + \sum_{i \geq 1} i m_i(\alpha) \varphi_{\alpha \setminus (i) \cup (i+1)}^{P_{k-1}} \\ \sum_{i \geq 1} i m_i(\alpha) \varphi_{\alpha \setminus (i) \cup (i+1)}^{P_k} &= -nz \varphi_\alpha^{P_{k-1}} \\ &\quad + \sum_{i,j \geq 1} \Big( ij m_i(\alpha)(m_j(\alpha) - \delta_{ij}) \varphi_{\alpha \setminus (i,j) \cup (i+j+1)}^{p_{k-1}} \\ &\qquad + (i+j-1) m_{i+j-1}(\alpha) \varphi_{\alpha \setminus (i+j-1) \cup (i,j)}^{p_{k-1}} \Big). \end{aligned}$$

Generalized class expansion can also be thought of from partial permutations and shifted symmetric functions point of view. These objects have origin in representation theory [19]. Each of the above expansion coefficients can be studied in terms of shifted symmetric functions. For discussion in this direction, see Section 2.8 of [24]. For generating series of expansion coefficients of complete shifted symmetric functions, see Section 9 of [24].

# Chapter 5

# Revisiting with Lassalle's Operators

Féray's argument is reminiscent of an argument by Goulden and Jackson. The combinatorial analysis of transposition acting on a permutation was used in [16] to count transitive factorizations into transpositions. In this Chapter, we derive finer details of $\mathcal{L}$ operators from this point of view.

We distill a family of operators $\mathcal{U}$ due to Carrell and Goulden [4]. The combinatorics of these operators describes "lifting" actions moving from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$ involving Jucys-Murphy elements. More interestingly, the $\mathcal{U}$ operators describe these "lifting" operations simultaneously for all symmetric groups. We will also see that a parameter $h$ naturally arises as an elementary property from both algebraic and combinatorial points of view. In the closing section of this chapter, the development of a generating series targeting this $h$ parameter is treated using some tools motivated by mathematical physics.

## 5.1 Joins and Cuts

We first consider an operator due to Goulden [11]. Choose arbitrarily some $n \geq 1$ and let $\Delta = J_1 + \cdots + J_n$. We omit $n$ from the notation as it is unnecessary. We will see shortly its algebraic counterpart describes its behaviour simultaneously for all symmetric groups. Consider the multiplication $\Delta \cdot \sigma$ in $\mathbb{C}[\mathcal{S}_n]$. It turns out this action has a neat description as a differential operator on Sym. Jucys-Murphy elements provide a quick proof of its action on Schur functions.

**Theorem 5.1.** *For each $\lambda \in \mathcal{P}$, we have*

$$\Delta s_\lambda = p_1(\mathbf{c}_\lambda) s_\lambda.$$

*Proof.* Note $\Delta s_\lambda = \mathrm{ch}\left((J_1 + \cdots + J_n) \cdot \chi^\lambda\right)$. The result follows from Theorem 2.12 since

$$(J_1 + \cdots + J_n)\chi^\lambda = p_1(\mathbf{c}_\lambda)\chi^\lambda.$$

□

We only sketch the proof of the following result since it is very similar to the proof of Theorem 4.5.

**Theorem 5.2.**

$$\Delta = \frac{1}{2} \sum_{i,j \geq 1} \left( p_{i+j} p_i^{\perp} p_j^{\perp} + p_i p_j p_{i+j}^{\perp} \right).$$

*Proof Sketch.* Let $\sigma \in \mathcal{C}_\alpha$ be any permutation. Choose distinct vertices $u, v \in 1, \ldots, n$. The cycle structure of $(uv) \cdot \sigma$ only differs from that of $\sigma$ by the cycles in $\sigma$ containing $u, v$. There are 2 cases.

If $u, v$ appear in the same cycle in $\sigma$, then $(uv)$ cuts the cycles into 2 smaller cycles, one of length $i$ containing $u$ and the other of length $j$ containing $v$. The cycle type of $(uv) \cdot \sigma$ is hence $\alpha \setminus (i + j) \cup (i, j)$. This case is captured by $p_j p_i p_{i+j}^{\perp}$.

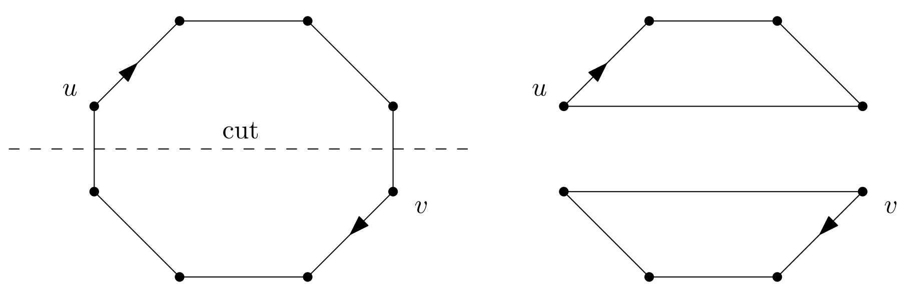



Figure 5.1: Cut Case

If $u, v$ appear in different cycles, say of lengths $i$ and $j$ respectively, then $(uv)$ joins these 2 cycles into a bigger cycle of length $i + j$. The cycle type of $(uv) \cdot \sigma$ is hence $\alpha \setminus (i, j) \cup (i + j)$. This case is captured by $p_{i+j} p_i^{\perp} p_j^{\perp}$.

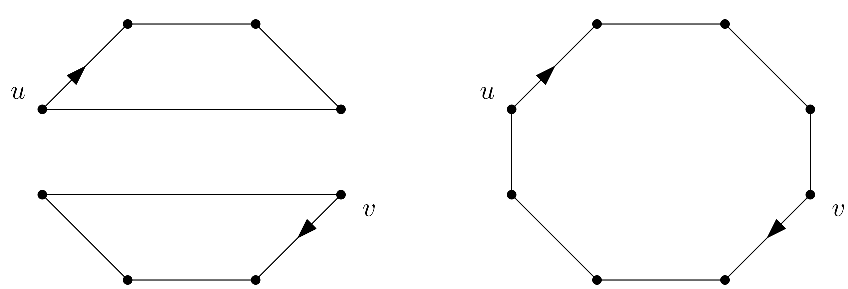



Figure 5.2: Join Case

The $\frac{1}{2}$ accounts for the symmetry that each $(uv)$ could be chosen in 2 different ways. $\square$

We observe the following obvious properties from the above 2 theorems.

(i) $\Delta$ is an eigenoperator on Schur functions, and

(ii) $\Delta$ is self-adjoint since $(p_i p_j p_{i+j}^{\perp})^{\perp} = p_{i+j} p_i^{\perp} p_j^{\perp}$.

Note the expression in Theorem 5.2 does not depend on $n$. So $\Delta$ describes the action $J_1 + \cdots + J_n$ simultaneously for all $n \geq 1$. Furthermore, comparing this expression with $\boldsymbol{D}(1)$, we get $\boldsymbol{D}(1) = \Delta + (n-1)\mathcal{E}$ where

$$\mathcal{E} = \sum_{i \geq 1} p_i p_i^{\perp},$$

since $\mathcal{E} p_\alpha = |\alpha| p_\alpha$. Now let $t$ be an indeterminate. Define $\mathcal{L}_0(t) = p_1$ and

$$\mathcal{L}_k(t) = [\Delta + t\mathcal{E}, \mathcal{L}_{k-1}], \quad k \geq 1.$$

Then it follows immediately that $\mathcal{L}_k(n-1) = \mathcal{L}_k$ for all $n, k \geq 1$.

## 5.2 Ups and Downs

We can pick apart $\mathcal{L}(t)$ operators to obtain a finer description of $\mathcal{L}$ operators. Such results will be used to prove Theorem 3.7 in Section 5.3. Define $\mathcal{U}_0 = p_1$ and $\mathcal{D}_0 = p_1^{\perp}$ and

$$\mathcal{U}_k = [\Delta, \mathcal{U}_{k-1}] \quad \text{and} \quad \mathcal{D}_k = [\mathcal{D}_{k-1}, \Delta], \quad k \geq 1. \tag{5.1}$$

These operators were first studied by Carrell and Goulden to count transitive factorizations [4].

Note by definition, $\mathcal{U}_k^{\perp} = \mathcal{D}_k$. Similar to that of $\mathcal{L}$ operators, their actions on Schur functions are easy to compute.

**Lemma 5.3.** *For $k \geq 0$, we have*

$$\mathcal{U}_k s_\lambda = \sum_{\mu = \lambda + \square} c(\square)^k s_\mu \quad \textit{and} \quad \mathcal{D}_k s_\lambda = \sum_{\mu = \lambda - \square} c(\square)^k s_\mu.$$

*Proof.* Let $\lambda \vdash n$. By Murnaghan-Nakayama rule, we have

$$\mathcal{U}_0 s_\lambda = \sum_{\mu = \lambda + \square} s_\mu \quad \text{and} \quad \mathcal{D}_0 s_\lambda = \sum_{\mu = \lambda - \square} s_\mu.$$

Then by induction, we have

$$\mathcal{U}_k s_\lambda = \sum_{\mu=\lambda+\Box} p_1(\mathbf{c}_\mu)c(\Box)^{k-1} - p_1(\mathbf{c}_\lambda)c(\Box)^{k-1} s_\mu = \sum_{\mu=\lambda+\Box} c(\Box)^k s_\mu.$$

Similarly we get the action for $\mathcal{D}_k s_\lambda$. □

Using the commutation relation $[p_a^\perp, p_b] = a\delta_{ab}$, we can compute the first few values of $\mathcal{U}_k$ directly from the definition. However, the process quickly becomes complicated and unmanageable. Its computation heavy proof can be found in the Appendices.

**Lemma 5.4.**

$$\mathcal{U}_0 = p_1,$$

$$\mathcal{U}_1 = \sum_{i\geq 1} p_{i+1}p_i^\perp,$$

$$\mathcal{U}_2 = \sum_{i,j\geq 1} \left(p_i p_j p_{i+j-1}^\perp + p_{i+j+1}p_i^\perp p_j^\perp\right),$$

$$\mathcal{U}_3 = \frac{1}{2}\left(\sum_{i,j\geq 1}(i+j)p_{i+j}p_{i+j-1}^\perp + \sum_{i,j\geq 1}\sum_{\substack{i',j'\geq 1\\ i+j=i'+j'+1}} p_i p_j p_{i'}^\perp p_{j'}^\perp\right) + \sum_{i,j,k\geq 1} p_{i+j}p_k p_j^\perp p_{i+k-1}^\perp$$
$$+ \sum_{i,j\geq 1} p_{i+j}p_i^\perp \sum_{\substack{i',j'\geq 1\\ i'+j'+1=j}} p_{i'}^\perp p_{j'}^\perp.$$

Note $\mathcal{U}_2$ has already appeared before in Section 4.3. Equation (4.13) can be rewritten as

$$\mathcal{U}_0\Phi^H = z\Phi^H + t^2\mathcal{U}_2\Phi^H.$$

We only treat $\mathcal{U}$ operators in the following sections. Analogous results can be obtained for $\mathcal{D}$ operator by simply taking adjoints.

## 5.3 Proof of Theorem 3.7

We need to express $\mathcal{L}_k(t)$ in terms of $\mathcal{U}_i$'s. We do so by comparing their actions on Schur functions.

**Theorem 5.5.** *For each $k \geq 1$, we have*

$$\mathcal{L}_k(t) = \sum_{i=0}^{k} \binom{k}{i} \mathcal{U}_i t^{k-i}.$$

*Proof.* By repeating the computation in the proof of Lemma 3.4, we get

$$\begin{aligned}
\mathcal{L}_k(t) s_\lambda &= \sum_{\mu=\lambda+\square} (c(\square)+t)^t s_\mu \\
&= \sum_{i=0}^{k} \binom{k}{i} t^{k-i} \sum_{\mu=\lambda+\square} c(\square)^i s_\mu \\
&= \sum_{i=0}^{k} \binom{k}{i} t^{k-i} \mathcal{U}_i s_\lambda.
\end{aligned}$$

The result follows since Schur functions form a basis of Sym. □

We are now in a position to get a short proof of Theorem 3.7. The grunt work has already been carried out in Lemma 5.4.

*Proof of Theorem 3.7.* Consider applying $\mathcal{L}_2(t)$ to $H_\lambda s_\lambda$ in 2 different ways. On one hand,

$$\mathcal{L}_2(t) H_\lambda s_\lambda = \sum_{i=0}^{2} \binom{2}{i} \sum_{\alpha \vdash n} \theta_\alpha^\lambda \mathcal{U}_i p_\alpha t^{2-i}.$$

On the other hand,

$$\mathcal{L}_2(t) H_\lambda s_\lambda = \sum_{i=0}^{2} \binom{2}{i} H_\lambda \mathcal{U}_i s_\lambda t^{2-i}.$$

Identify coefficients of $i$. For $i = 0, 1, 2$, we have

$$H_\lambda \mathcal{U}_i s_\lambda = \sum_{\alpha \vdash n} \theta_\alpha^\lambda \mathcal{U}_i p_\alpha.$$

After applying $\mathcal{U}_i$, the left-hand side becomes

$$\sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square)^i H_\mu s_\mu = \sum_{\beta \vdash (n+1)} \left( \sum_{\mu=\lambda+\square} \gamma_\lambda(\square) c(\square)^i \theta_\beta^\mu \right) p_\beta.$$

To conclude the proof, we simply replace $\mathcal{U}_i p_\alpha$ with corresponding expression from Lemma 5.4 and identify coefficients of $p_\beta$ for $\beta \vdash (n+1)$. □

This proof follows the approach in Lassalle's original proof. Our improvement is a cleaner way to package and compute the commutator relations using Theorem 5.5.

## 5.4 Combinatorial Interpretation

We observe that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ are all differential operators. This is of course true in general. Consider the generating series

$$\mathcal{U}(x) = \sum_{k\geq 0} \mathcal{U}_k \frac{x^k}{k!}.$$

**Theorem 5.6.** *We have* $\mathcal{U}(x) = \exp(x\Delta) p_1 \exp(-x\Delta)$*. Furthermore,*

$$\mathcal{U}(x, w) = \sum_{k\geq 0} \frac{x^k}{k!} \sum_{h\geq 0} w^h \sum_{\substack{\alpha,\beta\in\mathcal{P} \\ |\alpha|=|\beta|+1 \\ \ell(\alpha)+\ell(\beta)=k+1-2h}} u(k,h,\alpha,\beta) p_\alpha p_\beta^\perp.$$

*for some constants* $u(k, h, \alpha, \beta)$*.*

*Proof.* The recursive definition of $\mathcal{U}_k$ can be instead written as nested brackets. We have

$$\mathcal{U}_k = [\Delta, [\cdots, [\Delta, p_1]\cdots]] = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \Delta^{k-i} p_1 \Delta^i. \tag{5.2}$$

Then

$$\begin{aligned}
\mathcal{U}(x) &= \sum_{k\geq 0} \sum_{i=0}^{k} k! \frac{x^k}{k!} \frac{\Delta^{k-i}}{(k-i)!} p_1 \frac{(-\Delta)^i}{i!} \\
&= \sum_{i,j\geq 0} \frac{(\Delta x)^i}{i!} p_1 \frac{(-x\Delta)^j}{j!} \\
&= \exp(x\Delta) p_1 \exp(-x\Delta).
\end{aligned}$$

We prove the restrictions $|\alpha| = |\beta| + 1$ and $\ell(\alpha) + \ell(\beta) \leq k + 1 - 2h$ for some $h \geq 0$ by induction on $k$. The process is rather straightforward despite the appearances of long expressions. The base case has already been verified in Lemma 5.4.

Now suppose $u(k, h, \alpha, \beta) p_\alpha p_\beta^\perp$ is a monomial in $\mathcal{U}_k$ for some $k, h \geq 0$. Then $|\alpha| = |\beta| + 1$ and $\ell(\alpha) + \ell(\beta) = k + 1 - 2h$. Let $\mathcal{J} = \sum_{i,j\geq 1} p_{i+j} p_i^\perp p_j^\perp$ so that $\Delta = \frac{1}{2}(\mathcal{J} + \mathcal{J}^\perp)$

Note

$$[\mathcal{J}, p_\alpha p_\beta^\perp] = \sum_{i,j\geq 1} \left( p_{i+j} p_i^\perp p_j^\perp p_\alpha p_\beta^\perp - p_\alpha p_\beta^\perp p_{i+j} p_i^\perp p_j^\perp \right).$$

Since $p_i^\perp$'s are differential operators, we apply the product rule to get

$$\begin{aligned}[\mathcal{J}, p_\alpha p_\beta^\perp] = \sum_{i,j\geq 1} \Bigg( & \sum_{\substack{s,t=1\\ s\neq t}}^{\ell(\alpha)} \left(p_i^\perp p_{\alpha_s}\right) \left(p_j^\perp p_{\alpha_t}\right) p_{i+j} p_{\alpha\setminus(\alpha_s,\alpha_t)} p_\beta^\perp \\ & + \sum_{s=1}^{\ell(\alpha)} \left(p_i^\perp p_{\alpha_s}\right) p_{i+j} p_{\alpha\setminus(\alpha_s)} p_j^\perp p_\beta^\perp + \sum_{t=1}^{\ell(\alpha)} \left(p_j^\perp p_{\alpha_t}\right) p_{i+j} p_{\alpha\setminus(\alpha_t)} p_i^\perp p_\beta^\perp \\ & - \sum_{r=1}^{\ell(\beta)} \left(p_{\beta_r}^\perp p_{i+j}\right) p_\alpha p_{\beta\setminus(\beta_r)}^\perp p_i^\perp p_j^\perp \Bigg).\end{aligned}$$

We look at each monomial $p_\lambda p_\mu^\perp$ appearing in the above expression. There are 3 cases because the 2 expressions in the second line have the same form.

**Line 1.** We have $\lambda = \alpha \setminus (\alpha_s, \alpha_t) \cup (i+j)$ and $\mu = \beta$. If $(p_i^\perp p_{\alpha_s})(p_j^\perp p_{\alpha_t}) \neq 0$, then $\alpha_s = i$ and $\alpha_t = j$. Hence, $|\lambda| = |\mu| + 1$. It follows that $|\lambda| = |\alpha|$. Furthermore,

$$\ell(\lambda) + \ell(\mu) = k + 2 - 2(h+1).$$

**Line 2.** We have $\lambda = \alpha \setminus (\alpha_s)$ and $\mu = \beta \cup (j)$. If $p_i^\perp p_{\alpha_s} \neq 0$, then $\alpha_s = i$. It follows that $|\lambda| = |\alpha| + j$ and $|\mu| = |\beta| + j$. Hence, $|\lambda| = |\mu| + 1$. Furthermore, $\ell(\lambda) + \ell(\mu) = k + 2 - 2h$.

**Line 3.** We have $\lambda = \alpha$ and $\beta = \beta \setminus (\beta_r) \cup (i,j)$. If $p_{\beta_r}^\perp p_{i+j} \neq 0$, then $\beta_r = i+j$. It follows that $\mu = |\beta|$. Hence, $|\lambda| = |\mu| + 1$. Furthermore, $\ell(\lambda) + \ell(\mu) = k + 2 - 2h$.

Similarly, note

$$[\mathcal{J}^\perp, p_\alpha p_\beta^\perp] = \sum_{i,j\geq 1} \left( p_i p_j p_{i+j}^\perp p_\alpha p_\beta^\perp - p_\alpha p_\beta p_i p_j p_{i+j}^\perp \right).$$

Apply the product rule to get

$$[\mathcal{J}^\perp, p_\alpha p_\beta^\perp] = \sum_{i,j} \left( \sum_{r=1}^{\ell(\alpha)} \left(p_{i+j}^\perp p_{\alpha_r}\right) p_i p_j p_{\alpha\setminus(\alpha_r)} p_\beta^\perp \right.$$
$$- \sum_{\substack{s,t=1\\ s\neq t}}^{\ell(\beta)} \left(p_{\beta_s}^\perp p_i\right)\left(p_{\beta_t}^\perp p_j\right) p_\alpha p_{\beta\setminus(\beta_s,\beta_t)}^\perp p_{i+j}^\perp$$
$$\left. - \sum_{s=1}^{\ell(\beta)} \left(p_{\beta_s}^\perp p_i\right) p_\alpha p_j p_{\beta\setminus(\beta_t)}^\perp p_{i+j}^\perp - \sum_{t=1}^{\ell(\beta)} \left(p_{\beta_t}^\perp p_j\right) p_\alpha p_i p_{\beta\setminus(\beta_t)}^\perp p_{i+j}^\perp \right).$$

We look at each monomial $p_\lambda p_\mu^\perp$ appearing in the above expression. There are again 3 cases because the 2 expressions in the last line have the same form.

**Line 1.** We have $\lambda = \alpha \setminus (\alpha_r) \cup (i,j)$ and $\mu = \beta$. If $p_{i+j}^\perp p_{\alpha_r} \neq 0$, then $\alpha_r = i+j$. It follows that $|\lambda| = |\alpha|$. Hence $|\lambda| = |\mu| + 1$. Furthermore, $\ell(\lambda) + \ell(\mu) = k+2-2h$.

**Line 2.** We have $\lambda = \alpha$ and $\mu = \beta \setminus (\beta_s, \beta_t) \cup (i+j)$. If $(p_{\beta_s}^\perp p_i)(p_{\beta_t}^\perp p_j) \neq 0$, then $\beta_s = i$ and $\beta_t = j$. It follows that $|\mu| = |\beta|$. Hence $|\lambda| = |\mu| + 1$. Furthermore,

$$\ell(\lambda) + \ell(\mu) = k + 2 - 2(h+1).$$

**Line 3.** We have $\lambda = \alpha \cup (i)$ and $\mu = \beta \setminus (\beta_t) \cup (i+j)$. If $p_{\beta_t}^\perp p_i \neq 0$, then $\beta_t = i$. It follows that $|\lambda| = \alpha + j$ and $|\mu| = \beta + j$. Hence, $|\lambda| = |\mu| + 1$. Furthermore, $\ell(\lambda) + \ell(\mu) = k+2-2h$.

The result follows by induction. □

By taking adjoints, we get the dual result for $\mathcal{D}$ operators. The $\mathcal{U}$ operators bump up the total degree by 1 and $\mathcal{D}$ operators knock down the total degree by 1.

The parameter $h$ emerges naturally from matching pairs of $p_i^\perp$'s and $p_i$'s. The actions of $\mathcal{U}$ operators on power sums bring us closer to a combinatorial interpretation. We introduce a notation: If $\sigma \in \mathcal{S}_n$, then we write $\sigma' \in \mathcal{S}_{n+1}$ for its canonical embedding by adding $n+1$ as a fixed point.

**Lemma 5.7.** *If $\alpha \vdash n$, then*

$$\mathcal{U}_k p_\alpha = \sum_{\tau \in J_{n+1}^k} p_{\mathrm{cyc}(\tau\sigma')},$$

*where $\tau \in J_{n+1}^k$ is a monomial and $\sigma \in \mathcal{S}_n$ can be any permutation with cycle type $\alpha$.*

*Proof.* For each partition $\lambda$, denote $V^\lambda$ by the vector space carrying the corresponding irreducible representation of $\mathcal{S}_n$. For each $\sigma \in \mathcal{S}_n$, we denote its representing matrix with respect to Young's basis by $M^\lambda_\sigma$ and its character by $\chi^\lambda(\sigma) = \chi^\lambda_{\mathrm{cyc}(\sigma)} = \mathrm{tr}(M^\lambda_\sigma)$.

Consider $\mathcal{U}_k p_\alpha$ for some $\alpha \vdash n$. Expand $p_\alpha$ in terms of Schur functions, apply the definition of $\mathcal{U}_k$, and rearrange summations to get

$$\mathcal{U}_k p_\alpha = \sum_{\lambda \vdash n} \chi^\lambda_\alpha \sum_{\mu = \lambda + \square} c(\square)^k s_\mu = \sum_{\mu \vdash (n+1)} \mathrm{tr}\left( \sum_{\lambda = \mu - \square} c(\square)^k M^\lambda_\sigma \right) s_\mu,$$

where $\sigma$ is a permutation of cycle type $\alpha$. But $M^\mu_{J_{n+1}}$ is a diagonal matrix with contents of $n+1$ in corresponding tableaux of shape $\mu$ on its diagonal. And the set of standard tableaux of shape $\mu$ with $n+1$ in a common cell $\square$ forms the basis $V^{\mu-\square}$. So the inner sum above simplifies to

$$\sum_{\lambda = \mu - \square} c(\square)^k M^\lambda_\sigma = M^\mu_{J^k_{n+1}} M^\mu_{\sigma'} = M^\mu_{J^k_{n+1}\sigma'}, \tag{5.3}$$

where $\sigma'$ is induced from $\sigma$ by adding $n+1$ as a fixed point. By linearity of trace, we conclude

$$\mathcal{U}_k p_\alpha = \sum_{\mu \vdash (n+1)} \mathrm{tr}(M^\mu_{J^k_{n+1}\sigma'}) s_\mu = \sum_{\tau \in J^k_{n+1}\sigma'} \sum_{\mu \vdash n} \chi^\mu(\tau) s_\mu = \sum_{\tau \in J^k_{n+1}\sigma'} p_{\mathrm{cyc}(\tau)}.$$

$\square$

Note if $\mathrm{cyc}(\sigma) = \alpha$ then $\mathrm{ch}(\sigma) = p_\alpha / |\alpha|!$. It follows that

$$\mathrm{ch}(J^k_{n+1}\sigma') = \sum_{\tau \in J^k_{n+1}} \mathrm{ch}(\tau\sigma') = \mathcal{U}_k \frac{p_\alpha}{n!}.$$

In other words, $\mathcal{U}_k$ is the operator that describes the action $J^k_{n+1} \circ p_1$ where $p_1 : \mathbb{C}[\mathcal{S}_n] \to \mathbb{C}[\mathcal{S}_{n+1}]$ is the canonical embedding operator defined as $p_1(\sigma) = \sigma'$.

As operators on group algebras, the canonical projection $p_1^\perp$ and canonical embedding $p_1$ indeed form an adjoint pair since

$$\langle p_1^\perp \pi, \sigma \rangle = \delta_{\pi,\sigma'} = \langle \pi, p_1 \sigma \rangle,$$

for all $\pi \in \mathcal{S}_{n+1}$ and $\sigma \in \mathcal{S}_n$. It follows that $\mathcal{D}_k$ describes the action of $p_1^\perp \circ (J^k_{n+1})^\perp$. Note $p_1 p_\alpha = p_{\alpha \cup (1)}$ and $p_1^\perp p_\alpha = p_{\alpha \setminus (1)}$ for any $\alpha \in \mathcal{P}$. Hence as operators on symmetric functions they faithfully describe their group algebra counterparts.

We summarize the interpretation for $p_k$ and $p_k^\perp$ below for $k \geq 1$. Fix a canonical choice of permutation $\kappa$ on the ground set $\{n+1, \ldots, n+k\}$. Then multiplication by $p_k$ is the canonical embedding (with respect to $\sigma$) operator adding to each permutation in $\mathcal{S}_n$ the cycle $\kappa$ and $p_k^\perp$ is the canonical projection (with respect to $\kappa$) filtering permutations not having $\kappa$ in their cycles. Furthermore, $p_k$ and $p_k^\perp$ with respect to the same $\kappa$ are adjoint operators. In particular, when $k = 1$ then $\kappa = (n+1)$ is a fixed point and $p_1, p_1^\perp$ are the adjoint pair describing add-a-point embedding and remove-a-point projection simultaneously for all symmetric groups.

We now turn to the parameter $h$. The appearance of Jucys-Murphy elements suggests we consider joins and cuts as we did with multiplication by transpositions in Section 5.1. We shall see that $h$ counts the number of a certain type of joins. The combinatorial interpretation of monomials in the Join-Cut operator $\Delta$ naturally lends themselves to this application.

Recall a monomial $p_{i+j}p_i^\perp p_j^\perp$ in $\Delta$ describes the action of joining an $i$-cycle to a $j$-cycle to form an $(i+j)$-cycle marked by $p_{i+j}$. The subscript in the monomial (after differentiation and multiplication) $p_{i+j}p_i^\perp p_j^\perp p_\alpha$ is the resulting cycle type. We say an operator $p_i^\perp$ *grabs* an $i$-cycle. Similarly, a monomial $p_ip_jp_{i+j}^\perp$ describes the action of cutting an $(i+j)$-cycle into an $i$-cycle and a $j$-cycle. The subscript in the monomial (after differentiation and multiplication) $p_ip_jp_{i+j}^\perp p_\alpha$ is the resulting cycle type.

Let $k \geq 0$ and $n \geq 1$ and $\alpha \vdash n$. To understand the $h$ parameter, we interpret each monomial in $\mathcal{U}_k p_\alpha$. An example is provided after the statement of Theorem 5.8.

Consider the cycle type of $\tau\sigma'$ where $\tau \in J_{n+1}^k \sigma'$ and $\sigma \in \mathcal{C}_\alpha$. Write as a product of transpositions $\tau = \tau_k \cdots \tau_1$ where $\tau_1, \ldots, \tau_k \in J_{n+1}$. We consider the multiplication $\tau\sigma'$ in stages. Note the 0-th stage is the embedding $\sigma' = p_1\sigma$. Hence define $\sigma^{(0)} = \sigma'$ and

$$\sigma^{(i)} = \tau_{i-1} \cdot \sigma^{(i-1)}, \quad \text{for } i = 1, \ldots, k.$$

For $i = 0, \ldots, k$, the *canonical cycle* in $\sigma^{(i)}$ is the one containing the vertex $n+1$. The cycles in $\sigma$ are called *initial* cycles. We denote $s_i$ the length of the canonical cycle in $\sigma^{(i)}$. In particular, $s_1 = 1$.

The action of $\tau_i \cdot \sigma^{(i-1)}$ is split into 2 cases:

- The canonical cycle is cut into 2 cycles. The one not containing $n+1$ is called the *spare* cycle (at stage $i$). Denote the length of the spare cycle by $t_i$. We grab a canonical cycle by $p_{s_i}^\perp$ and split it into a spare cycle marked by $p_{t_i}$ and the canonical cycle of the next stage marked by $p_{s_{i+1}}$. The operator monomial describing this action is $p_{s_{i+1}}p_{t_i}p_{s_i}^\perp$.
- The canonical cycle is joined to another cycle of length $-t_i$. We grab the canonical cycle by $p_{s_i}^\perp$ and another cycle by $p_{-t_i}^\perp$ and join them to form the canonical cycle of the next stage marked by $p_{s_{i+1}}$. The operator monomial describing this action is $p_{s_{i+1}}p_{-t_i}^\perp p_{s_i}^\perp$.

Denote[1] $p_{-a} = p_a^\perp$ for $a \geq 1$. It follows that in either case the actions are described by

$$p_{s_{i+1}} p_{t_i} p_{s_i}^\perp.$$

Therefore the cycle type of $\sigma^{(k)}$ is the subscript in the monomial (after differentiations and multiplications)

$$\left(p_{s_{k+1}} p_{t_k} p_{s_k}^\perp\right) \cdots \left(p_{s_2} p_{t_1} p_{s_1}^\perp\right) p_1 p_\alpha. \tag{5.4}$$

Note $p_i^\perp p_i$ is a scalar for $i > 0$. The above contributes to $\mathcal{U}_k p_\alpha$ a monomial

$$\begin{aligned}(5.4) &= u(k,\tau,\alpha) p_{s_{k+1}} p_{t_k} \left(p_{s_k}^\perp p_{s_k}\right) p_{t_{k-1}} \cdots p_{t_1} \left(p_{s_1}^\perp p_1\right) p_\alpha \\ &= u(k,\tau,\alpha) p_{s_{k+1}} p_{t_k} \cdots p_{t_1} p_\alpha \end{aligned} \tag{5.5}$$

for some scalar $u(k,\tau,\alpha)$. Note that there are $k+1$ operators in the last line. This expression also preserves the action performed at each stage. A join is described by $p_{t_j}$ for some $t_j \leq -1$ and a cut is described by $p_{t_i}$ for some $t_i \geq 1$.

We now reduce Expression (5.5) by commuting into the form $p_\lambda p_\mu^\perp$. Consider 2 cases to a join action.

- If the canonical cycle is joined to an essential cycle, then we call the action an *essential join*.
- If the canonical cycle is joined to a spare cycle created at some previous stage, then we call the action a *spare join*.

If at stage $i$ we have a spare join, then by definition there exists some $i' < i$ such that $t_i < 0$ and $t_{i'} + t_i = 0$. Commute operators so that $p_{t_i}$ appears immediately to the left of $p_{t_{i'}}$. If there is some $i''$ such that $i' < i'' < i$ and $t_{i'} = t_{i''}$, the we chooses the largest one to be $i'$. We get

$$p_{s_{k+1}} p_{t_k} \cdots p_{t_i} \cdots p_{t_{i'}} \cdots p_{t_1} = p_{s_{k+1}} p_{t_k} \cdots \left(p_{t_i} p_{t_{i'}}\right) \cdots p_{t_1}.$$

The total number of operators in the right-hand side goes down by 2 since $p_{t_i} p_{t_{i'}} = p_{t_i} p_{-t_i}$ is a scalar.

Suppose there are $h$ number of spare joins and $j$ number of essential joins. Then the above reduction results in

$$p_{s_{k+1}} p_{t_k} \cdots p_{t_1} p_\alpha = u(k,\tau,\alpha,h) p_{a_1} \cdots p_{a_{k+1-2h-j}} p_{b_1}^\perp \cdots p_{b_j}^\perp p_\alpha$$

where $a_i, b_i \geq 1$ and $u(\tau,\alpha,k,h) \neq 0$ is some scalar. We have just derived the following result.

[1] This is common in mathematical physics literatures on vertex operators. They like to use $\alpha_i = p_i^\perp$ and $\alpha_{-i} = p_i$ for $i \geq 1$.

**Theorem 5.8.** *Suppose $\alpha \vdash n$ and $\sigma \in \mathcal{C}_\alpha$. If $\tau_k, \dots, \tau_1 \in J_{n+1}$, then $\tau_k \cdots \tau_1 \sigma'$ contributes to $\mathcal{U}_k p_\alpha$ a monomial*

$$u(k, \tau, \alpha, h) p_{a_1} \cdots p_{a_{k+1-2h-j}} p_{b_1}^\perp \cdots p_{b_j}^\perp p_\alpha$$

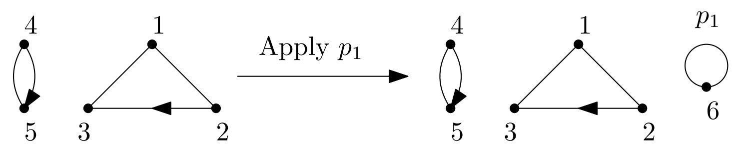
*for some $h \geq 0$ and some scalar $u(k, \tau, \alpha, h) \neq 0$. The parameter $h$ counts the number of spare joins. Moreover, we have*

$$(b_1 + \cdots + b_j) - (a_1 + \cdots + a_{k+1-2h-j}) = -1. \tag{5.6}$$

*Proof.* We have already proved the first part. For the second part, note $s_{i+1} = s_i + t_i$ implies $s_{i+1} = 1 + t_1 + \cdots + t_i$. In particular, $s_{k+1} = 1 + t_1 + \cdots + t_k$ implies Equation (5.6). □

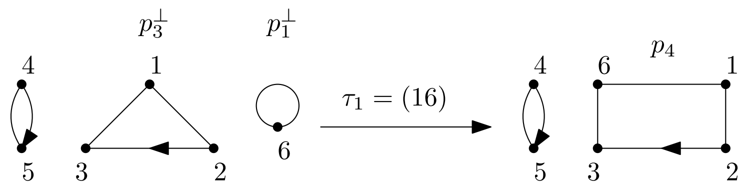
To illustrate the above process, we consider an example. Let $k = 4$ and $\sigma = (123)(45) \in \mathcal{S}_5$. Consider $\tau = (26)(46)(26)(16)$. Let $\alpha = \text{cyc}(\sigma) = (32)$.

(0). We embed $\sigma$ into $\mathcal{S}_6$ using the canonical embedding $p_1$.



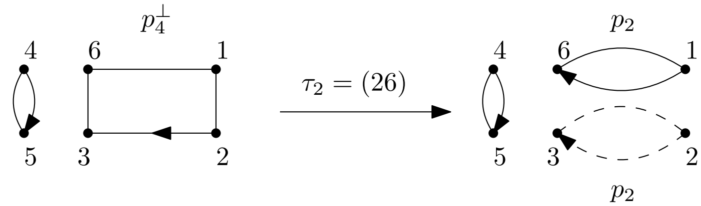

The change in cycle type is recorded in the calculation $p_1 p_\alpha = p_{(32)\cup(1)} = p_{(321)}$.

(1). We have $\sigma_0 = (123)(45)(6)$. The action $\tau_1 = (16)$ on $\sigma_0$ is a join.



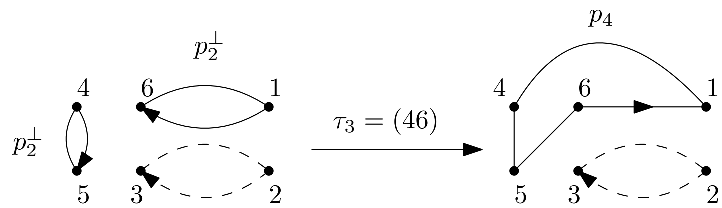

The change in cycle type is recorded in the calculation $p_4 p_3^\perp p_1^\perp p_\alpha = p_{(321)\setminus(31)\cup(4)} = p_{(42)}$.

(2). We have $\sigma_1 = (1234)(36)$. The action $\tau_2 = (26)$ on $\sigma_1$ is a cut. The spare cycle (23) is drawn with dash lines.



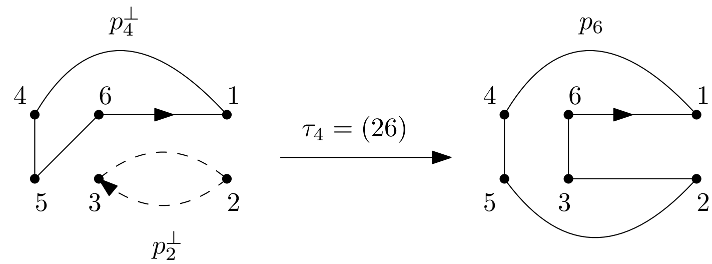
The change in cycle type is recorded in the calculation $p_2p_2p_4^\perp p_{(42)} = p_{(42)\setminus(4)\cup(22)} = p_{(222)}$.

(3). We have $\sigma_2 = (45)(23)(16)$. The action $\tau_3 = (46)$ on $\sigma_2$ is an essential join.




The change in cycle type is recorded in the calculation $p_4p_2^\perp p_2^\perp p_{(222)} = p_{(222)\setminus(22)\cup(4)} = p_{(42)}$.

(4). We have $\sigma_3 = (23)(1456)$. The action $\tau_4 = (26)$ on $\sigma_3$ is a spare join.




The change in cycle type is recorded in the calculation $p_6p_2^\perp p_4^\perp p_{(42)} = p_{(42)\setminus(4,2)\cup(6)} = p_{(6)}$.

So $\tau\sigma' = (145236)$. Note the 5 stages of multiplication $\tau\sigma'$ are captured by

$$\left(p_6p_2^\perp p_4^\perp\right)\left(p_4p_2^\perp p_2^\perp\right)\left(p_2p_2p_4^\perp\right)\left(p_4p_3^\perp p_1^\perp\right)p_1\cdot p_\alpha$$

which reduces to (ignoring scalar multiples)

$$\begin{aligned} p_6p_2^\perp\left(p_4^\perp p_4\right)p_2^\perp\left(p_2^\perp p_2\right)p_2\left(p_4^\perp p_4\right)p_3^\perp\left(p_1^\perp p_1\right)\cdot p_\alpha &= p_6p_2^\perp p_2^\perp p_2p_3^\perp\cdot p_\alpha \\ &= p_6p_2^\perp p_3^\perp\cdot p_\alpha. \end{aligned}$$

Note that the right-hand side of the first equality is what we expect from Equation (5.5). The contribution to $\mathcal{U}_4p_\alpha$ is $p_6p_2^\perp p_3^\perp p_\alpha$ with $h = 1$. This completes the example.

**Corollary 5.9.** *For each $k \geq 0$ we have*

$$\mathcal{U}_k^{(0)} = \sum_{j=1}^{k}\frac{1}{k+1}\binom{k+1}{j}\sum_{\substack{a_1,\ldots,a_{k+1-j},b_1,\ldots,b_j\geq 1\\ a_1+\cdots+a_{k+1-j}+1=b_1+\cdots+b_j}} p_{a_1}\cdots p_{a_{k+1-j}}p_{b_1}^\perp\cdots p_{b_j}^\perp.$$

*Proof.* Note $h = 0$ means there is no spare joins. In the notations above (replacing $s_{k+1}$ by $t_{k+1}$ for convenience), this amounts to counting sequences of non-zero integers $t_{k+1}, t_k, \ldots, t_1$ such that

$$t_{k+1} + t_k + \cdots + t_1 = -1$$

such that each partial sum represeting the length of the canonical cycle at stage $i$ satisfies

$$t_i + \cdots + t_1 + 1 \geq 1, \quad i \geq 1$$

Suppose there are $j$ number of essential joins. We choose $j$ number of $t_i$'s to be negative. There are clearly $\binom{k+1}{j}$ such choices. By the Cycle Lemma, we must divide by $k+1$ so that the above 2 conditions are satisfied. Assign $a_i$ and $b_j$ such that as multisets $\{a_1, \ldots, a_{k+1-j}\} = \{t_i > 0 : i \geq 1\}$ and $\{b_1, \ldots, b_j\} = \{t_i < 0 : i \geq 1\}$.

Finally, note the first stage is always an essential join so that $j \geq 1$. Since $s_{k+1} = t_{k+1} \geq 1$ so $k + 1 - j \geq 1$. We conclude that $j$ must be chosen from $\{1, \ldots, k\}$. □

## 5.5 Operator Generating Series

We wish to find expressions for $\mathcal{U}^{(h)}(x) = [w^h]\mathcal{U}(x, w)$. The proof of Theorem 5.6 doesn't seem useful. In this section, we describe a different method to get explicit expressions for $\mathcal{U}^{(h)}(x)$.

The Join-Cut operator $\Delta$ can be thought of as a simultaneous description of multiplication actions by

$$p_1(J_1, \ldots, J_n)$$

in all symmetric groups. Lascoux and Thibon considered a generalization [23]. The pair were interested in finding an operator $D$ on symmetric functions simultaneously describing multiplication actions by

$$F_n(t) = \sum_{k \geq 1} p_k(J_1, \ldots, J_n) \frac{t^n}{n!} = \sum_{i=1}^{n} (\exp(J_i) - 1)$$

in all symmetric groups. A method called *Bosonisation* from mathematical physics was proven effective. The Bosonisation of an operator on symmetric functions uses its action on Schur functions to find an expression in terms of $p_i$'s and $p_i^\perp$'s. They found a generating series

$$V = V(q, t, \mathbf{p}, \mathbf{p}^\perp) = \exp\left(\sum_{k \geq 1} \left(q^{k/2} - q^{-k/2}\right) p_k \frac{t^k}{k}\right) \exp\left(\sum_{k \geq 1} \left(q^{k/2} - q^{-k/2}\right) p_k^\perp \frac{t^{-k}}{k}\right)$$

such that

$$D = \frac{[t^0]V - 1}{\left(q^{1/2} - q^{-1/2}\right)^2} - \mathcal{E}.$$

Note $\mathcal{E} = \sum_{i \geq 1} p_i p_i^{\perp}$ is used in Section 5.1 to define $\mathcal{L}(t)$ operators.

A related family of operators is called the Bernstein operators [25, Example 29, p. 95]. Recall $H(t), E(t), P(t)$ denote generating series for complete, elementary and power sum symmetric functions. Define $B(t) = H(t)E^{\perp}(-t^{-1})$. The Berstein operators are $B_n = [t^n]B(t), n \in \mathbb{Z}$. Their adjoints are denoted as $B_n^{\perp}, n \in \mathbb{Z}$. Since $H(t) = \log P(t)$ and $H(t)E(-t) = 1$, we immediately have

$$B(t) = \sum_{n \in \mathbb{Z}} B_n t^n = \exp\left(\sum_{k \geq 0} p_k \frac{t^k}{k}\right) \exp\left(-\sum_{k \geq 0} p_k^{\perp} \frac{t^{-k}}{k}\right)$$

$$B^{\perp}(t) = \sum_{n \in \mathbb{Z}} B_n^{\perp} t^n = \exp\left(-\sum_{k \geq 0} p_k \frac{t^k}{k}\right) \exp\left(\sum_{k \geq 0} p_k^{\perp} \frac{t^{-k}}{k}\right).$$

It is well known that Berstein operators are *creation* operators for Schur function, i.e.,

$$B_{\lambda_k} B_{\lambda_{k-1}} \cdots B_{\lambda_1} \cdot 1 = s_{\lambda}.$$

It does so by Pieri's rule to create each row successively starting from the longest one. We now introduce some operations on partitions then state a result due to Carrell and Goulden [5] that describe their actions on Schur functions.

Given a partition $\lambda$, its *rim hook* is the skew shape $\mu/\lambda$ where $\mu$ is the smallest partition containing $\lambda$ and the cells $(1, \lambda_1 + 1)$, $(\lambda_1' + 1, 1)$ such that $\mu/\lambda$ is edge connected. Note each partition is determined uniquely by its rim hook. A cell on the rim hook of $\lambda$ *lies* below $\lambda$ if it appears below $\lambda$ on its Ferrers diagram. Similarly, a cell on the rim hook of $\lambda$ *lies* to the right of $\lambda$ if it appears to the right of $\lambda$ on its Ferrers diagram.

A *border strip* $\mu/\lambda$ is a skew shape that is also an edge-connected subset of the rim hook of $\lambda$. We can simply specify a border strip by the contents of its cells. Its *length* is the number of boxes and its *height* $\mathrm{ht}(\mu/\lambda)$ is the number of rows minus 1.

For example, the rim hook of $\lambda = (4, 3, 3, 1)$ with its contents filled in is shown in the following diagram. The cells labelled $-2, 1, 2, 4$ lay to the right of $\lambda$ and cells $-4, -2, -1, 2$ lay below $\lambda$. Note the subset $\{-2, \ldots, 3\}$ of the rim hook is not a border strip since it is not a skew shape (cannot be obtained by $\mu/\lambda$ for any $\mu \in \mathcal{P}$).

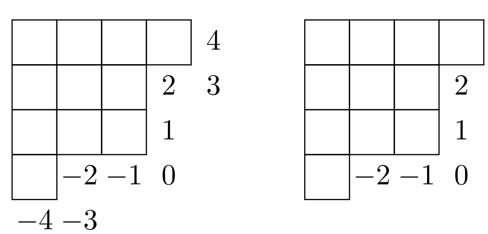



Figure 5.3: The rim hook and a border strip (having height 2) of $\lambda = (4, 3, 3, 1)$.

Suppose $\lambda \vdash n$ is a partition and $c$ is an integer. If the cell in the rim hook of $\lambda$ with content $c$ lies below $\lambda$, then define $b_c\lambda$ to be the partition obtained from $\lambda$ by removing the last cell from each row ending in a cell with content greater than $c$ and adding a cell to the end of each column ending in a cell with content is less or equal to $c$. Otherwise $b_c\lambda = 0$. Define $r_c(\lambda)$ to be the number of rows of $\lambda$ above $\square$.

The following diagram illustrates this operation in a 2-step process using $\lambda = (4, 3, 3, 1)$ from the above example. Cells to be removed are marked with $-$ and cells to be inserted are marked with $+$. If $b_c\lambda \neq 0$, then $|b_c\lambda| = |\lambda| + c$.

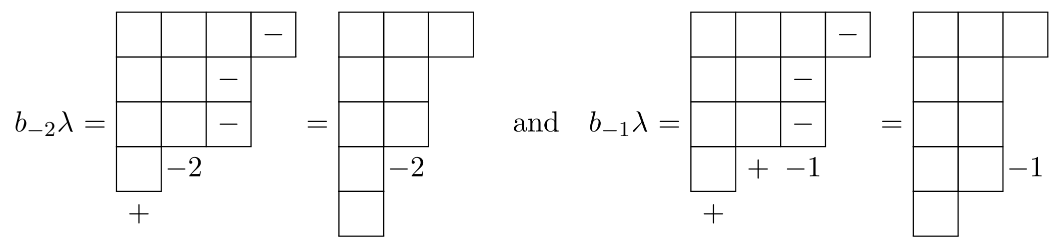



Figure 5.4: $b_{-2}\lambda = (3, 2, 2, 1, 1)$ and $b_{-1}\lambda = (3, 2, 2, 2, 1)$

Similarly, define an operation $b_c^*$ as follows. If the cell in the rim hook of $\lambda$ with content $c$ lies to the right of $\lambda$, then define $b_c^*\lambda$ to be the partition obtained from $\lambda$ by removing the last cell from each column ending with a cell having content less than $c$ and adding a cell to the end of each row ending with a cell having content greater or equal to $c$. If $b_c^*\lambda \neq 0$, then $|b_c^*\lambda| = |\lambda| - c$. Define $r_c^*(\lambda)$ to be the number of rows in $\lambda$ above $\square$.

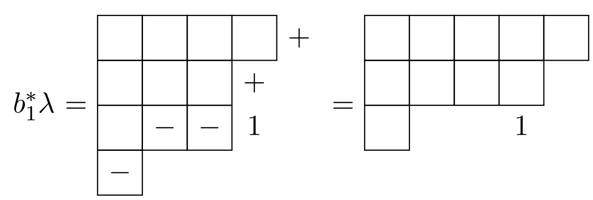



Figure 5.5: $b_1^*(\lambda) = (5, 4, 1)$

**Theorem 5.10.** *If $\lambda \in \mathcal{P}$ and $n, m \in \mathbb{Z}$, then*

$$B_n s_\lambda = (-1)^{r_n(\lambda)} s_{b_n\lambda} \quad \textit{and} \quad B_{-m}^{\perp} s_\lambda = (-1)^{r_m^*(\lambda)} s_{b_m^*\lambda}.$$

Note the above statement is modified slightly to use the language of $b$-operations instead of codes of partitions (also known as the abacus model of partitions). Now Consider applying $b_n b_m^*$ to a partition $\lambda$. We again use the partition $\lambda = (3, 2, 2, 1, 1)$ as an example. The difference between $\lambda$ and $b_n b_m^*\lambda$ are labelled with $\circ$ denoting cells added to $\lambda$ and $\times$ denoting cells removed from $\lambda$.

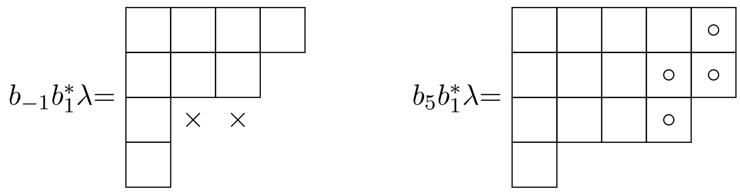



Figure 5.6: $b_n b_m^* \lambda$

Observe that in both cases, the difference between $b_n b_m^* \lambda$ and $\lambda$ is a border strip of length $|n-m|-1$. This in fact is a general phenomenon.

**Theorem 5.11.** *Suppose $\lambda \in \mathcal{P}$ and $n, m \in \mathbb{Z}$ such that $b_n b_m^* \lambda \neq 0$.*

- *If $n > m$, then $B_n B_{-m}^{\perp} s_\lambda = (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu$ where $\mu/\lambda$ is a border strip of length $n-m$ such that $\mathbf{c}_{\mu/\lambda} = \{m, m+1, \ldots, n-1\}$.*
- *If $n < m$, then $B_n B_{-m}^{\perp} s_\lambda = (-1)^{\mathrm{ht}(\lambda/\mu)} s_\mu$ where $\lambda/\mu$ is a border strip of length $m-n$ such that $\mathbf{c}_{\lambda/\mu} = \{n, n+1, \ldots, m-1\}$.*

*If $n = m$, then $B_n B_{-m}^{\perp} s_\lambda = s_\lambda$.*

*Proof.* Note by definition we have

$$B_n B_{-m}^{\perp} s_\lambda = (-1)^{r_n(b_m^*\lambda) + r_m^*(\lambda)} s_{b_n b_m^* \lambda}.$$

Denote $\nu = b_m^* \lambda$ and $\mu = b_n b_m^* \lambda$. Let $\square = (i, j)$ be the cell in the rim hook of $\lambda$ with content $m$ and let $\square' = (i', j')$ be the cell in the rim hook of $\nu$ with content $n$.

We prove the first case but only sketch the other 2 because they are similar.

- Suppose $m < n$.

  Because $\square$ lies to the right of $\lambda$, every column to the left of $\square$ in $\lambda$ has length at least $i$. Then every column to the left of $\square$ in $\nu$ has length at least $i-1$. So the last cell in each column to the left of $\square$ has content less or equal to $(j-1)-(i-1) = m$. But $m < n$. So cells removed by $b_m^*$ are put back by $b_n$. Since $\mu \neq 0$, the cell $\square'$ must lay below $\nu$. Then each row above $\square'$ has length at least $j'$. The last cell in each row has content at least $j' - i' = n$. But $n > m$. So cells in rows above $\square'$ in $\nu$ added by $b_m^*$ are removed by $b_n$. The cell in $\nu$ above $\square$ is the last cell in a column with content $m+1 \leq n$. So $b_n$ adds $\square$ to $\mu$. Hence, the number of cells in $\mu/\lambda$ is $n-m$.

Note $r^*_m(\lambda)$ is the number of rows above $\square$ and $r_n(\nu)$ is the number of rows above $\square'$. So

$$(-1)^{r_n(\nu)+r^*_m(\lambda)} = (-1)^{r^*_m(\lambda)-r_n(\nu)} = (i-1)-(i'-1)-1$$

is the number of rows between $\square$ and $\square'$ (non-inclusive). This is precisely the height of $\mu/\lambda$. Note $\mu/\lambda$ is clearly a border strip.

- Suppose $m > n$. Note $i \le i'i$. So every cell added by $b^*$ is removed by $b$. Similarly, cells added by $b$ form a subset of the cells added by $b^*$. Note

$$(-1)^{r_n(\nu)+r^*_m(\lambda)} = (-1)^{r_n(\nu)-r^*_m(\lambda)} = (i'-1)-(i-1)-1$$

is the number of rows between $\square'$ and $\square$. This is exactly the height of $\lambda/\mu$.

- Suppose $m = n$. Note that cells in $\lambda$ removed by $b^*_m$ is put back by $b_n$ because their are the last cells in columns of $\nu$ to the left of $\square$ and their contents are at most $m$. Similarly, cells in $\nu$ added by $b^*_m$ are removed by $b_n$ because they are the last cells in rows above $\square$ and their contents are greater than $m$.

This completes the proof. $\square$

The following corollary leads to an expression of $\mathcal{U}(x)$.

**Corollary 5.12.** *Let $q$ be an indeterminate. If $\lambda \in \mathcal{P}$ and $k \ge 1$, then*

$$[t^k]q^{-1/2}B(tq^{1/2})B^{\perp}(tq^{-1/2})s_\lambda = \sum_{\mu}(-1)^{\mathrm{ht}(\mu/\lambda)}q^{\left(\sum_{\square\in\mu/\lambda}c(\square)/k\right)}s_\mu,$$
$$[t^{-k}]q^{-1/2}B(tq^{1/2})B^{\perp}(tq^{-1/2})s_\lambda = \sum_{\mu}(-1)^{\mathrm{ht}(\lambda/\mu)}q^{\left(\sum_{\square\in\lambda/\mu}c(\square)/k\right)}s_\mu,$$

*where the sum run over all $\mu$ such that $\mu/\lambda$ (or $\lambda/\mu$ respectively) is a rim hook of size $k$.*

When $k = 1$ and $q = e^x$, we have

$$[t^1]e^{-x/2}B(te^{x/2})B^{\perp}(te^{-x/2})s_\lambda = \sum_{\mu=\lambda+\square}\sum_{k\ge 0}c(\square)^k s_\mu \frac{x^k}{k!}.$$

It follows that

$$\mathcal{U}(x) = [t^1]e^{-x/2}B(te^{x/2})B^{\perp}(te^{-x/2}).$$

Using the $V$ series introduced by Lascoux and Thibon, we can find an useful expression for $\mathcal{U}(x)$.

**Theorem 5.13.**

$$\mathcal{U}(x) = [t^1]\frac{1}{2\sinh(x/2)}\exp\left(\sum_{k\geq 1} 2\sinh(kx/2)p_k\frac{t^k}{k}\right)\exp\left(2\sinh(kx/2)p_k^{\perp}\frac{t^{-k}}{k}\right).$$

*Proof.* We use the following 2 elementary properties of exponentials of differential operators. For any scalars $a, b$ and indeterminate $z$, we have

$$\exp\left(a\frac{\partial}{\partial z}\right)f(z) = \left(\exp\left(a\frac{\partial}{\partial z}f(z)\right)\right)\exp\left(a\frac{\partial}{\partial z}\right) \tag{5.7}$$

where we treat $f(z)$ as a multiplication operator and

$$\exp(a\frac{\partial}{\partial z})\exp(bz) = \exp(ab)\exp(bz). \tag{5.8}$$

Using these properties with $z = p_k$, we have the following operator identity

$$\begin{aligned}
\exp\left(\sum_{k\geq 1} a_k p_k^{\perp}\right)\exp\left(\sum_{m\geq 1} b_m p_m\right) &= \prod_{k,m\geq 1}\exp\left(a_k k\frac{\partial}{\partial p_k}\right)\exp(b_m p_m)\\
&= \prod_{k,m\geq 1}\left(\exp\left(a_k k\frac{\partial}{\partial p_k}\right)\exp(b_m p_m)\right)\exp\left(a_k k\frac{\partial}{\partial p_k}\right)\\
&= \prod_{k,m\geq 1}\delta_{k,m}\exp\left(k a_k b_m\right)\exp(b_m p_m)\exp(a_k p_k^{\perp})\\
&= \exp\left(\sum_{k\geq 1} k a_k b_k\right)\exp\left(\sum_{m\geq 1} b_m p_m\right)\exp\left(\sum_{k\geq 1} a_k p_k^{\perp}\right)
\end{aligned}$$

for scalars $a_k, b_k$. We get the second equality by applying Equation (5.7) with $f = \exp(b_m p_m)$ and the third equality by applying Equation (5.8) to the first term.

Note $B(tq^{1/2})B^{\perp}(tq^{-1/2})$ can be written as

$$\exp\left(\sum_{k\geq 1}\frac{q^{k/2}t^k}{k}p_k\right)\exp\left(\sum_{k\geq 1} -\frac{q^{-k/2}t^{-k}}{k}p_k^{\perp}\right)\exp\left(\sum_{m\geq 1} -\frac{q^{-m/2}t^m}{m}p_m^{\perp}\right)\exp\left(\sum_{m\geq 1}\frac{q^{m/2}t^{-m}}{m}p_m^{\perp}\right).$$

Apply the above operator identity to the middle 2 terms. We have

$$\begin{aligned}
&\exp\left(\sum_{k\geq 1} -\frac{q^{-k/2}t^{-k}}{k}p_k^{\perp}\right)\exp\left(\sum_{m\geq 1} -\frac{q^{-m/2}t^{m}}{m}p_m^{\perp}\right)\\
=&\exp\left(\sum_{k\geq 1} k\frac{q^{-k/2}t^{-k}}{k}\frac{q^{-k/2}t^{k}}{k}\right)\exp\left(\sum_{m\geq 1} -\frac{q^{-m/2}t^{m}}{m}p_m\right)\exp\left(\sum_{k\geq 1} -\frac{q^{-k/2}t^{-k}}{k}p_k^{\perp}\right)\\
=&\frac{1}{1-q^{-1}}\exp\left(\sum_{m\geq 1} -\frac{q^{-m/2}t^{m}}{m}p_m\right)\exp\left(\sum_{k\geq 1} -\frac{q^{-k/2}t^{-k}}{k}p_k^{\perp}\right).
\end{aligned}$$

Note $q^{-1/2}\frac{1}{1-q^{-1}} = \frac{1}{q^{1/2}-q^{-1/2}}$. Putting everything together, we have

$$q^{-1/2}B(tq^{1/2})B^{\perp}(tq^{-1/2}) = \frac{1}{q^{1/2}-q^{-1/2}}V(q,t,\mathbf{p},\mathbf{p}^{\perp}).$$

We conclude by substituting $q = e^x$ and recalling $\sinh(x) = \frac{e^x - e^{-x}}{2}$. □

By expanding sinh, we can get a more detailed expression of $\mathcal{U}_k$. Note

$$\begin{aligned}
\mathcal{U}(x) &= [t^1]\left(\sum_{i\geq 0} 2\frac{(x/2)^{2i+1}}{(2i+1)!}\right)^{-1}\exp\left(\sum_{k\geq 1}\sum_{i\geq 0} 2\frac{(kx/2)^{2i+1}}{(2i+1)!}p_k\frac{t^k}{k}\right)\exp\left(\sum_{k\geq 1}\sum_{i\geq 0} 2\frac{(kx/2)^{2i+1}}{(2i+1)!}p_k^{\perp}\frac{t^{-k}}{k}\right)\\
&= [t^1]x^{-1}\left(\sum_{i\geq 0}\frac{x^{2i}}{2^{2i}(2i+1)!}\right)^{-1}\exp\left(\sum_{i\geq 0}\frac{x^{2i+1}}{2^{2i}(2i+1)!}Q_{2i}(t)\right)\exp\left(\sum_{i\geq 0}\frac{x^{2i+1}}{2^{2i}(2i+1)!}Q_{2i}^{\perp}(t^{-1})\right),
\end{aligned}$$

where

$$Q_j(t) = \sum_{k\geq 1} k^j p_k t^k \quad \text{and} \quad Q_j^{\perp}(t^{-1}) = \sum_{k\geq 1} k^j p_k^{\perp} t^{k^{-1}}, \quad j \geq 0.$$

The second term is (multiplicatively) invertible because its constant term is 1. To pick up a monomial $p_\alpha p_\beta^{\perp} x^k$ in the coefficient of $t^1$, we take $x^{-1+2h_1}$ from the first 2 terms, $p_\alpha x^{\ell(\alpha)+2h_2}$ from the second, and $p_\beta^{\perp} x^{\ell(\beta)+2h_3}$ from the third term. Then we arrive at a different proof of Theorem 5.6 which states for all $k, h \geq 0$ we have

$$\mathcal{U}_k^{(h)} = \sum_{\substack{\alpha,\beta\in\mathcal{P}\\ |\alpha|=|\beta|+1\\ \ell(\alpha)+\ell(\beta)=k+1-2h}} u(k,h,\alpha,\beta)p_\alpha p_\beta^{\perp},$$

where $u(k, h, \alpha, \beta)$ are scalars.

From the above paragraph, the contribution to the $h$ parameter comes from the last 3 terms. Hence we insert $w^{2i}$ in the last 3 terms to mark genus by $w$. Let $\tilde{Q}_i(t) = Q_i(t) + Q_i^{\perp}(t)$. We then get for each $h \geq 0$

$$\mathcal{U}^{(h)}(x) = [w^{2h}t^1]x^{-1}\left(\sum_{i\geq 0}\frac{w^{2i}x^{2i}}{2^{2i}(2i+1)!}\right)^{-1}\exp\left(\sum_{i\geq 0}\frac{w^{2i}x^{2i+1}}{2^{2i}(2i+1)!}\tilde{Q}_{2i}(t)\right).$$

We can read off $\mathcal{U}^{(h)}(x)$ for small values of $h$.

$$\begin{aligned}
\mathcal{U}^{(0)}(x) &= [t^1]x^{-1}\exp\left(x\tilde{Q}_0(t)\right)\\
\mathcal{U}^{(1)}(x) &= [t^1]\frac{1}{24}\left(\tilde{Q}_2(t) - x\right)\exp\left(x\tilde{Q}_0(t)\right)\\
\mathcal{U}^{(2)}(x) &= [t^1]\frac{1}{5760}\left(5x^5\tilde{Q}_2(t)^2 + 3x^4\tilde{Q}_4(t) - 10x^4\tilde{Q}_2(t) - 3x^3\right)\exp\left(x\tilde{Q}_0(t)\right).
\end{aligned}$$

In particular, we have an algebraic proof of Corollary 5.9.

# Chapter 6

# Enumeration of Transitive Factorizations

We have all the bells and whistles to take on the transitivity condition in permutation factorization problems. In Section 1.2, we introduced 3 classes of transitive factorizations problems. Each problem was solved using a different method. Using the tools introduced in last chapter, Carrell and Goulden found a unified algebraic approach to all 3 problems. Our intent is to describe this method.

Being eigenvalues of Jucys-Murphys elements, contents of partitions show up in an interesting way. We first develop a generating series involving contents of partitions. The $\mathcal{U}$ operators, whose action on Schur functions involve contents, are used to determine a partial differential equation. The results on the $h$ parameter are used to derive a partial differential equation for genus 0 content series.

We should mention that the combinatorial analysis of multiplication of arbitrary permutations remains a open problem. The inductive nature of $\mathcal{U}$ operators and the success of this method seem to suggest a new avenue. However, we were unsuccessful in this persuit.

## 6.1 Content Series

In this section, we derive 3 generating series for well-known classes of transitive factorizations introduced in Section 1.2. We then distill a family of series indexed by a univariate series involving contents of partitions. Once again, we use Jucys-Murphy elements to find a combinatorial interpretation.

We first consider transitive factorizations with no restrictions. Their counts are known as the $m$-hypermap numbers where $m$ is the number of factors. We have already derived their generating series in Example 2.17. It is

$$\mathbf{G}_m(y, z, \mathbf{p}) = \log\left(\sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \chi^\lambda_{(1^n)} s_\lambda \prod_{\Box\in\lambda} (1 + yc(\Box))^m\right). \tag{6.1}$$

We now consider transitive factorizations into transpositions. Their counts are the Hurwitz numbers. Consider an expression in the group algebra for not-necessarily-transitive factorizations into transpositions. If $\tau_1, \ldots, \tau_m$ are transpositions in $\mathcal{S}_n$, then we can naturally associate a graph on vertices $\{1, \ldots, n\}$ using $\tau_i$'s as edges. Since order matters in a factorization, an edge $\tau_i$ is labelled with its position $i$. Hence, if $n \geq 0$ then

$$\Theta^{\mathbf{H}}_n = \sum_{m\geq 0} \frac{1}{m!}(J_1 + \cdots + J_n)^m$$

is the group algebra expression for all not-necessarily-transitive factorizations into transpositions in $\mathcal{S}_n$. Note the inner summation can be written as an exponential function. Then we can rewrite $\Theta^{\mathbf{H}}_n = \exp(p_1(J_1, \ldots, J_n))$. This is a central element in $\mathbb{C}[\mathcal{S}_n]$. Hence, it has a class expansion

$$\Theta^{\mathbf{H}}_n = \sum_{\alpha\vdash n} \varphi^{e^{p_1}}_\alpha \mathcal{C}_\alpha.$$

Take its image under the characteristic map. Then $\mathrm{ch}^n\,\Theta^{\mathbf{H}}_n$ is its generating series in power sums with $p_\alpha$ marking the cycle type $\alpha$ of a product $\sigma_0 = \tau_1 \cdots \tau_m$ for some $m \geq 0$. We now take advantage of Theorem 2.12. Note $\sum_{\lambda\vdash n} \mathcal{F}^\lambda = 1$ where $\mathcal{F}^\lambda$ are central orthogonal idempotents. Multiply by $z^n$ and sum over $n \geq 0$ to get an expression for all transposition factorizations for all groups. Following the computation for hypermap numbers, we have

$$\begin{aligned}
\mathrm{ch} \sum_{n\geq 0} z^n \Theta^{\mathbf{H}}_n \left(\sum_{\lambda\vdash n} \mathcal{F}^\lambda\right) &= \sum_{n\geq 0} z^n \sum_{\lambda\vdash n} \mathrm{ch}^n \exp(p_1(J_1, \ldots, J_n)) \frac{\chi^\lambda}{H_\lambda} \\
&= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \frac{n!}{H_\lambda} \mathrm{ch}^n \exp(p_1(J_1, \ldots, J_n))\chi^\lambda \\
&= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \chi^\lambda_{(1^n)} \exp(p_1(\mathbf{c}_\lambda)) s_\lambda \\
&= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \chi^\lambda_{(1^n)} s_\lambda \prod_{\Box\in\lambda} \exp(c(\Box)).
\end{aligned}$$

Let $y$ mark the length of a factorization. Note connected objects in factorizations are exactly the transitive ones. Hence the generating series for transitive factorizations into transpositions is

$$\mathbf{H}(y, z, \mathbf{p}) = \log\left(\sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \chi^{\lambda}_{(1^n)} s_\lambda \prod_{\square\in\lambda} \exp(yc(\square))\right). \tag{6.2}$$

Lastly, we consider transitive monotone factorizations into transpositions. Their counts are known as monotone Hurwitz Numbers. Recall from Section 1.2, a factorization into transpositions $\sigma_0 = (a_1b_1)\cdots(a_mb_m)$ is monotone if $a_i < b_i$ for all $i = 1, \ldots, m$ and $b_1 \leq \cdots \leq b_m$. We call each distinct $b_i$ a *pivot*. The multiplicity of a pivot $b_i$ is the number of times it appears as some $b_j$. A monotone transposition factorization *pivoted* at $b_1, \cdots, b_k$ is one with pivots $b_1, \ldots, b_k$ and $b_1 \leq \cdots \leq b_k$. We first write down a group algebra expression for not-necessarily connected ones in $\mathcal{S}_n$. Let $a_1, \ldots, a_n \geq 0$ be integers. Then

$$J_1^{a_1} \cdots J_n^{a_n}$$

is the group algebra expression for monotone transposition factorizations pivoted at $\{i \geq 1 : a_i > 0\}$. Then

$$\Theta_n^{\vec{\mathbf{H}}} = \sum_{a_1,\ldots,a_n\geq 0} J_1^{a_1} \cdots J_n^{a_n}$$

is the group algebra expression for all not-necessarily-transitive monotone transposition factorizations in $\mathcal{S}_n$. Note the summation is secretly

$$\Theta_n^{\vec{H}} = \sum_{k\geq 0} h_k(J_1, \ldots, J_n).$$

Hence $\Theta_n^{\vec{H}}$ is a central element in $\mathbb{C}[\mathcal{S}_n]$ with class expansion

$$\Theta_n^{\vec{\mathbf{H}}} = \sum_{k\geq 0} \sum_{\alpha\vdash n} \varphi_\alpha^{h_k} \mathcal{C}_\alpha.$$

It follows that its image under the characteristic map is a generating series in power sums with $p_\alpha$ marking the cycle type $\alpha$ of a monotone transposition factorization $\sigma_0 = (a_1b_1)\cdots(a_mb_m)$. We again take advantage of Theorem 2.12 to get

$$\begin{aligned}
\operatorname{ch} \sum_{n\geq 0} \Theta^{\vec{\mathbf{H}}} \left(\sum_{\lambda\vdash n} \mathcal{F}^\lambda\right) &= \sum_{n\geq 0} z^n \sum_{\lambda\vdash n} \operatorname{ch}^n \sum_{k\geq 0} h_k(J_1, \ldots, J_n) \frac{\chi^\lambda}{H_\lambda} \\
&= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \chi^\lambda_{(1^n)} \operatorname{ch} \prod_{i=1}^{n} \frac{1}{1 - J_i} \chi^\lambda \\
&= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda\vdash n} \chi^\lambda_{(1^n)} s_\lambda \prod_{\square\in\lambda} \frac{1}{1 - c(\square)}.
\end{aligned}$$

Let $y$ mark the length of a factorization. It follows that the generating series for transitive monotone transposition factorization is

$$\vec{\mathbf{H}}(y,z,\mathbf{p}) = \log\left(\sum_{n\geq 0}\frac{z^n}{n!}\sum_{\lambda\vdash n}\chi^{\lambda}_{(1^n)}s_\lambda\prod_{\square\in\lambda}\frac{1}{1-yc(\square)}\right). \tag{6.3}$$

The generating series for all 3 special classes contain a product involving contents of partitions. Let $f(x)$ be a formal power series. A *content series* indexed by $f(x)$ is

$$\Phi^{f(x)}(y,z,\mathbf{p}) = \sum_{n\geq 0}\frac{z^n}{n!}\sum_{\lambda\vdash n}\chi^{\lambda}_{(1^n)}s_\lambda\prod_{\square\in\lambda}f(yc(\square)).$$

We also define the *connected* content series to be $\Psi^{f(x)}(y,z,\mathbf{p}) = \log\Phi(y,z,\mathbf{p})$. It follows immediately that the content series generalize the generating series of the 3 aforementioned special classes:

$$\begin{aligned}\mathbf{G} &= \Psi^{(1+x)^m},\\ \mathbf{H} &= \Psi^{e(x)},\\ \vec{\mathbf{H}} &= \Psi^{(1-x)^{-1}}.\end{aligned}$$

The content series $\Phi^{f(x)}$ is a generating of a family of central elements parameterize by $f$. The expression in the above definition can be thought of as a "decomposition" into irreducible representations in the following sense. Let $F(\mathbf{x}) = \prod_{i\geq 1}f(x_i)$. Then its specialization $F(J_1,\ldots,J_n)$ describes some central element in $\mathcal{S}_n$ for each $n\geq 1$. So $F$ describes a family of central elements simultaneously for all symmetric groups. Then by Theorem 2.12, we get

$$\Phi^{f(x)}(y,z,\mathbf{p}) = \sum_{n\geq 0}z^n\,\mathrm{ch}\sum_{\lambda\vdash n}\mathcal{F}^\lambda F(yJ_1,\ldots,yJ_n).$$

Recall $\mathcal{F}^\lambda$ is the projection onto irreducible representation indexed by $\lambda$. Hence, $\mathcal{F}^\lambda F(J_1,\ldots,J_n)$ is the irreducible portion of the generating series group algebra expression $F(J_1,\ldots,J_n)$. But $\sum_{\lambda\vdash n}\mathcal{F}^\lambda = 1$ so we get

$$\Phi^{f(x)}(y,z,\mathbf{p}) = \mathrm{ch}\sum_{n\geq 0}z^n F(yJ_1,\ldots,yJ_n). \tag{6.4}$$

Recall from Section 1.2 that the genus of a transitive factorization is given by the Riemann-Hurwitz formula. Conveniently, if $f(x) = (1-x)^{-1}$, then $F(\mathbf{x}) = H(\mathbf{x};1)$ is the generating series for complete symmetric functions which, when evaluated at Jucys-Murphy elements, encode monotone transposition factorizations.

**Lemma 6.1.** *Let $f_0, f_1, \ldots$ be indeterminate and let $f(x) = \sum_{i\geq 0} f_i x^i$. Then*

$$\Psi^{f(x)}(y,z,\mathbf{p}) = \sum_{g\geq 0} \Psi_g^{f(x)}(y,z,\mathbf{p}),$$

*where*

$$\Psi_g^{f(x)}(y,z,\mathbf{p}) = \sum_{n\geq 1} \frac{z^n}{n!} \sum_{\alpha\vdash n} \psi(g,\alpha) p_\alpha y^{n+\ell(\alpha)-2+2g},$$

*for some polynomial $\psi(g,\alpha)$ in $f_1, f_2, \ldots$.*

*Proof.* Consider monotone transposition factorizations. Mark a monotone factorization in

$$\sigma_0 = (a_1 b_1)\cdots(a_m b_m)$$

by $f_1^{i_1}\cdots f_m^{i_m} p_\alpha$ if the multiplicity of pivot $j$ is $i_j$ and $\mathrm{cyc}(\sigma_0) = \alpha$. Sum over all possible choices of $i_1, i_2, \ldots$ and $\alpha \in \mathcal{P}$. It follows immediately from Equation (6.4) that

$$\Phi^{f(x)}(y,z,\mathbf{p}) = \sum_{z\geq 0} \frac{z^n}{n!} \sum_{\alpha\vdash n} \varphi_\alpha^{h_k}(f_1,\ldots) p_\alpha$$

where $\varphi_\alpha^{h_k}(f_1,\ldots)$ is a series in $f_i$'s and it marks all monotone transpositions factorizations whose products have cycle type $\alpha$.

It follows that $\Psi^{f(x)}(y,z,\mathbf{p})$ is the generating series for transitive monotone transposition factorizations. If $\sigma_0 = (a_1b_1)\cdots(a_mb_m)$ is a genus $g$ transitive monotone transpositions factorization, then by Riemann-Hurwitz formula we have

$$n - \ell(\alpha) + m = n - \ell(\alpha) + \sum_{i=1}^{m} \Big(n - \ell\big(\mathrm{cyc}((a_ib_i))\big)\Big) = 2n - 2 + 2g.$$

In other words, $m = n + 2 + \ell(\alpha) - 2g$. Since $y$ marks the length of a factorization, we immediately have

$$\Psi^{f(x)}(y,z,\mathbf{p}) = \sum_{g\geq 0}\sum_{n\geq 1} \frac{z^n}{n!} \sum_{\alpha\vdash n} \psi(g,\alpha) p_\alpha y^{n+\ell(\alpha)-2+2g} \tag{6.5}$$

as desired. □

We say that $\Psi_g^{f(x)}(y,z,\mathbf{p})$ is the *genus $g$ connected* content series.

## 6.2 A Partial Differential Equation

The $\mathcal{U}$ operators are related to Sekiguchi-Debiard operators because they have a similar range of summation. Define for $k \geq 0$ eigenoperators for Schur functions

$$\mathcal{C}_k s_\mu = \left( \frac{n}{\chi^\mu_{(1^n)}} \sum_{\mu=\lambda+\Box} c(\Box)^k \chi^\lambda_{(1^{n-1})} \right) s_\mu.$$

We only need to use the fact that for any $\lambda \in \mathcal{P}$

$$\mathcal{C}_0 s_\lambda = |\lambda| s_\lambda.$$

**Theorem 6.2.** *Let $f(x) = \sum_{i\geq 0} f_i x^i$ and $g(x) = \sum_{i \geq 0} g_i x^i$ with $g_0 \neq 0$ be formal power series. Then $g^{-1}$ exists and $\Phi^{fg^{-1}(x)}$ is the unique solution to the partial differential equation*

$$\left( \sum_{i\geq 0} f_i y^i \mathcal{U}_i \right) \Phi^{fg^{-1}(x)} = z^{-1} \left( \sum_{i \geq 0} g_i y^i \mathcal{C}_i \right) \Phi^{fg^{-1}}$$

*with initial condition $\Phi^{fg^{-1}}(y, 0, \mathbf{p}) = 1$.*

*Proof.* We start with the left-hand side.

$$\begin{aligned}
z^{-1} \left( \sum_{i\geq 0} g_i y^i \mathcal{C}_i \right) \Phi^{fg^{-1}(x)} &= z^{-1} \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\mu \vdash n} \prod_{\Box \in \mu} \frac{f(yc(\Box))}{g(yc(\Box))} \sum_{i \geq 0} g_i y^i \frac{n}{\chi^\mu_{(1^n)}} \sum_{\mu=\Box+\Box''''} c(\Box')\chi^\lambda_{(1^{n-1})} s_\mu \\
&= \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\mu \vdash (n+1)} \sum_{\mu=\lambda+\Box'} \prod_{\Box\in\mu} \frac{f(yc(\Box))}{g(yc(\Box))} g(yc(\Box)) \chi^\lambda_{(1^n)} s_\mu \\
&= \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\mu \vdash (n+1)} \sum_{\mu=\lambda+\Box'} \prod_{\substack{\Box\in\mu \\ \Box \neq \Box'}} \frac{f(yc(\Box))}{g(yc(\Box))} f(yc(\Box)) \chi^\lambda_{(1^n)} s_\mu \\
&= \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\lambda \vdash n} \chi^\lambda_{(1^n)} \prod_{\Box \in \lambda} \frac{f(yc(\Box))}{g(yc(\Box))} \sum_{\mu = \lambda+\Box'} f(yc(\Box')) s_\mu \\
&= \left( \sum_{i \geq 0} f_i y^i \mathcal{U}_i \right) \Phi^{fg^{-1}(x)}.
\end{aligned}$$

We can read off the initial condition from the definition of $\Phi^{fg^{-1}(x)}$. $\Box$

**Corollary 6.3.** *Let $f(x) = \sum_{i \geq 0} f_i x^i$ be a formal power series. Then the content series $\Phi^{f(x)}$ is the unique solution to the partial differential equation*

$$\left( \sum_{i \geq 0} f_i y^i \mathcal{U}_i \right) \Phi^{f(x)} = \frac{\partial}{\partial z} \Phi^{f(x)} \tag{6.6}$$

*with initial condition $\Phi^{f(x)}(y, 0, \mathbf{p}) = 1$.*

*Proof.* Apply the above theorem with $g = 1$. Note $\mathcal{C}_0 s_\mu = |\mu| s_\mu$ for any $\mu \in \mathcal{P}$. We get

$$\begin{aligned}
\left( \sum_{i \geq 0} f_i y^i \mathcal{U}_i \right) \Phi^{f(x)} &= z^{-1} \mathcal{C}_0 \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\lambda \vdash n} \chi^\lambda_{(1^n)} s_\lambda \prod_{\square \in \lambda} f(yc(\square)) \\
&= \frac{\partial}{\partial z} \Phi^{f(x)}
\end{aligned}$$

Again, we can read off the initial condition from the definition of $\Phi^{f(x)}$. $\square$

## 6.3 Genus Specific PDE

The genus 0 case of the 3 special classes mentioned in derivation of the content series have all been enumerated. As a final application of the $\mathcal{U}$ operators, we consider Equation (6.6) from a genus point of view. We illustrate the process of getting genus specific partial differential equations by obtaining the genus 0 one.

We need a little lemma on differentiation.

**Lemma 6.4.** *Let $F = F(\mathbf{p})$ be a formal power series in power sums with no constant term. Suppose $\beta \in \mathcal{P}$ with $\ell(\beta) = m$. Denote $B_1 \sqcup \cdots \sqcup B_k = [m]$ a partition of $\{1, \ldots, m\}$ into pairwise disjoint and non-empty sets. Then*

$$p_\beta^\perp \exp(F) = \sum_{k=1}^{m} \sum_{B_1 \sqcup \cdots \sqcup B_k = [m]} \left( p^\perp_{\beta(B_1)} F \right) \cdots \left( p^\perp_{\beta(B_k)} F \right) \exp(F).$$

*Proof.* Note $\exp(F)$ is well-defined and by the chain rule we have $p_i^\perp \exp(F) = (p_i^\perp F) \exp(F)$ and by the product rule

$$p_i^\perp p_j^\perp \exp(F) = \left( (p_i^\perp p_j^\perp F) + \left( p_i^\perp F \right) \left( p_j^\perp F \right) \right) \exp(F).$$

If $B \subseteq [m]$ is non-empty, then we denote $p_{\beta(B)} = \prod_{i \in B} p_{\beta_i}$. Let $B_1 \sqcup \cdots \sqcup B_k = [m]$. Then by product rule we have

$$p_{\beta_m}^{\perp} \left( p_{\beta(B_1)}^{\perp} F \right) \cdots \left( p_{\beta(B_k)}^{\perp} F \right) \exp(F)$$
$$= \left( \left( p_{\beta_m}^{\perp} F \right) \left( p_{\beta(B_1)}^{\perp} F \right) \cdots \left( p_{\beta(B_k)}^{\perp} F \right) + \sum_{i=1}^{k} \left( p_{\beta_m}^{\perp} p_{\beta(B_i)}^{\perp} F \right) \prod_{\substack{1 \leq j \leq k \\ i \neq j}} \left( p_{\beta(B_j)}^{\perp} F \right) \right) \exp(F).$$

Note $p_{\beta_m}^{\perp} p_{\beta(b_i)}^{\perp} = p_{\beta(B_i \cup \{m\})}^{\perp}$. Since partitions of $[m]$ can be partitioned into partitions where $m$ is a singleton or not, we conclude by induction on $m$. □

Now consider Equation (6.6). We expand $\mathcal{U}_k$ into its monomials and focus on its action using the above lemma. We will find that monomials from both sides have weight $y^{|\alpha|+\ell(\alpha)-2+2h}$ for some $h$.

Note $\Phi^{f(x)} = \exp \Psi^{f(x)}$. Then Equation (6.6) can be written as

$$\sum_{k \geq 0} f_k y^k \mathcal{U}_k \exp \Psi^{f(x)} = \frac{\partial}{\partial z} \exp \Psi^{f(x)}. \tag{6.7}$$

Expand $\mathcal{U}_k$'s by Theorem 5.6 and apply above Lemma with $F = \Psi^{f(x)}$. Multiply through by $z \exp(-\Psi^{f(x)})$ to get rid of the byproduct $\exp(\Psi^{f(x)})$. The left-hand side of the above equation becomes

$$zp_1 + \sum_{\substack{k \geq 1 \\ h \geq 0}} \sum_{\substack{\alpha, \beta \in \mathcal{P} \\ |\alpha| = |\beta|+1 \\ \ell(\alpha)+\ell(\beta)=k+1-2h}} \sum_{i=1}^{\ell(\beta)} \sum_{B_1 \sqcup \cdots \sqcup B_i = [\ell(\beta)]} v(k,h,\alpha,\beta,i) z y^k p_\alpha \left( p_{\beta(B_1)}^{\perp} \Psi^{f(x)} \right) \cdots \left( p_{\beta(B_i)}^{\perp} \Psi^{f(x)} \right).$$

Expand $\Psi^{f(x)}$ by Equation (6.5). Every monomial in $p_{\beta(B_j)}^{\perp} \Psi^{f(x)}$ has the form (up to scaling)

$$p_{\beta(B_j)}^{\perp} p_\gamma z^{|\gamma|} y^{|\gamma|+\ell(\gamma)-2+2g}.$$

It follows that a monomial in the left-hand side has the form (up to scaling)

$$z y^k p_\alpha \left( p_{\beta(B_1)}^{\perp} p_{\gamma^{(1)}} z^{|\gamma^{(1)}|} y^{|\gamma^{(1)}|+\ell(\gamma^{(1)})-2+2g_1} \right) \cdots \left( p_{\beta(B_i)}^{\perp} p_{\gamma^{(i)}} z^{|\gamma^{(i)}|} y^{|\gamma^{(i)}|+\ell(\gamma^{(i)})-2+2g_i} \right)$$

for some $k \geq 1$ and $g_1, \ldots, g_i \geq 0$. Note the $p_\gamma$'s are power sum symmetric functions *not* multiplication operators. Hence if $p_\mu = p_\beta^{\perp} p_\gamma \neq 0$ (ignoring scalars), then $|\mu| = |\gamma| - |\beta|$ and

$\ell(\mu) = \ell(\gamma) - \ell(\beta)$. It follows that if $p_\mu z^N y^K$ is a monomial after applying $p^\perp$, then we must have

$$\begin{aligned}
N &= |\gamma^{(1)}| + \cdots + |\gamma^{(i)}| + 1, \\
|\mu| &= |\alpha| + \sum_{j=1}^{i} \Big( |\gamma^{(1)}| - |\beta(B_j)| \Big) \\
&= |\alpha| - |\beta| + R - 1 \\
&= N, \\
\ell(\mu) &= \ell(\alpha) + \sum_{j=1}^{i} \Big( \ell(\gamma^{(i)}) - \ell(B_j) \Big) = \ell(\alpha) - \ell(\beta) + \sum_{i=1}^{j} \ell(\gamma^{(i)}) \\
&= k + 1 - 2h - 2\ell(\beta) + \sum_{j=1}^{i} \ell(\gamma^{(j)}).
\end{aligned}$$

It follows that

$$\begin{aligned}
K &= k + \sum_{j=1}^{\ell} |\gamma^{(j)}| + \ell(\gamma^{(j)})| - 2 + 2g_i \\
&= k + (|\mu| - 1) + (\ell(\mu) - k - 1 + 2h + 2\ell(\beta)) - 2i + 2g' \\
&= |\mu| + \ell(\mu) - 2 + 2(h + \ell(\beta) - i + g'),
\end{aligned}$$

where $g' = g_1 + \cdots + g_i$. Note $\ell(\beta) - i \geq 0$ so $h' = h + \ell(\beta) - i + g' \geq 0$. In summary, every monomial in the left-hand side of Equation (6.7) has the form

$$p_\mu z^N y^K = p_\mu z^{|\mu|-1} y^{|\mu|+\ell(\mu)-2+2h'} \quad \text{and} \quad h' \geq 0.$$

A monomial in the RHS is much easier to compute. Multiply by $z\exp(-\Psi^{f(x)})$ and apply the chain rule. We have

$$z\exp(-\Psi^{f(x)}) \frac{\partial}{\partial z} \exp(\Psi^{f(x)}) = z \frac{\partial}{\partial z} \Psi^{f(x)}.$$

**Theorem 6.5.** *Let $f(x) = \sum_{k\geq 0} f_k x^k$ and $\hat{\Psi}_0 = \hat{\Psi}_0(z, \mathbf{p}) = \log \Phi_0^{f(x)}(1, z, \mathbf{p})$. Then $\hat{\Psi}_0^{f(x)}$ is the unique solution to the partial differential equation*

$$zp_1 + z \sum_{k\geq 1} \frac{f_k}{k+1} \sum_{j=1}^{k} \binom{k+1}{j} \sum_{s\geq 0} \sum_{\substack{a_1,\ldots,a_{k+1-j}\geq 1 \\ a_1+\cdots+a_{k+1-j}=s+1}} p_{a_1} \cdots p_{a_{k+1-j}} [u^s] \left( \sum_{i\geq 1} \Big( p_i^\perp \hat{\Psi}_0^{f(x)} u^i \Big) \right)^j = \mathcal{E} \hat{\Psi}_0^{f(x)}.$$

*with initial condition $\hat{\Psi}_0^{f(x)}(0, \mathbf{p}) = 0$ and $\mathcal{E} = \sum_{i\geq 1} p_i p_i^\perp$.*

*Proof.* Compare coefficients for monomials in Equation (6.7) with $h' = h'' = 0$.

Since $\mathcal{E} = \frac{\partial}{\partial z}$, the RHS of Equation (6.7) is

$$z\frac{\partial}{\partial z}\Psi^{f(x)} = \sum_{h''\geq 0} \frac{\partial}{\partial z}\Psi^{f(x)}_{h''}.$$

Take coefficient of $h'' = 0$ we get $\frac{\partial}{\partial z}\Psi^{f(x)}_{0}$.

On the left-hand side, these correspond to monomials with

$$h + \ell(\beta) - i + g' = 0.$$

But $h \geq 0$ and $g' \geq 0$. So we must have

$$\ell(\beta) = i, \quad h = 0, \quad \text{and} \quad g_j = 1, \ldots, i.$$

Note $h = 0$ implies we are in fact only applying $\mathcal{U}^{(0)}_k$'s in the left-hand side of Equation (6.7). Note $g_1 = \cdots = g_i = 0$ implies $\mathcal{U}^{(0)}_k$ acts on $\Psi^{(f)}_0$. Finally, note $\ell(\beta) = i$ implies $|B_1| = \cdots |B_i| = 1$ and we get the left-hand side of the genus 0 PDE after setting $y = 1$ as desired. □

From this proof, we see that the $h$ parameter from the $\mathcal{U}$ operators can be interpreted as genus in the sense of Riemann-Hurwitz formula. Using sophisticated tools in symmetric functions and eventually Lagrange Inversion, Carrell and Goulden [4] solved this PDE for the aforementioned 3 special classes of transitive factorizations verifying that their coefficients are indeed the desirable counts.

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# APPENDICES

## Computation of $\mathcal{U}_2$

Recall $\mathcal{U}_0 = p_1$ and $\mathcal{U}_k = [\Delta, \mathcal{U}_{k-1}]$ for $k \geq 1$. Let

$$\mathcal{J} = \sum_{i,j\geq 1} p_{i+j} p_i^{\perp} p_j^{\perp} \quad \text{and} \quad \mathcal{J}^{\perp} = \sum_{i,j\geq 1} p_i p_j p_{i+j}^{\perp}.$$

Then we use $\Delta = \frac{1}{2}(\mathcal{J} + \mathcal{J}^{\perp})$ break up the computation into 2 parts.

$$\begin{aligned}
[\mathcal{J}, \mathcal{U}_1] &= \sum_{i,j,k\geq 1} p_{i+j} p_i^{\perp} p_j^{\perp} p_{k+1} p_k^{\perp} - p_k p_k^{\perp} p_{i+j} p_i^{\perp} p_j^{\perp} \\
&= \sum_{i,j,k\geq 1} p_{i+j} p_i^{\perp} (p_j^{\perp} p_{k+1}) p_k^{\perp} + p_{i+j} (p_i^{\perp} p_{k+1}) p_j^{\perp} p_k^{\perp} - p_k (p_k^{\perp} p_{i+j}) p_i^{\perp} p_j^{\perp} \\
&= \sum_{\substack{i\geq 1\\ j\geq 2}} j p_{i+j} p_i^{\perp} p_{j-1}^{\perp} + \sum_{\substack{i\geq 2\\ j\geq 1}} i p_{i+j} p_j^{\perp} p_{i-1}^{\perp} - \sum_{i,j\geq 1} (i+j) p_{i+j+1} p_{i+j+1} p_i^{\perp} p_j^{\perp} \\
&= 2 \sum_{i,j\geq 1} p_{i+j+1} p_i^{\perp} p_j^{\perp}. \\
[\mathcal{J}^{\perp}, \mathcal{U}_1] &= \sum_{i,j,k\geq 1} p_i p_j p_{i+j}^{\perp} p_{k+1} p_k^{\perp} - p_{k+1} p_k^{\perp} p_i p_j p_{i+j}^{\perp} \\
&= \sum_{i,j,k\geq 1} p_i p_j (p_{i+j}^{\perp} p_{k+1}) p_k^{\perp} - p_{k+1} (p_k^{\perp} p_i) p_j p_{i+j}^{\perp} - p_{k+1} p_i (p_k^{\perp} p_j) p_{i+j}^{\perp} \\
&= \sum_{i,j\geq 1} (i+j) p_i p_j p_{i+j-1}^{\perp} - i p_{i+1} p_j p_{i+j}^{\perp} - j p_i p_{j+1} p_{i+j}^{\perp} \\
&= \sum_{i,j\geq 1} (i+j) p_i p_j p_{i+j-1}^{\perp} - \sum_{\substack{i\geq 2\\ j\geq 1}} (i-1) p_i p_j p_{i+j-1}^{\perp} - \sum_{\substack{i\geq 1\\ j\geq 2}} (j-1) p_i p_j p_{i+j-1}^{\perp} \\
&= 2 \sum_{i,j\geq 1} p_i p_j p_{i+j-1}^{\perp}.
\end{aligned}$$

To get the last equality in $[\mathcal{J}^\perp, \mathcal{U}_1]$, we throw $i = 1$ into the second summation in the second last expression since $(i-1) = 0$. Ditto for the rightmost sum. Hence, we have

$$\mathcal{U}_2 = \sum_{i,j\geq 1} p_{i+j+1} p_i^\perp p_j^\perp + p_i p_j p_{i+j-1}^\perp.$$

## Computation of $\mathcal{U}_3$.

We compute the 4 terms of $\mathcal{U}_3 = [\Delta, \mathcal{U}_2]$.

If $i, j, i', j' \geq 1$, then

$$\begin{aligned}
&[p_i p_j p_{i+j}^\perp, p_{i'} p_{j'} p_{i'+j'-1}^\perp] \\
&= \left(p_{i+j}^\perp p_{i'}\right) p_i p_j p_{j'} p_{i'+j'-1}^\perp + \left(p_{i+j}^\perp p_{j'}\right) p_i p_j p_{i'} p_{i'+j'-1}^\perp \\
&\quad - \left(p_{i'+j'-1}^\perp p_i\right) p_j p_{i'} p_{j'} p_{i+j}^\perp - \left(p_{i'+j'-1}^\perp p_j\right) p_i p_{i'} p_{j'} p_{i+j}^\perp.
\end{aligned}$$

Sum over all $i, j, i', j' \geq 1$ we have

$$\begin{aligned}
&\sum_{i,j,i',j'\geq 1} [p_i p_j p_{i+j}^\perp, p_{i'} p_{j'} p_{i'+j'-1}^\perp] \\
&= \sum_{i,j\geq 1} \left( \sum_{j'\geq 1} (i+j) p_i p_j p_{j'} p_{i+j+j'-1}^\perp \right) + \left( \sum_{i'\geq 1} (i+j) p_i p_j p_{i'} p_{i+j+i'-1}^\perp \right) \\
&\quad - \sum_{i',j'\geq 1} \left( \sum_{j\geq 1} (i'+j'-1) p_j p_{i'} p_{j'} p_{i+i'+j'-1}^\perp \right) - \left( \sum_{j\geq 1} (i'+j'-1) p_i p_{i'} p_{j'} p_{i+i'+j'-1}^\perp \right) \\
&= 2 \sum_{i,j,k\geq 1} p_i p_j p_k p_{i+j+k-1}^\perp.
\end{aligned}$$

If $i, j, i', j' \geq 1$, then

$$\begin{aligned}
&[p_{i+j} p_i^\perp p_j^\perp, p_{i'} p_{j'} p_{i'+j'-1}^\perp] \\
&= \left(p_j^\perp p_{i'}\right) \left(p_i^\perp p_{j'}\right) p_{i+j} p_{i'+j'-1}^\perp + \left(p_j^\perp p_{j'}\right) \left(p_i^\perp p_{i'}\right) p_{i+j} p_{i'+j'-1} \\
&\quad + \left(p_i^\perp p_{i'}\right) p_{i+j} p_{j'} p_j^\perp p_{i'+j'-1}^\perp + \left(p_i^\perp p_{j'}\right) p_{i+j} p_{i'} p_j^\perp p_{i'+j'-1}^\perp \\
&\quad + \left(p_j^\perp p_{i'}\right) p_{i+j} p_{j'} p_i^\perp p_{i'+j'-1}^\perp + \left(p_j^\perp p_{j'}\right) p_{i+j} p_{i'} p_j^\perp p_{i'+j'-1}^\perp \\
&\quad - \left(p_{i'+j'-1}^\perp p_{i+j}\right) p_{i'} p_{j'} p_i^\perp p_j^\perp,
\end{aligned}$$

and

$$
\begin{aligned}
&[p_{i'}p_{j'}p^{\perp}_{i'+j'-1}, p_{i+j}p^{\perp}_{i}p^{\perp}_{j}] \\
=& -\left(p^{\perp}_{j'}p_i\right)\left(p^{\perp}_{i'}p_j\right)p_{i'+j'+1}p^{\perp}_{i+j} - \left(p^{\perp}_{j'}p_j\right)\left(p^{\perp}_{i'}p_i\right)p_{i'+j'+1}p^{\perp}_{i+j} \\
&-\left(p^{\perp}_{i'}p_i\right)p_{i'+j'+1}p_jp^{\perp}_{j'}p^{\perp}_{i+j} - \left(p^{\perp}_{i'}p_j\right)p_{i'+j'+1}p_ip^{\perp}_{j'}p^{\perp}_{i+j} \\
&-\left(p^{\perp}_{j'}p_i\right)p_{i'+j'+1}p_jp^{\perp}_{i'}p^{\perp}_{i+j} - \left(p^{\perp}_{j'}p_j\right)p_{i'+j'+1}p_ip^{\perp}_{i'}p^{\perp}_{i+j} \\
&+\left(p^{\perp}_{i+j}p_{i'+j'+1}\right)p_ip_jp^{\perp}_{i'}p^{\perp}_{j'}.
\end{aligned}
$$

Sum over all $i, j, i', j' \geq 1$. We have

$$
\begin{aligned}
&\sum_{i,j,i',j'\geq 1}[p_{i+j}p^{\perp}_{i}p^{\perp}_{j}, p_{i'}p_{j'}p^{\perp}_{i'+j'-1}] + [p_{i'}p_{j'}p^{\perp}_{i'+j'-1}, p_{i+j}p^{\perp}_{i}p^{\perp}_{j}] \\
=&\sum_{i,j\geq 1}\left(ijp_{i+j}p^{\perp}_{i+j-1} + ijp_{i+j}p^{\perp}_{i+j-1}\right) - \sum_{i',j'\geq 1}\left(i'j'p_{i'+j'+1}p^{\perp}_{i'+j'} - i'j'p_{i'+j'+1}p_{i'+j'}\right) \\
&+\sum_{i,j\geq 1}\left(\sum_{j'\geq 1} ip_{i+j}p_{j'}p^{\perp}_{j}p^{\perp}_{i+j'-1}\right) + \left(\sum_{i'\geq 1} ip_{i+j}p_{i'}p^{\perp}_{j}p^{\perp}_{i'+i-1}\right) \\
&-\sum_{i',j'\geq 1}\left(\sum_{j\geq 1} i'p_{i'+j'+1}p_jp^{\perp}_{j'}p^{\perp}_{i'+j}\right) - \left(\sum_{i\geq 1} i'p_{i'+j'+1}p_ip^{\perp}_{j'}p^{\perp}_{i+i'}\right) \\
&+\sum_{i,j\geq 1}\left(\sum_{j'\geq 1} jp_{i+j}p_{j'}p^{\perp}_{j}p^{\perp}_{j+j'-1}\right) + \left(\sum_{i'\geq 1} jp_{i+j}p_{i'}p^{\perp}_{j}p^{\perp}_{i'+j-1}\right) \\
&-\sum_{i',j'\geq 1}\left(\sum_{i\geq 1} j'p_{i'+j'+1}p_jp^{\perp}_{i'}p^{\perp}_{j+j'}\right) - \left(\sum_{j\geq 1} j'p_{i'+j'+1}p_ip^{\perp}_{i'}p^{\perp}_{i+j'}\right) \\
&+\sum_{i,j\geq 1}\left(\sum_{i'+j'=i+j-1}(i+j)p_ip_jp^{\perp}_{i'}p^{\perp}_{j'}\right) - \sum_{i',j'\geq 1}\left(\sum_{i+j=i'+j'-1}(i'+j'-1)p_{i'}p_{j'}p^{\perp}_{i}p^{\perp}_{j}\right) \\
=&\sum_{i,j\geq 1}(i+j)p_{i+j}p^{\perp}_{i+j-1} + \sum_{i,j\geq 1}\sum_{\substack{i',j'\geq 1\\ i+j=i'+j'+1}} p_ip_jp^{\perp}_{i'}p^{\perp}_{j'} \\
&+2\sum_{i,j,k\geq 1}p_{i+j}p_kp^{\perp}_{j}p^{\perp}_{i+k-1} + 2\sum_{i,j,k\geq 1}p_{i+j}p_kp^{\perp}_{i}p^{\perp}_{j+k-1}.
\end{aligned}
$$

If $i, j, i', j' \geq 1$, then

$$[p_{i+j}p_i^\perp p_j^\perp, p_{i'+j'+1}p_{i'}^\perp p_{j'}^\perp] = p_{i+j}p_i^\perp \left(p_j^\perp p_{i'+j'+1}\right) p_{i'}^\perp p_{j'}^\perp - p_{i'+j'+1}p_{i'}^\perp \left(p_{j'}^\perp p_{i+j}\right) p_i^\perp p_j^\perp + p_{i+j}\left(p_i^\perp p_{i'+j'+1}\right) p_j^\perp p_{i'}^\perp p_{j'}^\perp - p_{i'+j'+1}\left(p_{i'}^\perp p_{i+j}\right) p_{j'}^\perp p_i^\perp p_j^\perp.$$

Summing over $i, j, i', j' \geq 1$, we have

$$\begin{aligned}
&\sum_{i,j,i',j'\geq 1} p_{i+j}p_i^\perp \left(p_j^\perp p_{i'+j'+1}\right) p_{i'}^\perp p_{j'}^\perp - p_{i'+j'+1}p_{i'}^\perp \left(p_{j'}^\perp p_{i+j}\right) p_i^\perp p_j^\perp \\
&= \sum_{i,j\geq 1} jp_{i+j}p_i^\perp \sum_{\substack{i',j'\geq 1\\ i'+j'+1=j}} p_{i'}^\perp p_{j'}^\perp - \sum_{i',j'\geq 1} jp_{i'+j'+1}p_{i'}^\perp \sum_{\substack{i,j\geq 1\\ j'=i+j}} p_i^\perp p_j^\perp \\
&= \sum_{i,j\geq 1} jp_{i+j}p_i^\perp \left(\sum_{\substack{i',j'\geq 1\\ i'+j'+1=j}} p_{i'}^\perp p_{j'}^\perp\right) - jp_{i+j+1}p_i^\perp \left(\sum_{\substack{i',j'\geq 1\\ j=1'+j'}} p_{i'}^\perp p_{j'}^\perp\right) \\
&= \sum_{\substack{i\geq 1\\ j\geq 0}} p_{i+j+1}p_i^\perp \sum_{\substack{i',j'\geq 1\\ i'+j'=j}} \left((j+1)p_{i'}^\perp p_{j'}^\perp - p_{i'}^\perp p_{j'}^\perp\right) \\
&= \sum_{i,j\geq 1} p_{i+j}p_i^\perp \sum_{\substack{i',j'\geq 1\\ i'+j'+1=j}} p_{i'}^\perp p_{j'}^\perp.
\end{aligned}$$

By symmetry, we have

$$\sum_{i,j,i',j'\geq 1} [p_{i+j}p_i^\perp p_j^\perp, p_{i'+j'+1}p_{i'}^\perp p_{j'}^\perp] = 2\sum_{i,j\geq 1} p_{i+j}p_i^\perp \sum_{\substack{i',j'\geq 1\\ i'+j'+1=j}} p_{i'}^\perp p_{j'}^\perp.$$