On the Robustness of Holographic Dualities: AdS Black Holes and Quotient Spaces

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of contributions

The present dissertation results from a serie of collaborative works and co-authored publications as follows:

Chap 3: Musema Sinamuli and Robert B. Mann, *"Super-Entropic Black Holes and the Kerr-CFT correspondence"*, Journal of High Energy Physics **08** (2016) 148.

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Chap 5: Musema Sinamuli and Robert B. Mann, *"Geons and the Quantum Information Metric"*, Physical Review D**96**. 026014 (2017).

Chap 6: Musema Sinamuli and Robert B. Mann, *"Topological and Time Dependence of the Action-Complexity Relation"*, Physical Review D**98**. 026005 (2018).

Abstract

This dissertation is a collection of works on different topics based on holographic dualities. The dualities studied here are the Kerr/CFT and AdS/CFT correspondence and the researches carried on aim to check their robustness for some particular theories of gravity.

We first demonstrate that the Kerr/CFT duality can be extended to superentropic black holes, which have non-compact horizons with finite area. The duality is robust as the near horizon limit of these black holes commutes with their ultraspinning limit. We notice that the duality holds as well for both the singly-spinning superentropic black holes in four dimensions and the double-spinning superentropic black holes of gauged supergravity in five dimensions.

Second, we test the AdS/CFT duality in Lovelock gravity theories or higher curvature theories of gravity in which we investigate the holographic Smarr relation beyond the large N limit. By making use of the holographic dictionary, we obtain a holographic equation of state in the conformal field theories (CFTs) dual to AdS spacetimes. We check the validity of this equation of state for a variety of non-trivial black holes including rotating planar black holes in Gauss-Bonnet-Born-Infeld gravity and non-extremal rotating black holes in minimal $5d$ gauged supergravity.

In the remaining part of this dissertation we return to investigate AdS/CFT duality, but now focus on computational complexities defined in the CFTs dual to AdS black holes.

The first one is the volume-complexity, which consists in a duality between a quantum information metric or Bures metric in a $(d+1)$- dimension CFT and the volume of a maximum time slice in the dual $(d+2)$- dimension AdS spacetime. We examine specific cases of black holes such as the Banados-Teitelboim-Zanelli (BTZ) and the planar Schwarzschild-AdS black holes in $(d+2)$- dimensions, along with their geon counterparts. Geons being quotient spaces of AdS black holes obtained from the identification of the left and right boundaries of their conformal diagrams. We find that the proposed duality relation remains the same for the geon space with a topological factor of 4.

The second one is the action-complexity, which conjectures a duality between the action of an AdS bulk evaluated on a Wheeler-De Witt (WDW) patch and a CFT computational complexity providing a measure of the minimum number of gates necessary to reach a target state from a reference state. We compute the dependence of the CFT complexity on a boundary temporal parameter (time) and find that its variation with time is commensurate with the rate of change of the bulk action on the WDW patch. We remark that the action-complexity duality holds for the geons associated to these black holes (with a topological factor up to 4) as well.

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As a graduate student in the past five years I experienced a number of events that helped me to grow in many aspects of my life. One of those aspects concerns my research skills.

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Chapter 1

Introduction

The holographic principle, as a principle of string theory, states that a description of physics in the volume of a region of space is encoded on a lower-dimensional boundary (such as a gravitational horizon or an asymptotic region) of this region.

The principle was first proposed by 't Hooft and was later reformulated by Susskind, where he brought new ideas to 't Hooft and Thorn's previous works [1, 2]. In his work, Thorn observed that gravity emerges in a holographic way from a lower-dimension description of string theory.

The holographic principle has also been inspired by black hole thermodynamics in which it is conjectured that the maximal entropy in a given region scales as the square of the radius of that region. In fact, following Hawking's theorems [3], Bekenstein thought of black holes as maximum entropy objects, i.e. objects that contain more entropy than any other object of the same volume [4]. In his work an upper bound was put on the entropy in a given region of space and that bound was proportional to the area of that region. He thus realized that the entropy of a black hole is proportional to the area of its event horizon. This entropy is also referred to as the Bekeinstein-Hawking entropy.

In the black hole context, the information content of an object falling into the black hole is retrieved in surface fluctuations of the event horizon. This feature implies that the black hole information paradox finds an answer in the framework of string theory [5]. The aforementioned papers on the holographic principle or dualities were landmarks that spurred numerous investigations in the field.

Later, Brown and Henneaux proved that the asymptotic symmetry of a $2 + 1$ gravitation theory gives rise to a Virasoro algebra, whose two dimensional conformal field theory is the corresponding quantum theory. Their work laid the foundation that has led to the Kerr/CFT duality which will be discussed in detail later in this chapter.

The greatest realization so far of the holographic principle is the AdS/CFT correspondence introduced by Maldacena [6]. The AdS/CFT correpsondence or gauge/gravity duality, is a conjectured duality between an Anti-de Sitter space in $d+1$ dimensions and a conformal field theory in d dimensions in its boundary. It cannot be overlooked in the understanding of string theory (and quantum gravity) because it provides a non-perturbative formulation of string theory. It also provides useful material in the study of strongly coupled quantum field theories.

This duality is very useful as it represents a strong-weak duality, which means that the fields in the quantum field theory are strongly coupled while the ones in the dual gravitational theory are weakly coupled. This consideration implies that the fields in the gravitational theory are more mathematically tractable. The fact of translating problems into a more mathematically tractable form in string theory is a trick used in several aspects of other fields of theoretical physics such as condensed matter physics.

Many examples of the AdS/CFT correspondence have been proposed and the most famous of them states that [6] *"A type IIB string theory in* $(AdS_5 \times S^5)_N$ *plus some appropriate boundary conditions (and possibly also some boundary degree of freedom) is dual to* $\mathcal{N} = 4 \quad d = 3+1 \quad U(N)$ *super Yang-Mills theory"*. The subindex indicates the dependence of the radius of the AdS on N. Another example of the correspondence states that [6] *"A (0,2) conformal field theory is dual to M-theory on* $(AdS_7 \times S^4)_N$*"*.

As stated earlier in this chapter, for holographic dualities in general, the AdS/CFT correspondence solves the black hole information paradox, to some extent, as it shows the evolution of a black hole in a fashion that is consistent with quantum mechanics. In fact, in the context of the AdS/CFT correspondence black holes are equivalent to a configuration of particles in the dual conformal field theory. Since these particles are subject to the laws of quantum mechanics and thus evolve in a unitary way, the black hole has to evolve in a unitary way as well.

The main weakness of the AdS/CFT correspondence resides in the fact that most of the black holes considered are physically unrealistic, since most of the versions of the correspondence involve high dimension spacetimes with unphysical supersymmetry [6, 7].

In [8] M. Guica et.al. proposed another form of holographic duality known as the Kerr/CFT correspondence, which was a step towards understanding astrophysical black holes. The Kerr/CFT correspondence applies to black holes that can be approximated to extreme Kerr black holes, i.e. black holes with the largest possible angular momentum that is compatible with their mass. The correspondence posits an equality relation between the Bekenstein-Hawking entropy of a Kerr black hole and the Cardy entropy of a two-dimensional chiral field theory near its horizon [8].

In the past two decades many fields of theoretical physics have been orientated towards holographic dualities. These dualities establish equivalence relations between observables existing in a theory of gravity and those in its dual field theory. In other words, the dynamics of observables defined in a field theory can tell us about the way their corresponding observables in a dual gravitational theory evolve and vice-versa. This fact is of tremendous interest as it becomes possible to get enough information on the dynamics of a theory of gravity just by investigating some observables associated with its dual field theory. This actually makes the field of holographic dualities very interesting and appealing.

This dissertation concentrates on providing additional theoretical evidence pertinent to the holographic principle. We extend these holographic dualities to new classes of black holes and gravitation theories. These new classes of black holes include super-entropic black holes (or ultraspinning Kerr-AdS-like black holes) and AdS quotient spaces known as geons. Their study provides new information about holographic dualities since they have particular topologies that can either modify the form of (or perhaps even prevent) holographic dualities or provide more evidence for their robustness.

In chapter 3, we study the Kerr/CFT correspondence for a new class of black holes referred to as super-entropic black holes, which are new ultraspinning limits of Kerr-AdS black holes, in which the rotation parameter a approaches the AdS radius l. Emparan and Myers [9] studied ultraspinning black holes in the large angular momentum limit [10]. But this new class of four dimensional rotating black hole solutions were elaborated in [11, 12] for both $\mathcal{N} = 2$ gauged supergravity coupled to a vector multiplet and Einstein-Maxwell-Λ theory. These black holes have the characteristic that their horizons are non-compact but have finite area. Topologically the event horizon is a sphere with two punctures. This feature provides a broader range of possible horizon topologies than what was previously thought.

Because of the features that characterize these black holes, an interesting question we investigate concerns the existence of the Kerr/CFT correspondence near their horizons. Can the presence of the punctures prevent them from exhibiting the Kerr/CFT correspondence when taking the near horizon limit?

In chapter 4, we test the robustness of one of these holographic dualities namely, the AdS/CFT correspondence in higher order curvature theories or Lovelock theories. A Lovelock theory of gravity is a generalisation of Einstein's general relativity to higher dimensions introduced by David Lovelock [13]. This class of theories is said to resemble string-theory-inspired models of gravity because the action contains, among others, a quadratic Gauss-Bonnet term. This quadratic term is present in the low energy effective action of the heterotic string theory, and also in six dimensional Calabi-Yau compactification of M-theory [14]. Despite being quadratic, the Gauss-Bonnet actions leads to differential equations of second order and has the property of being ghost-free [15]. These considerations make this class of theories interesting (as opposed to arbitrary higher-curvature theories of gravity).

Basically, Lovelock theories of gravity appear to be useful in understanding how a theory of gravity is corrected at short distances due to the presence of higher order curvature terms in their action. This theory has also been considered as a testing ground to study the effects of introducing higher-curvature terms in the context of AdS/CFT correspondence.

Now turning into what drives our attention, it is known that the introduction of the cosmological constant as pressure (along with an associated volume) has led to the extension of the thermodynamic phase space in the domain of black hole thermodynamics. The two new variables have led to the realization that black holes can exhibit diverse phase behaviour comparable to those observed in gases, superfluids and so on [16, 17, 18, 19, 20]. In fact, it has been found in black holes new phase behaviour such as those seen in gels and polymers as well as triple points such as those observed in water. It was also noticed that black holes in general are analogous to Van der Waals superfluids. The aforementioned features have given rise to an emerging field known as *black hole chemistry* [21].

Since the introduction of a negative cosmological constant has in a certain way led to the holographic depiction of black holes as systems dual to those in conformal field theories [6, 22, 23], it is of interest to see how the variety of phase behaviours observed in black holes [21] is manifest in the dual field theories.

In [24] the question of what the implications of black hole chemistry would be on the variables in the boundary field theory of an AdS space has been studied. A holographic Smarr relation was established in Einstein gravity via a holographic dictionary. The thermodynamic first law combined with the Smarr relation in the bulk correspond to an equation of state in the boundary conformal field theory.

In chapter 4, we aim to establish a holographic Smarr relation in the context of Lovelock gravity by extending the holographic dictionary in order to include higher-order curvature contributions. We shall also check the validity of the equation of state for some cases of Lovelock theories or black holes.

Prior to moving to the content of the next chapter we recall some important points on holographic dualities. Dual connections between entanglement and geometry is a topic that researchers from many fields of theoretical physics have been working on in recent years [25, 26, 27, 28, 29, 30]. Eternal AdS black holes are known to be dual to the thermofield double (TFD) state, which is a state that entangles two copies of the CFTs defined on the AdS boundaries. These copies of the CFTs are connected by a wormhole or a Einstein-Rosen-Bridge (ERB). Since the boundary field reaches thermal equilibrium in a (short) time that is of the order of the thermalization time $\beta = 1/T$ (due certainly to entanglement), whereas the ERB grows on a much larger time [31] , it implies that there must exist some quantities in the field theory that evolve even after thermal equilibrium has been reached [32, 33].

These hypotheses led Susskind to think of a holographic complexity as a quantity in the boundary CFT whose the growth is equivalent to the evolution of the ERB [34, 35, 36]. Along with his collaborators, he proposed two new quantities in a theory of gravity that both follow the growth of the ERB at late time.

The first one is the volume-complexity, which states that the complexity of a state in the boundary CFT is proportional to a maximal volume of a time slice connecting the boundaries (CFTs) [31, 34]. The second one is the action complexity, in which a computational complexity of the boundary state is identified to the gravitational action evaluated in a particular region of the bulk referred to as the Wheeler-De Witt (WDW) patch [37, 38]. But questions still remain regarding the interpretation of these complexities in the boundary CFT.

In chapter 5, we study the volume-complexity conjecture based on the work done by Miyaji et al [39] in which they introduced the CFT complexity as a quantity called the information metric or Bures metric. The information metric is defined as the *fidelity susceptibility* of a state in the CFT. Indeed, a marginal deformation was performed on the state and the scalar product between the state and its deformation led to the fidelity susceptibility. It appeared that the late time growth of the information metric was proportional to the one of the gravitational volume-complexity.

We probe this conjecture in chapter 5 by considering quotient spaces of AdS black holes called *geons*. Geons are obtained through the identification of the two sides of the conformal diagram of eternal AdS black holes. We compute the CFT volume-complexity of the geons along with its gravitational counterpart and thus check the robustness of the conjecture for the geon case.

Moving from understanding the nature of a computational complexity in the CFT dual to a codimension-one volume (volume) in the bulk, more recent work has suggested a deeper relationship between CFT complexity and the bulk spacetime [37, 38] that we explore in chapter 6. Here we extend our investigations of holographic complexity by considering the time-dependent action-complexity in the CFT. This quantity was introduced as a measure of the minimum number of operations necessary to approximate a unitary operator between a reference and target states of the CFT. We propose a time-dependent action-complexity in the CFT whose the growth at late time is proportional to the one associated to the gravitational action-complexity. We verify the conjecture for the geons case as well.

In chapter 7, we summarize the most important results of this thesis and give some directions for future work.

Chapter 2

Some important background

This chapter is mainly intended to provide a brief background on conformal field theories as they play a central role in the elaboration and understanding of what will follow in the present dissertation.

We shall start with the definition of a conformal transformation, then derive the conformal group in d dimensions and finally study how a field theory transforms under a conformal map. We shall also review the conformal invariance in $d = 2$ as this particular case is the smallest one and finds applications in quantum field and string theories. In the context of the current work $d = 2$ conformal field theories are connected to theories of gravity.

To end the chapter, we shall briefly review holographic dualities, the Kerr/CFT and AdS/CFT correspondences as they are the ones which matter the most in this work, while insisting more on the conformal field theory side. A short review on black hole chemistry shall appear as well.

2.1 Conformal transformation and global conformal invariance

A conformal transformation is defined as a map: $x \to x^{'}$ mapping a point x to another $x^{'}$ and under which the metric tensor transforms as [40]

$$g^{'}_{\mu\nu}(x^{'}) = \Lambda(x) g_{\mu\nu}. \tag{2.1}$$

From (2.1) we can infer that more generally a transformation is said to be conformal if it does not affect the angle between two arbitrary directions.

A conformal transformation can be locally thought of as a (pseudo) rotation and a dilation. The set of conformal transformations forms a group which admits the Poincare group as a subgroup. The

Poincare group corresponds to ($\Lambda(x) = 1$) in (2.1). In d dimensions the conformal group is identified with the non-compact group $SO(1, d+1)$.

Let us consider an infinitesimal transformation defined as

$$x^\mu \to x^{'\mu} = x^\mu + \epsilon^\mu(x) \tag{2.2}$$

where $\epsilon^\mu(x)$ is the parameter of the transformation such that the metric under this transformation changes as

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{2.3}$$

In order that (2.3) be conformal, it must obey the condition

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} \tag{2.4}$$

with $f(x)$ a function determined by

$$f(x) = \frac{2}{d} \partial_\mu \epsilon^\mu. \tag{2.5}$$

For the sake of simplicity, we assume that a conformal transformation is an infinitesimal deformation of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} \tag{2.6}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$.

Appying a derivation on (2.4), then permuting indices and taking linear combinations, we have

$$\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\mu\rho} \partial_\nu f + \eta_{\nu\rho} \partial_\mu f - \eta_{\mu\nu} \partial_\rho f. \tag{2.7}$$

After tracing the above equation over μ, ν

$$2\partial^2 \epsilon_\mu = (2 - d) \partial_\mu f \tag{2.8}$$

and using (2.5), it follows

$$(2 - d) \partial_\mu \partial_\nu f = \eta_{\mu\nu} \partial^2 f. \tag{2.9}$$

Hence

$$(d - 1) \partial^2 f = 0. \tag{2.10}$$

From (2.8) and (2.9), we get

$$\partial_\mu \partial_\nu f = 0. \tag{2.11}$$

The above equation is solved by the function

$$f(x) = A + B_\mu x^\mu \tag{2.12}$$

A and B_μ are constants.

Making use of (2.11) and (2.12) it results that the parameter of the conformal transformation obeys

$$\partial_\mu \partial_\nu \epsilon_\rho = C \tag{2.13}$$

with C a constant. This leads to the expression

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho . \tag{2.14}$$

A particular form of (2.14) gives rise to the infinitesimal transformation

$$x^{'\mu} = x^\mu + 2(x.b)x^\mu - b^\mu x^2 \qquad (b^\mu \ \text{ constant}) \tag{2.15}$$

which is known as the special conformal transformation (SCT).

The different possible infinitesimal transformations we can generate are

$$\begin{aligned} x^{'\mu} &= x^\mu + a^\mu \\ x^{'\mu} &= \alpha x^\mu \\ x^{'\mu} &= M^\mu_\nu x^\nu \\ x^{'\mu} &= \frac{x^\mu - b^\mu x^2}{1 - 2b.x + b^2 x^2} \end{aligned} \tag{2.16}$$

and correspond to the translation, dilation, rigid rotation and SCT respectively. The SCT in the last line of (2.16) corresponds to the composition of an inversion ($x^\mu \to \frac{x^\mu}{x^2}$), a translation ($\frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} - b^\mu$) and another inversion ($\frac{x^\mu}{x^2} - b^\mu \to \frac{x^\mu - b^\mu x^2}{1-2b.x+b^2x^2}$). For an infinitesimal parameter b^μ (2.16) is equivalent to (2.15).

These infinitesimal transformations are generated by the operators

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ D &= -ix^\mu \partial_\mu \\ L_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) \\ K_\mu &= -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \end{aligned} \tag{2.17}$$

which correspond to the translation, dilation, rigid rotation and SCT generators respectively. These operators obey the commutation rules given by

$$\begin{aligned}
\left[D, P_\mu\right] &= iP_\mu \\
\left[D, K_\mu\right] &= -iK_\mu \\
\left[K_\mu, P_\nu\right] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
\left[K_\rho, L_{\mu\nu}\right] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
\left[P_\rho, L_{\mu\nu}\right] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
\left[L_{\mu\nu}, L_{\rho\sigma}\right] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\nu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\sigma}). \qquad (2.18)
\end{aligned}$$

The next step of this chapter consists in the construction of conformal invariants. To this end, we consider functions $\Gamma(x_i)$ of N points x_i that are left unchanged under all types of conformal transformations.

First, imposing translation and rotation invariance implies that any function $\Gamma(x_i)$ can depend only on distances $|x_i - x_j|$. Scale invariance then implies that only ratios of distances, such as

$$\frac{|x_i - x_j|}{|x_k - x_l|}$$

are physically meaningful. Under SCT the distances $|x_i - x_j|$ becomes

$$|x_i^{'} - x_j^{'}| = \frac{|x_i - x_j|}{(1 - 2b.x_i + b^2 x_i^2)^{1/2}(1 - 2b.x_j + b^2 x_j^2)^{1/2}}.$$

As we can clear see the functions $\frac{|x_i - x_j|}{|x_k - x_l|}$ are not invariant under SCT.

Thus, we are led to the conclusion that it is impossible to construct an invariant function $\Gamma(x_i)$ with only 2 or 3 points and the simplest possibilities are the functions of 4 points

$$\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \qquad \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|}.$$

These expressions are known as *anharmonic ratios* or *cross-ratios*. For N distinct points there exist $N(N-3)/2$ independent ratios.

2.2 Conformal invariance in field theory

Here we review some basic notions on how a field theory transforms under conformal transformations and its implications on quantities such as the two-point functions.

Under a conformal transformation $x \to x^{'}$, a spinless field transforms as [40]

$$\Phi(x) \to \Phi^{'}(x^{'}) = \left|\frac{\partial x^{'}}{\partial x}\right|^{-\Delta/d} \Phi(x) \qquad (2.19)$$

with Δ the scaling dimension, d the dimension of the spacetime and $|\partial x^{'}/\partial x|$ the Jacobian of the conformal transformation of coordinates. A field transforming like (2.19) is called *quasi-primary.* We infer that

$$\left|\frac{\partial x^{'}}{\partial x}\right| = \Lambda(x)^{-d/2} \tag{2.20}$$

with $\Lambda(x)$ introduced in (2.1).

Focusing now on quantum field theories, in which we define the two-point function of two fields $\Phi_1(x_1)$ at the point x_1 and $\Phi_2(x_2)$ at x_2 is

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{1}{Z}\int [d\Phi]\Phi_1(x_1)\Phi_2(x_2)\exp[-S[\Phi]] \tag{2.21}$$

where Z is the partition function, $\Phi_1(x_1)$ and $\Phi_2(x_2)$ are quasi-primary fields, Φ the set of all functionally independent fields in the theory and $S[\Phi]$ the action, which we assume to be conformally invariant.

Under a conformal transformation, the two-point function (2.21) transforms as

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = \left|\frac{\partial x^{'}}{\partial x}\right|_{x=x_1}^{-\Delta_1/d}\left|\frac{\partial x^{'}}{\partial x}\right|_{x=x_2}^{-\Delta_2/d}\langle\Phi_1(x_1^{'})\Phi_2(x_2^{'})\rangle. \tag{2.22}$$

The scale transformation ($x \to \lambda x$) invariance puts (2.22) into the form

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle\Phi_1(\lambda x_1)\Phi_2(\lambda x_2)\rangle \tag{2.23}$$

Rotation and translation invariances gives to the two-point function the form

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = f(|x_1 - x_2|) \tag{2.24}$$

where $f(x) = \lambda^{\Delta_1+\Delta_2}f(\lambda x)$. Equation (2.24) can also be expressed as

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.25}$$

with C_{12} constant coefficient.

The SCT invariance

$$\left|\frac{\partial x^{'}}{\partial x}\right| = \frac{1}{(1 - 2b.x + b^2x^2)^d} \tag{2.26}$$

imposes the constraint

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.27}$$

with $\gamma_i = 1 - 2b.x_i + b^2x_i^2$ and which is identically satisfied only if $\Delta_1 = \Delta_2$.

It thus results that two quasi-primary fields are correlated if and only if they have the same scaling dimension. Therefore, the two-point function reads as

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = \begin{cases} \dfrac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{otherwise} \end{cases}. \tag{2.28}$$

2.3 Conformal invariance in two dimensions

In this section we restrict our study to conformal field theories in two dimensions and overview some of their important features.

Let us consider coordinates (z^0, z^1) on the plane, which transform under conformal transformations as

$$z^\mu \to \omega^\mu(z^\nu) \qquad \text{with} \quad \mu, \nu = 0, 1 \tag{2.29}$$

which implies the metric transformation

$$g^{\mu\nu} = \left(\frac{\partial \omega^\mu}{\partial z^\alpha}\right)\left(\frac{\partial \omega^\nu}{\partial z^\beta}\right) g^{\alpha\beta}. \tag{2.30}$$

Introducing complex coordinates z and $\bar{z}$ such as

$$\begin{aligned} z &= z^0 + iz^1 \\ \bar{z} &= z^0 - iz^1 \end{aligned} \tag{2.31}$$

It follows (2.29) the transformation

$$z \to \omega(z). \tag{2.32}$$

The set of global conformal transformations forms a group that is referred to as the special conformal group.

The complete set of conformal mappings on the complex plane is

$$f(z) = \frac{az + b}{cz + d} \qquad \text{with} \qquad ad - bc = 1 \tag{2.33}$$

and a, b, c, d are complex numbers.

Therefore the global conformal group on the complex plane ($d = 2$) is isomorphic to the group of complex invertible 2×2 matrices with determinant equal to 1, that is, to $Sl(2, \mathbb{C})$. Note that $Sl(2, \mathbb{C})$ is isomorphic to the Lorentz group $SO(3, 1)$ in four dimensions.

Prior to proceed further, we recall that a holomorphic infinitesimal transformation is expressed as

$$z^{'} = z + \epsilon(z) \qquad \text{with} \qquad \epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1} \tag{2.34}$$

assuming that the infinitesimal mapping admits a Laurent expansion around $z = 0$.

Considering that a spinless and dimensionless field $\Phi(z, \bar{z})$ on the complex plane satisfies

$$\begin{aligned} \Phi^{'}(z^{'}, \bar{z}^{'}) &= \Phi(z, \bar{z}) \\ &= \Phi(z^{'}, \bar{z}^{'}) - \epsilon(z^{'})\partial^{'}\Phi(z^{'}, \bar{z}^{'}) - \bar{\epsilon}(\bar{z}^{'})\bar{\partial}^{'}\Phi(z^{'}, \bar{z}^{'}) \end{aligned} \tag{2.35}$$

we are led to the relation

$$
\begin{aligned}
\delta\Phi &= -\epsilon(z)\partial\Phi - \bar{\epsilon}(\bar{z})\partial\Phi \\
&= \sum_n \left[\epsilon_n l_n \Phi(z,\bar{z}) + \bar{\epsilon}_n \bar{l}_n \Phi(z,\bar{z})\right] \qquad (2.36)
\end{aligned}
$$

in which the generators

$$
l_n = -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \qquad (2.37)
$$

have been introduced.

These generators satisfy the commutation rules

$$
\begin{aligned}
\left[l_n, l_m\right] &= (n-m)l_{n+m} \\
\left[\bar{l}_n, \bar{l}_m\right] &= (n-m)\bar{l}_{n+m} \\
\left[l_n, \bar{l}_m\right] &= 0 \qquad (2.38)
\end{aligned}
$$

and yield a conformal algebra which is the direct sum of two isomorphic algebras also known as the Witt algebra.

In two dimensions, the notion of quasi-primary fields applies also to the fields with spins.

Let Δ be the scaling dimension and s the planar spin. By defining a holomorphic transformation h and his counterpart $\bar{h}$ as

$$
h = \frac{1}{2}(\Delta + s) \quad \text{and} \quad \bar{h} = \frac{1}{2}(\Delta - s) \qquad (2.39)
$$

we notice that under a conformal map $z \to \omega(z)$ and $\bar{z} \to \bar{\omega}(\bar{z})$, the field theory transforms as

$$
\Phi^{'}(\omega,\bar{\omega}) = \left(\frac{d\omega}{dz}\right)^{-h}\left(\frac{d\bar{\omega}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}). \qquad (2.40)
$$

In the case of an infinitesimal map

$$
z \to \omega(z) \ \text{ such that } \ \omega = z + \epsilon(z) \ \text{ and } \ \bar{\omega} = \bar{z} + \bar{\epsilon}(\bar{z}) \qquad (2.41)
$$

with ϵ and $\bar{\epsilon}$ very small, we obtain

$$
\begin{aligned}
\delta_{\epsilon,\bar{\epsilon}}\Phi &= \Phi^{'}(z,\bar{z}) - \Phi(z,\bar{z}) \\
&= -(h\Phi\partial_z\epsilon + \epsilon\partial_z\Phi) - (\bar{h}\Phi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\Phi). \qquad (2.42)
\end{aligned}
$$

Thus, fields whose variations under local transformations in two dimension are expressed as in (2.42) are defined as primary fields.

The correlation functions under a conformal transformation change as [40]

$$
\langle\Phi_1(\omega_1,\bar{\omega}_1)...\Phi_n(\omega_n,\bar{\omega}_n)\rangle = \prod_{i=1}^{n}\left(\frac{d\omega}{dz}\right)^{-h_i}_{\omega=\omega_i}\left(\frac{d\bar{\omega}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{\omega}=\bar{\omega}_i}\langle\Phi_1(z_1,\bar{z}_1)...\Phi_n(z_n,\bar{z}_n)\rangle \qquad (2.43)
$$

in which the distance x_{ij} between the points z_i and z_j is equal to $(z_{ij}\bar{z}_{ij})^{1/2}$.

Hence the two-point function on the complex plane takes the form

$$\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\rangle = \frac{C_{12}}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \quad \text{when} \quad (h_1 = h_2 = h \ \ or \ \ \bar{h}_1 = \bar{h}_2 = \bar{h}). \tag{2.44}$$

2.4 Holographic dualities

2.4.1 Kerr black holes and chiral conformal field theories in two dimensions

It has been recently conjectured that a duality exists that links extreme Kerr black holes to chiral conformal field theories in two dimensions [8]. A key implication of the aforementioned conjecture is the existence of an equality relation between the Bekenstein-Hawking entropy of an extreme Kerr black hole and the Cardy entropy of a two dimensional conformal field theory defined near its horizon. Here we do not need to be very explicit about the technical details on the subject and we leave them to the next chapter.

Let us consider the near horizon limit of an extreme Kerr black hole (the angular momentum is related to the mass $J = M^2$). The resulting metric admits asymptotic symmetries preserved by some diffeomorphisms. There exist consistent boundary conditions for which the asymptotic symmetry generators (of the diffeomorphisms) form one copy of the Virasoro algebra [1] with central charge $c_L = 12J$ [8].

We ensue from the above statement that the near horizon states can be identified with states of a chiral half of a two-dimensional CFT.

In the extreme limit, the Frolov-Thorn [2] vacuum state reduces to a dimensionless temperature $(T_L = 1/2\pi)$ thermal density matrix whose the conjugate energy is the zero mode generator l_0 of the Virasoro algebra.

The Cardy formula, under unitarity assumptions, gives a microscopic entropy for the CFT $S_{micro} = \frac{\pi^2}{3}c_L T_L = 2\pi J$ which equates the macroscopic Bekenstein-Hawking entropy $S_{macro} = A/4 = 2\pi J$ where A is the horizon area of the Kerr black hole.

[1] The Virasoro algebra is widely used in two dimensional CFT and string theory and reads as the Witt algebra (2.38) to which is added a central charge term as follows $\left[l_n, l_m\right] = (n-m)l_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$.

[2] The Frolov-Thorn vacuum for the Kerr black hole is equivalent to the Hartle-Hawking vacuum for the Schwarzschild black hole. The Frolov-Thorn vacuum is obtained by expanding the quantum field theory in eigenmodes of the asymptotic energy ω and angular momentum m. After tracing over the states inside the horizon, the vacuum reads as a diagonal density matrix in the energy-angular momentum eigenbasis.

2.4.2 Anti-de Sitter spacetimes and conformal field theories

Since most of the topics covered in this dissertation are built around the AdS/CFT correspondence, it is important in the current subsection to review some basic properties of Anti-de Sitter (AdS) spaces that lead to the statement of a correspondence relation between these spacetimes and the conformal field theories defined on their boundaries.

In order to derive the metric of an AdS spacetime we start with a five dimensional flat manifold whose metric reads as [41]

$$ds_5^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2 \tag{2.45}$$

and embeds a hyperboloid given by

$$-u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2 \tag{2.46}$$

where α is a real constant.

In terms of the coordinates $(t^{'}, \rho, \theta, \phi)$ one the hyperboloid defined via the relations

$$\begin{aligned} u &= \alpha \sin t^{'} \cosh \rho \\ v &= \alpha \cos t^{'} \cosh \rho \\ x &= \alpha \sinh \rho \cos \theta \\ y &= \alpha \sinh \rho \sin \theta \cos \phi \\ z &= \alpha \sinh \rho \sin \theta \sin \phi \end{aligned} \tag{2.47}$$

(2.45) takes the form

$$ds^2 = \alpha^2 \big[-\cosh^2 \rho dt^{'2} + d\rho^2 + \sinh^2 \rho d\Omega_2^2 \big]. \tag{2.48}$$

This corresponds to the metric of an AdS_4 spacetime. In d dimensions, the AdS metric (2.48) becomes

$$ds^2 = \alpha^2 \big[-\cosh^2 \rho dt^{'2} + d\rho^2 + \sinh^2 \rho d\Omega_{d-2}^2 \big]. \tag{2.49}$$

Introducing the variables

$$\begin{aligned} r &= \alpha \sinh \rho \\ t &= \alpha t^{'} \end{aligned} \tag{2.50}$$

eq.(2.49) now reads

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \tag{2.51}$$

where $f(r) = 1 + r^2/\alpha^2$.

AdS manifolds with metrics (2.49) or (2.51) are maximally symmetric Lorentzian solutions to Einstein equation with constant negative curvature [3].

From the d dimensional form of (2.45) we infer that the large isometry group of AdS$_d$ is the non-compact group $SO(d-1,2)$.

Let us now turn our attention to field theories. Generally in these theories a ground state around which is performed a perturbation expansion is sought-after. An AdS$_d$ spacetime along with its isometry group $SO(d-1,2)$ are a nice choice of maximally symmetric group states.

AdS$_d$ spacetimes appear to be the near horizon geometry of the extreme black holes and p-branes[4], which are very central to the understanding of the theory. p-branes are referred to as extended objects with p spatial dimensions that move in a higher (particularly in 11 dimensions) dimensional spacetime. $p=0$ objects are point particles, $p=1$ are strings, $p=2$ are membranes and so on.

It is known that a supergravity theory is solved by a supersymmetric solution that admits one or more spinor ϵ_I satisfying [42]

$$\nabla_\alpha \epsilon_I + N_\alpha \epsilon_I = 0 \tag{2.52}$$

with ∇ the Levi-Civita connection and N_α a Clifford algebra valued one-form.

In an AdS$_d$

$$N_\alpha = \pm \frac{1}{2R}\gamma_\alpha \qquad\qquad \alpha = 0, 1, ..., d-1 \tag{2.53}$$

where R is the AdS radius and γ_α are the generators of the Clifford algebra.

We thus notice that when we particularly consider the case of N 3-branes with large N, we find that they have a supergravity description as a classical BPS [5] spacetime solution of a 10 dimension type IIB supergravity theory admitting 16 Majorana-Weyl killing spinors ϵ_I. That is [43, 44],

$$ds^2 = H^{-1/2}(-dt^2 + dx^2) + H^{1/2} dy^2 \tag{2.54}$$

with $x \in E^3$, $y \in E^6$ and $H(y)$ an harmonic function on E^6. E^3 and E^6 are the $3d$ and $6d$ Euclidean spacetime respectively.

[3]A maximally symmetric d dimensional manifold with metric $g_{\mu\nu}$ has a Riemann tensor $R_{\rho\sigma\mu\nu} = \kappa(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$ with κ a normalized measure of the Ricci curvature gieven by $\kappa = \frac{R}{d(d-1)}$ and R the Ricci scalar. $\kappa < 0$ for AdS spacetime.

[4]A Dp-brane is an extended object with p spatial dimensions. It can be thought of as a p-dimensional hyperplane moving in a d-dimensional space. Let $x^i (i = 1, ..., p)$ be the directions on the Dp-brane, it position is specified by $x^a = 0$ $(a = p+1, ..., d)$. The fact that the endpoints of the open string lie on the Dp-brane implies that the string coordinates normal to the brane satisfy Dirichlet boundary conditions $x^a(\tau,\sigma)|_{\sigma=0} = x^a(\tau,\sigma)|_{\sigma=\pi} = \bar{x}^a$.

[5]BPS states are named after Bogomolnyi, Prasad and Sommerfeld. Extremal (stable) black holes, which correspond to the endpoint of the Hawking evaporation, are BPS states for extended supergravity theories. Extremal black holes are characterized by the equivalence of their mass and charge [45].

For N coinciding 3-branes,

$$H = 1 + \left(\frac{R}{r}\right)^4 \tag{2.55}$$

with $R = (g_{YM}^2 N)^{1/4} l_s$ and $r = |y|$. g_{YM} is the gauge field coupling constant and l_s the string length.

Let us now focus on the near horizon geometry of (2.54) and (2.55). Near the horizon the isotropic coordinates break down and for small r (2.54) becomes

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx^2) + \frac{R^2}{r^2}dr^2 + R^2 dx_5^2. \tag{2.56}$$

Using the change of coordinates $z = R/r$ we obtain from (2.56) the metric

$$ds^2 = \frac{1}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) + R^2 dx_5^2 \tag{2.57}$$

with $\mu = 0, 1, ..., 3$ and $\eta_{\mu\nu}$ the Minkowski metric. We notice that the near horizon metric (2.56) or (2.57) correpsonds to that of an $\mathrm{AdS}_5 \times S^5$ spacetime. It is also obvious to notice that the symmetry group of the $\mathrm{AdS}_5 \times S^5$ is the same that the superconformal group in 3+1 spacetime dimension [46]. Indeed, the conformal symmetry group in $4d$ is isomorphic to the isometry group of the AdS_5 spacetime and the R-symmetry group is isomorphic to the $SO(6)$ symmetry group of S^5 as well.

From a couple of arguments (that we will not enumerate here) among which the above symmetry arguments, it results a reasonable AdS/CFT duality conjectured in [6]: *A type IIB* [6] *string theory in* $(AdS_5 \times S^5)_N$ *plus some appropriate boundary conditions (and possibly also some boundary degree of freedom) is dual to* $\mathcal{N} = 4$ $d = 3 + 1$ $U(N)$ *super Yang-Mills theory.*

2.5 Black hole chemistry

In this section we cover some key notions of black hole chemistry [21], as they represent an important part in the understanding of the next two chapters particularly.

A fundamental relationship between gravitation, thermodynamics and quantum mechanics has been established due to a number of theoretical arguments and calculations carried out over the past several decades. Originally black holes were thought of as objects characterized by few basic parameters such as their mass, charge and angular momentum [50]. They were also seen as entities absorbing all matter and emitting nothing and had neither temperature nor entropy.

[6] One of the four ground states of closed string theories (type II) has the form $|R_i^a\rangle \otimes |R_i^b\rangle \otimes |p^+, \vec{p}_T\rangle$. $|R_i^a\rangle$ ($i = 1, 2$ and $a = 1, ..., 8$) are the Ramond ground states or those subjected to fermionic boundary condition $\Psi^I(\tau, \pi) = +\Psi^I(\tau, -\pi)$. Type IIA strings are those for which the left and right Ramond ground states are different, i.e. $i \neq j$. Type IIB strings are those whose the left and right Ramond ground states are the same, i.e. $i = j$.

The study of quantum field theory in curved spaces has led to major changes in the understanding of black holes. Indeed, it was found that black holes can radiate heat comparable to black body radiation [7] and that the area of a black hole never decreases. Black hole radiation, while rooted in the basic foundations of quantum physics, paradoxically leads to a loss of information [47, 48].

Following this approach, it was possible to interpret the area of the black hole horizon as the thermodynamic entropy and its surface gravity as the thermodynamic temperature [51, 49]. In [52] Hawking stated that the area of the event horizon of a black hole can never decrease. It appears that this area law is similar to the second law of thermodynamics as noticed by Bekenstein.

Bardeen, Carter, and Hawking [51] subsequently formulated the 4 laws of black hole mechanics under the assumption that the event horizon of a black hole is a Killing horizon (which is a null hypersurface generated by a corresponding Killing vector field). These laws are:

1. The surface gravity κ of a stationary black hole is constant over its event horizon.
2. A rotating charged black hole with a mass M, angular momentum J and charge Q obeys

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J + \Phi\delta Q.$$

3. Hawking's area theorem states that the area A of a black hole event horizon can never decrease, i.e. $\delta A \geq 0$.
4. The surface gravity κ of a black hole cannot be reduced to zero in a finite number of steps.

From these laws it follows that black holes are physical thermodynamic systems that have temperatures and entropies.

Recently, the introduction of the cosmological constant as the pressure of the black hole, along with a volume, has given rise to a new set of techniques for studying the behaviour of black holes. This approach has led to the discovery of new phase behaviour analogous to that observed in gels and polymers. Triple points analogous to those in water have been observed as well in black holes.
These black holes have been understood as heat engines. It was also found that these black holes are

[7] Hawking discovered, when taking into account quantum effects, that black holes emit radiation [49] that has a black body spectrum with a characteristic temperature

$$k_B T = \frac{\hbar\kappa}{2\pi c}$$

where k_B is the Boltzmann's constant, c the speed of light, $\hbar$ the Planck's constant and κ the black hole surface gravity. The surface gravity κ is defined as

$$\xi^a \nabla_a \xi^b = \kappa \xi^b$$

where ξ^a is a normalized Killing vector that generates the horizon.

analogous to Van der Waals fluid and exhibit various behaviour of substances encountered in everyday life. These black hole features resulted in a field known as black hole chemistry [21, 53].

In the black hole thermodynamics language the thermodynamic energy E, temperature T and entropy S correspond to the mass M, the surface gravity κ and the horizon area A of the black hole respectively. These quantities are related as follows

$$\begin{aligned} E &= M \\ T &= \frac{\kappa}{2\pi} \\ S &= \frac{A}{4}. \end{aligned} \tag{2.58}$$

However, the correspondence breaks down when it comes to the first law of thermodynamics. Indeed, the first law of thermodynamics and the corresponding black hole first law read as

$$\begin{aligned} dE &= TdS - pdV + work\ terms \qquad \text{and} \\ dM &= \frac{\kappa}{8\pi}dA + \Omega dJ + \Phi dQ \end{aligned} \tag{2.59}$$

where $\Omega dJ + \Phi dQ$ is regarded as the work terms.

It is clear to notice that the term pdV in the first equation of (2.59) does not have a counterpart in the second equation of (2.59) [54]. To solve this problem, the mass of the black hole was interpreted as the enthalpy of the spacetime [55].

This idea results from the consideration of the Smarr relation [56, 57], where in d dimensions [58] it is given by

$$(d-3)M = (d-2)TS \tag{2.60}$$

for a static asymptotically flat d-dimensional black holes.

The fact of introducing the cosmological constant as the pressure modifies the Smarr relation. In fact, regarding the mass M as a function of both the horizon area and the cosmological constant, i.e. $M = M(A, \Lambda)$ and by the Euler's theorem on homogeneous functions we obtain

$$(d-3)M = (d-2)\frac{\partial M}{\partial A}A - \frac{\partial M}{\partial \Lambda}\Lambda. \tag{2.61}$$

Knowing that $T = 4\frac{\partial M}{\partial A}$, equation (2.61) implies that the pressure p as a thermodynamic variable [59, 60] takes the form

$$p = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2} \tag{2.62}$$

with the conjugate variable $V = -8\pi\frac{\partial M}{\partial \Lambda}$.

Thus the modified Smarr relation and the extended first law of thermodynamics become

$$\begin{aligned}(d-3)M &= (d-2)TS - 2pV \qquad\qquad \text{and} \\ dM &= TdS + Vdp. \end{aligned} \tag{2.63}$$

Chapter 3

Super-entropic black holes and the Kerr-CFT correspondence

This chapter is based on the exploration of the extension of the Kerr-CFT correspondence to a new class of black holes known as super-entropic black holes. Its main purpose is to extend the domain of application of the correspondence and prove that it is a reliable one (robust). The elaboration of this chapter depends strongly on some basic notions on the Kerr-CFT correspondence that we have briefly reviewed in section 2.4.1 and will see in more detail in the next few sections and the black hole chemistry covered in section 2.5.

3.1 Introduction

In the past few years dualities between black holes and conformal field theories (CFT) have been of considerable interest, yielding new insights into our understanding of gravity. One of the most prominent among them is the "Kerr/CFT correspondence" [8], which established a duality between the horizon of a Kerr black hole (bulk) and a $2d$ CFT. Many other intriguing results [61] have been obtained since then.

Recently a new class of rotating black hole solutions was obtained from the Kerr–Newman-AdS metrics [62, 11, 12]. These black holes have been called *super-entropic* insofar as their entropy is larger than that expected from the reverse isoperimetric inequality conjecture [63], which states that the thermodynamic volume of a black hole provides an upper bound on its entropy. The basic idea of the procedure is to transform an azimuthal coordinate of a d-dimensional Kerr-AdS metric (written in rotating-at-infinity coordinates) and then take the limit as its associated rotation parameter approaches the AdS length. The new azimuthal coordinate is then compactified, yielding a black hole whose horizon is topologically a $(d-2)$-sphere with two punctures. These black holes form a new ultraspinning limit of the Kerr-AdS class of metrics, and possess a non-compact event horizon of finite area (and hence finite entropy), the first example of such objects in the literature to date.

In this chapter we study the Kerr-CFT correspondence for super-entropic black holes. It is not a-priori obvious that the correspondence exists, given the non-compactness of their event horizons; as we shall see some but not all super-entropic black holes exhibit this correspondence. We will consider specifically the superentropic limit of singly-spinning Kerr-Newmann-AdS black holes in d dimensions [12] as well as the ultraspinning $d = 5$ black holes of minimal gauged supergravity [65, 64]. Similar studies of Kerr-Newman-AdS black holes [61] found that there exists a correspondence between the extremal versions of these black holes and a $d = 2$ (chiral half of a) CFT at its boundary. A key feature of these metrics is that in their near-horizon (NH) limits, the resultant metrics acquire a new form that has an extended asymptotic symmmetry group (ASG) whose generators depend strongly on the boundary conditions imposed. These boundary conditions have to be chosen in such a way that the charges associated with the diffeomorphisms preserving them satisfy a number of conditions such as finiteness, integrability and so on. These constraints ensure the existence of the charges (generators of diffeomorphisms); indeed, the consistency of the theory relies on them. Ignoring these constraints will certainly lead to ill-defined diffeomorphism generators from which no physics can be deduced.

Moreover, the boundary conditions allow us to compute the central charges that stem from the Virasoro algebra satisfied by the charges associated with the diffeomorphisms preserving the boundary conditions. Once obtained, the central charges yield the CFT entropy S_{CFT} via the Cardy formula, which is expected to be identical to the Bekenstein-Hawking S entropy of the black hole, thereby establishing the correspondence. It is not *a-priori* obvious that super-entropic black holes admit such a correspondence, given their properties. It is the purpose of this chapter to demonstrate that such a correspondence exists and indeed is robust.

The present chapter is organised as follows. In section two, we provide a brief review of the super-entropic black hole, highlighting its thermodynamic quantities and its relationship to the Smarr relation and the reverse isoperimetric inequality. In section three, we review the Kerr-CFT correspondence for Kerr-AdS black holes, along with the choice of boundary conditions. We then consider the Super-entropic-CFT correspondence, beginning with a Kerr-Newman-AdS black hole and taking both its super-entropic and near horizon limits in either order, and obtain the same results in either case. In section five, we perform a similar analysis for super-entropic black holes of minimal gauged supergravity in $d = 5$ [65]. We conclude with some remarks on our results.

3.2 Super-entropic black holes

In this section we review some important notions on super-entropic black holes and their construction. In other words, we shall derive the thermodynamic quantities associated to these black holes and the Smarr relation obeyed by these quantities. Let us start with the Kerr-Newman-AdS black hole metric [64, 12]

$$\begin{aligned} ds^2 &= -\frac{\Delta_a}{\Sigma_a}[d\bar{t} - a\sin^2\bar{\theta}\frac{d\bar{\phi}}{\Xi}]^2 + \frac{\Sigma_a}{\Delta_a}d\bar{r}^2 + \frac{\Sigma_a}{\mathcal{S}}d\bar{\theta}^2 + \frac{\mathcal{S}\sin^2\bar{\theta}}{\Sigma_a}[ad\bar{t} - (\bar{r}^2+a^2)\frac{d\bar{\phi}}{\Xi}]^2 \\ \mathcal{A} &= -\frac{q\bar{r}}{\Sigma_a}(d\bar{t} - a\sin^2\bar{\theta}\frac{d\bar{\phi}}{\Xi}) \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} \Xi &= 1 - \frac{a^2}{l^2}, \quad \mathcal{S} = 1 - \frac{a^2}{l^2}\cos^2\bar{\theta}, \quad \Sigma_a = \bar{r}^2 + a^2\cos^2\bar{\theta} \\ \omega_h &= \frac{a}{a^2+r_+^2}, \quad \Delta_a = (\bar{r}^2+a^2)(1+\frac{\bar{r}^2}{l^2}) - 2m\bar{r} + q^2. \end{aligned}$$

ω_h is the angular velocity and a is the rotation parameter of the black hole. Taking the ultraspinning limit $a \to l$ yields the super-entropic black hole, whose metric is given by [64, 12]

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\Sigma}[d\bar{t} - l\sin^2\bar{\theta}d\bar{\psi}]^2 + \frac{\Sigma}{\Delta}d\bar{r}^2 + \frac{\Sigma}{\sin^2\bar{\theta}}d\bar{\theta}^2 + \frac{\sin^4\bar{\theta}}{\Sigma}[ld\bar{t} - (\bar{r}^2+l^2)d\bar{\psi}]^2 \\ \mathcal{A} = &- \frac{q\bar{r}}{\Sigma}(d\bar{t} - l\sin^2\bar{\theta}d\bar{\psi}) \end{aligned} \quad (3.2)$$

where

$$\Sigma = \bar{r}^2 + l^2\cos^2\bar{\theta}, \quad \Delta(\bar{r}) = (l + \frac{\bar{r}^2}{l})^2 - 2m\bar{r} + q^2 \quad (3.3)$$

provided the rescaled coordinate $\bar{\psi}$

$$\bar{\psi} = \frac{\bar{\phi}}{\Xi}, \quad \Xi = 1 - \frac{a^2}{l^2} \;\text{ with }\; a \to l \quad (3.4)$$

is taken to be periodic, identifying $\bar{\psi} \sim \bar{\psi} + \mu$. The quantity μ can be regarded as the chemical potential of the black hole [12].

It is important to stress that the chemical potential defined here is not that used in the string literature [66]; indeed it is independent of the electric charge. Rather this terminology arose from considerations of holographic duality for non-relativistic field theories – the holographic dual of a d dimensional spatial CFT is a gravitational solution in a bulk space of $d+3$ dimensions[1]. This dual spacetime (solution) realizes Galilean scaling as an isometry. For example, starting with the $5d$ planar

[1] In [67, 68], this spacetime has the form $ds^2 = r^2(-2du\, dv - r^{2\nu}du^2 + dx^2) + dr^2/r^2$, where $x = (x_1 ... x_d)$ are the spatial coordinates of the Galilean field theory and ν a positive integer. The light-cone coordinate u is the boundary time coordinate and v is proposed to be treated as a compact coordinate.

Schwarzschild-AdS black hole (times S^5, with the geometry supported by the five-form flux $F_{(5)}$) [69], upon applying a transformation known as a Null Melvin Twist to this geometry [70] a new metric is obtained[2]. The Killing generator of the horizon of the non-extremal solution takes a form, from the gravitational perspective, that leads to a charge independent chemical potential[3]. It is in this latter sense that the term ‘chemical potential is being employed for the solution under consideration in this chapter. The asymptotic form of the metric (3.2) is AdS_3, but with the compactified coordinate $\bar{\psi}$ becoming null. This feature is similar to that of the asymptotically Schroedinger and Galilean spacetimes [69], and so the quantity μ was referred to as a chemical potential [64].

Trying to find roots of the function $\Delta(r)$, we notice that it admits roots $r_\pm$ only when

$$m \geq m_* \equiv 2r_*(\frac{r_*^2}{l^2}+1) \tag{3.5}$$

where

$$r_*^2 \equiv \frac{l^2}{3}[-1+\sqrt{4+3\frac{q^2}{l^2}}] \tag{3.6}$$

For an extremal black hole

$$r_+ = r_* \qquad m = 2r_+(\frac{r_+^2}{l^2}+1) \tag{3.7}$$

and we note that extremality can occur even if $q=0$.

The super-entropic character of these black holes can be understood in the context of an *extended phase space* framework [71], where the cosmological constant is identified with the thermodynamic pressure according to

$$P = -\frac{\Lambda}{8\pi} \tag{3.8}$$

in d spacetime dimensions, with its conjugate quantity treated as thermodynamic volume V. The

---

[2]The metric reads

$$\begin{aligned} ds^2 &= r^2\big[-\frac{\beta^2 r^2 f}{k}(dt+dy)^2 - \frac{f}{k}dr^2 + \frac{dy^2}{k} + dx^2\big] + \frac{dr^2}{r^2 f} + \frac{(d\psi+\tilde{A})}{k} + d\Sigma_4^2 \\ e^\phi &= \frac{1}{\sqrt{k}}, \qquad f = 1 - r_+^4/r^4, \qquad d\tilde{A} = 2J, \qquad k = 1+\beta^2\frac{r_+^4}{r^4} \\ F_{(5)} &= dC_{(4)} = 2(1+*)d\psi \times J \times J, \qquad B_{(2)} = \frac{r^2\beta}{k}(fdt+dy)\times(d\phi+\tilde{A}) \end{aligned}$$

β is a fixed parameter and $r_+$the horizon radius. Performing a Kaluza-Klein reduction on that new metric [69], we obtain a non-extremal solution which, in the light-cone coordinates, asymptotically approaches the extremal solution (which has the Galilean scaling as an isometry).

[3] The Killing generator of the horizon $\xi = \frac{1}{\beta}\frac{\partial}{\partial t} = \frac{\partial}{\partial u} + \frac{1}{2\beta^2}\frac{\partial}{\partial v}$. The chemical potential is $\mu = \frac{1}{2\beta^2}$.

thermodynamic parameters of the super-entropic black hole are

$$\begin{aligned}
M &= \frac{\mu m}{2\pi}, \\
J &= Ml, \\
Q &= \frac{\mu q}{2\pi}, \\
\Phi &= \frac{(q/l)x}{1+x^2}, \\
\omega_h &= \frac{1}{l(1+x^2)}, \\
K &= l\frac{(1-x^2)[(1+x^2)^2+q^2/l^2]}{8\pi x(1+x^2)}, \\
A &= 2\mu l^2(1+x^2), \\
V &= \frac{2}{3}l^3\mu x(1+x^2) = \frac{l}{3}xA, \\
T_H &= \frac{1}{4\pi l x}\big[3x^2 - 1 - \frac{q^2/l^2}{1+x^2}\big], \\
S &= \frac{\mu}{2}l^2(1+x^2) = \frac{A}{4}
\end{aligned} \tag{3.9}$$

with $x = r_+/l$. The quantities $M, J, Q, \Phi, \omega_h, V, K, T_H, A$ and S are the mass, the angular momentum, the charge, the electric potential, the angular velocity at the horizon, the volume, the conjugate variable to the chemical potential μ, the Hawking temperature, the area and the entropy of the black hole respectively. The first law of black hole thermodynamics is now extended to [57]

$$\delta M = T\delta S + \omega_h \delta J + \Phi\delta Q + V\delta P \tag{3.10}$$

with the quantities (3.9) satisfying the Smarr relation [57]

$$\frac{d-3}{d-2}M = TS + \omega_h J + \frac{d-3}{d-2}\Phi Q - \frac{2}{d-2}VP \tag{3.11}$$

with $d = 4$. The quantity S in (3.9) violates the *Reverse Isoperimetric Inequality*, which asserts [63]

$$\mathcal{R} \equiv \left(\frac{(d-1)V}{\omega_{d-2}}\right)^{\frac{1}{d-1}} \left(\frac{\omega_{d-2}}{A}\right)^{\frac{1}{d-2}} \geq 1 \tag{3.12}$$

where

$$\omega_d = \frac{\mu\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \tag{3.13}$$

It is straightforward to show that for the quantities in (3.9), we obtain

$$\mathcal{R} = \left(\frac{x^2}{1+x^2}\right)^{\frac{1}{6}} < 1 \tag{3.14}$$

which clearly violates the inequality. The entropy of this class of black holes exceeds, for a given thermodynamic volume, the bound set by (3.12), and hence are called super-entropic.

These singly spinning super-entropic black holes have extensions in any dimension, with metric [64]

$$ds^2 = -\frac{\Delta}{\rho^2}(d\bar{t} - l\sin^2\bar{\theta}d\bar{\psi})^2 + \frac{\rho^2}{\Delta}d\bar{r}^2 + \frac{\rho^2}{\sin^2\bar{\theta}}d\bar{\theta}^2 + \frac{\sin^4\bar{\theta}}{\rho^2}[ld\bar{t} - (\bar{r}^2 + l^2)d\bar{\psi}]^2 + \bar{r}^2\cos^2\bar{\theta}d\Omega^2_{d-4} \quad (3.15)$$

where

$$\Delta = \left(l + \frac{\bar{r}^2}{l}\right)^2 - 2m\bar{r}^{5-d}, \quad \rho^2 = \bar{r}^2 + l^2\cos^2\theta$$

and $d\Omega^2_d$ is the metric element on a d-dimensional sphere, where as before we identify $\bar{\psi} \sim \bar{\psi} + \mu$. Writing $\varpi_d = \frac{\mu\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})}$, their thermodynamic parameters are

$$\begin{aligned} M &= \frac{\varpi_{d-2}}{8\pi}(d-2)m, \\ \omega_h &= \frac{l}{r_+^2 + l^2}, \\ T &= \frac{1}{4\pi r_+ l^2}[(d-5)l^2 + r_+^2(d-1)], \\ J &= \frac{2}{d-2}Ml, \\ S &= \frac{\varpi_{d-2}}{4}(l^2 + r_+^2)r_+^{d-4} = \frac{A}{4}, \\ V &= \frac{r_+ A}{d-1} \end{aligned} \quad (3.16)$$

respectively denoting the mass, the angular velocity at the horizon, the Hawking temperature, the angular momentum, the Bekenstein-Hawking entropy and the volume of the black holes respectively.

While there exist horizons in any $d > 5$ only when $m > 0$ and for $d = 5$ when $m > l^2/2$, the extremal limit exists only in $d = 4$. Specifically, extremal super-entropic black holes are those for which

$$x^2 = \frac{5-d}{d-1} \quad (3.17)$$

whose only non-trivial solution is for $d = 4$. Henceforth we shall only consider this case.

3.3 The Kerr-CFT limit

Here we review the Kerr-CFT correspondence for the Kerr-Newman-AdS black hole. We intend to write the black hole metric in terms of the coordinates near its horizon. Returning to the metric (3.1) in the extremal case and carrying out the near-horizon (NH) transformation

$$\begin{aligned} \bar{t} &= r_0 t/\epsilon \\ \bar{\theta} &= \theta \\ \bar{r} &= r_+ + \epsilon r_0 r \\ \bar{\phi} &= \phi + \Xi\omega_h r_0 t/\epsilon \end{aligned} \quad (3.18)$$

where ϵ is a small parameter, it yields a metric of the form

$$
\begin{aligned}
ds^2 &= \Gamma(\theta)[-r^2dt^2 + \frac{dr^2}{r^2} + \alpha(\theta)d\theta^2 + \frac{\gamma(\theta)}{\Gamma(\theta)}(\frac{d\phi}{\Xi} + krdt)^2] \\
\mathcal{A} &= f(\theta)(\frac{d\phi}{\Xi} + krdt) \qquad (3.19)
\end{aligned}
$$

in the $\epsilon \to 0$ limit where k is constant. The quantities $\Gamma, \alpha, \gamma, f$ are functions of the variable θ and can be computed from (3.1). However (3.19) is quite general: the metric of any $d = 4$ stationary, axisymmetric extremal black hole, with a compact horizon section of non-toroidal topology, will have a near-horizon limit of the form (3.19), with $\Gamma, \alpha, \gamma, f$ taking specific values depending on the black hole studied [72, 73]. Note that the new coordinates (t, ϕ, θ, r) are dimensionless. The quantity r_0 is a parameter with dimension of length whose value will be subsequently be fixed.

The metric (3.19) is invariant under the isometries [61]

$$
\begin{aligned}
\bar{K}_1 &= \partial_t, \\
\bar{K}_2 &= t\partial_t - r\partial_r, \\
\bar{K}_3 &= (\frac{1}{2r^2} + \frac{t^2}{2})\partial_t - tr\partial_r - \frac{k}{r}\partial_\psi, \\
K_1 &= \partial_\psi \qquad (3.20)
\end{aligned}
$$

which generate a $Sl(2,\mathbb{R})_L \times U(1)_R$ symmetry group.

To study the corresponding CFTs, we will refer to the case of the Kerr-Newman- AdS black hole explored in [61] and perform a similar analysis. The metric function $\Delta(r)$ in (3.1) has a root r_+ when

$$
m \geq 2\mathtt{r}_*\left(\frac{\mathtt{r}_*^2}{l^2} + 1\right)
$$

with

$$
\mathtt{r}_*^2 = \frac{l^2}{6}[-1 - a^2/l^2 + \sqrt{1 + 14a^2/l^2 + a^4/l^4 + 12q^2/l^2}]
$$

The thermodynamic quantities of the black hole (3.1) are

$$\begin{aligned}
M &= m, \\
J &= Ma, \\
Q &= q, \\
\Phi &= \frac{qx}{l(a^2/l^2+x^2)}, \\
\omega_h &= \frac{a/l^2}{a^2/l^2+x^2}, \\
A &= \frac{4\pi}{\Xi} l^2(a^2/l^2+x^2), \\
T_H &= \frac{3x^2(a^2/l^2+x^2)+2x^2\Xi-(a^2/l^2+x^2)-q^2/l^2}{4\pi l x(a^2/l^2+x^2)}, \\
V &= \frac{2\pi l^3}{3}\frac{(\frac{a^2}{l^2}+x^2)(2x^2+\frac{a^2}{l^2}-x^2\frac{a^2}{l^2})+\frac{q^2a^2}{l^4}}{\Xi^2 x}
\end{aligned} \tag{3.21}$$

respectively being its mass, the angular momentum, the charge, the electric potential, the angular velocity at the horizon, the horizon area, the Hawking temperature and thermodynamic volume. For an extremal black hole $T_H = 0$ yielding

$$r_+ = \mathtt{r}_* \qquad m = 2\mathtt{r}_*(\frac{\mathtt{r}_*^2}{l^2}+1)$$

The Bekenstein-Hawking entropy as computed in [64, 12] takes the value

$$S = \frac{\pi}{\Xi}(a^2+r_+^2) = \frac{A}{4} \tag{3.22}$$

noting that ϕ is defined in the interval $[0, 2\pi)$.

We now aim to compute the CFT entropy of the Kerr-Newman-AdS black-hole (3.1). To this end, we first take the near horizon limit (3.18) and obtain the metric (3.19), with

$$\begin{aligned}
\Gamma(\theta) &= \frac{\Sigma_{a_+}}{1+a^2/l^2+6x^2}, \\
\alpha(\theta) &= \frac{1+a^2/l^2+6x^2}{\mathcal{S}}, \\
\gamma(\theta) &= \frac{\mathcal{S}}{\Sigma_{a_+}} l^4(a^2/l^2+x^2)^2\sin^2\theta, \\
k &= 2\frac{(a/l)x}{(a^2/l^2+x^2)(1+a^2/l^2+6x^2)}, \\
r_0^2 &= l^2\frac{a^2/l^2+x^2}{1+a^2/l^2+6x^2}, \\
f(\theta) &= q\frac{(1+x^2)}{2x}\frac{(x^2-a^2/l^2\cos^2\theta)}{x^2+a^2/l^2\cos^2\theta}
\end{aligned} \tag{3.23}$$

and $x = r_+/l$ as before. The asymptotic symmetries of this metric contain diffeomorphisms ζ such that [61]

$$\begin{aligned} \delta_\zeta A_\mu &= \mathcal{L}_\zeta A_\mu, \\ \delta_\zeta g_{\mu\nu} &= \mathcal{L}_\zeta g_{\mu\nu}, \end{aligned} \tag{3.24}$$

as well as a $U(1)$ gauge transformation

$$\delta_\Lambda A = d\Lambda. \tag{3.25}$$

We will shortly see that the diffeomorphism and gauge transformations (ζ, Λ) obey an algebra and that their associated charges $Q_{\zeta,\Lambda}$ obey the same algebra up to a central charge term. The charge difference between two neighbouring metrics $g_{\mu\nu}$ and $g_{\mu\nu} + \delta g_{\mu\nu}$ is [8, 61]

$$\delta Q_{\zeta,\Lambda} = \frac{1}{8\pi} \int (K_\zeta[h; g] + K_{\zeta,\Lambda}[h, a; g, A]) \tag{3.26}$$

with

$$a_\mu = \delta A_\mu, \quad h_{\mu\nu} = \delta g_{\mu\nu},$$

$$\begin{aligned} K_\zeta[h; g] &= \frac{1}{4}\epsilon_{\alpha\beta\mu\nu}[\zeta^\nu D^\mu h - \zeta^\nu D_\sigma h^{\mu\sigma} + \zeta_\sigma D^\nu h^{\mu\sigma} + \frac{1}{2} h D^\nu \zeta^\mu - h^{\nu\sigma} D_\sigma \zeta^\mu \\ &+ \frac{1}{2} h^{\sigma\nu}(D^\mu \zeta_\sigma + D_\sigma \zeta^\mu)] dx^\alpha \times dx^\beta \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} K_{\zeta,\Lambda} &= \frac{1}{8}\epsilon_{\alpha\beta\mu\nu}[(-\frac{1}{2} h F^{\mu\nu} + 2F^{\mu\sigma} {h_\sigma}^\nu - \delta F^{\mu\nu})(\zeta^\rho A_\rho + \Lambda) - F^{\mu\nu}\zeta^\sigma a_\sigma - 2F^{\alpha\mu}\zeta^\nu a_\alpha \\ &- g^{\mu\sigma} g^{\nu\rho} a_\sigma (\mathcal{L}_\zeta A_\rho + \partial_\rho \Lambda)] dx^\alpha \times dx^\beta \end{aligned} \tag{3.28}$$

with $\quad \delta F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} (\partial_\alpha a_\beta - \partial_\beta a_\alpha)$.

In order that the integral (3.26) be well defined we must impose suitable boundary conditions which fulfil the conditions given in [74, 75, 76]. The choice of boundary conditions determines the asymptotic symmetry group (ASG), which is nothing but the allowed symmetries (those preserving the boundary conditions) modulo the trivial ones (those for which the associated charges vanish). The idea is to make the boundary conditions as weak as we can whilst keeping the consistency of the theory [4]. Consistency implies that the charges associated to the diffeomorphisms have to be finite (or may vanish). We show in the appendix that the surface charges defined in (3.26) with the boundary conditions given in [8] are finite and integrable. Choosing these same boundary conditions for the NH metric (3.19) yields

$$h_{\mu\nu} \sim \begin{bmatrix} \mathcal{O}(r^2) & \mathcal{O}(1) & \mathcal{O}(1/r) & \mathcal{O}(1/r^2) \\ & \mathcal{O}(1) & \mathcal{O}(1/r) & \mathcal{O}(1/r) \\ & & \mathcal{O}(1/r) & \mathcal{O}(1/r^2) \\ & & & \mathcal{O}(1/r^3) \end{bmatrix}$$

[4]Making the boundary conditions weak amounts to choose them such that most of their contributions to the conserved charges associated to the diffeomorphisms ζ vanish at infinity ($r \to \infty$). See eqs. (A-3) and (A-4).

and for the gauge field

$$a_\mu \sim \mathcal{O}(r, 1/r, 1, 1/r^2) \tag{3.29}$$

all in the basis (t, ϕ, θ, r). Making the choice

$$\Lambda = -f(\theta)\epsilon(\phi)$$

for the compensating gauge transformation [61] satisfies the above boundary conditions.

The most general diffeomorphism preserving these boundary conditions is

$$\xi = [C + \mathcal{O}(1/r^3)]\partial_t + [-r\epsilon'(\phi) + \mathcal{O}(1)]\partial_r + \mathcal{O}(1/r)\partial_\theta + [\epsilon(\phi) + \mathcal{O}(1/r^2)]\partial_\phi \tag{3.30}$$

where C is an arbitrary constant and $\epsilon(\phi)$ an arbitrary smooth function of ϕ. This includes the diffeomorphism

$$\zeta = \epsilon\partial_\phi - r\epsilon'\partial_r \tag{3.31}$$

where $\epsilon' = d\epsilon/d\phi$. This yields the Virasoro algebra

$$i[\zeta_m, \zeta_n] = (m-n)\zeta_{m+n}, \qquad \zeta_n = \zeta(\epsilon_n) \quad \text{with} \quad \epsilon_n = -e^{-in\phi}$$

and

$$\begin{aligned} [\Lambda_m, \Lambda_n]_\zeta &= \zeta_m^\mu \partial_\mu \Lambda_n - \zeta_n^\mu \partial_\mu \Lambda_m \\ i[\Lambda_m, \Lambda_n]_\zeta &= (m-n)\Lambda_{m+n} \end{aligned}$$

which is the algebra of the ASG.

As noted above, the charges Q_n associated with these diffeomorphisms and gauge transformations (ζ_n, Λ_n) satisfy a similar algebra

$$\begin{aligned} i\{Q_m, Q_n\} &= (m-n)Q_{m+n} + \frac{1}{8\pi}\int (K_\zeta[h; g] + K_{\zeta,\Lambda}[h, a; g, A]) \\ &= (m-n)Q_{m+n} + \frac{c}{12}(m^3 - \alpha m)\delta_{m+n,0} \end{aligned} \tag{3.32}$$

the distinction being the central charge contribution, where α is a constant obtained after we parametrize Q_n. Proceeding as in [61] we therefore get a central charge

$$c = \frac{3k}{\Xi}\int \sqrt{\Gamma(\theta)\gamma(\theta)\alpha(\theta)}d\theta. \tag{3.33}$$

To determine the temperatures of the left and right-moving CFTs, we make use on one hand of the first law

$$TdS = dM - (\omega_h dJ + \Phi dQ) \tag{3.34}$$

and its extremality constraints

$$T^{ex}dS = dM - (\omega_h^{ex} dJ + \Phi^{ex} dQ) = 0 \qquad (3.35)$$

to obtain

$$TdS = -[(\omega_h - \omega_h^{ex})dJ + (\Phi - \Phi^{ex})dQ]. \qquad (3.36)$$

These variations can also been expressed as

$$dS = \frac{dJ}{T_L} + \frac{dQ}{T_e} \qquad (3.37)$$

We recall that for a scalar field its expansion in eigenmodes of the energy and angular momentum is [8]

$$\Phi = \sum_{E,J,s} \Phi_{E,J,s}\; e^{-iE\bar{t}+iJ\bar{\psi}} f_s(r,\theta) \qquad (3.38)$$

for a Kerr-AdS black hole. Near the horizon, the factor

$$e^{-iE\bar{t}+iJ\bar{\psi}}$$

becomes

$$\begin{aligned} e^{-iE\bar{t}+iJ\bar{\psi}} &= e^{-i(E-\omega_h^{ex}J)tr_0/\epsilon+iJ\psi} \\ &= e^{-in_R t+in_L\psi} \end{aligned} \qquad (3.39)$$

upon using (3.18), where

$$\begin{aligned} n_R &= (E - \omega_h^{ex} J) r_0/\epsilon, \\ n_L &= J. \end{aligned} \qquad (3.40)$$

For any system the density of states is $\rho = e^S$, with S the entropy. Using this fact we extend the preceding expressions of n_R and n_L for a Kerr-Newman-AdS black hole to

$$\begin{aligned} n_R &= (E - \omega_h^{ex} J - \Phi^{ex} Q) r_0/\epsilon, \\ n_L &= J \end{aligned} \qquad (3.41)$$

so that the density matrix in the energy and angular momentum eigenbasis has the Boltzmann weighting factor

$$e^{-(E-\omega_h J-\Phi Q)/T_H} = e^{-(n_R/T_R)-(n_L/T_L)-Q/T_e} \qquad (3.42)$$

and is a diagonal matrix when tracing over the modes inside the horizon. Comparing both sides of this equation yields the temperatures of the left and right-moving CFTs

$$\begin{aligned} T_L &= -\frac{\partial T_H/\partial x}{\partial \omega_h/\partial x}\bigg|_{ex}, \\ T_R &= \frac{T_H r_0}{\epsilon}\bigg|_{ex}, \\ T_e &= -\frac{\partial T_H/\partial x}{\partial \Phi/\partial x}\bigg|_{ex} \end{aligned} \tag{3.43}$$

as well as the T_e term in (3.42). For extremal black holes $T_H = 0$ we see that $T_R \to 0$ and a straightforward computation shows that the temperature of the left-moving CFT is $T_L = \frac{1}{2\pi k}$ and

$$T_e = \frac{1}{2\pi q}\frac{[3(a^2/l^2 + 2x^2) + 2\Xi - 1](a^2/l^2 + x^2)}{(x^2 - a^2/l^2)} \tag{3.44}$$

which is proportional to an inverse length ($\sim l^{-1}$).

The upshot of this exercise is that an extremal Kerr-Newmann-AdS black hole is dual to a $2d$ conformal field theory at its boundary with a mixed state whose density matrix is expressed below. The Hartle-Hawking vacuum state is generalized around the Kerr-Newman-AdS black hole with a density matrix

$$\rho = e^{-\frac{J}{T_L} - \frac{Q}{T_e}}. \tag{3.45}$$

Substituting our results into the CFT entropy from Cardy's formula, which states that the entropy of a unitary CFT at large T or with $T \gg c$ satisfies [5]

$$S_{CFT} = \frac{\pi^2}{3} c_L T_L \tag{3.46}$$

yields

$$S_{CFT} = \frac{\pi}{\Xi} l^2 (a^2/l^2 + x^2) \tag{3.47}$$

which is in agreement with the expression in (3.22). Conditions for the applicability of (3.46) have been given in [8] in situations where $T \gg c$ does not hold. We shall see that a sufficient condition for the applicability of Cardy's formula in the super-entropic case is to set the electric charge q to be large.

[5]For a conformal field theory on a Euclidean torus, the partition function with periodic time is $Z[\beta] = Tr\ e^{-\beta H} = e^{-\beta F}$. Very low temperature ($\beta \to \infty$) are characterized by the free energy dominated by the lowest energy state. It has been shown [77] that the vacuum state on the cylinder has the Casimir energy $H = -c/12$ (c is the central charge) which implies that the partition function takes the form $Z \to e^{c\beta/12}$ as $\beta \to \infty$. In an Euclidean space both directions of the torus are equivalent, i.e. we can assume that σ is the time and τ the space. Since we want the spatial direction to range in the interval $[0, 2\pi)$ we will use the transformation $\tau \to \frac{2\pi}{\beta}\tau$, $\sigma \to \frac{2\pi}{\beta}\sigma$. This yields the relation $Z[4\pi^2/\beta] = Z[\beta]$ and implies that the partition function now reads as $Z \to e^{\frac{c\pi^2}{3\beta}}$ as $\beta \to 0$. In the case of a $2d$ CFT, the entropy of a system at large energy scales as $S(E) \to N\sqrt{E}$ where N is the number of degree of freedom of the system. From the above equations, the free energy scales as $F \sim N^2 T^2$ with $N^2 = c$, so does E. It follows the Cardy formula $S \sim N^2 T = c\ T$.

$$
\begin{tikzcd}
\mathrm{KNDS} \arrow[r] \arrow[d] & \mathrm{KCFT} \arrow[d] \\
\mathrm{SE} \arrow[r] & \mathrm{SE\text{-}CFT}
\end{tikzcd}
$$

Figure 3.1: The above diagram indicates the various limits we are considering for the Kerr-Newman-AdS (KNDS) black hole: the Super-Entropic (SE) limit, the Kerr-CFT (KCFT) limit, and both limits together (SE-KCFT). The horizontal arrows (in red) are the near horizon limit (NH) and vertical ones (blue) the ultraspinning limit. The resulting black hole is obtained by taking both limits one after another; we find that the same SE-KCFT limit results, indicating that the square commutes.

3.4 Super-entropic-CFT correspondence

In this section we establish that the ultraspinning super-entropic limit (3.2)–(3.4) commutes with the Kerr-CFT limit (3.18), as shown in figure 3.1.

Let us first consider the lower path in figure 6.1a, which consists in starting with the Kerr-Newman-AdS black hole and taking the super-entropic limit, obtaining (3.2). The next step is to take the near-horizon limit of the extremal version of (3.2). We obtain the metric (3.19), now with

$$
\begin{aligned}
\Gamma(\theta) &= \frac{l^2}{2}\frac{x^2+\cos^2\theta}{1+3x^2}, \\
\alpha(\theta) &= \frac{2}{\sin^2\theta}(1+3x^2), \\
\gamma(\theta) &= l^2\sin^4\theta\frac{(1+x^2)^2}{x^2+\cos^2\theta}, \\
k &= \frac{x}{(1+x^2)(1+3x^2)}, \\
r_0^2 &= \frac{l^2}{2}\frac{1+x^2}{1+3x^2}, \\
f(\theta) &= q\frac{(1+x^2)}{x}\frac{x^2-\cos^2\theta}{x^2+\cos^2\theta}
\end{aligned}
\tag{3.48}
$$

whose functions are also given by the $a \to l$ limit of (3.23). The resulting metric likewise has the same topology as that of the metric (3.2), with punctures at the poles [64, 12].

To compute the entropy of the corresponding CFT, the steps remain the same as described in the previous sections. We impose the same boundary conditions and therefore the same diffeomorphisms and ASG. The central charge is now

$$\begin{aligned} c &= \frac{3k\mu}{2\pi}\int\sqrt{\Gamma(\theta)\gamma(\theta)\alpha(\theta)}d\theta \\ &= \frac{3k\mu}{\pi}l^2(1+x^2) \\ &= \frac{3x\mu l^2}{\pi(1+3x^2)} \end{aligned} \tag{3.49}$$

using (3.48). We remark that c is very small when the electric charge q becomes very large.

Considering the extremality constraint of the extended first law, we obtain

$$\begin{aligned} T^{ex}dS &= dM-(\omega_h^{ex}dJ+\Phi^{ex}dQ+K^{ex}d\mu) \\ &= 0 \end{aligned} \tag{3.50}$$

with μ the new thermodynamic variable whose conjugate is K in (3.9). The first law then takes the form

$$TdS = -[(\omega_h-\omega_h^{ex})dJ+(\Phi-\Phi^{ex})dQ+(K-K^{ex})d\mu] \tag{3.51}$$

or alternatively

$$dS = \frac{dJ}{T_L}+\frac{dQ}{T_e}+\frac{d\mu}{T_\mu}. \tag{3.52}$$

In this case the Boltzmann factor reads as

$$e^{-(E-\omega_h J-\Phi Q-K\mu)/T_H} \tag{3.53}$$

and we extend n_R and n_L as follows

$$\begin{aligned} n_R &= (E-\omega_h^{ex}J-\Phi^{ex}Q-K^{ex}\mu)r_0/\epsilon, \\ n_L &= J. \end{aligned} \tag{3.54}$$

It then takes the final form

$$e^{-n_R/T_R-n_L/T_L-Q/T_e-\mu/T_\mu}. \tag{3.55}$$

Evaluating n_R at the extremal limit, we find that it vanishes, unlike the situation for the Kerr-CFT case. Hence n_R can be interpreted as a quantity that measures a deviation from the extremal limit of a super-entropic black hole.

It follows from the preceding relations that

$$
\begin{aligned}
T_L &= -\left.\frac{\partial T_H/\partial r_+}{\partial \omega_h/\partial r_+}\right|_{ex}, \\
T_R &= \left.\frac{r_0}{\epsilon}T_H\right|_{ex}, \\
T_e &= -\left.\frac{\partial T_H/\partial r_+}{\partial \Phi/\partial r_+}\right|_{ex}, \\
T_\mu &= -\left.\frac{\partial T_H/\partial r_+}{\partial K/\partial r_+}\right|_{ex}
\end{aligned} \tag{3.56}
$$

which are explicitly

$$
\begin{aligned}
T_L &= \frac{1}{2\pi k} = \frac{(1+x^2)(1+3x^2)}{2\pi x}, \\
T_R &= 0, \\
T_e &= \frac{1}{\pi q}\frac{(3x^2+1)(x^2+1)}{(x^2-1)}, \\
T_\mu &= \frac{1}{2l^2}\frac{(1+3x^2)}{x^2}.
\end{aligned} \tag{3.57}
$$

We remark that T_μ is a quantity inversely proportional to the square of a length ($\sim l^{-2}$). Furthermore, $T_L \gg c$ for sufficiently large q, justifying the use of Cardy's formula (3.46) at least in this regime. We therefore find upon insertion of (3.49) into (3.46) that

$$
S_{CFT} = \frac{\mu}{2} l^2 (1+x^2) \tag{3.58}
$$

for the extremal super-entropic black hole.

Turning now to the upper path in 3.1, we must take the limit $a \to l$ in the metric (3.19) using the functions (3.23). This is straightforward and yields exactly the equations (3.48), where (as in (3.4)) we must rescale $\phi \to \psi\Xi$ in (3.19), identifying $\psi \sim \psi + \mu$ once the limit is taken. The Hartle-Hawking vacuum density matrix becomes

$$
\rho = e^{-\frac{J}{T_L} - \frac{Q}{T_e} - \frac{\mu}{T_\mu}} \tag{3.59}
$$

with the temperatures identical to those in (3.57). Likewise, the Cardy formula (3.46) yields (3.58) for the CFT entropy using central charge and the left temperature obtained in (3.33) and (3.57)respectively, thereby establishing the commutativity of the Kerr-CFT and super-entropic limits.

3.5 Super-entropic black holes of gauged supergravity

The second set of black holes are black holes of minimal gauged $5d$ supergravity whose metric is [65]

$$\begin{aligned} ds^2 &= d\gamma^2 - \frac{2q\nu\omega}{\Sigma} + \frac{f\omega^2}{\Sigma^2} + \frac{\Sigma}{\Delta}d\bar{r}^2 + \frac{\Sigma}{S}d\bar{\theta}^2 \\ A &= \frac{\sqrt{3}q\omega}{\Sigma} \end{aligned} \tag{3.60}$$

where

$$\begin{aligned} d\gamma^2 &= -\frac{S\rho^2 d\bar{t}^2}{\Xi_a\Xi_b l^2} + \frac{\bar{r}^2+a^2}{\Xi_a}\sin^2\bar{\theta}d\bar{\phi}^2 + \frac{\bar{r}^2+b^2}{\Xi_b}\cos^2\bar{\theta}d\bar{\bar{\psi}}^2, \\ \nu &= b\sin^2\bar{\theta}d\bar{\phi} + a\cos^2\bar{\theta}d\bar{\bar{\psi}}, \\ \omega &= \frac{Sd\bar{t}}{\Xi_a\Xi_b} - a\sin^2\bar{\theta}\frac{d\bar{\phi}}{\Xi_a} - b\cos^2\bar{\theta}\frac{d\bar{\bar{\psi}}}{\Xi_b}, \\ S &= \Xi_a\cos^2\bar{\theta} + \Xi_b\sin^2\bar{\theta}, \\ \Delta &= \frac{(\bar{r}^2+a^2)(\bar{r}^2+b^2)\rho^2/l^2 + q^2 + 2abq}{\bar{r}^2} - 2m, \\ \Sigma &= \bar{r}^2 + a^2\cos^2\bar{\theta} + b^2\sin^2\bar{\theta}, \\ \rho^2 &= \bar{r}^2 + l^2, \\ \Xi_a &= 1 - \frac{a^2}{l^2}, \\ \Xi_b &= 1 - \frac{b^2}{l^2}, \\ f &= (2m + \frac{2abq}{l^2})\Sigma - q^2. \end{aligned} \tag{3.61}$$

Considering coordinates that rotate at infinity

$$\begin{aligned} \bar{\phi} &= \bar{\phi}_R + \frac{a}{l^2}t, \\ \bar{\bar{\psi}} &= \bar{\psi}_R + \frac{b}{l^2}t \end{aligned} \tag{3.62}$$

we rewrite the metric in the more suitable form [78]

$$ds^2 = -e^0e^0 + \sum_{i=1}^{4} e^ie^i \tag{3.63}$$

with

$$
\begin{aligned}
e^0 &= \sqrt{\frac{\Delta}{\bar{r}^2+y^2}}\,\omega, \\
e^1 &= \sqrt{\frac{\bar{r}^2+y^2}{\Delta}}\,d\bar{r}, \\
e^2 &= \sqrt{\frac{Y}{\bar{r}^2+y^2}}\,(dt^{'}-\bar{r}^2 d\psi_1), \\
e^3 &= \sqrt{\frac{\bar{r}^2+y^2}{Y}}\,dy, \\
e^4 &= \frac{ab}{\bar{r}y}\left(dt^{'}+(y^2-\bar{r}^2)d\psi_1-\bar{r}^2y^2d\psi_2+\frac{qy^2}{ab(\bar{r}^2+y^2)}\varpi\right)
\end{aligned}
\tag{3.64}
$$

and

$$
\begin{aligned}
Y &= -\frac{(1+y^2/l^2)(a^2-y^2)(b^2-y^2)}{y^2}, \\
\varpi &= dt^{'}+y^2 d\psi_1, \\
t^{'} &= \bar{t}-(a^2+b^2)\psi_1-a^2b^2\psi_2, \\
\psi_1 &= \frac{a}{a^2-b^2}\frac{\bar{\phi}_R}{\Xi_a}+\frac{b}{b^2-a^2}\frac{\bar{\psi}_R}{\Xi_b}, \\
\psi_2 &= \frac{1}{a(b^2-a^2)}\frac{\bar{\phi}_R}{\Xi_a}+\frac{1}{b(a^2-b^2)}\frac{\bar{\psi}_R}{\Xi_b}, \\
y^2 &= a^2\cos^2\bar{\theta}+b^2\sin^2\bar{\theta}.
\end{aligned}
\tag{3.65}
$$

Let us introduce new notations $\bar{\varphi}=\bar{\phi}_R/\Xi_a$ and $\bar{\psi}=\bar{\psi}_R/\Xi_b$. The thermodynamics quantities for these black holes are

$$
\begin{aligned}
\omega_\varphi &= \frac{a(r_+^2+b^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq}, \\
\omega_\psi &= \frac{b(r_+^2+a^2)+bq}{(r_+^2+b^2)(r_+^2+b^2)+abq}, \\
T_H &= \frac{r_+^4[1+(2r_+^2+a^2+b^2)/l^2]-(q+ab)^2}{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]}, \\
S &= \frac{\pi^2[(r_+^2+a^2)(r_+^2+b^2)+abq]}{2\Xi_a\Xi_b r_+}
\end{aligned}
\tag{3.66}
$$

and are the angular velocity in the direction $\bar{\varphi}$ at the horizon, the angular velocity in the direction $\bar{\psi}$ at the horizon, the Hawking temperature and the entropy of the black hole respectively. In the case of an extremal black hole r_+ solves

$$
r_+^4[1+(2r_+^2+a^2+b^2)/l^2]-(q+ab)^2=0. \tag{3.67}
$$

This equation admits a positive root that grows as long as the charge q gets larger.

Let us consider first the upper path as in figure 3.1. Assuming extremality via (3.67), upon taking the NH limit

$$
\begin{aligned}
\bar{t} &= tr_0/\epsilon, \\
\bar{\theta} &= \theta, \\
\bar{r} &= r_+ + r_0 r\epsilon, \\
\bar{\varphi} &= \varphi + \omega_\varphi t r_0/\epsilon, \\
\bar{\psi} &= \psi + \omega_\psi t r_0/\epsilon
\end{aligned}
\tag{3.68}
$$

the vielbeins (3.64) take the form

$$
\begin{aligned}
e^0 &= \sqrt{\Gamma(\theta)} r dt, \\
e^1 &= \sqrt{\Gamma(\theta)} \frac{dr}{r}, \\
e^2 &= \alpha_1 e_1 + \alpha_2 e_2, \\
e^3 &= \sqrt{\Gamma(\theta)\alpha(\theta)} d\theta, \\
e^4 &= \beta_1 e_1 + \beta_2 e_2
\end{aligned}
\tag{3.69}
$$

where

$$
\Gamma(\theta) = \frac{l^2}{4} \frac{\Sigma_+}{3r_+^2 + l^2 + a^2 + b^2}, \tag{3.70}
$$

$$
\begin{aligned}
\alpha(\theta) &= \frac{\Sigma_+}{(\Xi_a \cos^2\theta + \Xi_b \sin^2\theta)\Gamma(\theta)}, \\
e_1 &= d\varphi + k_1 r dt, \\
e_2 &= d\psi + k_2 r dt, \\
r_0^2 &= \frac{l^2[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{4r_+^2(3r_+^2 + l^2 + a^2 + b^2)}, \\
k_1 &= \frac{l^2[[a(r_+^2 + b^2) + bq](r_+^2 + b^2) + bqr_+^2]}{2r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq][3r_+^2 + l^2 + a^2 + b^2]}, \\
k_2 &= \frac{l^2[[b(r_+^2 + a^2) + aq](r_+^2 + a^2) + aqr_+^2]}{2r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq][3r_+^2 + l^2 + a^2 + b^2]}
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_1 &= a\sqrt{\frac{Y}{r_+^2 + y^2} \frac{r_+^2 + a^2}{a^2 - b^2}}, \\
\alpha_2 &= b\sqrt{\frac{Y}{r_+^2 + y^2} \frac{r_+^2 + b^2}{b^2 - a^2}}, \\
\beta_1 &= \frac{(a^2 - y^2)[aqy^2 + b(r_+^2 + a^2)(r_+^2 + y^2)]}{r_+ y(a^2 - b^2)(r_+^2 + y^2)}, \\
\beta_2 &= \frac{(b^2 - y^2)[bqy^2 + a(r_+^2 + b^2)(r_+^2 + y^2)]}{r_+ y(b^2 - a^2)(r_+^2 + y^2)}.
\end{aligned}
\tag{3.71}
$$

The computation of the central charge requires the choice of boundary conditions

$$h_{\mu\nu} \sim \begin{bmatrix} \mathcal{O}(r^2) & \mathcal{O}(1) & \mathcal{O}(r^2) & \mathcal{O}(1/r) & \mathcal{O}(1/r^2) \\ & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1/r) & \mathcal{O}(1/r) \\ & & \mathcal{O}(1) & \mathcal{O}(1/r) & \mathcal{O}(1/r) \\ & & & \mathcal{O}(1/r) & \mathcal{O}(1/r^2) \\ & & & & \mathcal{O}(1/r^3) \end{bmatrix} \tag{3.72}$$

in the basis $(t, \varphi, \psi, \theta, r)$. The diffeomorphism that preserves these boundary conditions is

$$\begin{aligned} \xi &= [C + \mathcal{O}(1/r^3)]\partial_t + [-r(\epsilon'(\varphi) + \epsilon'(\psi)) + \mathcal{O}(1)]\partial_r \\ &+ \mathcal{O}(1/r)\partial_\theta + [\epsilon(\varphi) + \mathcal{O}(1/r^2)]\partial_\varphi + [\epsilon(\psi) + \mathcal{O}(1/r^2)]\partial_\psi. \end{aligned} \tag{3.73}$$

It clearly contains the diffeomorphisms

$$\begin{aligned} \zeta_\varphi &= \epsilon(\varphi)\partial_\varphi - r\epsilon'(\varphi)\partial_r, \\ \zeta_\psi &= \epsilon(\psi)\partial_\psi - r\epsilon'(\psi)\partial_r. \end{aligned} \tag{3.74}$$

When following the same steps than the previous cases the diffeormorphism ζ_φ gives rise to central charge

$$\begin{aligned} c_\varphi &= \frac{3k_1}{8\pi}\int \sqrt{\Gamma(\theta)\alpha(\theta)(\alpha_2\beta_1 - \alpha_1\beta_2)^2}\,d\theta d\varphi d\psi \\ &= \frac{3\pi k_1[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{2r_+\Xi_a\Xi_b}. \end{aligned} \tag{3.75}$$

Similarly for the diffeomorphism ζ_ψ

$$\begin{aligned} c_\psi &= \frac{3k_2}{8\pi}\int \sqrt{\Gamma(\theta)\alpha(\theta)(\alpha_2\beta_1 - \alpha_1\beta_2)^2}\,d\theta d\varphi d\psi \\ &= \frac{3\pi k_2[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{2r_+\Xi_a\Xi_b}. \end{aligned} \tag{3.76}$$

The first law of thermodynamics and the extremality constraint (3.67) give

$$TdS = -[(\omega_\varphi - \omega_\varphi^{ex})dJ_\varphi + (\omega_\psi - \omega_\psi^{ex})dJ_\psi + (\Phi - \Phi^{ex})dQ] \tag{3.77}$$

and can be rewritten as

$$dS = \frac{dJ_\varphi}{T_\varphi} + \frac{dJ_\psi}{T_\psi} + \frac{dQ}{T_e}. \tag{3.78}$$

The Boltzmann factor for black holes of gauged supergravity reads as [6]

$$e^{-(E-\omega_\varphi J_\varphi - \omega_\psi J_\psi - \Phi Q)/T_H} = e^{-n_R/T_R - n_\varphi/T_\varphi - n_\psi/T_\psi - Q/T_e} \tag{3.79}$$

[6] In order to get a correspondence with a $2d$ CFT, the ratio n_L/T_L is regarded as $n_L/T_L = n_\varphi/T_\varphi + n_\psi/T_\psi$.

where

$$
\begin{aligned}
n_R &= (E - \omega_\varphi^{ex} J_\varphi - \omega_\psi^{ex} J_\psi - \Phi^{ex} Q) r_0/\epsilon, \\
n_{\varphi,\psi} &= J_{\varphi,\psi}.
\end{aligned}
\tag{3.80}
$$

We then get the temperatures of the left and right-moving CFT's as well as the quantity T_e associated to the electric charge

$$
\begin{aligned}
T_\varphi &\equiv -\frac{\partial T_H/\partial r_+}{\partial \omega_\varphi/\partial r_+}\bigg|_{ex} = \frac{1}{2k_\varphi}, \\
T_\psi &\equiv -\frac{\partial T_H/\partial r_+}{\partial \omega_\psi/\partial r_+}\bigg|_{ex} = \frac{1}{2k_\psi}, \\
T_R &\equiv \frac{T_H r_0}{\epsilon}\bigg|_{ex} = 0, \\
T_e &\equiv -\frac{\partial T_H/\partial r_+}{\partial \Phi/\partial r_+}\bigg|_{ex}.
\end{aligned}
\tag{3.81}
$$

The applicability of the Cardy formula (3.46) again requires large q and small rotation parameter compared to the AdS raduis $a \ll l$. Thus, the CFT entropy is [7]

$$
\begin{aligned}
S_{CFT} &= \frac{\pi^2}{3} c_\varphi T_\varphi + \frac{\pi^2}{3} c_\psi T_\psi \\
&= \frac{[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{2r_+ \Xi_a \Xi_b}
\end{aligned}
\tag{3.82}
$$

for the extremal black hole (3.63).

The super-entropic limit of (3.63) can only be taken in one azimuthal direction [64]. Without loss of generality, we choose this be the ϕ-direction, setting $a \to l$ and requiring the new coordinate φ to be periodic with period μ. Replacing $\bar{\phi}_R/\Xi_a$ with φ, the vielbeins and associated parameters are obtained by $a = l$ in (3.64) and (3.65) respectively. The thermodynamic quantities of this super-entropic black

[7] In the corresponding $2d$ CFT the product of the left central charge and temperature is interpreted in term of the coordinates φ and ψ as $c_L T_L = c_\varphi T_\varphi + c_\psi T_\psi$.

hole are [64]

$$
\begin{aligned}
M &= \frac{\mu}{8}\frac{(m+bq/l)(2+\Xi_b)}{\Xi_b}, \\
J_\varphi &= \frac{\mu}{4}\frac{lm+bq}{\Xi_b}, \\
J_\psi &= \frac{\mu}{8}\frac{2bm+q(b^2+l^2)/l}{\Xi_b}, \\
\omega_\varphi &= \frac{l(r_+^2+b^2)+bq}{(r_+^2+l^2)(r_+^2+b^2)+lbq}, \\
\omega_\psi &= \frac{b(r_+^2+l^2)+lq}{(r_+^2+l^2)(r_+^2+b^2)+lbq}, \\
T_H &= \frac{r_+^4[2+(2r_+^2+b^2)/l^2]-(q+bl)^2}{2\pi r_+[(r_+^2+l^2)(r_+^2+b^2)+lbq]}, \\
S &= \frac{\mu\pi[(r_+^2+l^2)(r_+^2+b^2)+lbq]}{4r_+\Xi_b} = \frac{A}{4}, \\
\Phi &= \frac{\sqrt{3}qr_+^2}{(r_+^2+l^2)(r_+^2+b^2)+lbq}, \\
Q &= \frac{\mu\sqrt{3}q}{8\Xi_b}
\end{aligned} \tag{3.83}
$$

and are the mass, the angular momentum in the direction φ, the angular momentum in the direction ψ, the angular velocity in the direction φ at the horizon, the angular velocity in the direction ψ at the horizon, the Hawking temperature, the Bekeinstein-Hawking entropy, the electric potential and the electric charge of the black holes respectively.

Upon taking the NH limit (3.68) for the extremal case

$$
r_+^4[1+(2r_+^2+l^2+b^2)/l^2]-(q+lb)^2=0 \tag{3.84}
$$

we obtain (3.69,3.70,3.71) with $a=l$. This is the same result as that would be obtained if the super-entropic limit of (3.69) were taken.

After fixing the same boundary conditions (3.72) we find

$$
c_\varphi = \frac{3k_1\mu[(r_+^2+l^2)(r_+^2+b^2)+lbq]}{4r_+\Xi_b} \tag{3.85}
$$

and

$$
c_\psi = \frac{3k_2\mu[(r_+^2+l^2)(r_+^2+b^2)+lbq]}{4r_+\Xi_b} \tag{3.86}
$$

the central charges, this result is valid for both the upper and lower paths of figure 3.1.

For either path the first law of thermodynamics and its extremality constraint are

$$
TdS = -[(\omega_\varphi-\omega_\varphi^{ex})dJ_\varphi+(\omega_\psi-\omega_\psi^{ex})dJ_\psi+(\Phi-\Phi^{ex})dQ+(K-K^{ex})d\mu] \tag{3.87}
$$

which we write as

$$dS = \frac{dJ_\varphi}{T_\varphi} + \frac{dJ_\psi}{T_\psi} + \frac{dQ}{T_e} + \frac{d\mu}{T_\mu} \tag{3.88}$$

and the Boltzmann factor is

$$e^{-\frac{1}{T_H}(E-\omega_\varphi J_\varphi - \omega_\psi J_\psi - \Phi Q - K\mu)} = e^{-\frac{n_R}{T_R} - \frac{n_\varphi}{T_\varphi} - \frac{n_\psi}{T_\psi} - \frac{Q}{T_e} - \frac{\mu}{T_\mu}} \tag{3.89}$$

where

$$\begin{aligned} n_R &= (E - \omega_\varphi^{ex} J_\varphi - \omega_\psi^{ex} J_\psi - \Phi^{ex} Q - K^{ex}\mu)\frac{r_0}{\epsilon}, \\ n_{\varphi,\psi} &= J_{\varphi,\psi}. \end{aligned} \tag{3.90}$$

The temperatures of the two left and one right-moving CFT's and the quantities T_e and T_μ respectively associated with the electric charge and the chemical potential are

$$\begin{aligned} T_\varphi &\equiv -\frac{\partial T_H/\partial r_+}{\partial \omega_\varphi/\partial r_+}\bigg|_{ex} = \frac{1}{2k_\varphi}, \\ T_\psi &\equiv -\frac{\partial T_H/\partial r_+}{\partial \omega_{\psi'}/\partial r_+}\bigg|_{ex} = \frac{1}{2k_\psi}, \\ T_R &\equiv \frac{T_H r_0}{\epsilon}\bigg|_{ex} = 0, \\ T_e &\equiv -\frac{\partial T_H/\partial r_+}{\partial \Phi/\partial r_+}\bigg|_{ex}, \\ T_\mu &\equiv -\frac{\partial T_H/\partial r_+}{\partial K/\partial r_+}\bigg|_{ex} \end{aligned} \tag{3.91}$$

upon comparing the two equations in (3.90).

Finally, we find that the CFT entropy is

$$S_{CFT} = \frac{\mu\pi[(r_+^2 + l^2)(r_+^2 + b^2) + lbq]}{4r_+\Xi_b} \tag{3.92}$$

for both paths. As before, a sufficient condition for the Cardy formula (3.46) is that the electric charge q is sufficiently large, thereby ensuring that $T_\varphi \gg c$ and $T_\psi \gg c$.

3.6 Conclusion

We have demonstrated that the super-entropic black holes, despite the non-compactness of their horizons, have well-defined Kerr-CFT correspondence limits. These limits are robust: the CFT limit of a super-entropic black hole yields the same results as the super-entropic limit of an extremal near-horizon Kerr-AdS metric. Indeed, we verified that we always end up with the same outcome depending on whether we follow either the upper or lower path in figure 3.1. Our work enlarges the class of metrics respecting the Kerr-CFT correspondence in novel directions (black holes with non-compact horizons of finite area) whose implications remain to be explored.

A remarkable feature of super-entropic black holes is that the new variable μ, interpreted as the chemical potential and obtained from compactification of the azimuthal direction, not only enters into the Cardy formula to yield an entropy for the CFT, but also yields a new quantity T_μ that appears in the Hartle-Hawking density matrix. The latter scales as the inverse of the square of a length ($\sim l^{-2}$) in $4d$ or more generally as ($\sim l^{2-d}$) in all dimensions d.

We note that the Kerr-CFT correspondence only applies for singly-spinning super-entropic black holes in $d = 4$, since the extremality condition does not hold for $d > 4$. The $d = 5$ gauged supergravity super-entropic black holes, however, do exhibit the correspondence. In both cases, a sufficient condition for the applicability of the Cardy formula (3.46) is that the electric charge of such black holes is taken to be large. This is in contrast to both Kerr-Newman-AdS and gauged supergravity black holes in which both large electric charges and small rotation parameters compared to the AdS radii ($a \ll l$), are required.

We expect that the Kerr-CFT correspondence for the recently obtained multiply spinning super-entropic black holes [64] can be established using arguments similar to the ones that we have presented.

Chapter 4

Higher order corrections to holographic black hole chemistry

Unlike the previous chapter, we focus here on the AdS/CFT correspondence and try to extend it to gravitation theories beyond Einstein's theories that are referred to as Lovelock theories. Our study is made possible by the definition of a holographic dictionary, which connects thermodynamic quantities and laws in the bulk to those in the dual CFT. The validity of these holographic laws for Lovelock theories endorses the AdS/CFT correspondence. As in the previous chapter, some background on black hole chemistry (see section 2.5) is required to a better understanding of the content of this chapter.

4.1 Introduction

For nearly two decades the AdS/CFT correspondence [6] has been the subject of intense research, motivated by the fact that it posits a connection between an anti de Sitter (AdS) black hole and a conformal field theory (CFT) defined on its boundary. In the context of black hole thermodynamics, an understanding of the physics of the AdS black hole can be reinterpreted in terms of a thermal system on the boundary field theory and vice-versa.

The general assumption underlying nearly all investigations of the AdS/CFT correspondence is that the cosmological constant is a fixed parameter. However increasing interest has been focused on regarding the cosmological constant as a thermodynamic variable via the relation

$$P = -\frac{\Lambda}{8\pi G_d} = \frac{(d-1)(d-2)}{16\pi l^2 G_d} \tag{4.1}$$

where P is the pressure of the black hole system [57]. With this comes the associated concept of volume V for a black hole, which is the thermodynamic conjugate of pressure. The extension of thermodynamic phase space to include these two variables has led to the realization that black holes can exhibit enormously rich and diverse phase behaviour, including Van der Waals phase transitions for charged black holes [16, 17], triple points analogous to water [18], re-entrant phase transitions analogous to those seen in gels and polymers [19], and even superfluid phase transitions analogous to those in superfluid helium [20]. This burgeoning subfield is now referred to as black hole chemistry [21]

It is therefore of interest to ask what black hole chemistry implies for the variables in the boundary field theory. What do the first law of thermodynamics, the Smarr relation, and so on look like on the CFT side? The quantity l in (4.1) is the AdS radius and is related to the number of colours N in the dual gauge theory via a holographic relation of the form [24]

$$\frac{l^{d-2}}{G_d} \sim N^2 \tag{4.2}$$

where the d-dimensional gravitational constant G_d has a length dimension of $d-2$. This kind of relation was first introduced in the AdS/CFT correspondence from string theory [6], in which an $\mathrm{AdS}_5 \times S^5$ spacetime appears to be the near horizon geometry of N coincident D_3 branes in type IIB supergravity. The correspondence between an $\mathrm{AdS}_5 \times S^5$ spacetime and a $\mathcal{N} = 4$ SU(N) Yang-Mills theory on its boundary was expressed as follows

$$l^4 = \frac{\sqrt{2}\ell_{Pl}^4}{\pi^2} N \,, \tag{4.3}$$

where ℓ_{Pl} is the 10-dimensional Planck length. From the two preceding relations we can remark that the variation of the AdS radius l amounts to the variation of the color number N in the boundary Yang-Mills theory.

An interesting subject to think about is on the nature of the connection between the bulk and the CFT on its boundary as well as its implications when we are in presence of another theory of gravity.

The suggestion that varying the pressure, or Λ, is equivalent to varying the number of colors, N, in the boundary Yang–Mills theory has been proposed by a few authors [79, 80, 81], with V being interpreted in the boundary field theory as an associated chemical potential μ for colour. This has the consequence that the variation of Λ in the bulk moves on around the space of field theories in the boundary. Alternatively, one could keep N fixed, so that field theory remains the same, in which case varying Λ in the bulk corresponds to varying the curvature radius governing the space on which the field theory is defined [24].

From this latter perspective, a generalized Smarr relation can be derived by considering the thermodynamics of the dual field theory [24]. Noting that the free energy of the field theory scales simply as N^2, we have

$$\Omega(N,\mu,T,l) = N^2\Omega_0(\mu,T,l) \tag{4.4}$$

in the limit of large N. For a conformal field theory the equation of state reads

$$E = (d-2)p\mathcal{V} \tag{4.5}$$

and together with (4.4) can be used to obtain the standard Smarr relation $(d-3)M = (d-2)TS-2PV$ for an uncharged AdS black hole. In this sense equation (4.4) can be regarded as a '*holographic Smarr relation*'. Since $N^2 \sim \frac{l^{d-2}}{G_d}$, varying Λ is equivalent to varying the AdS length l, and since N is fixed, G_d must also be varied.

The purpose of this chapter is to investigate the holographic Smarr relation (4.4) beyond the large N limit, including sub-leading corrections to this relation. We will see that relation (4.4) can be generalized to a form that includes subleading $1/N$ corrections whose bulk correlates are related to the couplings in Lovelock gravity theories. Lovelock theories are higher curvature or derivative generalizations of Einstein's theory, and in the context of string theory are understood as quantum corrections to Einstein gravity. We shall show that the Lovelock couplings are related to a function of N, with variations of the Lovelock couplings in the bulk dictating the behaviour of the corresponding CFT via the variation of these functions.

Inquiries on how far this generalization extends (or what are the limits thereof) for a given black hole are also be of considerable interest. The bulk Smarr relation and the corresponding CFT equation of state are both expected to be satisfied at the lowest order (Einstein-Hilbert action). We shall look at what happens at higher order, considering especially the CFT equation of state to see whether or not it breaks down.

In section 2, we review some important notions and relations for Lovelock black holes, particularly the first law of thermodynamics and the Smarr relation. In section 3 we investigate how these important relations in the bulk theory are viewed in the CFT, particularly with regards to the derivation of the equation of state in the boundary field theory. Section 4 is devoted to the holographic derivation of the Smarr relation where we mostly make use of the equation (4.36) of the holographic dictionary and by regarding the grand canonical function Ω as a homogeneous function of functions of N. In section 5 we check the validity of the equation of state, introduced earlier in section 3, for some particular cases of black holes, including spherically symmetric AdS lovelock black holes, rotating planar black holes in Gauss-Bonnet-Born-Infeld gravity, and non-extremal rotating black holes in minimal $5d$ gauged supergravity. An explanation of the dependence of the function of N is in section 6, and in the last section we make some concluding remarks.

4.2 A Review of Lovelock black holes

In this current section we review the derivation of thermodynamic quantities associated with Lovelock black holes and some relations implied by these quantities.

Lovelock gravity is a generalization of Einstein's theory whose action and field equations are non-linear in the curvature whilst always maintaining second-order differential equations for the metric. Its Lagrangian has the form [13]

$$L = \frac{1}{16\pi G_d} \sum_{k=0}^{\frac{d-1}{2}} \hat{\alpha}_{(k)} L^{(k)} \tag{4.6}$$

with d the spacetime dimension, $\hat{\alpha}_{(k)}$ the Lovelock coupling constants for the k-th power of curvature, and $L^{(k)}$ the Euler density of dimension $2k$. These Euler densities are expressed as

$$L^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \dots a_k b_k}_{c_1 d_1 \dots c_k d_k} R^{c_1 d_1}_{a_1 b_1} \dots R^{c_k d_k}_{a_k b_k} \tag{4.7}$$

where the $\delta^{a_1 b_1 \dots a_k b_k}_{c_1 d_1 \dots c_k d_k}$ are the totally antisymmetric in both set of indices of the Kronecker delta functions and $R^{c_k d_k}_{a_k b_k}$ the Riemann tensors.

From the Lagrangian (4.6, 4.7), the variational principle yields the vacuum equations of motion for Lovelock gravity, which are

$$\mathcal{G}^a_b = \sum_{k=0}^{\frac{d-1}{2}} \hat{\alpha}_{(k)} G^{(k)}{}^a_b = 0 \tag{4.8}$$

with $G^{(k)}{}^a_b$ the Einstein-like tensors, which read as

$$G^{(k)}{}^a_b = -\frac{1}{2^{k+1}} \delta^{a a_1 b_1 \dots a_k b_k}_{b c_1 d_1 \dots c_k d_k} R^{c_1 d_1}_{a_1 b_1} \dots R^{c_k d_k}_{a_k b_k} \tag{4.9}$$

and each of them satisfy independently the conservation law $\nabla_a G^{(k)}{}^a_b = 0$.

If we minimally couple the theory to a Maxwell field F_{ab} the action is

$$S = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g} \big[\sum_{k=0}^{\frac{d-1}{2}} \hat{\alpha}_{(k)} L^{(k)} - 4\pi G_d F_{ab} F^{ab} \big]. \tag{4.10}$$

and yields

$$\sum_{k=0}^{\frac{d-1}{2}} \hat{\alpha}_{(k)} G^{(k)}{}^a_b = 8\pi G_d \big[F_{ac} F^c_b - \frac{1}{4} g_{ab} F_{cd} F^{cd} \big] \tag{4.11}$$

for the equations of motion. Without solving these equations, it can be shown for solutions of asymptotic constant curvature that the first law of thermodynamics and the Smarr relation respectively are

$$
\begin{aligned}
\delta M &= T\delta S + \mu\delta Q - \frac{1}{16\pi G_d}\sum_{k=0}^{\frac{d-1}{2}} \Psi^{(k)}\delta\hat{\alpha}_{(k)} \\
(d-3)M &= (d-2)TS + (d-3)\mu Q + \sum_{k=0}^{\frac{d-1}{2}} \frac{2(k-1)}{16\pi G_d}\Psi^{(k)}\hat{\alpha}_{(k)}
\end{aligned} \tag{4.12}
$$

where the solution is characterized by a mass M, a charge Q, Lovelock coupling constants $\hat{\alpha}_{(k)}$ each having thermodynamic conjugate $\Psi^{(k)}$, and (if it is a black hole) a temperature T , and an entropy S.

Restricting attention to spherically symmetric metrics

$$
\begin{aligned}
ds^2 &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{(\kappa)d-2} \\
F &= \frac{Q}{r^{d-2}} dt \wedge dr
\end{aligned} \tag{4.13}
$$

where $d\Omega^2_{(\kappa)d-2}$ is the line element of a compact space of dimension $(d-2)$ with constant curvature $(d-2)(d-3)\kappa$ ($\kappa = -1, 0, 1$), the equations of motion (4.11) for charged spherically symmetric black holes of mass M read [82, 83, 84, 85, 86, 87, 88]

$$
\sum_{k=0}^{\frac{d-1}{2}} \alpha_k \left(\frac{\kappa - f}{r^2}\right)^k = \frac{16\pi G_d M}{(d-2)\omega^{\kappa}_{d-2} r^{d-1}} - \frac{8\pi G_d Q^2}{(d-2)(d-3)r^{2(d-2)}} \tag{4.14}
$$

where the charge is given by

$$
Q = \frac{1}{2\omega^{(\kappa)}_{d-2}} \int *F \tag{4.15}
$$

and where $\omega^{(1)}_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}$ and

$$
\begin{aligned}
\alpha_0 &= \frac{\hat{\alpha}_{(0)}}{(d-1)(d-2)}, \qquad \alpha_1 = \hat{\alpha}_{(1)} \\
\alpha_k &= \hat{\alpha}_{(k)} \prod_{n=3}^{2k}(d-n) \qquad \text{for} \quad k \geq 2
\end{aligned} \tag{4.16}
$$

is a simple and useful rescaling of the Lovelock couplings.

Computing the temperature $T = \frac{f'(r_+)}{4\pi}$ via standard Wick rotation arguments, we need not explicitly to know $f(r)$ in order to determine the mass M, temperature T, entropy S and electric potential

μ of the black holes. These thermodynamic quantities are [85, 89]

$$\begin{aligned}
M &= \frac{\omega_{d-2}^{(\kappa)}(d-2)}{16\pi G_d}\sum_{k=0}\alpha_k\kappa^k {r_+}^{d-1-2k} + \frac{\omega_{d-2}^{(\kappa)}Q^2}{2(d-3){r_+}^{d-3}}, \\
T &= \frac{1}{4\pi r_+ D(r_+)}\left[\sum_{k=0}\kappa\alpha_k(d-2k-1)(\frac{\kappa}{r_+^2})^{k-1} - \frac{8\pi G_d Q^2}{(d-2)r_+^{2(d-3)}}\right], \\
S &= \frac{\omega_{d-2}^{(\kappa)}(d-2)}{4G_d}\sum_{k=0}\frac{k\kappa^{k-1}\alpha_k r_+^{d-2k}}{d-2k}, \\
\mu &= \frac{\omega_{d-2}^{(\kappa)}Q}{(d-3)r_+^{d-3}}
\end{aligned} \tag{4.17}$$

with $D(r_+) = \sum_{k=1} k\alpha_k(\kappa r_+^{-2})^{k-1}$ and r_+ the horizon radius. From the extended first law (4.12) and Smarr relation (4.12) we can obtain the thermodynamic conjugate quantities $\Psi^{(k)}$ [90, 91]

$$\Psi^{(k)} = \frac{\omega_{d-2}^{(\kappa)}(d-2)}{16\pi G_d}r_+^{d-2k}\left[\frac{\kappa}{r_+} - \frac{4\pi k T}{d-2k}\right], \qquad k \geq 0 \tag{4.18}$$

in terms of the rescaled coupling constants. The above quantities satisfy the Smarr relation (4.12). From these quantities it follows that the thermodynamic pressure and volume are given by

$$\begin{aligned}
P &= -\frac{\Lambda}{8\pi G_d} = \frac{(d-1)(d-2)}{16\pi G_d}\alpha_0, \\
V &= \omega_n^{(\kappa)}\frac{r_+^{n+1}}{n+1}
\end{aligned} \tag{4.19}$$

where $n = d-2$.

Before proceeding we pause to comment on the relationship between these quantities and the more standard notions of thermodynamics bulk pressure and volume in black hole thermodynamics, which are [92]

$$P_b = -\frac{\Lambda}{8\pi} \quad \text{and} \quad V_b = \frac{\partial M}{\partial P_b}\Big|_{S,Q_b,\alpha_{k\geq 1}}. \tag{4.20}$$

The first relation is the standard identification of α_0 with the cosmological constant [90]. Hence

$$P_b V_b = \alpha_0 \Psi^{(0)}. \tag{4.21}$$

The CFT pressure and volume can be defined as p and $v = \omega_n^{(\kappa)}R^n$. The pressure p has a length dimension of $-(n+1)$ and R is the radius of the sphere on which the CFT is defined.

4.3 Equation of state

In this section we derive the equation of state by looking at how transformations of parameters on the field theory lead to transformations in the bulk or vice versa. An example of this is the proposed correspondence between varying the cosmological constant $\Lambda \sim (\alpha_0)$ in the bulk and variations in the number of colors N in the field theory [81, 79, 93, 94, 95].

The Smarr relation (4.12) has been posited [24] to be derivable from the scaling properties of the free energy of the dual field theory in the limit of a large number of colors N. The free energy $\Omega(N, \mu, T, l)$ of the field theory dual to Einstein-AdS gravity scales as (4.4)

$$\Omega(N, \mu, T, l) = N^2 \Omega_0(\mu, T, l)\,. \tag{4.22}$$

where N^2 is the central charge. Extending $\Omega(N, \mu, T, l)$ to Lovelock gravity theory we posit

$$\Omega(N, \mu, T, \alpha_j, R) = \sum_{k=0} g_k(N) \Omega^k(\mu, T, \alpha_j, R) \tag{4.23}$$

where the $g_k(N)$ are assumed to be polynomial functions (as suggested in [24]) of N. We will see in the next section that this form is of great interest in the derivation of the holographic Smarr relation for Lovelock gravity.

Noting that the thermal properties of AdS black holes can be reinterpreted as those of a CFT at the same finite temperature [23], the grand canonical free energy and its density are expressible in terms of the on-shell action the (Euclidean) bulk solution as [24]

$$\begin{aligned} \Omega &= M - TS - \mu Q, \\ \tilde{\Omega} &= \tilde{M} - T\tilde{S} - \mu \tilde{Q} \end{aligned} \tag{4.24}$$

where the quantities $\tilde{M}, \tilde{S}$ and $\tilde{Q}$ are the respective mass, entropy and charge per unit volume of the CFT. Note that these thermodynamic quantities are defined on the boundary and have the following form: $Q \sim Q_b l$, $\mu \sim \mu_b/l$, $\alpha_k^F = \alpha_k l^{2(1-k)}$ and $\Psi_F^{(k)} = \Psi^{(k)} l^{2(k-1)}$, while others are kept unchanged.

Let us consider conformal field theories, whose equations of state is obtained by taking into account the behaviour of the thermodynamic quantities under an infinitesimal scale transformation

$$\begin{aligned} dS &= 0, \\ dQ &= 0, \\ d\alpha_k^F &= 0 \quad (k \geq 1), \\ dM &= M d\lambda, \\ dp &= (n+1) p d\lambda, \\ dv &= -n v d\lambda. \end{aligned} \tag{4.25}$$

λ is the parameter associated to the scale transformation. From these relations and the extended first law of thermodynamics

$$dM = TdS + \mu dQ + \sum_k \Psi_F^{(k)} d\alpha_k^F \tag{4.26}$$

we are led to the equation of state

$$\tilde{M} = (n+1)p. \tag{4.27}$$

Also the extended first law is reduced to

$$dM = vdp \tag{4.28}$$

and (under the constraints of (4.25)) knowing that

$$\begin{aligned} v &= \frac{\partial \Omega}{\partial p} \\ &= (\tilde{\Omega}\partial_R v + v\partial_R \tilde{\Omega})\frac{\partial R}{\partial p} \end{aligned} \tag{4.29}$$

we get

$$\begin{aligned} \partial_R p &= \frac{n}{R}\tilde{\Omega} + \partial_R \tilde{\Omega} \\ &\rightarrow n\tilde{\Omega} + R\partial_R \tilde{\Omega} \\ &= -(n+1)p \end{aligned} \tag{4.30}$$

where $R\partial_R p = -(n+1)p$ because of the length dimension of p. Inserting the equation of state (4.27) into (4.30) yields

$$\tilde{M} = -(n\tilde{\Omega} + R\partial_R \tilde{\Omega}) \tag{4.31}$$

or alternatively, using (4.24),

$$(n+1)\tilde{M} = n(T\tilde{S} + \mu\tilde{Q}) - R\partial_R \tilde{\Omega} \tag{4.32}$$

which is the holographic equation of state.

For rotating black holes (4.32) has to be slightly modified; we have to add one more condition to (4.25)

$$dJ_i = J_i d\lambda \tag{4.33}$$

with J_i the angular momentum associated to the i-th angular variable. The additional condition takes (4.32) to the new expression

$$(n+1)\tilde{M} = n(T\tilde{S} + \mu\tilde{Q}) + (n+1)\sum_i \omega_i \tilde{J}_i - Rd_R \tilde{\Omega} \tag{4.34}$$

where $\tilde{\Omega}_i$ is the angular velocity associated with the i-th angular variable and $\tilde{\Omega} = \tilde{M} - T\tilde{S} - \mu\tilde{Q} - \sum_i \omega_i \tilde{J}_i$.

4.4 Holographic Smarr relation

The grand canonical free energy (4.23) introduced in the previous section is a polynomial on the variable N^2. In Einstein gravity [24] only the first term of Ω is taken into account. This can be justified by the fact that the dual field theories to the black holes are considered to be in the large N limit.

For a Lovelock black hole nonzero additional terms appear due to contributions from the higher curvature terms. Without knowing explicitly their form, the dimensionality

$$[\alpha_k] = 2(k-1) \qquad \text{or} \qquad \alpha_k \sim l^{2(k-1)} \tag{4.35}$$

of the Lovelock couplings implies that

$$\beta_k(\alpha_k)^{\frac{d-2}{2(k-1)}} = g_k(N) \tag{4.36}$$

for which the $k = 0$ term is

$$\beta_0 l^{d-2} = N^2 \tag{4.37}$$

recovering the relationship (4.2) obtained previously [24]. Here $\beta_0 = \frac{\delta}{16\pi G_d}$, with δ an arbitrary dimensionless constant.

Equation (4.36) implies

$$\alpha_k \frac{\partial X}{\partial \alpha_k} = \frac{d-2}{2(k-1)} g_k \frac{\partial X}{\partial g_k} \tag{4.38}$$

for any arbitrary function X of the parameters α_k. Setting $X = \Omega$, equation (4.38) then becomes

$$\alpha_k \frac{\partial \Omega}{\partial \alpha_k} = \frac{d-2}{2(k-1)} g_k \frac{\partial \Omega}{\partial g_k} \tag{4.39}$$

After multiplying both sides by $2(k-1)$ and summing over k we have

$$\sum_{k=0} 2(k-1)\alpha_k \Psi^{(k)} = (d-2)\sum_{k=0} g_k \frac{\partial \Omega}{\partial g_k} \tag{4.40}$$

where $\Psi^{(k)} = \frac{\partial \Omega}{\partial \alpha_k}$. We thus have the general relation

$$l\frac{\partial}{\partial l} + \sum_{k=1} 2(k-1)\alpha_k \frac{\partial}{\partial \alpha_k} = (d-2)\sum_{k=0} g_k \frac{\partial}{\partial g_k} \tag{4.41}$$

noting that $-2\alpha_0 \partial_{\alpha_0} = l\partial_l$. Recalling that the Euler scaling relation $f(tx_1, \ldots, tx_m) = t^n f(x_1, \ldots, x_m)$ implies

$$nf(x_1, \ldots, x_m) = \sum_j x_j \frac{\partial f}{\partial x_j} \tag{4.42}$$

for a homogeneous function of order n, it is straightforward to see that equation (4.40) can be written as

$$\sum_{k=0} 2(k-1)\alpha_k \Psi^k = (d-2)\Omega \tag{4.43}$$

using

$$\Omega = \sum_{k=0} g_k \frac{\partial \Omega}{\partial g_k} \tag{4.44}$$

which holds since Ω is an homogeneous function of the g_k of degree 1.

More generally Ω is a function of (g_k, R, Q) and not just the g_k. For any function $f(l, Z)$, its derivative with respect to l will be

$$\partial_l f(l,Z)|_{Z_b} = \partial_l f|_Z + p\frac{Z}{l}\partial_Z f|_l. \tag{4.45}$$

if the quantity Z has scaling behaviour $Z = Z_0 l^p$ for some constant Z_0. For charged black holes,

$$A_b = lA, \qquad \mu_b = l\mu, \qquad Q_b = Q/l \tag{4.46}$$

after converting to a canonical normalized field strength of dimension 2, and the radius $R = R_0 l$ for the boundary CFT since

$$ds^2_{boundary} = -dt^2 + l^2 d\Omega^2_{d-2} \tag{4.47}$$

is the boundary metric [24]. Hence we obtain

$$l\frac{\partial}{\partial l} + \sum_{k=1} 2(k-1)\alpha_k \frac{\partial}{\partial \alpha_k} = (d-2)\sum_{k=0} g_k \frac{\partial}{\partial g_k} + R\frac{\partial}{\partial R} + Q\frac{\partial}{\partial Q} \tag{4.48}$$

and so (4.40) now reads as

$$\begin{aligned}
&\sum_{k=0} 2(k-1)\alpha_k \Psi^k \\
&= (d-2)\sum_{k=0} g_k \partial_{g_k}\Omega\big|_{\mu,T} + R\partial_R\Omega\big|_{\mu,T,\alpha_{k\geq 1}} + Q\partial_Q\Omega\big|_{\mu,T,\alpha_k} \\
&= (d-2)\Omega - M - \mu Q \\
&= (d-3)M - (d-2)TS - (d-3)\mu Q
\end{aligned} \tag{4.49}$$

upon using (4.24), which implies

$$d\Omega = -SdT - Qd\mu + vdp + \sum_{k\geq 1} \Psi^k d\alpha_k \tag{4.50}$$

so that $\partial_Q \Omega = -\mu$ and $\partial_R \Omega = v\partial_R p = -M$ from (4.27) and (4.30). We see that (4.49) is the Smarr relation (4.12).

4.5 Some cases

The main purpose of this section is to check the validity of the holographic equation of state (4.34) for a variety of special cases.

We shall consider spherically symmetric AdS Lovelolock black holes whose metrics are given in (4.13) and (4.14). The thermodynamic quantities for these black holes are given by

$$
\begin{aligned}
\tilde{M} &= \frac{d-2}{16\pi G_d}\frac{1}{R^{d-2}}\sum_{k=0}^{\frac{d-1}{2}}\alpha_k\kappa^k r_+^{d-1-2k} + \frac{Q^2}{2(d-3)r_+^{d-3}R^{d-2}} \\
T\tilde{S} &= \frac{d-2}{4G_d}\frac{T}{R^{d-2}}\sum_{k=1}^{\frac{d-1}{2}}\frac{k\kappa^{k-1}\alpha_k r_+^{d-2k}}{d-2k} \\
\mu\tilde{Q} &= \frac{Q^2}{(d-3)r_+^{d-3}R^{d-2}}
\end{aligned}
\tag{4.51}
$$

and

$$
\begin{aligned}
&\tilde{\Psi}^{(k)} = \frac{d-2}{16\pi G_d}\kappa^{k-1}\frac{r_+^{d-2k}}{R^{d-2}}\big[\frac{\kappa}{r_+} - \frac{4\pi kT}{d-2k}\big] \\
&V = \omega_{d-2}^{(\kappa)}R^{d-2}, \qquad R = l.
\end{aligned}
\tag{4.52}
$$

From these equations the free energy density looks like

$$
\tilde{\Omega} = \frac{d-2}{16\pi G_d l^{d-2}}\sum_{k=0}\alpha_k\kappa^{k-1}r_+^{d-2k-1}\big[\kappa - \frac{4\pi k r_+ T}{d-2k}\big] - \frac{Q^2}{2(d-3)r_+^{d-3}l^{d-2}} \tag{4.53}
$$

To compute $R\partial_R\tilde{\Omega}\big|_{N,\mu,T,\alpha_k^F}$ we have determine how the other quantities scale in term of l. It is easy to notice that $l^{d-2}/G_d \sim l^0$, $r_+ \sim l^2 T$, $Q \sim l^{3d/2-4}T^{d-2}$, $\alpha_k = \alpha_k^F l^{2(k-1)}$, $Q = lQ_b$ and so

$$
\begin{aligned}
R\partial_R\tilde{\Omega}\big|_{N,\mu,T,\alpha_k^F} &= l\partial_l\tilde{\Omega}\big|_{l^{d-2}/G_d,\mu,T,\alpha_k^F} \\
&= -\frac{d-2}{16\pi G_d l^{d-2}}\sum_{k=0}k\kappa^{k-1}\alpha_k r_+^{d-2k-1}\left(2\kappa - \frac{8\pi(k-1)r_+T}{d-2k}\right)
\end{aligned}
\tag{4.54}
$$

and using (4.17) and (4.51) it is also easy to check that

$$
\begin{aligned}
(d-1)\tilde{M} + R\partial_R\tilde{\Omega} &= (d-2)\left[\frac{(d-2)T}{4G_d l^{d-2}}\sum_{k=0}\frac{k\kappa^{k-1}\alpha_k r_+^{d-2k}}{d-2k} + \frac{Q^2}{(d-3)r_+^{d-3}l^{d-2}}\right] \\
&= (d-2)(T\tilde{S} + \mu\tilde{Q})
\end{aligned}
\tag{4.55}
$$

recovering the equation of state (4.32) with $n = d-2$.

We next consider rotating planar Lovelock black holes in Gauss-Bonnet-Born-Infeld Gravity. The action in d dimensions is given by [96]

$$
\begin{aligned}
I_G &= -\frac{1}{16\pi G_d}\int_{\mathcal{M}} d^dx\sqrt{-g}\big[R - 2\Lambda + \alpha(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2) + L(F)\big] \\
&- \frac{1}{8\pi G_d}\int_{\partial\mathcal{M}}\sqrt{-\gamma}\big[\Theta + 2\alpha(J - 2\hat{G}_{ab}\Theta^{ab})\big]
\end{aligned}
\tag{4.56}
$$

where $\Lambda = -(d-2)(d-1)/2l^2$ is the cosmological constant, α the Gauss-Bonnet coefficient and $L(F)$ the Born-Infeld Lagrangian

$$
L(F) = 4\beta^2\left(1 - \sqrt{1 + \frac{F^2}{2\beta^2}}\right) \tag{4.57}
$$

where β is the Born-Infeld parameter which has a dimension of mass, $F^2 = F^{\mu\nu}F_{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Regarding the boundary term, Θ is the trace of extrinsic curvature Θ^{ab} of the boundary, $\hat{G}^{ab}(\gamma)$ is the Einstein tensor on the boundary and J the trace of

$$J_{ab} = \frac{1}{3}(\Theta_{cd}\Theta^{cd}\Theta_{ab} + 2\Theta\Theta_{ac}\Theta^c_b - 2\Theta_{ac}\Theta^{cd}\Theta_{db} - \Theta^2\Theta_{ab}) \tag{4.58}$$

In order to solve the equations of motion derived from the action (4.56) we consider a d dimensional asymptotically AdS spacetime with k rotation parameters, whose metric reads [97, 98]

$$\begin{aligned} ds^2 &= -f(r)\left(\Xi dt - \sum_{i=1}^{k} a_i d\phi_i\right)^2 + \frac{r^2}{l^2}\sum_{i=1}^{k}(a_i dt - \Xi l^2 d\phi_i)^2 \\ &+ \frac{dr^2}{f(r)} - \frac{r^2}{l^2}\sum_{i<j}^{k}(a_i d\phi_j - a_j d\phi_i)^2 + r^2 dX^2 \end{aligned} \tag{4.59}$$

where $\Xi = \sqrt{1 + \sum_i^k a_i^2/l^2}$ and dX^2 a $(d-2-k)$ dimensional Euclidean metric. Using the ansatz

$$A_\mu = h(r)(\Xi\delta^0_\mu - \delta^i_\mu a_i) \tag{4.60}$$

the equations of motion for the vector potential yield

$$h(r) = -\sqrt{\frac{d-2}{2(d-3)}}\frac{q}{r^{d-3}}\,{}_2F_1\left(\frac{1}{2}, \frac{d-3}{2(d-2)}; \frac{3d-7}{2(d-2)}; -\eta\right) \tag{4.61}$$

where ${}_2F_1(a, b; c; z)$ is a hypergeometric function and

$$\eta = \frac{(d-3)(d-2)q^2}{2\beta^2 r^{2(d-2)}}. \tag{4.62}$$

Inserting (4.59) into the gravitational field equations gives

$$f(r) = \frac{r^2}{2(d-4)(d-3)\alpha}(1 - \sqrt{g(r)}) \tag{4.63}$$

where

$$\begin{aligned} g(r) &= 1 - 16\frac{(d-4)\alpha\beta^2\eta}{d-1}\,{}_2F_1\left(\frac{1}{2}, \frac{d-3}{2(d-2)}; \frac{3d-7}{2(d-2)}; -\eta\right) \\ &+ 4\frac{(d-4)(d-3)\alpha}{(d-2)(d-1)r^{d-1}}\left(2\Lambda r^{d-1} + (d-2)(d-1)m - 4\beta^2 r^{d-1}(1 - \sqrt{1+\eta})\right). \end{aligned} \tag{4.64}$$

Setting $f(r_+) = 0$, from the above expression it follows that m is given by

$$m = -\frac{2\Lambda r_+^{d-1}}{(d-2)(d-1)} + \frac{4\beta^2 r_+^{d-1}}{(d-2)(d-1)}(1 - \sqrt{1+\eta_+}) + 2(d-2)^2\frac{q^2}{r_+^{d-3}}\,{}_2F_1\left(\frac{1}{2}, \frac{d-3}{2(d-2)}; \frac{3d-7}{2(d-2)}; -\eta_+\right)$$

with r_+ the horizon radius.

The thermodynamic quantities associated with these black holes are

$$
\begin{aligned}
M &= \frac{1}{16\pi G_d} m((d-1)\Xi^2 - 1), \\
T &= \frac{r_+}{2(d-2)\pi\Xi}[2\beta^2(1-\sqrt{1+\eta_+}) - \Lambda], \\
S &= \frac{\Xi}{4G_d} r_+^{d-2}, \\
J_i &= \frac{1}{16\pi G_d}(d-1)\Xi m a_i, \\
\omega_i &= \frac{a_i}{\Xi l^2}, \\
Q &= \frac{\sqrt{2(d-3)(d-2)}\Xi}{8\pi G_d} q, \\
\mu &= \sqrt{\frac{d-2}{2(d-3)}} \frac{q}{\Xi r_+^{d-3}} {}_2F_1\left(\frac{1}{2}, \frac{d-3}{2(d-2)}; \frac{3d-7}{2(d-2)}; -\eta_+\right).
\end{aligned}
\tag{4.65}
$$

For the d dimensional black holes whose thermodynamic quantities given above we can see that

$$
\begin{aligned}
(d-1)\tilde{M} &= \frac{1}{16\pi G_d l^{d-2}}(d-2)(d-1)m + \frac{(d-1)^2}{16\pi G_d l^{d-2}} m \sum_i \frac{a_i^2}{l^2} \\
&= (d-2)(T\tilde{S} + \mu\tilde{Q}) + (d-1)\sum_i \omega_i \tilde{J}_i
\end{aligned}
\tag{4.66}
$$

which is (4.34) upon setting $n = d-2$, provided $R\partial_R\tilde{\Omega}$ vanishes.

To compute $R\partial_R\tilde{\Omega}$, we have to keep in mind that the bulk quantities $Q_b, \mu_b, J_i^b, \omega_i^b$ are redefined in the CFT as $Q = Q_b l,\ \mu = \mu_b/l,\ J_i = J_i^b/l,\ \omega_i = l\omega_i^b$; we also have $l^{d-2}/G_d \sim l^0,\ \ r_+ \sim l^2 T,\ \ q \sim \mu l r_+^{d-3}$ and $a_i \sim l\omega_i$ and $R = l$. Hence a direct computation of (4.24) yields

$$
\tilde{\Omega} = 0 \tag{4.67}
$$

and so

$$
\begin{aligned}
R\partial_R\tilde{\Omega}\big|_{N,\mu,T,\omega_i} &= l\partial_l\tilde{\Omega}\big|_{l^{n-1}/G_{n+1},\mu,T,\omega_i} \\
&= 0
\end{aligned}
\tag{4.68}
$$

as expected.

Finally we consider non-extremal rotating black holes in minimal $5d$ gauged supergravity. This provides an interesting non-trivial example with both charge and angular momentum. The metric in the Boyer- Lindquist coordinates $x^\mu=(t,r,\phi,\psi)$ reads [65]

$$\begin{aligned}
ds^2 &= -\frac{\Delta_\theta}{\Xi_a\Xi_b\rho^2}[(1+g^2r^2)\rho^2dt+2q\nu]dt+\frac{2q}{\rho^2}\nu\zeta+\frac{f}{\rho^4}\left(\frac{\Delta_\theta}{\Xi_a\Xi_b}dt-\zeta\right)^2+\frac{\rho^2}{\Delta_r}dr^2+\frac{\rho^2}{\Delta_\theta}d\theta^2\\
&+ \frac{r^2+a^2}{\Xi_a}\sin^2\theta d\phi^2+\frac{r^2+b^2}{\Xi_b}\cos^2\theta d\psi^2\\
A &= \frac{\sqrt{3}q}{\rho^2}\left(\frac{\Delta_\theta}{\Xi_a\Xi_b}dt-\zeta\right)
\end{aligned}\tag{4.69}$$

where

$$\begin{aligned}
\nu &= b\sin^2\theta d\phi+a\cos^2\theta d\psi,\\
\zeta &= a\sin^2\theta\frac{d\phi}{\Xi_a}+b\cos^2\theta\frac{d\psi}{\Xi_b},\\
\Delta_\theta &= 1-a^2g^2\cos^2\theta-b^2g^2\sin^2\theta,\\
\Delta_r &= \frac{(r^2+a^2)(r^2+b^2)(1+g^2r^2)+q^2+2abq}{r}-2m,\\
\rho^2 &= r^2+a^2\cos^2\theta+b^2\sin^2\theta,\\
\Xi_a &= 1-a^2g^2,\\
\Xi_b &= 1-b^2g^2,\\
f &= 2m\rho^2-q^2+2abqg^2\rho^2
\end{aligned}\tag{4.70}$$

with a,b the rotation parameters associated to the coordinates ϕ,ψ respectively and g is a constant with dimension of length.

The associated thermodynamic quantities are

$$\begin{aligned}
M &= \frac{m\pi(2\Xi_a+2\Xi_b-\Xi_a\Xi_b)+2\pi qabg^2(\Xi_a+\Xi_b)}{4\Xi_a^2\Xi_b^2G_5},\\
\omega_a &= \frac{a(r_+^2+b^2)(1+g^2r_+^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq},\\
\omega_b &= \frac{b(r_+^2+a^2)(1+g^2r_+^2)+aq}{(r_+^2+a^2)(r_+^2+b^2)+abq},\\
T &= \frac{r_+^4[1+g^2(2r_+^2+a^2+b^2)]-(q+ab)^2}{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]},\\
S &= \frac{\pi^2[(r_+^2+a^2)(r_+^2+b^2)+abq]}{2\Xi_a\Xi_b r_+G_5},\\
J_a &= \frac{\pi[2am+qb(1+a^2g^2)]}{4\Xi_a^2\Xi_bG_5},\\
J_b &= \frac{\pi[2bm+qa(1+b^2g^2)]}{4\Xi_a\Xi_b^2G_5},\\
Q &= \frac{\sqrt{3}\pi}{4\Xi_a\Xi_bG_5},\\
\mu &= \frac{\sqrt{3}qr_+^2}{(r_+^2+a^2)(r_+^2+b^2)+abq}
\end{aligned}\tag{4.71}$$

where r_+ is the horizon radius and

$$m = \frac{(r_+^2 + a^2)(r_+^2 + b^2)(1 + g^2 r_+^2) + q^2 + 2abq}{2r_+^2}.$$

In five dimensions $5d$ equation (4.34) takes the form

$$4\tilde{M} = 3(T\tilde{S} + \mu\tilde{Q}) + 4\omega_a \tilde{J}_a + 4\omega_b \tilde{J}_b - R\partial_R \tilde{\Omega}. \tag{4.72}$$

To check the validity of this equation, we need to underline how the CFT quantities are defined in term of the bulk ones as in the previous example. $Q = Q^b l,\ \mu = \mu^b/l,\ J_i = J_i^b/l,\ \omega_i = l\omega_i^b$. Also $l^3/G_5 \sim l^0,\ \ r_+ \sim l^2 T,\ \ q \sim \mu l r_+^2$ and $a_i \sim l \quad (a_1 = a,\ a_2 = b)$ and $R = l$. We compute $\tilde{\Omega}$ in the appendix and find from (B-4) that

$$\begin{aligned} R\partial_R \tilde{\Omega}\big|_{N,T,\mu,\omega_i} &= l\partial_l \tilde{\Omega}\big|_{l^3/G_5,T,\mu,\omega_i} \\ &= -4\tilde{M} + 3(T\tilde{S} + \mu\tilde{Q}) + 4\omega_a \tilde{J}_a + 4\omega_b \tilde{J}_b \end{aligned} \tag{4.73}$$

showing that (4.34) is satisfied.

4.6 On the g_k(N) dependence on N

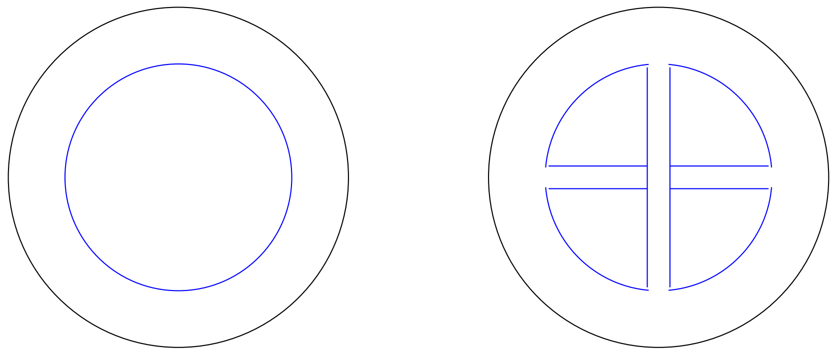
In the present section we investigate how the functions $g_k(N)$ can be approximated. To do so we employ the ansatz that the $g_k(N)$ appear as functions of powers of N^2 [24]. More explicitly, we assume

$$g_k(N) \equiv \mathcal{O}(N^{2(1-k)}) \tag{4.74}$$

which means that for higher order curvature theories ($k \geq 2$) the functions $g_k(N)$ are highly suppressed in the large N field limit. We shall here restrict our attention to the leading terms of each higher curvature contribution. When we look at a higher curvature theory of gravity, the additional contributions are seen as correction terms to the Einstein-Hilbert action.

To better illustrate what is meant, we can consider a pure Yang-Mills theory, or a field theory coupled to a Yang-Mills gauge theory, whose Lagrangian $L^{(1)}$ gives rise to planar and non planar diagrams. We can also consider a Kaluza-Klein-like model which couples gravity to a Yang-Mills gauge theory so that the general Lagrangian is similar to that of a pure gravity theory . In such theories gauge field self-interaction terms are part of the Lagrangian $L^{(1)}$, whereas higher order diagrams come explicitly from higher curvature terms ($R_{d+1} = R_d + gF^2 + ...$).

Considering only non-planar diagrams with 4 vertices, as shown in figure 4.1, we know that these kinds of diagrams bring a contribution to the scattering amplitude proportional to $N^2 g_{YM}^4 = \lambda^2$,



(a) (b)

Figure 4.1: (a) This planar diagram contributes to the scattering amplitude with an amount proportional to $N^2(g_{YM})^0 = N^2\lambda^0$ and corresponds to $L^{(0)}$ in the Lovelock theory. As we know that every closed loop comes with a factor N. (b) The non planar diagram, which is of great interest here, contributes to the scattering amplitude with a term proportional to $N^2(g_{YM})^4 = N^0\lambda^2$ and is linked to $L^{(1)}$. The Yang-Mills coupling g_{YM} appears at each vertex of the diagram.

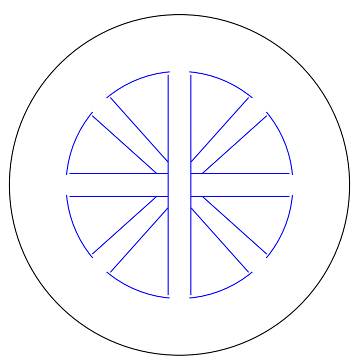


Figure 4.2: The following diagram can be thought of as two similar copies of the four vertices non planar diagram piled together one on top of the other. It gives rise to a contribution to the scattering amplitude proportional to $N^2(g_{YM})^8 = N^{-2}\lambda^4$ and corresponds to $L^{(2)}$ in the Lovelock theory. For the term in $L^{(k)}$ in the Lovelock theory the corresponding diagram will consist in a stack of k copies of the four vertices non planar diagram, whose the contribution to the scattering amplitude is proportional to $N^2(g_{YM})^{2k} = N^{2(1-k)}\lambda^{2k}$

where g_{YM} and λ are the Yang-Mills and 't Hooft couplings respectively. It clearly appears that these diagrams lead to contributions of the order of N^0 in the computation of the scattering amplitude. This reasoning generalizes to higher curvature terms ($k \geq 2$) in the following way:

$$\begin{aligned} L^{(0)} \sim \mathcal{R}^0 &\quad\rightarrow\quad N^2 \\ L^{(1)} \sim \mathcal{R}^1 &\quad\rightarrow\quad N^0 \\ L^{(2)} \sim \mathcal{R}^2 &\quad\rightarrow\quad N^{-2} \\ L^{(k)} \sim \mathcal{R}^k &\quad\rightarrow\quad N^{2(1-k)} \end{aligned} \tag{4.75}$$

with $\mathcal{R}$ some scalar measure of the curvature. Following the above construction we can infer that there should be a correspondence between the dependence of the functions $g_k(N)$ on N^2 and the contribution of non-planar diagrams to the scattering amplitude (see figure 4.2), where we infer $N^2(g_{YM})^{2k} = N^{2(1-k)}\lambda^{2k}$. This suggests the correspondence (4.74). These results are applicable to any higher-curvature theory of gravity.

4.7 Conclusion

By considering the grand canonical free energy Ω in both the bulk and the field theory on its boundary, we have derived a holographic Smarr relation valid beyond the large N limit, with subleading terms of the from $g_k(N)$ arising from higher-curvature corrections of the type found in k-th order Lovelock gravity. By assuming that Ω in the CFT is a homogeneous function of degree one of the $g_k(N)$ functions we were able to obtain the holographic equation of state (4.34). We illustrated its validity for several non-trivial cases in Lovelock gravity and in minimal gauged supergravity in 5 dimensions.

We expect that asymptotically AdS black holes will in general satisfy the relations we have derived, testifying to the robustness of the correspondence between the bulk relations such as the Smarr relation and the equation of state in the CFT. It was shown [24] that Einstein-gravity black holes whose dual field theories are the large N gauge theories with hyperscaling violation satisfy a modified equation of state in the large N limit. We therefore expect that many other non trivial examples of black holes exist where the equation of state beyond this limit has a slightly modified [1] form from the one we obtained. Some of these black holes are the black Dn branes, which are dual to maximally supersymmetric gauge theories in $n+1$ dimensions. Higher-curvature theories that are dual to gauge theories with hyperscaling violation should be good examples of such theories. An interesting project

[1] Field theories that are maximally supersymmetric gauge theories in $n+1$ dimensions dual to Dn-branes [24] verify an equation of state of the form $(n+1-\theta)\tilde{M} = (n-\theta)(T\tilde{S} + \mu\tilde{Q})$ with $\theta = -\frac{(n-3)^2}{5-n}$.

for future work would be to investigate for these black holes the form of the equation of state at higher order.

Chapter 5

Geons and the quantum information metric

We now turn our study of the AdS/CFT correspondence toward another aspect of holographic dualities known as holographic complexities. The complexity we explore here is the volume-complexity which consists in a conjectured relation between a computational complexity in the CFT introduced as the information metric [1] and a codimension-one volume in the dual bulk space. The point of this chapter is to investigate on the volume-complexity conjecture for $(d+2)$-dimensional AdS black holes and their quotient spaces called geons. Geon spacetimes, because of their topology, have a qualitatively distinct relationship between bulk and boundary as compared to their black hole counterparts. By studying them, we can see if the proposal in [39] is sensitive (or not) to this feature.

5.1 Introduction

Research in holography has been of interest since the advent of the AdS/CFT correspondence conjecture [6], and has since been extended to more general notions of gauge/gravity duality. A recent new example is the proposal that there exists a dual connection between the geometric length of an Einstein-Rosen bridge and the computational complexity of the dual Conformal Field Theory's

[1] The information metric is also known as the fidelity susceptiblity. By definition, the fidelity of a state in quantum information theory is a measure of the *closeness* of two quantum states. It can be thought of as the probability that a state will pass a test to identify as another. Considering a one parameter state $|\Psi(\lambda)\rangle$ and its deformation $|\Psi(\lambda+\delta\lambda)\rangle$, the fidelity of the state $|\Psi(\lambda+\delta\lambda)\rangle$ is defined by the scalar product $F \equiv |\langle\Psi(\lambda)|\Psi(\lambda+\delta\lambda)\rangle| = 1 - G_{\lambda\lambda}\,\delta\lambda^2 + O(\delta\lambda^3)$ where $G_{\lambda\lambda}$ is the fidelity susceptibility.

(CFT's) quantum states [34, 99, 36]. This in turn has led to a broad number of investigations on the topic, many concerned with its quantum informational aspects and whether there exists a CFT quantity that is dual to a volume of a codimension one time slice in anti-de Sitter(AdS), analogous to the relationship between holographic entanglement entropy and the area of codimension two extremal surfaces [100].

A recent proposal to this end has been that of a correspondence between a quantum information quantity referred to as the *information metric* (or the *fidelity susceptibility*) and the volume of a maximum time slice of an AdS-like black hole [39]. The notion of a quantum information metric $G_{\lambda\lambda}$ has been around for quite some time and consists of the comparison between a quantum state of the CFT and its counterpart in a *marginal* deformation [2] thereof, giving rise to a term proportional to the fidelity susceptibility. The proposed corresponding bulk quantity is the maximal volume of a time slice connecting the two boundaries (CFTs) of the dual AdS black hole. After comparing the computations of each, the following relation

$$G_{\lambda\lambda} = n_d \frac{\mathrm{Vol}(\Sigma_{\max})}{R^{d+1}} \tag{5.1}$$

was proposed [39], where n_d is the $\mathcal{O}(1)$ constant, Σ a time slice, d the dimension of the spacetime, and R the radius of the AdS spacetime.

Our intention here is to explore the proposal (5.1) for topologically nontrivial spacetimes. We specifically shall consider geons. Introduced by Wheeler in 1955 as bound gravitational and electromagnetic entities, geons were found to be unstable due to the tendency of a massless field to either disperse to infinity or collapse into a black hole. Topological geons were introduced somewhat later [101], generalizing the original construction by allowing nontrivial spatial topology with a black hole horizon. These objects provide an arena for advancing our understanding of black holes in both classical and quantum contexts and so are of considerable interest. Specific examples include topological censorship theorems [102], Hawking radiation [104, 103], and the behavior of Unruh de Witt detectors as probes of hidden topology [105]. The topological identification inherent in the construction of geons affects both the bulk interior and its posited relationship to the dual CFT, making these interesting objects of study insofar as understanding the proposal (5.1) is concerned.

We consider the corresponding geon space of two distinct black holes: the Banados-Teitelboim-Zannelli (BTZ) black hole in $d = 3$ and the AdS-Schwarzchild black brane in $(d + 2)$ dimensions. We compute the information metric and the maximal volume of a time slice in the bulk for each case and compare them. Our results suggest that the coefficient n_d is a function of spacetime topology: we find in both cases that the proportionality in (5.1) is preserved, but the coefficient is increased by a factor

[2] By marginal deformation, it is meant a deformation for which the scaling dimension Δ of the scalar field is equal to $d + 1$, i.e. $\Delta = d + 1$.

of 4. The information metric is indeed sensitive to spacetime topology.

The current chapter is organized as follows. In Sec. 2 we briefly review the notion and construction of geons. In Sec. 3, we describe the information metric and compute it for the black holes as well as for their geon counterparts. Section 4 consists of a computation of the maximal volume of a spatial slice for each case, and then extends these considerations to the planar Schwarzschild metric and its geon counterpart. In the concluding section we summarize our results and their implications. Appendix A contains some computational details of our work.

5.2 Review of geons

The construction of a geon generally makes use of a freely acting involutive isometry that acts on a (black hole) spacetime. A necessary condition for the construction is that the spacetime must be time orientable and foliated by spacelike hypersurfaces with a single asymptotic region [106]. The asymptotic region is generally also required to be stationary and allow conserved charges to be defined by appropriate integrals.

Let us illustrate this for the BTZ black hole, which is a quotient of AdS_3, and its geon counterpart (the RP_2 geon) is obtained via further quotienting as follows. The metric for the nonrotating BTZ black hole is [105]

$$\begin{aligned} ds^2 &= -f(r)dt^2 + dr^2/f(r) + r^2 d\phi^2 \\ f(r) &= -M + r^2/l^2 \qquad \text{with} \qquad r_+ = l\sqrt{M}. \end{aligned} \tag{5.2}$$

with l the AdS radius and r_+ the horizon radius. Writing Eq. (5.2) in Kruskal coordinates, in region (I) (figure 5.1a) where

$$\begin{aligned} &U = -e^{-ur_+/l^2}, \quad V = e^{vr_+/l^2} \\ &v = t + r^*, \quad u = t - r^* \qquad \text{with} \qquad r^* = \int \frac{dr}{f(r)} \end{aligned} \tag{5.3}$$

it takes the form

$$ds^2 = -\frac{l^2}{1+UV}\left[-4dUdV + M(1-UV)^2 d\phi^2 \right] \tag{5.4}$$

and we remark that the metric in the new coordinates is invariant under the interchange of U and V as well as under the translations of ϕ. The geon is the resulting space obtained via the identification (see figure 5.1a)

$$J : (U, V, \phi) \to (V, U, P(\phi)) \tag{5.5}$$

P is the antipodal transformation and

$$\Gamma = \{Id, J\} \simeq \mathbb{Z}_2 \tag{5.6}$$

is the group generated by the freely acting involutive isometry J. Equation (5.5) corresponds to

$$(t, \phi) \rightarrow (-t, \phi + \pi) \tag{5.7}$$

in the original coordinates (5.2).

5.3 2+1 Geon information metric

The proposed quantum information metric $G_{\lambda\lambda}$ [39] is defined by considering the fidelity susceptibility between neighboring states

$$|\langle\psi(\lambda)|\psi(\lambda + \delta\lambda)\rangle| = 1 - G_{\lambda\lambda}(\delta\lambda)^2 + O((\delta\lambda)^3) \tag{5.8}$$

in the dual CFT, where the parameter λ generates a one parameter family of states $|\psi(\lambda)\rangle$. In this section we shall compute this quantity for the states associated with the geon dual, considering only the time-dependent states. These states in the thermofield double (TFD) description of the finite temperature state in a $2d$ CFT dual to the AdS$_3$ read

$$|\Psi_{TFD}\rangle \equiv e^{-i(H_1+H_2)t} \sum_n e^{-\frac{\beta}{4}(H_1+H_2)} |n\rangle_1 |n\rangle_2 \tag{5.9}$$

where $H_{1,2}$ are the Hamiltonians of the CFT$_{1,2}$ respectively, and $|n\rangle_{1,2}$ are the unit norm eigenstates of the CFT$_{1,2}$ respectively. It is important to emphasize that these Hamiltonians are identical.

We shall work with the Euclidean path-integral formalism of the $2d$ CFT, in which case we must compute $\langle\Psi^{'}_{\text{TFD}}(\tau)|\Psi_{\text{TFD}}(\tau)\rangle$, where

$$\begin{aligned} |\Psi_{TFD}(\tau)\rangle &\equiv \sum_n e^{-(\frac{\beta}{4}+\tau)(H_1+H_2)} |n\rangle_1 |n\rangle_2 \\ |\Psi^{'}_{TFD}(\tau)\rangle &= \frac{1}{N} \sum_n e^{-(\frac{\beta}{4}+\tau+\epsilon)(H^{'}_1+H^{'}_2)} |n\rangle_1 |n\rangle_2 \end{aligned} \tag{5.10}$$

in terms of the Euclidean time. Here N is a normalization factor and the state $|\Psi^{'}_{\text{TFD}}(t)\rangle$ is an eigenstate of the Hamiltonian $H^{'}_1 + H^{'}_2$, which is an infinitesimal marginal deformation of the original Hamiltonians. Following the construction in [39], the scalar product (5.8) takes the form

$$\langle\Psi^{'}|\Psi\rangle = \frac{\langle \exp[-\int_{\frac{\beta}{4}+\tau+\epsilon}^{\frac{3\beta}{4}-\tau-\epsilon} d\tau_1 \int d^d x \delta L]\rangle}{\left[\langle \exp[-(\int_{\frac{\beta}{4}+\tau+\epsilon}^{\frac{3\beta}{4}-\tau-\epsilon} + \int_{-\frac{\beta}{4}-\tau+\epsilon}^{\frac{\beta}{4}+\tau-\epsilon}) d\tau_1 \int d^d x \delta L]\rangle\right]^{\frac{1}{2}}} \tag{5.11}$$

with $\delta L = L^{'} - L \equiv \delta\lambda \mathcal{O}(\tau, x)$. L and $L^{'}$ are Lagrangian densities associated with the Hamiltonian H and $H^{'}$ respectively, and we shall henceforth assume the perturbation $\delta\lambda \cdot \mathcal{O}(\tau, x)$ is marginal (of dimension $\Delta = d + 1 = 2$ for the present case). $\mathcal{O}(\tau, x)$ is an operator defined in the field theory. ϵ is a very small parameter that we regard as a cutoff.

In the Euclidean path-integral formalism the two point function for the BTZ black hole is defined on $S^1 \times S^1$ (where one S^1 is the thermal circle with period β) and takes the form [107, 108]

$$\langle \mathcal{O}(\phi_1, \tau_1)\mathcal{O}(\phi_2, \tau_2)\rangle_{BTZ} = \sum_n \frac{(\frac{\pi}{\beta})^{2\Delta}}{\left[\sinh^2\left(\frac{\pi(\phi_2-\phi_1+2\pi n)}{\beta}\right) + \sin^2\left(\frac{\pi(\tau_2-\tau_1)}{\beta}\right)\right]^{\Delta}} \tag{5.12}$$

where Δ is the total conformal dimension of the primary field $\mathcal{O}$. Upon expanding (5.11) and comparing to (5.8) we find

$$G^{BTZ}_{\lambda\lambda} = \frac{1}{2}\int_0^{2\pi} d\phi_1 d\phi_2 \int_{\frac{\beta}{4}+\tau+\epsilon}^{\frac{3\beta}{4}-\tau-\epsilon} d\tau_2 \int_{-\frac{\beta}{4}-\tau+\epsilon}^{\frac{\beta}{4}+\tau-\epsilon} d\tau_1 \langle \mathcal{O}(\phi_1, \tau_1)\mathcal{O}(\phi_2, \tau_2)\rangle_{BTZ}. \tag{5.13}$$

Noting the identity

$$\begin{aligned}\sum_n \int_0^{2\pi} d\phi_2 f(\phi_2 + 2\pi n) &= \sum_n \int_{2\pi n}^{2\pi(n+1)} dx_n f(x_n) \\ &= \int_{-\infty}^{\infty} dx f(x) \end{aligned}\tag{5.14}$$

with $x_n = \phi_2 + 2\pi n$, n integer, we can rewrite (5.13) as

$$G^{BTZ}_{\lambda\lambda} = \frac{1}{2}\int_0^{2\pi} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{\frac{\beta}{4}+\tau+\epsilon}^{\frac{3\beta}{4}-\tau-\epsilon} d\tau_2 \int_{-\frac{\beta}{4}-\tau+\epsilon}^{\frac{\beta}{4}+\tau-\epsilon} d\tau_1 \langle \mathcal{O}(x_1, \tau_1)\mathcal{O}(x_2, \tau_2)\rangle \tag{5.15}$$

where $x_1 \in [0, 2\pi)$, $x_2 \in \mathbb{R}$ and the new two point function reads

$$\langle \mathcal{O}(x_1, \tau_1)\mathcal{O}(x_2, \tau_2)\rangle = \frac{(\frac{\pi}{\beta})^{2\Delta}}{\left[\sinh^2\left(\frac{\pi(x_2-x_1)}{\beta}\right) + \sin^2\left(\frac{\pi(\tau_2-\tau_1)}{\beta}\right)\right]^{\Delta}}. \tag{5.16}$$

Now that we have assembled all the ingredients to compute the information metric (5.16) , we define $u = \frac{\pi(\tau_1-\tau_2)}{\beta}$ and likewise $\tilde{x} = \frac{\pi(x_2-x_1)}{\beta}$, noting the restriction $(0 \leq u \leq \pi)$. We obtain for the integration over x_2 of the integrand in (5.15) an integral of the form [39]

$$\int_{-\infty}^{\infty} d\tilde{x}\, \left(\sinh^2\tilde{x} + \sin^2 u\right)^{-2} = \frac{1}{\sin^2 u \cos^2 u} + (u - \pi/2)\frac{2\sin^2 u - 1}{\sin^3 u \cos^3 u}. \tag{5.17}$$

To find the information metric we have to integrate over all the variables in (5.15). Upon integration of (5.17) with respect to τ_1, we find

$$\begin{aligned} G^{BTZ}_{\lambda\lambda} &= -\frac{1}{2}\int_0^{2\pi} dx_1 \int_{\frac{\beta}{4}+\tau+\epsilon}^{\frac{3\beta}{4}-\tau-\epsilon} d\tau_2 \left(\cot 2u + 2\frac{(u-\pi/2)}{\sin^2 2u}\right)\Bigg|_{-\frac{\pi}{\beta}(\beta/4+\tau-\epsilon+\tau_2)}^{\frac{\pi}{\beta}(\beta/4+\tau-\epsilon-\tau_2)} \\ &= \frac{\pi V_1}{8\epsilon} - \frac{\pi V_1}{2\beta} + \frac{2\pi^2 V_1}{\beta^2}\tau\cot\left(\frac{4\pi\tau}{\beta}\right) \end{aligned}\tag{5.18}$$

where $V_1 = 2\pi$ is the finite volume obtained upon integration over x_1 from 0 to 2π.

For the geon the corresponding calculation is similar. The two point function is [108, 109] [3]

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle_{geon} = \langle \mathcal{O}(x)\mathcal{O}(x')\rangle_{BTZ} + \langle \mathcal{O}(x)\mathcal{O}(Jx')\rangle_{BTZ} \tag{5.19}$$

where the first term is the contribution (5.12) that appears in the BTZ case and the second term is the geon contribution to the two point function. Using (5.7) it reads

$$\langle \mathcal{O}(\phi_1,\tau_1)\mathcal{O}(J\phi_2,J\tau_2)\rangle_{BTZ} = \sum_n \frac{(\frac{\pi}{\beta})^{2\Delta}}{\left[\sinh^2\left(\frac{\pi(\phi_2-\phi_1+\pi+2\pi n)}{\beta}\right)+\sin^2\left(\frac{\pi(\tau_2+\tau_1)}{\beta}\right)\right]^{\Delta}}. \tag{5.20}$$

The information metric for the geon will be $G^{\rm geon}_{\lambda\lambda} = G^{\rm BTZ}_{\lambda\lambda} + \tilde{G}^{\rm BTZ}_{\lambda\lambda}$, the latter contribution coming from (5.20). Making use of the identity (5.14), we obtain from this term an integral of the form (5.15) but where

$$\langle \mathcal{O}(x_1,\tau_1)\mathcal{O}(x_2,\tau_2)\rangle = \frac{(\frac{\pi}{\beta})^{2\Delta}}{\left[\sinh^2\left(\frac{\pi(x_2-x_1)}{\beta}\right)+\sin^2\left(\frac{\pi(\tau_2+\tau_1)}{\beta}\right)\right]^{\Delta}}. \tag{5.21}$$

The integration over the second term in (5.20) proceeds as before, the only distinction being the sign of τ_1. We obtain

$$\tilde{G}^{BTZ}_{\lambda\lambda} = \frac{\pi V_1}{8\epsilon} - \frac{\pi V_1}{2\beta} + \frac{2\pi^2 V_1}{\beta^2}\tau\cot\left(\frac{4\pi\tau}{\beta}\right) \tag{5.22}$$

which is the same result as in (5.18).

We see that the quantum information metric for the geon is the sum of these two contributions and so is double of that of the original black hole

$$G^{geon}_{\lambda\lambda} = \frac{\pi V_1}{4\epsilon} - \frac{\pi V_1}{\beta} + \frac{4\pi^2 V_1}{\beta^2}\tau\cot\left(\frac{4\pi\tau}{\beta}\right). \tag{5.23}$$

Returning to the original (non-Euclidean) coordinates via $\tau = it$, the quantum information metric becomes

$$G^{geon}_{\lambda\lambda} = \frac{\pi V_1}{4\epsilon} - \frac{\pi V_1}{\beta} + \frac{4\pi^2 V_1}{\beta^2}t\coth\left(\frac{4\pi t}{\beta}\right). \tag{5.24}$$

In the late time limit $t \gg \beta$, it reduces to

$$G^{geon}_{\lambda\lambda} \simeq \frac{\pi V_1}{4\epsilon} + \frac{4\pi^2 V_1}{\beta^2}t. \tag{5.25}$$

In the early time limit $t \to 0$, we obtain

$$G^{geon}_{\lambda\lambda} \simeq \frac{\pi V_1}{4\epsilon} + \frac{16\pi^3}{3\beta^3}V_1 t^2. \tag{5.26}$$

[3]Since a point in the geon space has two images (due to a mirror reflection) in the corresponding BTZ black hole.

We now consider the bulk side of this calculation. As prescribed in [110, 111, 112, 113, 114] a perturbation of the parameter at the time slice $\tau = 0$ in the CFT is equivalent to adding a defect brane action

$$S = T \int_{\Sigma} \sqrt{g} \tag{5.27}$$

to the Einstein-Hilbert action. The above statement can also be generalized to time-dependent states. In the case of an infinitesimally small deformation, the quantity T can be approximated to [39]

$$T \simeq n_d \frac{(\delta\lambda)^2}{R^{d+1}} \tag{5.28}$$

with $d = 1$ for a $3d$ bulk and n_d an $O(1)$ constant fixed when normalizing the two point function.

Turning to the case of the BTZ black hole with the coordinates given in [32] as

$$ds^2 = R^2\big(- \sinh^2 \rho dt^2 + d\rho^2 + \cosh^2 \rho dx^2\big) \tag{5.29}$$

which can be obtained from (5.2) by setting $r = l\sqrt{M}\cosh\rho$ and appropriately rescaling t and x. Here we identify $x \to x + 2\pi$ and we do not have to unwrap the metric as in [39, 32].

In the region (II) of the Penrose diagram of the black hole figure 5.1a, we use the analytic continuation through the parametrization $\kappa = -i\rho$ and $\tilde{t} = t + i\pi/2$ and define Σ to be the space characterized by $\kappa = \kappa(\tilde{t})$ and such that

$$\mathrm{Vol}^{BTZ}(\Sigma) = R^2 V_1 \int d\tilde{t} \cos\kappa \sqrt{\sin^2\kappa - (\partial\kappa/\partial\tilde{t})^2}. \tag{5.30}$$

Denoting by κ_* the value of κ for which $\partial\kappa/\partial\tilde{t} = 0$ [defined within the interval $(0 \le \kappa_* < \pi/4)$], and noting that $\dot{\kappa}\frac{\partial L}{\partial \dot{\kappa}} - L$ is a constant of the motion, we can repeat the steps in [32] to obtain a maximum volume $\mathrm{Vol}(\Sigma_{\mathrm{max}})$. This result can be extended to the region (I) in figure 5.1a and we find

$$\frac{\mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)}{R^2 V_1} = 2\int_0^{\kappa_*} \frac{\cos\kappa d\kappa}{\sqrt{\sin^2(2\kappa_*)/\sin^2(2\kappa) - 1}} + 2\int_0^{\rho_\infty} \frac{\cosh\rho d\rho}{\sqrt{1 + \sin^2(2\kappa_*)/\sinh^2(2\rho)}} \tag{5.31}$$

and for the t coordinate in region (I)

$$t = \int_0^{\kappa_*} \frac{d\kappa}{\sin\kappa\sqrt{1 - \sin^2(2\kappa)/\sin^2(2\kappa_*)}} - \int_0^{\rho_\infty} \frac{d\rho}{\sinh\rho\sqrt{1 + \sinh^2(2\rho)/\sin^2(2\kappa_*)}} \tag{5.32}$$

with the integration contour shown in figure 5.2a, and where the factor of 2 comes from the symmetry in figure 5.1a.

If we define the UV cutoff ρ_∞ such that $e^{\rho_\infty} \propto \pi/4\epsilon$, we find for $\beta = 2\pi$ (computation details are given in the Appendix A)

$$\begin{aligned} \frac{\mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)}{R^2} &\simeq \frac{\pi V_1}{4\epsilon} + V_1 t \qquad \text{and} \\ 2G^{\mathrm{BTZ}}_{\lambda\lambda} &\simeq \frac{\pi V_1}{4\epsilon} + V_1 t \end{aligned} \tag{5.33}$$

for the late time limit ($t \gg \beta$ or $\kappa_* \to \pi/4$) as well as

$$\begin{aligned} \frac{\mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)}{R^2} &\simeq \frac{\pi V_1}{4\epsilon} + \frac{2}{\pi} V_1 t^2 \qquad \text{and} \\ 2G^{\mathrm{BTZ}}_{\lambda\lambda} &\simeq \frac{\pi V_1}{4\epsilon} + \frac{2}{3} V_1 t^2 \end{aligned} \tag{5.34}$$

for the early time limit ($t \to 0$ or $\kappa_* \to 0$), recovering the dual results for the BTZ black hole and (in both cases) the holography relation

$$2G^{\mathrm{BTZ}}_{\lambda\lambda} \simeq n_1^{\mathrm{BTZ}} \frac{\mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)}{R^2} \tag{5.35}$$

claimed in [39]. The above results have been obtained through computations expressed in details in the Appendix (C-2, C-4).

For the geon space the computation is similar, except that the symmetry of figure 5.1a is absent in figure 5.1b and so the volume Vol(Σ) [4] is reduced to half the value for the BTZ case ($\mathrm{Vol}^{\mathrm{GEON}}(\Sigma) = \mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)/2$) and the quantum information metric (5.24) is twice the BTZ value. We therefore obtain

$$\begin{aligned} 2G^{\mathrm{GEON}}_{\lambda\lambda} &= n_1^{\mathrm{GEON}} \frac{\mathrm{Vol}^{\mathrm{GEON}}(\Sigma)}{R^2} \\ &= 4n_1^{\mathrm{BTZ}} \frac{\mathrm{Vol}^{\mathrm{GEON}}(\Sigma)}{R^2} \end{aligned} \tag{5.36}$$

which is consistent with the holographic relation (5.35) but with a different factor. This suggests that the coefficient n_d is sensitive to the topology of spacetime.

5.4 Information metric for planar black holes and their geon counterparts

We next consider $(d+2)$-dimensional [5] Schwarzchild-AdS planar black holes, with the metric of the general form

$$\begin{aligned} ds^2 &= -f(r)dt^2 + dr^2/f(r) + r^2 d\Sigma_d^2 \\ f(r) &= -M/r^{d-1} + r^2/l^2 \end{aligned} \tag{5.37}$$

which in turn becomes

$$\begin{aligned} ds^2 &= \frac{1}{z^2}\Big[-h(z)dt^2 + \frac{dz^2}{h(z)} + d\Sigma_d^2\Big] \\ h(z) &= 1 - (z/z_0)^{d+1} \end{aligned} \tag{5.38}$$

[4] From the geon standpoint the volume $\mathrm{Vol}^{Geon}(\Sigma)$ at time $t = t_1 + t_2$ is interpreted as the sum of the corresponding BTZ volume $\mathrm{Vol}^{BTZ}(\Sigma)$ at $t = t_1 - t_2$ and $t = t_1 + t_2$ evaluated on half of the BTZ diagram, i.e. $\mathrm{Vol}^{Geon}(\Sigma)(t_1+t_2) = \mathrm{Vol}^{BTZ}(\Sigma)(t_1-t_2) + \mathrm{Vol}^{BTZ}(\Sigma)(t_1+t_2)$. It is clear to see that for a symmetric time evolution ($t_1 = t_2 = t/2$), the first term of the right-hand side is time independent.

[5] Here we use the index d instead of $d+2$ on the AdS, i.e. AdS_d for AdS_{d+2}, to simplify the notation.

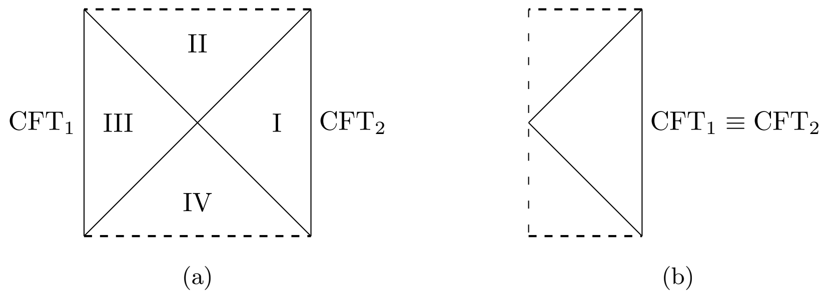


Figure 5.1: (a) Penrose diagram of the AdS (BTZ) black hole. The diagram shows two $2d$ CFTs each at one boundary of the AdS (BTZ) black hole as the AdS/CFT correspondence requires. (b) Penrose diagram of the AdS (BTZ) geon. This diagram appears to be the half of the AdS (BTZ) black hole one. In fact, the geon space is obtained via the identification (5.5) and consequently "splits" the original space into two pieces which are equivalent by a mirror symmetry. As the resulting space happens to be a one-sided black hole, an important and remarkable feature of this diagram is that the two CFTs are now identified and lie on the only remaining boundary.

via the change of variable $r = l/z$ and by setting $l = M = 1$.

The quantum information metric for CFT$_{d+1}$ associated to the $d+2$ dimension AdS-Schwarzschild planar black holes is given by

$$G_{\lambda\lambda}^{SAdS_d} = \frac{1}{2}\int_{\frac{\beta}{4}+\tau+\epsilon}^{\frac{3\beta}{4}-\tau-\epsilon} d\tau_2 \int_{-\frac{\beta}{4}-\tau+\epsilon}^{\frac{\beta}{4}+\tau-\epsilon} d\tau_1 \int d^d x_1 \int d^d x_2 \langle \mathcal{O}(x_1,\tau_1)\mathcal{O}(x_2,\tau_2)\rangle_{SAdS_d} \tag{5.39}$$

with

$$\langle \mathcal{O}(x_1,\tau_1)\mathcal{O}(x_2,\tau_2)\rangle_{SAdS_d} = \frac{C_{12}}{|(\tau_1-\tau_2)^2 + \sum_1^d (x_1-x_2)^2|^{\Delta}} \tag{5.40}$$

where the coordinates x are unwrapped (i.e. $x \in \mathbb{R}$) and C_{12} is a constant whose value will be subsequently fixed. Upon carrying out the integration over x_2 the information metric takes the form

$$\begin{aligned} G_{\lambda\lambda}^{SAdS_d} &= \frac{1}{2}\int_{\frac{\beta}{4}+\tau+\epsilon}^{\frac{3\beta}{4}-\tau-\epsilon} d\tau_2 \int_{-\frac{\beta}{4}-\tau+\epsilon}^{\frac{\beta}{4}+\tau-\epsilon} d\tau_1 \int d^d x_1 \int dx \\ &\times \frac{\Gamma(\Delta-\frac{d-1}{2})}{\Gamma(\Delta)} \frac{C_{12}\pi^{\frac{d-1}{2}}(\pi/\beta)^{2(\Delta-\frac{d-1}{2})}}{[\sinh^2(\frac{\pi x}{\beta}) + \sin^2(\frac{\pi(\tau_1-\tau_2)}{\beta})]^{\Delta-\frac{d-1}{2}}}. \end{aligned} \tag{5.41}$$

Choosing $C_{12} = \frac{\Gamma(\Delta)}{\pi^{\frac{d-1}{2}}\Gamma(\Delta-\frac{d-1}{2})}$, for an exactly marginal deformation $\Delta = d+1$ [39] the information metric can be put into the form

$$\begin{aligned} G_{\lambda\lambda}^{SAdS_d} &= \frac{1}{2}\int_{\frac{\beta}{4}+\tau+\epsilon}^{\frac{3\beta}{4}-\tau-\epsilon} d\tau_2 \int_{-\frac{\beta}{4}-\tau+\epsilon}^{\frac{\beta}{4}+\tau-\epsilon} d\tau_1 \int d^d x_1 \int dx \\ &\times \frac{(\frac{\pi}{\beta})^{d+3}}{[\sinh^2(\frac{\pi x}{\beta}) + \sin^2(\frac{\pi(\tau_1-\tau_2)}{\beta})]^{(d+3)/2}} \end{aligned} \tag{5.42}$$

which is quite similar to the CFT$_2$ case, the only difference is that the integral over x_1 is taken over a d-dimensional spacetime. Using the expression

$$\int_{-\infty}^{\infty} \frac{dx}{[\sinh^2 x + \sin^2 t]^n} = 2^{2n-1} \sin^n(2t)\ \beta(n,n)\ F_1(n,n,n,2n,1+1/a,1+1/b) \tag{5.43}$$

with $a = \cot 2t + i$ and $b = \cot 2t - i$, we are led finally for the late time limit ($\tau \gg \beta$ and d odd) to

$$G_{\lambda\lambda}^{SAdS_d} \simeq \frac{V_d}{\epsilon^d} + \left(\frac{2\pi}{\beta}\right)^{d+1} V_d \tau \tag{5.44}$$

and the early time limit ($\tau \to 0$ and d odd) to

$$G_{\lambda\lambda}^{SAdS_d} \simeq \frac{V_d}{\epsilon^d} + \left(\frac{2\pi}{\beta}\right)^{d+2} V_d \tau^2. \tag{5.45}$$

Here V_d is an infinite d-dimensional volume.

Proceeding as before, we define a hypersurface $z = z(t)$. Its volume is

$$\frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} = V_d \int \frac{dt}{z^{d+1}\sqrt{h(z)}} \sqrt{\dot{z}^2 - h(z)^2}. \tag{5.46}$$

Following the same steps as in the previous section, we are led to a maximum volume (with z_* the value of z such that $\partial z/\partial t = 0$)

$$\begin{aligned} \frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} &= 2V_d \int \frac{dz}{z^{d+1}\sqrt{h}\sqrt{1-(z/z_*)^{2(d+1)}(h_*/h)}} \quad \text{and} \\ t &= \int \frac{dz}{h\sqrt{1-(z_*/z)^{2(d+1)}(h/h_*)}}. \end{aligned} \tag{5.47}$$

In the late time limit ($t \gg \beta$ or $z_* \to 2^{\frac{1}{d+1}} z_0$), we obtain for $d = 1+4n$ (with n integer)

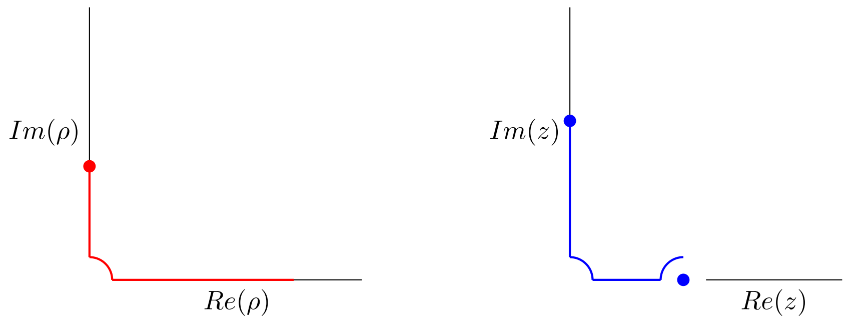
$$\begin{aligned} \frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} &= 2V_d \Big[-i \int_0^{2^{\frac{1}{d+1}} z_0} \frac{dz}{z^{d+1}\big[1+\frac{1}{2}(\frac{z}{z_0})^{d+1}\big]} + \int_0^{z_0} \frac{dz}{z^{d+1}\big[1-\frac{1}{2}(\frac{z}{z_0})^{d+1}\big]} \Big] \\ t &= \frac{1}{2z_0^{d+1}} \Big[-i \int_0^{2^{\frac{1}{d+1}} z_0} \frac{z^{d+1} dz}{(1+\frac{z^{d+1}}{z_0^{d+1}})\big[1+\frac{1}{2}(\frac{z}{z_0})^{d+1}\big]} + \int_0^{z_0} \frac{z^{d+1} dz}{(1-\frac{z^{d+1}}{z_0^{d+1}})\big[1-\frac{1}{2}(\frac{z}{z_0})^{d+1}\big]} \Big] \end{aligned} \tag{5.48}$$

with the integration contour given in figure 5.2b. Integrating out these expressions, we find the same behavior as in the previous case (with $z_0 = \beta/2\pi$)

$$\begin{aligned} \frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} &\simeq \frac{V_d}{d\epsilon^d} + \left(\frac{2\pi}{\beta}\right)^{d+1} V_d t \qquad \text{and} \\ 2G_{\lambda\lambda}^{SAdS_{d+2}} &\simeq \frac{V_d}{\epsilon^d} + \left(\frac{2\pi}{\beta}\right)^{d+1} V_d t \end{aligned} \tag{5.49}$$

When looking at the early time limit ($t \to 0$ or $z_* \to z_0$), we have

$$\begin{aligned} \frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} &= 2V_d \Big[\int_0^{z_0} \frac{1}{z^{d+1}\sqrt{h}} dz + \frac{1}{2} \frac{h_*}{z_*^{2(d+1)}} \int_0^{z_0} \frac{z^{d+1}}{\sqrt{h^3}} dz \Big] \\ t &= -i \frac{\sqrt{h_*}}{z_*^{d+1}} \Big[\int_0^{z_0} \frac{z^{d+1}}{\sqrt{h^3}} dz + \frac{1}{2} \frac{h_*}{z_*^{2(d+1)}} \int_0^{z_0} \frac{z^{3(d+1)}}{\sqrt{h^5}} dz \Big] \end{aligned} \tag{5.50}$$



(a) (b)

Figure 5.2: (a) The contour chosen on the complex plane ρ to compute $\mathrm{Vol}(\Sigma_{\max})$ for the BTZ black hole. κ_* is the point on the imaginary axis. (b) Represents the contour chosen on the complex plane z for the computation of $\mathrm{Vol}(\Sigma_{\max})$ for the AdS Schwarzschild planar black hole. z_* on the imaginary axis is at $\sqrt{2}z_0$ and at z_0 on the real axis.

after integrating

$$\begin{aligned}
\frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} &\simeq \frac{V_d}{d\epsilon^d} + \frac{(d+1)^2}{2\beta(\frac{1}{2},\frac{1}{d+1})}\left(\frac{2\pi}{\beta}\right)^{d+2} V_d t^2 \\
2G_{\lambda\lambda}^{SAdS_{d+2}} &\simeq \frac{V_d}{\epsilon^d} + \left(\frac{2\pi}{\beta}\right)^{d+2} V_d t^2
\end{aligned} \tag{5.51}$$

which is similar to the previous case with $(d = 1)$. Hence we obtain

$$2G_{\lambda\lambda}^{SAdS_d} = n_d^{SAdS}\frac{\mathrm{Vol}^{SAdS}(\Sigma_{\max})}{R^{d+1}} \tag{5.52}$$

which is an extension of (5.35) to d dimensions.

Turning now to the geon, the identification (5.7) acts on the same pair of coordinates with the other coordinates remaining invariant. For the holographic computations in $(d+2)$-dim the volume V_1 in $(1+2)$-dimensions is replaced by the (infinite) volume V_d associated with (5.38). Likewise, we find that the generalization of the two-point function (5.40) for the geon consists of two contributions of equal value, and so we obtain

$$G_{\lambda\lambda}^{SAdS_{\mathrm{geon}_d}} = 2G_{\lambda\lambda}^{SAdS_d}. \tag{5.53}$$

Putting all these aforementioned results together, we find

$$\begin{aligned}
2G_{\lambda\lambda}^{SAdS_{\mathrm{geon}_d}} &= n_d^{SAdS_{\mathrm{geon}}}\frac{\mathrm{Vol}^{\mathrm{geon}}(\Sigma_{\max})}{R^{d+1}} \\
&= 4n_d^{SAdS}\frac{\mathrm{Vol}^{\mathrm{geon}}(\Sigma_{\max})}{R^{d+1}}
\end{aligned} \tag{5.54}$$

for the planar Schwarzschild AdS black hole.

5.5 Conclusion

By investigating both bulk and boundary contributions for the BTZ black hole and the d-dimensional planar Schwarzschild AdS black hole, we have found that the relation (5.1) holds for their geon counterparts apart from a factor of 4. For each case, compared to its black hole counterpart the information metric of the corresponding geon is twice as large and on the bulk side the maximum volume of a time slice for the geon is half as large. We conclude that the relation (5.1) (and thus the coefficient n_d) is sensitive to the topological structure of the spacetime, with $n_d^{\text{geon}} = 4n_d$ for the cases we have considered. In this sense the information metric in the CFT is a "probe" of spacetime topology.

It would be interesting to explore this relationship further, extending the proposed relation (5.1) to spacetimes of more interesting topology, including rotation, solitons, and more generalized geometries.

Chapter 6

Topological and time dependence of the action-complexity relation

The focus of the present chapter is similar to the previous one, but here we deal with an action-complexity conjecture instead. It has been conjectured that there exists a computation complexity in the CFT (seen as a measure of the minimum number of gates necessary to approximate an unitary operator between two states known as the reference and target states) which is connected to an action evaluated on a particular region of the dual bulk space. We study the action-complexity conjecture on both the $(d+1)$-dimensional AdS black holes and their geons. We attempt to derive a time-dependent CFT action-complexity as it is expected to grow at late time similarly to the bulk action-complexity in order to have a consistent conjecture. Because geons have a qualitatively distinct relationship between bulk and boundary as compared to their black hole counterparts, we can test the sensitivity of the proposal in [115, 116] to this feature.

6.1 Introduction

The importance of dualities between quantum field and gravity theories is difficult to underestimate. The AdS/CFT correspondence [6], the first and most successful, posits the existence of a d-dimensional conformal field theory (CFT) on the boundary of a $(d+1)$-dimensional asymptotically anti-de-Sitter (AdS) spacetime, and has therefore led to several dualities between quantities observed in AdS (for example black holes in the bulk) and those in the CFTs defined on their boundaries.

We explored in chapter 5 the proposal of Watanabe et.al. [39], who introduced a duality between a quantum information metric defined in the CFT on the boundary of an AdS black hole, and the volume of a time slice in the AdS. Their work was motivated by Susskind's idea [34] that it would be interesting to find a quantity in a CFT that might be dual to a volume of a co-dimension-1 time slice of an AdS black hole spacetime.

More recently a similar idea was proposed suggesting a correspondence between computational complexity in a CFT and the action evaluated on a Wheeler-De Witt (WDW) patch in the bulk [38]. In specific terms the conjecture is

$$C = I_{\mathrm{WDW}}/\pi \tag{6.1}$$

where the WDW patch refers to the region enclosed by past and future light sheets that are sent into the bulk spacetime from a time slice on the boundary. Subsequent work [117, 115] was devoted to a better understanding of how one evaluates the right-hand side of this relation.

Complexity is concerned with quantifying the degree of difficulty of carrying out a computational task. However a sufficiently clear definition of its meaning in the CFT remains to be fully formulated. One attempt to this end [116] proposes a function providing a measure of the minimum number of gates necessary to reach a target state from a reference state in the CFT. This proposal is motivated by an earlier attempt [118] to provide a geometric interpretation of quantum circuits, which consisted of the definition of two states – a reference and a target state – along with a unitary operator mapping the former to the latter. The minimum number of gates required to synthesize the unitary operator has been interpreted as a minimum length between the identity operator and that unitary operator in the manifold of unitaries. This manifold is endowed with a local metric known as the Finsler metric. The aforementioned proposal [116] chose instead the Fubini-Study metric, and the computational complexity obtained from some fixed reference and target states (related by unitaries involving a squeezing operator) appeared to be somewhat similar to the action on a WDW patch in the bulk.

Furthermore, a time dependent expression of the complexity derived from the CFT computations remains to be derived, despite previous work computing the rate of change of the conjectured complexity in terms of the rate of change of the action on a Wheeler deWitt (WDW) patch at late time [38, 117, 115, 119]. It is of particular interest to determine how computational complexity grows in the late boundary-time limit. An attempt to build a time-dependent complexity from CFTs [120] yielded an expression for complexity that did not grow linearly at late time as conjectured. To this end, one goal of the current chapter is to compute from the CFT perspective the dependence of complexity on boundary time in the late time limit.

The other goal of is to understand if and how equation (6.1) is sensitive to topological effects. The simplest spacetimes that allow the most straightforward exploration of such effects is the AdS black hole in $(d+1)$ dimensions with an identification that renders it an $\mathbb{RP}^d$ geon [106]. The complexity of the AdS black hole spacetimes has been studied recently [115], but their geon counterparts have not (though there has been recent work incorporating a different form of topological identification in the BTZ case ($d = 2$) [121]). In the particular case $d = 2$, the BTZ-geon is obtained by placing further identifications on the BTZ black hole; the boundary of the Euclidean continuation of the BTZ spacetime is an $\mathbb{RP}^2$ space, whereas that of its geon counterpart is a Klein bottle [106, 122]. Previous work [123] demonstrated that the quantum information metric [39] was sensitive to spacetime topology in this case, and so it is reasonable to expect complexity to have a similar dual dependence on bulk topology.

This chapter is organized as follows. In section 2, the notion of complexity will be revisited and written in term of control functions, introduced as the Hamiltonian components in a basis of generalized Pauli matrices. The same steps will be followed in section 3, but here the manifold of unitaries will be taken to be $SU(1,1)$, which is non compact. A useful expression of the complexity will then be derived. Section 4 will specify our considerations to Gaussian states as they are very central in the understanding of quantum information processing with continuous variables. The reference and target states will both be taken to be Gaussian states. The complexity of a d dimensional CFT will be expressed in section 5, as well as its rate of change in the late time limit. To attain this, a time-dependent target state will be chosen, and thus the unitary map between the reference and target state will have time dependence. Section 6 will be devoted to the complexity of the Schwarzschild-AdS_{d+1} spacetime and its geon counterpart as a quotient space, along with its equivalent quantum system, and in section 7 the rate of change of the action in the bulk evaluated on a WDW patch for both the AdS_{d+1} black hole and the AdS_{d+1}-geon will be computed. The result will be two similar correspondence relations that illustrate the sensitivity of (6.1) to the topology of the bulk. The last section will be a conclusion and discussion, in which our results will be summarized in the context of previous work.

6.2 Complexity and cost function

Here we intend to define computational complexity in a quantum theory and study its evolution in terms of a single parameter. We revise the notion of complexity introduced in [118] as a quantity obtained from two fixed (in time) states and a unitary operator mapping one state to the other. We follow the same steps in the case where at least one of the states (from which the complexity is constructed) is time-dependent. This complexity can be understood as the minimum number of resources required to reach a given configuration of a quantum system starting from an initial configuration thereof.

We will be working with quantum systems (more specifically CFTs) whose set of unitary operators corresponds to $SU(2^n)$. To this end, let us consider a quantum system whose Hamiltonian in an $SU(2^n)$ basis takes the form [118]

$$H(t) = \sum_i \gamma^i(t)\sigma_i \tag{6.2}$$

where σ_i are the $4^n - 1$ basis matrices of $SU(2^n)$ and $\gamma^i(t)$ are the components of the Hamiltonian in that basis. These are functions of the variable t defined in the interval $[s_i, s_f]$, and are referred to as control functions.

The evolution of an arbitrary operator V in the manifold $SU(2^n)$, whose Hamiltonian is of the form (6.2), satisfies the equation [118]

$$\frac{dV}{dt} = -iH(t)V \quad \text{with} \quad V(0) = I \quad \text{and} \quad V(1) = U \tag{6.3}$$

where I is the identity operator. We have also defined t in the interval $[s_i = 0, s_f = 1]$.

We now introduce two states, an initial reference state $|R\rangle$ and a final target state $|T\rangle$, whose relationship is given by

$$|T\rangle = U|R\rangle \tag{6.4}$$

with U the unitary operator introduced in (6.3). It can be reached or approximated by a combination of unitary gates of $SU(2^n)$. In this context, computational complexity is defined as an expression quantifying the minimum number of gates or operators required to synthesize U.

To make this concrete we introduce a cost function as a functional of the control function via the relation [118]

$$C_f(\gamma) = \int_0^1 f(\gamma(t))dt \tag{6.5}$$

where the function f is a given distance function. We define complexity by minimizing the cost function via

$$C_f(U) \equiv \inf_\gamma C_f(\gamma). \tag{6.6}$$

In order to be more specific on the nature of the function $f(\gamma)$, let us define the tangent space to the unitary manifold $SU(2^n)$ at the point U as $T_U SU(2^n)$ (or T to be short). Thus, we identify $f(\gamma)$ with a metric function mapping elements of the tangent bundle TM ($M = SU(2^n)$) at a point U to elements of the set of scalars $\mathbb{R}$. That is, $f : TM \to \mathbb{R}$. We can reformulate $f(\gamma)$ in terms of a new metric function via [118]

$$F(U, y) \equiv f(\gamma) \qquad \text{with} \qquad y \in T_U \mathrm{SU}(2^n) \tag{6.7}$$

and

$$
\begin{aligned}
&y = \sum_i y^i (\partial/\partial x^i)_U, \\
&y^i = i\mathrm{Tr}(\sigma_i dU/dt\ U^\dagger)/2^n \qquad \text{or} \qquad y.\sigma_i = idU/dt\ U^\dagger.
\end{aligned} \tag{6.8}
$$

The coordinates y^i are determined for a given unitary operator in equation (D-2) in the appendix.

The cost function (6.5) is proportional to the length associated with the metric function $F(U, y)$, and will have the form [118]

$$
l_F(s) = \int_I dt F(s(t), [s]_t) \tag{6.9}
$$

where $s : I \to M$ maps elements of an interval I to those of the manifold $M = SU(2^n)$, $s(t)$ is a point on the manifold and $[s]_t$ the tangent space to the manifold at that point. The complexity measure (6.6) is obtained by minimizing $l_F(s)$ over the interval from reference to target state.

There are various different types of functions $F(U, y)$ that one can employ to compute (6.9). We will only enumerate those that involve an $L^{(1)}$-norm and an $L^{(2)}$-norm along the path, namely [118]

$$
\begin{aligned}
F_1(U, y) &\equiv \sum_i |y^i|, \\
F_2(U, y) &\equiv \sqrt{\sum_i (y^i)^2}, \\
F_p(U, y) &\equiv \sum_\sigma p(wt(\sigma_i))|y^i|, \\
F_q(U, y) &\equiv \sqrt{\sum_i q(wt(\sigma_i))(y^i)^2}
\end{aligned} \tag{6.10}
$$

where $p(wt(\sigma_i))$ and $q(wt(\sigma_i))$ are weight functions.

Suppose that the target state is a state that depends on a parameter σ (not to be confused with the basis functions σ_i) defined in the interval $[s_i, s_f]$. The expression (6.4) in this case takes the form

$$
|\Psi(\sigma)\rangle = U(\sigma)|R\rangle. \tag{6.11}
$$

Introducing the Fubini-Study metric [116]

$$
ds_{FS}(\sigma) = d\sigma \sqrt{|\partial_\sigma|\Psi(\sigma)\rangle|^2 - |\langle\Psi(\sigma)|\partial_\sigma|\Psi(\sigma)\rangle|^2} \tag{6.12}
$$

we find

$$
l(|\Psi(s_i)\rangle, |\Psi(s_f)\rangle) = \int_{s_i}^{s_f} ds_{FS}(\sigma) \tag{6.13}
$$

yielding the length as function of σ associated with the FS metric. The above expression tells us about the evolution of the computational complexity as a function of σ. We shall postpone the question as to whether the current metric is an $L^{(1)}$-or $L^{(2)}$-norm in the coming sections.

6.3 $SU(1,1)$ manifold and metric generation

We now review the steps required for the derivation of the unitary operator mapping the reference to the target state and thus the Fubini-Study metric that the unitary yields [116], but with complexity reformulated to be time-dependent. For simplicity we shall deal with quantum systems whose manifolds of unitaries are non compact and isomorphic to $SU(2)$. We shall specifically work with the group $SU(1,1)$ which admits the Poincare disk as the manifold associated with its coset $SU(1,1)/U(1)$.

Coherent states, which are either characterized by complex eigenvalues of a non compact generator of the group $SU(1,1)$ [124] or by points of a coset space of the same group [125], can be defined for a unitary irreducible representation of $SU(1,1)$. $SU(1,1)$ coherent states are the result of a two mode squeezing operator

$$S_2(\xi) = \exp[\xi^* K_- - \xi K_+] \tag{6.14}$$

acting on a Fock state. ξ is a complex parameter and $K_\pm$ are generators of the $SU(1,1)$ group that we will define explicitly in the next few steps.

We start with a target state $|\Psi(\sigma)\rangle$ (where σ is a parameter in the time interval $[s_i, s_f]$) in a d dimensional CFT, which obeys the equation (6.11) with a reference state being a two-mode state of some momentum spaces. This two-mode state consists of a product state $|\overrightarrow{k}, -\overrightarrow{k}\rangle$ of two basis states, one mode representing a state of positive momentum $\overrightarrow{k}$ and the other of negative momentum $-\overrightarrow{k}$. This can also be expressed in terms of the quantum numbers associated with the momenta $|n_k, n_{-k}\rangle$. We also consider the unitary operator $U(\sigma)$ to be of the form

$$U(\sigma) = e^{\int_\Lambda d^{d-1}k \; g(\overrightarrow{k},\sigma)} \tag{6.15}$$

with

$$g(\overrightarrow{k},\sigma) \quad = \quad \alpha_+(\overrightarrow{k},\sigma)K_+(\overrightarrow{k}) + \alpha_-(\overrightarrow{k},\sigma)K_-(\overrightarrow{k}) + \omega(\overrightarrow{k},\sigma)K_0(\overrightarrow{k}) \tag{6.16}$$

and Λ a momentum cut-off parameter. Note that the direction that only gives an overall phase to the state is modded out .

The quantities $\alpha_+(\overrightarrow{k},\sigma)$, $\alpha_-(\overrightarrow{k},\sigma)$, $\omega(\overrightarrow{k},\sigma)$ are arbitrary functions whereas $K_+(\overrightarrow{k})$, $K_-(\overrightarrow{k})$ and $K_0(\overrightarrow{k})$ are the generators of the $SU(1,1)$ algebra. These latter quantities can be written in term of annihilation operators $(b_{\overrightarrow{k}}, b_{-\overrightarrow{k}})$ and creation operators $(b^\dagger_{\overrightarrow{k}}, b^\dagger_{-\overrightarrow{k}})$ associated with the respective modes $(\overrightarrow{k}, -\overrightarrow{k})$ as [116]

$$K_+ = \frac{1}{2} b^\dagger_{\overrightarrow{k}} b^\dagger_{-\overrightarrow{k}}$$
$$K_- = \frac{1}{2} b_{\overrightarrow{k}} b_{-\overrightarrow{k}}$$
$$K_0 = \frac{1}{4}(b^\dagger_{\overrightarrow{k}} b_{\overrightarrow{k}} + b_{-\overrightarrow{k}} b^\dagger_{-\overrightarrow{k}}) \tag{6.17}$$

and satisfy the commutation relations

$$[K_+, K_-] = -K_0 \qquad \text{and} \qquad [K_0, K_\pm] = \pm\frac{1}{2}K_\pm \tag{6.18}$$

It is straightforward to show that (6.15) can be put into the form [126]

$$U(\sigma) \quad = \quad e^{\int_\Lambda d^{d-1}k \ \gamma_+(\overrightarrow{k},\sigma)K_+(\overrightarrow{k})} e^{\int_\Lambda d^{d-1}k \ \log(\gamma_0(\overrightarrow{k},\sigma))K_0(\overrightarrow{k})} e^{\int_\Lambda d^{d-1} \ k\gamma_-(\overrightarrow{k},\sigma)K_-(\overrightarrow{k})} \tag{6.19}$$

where the new functions $\gamma_+(\overrightarrow{k},\sigma), \gamma_-(\overrightarrow{k},\sigma)$ and $\gamma_0(\overrightarrow{k},\sigma)$ read as

$$\gamma_\pm = \frac{2\alpha_\pm \sinh \Xi}{2\Xi \cosh \Xi - \omega \sinh \Xi}$$
$$\gamma_0 = (\cosh \Xi - \frac{\omega}{2\Xi} \sinh \Xi)^{-2}$$
$$\Xi^2 = \frac{\omega^2}{4} - \alpha_+\alpha_-. \tag{6.20}$$

It is desirable to obtain the simplest possible form of (6.19). This can be done by imposing the conditions [116]

$$K_-|R\rangle = 0 \qquad \text{and} \qquad K_0|R\rangle = \frac{\delta^{d-1}(0)}{4}|R\rangle \tag{6.21}$$

on the reference state, yielding

$$|\Psi(\sigma)\rangle \quad = \quad N \ e^{\int_\Lambda d^{d-1}k \ \gamma_+(\overrightarrow{k},\sigma)K_+(\overrightarrow{k})}|R\rangle$$
$$N \quad = \quad e^{\frac{1}{4}\delta^{d-1}(0) \int_\Lambda d^{d-1}k \log(\gamma_0(\overrightarrow{k},\sigma))} \tag{6.22}$$

and so only the factor involving γ_+ needs to be taken into account. The quantity $\delta^{d-1}(0)$ comes from the commutation rules $[b_{-\overrightarrow{k}}, b^\dagger_{-\overrightarrow{k}'}] = \delta^{d-1}(\overrightarrow{k} - \overrightarrow{k}')$ obeyed by the operators $b_{-\overrightarrow{k}}$ that appear in the generator K_0.

Now that we have managed to find a reduced form of the unitary operator $U(\sigma)$, we will chose a reference state and attempt to derive the complexity using the Fubini-Study metric (6.12). By choosing a reference state annihilated by the $b_{\overrightarrow{k}}$ [1]

$$|R\rangle = |0, 0\rangle \tag{6.23}$$

[1]In another momentum sector the reference state is identified with a vacuum state, i.e. a state with no particle excitation for the modes $\overrightarrow{k}$ and $-\overrightarrow{k}$ according to the operator $b_{\overrightarrow{k}}$ and $b_{-\overrightarrow{k}}$.

we obtain, when omitting the variables and the integrals

$$|\Psi\rangle = Ne^{\gamma_+ K_+}|0,0\rangle. \tag{6.24}$$

We find that (6.24) becomes

$$|\Psi\rangle = \sqrt{1-|\gamma_+|^2}\sum_n (\gamma_+)^n |n,n\rangle \tag{6.25}$$

upon choosing N so that the target state is normalized. Inserting (6.25) in the Fubini-Study metric,

$$ds^2_{FS} = \langle\delta\Psi|\delta\Psi\rangle - \langle\delta\Psi|\Psi\rangle\langle\Psi|\delta\Psi\rangle \tag{6.26}$$

we get (see also appendix (D-4))

$$ds^2_{FS} = \frac{|\delta\gamma_+|^2}{(1-|\gamma_+|^2)^2}. \tag{6.27}$$

Restoring the variables and the integrals, we obtain a more general form of the complexity (6.13) with the expression

$$C^{(n)} \quad = \quad \min_{\gamma_+}\int_{s_i}^{s_f} d\sigma \sqrt[n]{\frac{V_{d-1}}{2}\int d^{d-1}k\ |ds_{FS}(\sigma)/d\sigma|^n} \tag{6.28}$$

with $\gamma_+^{'} = \partial\gamma_+/\partial\sigma$ and V_{d-1} the $(d-1)$-dimensional volume of a time slice. Upon comparison with (6.10) we see that (6.28) is an $L^{(n)}$-norm.

We will mostly use the case where $n = 1$

$$C^{(1)} \quad = \quad \min_{\gamma_+}\int_{s_i}^{s_f} d\sigma\ \frac{V_{d-1}}{2}\int d^{d-1}k\left|\frac{\gamma_+^{'}}{1-|\gamma_+|^2}\right| \tag{6.29}$$

as it leads to a function easier to integrate as well as to a complexity whose rate of change corresponds to that of the action evaluated in the bulk. Note that the gates for different k's are not allowed to act in parallel in order to obtain the $C^{(1)}$ norm.

6.4 Gaussian states

Here we briefly review the Gaussian states of a quantum system [116]. Such states play a central role in quantum information processing with continuous variables as well as in quantum field theory where the vacuum states of some field theories (for example, quantum electrodynamics) appear to be Gaussian states. We shall choose the reference and target states to be Gaussian states.

Consider a scalar field theory in a d dimensional spacetime with the Hamiltonian density

$$H_m = \frac{1}{2}\int d^{d-1}x\ [\pi^2 + (\partial_x\Phi)^2 + m^2\Phi^2] \tag{6.30}$$

where m is the mass of the field $\Phi(x)$ and $\pi(x)$ is its conjugate momentum. These obey the commutation rules

$$[\Phi(\overrightarrow{x}), \pi(\overrightarrow{x}')] = i\delta^{d-1}(\overrightarrow{x} - \overrightarrow{x}'). \tag{6.31}$$

The field and its conjugate momentum in terms of the annihilation a_k and creation operators $a_k^\dagger$ are explicitly given by

$$\begin{aligned} \Phi(x) &= \int d^{d-1}k \frac{1}{\sqrt{2\omega_k}}(a_k\ e^{-ikx} + a_k^\dagger\ e^{ikx}) \\ \pi(x) &= \int d^{d-1}k \frac{\sqrt{\omega_k}}{\sqrt{2}i}(a_k\ e^{-ikx} - a_k^\dagger\ e^{ikx}) \end{aligned} \tag{6.32}$$

with $\omega_k = \sqrt{k^2 + m^2}$. Substituting (6.32) into (6.31) we find

$$[a_{\overrightarrow{k}}, a^\dagger_{\overrightarrow{k}'}] = \delta^{d-1}(\overrightarrow{k} - \overrightarrow{k}') \tag{6.33}$$

with all other commutators zero.

It is helpful to write things in momentum space where the Hamiltonian can be expressed in a more elegant form as

$$H_m = \int d^{d-1}k\ \omega_k\ [a^\dagger_{\overrightarrow{k}} a_{\overrightarrow{k}} + \frac{1}{2}] \tag{6.34}$$

and the field and its associated momentum become

$$\begin{aligned} \Phi(\overrightarrow{k}) &= \frac{1}{\sqrt{2\omega_k}}(a_{\overrightarrow{k}} + a^\dagger_{-\overrightarrow{k}}) \\ \pi(\overrightarrow{k}) &= \frac{\sqrt{\omega_k}}{\sqrt{2}i}(a_{\overrightarrow{k}} - a^\dagger_{-\overrightarrow{k}}). \end{aligned} \tag{6.35}$$

In the sequel we consider a CFT for which the field is massless ($m = 0$).

A pure Gaussian state $|S\rangle$ is a state for which [116]

$$\Big[\sqrt{\frac{\alpha_k}{2}}\Phi(\overrightarrow{k}) + \frac{i}{\sqrt{2\alpha_k}}\pi(\overrightarrow{k})\Big]|S\rangle = 0 \tag{6.36}$$

where $\alpha_k = \omega_k$ corresponds to the ground state $|m\rangle$ of the theory. The ground state of the theory can be a good choice of the target state.

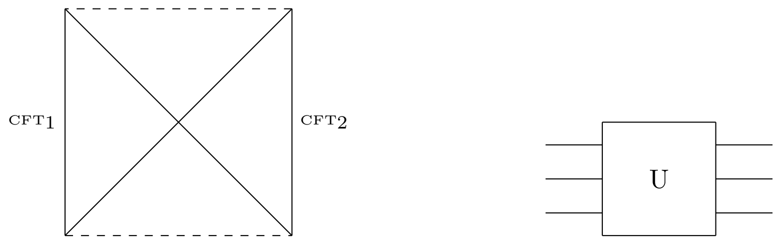
To construct the reference state $|R(M)\rangle$ we write the Bogoliubov transformation [116]

$$b_{\overrightarrow{k}} = \beta_k^+ a_{\overrightarrow{k}} + \beta_k^- a^\dagger_{-\overrightarrow{k}} \tag{6.37}$$

and require

$$b_{\overrightarrow{k}}|R(M)\rangle = 0. \tag{6.38}$$

where $\beta_k^+ = \cosh 2r_k$, $\beta_k^- = \sinh 2r_k$ and $r_k = \log(\sqrt[4]{M/\omega_k})$. This corresponds to a state with $\alpha_k = M$ in (6.36).



(a) (b)

Figure 6.1: (a) Conformal diagram of a BTZ ($d = 2$) black hole. As we can see, a CFT is defined at each boundary thereof. (b) Quantum circuit which consists in an unitary U acting on n qubits. In the context of the current work its associated complexity can be regarded as equivalent to the action integral evaluated on a WDW patch in BTZ black hole.

6.5 Conformal field theory in d dimensions

Employing the formalism of the previous sections, we now compute the complexity defined in the CFT dual of an AdS gravitational theory. The spacetimes we have in mind for the latter are AdS black holes which, according to the AdS/CFT correspondence, admit CFTs on their boundaries. The Penrose diagram for the AdS$_{d+1}$ black hole is illustrated in figure 6.1. The BTZ case can be described as a quotient space of AdS$_{d+1}$ with $d = 2$.

Here we aim to derive the computational complexity associated to quantum theories defined in the boundary CFTs. States on such CFTs are described by thermofield double (TFD) of finite temperature, defined in a thermal circle of period β [107]

$$\begin{aligned}|\mathrm{TFD(t)}\rangle &\equiv e^{-i(H_1+H_2)t}|\mathrm{TFD(0)}\rangle \\ &= e^{-i(H_1+H_2)t}\sum_n e^{-\beta E_n/2}|n\rangle_1|n\rangle_2 \end{aligned} \tag{6.39}$$

with $H_{1,2}$ the free Hamiltonians, $|n\rangle_{1,2}$ the eigenstates of the free Hamiltonians defined on the CFT$_{1,2}$ and E_n their corresponding energies. These states on the CFT$_1$ can be assigned to the positive momentum modes $\overrightarrow{k}$ and the ones on the CFT$_2$ to the negative momentum modes $-\overrightarrow{k}$ of a scalar field theory.

We see that

$$\begin{aligned}|\mathrm{TFD(0)}\rangle &\equiv \sum_n e^{-\beta E_n/2}|n\rangle_1|n\rangle_2 \\ &= e^{\int d^{d-1}k\; e^{-\beta\omega_k/2} a^\dagger_{\overrightarrow{k}} a^\dagger_{-\overrightarrow{k}}}|0\rangle \end{aligned} \tag{6.40}$$

for a free scalar field theory. The state $|TFD(0)\rangle$ is annihilated by operators $b_{\pm\overrightarrow{k}}$ defined via a

Bogoliubov transformation as

$$\begin{aligned} b_{\vec{k}} &= \cosh\theta_k a_{\vec{k}} - \sinh\theta_k a^\dagger_{-\vec{k}} \\ b_{-\vec{k}} &= \cosh\theta_k a_{-\vec{k}} - \sinh\theta_k a^\dagger_{\vec{k}} \end{aligned} \tag{6.41}$$

with $\tanh\theta_k = e^{-\beta\omega_k/2}$.

We can regard the states in the boundaries as two-mode states where one side of the diagram (figure 6.1a) corresponds to states of a conformal scalar field theory with positive momentum $\vec{k}$ and the other side to a scalar field theory with negative momentum states $-\vec{k}$. The total Hamiltonian of the system according to (6.34) will be

$$\begin{aligned} H &= H_1 + H_2 \\ &= \int d^{d-1}k\ \omega_k[a_1^\dagger a_1 + a_2^\dagger a_2 + 1] \end{aligned} \tag{6.42}$$

where $\omega_k = k$, $a_1 = a_{\vec{k}}$ and $a_2 = a_{-\vec{k}}$. Using (6.41), the total Hamiltonian (6.42) in the basis (6.17) has the form

$$a_1^\dagger a_1 + a_2^\dagger a_2 + 1 = 4\cosh(2\theta_k)K_0 + 2\sinh(2\theta_k)(K_+ + K_-) \tag{6.43}$$

and so (6.39) becomes

$$|\mathrm{TFD}\rangle \equiv e^{\alpha_+ K_+ + \alpha_- K_- + \omega K_0}\ |TFD(0)\rangle \tag{6.44}$$

with

$$\begin{aligned} \alpha_\pm &= -2i\ \omega_k\ t\sinh(2\theta_k) \\ \omega &= -4i\ \omega_k\ t\cosh(2\theta_k). \end{aligned} \tag{6.45}$$

Equation (6.44) will become [2]

$$|\mathrm{TFD}\rangle \equiv e^{\gamma_+ K_+} e^{\log(\gamma_0)K_0} e^{\gamma_- K_-}\ |TFD(0)\rangle \tag{6.46}$$

using the transformation of the unitary operator (6.19).

We obtain a state equivalent to (6.24) and (6.25), but where

$$\begin{aligned} &\gamma_\pm = \frac{-i\sinh(2\theta_k)\sin\Xi}{\cos\Xi + i\cosh(2\theta_k)\sin\Xi} \qquad \text{with} \\ &\Xi = 2\omega_k\ t \quad \text{ and } \quad \omega_k = k. \end{aligned} \tag{6.47}$$

[2]The computational complexity is evaluated for the reference and target states corresponding to the TFD at time $t = 0$ and t respectively.

In term of the parameter σ the control function γ_+ can be written as

$$
\begin{aligned}
&\gamma_\pm(k,\sigma) = \frac{-i\sinh(2\theta_k)\sin\Xi}{\cos\Xi + i\cosh(2\theta_k)\sin\Xi} \qquad \text{with} \\
&\Xi = 2kt\ \sigma.
\end{aligned} \tag{6.48}
$$

It is easy to check that $\gamma_+ = \gamma_+(k,\sigma)$ as a function of σ, satisfies the conditions

$$
\begin{aligned}
\gamma_+(k,s_i) &= 0 \quad \text{and} \\
\gamma_+(k,s_f) &= \frac{-i\sinh(2\theta_k)\sin(2kt)}{\cos(2kt) + i\cosh(2\theta_k)\sin(2kt)}
\end{aligned} \tag{6.49}
$$

corresponding to reference and target state respectively. It appears that the control function is time-dependent and this fact will imply a time-dependent complexity.

In order to compute the complexity in the simplest possible manner we consider situations in which the control function obeys the condition $|\gamma_+| < 1$, which is holds if the operator is unitary.

Now that we have assembled all the ingredients, the complexity (6.29) as a function of t is

$$
\begin{aligned}
C^{(1)}(t) &= \min_{\gamma_+} \int_{s_i}^{s_f} d\sigma\ \frac{V_{d-1}}{2} \int d^{d-1}k\ \frac{|\gamma_+^{'}|}{1-|\gamma_+|^2} \\
&= 2V_{d-1}\Omega_{\kappa,d-2}\beta^{-d}(2^d-1)\Gamma(d)\zeta(d)\ t
\end{aligned} \tag{6.50}
$$

as detailed in eq. (D-5) in the appendix. The computational complexity can be understood as the minimum number of gates needed to synthesize a unitary operator U (figure 6.1b).

Before proceeding further, we define the total energy of the scalar field as (see (D-15) in the appendix)

$$
\begin{aligned}
E &= V_{d-1} \int d^{d-1}k\ \omega_k e^{-\beta\omega_k} \\
&= V_{d-1}\Omega_{d-2}\beta^{-d}\Gamma(d).
\end{aligned} \tag{6.51}
$$

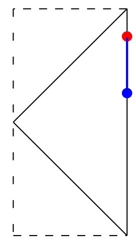
Hence the complexity (6.50) takes the form

$$
C^{(1)}(t) = 2(2^d-1)\zeta(d)E\ t. \tag{6.52}
$$

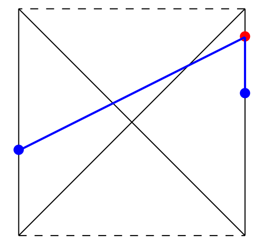
Note that the rate of change of the complexity for very large t is

$$
\frac{dC(t)}{dt}^{\mathrm{AdS}_{d+1}} = n_d E \tag{6.53}
$$

with $n_d = 2(2^d-1)\zeta(d)$ a dimensionless constant. Equation (6.53) means that the variation of the complexity with respect to time at late time is proportional to the total energy E of the CFT . This total energy E will later be identified with the mass of the AdS black hole dual to the CFT.



(a)



(b)

Figure 6.2: (a) The current diagram shows the thermofield single state on the boundary of the BTZ-geon. The red and blue points represent the right- and left-modes of the thermofield single (both modes or CFTs are superposed on the same boundary), respectively. The thick blue line corresponds to the complexity derived from the entangled state. (b) This diagram corresponds to the first one but here the situation is seen in the BTZ context. Unfolding the CFTs on the first diagram, the left-modes appear on both sides of the new diagram (which a BTZ one). It results a sum of two complexities.

6.6 Geon and direct products

In this section we repeat these computations in the context of the AdS$_{d+1}$-geon.

The AdS$_{d+1}$ black hole has the metric

$$\begin{aligned} ds^2 &= -f(r)dt^2 + dr^2/f(r) + r^2 d\Sigma^2_{\kappa,d-1} \\ f(r) &= \kappa - \omega^{d-2}/r^{d-2} + r^2/l^2 \end{aligned} \tag{6.54}$$

which, in Kruskal coordinates $(\tilde{U}, \tilde{V}, x^i)$ with $i = 1$ to $d-1$, takes the form

$$ds^2 = -f d\tilde{U} d\tilde{V} + r^2 d\Sigma^2_{\kappa,d-1}(x^i) \tag{6.55}$$

where f and r are smooth functions of $(\tilde{U}, \tilde{V})$.

The AdS$_{d+1}$-geon is the quotient spacetime resulting from a freely activing involutive isometry applied to the AdS$_{d+1}$ black hole [106]. It is obtained via the identification [106, 105]

$$J : (\tilde{U}, \tilde{V}, x^i) \to (\tilde{V}, \tilde{U}, P(x^i)) \tag{6.56}$$

which corresponds to the change

$$(t, x^i) \to (-t, -x^i) \tag{6.57}$$

in the spacetime coordinates. $P(x^i) = -x^i$ is the antipodal map on the $(d-1)$-dimensional sphere S^{d-1}, which corresponds to $\kappa = 1$ in (6.54).

The state associated with the CFT on the geon boundary is the thermofield single [109]

$$|\Psi_g\rangle = e^{-(\beta/4+it)H}|C\rangle \tag{6.58}$$

where $|C\rangle$ is the cross-cap state, consisting of an entangled state between left- and right- moving modes of a free boson CFT (see figure 6.2a). In terms of the modes j_n and $\bar{j}_n$ of the holomorphic and anti-holomorphic conserverd currents $J = i\partial X$ and $\bar{J} = i\bar{\partial} X$, respectively, it is solution to [127]

$$[j_n + (-1)^n \bar{j}_{-n}]|C\rangle = 0 \tag{6.59}$$

and thus takes the form

$$|C\rangle = \exp\left[-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} j_{-n}\bar{j}_{-n}\right]|0\rangle \tag{6.60}$$

which clearly shows entanglement between the left- and right-moving modes of the CFT.

In the case of the geon space, we claim that due to the reflection coming from the involution J the metric function $F(U, y)$, satisfies

$$F(U, y)_{\text{Geon}} \leq F(U, y)_{\text{BTZ}} + F(U^{'}, y^{'})_{\text{BTZ}}. \tag{6.61}$$

The right-hand side of (6.61) saturates the geon metric function. This make sense when the complexity is regarded as the minimum time required to approximate the unitary. The presence of first and second terms on the right hand side of (6.61) is depicted in figures 6.2a and 6.2b.

Thus the unitary operator $U^{'}$ and the tangent space vectors $y^{'}$ to the manifold of unitary operators at $U^{'}$ correspond to those where the spacetime coordinates for the left-modes are $(-t, -x^i)$ [3]. Equation (6.61) can be understood as the metric function of a quantum system consisting of the direct product of two other quantum systems (figures 6.3a and 6.3b). Indeed, let us suppose that F_A, F_B, and F_{AB} are the metrics given in equation (6.7) on $SU(2)^{n_A}$, $SU(2)^{n_B}$ and $SU(2)^{n_A+n_B}$, respectively. The metric F_{AB} of the system composed of a unitary U on the n_A qubit and a unitary V on the n_B qubits is [118]

$$F^2_{AB}(U \otimes V, H_A + H_B) = F^2_A(U, H_A) + F^2_B(V, H_B) \tag{6.62}$$

where $H_A \in SU(2)^{n_A}$ and $H_B \in SU(2)^{n_B}$ (omitting the tensor factors $I_A \otimes .$ and $. \otimes I_B$ acting trivially on V and U, respectively). The Finsler metrics F_A, F_B and F_{AB} are said to form an *additive triple* of Finsler metrics. Equation (6.62) leads to the inequality

$$F_{AB}(U \otimes V, H_A + H_B) \leq F_A(U, H_A) + F_B(V, H_B). \tag{6.63}$$

.

The quantity we are now going to compute is the complexity corresponding to the metric $F(U^{'}, y^{'})$ in (6.61). We first introduce the notion of an F-Isometry. A map $h : s(t) \to h(s(t))$ is an F-Isometry

[3]The coordinates on one of the CFTs (left or right) change as $(t, x^i) \to (-t, -x^i)$ while they remain unchanged on the other CFT.

if and only if the length (6.9) associated with the metric $F(s[t],[s]_t)$ satisfies the relation

$$l_F(s) = l_F(h\ o\ s) \tag{6.64}$$

and

$$F(s[t],[s]_t) = F((h\ o\ s)(t),[h\ o\ s]_t). \tag{6.65}$$

In the tangent space to the manifold at $s(t)$, it acts like

$$[h\ o\ s]_t = h_*[s]_t \tag{6.66}$$

with h_* defined as

$$h_* : T_{s(t)}M \to T_{h(s(t))}M \tag{6.67}$$

such that the F-Isometry reads as

$$F(x,y) = F(h(x),h_*y). \tag{6.68}$$

Under the identification (6.57), the momentum components transform as

$$\begin{aligned}
k_0 &\equiv \frac{\partial}{\partial t} \to \frac{\partial}{\partial(-t)} = -\frac{\partial}{\partial t} \equiv -k_0 \\
k_i &\equiv \frac{\partial}{\partial x^i} \to \frac{\partial}{\partial(-x^i)} = -\frac{\partial}{\partial x^i} \equiv -k_i.
\end{aligned} \tag{6.69}$$

From the above relations we infer that the quantities $k = \sqrt{\sum_{i=1}^{d-1} k_i^2}$, and $\Xi = 2\omega_k t$ with $(\omega_k \to -\omega_k,\ t \to -t)$ are invariant under these transformations. Hence the control function

$$\gamma_+ = \frac{-i\sinh(2\theta_k)\sin(2kt)}{\cos(2kt) + i\cosh(2\theta_k)\sin(2kt)} \tag{6.70}$$

is still invariant under these transformations. Thus, the geon transformation is an F-Isometry, and still obeys the condition $|\gamma_+| < 1$.

The complexity is therefore equal to twice that of the AdS$_{d+1}$ black hole since the two contributions from the geon metric contribute equally to the complexity

$$\begin{aligned}
C^{(1)}(t) &= \min_{\gamma_+} \int_{s_i}^{s_f} d\sigma\ V_{d-1} \int d^{d-1}k\ \frac{|\gamma_+'|}{1-|\gamma_+|^2} \\
&= 2n_d E\ t
\end{aligned} \tag{6.71}$$

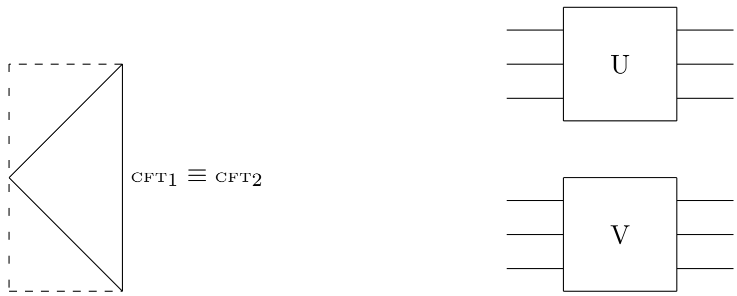
and the rate of change thereof is

$$\frac{dC(t)}{dt}^{\text{Geon}} = 2n_d E. \tag{6.72}$$

Equations (6.71) and (6.72) hold for any $(d+1)$ dimensional AdS geon with $d \geq 2$.

For any limiting value of t, the geon complexity is still twice the amount obtained in (6.52). More explicitly, we have

$$C^{\text{Geon}}(t) = 2\ C^{\text{AdS}_{d+1}}(t). \tag{6.73}$$



(a) (b)

Figure 6.3: (a) Conformal diagram of a BTZ geon. The two CFTs, one at each boundary are now identified in only one boundary. (b) Quantum circuit composed of unitaries U acting on n_A qubits and V acting on n_B qubits (when $V = I$, the n_B qubits are ancilla ones). This circuit complexity corresponds to the action integral evaluated on a WDW patch in the BTZ geon space.

6.7 Rate of variation of the action

In this section we verify the action-complexity conjecture in the context in which we have been working: between an action evaluated in the bulk (on a particular patch) and the complexity computed in the CFTs at the boundaries of the Schwarzschild AdS black holes and their geon counterparts.

Consider a Schwarzschild-AdS black hole in $d+1$ dimensions whose metric is given by

$$\begin{aligned} ds^2 &= -f dt^2 + dr^2/f + r^2 d\Sigma^2_{k,d-1} \\ f &= \frac{r^2}{l^2} + k - \frac{\omega^{d-2}}{r^{d-2}} \end{aligned} \tag{6.74}$$

where $k=0$ for planar black holes. We aim to compute the action evaluated on a WDW patch, as shown in the figure 6.4a, for this black hole. The different contributions to the action from the bulk and the boundary terms are [115, 117]

$$\begin{aligned} I &= \frac{1}{16\pi G_N}\int_M d^{d+1}x\sqrt{-g}\big(R + \frac{d(d-1)}{l^2}\big) + \frac{1}{8\pi G_N}\int_B d^d\sqrt{h}K - \frac{1}{8\pi G_N}\int_{B'} d\lambda d^{d-1}\theta\sqrt{\gamma}\ \kappa \\ &+ \frac{1}{8\pi G_N}\int_\Sigma d^{d-1}x\sqrt{\sigma}\eta + \frac{1}{8\pi G_N}\int_{\Sigma'} d^{d-1}x\sqrt{\sigma}a \end{aligned} \tag{6.75}$$

with the cosmological constant (not to be confused with the cut-off parameter in the CFTs) $\Lambda = -d(d-1)/(2l^2)$ and the curvature radius $R = -d(d+1)/l^2$.

The first term in (6.75) accounts for the bulk contribution. The other terms are the boundary contributions. The second term is the surface or Gibbons-Hawking-York term, in which K represents the extrinsic curvature. The third term comes from the null hypersurfaces with κ a parameter related to the tangent vector to these hypersurfaces. The fourth term (Hayward term) is a joint term involving the junctions of spacelike/timelike hypersurfaces [128, 129, 130, 131]. The last term is also a joint term involving the junctions of null hypersurfaces.

Evaluating the bulk contributions, we obtain for the four quadrants of figure 6.4a

$$\begin{aligned} I_{Bulk} &= \frac{1}{16\pi G_N}\int_M d^{d+1}x\sqrt{-g}\big(R+\frac{d(d-1)}{l^2}\big) \\ &= \frac{\Omega_{k,d-1}\ d}{8\pi G_N l^2}\int_0^{r_{max}} dr\ r^{d-1}(v_\infty - r^*(r)) \end{aligned} \tag{6.76}$$

where $v = t + r^*$ and $r^* = \int dr/f$. The surface contributions lead, for the four quadrants in figure 6.4a, to

$$\begin{aligned} I_{GHY} &= \frac{1}{8\pi G_N}\int_B d^dx\sqrt{|h|}K \\ &= \frac{\Omega_{k,d-1}\ d\ \omega^{d-2}}{16\pi G_N}(v_\infty - r^*(0)) \end{aligned} \tag{6.77}$$

with h the induced metric on the surface. The only nonzero contributions are those coming from the singularities ($r = 0$).

The null surface contributions are

$$I_{Null} = -\frac{1}{8\pi G_N}\int_{B'} d\lambda d^{d-1}\theta\sqrt{\gamma}\ \kappa \tag{6.78}$$

with $x^\mu = (\lambda, \theta^A)$ parametrizing the null hypersurfaces and γ the induced metric on them. κ satisfies the equation $k^\mu\nabla_\mu k_\nu = \kappa k_\nu$ and $k^\mu = \frac{\partial x^\mu}{\partial\lambda}$ are the tangent vectors to these surfaces. It is possible to choose everything to be affinely parametrized such that $\kappa = 0$. We thus can infer that the null surfaces do not contribute to the action. The joint term (Hayward) contributions have the form

$$I_{Hay} = \frac{1}{8\pi G_N}\int_\Sigma d^{d-1}x\sqrt{\sigma}\eta. \tag{6.79}$$

In our case there is no contribution coming from this term since there are no spacelike/timelike junctions for the chosen patch (figure 6.4a). The contribution of the last term for the four quadrants is

$$\begin{aligned} I_{jnt} &= \frac{1}{8\pi G_N}\int_{\Sigma'} d^{d-1}x\sqrt{\sigma}a \\ &= \frac{\Omega_{k,d-1}}{16\pi G_N}\epsilon_0^{d-1}\log(\epsilon_0^{d-2}/\omega^{d-2}). \end{aligned} \tag{6.80}$$

It is important to recall that here the only non zero contributions are those of the junctions at the region near the singularities ($r = \epsilon_0$ with ϵ_0 very small). And we also have to keep in mind that those contributions only appear when we consider black holes with hyperbolic metrics ($k = -1$) whose horizon radii are smaller than the AdS radius ($r_h < l$). We shall not consider these kinds of black holes any further; they lead to similar conclusions.

After summing up all these contributions we find that the rate of change of the action at late time is

$$\begin{aligned}\left.\frac{dI}{dt}\right|_{t\to\infty} &= \frac{1}{\pi}\frac{d}{dt}\left[I_{Bulk}+I_{GHY}\right]\Big|_{t\to\infty}\\ \left.\frac{dI}{dt}\right|_{t\to\infty} &= 2M_* \end{aligned}\tag{6.81}$$

with M_* given in appendix (D-11). We shall see in the next few steps that the mass term M_* can be identified with the total energy E of the scalar field.

Focusing now on the geon case, since in figure 6.4b only half of the patch (two quadrants) contributes to the action, it implies that the total action for the geon space will be the half of that of the AdS_{d+1} black hole.

In fact, the time in the geon conformal diagram (see figure 6.4b) is moving up for both the left and right CFTs. The geon action can be interpreted in the AdS context as

$$I_{Geon}(t_1+t_2) = I_{AdS}(t_1+t_2) + I_{AdS}(t_1-t_2). \tag{6.82}$$

This can be justified by the fact that a given point in the geon diagram has two images in the AdS diagram. For symmetric time evolution ($t_1 = t_2 = t/2$) the second term of the right-hand side of (6.82) is time independent whereas the first term is time dependent and is only evaluated on half the patch of the AdS black hole.

The rate of change at late time for the geon action then becomes

$$\left.\frac{dI}{dt}\right|_{t\to\infty} = M_* \tag{6.83}$$

We thus obtain for $d \geq 2$ the relation

$$I^{\text{Geon}} = \frac{1}{2}\, I^{\text{AdS}_{d+1}}. \tag{6.84}$$

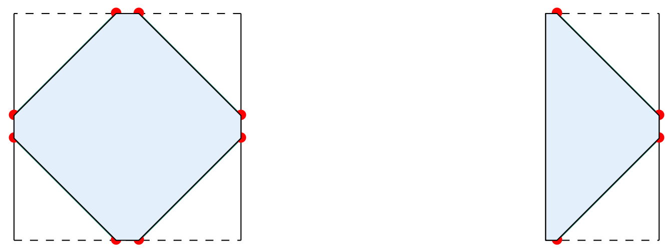
Setting the total energy E of the CFTs to be equal to the mass term M_* of the AdS_{d+1} black hole, we infer that the complexity (6.53) defined in the CFTs at the boundaries of the AdS_{d+1} black holes can be expressed in term of the AdS_{d+1} action (6.81) as follows

$$C^{\text{AdS}_{d+1}} = \frac{n_d}{2} I^{\text{AdS}_{d+1}}. \tag{6.85}$$

Equation (6.85) is the conjectured relation.

Making use of the equations (6.73) and (6.84) we find the same relation for the AdS_{d+1} geon

$$C^{\text{Geon}} = 2n_d I^{\text{Geon}} \tag{6.86}$$



(a) (b)

Figure 6.4: (a) Conformal diagram of a BTZ black hole with its WDW patch. The coloured area in light blue is the area over which we evaluated the bulk contribution. The green lines are the null hypersurfaces and the red points are the joints that involve null hypersurfaces with spacelike and timelike ones. (b) Conformal diagram of the geon space with the WDW patch on it. It is obvious to notice that only half of the coloured area, the green lines and red points in the BTZ diagram appear for the geon space.

except for a factor of 4, indicative of the sensitive of complexity to the underlying topology of the spacetime.

In [121] the action was computed at $t = 0$ for the BTZ-geon on a WDW patch partitioned into non-intersecting pieces associated with each boundary and a remaining interior piece. It was found that the action evaluated on each partition is precisely half the WDW patch-action of the corresponding two-sided BTZ wormhole ($t = 0$) and is independent of the black hole mass.

6.8 Conclusion

We have derived the computational complexity of a CFT defined on the boundary of an AdS_{d+1} black hole as a function of a temporal variable t, and have explicitly computed the small-t and large-t limits. The quantity t can be regarded as the boundary time parameter, yielding the rate of change of the CFT complexity. Up to a factor this equals n_d times the rate of change of the bulk action evaluated on a WDW patch as conjectured [117, 38].

Our results are commensurate with previous work [116], where the target state was defined for a fixed value of time and where a different control function was employed, resulting in a dimensionless complexity proportional to $V_{d-1}\Lambda^{d-1}$. Similar results have been derived in the context of the cMERA circuit [132, 133, 134].

In contrast to this, we began with a generic TFD state defined on the boundaries of an AdS_{d+1} black hole as the target state and obtained a more complex control function depending on the parameter t. This led us to a dimensionless expression (6.50) for the complexity that is a function of t, which is proportional to $V_{d-1}\Lambda^{d-1}$ as well.

We have also established a correspondence between the geon quotient space of the AdS_{d+1} black hole and a quantum system consisting of a product of two quantum systems. We found that the complexity of the CFT on the boundary of the AdS_{d+1} geon is twice that of the its AdS_{d+1} black hole counterpart. Furthermore, we found that the rate of change of the bulk action of the AdS_{d+1} geon evaluated on a WDW patch is half of that of the AdS_{d+1} black hole.

We therefore infer that the complexity/action relationship is sensitive to the topology of the bulk spacetime: there exists the same kind of correspondence relation between the complexity of a CFT and the bulk action of a geon evaluated on a WDW patch (6.86), but with the additional (topological) factor of 4.

It would be interesting to compute in future investigations the computational complexities $C^{(n)}$ (with $n > 1$) associated with the same control function $\gamma_+(\overrightarrow{k}, \sigma)$ and see whether they can lead to desired and more general forms of the complexity $C^{(1)}$. Likewise an exploration of the computational complexities $C^{(n)}$ (with $n \geq 1$) for charged and/or rotating AdS black holes (and their geon counterparts [106]) should also provide further insight.

Chapter 7

Conclusion

Here we summarize the main results derived in this dissertation, and provide some important remarks and directions for future study in the field of holographic dualities.

In chapter 3, we found that some, but not all super-entropic black holes exhibit the Kerr/CFT correspondence when considering their near extremal horizon limit despite their reputation to have non-compact horizons. It was also proven that starting with a Kerr-AdS-like black hole the outcome of the study is independent of limit that is taken first. In other words, the order between the super-entropic and the Kerr-CFT limits does not matter.

We explored the Kerr/CFT correspondence for singly-spinning super-entropic black holes and found that the correspondence exists in four dimensional spacetimes. We noticed that beyond that dimension, i.e. for $d \geq 5$, the Kerr-CFT correspondence does not apply to these black holes because they cannot be extremal. We were also able to show that the five dimensional gauged supergravity super-entropic black hole presents the Kerr/CFT correspondence near its horizon only when it carries a very large electric charge (this condition allows the applicability of Cardy's formula).

From these different cases of super-entropic black holes it appears that the Kerr/CFT correspondence is a robust holographic duality for this type of black hole. Whether the Kerr/CFT correspondence holds for other cases of super-entropic black holes, such as multiply spinning super-entropic black holes, remains to be tested.

In chapter 4, we managed to obtain the holographic equation of state for Lovelock theories of gravity by assuming that the grand canonical free energy in the CFT is a homogeneous function of the functions $g_k(N)$ of the number of degree of freedom N or central charge N^2 of the field theories. We checked its validity for many non-trivial cases of Lovelock theories and the $5d$ minimal gauged supergravity (which is a rotating black hole). The validity of the equation of state for many cases of Lovelock theories provides further supporting evidence for the robustness AdS/CFT correspondence.

An important question left for future investigation concerns Einstein gravity black holes whose dual field theories are the large N gauge theories with hyperscaling violation. For these theories the equation of state is slightly modified and it would be interesting to see whether this new equation of state holds in Lovelock theories.

In chapter 5, we explored the volume-complexity conjecture, which connects computational complexity in a CFT, known as the quantum information metric, to the maximal volume of a time slice in the bulk. We proved that the volume-complexity conjecture holds for both the planar Schwarzschild-AdS black holes and their geon counterparts. The fact that the conjecture applies to the geon quotient spaces gives a glimpse on the robustness of the AdS/CFT duality. The relation between the CFT and gravitational volume-complexity comes with a factor 4 due to the topology of the geons, showing that the volume-complexity proposal is sensitive to the topological structure of the spacetime. This suggests that an exploration of this conjecture for more interesting topologies, such as the ones including rotation, solitons and more generalized geometries, would be of interest.

In chapter 6 we derived an expression for the action-complexity in the CFT (the fields theories consist in free scalar field theories and quantum states of the field theories are Gaussian ones) as a function of a temporal variable. As in the volume-complexity conjecture in the previous chapter, we were to able to demonstrate that the action-complexity holds for both the $(d+1)$-dimensional AdS spacetimes and their geons. The action-complexity conjecture is sensitive to the topology of the spacetime, with a factor of 4 appearing in the action-complexity relation. Thus provides further evidence in support of the robustness of AdS/CFT holographic duality. An interesting problem to tackle concerns the nature of the action-complexity conjecture for charged and/or rotating black holes as well as for theirs geon counterparts as the computation of the complexity in the CFT for the aforementioned black holes remains an open question.

To end this concluding section, we recall that in the current dissertation we studied two holographic dualities namely the Kerr/CFT correspondence and the AdS/CFT correspondence. We found evidence in favour of the robustness of these holographic dualities by probing them in some new classes of black holes and theories of gravity. These classes of black holes include super-entropic (ultraspinning Kerr-AdS-like) black holes and quotient spaces of AdS black holes known as geons. The theories of gravity involve both Einstein and Lovelock theories. The holographic aspects of black hole chemistry were further extended and strengthened. Clearly, there is much more to be learned about holographic duality from the study of these black holes as they have unusual topologies (features).

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Appendices

A Vanishing charge difference between neighbouring metrics

Here we determine explicitly the charge difference between two neighbouring metrics (3.26), which in turn gives rise to the central charge.

Prior to computing the integrals that contribute to the central charge we have to determine the non zero components of $h_{\mu\nu}$ (at leading order), which are

$$\begin{aligned} h_{tt} &= -2\epsilon^{'} r^2(k^2\gamma - \Gamma), \\ h_{r\phi} &= -\frac{\epsilon^{''}}{r}\Gamma, \\ h_{\phi\phi} &= 2\epsilon^{'}\gamma. \end{aligned} \tag{A-1}$$

Upon inserting these expressions into (3.27,3.28), we obtain

$$K_\zeta + K_{\zeta,\Lambda} = \frac{ik}{2}\left(\frac{1}{\Gamma}\epsilon^{''}_m\epsilon^{'}_n + \frac{\gamma}{\Gamma^2}\epsilon^{'}_m\epsilon_n + \frac{f}{\Gamma^2}\epsilon^{'}_m(f\epsilon_n + \Lambda) - (m \leftrightarrow n)\right)d\theta \times d\phi + \cdots \tag{A-2}$$

where α, γ, Γ and f are functions of θ and k is a constant. The first two terms in the above equation are the gravitational and gauge contributions to the integrand of (3.26), and respectively yield the two central charge terms in (3.32); the third term is eliminated via the gauge choice $\Lambda = -f(\theta)\epsilon(\phi)$.

The remaining terms in the ellipsis are those that vanish when $r \to \infty$ or are not tangent to the surface over which the integration (3.26) is taken [8]; hence they do not contribute to the integral. We have checked that all such terms vanish. Rather than providing an exhaustive list of these terms, we shall provide only a few examples. The terms that stem from the boundary conditions (3.29) are of the form

$$\begin{aligned} K_\zeta &\supset \frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\ h^{\sigma\nu}D^\mu\zeta_\sigma\ dx^\alpha \times dx^\beta \\ &\supset -\frac{1}{2}g^{13}g^{22}h_{13}\Gamma^1_{23}\zeta^3 d\theta \times d\phi \\ &= -\frac{k^2}{4}\frac{\gamma}{\Gamma}\mathcal{O}(\frac{1}{r})\epsilon\ d\theta \times d\phi \end{aligned} \tag{A-3}$$

whereas the contributions coming from the gauge field have the form

$$\begin{aligned} K_{\zeta,\Lambda} &\supset \frac{1}{8}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}\zeta^{\sigma}a_{\sigma}\ dx^{\alpha}\times dx^{\beta} \\ &\supset -\frac{1}{2}g^{11}g^{22}F_{12}\zeta^{\sigma}a_{\sigma}\ d\theta\times d\phi \\ &= -\frac{kf}{2\Gamma^2}\mathcal{O}(\frac{1}{r^2})\epsilon\ d\theta\times d\phi \end{aligned} \tag{A-4}$$

where the coordinates (x^1,x^2,x^3,x^4) are (t,r,ϕ,θ) respectively. It is clear that all these contributions are finite and vanish for $r\to\infty$.

B Equation of state in $5d$ gauged supergravity

We consider here a computation of the free energy density of rotating black holes in minimal $5d$ gauged supergravity. From the thermodynamic quantities given in (4.71) we see that

$$\begin{aligned} 4\tilde{M}-3(T\tilde{S}+\mu\tilde{Q})-4\omega_a\tilde{J}_a-4\omega_b\tilde{J}_b &= \frac{\pi}{\Xi_a^2\Xi_b^2}m(2\Xi_a+2\Xi_b-\Xi_a\Xi_b)\frac{1}{G_5l^3} \\ &+ \frac{2\pi}{\Xi_a^2\Xi_b^2}qabg^2(\Xi_a+\Xi_b)\frac{1}{G_5l^3}-\frac{3}{G_5l^3}TS-\frac{3}{G_5l^3}\mu Q \\ &- \frac{\pi}{l\Xi_a^2\Xi_b}[2am+qb(1+a^2g^2)]\frac{\omega_a}{G_5l^3} \\ &- \frac{\pi}{l\Xi_a\Xi_b^2}[2bm+qa(1+b^2g^2)]\frac{\omega_b}{G_5l^3}. \end{aligned} \tag{B-1}$$

The free energy density reads as

$$\begin{aligned} \tilde{\Omega} &= \frac{\pi m}{4\Xi_a^2\Xi_b^2G_5l^3}(2\Xi_a+2\Xi_b-\Xi_a\Xi_b)+\frac{\pi qabg^2(\Xi_a+\Xi_b)}{2\Xi_a^2\Xi_b^2G_5l^3}-\frac{\pi^2}{2\Xi_a\Xi_bG_5l^3r_+}T(r_+^2+a^2)(r_+^2+b^2) \\ &- \frac{\pi^2}{2\Xi_a\Xi_bG_5l^3r_+}Tabq-\frac{\sqrt{3}\pi}{4\Xi_a\Xi_bG_5l^3r_+}\mu ql-\frac{\pi am}{2\Xi_a^2\Xi_bG_5l^3r_+}\frac{\omega_a}{l}-\frac{\pi bm}{2\Xi_a\Xi_b^2G_5l^3r_+}\frac{\omega_b}{l} \\ &- \frac{\pi qb(1+a^2g^2)}{4\Xi_a^2\Xi_bG_5l^3r_+}\frac{\omega_a}{l}-\frac{\pi qa(1+b^2g^2)}{4\Xi_a\Xi_b^2G_5l^3r_+}\frac{\omega_b}{l}. \end{aligned} \tag{B-2}$$

Therefore, we have

$$\begin{aligned} l\partial_l\tilde{\Omega}\big|_{l^3/G_5,T,\mu,\omega_i} &= \frac{\pi}{4\Xi_a^2\Xi_b^2}[-4m+\frac{4\Xi_a\Xi_b}{\pi}TS+\frac{3}{r_+^2}(q^2+abq)](2\Xi_a+2\Xi_b-\Xi_a\Xi_b)\frac{1}{G_5l^3} \\ &+(\frac{3}{2}-2)\frac{\pi}{\Xi_a^2\Xi_b^2}qabg^2(\Xi_a+\Xi_b)\frac{1}{G_5l^3}+\frac{3}{G_5l^3}TS \\ &-\frac{\pi^2}{2\Xi_a\Xi_br_+}[3r_+^4+(a^2+b^2)r_+^2-a^2b^2+2abq]\frac{1}{G_5l^3} \\ &+\frac{\pi}{2l\Xi_a^2\Xi_b}[4am+2qb(1+a^2g^2)]\frac{\omega_a}{G_5l^3}+\frac{\pi}{2l\Xi_a\Xi_b^2}[4bm+2qa(1+b^2g^2)]\frac{\omega_b}{G_5l^3} \\ &-\frac{\pi}{2l\Xi_a^2\Xi_b}[\frac{4a\Xi_a\Xi_b}{\pi}TS+\frac{3a}{r_+^2}(q^2+abq)+\frac{3}{2}qb(1+a^2g^2)]\frac{\omega_a}{G_5l^3} \\ &-\frac{\pi}{2l\Xi_a\Xi_b^2}[\frac{4b\Xi_a\Xi_b}{\pi}TS+\frac{3b}{r_+^2}(q^2+abq)+\frac{3}{2}ab(1+b^2g^2)]\frac{\omega_b}{G_5l^3}. \end{aligned} \tag{B-3}$$

and which can finally be written as

$$
\begin{aligned}
l\partial_l\tilde{\Omega}\big|_{l^3/G_5,T,\mu,\omega_i} &= -4\tilde{M}+3T\tilde{S}+4\omega_a\tilde{J}_a+4\omega_b\tilde{J}_b+\frac{9\pi}{4\Xi_a\Xi_b}\frac{q^2r_+^2}{[(r_+^2+a^2)(r_+^2+b^2)+abq]}\frac{1}{G_5l^3} \\
&= -4\tilde{M}+3(T\tilde{S}+\mu\tilde{Q})+4\omega_a\tilde{J}_a+4\omega_b\tilde{J}_b. \qquad \text{(B-4)}
\end{aligned}
$$

using (4.71).

C Complexity-Volume: computations in the bulk

Here we provide more details on some of the computations in the bulk.

Let us start with the BTZ black hole; in this case we obtain from (5.31) for the late time limit

$$
\begin{aligned}
\frac{\mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)}{R^2V_1} &= -\int_0^{\pi/4}\frac{\cos 2\kappa}{\sin\kappa}d\kappa+\int_0^{\rho_\infty}\frac{\cosh 2\rho}{\sinh\rho}d\rho+t \\
t &= \int_0^{\pi/4}\frac{d\kappa}{\sin\kappa\cos 2\kappa}-\int_0^{\rho_\infty}\frac{d\rho}{\sinh\rho\cosh 2\rho}
\end{aligned}
\qquad \text{(C-1)}
$$

or, more explicitly,

$$
\begin{aligned}
\frac{\mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)}{R^2V_1} &= 2\cosh\rho_\infty-\sqrt{2}-\log(1-\sqrt{2})+t \\
t &= \int_0^{\pi/4}\frac{d\kappa}{\sin\kappa\cos 2\kappa}-\int_0^{\rho_\infty}\frac{d\rho}{\sinh\rho\cosh 2\rho}
\end{aligned}
$$

Setting $e^{\rho_\infty}\simeq\pi/4\epsilon$, this can be approximated to the asymptotic form

$$
\frac{\mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)}{R^2}\simeq\frac{\pi V_1}{4\epsilon}+V_1t+\cdots \qquad \text{(C-2)}
$$

with $t\sim-\log\epsilon$, which is congruent with the results we got from the CFT computations. We can also see that the closer the parameter ϵ gets to 0 the greater is the time t.

When looking at the early time limit, we find from (5.31) that

$$
\begin{aligned}
\frac{\mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)}{R^2V_1} &= 2\int_0^{\rho_\infty}\cosh\rho d\rho+\sin^2 2\kappa_*\int_0^{\rho_\infty}\frac{d\rho}{\sinh\rho\sinh 2\rho} \\
t &= -\sin 2\kappa_*\int_0^{\rho_\infty}\frac{d\rho}{\sinh\rho\sinh 2\rho}
\end{aligned}
\qquad \text{(C-3)}
$$

which can also take the form

$$
\frac{\mathrm{Vol}^{\mathrm{BTZ}}(\Sigma)}{R^2}\simeq\frac{\pi V_1}{4\epsilon}+\frac{2}{\pi}t^2 \qquad \text{(C-4)}
$$

where now $t\simeq-(\frac{1}{\epsilon}-\frac{\pi}{2})\sin(2\kappa_*)$. Here we notice that the term in $1/\epsilon$ can be eliminated by means of renormalization of the time t in (D-6), rendering the latter finite and converging to 0 when κ_* approaches 0.

Concerning the AdS Schwarzschild planar black hole, we obtain in the late time limit

$$\frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} = 2\frac{V_d}{z_0^d}\Big[-\frac{i}{2^{\frac{d}{d+1}}}\int_\epsilon^1 \frac{dx}{x^{d+1}\big[1+x^{d+1}\big]} + \int_\epsilon^{1-\epsilon}\frac{dx}{x^{d+1}\big[1-\frac{1}{2}x^{d+1}\big]}\Big] \tag{C-5}$$

from (5.47), or explicitly

$$\begin{aligned}\frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} &= 2\frac{V_d}{z_0^d}\big[-2^{\frac{d}{d+1}}i\int_\epsilon^1 \frac{x^{d+1}dx}{(1+x^{d+1})\Big[1+2x^{d+1}\Big]} - i\int_\epsilon^1 \frac{\beta_1 dx}{1+x^{d+1}}\\ &+ \int_\epsilon^{1-\epsilon}\frac{x^{d+1}dx}{(1-x^{d+1})\big[1-\frac{1}{2}x^{d+1}\big]} + \int_\epsilon^{1-\epsilon}\frac{\beta_2 dx}{1-\frac{1}{2}x^{d+1}}\Big]\\ t &= z_0\Big[-2^{\frac{d}{d+1}}i\int_\epsilon^1 \frac{x^{d+1}dx}{(1+x^{d+1})\big[1+2x^{d+1}\big]} + \int_\epsilon^{1-\epsilon}\frac{x^{d+1}dx}{(1-x^{d+1})\big[1-\frac{1}{2}x^{d+1}\big]}\Big]\end{aligned} \tag{C-6}$$

with

$$\begin{aligned}\beta_1 &= \frac{2^{\frac{1}{d+1}}(1+2x^{d+1}) - 2^{\frac{d}{d+1}}x^{2d+2}}{x^{d+1}(1+2x^{d+1})} \quad \text{and}\\ \beta_2 &= \frac{2-2x^{d+1}-x^{2d+2}}{x^{d+1}(1-x^{d+1})}.\end{aligned}$$

This can finally be put into the asymptotic form

$$\frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} \simeq \frac{V_d}{d\epsilon^d} + \frac{V_d t}{z_0^{d+1}} \tag{C-7}$$

with $\quad t \sim \epsilon^{\frac{1}{d+1}}\,{}_2F_1\big[\frac{-1}{d+1}, \frac{-1}{d+1}, \frac{d}{d+1}, \frac{1}{(d+1)\epsilon}\big]$.

The early time limit reads

$$\begin{aligned}\frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} &= 2\frac{V_d}{z_0^d}\Big[\int_\epsilon^{1-\epsilon}\frac{dx}{x^{d+1}\sqrt{1-x^{d+1}}} + \frac{1}{2}h_*\int_\epsilon^{1-\epsilon}\frac{x^{d+1}dx}{\sqrt{(1-x^{d+1})^3}}\Big]\\ t &= -iz_0\sqrt{h_*}\int_\epsilon^{1-\epsilon}\frac{x^{d+1}dx}{\sqrt{(1-x^{d+1})^3}}.\end{aligned} \tag{C-8}$$

Hence

$$\frac{\mathrm{Vol}^{SAdS_{d+2}}(\Sigma)}{R^{d+1}} \simeq 2V_d\Big[\frac{1}{d\epsilon^d z_0^d} + \frac{(d+1)^2}{4\beta(\frac{1}{2},\frac{1}{d+1})}\frac{t^2}{z_0^{d+2}}\Big] \tag{C-9}$$

where now $t \simeq -iz_0\sqrt{h_*}\big[\frac{2}{\sqrt{(d+1)^3\epsilon}} - \frac{2}{(d+1)^2}\beta(\frac{1}{2},\frac{1}{d+1})\big]$. The term in $1/\sqrt{\epsilon}$ can be removed by renormalizing this expression so that the time t goes to 0 when $z_* = z_0$.

D Supplementary material for the action-complexity relation

D.1 Coordinates on the tangent plane

Consider the manifold of unitaries $SU(2^n)$ and the unitary operator [118]

$$U = \exp[-i \sum_j \gamma^j \sigma_j] \tag{D-1}$$

thereof, the tangent to $SU(2^n)$ at this point U, admits the coordinates

$$\begin{aligned} y^i &= i\mathrm{Tr}(\sigma_i dU/dt\ U^\dagger)/2^n \\ &= d\gamma^i/dt. \end{aligned} \tag{D-2}$$

For the metric function $F_1(U, y) = f(\gamma)$, the complexity or length (Euclidean distance) associated with it reads

$$\begin{aligned} C_f(U) &= \inf_\gamma \int_I f(\gamma(t))\ dt \\ &= \inf_\gamma \int_I \sum_i d\gamma^i. \end{aligned} \tag{D-3}$$

In the Poincare disk model (with γ^i $(i = +)$), the complexity or length (hyperbolic distance) associated with the metric $F_1(U, y)$ has the form

$$C_f(U) = \inf_{\gamma_+} \int \frac{d\gamma_+}{1 - |\gamma_+|^2}. \tag{D-4}$$

D.2 Complexity

This subsection is devoted to the derivation of the final form of the computational complexity $C^{(1)}(t)$. As introduced earlier in the previous sections, it has the form

$$\begin{aligned} C^{(1)}(t) &= \min_{\gamma_+} \int_{s_i}^{s_f} d\sigma\ \frac{V_{d-1}}{2} \int d^{d-1}k\ \frac{|\gamma_+^{'}|}{1 - |\gamma_+|^2} \\ &= \int_{s_i}^{s_f} d\sigma\ \frac{V_{d-1}}{2} \int d^{d-1}k\ |2\omega_k t \sinh(2\theta_k)| \\ &= 2V_{d-1}\ t\ \Omega_{\kappa,d-2} \int k^{d-1} \frac{e^{-\beta k/2}}{1 - e^{-\beta k}} dk \\ &= 2V_{d-1}\Omega_{\kappa,d-2}(2^d - 1)\beta^{-d}\Gamma(d)\zeta(d)\ t \end{aligned} \tag{D-5}$$

where we employ the control function

$$\gamma_+(\overrightarrow{k}, \sigma) = \frac{-i \sinh(2\theta_k) \sin(2kt\sigma)}{\cos(2kt) + i \cosh(2\theta_k) \sin(2kt\sigma)} \tag{D-6}$$

yielding in turn

$$\frac{|\gamma_+^{'}|}{1-|\gamma_+|^2} = 2\omega_k t \sinh(2\theta_k) \tag{D-7}$$

with

$$\sinh(2\theta_k) = \frac{2e^{-\beta\omega_k/2}}{1-e^{-\beta\omega_k}}. \tag{D-8}$$

D.3 AdS/CFT (Planar black holes)

Here we review some useful notions on the metric of Schwarzschild-AdS black hole, particularly the planar one, as well as the metric of its boundary CFT.

A planar Schwarzschild-AdS black hole in $d+1$ dimension has the metric

$$\begin{aligned} ds^2 &= -fdt^2 + dr^2/f + r^2 d\Sigma^2_{\kappa,d-1} \\ f &= -\omega^{d-2}/r^{d-2} + r^2/l^2. \end{aligned} \tag{D-9}$$

Changing variables to $z = l/r$, (D-9) becomes

$$\begin{aligned} ds^2 &= \frac{l^2}{z^2}[-hd\tilde{t}^2 + dz^2/h + d\Sigma^2_{\kappa,d-1}] \\ h &= 1-(z/z_0)^d \end{aligned} \tag{D-10}$$

where $\tilde{t} = t/l,\ R = l,\ z_0^d = l^{d-2}/\omega^{d-2}$ and $\omega^{d-2} = r_h^d/l^2$.

The mass of this black hole is

$$M_* = \frac{d-1}{16\pi G_N}\Omega_{0,d-1}\omega^{d-2}. \tag{D-11}$$

The metric of the CFT on the boundary of the black hole is of the form

$$ds^2_{boundary} = -dt^2 + l^2 d\Sigma^2_{\kappa,d-1} \tag{D-12}$$

$(\kappa = 0)$ for planar black holes. It can be rewritten as

$$ds^2_{boundary} = -l^2[d\tilde{t}^2 + d\Sigma^2_{\kappa,d-1}] \tag{D-13}$$

and we can label $\tilde{t}$ as t.

D.4 Total energy of the scalar field

Here we compute the total energy of the scalar field knowing the probability densities of the Hamiltonian eigenstates $|n,n\rangle$.

Starting with the state $|TFD(0)\rangle$ in (6.40) we find that the density matrix is obtained from the expression

$$\begin{aligned} \rho &= Tr(|TFD(0)\rangle\langle TFD(0)|) \\ &= \sum_{n_k} e^{-\beta\omega_k} |n_k\rangle\langle n_k| \end{aligned} \tag{D-14}$$

after tracing over the states $|n_k\rangle_2$, where $e^{-\beta\omega_k}$ are clearly the probability densities of the Hamiltonian eigenstates. From the above expression we infer that the total energy of the scalar field reads as

$$\begin{aligned} E &= V_{d-1} \int d^{d-1}k \; \omega_k \; e^{-\beta\omega_k} \\ &= V_{d-1} \int d^{d-1}k \; k \; e^{-\beta k} \\ &= V_{d-1} \Omega_{\kappa,d-2} \beta^{-d} \Gamma(d). \end{aligned} \tag{D-15}$$