# Regular Dilation on Semigroups

by

Boyu Li

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## Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner: Marcelo Laca
Professor, Dept. of Mathematics and Statistics,
University of Victoria

Supervisor(s): Kenneth Davidson
University Professor, Dept. of Pure Mathematics,
University of Waterloo

Internal Member: Alexandru Nica
Professor, Dept. of Pure Mathematics,
University of Waterloo

Internal Member: Laurent Marcoux
Professor, Dept. of Pure Mathematics,
University of Waterloo

Internal-External Member: Henry Wolkowicz
Professor, Dept. of Combinatorics and Optimization,
University of Waterloo

## Abstract

Dilation theory originated from Sz.Nagy's celebrated dilation theorem which states that every contractive operator has an isometric dilation. Regular dilation is one of many fruitful directions that aims to generalize Sz.Nagy's dilation theorem to the multi-variate setting. First studied by Brehmer in 1961, regular dilation has since been generalized to many other contexts in recent years.

This thesis is a compilation of my recent study of regular dilation on various semigroups. We start from studying regular dilation on lattice ordered semigroups and shows that contractive Nica-covariant representations are regular. Then, we consider the connection between regular dilation on graph products of $\mathbb{N}$, which unifies Brehmer's dilation theorem and the well-known Frazho-Bunce-Popescu's dilation theorem. Finally, we consider regular dilation on right LCM semigroups and study its connection to Nica-covariant dilation.

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## Dedication

This is dedicated to Betty Xing.

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# Chapter 1

# Introduction

The study of dilation theory originated from Sz.Nagy's celebrated dilation theorem [67]. It states that every contractive operator $T \in \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ can be embedded in an isometric operator $V \in \mathcal{B}(\mathcal{K})$ on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$, so that for every $n \geq 1$,

$$P_{\mathcal{H}} V^n\big|_{\mathcal{H}} = T^n.$$

The operator $V$ is often called the isometric dilation of $T$. There are many attempts to generalize Sz.Nagy's result to the multi-variate setting. Ando [4] proved that a pair of commuting contractions can be simultaneously dilated to a pair of commuting isometries. However, it cannot be extended further due to a counterexample of Parrott [51], where he found a triple of commuting contractions that do not have a commuting isometric dilation. A natural question to ask is when does a family of contractions have isometric dilations?

There are many results that seek to generalize Sz.Nagy's result to this setting. Brehmer [9] first considered a special type of isometric dilation called the regular dilation. For each $m = (m_1, \cdots, m_k) \in \mathbb{Z}^n$, denote $(m_i^+) = (\max\{m_i, 0\})$ and $(m_i^-) = (\max\{-m_i, 0\})$. Brehmer considered the question: when does a commuting family of contractions $T_1, \cdots, T_k$ has a commuting isometric dilation $V_1, \cdots, V_k$ that satisfy a stronger condition,

$$T(m^-)^* T(m^+) = P_{\mathcal{H}} V(m^-)^* V(m^+)\big|_{\mathcal{H}}, \forall m \in \mathbb{Z}^k.$$

Brehmer called such dilation a regular dilation for the family $(T_i)$ and established that $(T_i)$ has a regular dilation if and only if for every subset $W \subseteq \{1, \cdots, k\}$,

$$\sum_{U \subseteq W} (-1)^{|U|} T_U^* T_U \geq 0.$$

Here, $T_U = \prod_{i \in U} T_i$ and by convention $T_\emptyset = I$.

The family of commuting contractions $T = (T_1, \cdots, T_k)$ can be viewed as a contractive representation of the abelian semigroup $\mathbb{N}^k$. The corresponding representation $T : \mathbb{N}^k \to \mathcal{B}(\mathcal{H})$ can be defined by sending the $i$-th generator $e_i$ to $T(e_i) = T_i$. Therefore, it is natural to consider isometric dilation of contractive representations of semigroups. Given a semigroup $P$, one can consider a contractive representation $T : P \to \mathcal{B}(\mathcal{H})$, where each $T(p)$ is a contractive operator. We can ask the question when does $T$ have an isometric dilation in the sense that there exists an isometric representation $V : P \to \mathcal{B}(\mathcal{K})$ on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$, so that for all $p \in P$,

$$P_{\mathcal{H}} V(p)\big|_{\mathcal{H}} = T(p).$$

It is not immediately clear on how one can extend Brehmer's regular dilation to representations on semigroups. Indeed, one has to first define a notion of $m^+ = \max\{m, 0\}$ and $m^- = \max\{-m, 0\}$. Nevertheless, for a special class of semigroups called the lattice-ordered semigroups, this notion can be defined. This allows us to study regular dilation on this special class of semigroups. Davidson, Fuller and Kakariadis [21] study regular dilation in relation to $C^*$-envelopes of semicrossed products of operator algebras. It was not known how to extend Brehmer's condition to an arbitrary lattice-ordered semigroup. In particular, it was an open question in [21] whether every contractive Nica-covariant representation on an abelian lattice ordered semigroup has a regular dilation.

This question initiated our study of regular dilation. This thesis is a compilation of recent works on regular dilation on various semigroups, its relation with isometric Nica-covariant representation, and its application in the study of operator theory and operator algebras.

In Chapter 3, which is based on [40], I establish equivalent conditions for a contractive representation of any lattice ordered semigroup to have regular dilation. In particular, I show every contractive Nica-covariant representation on any lattice ordered semigroup has a regular dilation. This answers the question posed in [21] positively. We also prove the minimal isometric dilation for a contractive Nica-covariant representation is Nica-covariant. This provides a glimpse of the relation between regular dilation and isometric Nica-covariant representation. Indeed, we eventually show that having a regular dilation is equivalent to having a Nica-covariant dilation. We also define and study row contractive representations on a lattice ordered semigroup as a generalization of the commuting row contractive family studied by Brehmer. Finally, we investigate the relation between Brehmer's condition and the condition I derive for lattice ordered semigroups. This leads to a nice Cholesky decomposition for certain operator matrix. This Cholesky decomposition technique becomes a crucial tool in the analysis of regular dilation of other semigroups.

However, the lattice ordered semigroups have many limitations. Many interesting classes of semigroups (the free semigroup, graph product of $\mathbb{N}$) are not included. The goal is to extend regular dilation further to a larger class of semigroups. One potential candidate is the class of quasi-lattice ordered semigroups. Quasi-lattice ordered semigroups were first studied by Nica [48] where he studied $C^*$-algebras generated by certain covariant representations on the semigroup. These representations are now called isometric Nica-covariant representations.

The first step towards generalizing regular dilation to quasi-lattice ordered groups starts with considering a very concrete setting on graph product of $\mathbb{N}$. There are many advantages behind considering this class of semigroups. They are an important class of quasi-lattice ordered semigroups intensively studied in [17]. They are also interpolating the commutative lattice ordered semigroup $\mathbb{N}^k$ and the non-commutative quasi-lattice ordered semigroup $\mathbb{F}_k^+$. On the free semigroup $\mathbb{F}_k^+$, there is another well-known theorem due to Frazho, Bunce, and Popescu that generalizes Sz.Nagy's dilation to non-commutative operators. Given $T = (T_1, \cdots, T_n)$ ($n \geq 2$, and can be $\infty$), it is called a row contraction if

$$\sum_{i=1}^{n} T_i T_i^* \leq I.$$

Equivalently, $T$ can be viewed as a contractive operator from $\mathcal{H}^{(n)}$ to $\mathcal{H}$. Frazho-Bunce-Popescu's dilation states that every row contractions can be dilated to a row isometry $V = (V_1, \cdots, V_n)$. Here, $V_i$ are isometries with orthogonal ranges. The $n$-tuple $T = (T_1, \cdots, T_n)$ can be thought as a representation of the free semigroup $\mathbb{F}_n^+$. One may notice that an isometric row contraction precisely corresponds to an isometric Nica-covariant representation of the free semigroup. This inspires us to wonder about the connection between two seemingly unrelated results: the Brehmer dilation on the abelian $\mathbb{N}^k$ and the Frazho-Bunce-Popescu dilation on the non-abelian $\mathcal{F}_k^+$.

In Chapter 4, which is based on [42], we explore the connection between these two results by studying regular dilation on graph products of $\mathbb{N}$. Given a simple graph on $k$ vertices, we can define the graph product of $\mathbb{N}$ as the unital semigroup generated by $k$ generators $e_1, \cdots, e_k$, where $e_i, e_j$ commute when $(i, j)$ is an edge in the graph. This class of semigroups naturally connects the free abelian semigroup $\mathbb{N}^k$ and the free semigroup $\mathbb{F}_k$. Indeed, on one extreme, if the graph has no edges, then the graph product is the free semigroup. On the other extreme, when the graph contains all possible edges, then the graph product is the free abelian semigroup.

We established a Brehmer type condition for contractive representations of the graph product of $\mathbb{N}$, which unifies the Brehmer's dilation and Frazho-Bunce-Popescu's dilation. Moreover, we make an important observation that having a regular dilation is equivalent to having a minimal isometric Nica-covariant dilation. This is a crucial step when we extend regular dilation further and beyond quasi-lattice ordered semigroup in Chapter 5.

Nica-covariance condition has already been generalized beyond quasi-lattice ordered semigroups. For example, the Nica-covariance condition can be defined on right LCM semigroups, and more recently any cancellative semigroup following Xin Li's construction via constructible right ideals [44]. The dilation result on graph product of $\mathbb{N}$ connects regular dilation with isometric Nica-covariant representation, which allows us to further generalize the notion of regular dilation to a wider class of semigroups.

In Chapter 5, which is based on [41], we extend the regular dilation results to right LCM semigroups. Right LCM semigroups are a natural generalization of quasi-lattice ordered semigroups that attracted much research interest recently. The main result establishes the equivalence among having regular dilation, having an isometric Nica-covariant dilation, and a Brehmer-type condition. In particular, we focus on a few examples of right LCM semigroups and derive their corresponding Brehmer-type condition. This proof avoids many technical lemmas that we used in Chapter 4, and gives a shorter proof of the result in the case of graph product of $\mathbb{N}$. This concludes our study of regular dilation.

Dilation theory has many applications in the study of operator algebra and operator theory. We go over a few applications of regular dilation in Chapter 6. In Section 6.1, we show how regular dilation plays a role in the study of semicrossed product algebra. In Section 6.2, which is based on [39], we study the relation between regular dilation and subnormal operators.

# Chapter 2

# Preliminaries

This chapter briefly introduces the background for this thesis. The goal of this thesis is to present many recent results that characterize regular dilation on various semigroups and relates regular dilation to the Nica-covariance condition.

We start by reviewing the theory of semigroups. In particular, we mostly focus on the structure of lattice ordered and quasi-lattice ordered semigroups. We will briefly go over the left-cancellative semigroups, recently studied by Xin Li. One particular left-cancellative semigroup that we focus on is called the right LCM semigroup. Finally, we review the graph product of semigroups, which is a useful way to construct new semigroups from existing ones.

We then explore many dilation results for various families of operators and representations. Dilation theory started from Sz.Nagy's celebrated dilation theorem. Initially, people focused on studying dilation of commuting contractions, especially after Ando showed a pair of commuting contractions have commuting isometric dilation. However, there are counterexamples where a triple of commuting contractions fails to have commuting isometric dilation. This motivated Brehmer's work on regular dilation which gives a nice condition on whether a family of commuting contractions can have a stronger dilation known as regular dilation. Meanwhile, dilation of non-commuting contractions is also very fruitful, following Frazho-Bunce-Popescu's dilation theorem on row contractions. These dilation results can be seen as dilation of semigroup representations. We will review some earlier results of regular dilation on lattice ordered groups that motivated our study.

Finally, we review the Nica-covariance condition on quasi-lattice ordered semigroups and its generalization on right LCM-semigroups.

## 2.1 Semigroups

This section gives a brief overview of various classes of semigroups. We will go over the basics of lattice ordered semigroups, quasi-lattice ordered semigroups, and the more general left-cancellative semigroups including the right LCM semigroups. The structure of lattice ordered semigroups allows us to easily extend the definition of Brehmer's regular dilation. However, this is no longer the case for quasi-lattice ordered semigroups. Among many classes of quasi-lattice ordered semigroups, our focus is on the graph product of $\mathbb{N}$, an important class of semigroups interpolating the free abelian semigroup and the free semigroup. We will also briefly go through some recent development of left-cancellative semigroups that further generalizes the quasi-lattice ordered semigroups.

Throughout this thesis, a semigroup $P$ is a set with an associative binary operation $\cdot : P \times P \to P$. The semigroup is always assumed to be left-cancellative, meaning if $a, x, y \in P$ and $ax = ay$, then $x = y$. The semigroup is always assumed to be unital (often called monoid), meaning that there exists a unit $e \in P$ so that for any $x \in P$, $ex = xe = x$. The semigroup $P$ does not have to be embedded in a group $G$. In fact, checking whether certain semigroup can be embedded in a group can be difficult (see for example of Artin monoids [50]). We only make the assumption that $P$ is embedded in a group $G$ for sections 2.1.1 and 2.1.2. For section 2.1.3, we discuss some recent development of left-cancellative semigroups, focusing on the case of right LCM semigroups. Finally, in section 2.1.4, we discuss the graph product of semigroups, which gives a rich class of semigroups from existing ones.

### 2.1.1 Lattice-Ordered Semigroups

Let $G$ be a group. A unital semigroup $P \subseteq G$ is called a *cone*. A cone $P$ is *spanning* if $PP^{-1} = G$, and is *positive* when $P \bigcap P^{-1} = \{e\}$. A positive cone $P$ defines a partial order on $G$ via $x \leq y$ if $x^{-1}y \in P$. We call this partial order *compatible with the group* if for any $x \leq y$ and $g \in G$, we always have $gx \leq gy$ and $xg \leq yg$. Equivalently, the corresponding positive cone satisfies a normality condition that $gPg^{-1} \subseteq P$ for any $g \in G$, and thus $x \leq y$ whenever $yx^{-1} \in P$ as well. When $P$ is a positive spanning cone of $G$ whose partial order is compatible with the group, if every two elements $x, y \in G$ have a least upper bound (denoted by $x \vee y$) and a greatest lower bound (denoted by $x \wedge y$), the pair $(G, P)$ is called a *lattice ordered group*. Conversely, if $\leq$ is a lattice order on $G$ that is compatible with the group, it is not hard to check that $P = \{p : e \leq p\}$ defines a positive spanning cone. When there is no ambiguity, we may refer $P$ as a lattice ordered semigroup.

In the special case when the partial order $\leq$ defines a total order on $G$, $G$ is called a totally ordered group. Equivalently, $G$ is totally ordered if and only if the corresponding semigroup $P$ is a positive cone that satisfies $P \bigcup P^{-1} = G$.

Lattice ordered groups are also called $\ell$-groups. One has to be cautious that there is a different notion of lattice ordered groups/semigroups defined in [17, Definition 26], where the normality condition on $P$ is removed.

**Example 2.1.1.** *(Examples of Lattice Ordered Groups)*

1. $(\mathbb{Z}, \mathbb{Z}_{\geq 0})$ *is a lattice ordered group. In fact, this partial order is also a total order. More generally, any totally ordered group* $(G, P)$ *is also a lattice ordered group.*
2. *Let* $(G_i, P_i)_{i\in I}$ *be a family of lattice ordered groups. Their direct product* $(\prod G_i, \prod P_i)$ *is also a lattice ordered group.*
3. *Let* $\mathcal{T}$ *be a totally ordered set. A permutation* $\alpha$ *on* $\mathcal{T}$ *is called order preserving if for any* $p, q \in \mathcal{T}$, $p \leq q$, *we also have* $\alpha(p) \leq \alpha(q)$. *Let* $G$ *be the set of all order preserving permutations, which is clearly a group under composition. Let* $P = \{\alpha \in G : \alpha(t) \geq t, \text{ for all } t \in \mathcal{T}\}$. *Then* $(G, P)$ *is a non-abelian lattice ordered group [3].*
4. *Let* $\mathbb{F}_n$ *be the free group on* $n$ *generators, and* $\mathbb{F}_n^+$ *be the semigroup generated by the* $n$*-generators. Then* $(\mathbb{F}_n, \mathbb{F}_n^+)$ *defines a quasi-lattice ordered group [48, Examples 2.3]. However, this is not a lattice ordered group since* $\mathbb{F}_n^+$ *is not spanning.*
5. *Consider the Braid monoid on 4 strings:*

$$\mathbb{B}_4^+ = \langle e_1, e_2, e_3 : e_1e_2e_1 = e_2e_1e_2, e_2e_3e_2 = e_3e_2e_3, e_1e_3 = e_3e_1\rangle.$$

*We can similarly define the Braid group* $\mathbb{B}_4$ *on 4 strings to be the group generated by the same set of generators.* $(\mathbb{B}_4, \mathbb{B}_4^+)$ *is a quasi-lattice ordered group, and it is not hard to verify that any finite subset of* $\mathbb{B}_4$ *has a least upper bound. Hence it is a "lattice ordered semigroup" according to the definition in [17]. However, this is not a "lattice ordered group" according to our definition since the partial order it defined is not compatible with the group. For example, take* $x = e_1$ *and* $y = e_1e_2$. *It is simple to check that* $x \leq y$ *but* $xe_3 \not\leq ye_3$.

One important feature of lattice ordered groups is that every element $g$ can be decomposed as a product of a positive part and the inverse of a negative part in a unique way. For example, when $(G, P) = (\mathbb{Z}^k, \mathbb{N}^k)$, every element $n = (n_i) \in G$ can be written as

$n^+ - n^-$, where $(n_i^+) = (\max\{n_i, 0\})$ and $(n_i^-) = (\max\{-n_i, 0\})$. For any element $g \in G$ of a lattice ordered group $(G, P)$, $g$ can be written uniquely as $g = g_+ g_-^{-1}$ where $g_+, g_- \in P$, and $g_+ \wedge g_- = e$. In fact, $g_+ = g \vee e$ and $g_- = g^{-1} \vee e$. This property is essential in our definition of regular dilation on lattice ordered semigroups.

Lattice ordered groups have many nice properties.

**Lemma 2.1.2.** *Let $(G, P)$ be a lattice order group, and $a, b, c \in G$.*

1. $a(b \vee c) = (ab) \vee (ac)$ *and* $(b \vee c)a = (ba) \vee (ca)$. *A similar distributive law holds for* $\wedge$.
2. $(a \wedge b)^{-1} = a^{-1} \vee b^{-1}$ *and similarly* $(a \vee b)^{-1} = a^{-1} \wedge b^{-1}$.
3. $a \geq b$ *if and only if* $a^{-1} \leq b^{-1}$.
4. $a(a \wedge b)^{-1} b = a \vee b$. *In particular, when* $a \wedge b = e$, $ab = ba = a \vee b$.
5. *If* $a, b, c \in P$, *then* $a \wedge (bc) \leq (a \wedge b)(a \wedge c)$.

One may refer to [3] for a detailed discussion of this subject. Notice by statement (4) of Lemma 2.1.2 $g_+, g_-$ commute and thus $g = g_+ g_-^{-1} = g_-^{-1} g_+$.

Here are some technical lemma that are very useful later.

**Lemma 2.1.3.** *Let $p, q \in P$. Then,*

$$\begin{aligned} (pq^{-1})_+ &= p(p \wedge q)^{-1} \text{ and,} \\ (pq^{-1})_- &= q(p \wedge q)^{-1}. \end{aligned}$$

*Proof.* By property (1) and (2) in Lemma 2.1.2,

$$\begin{aligned} (pq^{-1})_+ &= (pq^{-1} \vee e) \\ &= p(q^{-1} \vee p^{-1}) \\ &= p(p \wedge q)^{-1}. \end{aligned}$$

Similarly, $(pq^{-1})_- = q(p \wedge q)^{-1}$. □

**Lemma 2.1.4.** *Let $p, q, g \in P$ such that $g \wedge q = e$. Then $(pg) \wedge q = p \wedge q$.*

*Proof.* By the property (5) of Lemma 2.1.2, we have that

$$(pg) \wedge q \leq (p \wedge q)(g \wedge q) = p \wedge q.$$

On the other hand, $p \wedge q$ is clearly a lower bound for both $p \leq pg$ and $q$, and hence $p \wedge q \leq (pg) \wedge q$. This proves the equality. □

**Lemma 2.1.5.** *Let $p, q \in P$. If $g \in P$ is another element where $g \wedge q = 0$, then*

$$\begin{aligned}(pgq^{-1})_- &= (pq^{-1})_- \text{ and,}\\ (pgq^{-1})_+ &= (pq^{-1})_+ g.\end{aligned}$$

*In particular, if $0 \leq g \leq p$, then*

$$\begin{aligned}(pg^{-1}q^{-1})_- &= (pq^{-1})_- \text{ and,}\\ (pg^{-1}q^{-1})_+ &= (pq^{-1})_+ g^{-1}.\end{aligned}$$

*Proof.* By Lemma 2.1.3, we get $(pgq^{-1})_+ = pg(q \wedge pg)^{-1}$. Apply Lemma 2.1.4 to get

$$(q \wedge pg)^{-1} = (q \wedge p)^{-1}.$$

Now $g \wedge (p \wedge q) = e$ and thus $g$ commutes with $p \wedge q$ by property (4) of Lemma 2.1.2. Therefore,

$$\begin{aligned}(pgq^{-1})_+ &= pg(q \wedge pg)^{-1}\\ &= p(q \wedge p)^{-1} g\\ &= (pq^{-1})_+ g.\end{aligned}$$

The statement $(pgq^{-1})_- = (pq^{-1})_- g$ can be proven in a similar way.

Finally, for the case where $0 \leq g \leq p$, it follows immediately by considering $p' = pg^{-1}$ and thus $p = p'g$. □

**Lemma 2.1.6.** *If $p_1, p_2, \cdots, p_n \in P$ and $g_1, \cdots, g_n \in P$ be such that $g_i \leq p_i$ for all $i = 1, 2, \cdots, n$. Then $\wedge_{i=1}^n p_i g_i^{-1} \leq \wedge_{i=1}^n p_i$. In particular, when $\wedge_{i=1}^n p_i = e$, we have $\wedge_{i=1}^n p_i g_i^{-1} = e$.*

*Proof.* It is clear that $e \leq p_i g_i^{-1} \leq p_i$, and thus

$$e \leq \wedge_{i=1}^n p_i g_i^{-1} \leq \wedge_{i=1}^n p_i.$$

Therefore, the equality holds when the last term is $e$. □

### 2.1.2 Quasi-lattice Ordered Groups

Quasi-lattice ordered groups were first defined by Nica in [48], where he studied isometric covariant representations and their $C^*$-algebras. These representations are now known as isometric Nica-covariant representations, and they have been intensively studied since then [37, 35, 17, 18, 43].

Suppose $P$ is a unital positive cone inside a group $G$. Similar to the case of lattice ordered group, $P$ defines a left-invariant partial order $\leq$ on $G$ via $x \leq y$ whenever $x^{-1}y \in P$. The partial order $\leq$ defined by $P$ on $G$ is called a quasi-lattice order if any finite set $F \subset G$ with an upper bound in $G$ has a least upper bound in $G$, denoted by $\vee F$. In this case, the pair $(G, P)$ is called a *quasi-lattice ordered group.* We often refer $P$ as a quasi-lattice ordered semigroup.

Quasilattice ordered groups differ from lattice ordered group in two ways. First, not every finite subset $F \subset G$ has an upper bound. It is often convenient to add an $\infty$ to $P$ where $x\infty = \infty = \infty x$ for all $x \in G$, and when $F$ has no upper bound, we often denote $\vee F = \infty$. Second, we do not require the partial order to be compatible with $G$. The partial order we defined is only left-invariant, meaning that if $x \leq y$ when $gx \leq gy$ for all $g, x, y \in P$. Dually, we can define a right-invariant partial order $\leq_r$ by $x \leq_r y$ if $yx^{-1} \in P$.

**Example 2.1.7.** *Quasi-lattice ordered semigroups cover a wide range of important classes of semigroups.*

1. *Every lattice ordered group $(G, P)$ is also quasi-lattice ordered.*
2. *Given a simple graph $\Gamma$ on $k$ vertices, one can define $P_\Gamma$, the graph product of $\mathbb{N}$ associated with the graph to be the unital semigroup generated by $k$ generators where $e_i, e_j$ commute whenever there is an edge between the vertices $i, j$. This is also known as the right angled Artin monoid or the graph semigroup. It is a quasi-lattice ordered semigroup inside the group generated by the same set of generators. Notice that in the special case when the graph is the complete graph, $P_\Gamma$ is simply $\mathbb{N}^k$. When the graph contains no edge, $P_\Gamma$ is the free semigroup on $n$ generators.*

   *We can similarly define $G_\Gamma$ to be the group generated by the same set of generators. $(G_\Gamma, P_\Gamma)$ forms a quasi-lattice ordered group for each simple graph $\Gamma$.*

The graph product of $\mathbb{N}$ is a special case of a large class of quasi-lattice ordered semigroups known as the Artin monoids.

**Example 2.1.8.** *We first denote* $\langle s,t\rangle_m = stst\cdots$, *where we write* $s,t$ *alternatively for a total of* $m$ *times. For example,* $\langle s,t\rangle_3 = sts$.

*Consider a symmetric* $n\times n$ *matrix* $M$ *where* $m_{i,i}=1$ *for all* $i$, *and* $m_{i,j}\in\{2,\cdots,+\infty\}$ *when* $i\neq j$. *One can define* $A_M^+$, *the Artin monoid associated with* $M$ *to be the unital semigroup generated by* $e_1,\cdots,e_n$, *where each* $e_i,e_j$, $i\neq j$, *satisfy the relation* $\langle e_i,e_j\rangle_{m_{i,j}} = \langle e_j,e_i\rangle_{m_{i,j}}$. *In particular, when* $m_{i,j}=+\infty$, *this means there is no relation between* $e_i$ *and* $e_j$. *One can similarly define the Artin group* $A_M$ *be the group generated by the same set of generators.*

*The Artin monoid is said to be right-angled if each* $m_{i,j}=2$ *or* $+\infty$ *for all* $i\neq j$. *One may define a graph* $\Gamma$ *on* $n$ *vertices where* $i,j$ *are adjacent whenever* $m_{i,j}=2$. *The graph product associated with* $\Gamma$ *discussed in the Example 2.1.7 (2) is precisely the right-angled Artin monoid.*

*The Artin monoid is said to be of finite type if each* $m_{i,j}<\infty$. *For example, if for all* $i\neq j$, $m_{i,j}=3$ *when* $|i-j|=1$ *and* $m_{i,j}=2$ *otherwise, then the Artin group is the familiar Braid group on* $(n+1)$*-strings.*

*It is known that* $(A_M, A_M^+)$ *is a quasi-lattice ordered group when it is right angled or of finite type. In fact, these two cases are the only known Artin monoids to form a quasi-lattice ordered group [17].*

### 2.1.3 Left-Cancellative Semigroups

Very recently, there has been a lot of research interest on the $C^*$-algebra of left-cancellative semigroups, following Xin Li's work on semigroup $C^*$-algebras [43, 44]. Li's construction can be seen as a generalization of Nica's study of quasi-lattice ordered semigroup.

**Definition 2.1.9.** *A semigroup* $P$ *is called left cancellative if for any* $p,a,b\in P$ *with* $pa=pb$, *we have* $a=b$.

Xin Li generalized Nica-covariant representations on quasi-lattice ordered groups by considering the so-called constructible right ideal.

**Definition 2.1.10.** *Given a left cancellative semigroup* $P$, *a set* $I\subseteq P$ *is called a right ideal if for any* $p\in P$,

$$I\cdot p=\{xp : x\in I\}\subseteq I.$$

*A right ideal* $I$ *is called a principal right ideal if* $I=pP$ *for some* $p\in P$.

Given a right ideal $I$ and $p \in P$, one can define

$$\begin{aligned} pI &= \{px : x \in I\} \\ p^{-1}I &= \{y : py \in I\} \end{aligned}$$

It is not hard to check that when $I$ is a right ideal, both $pI$ and $p^{-1}I$ are also right ideals for all $p \in P$. Moreover, if $I, J$ are two right ideals in $P$, then their intersection $I \cap J$ is also a right ideal in $P$.

**Definition 2.1.11.** *The set of constructible ideals $\mathcal{J}(P)$ of a left-cancellative semigroup $P$ is the smallest collection of right ideals of $P$ so that*

1. *Every principal right ideal is in $\mathcal{J}(P)$.*
2. *$\mathcal{J}(P)$ is closed under finite intersection.*
3. *For each $I \in \mathcal{J}(P)$ and $p \in P$, $pI, p^{-1}I$ are also in $\mathcal{J}(P)$.*

In this section, we focus on a special case of left-cancellative semigroup.

**Definition 2.1.12.** *A unital semigroup $P$ is called right LCM if it is left cancellative and for any $p, q \in P$, either $pP \bigcap qP = rP$ for some $r \in P$ or $pP \bigcap qP = \emptyset$.*

In the case when $pP \bigcap qP = rP$, we can treat $r$ as a least common multiple of $p, q$. There might be many such least common multiples, but it is clear that if $r, r'$ are both least common multiples of $p, q$, then there exists an invertible $u$ with $r \cdot u = r'$. For each $p, q \in P$, let us denote $p \vee q = \{r : pP \bigcup qP = rP\}$. Similarly, for a finite subset $F \subset P$, let $\vee F = \{r : \bigcup_{x \in F} xP = rP\}$. In the case when $\vee F = \emptyset$, we often write $\vee F = \infty$ (in the case of quasi-lattice ordered groups, this corresponds to $F$ having no common upper bound). We also denote $P^*$ the set of invertible elements in $P$.

**Example 2.1.13.** *The Thompson's monoid is closely related to the well-known Thompson's group. There is a great interest in whether the Thompson's group is amenable or not. The Thompson's monoid can be written as*

$$F^+ = \langle x_0, x_1, \cdots | x_n x_k = x_k x_{n+1}, k < n \rangle .$$

*The Thompson's monoid embeds injectively in the Thompson group, and it is a right LCM semigroup [44] (it follows from the discussion after [44, Lemma 6.32] that every constructible right ideal of $F^+$ is principal and thus it has the right LCM property).*

**Example 2.1.14.** *For an Artin monoid $A_M^+$ that is neither right-angled nor finite type, it is known that $A_M^+$ embeds injectively inside $A_M$ [50]. It is an open question on whether $(A_M, A_M^+)$ forms a quasi-lattice ordered group. However, it is known that $A_M^+$ is a right LCM semigroup.*

*Notice that being a right LCM semigroup only requires that every finite subset $F \subset A_M^+$ with an upper bound to have a least upper bound. Being a quasi-lattice ordered group requires that every finite subset $F \subset A_M$ with an upper bound to have a least upper bound. In general, right LCM is a much easier condition to check.*

In [13], it is shown that the Zappa-Szép product of semigroups provide a way to construct a rich class of right LCM semigroups. Let $U, A$ be two unital semigroup with identities $e_A, e_U$ respectively. Suppose there are two maps $U \times A \to U$ by $(u, a) \to a \cdot u$ and $U \times A \to A$ by $(u, a) \to a|_u$ that satisfy:

$$\begin{array}{ll}
(B1) e_A \cdot u = u; & (B5) a \cdot (uv) = (a \cdot u)(a|_u \cdot v); \\
(B2) (ab) \cdot u = a \cdot (b \cdot u); & (B6) a|_{uv} = (a|_u)|_v; \\
(B3) a \cdot e_U = e_U; & (B7) e_A|u = e_A; \\
(B4) a|_{e_U} = a; & (B8) (ab)|u = a|_{b \cdot u} b|_u.
\end{array}$$

Then the external Zappa-Szép product $U \bowtie A$ is the Cartesian product $U \times A$ with multiplication defined by

$$(u, a)(v, b) = (u(a \cdot v), (a|_v)b).$$

This allows us to build more right LCM semigroups from existing ones.

**Lemma 2.1.15** (Lemma 3.3, [13])**.** *Suppose $U, A$ are left cancellative semigroups with maps $(a, u) \to a \cdot u$ and $(a, u) \to a|_u$ that defines a Zappa-Szép product $U \bowtie A$. Suppose $U$ is a right LCM semigroup, and the set of constructible right ideals of $A$ is totally ordered by inclusion, and $u \to a \cdot u$ is a bijection from $U$ to $U$ for each $a \in A$. Then $U \bowtie A$ is a right LCM semigroup.*

**Example 2.1.16.** *Zappa-Szép products provide more examples of right LCM semigroups.*

1. *Baumslag-Solitar monoids form another class of quasi-lattice ordered groups recently studied in [65, 15]. For $n, m \geq 1$, the Baumslag-Solitar monoid $B_{n,m}$ is the monoid generated by $a, b$ with the relation $ab^n = b^m a$. It is pointed out in [13, Section 3.1] that they are the Zappa-Szép product of*

$$U = \langle e, a, ba, \cdots, b^{m-1}a \rangle, \text{ and } A = \langle e, b \rangle.$$

2. *The semigroup* $\mathbb{N} \rtimes \mathbb{N}^{\times}$ *where*

$$(x,a)(y,b) = (x+qy, ab).$$

*One can similarly define* $\mathbb{Q} \rtimes \mathbb{Q}_{+}^{\times}$*. It is known that the pair* $(\mathbb{Q} \rtimes \mathbb{Q}_{+}^{\times}, \mathbb{N} \rtimes \mathbb{N}^{\times})$ *is quasi-lattice ordered [38, Proposition 2.1]. It is also shown in [13, Section 3.2] that this semigroup is a Zappa-Szép product.*

3. *One can construct a right LCM semigroup that is not quasi-lattice ordered using Zappa-Szép product. Take* $U = \mathbb{N}^{\times}$ *and* $A = \mathbb{T}$*, and let* $a \cdot u = u$*,* $a|_u = a^u$ *for all* $a \in A, u \in U$*. Their Zappa-Szép product can be described as*

$$(n, e^{i\alpha})(m, e^{i\beta}) = (nm, e^{i(m\alpha+\beta)}).$$

*One can easily verify that* $U \bowtie A$ *is a right LCM semigroup using Lemma 2.1.15. In* $U \bowtie A$*,* $(1,1)$ *is the identity. Moreover, the set of invertible elements consists of* $(1, e^{i\alpha})$*, where the inverse of* $(1, e^{i\alpha})$ *is* $(1, e^{-i\alpha})$*. Since it has non-trivial invertible elements,* $U \bowtie A$ *cannot be a quasi-lattice ordered semigroup since it is not a positive cone.*

We now briefly discuss a few important properties of right LCM semigroups which will be useful later. For the rest of this section, we fix a right LCM semigroup $P$.

Let $a \in P$ and let $F \subset P$ be a finite subset. Denote $a \cdot F = \{a \cdot p : p \in F\}$. If $bP \supseteq \bigcap_{x\in F} xP$, we often write $b^{-1}\vee F = \{b^{-1}r : r \in \vee F\}$. Notice that since $bP \supseteq \bigcap_{x\in F} xP$, for each $r \in \vee F$, $bP \supseteq rP$ and $r = bp$ for some $p \in P$. This implies that $b^{-1}r \in P$ and $b^{-1} \vee F \subset P$, even though $b^{-1}$ is not part of the semigroup.

**Lemma 2.1.17.** *Let* $a \in P$ *and* $F \subset P$ *be a finite subset,* $\vee(a \cdot F) = a \cdot \vee F$*.*

*Proof.* It suffices to show $\bigcap_{x\in F} axP = a \cdot \bigcap_{x \in F} xP$. The containment $\supseteq$ is obvious. For the $\subseteq$ direction, take $r \in \bigcap_{x\in F} axP$ and let $F = \{x_1, \cdots, x_n\}$. We can find $p_1, \cdots, p_n \in P$ so that $r = ax_ip_i$. By the left cancellative property, $x_ip_i = x_jp_j$ for all $i,j$, and thus $r \in a \cdot \bigcap_{x\in F} xP$. □

Let $F$ be a finite subset of $P$ and $x \in \vee F$. Consider the set $x \vee y$ for some $y \in P$. Notice for any $s \in \vee F$, $x = su$ for some invertible element $u \in P^*$. Therefore, $xP = suP = sP$ and thus

$$x \vee y = \{r : rP = xP \cap yP\} = \{r : rP = sP \cap yP\}.$$

Therefore, $x \vee y$ is independent on the choice of $x \in \vee F$. For simplicity, we shall write it as $(\vee F) \vee y$.

**Lemma 2.1.18.** *Let $F_1, F_2 \subset P$ be two finite sets. Then*

$$\vee(F_1 \cup F_2) = (\vee F_1) \vee (\vee F_2)\,.$$

*Proof.* Fix $s_i \in \vee F_i$, we have

$$\begin{aligned}(\vee F_1) \vee (\vee F_2) &= s_1 \vee s_2 \\ &= \{r : rP = s_1 P \cap s_2 P\} \\ &= \{r : rP = \Big(\bigcap_{x \in F_1} xP\Big) \cap \Big(\bigcap_{x \in F_2} xP\Big)\} \\ &= \{r : rP = \bigcap_{x \in F_1 \cup F_2} xP\} \\ &= \vee(F_1 \cup F_2).\end{aligned}$$

The argument still works when one of $\vee F_i = \emptyset$. □

**Lemma 2.1.19.** *Let $p_1, \cdots, p_n \in P$ and $a \in P$. Let $F_1 = \{p_1 \cdot a, p_2, \cdots, p_n\}$ and $F_2 = \{a, p_1^{-1}(p_1 \vee p_2), \cdots, p_1^{-1}(p_1 \vee p_n)\}$. Then*

$$\vee F_1 = p_1 \cdot \vee F_2.$$

*Proof.* Take $s_i \in p_1 \vee p_i$ for all $2 \leq i \leq n$. Since $s_i \in p_1 P$, $p_1^{-1} s_i \in P$ for all $i$.

If $s_i' \in p_1 \vee p_i$, then $s_i = s_i' u$ for some invertible $u$, and thus $s_i P = s_i' P$. Therefore, $\vee F_2 = \vee\{a, p_1^{-1} s_i\}$. Hence, by Lemma 2.1.17,

$$\begin{aligned}p_1 \cdot \vee F_2 &= p_1 \cdot \vee\{a, p_1^{-1} s_i\} \\ &= \vee\big(p_1 \cdot \{a, p_1^{-1} s_i\}\big) \\ &= \vee\{p_1 a, s_i\}\end{aligned}$$

But $s_i P = p_1 P \cap p_i P$ since $s_i \in p_1 \vee p_i$. Therefore,

$$\begin{aligned}p_1 \cdot \vee F_2 &= \vee\{p_1 a, s_i\} \\ &= \big\{r : rP = p_1 a P \cap \big(\bigcap_{i=2}^{n} s_i P\big)\big\} \\ &= \big\{r : rP = p_1 a P \cap \big(\bigcap_{i=2}^{n} p_1 P \cap p_i P\big)\big\} \\ &= \big\{r : rP = p_1 a P \cap \big(\bigcap_{i=1}^{n} p_i P\big)\big\}\end{aligned}$$

Notice that $p_1 aP \subseteq p_1 P$, and thus

$$\left\{r : rP = p_1 aP \cap \big(\bigcap_{i=1}^{n} p_i P\big)\right\} = \left\{r : rP = p_1 aP \cap \big(\bigcap_{i=2}^{n} p_i P\big)\right\} = \vee F_1. \quad \square$$

### 2.1.4 Graph Product of Semigroups

Let $\Gamma = (V, E)$ be a countable simple undirected graph (i.e. the vertex set $V$ is countable, and there is no 1-loop or multiple edges in the graph). Suppose $P = (P_v)_{v \in V}$ is a countable collection of right LCM semigroups. The graph product $\Gamma_{v \in V} P_v$ is the semigroup defined by taking the free product $*_{v \in V} P_v$ modulo the relation $p \in P_v$ commutes with $q \in P_u$ whenever $(u, v)$ is an edge in the graph $\Gamma$. For simplicity, we shall denote $P_\Gamma = \Gamma_{v \in V} P_v$.

The graph product of groups was first studied in Green's thesis [30]. Subsequently, it was used to construct new quasi-lattice ordered groups [17]. A graph product of quasi-lattice ordered groups is also quasi-lattice ordered [17, Theorem 10]. This is generalized to graph products of right LCM semigroups. A graph product of right LCM semigroups is still right LCM [25, Theorem 2.6] (though the original statement concerns left LCM semigroups, this can be easily translated into right LCM semigroups).

Given $x \in P_\Gamma$, if we can write $x = x_1 x_2 \cdots x_n$ where each $e \neq x_j \in P_{v_j}$, this is called an expression of $x$. Each $x_j$ is called a syllable in the expression. For $e \neq p \in \bigcup_{v \in V} P_v$, let $I(p) = v$ if $p \in P_v$.

Let $x = x_1 x_2 \cdots x_n$ be an expression of $x$. Suppose $I(x_j)$ is adjacent to $I(x_{j+1})$, then $x_j x_{j+1} = x_{j+1} x_j$ and thus we can write

$$x = x_1 \cdots x_{j-1} x_{j+1} x_j x_{j+2} \cdots x_n.$$

This is called a shuffle of $x$. Two expressions of $x$ are called shuffle equivalent if one expression can be obtained from the other via finitely many shuffles.

In the case when $I(x_j) = I(x_{j+1})$, we can let $x_j' = x_j x_{j+1}$ and write

$$x = x_1 \cdots x_{j-1} x_j' x_{j+2} \cdots x_n.$$

This is called an amalgamation.

An expression $x = x_1 \cdots x_n$ is called a reduced expression for $x$ if it is not shuffle equivalent to an expression that admits an amalgamation. Equivalently, this implies whenever $I(p_i) = I(p_j)$ for some $i < j$, there exists $i < k < j$ so that $I(p_k)$ is not adjacent to $I(p_i)$.

A result of Green [30] states that every element $x$ has a reduced expression, and any two reduced expression of $x$ are shuffle equivalent. Therefore, one can define $\ell(x)$ to be the number of syllables in a reduced expression of $x$. $\ell(x)$ is the least number of syllables in an expression of $x$. By convention, if $x = e$, $\ell(x) = 0$.

Given a reduced expression $x = x_1 x_2 \cdots x_n$, a syllable $x_i$ is called an initial syllable if we can shuffle this reduced expression as $x = x_i x_2' \cdots x_n'$. Notice that we can shuffle $x_i$ to the front if and only if $x_i$ commutes with all the syllables $x_1, \cdots, x_{i-1}$. Therefore, if $x_i, x_j$ are two distinct initial syllables of $x$, they have to commute. We call a vertex $v$ an initial vertex of $x$ if there exists an initial syllable $x_i$ of $x$ with $I(x_i) = v$.

It is clear that when $x = x_1 \cdots x_n$, $x_1$ is always an initial syllable of $x$ and $v = I(x_1)$ is an initial vertex. Moreover, even if the expression $x = x_1 \cdots x_n$ is not a reduced expression, $I(x_1)$ is still an initial vertex of $x$ (as long as $x_1 \neq e$). This follows from the fact that $y = x_2 \cdots x_n$ admits a reduced expression $y_1 \cdots y_k$, and $x = x_1 \cdot y_1 \cdots y_k$. Either $v$ is not an initial vertex of $y$ and $x_1$ is an initial syllable, or $v$ is an initial vertex of $y$ and $x_1$ amalgamate with this initial vertex and form an initial syllable from $P_v$.

The graph product of right LCM semigroups has some nice properties. Let us fix a simple graph $\Gamma = (V, E)$ and a collection of right LCM semigroups $(P_v)_{v \in V}$. Let their graph product be $P_\Gamma$. The next two lemmas are directly taken from [25].

**Lemma 2.1.20** ([25, Lemma 2.5])**.** *Let $e \neq p \in P_u$ and $e \neq q \in P_v$ where $(u, v) \in E$. Then*

$$pP_\Gamma \cap qP_\Gamma = pqP_\Gamma.$$

**Lemma 2.1.21** ([25, Lemma 2.7])**.** *Let $x, y \in P_v$ for some $v \in V$. Then*

1. *$xP_v \cap yP_v = \emptyset$ if and only if $xP_\Gamma \cap yP_\Gamma = \emptyset$.*
2. *If $xP_v \cap yP_v = zP_v$ (i.e. $z \in x \vee y$), then $xP_\Gamma \cap yP_\Gamma = zP_\Gamma$.*

Lemma 2.1.20 implies that for $e \neq p \in P_u$ and $e \neq q \in P_v$ where $(u, v) \in E$, $pq \in p \vee q$. Following the proof of [25, Lemma 2.5], one can deduce that this is true for more than 2 vertices. Recall a finite subset $W \subset V$ is called a clique if every two vertices in $W$ are adjacent in $\Gamma$.

**Lemma 2.1.22.** *If $W \subseteq V$ is a clique in $\Gamma$, and $e \neq p_v \in P_v$ for all $v \in W$. Then $\prod_{v \in W} p_v \in \vee\{p_v : v \in W\}$. In other words,*

$$\Big(\prod_{v \in W} p_v\Big) P_\Gamma = \bigcap_{v \in W} p_v P_\Gamma.$$

**Example 2.1.23.** *The graph product is a useful tool in constructing new semigroups.*

1. *In the case when the graph* $\Gamma$ *contains no edges, the graph product is simply the free product.*
2. *In the case when the graph* $\Gamma$ *is a complete graph (i.e. there is an edge between any distinct pair of vertices), the graph product is simply the direct sum.*
3. *When each semigroup* $P_v = \mathbb{N}$*, the graph product of* $\mathbb{N}$ *is precisely the corresponding right-angled Artin monoid.*

## 2.2 Dilation Theorems

Since Sz.Nagy's celebrated dilation theorem, dilation has become an active area research in operator theory and operator algebra. This section gives a brief survey of many dilation theorems in the literature.

### 2.2.1 Dilation of Commuting Contractions

Ando first extended Sz.Nagy's result to two commuting contractions.

**Theorem 2.2.1** (Ando). *For a pair of commuting contractions* $T_1, T_2 \in \mathcal{B}(\mathcal{H})$*, there exists a pair of commuting isometries* $V_1, V_2 \in \mathcal{B}(\mathcal{K})$ *on a larger Hilbert space* $\mathcal{K} \supset \mathcal{H}$*, so that for any* $n_1, n_2 \geq 0$*,*

$$P_{\mathcal{H}} V_1^{n_1} V_2^{n_2}\big|_{\mathcal{H}} = T_1^{n_1} T_2^{n_2}.$$

Isometric dilations are important due to a theorem of Ito where he proved isometries can be further dilated to unitaries ([32], see also [52, Theorem 5.1]).

**Theorem 2.2.2** (Itô). *For* $k$ *commuting isometries* $T_1, \cdots, T_k \in \mathcal{B}(\mathcal{H})$*, there exists* $k$ *commuting unitaries* $U_1, \cdots, U_k \in \mathcal{B}(\mathcal{K})$ *on a larger Hilbert space* $\mathcal{K} \supset \mathcal{H}$*, so that for any* $n_1, \cdots, n_k \geq 0$*,*

$$P_{\mathcal{H}} U_1^{n_1} \cdots U_k^{n_k}\big|_{\mathcal{H}} = T_1^{n_1} \cdots T_k^{n_k}.$$

Therefore, whenever $T_1, \cdots, T_k$ have commuting isometric dilations, we have for every polynomial $p$ in $k$ variables,

$$p(T_1, \cdots, T_k) = P_{\mathcal{H}} p(U_1, \cdots, U_k)\big|_{\mathcal{H}}.$$

Hence,

$$\|p(T_1, \cdots, T_k)\| \leq \|p(U_1, \cdots, U_k)\| = \|p\|_{\mathbb{D}^k,\infty}.$$

This is known as the von-Neumann inequality. This still holds true if we take $p$ to be any matrix-valued polynomial. A theorem of Arveson [5] stated that $T_1, \cdots, T_k$ have isometric dilation if and only if the matrix-valued von-Neumann inequality holds true for every matrix-valued polynomial $p$.

However, Parrott found a triple of commuting contractions $T_1, T_2, T_3$ that fails to satisfy the scalar valued von-Neumann inequality ([51], see also [71]), and thus fails to be simultaneously dilated to a triple of commuting isometries. Therefore, for a family of three or more commuting contractions, some extra condition is necessary to guarantee commuting isometric dilations.

Given a contraction $T$, there is a minimal isometric dilation $V$ for $T$ that has the form

$$V = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix}$$

Therefore, the minimal Sz.Nagy dilation also satisfies

$$P_{\mathcal{H}} V^{*n}\big|_{\mathcal{H}} = T^{*n}.$$

for all $n \geq 1$. This motivated Brehmer to consider a stronger type of dilation for commuting contractions $T_1, \cdots, T_k$. For $m = (m_i) \in \mathbb{N}^k$, we let $T^m$ to be the product of $T_i^{m_i}$. Since $T_i$ are commuting, the order of multiplication does not matter. Now, for $n = (n_i) \in \mathbb{Z}^k$, denote $n^+ = (\max\{n_i, 0\})$ and $n^- = (\max\{-n_i, 0\})$. Brehmer considered when this family $T$ can be dilated to a commuting family of isometries $V$ so that for every $n \in \mathbb{Z}^k$,

$$P_{\mathcal{H}} V^{*n^-} V^{n^+}\big|_{\mathcal{H}} = T^{*n^-} T^{n^+}.$$

He called such dilation $V$ a regular dilation for $T$. Brehmer showed that having regular dilation is equivalent to certain operator are positive.

**Theorem 2.2.3** (Brehmer). *Let $\{T_1, \cdots, T_k\}$ be a family of commuting contractions. For a finite set $U \subset \{1, \cdots, k\}$, denote $T_U = \prod_{i \in U} T_i$. Then, $T$ has a regular dilation if and only if for any finite $W$, the operator*

$$\sum_{U \subseteq W} (-1)^{|V|} T_U^* T_U \geq 0. \tag{2.1}$$

As an application, Brehmer showed the following result:

**Corollary 2.2.4** (Brehmer). *Let $\{T_1, \cdots, T_k\}$ be a family of commuting contractions. Then $T$ has a regular dilation if:*

1. *$T$ is doubly commuting, meaning for all $i \neq j$, $T_i$ commutes with both $T_j$ and $T_j^*$. Or,*

2. *$T$ is a column contraction, meaning $\sum_{i=1}^k T_i^* T_i \leq I$.*

Dually, we can define $*$-regular dilation of $T$ to be an isometric representation $V$ so that for all $n \in \mathbb{Z}^k$,

$$P_{\mathcal{H}} V^{*n^-} V^{n^+}\big|_{\mathcal{H}} = T^{n^+} T^{*n^-}.$$

The role of $*$-regular dilation has not been studied much. It is shown in [29, Theorem 1] that a pair of commuting contractions $T_1, T_2$ have a $*$-regular dilation if and only if there exists a $*$-regular dilation $V_1, V_2$ of $T$ that are $*$-commuting. However, their proof requires a Wold-decomposition of $*$-commuting isometries that is hard to generalize to arbitrary commuting contractions. We study $*$-regular dilation from a different approach and establish an analogue of this result in Theorem 4.5.5 and Theorem 5.1.7.

### 2.2.2 Dilation of Non-commuting Contractions

Along another fruitful path to generalize Sz.Nagy dilation, people started to look at contractions that are not commuting. Frazho considered a pair of non-commuting contractions $T_1, T_2$ that satisfied $T_1 T_1^* + T_2 T_2^* \leq I$. He showed they can be dilated to non-commuting isometries $V_1, V_2$ that satisfy $V_1 V_1^* + V_2 V_2^* \leq I$. Bunce further extended Frazho's result to a finite family $T_1, \cdots, T_k$, and finally Popescu established the case for $k = \infty$. This result is now known as the Frazho-Bunce-Popescu dilation.

**Theorem 2.2.5** (Frazho-Bunce-Popescu). *Suppose $T_1, \cdots, T_k \in \mathcal{B}(\mathcal{H})$ ($k \in \mathbb{N} \bigcup \{\infty\}$) satisfy*

$$\sum_{i=1}^k T_i T_i^* \leq I.$$

*Then, we can find isometries $V_1, \cdots, V_k \in \mathcal{B}(\mathcal{K})$ on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$, so that each $V_i$ dilates $T_i$ in the sense that*

$$P_{\mathcal{H}} V_i^n\big|_{\mathcal{H}} = T_i^n,$$

*and $V_i$ also satisfies*

$$\sum_{i=1}^{k} V_i V_i^* \leq I.$$

**Remark 2.2.6.** *The row contractive condition for isometries $V_1, \cdots, V_k$ is precisely equivalent of saying these isometries have orthogonal ranges.*

We can also consider a mixture of commuting and non-commuting contractions. For example, Popescu [55] considers a family of operators $\{T_{i,j} : 1 \leq i \leq k, 1 \leq j \leq n_i\}$ where for each fixed $i$, $\{T_{i,j} : 1 \leq j \leq n_i\}$ is a non-commutative family of row contractions, and for each $i_1 \neq i_2$, $T_{i_1,j_1}$ commutes with $T_{i_2,j_2}$ for all $1 \leq j_1 \leq n_{i_1}$ and $1 \leq j_2 \leq n_{j_2}$. He provided an equivalent condition for such families of contractions to be dilated to isometries that satisfies similar conditions. We shall discuss this in more detail in Section 4.6.

More generally, one can consider a simple graph $\Gamma$ on $n$ vertices that dictates the commutation relations of $n$ contractions, where two contractions $T_i, T_j$ commutes whenever $(i,j)$ is an edge of the graph. This family of contractions can be seen as a representation of the graph product of $\mathbb{N}$. Opela [49] showed that when the graph is acyclic, one can dilate these contractions into unitaries that satisfy the same commutation relations. This is an generalization of Ando-type dilation. In Chapter 4, we will study regular dilation for such family of contractions.

### 2.2.3 Dilation on Semigroups

Many dilation results can be viewed as dilating a contractive representation of a semigroup.

**Definition 2.2.7.** *Let $P$ be any semigroup, and consider a representation $T : P \to \mathcal{B}(\mathcal{H})$. A representation $V : P \to \mathcal{B}(\mathcal{K})$ on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ is an isometric dilation of $T$ if for any $p \in P$,*

$$P_{\mathcal{H}} V(p)\big|_{\mathcal{H}} = T(p).$$

A result of Sarason [60] states that $\mathcal{K}$ decomposes as $\mathcal{K} = \mathcal{H}_- \oplus \mathcal{H} \oplus \mathcal{H}^+$, so that under such decomposition, the isometric dilation $V(p)$ has the form:

$$V(p) = \begin{bmatrix} * & 0 & 0 \\ * & T(p) & 0 \\ * & * & * \end{bmatrix}$$

$V$ is called an extension of $T$ if $\mathcal{H}$ is invariant, in which case, $\mathcal{H}^+ = \{0\}$. $V$ is called a co-extension of $T$ if $\mathcal{H}$ is invariant for $V^*$, in which case $\mathcal{H}_- = \{0\}$.

$V$ is called minimal if

$$\mathcal{K} = \overline{\operatorname{span}}\{V(p)h : p \in P, h \in \mathcal{H}\}$$

When $V$ is minimal, $\mathcal{H}_-$ must be $\{0\}$ and thus $V$ is a co-extension of $T$. For each $p \in P$, we can write $V(p)$ as a $2 \times 2$ block matrix with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$:

$$V(p) = \begin{bmatrix} T(p) & 0 \\ * & * \end{bmatrix}.$$

Notice that when $V$ is a dilation of $T$, $\|T(p)\| \leq \|V(p)\| = 1$, and thus $T$ is always a contractive representation.

**Example 2.2.8.** *There have been studies of dilation theory on various types of semigroups.*

1. *The Sz.Nagy's dilation can be restated as saying that every contractive representation of the semigroup $\mathbb{N}$ has an isometric dilation. Mlak [46] extended Sz.Nagy's dilation to any totally ordered abelian semigroup, where he showed every contractive representations on such semigroup has an isometric dilation.*

2. *If we take $P = \{0, 2, 3, \cdots\}$ as a semigroup embedded inside $\mathbb{Z}$, it is shown in [23] that not every contractive representation of $P$ has an isometric dilation.*

3. *If we take $P$ to be the direct product of a family of totally ordered semigroups, Fuller [28] showed a contractive Nica-covariant representation of such semigroup has an isometric dilation.*

The problem of finding an isometric dilation for a contractive representation $T$ turns out to be equivalent to showing that a certain kernel satisfies a completely positive definite condition. Structures of completely positive definite kernels are studied in [54, 56], and we shall give a brief overview of these results.

Let $P$ be a unital semigroup. A *unital Toeplitz kernel* on $P$ is a map $K : P \times P \to \mathcal{B}(\mathcal{H})$ with the property that $K(e,e) = I$, $K(p,q) = K(q,p)^*$, and $K(ap, aq) = K(p,q)$ for all $a, p, q \in P$. We call such a kernel *completely positive definite* if for each $n \geq 1$, and any $p_1, \cdots, p_n \in P$ and $h_1, \cdots, h_n \in \mathcal{H}$, we have

$$\sum_{i,j=1}^{n} \langle K(p_i, p_j) h_j, h_i \rangle \geq 0.$$

Equivalently, this is saying that for each $n \geq 1$, the $n \times n$ operator matrix $[K(p_i, p_j)]$, viewed as an operator on $\mathcal{H}^n$, is positive. We shall abbreviate unital completely positive definite Toeplitz kernel as completely positive definite kernel.

Existence of a completely positive definite kernel is closely related to the existence of an isometric dilation. A classical result known as Naimark dilation theorem [47] can be restated as the following theorem ([56, Theorem 3.2]):

**Theorem 2.2.9.** *If $K : P \times P \to \mathcal{B}(\mathcal{H})$ is a completely positive definite kernel, then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and an isometric representation $V : P \to \mathcal{B}(\mathcal{K})$ so that*

$$K(p,q) = P_{\mathcal{H}} V(p)^* V(q)\big|_{\mathcal{H}} \text{ for all } p, q \in P.$$

*Moreover, there is a unique minimal dilation $V$, up to unitary equivalence, that satisfies*

$$\overline{\operatorname{span}}\{V(p)h : p \in P, h \in \mathcal{H}\} = \mathcal{K},$$

*and $\mathcal{H}$ is co-invariant for $V$. The minimal dilation $V$ is called the Naimark dilation of $K$.*

*Conversely, if $V : P \to \mathcal{B}(\mathcal{K})$ is a minimal isometric dilation of $T : P \to \mathcal{B}(\mathcal{H})$, then let*

$$K(p,q) = P_{\mathcal{H}} V(p)^* V(q)\big|_{\mathcal{H}}$$

*We have $K(p,q)$ is a completely positive definite Toeplitz kernel with $K(e,p) = T(p)$ for all $p \in P$.*

*Proof.* The proof of the theorem can be found in [56, Theorem 3.2]. However, it is worthwhile to briefly go over the proof since it explicitly constructs the minimal Naimark dilation that is useful later.

First let $\mathcal{K}_0 = P \otimes \mathcal{H}$ and define a degenerate inner product by

$$\left\langle \sum \delta_p \otimes h_p, \sum \delta_q \otimes k_q \right\rangle = \sum_{p,q} \langle K(q,p) h_p, k_q \rangle.$$

Let $\mathcal{N} = \{k \in \mathcal{K}_0 : \langle k, k \rangle = 0\}$ and $\mathcal{K}$ be the completion of $\mathcal{K}_0 / \mathcal{N}$ with respect to the inner product. $\mathcal{H}$ is naturally embedded in $\mathcal{K}$ as $\delta_e \otimes \mathcal{H}$. For each $p \in P$, define $V(p)\delta(q) \otimes h = \delta(pq) \otimes h$. One can check $V : P \to \mathcal{B}(\mathcal{K})$ is the minimal Naimark dilation of $T$.

For the converse, it is simple to check that $K$ is indeed a Toeplitz kernel. To show it is completely positive definite, take any $p_1, \cdots, p_n \in P$, the operator matrix

$$\begin{aligned} & [K(p_i, p_j)] \\ =& P_{\mathcal{H}^n} [V(p_i)^* V(p_j)] \big|_{\mathcal{H}^n} \\ =& P_{\mathcal{H}^n} \left( \begin{bmatrix} V(p_1)^* \\ \vdots \\ V(p_n)^* \end{bmatrix} \begin{bmatrix} V(p_1) & \cdots & V(p_n) \end{bmatrix} \right) \bigg|_{\mathcal{H}^n} \geq 0. \end{aligned}$$

Therefore, $K$ is a completely positive definite Toeplitz kernel. Moreover, $K(e,p) = P_{\mathcal{H}} V(p)\big|_{\mathcal{H}} = T(p)$ for all $p \in P$. □

Notice that in Theorem 2.2.9, if we set $p = e$, we get $K(e,q) = P_{\mathcal{H}} V(q)\big|_{\mathcal{H}}$. Assume now that $T : P \to \mathcal{B}(\mathcal{H})$ is a contractive representation. If we can find a completely positive definite kernel $K$ so that $K(e,q) = T(q)$ for all $q \in P$, then Theorem 2.2.9 gives us an isometric representation $V$ so that $T(q) = P_{\mathcal{H}} V(q)\big|_{\mathcal{H}}$. In other words, $V$ is an isometric dilation for $T$. Therefore, we reach the following conclusion:

**Corollary 2.2.10.** *Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation, for which there exists a completely positive definite kernel $K$ so that $K(e,q) = T(q)$. Then $T$ has an isometric dilation $V : P \to \mathcal{B}(\mathcal{K})$, which can be taken as minimal in the sense that*

$$\overline{\operatorname{span}}\{V(p)h : p \in P, h \in \mathcal{H}\} = \mathcal{K}.$$

Such a kernel $K$ may not always exist. Indeed, if $P = \mathbb{N}^3$, let $T$ send three generators to the three commuting contractions as in Parrott's example [51]. Such $T$ can never have an isometric dilation and thus there is no completely positive definite kernel $K$ so that $K(e,q) = T(q)$. Even when $T$ has an isometric dilation, it may be extremely hard to express $K$ in terms of $T$.

In many circumstances, we want to study the unitary dilation of a contractive representation, instead of isometric dilation. Given a representation $T : P \to \mathcal{B}(\mathcal{H})$ where $P$ embeds in a group $G$, we say a representation $U : G \to \mathcal{B}(\mathcal{K})$ on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ is a unitary dilation of $T$ if for any $p \in P$,

$$P_{\mathcal{H}} U(p)\big|_{\mathcal{H}} = T(p).$$

$U$ is called minimal if

$$\mathcal{K} = \overline{\operatorname{span}}\{U(g)h : g \in G, h \in \mathcal{H}\}.$$

Having an isometric dilation is often an intermediate step in obtaining a unitary dilation. For example, Itô's dilation theorem states that a family of commuting isometries can be dilated to commuting unitaries (Theorem 2.2.2). Laca further showed that an isometric representation of an Ore semigroup has a unitary dilation [36].

In this thesis, we mostly study isometric dilation instead of unitary dilation. The main advantage of isometric dilation is that it allows us to study certain property (namely the Nica-covariance in Section 2.3) that is impossible to study for unitary dilation. For example, take a row-contractive representation $T$ of $\mathbb{F}_2^+$. Frazho-Bunce-Popescu's dilation states that $T$ has an isometric row contractive dilation. However, it is impossible for any unitary dilation to be row contractive.

Closely related to the completely positive definite kernel is a concept called completely positive map on semigroups. Let $P$ be a semigroup embedded inside a group $G$, and a contractive map $T : P^{-1}P \to \mathcal{B}(\mathcal{H})$ is called a completely positive definite if for each $n \geq 1$ and any $p_1, \cdots, p_n \in P$, the operator matrix $[T(p_i^{-1}p_j)]$ is non-negative.

This is closely related to the concept of completely positive definite kernel. Indeed, given a completely positive definite Toeplitz kernel $K$, one can define a map $T : P^{-1}P \to \mathcal{B}(\mathcal{H})$ so that $T(p_i^{-1}p_j) = K(p_i, p_j)$. The Toeplitz condition guarantees that this map $T$ is well defined. The converse also holds true: for each completely positive definite map $T : P^{-1}P \to \mathcal{B}(\mathcal{H})$, one can simply define a kernel $K(p, q) = T(p^{-1}q)$. This kernel is always a completely positive definite Toeplitz kernel.

Similarly, if $G$ is a group, a contractive map $T : G \to \mathcal{B}(\mathcal{H})$ is called completely positive definite if for each $n \geq 1$ and for any $g_1, \cdots, g_n \in P$,

$$[T(g_i^{-1}g_j)] \geq 0.$$

Here, the map $T$ on the group $G$ need not be a representation of $G$. We have the following version of Naimark's dilation theorem for completely positive definite maps on semigroups and groups:

**Theorem 2.2.11.** *Let $P$ be a positive cone embedded inside a group $G$.*

1. *If $T : P^{-1}P \to \mathcal{B}(\mathcal{H})$ is a map that is completely positive definite, then there exists an isometric representation $V : P \to \mathcal{B}(\mathcal{K})$ so that for all $p \in P$,*

$$P_{\mathcal{H}} V(p)\big|_{\mathcal{H}} = T(p).$$

2. *If* $S : G \to \mathcal{B}(\mathcal{H})$ *is a map that is completely positive definite, then there exists a unitary representation* $U : G \to \mathcal{B}(\mathcal{K})$ *so that for all* $g \in G$*,*

$$P_{\mathcal{H}} U(g)\big|_{\mathcal{H}} = T(g).$$

In the case of a lattice ordered semigroup, the completely positive maps on the semigroup $P^{-1}P = G$ coincide with the completely positive map on the group $G$. Moreover, whether we consider $T : P^{-1}P \to \mathcal{B}(\mathcal{H})$ or $T : PP^{-1} \to \mathcal{B}(\mathcal{H})$ does not matter, as we see in the following Lemma.

**Lemma 2.2.12.** *Let* $S : G \to \mathcal{B}(\mathcal{H})$ *be a map and let* $n \geq 1$*, then the following are equivalent:*

1. $\left[S(g_i^{-1}g_j)\right]_{1\leq i,j\leq n} \geq 0$ *for any* $g_1, g_2, \cdots, g_n \in G$*;*
2. $\left[S(g_i g_j^{-1})\right]_{1\leq i,j\leq n} \geq 0$ *for any* $g_1, g_2, \cdots, g_n \in G$*;*
3. $\left[S(p_i^{-1}p_j)\right]_{1\leq i,j\leq n} \geq 0$ *for any* $p_1, p_2, \cdots, p_n \in P$*;*
4. $\left[S(p_i p_j^{-1})\right]_{1\leq i,j\leq n} \geq 0$ *for any* $p_1, p_2, \cdots, p_n \in P$*.*

*Proof.* Since $G$ is a group, by considering $g_i$ and $g_i^{-1}$, it is clear that (1) and (2) are equivalent. Statement (1) clearly implies statement (3), and conversely when statement (3) holds true, for any $g_1, \cdots, g_n \in G$, take $g = \vee_{i=1}^n (g_i)_-$. Denote $p_i = g \cdot g_i$ and notice that from our choice of $g$, $g \geq (g_i)_-$. Hence,

$$p_i = g \cdot (g_i)_-^{-1} (g_i)_+ \in P.$$

But notice that for each $i, j$, $p_i^{-1}p_j = g_i^{-1}g^{-1}gg_j = g_i^{-1}g_j$. Therefore,

$$\left[S(g_i^{-1}g_j)\right]_{1\leq i,j\leq n} = \left[S(p_i^{-1}p_j)\right]_{1\leq i,j\leq n} \geq 0.$$

Similarly, statements (2) and (4) are equivalent. □

## 2.3 Nica-Covariance Condition

The study of isometric Nica-covariant representations originated from Nica's work on certain representations of quasi-lattice ordered groups, as a generalization to the well-known Toeplitz-Cuntz algebras. Given a quasi-lattice ordered group $(G, P)$, an isometric representation $W : P \to \mathcal{B}(\mathcal{H})$ is Nica-covariant if for any $x, y$ with an upper bound,

$$W(x)W(x)^*W(y)W(y)^* = W(x \vee y)W(x \vee y)^*.$$

and $W(x)W(x)^*W(y)W(y)^* = 0$ if $x, y$ have no common upper bound.

Equivalently,

$$W(x)^*W(y) = \begin{cases} W(x^{-1}(x \vee y))W(y^{-1}(x \vee y))^*, \text{ if } x \vee y \in P \\ 0, \text{ if } x \vee y = \infty. \end{cases}$$

In the special case when $P$ is a lattice ordered semigroup, the Nica-covariance condition is equivalent to the property that $W_s, W_t^*$ commute whenever $s \wedge t = e$. Motivated from this observation, [21] first defined the contractive Nica-covariant representation of a lattice ordered semigroup.

**Definition 2.3.1.** *A contractive representation $T : P \to \mathcal{B}(\mathcal{H})$ is called a contractive Nica-covariant representation if for any $p, q$ with $p \wedge q = e$, $T(p)T(q)^* = T(q)^*T(p)$.*

Recall in a lattice ordered semigroup, whenever $p \wedge q = e$, we have $p, q$ commute and thus $T(p), T(q)$ actually $*$-commute.

It observed in [21] that contractive Nica-covariant representations have isometric dilations that are analogue of Brehmer's regular dilation. Recall that Brehmer defined a representation $T : \mathbb{N}^k \to \mathcal{B}(\mathcal{H})$ to have regular dilation if it has a dilation $V : \mathbb{N}^k \to \mathcal{B}(\mathcal{H})$, where for every $n \in \mathbb{Z}^k$,

$$T(n_-)^*T(n_+) = P_{\mathcal{H}}V(n_-)^*V(n_+)\big|_{\mathcal{H}}.$$

We can replace $(\mathbb{Z}^k, \mathbb{N}^k)$ by any lattice ordered group $(G, P)$ to define regular dilation on lattice ordered group. Every element $g$ in the lattice ordered group $G$ can be written as $g = (g_-)^{-1}g_+$. It is natural to replace $n_+, n_-$ by $g_+, g_-$.

**Definition 2.3.2.** *Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation of a lattice ordered semigroup. We say an isometric representation $V : P \to \mathcal{B}(\mathcal{K})$ is a regular dilation of $T$ if for any $g \in G$,*

$$T(g_-)^*T(g_+) = P_{\mathcal{H}}V(g_-)^*V(g_+)\big|_{\mathcal{H}}.$$

**Example 2.3.3.** *(Examples of Nica covariant representations)*

1. *On $(\mathbb{Z}, \mathbb{Z}_+)$, a contractive representation $T$ on $\mathbb{Z}_+$ only depends on $T_1 = T(1)$ since $T(n) = T_1^n$. This representation is always Nica-covariant since for any $s, t \geq 0$, $s \wedge t = 0$ if and only if one of $s, t$ is $0$. A well known result due to Sz.Nagy [67] shows that its extension to $\mathbb{Z}$ by $\tilde{T}(-n) = T^{*n}$ is completely positive definite and thus $T$ has regular dilation.*

2. *Similarly, any contractive representation of a totally ordered abelian group $(G, P)$ is Nica-covariant. Mlak [46] shows that such representations have regular dilations.*

3. *$(\mathbb{Z}^n, \mathbb{Z}_+^n)$, the finite Cartesian product of $(\mathbb{Z}, \mathbb{Z}_+)$ is a lattice ordered group. A representation $T$ on $\mathbb{Z}_+^n$ depends on $n$ contractions $T_1 = T(1, 0, \cdots, 0)$, $T_2 = T(0, 1, 0, \cdots, 0), \cdots$, $T_n = T(0, \cdots, 0, 1)$. Notice $T$ is Nica covariant if and only if $T_i, T_j$ $*$-commute whenever $i \neq j$. Such $T$ is often called doubly commuting. Brehmer's result implies doubly commuting contractive representations always have regular dilations.*

4. *For a lattice ordered group made from a direct product of totally ordered groups, Fuller [28] showed that their contractive Nica-covariant representations have regular dilations.*

A question posed in [21, Question 2.5.11] asks whether contractive Nica-covariant representations on abelian lattice ordered groups have regular dilations in general. For example, for $G = C_{\mathbb{R}}[0, 1]$ and $P$ equal to the set of non-negative continuous functions, there were no known results on whether contractive Nica-covariant representations have regular dilations on such semigroup. Little was known for the non-abelian lattice ordered groups. I was able to answer this question in [40] by giving an equivalent condition for a representation of lattice ordered semigroup to have regular dilation. We will cover these results in Chapter 3

Nica-covariant condition has since been generalized to other contexts. Nica-covariant representations have also been generalized to left cancellative semigroups by Xin Li [43] via constructible ideals. In the case of the right LCM semigroups, all constructible ideals are right principle ideals. Xin Li's generalization of Nica-covariant representations on right LCM semigroups can be interpreted as the following: an isometric representation $V : P \to \mathcal{B}(\mathcal{K})$ is called Nica-covariant if for any $p, q \in P$,

$$V(p)V(p)^*V(q)V(q)^* = \begin{cases} V(r)V(r)^*, r \in p \vee q \neq \emptyset \\ 0, p \vee q = \infty \end{cases}$$

Here, since $pP\bigcap qP=rP$, we can treat $r$ as a least common multiple of $p, q$. There might be many such least common multiples, but it is clear that if $r, r'$ are both least common multiples of $p, q$, then there exists an invertible $u$ with $r\cdot u=r'$. Denote $P^*$ the set of invertible elements in $P$. Since $V$ is a contractive representation, each $u\in P^*$ is represented by a unitary $V(u)$. Therefore,

$$V(r')V(r')^*=V(r)V(u)V(u)^*V(r)^*=V(r)V(r)^*.$$

So the Nica-covariance condition is indeed well-defined.

# Chapter 3

# Regular Dilation on Lattice Ordered Semigroups

Our first step in studying regular dilation arises from the study of lattice ordered semigroups. Regular dilation is found to be an important property when Davidson-Fuller-Kakariadis studied the $C^*$-envelope of certain semi-crossed products in [21]. In particular, it was an open question in [21] whether a contractive Nica-covariant representation of an abelian lattice ordered semigroup has a regular dilation.

Throughout this chapter, we fix a lattice ordered group $(G, P)$. We first work towards a characterization of contractive presentations of $P$ that have regular dilation (Theorem 3.2.1). This allows us to give an affirmative answer to the question posed in [21]. In fact every contractive Nica-covariant representation of any lattice ordered semigroup (not necessarily abelian) has a regular dilation (Theorem 3.3.1). Moreover, the minimal regular dilation is isometric Nica-covariant (Theorem 3.3.2). This gives us a little glimpse of the relation between regular dilation and isometric Nica-covariant dilation. As we shall see in later chapters, having isometric Nica-covariant dilation is in fact equivalent to having $*$-regular dilation.

We also define and study column and row contractive representations of lattice ordered semigroups. This generalizes a corollary of Brehmer's result that every commuting column contraction has a regular dilation (Theorem 3.3.5).

Finally, we notice that the characterization we establish is different from Brehmer's in the sense that Brehmer's condition involves the positivity of an operator whereas our condition involves the positivity of an operator matrix. To understand fully the relation between our characterization and Brehmer's condition, we study the matrix decomposition of certain operator matrix. It turns out that Brehmer's condition allows us to do a Cholesky decomposition of certain operator matrices (Proposition 3.4.4). This technique becomes an essential tool in the analysis of regular dilation in later chapters.

## 3.1 Regular Dilation

Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation of a lattice ordered semigroup. Recall from Definition 2.3.2, an isometric representation $V : P \to \mathcal{B}(\mathcal{K})$ is a regular dilation of $T$ if for every $g \in G$,

$$T(g_-)^*T(g_+) = P_\mathcal{H} V(g_-)^*V(g_+)\big|_\mathcal{H}.$$

Dually, we say $V$ is a $*$-regular dilation of $T$ if for any $g \in G$,

$$T(g_+)T(g_-)^* = P_\mathcal{H} V(g_-)^*V(g_+)\big|_\mathcal{H}.$$

For all $p, q \in P$ with $p \wedge q = e$, we can define a Toeplitz kernel $K(p, q) = T(p)^*T(q)$. Suppose $K$ is completely positive definite, then the Naimark dilation $V$ for $K$ is a regular dilation of $T$. Indeed, for all $g \in G$, the decomposition $g = g_-^{-1} g_+$ satisfies $g_-, g_+ \in P$ and $g_- \wedge g_+ = e$. Hence,

$$T(g_-)^*T(g_+) = K(g_-, g_+) = P_\mathcal{H} V(g_-)^*V(g_+)\big|_\mathcal{H}.$$

The kernel $K$ corresponds to a map $\tilde{T}$ on $G$ by $\tilde{T}(g) = T(g_-)^*T(g_+)$. $T$ has a regular dilation if and only if the map $\tilde{T} : G \to \mathcal{B}(\mathcal{H})$ is completely positive definite.

The definition of $*$-regular dilation will be very useful when we consider regular dilation of representations of other semigroups. For lattice ordered groups, $*$-regular and regular are closely related. Indeed, if $(G, P)$ is a lattice ordered semigroup, then $(G, P^{-1})$ naturally inherits a lattice order group structure. Here the partial order $\leq'$ on $(G, P^{-1})$ has $x^{-1} \leq' y^{-1}$ whenever $xy^{-1} \in P^{-1}$. By the normally of $P$, one can show that this is equivalent to $x \leq y$. Therefore, the lattice $\wedge', \vee'$ on $(G, P^{-1})$ satisfies $x^{-1} wedge' y^{-1} = (x \wedge y)^{-1}$ and $x^{-1} \vee' y^{-1} = (x \vee y)^{-1}$.

A representation $T : P \to \mathcal{B}(\mathcal{H})$ give raise to a dual representation $T^* : P^{-1} \to \mathcal{B}(\mathcal{H})$ where $T^*(p^{-1}) = T(p)^*$. Consider $g = g_+ g_-^{-1} = g_-^{-1}\left(g_+^{-1}\right)^{-1}$. Therefore, in the unique decomposition of $g$ as an element of $(G, P^{-1})$, the positive part is $g_-^{-1}$ and the negative part is $g_+^{-1}$. Define

$$\overline{T^*}(g) = T^*(g_-^{-1}) T^*(g_+^{-1})^*.$$

$T^*$ has a $*$-regular dilation if and only if $\overline{T^*}$ is completely positive definition. By the definition of $T^*$,

$$\overline{T^*}(g) = T^*(g_-^{-1})T^*(g_+^{-1})^* = T(g_-)^*T(g_+) = \tilde{T}(g).$$

Since $\overline{T^*}$ agrees with $\tilde{T}$ on $G$, $\overline{T^*}$ is completely positive if and only if $\tilde{T}$ is completely positive definite, which is equivalent to $T$ having a regular dilation. Therefore, we obtain the following Proposition.

**Proposition 3.1.1.** *Let $(G, P)$ be a lattice ordered group, and $T : P \to \mathcal{B}(\mathcal{H})$ be a representation and $T^*$ defined as above. Then the following are equivalent*

1. *$T$ has a regular dilation.*
2. *$T^*$ has a $*$-regular dilation.*
3. *$\tilde{T}(g) = T(g_-)^*T(g_+)$ is a completely positive definite map on $G$.*

Regular dilation is a stronger condition than isometric dilation. Not every isometric dilation has regular dilation.

**Example 3.1.2.** *It follows from Brehmer's theorem that a representation $T$ on $\mathbb{Z}_+^2$ has regular dilation if and only if $T_1 = T(e_1), T_2 = T(e_2)$ are contractions that satisfy*

$$I - T_1^*T_1 - T_2^*T_2 + (T_1T_2)^*T_1T_2 \geq 0.$$

*Take* $T_1 = T_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and notice,*

$$I - T_1^*T_1 - T_2^*T_2 + (T_1T_2)^*T_1T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

*Brehmer's result implies that $T$ is not regular. However, from Ando's theorem [4], any contractive representation on $\mathbb{Z}_+^2$ has a unitary dilation and thus is completely positive definite.*

## 3.2 Main Theorem

When $T : P \to \mathcal{B}(\mathcal{H})$ is a representation of a lattice ordered semigroup, we denote $\tilde{T}(g) = T(g^-)^* T(g^+)$. Recall that $T$ has regular dilation if $\tilde{T}$ is completely positive definite. We often say $T$ is regular (or $*$-regular) when $T$ has regular dilation (or $*$-regular dilation). The main result is the following necessary and sufficient condition for regularity:

**Theorem 3.2.1.** *Let $(G, P)$ be a lattice ordered group and $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Then $T$ has regular dilation if and only if for each $n \geq 1$ and for any $p_1, \cdots, p_n \in P$ and $g \in P$ where $g \wedge p_i = e$ for all $i = 1, 2, \cdots, n$, we have*

$$\left[T(g)^* \tilde{T}(p_i p_j^{-1}) T(g)\right] \leq \left[\tilde{T}(p_i p_j^{-1})\right]. \tag{$\star$}$$

**Remark 3.2.2.** *If we denote*

$$X = \left[\tilde{T}(p_i p_j^{-1})\right]$$

*and $D = \mathrm{diag}(T(g), T(g), \cdots, T(g))$, Condition ($\star$) is equivalent to saying that $D^* X D \leq X$. Notice that we make no assumption on $X \geq 0$. Indeed, it follows from the main result that Condition ($\star$) is equivalent to saying the representation $T$ has regular dilation, which in turn implies $X \geq 0$. Therefore, when checking Condition ($\star$), we may assume $X \geq 0$.*

**Remark 3.2.3.** *By setting $p_1 = e$ and picking any $g \in P$, Condition ($\star$) implies that $T(g)^* T(g) \leq I$, and thus $T$ must be contractive.*

The following Lemma is taken from [20, Lemma 14.13].

**Lemma 3.2.4.** *If $A, X, D$ are operators in $\mathcal{B}(\mathcal{H})$ where $A \geq 0$. Then a matrix of the form $\begin{bmatrix} A & A^{1/2} X \\ X^* A^{1/2} & D \end{bmatrix}$ is positive if and only if $D \geq X^* X$.*

Condition ($\star$) can thus be interpreted in the following equivalent form.

**Lemma 3.2.5.** *Condition ($\star$) is equivalent to for each $n \geq 1$ and for all $p_1, \cdots, p_n \in P$, $g \in P$ with $g \wedge p_i = e$, $\left[\tilde{T}(q_i q_j^{-1})\right] \geq 0$. Here, $q_1 = p_1 g, \cdots, q_n = p_n g$ and $q_{n+1} = p_1, \cdots, q_{2n} = p_n$.*

*Proof.* Let $X = [\tilde{T}(p_i p_j^{-1})] \geq 0$ and $D = \mathrm{diag}(T(g), T(g), \cdots, T(g))$. Notice by Lemma 2.1.5 that

$$(p_i g p_j^{-1})_+ = (p_i p_j^{-1})_+ g$$
$$(p_i g p_j^{-1})_- = (p_i p_j^{-1})_-,$$

and thus $\tilde{T}(p_i g p_j^{-1}) = \tilde{T}(p_i p_j^{-1})T(g)$. Therefore,

$$\left[\tilde{T}(q_i q_j^{-1})\right] = \begin{bmatrix} X & XD \\ D^*X & X \end{bmatrix}.$$

Lemma 3.2.4 implies that this matrix is positive if and only if $D^*XD \leq X$, which is Condition ($\star$). □

The following lemma will serve as a base case in the proof of the main result.

**Lemma 3.2.6.** *Let $(G, P)$ be a lattice ordered group, and $T$ be a representation on $P$ that satisfies Condition ($\star$). If $p_i \wedge p_j = e$ for all $i \neq j$, then $[\tilde{T}(p_i p_j^{-1})] \geq 0$.*

*Proof.* Let $q_1 = e, q_2 = p_1$ and for each $1 < m \leq n$, recursively define $q_{2^{m-1}+k} = p_m q_k$ where $1 \leq k \leq 2^{m-1}$. Since $T$ is contractive,

$$[\tilde{T}(q_i q_j^{-1})]_{1\leq i,j\leq 2} = \begin{bmatrix} I & \tilde{T}(q_1 q_2^{-1}) \\ \tilde{T}(q_2 q_1^{-1}) & I \end{bmatrix} \geq 0.$$

By Lemma 3.2.5, for each $m$, $[\tilde{T}(q_i q_j^{-1})]_{1\leq i,j\leq 2^m} \geq 0$. Notice that $q_{2^{m-1}} = p_m$ for each $1 \leq m \leq n$. Therefore, $[\tilde{T}(p_i p_j^{-1})]$ is a corner of $[\tilde{T}(q_i q_j^{-1})] \geq 0$, and thus must be positive. □

For arbitrary choices of $p_1, \cdots, p_n \in P$, the goal is to reduce it to the case where $p_i \wedge p_j = e$. The following lemma does the reduction.

**Lemma 3.2.7.** *Let $(G, P)$ be a lattice ordered group and let $T$ be a representation that satisfies Condition ($\star$).*

*Assume there exists $2 \leq k < n$ where for each $J \subset \{1, 2, \cdots, n\}$ with $|J| > k$, $\wedge_{j\in J} p_j = e$. Then let $g = \wedge_{j=1}^k p_j$ and $q_1 = p_1 g^{-1}, \cdots, q_k = p_k g^{-1}$, and $q_{k+1} = p_{k+1}, \cdots, q_n = p_n$. Then $[\tilde{T}(p_i p_j^{-1})] \geq 0$ if $[\tilde{T}(q_i q_j^{-1})] \geq 0$.*

*Proof.* Let us denote $X = [\tilde{T}(q_j q_i^{-1})] \geq 0$ and its lower right $(n-k)\times(n-k)$ corner to be $Y$. Notice first of all, when $i, j \in \{1, 2, \cdots, k\}$,

$$q_i q_j^{-1} = p_i g^{-1} g p_j^{-1} = p_i p_j^{-1}.$$

So the upper left $k \times k$ corner of $[\tilde{T}(q_i q_j^{-1})]$ and the lower right $(n-k)\times(n-k)$ corner of $X$ are both the same as those in $[\tilde{T}(p_i p_j^{-1})]$.

Now consider $i \in \{1, 2, \cdots, k\}$ and $j \in \{k+1, \cdots, n\}$. It follows from the assumption that $g \wedge p_j = \left(\wedge_{s=1}^{k} p_s\right) \wedge p_j = e$ and $g \leq p_i$. Therefore, we can apply Lemma 2.1.5 to get

$$\begin{aligned} (p_i g^{-1} p_j^{-1})_- &= (p_i p_j^{-1})_- \\ (p_i g^{-1} p_j^{-1})_+ &= (p_i p_j^{-1})_+ g^{-1}. \end{aligned}$$

Now $g \in P$, so that

$$\begin{aligned} T((q_i q_j^{-1})_+)T(g) &= T((p_i p_j^{-1})_+) \\ T((q_i q_j^{-1})_-) &= T((p_i p_j^{-1})_-). \end{aligned}$$

Hence,

$$\tilde{T}(q_i q_j^{-1})T(g) = \tilde{T}(p_i p_j^{-1}).$$

Similarly, for $i \in \{k+1, \cdots, n\}$, $j \in \{1, 2, \cdots, k\}$, we have

$$\tilde{T}(p_i p_j^{-1}) = T(g)^* \tilde{T}(q_j q_i^{-1}).$$

Now define $D = \operatorname{diag}(I, \cdots, I, T(g), \cdots, T(g))$ be the block diagonal matrix with $k$ copies of $I$ followed by $n-k$ copies of $T(g)$. Consider $DXD^*$: it follows immediately from the assumption that $D^*XD \geq 0$. We have,

$$D^*[\tilde{T}(q_i q_j^{-1})]D = \left[\begin{array}{ccc|c} \cdots & \cdots & \cdots & \vdots \\ \cdots & \tilde{T}(p_i p_j^{-1}) & \cdots & \tilde{T}(q_i q_j^{-1})T(g) \\ \cdots & \cdots & \cdots & \vdots \\ \hline \cdots & T(g)^* \tilde{T}(q_i q_j^{-1}) & \cdots & [T(g)^* \tilde{T}(p_i p_j^{-1}) T(g)] \end{array}\right] \geq 0.$$

It follows from previous computation that each entry in the lower left $(n-k) \times k$ corner and upper right $k \times (n-k)$ corner are the same as those in $[\tilde{T}(p_i p_j^{-1})]$. T, $DXD^*$ only differs from $[\tilde{T}(p_i p_j^{-1})]$ on the lower right $(n-k) \times (n-k)$ corner. It follows from Condition $(\star)$ that

$$[T(g)^* \tilde{T}(p_i p_j^{-1}) T(g)] \leq [\tilde{T}(p_i p_j^{-1})].$$

Therefore, the matrix remains positive when the lower right corner in $D^*XD$ is changed from $[T(g)^* \tilde{T}(p_i p_j^{-1}) T(g)]$ to $[\tilde{T}(p_i p_j^{-1})]$. The resulting matrix is exactly $[\tilde{T}(p_i p_j^{-1})]$, which must be positive. $\square$

Now the main result (Theorem 3.2.1) can be deduced inductively:

*Proof of Theorem 3.2.1* First assume that $T : P \to \mathcal{B}(\mathcal{H})$ is a representation that satisfies Condition $(\star)$, which has to be contractive (by Remark 3.2.3). The goal is to show for any $n$ elements $p_1, p_2, \cdots, p_n \in P$, the operator matrix $[\tilde{T}(p_i p_j^{-1})]$ is positive and thus $T$ has regular dilation. We proceed by induction on $n$.

For $n = 1$, $\tilde{T}(p_1 p_1^{-1}) = I \geq 0$.

For $n = 2$, we have,

$$[\tilde{T}(p_i p_j^{-1})] = \left[ \begin{array}{cc} I & \tilde{T}(p_1 p_2^{-1}) \\ \tilde{T}(p_2 p_1^{-1}) & I \end{array} \right].$$

Here, $\tilde{T}(p_2 p_1^{-1}) = \tilde{T}(p_1 p_2^{-1})^*$, and they are contractions since $T$ is contractive. Therefore, this $2 \times 2$ operator matrix is positive.

Now assume that there is an $N$ such that for any $n < N$, we have $[\tilde{T}(p_i p_j^{-1})]$ is positive for any $p_1, p_2, \cdots, p_n \in P$. Consider the case when $n = N$:

For arbitrary choices $p_1, \cdots, p_N \in P$, let $g = \wedge_{i=1}^N p_i$, and replace $p_i$ by $p_i g^{-1}$. By doing so, $p_i g^{-1} \left(p_j g^{-1}\right)^{-1} = p_i p_j^{-1}$, and thus they give the same matrix $[\tilde{T}(p_i p_j^{-1})]$. Moreover, $\wedge_{i=1}^n p_i g^{-1} = (\wedge_{i=1}^N p_i) g^{-1} = e$. Hence, without loss of generality, we may assume $\wedge_{i=1}^N p_i = e$.

Let $m$ be the smallest integer such that for all $J \subseteq \{1, 2, \cdots, N\}$ and $|J| > m$, we have $\wedge_{j \in J} p_j = e$. It is clear that $m \leq N - 1$. Now do induction on $m$:

For the base case when $m = 1$, we have $p_i \wedge p_j = e$ for all $i \neq j$. Lemma 3.2.6 tells that Condition $(\star)$ implies $[\tilde{T}(p_i p_j^{-1})] \geq 0$.

Now assume $[\tilde{T}(p_i p_j^{-1})] \geq 0$ whenever $m \leq M - 1 < N - 1$ and consider the case when $m = M$: For a subset $J \subseteq \{1, 2, \cdots, n\}$ with $|J| = M$, let $g = \wedge_{j \in J} p_j$ and set $q_j = p_j g^{-1}$ for all $j \in J$, and $q_j = p_j$ otherwise. Lemma 3.2.7 concluded that $[\tilde{T}(p_i p_j^{-1})] \geq 0$ whenever $[\tilde{T}(q_i q_j^{-1})] \geq 0$ and the sub-matrix $[\tilde{T}(p_i p_j^{-1})]_{i,j \notin J} \geq 0$.

Since $|\{1, 2, \cdots, N\} \backslash J| = N - M < N$, the induction hypothesis on $n$ implies that $[\tilde{T}(p_i p_j^{-1})]_{i,j \notin J} \geq 0$. Therefore, $[\tilde{T}(p_i p_j^{-1})] \geq 0$ whenever $[\tilde{T}(q_i q_j^{-1})] \geq 0$, and by dropping from $p_i$ to $q_i$, we may, without loss of generality, assume that $\wedge_{j \in J} p_j = e$. Repeat this process for all subsets $J \subset \{1, 2, \cdots, n\}$ where $|J| = M$, and with Lemma 2.1.6, we eventually reach a state when $\wedge_{j \in J} p_j = e$ for all $J \subseteq \{1, 2, \cdots, N\}$, $|J| = M$. But in such case, for all $|J| \geq M$, we have $\wedge_{j \in J} p_j = e$. Therefore, we are in a situation where $m \leq M - 1$. The result follows from the induction hypothesis on $m$.

Conversely, suppose that $T$ has regular dilation. Fix $g \in P$ and $p_1, p_2, \cdots, p_k \in P$ where $g \wedge p_i = e$ for all $i = 1, 2, \cdots, k$. Denote $q_1 = p_1 g, q_2 = p_2 g, \cdots, q_k = p_k g$, and $q_{k+1} = p_1, q_{k+2} = p_2, \cdots, q_{2k} = p_k$. It follows from regularity that $[\tilde{T}(q_i q_j^{-1})] \geq 0$, which is equivalent to Condition $(\star)$ by Lemma 3.2.5.□

## 3.3 Applications

There are two immediate corollaries of Brehmer's theorem. Brehmer showed that the family of commuting contractions $T_1, \cdots, T_n$ has a regular dilation automatically in the following two cases:

1. When $T_i$ are $*$-commuting. In other words, for every $i \neq j$, $T_i$ commutes with both $T_j, T_j^*$.
2. When $T_i$ is a column contraction. In other words,

$$\sum_{i=1}^{k} T_i^* T_i \leq I.$$

In the first case, $*$-commuting can be generalized as contractive Nica-covariant representations, defined by [21]. In the second case, we need to first define an analogue of column contraction in the context of lattice ordered groups.

### 3.3.1 Contractive Nica-covariant Representation

We first answer the open question in [21].

**Theorem 3.3.1.** *A contractive Nica-covariant representation of a lattice ordered group has regular dilation.*

*Proof.* Suppose $T$ is a contractive Nica-covariant representation of a lattice ordered group $(G, P)$. Let $p_1, \cdots, p_k \in P$ and $g \in P$ with $g \wedge p_i = e$ for all $i = 1, 2, \cdots, k$. Let $X = [\tilde{T}(p_i p_j^{-1})]$ and $D = \mathrm{diag}(T(g), T(g), \cdots, T(g))$. By Remark 3.2.2, we may assume $X \geq 0$.

Since for each $p_i, p_j \in P$, $\tilde{T}(p_i p_j^{-1}) = T(p_{i,j}^-)^* T(p_{i,j}^+)$ where $e \leq p_{i,j}^{\pm} \leq p_i, p_j$. Hence, $g \wedge p_{i,j}^{\pm} = e$ and thus $g$ commutes with $p_{i,j}^{\pm}$. Therefore $T(g)$ commutes with $T(p_{i,j}^+)$ because $T$ is a representation and it also commutes with $T(p_{i,j}^-)^*$ by the Nica-covariant condition. As a result, $T(g)$ commutes with each entry in $X$, and thus $D$ commutes with $X$. Similarly, $D^*$ commutes with $X$ as well.

By continuous functional calculus, since $X \geq 0$, we know $D, D^*$ also commutes with $X^{1/2}$. Hence, in such case,

$$D^* X D = D^* X^{1/2} X^{1/2} D = X^{1/2} D^* D X^{1/2} \leq X. \qquad \square$$

It was shown in [21, Proposition 2.5.10] that a contractive Nica-covariant representation on abelian lattice ordered groups can be dilated to an isometric Nica-covariant representation. Here, we shall extend this result to non-abelian case.

**Corollary 3.3.2.** *Let $(G,P)$ be a lattice ordered group. Any minimal isometric dilation $V : P \to \mathcal{B}(\mathcal{K})$ of a contractive Nica-covariant representation $T : P \to \mathcal{B}(\mathcal{H})$ is also Nica-covariant.*

*Proof.* Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive Nica-covariant representation. Theorem 3.3.1 implies that $T$ has regular dilation where $\tilde{T}$ is completely positive definite on the group $G$. Therefore, it has a minimal unitary dilation $U : G \to \mathcal{B}(\mathcal{L})$, which gives rise to a minimal isometric dilation $V : P \to \mathcal{B}(\mathcal{K})$. Here $\mathcal{K} = \bigvee_{p\in P} V(p)\mathcal{H}$ and $V(p) = P_{\mathcal{K}} U(p)|_{\mathcal{K}}$. Notice that $\mathcal{K}$ is invariant for $U$ and therefore, $P_{\mathcal{K}} U(p)^* U(q)|_{\mathcal{K}} = V(p)^* V(q)$ for any $p, q \in P$. In particular, if $p \wedge q = e$, $p, q \in P$, we have from the regularity that

$$\begin{aligned} T(p)^*T(q) &= P_{\mathcal{H}} U(p)^* U(q)|_{\mathcal{H}} \\ &= P_{\mathcal{H}} (P_{\mathcal{K}} U(p)^* U(q)|_{\mathcal{K}})|_{\mathcal{H}} \\ &= P_{\mathcal{H}} V(p)^* V(q)|_{\mathcal{H}}. \end{aligned}$$

Now let $s, t \in P$ be such that $s \wedge t = e$. First, we shall prove $V(s)^*V(t)|_{\mathcal{H}} = V(t)V(s)^*|_{\mathcal{H}}$: Since $\{V(p)h : p \in P, h \in \mathcal{H}\}$ is dense in $\mathcal{K}$, it suffices to show for any $h, k \in \mathcal{H}$ and $p \in P$,

$$\langle V(s)^*V(t)h, V(p)k\rangle = \langle V(t)V(s)^*h, V(p)k\rangle .$$

Start from the left,

$$\begin{aligned} &\langle V(s)^*V(t)h, V(p)k\rangle \\ &= \langle V(p)^*V(s)^*V(t)h, k\rangle = \langle V(sp)^*V(t)h, k\rangle \\ &= \left\langle V((sp\wedge t)^{-1}sp)^*V(sp\wedge t)^*V(sp\wedge t)V((sp\wedge t)^{-1}t)h, k\right\rangle \\ &= \left\langle V((sp\wedge t)^{-1}sp)^*V((sp\wedge t)^{-1}t)h, k\right\rangle \\ &= \left\langle T((sp\wedge t)^{-1}sp)^*T((sp\wedge t)^{-1}t)h, k\right\rangle . \end{aligned}$$

The last equality follows from $((sp\wedge t)^{-1}sp) \wedge ((sp\wedge t)^{-1}t) = e$ and thus,

$$T((sp\wedge t)^{-1}sp)^*T((sp\wedge t)^{-1}t) = P_{\mathcal{H}} V((sp\wedge t)^{-1}sp)^*V((sp\wedge t)^{-1}t)|_{\mathcal{H}}.$$

Since $s\wedge t = e$, Lemma 2.1.4 implies that $sp\wedge t = p\wedge t$. Notice $(p\wedge t)\wedge s \leq t\wedge s = e$, and thus by Property (4) of Lemma 2.1.2, $s$ commutes with $p\wedge t$. By the Nica-covariance of $T$, this also implies $T(s)^*$ commutes with $T((p\wedge t)^{-1}t)$. Put all these back to the equation:

$$\begin{aligned}
&\left\langle T((sp\wedge t)^{-1}sp)^*T((sp\wedge t)^{-1}t)h,k\right\rangle\\
&=\left\langle T(s(p\wedge t)^{-1}p)^*T((p\wedge t)^{-1}t)h,k\right\rangle\\
&=\left\langle T((p\wedge t)^{-1}p)^*T(s)^*T((p\wedge t)^{-1}t)h,k\right\rangle\\
&=\left\langle T((p\wedge t)^{-1}p)^*T((p\wedge t)^{-1}t)\left(T(s)^*h\right),k\right\rangle\\
&=\left\langle V((p\wedge t)^{-1}p)^*V((p\wedge t)^{-1}t)\left(T(s)^*h\right),k\right\rangle\\
&=\left\langle V((p\wedge t)^{-1}p)^*V((p\wedge t)^{-1}t)\left(V(s)^*h\right),k\right\rangle\\
&=\left\langle V(p)^*V(t)\left(V(s)^*h\right),k\right\rangle=\left\langle V(t)V(s)^*h,V(p)k\right\rangle.
\end{aligned}$$

Here we used the fact that $P_{\mathcal{H}}V(p)^*V(q)|_{\mathcal{H}}=T(p)^*T(q)$ whenever $p\wedge q=e$. Also, that $\mathcal{H}$ is invariant under $V(s)^*$, so that $T(s)^*h\in\mathcal{K}$ is the same as $V(s)^*h$.

Now to show $V(s)^*V(t)=V(t)V(s)^*$ in general, it suffices to show for every $p\in P$, $V(s)^*V(t)V(p)|_{\mathcal{H}}=V(t)V(s)^*V(p)|_{\mathcal{H}}$. Start with the left hand side and repeatedly use similar argument as above,

$$\begin{aligned}
&V(s)^*V(t)V(p)|_{\mathcal{H}}\\
&=V(s)^*V_{tp}|_{\mathcal{H}}=V((s\wedge tp)^{-1}s)^*V((s\wedge tp)^{-1}tp)|_{\mathcal{H}}\\
&=V(t(s\wedge p)^{-1}p)V((s\wedge p)^{-1}s)^*|_{\mathcal{H}}\\
&=V(t)V((s\wedge p)^{-1}s)^*V((s\wedge p)^{-1}p)|_{\mathcal{H}}=V(t)V(s)^*V(p)|_{\mathcal{H}}.
\end{aligned}$$

This finishes the proof. □

### 3.3.2 Column Contraction

We first generalizes the notion of row and column contraction to arbitrary lattice ordered groups.

**Definition 3.3.3.** *Let $T:P\to\mathcal{B}(\mathcal{H})$ be a contractive representation of a lattice ordered group $(G,P)$. $T$ is called* row contractive *if for each $n\geq 1$ and for any $p_1,\cdots,p_n\in P$ where $p_i\neq e$ and $p_i\wedge p_j=e$ for all $i\neq j$,*

$$\sum_{i=1}^n T(p_i)T(p_i)^*\leq I.$$

*Dually, $T$ is called* column contractive *if*

$$\sum_{i=1}^{n} T(p_i)^* T(p_i) \leq I.$$

*for any collection of such $p_i$.*

**Remark 3.3.4.** *Definition 3.3.3 indeed generalizes the notion of commuting row contractions: when the group is $(\mathbb{Z}^\Omega, \mathbb{Z}_+^\Omega)$ where $\Omega$ is countable, a representation $T : \mathbb{Z}_+^\Omega \to \mathcal{B}(\mathcal{H})$ is uniquely determined by its value on the generators $T_\omega = T(e_\omega)$. $T$ is called a commuting row contraction when $\sum_{\omega\in\Omega} T_\omega T_\omega^* \leq I$. Suppose $p_1, \cdots, p_k \in \mathbb{Z}_+^\Omega$ where $p_i \wedge p_j = 0$ for all $i \neq j$ and $p_i \neq 0$, each $p_i$ can be seen as a function from $\Omega$ to $\mathbb{Z}_+$ with finite support. Let $S_i \subseteq \Omega$ be the support of $p_i$, which is non-empty since $p_i \neq 0$. We have $S_i \bigcap S_j = \emptyset$ since $p_i \wedge p_j = 0$. For any $\omega_i \in S_i$, $w_i \leq p_i$. Since $T$ is contractive, $T(\omega_i)T(\omega_i)^* \geq T(p_i)T(p_i)^*$. Since $S_i$ are pairwise-disjoint, $\omega_i$ are distinct. Therefore, we get that*

$$\sum_{i=1}^{n} T(p_i)T(p_i)^* \leq \sum_{i=1}^{n} T(\omega_i)T(\omega_i)^* \leq I.$$

*and thus $T$ satisfies Definition 3.3.3. Hence, on $(\mathbb{Z}^\Omega, \mathbb{Z}_+^\Omega)$, the two definitions coincide.*

Our goal is to prove the following result:

**Theorem 3.3.5.** *A column contractive representation of a lattice ordered semigroup has a regular dilation. Consequently, a row contractive representation has a $*$-regular dilation.*

We shall proceed with a method similar to the proof of Theorem 3.2.1.

**Lemma 3.3.6.** *Let $T$ be a column contractive representation. Let $p_1, \cdots, p_n \in P$ and $g_1, \cdots, g_k \in P$ where $p_i \wedge p_{i'} = p_i \wedge g_j = g_j \wedge g_{j'} = e$ for all $1 \leq i \neq i' \leq n$ and $1 \leq j \neq j' \leq k$. Moreover, assume that $g_i \neq e$. Denote $X = [\tilde{T}(p_i p_j^{-1})]$ and $D_i = \operatorname{diag}(T(g_i), \cdots, T(g_i))$. Then,*

$$\sum_{i=1}^{k} D_i^* X D_i \leq X.$$

*Proof.* The statement is clearly true for all $k$ when $n = 1$. Now assuming it is true for all $k$ whenever $n < N$, and consider the case when $n = N$:

It is clear that when all of the $p_i$ are equal to $e$, then $X - \sum_{i=1}^{k} D_i^* X D_i$ is a $n \times n$ matrix whose entries are all equal to $I - \sum_{i=1}^{k} T(g_i)^* T(g_i) \geq 0$, and thus the statement is true. Otherwise, we may assume without loss of generality that $p_1 \neq e$. Let $q_1 = e$ and $q_2 = p_2, \cdots, q_n = p_n$. Denote $X_0 = [\tilde{T}(q_i q_j^{-1})]$ and $E = \mathrm{diag}(I, T(p_1), \cdots, T(p_1))$ be a $n \times n$ block diagonal matrix.

Denote $Y = [\tilde{T}(p_i p_j^{-1})]_{2 \leq i,j \leq n}$ and set $E_i = \mathrm{diag}(T(g_i), \cdots, T(g_i))$ be a $(n-1) \times (n-1)$ block diagonal matrix. Finally, set $E_{k+1} = \mathrm{diag}(T(p_1), \cdots, T(p_1))$ be a $(n-1) \times (n-1)$ block diagonal matrix.

From the proof of Theorem 3.2.1,

$$X = E^* X_0 E + \begin{bmatrix} 0 & 0 \\ 0 & Y - E_{k+1}^* Y E_{k+1} \end{bmatrix}.$$

Now $Y$ is a matrix of smaller size and thus by induction hypothesis,

$$\sum_{i=1}^{k+1} E_i^* Y E_i \leq Y.$$

Hence,

$$\begin{aligned} Y - E_{k+1}^* Y E_{k+1} &\geq \sum_{i=1}^{k} E_i^* Y E_i \\ &\geq \sum_{i=1}^{k} E_i^* (Y - E_{k+1}^* Y E_{k+1}) E_i. \end{aligned}$$

Also notice that $E$ commutes with $D_i$ and therefore, if $\sum_{i=1}^{k} D_i^* X_0 D_i \leq X_0$, we have

$$\begin{aligned} &\sum_{i=1}^{k} D_i^* X D_i \\ &= E^* \left( \sum_{i=1}^{k} D_i^* X_0 D_i \right) E + \begin{bmatrix} 0 & 0 \\ 0 & \sum_{i=1}^{k} E_i^* (Y - E_{k+1}^* Y E_{k+1}) E_i \end{bmatrix} \\ &\leq E^* X_0 E + \begin{bmatrix} 0 & 0 \\ 0 & Y - E_{k+1}^* Y E_{k+1} \end{bmatrix} = X. \end{aligned}$$

Hence, $\sum_{i=1}^{k} D_i^* X D_i \leq X$ if $\sum_{i=1}^{k} D_i^* X_0 D_i \leq X_0$. This reduction from $X$ to $X_0$ changes one $p_i \neq e$ to $e$, and therefore by repeating this process, we eventually reach a state where all $p_i = e$. □

Theorem 3.3.5 can be deduced immediately from the following Proposition and Theorem 3.2.1:

**Proposition 3.3.7.** *Let $T$ be a column contractive representation of a lattice ordered semigroup $P$. Let $p_1, \cdots, p_n \in P$ and $g_1, \cdots, g_k \in P$ where for all $1 \leq i \neq l \leq k$ and $1 \leq j \leq n$, $g_i \wedge p_j = e$ and $g_i \wedge g_l = e$. Assuming $g_i \neq e$ and denote $X = [\tilde{T}(p_i p_j^{-1})]$ and $D_i = \operatorname{diag}(T(g_i), \cdots, T(g_i))$. Then*

$$\sum_{i=1}^{k} D_i^* X D_i \leq X.$$

*In particular, $T$ satisfies Condition ($\star$) when $k = 1$.*

*Proof.* The statement is clear when $n = 1$. Assume it's true for $n < N$, and consider the case when $n = N$. Let $m$ be the smallest integer such that for all $J \subseteq \{1, 2, \cdots, N\}$ and $|J| > m$, $\wedge_{j \in J} p_j = e$. It was observed in the proof of Theorem 3.2.1 that $m \leq N - 1$. Proceed by induction on $m$:

In the base case when $m = 1$, $p_i \wedge p_j = e$ for all $i \neq j$, the statement is proved in Lemma 3.3.6. Assuming the statement is true for $m < M - 1 < N - 1$ and consider the case when $m = M$. For each $J \subseteq \{1, 2, \cdots, N\}$ with $|J| = M$ and $\wedge_{j=1}^{M} p_j = g \neq e$, denote $q_i = p_i$ when $i \notin J$ and $q_i = q_i g^{-1}$ when $i \in J$. Let $X_0 = [\tilde{T}(q_i q_j^{-1})]$ and $E$ be a block diagonal matrix whose $i$-th diagonal entry is $I$ when $i \notin J$ and $T(g)$ otherwise. Denote $Y = [\tilde{T}(q_i q_j^{-1})]_{i,j \notin J}$ and $E_i = \operatorname{diag}(T(g_i), \cdots, T(g_i))$ with $N - M$ copies of $T(g_i)$. Finally, let $E_{k+1} = \operatorname{diag}(T(g), \cdots, T(g))$ with $N - M$ copies of $T(g)$.

From the proof of Theorem 3.2.1, by assuming without loss of generality that $J = \{1, 2, \cdots, M\}$, we have

$$X = E^* X_0 E + \begin{bmatrix} 0 & 0 \\ 0 & Y - E_{k+1}^* Y E_{k+1} \end{bmatrix}.$$

Now $Y$ has a smaller size and thus by induction hypothesis on $n$,

$$\sum_{i=1}^{k+1} E_i^* Y E_i \leq Y.$$

and thus

$$\begin{aligned}Y - E_{k+1}^* Y E_{k+1} &\geq \sum_{i=1}^k E_i^* Y E_i \\ &\geq \sum_{i=1}^k E_i^* (Y - E_{k+1}^* Y E_{k+1}) E_i.\end{aligned}$$

Therefore, if $\sum_{i=1}^k D_i^* X_0 D_i \leq X_0$,

$$\begin{aligned}&\sum_{i=1}^k D_i^* X D_i \\ &= E^* \left( \sum_{i=1}^k D_i^* X_0 D_i \right) E + \begin{bmatrix} 0 & 0 \\ 0 & \sum_{i=1}^k E_i^* (Y - E_{k+1}^* Y E_{k+1}) E_i \end{bmatrix} \\ &\leq E^* X_0 E + \begin{bmatrix} 0 & 0 \\ 0 & Y - E_{k+1}^* Y E_{k+1} \end{bmatrix} = X.\end{aligned}$$

Hence, the statement is true for $p_i$ if it is true for $q_i$, where $\wedge_{j \in J} q_j = e$. Repeat the process until all such $|J| = M$ has $\wedge_{j \in J} p_j = e$, which reduces to a case where $m < M$. This finishes the induction. Notice Condition $(\star)$ is clearly true when $g = e$, and when $g \neq e$, it is shown by the case when $m = 1$. This finishes the proof. □

## 3.4 Brehmer's Condition

Brehmer's condition for regular dilation is quite different from the main result that we derived. Indeed, Brehmer's condition involves checking certain operator are positive, whereas condition $(\star)$ involves checking certain operator matrix inequalities. It is extremely beneficial to study how these two conditions relate. We found that Brehmer's condition allows us to derive a Cholesky decomposition of a certain operator matrix. This technique is an essential tool in the analysis of regular dilation on other semigroups.

Let $\{T_\omega\}_{\omega \in \Omega}$ be a family of commuting contractions, which leads to a contractive representation on $\mathbb{Z}_+^\Omega$ by sending each $e_\omega$ to $T_\omega$. For each $U \subseteq \Omega$, denote

$$Z_U = \sum_{V \subseteq U} (-1)^{|V|} T(e_V)^* T(e_V).$$

For example,

$$\begin{aligned}
Z_\emptyset &= I\\
Z_{\{1\}} &= I - T_1^*T_1\\
Z_{\{1,2\}} &= Z_{\{1\}} - T_2^*Z_{\{1\}}T_2 = I - T_1^*T_1 - T_2^*T_2 + T_2^*T_1^*T_1T_2\\
&\vdots
\end{aligned}$$

Brehmer's theorem stated that $T$ has regular dilation if and only if $Z_U \geq 0$ for any finite subset $U \subseteq \Omega$. We shall first transform Brehmer's condition into an equivalent form.

**Lemma 3.4.1.** *$Z_U \geq 0$ for each finite subset $U \subseteq \Omega$ if and only if for any finite set $J \subseteq \Omega$ and $\omega \in \Omega$, $\omega \notin J$,*

$$T_\omega^* Z_J T_\omega \leq Z_J.$$

*Proof.* Take any finite subset $J \subseteq \Omega$ and $\omega \in \Omega$, $\omega \notin J$.

$$\begin{aligned}
&Z_J - T_\omega^* Z_J T_\omega\\
&= \sum_{V\subseteq J} (-1)^{|V|} T(e_V)^* T(e_V) + \sum_{V\subseteq J} (-1)^{|V|+1} T_\omega^* T(e_V)^* T(e_V) T_\omega\\
&= \sum_{V\subseteq\{\omega\}\bigcup J, \omega\notin V} (-1)^{|V|} T(e_V)^* T(e_V) + \sum_{V\subseteq\{\omega\}\bigcup J, \omega\in V} (-1)^{|V|} T(e_V)^* T(e_V)\\
&= Z_{\{\omega\}\bigcup J}.
\end{aligned}$$

Therefore, $T_\omega^* Z_J T_\omega \leq Z_J$ if and only if $Z_{\{\omega\}\bigcup J} \geq 0$. This finishes the proof. □

A major tool is the following version of Douglas Lemma [22]:

**Lemma 3.4.2** (Douglas)**.** *For $A, B \in \mathcal{B}(\mathcal{H})$, $A^*A \leq B^*B$ if and only if there exists a contraction $C$ such that $A = CB$.*

As an immediate consequence of Lemma 3.4.2, $T_\omega^* Z_J T_\omega \leq Z_J$ is satisfied if and only if there is a contraction $W_{\omega,J}$ such that $Z_J^{1/2} T_\omega = W_{\omega,J} Z_J^{1/2}$. Therefore, it would suffices to find such contraction $W_{\omega,J}$ for each finite subset $J \subseteq \Omega$ and $\omega \in \Omega$, $\omega \notin J$. By symmetry, it would suffices to do so for each $J_n = \{1, 2, \cdots, n\}$ and $\omega_n = n + 1$. Without loss of generality, we shall assume that $\Omega = \mathbb{N}$.

Consider $\mathcal{P}(J_n) = \{U \subseteq J_n\}$, and denote $p_U = \sum_{i\in U} e_i \in \mathbb{Z}_+^\Omega$. Denote $X_n = [\tilde{T}(p_U - p_V)]$ where $U$ is the row index and $V$ is the column index.

**Lemma 3.4.3.** *Assuming $Z_J \geq 0$ for all $J \subseteq J_n$. Then for a fixed $F \subseteq J_n$, we have,*

$$\sum_{U \subseteq F} T_U^* Z_{F \setminus U} T_U = I.$$

*Proof.* We first notice that by definition, $Z_J = \sum_{U \subseteq J} (-1)^{|U|} T_U^* T_U$. Therefore,

$$\sum_{U \subseteq F} T_U^* Z_{F \setminus U} T_U = \sum_{U \subseteq F} \sum_{V \subseteq F \setminus U} (-1)^{|V|} T_{U \bigcup V}^* T_{U \bigcup V}.$$

For a fixed set $W \subseteq F$, consider the coefficient of $T_W^* T_W$ in the double summation. It appears in the expansion of every $T_U^* Z_{F \setminus U} T_U$, where $U \subseteq W$, and its coefficient in the expansion of such term is equal to $(-1)^{|W \setminus U|}$. Therefore, the coefficient of $T_W^* T_W$ is equal to

$$\sum_{U \subseteq W} (-1)^{|W \setminus U|} = \sum_{i=0}^{|W|} \binom{|W|}{i} (-1)^i.$$

This evaluates to 0 when $|W| > 0$ and 1 when $|W| = 0$, in which case, $W = \emptyset$ and $T_W = I$. $\square$

Now can now decompose $X_n = R_n^* R_n$ explicitly.

**Proposition 3.4.4.** *Assuming $Z_J \geq 0$ for all $J \subseteq J_n$. Define a block matrix $R_n$, whose rows and columns are indexed by $\mathcal{P}(J_n)$, by $R_n(U, V) = Z_{J_n \setminus U}^{1/2} T_{U \setminus V}$ whenever $V \subseteq U$ and 0 otherwise. Then $X_n = R_n^* R_n$*

*Proof.* Fix $U, V \subseteq J_n$, the $(U, V)$-entry in $X_n$ is $\tilde{T}(p_U - p_V) = T_{V \setminus U}^* T_{U \setminus V}$. Now the $(U, V)$-entry in $R_n^* R_n$ is equal to

$$\sum_{W \subseteq J_n} R_n(W, U)^* R_n(W, V).$$

It follows from the definition that $R_n(W, U)^* R_n(W, V) = 0$ unless $U, V \subseteq W$, and thus

$U \bigcup V \subseteq W$. Hence,

$$\begin{aligned}
&\sum_{W \in \mathcal{P}(J_n)} R_n(W,U)^* R_n(W,V) \\
&= \sum_{U \bigcup V \subseteq W} T^*_{W \backslash U} Z_{J_n \backslash W} T_{W \backslash V} \\
&= \sum_{U \bigcup V \subseteq W} T^*_{V \backslash U} T^*_{W \backslash (U \bigcup V)} Z_{J_n \backslash W} T_{W \backslash (U \bigcup V)} T_{W \backslash U} \\
&= T^*_{V \backslash U} \left( \sum_{U \bigcup V \subseteq W} T^*_{W \backslash (U \bigcup V)} Z_{J_n \backslash W} T_{W \backslash (U \bigcup V)} \right) T_{W \backslash U}.
\end{aligned}$$

If we denote $F = J_n \backslash (U \bigcup V)$ and $W' = W \backslash (U \bigcup V)$, since $U \bigcup V \subseteq W$, we have $J_n \backslash W = F \backslash W'$. Hence the summation becomes

$$\sum_{U \bigcup V \subseteq W} T^*_{W \backslash (U \bigcup V)} Z_{J_n \backslash W} T_{W \backslash (U \bigcup V)} = \sum_{W' \subseteq F} T^*_{W'} Z_{F \backslash W'} T_{W'},$$

which by Lemma 3.4.3 is equal to $I$. Therefore, the $(U,V)$-entry in $R_n^* R_n$ is equal to $T^*_{V \backslash U} T_{W \backslash U}$ and $X_n = R_n^* R_n$ ☐

**Remark 3.4.5.** *If we order the subsets of $J_n$ by cardinality and put larger sets first, then since $R_n(U,V) \neq 0$ only when $V \subseteq U$, $R_n$ becomes a lower triangular matrix. In particular, the row of $\emptyset$ contains exactly one non-zero entry, which is $Z_{J_n}^{1/2}$ at $(\emptyset, \emptyset)$.*

**Example 3.4.6.** *Let us consider the case when $n = 2$, and $J_2$ has 4 subsets $\{1,2\}$, $\{2\}$,$\{1\}$,$\emptyset$. Under this ordering,*

$$X_n = \begin{bmatrix} I & T_1 & T_2 & T_1T_2 \\ T_1^* & I & T_1^*T_2 & T_2 \\ T_2^* & T_2^*T_1 & I & T_1 \\ T_1^*T_2^* & T_2^* & T_1^* & I \end{bmatrix}.$$

*Proposition 3.4.4 gives that*

$$R_n = \begin{bmatrix} I & T_1 & T_2 & T_1T_2 \\ 0 & Z_1^{1/2} & 0 & Z_1^{1/2}T_2 \\ 0 & 0 & Z_2^{1/2} & Z_2^{1/2}T_1 \\ 0 & 0 & 0 & Z_{1,2}^{1/2} \end{bmatrix}$$

*satisfies $R_n^* R_n = X_n$.*

We can now prove Brehmer's condition from Condition $(\star)$ without invoking their equivalence to regularity.

**Proposition 3.4.7.** *In the case of $T : \mathbb{Z}_+^\Omega \to \mathcal{B}(\mathcal{H})$, Condition $(\star)$ implies the Brehmer's condition.*

*Proof.* Without loss of generality, we may assume $\Omega = \mathbb{N}$. We shall proceed by induction on the size of $J \subseteq \mathbb{N}$.

For $|J| = 1$ (i.e. $J = \{\omega\}$), Condition $(\star)$ implies $T$ is contractive. Hence, $Z_J = I - T_\omega^* T_\omega \geq 0$. Assuming $Z_J \geq 0$ for all $|J| \leq n$, and consider the case when $|J| = n+1$. By symmetry, it would suffices to show this for $J = J_{n+1} = \{1, 2, \cdots, n+1\}$.

By Proposition 3.4.4, $X_n = R_n^* R_n$ where the $(\emptyset, \emptyset)$-entry of $R_n$ is equal to $Z_{J_n}^{1/2}$. Let $D_n$ be a block diagonal matrix with $2^n$ copies of $T_{n+1}$ along the diagonal. Condition $(\star)$ implies that

$$D_n^* X_n D_n = D_n^* R_n^* R_n D_n \leq X_n = R_n^* R_n.$$

Therefore, by Lemma 3.4.2, there exists a contraction $W_n$ such that $W_n R_n = R_n D_n$. By comparing the $(\emptyset, \emptyset)$-entry on both sides, there exists $C_n$ such that $C_n Z_{J_n}^{1/2} = Z_{J_n}^{1/2} T_{n+1}$, where $C_n$ is the $(\emptyset, \emptyset)$-entry of $W_n$, which must be contractive as well. Hence, by Lemma 3.4.1 and 3.4.2,

$$Z_{J_{n+1}} = Z_{J_n} - T_{n+1}^* Z_{J_n} T_{n+1} \geq 0.$$

This finishes the proof. $\square$

# Chapter 4

# Regular Dilation on Graph Products of $\mathbb{N}$

The lattice ordered semigroup only provides a limited number of examples. The analysis of the Nica-covariant condition is closely related to a larger class of semigroups called the quasi-lattice ordered semigroups, which covers many more interesting classes of semigroups. For example, the free semigroup $\mathbb{F}_k^+$ is quasi-lattice ordered but not lattice ordered. Therefore, one may wonder whether we can extend our analysis of regular dilation to quasi-lattice ordered semigroups.

There is one immediate difficulty that we have to overcome: in a quasi-lattice ordered group $(G, P)$, we cannot always write $g \in G$ as $(g_-)^{-1}g_+$. Without the positive and negative part $g_+, g_-$, it is not clear how to even define regular dilation in this context.

In this chapter, we approach this problem by first considering a very special class of quasi-lattice ordered semigroups called the graph products of $\mathbb{N}$. First, we give a brief overview of the graph product of $\mathbb{N}$. We then proceed to prove a few technical lemmas and the main theorem (Theorem 4.2.11), which connects the Brehmer's condition with the Frazho-Bunce-Popescu's dilation. Then, we discuss the connection between $*$-regular dilation and isometric Nica-covariant representations (Theorem 4.5.5).

In the special case when the graph is a complete multi-partite graph, Popescu studied a similar program of dilation for such representations. In his study, he used a condition called the Property (P) to establish the dilation. We briefly discuss the relation between our work and Popescu's Property (P), and explains how Property (P) arises naturally from the Nica-covariant condition.

After realizing the connection between $*$-regular dilation and isometric Nica-covariant representations, we are able to further extend the theory of regular dilation beyond quasi-lattice ordered semigroups. Many technical lemma can be greatly shortened. We will discuss this in the next chapter.

## 4.1 Graph Product of $\mathbb{N}$

Fix a simple graph $\Gamma$ with a countable vertex set $\Lambda$. Recall that a graph product of $\mathbb{N}$ is a unital semigroup $P_\Gamma = \Gamma_{i\in\Lambda}\mathbb{N}$, generated by generators $\{e_i\}_{i\in\Lambda}$ where $e_i, e_j$ commute whenever $i, j$ are adjacent in $\Gamma$. We also call $P_\Gamma$ the graph semigroup or the right-angled Artin monoid. It is also closely related to the Cartier-Foata monoid [34] where $e_i, e_j$ commute whenever $i, j$ are not adjacent.

We can similarly define the graph product of $\mathbb{Z}$, $G_\Gamma = \Gamma_{i\in\Lambda}\mathbb{Z}$. It is defined to be the free product of $\mathbb{Z}$ modulo the rule that elements in the $i$-th and $j$-th copies of $\mathbb{Z}$ commute whenever $(i, j)$ is an edge of $\Gamma$. $G_\Gamma$ is a group, which is also called the graph group or the right-angled Artin group. $G_\Gamma$ together with $P_\Gamma$ is an important example of a quasi-lattice ordered group that is studied by Crisp and Laca [17].

**Example 4.1.1.** *[Examples of Graph Products]*

1. *Consider the complete graph $\Gamma$ that contains every possible edge $(i, j)$ $i \neq j$. The graph product $\Gamma_{i\in\Lambda}\mathbb{N}$ is equal to the abelian semigroup $\mathbb{N}_k^+$, since any two generators $e_i, e_j$ commute.*

2. *Consider the graph $\Gamma$ that contains no edges. The graph product $P_\Gamma = \Gamma_{i\in\Lambda}\mathbb{N}$ is equal to the free product $\mathbb{F}_k^+$.*

3. *Consider the following graph product associated with the graph in Figure 4.1.*

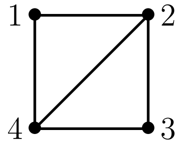

Figure 4.1: A simple graph of 4 vertices

*The graph product semigroup is a unital semigroup generated by* $4$ *generators* $e_1, \cdots, e_4$, *where the commutation relation is dictated by the edges of the graph. In this example,* $e_i, e_j$ *pairwise commute except for the pair* $e_1, e_3$.

We covered the basics of graph products of semigroups in Section 2.1.4. We now discuss a few technical lemmas. Recall in an expression $x = x_1 \cdots x_n$, $I(x_i)$ is defined to be the vertex $v$ so that $x_i$ belongs to the copy corresponding to the vertex $v$.

**Lemma 4.1.2.** *An expression* $x = x_1 \cdots x_n$ *is reduced (*$x$ *can be in either* $P_\Gamma$ *or* $G_\Gamma$*) if and only if for all* $i < j$ *such that* $I(x_i) = I(x_j)$*, there exists an* $i < t < j$ *so that* $I(x_t)$ *is not adjacent to* $I(x_i)$.

The idea is that when $I(x_i) = I(x_j)$, as long as everything between $x_i$ and $x_j$ commute with $x_i$ and $x_j$, we can shuffle $x_j$ to be adjacent to $x_i$ and amalgamate the two. It is observed in [30] that reduced expressions are shuffle equivalent:

**Theorem 4.1.3** (Green [30])**.** *If* $x = x_1 \cdots x_n = x'_1 \cdots x'_m$ *are two reduced expressions for* $x \in G_\Gamma$ *(or* $P_\Gamma$*). Then two expressions are shuffle equivalent. In particular* $m = n$.

This allows us to define *the length of an element* $x$ to be $\ell(x) = n$, when $x$ has a reduced expression $x_1 \cdots x_n$.

Given a reduced expression $x = x_1 \cdots x_n$, a syllable $x_i$ is called *an initial syllable* if $x$ can be shuffled as $x = x_i x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$. Equivalently, it means the vertex $I(x_i)$ is adjacent to any previous vertices $I(x_j)$, $j < i$. The vertex $I(x_i)$ of an initial syllable is called *an initial vertex.* The following lemma is partially taken from [17, Lemma 2.3].

**Lemma 4.1.4.** *Let* $x = x_1 \cdots x_n$ *be a reduced expression. Then,*

1. *If* $i \neq j$ *and* $x_i, x_j$ *are two initial syllables, then* $I(x_i) \neq I(x_j)$.
2. *The initial vertices of* $X$ *are pairwise adjacent.*
3. *Let* $J = \{i : x_i \text{ is an initial syllable}\}$*. Then* $x = \prod_{j \in J} x_j \prod_{j \notin J} x_j$*, where the second product is taken in the same order as in the original expression.*

*Proof.* If $I(x_i) = I(x_j)$ in a reduced expression, by Lemma 4.1.2, there has to be an index $i < t < j$ so that $I(x_t)$ is not adjacent to $I(x_i) = I(x_j)$. Therefore, it is impossible to shuffle $x_j$ to the front. Therefore, any two initial syllables have different vertices.

If $x_i, x_j$ are two initial syllables where $i < j$. Then to shuffle $x_j$ to the front, it must be the case that $x_j$ can commute with $x_i$, and thus $I(x_i)$ is adjacent with $I(x_j)$. This shows initial vertices are pairwise adjacent.

Now let $J = \{1 < j_1 < j_2 < \cdots < j_m\}$ be all $i$ where $x_i$ is an initial syllable. Then, we can recursively shift each $x_{j_s}$ to the front. The result is that we can shuffle all the initial vertices to the front as $\prod_{j \in J} x_j$, while all the other syllables are multiplied subsequently in the original order. $\square$

Lemma 4.1.4 shows that the initial vertices are pairwise adjacent and thus form a clique of the graph $\Gamma$.

Lemma 4.1.4 allows us to further divide a reduced expression of $x$ into blocks. Given a reduced expression $x = x_1 \cdots x_n$, we define the first block $b_1$ of $x$ to be the product of all initial syllables. Since any two initial syllables commute, there is no ambiguity in the order of this product. We simply denote $I_1(x) = \{i : x_i \text{ is an initial syllable}\}$, and $b_1 = \prod_{j \in I_1(x)} x_j$. Since $x_1$ is always an initial syllable, $I_1(x) \neq \emptyset$ and $b_1 \neq e$.

Now $x = b_1 x^{(1)}$, where $x^{(1)}$ has strictly shorter length compared to $x$. We can define the second block $b_2$ of $x$ to be the first block of $x^{(1)}$ when $x^{(1)} \neq e$. Of course, if $x^{(1)} = e$, we are finished since $x = b_1$. Repeat this process, and let each $x^{(t)} = b_{t+1} x^{(t+1)}$, where $b_{t+1}$ is the first block of $x^{(t)}$. Since the length of $x^{(t)}$ is always strictly decreasing, we eventually reach a state when $x^{(m-1)} = b_m x^{(m)}$ and $x^{(m)} = e$. In such case, $x$ is written as a product of $m$ blocks $x = b_1 b_2 \cdots b_m$. Here, each $b_j$ is the first block of $b_j b_{j+1} \cdots b_m$. We call this a block representation of $x$. We shall denote $I_t(x)$ be the vertex of all syllables in the $t$-th block $b_t$.

Since any two reduced expressions are shuffle equivalent, it is easy to see this block representation is unique.

**Lemma 4.1.5.** *Let a reduced expression $x = x_1 \cdots x_n$ have a block representation $b_1 \cdots b_m$*

1. *Two adjacent $I_t(x), I_{t+1}(x)$ are disjoint.*
2. *For any vertex $\lambda_2 \in I_{t+1}(x)$, there exists another vertex $\lambda_1 \in I_t(x)$ so that $\lambda_1, \lambda_2$ are not adjacent.*

*Proof.* For (1), if $I_t(x), I_{t+1}(x)$ share some common vertex $\delta$, then the syllable corresponding to $\delta$ in the $(t+1)$-th block can be shuffled to the front of the $(t+1)$-th block, and since $\delta \in I_t(x)$, this syllable commutes with all syllable in the $t$-th block. Therefore, it can be amalgamated into the $t$-th block, leading to a contradiction that the expression is reduced.

For (2), if otherwise, we can pick a vertex $\lambda_2 \in I_{t+1}(x)$ that is adjacent to every vertex in $I_t(x)$. The syllable corresponding to $\lambda_2$ can be shuffled to the front of $(t+1)$-th block, and commutes with everything in the $t$-th block. Therefore, it must be an initial syllable for $b_t b_{t+1} \cdots b_m$. But in such case, $\delta \in I_t(x)$ and cannot be in $I_{t+1}(x)$ by (1). $\square$

Studying regular dilations often requires a deep understanding of elements of the form $x^{-1}y$ for $x, y$ from the semigroup.

**Lemma 4.1.6.** *Let $x, y \in P_\Gamma$. Then there exist $u, v \in P_\Gamma$ with $x^{-1}y = u^{-1}v$, and $I_1(u)$ disjoint from $I_1(v)$. Moreover, $u, v$ are unique.*

*Proof.* Suppose that there exists a vertex $\lambda \in I_1(x) \bigcap I_1(y)$. Then we can find initial syllables $e_\lambda^{m_1}$ and $e_\lambda^{m_2}$ from reduced expressions of $x, y$. We may without loss of generality assume that $x_1 = e_\lambda^{m_1}$ and $y_1 = e_\lambda^{m_2}$.

Set $u_1 = e_\lambda^{-\min\{m_1,m_2\}} x$ and $v_1 = e_\lambda^{-\min\{m_1,m_2\}} y$. We have the relation $u_1^{-1} v_1 = x^{-1}y$. Notice that at least one of $x_1$ and $y_1$ is removed in this process, and thus the total length $\ell(u_1) + \ell(v_1)$ is strictly less than $\ell(x) + \ell(y)$. Repeat this process whenever $I_1(u_j) \bigcap I_1(v_j) \neq \emptyset$, and recursively define $u_{j+1}, v_{j+1}$ in the same manner to keep $u_j^{-1} v_j = u_{j+1}^{-1} v_{j+1}$. Since the total length $u_j, v_j$ is strictly decreasing in the process, we eventually stop in a state when $I_1(u_j)$ is disjoint from $I_1(v_j)$. This gives a desired $u = u_j, v = v_j$.

Suppose that $u^{-1}v = s^{-1}t$ for some other $s, t \in P_\Gamma$ with $I_1(s) \bigcap I_1(t) = \emptyset$. Let reduced expressions for $u, v, s, t$ be,

$$\begin{aligned} u &= u_1 \cdots u_m \\ v &= v_1 \cdots v_n \\ s &= s_1 \cdots s_l \\ t &= t_1 \cdots t_r \end{aligned}$$

We first show $u^{-1}v = u_m^{-1} \cdots u_1^{-1} v_1 \cdots v_n$ is a reduced expression in $G_\Gamma$, and so is $s^{-1}t = s_l^{-1} \cdots s_1^{-1} t_1 \cdots t_r$. Assume otherwise, by Lemma 4.1.2, there exists two syllables from the same vertex that commute with everything in between. These two syllables must have one from $u$ and the other from $v$, since $u_1 \cdots u_m$ and $v_1 \cdots v_n$ are both reduced. Let $u_i, v_j$ be two such syllables that come from the same vertex that commutes with everything in between. In that case, by Lemma 4.1.4, $u_i, v_j$ are both initial syllables for $u, v$. But $u, v$ have no common initial syllables, this leads to a contradiction.

Therefore, $u_m^{-1}\cdots u_1^{-1}v_1\cdots v_n = s_l^{-1}\cdots s_1^{-1}t_1\cdots t_r$ are both reduced expressions for $u^{-1}v = s^{-1}t$, and thus by Theorem 4.1.3 are shuffle equivalent. Notice each individual syllable $u_i, v_i, s_i, t_i$ is from the graph semigroup. To shuffle from $u_m^{-1}\cdots u_1^{-1}v_1\cdots v_n$ to $s_l^{-1}\cdots s_1^{-1}t_1\cdots t_r$, each $s_i^{-1}$ must be some $u_j^{-1}$, and $t_i$ must be some $v_j$. Therefore, $v_1\cdots v_n$ must be a shuffle of $t_1\cdots t_r$, and also $u_1\cdots u_m$ is a shuffle of $s_1\cdots s_l$. Hence, $s = u, t = v$. ☐

**Lemma 4.1.7.** *Suppose $u, v \in \Gamma_{i\in\Lambda}\mathbb{N}$. Then the following are equivalent:*

1. *$u, v$ commute.*
2. *Every syllable $v_j$ of $v$ commutes with $u$.*

*Proof.* (2)$\Longrightarrow$(1) is trivial. Assuming (1) and let $v = v_1\cdots v_m$. Consider the first syllable $v_1$ of $v$. Since $uv = vu$, $v_1$ is a initial syllable of $uv$. Therefore, $v_1$ commutes with $u$. By canceling $v_1$, one can observe that $v_2\cdots v_m$ also commutes with $u$, and recursively each $v_j$ commutes with $u$. ☐

**Lemma 4.1.8.** *Suppose $p \in P_\Gamma$, $\lambda \in \Lambda$ so that $\lambda \notin I_1(p)$ and $e_\lambda$ does not commute with $p$. Let $x, y \in P_\Gamma$ and apply the procedure in Lemma 4.1.6 to repeatedly remove common initial vertex of $e_\lambda x$ and $py$ until $(e_\lambda x)^{-1}py = u^{-1}v$ with $I_1(u)\bigcap I_1(v) = \emptyset$. Then $u, v$ do not commute.*

*Proof.* Let $p = p_1\cdots p_n$ be a reduced expression of $p$. By Lemma 4.1.7, there exists a smallest $i$ so that $e_\lambda$ does not commute with $p_i$. We first observe that none of $p_1, \cdots, p_{i-1}$ come from the vertex $\lambda$. Otherwise, if some $p_s$ comes from the vertex $\lambda$, it must commute with every $p_1, \cdots, p_{i-1}$ as $e_\lambda$ does. Therefore, $p_s$ is an initial syllable and $\lambda \in I_1(p)$, which contradicts to our assumption.

Let $p_i$ be a syllable corresponding to vertex $\lambda'$, where $\lambda'$ is certainly not adjacent to $\lambda$.

Consider the procedure of removing a common initial vertex for $u_0 = e_\lambda x$ and $v_0 = py$. At each step, we removed a common initial vertex $\lambda_i$ for $u_i, v_i$ and obtained $u_{i+1}^{-1}v_{i+1} = u_i^{-1}v_i$, until we reach $u_m = u, v_m = v$ that shares no common initial vertex. It is clear that $\lambda \notin I_1(v_0)$ and $\lambda' \notin I_1(u_0)$.

Observe that $\lambda_0 \neq \lambda'$ since $\lambda \in I_1(e_\lambda x)$ and $\lambda'$ cannot be an initial vertex of $e_\lambda x$. Therefore, the syllable $p_i$ remains in $u_1$ after the first elimination step, while no syllable before $p_i$ belongs to the vertex $\lambda$. Hence, $\lambda \notin I_1(v_1)$ and $\lambda' \notin I_1(u_1)$. Inductively, $\lambda \notin I_1(v_j)$ and $\lambda' \notin I_1(u_j)$, and thus $e_\lambda$ is still an initial syllable of $u$ and $p_i$ is still a syllable of $v$. Therefore, $u, v$ do not commute. ☐

## 4.2 Regular Dilation

Let us now turn our attention to contractive representations on a graph product $P_\Gamma = \Gamma_{i\in\Lambda}\mathbb{N}$. This semigroup is the free semigroup generated by $e_1, \cdots, e_n$ with additional rules that $e_ie_j = e_je_i$ whenever $(i,j) \in E(\Gamma)$. Therefore, a representation $T$ of $P_\Gamma$ is uniquely determined by its values on generators $T_i = T(e_i)$, where they have to satisfy $T_iT_j = T_jT_i$ whenever $(i,j) \in E(\Gamma)$.

Our goal is to define an analogue of Brehmer's regular dilation. However, not every $g \in G$ can be written as $g = g_+g_-^{-1}$. In fact, in a quasi-lattice ordered group $(G, P)$, $G \neq PP^{-1}$ in most cases.

To overcome this difficulty, we start by considering how we can define a Toeplitz kernel $K$ on $P$ that is analogous to Brehmer's definition. For any $p, q \in P$, if there exists a common initial vertex $i$ for $p, q$, we can write $p = e_ip'$ and $q = e_iq'$. Since $K$ is a Toeplitz kernel, $K(p,q) = K(p', q')$. Therefore, by repeatedly removing common initial vertices and applying Lemma 4.1.6, it suffices to consider how we can define $K(p,q)$ when $p, q$ share no common initial vertex.

**Definition 4.2.1.** *Given a contractive representation $T$ of the graph product $\Gamma_{i\in\Lambda}\mathbb{N}$, we define* the Toeplitz kernel $K$ associated with $T$ *using the following rules:*

1. *$K(p,q) = T(q)T(p)^*$ whenever $I_1(p) \bigcap I_1(q) = \emptyset$ and $p, q$ commute.*
2. *$K(p,q) = 0$ whenever $I_1(p) \bigcap I_1(q) = \emptyset$ and $p, q$ do not commute.*
3. *Otherwise, Lemma 4.1.6 shows that we can find unique $u, v$ with $p^{-1}q = u^{-1}v$ where $u, v$ share no common initial vertex. In this case, define $K(p,q) = K(u,v)$.*

**Remark 4.2.2.** *We may observe that since $I_1(e) = \emptyset$, and $e$ commutes with any $q$. $K(e,q) = T(q)$ by (1). Therefore, if $K$ is completely positive definite, the isometric Naimark dilation $V$ will be a dilation for $T$.*

One can verify that the kernel $K$ is indeed a Toeplitz kernel. In fact, it satisfies a stronger property.

**Lemma 4.2.3.** *If $p, q, x, y \in P_\Gamma$ satisfies $p^{-1}q = x^{-1}y$, then $K(p,q) = K(x,y)$.*

*Proof.* Repeatedly removing common initial vertices for the pairs $p, q$ and $x, y$ using the procedure in Lemma 4.1.6, we end up with $p^{-1}q = u^{-1}v$, $x^{-1}y = s^{-1}t$, where $u, v$ has no common initial vertex; $s, t$ has no common initial vertex. Then, $K(p,q) = K(u,v)$ and $K(x,y) = K(s,t)$. By Lemma 4.1.6, $u = s, t = v$. Therefore, $K(p,q) = K(x,y)$. □

There is in fact another description of this kernel $K$, inspired by later studies of regular dilation on right LCM semigroups.

**Lemma 4.2.4.** *For any $p, q \in P_\Gamma$, $K(p,q) = T(p^{-1}(p \vee q))T(q^{-1}(p \vee q))$ if $p \vee q \neq \infty$, and $K(p,q) = 0$ if $p \vee q = \infty$.*

*Proof.* Let $p = sp'$ and $q = sq'$ where $s$ is the product of all common initial vertices of $p, q$. It follows from the Definition 4.2.1 that $K(p,q) = K(p',q')$. In the case when $p', q'$ commute, it is clear that $p' \vee q' = p'q'$ and hence

$$K(p,q) = T(q')T(p')^* = T(p^{-1}p \vee q)T(q^{-1}p \vee q)^*.$$

In the case when $p', q'$ do not commute, one can check that $p' \vee q' = \infty$ and hence $p \vee q = \infty$. In this case, $K(p,q) = 0$. $\square$

**Definition 4.2.5.** *We say that $T$ is* $*$-regular *if the Toeplitz kernel $K$ associated with $T$ as defined in Definition 4.2.1 is completely positive definite. A Naimark dilation $V$ for this kernel $K$ is called a $*$-regular dilation for $T$. Dually, we say that $T$ is* regular *if $T^*$ has $*$-regular dilation. Here, $T^*(e_i) = T(e_i)^*$.*

**Remark 4.2.6.** *Our definition of regular dilation is slightly different from that of Brehmer's. When the graph semigroup is the abelian semigroup $\mathbb{N}^k$, Brehmer defined $T$ to be regular if a kernel $K^*$ is completely positive definite, where $K^*$ is the Toeplitz kernel by replacing Condition (1) in the Definition 4.2.1 by $K^*(p,q) = T(p)^*T(q)$. In general, the kernel $K^*$ is different from the kernel we defined in Definition 4.2.1. However, it turns out when the semigroup is the abelian semigroup $\mathbb{N}^k$, our definition of regular dilation (Definition 4.2.5) coincides with Brehmer's definition (Definition 5.1.2).*

*However, on a general graph semigroup, when the kernel $K^*$ is completely positive definite is hard to characterize. For example, when the graph $\Gamma$ contains no edge and the graph semigroup corresponds to the free semigroup, the only chance that $p, q$ commute and $I_1(p) \bigcap I_1(q) = \emptyset$ is when at least one of $p, q$ is $e$. Therefore, in such case, $K^* = K$ and $K^*$ is completelely positive definite whenever $K$ is.*

*Our definition of regular dilation implies there are isometric dilations for $T_i^*$ and thus co-isometric extensions for $T_i$. This coincides with the literature on the dilation of row contractions: for example, dilations for column contractions considered by Bunce [14] can be thought as regular dilation on the free semigroup $\mathbb{F}_+^k$.*

The $*$-regular representations are precisely those with a certain minimal Naimark dilation due to Theorem 2.2.9.

**Theorem 4.2.7.** *$T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ has $*$-regular dilation if and only if it has a minimal isometric Naimark dilation $V : P_\Gamma \to \mathcal{B}(\mathcal{K})$ so that for all $p, q \in P_\Gamma$, $K(p,q) = P_\mathcal{H} V(p)^* V(q)\big|_\mathcal{H}$.*

**Remark 4.2.8.** *Given a representation $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$, there might be kernels different from the kernel we defined in Definition 4.2.1 that are also completely positive definite. For example, it is pointed out in [49] that when $\Gamma$ is acyclic, $T$ always has a unitary dilation. By restricting to $\mathcal{H}$, such a unitary dilation defines a completely positive definite kernel that is generally different from the kernel we defined. Popescu [56] has also considered many ways to construct completely positive definite kernels on the free semigroup.*

The goal of the next two sections is to provide a necessary condition for $*$-regularity of a contractive representation of a graph semigroup, which turns out to be also a sufficient condition. We draw our inspiration from two special cases where the graph is the complete graph and where the graph is the empty graph.

**Example 4.2.9.** *In the case when $\Gamma$ is a complete graph on $k$ vertices. The graph semigroup $P_\Gamma$ is simply the abelian semigroup $\mathbb{N}^k$. It forms a lattice ordered semigroup. Each element in this semigroup can be written as a $k$ tuple $(a_1, \cdots, a_k)$. Since this semigroup is abelian, the set of initial vertex is precisely $\{i : a_i \neq 0\}$.*

*Two elements $p = (p_i), q = (q_i)$ have disjoint initial vertex sets if and only if at least one of $p_i, q_i$ is zero for all $i$. In the terminology of the lattice order, this implies the greatest lower bound $p \wedge q = e$. As it is first defined in [9], a representation $T : \mathbb{N}^k \to \mathcal{B}(\mathcal{H})$ is called $*$-regular if the kernel $K(p,q)$ is completely positive definite.*

*Brehmer's result (Theorem 2.2.3) shows that $K$ is completely positive definite if and only if for every subset $V \subseteq \{1, 2, \cdots, k\}$,*

$$\sum_{U \subseteq V} (-1)^{|U|} T_U T_U^* \geq 0.$$

*Here $|U|$ is the cardinality of $U$, and $T_U = \prod_{i \in U} T(e_i)$ with the convention that $T_\emptyset = I$.*

**Example 4.2.10.** *In the case when $\Gamma$ is a graph on $k$ vertices with no edge. The graph semigroup $\Gamma_{i\in\Lambda}\mathbb{N}$ is simply the free semigroup $\mathbb{F}_k^+$. Fix a contractive representation $T : \mathbb{F}_k^+ \to \mathcal{B}(\mathcal{H})$, which is uniquely determined by its value on generators $T_i = T(e_i)$. The Toeplitz kernel associated with $T$ defined in Definition 4.2.1 is the same as the kernel considered in [54, 56], where it is shown that $K$ is completely positive definite if and only* *if $T$ is row contractive in the sense that*

$$I-\sum_{i=1}^{k} T_i T_i^* \geq 0.$$

*It turns out the minimal Naimark dilation for $K$ in this case is also a row contraction, and thus proves the Frazho-Bunce-Popescu dilation.*

Inspired by both Example 4.2.9 and 4.2.10, our first main result unifies the Brehmer's dilation and the Frazho-Bunce-Popescu dilation. Recall that a set of vertices $U \subseteq \Lambda$ is called a clique if the subgraph induced on $U$ is a complete subgraph.

**Theorem 4.2.11.** *Let $T$ be a contractive representation of a graph semigroup $P_\Gamma$. Then, $T$ has $*$-regular dilation if for every finite $W \subseteq \Lambda$,*

$$\sum_{\substack{U\subseteq W \\ U \text{ is a clique}}} (-1)^{|U|} T_U T_U^* \geq 0. \tag{4.1}$$

The proof of Theorem 4.2.11 requires a few technical lemmas that we need to develop in the next section.

**Remark 4.2.12.** *Condition (4.1) coincides with conditions in both Example 4.2.9 and 4.2.10. Indeed, when $\Gamma$ is a complete graph, any $U \subseteq V$ is a clique. When $\Gamma$ contains no edge, the only cliques in $\Gamma$ are singletons $\{i\}$.*

## 4.3 Technical Lemmas

Since we are dealing with positive definiteness of operator matrices, the following lemma, taken from [20, Lemma 14.13], is extremely useful.

**Lemma 4.3.1.** *If an operator matrix $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is positive, then there exists an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ so that $B = XA^{1/2}$. Moreover, if $B$ has this form, then the operator matrix is positive if and only if $C \geq XX^*$.*

**Lemma 4.3.2.** *Let $X, L \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$. Define an $n \times n$ operator matrix*

$$A_n = \begin{bmatrix} X & XL^* & XL^{*2} & \cdots & XL^{*(n-1)} \\ LX & X & XL^* & \cdots & XL^{*(n-2)} \\ L^2X & LX & X & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & XL^* \\ L^{n-1}X & L^{n-2}X & \cdots & LX & X \end{bmatrix}.$$

*If $LXL^* \leq X$, then every $A_n$ is positive.*

*Proof.* Assuming $LXL^* \leq X$, we shall inductively show each $A_n$ is positive. Since the case when $n = 1$, $A_1 = X \geq 0$ is given. Suppose $A_n \geq 0$, and rewrite $A_{n+1}$ as

$$A_{n+1} = \left[\begin{array}{c|c} & XL^{*n} \\ & XL^{*(n-1)} \\ A_n & \vdots \\ & \vdots \\ & XL^* \\ \hline L^nX \quad L^{n-1}X \quad \cdots \quad \cdots \quad LX & X \end{array}\right].$$

Now notice that the row operator $[L^nX, \cdots, LX] = [0, \cdots, 0, L]A_n$. Therefore, by Lemma 4.3.1, $A_{n+1} \geq 0$ if

$$[0, \cdots, 0, L]A_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ L^* \end{bmatrix} \leq X.$$

Expand the left hand side gives $LXL^* \leq X$. $\square$

**Corollary 4.3.3.** *The matrix $A_n$ defined in Lemma 4.3.2 is positive if and only if $A_0 = X \geq 0$ and $A_1 \geq 0$.*

*Proof.* Indeed, by Lemma 4.3.1,

$$A_1 = \begin{bmatrix} X & X^{1/2}X^{1/2}L^* \\ LX^{1/2}X^{1/2} & X \end{bmatrix} \geq 0$$

if and only if $X \geq 0$ and $\left(LX^{1/2}\right)\left(X^{1/2}L\right) = LXL^* \leq X$. This is sufficient for every $A_n \geq 0$ by Lemma 4.3.2. $\square$

We now turn our attention to the contractive representation $T$ of a graph semigroup $P_\Gamma = \Gamma_{i\in\Lambda}\mathbb{N}$. Throughout this section, we fix such a representation $T$ and its associated Toeplitz kernel $K$ defined in Definition 4.2.1. For two finite sequences $F_1, F_2 \subset P_\Gamma$, where $F_1 = \{p_1, \cdots, p_m\}$ and $F_2 = \{q_1, \cdots, q_n\}$, we denote $K[F_1, F_2]$ to be the $m \times n$ operator matrix, whose $(i, j)$-entry is equal to $K(p_i, q_j)$. When $F_1 = F_2$, we simply write $K[F_1] = K[F_1, F_1]$. Recall $K$ is completely positive definite if and only if for all finite subsets $F \subseteq P_\Gamma$, $K[F] \geq 0$. If $F$ is a collection of elements that may contain duplicates, we may similarly define $K[F]$. It turns out duplicated elements will not affect the positivity of $K[F]$.

**Lemma 4.3.4.** *Let* $F = \{p_1, p_1, p_2, \cdots, p_m\}$ *and* $F_1 = \{p_1, p_2, \cdots, p_m\}$. *Then* $K[F] \geq 0$ *if and only if* $K[F_1] \geq 0$.

*Proof.* Denote $F_2 = \{p_2, \cdots, p_m\}$. We have,

$$K[F] = \left[\begin{array}{c|cc} I & I & K[p_1, F_2] \\ \hline I & I & K[p_1, F_2] \\ K[F_2, p_1] & K[F_2, p_1] & K[F_2] \end{array}\right].$$

Here, the lower right corner is $K[F_1]$.

By Lemma 4.3.1, $K[F] \geq 0$ if and only if $K[F_2, p_1]K[p_1F_2] \leq K[F_2]$. By Lemma 4.3.1 again, this happens if and only if $K[F_1] \geq 0$. $\square$

**Lemma 4.3.5.** *Let* $F_1 = \{p_1, \cdots, p_m\}$ *and* $F_2 = \{q_1, \cdots, q_n\}$ *and fix a vertex* $\lambda \in \Lambda$ *so that* $\lambda$ *is not an initial vertex for any of the* $p_i$. *Let* $D(\lambda, F_1)$ *be a diagonal* $m \times m$ *operator matrix whose* $i$*-th diagonal entry is equal to* $T(e_\lambda)^m$ *if* $e_\lambda$ *commutes with* $p_i$ *and* $0$ *otherwise. Then,* $K[F_1, e_\lambda^m \cdot F_2] = D(\lambda, F_1) \cdot K[F_1, F_2]$.

*Proof.* This is essentially proving that $K(p_i, e_\lambda^m q_j) = T(e_\lambda)^m K(p_i, q_j)$ if $e_\lambda$ commutes with $p_i$ and 0 otherwise.

Assuming first that $e_\lambda$ commutes with $p_i$. Then $p_i^{-1} e_\lambda^m q_j = e_\lambda^m p_i^{-1} q_j$. A key observation here is that when this happens, $p_i$ contains no syllable from the vertex $\lambda$. Since $e_\lambda$ commutes with every syllable of $p_i$, if there is a syllable of $p_i$ from the vertex $\lambda$, it must be an initial syllable, which contradicts to our selection of $p_i$.

Repeatedly removing common initial vertices for $p_i, q_j$ using Lemma 4.1.6, we end up with $p_i^{-1} q_j = u^{-1} v$, where $u, v$ have no common initial vertex. It follows from the Definition 4.2.1 that $K(p_i, q_j) = K(u, v)$. Notice that $I_1(e_\lambda^m v)$ includes $\lambda$ and every vertex in $I_1(v)$ that is adjacent to $\lambda$. Moreover, we observed that $\lambda \notin I_1(u)$. Therefore, we have $I_1(e_\lambda^m v) \bigcap I_1(u) = \emptyset$.

Suppose $u, v$ commute. Then $p_i^{-1} e_\lambda^m p_j = e_\lambda^m v u^{-1} = u^{-1} e_\lambda v$. Therefore, by Lemma 4.2.3, $K(p_i, e_\lambda^m q_j) = K(u, e_\lambda^m v)$. Hence, in this case,

$$K(u, e_\lambda v) = T(e_\lambda)^m T(v) T(u)^* = T(e_\lambda)^m K(u, v).$$

If $u, v$ does not commute, $e_\lambda^m v$ also does not commute with $u$. Therefore, $K(u, v) = K(u, e_\lambda v) = 0$.

Assume now that $e_\lambda$ does not commute with $p_i$. Consider the procedure of removing common initial syllables in $p_i$ and $e_\lambda^m q_j$: since $\lambda$ is not an initial vertex of $p_i$, each step we have to cancel out a syllable from $p_i$ and $q_j$ that both commute with $e_\lambda^m$. After each step of removing a common initial vertex, we removed some syllable from $p_i$ that commute with $e_\lambda$. Since $\lambda$ is not an initial vertex of $p_i$, each step will not cancel out any $e_\lambda^m$. Eventually, we always end up with $p_i^{-1} q_j = u^{-1} e_\lambda^m v$, where $u, e_\lambda^m v$ do not share any common initial vertex.

By Lemma 4.1.7, some syllable in $p_i$ does not commute with $e_\lambda$. Since all the syllables that got canceled commute with $e_\lambda$, there has to be some syllable in the left over $u$ that does not commute with $e_\lambda$. Therefore, $u$ and $e_\lambda^m v$ do not commute. Hence, $K(u, e_\lambda^m v) = 0$. □

As an immediate corollary,

**Corollary 4.3.6.** *Let $F = \{p_1, \cdots, p_n\}$ be a finite subset of $P_\Gamma$, and $\lambda \in \Lambda$ is a vertex that is not an initial vertex for any of $p_i$. For every $m \geq 0$, denote $F_m = \bigcup_{j=0}^m e_\lambda^j \cdot F$. Then $K[F_m] \geq 0$ if and only if $K[F] \geq 0$ and $K[F_1] \geq 0$.*

*Proof.* For each $i \leq j$, $K[e_\lambda^i F, e_\lambda^j F] = K[F, e_\lambda^{j-i} F]$. Let $D = D(\lambda, F)$ be the $n \times n$ diagonal operator matrix, whose $(i,i)$-entry is $T(e_\lambda)$ if $e_\lambda$ commutes with $p_i$ and 0 otherwise. It follows from Lemma 4.3.5 that $K[F, e_\lambda^{j-i} F] = D^{j-i} K[F]$. Similarly, for each $i > j$,

$$K[e_\lambda^i F, e_\lambda^j F] = K[e_\lambda^j F, e_\lambda^i F]^* = K[F] D^{*(i-j)}.$$

Therefore,

$$K[F_m] = \begin{bmatrix} K[F] & K[F]D^* & K[F]D^{*2} & \cdots & K[F]D^{*m} \\ DK[F] & K[F] & K[F]D^* & \cdots & K[F]D^{*(m-1)} \\ D^2K[F] & DK[F] & K[F] & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & K[F]D^* \\ D^mK[F] & D^{m-1}X & \cdots & DK[F] & K[F] \end{bmatrix}.$$

Corollary 4.3.3 can be applied so that $K[F_m] \geq 0$ if and only if $K[F] \geq 0$ and $K[F_1] \geq 0$. ☐

**Lemma 4.3.7.** *Let* $F_1 = \{p_1, \cdots, p_n\}$, $F_2 = \{q_1, \cdots, q_m\}$ *be finite subsets of* $P_\Gamma$, *and* $\lambda \in \Lambda$ *is a vertex that is not an initial vertex for any of* $p_i$ *nor* $q_j$. *Suppose that* $e_\lambda$ *commutes with every* $q_j$, *but not with any* $p_i$. *Denote,*

$$\begin{aligned} F_0 &= F_1 \bigcup F_2 \\ F &= e_\lambda \cdot \left(F_1 \bigcup F_2\right) \bigcup \left(F_1 \bigcup F_2\right) \\ &= e_\lambda F_0 \bigcup F_0 \\ F' &= e_\lambda \cdot F_2 \bigcup F_1 \bigcup F_2 \end{aligned}$$

*Then,* $K[F] \geq 0$ *if and only if* $K[F'] \geq 0$.

*Proof.* Let $D$ denote an $m \times m$ diagonal operator matrix whose diagonal entries are all $T(e_\lambda)$. Repeatedly apply Lemma 4.3.5,

$$K[F] = \begin{bmatrix} K[F_1] & K[F_1, F_2] & 0 & K[F_1, F_2]D^* \\ K[F_2, F_1] & K[F_2] & 0 & K[F_2]D^* \\ 0 & 0 & K[F_1] & K[F_1, F_2] \\ DK[F_2, F_1] & DK[F_2] & K[F_2, F_1] & K[F_2] \end{bmatrix}.$$

Denote the upper left $2 \times 2$ corner by $X = \begin{bmatrix} K[F_1] & K[F_1, F_2] \\ K[F_2, F_1] & K[F_2] \end{bmatrix}$. It is clear that $X = K[F_0]$. Let $L$ be a $(n+m) \times (n+m)$ diagonal operator matrix, whose first $n$ diagonal entries are 0, and the rest $m$ diagonal entries be $T(e_\lambda)$. Then, the lower left $2 \times 2$ corner can be written as $LX$, and $K[F] = \begin{bmatrix} X & XL^* \\ LX & X \end{bmatrix}$.

Lemma 4.3.2 states that $K[F] \geq 0$ if and only if $X = K[F_0] \geq 0$ and $LXL^* \leq X$. Explicitly writing out $X - LXL^*$, we get,

$$X - LXL^* = \begin{bmatrix} K[F_1] & K[F_1, F_2] \\ K[F_2, F_1] & K[F_2] - DK[F_2]D^* \end{bmatrix}. \tag{4.2}$$

Now consider $K[F']$:

$$K[F'] = \begin{bmatrix} K[F_2] & 0 & K[F_2]D^* \\ 0 & K[F_1] & K[F_1, F_2] \\ DK[F_2] & K[F_2, F_1] & K[F_2] \end{bmatrix}. \tag{4.3}$$

Notice here $\begin{bmatrix} 0 \\ DK[F_2] \end{bmatrix} = \begin{bmatrix} 0 \\ D \end{bmatrix} K[F_2]$. By Lemma 4.3.1, $K[F'] \geq 0$ if and only if $K[F_2] \geq 0$ and

$$\begin{bmatrix} 0 \\ D \end{bmatrix} K[F_2] \begin{bmatrix} 0 & D^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & DK[F_2]D^* \end{bmatrix} \leq \begin{bmatrix} K[F_1] & K[F_1, F_2] \\ K[F_2, F_1] & K[F_2] \end{bmatrix}.$$

This is precisely the condition required in Condition (4.2). Therefore, combing the results from above, $K[F] \geq 0$ if and only if $K[F'] \geq 0$, $K[F_0] \geq 0$ and $K[F_2] \geq 0$. But notice $F_0, F_2$ are subset of $F'$, the later condition is equivalent to $K[F'] \geq 0$. □

## 4.4 Proof of The Main Result

We prove the first main result (Theorem 4.2.11) in this section. The goal is to show that for every finite $F = \{p_1, \cdots, p_n\} \subset P_\Gamma$, $K[F] \geq 0$ where $K$ is the Toeplitz kernel associated with a contractive representation $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ that satisfy condition 4.1.

The proof of the main result Theorem 4.2.11 is divided into 2 steps. In the first step, we define an order on finite subsets of $P_\Gamma$, and show that for each $F \subset P_\Gamma$, $K[F] \geq 0$ follows from $K[F'] \geq 0$ for some $F' < F$ under this order. This allows us to make an induction along finite subsets of $P_\Gamma$.

The base case of the induction turns out to be the case when every element in $F$ has precisely one block. The second step is to show for all such $F$, $K[F] \geq 0$. Inspired by [40, Section 6], we shall then use an argument to show such $K[F]$ can be decomposed as $RR^*$ for some operator matrix $R$ explicitly.

For the first step, we show that as long as $F$ contains some element that has more than 1 block, one can find another finite subset $F' \subset P_\Gamma$ so that $K[F] \geq 0$ if $K[F'] \geq 0$. The key is then to show that this process of finding $F'$ will terminate after finitely many steps.

**Definition 4.4.1.** *For each $\lambda \in \Lambda$, and $p \in P_\Gamma$, define $d_\lambda(p)$ to be:*

1. *If $p = e_\lambda^{n_1} p' \in F$ where $e_\lambda$ does not commute with $p'$, then $d_\lambda(p) = \{p'\}$.*
2. *If $p = e_\lambda^{n_1} p' \in F$ where $e_\lambda$ commutes with $p'$, then $d_\lambda(p) = \{e_\lambda p', p'\}$.*
3. *If $\lambda$ is not an initial vertex of $p$ and $e_\lambda$ does not commute with $p$, then $d_\lambda(p) = \{p\}$.*
4. *If $\lambda$ is not an initial vertex of $p$ and $e_\lambda$ commutes with $p$,, then $d_\lambda(p) = \{e_\lambda p, p\}$.*

*For any finite set $F \subseteq P_\Gamma$, denote $d_\lambda(F) = \bigcup_{p \in F} d_\lambda(p)$.*

**Lemma 4.4.2.** *Let $F = \{p_1, \cdots, p_n\} \subset P_\Gamma$ with some $p_i$ containing at least 2 blocks. Pick a $\lambda$ that is an initial vertex for some $p_i$, but $e_\lambda$ does not commute with $p_i$.*

*Then $K[F] \geq 0$ if $K[d_\lambda(F)] \geq 0$.*

*Proof.* Without loss of generality, assume $p_1$ has at least two blocks. First of all, by Lemma 4.1.5, there exists an initial vertex $\lambda$ of $p_1$ that is not adjacent to some vertex $\lambda'$ in the second block of $p_1$. Therefore, $e_\lambda$ does not commute with $p_1$. We fix this vertex $\lambda$, and reorder $p_1, \cdots, p_n$ so that $\lambda$ is an initial vertex for $p_1, \cdots, p_m$ but not $p_{m+1}, \cdots, p_n$.

Write $p_i = e_\lambda^{n_i} p'_i$ for all $1 \leq i \leq m$. Denote $F_0 = \{p'_1, \cdots, p'_m, p_{m+1}, \cdots, p_n\}$. None of elements in $F_0$ has $\lambda$ as an initial vertex. Let $N = \max\{n_i\}$ and denote $F_N = \bigcup_{j=0}^{N} e_\lambda^j \cdot F_0$. It is clear that $F \subseteq F_N$, and thus $K[F] \geq 0$ if $K[F_N] \geq 0$. By Corollary 4.3.6, $K[F_N] \geq 0$ if and only if $K[F_1] \geq 0$ where $F_1 = (e_\lambda \cdot F_0) \bigcup F_0$.

We may further split $F_0$ into two subsets $F_0 = C \bigcup N$, where $C = \{f \in F : f \text{ commutes with } e_\lambda\}$ and $N = \{f \in F : f \text{ does not commute with } e_\lambda\}$. Now apply Lemma 4.3.7, $K[F_1] \geq 0$ if and only if $K[(e_\lambda \cdot C) \bigcup F_0] \geq 0$. Denote

$$F' = (e_\lambda \cdot C) \bigcup F_0 = (e_\lambda \cdot C) \bigcup C \bigcup N.$$

This proves that $K[F'] \geq 0$ implies $K[F] \geq 0$.

To see $F' = d_\lambda(F)$: fix an element $p_i \in F$ and consider 4 possibilities:

1. If $p_i = e_\lambda^{n_1} p'_i \in F$ where $e_\lambda$ does not commute with $p'_i$, then $d_\lambda(p_i) = \{p'_i\}$ is contained in $N \subseteq F_0 \subseteq F'$;
2. If $p_i = e_\lambda^{n_1} p'_i \in F$ where $e_\lambda$ commutes with $p'_i$, then $p'_i$ is an element of $C$ and thus $d_\lambda(p_i) = \{e_\lambda p'_i, p'_i\}$ is contained in $(e_\lambda \cdot C) \bigcup C \subseteq F'$;
3. If $\lambda$ is not an initial vertex of $p_i$ and $e_\lambda$ does not commute with $p_i$, then $p_i$ is in the set $N$ and $d_\lambda(p_i) = \{p_i\}$ is contained in $N \subseteq F'$;
4. If $\lambda$ is not an initial vertex of $p_i$ and $e_\lambda$ commutes with $p_i$, then $p_i$ is in the set $C$ and $d_\lambda(p_i) = \{e_\lambda p_i, p_i\}$ is contained in $(e_\lambda \cdot C) \bigcup C \subseteq F'$.

One can now observe that $F' = d_\lambda(F)$. This finishes the proof. □

**Remark 4.4.3.** *One may observe that due to (2) and (4), the set $F'$ might be a larger set compared to $F$. The idea here is we remove $e_\lambda$ where it does not commute with some later syllables, this should make syllables of each element in $F'$ more commutative with one another. Therefore repeating this process will end up with an $F'$ where every element has only one block. This motivates the Definition 4.4.4.*

**Definition 4.4.4.** *For each element $p \in P_\Gamma$ with $m$ blocks, we define* the block-vertex sequence *of $p$ to be $m$ sets of vertices $B_1(p), \cdots, B_m(p)$, where $B_1(p) = \{\lambda \in I_1(p) : e_\lambda \text{ does not commute with } p\}$, and $B_j(p) = I_j(p)$ for all $2 \leq j \leq m$. In other words, the $j$-th set is equal to the vertex set of $j$-th block of $p$, except for the first block, where we only include any vertex that does not commutes with the rest of the blocks. We also define $B_0(p) = \{\lambda \in I_1(p) : e_\lambda \text{ commutes with } p\}$, the set of all initial vertices that are adjacent to every vertex that appears in $p$.*

*Define the block-vertex length of $p$ be $c(p) = \sum_{j=1}^{m} |B_j(p)|$.*

**Remark 4.4.5.** *In the case that $p$ has only one block, then every syllable is initial and thus commuting. In such case, $B_1(p) = \emptyset$ and $c(p) = 0$. This is the only case when $c(p) = 0$.*

*Also observe that for $p = e_{\lambda_1}^{m_1} \cdots e_{\lambda_n}^{m_n}$, the power $m_i \geq 1$ does not affect the block-vertex sequence of $p$. The only thing that matters is what kind of vertex appears in each block.*

*In a reduced expression of $p$, each syllable uniquely corresponds to some vertex in one of $B_0(p), \cdots, B_m(p)$. Therefore, the length $\ell(p) = \sum_{j=0}^{m} |B_j(p)|$. The quantity $c(p) = \ell(p) - |B_0(p)|$ counts the number of syllables that do not commute with the rest.*

**Lemma 4.4.6.** *Let $p \in P_\Gamma$ and $\lambda \in \Lambda$.*

1. *If $\lambda \in B_1(p)$, and $p = e_\lambda^n p'$. Then $c(p') < c(p)$.*
2. *If $e_\lambda$ commutes with $p$, then the block vertex sequence of any element in $d_\lambda(p)$ is the same as that of $p$. Here, $d_\lambda(p)$ is defined as in the Definition 4.4.1.*
3. *If $e_\lambda$ does not commute with $p$ and $\lambda$ is not an initial vertex of $p$, then the block vertex sequence of any element in $d_\lambda(p)$ is the same as that of $p$.*

*Proof.* For (1), every vertex in $B_0(p)$ is still in $B_0(p')$. Since we removed the syllable $e_\lambda^n$, $\ell(p') \leq \ell(p) - 1$, it is observed by Remark 4.4.5 that $c(p') < c(p)$.

For (2), there are two cases: either $\lambda \in B_0(p)$ or not. In the first case, write $p = e_\lambda^n p'$ and $d_\lambda(p) = \{p, p'\}$. Since we only removed an initial vertex that commutes with the rest of the word, $p'$ has the same block-vertex sequence as $p$. In the later case when $\lambda \notin B_0(p)$,

$d_\lambda(p) = \{p, e_\lambda p\}$. Since $e_\lambda$ commutes with $p$, $\lambda$ will be added to $B_0(e_\lambda p)$ and thus will not change the block-vertex sequence of $e_\lambda p$. In any case, the block vertex sequence of any element in $d_\lambda(p)$ is the same as that of $p$.

For (3), $d_\lambda(p) = \{p\}$, and it is clear. $\square$

**Lemma 4.4.7.** *If $p_1, p_2$ have the same block-vertex sequence, then so does every element of $d_\lambda(p_1), d_\lambda(p_2)$.*

*Proof.* If $\lambda \in B_1(p_1) = B_1(p_2)$, write $p_i = e_\lambda^{n_i} p_i'$ and $d_\lambda(p_i) = \{p_i'\}$. Then $p_i'$ is $p_i$ with the syllable $e_\lambda^{n_i}$ removed, and since $p_1, p_2$ have the same block-vertex sequence, $p_1', p_2'$ must also have the same block-vertex sequence. In any other case, by Lemma 4.4.6, every element in $d_\lambda(p_i)$ has the same block-vertex sequence as $p_i$. $\square$

**Definition 4.4.8.** *Let $F \subset P_\Gamma$ be a finite set. Define $c(F) = \sum c(f)$, where the summation is over all $f \in F$, but multiple elements with the same block-vertex sequence are only summed once.*

**Lemma 4.4.9.** *$c(d_\lambda(F)) < c(F)$.*

*Proof.* Without loss of generality, let $f_1, \cdots, f_t$ have distinct block-vertex sequences while $f_{t+1}, \cdots, f_n$ have the same block vertex sequence as some $f_i$, $1 \leq i \leq t$, where $f_1 = p_1 = e_\lambda^{n_1} p_1'$ and $e_\lambda$ not commuting with $p_1'$. Then $c(F) = \sum_{i=1}^t c(p_i)$.

Now, from Lemma 4.4.2, $\lambda \in B_1(p_1)$. Therefore, $d_\lambda(f_1) = \{p_1'\}$, and $c(p_1') < c(f_1)$. Now apply Lemma 4.4.7, the block-vertex sequence of each $d_\lambda(f_{t+1}), \cdots, d_\lambda(f_n)$ is the same as that of some $d_\lambda(f_1), \cdots, d_\lambda(f_t)$. Moreover, by Lemma 4.4.6, $c(d_\lambda(f_i)) \leq c(f_i)$. Therefore, since $d_\lambda(F) = \bigcup_{i=1}^n d_\lambda(f_i)$, we have,

$$c(d_\lambda(F)) \leq \sum_{i=1}^t c(d_\lambda(f_i)) < \sum_{i=1}^t c(f_i) = c(F). \quad \square$$

To summarize the first step towards the proof of the main theorem,

**Proposition 4.4.10.** *For every finite subset $F \subset P_\Gamma$, there exists finite subset $\tilde{F} \subset P_\Gamma$, where every element in $\tilde{F}$ contains exactly one block, and $K[F] \geq 0$ if $K[\tilde{F}] \geq 0$.*

*Proof.* We start with $F = F_0$ and repeatedly apply Lemma 4.4.2 to obtain $F_1 = d_\lambda(F)$, $F_2 = d_\lambda(F_1), \cdots$. Lemma 4.4.2 proves that $K[F_n] \geq 0$ if $K[F_{n+1}] \geq 0$. Lemma 4.4.9 shows that $c(F_n)$ is a strictly decreasing integral sequence, and thus must stop at some $F_N = \tilde{F}$.

If $c(\tilde{F}) \neq 0$, some elements in $\tilde{F}$ has at least 2 blocks and Lemma 4.4.2 can still be applied to obtain another set $\tilde{F}' = d_\lambda(\tilde{F})$ with $c(\tilde{F}') < c(\tilde{F})$. Therefore, the last $F_N = \tilde{F}$ must have $c(F_N) = 0$, which is equivalent of saying every element in $\tilde{F}$ contains exactly one block. It is also clear that $K[F] \geq 0$ if $K[F_N] \geq 0$. □

Our second step shall prove that for every finite subset $F$ where every element has exactly one block, $K[F] \geq 0$. Since $F$ only contain finitely many syllables, we may consider only the case when $\Gamma$ is a finite graph. If an element has exactly one block, then every syllable commutes with all other syllables, and thus their vertices corresponds to a clique in $\Lambda$. For a clique $U$, denote $e_U = \prod_{\lambda \in J} e_\lambda$. Since $U$ is a clique, there is no ambiguity in the order of this product. One exception to the definition is that we shall consider the empty set as a clique as well, and denote $e_\emptyset = e$. When $\Gamma$ is a finite graph, there are only finitely many cliques. Denote $F_c = \{e_U : U \text{ is a clique}\}$. The first lemma shows that it suffices to prove $K[F_c] \geq 0$.

**Lemma 4.4.11.** *If $K[F_c] \geq 0$, then for any finite subset $F$ of $P_\Gamma$ whose elements all have one block, $K[F] \geq 0$.*

*Proof.* Suppose $F = \{p_1, \cdots, p_n\}$ contains an element $e_\lambda^n p'$ with $n \geq 2$, then reorder $p_1, \cdots, p_n$ so that $\lambda$ is an initial vertex for $p_1, \cdots, p_m$ but not $p_{m+1}, \cdots, p_n$. Let $p_i'$ be the $p_i$ with the syllable corresponding to $\lambda$ removed. Let $F_0 = \{p_1', \cdots, p_m', p_{m+1}, \cdots, p_n\}$ and let $C \subseteq F_0$ be all elements that commute with $e_\lambda$. Lemma 4.4.2 proves that $K[F_0] \geq 0$ if $K[F'] = K[(e_\lambda \cdot C) \bigcup F_0] \geq 0$. Since elements in $F_0$ contain exactly one block, and elements in $C$ commute with $F_0$, we have every element in $F'$ contains exactly one block.

Moreover, each syllable corresponding to the vertex $\lambda$ is $e_\lambda$. Repeat this process until we reach $\tilde{F}$ where for all $\lambda$, all syllables corresponding to $\lambda$ are $e_\lambda$. In such case, every element has the form $e_U$ for some clique $U$. It is clear that $\tilde{F} \subset F_c$ and thus if $K[F_c] \geq 0$, then $K[\tilde{F}] \geq 0$ and thus $K[F] \geq 0$. □

To show $K[F_c] \geq 0$, it suffices to show $K[F_c]$ can be decomposed as $R_c R_c^*$. Following the technique outlined in [40, Section 6], we can explicitly find such $R_c$. Moreover, under a certain ordering, $R_c$ can be chosen to be a lower triangular matrix, and can thus be viewed as a Cholesky decomposition of $K[F_c]$. This will be done in Proposition 4.4.14, where we shall see where the conditions in Condition (4.1) come from.

From Condition (4.1), denote

$$Z_V = \sum_{\substack{U \subseteq V \\ U \text{ is a clique}}} (-1)^{|U|} T_U T_U^* \geq 0. \tag{4.4}$$

Here, $V$ is any subset of the vertex set $\Lambda$, and $T_U = T(e_U)$. Assuming Condition (4.1) holds true for a contractive representation $T$, each $Z_V \geq 0$ and we can thus take its square root $Z_V^{1/2} \geq 0$.

**Definition 4.4.12.** *For a clique $V$, we define the neighborhood of $V$, denoted by $N_V$, to be*

$$N_V = \{\lambda \in \Lambda : \lambda \notin V, \text{ and } \lambda \text{ is adjacent to every vertex in } V\}.$$

*In particular, we define $N_\emptyset = \Lambda$.*

**Lemma 4.4.13.** *Fix a clique $F$, then*

$$\sum_{\substack{F \subseteq W \\ W \text{ is a clique}}} T_{W \backslash F} Z_{N_W} T^*_{W \backslash F} = I.$$

*Proof.* Replace $Z_{N_W}$ using Equation (4.4),

$$\begin{aligned} & \sum_{\substack{W \supseteq F \\ W \text{ is a clique}}} T_{W \backslash F} Z_{N_W} T^*_{W \backslash F} \\ = & \sum_{\substack{W \supseteq F \\ W \text{ is a clique}}} T_{W \backslash F} \left( \sum_{\substack{U \subseteq N_W \\ U \text{ is a clique}}} (-1)^{|U|} T_U T_U^* \right) T^*_{W \backslash F} \\ = & \sum_{\substack{W \supseteq F \\ W \text{ is a clique}}} \left( \sum_{\substack{U \subseteq N_W \\ U \text{ is a clique}}} (-1)^{|U|} T_{(U \bigcup W) \backslash F} T^*_{(U \bigcup W) \backslash F} \right) \end{aligned}$$

Suppose $U \subseteq N_W$ is a clique, then every vertex of $U$ is adjacent to every vertex in $W$, and vertices in $U$ are adjacent to one another. Therefore, $U \bigcup W$ is also a clique. The converse is true as well: if $U \bigcup W$ is a clique where $U \bigcap W = \emptyset$, then $U \subseteq N_W$ is a clique. Hence, we can rearrange the double summation so that we first sum over all possible cliques $V = U \bigcup W$, and then sum over all possible $U$. For a fixed clique $V = U \bigcup W$, the set $W = V \backslash U$ and the only requirement is that $F \subseteq W$. Therefore, we only sum those $U$ so that $U \subseteq V \backslash F$. Rewrite the double summation as:

$$\sum_{\substack{V=U\bigcup W\\ V \text{ is a clique}}} \left( \sum_{\substack{U\subseteq V\backslash F\\ U \text{ is a clique}}} (-1)^{|U|} T_{V\backslash F} T^*_{V\backslash F} \right).$$

For a fixed clique $V = U \bigcup W$ where $U \bigcap W = \emptyset$, consider the inner summation over all clique $U \subseteq V\backslash F$. $|U|$ can take any value between 0 and $|V\backslash F|$. Moreover, for a fixed size $|U| = k$, there are precisely $\binom{|V\backslash F|}{|U|}$ possibilities for $U$ where $U \subseteq V\backslash F$ with size $k$.

Therefore, the coefficient for $T_{V\backslash F}T^*_{V\backslash F}$ where $V$ is a clique containing $F$, is equal to

$$\sum_{j=0}^{|V\backslash F|} \binom{|V\backslash F|}{j} (-1)^j$$

This summation is equal to 1 if $V = F$ and $|V\backslash F| = 0$. Otherwise, this is equal to $(1-1)^{|V\backslash F|} = 0$. This proves the double summation is equal to $T_{F\backslash F}T^*_{F\backslash F} = I$. $\square$

We are now ready to show $K[F_c] \geq 0$. $K[F_c]$ is a $|F_c| \times |F_c|$ operator matrix, whose rows and columns are indexed by cliques $U, V$. Its $(U, V)$-entry is equal to $K[e_U, e_V]$. Eliminating common initial vertices, $K[e_U, e_V] = K[e_{U\backslash V}, e_{V\backslash U}]$. Now $e_{U\backslash V}$ commutes with $e_{V\backslash U}$ if and only if all vertices in $U\backslash V$ are adjacent to all vertices in $V\backslash U$. In other words, $U \bigcup V$ is a clique. Therefore, we have,

$$K[e_U, e_V] = \begin{cases} T_{V\backslash U} T^*_{U\backslash V}, \text{ if } U \bigcup V \text{ is a clique;} \\ 0, \text{ otherwise.} \end{cases} \tag{4.5}$$

Let $R_c$ be a $|F_c| \times |F_c|$ operator matrix, where

$$R_c[U, W] = \begin{cases} T_{W\backslash U} Z^{1/2}_{N_W}, \text{ if } U \subseteq W \\ 0, \text{ otherwise.} \end{cases} \tag{4.6}$$

**Proposition 4.4.14.** $K[F_c] = R_c \cdot R_c^*$. *In particular,* $K[F_c] \geq 0$.

*Proof.* The $(U, V)$-entry for $R_c \cdot R_c^*$ is equal to $\sum_W R_c[U, W] R_c[V, W]^*$.

If $U \bigcup V$ is not a clique, we cannot find a clique $W$ that contains both $U$ and $V$. Therefore, for every clique $W$, we cannot have both $U, V$ contained in $W$. By Equation (4.6), this implies at least one of $R_c[U,W], R_c[V,W]$ is 0. Hence, the $(U,V)$-entry for $R_c \cdot R_c^*$ is 0, which agrees with the $(U,V)$-entry of $K[F_c]$ by Equation (4.5).

If $U \bigcup V$ is a clique, then $R_c[U,W]R_c[V,W]^*$ may be non-zero only when $W$ is a clique containing both $U, V$. Therefore, in such case,

$$\begin{aligned}
&\sum_W R_c[U,W]R_c[V,W]^* \\
=&\sum_{U \bigcup V \subseteq W} R_c[U,W]R_c[V,W]^* \\
=&\sum_{U \bigcup V \subseteq W} T_{W \backslash U} Z_{N_W} T_{W \backslash V}^* \\
=&T_{V \backslash U} \left( \sum_{U \bigcup V \subseteq W} T_{W \backslash (U \bigcup V)} Z_{N_W} T_{W \backslash (U \bigcup V)}^* \right) T_{U \backslash V}^*
\end{aligned}$$

The summation in the middle is equal to $I$ by Lemma 4.4.13, in which $F$ is the fixed clique $U \bigcup V$. This proves that the $(U,V)$-entry for $R_c \cdot R_c^*$ is equal to $T_{V \backslash U} T_{U \backslash V}^* = K[e_U, e_V]$ in this case.

Therefore, we conclude that $K[F_c] = R_c \cdot R_c^*$ and $K[F_c] \geq 0$. □

**Remark 4.4.15.** *We can regard $R_c$ as a Cholesky decomposition of $K[F_c]$ by rearranging $R_c$ as a lower triangular matrix. We first notice that whenever $U$ contains more elements than $W$, $R_c[U,W] = 0$. Moreover, when $|U| = |W|$, $U \subseteq W$ is equivalent to $U = W$. Therefore, $R_c[U,W] = 0$ whenever $|U| \leq |W|$ and $U \neq W$. Therefore, if we rearrange $F_c$ according to the size of cliques (larger cliques come first), $R_c$ becomes a lower triangular matrix.*

**Example 4.4.16.** *Let us consider the graph product of $\mathbb{N}$ associated with the graph in Figure 4.2:*

*The graph semigroup is the unital semigroup generated by $e_1, e_2, e_3$ where $e_1, e_2$ commute. There are 5 cliques in this graph: $\{1,2\}$, $\{1\}$, $\{2\}$, $\{3\}$, and $\emptyset$. Under this ordering,*

$$K[F_c] = \begin{bmatrix} I & T_2^* & T_1^* & 0 & T_2^* T_1^* \\ T_2 & I & T_1^* T_2 & 0 & T_1^* \\ T_1 & T_2^* T_1 & I & 0 & T_2^* \\ 0 & 0 & 0 & I & T_3^* \\ T_1 T_2 & T_1 & T_2 & T_3 & I \end{bmatrix}.$$

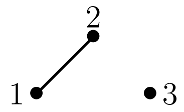

Figure 4.2: A Simple Graph on 3 Vertices

*We can write out the matrix $R_c$ using Equation (4.6):*

$$R_c = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ T_2 & Z_2^{1/2} & 0 & 0 & 0 \\ T_1 & 0 & Z_1^{1/2} & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ T_1T_2 & T_1Z_2^{1/2} & T_2Z_1^{1/2} & T_3 & Z_{\{1,2,3\}}^{1/2} \end{bmatrix}.$$

*One can verify that $K[F_c] = R_c \cdot R_c^*$.*

We are now ready to prove the main Theorem 4.2.11.

*Proof.* It suffices to prove that the Toeplitz kernel $K$ in Definition 4.2.1 is completely positive definite. For any finite subset $F \subset P_\Gamma$, it suffices to prove $K[F] \geq 0$. Proposition 4.4.10 shows that it suffices to prove $K[\tilde{F}] \geq 0$ for some finite subset $\tilde{F} \subset P_\Gamma$, where each element in $\tilde{F}$ has precisely one block. Let $\Lambda_0$ be all the vertices that appears in a syllable of some element of $\tilde{F}$, which is a finite set. Denote $F_c = \{e_J \in \Lambda_0 : J \text{ is a clique}\}$. By Lemma 4.4.11, $K[\tilde{F}] \geq 0$ if $K[F_c] \geq 0$. Finally, by Proposition 4.4.14, $K[F_c] \geq 0$. □

**Remark 4.4.17.** *The converse of Theorem 4.2.11 is also true (see Corollary 4.5.4).*

## 4.5 Nica-Covariant Representation on Graph Products

Isometric Nica-covariant representations on quasi-lattice ordered groups are first studied in [48], and were soon found to be an important concept in the study of operator algebras. Isometric Nica-covariant representations on graph semigroups, in particular graph products of ℕ, are intensively studied in [17]. It is observed in [17, Theorem 24] that an isometric representation $V$ of the graph semigroup is isometric Nica-covariant if

1. for any two adjacent vertices $i, j$, $V_i$ and $V_j$ $*$-commute.
2. for any two non-adjacent vertices $i, j$, $V_i$ and $V_j$ have orthogonal ranges. In other words, $V_i^* V_j = 0$.

Contractive Nica-covariant representations on lattice ordered semigroups are first defined and studied in [28, 21]. However, lattice order is quite restrictive compared to quasi-lattice order. For example, the free semigroup $\mathbb{F}_m^+$ is quasi-lattice ordered, but not lattice ordered. In particular, the graph product $P_\Gamma$ is only lattice ordered when the graph $\Gamma$ is the complete graph, which corresponds to the abelian semigroup $\mathbb{N}^k$. This leads to a question of which representations of the graph product $P_\Gamma$ have isometric Nica-covariant dilations.

In [29], it is shown that a pair of commuting contractions has a $*$-regular dilation if and only if they have a $*$-commuting isometric dilation, which is an equivalent way of saying a Nica-covariant dilation. The contractive Nica-covariant representations defined in [28, 21, 40] are always $*$-regular. It turns out that $*$-regular is equivalent of having an isometric Nica-covariant dilation.

**Theorem 4.5.1.** *If $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ has $*$-regular dilation, then its minimal Naimark dilation is an isometric Nica-covariant representation of the graph semigroup.*

The minimal Naimark dilation in Theorem 2.2.9 can be constructed explicitly. We loosely follow the construction in [56, Theorem 3.2]. Given a completely positive definite kernel $K : P \times P \to \mathcal{B}(\mathcal{H})$, define $\mathcal{K}_0 = P \otimes \mathcal{H}$ with a semi-inner product defined by

$$\left\langle \sum \delta_p \otimes h_p, \sum \delta_q \otimes k_q \right\rangle = \sum_{p,q} \langle K(q,p) h_p, k_q \rangle.$$

The original Hilbert space $\mathcal{H}$ can be embedded into $\mathcal{K}_0$ as $\delta_e \otimes \mathcal{H}$. The minimal Naimark dilation $V$ of $T$ acts on the $\mathcal{K}$ by $V(p)\delta_q \otimes h = \delta_{pq} \otimes h$, which are clearly isometries. Moreover, for any $h_1, h_2 \in \mathcal{H}$,

$$\begin{aligned} \langle V(q)^* V(p) h_1, h_2 \rangle =& \langle \delta_p \otimes h_1, \delta_q \otimes h_2 \rangle \\ =& \langle K(q,p) h_1, h_2 \rangle \end{aligned}$$

Therefore, $P_{\mathcal{H}} V(q)^* V(p)\big|_{\mathcal{H}} = K(q,p)$. Let $\mathcal{N} = \{k \in \mathcal{K}_0 : \langle k, k \rangle = 0\}$. One can show that $\mathcal{N}$ is invariant for all $V(p)$, and thus we can let $\mathcal{K} = \overline{\mathcal{K}_0 / \mathcal{N}}$, which is a Hilbert space. $V$ can be defined as isometries on $\mathcal{K}$, and it turns out that it is a minimal Naimark dilation. For technical details, one may refer to [56, Theorem 3.2]. It is worth noting that $\mathcal{H}$ is coinvariant for the minimal Naimark dilation $V$, and thus invariant for $V^*$.

Throughout the rest of this section, we fix a contractive representation $T$ on $P_\Gamma$ that is $*$-regular, and let $V : P_\Gamma \to \mathcal{B}(\mathcal{K})$ be the minimal Naimark dilation for $T$ described as above.

**Lemma 4.5.2.** *Suppose $p \in P_\Gamma$, $\lambda \in \Lambda$ so that $\lambda \notin I_1(p)$ and $e_\lambda$ does not commute with $p$. Then $V(e_\lambda)$ and $V(p)$ have orthogonal ranges. In other words, $V(e_\lambda)^* V(p) = 0$.*

*Proof.* It suffices to prove for any $h = \sum_i \delta_{x_i} \otimes h_i \in \mathcal{K}_0 = P_\Gamma \otimes \mathcal{H}$ and $h = \sum_j \delta_{y_j} \otimes k_i \in \mathcal{K}_0 = P_\Gamma \otimes \mathcal{H}$, $\langle V(p)h, V(e_\lambda)k\rangle = 0$.

By the definition of the pre-inner product on $\mathcal{K}_0$,

$$\begin{aligned}\langle V(p)h, V(e_\lambda)k\rangle &= \langle \sum_i \delta_{p\cdot x_i} \otimes h_i, \sum_j \delta_{e_\lambda \cdot y_j} \otimes k_i\rangle \\ &= \sum_{i,j} \langle K(e_\lambda \cdot y_j, p\cdot x_i)h_i, k_j\rangle\end{aligned}$$

Suppose $(e_\lambda \cdot y_j)^{-1} p \cdot x_i = u^{-1}v$ for some $u, v \in P_\Gamma$, where $u, v$ share no common initial vertices. By Lemma 4.1.8, $u, v$ do not commute. Therefore, $K(e_\lambda \cdot y_j, p \cdot x_i) = 0$ for all $i, j$. Hence, the inner product is equal to 0. □

**Lemma 4.5.3.** *Let $p \in P_\Gamma$ and $\lambda \in \Lambda$ be a vertex such that $\lambda \notin I_1(p)$ and $e_\lambda$ commutes with $p$. Then $V(e_\lambda)^* V(p)\big|_{\mathcal{H}} = V(p)V(e_\lambda)^*\big|_{\mathcal{H}}$*

*Proof.* By the minimality of $V$, $\operatorname{span}\{V(q)k : q \in P_\Gamma, k \in \mathcal{H}\}$ is dense in $\mathcal{K}$. Therefore, it suffices to prove for all $q \in P_\Gamma$, $h, k \in \mathcal{H}$,

$$\langle V(e_\lambda)^* V(p)h, V(q)k\rangle = \langle V(p)V(e_\lambda)^* h, V(q)k\rangle \tag{4.7}$$

Starting from the left hand side of Equation (4.7),

$$\begin{aligned}\langle V(e_\lambda)^* V(p)h, V(q)k\rangle =& \langle V(e_\lambda q)^* V(p)h, k\rangle \\ =& \langle K(e_\lambda q, p)h, k\rangle \\ =& \langle K(q,p)T(e_\lambda)^* h, k\rangle\end{aligned}$$

Here we used Lemma 4.3.5 to show $K(e_\lambda q, p) = K(q,p)T(e_\lambda)^*$. Now since $V(e_\lambda) = \begin{bmatrix} T(e_\lambda) & 0 \\ * & * \end{bmatrix}$ with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, $V(e_\lambda)^* h = T(e_\lambda)^* h \in \mathcal{H}$. Therefore,

$$\begin{aligned} \langle K(q,p)T(e_\lambda)^*h, k\rangle =& \langle K(q,p)V(e_\lambda)^*h, k\rangle \\ =& \langle V(q)^*V(p)V(e_\lambda)^*h, k\rangle \\ =& \langle V(p)V(e_\lambda)^*h, V(q)k\rangle \end{aligned}$$

This proves Equation (4.7). □

We now prove the main result of this section:

*Proof of Theorem 4.5.1.* It suffices to pick any two vertices $\lambda_1, \lambda_2$ and consider two cases when they are adjacent or not.

If $\lambda_1, \lambda_2$ are not adjacent, by Lemma 4.5.2, $V(e_{\lambda_1})$ and $V(e_{\lambda_2})$ are isometries with orthogonal ranges.

If $\lambda_1, \lambda_2$ are adjacent, it suffices to prove for all $p \in P_\Gamma$,

$$V(e_{\lambda_1})^*V(e_{\lambda_2})V(p)\big|_{\mathcal{H}} = V(e_{\lambda_2})V(e_{\lambda_1})^*V(p)\big|_{\mathcal{H}}. \tag{4.8}$$

Indeed, since $\operatorname{span}\{V(p)h : p \in P_\Gamma, h \in \mathcal{H}\}$ is dense in $\mathcal{K}$, Equation (4.8) implies that $V(e_{\lambda_1})^*V(e_{\lambda_2}) = V(e_{\lambda_2})V(e_{\lambda_1})^*$.

There are now several possibilities:

If $\lambda \in I_1(p)$, we can write $p = e_{\lambda_1}p'$, and thus $V(p) = V(e_{\lambda_1})V(p')$. Since $\lambda_1, \lambda_2$ are adjacent, $V(e_{\lambda_1})$ commutes with $V(e_{\lambda_2})$. Hence, both sides of the Equation (4.8) are equal to $V(e_{\lambda_2})V(p')\big|_{\mathcal{H}}$.

If $\lambda \notin I_1(p)$ and $e_{\lambda_1}$ does not commute with $p$, then $\lambda_1 \notin I_1(e_{\lambda_2}p)$ and $e_{\lambda_1}$ does not commute with $e_{\lambda_2}p$ as well. Therefore, by Lemma 4.5.2, $V(e_{\lambda_1})$ and $V(p)$ are isometries with orthogonal ranges, and $V(e_{\lambda_1})^*V(p) = 0$. Similarly, $V(e_{\lambda_1})$ and $V(e_{\lambda_2}p)$ are isometries with orthogonal ranges, and $V(e_{\lambda_1})^*V(e_{\lambda_2}p) = 0$. Both sides of the Equation (4.8) are 0.

Lastly, if $\lambda \notin I_1(p)$ and $e_{\lambda_1}$ commutes with $p$. Then $e_{\lambda_2}p$ and $p$ are both element in $P_\Gamma$ that commutes with $e_{\lambda_1}$ without $\lambda_1$ as an initial vertex. By Lemma 4.5.3, for every $h \in \mathcal{H}$,

$$\begin{aligned} V(e_{\lambda_1})^*V(e_{\lambda_2})V(p)h =& V(e_{\lambda_2})V(p)V(e_{\lambda_1})^*h \\ =& V(e_{\lambda_2})V(e_{\lambda_1})^*V(p)h \end{aligned}$$

This is precisely the Equation (4.8), and thus we finished the proof. □

**Corollary 4.5.4.** *Let $T$ be a contractive representation of a graph product of $\mathbb{N}$. If $T$ is has a minimal isometric Nica-covariant dilation, then,*

$$\sum_{\substack{U\subseteq W \\ U \text{ is a clique}}} (-1)^{|U|} T_U T_U^* \geq 0.$$

*Proof.* Let $V : P_\Gamma \to \mathcal{B}(\mathcal{K})$ be the minimal Naimark dilation for $T$. We have $\mathcal{H}$ is co-invariant for $V$, and thus with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, $V(p) = \begin{bmatrix} T(p) & 0 \\ * & * \end{bmatrix}$. Therefore, for every clique $U$ in $\Gamma$,

$$T_U T_U^* = P_{\mathcal{H}} V(e_U) V(e_U)^* \big|_{\mathcal{H}}.$$

It suffices to show for every $W \subseteq \Lambda$,

$$\sum_{\substack{U\subseteq W \\ U \text{ is a clique}}} (-1)^{|U|} V(e_U) V(e_U)^* \geq 0. \tag{4.9}$$

For each vertex $i \in \Lambda$, denote $P_i = V(e_i)V(e_i)^*$ the range projection of the isometry $V(e_i)$. Since $V$ is Nica-covariant, $P_i, P_j$ commutes and

$$P_i P_j = \begin{cases} V_i V_j V_j^* V_i^*, & \text{if } i \text{ is adjacent to } j; \\ 0, & \text{otherwise.} \end{cases}$$

For each $U \subseteq W$, denote $P_U = \prod_{i\in U} P_i$ and in particular let $P_\emptyset = I$. If $U \subseteq W$ is not a clique, then we can find two vertices $i, j \in U$ that are not adjacent. Since $P_i P_j = 0$, it follows that $P_U = 0$. If $U \subseteq W$ is a clique, then it follows from that Nica-covariant condition that $P_U = V(e_U)V(e_U)^*$.

Consider the projection $R = \prod_{i\in W}(I - P_i)$:

$$\begin{aligned} R &= \prod_{i\in W}(I - P_i) \\ &= \sum_{U\subseteq W} (-1)^{|U|} P_U \\ &= \sum_{\substack{U\subseteq W \\ U \text{ is a clique}}} (-1)^{|U|} P_U \\ &= \sum_{\substack{U\subseteq W \\ U \text{ is a clique}}} (-1)^{|U|} V(e_U)V(e_U)^*. \end{aligned}$$

Since $R$ is a projection, $R \geq 0$ and this proves condition (4.9). □

We have now established the equivalence among Condition (4.1), $*$-regular, and having a minimal isometric dilation that is Nica-covariant.

**Theorem 4.5.5.** *Let $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ be a representation of a graph product of $\mathbb{N}$. Then the following are equivalent:*

1. *$T$ has $*$-regular dilation,*
2. *$T$ has a minimal isometric dilation that is Nica-covariant,*
3. *$T$ satisfies Condition (4.1).*

*Proof.* (1) $\Longrightarrow$ (2) is established in Theorem 4.5.1. (1) $\Longrightarrow$ (2) is established in Corollary 4.5.4. Finally, (3) $\Longrightarrow$ (1) is established in Theorem 4.2.11. □

## 4.6 The Property (P)

Popescu [55] first studied the noncommutative Poisson transform associated to a certain class of operators that satisfies the property (P). The property (P) has recently been generalized to higher rank graphs [62, 63]. It turns out that the class of operators Popescu studied can be viewed as a representation of a graph product of $\mathbb{N}$, and we thereby extend the Property (P) to representations of graph products of $\mathbb{N}$. This section proves that $*$-regular condition implies the property (P), and they are equivalent under certain conditions.

Throughout this section, we fix a finite simple graph $\Gamma$ whose vertex set is denoted by $\Lambda$.

**Definition 4.6.1.** *A contractive representation* $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ *is said to have* the Property (P) *if there exists* $0 \leq \rho < 1$ *so that for all* $\rho \leq r \leq 1$,

$$\sum_{\substack{U \subseteq \Lambda \\ U \text{ is a clique}}} (-1)^{|U|} r^{|U|} T(e_U) T(e_U)^* \geq 0. \tag{4.10}$$

**Example 4.6.2.** *Let* $\Gamma$ *be a complete* $k$*-partite graph* $K_{n_1,n_2,\cdots,n_k}$. *In other words, denote* $\Lambda = \{(i,j) : 1 \leq i \leq k, 1 \leq j \leq n_i\}$ *be the vertex set, and* $(i_1, j_1)$ *is adjacent to* $(i_2, j_2)$ *in* $\Gamma$ *if and only if* $i_1 \neq i_2$. *A contractive representation* $T$ *of this graph semigroup* $P_\Gamma$ *is uniquely determined by* $T_{i,j} = T(e_{i,j})$. *Here, for each* $i$, $T_{i,1}, \cdots, T_{i,n_i}$ *are not necessarily commuting contractions. However, for each* $i_1 \neq i_2$, $T_{i_1,j_1}$ *commutes with* $T_{i_2,j_2}$.

*In [55], Popescu considered such class of operators* $\{T_{i,j}\}$ *where for each* $i$, $\{T_{i,j}\}_{j=1}^{n_j}$ *forms a row contraction in the sense that,*

$$\sum_{j=1}^{n_i} T_{i,j} T_{i,j}^* \leq I.$$

*This family of operators is also considered in many subsequent papers on non-commutative polyballs (see also [57, 59]). For such family of operators, Popescu says it has the property (P) if condition (4.10) is satisfied. It is observed in [55] that the property (P) allows one to obtain a Poisson transform and subsequently a dilation of the family of operators* $\{T_{i,j}\}$.

*One may observe that Definition 4.6.1 of the property (P) does not require the row contractive condition. Instead, this paper mostly considers a contractive representation* $T$ *of the graph product* $P_\Gamma$ *that satisfies condition (4.1) and thus has a* $*$*-regular dilation. The row contractive condition is embedded in condition (4.1).*

Our first result shows that if $T$ satisfies condition (4.1), then it has the property (P). Let $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ be a representation that satisfies condition (4.1). By Theorem 4.5.5, it has a minimal isometric Nica-covariant dilation $V : P_\Gamma \to \mathcal{B}(\mathcal{K})$. Moreover, $\mathcal{H}$ is co-invariant for $V$, and thus

$$P_\mathcal{H} V(e_U) V(e_U)^* \big|_\mathcal{H} = T(e_U) T(e_U)^*.$$

Therefore, to show $T$ has the property (P), it suffices to show $V$ has the property (P). For $r \in \mathbb{R}$, let us denote

$$f(r) = \sum_{\substack{U \subseteq \Lambda \\ U \text{ is a clique}}} (-1)^{|U|} r^{|U|} V(e_U) V(e_U)^*.$$

It follows from the proof of Corollary 4.5.4 that $f(1) \geq 0$. In fact, $f(1)$ is a projection onto the subspace that is orthogonal to all the ranges of $V(e_i)$. Following the notation we used in the proof of Corollary 4.5.4, for each vertex $i \in \Lambda$, denote $P_i = V_i V_i^*$. Since $V$ is Nica-covariant, $P_i, P_j$ commute, and

$$P_i P_j = \begin{cases} V_i V_j V_j^* V_i^*, & \text{if } i \text{ is adjacent to } j; \\ 0, & \text{otherwise.} \end{cases}$$

For each $U \subseteq \Lambda$, denote $P_U = \prod_{i \in U} P_i$, the projection onto the intersection of the ranges of all $\{P_i\}_{i \in U}$. In particular, we let $P_\emptyset = I$. Notice that if there are two vertices $i, j \in U$ that are not adjacent, $P_i P_j = 0$ and thus $P_U = 0$. Therefore, $P_U \neq 0$ only if $U$ is a clique. The function $f(r)$ can be rewritten as

$$\begin{aligned} f(r) &= \sum_{\substack{U \subseteq \Lambda \\ U \text{ is a clique}}} (-1)^{|U|} r^{|U|} P_U \\ &= \sum_{U \subseteq \Lambda} (-1)^{|U|} r^{|U|} P_U \\ &= \sum_{k=0}^{|\Lambda|} \left( \sum_{\substack{U \subseteq \Lambda \\ |U| = k}} (-1)^k P_U \right) r^k \end{aligned}$$

For each $U \subseteq \Lambda$, denote $R_U = P_U \cdot \prod_{i \notin U} P_i^\perp$. The range of $R_U$ are those vectors that are contained in the range of $P_U$ but orthogonal to the range of $P_i$ where $i \notin U$. In particular, $R_\emptyset = \prod_{i \in \Lambda} P_i^\perp$, which is the projection onto those vectors that are orthogonal to the ranges of all $P_i$. It was observed in Corollary 4.5.4 that

$$R_\emptyset = \sum_{\substack{U \subseteq \Lambda \\ U \text{ is a clique}}} (-1)^{|U|} V(e_U) V(e_U)^* = f(1).$$

Finally, denote

$$Q_m = \sum_{\substack{U \subseteq \Lambda \\ |U|=m}} R_U. \tag{4.11}$$

In particular, $Q_0 = R_\emptyset = f(1)$. Notice that if two distinct subsets $U_1, U_2 \subseteq \Lambda$ and $|U_1| = |U_2| = m$, then at least one vertex in $U_1$ is not in $U_2$ and vice versa. Therefore, $R_{U_1} R_{U_2} = 0$ and thus $R_{U_1}, R_{U_2}$ are projections onto orthogonal subspaces. Hence, $Q_m$ is a projection. Intuitively, the range of $Q_m$ are those vectors that are contained in the range of $m$ of $P_i$ and orthogonal to the range of all other $P_i$. Therefore, $\{Q_m\}_{m=0}^{|\Lambda|}$ are pairwise orthogonal projections and

$$\sum_{m=0}^{|\Lambda|} Q_m = I.$$

We first obtain a Taylor expansion of $f$ about $r = 1$. For each $1 \leq m \leq |\Lambda|$, the $m$-th derivative of $f$ is equal to:

$$\begin{aligned} f^{(m)}(r) &= \sum_{k=m}^{|\Lambda|} \sum_{\substack{U \subseteq \Lambda \\ |U|=k}} (-1)^k \frac{k!}{(k-m)!} r^{k-m} P_U \\ &= (-1)^m m! \sum_{k=m}^{|\Lambda|} \sum_{\substack{U \subseteq \Lambda \\ |U|=k}} (-1)^{k-m} \binom{k}{m} r^{k-m} P_U \end{aligned}$$

**Lemma 4.6.3.** *$f^{(m)}(1) = (-1)^m m! \cdot Q_m$. Moreover, $f$ has the Taylor series expansion*

$$f(r) = \sum_{m=0}^{|\Lambda|} (-1)^m (r-1)^m Q_m.$$

*Proof.* It suffices to prove

$$Q_m = \sum_{k=m}^{|\Lambda|} \sum_{\substack{U \subseteq \Lambda \\ |U|=k}} (-1)^{k-m} \binom{k}{m} P_U.$$

Denote the right hand side of the summation $S_m$. It suffices to prove

$$S_m Q_i = Q_i S_m = \begin{cases} Q_i, & \text{if } i = m; \\ 0, & \text{if } i \neq m. \end{cases}$$

From Equation (4.11), $Q_m$ is the sum of all $R_W$ where $|W| = m$. Since $\{R_W\}_{|W|=m}$ are pairwise orthogonal projections, it suffices to prove

$$S_m R_W = R_W S_m = \begin{cases} R_W, & \text{if } |W| = m; \\ 0, & \text{if } |W| \neq m. \end{cases}$$

First of all, since $\{P_i\}_{i\in\Lambda}$ are commuting orthogonal projections, $R_W, S_m$ commute for all $W \subseteq \Lambda$ and $0 \leq m \leq |\Lambda|$. Fix $W$ and consider $S_m R_W$.

If $|W| < m$, then every $|U| \geq m$ contains some vertex not in $W$. Therefore, $P_U R_W = 0$, and hence $S_m R_W = 0$.

If $|W| \geq m$, then for each $|U| \geq m$,

$$P_U R_W = \begin{cases} R_W, & \text{if } U \subseteq W; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}
S_m R_W &= \left( \sum_{k=m}^{|\Lambda|} \sum_{\substack{U \subseteq \Lambda \\ |U|=k}} (-1)^{k-m} \binom{k}{m} P_U \right) \cdot R_W \\
&= \sum_{k=m}^{|W|} \sum_{\substack{U \subseteq W \\ |U|=k}} (-1)^{k-m} \binom{k}{m} R_W \\
&= \sum_{k=m}^{|W|} (-1)^{k-m} \binom{|W|}{k} \binom{k}{m} \binom{k}{m} R_W \\
&= \sum_{k=m}^{|W|} (-1)^{k-m} \frac{|W|!}{k!(|W|-k)!} \frac{k!}{m!(k-m)!} R_W \\
&= \binom{|W|}{m} \sum_{k=m}^{|W|} (-1)^{k-m} \binom{|W|-m}{k-m} R_W \\
&= \binom{|W|}{m} \sum_{j=0}^{|W|-m} (-1)^{j} \binom{|W|-m}{j} R_W.
\end{aligned}$$

Here, $\sum_{j=0}^{|W|-m}(-1)^j \binom{|W|-m}{j}$ is equal to $(1-1)^{|W|-m} = 0$ if $|W| > m$, and 1 if $|W| = m$. Therefore,

$$S_m R_W = \begin{cases} R_W, & \text{if } |W| = m; \\ 0, & \text{otherwise.} \end{cases}$$

This proves $S_m = Q_m$. Since the graph $\Gamma$ is assumed to be a finite graph, $f(r)$ is a finite operator-valued polynomial. Its Taylor series expansion about 1 is equal to:

$$\begin{aligned}
f(r) &= \sum_{m=0}^{|\Lambda|} \frac{f^{(m)}(1)}{m!} (r-1)^m \\
&= \sum_{m=0}^{|\Lambda|} (-1)^m (r-1)^m Q_m.
\end{aligned}$$

□

**Theorem 4.6.4.** *If a representation $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ has $*$-regular dilation, then $T$ satisfies property (P). Moreover, the constant $\rho$ in property (P) can be chosen to be* 0.

*Proof.* Let $V : P_\Gamma \to \mathcal{B}(\mathcal{K})$ be the minimal isometric $*$-regular dilation for $T$. By Lemma 4.6.3, for each $0 \le r \le 1$,

$$\begin{aligned} f(r) &= \sum_{\substack{U \subseteq \Lambda \\ U \text{ is a clique}}} (-1)^{|U|} r^{|U|} P_U \\ &= \sum_{m=0}^{|\Lambda|} (-1)^m (r-1)^m Q_m \end{aligned}$$

For $0 \le r \le 1$, $(-1)^m (r-1)^m \ge 0$. Since each $Q_m$ is an orthogonal projection, $f(r) \ge 0$. Notice when $U$ is a clique, $P_U = V_U V_U^*$, where $V_U = \begin{bmatrix} T_U & 0 \\ * & * \end{bmatrix}$ with respect to $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. Therefore, by projecting onto the corner corresponding to $\mathcal{H}$, we obtain that for all $0 \le r \le 1$,

$$\sum_{\substack{U \subseteq \Lambda \\ U \text{ is a clique}}} (-1)^{|U|} r^{|U|} T_U T_U^* \ge 0.$$

This implies $T$ satisfies the property (P) with $\rho = 0$. $\square$

It is not clear when the converse of Theorem 4.6.4 also holds. Popescu established in [55, Corollary 5.2] the converse for a special class of operators. Recall a complete $k$-multipartite graph $K_{n_1,\cdots,n_k}$ is a graph with vertices $V = \{(i,j) : 1 \le i \le k, 1 \le j \le n_i\}$ and each vertex $(i,j)$ is adjacent to all other vertices except $(i,j')$.

**Proposition 4.6.5** (Corollary 5.2, [55])**.** *Let $\Gamma = K_{n_1,\cdots,n_k}$ be a complete $k$-multipartite graph. Let $\{T_{i,j} \in \mathcal{B}(\mathcal{H}) : 1 \le i \le k, 1 \le j \le n_i\}$ be a family of operators such that:*

1. *For each $i$, $\sum_{j=1}^{n_i} T_{i,j} T_{i,j}^* \le I$,*
2. *The associated representation $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ has property (P).*

*Then the associated representation $T$ has a minimal isometric Nica-covariant dilation.*

However, for a representation of an arbitrary graph semigroup, it is not clear how one can replace Condition (1) in Proposition 4.6.5.

**Example 4.6.6.** *Let us consider the special case when $n_1 = \cdots = n_k = 1$ and the graph $\Gamma$ is the complete graph on $k$-vertices. Let $\{T_i\}_{i=1}^k$ be a family of operators as in Proposition 4.6.5. Notice that Condition (1) is simply saying that each $T_i$ is a contraction. Proposition 4.6.5 states that such $T_i$ has a minimal isometric Nica-covariant dilation, and thus by Theorem 4.5.5, $T_i$ has to satisfy Condition (4.1). Note that in a complete graph, Condition 4.1 is the same as Brehmer's Condition (2.1).*

*In fact, we can derive condition (4.1) directly from the property (P), without invoking the minimal isometric Nica-covariant dilation.*

*For any subset $W \subseteq \{1, 2, \cdots, n\}$, denote*

$$\Delta_W(r) = \sum_{U \subseteq W} (-1)^{|U|} r^{|U|} T_U T_U^*.$$

*The property (P) implies for some $0 \leq \rho < 1$ and all $\rho \leq r \leq 1$, $\Delta_{\{1,2,\cdots,n\}}(r) \geq 0$. For any $1 \leq i \leq n$, let $W_i = \{1, \cdots, i-1, i+1, \cdots, n\}$. Notice that,*

$$\Delta_{\{1,2,\cdots,n\}}(r) = \Delta_{W_i}(r) - rT_i \Delta_{W_i}(r) T_i^*.$$

*We claim that $\Delta_{W_i}(r) \geq 0$ for all $\rho \leq r < 1$. If otherwise, since $\Delta_{W_i}(r)$ is a self-adjoint operator, let*

$$-M = \inf\{\langle \Delta_{W_i}(r)h, h\rangle : \|h\| = 1\} < 0.$$

*Pick a unit vector $h$ so that $-M \leq \langle \Delta_{W_i}(r)h, h\rangle < -M \cdot r$. Then,*

$$\begin{aligned} \langle rT_i \Delta_{W_i}(r) T_i^* h, h\rangle &= r \cdot \langle \Delta_{W_i}(r) T_i^* h, T_i^* h\rangle \\ &\geq -M \cdot r. \end{aligned}$$

*Therefore,*

$$\begin{aligned} \langle \Delta_{\{1,2,\cdots,n\}}(r)h, h\rangle &= \langle \Delta_{W_i}(r)h, h\rangle - \langle rT_i \Delta_{W_i}(r) T_i^* h, h\rangle \\ &< -M \cdot r + M \cdot r = 0. \end{aligned}$$

*This contradicts that $\Delta_{\{1,2,\cdots,n\}}(r) \geq 0$. Hence, we can conclude that $\Delta_{W_i}(r) \geq 0$. In other words, $\{T_1, \cdots, T_{i-1}, T_{i+1}, \cdots, T_n\}$ satisfies the property (P). Similarly, by removing one element each time, we obtain that for any $W \subseteq \{1, 2, \cdots, n\}$, $\Delta_W(r) \geq 0$ for all $\rho \leq r < 1$. In particular, let $r \to 1$, we obtain that for every $W \subseteq \{1, 2, \cdots, n\}$,*

$$\sum_{U \subseteq W} (-1)^{|U|} T_U T_U^* \geq 0.$$

*This is exactly Condition (4.1) on the complete graph (equivalently, Brehmer's Condition (2.1)).*

**Remark 4.6.7.** *For an arbitrary graph* $\Gamma$*, it is not clear how we can replace Condition (2) in Proposition 4.6.5 to guarantee a minimal isometric Nica-covariant dilation for a representation* $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$*.*

# Chapter 5

# Regular Dilation on Other Semigroups

In Chapter 4, Theorem 4.5.1 stated that a contractive representation on graph product of $\mathbb{N}$ has $*$-regular dilation if and only if it has a minimal isometric Nica-covariant dilation. This motivated us to consider $*$-regular dilation on more general semigroups, where we treat having a $*$-regular dilation as being a compression of an isometric Nica-covariant representation. This allows us to extend the definition of $*$-regular dilation to any right LCM semigroups.

The difficulty, however, comes from the lack of a satisfactory analogue of the matrix reduction tricks that we used in the case of graph product of $\mathbb{N}$ (e.g. Corollary 4.3.6). Instead, we work around this difficulty by directly studying the Cholesky decomposition of the operator matrix arising from the Toeplitz kernel and obtain Brehmer-type conditions (Theorem 5.1.8).

The condition we obtain requires that for every finite subset $F$ of the semigroup $P$, a certain operator $Z_F$ must be positive. This can be a difficult condition to check, which motivates us to reduce this condition to a smaller collection of subsets. With the help of a few technical lemmas (Lemma 5.2.2), we show the condition can be reduced to checking finite subsets of the set of minimal elements when the semigroup satisfies the descending chain condition (Theorem 5.2.8).

We then apply our result to study regular dilation on many examples of right LCM semigroups and derive their corresponding $*$-regular condition.

## 5.1 Regular Dilation on Right LCM Semigroups

Fix a right LCM semigroup $P$. Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Suppose $T$ has a minimal isometric Nica-covariant dilation $V$, then the Toeplitz kernel $K$ defined by $V$ can be written out in terms of $T$ in an explicit way.

**Proposition 5.1.1.** *Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation of a right LCM semigroup $P$. Suppose $T$ has a minimal isometric Nica-covariant dilation $V$. Then,*

$$P_{\mathcal{H}} V(p)^* V(q)\big|_{\mathcal{H}} = T(p^{-1}s)T(q^{-1}s)^*$$

*for all $p, q \in P$, $s \in p \vee q$.*

*Proof.* By the Nica-covariance, $V(p)V(p)^*V(q)V(q)^* = V(s)V(s)^*$ for $s \in p \vee q$, where by convention, $V(s) = 0$ if $p \vee q = \infty$. Multiplying $V(p)^*$ on the left and $V(q)$ on the right gives us

$$V(p)^*V(q) = V(p^{-1}s)V(q^{-1}s)^*.$$

Since $V$ is minimal, $\mathcal{H}$ is co-invariant. With respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, each $V(a)$ can be written as

$$V(a) = \begin{bmatrix} T(a) & 0 \\ * & * \end{bmatrix}.$$

Therefore, for any $a, b \in P$, $V(a)V(b)^*$ can be written as

$$V(a)V(b)^* = \begin{bmatrix} T(a)T(b)^* & * \\ * & * \end{bmatrix}.$$

Therefore,

$$\begin{aligned} & P_{\mathcal{H}} V(p)^* V(q)\big|_{\mathcal{H}} \\ & = P_{\mathcal{H}} V(p^{-1}s) V(q^{-1}s)^*\big|_{\mathcal{H}} \\ & = T(p^{-1}s)T(q^{-1}s)^* \end{aligned}$$

This proves the desired result. □

This motivates our definition of $*$-regular dilation.

**Definition 5.1.2.** *Let $P$ be a right LCM semigroup and $T : P \to \mathcal{B}(\mathcal{H})$ a unital contractive representation. Define a kernel $K : P \times P \to \mathcal{B}(\mathcal{H})$ by*

$$K(p,q) = T(p^{-1}s)T(q^{-1}s)^*$$

*for all $p, q \in P$, $s \in p \vee q$. Here, we assume by convention that when $p \vee q = \infty$, $K(p,q) = 0$.*

*We say $T$ has a $*$-regular dilation if this kernel $K$ is completely positive definite. In such case, the minimal Naimark dilation $V$ of the kernel $K$ is called the $*$-regular dilation of $T$.*

**Remark 5.1.3.** *This kernel $K$ is well defined since for any $s, t \in p \vee q$, there exists an invertible $u$ with $s = tu$. Therefore,*

$$T(p^{-1}s)T(q^{-1}s)^* = T(p^{-1}t)T(u)T(u)^*T(q^{-1}t)^* = T(p^{-1}t)T(q^{-1}t)^*.$$

**Remark 5.1.4.** *This kernel $K$ is a Toeplitz kernel. It is clear that $K(e,e) = I$, $K(p,q) = K(q,p)^*$. If $a \in P$, by Lemma 2.1.17, we have $ap \vee aq = a(p \vee q)$ and therefore $(ap)^{-1}(ap \vee aq) = p^{-1}(p \vee q)$ and similarly $(aq)^{-1}(ap \vee aq) = q^{-1}(p \vee q)$.*

It is now evident from Proposition 5.1.1 that the kernel in the Definition 5.1.2 is our only choice if we desire $T$ to have a minimal isometric Nica-covariant dilation. We shall soon see that if this kernel $K$ is completely positive definite, then its minimal Naimark dilation is Nica-covariant (Theorem 5.1.7). We first note that our definition of $*$-regular dilation coincides with the definition in the context of $\ell$-semigroups and graph products of $\mathbb{N}$.

**Example 5.1.5.** *In the case that $P$ is an $\ell$-semigroup, regular dilation was first defined and studied in [21] and a necessary and sufficient condition was given in [40]. In such case, for every $p, q \in P$, there exists a unique pair $g_+, g_- \in P$ with $p^{-1}q = g_-^{-1}g_+$ and $g_- \wedge g_+ = e$. The definition of $*$-regularity on an $\ell$-semigroup is equivalent to the kernel $K(p,q) = T(g_+)T(g_-)^*$ being completely positive definite.*

*In fact, $g_+ = (p \wedge q)^{-1}q = p^{-1}(p \vee q)$ and $g_- = (p \wedge q)^{-1}p = q^{-1}(p \vee q)$, and it is clear that these two definitions coincide.*

*Historically, Brehmer's original definition of regular dilation on $\mathbb{N}^k$ requires the kernel $K(p,q) = T(g_-)^*T(g_+)$ to be completely positive, which is equivalent to $T^*$ being $*$-regular. This is why we adopt the notion of $*$-regular dilation instead of regular dilation to be consistent with Brehmer's definition.*

**Example 5.1.6.** *In the case that $P$ is a graph product of $\mathbb{N}$, $*$-regular dilation was recently defined in [42] as a generalization of the Brehmer dilation and Frazho-Bunce-Popescu dilation. The definition of $*$-regular dilation in this case can be summarized as follow: given $p, q \in P$, one first identifies the largest $a \in P$ so that $p = a \cdot p', q = a \cdot q'$ via repeatedly removing a common initial syllable. This procedures ends when there is no $e \neq b \in P$ with $p' = b \cdot p''$ and $q' = b \cdot q''$. Then the kernel is defined as*

$$K(p,q) = K(p',q') = \begin{cases} T(q')T(p')^*, & \text{if } p', q' \text{ commute;} \\ 0, & \text{otherwise.} \end{cases}$$

*Now if $p', q'$ do not commute, then $p' \vee q' = \infty$ and similarly $p \vee q = \infty$. Otherwise, since they have no common initial syllable, $p' \vee q' = p'q'$. Therefore,*

$$\begin{aligned} p^{-1}(p \vee q) &= p'^{-1}(p' \vee q') \\ &= p'^{-1}p' \cdot q' = q' \end{aligned}$$

*Similarly, $q^{-1}(p \vee q) = p'$. Again, the Definition 5.1.2 coincides with that in [42].*

**Theorem 5.1.7.** *$T$ has a $*$-regular dilation if and only if it has a minimal isometric Nica-covariant dilation.*

*Proof.* It follows from Proposition 5.1.1 that if $V$ is a minimal isometric Nica-covariant dilation, then for any $p, q \in P$ and $s \in p \vee q$,

$$K(p,q) = T(p^{-1}s)T(q^{-1}s)^* = P_{\mathcal{H}} V(p)^* V(q)\big|_{\mathcal{H}}.$$

Since $V$ is a minimal isometric dilation of $T$, it follows from the second half of Theorem 2.2.9 that $K$ is completely positive definite, which is exactly what it mean for $T$ to have a $*$-regular dilation.

Conversely, suppose that $T$ has a $*$-regular dilation so that the kernel $K$ in the Definition 5.1.2 is completely positive definite. Let $V : P \to \mathcal{B}(\mathcal{K})$ be the minimal Naimark dilation as constructed in the proof of Theorem 2.2.9. We first show that for any $p, q \in P$ and $s \in p \vee q$, $V(p)^*V(q)|_{\mathcal{H}} = V(p^{-1}s)V(q^{-1}s)^*|_{\mathcal{H}}$ (in case of $p \vee q = \infty$, the right hand side is 0 by convention).

Since $\text{span}\{V(r)h : r \in P, h \in \mathcal{H}\}$ is dense in $\mathcal{K}$, it suffices to prove for any $r \in P$ and $h, k \in \mathcal{H}$, we have

$$\langle V(p)^*V(q)h, V(r)k \rangle = \langle V(p^{-1}s)V(q^{-1}s)^*h, V(r)k \rangle.$$

Starting from the left hand side:

$$\begin{aligned}\langle V(p)^* V(q)h, V(r)k\rangle &= \langle V(pr)^* V(q)h, k\rangle \\ &= \langle K(pr,q)h, k\rangle_{\mathcal{H}}\end{aligned}$$

When $p \vee q = \infty$, $pr \vee q = \infty$ and thus $K(pr,q) = 0$ which coincides with the right hand side (which is assumed to be 0 in this case). Otherwise, there are two cases,

Case 1: If $t \in pr \vee q \neq \emptyset$, $K(pr,q) = T((pr)^{-1}t)T(q^{-1}t)^*$. Since $pr \vee q \neq \emptyset$, $p \vee q \neq \emptyset$ and we can take $s \in p \vee q$ and $w = p^{-1}s$. Notice now, by Lemma 2.1.19,

$$p^{-1}(pr \vee q) = r \vee (p^{-1}s) = r \vee w.$$

Hence,

$$\begin{aligned}q^{-1}(pr \vee q) &= q^{-1}s \cdot s^{-1}(pr \vee q) \\ &= q^{-1}s \cdot w^{-1}p^{-1}(pr \vee q) \\ &= q^{-1}s \cdot w^{-1}(r \vee w).\end{aligned}$$

Therefore, take $v = p^{-1}t \in r \vee w$,

$$\begin{aligned}&\langle T((pr)^{-1}t)T(q^{-1}t)^* h, k\rangle_{\mathcal{H}} \\ =&\langle T(r^{-1}v)T(w^{-1}v)^* T(q^{-1}s)^* h, k\rangle_{\mathcal{H}} \\ =&\langle K(r,w)V^*(q^{-1}s)^* h, k\rangle_{\mathcal{H}} \\ =&\langle V(p^{-1}s)V^*(q^{-1}s)^* h, V(r)k\rangle.\end{aligned}$$

Here, we used the fact that for all $s \in P$, $\mathcal{H}$ is co-invariant for $V$ and thus $h' = V^*(q^{-1}s)^* h \in \mathcal{H}$. Since $h', k \in \mathcal{H}$,

$$\langle K(r,w)h', k\rangle = \langle V(r)^* V(w)h', k\rangle = \langle V(w)h', V(r)k\rangle.$$

Case 2: In the case when $pr \vee q = \emptyset$, $K(pr,q) = 0$. Hence, $\langle V(p)^* V(q)h, V(r)k\rangle = 0$. On the right hand side, in $\mathcal{H}$ is invariant for $V^*$, we have,

$$\begin{aligned}&\langle V(p^{-1}s)V(q^{-1}s)^* h, V(r)k\rangle \\ =&\langle V(r)^* V(p^{-1}s)T(q^{-1}s)^* h, k\rangle \\ =&\langle K(r,p^{-1}s)T(q^{-1}s)^* h, k\rangle\end{aligned}$$

Since $pr \vee q = \emptyset$, we have $prP \cap qP = \emptyset$ and thus $prP \cap qP \cap pP = prP \cap sP = \emptyset$. Multiply by $p^{-1}$, we obtain $rP \cap p^{-1}sP = \emptyset$. Hence $r \vee (p^{-1}s) = \emptyset$ and $K(r, p^{-1}s) = 0$ by definition. Both sides are 0 in this case.

Now it suffices to show for all $r \in P$ and $s \in p \vee q$,

$$V(p)^* V(q) V(r)|_{\mathcal{H}} = V(p^{-1}s) V(q^{-1}s)^* V(r)|_{\mathcal{H}}.$$

Denote $w = q^{-1}s$ and similar to the computation earlier, observe that $w \vee r = q^{-1}(p \vee qr)$. Take $t \in p \vee qr$ and $v = q^{-1}t \in w \vee r$, and start from the left,

$$\begin{aligned} V(p)^* V(q) V(r)|_{\mathcal{H}} =& V(p^{-1}t) V(r^{-1}q^{-1}t)^*|_{\mathcal{H}} \\ =& V(p^{-1}s) V(w^{-1}v) V^*(r^{-1}v)|_{\mathcal{H}} \\ =& V(p^{-1}s) V(w)^* V(r)|_{\mathcal{H}} \\ =& V(p^{-1}s) V(q^{-1}s)^* V(r)|_{\mathcal{H}} \end{aligned}$$

This proves for any $p, q \in P$ and $s \in p \vee q$, $V(p)^* V(q) = V(p^{-1}s) V(q^{-1}s)^*$. Multiplying $V(p)$ on the left and $V(q)^*$ on the right proves that $V$ is Nica-covariant. □

It has been observed that the kernel $K$ being completely positive is often equivalent to a Brehmer-type condition where a collection of operators (instead of a collection of operator matrices) are positive. This is the case in Brehmer's dilation, Frazho-Bunce-Popescu's dilation, and more recently, dilation on graph products of $\mathbb{N}$. We first establish a Brehmer-type condition in the case of an arbitrary right LCM semigroup.

For simplicity, we shall denote $TT^*(p) = T(p)T(p)^*$. It is clear that $TT^*(pq) = T(p)TT^*(q)T(p)^*$. Since $T$ is contractive, for each invertible $u \in P^*$, $T(u)$ must be an unitary. For a finite subset $F \subset P$, we define $TT^*(\vee F) = TT^*(p)$ for some $p \in \vee F$. This is well-defined since for any two $p, q \in \vee P$, $p = qu$ for some invertible $u \in P^*$. Therefore, $TT^*(p) = T(q)TT^*(u)T(q)^* = TT^*(q)$.

**Theorem 5.1.8.** *Let $T : P \to \mathcal{B}(\mathcal{H})$ be a unital representation of a right LCM semigroup. The following are equivalent:*

1. *$T$ has a $*$-regular dilation;*
2. *$T$ has a minimal isometric Nica-covariant dilation;*
3. *For any finite set $F \subset P$,*

$$Z(F) = \sum_{U \subseteq F} (-1)^{|U|} TT^*(\vee U) \geq 0.$$

*Proof.* First of all, the equivalence between (1) and (2) is shown in Theorem 5.1.7.

To show (2) implies (3), let $V : P \to \mathcal{B}(\mathcal{K})$ be the minimal isometric Nica-covariant dilation for $T : P \to \mathcal{B}(\mathcal{H})$. Consider the product $\prod_{p\in F}(I - V(p)V(p)^*)$: notice that for any subset $U \subseteq F$, by the Nica-covariance,

$$\prod_{p\in U} V(p)V(p)^* = V(\vee U)V(\vee U)^*.$$

Hence,

$$\prod_{p\in F}(I - V(p)V(p)^*) = \sum_{U\subseteq F}(-1)^{|U|}V(\vee U)V(\vee U)^*.$$

Now since $\mathcal{H}$ is co-invariant for $V$, we have

$$P_{\mathcal{H}}V(\vee U)V(\vee U)^*\big|_{\mathcal{H}} = T(\vee U)T(\vee U)^*.$$

By restricting to $\mathcal{H}$, we have

$$\begin{aligned} Z(F) &= \sum_{U\subseteq F}(-1)^{|U|}T(\vee U)T(\vee U)^* \\ &= P_{\mathcal{H}}\Big(\sum_{U\subseteq F}(-1)^{|U|}V(\vee U)V(\vee U)^*\Big)\Big|_{\mathcal{H}} \\ &= P_{\mathcal{H}}\Big(\prod_{p\in F}(I - V(p)V(p)^*)\Big)\Big|_{\mathcal{H}} \geq 0 \end{aligned}$$

Now to show (3) implies (1), it suffices to show for any $F_0 = \{p_1, \cdots, p_n\} \subset P$, the operator matrix $K[F_0]$ is positive. Now for each $U \subseteq F_0$, pick $s_U \in \vee U$. Since

$$\vee\{p_i\} = \{r : rP = p_iP\} = p_iP^*,$$

we can pick $s_{p_i} = p_i$ for all $i$. Let $F_1 = \{s_U : U \subseteq F_0\}$. $F_1$ is still a finite subset of $P$, and $F_0 \subset F_1$ since each $s_{p_i} = p_i \in F_1$. Therefore, it suffices to show $K[F_1] \geq 0$. Let us now show that $K[F_1] \geq 0$ given condition (3).

First, rows and columns of $K[F_1]$ are indexed by subsets of $F_0$. For any subsets $A_i, A_j \subseteq F_0$, the $(A_i, A_j)$-entry of $K[F_1]$ can be expressed as

$$K(s_{A_i}, s_{A_j}) = T\left(s_{A_i}^{-1}s\right)T\left(s_{A_j}^{-1}s\right)^*$$

for some $s \in s_{A_i} \vee s_{A_j}$ (the choice does not affect the value). By Lemma 2.1.18,

$$s_{A_i \cup A_j} \in \vee(A_i \cup A_j) = (\vee A_i) \vee (\vee A_j) = s_{A_i} \vee s_{A_j}.$$

Hence,

$$K(s_{A_i}, s_{A_j}) = T\left(s_{A_i}^{-1} s_{A_i \cup A_j}\right) T\left(s_{A_j}^{-1} s_{A_i \cup A_j}\right)^*$$

Now define an operator matrix $R$ with the same dimension as $K[F_1]$. For any subsets $A_i, A_j \subseteq F_0$, define the $(A_i, A_j)$-entry of $R$ to be 0 if $A_i$ is not a subset of $A_j$. Otherwise, define $R(A_i, A_j)$ to be:

$$T\left(s_{A_i}^{-1} s_{A_j}\right) \Big( \sum_{A_j \subseteq U \subseteq F_0} (-1)^{|U \backslash A_j|} TT^*\left(s_{A_j}^{-1} s_U\right) \Big)^{1/2}$$

We first show that this is well defined given condition (3). For a fixed $A \subseteq F_0$, let $F_0 \backslash A = \{q_1, \cdots, q_k\}$ and define

$$F_A = \{s_A^{-1} s_{A \cup \{q_j\}} : 1 \leq j \leq k\}.$$

Then,

$$\begin{aligned} Z(F_A) &= \sum_{W \subseteq F_A} (-1)^{|W|} TT^*(\vee W) \\ &= \sum_{W_0 \subseteq F_0 \backslash A} (-1)^{|W_0|} TT^*\Big( \bigvee_{q \in W_0} s_A^{-1} s_{A \cup \{q\}} \Big) \\ &= \sum_{W_0 \subseteq F_0 \backslash A} (-1)^{|W_0|} TT^*\Big( s_A^{-1}\left(\vee(A \cup W_0)\right) \Big) \\ &= \sum_{A \subseteq U \subseteq F_0} (-1)^{|U \backslash A|} TT^*\left(s_A^{-1}(\vee U)\right) \\ &= \sum_{A \subseteq U \subseteq F_0} (-1)^{|U \backslash A|} TT^*\left(s_A^{-1} s_U\right) \end{aligned}$$

Therefore, $R(A_i, A_j)$ is in fact equal to $T\left(s_{A_i}^{-1} s_{A_j}\right) Z(F_{A_j})^{1/2}$, where $Z(F_{A_j}) \geq 0$ by condition (3). We now claim that

$$K[F_1] = R \cdot R^* \geq 0.$$

Fix $A_i, A_j \subset F_0$ for which we compute the $(A_i, A_j)$-entry of $R \cdot R^*$, which is equal to $\sum_{U \subseteq F_0} R(A_i, U)R(A_j, U)^*$. By the construction of $R$, $R(A_i, U)R(A_j, U)^* \neq 0$ only when $A_i, A_j$ are subsets of $U$. Therefore,

$$\begin{aligned} RR^*[A_i, A_j] &= \sum_{U \subseteq F_0} R(A_i, U)R(A_j, U)^* \\ &= \sum_{A_i \cup A_j \subseteq U \subseteq F_0} R(A_i, U)R(A_j, U)^* \\ &= \sum_{A_i \cup A_j \subseteq U \subseteq F_0} T\left(s_{A_i}^{-1} s_U\right) Z(F_U) T\left(s_{A_j}^{-1} s_U\right)^* \end{aligned}$$

Replacing $Z(F_U)$ using the earlier computation, we obtain

$$\begin{aligned} &RR^*[A_i, A_j] \\ =&\sum_{U} \sum_{U \subseteq W} (-1)^{|W \backslash U|} T\left(s_{A_i}^{-1} s_U\right) TT^*\left(s_U^{-1} s_W\right) T\left(s_{A_j}^{-1} s_U\right)^* \\ =&\sum_{U} \sum_{U \subseteq W} (-1)^{|W \backslash U|} T\left(s_{A_i}^{-1} s_W\right) T\left(s_{A_j}^{-1} s_W\right)^* \end{aligned}$$

Consider the term $T\left(s_{A_i}^{-1} s_W\right) T\left(s_{A_j}^{-1} s_W\right)^*$ in the double summation. It occurs whenever $A_i \cup A_j \subseteq U \subseteq W$. Let $m = |W \backslash (A_i \cup A_j)|$ and $k = |W \backslash U|$, $U$ has to contain all the elements in $A_i \cup A_j$ and $m - k$ elements in $W \backslash (A_i \cup A_j)$. There are precisely $\binom{m}{k}$ choices of $U$. Therefore,

$$\begin{aligned} &RR^*[A_i, A_j] \\ =&\sum_{W : U \subseteq W} \left( \sum_{k=0}^{m} (-1)^k \binom{m}{k} \right) T\left(s_{A_i}^{-1} s_W\right) T\left(s_{A_j}^{-1} s_W\right)^* \end{aligned}$$

Notice that

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} = \begin{cases} 1, & \text{if } m = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the only non-zero term in the summation occurs when $m = 0$ and thus $W_0 =$

$A_i \cup A_j$. Therefore,

$$\begin{aligned}&\sum_{U\subseteq F_0} R(A_i,U)R(A_j,U)^*\\ =&T\left(s_{A_i}^{-1}s_{W_0}\right)T\left(s_{A_j}^{-1}s_{W_0}\right)^*\\ =&T\left(s_{A_i}^{-1}s_{A_i\cup A_j}\right)T\left(s_{A_j}^{-1}s_{A_i\cup A_j}\right)^*\\ =&K(s_{A_i},s_{A_j})\end{aligned}$$

This finishes the proof. □

**Remark 5.1.9.** *As observed in [40, 42], the matrix $R$ is a Cholesky decomposition of the operator matrix $K[F_1]$. Given two subsets $A_i, A_j \subseteq F_0$, $R(A_i, A_j) = 0$ whenever $|A_j| > |A_i|$. When $|A_j| = |A_i|$, the only case when $R(A_i, A_j) \neq 0$ is when $A_i \subseteq A_j$ and thus $A_i = A_j$. Hence by arranging $F_1 = \{\vee A : A \subseteq F_0\}$ according to $|A|$ in decreasing order, the matrix $R$ becomes a lower triangular matrix.*

As a quick corollary, every co-isometric representation of a lattice ordered semigroup has $*$-regular dilation. This generalizes [40, Corollary 3.8] in the case of $\ell$-semigroups.

**Corollary 5.1.10.** *Suppose that any finite subset of $P$ has a least upper bound. If $T : P \to \mathcal{B}(\mathcal{H})$ is a co-isometric representation (i.e. $T(p)T(p)^* = I$ for all $p \in P$), then $T$ has $*$-regular dilation.*

*Proof.* It suffices to check that $T$ satisfies condition (3) in Theorem 5.1.8. For any finite set $F \subset P$ and any $U \subseteq F$, since $P$ is lattice ordered, $\vee U \in P$ and thus $T(\vee U)T(\vee U)^* = I$. Therefore,

$$Z(F) = \sum_{U\subseteq F}(-1)^{|U|}T(\vee U)T(\vee U)^* = \sum_{U\subseteq F}(-1)^{|U|}I = 0. \quad \square$$

## 5.2 Descending Chain Condition

In general, Condition (3) in Theorem 5.1.8 can be very difficult to verify since it requires $Z(F) \geq 0$ for all finite subsets of $P$. Our goal is to reduce it to a smaller collection of finite subsets.

### 5.2.1 Reduction Lemmas

We first prove a few technical lemmas that help us with the reduction.

**Lemma 5.2.1.** *Let $F \subseteq P$ be a finite subset.*

1. *If $F = \{p_1, p_2, \cdots, p_n\}$ where $p_1 P = p_2 P$, then let $F_0 = \{p_2, \cdots, p_n\}$. Then $Z(F) = Z(F_0)$ and thus $Z(F) \geq 0$ if and only if $Z(F_0) \geq 0$.*
2. *If $F = \{p_1, p_2, \cdots, p_n\}$ and $p_1 \in P^*$, then $Z(F) = 0$.*

*Proof.* For (1): Consider $Z(F) = \sum_{U \subseteq F} (-1)^{|U|} TT^*(\vee U)^*$. For any $U_0 \subseteq \{p_3, \cdots, p_n\}$ and consider the terms $U_1 = \{p_1\} \cup U_0$ and $U_2 = \{p_1, p_2\} \cup U_0$. Since $p_1 P = p_2 P$, it is clear that $\vee U_1 = \vee U_2$ and $|U_2| = |U_1| + 1$. Therefore,

$$(-1)^{|U_1|} TT^*(\vee U_1) + (-1)^{|U_1|} TT^*(\vee U_2) = 0.$$

Hence,

$$Z(F) - Z(F_0) = \sum_{p_1 \in U \subset F} (-1)^{|U|} TT^*(\vee U)^* = 0.$$

For (2): Since $p_1 \in P^*$ is invertible, $p_1 P = P$. Hence, for any $U_0 \subseteq \{p_2, \cdots, p_n\}$, $\vee U_0 = \vee \{p_1\} \cup U_0$. It follows from a similar argument that $Z(F) = 0$. $\square$

**Lemma 5.2.2.** *Let $T : P \to \mathcal{B}(\mathcal{H})$ be a unital representation of a right LCM semigroup. Let $p_1, \cdots, p_n, q \in P$. Define:*

$$\begin{aligned} F &= \{p_1 \cdot q, p_2, \cdots, p_n\}, \\ F_1 &= \{p_1, \cdots, p_n\}, \\ F_2 &= \{q, p_1^{-1} s_2, \cdots, p_1^{-1} s_n\}. \end{aligned}$$

*where $s_i \in p_1 \vee p_i$ for all $2 \leq i \leq n$. Here, when $p_1 \vee p_i = \emptyset$, we can exclude the term $p_1^{-1} s_i$ in $F_2$.*

*Then $Z(F) = Z(F_1) + T(p_1) Z(F_2) T(p_1)^*$. In particular, $Z(F) \geq 0$ if $Z(F_1), Z(F_2) \geq 0$.*

*Proof.* Let $F_0 = \{p_2, \cdots, p_n\}$ and consider $Z(F) - Z(F_1)$:

$$\begin{aligned}&Z(F) - Z(F_1)\\ =&\sum_{U\subseteq F}(-1)^{|U|}TT^*(\vee U) - \sum_{U\subseteq F_1}(-1)^{|U|}TT^*(\vee U)\end{aligned}$$

The only difference between $F$ and $F_1$ is their first element, and therefore the only difference between $Z(F)$ and $Z(F_1)$ occurs when $U$ contains the first element. Hence,

$$\begin{aligned}&Z(F) - Z(F_1)\\ =&\sum_{U\subseteq F_0}(\text{-}1)^{|U|+1}\left(TT^*\left(\vee(\{p_1q\}\cup U)\right) - TT^*\left(\vee(\{p_1\}\cup U)\right)\right)\\ =&T(p_1)\Big(\sum_{U\subseteq F_0}(\text{-}1)^{|U|+1}TT^*\big(p_1^{-1}\vee(\{p_1q\}\cup U)\big) - TT^*\big(p_1^{-1}\vee(\{p_1\}\cup U)\big)\Big)T(p_1)^*\\ =&T(p_1)\Big(\sum_{U\subseteq F_0}(\text{-}1)^{|U|+1}TT^*\big(q\vee\bigvee_{p\in U}p_1^{-1}(p_1\vee p)\big) - TT^*\big(\bigvee_{p\in U}p_1^{-1}(p_1\vee p)\big)\Big)T(p_1)^*\\ =&T(p_1)\Big(\sum_{q\in U\subseteq F_2}(\text{-}1)^{|U|}TT^*(\vee U) + \sum_{q\notin U\subseteq F_2}(\text{-}1)^{|U|}TT^*(\vee U)\Big)T(p_1)^*\\ =&T(p_1)Z(F_2)T(p_1)^*\end{aligned}$$

Now it is clear that $Z(F)\geq 0$ if $Z(F_1)\geq 0$ and $Z(F_2)\geq 0$. In the case when $p_1\vee p_i = \emptyset$, $\vee U = \emptyset$ whenever $p_1, p_i \in U \subset F_1$ or $p_1q, p_i\in U\subset F$. Therefore, we can simply pretend that the term $p_1^{-1}s_i$ does not exist in $F_2$. The calculation will not be affected. □

**Remark 5.2.3.** *Lemma 5.2.2 allows us to reduce the positivity of $Z(F)$ to the positivity of $Z(F_1), Z(F_2)$. $F_1$ replaces the element $p_1q\in F$ by $p_1\in F_1$ while keeping the rest of it unchanged. Moreover, since $p_1qP\subseteq p_1P$, take $r_1\in\vee F_1$ and $r\in\vee F$, we have $rP\subseteq r_1P$ and thus $r = r_1v$ for some $v\in P$. For $F_2$, observe that*

$$\begin{aligned}\vee F &= (p_1q\vee p_2\vee\cdots\vee p_n\vee e)\\ &= p_1\cdot\left(q\vee\left(p_1^{-1}(p_1\vee p_2)\right)\vee\cdots\left(p_1^{-1}(p_1\vee p_n)\right)\right)\\ &= p_1\cdot\vee F_2\end{aligned}$$

*Intuitively, elements are 'smaller' in $F_1, F_2$ compared to $F$.*

**Remark 5.2.4.** *In the case when $T$ is an isometric Nica-covariant representation,*

$$Z(F) = (I - TT^*(p_1q))\cdot\prod_{i=2}^{n}(I - TT^*(p_i))$$

*Observe that*

$$I - TT^*(p_1 q) = (I - TT^*(p_1)) + T(p_1)(I - TT^*(q))T(p_1)^*.$$

*Therefore,*

$$\begin{aligned} Z(F) =& (I - TT^*(p_1)) \cdot \prod_{i=2}^{n} (I - TT^*(p_i)) \\ &+ T(p_1) \left( (I - TT^*(q)) \cdot \prod_{i=2}^{n} (I - TT^*(p_i)) \right) T(p_1)^* \\ =& Z(F_1) + T(p_1) Z(F_2) T(p_1)^*. \end{aligned}$$

### 5.2.2 Ore LCM semigroups

We say the right LCM semigroup $P$ is an Ore semigroup if for any $p, q \in P$, $pP \cap qP \neq \emptyset$. In the case of quasi-lattice ordered group, this corresponds to the lattice order condition discussed in [17] where every finite subset $F$ of $P$ always has a least upper bound.

**Definition 5.2.5.** *We say that $P$ satisfies* the descending chain condition *if there is no infinite sequence $x_n \in P$ and $y_n \notin P^*$ so that $x_n = x_{n+1} y_n$ or $x_n = y_n x_{n+1}$.*

*An element $x \in P$ is called* minimal *if $x \notin P^*$ and whenever $x = yz$ for $y, z \in P$, either $y \in P^*$ or $z \in P^*$. We let $P_{min}$ be the set of all minimal elements in $P$.*

Intuitively, $P$ has the descending chain property if we cannot cancel non-invertible factors from each $x \in P$ from the left or the right infinitely many times.

**Remark 5.2.6.** *In the case when $(G, P)$ is a quasi-lattice ordered group, the descending chain condition is saying there is no infinite sequence $x_n$ so that $x_{n+1} < x_n$ (i.e. when there is $y_n \neq e$, $x_n = x_{n+1} y_n$) or $x_{n+1} <_r x_n$ (i.e. when there is $y_n \neq e$, $x_n = y_n x_{n+1}$). We are not sure if the descending chain property of the partial order $<$ (or $<_r$) alone would be sufficient.*

Suppose $P$ satisfies the descending chain condition, it is clear that $P_{min} \neq \emptyset$ since otherwise we can build an infinite descending chain starting from any element $x \neq e$. It turns out that testing subsets of $P_{min}$ is sufficient for Condition (3) in Theorem 5.1.8.

**Proposition 5.2.7.** *Let $P$ be a right LCM Ore semigroup that satisfies the descending chain condition. Suppose $Z(F) \geq 0$ for all finite $F \subset P_{min}$. Then $Z(F) \geq 0$ for all finite $F \subset P$.*

*Proof.* Pick any finite $F \subset P$. If $F \cap P^* \neq \emptyset$, we have $Z(F) = 0 \geq 0$ by Lemma 5.2.1. If $F \not\subseteq P_{min}$, we can pick some element $x \in F$ that is not minimal. Therefore, we can write $x = p_1 \cdot q$ for $p_1, q \notin P^*$ and write $F = \{p_1 q, p_2, \cdots, p_n\}$. We have $Z(F) \geq 0$ if $Z(F_1) \geq 0$ and $Z(F_2) \geq 0$ where $F_1, F_2$ are defined in Lemma 5.2.2.

This process allows us to build a binary tree rooted at $F$. Let $\mathbb{F}_2^+$ be the free semigroup generated by $\{1, 2\}$, and let $\epsilon \in \mathbb{F}_2^+$ be the empty word. We start with $F_\epsilon = F$. Suppose for a word $\omega \in \mathbb{F}_2^+$ where $F_\omega \not\subseteq P_{min} \cup P^*$, we can pick an element $x = p_1 \cdot q \in F_\omega$ where $p_1, q \notin P^*$. This allows us to define $F_{\omega 1}$ and $F_{\omega 2}$ as in Lemma 5.2.2. We have $Z(F_\omega) \geq 0$ whenever $Z(F_{\omega 1}) \geq 0$ and $Z(F_{\omega 2}) \geq 0$.

Suppose the binary tree is finite, its leaves contain finite subsets $\overline{F} \subset P_{min} \cup P^*$. We know such $\overline{F}$ satisfies $Z(\overline{F}) \geq 0$ by the hypothesis (in the case when $\overline{F} \subset P_{min}$) or Lemma 5.2.1 (in the case when $\overline{F} \cap P^* \neq \emptyset$). Therefore, it suffices to show the binary tree is finite.

Assume otherwise that the binary tree is infinite. By the König Lemma, this tree has an infinite path $s_1 s_2 \cdots s_n \cdots$, $s_i \in \{1, 2\}$. Let $\omega_n = s_1 s_2 \cdots s_n$ so that $F_{\omega_n}$ are nodes in the binary tree. Pick $t_\omega \in \vee F_\omega$ for each node of the binary tree. Here, we are using the Ore condition to ensure that $\vee F_\omega \neq \emptyset$. As we observed in Remark 5.2.3, there exists $p_{\omega 2} \notin P^*$ so that $p_{\omega 2} \cdot t_{\omega 2} = t_\omega$ and some element $u_{\omega 1} \in P$ so that $t_{\omega 1} u_{\omega 1} = t_\omega$. By the descending chain condition, this implies there is only finitely many $s_i = 2$ and hence there is $N$ so that $s_i = 1$ for all $i > N$.

For $n > N$, the only difference between $F_{\omega_n}$ and $F_{\omega_{n+1}} = F_{\omega_n 1}$ is an element $p_1 q \in F_{\omega_n}$ and $p_1 \in F_{\omega_{n+1}}$ where $q \notin P^*$. By the descending chain condition again, this process cannot continue infinitely many times. This proves the binary tree has to be finite which finishes the proof. □

As an immediate consequence, we can replace condition (3) in Theorem 5.1.8 by a much smaller collection of subsets when the Ore semigroup has the descending chain property.

**Theorem 5.2.8.** *Let $T : P \to \mathcal{B}(\mathcal{H})$ be a unital representation of a right LCM Ore semigroup with the descending chain property. Let $P_{min}$ be the set of all minimal elements in $P$. The following are equivalent:*

1. *$T$ has a $*$-regular dilation;*

*2. $T$ has a minimal isometric Nica-covariant dilation;*

*3. For any finite set $F \subset P_{min}$,*

$$Z(F) = \sum_{U \subseteq F} (-1)^{|U|} TT^*(\vee U)^* \geq 0.$$

### 5.2.3 Non-Ore LCM Semigroups

In the case of a right LCM semigroup that fails to satisfy the Ore condition, the proof of Proposition 5.2.7 fails due to the fact that $\vee F$ can be $\infty$. Nevertheless, a similar argument can be applied.

**Definition 5.2.9.** *We say a subset $P_0$ of a right LCM semigroup is a minimal set if*

1. *$P_{min} \subseteq P_0$*
2. *For any $x \in P_{min}$ and $y \in P_0$, we have*

$$x^{-1}(x \vee y) \subseteq P_0 \cup P^*.$$

It is clear that $P_0 = P$ is always a minimal set. However, in many cases, we can choose $P_0$ to be a much smaller set.

**Proposition 5.2.10.** *Let $P$ be a right LCM semigroup that satisfies the descending chain condition. Let $P_0$ be any minimal set of $P$. Suppose $Z(F) \geq 0$ for all finite $F \subset P_0$. Then $Z(F) \geq 0$ for all finite $F \subset P$.*

*Proof.* For every finite $F \subset P$, denote $m(F) = |F \cap P_0|$ which counts the number of elements in $F$ that are from $P_0$. In the case when $m(F) = |F|$, we have $F \subset P_0$ and thus $Z(F) \geq 0$. Otherwise, we will show that we can find a collection $F_1, \cdots, F_k$ with $m(F_i) > m(F)$, and $Z(F) \geq 0$ whenever $Z(F_i) \geq 0$ for all $i$. This allows us to proceed with induction with $m(F)$.

Suppose $m(F) < |F|$, pick $x \in F$ so that $x$ is not in $P_0$. Since $P$ has the descending chain condition, we can repeatedly remove a minimal element from $x$ for a finite number of times. Hence, we can write $x = x_1 x_2 \cdots x_n$ where $x_i \in P_{min}$. Write $F = \{x, p_2, p_3, \cdots, p_n\}$. Apply Lemma 4.3.7, $Z(F) \geq 0$ if $Z(F_1), Z(F_2) \geq 0$, where

$$\begin{aligned} F_1 &= \{x_1, p_2, p_3, \cdots, p_n\} \\ F_2 &= \{x_2 x_3 \cdots x_n, x_1^{-1}(x_1 \vee p_2), \cdots, x_1^{-1}(x_1 \vee p_n)\} \end{aligned}$$

Notice that $x_1 \in P_{min} \subset P_0$, and thus $m(F_1) = m(F) + 1$. For each $p_i \in F \cap P_0$, $x_1^{-1}(x_1 \vee p_i) \in P_0 \cup P^*$. If $x_1^{-1}(x_1 \vee p_i) \in P^*$, then it follows from Lemma 5.2.1 that $Z(F_2) = 0$. Otherwise, we must have $m(F_2) \geq m(F)$. In the case when $n = 2$, $x_n \in P_{min} \subset P_0$ and we get $m(F_2) > m(F)$, which we can proceed with induction. Otherwise, notice that though $m(F_2) = m(F)$, the element $x = x_1 \cdots x_n \in F$ is replaced by $x' = x_2 x_3 \cdots x_n$ in $F_2$, where $x'$ is a product of $(n-1)$ minimal elements. Repeat the same procedure again for $F_2$, we get $Z(F_2) \geq 0$ if $Z(F_{21}) \geq 0$ and $Z(F_{22}) \geq 0$, where $m(F_{21}) > Z(F_2) \geq Z(F)$ and $m(F_{22}) \geq m(F_2) \geq Z(F)$. The inequality is strict when $n = 3$ since $x_3 \in F_{22} \cap P_0$. Otherwise, repeat the same procedure again. Eventually, we can reduce the positivity of $Z(F)$ to the positivity of $Z(F_i)$ with $m(F_i) > m(F)$. This finishes the proof. $\square$

We now reach a nice condition for $*$-regularity in the case of an arbitrary right LCM semigroup with descending chain condition.

**Theorem 5.2.11.** *Let $T : P \to \mathcal{B}(\mathcal{H})$ be a unital representation of a right LCM with the descending chain condition. Let $P_0$ be a minimal set. The following are equivalent:*

1. *$T$ has a $*$-regular dilation;*
2. *$T$ has a minimal isometric Nica-covariant dilation;*
3. *For any finite set $F \subset P_0$,*

$$Z(F) = \sum_{U \subseteq F} (-1)^{|U|} TT^*(\vee U)^* \geq 0.$$

## 5.3 Examples

We now examine several classes of right LCM semigroups that satisfy the descending chain condition. For each class of semigroups, we derive the corresponding conditions for $*$-regularity.

### 5.3.1 Artin Monoids

Artin monoids (see Example 2.1.8) form an important class of right LCM semigroups. Their Nica-covariant representations and related $C^*$-algebras are studied in [17]. In the case of finite type or right-angled Artin monoids $P_M$, it is known that they are embedded injectively in the corresponding Artin group $G_M$, and $(G_M, P_M)$ form a quasi-lattice ordered group [17]. In general, Artin monoids are shown to embed injectively inside the corresponding artin group [50]. The semigroup $P_M$ is known to be a right LCM semigroup, but it is unknown whether $(G_M, P_M)$ is quasi-lattice ordered.

Let $\{e_1, \cdots, e_n\}$ be the set of generators for $P_M$. Each element $p \in P_M$ can be written as $p = e_{i_1} e_{i_2} \cdots e_{i_n}$, and we define the length of $p$ to be $\ell(p) = n$ when $p$ can be expressed as a product of $n$ generators. Though there may be multiple ways to express $p$ as a product of generators, the relations on an Artin monoid are always homogeneous and thus it always takes the same number of generators to express $p$. Therefore, $\ell(p)$ is well-defined.

**Lemma 5.3.1.** *Every Artin monoid $P_M$ has the descending chain property. The set of minimal elements is precisely the set of generators $\Gamma$.*

*Proof.* Once we defined the length of an element $\ell(p)$ to be the number of generators requires to express $p$. We have for any $p, q \in P_M$, $\ell(pq) = \ell(p) + \ell(q)$. It is clear that we can not find infinite sequences $x_n$ and $y_n \neq e$ with $x_n = y_n x_{n+1}$ or $x_n = x_{n+1} y_n$ since otherwise, $\ell(x_n) \in \mathbb{Z}_{\geq 0}$ is strictly decreasing.

Its set of minimal elements are precisely the set of elements with length 1, which is exactly the set of generators. □

The Artin monoids of finite types are all lattice ordered. Therefore, Theorem 5.2.8 applies.

**Theorem 5.3.2.** *A contractive representation $T$ of finite-type Artin monoids are $*$-regular if and only if $Z(F) \geq 0$ for all finite subset $F$ of the set of generators.*

**Example 5.3.3.** *Let us consider the Braid monoid on 3 strands:*

$$B_3^+ = \langle e_1, e_2 : e_1 e_2 e_1 = e_2 e_1 e_2 \rangle.$$

*A representation $T : B_3^+ \to \mathcal{B}(\mathcal{H})$ is uniquely determined by $T_i = T(e_i)$, $i = 1, 2$, which satisfies $T_1 T_2 T_1 = T_2 T_1 T_2$. Theorem 5.3.2 states that $T$ has $*$-regular dilation if and only if $T_1, T_2$ are contractions, and*

$$\begin{aligned} & I - T_1 T_1^* - T_2 T_2^* + T(e_1 \vee e_2) T(e_1 \vee e_2)^* \\ = & I - T_1 T_1^* - T_2 T_2^* + T_1 T_2 T_1 T_1^* T_2^* T_1^* \geq 0. \end{aligned}$$

When the Artin monoid is infinite, it is hard to find a minimal set in general. Recall that an Artin monoid is called right-angled if entries in $M$ are either 2 or $\infty$. This is also known as the graph product of $\mathbb{N}$. Regular dilations of right-angled Artin monoids were studied in [42].

**Proposition 5.3.4.** *Given a right-angled Artin monoid $A_M^+$, the set of generators $P_{min} = \{e_1, \cdots, e_n\}$ is also a minimal set.*

*Proof.* Pick any $e_i \in P_{min}$ and $e_j$ with $e_j \neq e_i$. Either $m_{ij} = 2$, in which case $e_i^{-1}(e_i \vee e_j) = e_i^{-1} e_i e_j = e_j$. Or $m_{ij} = \infty$, in which case $e_i \vee e_j = \infty$. In either case, we can see $P_{min}$ is a minimal set. ☐

**Remark 5.3.5.** *Combining Proposition 5.3.4 with Theorem 5.2.11, this recovers our main result on $*$-regular dilation on graph products of $\mathbb{N}$ (Theorem 4.5.5).*

### 5.3.2 Thompson's Monoid

Recall the Thompson's monoid from Example 2.1.7 (2.1.13):

$$F^+ = \langle x_0, x_1, \cdots | x_n x_k = x_k x_{n+1}, k < n \rangle .$$

Our result of $*$-regular dilation can help us generate isometric Nica-covariant representations for the Thompson's monoid. We first show that $F^+$ has the descending chain property.

**Lemma 5.3.6.** *Thompson's monoid $F^+$ has the descending chain property. The set of minimal elements is the set of generators $\{x_0, x_1, \cdots\}$. The set of generators is also a minimal set for $F^+$.*

*Proof.* Similar to the case of Artin monoids, since the relations that define the Thompson's monoid $F^+$ are homogeneous, we can define $\ell(p) = n$ if we can write $p$ as a product of $n$ generators $p = x_{i_1} x_{i_2} \cdots x_{i_n}$. It is clear that for all $p, q \in F^+$, $\ell(p) + \ell(q) = \ell(pq)$. Therefore, $F^+$ has the descending chain property (otherwise, we can obtain a strictly decreasing sequence of $\ell(p_n)$). It is clear that the set of minimal elements are precisely the set of generators.

Now for any $x_i, x_j$, $i < j$. It follows from the relation $x_j x_i = x_i x_{j+1}$ that $x_i \vee x_j = x_j x_i$ and thus both $x_i^{-1}(x_i \vee x_j) = x_j$ and $x_j^{-1}(x_i \vee x_j) = x_{j+1}$ are again minimal elements. Therefore, $P_{min}$ is also a minimal set. ☐

Again, Theorem 5.2.8 applies to the Thompson's monoid.

**Theorem 5.3.7.** *Let $T : F^+ \to \mathcal{B}(\mathcal{H})$ be a unital representation uniquely determined by the generators $T_i = T(e_i)$. Then $T$ has a $*$-regular dilation if and only if for any finite subset $F$ of the generators, $Z(F) \geq 0$.*

### 5.3.3 $\mathbb{N} \rtimes \mathbb{N}^\times$

Recall the semigroup $\mathbb{N} \rtimes \mathbb{N}^\times$ (Example 2.1.16 (2)) is the monoid $\{(a,p) : a \in \mathbb{N}, p \in \mathbb{N}^\times\}$ with the multiplication

$$(a,p)(b,q) = (a+bp, pq).$$

It embeds in $\mathbb{Q} \rtimes \mathbb{Q}^\times$, and they form a quasi-lattice ordered group [38]. The semigroup $\mathbb{N} \rtimes \mathbb{N}^\times$ has $(0,1)$ as the identity, and it is generated by $P_0 = \{(1,1),(0,p) : p \text{ is a prime}\}$ with the relations:

$$\begin{aligned}(0,p)(1,1) &= (p,p) = (1,1)^p(0,p),\\ (0,p)(0,q) &= (0,pq).\end{aligned}$$

It is obvious that $\mathbb{N} \rtimes \mathbb{N}^\times$ has the descending chain property and the set of minimal elements are precisely the set of its generators $P_{min}$. However, it is not a Ore-semigroup. For example, consider the principal right ideal generated by $(0,2)$ and $(1,2)$. For all $(b,q) \in P$, $(i,2)(b,q) = (i+2b,q)$, and thus the first coordinate always has the same parity as $i$. Therefore, $(0,2)P \cap (1,2)P = \emptyset$. In general, given $(a,m),(b,n) \in \mathbb{N} \rtimes \mathbb{N}^\times$, one can compute [38, Remark 2.3]:

$$(a,m) \vee (b,n) = \begin{cases}(\ell, \operatorname{lcm}(m,n)) : (a+m\mathbb{N}) \cap (b+n\mathbb{N}) \neq \emptyset;\\ \infty, (a+m\mathbb{N}) \cap (b+n\mathbb{N}) = \emptyset\end{cases}$$

Here, $\ell = \min\{(a+m\mathbb{N}) \cap (b+n\mathbb{N})\}$.

**Proposition 5.3.8.** *Let $P_0 = \{(1,1),(i,p) : 0 \leq i < p, p \text{ is a prime}\}$. Then $P_0$ is a minimal set.*

*Proof.* We need to show for all $x \in P_{min}$ and $y \in P_0$, $x^{-1}(x \vee y) \in P_0$. We divide the proof into several cases.

**Case 1**: take $y = (1,1)$. It is clear that if we take $x = (1,1)$, then $x^{-1}(x \vee y) = (0,1) \in P^*$. Suppose we take $x = (0,p)$ for a prime $p$, then one can check that $x \vee y = (p,p) = x \cdot (1,1)$. Thus, $x^{-1}(x \vee y) = (1,1) \in P_0$.

**Case 2**: Take $y = (i,p)$ for some prime $p$ and $0 \leq i < p$. We divide the choices of $x$ into three cases:

When $x = (1,1)$, we have $x \vee y = (p,p) = (1,1)(p-1,p)$. Therefore, $x^{-1}(x \vee y) = (p-1,p) \in P_0$.

When $x = (0,p)$, we have $x \vee y = \infty$ unless $y = (0,p)$, in which case $x^{-1}(x \vee y) = (0,1) \in P^*$.

When $x = (0,q)$ for some prime $q \neq p$, we have $x \vee y = (\ell, pq)$, where $\ell = \min\{(i + p\mathbb{N}) \cap q\mathbb{N}\}$. Notice that by the Chinese remainder theorem, there always exists a solution $\ell \in [0, pq-1)$, and thus $\ell = kq$ for some $0 \leq k < p$. Hence, $x \vee y = (kq, pq) = (0,q)(k,p)$ and thus $x^{-1}(x \vee y) = (k,p) \in P_0$. This finishes the last case of the proof. ☐

Therefore, we obtain the following characterization:

**Theorem 5.3.9.** *Let $T : \mathbb{N} \rtimes \mathbb{N}^\times \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Then, $T$ has a $*$-regular dilation if and only if $Z(F) \geq 0$ for any $F \subseteq P_0 = \{(1,1), (i,p) : 0 \leq i < p, \forall p \text{ is a prime}\}$*

### 5.3.4 Baumslag-Solitar monoids

The Baumslag-Solitar monoid $B_{n,m}$ (Example 2.1.16 (1)) is the monoid generated by $a, b$ with the relation $ab^n = b^m a$. Each $B_{n,m}$ is a right LCM semigroup.

**Lemma 5.3.10.** *Every Baumslag-Solitar monoid $B_{n,m}$ has the descending chain property. The set of minimal elements is precisely $\{a, b\}$.*

*Proof.* Every elements $p \in P$ can have many different expressions as product of $a, b$. We let $\ell(p)$ to be the maximum number of $a, b$ we can use to express $p$. $\ell(p)$ is always bounded [33, Lemma 2.2]. It is clear that for any $p, q \in B_{n,m}$, $\ell(pq) \geq \ell(p) + \ell(q)$. Therefore, whenever $p, q \neq e$, we have $\ell(p), \ell(q) < \ell(pq)$. Since $\ell(p) \geq 1$ are integer-valued, $B_{n,m}$ has the descending chain property. It is clear that the set of minimal elements are $\{a, b\}$. ☐

We first find a minimal set for $B_{n,m}$.

**Proposition 5.3.11.** *$P_0 = \{b^i a : 0 \leq i\} \cup \{b^j : 1 \leq j\}$ is a minimal set for $B_{n,m}$.*

*Proof.* We need to show for all $x \in P_{min}$ and $y \in P_0$, $x^{-1}(x \vee y) \in P_0$. We divide the proof into several cases.

**Case 1**. Suppose $y = b^i a$ for some $0 \leq i$. If $x = a$, then either $i$ is a multiple of $m$ in which case $x^{-1}(x \vee y) = b^i \in P_0 \cup P^*$, or $i \neq 0$ in which case $x \vee y = \emptyset$. If $x = b$, then either $i = 0$ in which case $x^{-1}(x \vee y) = b^{m-1}a \in P_0$, or $i \neq 0$ in which case $x^{-1}(x \vee y) = b^{i-1}a \in P_0$.

**Case 2**. Suppose $y = b^j$ for some $1 \leq j$. If $x = b$, then $x^{-1}y = b^{j-1} \in P_0 \cup P^*$. If $x = a$, then $x \vee y = b^\ell a$ where $\ell = min\{m\mathbb{N} \cap \mathbb{N}_{\geq j}\}$. Assume $\ell = km$, we have $x \vee y = ab^{kn}$. Hence, $x^{-1}(x \vee y) = b^{kn} \in P_0$. This finishes the proof. $\square$

In fact, we can further reduce this set $P_0$ to a smaller set. Let $P_{00} = \{b, b^i a : 0 \leq i \leq m-1\}$.

**Proposition 5.3.12.** *The following are equivalent:*

1. $Z(F) \geq 0$ *for all finite* $F \subset P_0$.
2. $Z(F) \geq 0$ *for all finite* $F \subset P_{00}$.

*Proof.* It is clear that $P_{00} \subset P_0$ and thus one direction is trivial. Now suppose $Z(E) \geq 0$ for all finite $E \subset P_{00}$. Now take a finite $F \subset P_0$ and let $k(F) = \max\{i : b^i a \in F\}$ and $\ell(F) = \max\{j : b^j \in F\}$. We know $F \subset P_{00}$ when $k(F) < m$ and $\ell(F) \leq 1$.

Suppose $k(F) = k \geq m$, then write $F = \{b^k a, p_2, \cdots, p_n\}$. Due to Lemma 5.2.1, we may assume all other elements in $F$ with the form $b^i a$ has $i < k$. Denote

$$\begin{aligned} F_1 &= \{b, p_2, \cdot, p_n\}, \\ F_2 &= \{b^{k-1}a, b^{-1}(b \vee p_2), \cdots, b^{-1}(b \vee p_n)\}. \end{aligned}$$

It follows from Lemma 5.2.2 that $Z(F) \geq 0$ if both $Z(F_1) \geq 0$ and $Z(F_2) \geq 0$. Notice that we replaced $b^k a$ by $b$ in $F_1$, so that $k(F_1) < k(F)$ and $\ell(F_1) = \ell(F)$. For $F_2$, it follows from the calculation in Proposition 5.3.11 that if $p_i = b^i a$, then

$$b^{-1}(b \vee p_i) = \begin{cases} b^{i-1}a, i \geq 1 \\ b^{m-1}a, i = 0 \end{cases}$$

If $p_i = b^j$, then $b^{-1}(b \vee p_i) = b^{j-1}$. Therefore, $k(F_2) < \max\{k(F), m-1\}$ and $\ell(F_2) \leq \ell(F_1)$.

Suppose $\ell(F) = \ell > 1$, then write $F = \{b^\ell, p_2, \cdots, p_n\}$. Denote

$$\begin{aligned} F_1 &= \{b, p_2, \cdot, p_n\}, \\ F_2 &= \{b^{\ell-1}, b^{-1}(b \vee p_2), \cdots, b^{-1}(b \vee p_n)\}. \end{aligned}$$

It follows from Lemma 5.2.2 that $Z(F) \geq 0$ if both $Z(F_1) \geq 0$ and $Z(F_2) \geq 0$. A similar computation shows that $k(F_1) = k(F)$, $k(F_2) \leq \max\{k(F), m-1\}$, and $\ell(F_1), \ell(F_2) < \ell(F)$.

Combining these two cases, we able to repeated use Lemma 5.2.2 and induction on $(k(F), \ell(F))$ to show $Z(F) \geq 0$ assuming $Z(E) \geq 0$ for all finite $E \subset P_{00}$. □

**Theorem 5.3.13.** *Let $B_{n,m}$ be a Baumslag-Solitar monoid for $n, m \geq 1$ and let $a, b$ be its generators. Let $P_{00} = \{b, b^i a : 0 \leq i \leq m-1\}$. Then $T$ has $*$-regular dilation if and only if $Z(F) \geq 0$ for all finite $F \subset P_{00}$.*

**Remark 5.3.14.** *This set $P_{00}$ arise naturally in the study of Baumslag-Solitar monoids. A set $E \subset P$ is called a foundation set for every $p \in P$, there exists $e \in E$ so that $eP \cap pP \neq \emptyset$. Foundation set naturally arises from the study of boundary quotient of various semigroup $C^*$-algebras [18, 11, 66].*

*One may also notice that the minimal sets in the case of finite Artin monoids, Thompson's monoid, $\mathbb{N} \rtimes \mathbb{N}^\times$, the minimal set $P_0$ is in fact also a foundation set in the corresponding semigroups. However, it is unknown if this is true in general: namely, whether $T$ has $*$-regular dilation if and only if $Z(F) \geq 0$ for all finite $F \subset E$ where $E$ is a foundation set.*

## 5.4 The Graph Product of Right LCM Semigroups

Recall the graph product construction discussed in the Section 2.1.4 provides a way to construct right LCM semigroups from existing ones. Our goal is to study the relation between $*$-regular condition on the graph product of right LCM semigroups.

We first prove a key technical lemma in our analysis of the $*$-regular condition on graph product of right LCM semigroup.

**Lemma 5.4.1.** *Let $p \in P_v$ and $x \in P_\Gamma$ so that $x \vee p \neq \emptyset$. Then there exists $s \in x \vee p$ with $\ell(p^{-1}s) \leq \ell(s)$.*

*Proof.* The statement is trivially true if $p = e$ since we can simply pick $s = x \in x \vee e$. Suppose otherwise, let $x = x_1 x_2 \cdots x_n$ be a reduced expression of $x$ and let $x_1 \in P_u$. Here, $\ell(x) = n$. Let $y = x_2 \cdots x_n$ and $x = x_1 y$. First of all, since $x \vee p \neq \emptyset$, there exists $q = p \cdot p' = x \cdot x' = x_1 y \cdot x'$ for some $x', p' \in P_\Gamma$. Since $p$ and $x_1$ are in the front of this expression, $u, v$ are both initial vertices of $q$ and thus either $u = v$ or $(u, v)$ is an edge of $\Gamma$.

Let us do an induction on $\ell(x)$. In the base case when $\ell(x) = 1$, $x = x_1$ has only one syllable in its reduced expression. There are two cases:

Case 1: if $u = v$, then $x_1, p \in P_v$. By Lemma 2.1.21, $xP_\Gamma \cap pP_\Gamma \neq \emptyset$ implies $xP_v \cap pP_v \neq \emptyset$. Therefore, we can pick $s \in x \vee p \subset P_v$ and $p^{-1}s \in P_v \subset P_\Gamma$. $p^{-1}s$ has only one syllable, and its length is either 0 (when $p^{-1}s = e$) or 1. Hence, $\ell(p^{-1}s) \leq 1 = \ell(x)$.

Case 2: if $u \neq v$, then $(u, v)$ must be an edge of $\Gamma$. By Lemma 2.1.20, we can pick $s = px_1 \in p \vee x$ and thus $\ell(p^{-1}s) = \ell(x)$.

Suppose now the statement holds true for all $x$ with $\ell(x) < n$. Now consider the case when $\ell(x) = n$ and $x = x_1 \cdots x_n$ is an reduced expression of $x$. Let $y = x_2 \cdots x_n$. There are again two cases.

Case 1: if $u = v$, then $xP_\Gamma \cap pP_\Gamma \neq \emptyset$ implies $x_1P_v \cap pP_v \neq \emptyset$. Pick $t \in x_1 \vee p \in P_v$ and let $q = x_1^{-1}t \in P_v$. We first prove that

$$tP_\Gamma \cap xP_\Gamma = pP_\Gamma \cap xP_\Gamma.$$

First, by Lemma 2.1.21, $t \in x_1 \vee p$ implies $tP_\Gamma = x_1P_\Gamma \cap pP_\Gamma$. Hence

$$tP_\Gamma \cap xP_\Gamma \subseteq pP_\Gamma \cap xP_\Gamma.$$

Conversely, by Lemma 2.1.21,

$$pP_\Gamma \cap xP_\Gamma \subset pP_\Gamma \cap x_1P_\Gamma = tP_\Gamma.$$

This proves the other inclusion.

Now $t \vee x = p \vee x$. But $t = x_1q$ and $x = x_1y$, by Lemma 2.1.17, $x_1 \cdot (q \vee y) = p \vee x \neq \emptyset$. In particular, $q \vee y \neq \emptyset$. Notice that $y$ is obtained by removing the initial syllable $x_1$ from a reduced expression $x = x_1y$. Hence, $\ell(y) = \ell(x) - 1 < n$. By the induction hypothesis, there exists $s \in q \vee y$ and $\ell(q^{-1}s) \leq \ell(y)$.

Let $w = q^{-1}s$ and $s = qw \in q \vee y$. Let $s' = x_1s \in x_1(q \vee y) = p \vee x$. $s' = x_1qw = tw$ and $p^{-1}s' = (p^{-1}t)w$. The induction hypothesis gives $\ell(w) \leq \ell(y)$. Now $p^{-1}t \in P_v$ and thus $\ell(p^{-1}s') = \ell(p^{-1}tw) \leq \ell(w) + 1$. Hence

$$\ell(p^{-1}s') \leq \ell(w) + 1 \leq \ell(y) + 1 = \ell(x),$$

where $s' \in p \vee x$. This finishes the induction step for this case.

Case 2: if $u \neq v$, then $(u, v)$ must be an edge of $\Gamma$ and $x_1, p$ commute. We first prove that

$$x_1pP_\Gamma \cap xP_\Gamma = pP_\Gamma \cap xP_\Gamma.$$

The $\subseteq$ direction is trivial as $x_1 p P_\gamma = p x_1 P_\Gamma \subset p P_\Gamma$. Conversely, by Lemma 2.1.20,

$$pP_\Gamma \cap xP_\Gamma \subset pP_\Gamma \cap x_1 P_\Gamma = x_1 p P_\Gamma.$$

This proves the other inclusion.

By Lemma 2.1.17, $p \vee x = x_1 p \vee x_1 y = x_1(p \vee y) \neq \emptyset$. Hence $p \vee y \neq \emptyset$. Moreover, $\ell(y) = \ell(x) - 1 < n$. By the induction hypothesis, there exists $s \in p \vee y$ so that $\ell(p^{-1}s) \leq \ell(y)$. Let $w = p^{-1}s$ and $s' = x_1 s = x_1 p w \in x_1(p \vee y) = p \vee x$. Now

$$\ell(p^{-1}s') = \ell(p^{-1}x_1 p w) = \ell(x_1 w) \leq \ell(w) + 1 \leq \ell(y) + 1 = \ell(x),$$

where $s' \in p \vee x$. This finishes the induction step for this case and thus the entire proof. $\square$

Now consider a collection of representations $T_v : P_v \to \mathcal{B}(\mathcal{H})$. Suppose for any edge $(u, v)$ of $\Gamma$, $T_u(p)$ commutes with $T_v(q)$ for all $p \in P_u$ and $q \in P_v$. Then we can build a representation $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ where for any $x = x_1 \cdots x_n$, $x_i \in P_{v_i}$,

$$T(x) = T_{v_1}(x_1) T_{v_2}(x_2) \cdots T_{v_n}(x_n).$$

Since the commutation relations of $T_v$ coincide with the commutation relations in $P_\Gamma$, this defines a representation $T$ on the graph product $P_\Gamma$. In fact, every representation $T$ of $P_\Gamma$ arises in this way since we can simply let $T_v$ be the restriction of $T$ on $P_v$. We are interested in when the representation $T$ has $*$-regular dilation.

**Example 5.4.2.** *Take $P_v = \mathbb{N}$ for all $v \in V$. This semigroup $P_\Gamma$ is the graph product of $\mathbb{N}$, also known as a right-angled Artin monoid as discussed previously (Example 2.1.7(2)). Each representation $T_v$ of $P_v = \mathbb{N}$ is uniquely determined by the value $T_v = T_v(1_v)$. The commutation relations require that $T_u, T_v$ commute whenever $(u, v)$ is an edge of $\Gamma$.*

*The $*$-regular dilation for such representation $T$ of graph product of $\mathbb{N}$ was the focus of [42]. A Brehmer-type condition is established in [42, Theorem 2.4]. It is shown that the following are equivalent:*

1. *$T$ has a $*$-regular dilation;*
2. *$T$ has a minimal isometric Nica-covariant dilation;*
3. *For every finite $W \subset V$,*

$$\sum_{\substack{U \subseteq W \\ U \text{ is a clique}}} (-1)^{|U|} T_U T_U^* \geq 0.$$

*Here, $T_U = \prod_{v \in U} T_v$.*

We would like to extend our result of $*$-regular dilation on graph product of $\mathbb{N}$ to graph product of any right LCM semigroup. We have derived in Theorem 5.1.8 that $T$ has a $*$-regular dilation if and only if for every finite set $F \subset P_\Gamma$,

$$Z(F) = \sum_{U \subseteq F} (-1)^{|U|} TT^*(\vee U) \geq 0.$$

The goal is to reduce $F$ further to a much smaller collection of subsets.

**Proposition 5.4.3.** *Let $P_\Gamma$ be a graph product of a collection of right LCM semigroups $(P_v)_{v \in V}$, and $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Then the following are equivalent:*

1. *For every finite set $F \subset P_\Gamma$, $Z(F) \geq 0$.*
2. *For every finite set $e \notin F \subset \bigcup_{v \in V} P_v$, $Z(F) \geq 0$.*

*Proof.* The direction (1) $\Rightarrow$ (2) is obvious. To show the converse, notice that a finite set $e \notin F \subset \bigcup_{v \in V} P_v$ if and only if every element $x \in F$ is inside some $P_v$ and thus $\ell(x) = 1$ for all $x \in F$. Denote $c(F) = \sum_{x \in F} (\ell(x) - 1)$. Then for a finite subset $e \notin F \subset P_\Gamma$, $c(F) \geq 0$ and $F \subset \bigcup_{v \in V} P_v$ if and only if $c(F) = 0$.

If $e \notin F$ has $c(F) > 0$, then there exists $x \in F$ with $\ell(x) \geq 2$. Write $x = p_1 q$ for some $p_1 \in P_v$ and $\ell(q) = \ell(x) - 1$. Let $F = \{p_1 q, p_2, \cdots, p_n\}$. Let

$$\begin{aligned} F_1 &= \{p_1, p_2, \cdots, p_n\} \\ F_2 &= \{q, p_1^{-1} s_2, \cdots, p_1^{-1} s_n\} \end{aligned}$$

where $s_i \in p_1 \vee p_i$ for all $2 \leq i \leq n$. By Lemma 5.2.2, $Z(F) \geq 0$ if $Z(F_1) \geq 0$ and $Z(F_2) \geq 0$.

Since $\ell(p_1) < \ell(p_1 q)$, we have $c(F_1) < c(F)$. By Lemma 5.4.1, for each $2 \leq i \leq n$, either $p_1 \vee p_i = \emptyset$ or there exists $s_i \in p_1 \vee p_i$ with $\ell(p_1^{-1} s_i) \leq \ell(p_i)$. Therefore, compare elements in $F$ with $F_2$: either an element $p_i$ is removed when $p_1 \vee p_i = \emptyset$, or $p_i$ is replaced by $p_1^{-1} s_i$ with $\ell(p_1^{-1} s_i) \leq \ell(s_i)$. Moreover, the element $p_1 q$ in $F$ is replaced by $q$ where $\ell(q) = \ell(p_1 q) - 1$. Hence, $c(F_2) < c(F)$.

Now $c(F_1), c(F_2) < c(F)$. We can only repeat this process finitely many times. The positivity of any finite $e \notin F \subset P_\Gamma$ is therefore reduced to the positivity of sets of the form $F \subset P_\Gamma$ where $\ell(x) \leq 1$ for all $x \in F$. Notice that $Z(F) \geq 0$ whenever $e \in F$ (Lemma 5.2.1(2)). Hence, condition (2) is sufficient. $\square$

For a finite set $e \notin U \subset \bigcup_{v\in V} P_v$, we denote $I(U) = \{I(x) : x \in U\}$. Suppose $(u, v)$ is not an edge of $\Gamma$, and $e \neq p \in P_u$, $e \neq q \in P_v$, then $pP_\Gamma \cap qP_\Gamma$ must be $\emptyset$ since $u, v$ are both initial vertices of any element $r \in p \vee q$. Therefore, $\vee U = \emptyset$ unless any two vertices in $U$ are adjacent to one another. In other words, $\vee U = \emptyset$ unless $I(U)$ is a clique in $\Gamma$. Hence, we can simplify $Z(F)$ as:

$$\begin{aligned} Z(F) &= \sum_{U\subseteq F} (-1)^{|U|} TT^*(\vee U) \\ &= \sum_{\substack{U\subseteq F \\ I(U) \text{ is a clique}}} (-1)^{|U|} TT^*(\vee U). \end{aligned}$$

Take a finite set finite $e \notin F \subset \bigcup_{v\in V} P_v$. Each $x \in F$ belongs to a certain copy of $P_v$. If $x = p_1 q$ with $p_1, q \in P_v$ and $F = \{p_1 q, p_2, \cdots, p_n\}$. Let $F_1, F_2$ be the subsets defined in Lemma 5.2.2:

$$\begin{aligned} F_1 &= \{p_1, p_2, \cdots, p_n\} \\ F_2 &= \{q, p_1^{-1} s_2, \cdots, p_1^{-1} s_n\} \end{aligned}$$

where $s_i \in p_1 \vee p_i$ for all $2 \leq i \leq n$. For each $2 \leq i \leq n$, apply Lemma 5.4.1, either $p_1 \vee p_i = \emptyset$ or we can pick $s_i \in p_1 \vee p_i$ with $\ell(p_1^{-1} s_i) \leq \ell(p_i)$. Hence, $F_1, F_2 \subset \bigcup_{v\in V} P_v$. In the case when a semigroup $P_u$ satisfies the descending chain condition, the procedure described in Proposition 5.2.7 still applies. This can further reduce $F$ to a subset $F \subset \bigcup_{v\in V} P_v$ where every element in $F \cap P_u$ is a minimal element (i.e. $F \cap P_u = (P_u)_0$).

Therefore, we obtain the following characterization of $*$-regular representations of a graph product of right LCM semigroups.

**Theorem 5.4.4.** *Let $P_\Gamma$ be a graph product of right LCM semigroups and $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Then the following are equivalent:*

1. *$T$ has a $*$-regular dilation;*
2. *$T$ has a minimal isometric Nica-covariant dilation;*
3. *For every finite $F \subset P_\Gamma$,*

$$Z(F) = \sum_{U\subseteq F} (-1)^{|U|} TT^*(\vee U) \geq 0;$$

*4. For every finite* $e \notin F \subset \bigcup_{v\in V} P_v$,

$$Z(F) = \sum_{\substack{U\subseteq F \\ I(U) \text{ is a clique}}} (-1)^{|U|} TT^*(\vee U) \geq 0.$$

*In particular, in the case when* $P_u$ *satisfies the descending chain condition, we may assume* $F \cap P_u \subset (P_u)_0$, *where* $(P_u)_0$ *is the set of minimal elements of* $P_u$.

**Example 5.4.5.** *In the case when* $P_v = \mathbb{N}$ *for all* $v \in V$. $P_\Gamma$ *is a graph product of* $\mathbb{N}$, *which is generated by* $\{e_v : v \in V\}$ *with the relation* $e_u e_v = e_v e_u$ *whenever* $(u, v)$ *is an edge of* $\Gamma$. *Let* $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ *be a contractive representation which is uniquely determined by* $T_v = T(e_v)$. *For each finite* $W \subset V$, *if* $W$ *is a clique, let* $T_W = \prod_{v\in W} T_W$. *Each* $P_v = \mathbb{N}$ *satisfies the descending chain condition, and the set of minimal elements* $(P_v)_0 = \{e_v\}$.

*By Theorem 5.4.4,* $T$ *has a* $*$*-regular dilation if and only if for every finite* $F \subseteq \{e_v : v \in V\}$,

$$Z(F) = \sum_{\substack{E\subseteq F \\ I(E) \text{ is a clique}}} (-1)^{|E|} TT^*(\vee E) \geq 0.$$

*Notice that each finite* $E \subseteq \{e_v : v \in V\}$ *corresponds to a finite set* $U = \{v : e_v \in E\} \subset V$. *It is easy to see that* $I(E) = U$ *and* $\vee E = \prod_{v\in U} e_v$. *Therefore,* $T$ *has a* $*$*-regular dilation if and only if for every finite* $W \subset V$,

$$\sum_{\substack{U\subseteq W \\ U \text{ is a clique}}} (-1)^{|U|} T_U T_U^* \geq 0.$$

*Here,* $T_U = \prod_{u\in U} T(e_u)$. *This gives an another proof of Theorem 4.5.5. Two proof differs in the following manner: the proof in Chapter 4 reduces the positivity of* $K[F]$ *to subsets* $F \subset \{e_v\}$ *by exploiting the structure of graph product of* $\mathbb{N}$. *A Cholesky decomposition is then applied to such* $K[F]$ *using the positivity of* $Z(F)$. *In this chapter, we first make reduce the positivity of* $K[F]$ *to the positivity of* $Z(F)$ *via Cholesky decomposition. Then, we use the descending chain properties and few reduction lemmas to reduce* $Z(F)$ *to subsets* $F \subset \{e_v\}$.

We would like to consider an application of Theorem 5.4.4. Let $\Gamma$ be a complete graph. In other words, $(u, v) \in E$ for all $u \neq v$ in $V$. The graph product $P_\Gamma$ is simply the direct sum $\oplus_{v\in V} P_v$.

**Definition 5.4.6.** *A family of contractive representation $T_v : P_v \to \mathcal{B}(\mathcal{H})$ are called doubly commuting if for any $u \neq v$ and $p \in P_u$, $q \in P_v$, $T_u(p)$ commutes with both $T_v(q)$ and $T_v(q)^*$.*

*Suppose that $\Gamma$ is a complete graph. A representation $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ is called doubly commuting if $T_v : P_v \to \mathcal{B}(\mathcal{H})$ given by restricting $T$ on $P_v$ form a doubly commuting family of contractive representations.*

Doubly commuting representations on products of special semigroups have been previously studied. A doubly commuting representation of $\mathbb{N}^k$ is always regular ([9], see also [52] for an alternative proof using $C^*$-algebra and completely positive maps). Fuller [28, Theorem 2.4] proved that a doubly commuting representation of $\oplus S_i$ is always regular, where $S_i$ is a countable additive subgroup of $\mathbb{R}^+$. We are now going to extend all these results to direct sums of right LCM semigroups.

**Lemma 5.4.7.** *Fix a finite subset $W \subset V$ and for each $w \in W$, $F_w \subset P_w$ is a finite subset. Let $F = \bigcup_{w \in W} F_w$. Let $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ be a doubly commuting contractive representation and $T_v : P_v \to \mathcal{B}(\mathcal{H})$ be the representation by restricting $T$ on $P_v$. Then*

$$TT^*(\vee F) = \prod_{w \in W} T_w T_w^*(\vee F_w)$$

.

*Moreover, let*

$$Z(F) = \sum_{U \subseteq F} (-1)^{|U|} TT^*(\vee U),$$

$$Z_w(F_w) = \sum_{U_w \subseteq F_w} (-1)^{|U_w|} T_w T_w^*(\vee U_w).$$

*Then $\{Z_w(F_w)\}_{w \in W}$ is a collection of commuting operators, and*

$$Z(F) = \prod_{w \in W} Z_w(F_w).$$

*Proof.* For each $w \in W$, pick $p_w \in \vee F_w$. By Lemma 2.1.18, $\vee F = \vee \{p_w\}_{w \in W}$. By Lemma 2.1.22,

$$\prod_{w \in W} p_w \in \vee \{p_w\}_{w \in W} = \vee F.$$

Hence,

$$\begin{aligned} TT^*(\vee F) &= TT^*\big(\prod_{w\in W} p_w\big) \\ &= \big(\prod_{w\in W} T_w(p_w)\big)\big(\prod_{w\in W} T_w(p_w)^*\big) \\ &= \prod_{w\in W} T_w(p_w)T_w(p_w)^*. \end{aligned}$$

Now for each $U \subseteq F$, let $U_w = U \cap F_w$ be disjoint subsets. Then,

$$(-1)^{|U|}TT^*(\vee U) = \prod_{w\in W} (-1)^{|U_w|}T_wT_w^*(\vee U_w).$$

It is now easy to check $Z(F) = \prod_{w\in W} Z_w(F_w)$. □

As a direct consequence of Lemma 5.4.7 and Theorem 5.4.4:

**Theorem 5.4.8.** *Let $T : P_\Gamma \to \mathcal{B}(\mathcal{H})$ be a doubly commuting contractive representation of $P_\Gamma$ and $T_v : P_v \to \mathcal{B}(\mathcal{H})$ be the representation by restricting $T$ on $P_v$. Then $T$ has a $*$-regular dilation if and only if each $T_v$ has a $*$-regular dilation as a representation of $P_v$.*

# Chapter 6

# Application

This chapter discusses two applications of regular dilation. In the Section 6.1, which is based on Section 7 of [40], we look into the application of regular dilation in the study of semi-crossed product. The main result (Theorem 6.1.3) states that a contractive Nica-covariant pair can be dilated toan isometric Nica-covariant pair. Therefore, certain universal semi-crossed product algebra generated by contractive Nica-covariant pairs coincides with those generated by isometric Nica-covariant pairs (Corollary 6.1.4). Finally, in the Section 6.2, which is based on [39], we look into the surprising relation between regular dilation and subnormal operators.

## 6.1 Covariant Representations

The semicrossed products of a dynamical system by Nica-covariant representations was discussed in [28, 21], where its regularity is seen as a key to many results. Our result on the regularity of Nica-covariant representations (Theorem 3.3.1 and Corollary 3.3.2) allows us to generalize some of the results to arbitrary lattice ordered abelian groups.

**Definition 6.1.1.** *A $C^*$-dynamical system is a triple $(A, \alpha, P)$ where*

1. *$A$ is a $C^*$-algebra;*
2. *$\alpha : P \to \mathrm{End}(A)$ maps each $p \in P$ to a $*$-endomorphism on $A$;*
3. *$P$ is a spanning cone of some group $G$.*

**Definition 6.1.2.** *A pair* $(\pi, T)$ *is called a* covariant pair *for a* $C^*$*-dynamical system if*

1. $\pi : A \to \mathcal{B}(\mathcal{H})$ *is a* $*$*-representation;*
2. $T : P \to \mathcal{B}(\mathcal{H})$ *is a contractive representation of* $P$*;*
3. $\pi(a)T(s) = T(s)\pi(\alpha_s(a))$ *for all* $s \in P$ *and* $a \in A$*.*

*In particular, a covariant pair* $(\pi, T)$ *is called Nica-covariant/isometric, if* $T$ *is Nica-covariant/isometric.*

The main goal is to prove that Nica-covariant pairs on $C^*$-dynamical systems can be lifted to isometric Nica-covariant pairs. This can be seen from [21, Theorem 4.1.2] and Corollary 3.3.2. However, we shall present a slightly different approach by taking the advantage of the structure of lattice ordered abelian group.

**Theorem 6.1.3.** *Let* $(A, \alpha, P)$ *be a* $C^*$*-dynamical system over a positive cone* $P$ *of a lattice ordered abelian group* $G$*. Let* $\pi : A \to \mathcal{B}(\mathcal{H})$ *and* $T : P \to \mathcal{B}(\mathcal{H})$ *form a Nica-covariant pair* $(\pi, T)$ *for this* $C^*$*-dynamical system. If* $V : P \to \mathcal{K}$ *is a minimal isometric dilation of* $T$*, then there is an isometric Nica-covariant pair* $(\rho, V)$ *such that for all* $a \in A$*,*

$$P_{\mathcal{H}}\rho(a)\big|_{\mathcal{H}} = \pi(a).$$

*Moreover,* $\mathcal{H}$ *is invariant for* $\rho(a)$*.*

*Proof.* Fix a minimal dilation $V$ of $T$ and consider any $h \in \mathcal{H}$, $p \in P$, and $a \in A$: define

$$\rho(a)V(p)h = V(p)\pi(\alpha_p(a))h$$

We shall first show that this is a well defined map. First of all, since $V$ is a minimal isometric dilation, the set $\{V(p)h\}$ is dense in $\mathcal{K}$. Suppose $V(p)h_1 = V(s)h_2$ for some $p, s \in P$ and $h_1, h_2 \in \mathcal{H}$. It suffices to show that for any $t \in P$ and $h \in \mathcal{H}$, we have

$$\langle V(p)\pi(\alpha_p(a))h_1, V(t)h\rangle = \langle V(s)\pi(\alpha_s(a))h_2, V(t)h\rangle\,. \tag{6.1}$$

Since $A$ is a $C^*$-dynamical system, it follows from the covariant condition $\pi(a)T(s) = T(s)\pi(\alpha_s(a))$ that $T(s)^*\pi(a) = \pi(\alpha_s(a))T(s)^*$. Hence,

$$\begin{aligned}
& \langle V(p)\pi(\alpha_p(a))h_1, V(t)h\rangle \\
= \;& \langle V(t)^*V(p)\pi(\alpha_p(a))h_1, h\rangle \\
= \;& \langle V(t-t\wedge p)^*V(p-t\wedge p)\pi(\alpha_p(a))h_1, h\rangle \\
= \;& \langle T(t-t\wedge p)^*T(p-t\wedge p)\pi(\alpha_p(a))h_1, h\rangle \\
= \;& \left\langle \pi(\alpha_{p-(p-t\wedge p)+(t-t\wedge p)}(a))T(t-t\wedge p)^*T(p-t\wedge p)h_1, h\right\rangle \\
= \;& \langle \pi(\alpha_t(a))T(t-t\wedge p)^*T(p-t\wedge p)h_1, h\rangle\,.
\end{aligned}$$

Here we used that fact that $V$ has regular dilation and thus

$$P_{\mathcal{H}}V(t-t\wedge p)^*V(p-t\wedge p)\big|_{\mathcal{H}} = T(t-t\wedge p)^*T(p-t\wedge p).$$

Now notice that

$$\begin{aligned} T(t-t\wedge p)^*T(p-t\wedge p)h_1 &= P_{\mathcal{H}}V(t-t\wedge p)^*V(p-t\wedge p)h_1 \\ &= P_{\mathcal{H}}V(t)^*V(p)h_1. \end{aligned}$$

Similarly,

$$\langle V(s)\pi(\alpha_s(a))h_2, V(t)h\rangle = \langle \pi(\alpha_t(a))T(t-t\wedge s)^*T(s-t\wedge s)h_2, h\rangle ,$$

where

$$T(t-t\wedge s)^*T(s-t\wedge s)h_2 = P_{\mathcal{H}}V(t)^*V(s)h_2 = P_{\mathcal{H}}V(t)^*V(p)h_1.$$

Therefore, $\rho$ is well defined on the dense subset $\{V(p)h\}$.

Since $V(p)$ is isometric and $\pi, \alpha$ are completely contractive,

$$\|V(p)\pi(\alpha_p(a))h\| = \|\pi(\alpha_p(a))h\| \leq \|h\| = \|V(p)h\|,$$

and thus $\rho(a)$ is contractive on $\{V(p)h\}$. Hence, $\rho(a)$ can be extended to a contractive map on $\mathcal{K}$. Moreover, for any $h\in\mathcal{H}$ and $a\in A$, we have $\rho(a)h=\pi(a)h\in\mathcal{H}$, and thus $\mathcal{H}$ is invariant for $\rho$. For any $a,b\in A$, $p\in P$, and $h\in\mathcal{H}$,

$$\begin{aligned} \rho(a)\rho(b)V(p)h &= V(p)\pi(\alpha_p(a))\pi(\alpha_p(b))h \\ &= V(p)\pi(\alpha_p(ab))h \\ &= \rho(ab)V(p)h. \end{aligned}$$

Therefore, $\rho$ is a contractive representation of $A$ and thus a $*$-representation. Now for any $p,t\in P$ and $h\in\mathcal{H}$,

$$\begin{aligned} \rho(a)V(p)V(t)h &= V(p+t)\pi(\alpha_{p+t}(a))h \\ &= V(p)V(t)\rho(\alpha_{p+t}(a))h \\ &= V(p)\rho(\alpha_p(a))V(t)h. \end{aligned}$$

Hence, $(\rho, V)$ is an isometric Nica-covariant pair. □

This lifting of contractive Nica-covariant pairs to isometric Nica-covariant pairs has significant implication in its associated semi-crossed product. A family of covariant pairs gives rise to a semi-crossed product algebra in the following way [28, 21]. For a $C^*$-dynamical system $(A, \alpha, P)$, denote $\mathcal{P}(A, P)$ be the algebra of all formal polynomials $q$ of the form

$$q = \sum_{i=1}^{n} e_{p_i} a_{p_i},$$

where $p_i \in P$ and $a_{p_i} \in A$. The multiplication on such polynomials follows the rule that $ae_s = e_s\alpha(a)$ and $e_p e_q = e_{pq}$. For a covariant pair $(\sigma, T)$ on this dynamical system, define a representation of $\mathcal{P}(A, P)$ by

$$(\sigma \times T)\left(\sum_{i=1}^{n} e_{p_i} a_{p_i}\right) = \sum_{i=1}^{n} T(p_i)\sigma(a_{p_i}).$$

Now let $\mathcal{F}$ be a family of covariant pairs on this dynamical system. We may define a norm on $\mathcal{P}(A, S)$ by

$$\|p\|_{\mathcal{F}} = \sup\{(\sigma \times T)(p) : (\sigma, T) \in \mathcal{F}\},$$

and the semi-crossed product algebra is defined as

$$A \times_{\alpha}^{\mathcal{F}} P = \overline{\mathcal{P}(A, S)}^{\|\cdot\|_{\mathcal{F}}}.$$

In particular, $A \times_{\alpha}^{nc} P$ is determined by the Nica-covariant representations, and $A \times_{\alpha}^{nc,iso} P$ is determined by the isometric Nica-covariant representation. As an immediate corollary from Theorem 5.1.8 and 6.1.3,

**Corollary 6.1.4.** *For a $C^*$-dynamical system $(A, \alpha, P)$, the semi-crossed product algebra given by Nica-covariant pairs agrees with that given by isometric Nica-covariant pairs. In other words,*

$$A \times_{\alpha}^{nc} P \cong A \times_{\alpha}^{nc,iso} P.$$

## 6.2 Subnormal Representations

An operator $T \in \mathcal{B}(\mathcal{H})$ is called subnormal if there exists a normal extension $N \in \mathcal{B}(\mathcal{K})$ where $\mathcal{H} \subseteq \mathcal{K}$ and $N|_{\mathcal{H}} = T$. There are many equivalent conditions for an operator being subnormal, for example, Agler showed a contractive operator $T$ is subnormal if and only if for any $n \geq 0$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \geq 0.$$

One may refer to [16, Chapter II] for many other characterizations of subnormal operators.

A commuting pair of subnormal operators $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ might not have commuting normal extensions [45, 1], and a necessary and sufficient condition was given by Itô in [32]. Athavale obtained a necessary and sufficient condition for $n$ commuting operators $T_1, \cdots, T_n \in \mathcal{B}(\mathcal{H})$ to have commuting normal extensions in terms of operator polynomials [6, 8].

This section consider the question as to when a contractive representation of a unital abelian semigroup can be extended to a contractive normal representation. Athavale's result can be applied to the set of generators, and obtain a map that sends the semigroup into a family of commuting normal operators. Our first result shows that such normal map guarantees the existence of a normal representation. It is also observed that Athavale's result is equivalent to a certain representation being regular, and we further extend Athavale's result to abelian lattice ordered semigroups.

### 6.2.1 Involution Semigroup and Subnormal Map

Itô [32] established a necessary and sufficient condition for a commuting family of subnormal operators to have commuting normal extensions. Athavale [6] generalized Agler's result to a family of commuting contractions:

**Theorem 6.2.1** (Athavale)**.** *Let $T = (T_1, T_2, \cdots, T_m)$ be a family of $m$ commuting contractions. Then $T$ has a commuting normal extension $N$ if and only if for any $n_1, n_2, \cdots, n_m \geq 0$, we have*

$$\sum_{0 \leq k_i \leq n_i} (-1)^{k_1+k_2+\cdots+k_m} \binom{n_1}{k_1} \cdots \binom{n_m}{k_m} T_1^{*k_1} T_2^{*k_2} \cdots T_m^{*k_m} T_m^{k_m} \cdots T_1^{k_1} \geq 0. \qquad (\star)$$

One may observe that a family of $m$ commuting contractions defines a contractive representation $T : \mathbb{N}^m \to \mathcal{B}(\mathcal{H})$ that sends each generator $e_i$ to $T_i$. A commuting normal extension $N = (N_1, \cdots, N_m)$ can be seen as a contractive normal representation $N : \mathbb{N}^m \to \mathcal{B}(\mathcal{K})$ that extends $T$. Athavale's result gives a necessary and sufficient condition for the existence of a normal representation that extends $T$. If $P$ is a unital abelian semigroup and $T : P \to \mathcal{B}(\mathcal{H})$ is a contractive representation, we may also ask the question when there exists a normal representation $N : P \to \mathcal{B}(\mathcal{K})$ that extends $T$.

**Example 6.2.2.** *Consider $P = \mathbb{N}\backslash\{1\}$ which is a unital semigroup generated by 2 and 3. A contractive representation $T : P \to \mathcal{B}(\mathcal{H})$ is uniquely determined by $T(2), T(3)$, which satisfies $T(2)^3 = T(3)^2$. We may use Theorem 6.2.1 to test if $T(2), T(3)$ has commuting normal extensions $N_2, N_3$. However, even if they do have such extensions, there is no guarantee that $N_2^3 = N_3^2$ and therefore it is not clear if we can get a normal representation $N : P \to \mathcal{B}(\mathcal{K})$ that extends $T$. Nevertheless, since $N_2, N_3$ extend $T(2), T(3)$ respectively, we may define a normal map $N : P \to \mathcal{B}(\mathcal{K})$ using $N_2, N_3$ such that $\{N(p)\}_{p\in P}$ is a family of commuting normal operators where $N(p)$ extends $T(p)$. As we shall see soon, in Theorem 6.2.6, the existence of such normal map guarantees a normal representation that extend $T$.*

*We shall also note that this semigroup $P = \mathbb{N}\backslash\{1\}$ is closely related to the so-called Neil algebra $\mathcal{A} = \{f \in A(\mathbb{D}) : f'(0) = 0\}$. Dilation on Neil algebra has been studied in [23, 10]. Unlike $\mathbb{N}$ where every contractive representation has a unitary dilation due to Sz.Nagy's dilation, contractive representations of $P$ may not have a unitary dilation. Even so, for a contractive representation $T : P \to \mathcal{B}(\mathcal{H})$, we may apply Ando's theorem to dilate $T(2), T(3)$ into commuting unitaries $U_2, U_3$, and therefore there exists a family $\{U_n\}_{n\in P}$ of commuting unitaries where $P_{\mathcal{H}} U_n|_{\mathcal{H}} = T(n)$ for each $n$ [23, Example 2.4]. However, existence of such unitary maps does not guarantees a unitary dilation of $T$.*

One of the main tools for the proof is the involution semigroup. Sz.Nagy used such a technique and proved a subnormality condition of a single operator due to Halmos [68], and Athavale also used this technique in [6]. We shall extend this technique to a more general setting.

**Definition 6.2.3.** *A semigroup $P$ is called an involution semigroup (or a $*$-semigroup) if there is an involution $* : P \to P$ that satisfies $p^{**} = p$ and $(pq)^* = q^* p^*$.*

For example, any group $G$ can be seen as an involution semigroup where $g^* = g^{-1}$. Any abelian semigroup can be seen as involution semigroup where $p^* = p$. A representation $D$ of a unital involution semigroup $P$ is a unital $*$-homomorphism. It is obvious that if $pp^* = p^* p$, then $D(p)$ is normal. Sz.Nagy established a condition which guarantees that a map on an involution semigroup has a dilation to a representation of the semigroup [68].

**Theorem 6.2.4.** *Let $P$ be a $*$-semigroup and $T : P \to \mathcal{B}(\mathcal{H})$ satisfies the following conditions:*

1. $T(e) = I, T(p^*) = T(p)^*$,

2. *For any* $p_1, \cdots, p_n \in P$*, the operator matrix* $[T(p_i^* p_j)]$ *is positive,*
3. *There exists a constant* $C_a > 0$ *for each* $a \in P$ *such that for all* $p_1, \cdots, p_n \in P$*,*
$$[T(p_i^* a^* a p_j)] \le C_a^2 [T(p_i^* p_j)].$$

*Then, there exists a representation* $D : P \to \mathcal{B}(\mathcal{K})$ *that satisfies* $T(p) = P_{\mathcal{H}} D(p)|_{\mathcal{H}}$ *and* $\|D(p)\| \le C_p$*.*

Now let $P$ be a unital abelian semigroup and consider $Q = \{(p, q) : p, q \in P\}$. $Q$ is a unital semigroup under the point-wise semigroup operation

$$(p_1, q_1) + (p_2, q_2) = (p_1 + p_2, q_1 + q_2).$$

Define a involution operation of $Q$ by $(p, q)^* = (q, p)$, which turns $Q$ into an involution semigroup. Notice since $P$ is abelian, $Q$ is also abelian. Moreover, any element $(p, q) = (0, q) + (0, p)^*$. If $D : Q \to \mathcal{B}(\mathcal{K})$ is a representation, then

$$D(0, p)^* D(0, p) = D(p, p) = D(0, p) D(0, p)^*,$$

and therefore $D(0, p)$ is normal.

**Lemma 6.2.5.** *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $N \in \mathcal{B}(\mathcal{K})$ *where* $\mathcal{H}$ *is a subspace of* $\mathcal{K}$*. Suppose* $T = P_{\mathcal{H}} N|_{\mathcal{H}}$ *and* $T^*T = P_{\mathcal{H}} N^* N|_{\mathcal{H}}$*, then* $N$ *is an extension of* $T$*.*

*Proof.* From the conditions, we have for any $h \in \mathcal{H}$, $\|Th\|^2 = \langle Th, Th \rangle = \langle T^*Th, h \rangle$. Since $T^*T = P_{\mathcal{H}} N^* N|_{\mathcal{H}}$, $\langle T^*Th, h \rangle = \langle N^*Nh, h \rangle = \|Nh\|^2$.

On the other hand, $\|Th\| = \sup_{\|k\| \le 1, k \in \mathcal{H}} \langle Th, k \rangle$. But $T = P_{\mathcal{H}} N|_{\mathcal{H}}$, and thus $\langle Th, k \rangle = \langle Nh, k \rangle$. Therefore,

$$\begin{aligned} \|Th\| &= \sup_{\|k\| \le 1, k \in \mathcal{H}} \langle Th, k \rangle \\ &= \sup_{\|k\| \le 1, k \in \mathcal{H}} \langle Nh, k \rangle \\ &= \|P_{\mathcal{H}} Nh\| \end{aligned}$$

Therefore, $\|Th\| = \|Nh\| = \|P_{\mathcal{H}} Nh\|$ and thus $\mathcal{H}$ is invariant for $N$. Hence, $N$ is an extension of $T$. □

**Theorem 6.2.6.** *Let $P$ be any unital abelian semigroup and let $T : P \to \mathcal{B}(\mathcal{H})$ be a unital contractive representation of $P$. Then the following are equivalent:*

1. *There exists a contractive normal map $N : P \to \mathcal{B}(\mathcal{K})$ that extends $T$, where the family $\{N(p)\}_{p\in P}$ is a commuting family of normal operators.*
2. *There exists a contractive normal representation $N : P \to \mathcal{B}(\mathcal{L})$ that extends $T$.*

*Proof. (ii)$\Longrightarrow$(i)* is trivial. For the other direction, denote $Q$ be the $*$-semigroup constructed before and let $\tilde{T} : Q \to \mathcal{B}(\mathcal{H})$ defined by $\tilde{T}(p,q) = T(p)^*T(q)$. For each $p \in P$, denote $N(p) = \begin{bmatrix} T(p) & X_p \\ 0 & Y_p \end{bmatrix}$. Pick $s_i = (p_i, q_i) \in Q$ and $t = (a,b) \in Q$. We shall show that $\tilde{T}$ satisfies all the conditions in Theorem 6.2.4.

The first condition of Theorem 6.2.4 is clearly valid. For the second condition:

$$\begin{aligned}
&[\tilde{T}(s_i^* s_j)] \\
=&[\tilde{T}(q_i p_j, p_i q_j)] \\
=&[T(q_i)^* T(p_j)^* T(p_i) T(q_j)] \\
=&\operatorname{diag}(T(q_1)^*, T(q_2)^*, \cdots, T(q_n)^*)[T(p_j)^*T(p_i)]\operatorname{diag}(T(q_1), \cdots, T(q_n))
\end{aligned}$$

It suffices to show $[T(p_j)^*T(p_i)] \geq 0$. Notice that $\{N(p_i)\}$ is a commuting family of normal operators and thus they also doubly commute (by Fuglede's Theorem).

$$[N(p_j)^*N(p_i)] = [N(p_i)N(p_j)^*] = \begin{bmatrix} N(p_1) \\ N(p_2) \\ \vdots \\ N(p_n) \end{bmatrix} \begin{bmatrix} N(p_1)^* & N(p_2)^* & \cdots N(p_n)^* \end{bmatrix} \geq 0.$$

$N(p_i)$ extends $T(p_i)$ and therefore $P_{\mathcal{H}} N(p_j)^* N(p_i)|_{\mathcal{H}} = T(p_j)^*T(p_i)$. By projecting on $\mathcal{H}^n$, we get the desired inequality.

For the third condition:

$$\begin{aligned}
&[\tilde{T}(s_i^* t^* t s_j)] \\
=&[\tilde{T}(q_i p_j ab, ab p_i q_j)] \\
=&[T(ab)^* T(q_i)^* T(p_j)^* T(p_i) T(q_j) T(ab)] \\
=&\operatorname{diag}(T(q_1)^*, T(q_2)^*, \cdots, T(q_n)^*)[T(ab)^*T(p_j)^*T(p_i)T(ab)]\operatorname{diag}(T(q_1), \cdots, T(q_n))
\end{aligned}$$

Therefore, it suffices to show (with $C_t = 1$ in the condition)

$$[T(ab)^*T(p_j)^*T(p_i)T(ab)] \leq [T(p_j)^*T(p_i)]$$

Similar to the previous case, it suffices to show

$$[N(ab)^*N(p_j)^*N(p_i)N(ab)] \leq [N(p_j)^*N(p_i)]$$

Let $X = [N(p_j)^*N(p_i)] \geq 0$ and $D = \mathrm{diag}(N(ab), \cdots, N(ab))$. Since $D$ and $X$ $*$-commute, and thus $D$ and $X^{1/2}$ also $*$-commute. We have

$$D^*XD = X^{1/2}D^*DX^{1/2} \leq \|N(ab)\|X.$$

Since $N$ is contractive, this shows $D^*XD \leq X$. Therefore, all conditions in Theorem 6.2.4 are met, and thus there exists a contractive representation $S : Q \to \mathcal{B}(\mathcal{L})$ such that $\tilde{T}(p,q) = P_{\mathcal{H}}S(p,q)|_{\mathcal{H}}$. Denote $M(p) = S(0,p)$. Then $M : P \to \mathcal{B}(\mathcal{L})$ is a representation of $P$, and moreover,

$$T(p)^*T(p) = P_{\mathcal{H}}S(p,p)|_{\mathcal{H}} = P_{\mathcal{H}}M(p)^*M(p)|_{\mathcal{H}}.$$

By Lemma 6.2.5, we know $M(p)$ extends $T(p)$ and therefore $M$ is a normal extension. □

**Remark 6.2.7.** *When the semigroup is $P = \mathbb{N}^k$, Theorem 6.2.6 is trivial: for a normal map $N : \mathbb{N}^k \to \mathcal{B}(\mathcal{K})$, one may define a normal representation by sending each generator $e_i$ to $N(e_i)$. However, it is not clear how we can derive a normal representation from a normal map when the semigroup does not have nice generators. For example, we have seen this issue in Example 6.2.2 where the semigroup $P = \mathbb{N}\backslash\{1\}$ is finitely generated. This result shows that finding a commuting family of normal extensions for $\{T(p)\}_{p\in P}$ is equivalent of finding a normal representation that extends $T$.*

**Corollary 6.2.8.** *Let $P$ be a commutative unital semigroup generated by $\{p_i\}_{i\in I}$, and $T : P \to \mathcal{B}(\mathcal{H})$ a unital contractive representation. Then the family $\{T(p_i)\}_{i\in I}$ has commuting normal extensions $\{N_i\}_{i\in I}$ if and only if there exists a normal representation $N : P \to \mathcal{B}(\mathcal{K})$ such that each $N(p)$ extends $T(p)$.*

*Proof.* The backward direction is obvious. Now assuming $\{T(p_i)\}_{i\in I}$ has commuting normal extension $\{N_i\}_{i\in I}$. For each element $p \in P$, write $p$ as a finite product of $\{p_i\}_{i\in I}$ and define $N(p)$ to be the corresponding product of $T(p_i)$. Since $N_i$ commutes with one another, we obtain a normal map $\overline{N} : P \to \mathcal{B}(\mathcal{L})$ where $\{\overline{N}(p)\}_{p\in P}$ is a family of commuting normal operators where $\overline{N}(p)$ extends $T(p)$. Theorem 6.2.6 implies the existence of the desired normal representation $N$. □

**Remark 6.2.9.** *Corollary 6.2.8 shows that for a contractive representation* $T : P \to \mathcal{B}(\mathcal{H})$, *it suffices to extends the image of* $T$ *on a set of generators. Since Athavale's result still holds for an infinite family of operators (Corollary 6.2.12), we may use Condition* $(\star)$ *to check if the set of generators have a commuting normal extension. However, when the semigroup has too many generators, Condition* $(\star)$ *is hard to check. We shall give another equivalent condition for an abelian lattice ordered group in the next section.*

### 6.2.2 Normal Extensions For Lattice Ordered Semigroups

Although it is observed that Condition $(\star)$ implies a representation $T : \mathbb{N}^m \to \mathcal{B}(\mathcal{H})$ has regular dilation [7], the converse is not true. However, we shall prove that Athavale's result is equivalent to saying that a certain representation $T^\infty$ has regular dilation. First of all, define $\mathbb{N}^{m\times\infty}$ by taking the product of infinitely many copies of $\mathbb{N}^m$, in other words, $\mathbb{N}^{m\times\infty}$ is the abelian semigroup generated by $(e_{i,j})_{\substack{1\le i\le m\\ j\in\mathbb{N}}}$. Consider $T^\infty : \mathbb{N}^{m\times\infty} \to \mathcal{B}(\mathcal{H})$ where $T^\infty$ sends each generator $e_{i,j}$ to $T_i$.

**Lemma 6.2.10.** *As defined above,* $T^\infty$ *has regular dilation if and only if* $T$ *satisfies condition* $(\star)$.

*Proof.* It suffices to verify Condition $(\star)$ is equivalent to Brehmer's condition on $\mathbb{N}^{m\times\infty}$ in Theorem 2.2.3. For any finite set $U \subseteq \{1, 2, \cdots, m\} \times \mathbb{N}$, denote by $n_i$ the number of $u \in U$ whose first coordinate is $i$. For any subset $V \subseteq U$, denote by $k_i$ the number of $v \in V$ whose first coordinate is $i$. It is clear that $0 \le k_i \le n_i$. Notice that $T(e_V) = T_1^{k_1} T_2^{k_2} \cdots T_m^{k_m}$, and among all subsets of $U$, there are exactly $\binom{n_1}{k_1} \cdots \binom{n_m}{k_m}$ subsets $V$ that have $k_i$ elements whose first coordinate is $i$. Therefore,

$$
\begin{aligned}
&\sum_{V\subseteq U} (-1)^{|V|} T(e_V)^* T(e_V) \\
=& \sum_{0\le k_i\le n_i} (-1)^{k_1+k_2+\cdots+k_m} \binom{n_1}{k_1} \cdots \binom{n_m}{k_m} T_1^{*k_1} T_2^{*k_2} \cdots T_m^{*k_m} T_m^{k_m} \cdots T_1^{k_1}.
\end{aligned}
$$

Hence, Brehmer's condition holds if and only if $T$ satisfies Condition $(\star)$. □

Notice that Condition $(\star)$ cannot be generalized directly to arbitrary abelian lattice ordered semigroups when the semigroup lacks generators. However, Lemma 6.2.10 motivates us to consider $T^\infty$ in an abelian lattice ordered semigroup: for a lattice ordered semigroup $P$ inside a group $G$, define $P^\infty = \prod_{i=1}^\infty P$ to be the abelian semigroup generated by infinitely many identical copies of $P$. We shall denote $p \otimes \delta_n$ to be $p$ inside the $n$-th copy of $P^\infty$. A typical element of $P^\infty$ can be denoted by $\sum_{i=1}^N p_i \otimes \delta_i$ for some large enough $N$. $P^\infty$ is naturally a lattice ordered semigroup inside the group $G^\infty$, where

$$\left(\sum_{i=1}^N p_i \otimes \delta_i\right) \wedge \left(\sum_{i=1}^N q_i \otimes \delta_i\right) = \sum_{i=1}^N p_i \wedge q_i \otimes \delta_i.$$

Our main result shows that $T^\infty$ being regular is equivalent to having a normal extension.

**Theorem 6.2.11.** *Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation on an abelian lattice ordered semigroup. Define $T^\infty : P^{\mathbb{N}} \to \mathcal{B}(\mathcal{H})$ by $T^\infty(p,n) = T(p)$ for any $n$. Then the following are equivalent:*

1. *$T$ has a contractive normal extension to a representation $N : P \to \mathcal{B}(\mathcal{K})$. In other words, there exists a contractive normal representation $N : P \to \mathcal{B}(\mathcal{K})$ such that for all $p \in P$, $T(p) = N(p)|_{\mathcal{H}}$.*
2. *$T^\infty$ has regular dilation.*

*Proof.* *(i)⇒(ii)*: First of all notice that the family $\{N(p)\}_{p\in P}$ $*$-commutes due to Fuglede's theorem. Define $N^\infty$ by sending $N^\infty(p,n) = N(p)$ for all $p \in P, n \in \mathbb{N}$. Then for any $s,t \in P^\infty$, $N^\infty(s), N^\infty(t)$ are a finite product of operators in $\{N(p)\}_{p\in P}$ and therefore they also $*$-commute. In particular, $N^\infty$ is Nica-covariant and therefore has regular dilation [40, Theorem 4.1]. Since $N$ extends $T$, $N^\infty$ also extends $T^\infty$, and therefore for any $s,t \in P^\infty$,

$$P_{\mathcal{H}} N^\infty(t)^* N^\infty(s)|_{\mathcal{H}} = T^\infty(t)^* T^\infty(s).$$

$N^\infty$ satisfies the condition in Theorem 3.2.1, and by projecting onto $\mathcal{H}$, $T^\infty$ also satisfies this condition and thus has regular dilation.

*(ii)⇒(i)*: Let $U : G^\infty \to \mathcal{B}(\mathcal{K})$ be a regular unitary dilation of $T^\infty$, and decompose $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$ so that under such decomposition, for each $w \in P^\infty$,

$$U(w) = \begin{bmatrix} * & 0 & 0 \\ * & T(w) & 0 \\ * & * & * \end{bmatrix}.$$

Fix $p \in P$, denote $U_i(p) = U(p \otimes \delta_i)$. Under the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$, let

$$U_i(p) = \begin{bmatrix} A_i & 0 & 0 \\ B_i & T(p) & 0 \\ C_i & D_i & E_i \end{bmatrix}.$$

First by regularity of $U$, for any $i \neq j$,

$$\begin{aligned} T(p)^*T(p) &= P_{\mathcal{H}} U(p\otimes\delta_i - p\otimes\delta_j)|_{\mathcal{H}} \\ &= P_{\mathcal{H}} U_j(p)^* U_i(p)|_{\mathcal{H}} \\ &= P_{\mathcal{H}} \left.\begin{bmatrix} A_j^* & B_j^* & C_j^* \\ 0 & T(p)^* & D_j^* \\ 0 & 0 & E_j^* \end{bmatrix} \begin{bmatrix} A_i & 0 & 0 \\ B_i & T(p) & 0 \\ C_i & D_i & E_i \end{bmatrix}\right|_{\mathcal{H}} \\ &= P_{\mathcal{H}} \left.\begin{bmatrix} * & * & * \\ * & T(p)^*T(p) + D_j^* D_i & * \\ * & * & * \end{bmatrix}\right|_{\mathcal{H}} \end{aligned}$$

Therefore, each $D_j^* D_i = 0$ whenever $i \neq j$. When $i = j$, since $U$ is a unitary representation, $U_i(p)$ is a unitary, and thus $D_i^* D_i = I - T(p)^*T(p)$. Now fix $\epsilon > 0$, denote

$$\Lambda_\epsilon = \{\lambda = (\lambda_i)_{i=1}^\infty \in c_{00} : \sum_{i=1}^\infty \lambda_i = 1, 0 \le \lambda_i \le 1, \|\lambda\|_2 < \epsilon\}.$$

This set is non-empty since we may let $\lambda_i = \frac{1}{n}$ for $1 \le i \le n$, and 0 otherwise. This gives $\|\lambda\|_2 = \frac{1}{\sqrt{n}}$, which can be arbitrarily small as $n \to \infty$. For each $\lambda \in \Lambda_\epsilon$, denote $N_\lambda = \sum_{i=1}^\infty \lambda_i U_i(p)$, which converges since $\lambda$ has finite support. Denote

$$\mathcal{N}_\epsilon = \{N_\lambda : \lambda \in \Lambda_\epsilon\}$$

Notice that $P_{\mathcal{H}} N_\lambda|_{\mathcal{H}} = \sum_{i=1}^\infty \lambda_i T(p) = T(p)$. Therefore, under the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$,

$$N_\lambda = \begin{bmatrix} A_\lambda & 0 & 0 \\ B_\lambda & T(p) & 0 \\ C_\lambda & D_\lambda & E_\lambda \end{bmatrix}.$$

Here, $D_\lambda = \sum_{i=1}^\infty \lambda_i D_i$ and thus

$$\begin{aligned} D_\lambda^* D_\lambda =& \sum_{i,j=1}^\infty \overline{\lambda_i} \lambda_j D_i^* D_j \\ =& \sum_{i=1}^\infty |\lambda_i|^2 D_i^* D_i \end{aligned}$$

Here we used the fact that $D_i^* D_j = 0$ whenever $i \neq j$. Note that each $D_i^* D_i = I - T(p)^* T(p)$, which is contractive. Hence,

$$\|D_\lambda^* D_\lambda\| \leq \|\lambda\|_2^2 < \epsilon^2$$

Each $N_\lambda$ is a convex combination of $U_i$ and thus is contained in the convex hull of $U_i$, which is also contained in the unit ball in $\mathcal{B}(\mathcal{K})$. Observe that each $\mathcal{N}_\epsilon$ is also convex. Therefore, the convexity implies their SOT$^*$ and WOT closures agree (here, $SOT^* - limT_n = T$ if $T_n$ and $T_n^*$ converges to $T$ and $T^*$ respectively in SOT.). Hence,

$$\overline{\mathcal{N}_\epsilon}^{SOT^*} = \overline{\mathcal{N}_\epsilon}^{WOT} \subseteq \overline{\text{conv}}^{WOT}\{U_i\} \subseteq b_1(\mathcal{B}(\mathcal{K}))$$

The Banach Alaoglu theorem gives $b_1(\mathcal{B}(\mathcal{K}))$ is WOT-compact, and therefore $\overline{\mathcal{N}_\epsilon}^{WOT}$ is a decreasing nest of WOT-compact sets. By the Cantor intersection theorem,

$$\bigcap_{\epsilon>0} \overline{\mathcal{N}_\epsilon}^{SOT^*} = \bigcap_{\epsilon>0} \overline{\mathcal{N}_\epsilon}^{WOT} \neq \emptyset$$

Pick $N(p) \in \bigcap_{\epsilon>0} \overline{\mathcal{N}_\epsilon}^{SOT^*}$. Then for any $\epsilon > 0$, we can choose a net $(N_\lambda)_{\lambda \in I_\epsilon}$, where $I_\epsilon \subseteq \Lambda_\epsilon$, such that $SOT^* - \lim_{I_\epsilon} N_\lambda = N(p)$ and thus $SOT^* - \lim_{I_\epsilon} N_\lambda^* = N(p)^*$. Now both $N_\lambda, N_\lambda^*$ are uniformly bounded by 1 since they are all contractions. Hence, their product is SOT-continuous.

$$SOT - \lim_{\Lambda} N_\lambda^* N_\lambda = N(p)^* N(p)$$
$$SOT - \lim_{\Lambda} N_\lambda N_\lambda^* = N(p) N(p)^*$$

But since $U_i$ are commuting unitaries and thus $*$-commute, $N_\lambda$ is normal. Hence, $N(p)^* N(p) = N(p) N(p)^*$ and $N(p)$ is normal.

Consider $N(p) \in \mathcal{B}(\mathcal{K})$ under the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$, each entry must be the WOT-limit of $(N_\lambda)_{\lambda \in I_\epsilon}$ and therefore it has the form

$$N(p) = \begin{bmatrix} A(p) & 0 & 0 \\ B(p) & T(p) & 0 \\ C(p) & D(p) & E(p) \end{bmatrix}.$$

Since $(D_\lambda)_{\lambda \in I_\epsilon}$ WOT-converges to $D(p)$, and for each $\lambda \in \Lambda_\epsilon$, $\|D_\lambda\| < \epsilon$. Therefore, $\|D(p)\| < \epsilon$ for every $\epsilon > 0$ and thus $D(p) = 0$. Hence $\mathcal{H}$ is invariant for $N(p)$, whence $N(p)$ is a normal extension for $T(p)$.

The procedure above gives a normal map $N : P \to \mathcal{B}(\mathcal{K})$ where each $N(p)$ is a normal contraction that extends $T(p)$. Notice $N(p)$ is a WOT-limit of convex combinations of $\{U_i(p)\}_{i\in\mathbb{N}}$, where the family $\{U_i(p)\}_{i,p}$ is commuting since $P$ is abelian. Any convex combination of $\{U_i(p)\}_{i\in\mathbb{N}}$ also commutes with any convex combination of $\{U_i(q)\}_{i\in\mathbb{N}}$. Therefore, $\{N(p)\}_{p\in P}$ is also a commuting family of normal operators. By Theorem 6.2.6, there exists a normal representation $N : P \to \mathcal{B}(\mathcal{L})$ that extends $T$. $\square$

As an immediate corollary, Theorem 6.2.1 can be extended to any family of commuting contractions $\{T(\omega)\}_{\omega\in\Omega}$ by considering Brehmer's condition on $\mathbb{N}^{\Omega\times\infty}$.

**Corollary 6.2.12.** *Let* $\{T_i\}_{i\in I}$ *be a family of commuting contractions. Then there exists a family of commuting normal contractions* $\{N_i\}_{i\in I}$ *that extends* $\{T_i\}_{i\in I}$ *if and only if for any finite set* $F \subseteq I$*,* $\{T_i\}_{i\in F}$ *satisfies Condition (*$\star$*).*

It is known that isometric representations of lattice ordered semigroups are automatically regular [40, Corollary 3.8]. Therefore, if $T : P \to \mathcal{B}(\mathcal{H})$ is an isometric representation, then $T^\infty : P^\infty \to \mathcal{B}(\mathcal{H})$ is also an isometric representation and thus $T$ has a subnormal extension.

**Corollary 6.2.13.** *Every isometric representation of an abelian lattice ordered semigroup has a contractive subnormal extension.*

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