

Accepted Manuscript

Existence and uniqueness of weak moment maps in multisymplectic geometry

Jonathan Herman

PII: S0393-0440(18)30285-7

DOI: <https://doi.org/10.1016/j.geomphys.2018.05.001>

Reference: GEOPHY 3208

To appear in: *Journal of Geometry and Physics*

Received date: 14 November 2017

Revised date: 19 March 2018

Accepted date: 2 May 2018

Please cite this article as: J. Herman, Existence and uniqueness of weak moment maps in multisymplectic geometry, *Journal of Geometry and Physics* (2018), <https://doi.org/10.1016/j.geomphys.2018.05.001>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



Existence and Uniqueness of Weak Homotopy Moment Maps

Jonathan Herman

Department of Pure Mathematics, University of Waterloo
j3herman@uwaterloo.ca

Abstract

We show that the classical results on the existence and uniqueness of moment maps in symplectic geometry generalize directly to weak homotopy moment maps in multisymplectic geometry. In particular, we show that their existence and uniqueness is governed by a Lie algebra cohomology complex which reduces to the Chevalley-Eilenberg complex in the symplectic setup.

Contents

1	Introduction	2
2	Cohomology	4
2.1	Group Cohomology	4
2.2	Lie Algebra Cohomology	5
3	Multisymplectic Geometry	7
3.1	Multisymplectic Manifolds	7
3.2	Hamiltonian forms	8
3.3	Weak Homotopy Moment Maps	8
4	Equivariance of Weak Moment Maps	9
4.1	Equivariance in Multisymplectic Geometry	9
4.2	Infinitesimal Equivariance in Multisymplectic Geometry	13
5	Existence of Not Necessarily Equivariant Weak Moment Maps	14
6	Obtaining an Equivariant Moment Map from a Non-Equivariant Moment Map	17
7	Uniqueness of Weak moment Maps	18
8	Weak moment Maps as Morphisms	18
9	Open Questions	20

1 Introduction

Recall that for a symplectic manifold (M, ω) , a Lie algebra \mathfrak{g} is said to act symplectically if $\mathcal{L}_{V_\xi} \omega = 0$, for all $\xi \in \mathfrak{g}$, where V_ξ is its infinitesimal generator. A symplectic group action is called Hamiltonian if one can find a moment map, that is, a map $f : \mathfrak{g} \rightarrow C^\infty(M)$ satisfying

$$df(\xi) = V_\xi \lrcorner \omega,$$

for all $\xi \in \mathfrak{g}$.

In multisymplectic geometry, ω is replaced by a closed, non-degenerate $(n+1)$ -form, where $n \geq 1$. A Lie algebra action is called multisymplectic if $\mathcal{L}_{V_\xi}\omega = 0$ for each $\xi \in \mathfrak{g}$. A generalization of moment maps from symplectic to multisymplectic geometry is given by a (homotopy) moment map. These maps are discussed in detail in [3]. A homotopy moment map is a collection of maps, $f_k : \Lambda^k \mathfrak{g} \rightarrow \Omega^{n-k}(M)$, with $1 \leq k \leq n+1$, satisfying

$$df_k(p) = -f_{k-1}(\partial_k(p)) + (-1)^{\frac{k(k+1)}{2}} V_p \lrcorner \omega, \quad (1.1)$$

for all $p \in \Lambda^k \mathfrak{g}$, where V_p is its infinitesimal generator (see Definition 2.5). A weak (homotopy) moment map is a collection of maps $f_k : \mathcal{P}_{\mathfrak{g},k} \rightarrow \Omega^{n-k}(M)$ satisfying

$$df_k(p) = (-1)^{\frac{k(k+1)}{2}} V_p \lrcorner \omega, \quad (1.2)$$

for $p \in \mathcal{P}_{\mathfrak{g},k}$. Here $\mathcal{P}_{\mathfrak{g},k}$ is the Lie kernel, which is the kernel of the k -th Lie algebra cohomology differential $\partial_k : \Lambda^k \mathfrak{g} \rightarrow \Lambda^{k-1} \mathfrak{g}$, defined by

$$\partial_k : \Lambda^k \mathfrak{g} \rightarrow \Lambda^{k-1} \mathfrak{g} \quad \xi_1 \wedge \cdots \wedge \xi_k \mapsto \sum_{1 \leq i < j \leq k} (-1)^{i+j} [\xi_i, \xi_j] \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_k,$$

for $k \geq 1$ and $\xi_1, \dots, \xi_k \in \mathfrak{g}$.

We see that any collection of functions satisfying equation (1.1) must also satisfy (1.2). That is, any homotopy moment map induces a weak homotopy moment map.

Weak moment maps generalize the moment maps of Madsen and Swann in [6] and [7], and were also used to give a multisymplectic version of Noether's theorem in [5]. In this paper, we study the existence and uniqueness of weak homotopy moment maps and show that the theory is a generalization from symplectic geometry. We also show that the equivariance of a weak moment map can be characterized in terms of \mathfrak{g} -module morphisms, analogous to symplectic geometry.

Recall that in symplectic geometry we have the following well-known results on the existence and uniqueness of moment maps.

Proposition 1.1. *Consider the symplectic action of a connected Lie group G acting on a symplectic manifold (M, ω) .*

- *If the first Lie algebra cohomology vanishes, i.e. $H^1(\mathfrak{g}) = 0$, then a not necessarily equivariant moment map exists.*
- *If the second Lie algebra cohomology vanishes, i.e. $H^2(\mathfrak{g}) = 0$, then any non-equivariant moment map can be made equivariant.*
- *If the first Lie algebra cohomology vanishes, i.e. $H^1(\mathfrak{g}) = 0$, then equivariant moment maps are unique,*

and combining these results,

- *If both the first and second Lie algebra cohomology vanish, i.e. $H^1(\mathfrak{g}) = 0$ and $H^2(\mathfrak{g}) = 0$, then there exists a unique equivariant moment map.*

We generalize these results with the following theorems. Letting Ω_{cl}^{n-k} denote the set of closed $(n-k)$ -forms on M , we get the above propositions, in their respective order, by taking $n = k = 1$.

Theorem 1.2. *If $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^*) = 0$, then there exists a not necessarily equivariant weak homotopy k -moment map. The same result holds if $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{cl}^{n-k}) = 0$ and $H^0(\mathfrak{g}, \Omega_{cl}^{n-k}) \neq 0$.*

Theorem 1.3. *If $H^1(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{cl}^{n-k}) = 0$, then any non-equivariant weak homotopy k -moment map can be made equivariant.*

Theorem 1.4. *If $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{cl}^{n-k}) = 0$ then an equivariant weak homotopy k -moment map is unique,*

and combining these results,

Theorem 1.5. *If $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^*) = 0$, and $H^1(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{cl}^{n-k}) = 0$, then there exists a unique equivariant weak k -moment map $f_k : \mathcal{P}_{\mathfrak{g},k} \rightarrow \Omega^{n-k}$. Moreover, if $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^*) = 0$, and $H^1(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{cl}^{n-k}) = 0$ for all $1 \leq k \leq n$, then a full equivariant weak moment map exists and is unique.*

We also show that the morphism properties of moment maps from symplectic geometry are preserved in multisymplectic geometry. More specifically, recall that in symplectic geometry the equivariance of a moment map $f : \mathfrak{g} \rightarrow C^\infty(M)$ is characterized by whether or not f is a Lie algebra morphism. That is, f is equivariant if and only if

$$f([\xi, \eta]) = \{f(\xi), f(\eta)\},$$

for all $\xi, \eta \in \mathfrak{g}$. However, as shown in Theorem 4.2.8 of [1] it is always true that f induces a Lie algebra morphism between \mathfrak{g} and $C^\infty(M)/\text{constant}$, because $df([\xi, \eta]) = d\{f(\xi), f(\eta)\}$.

We generalize these results to multisymplectic geometry by showing that:

Theorem 1.6. *For any $1 \leq k \leq n$, a weak k -moment map is always a \mathfrak{g} -module morphism from $\mathcal{P}_{\mathfrak{g},k} \rightarrow \Omega_{\text{Ham}}^{n-k}(M)/\text{closed}$. A weak k -moment map is equivariant if and only if it is a \mathfrak{g} -module morphism from $\mathcal{P}_{\mathfrak{g},k} \rightarrow \Omega_{\text{Ham}}^{n-k}(M)$.*

Here $\Omega_{\text{Ham}}^{n-k}(M)$ is denoting the space of multi-Hamiltonian forms, which are differential forms $\alpha \in \Omega^{n-k}(M)$ satisfying $d\alpha = X_\alpha \lrcorner \omega$ for some $X_\alpha \in \Gamma(\Lambda^k(TM))$ (see Definition 3.5). These forms were introduced in [5], and give a notion of a multi-symmetry which occurs when there is a given Hamiltonian $(n-1)$ -form $H \in \Omega_{\text{Ham}}^{n-1}(M)$, (see Definition 3.2).

2 Cohomology

We briefly recall some basic notions from group and Lie algebra cohomology.

2.1 Group Cohomology

Let G be a group and S a G -module. For $g \in G$ and $s \in S$, let $g \cdot s$ denote the action of G on S . Let $C^k(G, S)$ denote the space of smooth alternating functions from G^k to S and consider the differential $\partial_k : C^k(G, S) \rightarrow C^{k+1}(G, S)$ defined as follows. For $\sigma \in C^k(G, S)$ and $g_1, \dots, g_{k+1} \in G$ define

$$\begin{aligned} \partial_k \sigma(g_1, \dots, g_{k+1}) := \\ g_1 \cdot \sigma(g_2, \dots, g_{k+1}) + \sum_{i=1}^k (-1)^i \sigma(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{k+1}) - (-1)^k \sigma(g_1, \dots, g_k). \end{aligned} \quad (2.1)$$

A computation shows that $\partial_k^2 = 0$ so that $C^0(G, S) \rightarrow C^1(G, S) \rightarrow \dots$ is a cochain complex. This cohomology is known as the differentiable cohomology of G with coefficients in S . We let $H^k(G, S)$ denote the k -th cohomology group and will call an equivalence class representative a k -cocycle.

2.2 Lie Algebra Cohomology

Let \mathfrak{g} be a Lie algebra and R a \mathfrak{g} -module. Given $\xi \in \mathfrak{g}$ and $r \in R$, let $\xi \cdot r$ denote the action of \mathfrak{g} on R . We let $C^k(\mathfrak{g}, R)$ denote the space of multilinear alternating functions from \mathfrak{g}^k to R and consider the differential $\delta_k : C^k(\mathfrak{g}, R) \rightarrow C^{k+1}(\mathfrak{g}, R)$ defined as follows. For $f \in C^k(\mathfrak{g}, R)$ and $\xi_1, \dots, \xi_{k+1} \in \mathfrak{g}$ define

$$\delta_k f(\xi_1, \dots, \xi_{k+1}) := \sum_i (-1)^{i+1} \xi_i \cdot f(\xi_1, \dots, \widehat{\xi}_i, \dots, \xi_{k+1}) + \sum_{i < j} (-1)^{i+j} f([\xi_i, \xi_j], \xi_1, \dots, \widehat{\xi}_i, \dots, \widehat{\xi}_j, \dots, \xi_{k+1}). \quad (2.2)$$

A computation shows that $\delta_k^2 = 0$. We let $H^k(\mathfrak{g}, R)$ denote the k -th cohomology group and call an equivalence class representative a (Lie algebra) k -cocycle. Note that for $k = 0$ the map $\delta_0 : R \rightarrow C^1(\mathfrak{g}, R)$ is given by $(\delta_0 r)(\xi) = \xi \cdot r$, where $r \in R$ and $\xi \in \mathfrak{g}$. For $k = 1$ the map $\delta_1 : C^1(\mathfrak{g}, R) \rightarrow C^2(\mathfrak{g}, R)$ is given by $\delta_1(f)(\xi_1, \xi_2) = \xi_1 \cdot f(\xi_2) - \xi_2 \cdot f(\xi_1) - f([\xi_1, \xi_2])$, where $f \in C^1(\mathfrak{g}, R)$ and ξ_1 and ξ_2 are in \mathfrak{g} .

The standard example of Lie algebra cohomology is given when $R = \mathbb{R}$:

Example 2.1. (Exterior algebra of \mathfrak{g}^*) Consider the trivial \mathfrak{g} -action on \mathbb{R} . Then $C^k(\mathfrak{g}, \mathbb{R}) = \Lambda^k \mathfrak{g}^*$, and the Lie algebra cohomology differential $\delta_k : \Lambda^k \mathfrak{g}^* \rightarrow \Lambda^{k+1} \mathfrak{g}^*$ is given by

$$\delta_k \alpha(\xi_1 \wedge \dots \wedge \xi_k) := \alpha \left(\sum_{1 \leq i < j \leq k} (-1)^{i+j} [\xi_i, \xi_j] \wedge \xi_1 \wedge \dots \wedge \widehat{\xi}_i \wedge \dots \wedge \widehat{\xi}_j \wedge \dots \wedge \xi_k \right) \quad (2.3)$$

where $\alpha \in \Lambda^k \mathfrak{g}^*$, and $\xi_1 \wedge \dots \wedge \xi_k$ is a decomposable element of $\Lambda^k \mathfrak{g}$, and extended by linearity to non-decomposables. It is easy to check that $\delta^2 = 0$. We will also make frequent reference to the corresponding Lie algebra homology differential which is given by

$$\partial_k : \Lambda^k \mathfrak{g} \rightarrow \Lambda^{k-1} \mathfrak{g} \quad \xi_1 \wedge \dots \wedge \xi_k \mapsto \sum_{1 \leq i < j \leq k} (-1)^{i+j} [\xi_i, \xi_j] \wedge \xi_1 \wedge \dots \wedge \widehat{\xi}_i \wedge \dots \wedge \widehat{\xi}_j \wedge \dots \wedge \xi_k, \quad (2.4)$$

for $k \geq 1$. We define $\Lambda^{-1} \mathfrak{g} = \{0\}$ and ∂_0 to be the zero map.

For the rest of this section we only consider the exterior algebra homology complex.

Definition 2.2. We follow the terminology and notation of [6] and call $\mathcal{P}_{\mathfrak{g},k} = \ker \partial_k$ the k -th Lie kernel, which is a vector subspace of $\Lambda^k \mathfrak{g}$. Notice that if \mathfrak{g} is abelian then $\mathcal{P}_{\mathfrak{g},k} = \Lambda^k \mathfrak{g}$. We will let $\mathcal{P}_{\mathfrak{g}}$ denote the direct sum of all the Lie kernels;

$$\mathcal{P}_{\mathfrak{g}} = \bigoplus_{k=0}^{\dim(\mathfrak{g})} \mathcal{P}_{\mathfrak{g},k},$$

and denote $H^k(\mathfrak{g}, \mathbb{R})$ simply by $H^k(\mathfrak{g})$.

We now recall the Schouten Bracket.

Definition 2.3. On decomposable multivectors $X = X_1 \wedge \dots \wedge X_k \in \Lambda^k \mathfrak{g}$ and $Y = Y_1 \wedge \dots \wedge Y_l \in \Lambda^l \mathfrak{g}$, the Schouten bracket $[\cdot, \cdot]$ is given by

$$[X, Y] := \sum_{i=1}^k \sum_{j=1}^l (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge X_k \wedge Y_1 \wedge \dots \wedge \widehat{Y}_j \wedge \dots \wedge Y_l,$$

and extended by linearity to all multivector fields.

The next proposition shows that the Schouten bracket and the Lie algebra differential are equal, when restricted to elements of a certain form.

Proposition 2.4. *For $p \in \mathcal{P}_{\mathfrak{g},k}$ and $\xi \in \mathfrak{g}$ we have that*

$$\partial(p \wedge \xi) = [p, \xi].$$

Proof. A computation using the definition of ∂ shows that

$$\begin{aligned} \partial(p \wedge \xi) &= \partial(p) \wedge \xi + p \wedge \partial(\xi) + [p, \xi] \\ &= [p, \xi] \end{aligned} \quad \text{since } p \in \mathcal{P}_{\mathfrak{g},k} \text{ and } \partial(\xi) = [\xi, \xi] = 0.$$

□

Let \mathfrak{g} be a Lie algebra acting on a manifold M . For $\xi \in \mathfrak{g}$, we let $V_\xi \in \Gamma(TM)$ denote its infinitesimal generator.

Definition 2.5. For a decomposable element $p = \xi_1 \wedge \cdots \wedge \xi_k$ of $\Lambda^k \mathfrak{g}$, its infinitesimal generator, denoted V_p , is the multivector field $V_{\xi_1} \wedge \cdots \wedge V_{\xi_k}$.

Lemma 2.6. (Extended Cartan Lemma) For a decomposable multivector field $p = \xi_1 \wedge \cdots \wedge \xi_k$ in $\Lambda^k \mathfrak{g}$ and differential form τ we have that

$$(-1)^k d(V_p \lrcorner \tau) = V_{\partial p} \lrcorner \tau + \sum_{i=1}^k (-1)^i (V_{\xi_1} \wedge \cdots \wedge \widehat{V_{\xi_i}} \wedge \cdots \wedge V_{\xi_k}) \lrcorner \mathcal{L}_{V_{\xi_i}} \tau + V_p \lrcorner d\tau.$$

Proof. This is Lemma 3.4 of [6] or Lemma 2.18 of [10].

□

Let $\Phi : G \times M \rightarrow M$ be a Lie group action on M .

Definition 2.7. For $A \in \Gamma(TM)$ we let $\Phi_g^* A$ denote the vector field given by the push-forward of A by Φ_g^{-1} . That is,

$$(\Phi_g^*(A))_x := (\Phi_{g^{-1}})_* (A_{\Phi_g(x)}),$$

where $x \in M$. For a decomposable multivector field $Y = Y_1 \wedge \cdots \wedge Y_k$ in $\Gamma(\Lambda^k(TM))$ we will let $\text{Ad}_g Y$ denote the extended adjoint action

$$\text{Ad}_g Y = \text{Ad}_g Y_1 \wedge \cdots \wedge \text{Ad}_g Y_k$$

and we will let $\Phi_g^* Y$ denote the multivector field

$$\Phi_g^* Y = \Phi_g^* Y_1 \wedge \cdots \wedge \Phi_g^* Y_k.$$

We also extend ad to a map $\text{ad} : \mathfrak{g} \times \Lambda^k \mathfrak{g} \rightarrow \Lambda^k \mathfrak{g}$ by

$$\text{ad}_\xi(Y_1 \wedge \cdots \wedge Y_k) = \sum_{i=1}^k Y_1 \wedge \cdots \wedge \text{ad}_\xi(Y_i) \wedge \cdots \wedge Y_k. \quad (2.5)$$

Corollary 2.8. *For $\xi \in \mathfrak{g}$ we have that ad_ξ preserves the Lie kernel. That is, if p is in $\mathcal{P}_{\mathfrak{g},k}$ then $\text{ad}_\xi(p)$ is in $\mathcal{P}_{\mathfrak{g},k}$.*

Proof. A computation shows that $\text{ad}_\xi(p) = [\xi, p]$. Hence, by Proposition 2.4, we see $[\xi, p]$ is exact. Thus it is closed. \square

The next proposition shows that the infinitesimal generator of the extended adjoint action agrees with the pull back action.

Proposition 2.9. *Let $\Phi : G \times M \rightarrow M$ be a group action. For every $g \in G$ and $p \in \Lambda^k \mathfrak{g}$ we have that*

$$V_{\text{Ad}_g p} = \Phi_{g^{-1}}^* V_p.$$

Equivalently, the map $\Lambda^k \mathfrak{g} \rightarrow \Gamma(\Lambda^k(TM))$ given by $\xi_1 \wedge \cdots \wedge \xi_k \mapsto V_{\xi_1} \wedge \cdots \wedge V_{\xi_k}$ is equivariant with respect to the extended adjoint and pull back action.

Proof. Fix $q \in M$, $g \in G$. First suppose that $\xi \in \mathfrak{g}$. Then by Proposition 4.1.26 of [1] we have that

$$V_{\text{Ad}_g \xi} = \Phi_{g^{-1}}^* V_\xi.$$

The claim now follows since for $p = \xi_1 \wedge \cdots \wedge \xi_k$ in $\Lambda^k \mathfrak{g}$,

$$\begin{aligned} V_{\text{Ad}_g p} &:= V_{\text{Ad}_g \xi_1} \wedge \cdots \wedge V_{\text{Ad}_g \xi_k} \\ &= \Phi_{g^{-1}}^* V_{\xi_1} \wedge \cdots \wedge \Phi_{g^{-1}}^* V_{\xi_k} \\ &= \Phi_{g^{-1}}^* V_p \end{aligned} \quad \text{by definition.}$$

\square

3 Multisymplectic Geometry

Here we recall some concepts and tools used in multisymplectic geometry.

3.1 Multisymplectic Manifolds

Definition 3.1. A manifold M equipped with a closed $(n+1)$ -form ω is called a pre-multisymplectic manifold. If in addition the map $T_p M \rightarrow \Lambda^n T_p^* M$, $V \mapsto V \lrcorner \omega$ is injective, then (M, ω) is called a multisymplectic, or n -plectic, manifold.

Definition 3.2. An $(n-1)$ -form $\alpha \in \Omega^{n-1}(M)$ is called Hamiltonian if there exists a vector field $V_\alpha \in \Gamma(TM)$ such that $d\alpha = -V_\alpha \lrcorner \omega$. Note that the non-degeneracy of ω insures uniqueness of the corresponding Hamiltonian vector field. We let $\Omega_{\text{Ham}}^{n-1}(M)$ denote the space of Hamiltonian $(n-1)$ -forms, which is a subspace of $\Omega^{n-1}(M)$.

As in symplectic geometry, we are interested in Lie group actions which preserve the n -plectic form.

Definition 3.3. A Lie group action $\Phi : G \times M \rightarrow M$ is called multisymplectic if $\Phi_g^* \omega = \omega$. A Lie algebra action $\mathfrak{g} \times \Gamma(TM) \rightarrow \Gamma(TM)$ is called multisymplectic if $\mathcal{L}_{V_\xi} \omega = 0$ for all $\xi \in \mathfrak{g}$. We remark that a multisymplectic Lie group action induces a multisymplectic Lie algebra action. Conversely, a multisymplectic Lie algebra action induces a multisymplectic group action if the Lie group is connected.

In [8] it was shown that to any multisymplectic manifold one can associate the following L_∞ -algebra.

Definition 3.4. The Lie n -algebra of observables, $L_\infty(M, \omega)$ is the following L_∞ -algebra. Let $L = \oplus_{i=0}^n L_i$ where $L_0 = \Omega_{\text{Ham}}^{n-1}(M)$ and $L_i = \Omega^{n-1-i}(M)$ for $1 \leq i \leq n-1$. The maps $l_k : L^{\otimes k} \rightarrow L$ of degree $k-2$ are defined as follows: For $k=1$,

$$l_1(\alpha) = \begin{cases} d\alpha & \text{if } \deg \alpha > 0, \\ 0 & \text{if } \deg \alpha = 0. \end{cases}$$

For $k > 1$,

$$l_k(\alpha_1, \dots, \alpha_k) = \begin{cases} \zeta(k) X_{\alpha_k} \lrcorner \dots X_{\alpha_1} \lrcorner \omega & \text{if } \deg \alpha_1 \otimes \dots \otimes \alpha_k = 0, \\ 0 & \text{if } \deg \alpha_1 \otimes \dots \otimes \alpha_k > 0. \end{cases}$$

Here $\zeta(k)$ is defined to equal $-(-1)^{\frac{k(k+1)}{2}}$. We introduce this notation as this sign comes up frequently.

3.2 Hamiltonian forms

Let (M, ω) be an n -plectic manifold. The following definition generalizes the concept of a Hamiltonian 1-form from symplectic geometry.

Definition 3.5. A differential form $\alpha \in \Omega^{n-k}(M)$ is called Hamiltonian if there exists a multivector field $X_\alpha \in \Gamma(\Lambda^k(TM))$ such that $d\alpha = -X_\alpha \lrcorner \omega$.

Note that the Hamiltonian multivector field corresponding to a Hamiltonian form is not unique; however, the difference of any two Hamiltonian vector fields is in the kernel of ω . The next proposition shows that the Hamiltonian forms are an L_∞ subalgebra of the Lie n -algebra of observables

Proposition 3.6. Let $\hat{L}_i = \Omega_{\text{Ham}}^{n-i}(M)$ and $\hat{L} = \oplus_{i=0}^n \hat{L}_i$. Let $\hat{L}_\infty(M, \omega)$ denote the space \hat{L} together with the mappings l_k defined above in the definition of the Lie- n -algebra of observables. Then $\hat{L}_\infty(M, \omega)$ is an L_∞ -subalgebra of $L_\infty(M, \omega)$.

Proof. This is Theorem 4.15 of [5]. □

3.3 Weak Homotopy Moment Maps

For a group acting on a symplectic manifold M , a moment map is a Lie algebra morphism between $(\mathfrak{g}, [\cdot, \cdot])$ and $(C^\infty(M), \{\cdot, \cdot\})$, where $\{\cdot, \cdot\}$ is the Poisson bracket. In multisymplectic geometry, a moment map is an L_∞ -morphism from the exterior algebra of \mathfrak{g} to the Lie n -algebra of observables. We direct the reader to [3] for more information on L_∞ -algebras and morphisms.

Definition 3.7. A (homotopy) moment map is an L_∞ -morphism (f) between \mathfrak{g} and the Lie n -algebra of observables. This means that (f) is a collection of maps $f_1 : \Lambda^1 \mathfrak{g} \rightarrow \Omega_{\text{Ham}}^{n-1}(M)$ and $f_k : \Lambda^k \mathfrak{g} \rightarrow \Omega^{n-k}(M)$ for $k \geq 2$ satisfying, for $p \in \Lambda^k \mathfrak{g}$

$$-f_{k-1}(\partial p) = df_k(p) + \zeta(k) V_p \lrcorner \omega. \quad (3.1)$$

It follows immediately from equation (3.1) that if p is in $\mathcal{P}_{\mathfrak{g},k}$ then $f_k(p)$ is a Hamiltonian form. That is, if the domain of a homotopy moment map (f) is restricted to the Lie kernel, then the image of (f) is completely contained in the space of Hamiltonian forms. This motivates the definition of a weak homotopy moment map:

Definition 3.8. A weak (homotopy) moment map, is a collection of maps (f) with $f_k : \mathcal{P}_{\mathfrak{g},k} \rightarrow \Omega_{\text{Ham}}^{n-k}(M)$ satisfying

$$df_k(p) = -\zeta(k)V_p \lrcorner \omega. \quad (3.2)$$

We refer to the component f_k as a weak k -moment map.

Remark 3.9. Notice that any moment map gives a weak moment map. Indeed, if (f) satisfies equation (3.1) then it satisfies equation (3.2).

Remark 3.10. Notice that a weak homotopy moment map coincides with the moment map from symplectic geometry in the case $n = 1$. Indeed, setting $n = 1$ in equations (3.1) and (3.2) yields $f : \mathfrak{g} \rightarrow C^\infty(M)$ such that $df(\xi) = -V_\xi \lrcorner \omega$. Also notice that the n -th component of a weak moment map is precisely the moment map introduced by Madsen and Swann in [6] and [7].

The next proposition says that a weak moment map is still an L_∞ -morphism.

Proposition 3.11. *A weak moment map is an L_∞ -morphism from \mathfrak{g} to $\widehat{L}_\infty(M, \omega)$.*

Proof. This is Proposition 5.9 of [5]. □

Definition 3.12. A homotopy moment map (f) is equivariant if each component $f_k : \Lambda^k \mathfrak{g} \rightarrow \Omega^{n-k}(M)$ is equivariant with respect to the adjoint and pullback actions respectively. That is, for all $g \in G$, $p \in \Lambda^k \mathfrak{g}$ and $1 \leq k \leq n$

$$f_k(\text{Ad}_g^* p) = \Phi_g^* f_k(p). \quad (3.3)$$

Similarly, a weak moment map is equivariant if equation (3.3) holds for all $p \in \mathcal{P}_{\mathfrak{g},k}$ and $1 \leq k \leq n$.

We study the equivariance of moment maps further in the following section.

4 Equivariance of Weak Moment Maps

In this section we show how the theory of equivariance of moments maps in symplectic geometry generalizes to multisymplectic geometry.

4.1 Equivariance in Multisymplectic Geometry

We first recall the theory from symplectic geometry without proof and then generalize to the multisymplectic setting. The results from symplectic geometry can all be found in Chapter 4.2 of [1] for example. Let (M, ω) be a symplectic manifold, and $\Phi : G \times M \rightarrow M$ a symplectic Lie group action by a connected Lie group G . We consider the induced symplectic Lie algebra action $\mathfrak{g} \times \Gamma(TM) \rightarrow \Gamma(TM)$. Suppose that a moment map $f : \mathfrak{g} \rightarrow C^\infty(M)$ exists. That is, $df(\xi) = V_\xi \lrcorner \omega$ for all $\xi \in \mathfrak{g}$. By definition, f is equivariant if

$$f(\text{Ad}_{g^{-1}} \xi) = \Phi_g^* f(\xi).$$

Following Chapter 4.2 of [1], for $g \in G$ and $\xi \in \mathfrak{g}$ define $\psi_{g,\xi} \in C^\infty(M)$ by

$$\psi_{g,\xi}(x) := f(\xi)(\Phi_g(x)) - f(\text{Ad}_{g^{-1}} \xi)(x). \quad (4.1)$$

Proposition 4.1. *For each $g \in G$ and $\xi \in \mathfrak{g}$, the function $\psi_{g,\xi} \in C^\infty(M)$ is constant.*

Since $\psi_{g,\xi}$ is constant, we may define the map $\sigma : G \rightarrow \mathfrak{g}^*$ by

$$\sigma(g)(\xi) := \psi_{g,\xi},$$

where the right hand side is the constant value of $\psi_{g,\xi}$.

Proposition 4.2. *The map $\sigma : G \rightarrow \mathfrak{g}^*$ is a cocycle in the chain complex*

$$\mathfrak{g}^* \rightarrow C^1(G, \mathfrak{g}^*) \rightarrow C^2(G, \mathfrak{g}^*) \rightarrow \cdots.$$

That is, $\sigma(gh) = \sigma(g) + \text{Ad}_{g^{-1}}^ \sigma(h)$ for all $g, h \in G$.*

The map σ is called the cocycle corresponding to f . The following proposition shows that for any symplectic group action, the cocycle gives a well defined cohomology class.

Proposition 4.3. *For any symplectic action of G on M admitting a moment map, there is a well defined cohomology class. More specifically, if f_1 and f_2 are two moment maps, then their corresponding cocycles σ_1 and σ_2 are in the same cohomology class, i.e. $[\sigma_1] = [\sigma_2]$.*

By definition, we see that σ is measuring the equivariance of f . That is, $\sigma = 0$ if and only if f is equivariant. Moreover, if the cocycle corresponding to a moment map vanishes in cohomology, the next proposition shows that we can modify the original moment map to make it equivariant.

Proposition 4.4. *Suppose that f is a moment map with corresponding cocycle σ . If $[\sigma] = 0$ then $\sigma = \partial\theta$ for some $\theta \in \mathfrak{g}^*$, and $f + \theta$ is an equivariant moment map.*

We now show how this theory generalizes to multisymplectic geometry. For the rest of this section we let (M, ω) denote an n -plectic manifold and $\Phi : G \times M \rightarrow M$ a multisymplectic connected group action. We consider the induced multisymplectic Lie algebra action $\mathfrak{g} \times \Gamma(TM) \rightarrow \Gamma(TM)$. Assume that we have a weak homotopy moment map (f) , i.e. a collection of maps $f_k : \mathcal{P}_{\mathfrak{g},k} \rightarrow \Omega_{\text{Ham}}^{n-k}(M)$ satisfying equation (3.2).

To extend equation (4.1) to multisymplectic geometry, for $g \in G$ and $p \in \mathcal{P}_{\mathfrak{g},k}$, we define the following $(n-k)$ -form:

$$\psi_{g,p}^k := f_k(p) - \Phi_{g^{-1}}^* f_k(\text{Ad}_{g^{-1}}(p)). \quad (4.2)$$

The following proposition generalizes Proposition 4.1.

Proposition 4.5. *The $(n-k)$ -form $\psi_{g,p}^k$ is closed.*

Proof. Since Φ_g^* is injective and commutes with the differential, our claim is equivalent to showing that $\Phi_g^*(\psi_{g,p}^k)$ is closed. Indeed we have that

$$\begin{aligned} d(\Phi_g^*(\psi_{g,p}^k)) &= d(\Phi_g^* f_k(p) - f_k(\text{Ad}_{g^{-1}}(p))) \\ &= \Phi_g^*(df_k(p)) - d(f_k(\text{Ad}_{g^{-1}}(p))) \\ &= -\zeta(k)\Phi_g^*(V_p \lrcorner \omega) + \zeta(k)V_{\text{Ad}_{g^{-1}}p} \lrcorner \omega && \text{since } (f) \text{ is moment map} \\ &= -\zeta(k)\Phi_g^*(V_p \lrcorner \omega) + \zeta(k)(\Phi_g^* V_p) \lrcorner \omega && \text{by Proposition 2.9} \\ &= -\zeta(k)\Phi_g^*(V_p \lrcorner \omega) + \zeta(k)\Phi_g^*(V_p \lrcorner \omega) && \text{since } G \text{ preserves } \omega \\ &= 0. \end{aligned}$$

□

In analogy to symplectic geometry, we now show that each component of a weak moment map gives a cocycle.

Definition 4.6. We call the map $\sigma_k : G \rightarrow \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}$ defined by

$$\sigma_k(g)(p) := \psi_{g,p}^k$$

the cocycle corresponding to f_k .

As a generalization of Proposition 4.2 we obtain:

Proposition 4.7. *The map σ_k is a 1-cocycle in the chain complex*

$$\mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k} \rightarrow C^1(G, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) \rightarrow C^2(G, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) \rightarrow \dots,$$

where the action of G on $\mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}$ is given by the tensor product of the co-adjoint and pullback actions. The induced infinitesimal action of \mathfrak{g} on $\mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}$ is defined as follows: for $f \in \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{Ham}}^{n-k}$, $p \in \mathcal{P}_{\mathfrak{g},k}$ and $\xi \in \mathfrak{g}$,

$$(\xi \cdot f)(p) := f(\text{ad}_\xi(p)) + \mathcal{L}_{V_\xi} f(p). \quad (4.3)$$

Proof. By equation (2.1) we know that $(\partial(\sigma)(g, h))(p) := \sigma(gh)(p) - \sigma(g)(p) - g \cdot \sigma(h)(p)$. For arbitrary $p \in \mathcal{P}_{\mathfrak{g},k}$ we have

$$\begin{aligned} \sigma_k(gh)(p) &= f_k(p) - \Phi_{(gh)^{-1}}^*(f_k \text{Ad}_{(gh)^{-1}} p) \\ &= f_k(p) - \Phi_{g^{-1}}^* \Phi_{h^{-1}}^*(f_k(\text{Ad}_{h^{-1}} \text{Ad}_{g^{-1}} p)) \\ &= f_k(p) - \Phi_{g^{-1}}^*(f_k(\text{Ad}_{g^{-1}} p)) + \Phi_{g^{-1}}^*(f_k(\text{Ad}_{g^{-1}} p)) - \Phi_{g^{-1}}^*(\Phi_{h^{-1}}^*(f_k(\text{Ad}_{h^{-1}} \text{Ad}_{g^{-1}} p))) \\ &= \sigma_k(g)(p) + \Phi_{g^{-1}}^*(\sigma_k(h)(\text{Ad}_{g^{-1}} p)) \\ &= \sigma_k(g)(p) + g \cdot \sigma_k(h)(p). \end{aligned}$$

□

Definition 4.8. Let

$$\mathfrak{C} = \bigoplus_{k=1}^n \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}$$

Let $\sigma = \sigma_1 + \sigma_2 + \dots$. We call the map σ the cocycle corresponding to (f) .

Since the components of a weak moment map do not interact, as a corollary to Proposition 4.7 we obtain

Proposition 4.9. *The map σ is a cocycle in the complex*

$$\mathfrak{C} \rightarrow C^1(G, \mathfrak{C}) \rightarrow C^2(G, \mathfrak{C}) \rightarrow \dots$$

The next theorem shows that multisymplectic Lie algebra actions admitting weak moment maps give a well defined cohomology class, generalizing Proposition 4.3.

Theorem 4.10. *Let G act multisymplectically on (M, ω) . To any weak moment map, there is a well defined cohomology class $[\sigma]$ in $H^1(G, \mathfrak{C})$. More precisely if (f) and (g) are two weak moment maps with cocycles σ and τ , then $\sigma - \tau$ is a coboundary.*

Proof. We need to show that $\sigma_k - \tau_k$ is a coboundary for each k . We have that

$$\sigma_k(g)(p) - \tau_k(g)(p) = f_k(p) - g_k(p) - \Phi_{g^{-1}}^*(f_k(\text{Ad}_{g^{-1}}p) - g_k(\text{Ad}_{g^{-1}}(\xi))).$$

However, (f) and (g) are both moment maps and so $d(f_k(p) - g_k(p)) = 0$. Thus $f_k - g_k$ is in \mathfrak{C} . Moreover, by equation (2.2), we see that $\sigma_k - \tau_k = \partial(f_k - g_k)$. \square

If (f) is not equivariant but its cocycle vanishes, then we can define a new equivariant moment map from (f) , in analogy to Proposition 4.4.

Proposition 4.11. *Let (f) be a weak moment map with cocycle satisfying $[\sigma] = 0$. This means that $\sigma = \partial\theta$ for some $\theta \in \mathfrak{C}$. The map $(f) + \theta$ is a weak moment map that is equivariant.*

Proof. We have that $(f) + \theta$ is a moment map since $\theta(p)$ is closed for all $p \in \mathcal{P}_{\mathfrak{g},k}$. Let $\tilde{\sigma}$ denote the corresponding cocycle. Note that by equation (2.2) we have $(\partial(\theta)(g))(p) = \theta(\text{Ad}_{g^{-1}}p) - \Phi_g^*\theta(p)$. By the injectivity of Φ_g^* , to show that $\tilde{\sigma} = 0$, it is sufficient to show that $\Phi_g^*(\tilde{\sigma}(g)(p)) = 0$ for all $g \in G$ and $p \in \mathcal{P}_{\mathfrak{g},k}$. Indeed,

$$\begin{aligned} \Phi_g^*(\tilde{\sigma}_k(g)(p)) &= \Phi_g^*f(p) + \Phi_g^*\theta(p) - f(\text{Ad}_{g^{-1}}p) - \theta(\text{Ad}_{g^{-1}}p) \\ &= \sigma(g)(\xi) - \partial\theta(g)(\xi) \\ &= \sigma(g)(\xi) - \sigma(g)(\xi) && \text{since } \partial\theta = \sigma \\ &= 0. \end{aligned}$$

\square

If (f) is not equivariant with respect to the G -action, then we can define a new action for which (f) is equivariant.

Proposition 4.12. *For $g \in G$ define $\Upsilon_g : \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{Ham}}^{n-k} \rightarrow \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{Ham}}^{n-k}$ by*

$$\Upsilon_g(\theta)(p) := \Phi_{g^{-1}}^*\theta(\text{Ad}_{g^{-1}}p) + \sigma(g)(p)$$

where θ is in $\mathcal{P}_{\mathfrak{g},k}^ \otimes \Omega_{\text{Ham}}^{n-k}$ and p is in $\mathcal{P}_{\mathfrak{g},k}$. Then Υ_g is a group action and (f) is Υ_g -equivariant.*

Proof. The proof is a direct extension from the proof of Proposition 4.2.7 in [1]. We first show that Υ_g is a group action. Indeed, $\sigma(e) = 0$ and Ad_e is the identity showing that $\Upsilon_e(\theta) = \theta$. For the multiplicative property of the group action we have

$$\begin{aligned} \Upsilon_{gh}(\theta)(p) &= \Phi_{(gh)^{-1}}^*\theta(\text{Ad}_{(gh)^{-1}}p) + \sigma(gh)(p) \\ &= \Phi_{g^{-1}}^*(\Phi_{h^{-1}}^*(\theta(\text{Ad}_{h^{-1}}\text{Ad}_{g^{-1}}p))) + \sigma(g)(p) + \Phi_{g^{-1}}^*(\sigma(h)(\text{Ad}_{g^{-1}}p)) \quad \text{by Proposition 4.2} \\ &= \Phi_{g^{-1}}^*(\Upsilon_h(\theta)(\text{Ad}_{g^{-1}}p)) + \sigma(g)(p) \\ &= \Upsilon_g(\Upsilon_h(\theta))(p). \end{aligned}$$

To show that f_k is equivariant The moment map f_k is equivariant with respect to this action because

$$\begin{aligned} \Upsilon_g(f_k)(p) &= \Phi_{g^{-1}}^*(f_k(\text{Ad}_{g^{-1}}p)) + \sigma(g)(p) \\ &= \Phi_{g^{-1}}^*f_k(\text{Ad}_{g^{-1}}p) + f_k(p) - \Phi_{g^{-1}}^*f_k(\text{Ad}_{g^{-1}}p) \\ &= f_k(p). \end{aligned}$$

\square

4.2 Infinitesimal Equivariance in Multisymplectic Geometry

Next we recall the notion of infinitesimal version of equivariance in symplectic geometry. That is, we differentiate equation (4.1) to obtain the map $\Sigma : \mathfrak{g} \times \mathfrak{g} \rightarrow C^\infty(M)$ defined by $\Sigma(\xi, \eta) := \frac{d}{dt}\big|_{t=0} \psi_{\exp(t\eta), \xi}$. A straightforward computation, which we generalize in Proposition 4.16, gives that

$$\Sigma(\xi, \eta) = f([\xi, \eta]) - \{f(\xi), f(\eta)\}.$$

Another quick computation shows that $df([\xi, \eta]) = d\{f(\xi), f(\eta)\}$, showing $\Sigma(\xi, \eta)$ is a constant function for every $\xi, \eta \in \mathfrak{g}$. That is, Σ is a function from $\mathfrak{g} \times \mathfrak{g}$ to \mathbb{R} .

Proposition 4.13. *The map $\Sigma : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ is a Lie algebra 2-cocycle in the chain complex*

$$\mathbb{R} \rightarrow C^1(\mathfrak{g}, \mathbb{R}) \rightarrow C^2(\mathfrak{g}, \mathbb{R}) \rightarrow \cdots.$$

Definition 4.14. A moment map $f : \mathfrak{g} \rightarrow C^\infty(M)$ is infinitesimally equivariant if $\Sigma = 0$, i.e. if

$$f([\xi, \eta]) = \{f(\xi), f(\eta)\} \quad (4.4)$$

for all $\xi, \eta \in \mathfrak{g}$.

Proposition 4.15. *For a connected Lie group, infinitesimal equivariance and equivariance are equivalent.*

Proof. This is clear since Σ is just the derivative of σ . □

Since we will always be working with connected Lie groups, we will abuse terminology and call a moment map equivariant if it satisfies equation (3.3) or (4.4).

Now we turn our attention towards the multisymplectic setting. As in symplectic geometry, the infinitesimal equivariance of a weak moment map comes from differentiating $\psi_{\exp(t\xi), p}$ for fixed $\xi \in \mathfrak{g}$ and $p \in \mathcal{P}_{g,k}$.

Proposition 4.16. *Let Σ_k denote $\frac{d}{dt}\big|_{t=0} \psi_{\exp(t\xi), p}$. Then we have that Σ_k is a map from \mathfrak{g} to $\mathcal{P}_{g,k}^* \otimes \Omega_{cl}^{n-k}$ and is given by*

$$\Sigma_k(\xi, p) = f_k([\xi, p]) + L_{V_\xi} f_k(p).$$

Proof. We have that

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(t\xi), p} &= \frac{d}{dt}\bigg|_{t=0} f_k(p) - \frac{d}{dt}\bigg|_{t=0} \Phi_{\exp(-t\xi)}^*(f_k(\text{Ad}_{\exp(-t\xi)}(p))) \\ &= -\frac{d}{dt}\bigg|_{t=0} \Phi_{\exp(-t\xi)}^*(f_k(\text{Ad}_{\exp(-t\xi)}(p))) \\ &= -f_k\left(\frac{d}{dt}\bigg|_{t=0} \text{Ad}_{\exp(-t\xi)}(p)\right) - \left(\frac{d}{dt}\bigg|_{t=0} \Phi_{\exp(-t\xi)}^*\right)(f_k(p)) \\ &= -f_k(-[\xi, p]) + L_{\xi_M} f_k(p) \end{aligned} \quad \text{by Corollary 2.8.}$$

□

Let $R_k = \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}$. Then R_k is a \mathfrak{g} -module under the induced action from the tensor product of the adjoint and Lie derivative actions. Concretely, for $\alpha \in R_k$, $\xi \in \mathfrak{g}$ and $p \in \mathcal{P}_{\mathfrak{g},k}$,

$$(\xi \cdot \alpha)(p) = \alpha([\xi, p]) + L_{V_\xi} \alpha.$$

Consider the cohomology complex

$$R_k \rightarrow C^1(\mathfrak{g}, R_k) \rightarrow C^2(\mathfrak{g}, R_k) \rightarrow \cdots,$$

where the differential is the usual one from equation (2.2).

The following is a generalization of Proposition 4.13.

Proposition 4.17. *The map Σ_k is in the kernel of ∂_k . That is, Σ_k is a cocycle.*

Proof. We need to show that $\partial \Sigma_k = 0$. Indeed, for $\xi, \eta \in \mathfrak{g}$ and $p \in \mathcal{P}_{\mathfrak{g},k}$, we have that

$$\begin{aligned} \partial \Sigma_k(\xi, \eta)(p) &= \xi \cdot (\Sigma_k(\eta)(p)) - \eta \cdot (\Sigma_k(\xi)(p)) + \Sigma_k([\xi, \eta])(p) && \text{by equation (2.2)} \\ &= \Sigma_k(\eta)(\text{ad}_\xi(p)) + \mathcal{L}_{V_\xi}(\Sigma_k(\eta)(p)) - \Sigma_k(\xi)(\text{ad}_\eta(p)) \\ &\quad - \mathcal{L}_{V_\eta}(\Sigma_k(\xi)(p)) + \Sigma_k([\xi, \eta])(p) \\ &= \Sigma_k(\eta)([\xi, p]) + \mathcal{L}_{V_\xi}(\Sigma_k(\eta)(p)) - \Sigma_k(\xi)([\eta, p]) \\ &\quad - \mathcal{L}_{V_\eta}(\Sigma_k(\xi)(p)) + \Sigma_k([\xi, \eta])(p) && \text{by definition of ad} \\ &= f_k([\eta, [\xi, p]]) + \mathcal{L}_{V_\eta} f_k([\xi, p]) + \mathcal{L}_{V_\xi} f_k([\eta, p]) + \mathcal{L}_{V_\xi} \mathcal{L}_{V_\eta} f_k(p) \\ &\quad - f_k([\xi, [\eta, p]]) - \mathcal{L}_{V_\xi} f_k([\eta, p]) - \mathcal{L}_{V_\eta} f_k([\xi, p]) - \mathcal{L}_{V_\eta} \mathcal{L}_{V_\xi} f_k(p) \\ &\quad + f_k([\xi, \eta], p) - \mathcal{L}_{V_{[\xi, \eta]}} f_k(p) \\ &= f_k([\eta, [\xi, p]]) - f_k([\xi, [\eta, p]]) + f_k([\xi, \eta], p) \\ &\quad + \mathcal{L}_{V_\xi} \mathcal{L}_{V_\eta} f_k(p) - \mathcal{L}_{V_\eta} \mathcal{L}_{V_\xi} f_k(p) - \mathcal{L}_{V_{[\xi, \eta]}} f_k(p) \\ &= \mathcal{L}_{V_\xi} \mathcal{L}_{V_\eta} f_k(p) - \mathcal{L}_{V_\eta} \mathcal{L}_{V_\xi} f_k(p) - \mathcal{L}_{V_{[\xi, \eta]}} f_k(p) && \text{by the Jacobi identity} \\ &= 0 && \text{by the Lie derivative property.} \end{aligned}$$

□

As in symplectic geometry, we have that for a connected Lie group, a weak homotopy moment map is equivariant if and only if it is infinitesimally equivariant. That is, the weak homotopy k -moment map is equivariant if and only if $\sigma_k = 0$ or $\Sigma_k = 0$. A weak homotopy moment map is equivariant if $\sigma_k = 0$ or $\Sigma_k = 0$ for all $1 \leq k \leq n$.

Now that we have generalized the notions of equivariance from symplectic to multisymplectic geometry, we move on to study the existence and uniqueness of these weak homotopy moment maps.

5 Existence of Not Necessarily Equivariant Weak Moment Maps

In this section we show how the results on the existence of not necessarily equivariant moment maps in symplectic geometry generalizes to multisymplectic geometry.

For a connected Lie group G acting symplectically on a symplectic manifold (M, ω) , recall the following standard results from symplectic geometry.

Proposition 5.1. *For any $\xi, \eta \in \mathfrak{g}$ we have*

$$[V_\xi, V_\eta] \lrcorner \omega = d(V_\xi \lrcorner V_\eta \lrcorner \omega).$$

Proposition 5.2. *We have that $H^1(\mathfrak{g}) = 0$ if and only if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

and

Combining these two propositions

Proposition 5.3. *If $H^1(\mathfrak{g}) = 0$, then any symplectic action admits a moment map, which is not necessarily equivariant.*

We now show how these results generalize to multisymplectic geometry. Let a connected Lie group act multisymplectically on an n -plectic manifold (M, ω) .

Proposition 5.4. *For arbitrary q in $\mathcal{P}_{\mathfrak{g},k}$ and $\xi \in \mathfrak{g}$ we have that*

$$[V_q, V_\xi] \lrcorner \omega = -(-1)^k d(V_q \lrcorner V_\xi \lrcorner \omega).$$

Proof. By linearity it suffices to consider decomposable $q = \eta_1 \wedge \cdots \wedge \eta_k$. A quick computation shows that $[V_q, V_\xi] \lrcorner \omega = -V_{[q, \xi]} \lrcorner \omega$. It follows that

$$\begin{aligned} V_{[q, \xi]} \lrcorner \omega &= V_{\partial(q \wedge \xi)} \lrcorner \omega && \text{by Proposition 2.4} \\ &= (-1)^k d(V_q \wedge \xi \lrcorner \omega) - \sum_{i=1}^k (-1)^i \eta_1 \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_k \wedge \xi \lrcorner \mathcal{L}_{\eta_i} \omega - V_{q \wedge \xi} \lrcorner d\omega && \text{by Lemma 2.6} \\ &= (-1)^k d(V_q \lrcorner V_\xi \lrcorner \omega). \end{aligned}$$

The claim now follows. \square

The next proposition is a generalization of Proposition 5.2.

Proposition 5.5. *If $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^*) = 0$ then $\mathcal{P}_{\mathfrak{g},k} = [\mathcal{P}_{\mathfrak{g},k}, \mathfrak{g}]$.*

Proof. By equation (2.2), an element $c \in H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^*)$ satisfies $c([\xi, p]) = 0$ for all $\xi \in \mathfrak{g}$. That is,

$$H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^*) = [\mathcal{P}_{\mathfrak{g},k}, \mathfrak{g}]^0,$$

where $[\mathcal{P}_{\mathfrak{g},k}, \mathfrak{g}]^0$ is the annihilator of $[\mathcal{P}_{\mathfrak{g},k}, \mathfrak{g}]$. \square

We now arrive at our main theorem on the existence of not necessarily equivariant weak moment maps. The following is a generalization of Proposition 5.3.

Theorem 5.6. *Let G act multisymplectically on (M, ω) . If $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) = 0$, and $H^0(\mathfrak{g}, \Omega_{\text{cl}}^{n-k}) \neq 0$, then the k -th component of a not necessarily equivariant moment map exists.*

Proof. If $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) = 0$ and $H^0(\mathfrak{g}, \Omega_{\text{cl}}^{n-k}) \neq 0$, then $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^*) = 0$ by the Kunneth formula (see for example Theorem 3.6.3 of [11]). The claim now follows from Proposition 5.5 and Proposition 5.4. Indeed, Proposition 5.4 says we may define a weak moment map on elements of the form $[p, \xi]$ by $(-1)^k V_p \lrcorner V_\xi \lrcorner \omega$, where $p \in \mathcal{P}_{\mathfrak{g},k}$ and $\xi \in \mathfrak{g}$, and Proposition 5.5 says every element in $\mathcal{P}_{\mathfrak{g},k}$ is a sum of elements of this form. \square

Remark 5.7. Notice that for the case $n = k$, it is always true that $H^0(\mathfrak{g}, \Omega_{\text{cl}}^{n-k}) \neq 0$ since any non zero constant function is closed. Hence Theorem 5.6 gives a generalization of Theorems 3.5 and 3.14 of [6] and [7] respectively. Moreover, by taking $n = k = 1$, we see that we are obtaining a generalization from symplectic geometry.

Example 5.8. Consider the multisymplectic manifold (\mathbb{R}^4, ω) where $\omega = \text{vol}$ is the standard volume form. That is, we are working in the case $n = 3$. Let x_1, \dots, x_4 denote the standard coordinates. Let $G = U(2)$ act on \mathbb{R}^4 by rotations. The corresponding Lie algebra action generates the vector fields

$$\begin{aligned} E_0 &= x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^4}, \\ E_1 &= x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^4}, \\ E_2 &= -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}, \end{aligned}$$

and

$$E_3 = x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} - x^1 \frac{\partial}{\partial x^4}.$$

For the case $k = 2$, consider the distance function $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$. It is clear that the distance function is invariant under rotations and hence $\mathcal{L}_{E_i} dr = 0$ for $i = 0, 1, 2, 3$. Since dr is a closed 1-form, it follows that dr is a non-zero element of $H^0(\mathfrak{g}, \Omega_{\text{cl}}^1(M))$. That is, $H^0(\mathfrak{g}, \Omega_{\text{cl}}^1(M)) \neq 0$.

For the case $k = 1$, consider $\alpha := dx^1 \wedge dx^2 + dx^3 \wedge dx^4$. A quick calculation shows that $E_i \lrcorner \alpha = 0$ for $i = 0, 1, 2, 3$ so that α is invariant under the $\mathfrak{u}(2)$ action. Since $d\alpha = 0$, it follows that $H^0(\mathfrak{g}, \Omega_{\text{cl}}^2(M)) \neq 0$ as well.

Hence, by Theorem 5.6, it follows that a weak moment map exists.

The next example gives a scenario for which Theorem 5.6 can only be applied to specific components of a weak moment map.

Example 5.9. Take the setup of Example 5.8 but instead consider the action of $SO(4)$. As in Example 5.8, dr is a non-zero closed 1-form which is invariant under the action. That is, $H^0(\mathfrak{g}, \Omega_{\text{cl}}^1(M)) \neq 0$. However, in this setup, $H^0(\mathfrak{g}, \Omega_{\text{cl}}^2(M)) = 0$. Indeed, the infinitesimal generators of $\mathfrak{so}(4)$ are of the form $x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$ where $1 \leq i, j \leq 4$. An arbitrary 2-form may be written as $\beta = \sum_{i,j} a_{ij} dx^i \wedge dx^j$. A computation shows that the condition $\mathcal{L}_{V_\xi} \beta = 0$ for all $\xi \in \mathfrak{so}(4)$ showing that necessarily $\beta = 0$. Hence $H^0(\mathfrak{g}, \Omega_{\text{cl}}^1(M)) = 0$.

It follows that, in this case, Theorem 5.6 guarantees the existence of the 2nd component of a weak moment map, but does not guarantee the existence of the 1st.

Another generalization of Proposition 5.3 to multisymplectic geometry is given by:

Proposition 5.10. *If $H^k(\mathfrak{g}) = 0$, then the k -th component of a not necessarily equivariant weak moment map exists.*

Proof. If $H^k(\mathfrak{g}) = 0$ then $\mathcal{P}_{\mathfrak{g},k} = \text{Im}(\partial_{k+1})$, since $\mathcal{P}_{\mathfrak{g},k} = \ker(\partial_k)$. But for $p \in \text{Im}(\partial_{k+1})$ we have that $p = \partial q$ for some $q \in \Lambda^{k+1} \mathfrak{g}$. Then by Lemma 2.6 we have

$$V_p \lrcorner \omega = (-1)^k d(V_q \lrcorner \omega).$$

Hence we may define $f_k(p)$ to be $(-1)^k V_q \lrcorner \omega$. □

Remark 5.11. Proposition 5.10 gives another generalization of the results of Madsen and Swann. Indeed, by taking $n = k$ we again arrive at Theorems 3.5 and 3.14 of [6] and [7] respectively.

Summarizing Theorem 5.6 and Proposition 5.10 we obtain:

Proposition 5.12. *If $H^1(\mathfrak{g}) = \dots = H^n(\mathfrak{g}) = 0$ then a not necessarily equivariant weak moment map (f) exists.*

Theorem 5.13. *If, for all $1 \leq k \leq n$, $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^*) = 0$, or equivalently $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) = 0$ and $H^0(\mathfrak{g}, \Omega_{\text{cl}}^{n-k}) \neq 0$, then a not necessarily equivariant weak moment map (f) exists.*

In the next section we study when a non-equivariant weak moment map can be made equivariant.

6 Obtaining an Equivariant Moment Map from a Non-Equivariant Moment Map

In this section we show that the theory involved in obtaining an equivariant moment map from a non-equivariant moment map extends from symplectic to multisymplectic geometry. We first recall the standard results from symplectic geometry.

Proposition 4.13 shows that the map Σ corresponding to a moment map f is a Lie algebra 2-cocycle. The next proposition says that if the cocycle is exact then f can be made equivariant.

Proposition 6.1. *Let f be a moment map and Σ its corresponding cocycle. If $\Sigma = \partial(l)$ for some l , then $f + l$ is equivariant,*

It follows from this that

Proposition 6.2. *If $H^2(\mathfrak{g}) = 0$ then one can obtain an equivariant moment map from a non-equivariant moment map.*

Now let G be a connected Lie group acting on an n -plectic manifold (M, ω) . The following proposition generalizes Proposition 6.1 to multisymplectic geometry.

Proposition 6.3. *Let f_k be the weak homotopy k -moment map, and let Σ_k denote its corresponding cocycle. If $\Sigma_k = \partial(l_k)$ for some $l_k \in H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k})$, then $f_k + l_k$ is equivariant.*

Proof. Fix $p \in \mathcal{P}_{\mathfrak{g},k}$ and $\xi \in \mathfrak{g}$. Then

$$\begin{aligned} (f_k + l_k)([\xi, p]) &= f_k([\xi, p]) + l_k([\xi, p]) \\ &= f_k([\xi, p]) - ((\partial l_k)(\xi))(p) + \mathcal{L}_{V_\xi} l_k(p) && \text{by equation (2.2)} \\ &= f_k([\xi, p]) - \Sigma_k([\xi, p]) + \mathcal{L}_{V_\xi} l_k(p) \\ &= \mathcal{L}_{V_\xi} f_k(p) + \mathcal{L}_{V_\xi} (l_k(p)) && \text{by definition of } \Sigma_k \\ &= \mathcal{L}_{V_\xi} ((f_k + l_k)(p)). \end{aligned}$$

□

We now arrive at our generalization of Proposition 6.2:

Theorem 6.4. *If $H^1(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) = 0$ then any weak k -moment map can be made equivariant. In particular, if $H^1(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) = 0$ for all $1 \leq k \leq n$, then any weak moment map (f) can be made equivariant.*

Proof. Let $f_k : \mathcal{P}_{\mathfrak{g},k} \rightarrow \Omega_{\text{Ham}}^{n-k}$ be a weak k -moment map. If $H^1(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) = 0$ then the corresponding cocycle Σ_k is exact, i.e. $\Sigma_k = \partial(l_k)$ for some $l_k \in H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^*)$. It follows from Proposition 6.3 that $f_k + l_k$ is equivariant. □

7 Uniqueness of Weak moment Maps

We first recall the results from symplectic geometry without explicit proof. A proof can be found by setting $n = 1$ (i.e. the symplectic case) in our more general Theorem 7.4. Let \mathfrak{g} be a Lie algebra acting on a symplectic manifold (M, ω) .

Proposition 7.1. *If f and g are two equivariant moment maps, then $f - g$ is in $H^1(\mathfrak{g})$.*

Proof. For $\xi, \eta \in \mathfrak{g}$ we have that $(f - g)([\xi, \eta]) = \{(f - g)(\xi), (f - g)(\eta)\}$ since f and g are equivariant. However, $(f - g)(\xi)$ is a constant function since both f and g are moment maps. The claim now follows since the Poisson bracket with a constant function vanishes. \square

From Proposition 7.1 it immediately follows that

Proposition 7.2. *If $H^1(\mathfrak{g}) = 0$ then equivariant moment maps are unique.*

The following is a generalization of Proposition 7.1.

Proposition 7.3. *If f_k and g_k are k -th components of two equivariant weak moment maps, then $f_k - g_k$ is in $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k})$.*

Proof. If f_k and g_k are equivariant then $(f_k - g_k)([\xi, p]) = \mathcal{L}_{V_\xi}((f_k - g_k)(p))$. Moreover, $(f_k - g_k)(p)$ is closed since both f_k and g_k are moment maps. \square

We now arrive at our generalization of Proposition 7.2. Let \mathfrak{g} be a Lie algebra acting on an n -plectic manifold (M, ω) .

Theorem 7.4. *If $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) = 0$, then equivariant weak k -moment maps are unique. In particular, if $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k}) = 0$ for all $1 \leq k \leq n$ then equivariant weak moment maps are unique.*

Proof. If f_k and g_k are two equivariant weak k -moment maps, then Proposition 7.3 shows that $f_k - g_k$ is in $H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k}^* \otimes \Omega_{\text{cl}}^{n-k})$. \square

Remark 7.5. This theorem gives a generalization of the results of Madsen and Swann. Indeed, by taking $n = k$ we again arrive at Theorems 3.5 and 3.14 of [6] and [7] respectively.

8 Weak moment Maps as Morphisms

Consider a symplectic action of a connected Lie group G acting on a symplectic manifold (M, ω) . Let $f : \mathfrak{g} \rightarrow C^\infty(M)$ be a moment map. By Definition 4.14, f is equivariant if and only if f is a Lie algebra morphism from $(\mathfrak{g}, [\cdot, \cdot])$ to $(C^\infty(M), \{\cdot, \cdot\})$. That is, if and only if

$$f([\xi, \eta]) = \{f(\xi), f(\eta)\}.$$

Taking d of both sides of this equation yields:

Proposition 8.1. *A moment map f induces a morphism onto the quotient of $C^\infty(M)$ by constant functions. That is, a moment map induces a Lie algebra morphism from $(\mathfrak{g}, [\cdot, \cdot])$ to $(C^\infty(M)/\text{constant}, \{\cdot, \cdot\})$, regardless of equivariance. If moreover, the moment map f is equivariant, then f is a morphism from $(\mathfrak{g}, [\cdot, \cdot])$ to $(C^\infty(M), \{\cdot, \cdot\})$.*

We now restate Proposition 8.1 in an equivalent way, but which will allow for a direct generalization to multisymplectic geometry: Notice that \mathfrak{g} is a \mathfrak{g} -module under the Lie bracket action and $C^\infty(M)$ is \mathfrak{g} -module under the action $\xi \cdot g = L_{V_\xi}g$, where $\xi \in \mathfrak{g}$ and $g \in C^\infty(M)$. Proposition 8.1 is equivalent to:

Proposition 8.2. *A moment map f always induces a \mathfrak{g} -module morphism from \mathfrak{g} to $C^\infty(M)/\text{constant}$. Moreover, if the moment map f is equivariant, then it is a \mathfrak{g} -module morphism from \mathfrak{g} to $C^\infty(M)$.*

Now let a connected Lie group G act multisymplectically on an n -plectic manifold (M, ω) .

Proposition 8.3. *For any $1 \leq k \leq n$, we have that $\mathcal{P}_{\mathfrak{g},k}$ is a \mathfrak{g} -module under the action $\xi \cdot p = [p, \xi]$, where $p \in \mathcal{P}_{\mathfrak{g},k}$, $\xi \in \mathfrak{g}$, and $[\cdot, \cdot]$ is the Schouten bracket.*

Proof. This follows since Proposition 2.4 shows that $[p, \xi]$ is in the Lie kernel. \square

Proposition 8.4. *For any $1 \leq k \leq n$, we have that $\Omega_{\text{Ham}}^{n-k}(M)$ is a \mathfrak{g} -module under the action $\xi \cdot \alpha = \mathcal{L}_{V_\xi}\alpha$, where $\alpha \in \Omega_{\text{Ham}}^{n-k}(M)$ and $\xi \in \mathfrak{g}$.*

Proof. Suppose that $\alpha \in \Omega_{\text{Ham}}^{n-k}(M)$ is a Hamiltonian $(n-k)$ -form. Then $d\alpha = -X_\alpha \lrcorner \omega$ for some $X_\alpha \in \Gamma(\Lambda^k(TM))$. Then, for $\xi \in \mathfrak{g}$,

$$\begin{aligned} d\mathcal{L}_{V_\xi}\alpha &= -\mathcal{L}_{V_\xi}(X_\alpha \lrcorner \omega) \\ &= -\mathcal{L}_{V_\xi}(X_\alpha \lrcorner \omega) + X_\alpha \lrcorner \mathcal{L}_{V_\xi}\omega && \text{since } \mathcal{L}_{V_\xi}\omega = 0 \\ &= [V_\xi, X_\alpha] \lrcorner \omega && \text{by the product rule} \end{aligned}$$

Hence $\mathcal{L}_{V_\xi}\alpha$ is in $\Omega_{\text{Ham}}^{n-k}(M)$. \square

Our generalization of Proposition 8.2 to multisymplectic geometry is:

Theorem 8.5. *For any $1 \leq k \leq n$, the k -th component of a moment map f_k is a \mathfrak{g} -module morphism from $\mathcal{P}_{\mathfrak{g},k}$ to $\Omega_{\text{Ham}}^{n-k}(M)/\text{closed}$. Moreover, a weak k -moment map f_k is equivariant if and only if it is a \mathfrak{g} -module morphism from $\mathcal{P}_{\mathfrak{g},k}$ to $\Omega_{\text{Ham}}^{n-k}(M)$.*

Proof. Suppose that (f) is a weak moment map. Then, by definition

$$\begin{aligned} df_k([\xi, p]) &= -\zeta(k)V_{[\xi, p]} \lrcorner \omega \\ &= -\zeta(k)[V_\xi, V_p] \lrcorner \omega \\ &= -\zeta(k)\mathcal{L}_{V_\xi}(V_p \lrcorner \omega) \\ &= \zeta(k)\zeta(k)d\mathcal{L}_{V_\xi}f_k(p) \\ &= d\mathcal{L}_{V_\xi}f_k(p). \end{aligned}$$

This proves the first statement of the theorem. Now suppose f_k is equivariant. It follows that $\Sigma_k = 0$. Thus, by Proposition 4.16 we have $f_k([\xi, p]) = \mathcal{L}_{V_\xi}f_k(p)$. Conversely, if f_k is a \mathfrak{g} -module morphism, that $f_k([\xi, p]) = \mathcal{L}_{V_\xi}f_k(p)$ for every $\xi \in \mathfrak{g}$ and $p \in \mathcal{P}_{\mathfrak{g},k}$. That is, $\Sigma_k = 0$. \square

9 Open Questions

We end by noting some open questions naturally posed by the results in this paper.

1. Consider Theorems 8.5. In symplectic geometry, Proposition 4.13 shows that a moment map $f : \mathfrak{g} \rightarrow C^\infty(M)$ induces a Lie algebra morphism from $(\mathfrak{g}, [\cdot, \cdot])$ to the quotient space $(C^\infty(M)/\text{constant}, \{\cdot, \cdot\})$, and if f is equivariant then it is a Lie algebra morphism from $(\mathfrak{g}, [\cdot, \cdot])$ to $(C^\infty(M)/\text{exact}, \{\cdot, \cdot\})$. Moreover, in [5], Proposition 4.10 showed that both $\Omega_{\text{Ham}}^\bullet(M)/\text{closed}$ and $\Omega_{\text{Ham}}^\bullet(M)/\text{exact}$ are graded Lie algebras while Proposition 5.9 of [5] showed that a weak homotopy moment map is always a graded Lie algebra morphism from $\mathcal{P}_{\mathfrak{g}}$ to $\Omega_{\text{Ham}}^\bullet(M)/\text{closed}$.

Hence, a natural question is:

If (f) is an equivariant weak moment map, does it induce a graded Lie algebra morphism from $(\mathcal{P}_{\mathfrak{g}}, [\cdot, \cdot])$ to $(\Omega_{\text{Ham}}^\bullet(M)/\text{exact}, \{\cdot, \cdot\})$? Conversely?

2. In our work, we provided a couple of examples of n -plectic group actions to which our theory of the existence and uniqueness of moment maps could be applied. There are many other interesting n -plectic geometries; see for example [3], [5] and [10]. What does the work done in our paper say about the existence and uniqueness of moment maps in these setups?
3. Given a weak moment map (f) with $f_k : \mathcal{P}_{\mathfrak{g},k} \rightarrow \Omega_{\text{Ham}}^{n-k}(M)$, does there exist a full homotopy moment map (h) whose restriction to the Lie kernel is (f) ? Something about equivariant cohomology. In particular, what is the relationship between the results on the existence and uniqueness of homotopy moment maps given in [9] and [4] to the results in this paper?

References

- [1] R. Abraham and J. E. Marsden. *Foundations of Mechanics, 2nd edition*, Reading, Massachusetts: Benjamin/Cummings Publishing Company, 1978.
- [2] A. Cannas da Silva *Lectures on Symplectic Geometry*, volume 1764 of Lectures Notes in Mathematics. Springer-Verlag, Berlin, 2001.
- [3] M. Callies, Y. Fregier, C. L. Rogers, and M. Zambon. *Homotopy moment maps*, Advances in Mathematics 303 (2016): 954-1043.
- [4] Y. Frégier, C. Laurent-Gengoux, and M. Zambon. *A cohomological framework for homotopy moment maps*, Journal of Geometry and Physics 97 (2015): 119-132.
- [5] J. Herman. *Noether's Theorem in Multisymplectic Geometry* Differ. Geom. Appl. (2017), <https://doi.org/10.1016/j.difgeo.2017.09.003>.
- [6] T. B. Madsen and A. Swann. *Multi-moment maps*, Advances in Mathematics 229.4 (2012): 2287-2309.
- [7] T. B. Madsen and A. Swann. *Closed forms and multi-moment maps*, Geometriae Dedicata (2013): 1-28.
- [8] C. L. Rogers. *L-infinity algebras from multisymplectic geometry*, Lett. Math. Phys., 100(1):29-50, 2012.

- [9] L. Ryvkin, and T. Wurzbacher. *Existence and unicity of moments in multisymplectic geometry*, Differential Geometry and its Applications 41 (2015): 1-11.
- [10] L. Ryvkin, T. Wurzbacher, and M. Zambon. *Conserved quantities on multisymplectic manifolds*, arXiv preprint arXiv:1610.05592 (2016).
- [11] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of Cambridge Studies in Advanced Mathematics, 1994.