

# On Geometric Drawings of Graphs

by

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## Abstract

This thesis is about geometric drawings of graphs and their topological generalizations.

First, we study pseudolinear drawings of graphs in the plane. A *pseudolinear* drawing is one in which every edge can be extended into an infinite simple arc in the plane, homeomorphic to  $\mathbb{R}$ , and such that every two extending arcs cross exactly once. This is a natural generalization of the well-studied class of *rectilinear drawings*, where edges are straight-line segments. Although, the problem of deciding whether a drawing is homeomorphic to a rectilinear drawing is NP-hard, in this work we characterize the minimal forbidden subdrawings for pseudolinear drawings and we also provide a polynomial-time algorithm for recognizing this family of drawings.

Second, we consider the problem of transforming a topological drawing into a similar rectilinear drawing preserving the set of crossing pairs of edges. We show that, under some circumstances, pseudolinearity is a necessary and sufficient condition for the existence of such transformation. For this, we prove a generalization of Tutte's Spring Theorem for drawings with crossings placed in a particular way.

Lastly, we study drawings of  $K_n$  in the sphere whose edges can be extended to an arrangement of pseudocircles. An *arrangement of pseudocircles* is a set of simple closed curves in the sphere such that every two intersect at most twice. We show that (i) there is drawing of  $K_{10}$  that cannot be extended into an arrangement of pseudocircles; and (ii) there is a drawing of  $K_9$  that can be extended to an arrangement of pseudocircles, but no extension satisfies that every two pseudocircles intersects exactly twice. We also introduce the notion *pseudospherical drawings of  $K_n$* , a generalization of *spherical drawings* in which each edge is a minor arc of a great circle. We show that these drawings are characterized by a simple local property. We also show that every pseudospherical drawing has an extension into an arrangement of pseudocircles where the "at most twice" condition is replaced by "exactly twice".

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Finally, I would like to thank my wife Diana for always reminding me of the truly important things in life.

## **Dedication**

Para Chichí, que me enseñó a disfrutar la vida [For Chichí, who taught me how to enjoy life].

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# Chapter 1

## Introduction

In 2014, Dan McQuillan and Gelasio Salazar visited the University of Waterloo when I was starting my PhD studies. The purpose of their visit was to begin an investigation with Bruce Richter and me, about one of the topics treated in the *Exact Crossing Numbers Workshop*, held at the American Mathematical Institute (AIM) in the same year.

The topic was the recent progress on the Harary-Hill conjecture, stating that the crossing number of  $K_n$  is given by the following formula:

$$H(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

At the AIM meeting we learned about new methods found by Ábrego et al. [1] showing that the Harary-Hill Conjecture holds for some interesting classes of drawings, such as rectilinear, 2-page, cylindrical and monotone drawings.

These methods emphasize the importance of understanding the special features of these classes of drawings. Perhaps, with this in mind, Gelasio Salazar asked us, in our first meeting in Waterloo, the following question: *Is there a simple way, in terms of local properties of the drawing, to characterize rectilinear drawings of  $K_n$ ?*

As an initial observation, we found that pseudolinear drawings (a class of drawings defined below that generalizes rectilinear drawings) have the property that, if one fixes a 3-cycle  $T$  and looks at the vertices drawn in the bounded side of  $T$ , then the edges connecting these interior vertices to the three vertices in  $T$  are contained in the bounded side of  $T$ . Not every topological drawing of  $K_n$  has this property; it was reasonable to think that drawings of  $K_n$  having this property are pseudolinear.

A *pseudolinear* drawing is one in which every edge can be extended into an infinite simple arc in the plane, homeomorphic to  $\mathbb{R}$ , and such that every two extending arcs cross exactly once. These extending arcs are called *pseudolines*, and they are the topological analog of the straight lines extending the edges of a rectilinear drawing.

Many of the properties satisfied by rectilinear drawings also hold for pseudolinear drawings. In fact, knowing that a property holds for pseudolinear drawings has its practical advantages: proofs and algorithms for pseudolinear drawings do not rely on geometric parameters such as angles or distances; they rely only on the combinatorial structure of how the extending pseudolines intersect and how they separate the vertices in the drawing.

Our group continued studying drawings of  $K_n$  satisfying the local property satisfied by the 3-cycles. We found that *every drawing of  $K_n$  satisfying this local property is pseudolinear* [8]. In fact, an equivalent result was independently discovered by Aichholzer et al. [5]. They showed that a drawing of  $K_n$  is pseudolinear if and only if it does not include one of the three drawings of  $K_4$  in the plane.

Hernández-Vélez presented a talk in the Crossing-Number Workshop held in Rio de Janeiro (2015) about investigating the difference between the crossing number  $\text{cr}(G)$ , the rectilinear crossing number  $\overline{\text{cr}}(G)$  and the pseudolinear crossing number  $\tilde{\text{cr}}(G)$  [18]. He explained how to find graphs where the difference between  $\text{cr}$  and  $\tilde{\text{cr}}$  is large. The key idea was to use some basic non-pseudolinear drawings, that he called *obstructions*. At that time, it was unclear how to construct new obstructions, or how many of them there were.

A few months later, I visited Carsten Thomassen and his group at DTU as part of an internship. I gave a talk about pseudolinear drawings of  $K_n$ , where I posed the problem of classifying the minimal forbidden obstructions of pseudolinear drawings of all graphs. Thomassen observed that one of the difficulties of this problem is the existence of infinitely many minimal obstructions (In fact, in [29], it was already observed that there are infinitely many obstructions for rectilinear drawings; these are also examples of obstructions for pseudolinear drawings).

This thesis work is about the research following the previous events. In Chapter 2 we show as one of our main results in this thesis, a characterization of the minimal forbidden subdrawings of pseudolinear drawings. In this joint work with Bensmail and Richter, we found a simple description for all the obstructions, leading to a polynomial-time algorithm to recognize pseudolinear drawings.

Understanding pseudolinear drawings can be helpful to know more about geometric drawings of graphs. In [13], Bienstock and Dean showed that a graph  $G$  with  $\text{cr}(G) \leq 3$ , has  $\overline{\text{cr}}(G) = \text{cr}(G)$ . With this result as our motivation, in Chapter 3 we study the problem of turning a topological drawing into a rectilinear drawing with the same number

of crossings. Our main result in Chapter 3 finds conditions where this can be done and, surprisingly, where pseudolinearity plays an essential part of these conditions. For this purpose, we obtained a generalization of Tutte’s Spring Theorem that allow some edges to be crossed. The content of Chapter 3 is based on the author’s original ideas, but has been somewhat influenced by what will be a more general work in collaboration with Richter and Thomassen.

Chapter 4 is motivated from studying *spherical* drawings of  $K_n$ , where vertices are represented as distinct points in the unit sphere and the edges are shortest-arcs connecting these points. Spherical and rectilinear drawings have some properties in common, however, it is still unknown if spherical drawings of  $K_n$  satisfy the Harary-Hill conjecture.

A problem in the spotlight of the graph drawing community was to find a class of drawings generalizing spherical drawings of  $K_n$  in the same way pseudolinear drawings generalize rectilinear drawings. In Chapter 4, we investigate this by answering two open problems about extending drawings of  $K_n$  into arrangements of pseudocircles. Moreover, we propose a definition for pseudospherical drawings that achieves many of the desired properties.

In connection to our previous work in pseudolinear drawings of  $K_n$ , we show that a drawing is pseudospherical if and only if for every 3-cycle  $T$  there is a disk  $\Delta_T$  in the sphere bounded by  $T$ , where all the edges connecting a vertex inside  $\Delta_T$  to a vertex in  $T$  are drawn inside  $\Delta_T$  and all the  $\Delta_T$ s are structured in a hereditary-way (for more technical details see Definition 4.4.1). Using this local property we show that every edge in a pseudospherical drawing induces a decomposition of the drawing into two pseudolinear pieces (this generalizes the observation that every edge in a spherical drawing of  $K_n$  induces a partition into two rectilinear drawings).

Part of the discussion of how to define pseudospherical drawings was focused on whether to include the condition of any two pseudocircles intersecting exactly twice or to simply let them intersect zero or two times. The main result of Chapter 4 ends this discussion by showing that a pseudospherical drawing of  $K_n$  admitting an extension where the pseudocircles cross zero or two times, also has one where they cross exactly twice. This last chapter is a joint work with Richter and Sunohara.

# Chapter 2

## Pseudolinear drawings of graphs

### 2.1 Introduction

When is a drawing of a graph in the plane homeomorphic to a drawing whose edges are straight line segments? A rectilinear drawing of a graph is one in which edges are drawn using straight line segments, and more generally, a *stretchable* drawing is one that is homeomorphic to a rectilinear drawing. Fáry's Theorem [14, 31, 27], a classic result in graph theory, asserts that drawings of simple graphs with no crossings between edges are stretchable.

A *good drawing* is one in which every two incident edges are not crossed, every pair of edges cross at most once, and no crossing point is in three edges. In [29], Thomassen extended Fáry's Theorem by characterizing stretchable good drawings of graphs in which every edge is crossed at most once. This characterization is in terms of forbidding two configurations, known as the *B* and *W* configurations, shown in Figure 2.1.

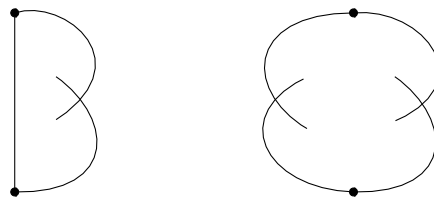


Figure 2.1: *B* and *W* configurations.

Thomassen’s characterization is a partial answer to the general problem of determining which drawings are stretchable. There is not likely to be a complete characterization, because the closely related problem of stretchability of arrangements of pseudolines is NP-hard.

A *pseudoline* is an unbounded open arc in the plane whose complement is disconnected. In particular, lines are pseudolines, and any pseudoline is the image of a line under a homeomorphism of the plane into itself. An *arrangement of pseudolines* is a set of pseudolines in which every two intersect in exactly one point, and their intersection point is a crossing. We allow multiple pseudolines crossing at one point.

Mnëv showed [21, 22] that it is NP-hard to determine whether an arrangement of pseudolines is stretchable (in fact, he showed that the stretchability problem is  $\exists\mathbb{R}$ -hard). The problem of stretching arrangements of pseudolines can be reduced to the problem of deciding whether the drawing of a graph is stretchable: given an arrangement of pseudolines, consider a big circle containing all the intersection points between the pseudolines. Cut off the two ends of each pseudoline outside this circle to obtain a drawing of multiple copies of  $K_2$ . This drawing is stretchable if and only if the arrangement is stretchable.

Our interest in stretchable drawings comes from studying crossing numbers. The *crossing number* of a graph  $G$  is the minimum number  $cr(G)$  of crossings in a drawing of  $G$  in the plane. If we restrict our drawings to be stretchable, then this minimum is the *rectilinear crossing number*  $\overline{cr}(G)$ .

One of the main open problems in the study of crossing numbers is the Harary-Hill Conjecture [16], which states that the crossing number of  $K_n$  is equal to

$$H(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor. \quad (2.1)$$

There are drawings of  $K_n$  with  $H(n)$  crossings. In 2004 it was shown in [2] and in [19] that  $\overline{cr}(K_n) \geq H(n)$ , and in fact, the tools used to show this lower bound come from the study of arrangements of pseudolines, so naturally the proof generalizes to pseudolinear drawings. A *pseudolinear* drawing of a graph is one in which each edge-arc can be extended to a pseudoline, resulting in an arrangement of pseudolines. Clearly stretchable drawings are pseudolinear.

In the recent study of crossing numbers, pseudolinear drawings have played a predominant role, as they are treated as a combinatorial generalization of rectilinear drawings. For instance, in parallel results from [6] and [9], it was shown that in rectilinear and pseudolinear drawings of  $K_n$  with as few crossings as possible, the outer face is bounded by a



3-cycle. This connection led Balogh et al. [9] to conjecture that  $\tilde{cr}(K_n) = \overline{cr}(K_n)$  (where  $\tilde{cr}(K_n)$  denotes the *pseudolinear crossing number* of  $K_n$ ).

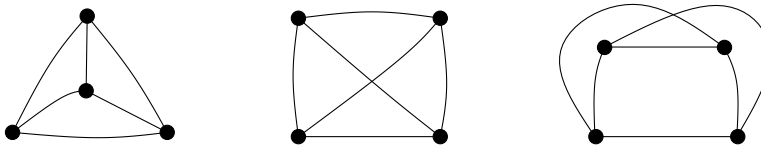


Figure 2.2: Non-isomorphic drawings of  $K_4$  in the plane.

Pseudolinear drawings of  $K_n$  can be characterized in terms of a forbidden minimal subdrawing. There are exactly three non-homeomorphic drawings of  $K_4$  (shown in Figure 2.2), and exactly one of them (the one on the right) is not pseudolinear. It was shown in [5] and [8] that a drawing of  $K_n$  is pseudolinear if and only if it does not include the non-pseudolinear drawing of  $K_4$ .

The aim of this chapter to generalize the previous result by characterizing pseudolinear drawings of all graphs by describing a set of minimal forbidden subdrawings. One of the difficulties to finding such a set of drawings is the fact that there are infinitely many of them, such as the ones in Figure 2.3. Thomassen already made this observation for rectilinear drawings [29].

The description of minimal forbidden drawings requires a refined understanding of drawings and also of what we meant by saying “minimal”, leading us to study sets of strings. A *string*  $\sigma$  is the image  $f([0, 1])$  of a continuous function  $f : [0, 1] \rightarrow \mathbb{R}^2$  that restricted to  $(0, 1)$  is injective; in other words, strings are arcs that are allowed to self-intersect only at their *ends*  $f(0)$  and  $f(1)$ . If no such self-intersection exists, then  $\sigma$  is *simple*. Most of the time we will consider simple strings, although considering non-simple strings will come in handy for technical reasons.

A set of strings  $\Sigma$  is in *general position* if, for every two strings  $\sigma, \sigma' \in \Sigma$  (i)  $\sigma \cap \sigma'$  is a finite set of points in  $\mathbb{R}^2$ ; and (ii) each point in  $\sigma \cap \sigma'$  is either a crossing between  $\sigma$  and  $\sigma'$ , or an end of either  $\sigma$  or  $\sigma'$ . For instance, the set of edge-arcs of a good drawing of a graph is a set of strings in general position, but not all the sets of strings in general position come in this fashion: a string might include end points of other strings in its interior.

For a set  $\Sigma$  of strings in general position, its *underlying plane graph*  $G(\Sigma)$  is the plane graph obtained from  $\Sigma$  by replacing the crossings between strings and the end points of every string in  $\Sigma$  by vertices. Our main result below characterizes when a set of strings in general position can be extended to an arrangement of pseudolines.

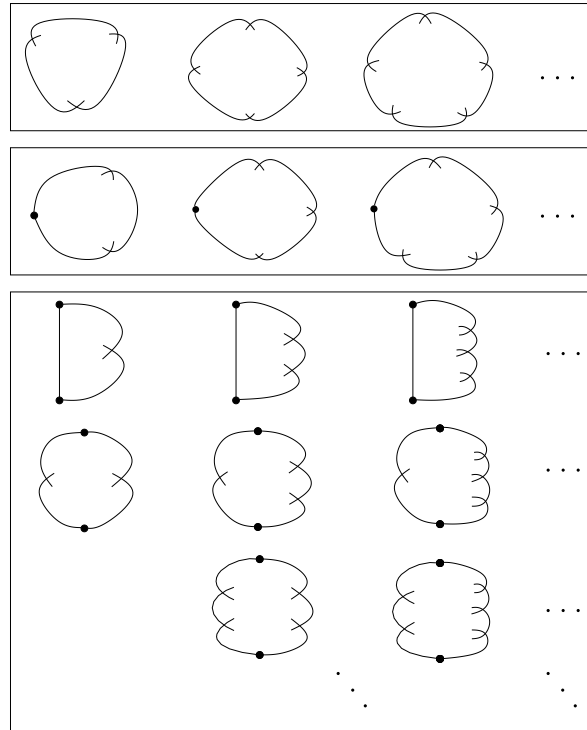


Figure 2.3: Obstructions to pseudolinear drawings.

**Theorem 2.1.1.** *A set of strings  $\Sigma$  in general position can be extended to an arrangement of pseudolines if and only if, for each cycle  $C$  in the underlying plane graph  $G(\Sigma)$  of  $\Sigma$ , there are at least three vertices with the property that the edges incident to the vertex that are included in the closed disk bounded by  $C$  belong to distinct strings in  $\Sigma$ .*

For instance, let  $C$  be the unique cycle in the underlying plane graph in any of the drawings in Figure 2.3. There are at most two vertices of  $G$  in  $C$ , represented as black dots. The strings incident with such a vertex are distinct and contained in the closed disk bounded by  $C$ . The vertices represented as crossings do not satisfy this property: they are incident with four edges in the disk bounded by  $C$ , and these four edges consist of two strings that cross at this vertex. Theorem 2.1.1 implies that none of the drawings in Figure 2.3 is pseudolinear. Surprisingly, we will show, as a consequence of Theorem 2.1.1, that every non-pseudolinear good drawing of a graph  $H$  contains one of the configurations in Figure 2.3 as a subdrawing.

**Theorem 2.1.2.** *Let  $D$  be a non-pseudolinear good drawing of a graph  $H$ . Then there is a*

subset  $S$  of edge-arcs in  $\{D[e] : e \in E(H)\}$ , such that each  $\sigma \in S$  has a substring  $\sigma' \subseteq \sigma$  for which  $\bigcup_{\sigma \in S} \sigma'$  is one of the drawings in Figure 2.3.

Cycles that have fewer than three vertices as in Theorem 1 are the *obstructions* of  $G(\Sigma)$  (this definition will be made more precise at the beginning of Section 2.2). Showing that when  $G(\Sigma)$  has obstructions, then  $\Sigma$  cannot be extended to an arrangement of pseudolines, is the first part of Section 2.2. The rest of Section 2.2 is devoted to show that if  $G(\Sigma)$  has no obstructions, then  $\Sigma$  can be extended to an arrangement of pseudolines. The proof of two technical lemmas used in the proof of Theorem 2.1.1 are deferred to Section 2.3. In Section 2.4, we describe a simple algorithm that finds an obstruction in polynomial time. In Section 2.5, by applying Theorem 2.1.1, we prove that a drawing of a complete graph  $K_n$  is pseudolinear if and only if it does not contain the  $B$  configuration in Figure 2.1. This result is equivalent to the characterizations of pseudolinear drawings of  $K_n$  given in [5] and [8], but its proof is simpler. At the end, in Section 2.6, we show how Theorem 2.1.2 easily follows from Theorem 2.1.1, together with some concluding remarks.

The present chapter is a collaborative work with Julien Bensmail and Bruce Richter.

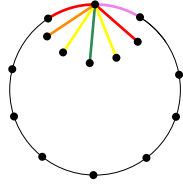
## 2.2 Proof of Theorem 2.1.1

In this section, we use Lemmas 2.2.4 and 2.2.5 (proved in the next section) to prove Theorem 2.1.1. As we enter into the subject, we need some notation that is useful in identifying an obstruction. Let  $C$  be a cycle of a plane graph  $G$  and let  $v$  be a vertex of  $C$ . The *rotation at  $v$  inside  $C$*  is the counterclockwise ordered list  $e_0, e_1, \dots, e_k$  of edges incident with  $v$  that are included in the closed disk bounded by  $C$ , with  $e_0$  and  $e_k$  both in  $C$ . Likewise, the *rotation at  $v$  outside  $C$*  is defined as the counterclockwise ordered list  $e_k, e_{k+1}, \dots, e_0$  of edges incident with  $v$  included in the closure of the exterior of  $C$ .

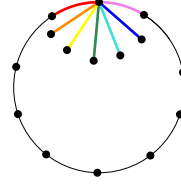
In the case  $G = G(\Sigma)$  for some set  $\Sigma$  of strings in general position, a vertex  $v$  in a cycle  $C$  of  $G(\Sigma)$  is *reflecting* in  $C$  if at least two edges in the rotation at  $v$  inside  $C$  belong to the same string (Figure 2.4a). The alternative is that  $v$  is a *rainbow*, in which case all the edges of its rotation inside  $C$  are in different strings (Figure 2.4b). In these terms, an *obstruction* is a cycle with at most two rainbows.

### 2.2.1 Sets of strings with obstructions are not extendible

The following observation will be used in this subsection and also in Theorem 2.5.1. If  $C$  is a cycle in  $G(\Sigma)$ , where  $\Sigma$  is a set of strings in general position, then  $\delta(C)$  is the set of



(a) A reflecting vertex



(b) A rainbow

Figure 2.4: A representation of reflecting and rainbow vertices, where each string in  $G(\Sigma)$  has assigned a unique colour.

vertices in  $C$  for which their two incident edges in  $C$  belong to two distinct strings in  $\Sigma$ . Note that if  $|\delta(C)| < 3$  for some cycle  $C$ , then either  $|\delta(C)| = 2$  and two strings intersect more than once, or  $|\delta(C)| \leq 1$  and some string is self-crossed. Both these possibilities are forbidden in good drawings.

**Observation 2.2.1.** *Let  $\Sigma$  be a set of simple strings in general position in which every two strings intersect at most once. Let:*

- (a)  $C$  be an obstruction of  $G(\Sigma)$  for which  $|\delta(C)|$  is as small as possible;
- (b)  $x \in \delta(C)$ ;
- (c)  $e$  be an edge in  $C$  incident to  $x$ ;
- (d)  $\sigma \in \Sigma$  be the string containing  $e$ ; and
- (e)  $\sigma'$  be the component of  $\sigma \setminus e$  containing  $x$ .

Then  $\sigma' \cap C = \{x\}$ .

*Proof.* By way of contradiction, suppose that  $\sigma' \cap C$  includes a point distinct from  $x$ . This in particular implies that  $\sigma' \neq \{x\}$ , and, because  $x \in \delta(C)$ , the points of  $\sigma' \setminus \{x\}$  near  $x$  are not in  $C$ . Let  $P$  be the path in  $G(\Sigma)$  obtained by traversing  $\sigma'$ , starting at  $x$ , and stopping the first time we encounter a point  $y \in C \cap (\sigma' \setminus \{x\})$ . Note that  $y \in V(C)$  and that  $P$  is drawn in either the interior or the exterior of  $C$ .

First, suppose that  $P$  is drawn in the interior of  $C$ . Let  $C_1$  and  $C_2$  be the cycles obtained from the union of  $P$  and one of the two  $xy$ -subpaths in  $C$ . We may assume  $C_1$  includes  $e$ . Each of  $C_1 - P$  and  $C_2 - P$  has a vertex in  $\delta(C)$ ; otherwise one of  $C_1$  or  $C_2$  would

be included in at most two strings, implying that a string is self-crossing or two strings intersect twice. Therefore  $|\delta(C_1)|$  and  $|\delta(C_2)|$  are strictly smaller than  $|\delta(C)|$ . Then, by assumption,  $C_1$  and  $C_2$  are not obstructions.

None of the vertices in  $P - y$  is a rainbow for  $C_1$  ( $P \subseteq \sigma'$  and  $x$  is reflecting in  $C_1$ , so the interior rotations of the vertices in  $P - y$  include two edges in  $\sigma$ ). Since all the vertices in  $C_1 - V(P)$  that are rainbow in  $C_1$  are also rainbow in  $C$ ,  $C_1$  has at most two rainbows in  $V(C_1) \setminus V(P)$ . These last two observations and the fact that  $C_1$  is not an obstruction, together imply that  $C_1$  has three rainbows: two of them are in  $V(C_1) \setminus V(P)$  and the other is  $y$ .

Now we look at the rainbows in  $C_2$ . Because  $C$  has two rainbows in  $C_1 - V(P)$  and any rainbow in  $V(C_2) \setminus V(P)$  for  $C_2$  is rainbow for  $C$ ,  $C_2$  has no rainbow in  $V(C_2) \setminus V(P)$ . All the interior vertices of  $P$  are reflecting in  $C_2$ , so  $C_2$  has at most two rainbows. This contradicts that  $C_2$  is not an obstruction.

Secondly, suppose that  $P$  is drawn in the exterior of  $C$ . Let  $C_{out}$  be the cycle bounding the outer face of  $C \cup P$ . The cycle  $C_{out}$  is the union of  $P$  and one of the two  $xy$ -paths in  $C$ , and, in both cases, as  $x \in \delta(C) \setminus \delta(C_{out})$  and  $P - y \subset \sigma$ ,  $|\delta(C_{out})| < |\delta(C)|$ . Every vertex in  $P - y$  is reflecting in  $C_{out}$  (this statement follows from the fact that the rotation of a vertex inside a cycle also includes the edges of the cycle incident with the vertex). Moreover, every vertex in  $V(C_{out}) \setminus (V(P - y))$  that is a rainbow in  $C_{out}$  is also a rainbow in  $C$ . These two facts imply that  $C_{out}$  has at most as many rainbows as  $C$ ; hence  $C_{out}$  is an obstruction. This contradicts the fact that  $C$  minimizes  $|\delta|$ .  $\square$

Next we show that, if a set of strings contains an obstruction, then it is not pseudolinear.

**Observation 2.2.2.** *If  $\Sigma$  is a set of strings in general position and  $G(\Sigma)$  has an obstruction, then  $\Sigma$  cannot be extended to an arrangement of pseudolines.*

*Proof.* By way of contradiction, suppose that there is a set of strings  $\Sigma$  that can be extended to an arrangement of pseudolines and  $G(\Sigma)$  has an obstruction  $C$ . Consider an extension of  $\Sigma$  to an arrangement of pseudolines, and then cut off the two infinite ends of each pseudoline to obtain a set of strings  $\Sigma'$  extending  $\Sigma$ , and in which every two strings in  $\Sigma'$  cross. In  $G(\Sigma')$ , there is a cycle  $C'$  that represents the same simple closed curve as  $C$ . Because  $C'$  is obtained from subdividing some edges of  $C$ ,  $C'$  has fewer than three rainbows. Therefore, we may assume that  $\Sigma = \Sigma'$  and  $C = C'$ . Now, the ends of every string in  $\Sigma$  are degree-one vertices in the outer face of  $G(\Sigma)$ .

As every string in  $\Sigma$  is simple, and no two strings intersect more than once,  $|\delta(C)| \geq 3$ . We will assume that  $C$  is chosen to minimize  $|\delta(C)|$ .

Since  $C$  is an obstruction, there is at least one vertex  $x \in \delta(C)$  reflecting inside  $C$ . Let  $e \in E(C)$  be an edge incident to  $x$ , and suppose that  $\sigma$  is the string including  $e$ . Traversing  $\sigma$  along  $e$  through  $x$ , we encounter another edge  $e' \subseteq \sigma$  incident to  $x$ . Because  $x \in \delta(C)$ ,  $e'$  is not in  $C$ . Suppose that  $e'$  is drawn in the outer face of  $C$ . As  $x$  is reflecting inside  $C$ , there exists a string  $\bar{\sigma}$  that includes two edges in the rotation at  $x$  inside  $C$ . However,  $\sigma$  and  $\bar{\sigma}$  tangentially intersect at  $x$ , contradicting that the strings in  $\Sigma$  are in general position. Therefore  $e'$  is drawn inside  $C$ .

Let  $y$  be the end of  $\sigma$  contained in the component of  $\sigma \setminus e$  containing  $x$ . Since  $|\delta(C)|$  is minimum, Observation 2.2.1 implies that the component of  $\sigma \setminus e$  having  $x$  and  $y$  as ends have all its points, with the exception of  $x$ , in the inner face of  $C$ . However,  $y$  is drawn in the inner face of  $C$ , contradicting that the ends of all the strings in  $\Sigma$  are incident with the outer face of  $G(\Sigma)$ .  $\square$

## 2.2.2 Extending sets of strings with no obstructions

In this subsection we prove that a set of strings with no obstructions can be extended to an arrangement of pseudolines. We restate Theorem 2.1.1 using our new terminology.

**Theorem 2.2.3.** *A set of strings  $\Sigma$  in general position can be extended to an arrangement of pseudolines if and only if  $G(\Sigma)$  has no obstructions.*

*Proof.* We showed in Observation 2.2.2 that if  $G(\Sigma)$  has an obstruction, then  $\Sigma$  cannot be extended to an arrangement of pseudolines. For the converse, suppose that  $G(\Sigma)$  has no obstructions.

We start by reducing the proof to the case in which the point set  $\bigcup \Sigma$  is connected. If  $\bigcup \Sigma$  is not connected, then we add a simple string to  $\Sigma$ , connecting two points in distinct components of  $G(\Sigma)$ , and so that it is included inside a face of  $G(\Sigma)$ . This operation: reduces the number of components; does not create obstructions; and ensures that any pseudolinear extension of the new set of strings shows the existence of one for  $\Sigma$ . We continue adding strings in this way until we obtain a connected set of strings and we redefine  $\Sigma$  to be this set. Thus, we may assume  $\bigcup \Sigma$  is connected.

Our proof is algorithmic, and consists of repeatedly applying one of the three steps described below.

- **Disentangling Step.** If a string  $\sigma \in \Sigma$  has an end  $a$  with degree at least 2 in  $G(\Sigma)$ , then we slightly extend the  $a$ -end of  $\sigma$  into one of the faces incident with  $a$ .

- **Face-Escaping Step.** If a string  $\sigma \in \Sigma$  has an end  $a$  with degree 1 in  $G(\Sigma)$ , and is incident with an inner face, then we extend the  $a$ -end of  $\sigma$  until we intersect some point in the boundary of this face.
- **Exterior-Meeting Step.** Assuming that all the strings in  $\Sigma$  have their two ends in the outer face and these ends have degree 1 in  $G(\Sigma)$ , we extend the ends of two disjoint strings so that they meet in the outer face.

We can always perform at least one of these steps, unless the strings are pairwise intersecting and all of them have their ends in the outer face (in this case we extend their ends to infinity to obtain the desired arrangement of pseudolines). Each step increases the number of pairwise intersecting strings. Henceforth, our aim is to show that, as long as there is a pair of non-intersecting strings, then one of these three steps may be performed without adding an obstruction. The proof is now divided into three parts that can be read independently.

**Disentangling Step.** *Suppose that  $\sigma \in \Sigma$  has an end  $a$  with degree at least 2 in  $G(\Sigma)$ . Then we can extend the  $a$ -end of  $\sigma$  into one of the faces incident to  $a$  without creating an obstruction.*

*Proof.* An edge  $f$  of  $G(\Sigma)$  incident with  $a$  is a *twin* if there exists another edge  $f' \neq f$  incident with  $a$  such that both  $f$  and  $f'$  are part of the same string in  $\Sigma$ . Observe that the edge  $e_0 \subseteq \sigma$  incident with  $a$  is not a twin.

The fact no pair of strings tangentially intersect at  $a$  tells us that if  $(f_1, f'_1)$  and  $(f_2, f'_2)$  are pairs of corresponding twins, then  $f_1, f_2, f'_1, f'_2$  occur in this cyclic order for either the clockwise or counterclockwise rotation at  $a$ . Thus, we may assume that the twins at  $a$  are labeled as  $f_1, \dots, f_t, f'_1, \dots, f'_t$ , and that this is their counterclockwise order occurrence when we follow the rotation at  $a$  starting at  $e_0$ . In such a case,  $(f_i, f'_i)$  is a pair of corresponding twins for  $i = 1, \dots, t$ .

In order to avoid tangential intersections when twins are present, every valid extension of  $\sigma$  at  $a$  must cross into the angle between  $f_t$  and  $f'_1$  not containing  $e_0$ .

Let  $(e_1, \dots, e_k)$  be the list of non-twin edges between  $f_t$  and  $f'_1$  in the counterclockwise rotation at  $a$ ; this list might be empty. In the case there are no twins, we set  $f_t$  and  $f'_1$  both equal to  $e_0$ , so  $(e_1, \dots, e_k)$  is all the edges incident with  $a$  other than  $e_0$ .

We consider all the feasible extensions for  $\sigma$ : for each  $i \in \{1, \dots, k-1\}$ , we let  $\Sigma_i$  be the set of strings obtained from extending  $\sigma$  by adding a small bit of arc  $\alpha_i$  starting at  $a$ , and continuing into the face between  $e_i$  and  $e_{i+1}$ . Let  $\Sigma_0$  be the set of strings obtained by

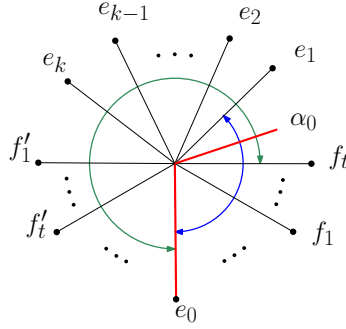


Figure 2.5: Substrings included in the disk bounded by  $C_0$ .

adding an arc  $\alpha_0$  in the face between  $f_t$  and  $e_1$ , and let  $\Sigma_k$  be obtained by adding an arc  $\alpha_k$  in the face between  $e_k$  and  $f'_1$ .

Seeking a contradiction, suppose that, for each  $i \in \{0, \dots, k\}$ ,  $G(\Sigma_i)$  contains an obstruction  $C_i$ . The cycle  $C_i$  does not include the bit of arc  $\alpha_i$  as an edge, so  $C_i$  is a cycle in  $G(\Sigma)$ . This cycle is not an obstruction in  $G(\Sigma)$ , although it becomes one when we add  $\alpha_i$ . The reason explaining this conversion is simple: in  $G(\Sigma)$ ,  $C_i$  has exactly three vertices not reflecting, and one of them is  $a$ . After  $\alpha_i$  is added,  $a$  is now reflecting in  $C_i$  (witnessed by  $\sigma$ ).

Understanding how cycles with exactly three rainbows may behave in an obstructionless set of strings is a crucial piece of the proof. In general, if  $v$  is a vertex in the underlying plane graph of a set of strings in general position, then a *near-obstruction at  $v$*  is a cycle with exactly three rainbows, and one of them is  $v$ . Each of the cycles  $C_0, C_1, \dots, C_k$  above is a near-obstruction at  $a$  in  $G(\Sigma)$ .

Both  $e_0$  and  $\alpha_0$  are on the disk bounded by  $C_0$ , and since  $\alpha_0$  is not part of  $C_0$ , either  $e_0, f_1, f_2, \dots, f_t, e_1$  are on the same side of  $C_0$  (blue bidirectional arrow in Figure 2.5) or all of  $f_t, e_1, \dots, e_k, f'_1, f'_2, \dots, f'_1, e_0$  are in the same side of  $C_0$  (green bidirectional arrow in Figure 2.5). Because  $\alpha_0$  is the only edge between  $f_t$  and  $e_1$ , we see that  $e_1$  belongs to the first sublist (the blue one) and  $f_t$  belongs to the second list (the green one). In the second case both  $f_t$  and  $f'_t$  are in the disk bounded by  $C_0$ , showing that  $a$  is not a rainbow for  $C_0$  in  $\Sigma$ . Therefore, all of  $e_0, f_1, f_2, \dots, f_t, e_1$  are in the disk bounded by  $C_0$ .

Regardless of the presence or absence of twins, we know that  $(e_0, e_1)$  occurs as a sublist of the rotation of  $a$  inside  $C_0$ . A symmetric argument shows that  $(e_k, e_0)$  occurs as a sublist of the rotation of  $a$  inside  $C_k$ .

Since  $(e_0, e_1)$  is a substring of the rotation at  $a$  inside  $C_0$  and  $(e_0, e_1, \dots, e_k, f'_1)$  is not inside  $C_k$ , there is a largest  $i \in \{0, 1, \dots, k-1\}$  such that  $(e_0, \dots, e_{i+1})$  is inside  $C_i$ . The



choice of  $i$  implies  $(e_{i+1}, \dots, e_k, e_0)$  is inside  $C_{i+1}$ .

The next lemma states that the existence of such a pair of cycles  $C_i$  and  $C_{i+1}$  is impossible, completing the proof.  $\square$

**Lemma 2.2.4.** *Let  $\Sigma$  be a set of strings in general position. Suppose that  $C_1$  and  $C_2$  are cycles in  $G(\Sigma)$  that are near-obstructions at  $v$ , so that the rotation at  $v$  inside  $C_1$  includes (as a sublist) the rotation at  $v$  outside  $C_2$ , and that the rotation at  $v$  inside  $C_2$  includes the rotation at  $v$  outside  $C_1$ . Then  $G(\Sigma)$  has an obstruction.*

We defer the proof of Lemma 2.2.4 to Section 2.3 as it is technical and it deviates our attention from the proof of Theorem 2.1.1.

**Face-Escaping Step.** *Suppose that there is a string  $\sigma$  that has an end  $a$  with degree 1 in  $G(\Sigma)$ , and  $a$  is incident to an inner face  $F$ . Then there is an extension  $\sigma'$  of  $\sigma$  from its  $a$ -end to a point in the boundary of  $F$  such that the set  $(\Sigma \setminus \{\sigma\}) \cup \{\sigma'\}$  has no obstruction.*

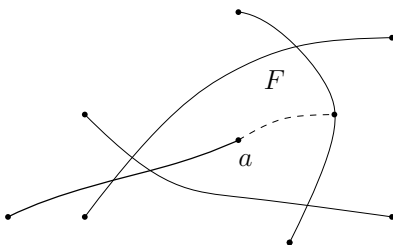


Figure 2.6: Face-Escaping Step.

*Proof.* Let  $W$  be the closed boundary walk  $(x_0, e_1, \dots, e_n, x_n)$  of  $F$  such that  $x_0 = x_n = a$  and  $F$  is to the left as we traverse  $W$ .

Sometimes it is only possible to extend  $\sigma$  by using an arc connecting  $a$  to a vertex in the middle of an edge. Figure 2.6 shows an example of this situation. Joining  $a$  to any  $x_i$  produces a second crossing between two strings; the simple closed curves enclosed by the union of these two strings is an obstruction. So for each edge  $e_i$ , let  $m_i$  be a point in  $e_i$ , between  $x_{i-1}$  and  $x_i$ .

Let  $P$  denote the list of points  $(m_1, x_1, m_2, x_2, \dots, x_{n-1}, m_n)$ . For each point  $p$  in  $P$ , let  $\alpha_p$  denote an arc in  $F$  connecting  $a$  to  $p$ . (A given element of the point set of  $G(\Sigma)$  might occur more than once in  $P$ . In particular, it is always the case that  $x_1 = x_{n-1}$ ; see Figure 2.7.)

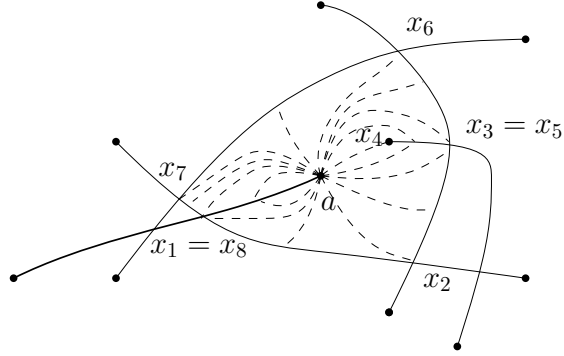


Figure 2.7: All possible extensions in the Face-Escaping Step.

For each point  $p$  in  $P$ , we consider the set of strings  $\Sigma_p$  obtained from  $\Sigma$  by extending  $\sigma$  using  $\alpha_p$ . Let  $f_p$  be the edge  $e_1 \cup \alpha_p$  in  $G(\Sigma_p)$ ; it has ends  $x_1$  and  $p$ . Also, let  $\sigma^p = \sigma \cup \alpha_p$ .

The existence of obstructions in  $G(\Sigma_p)$  is independent of how we draw  $\alpha_p$  inside  $F$ . We will take advantage of this fact later on in the proof.

Seeking a contradiction, suppose that each  $G(\Sigma_p)$  has an obstruction. Our next claim gives two sufficient conditions on  $p$  that imply that all the obstructions in  $G(\Sigma_p)$  contain  $f_p$ .

**Claim 1.** *Let  $p \in P$  be either one of  $m_1, \dots, m_n$  or not in  $\sigma$ . Then every obstruction in  $G(\Sigma_p)$  includes  $f_p$ .*

*Proof.* Let  $p \in P$  be such that there is an obstruction  $C$  in  $G(\Sigma_p)$  not including  $f_p$ .

First, we show that  $p$  is not a vertex in the middle of an edge in  $W$ . By contradiction, suppose that  $p = m_i$  for some  $i \in \{1, \dots, n\}$ . Since  $m_i$  is the only vertex whose rotation in  $G(\Sigma)$  differs from its rotation in  $G(\Sigma_{m_i})$ ,  $m_i \in V(C)$ . Consider the cycle  $C'$  of  $G(\Sigma)$  obtained by replacing the subpath  $x_{i-1}, m_i, x_i$  of  $C$  by the edge  $x_{i-1}x_i$ . The inside rotation of each vertex in  $C'$  is the same as their rotation inside  $C$ . This shows that  $C'$  is an obstruction in  $G(\Sigma)$ , a contradiction.

Now suppose that  $p$  is not in the middle of an edge in  $W$ . Then  $C$  is a cycle in  $G(\Sigma)$  and is not an obstruction in  $G(\Sigma)$ . The only vertex in  $G(\Sigma_p)$  that has a rotation that is different from its rotation in  $G(\Sigma)$  is  $p$ . Therefore  $p$  is a point in  $C$  that is reflecting inside  $C$  (witnessed by two edges included in  $\sigma^p$ ), and is not reflecting in  $C$  with respect to  $G(\Sigma)$ . Exactly one of the two witnessing edges is in  $G(\Sigma)$ . So  $p \in \sigma$ .  $\square$

More can be said about the obstructions in  $G(\Sigma_p)$  for each point in  $P$ , but for this we need some terminology. If we orient an edge  $e$  in a plane graph, then the *sides* of  $e$  are

either the points near  $e$  that are to the right of  $e$ , or the points near  $e$  to the left of  $e$ . Our next lemma shows that if  $p \in P$ , then all the obstructions in  $G(\Sigma_p)$  include the same side of  $f_p$  in its interior face. We defer its proof to Section 2.3 to keep the flow of the current proof. For the convenience of the reader, we provide all the hypotheses in the statement.

**Lemma 2.2.5.** *Let  $\Sigma$  be a set of strings in general position. Let  $C_1$  and  $C_2$  be obstructions in  $G(\Sigma)$  with  $e \in E(C_1) \cap E(C_2)$ . If  $C_1$  and  $C_2$  include distinct sides of  $e$  in their interior faces, then  $G(\Sigma)$  has an obstruction not including  $e$ .*

The condition on the two cycles  $C_1$  and  $C_2$  containing distinct sides of  $e$  implies that  $e$  is incident with only interior faces of  $C_1 \cup C_2$ . The perspective of the cycles being on distinct sides of  $e$  is useful in the application, but what we really use in the proof of Lemma 2.2.5 is that  $e$  is not incident with the outer face of  $C_1 \cup C_2$ .

For each point  $p \in P$ , we will consider an obstruction  $C_p$  containing  $f_p$ ; the choice of  $C_p$  will be more specific when  $p \in \sigma$  (see below). It is important to know what side of  $f_p$  our obstruction  $C_p$  includes in its interior, so from now on we will assume  $f_p$  is oriented from  $x_1$  to  $p$ . If  $p \notin \sigma$ , then we let  $C_p$  be any obstruction in  $G(\Sigma_p)$  (Claim 1 guarantees that  $f_p \in E(C_p)$ ). In case  $p \in \sigma$ , we choose  $C_p$  to be the unique cycle included in the drawing of  $\sigma^p$ . Note that for  $p = x_1$ , the interior of  $C_p$  includes the right of  $f_p$ , while for  $p = x_{n-1}$ , the interior of  $C_p$  includes the left of  $f_p$  (here we use the fact that the face  $F$  is bounded).

Our last observation implies that there are two consecutive vertices  $x_{i-1}, x_i$  in  $W - a$  such that the interior of  $C_{x_{i-1}}$  includes the right of  $f_{x_{i-1}}$  and the interior of  $C_{x_i}$  includes the left of  $f_{x_i}$ .

Without loss of generality, suppose that the interior of  $C_{m_i}$  includes the left of  $f_{m_i}$  (otherwise we reflect our drawing in a mirror). To make the notation simpler, we let  $x = x_{i-1}$  and  $m = m_i$ . We may assume that  $f_m$  is drawn near the left of  $f_x$ .

The next claim is the last ingredient to obtain a final contradiction.

**Claim 2.** *Exactly one of the following holds:*

- (a)  $x \in \sigma$  and  $G(\Sigma_m)$  has an obstruction containing  $f_m$  whose interior includes a side that is distinct from the side included by  $C_m$ ; or
- (b)  $x \notin \sigma$  and  $G(\Sigma_x)$  has an obstruction containing  $f_x$  whose interior includes a side of  $f_x$  that is distinct from the side included by  $C_x$ .

*Proof.* First, suppose that  $x \in \sigma$ . For (2.a) we have two cases depending on whether  $x_{i-1}x_i$  is an edge in  $C_x$ .

**Case a.1**  $x_{i-1}x_i$  is not in  $C_x$ .

In this case we consider the cycle  $C'_m$  obtained by replacing in  $C_x$  the edge  $f_x$  by the path  $P = (x_1, f_m, m, mx, x)$ . Since  $x \in \sigma$ , by the choice of  $C_x$ , all the edges in  $C_x$  are in  $\sigma^x$ . Therefore all the edges in  $C'_m$ , with the possible exception of  $mx$ , are in  $\sigma^m$ . Thus  $C'_m$  is an obstruction in  $G(\Sigma_m)$ .

It remains to show that the interior of  $C'_m$  includes the right side of  $f_m$ . Note that  $C_x \cup P$  consists of three internally disjoint  $x_1x$ -paths, and because some points in  $P$  are near the left side of  $f_x$ ,  $P$  is in the outer face of  $C_x$ . The face of  $f_x \cup P$  that is to the right of  $f_m$  is included in the inner face  $F$ , so it is bounded. This implies the interior face of  $C'_m$  includes the right of  $f_m$ . Since the interior of  $C_m$  includes the left of  $f_m$ ,  $C'_m$  and  $C_m$  are obstructions including distinct sides of  $f_m$ .

**Case a.2.**  $x_{i-1}x_i$  is in  $C_x$ .

In this case,  $(x_1, f_x, x, xx_i, x_i)$  is a subpath of  $C_x$ . We let  $C'_m$  be the cycle obtained by replacing this path by  $P = (x_1, f_m, m, mx_i, x_i)$ . Since  $x \in \sigma$ , the way we choose  $C_x$  implies that all the edges in  $C_x$  are in  $\sigma^x$ . So all the edges in  $C'_m$  are in  $\sigma^m$ , and  $C'_m$  is an obstruction. An argument similar to the one given in the previous case shows that the interior of  $C'_m$  includes the right side of  $f_m$ . Thus the interior of  $C_m$  and  $C'_m$  include distinct sides of  $f_m$ .

Turning to (2.b), let us suppose that  $p \notin \sigma$ . We split the proof into two cases depending on whether  $x$  is in  $C_m$ .

**Case b.1.**  $x$  is in  $C_m$ .

First, we redraw  $f_x$  and  $f_m$  inside  $F$  so that  $f_x \cap f_m = \{x_1\}$ . Let  $T$  be the triangle bounded by  $f_x$ ,  $f_m$  and  $xm$ . The interior face of  $T$  is to the left of  $f_x$  and to the right of  $f_m$ . Consider the  $mx$ -path  $P$  of  $C_m$  that does not include the edge  $f_m$ . Since the interior face of  $T$  is a subset of  $F$ ,  $P$  is drawn in the closure of the exterior of  $T$  (possibly  $P = (m, mx, x)$ ).

Let  $C$  be the simple closed curve bounded by  $P \cup f_x \cup f_m$ . We claim that the interior of  $C$  is on the left of  $f_x$ . In the alternative, suppose that the interior of  $C$  is on the right of  $f_x$ . Then  $C' = P + xm$  is a cycle of  $G(\Sigma_m)$  including  $f_x$  and  $f_m$  in its interior. The  $xx_1$ -path  $P'$  of  $C_m$  that does not include  $m$ , is an arc connecting  $x_1$  to  $x$  inside  $C'$ . Thus,  $V(C') \subseteq V(C_m)$  and the closed disk bounded by  $C'$  includes  $C_m$ . These two observations together imply that  $C'$  has at most as many rainbows as  $C_m$ , and hence,  $C'$  is an obstruction

of  $G(\Sigma_m)$  not including  $f_m$ . Claim 1 asserts that all the obstructions in  $G(\Sigma_m)$  include  $f_m$ , a contradiction. Thus the interior of  $C$  is on the left of  $f_m$ .

From our last observation, it follows that  $P'$  is an arc connecting  $x_1$  and  $x$  in the exterior of  $C$ . Because the interior of  $C_m = P' \cup f_m \cup P$  is on the left of  $f_m$ , the interior of the cycle  $C'_x = P' + f_x$  is on the left of  $f_x$ .

Now we show that  $C'_x$  is an obstruction. Note that  $V(C'_x) \subseteq V(C_m)$  and that the closed disk bounded by  $C'_x$  includes  $C_m$ . Then, every rainbow in  $C'_x$  is a rainbow in  $C_m$ , and hence  $C'_x$  is an obstruction. The cycles  $C_x$  and  $C'_x$  are obstructions including distinct sides of  $f_x$  in their interiors, as claimed.

**Case b.2.**  $x$  is not in  $C_m$ .

In this case we let  $C'_x$  be the cycle obtained by replacing the path  $(x_1, f_m, m, mx_i, x_i)$  in  $C_m$  by the path  $P = (x_1, f_x, x, xx_i, x_i)$  in  $G(\Sigma_x)$ . Let  $\alpha$  be the subarc of  $P$  joining  $x_1$  to  $m$ . As the points of  $\alpha$  near  $x_1$  are drawn on the left of  $f_m$ , and  $\alpha$  is internally disjoint to  $C_m$ ,  $\alpha$  connects  $x_1$  and  $m$  in the exterior of  $C_m$ . Since the interior face of  $\alpha \cup f_m$  is on the left of  $f_x$ , the interior face of  $C'_x$  is on the left of  $f_x$ .

To show that  $C'_x$  is an obstruction, note that the disk bounded by  $C'_x$  includes  $C_m$  and that  $V(C'_x) \setminus \{x\} \subseteq V(C_m)$ . Thus all the rainbows of  $C'_x$  in  $V(C'_x) \setminus \{x\}$  are also rainbows in  $C_m$ . The rotation of  $x$  inside  $C'_x$  is the list  $(xx_i, f_x)$ , and, because  $x \notin \sigma$ ,  $x$  is a rainbow in  $C'_x$ , and is not a vertex of  $C_m$ . To compensate, we note that  $m$  is a rainbow in  $C_m$  that is not in  $V(C_x)$ : if  $m$  is not rainbow, both  $f_m$  and  $xx_i$  are included in  $\sigma$ , implying that  $x \in \sigma$ . This shows that  $C'_x$  has at most as many rainbows as  $C_m$ . Thus  $C'_x$  is an obstruction. Again, the interiors of  $C_x$  and  $C'_x$  include distinct sides of  $f_x$ .  $\square$

By Claim 2, for some  $p \in \{x, m\}$ ,  $G(\Sigma_p)$  has obstructions including both sides of  $f_p$  (and when  $p = x$ , we can guarantee that  $p \notin \sigma$ ). Lemma 2.2.5 implies that  $G(\Sigma_p)$  has an obstruction not including  $f_p$ . Since either  $p \notin \sigma$  or  $p = m$ , this last statement contradicts Claim 1.  $\square$

**Exterior-Meeting Step.** *Suppose that all the strings in  $\Sigma$  have their ends on the outer face of  $G(\Sigma)$  and that all the ends have degree 1 in  $G(\Sigma)$ . Then either all the strings are pairwise intersecting, and then  $\Sigma$  can be extended to an arrangement of pseudolines, or we can extend two disjoint strings so that these strings intersect without creating an obstruction.*

*Proof.* We start by considering a simple closed curve  $\mathcal{O}$  containing all the ends of the strings in  $\Sigma$ , and that is otherwise disjoint from  $\bigcup \Sigma$ . We construct this curve by connecting each pair of vertices with degree 1 that are consecutive in the boundary walk of the outer face. To connect these pairs we use an arc whose interior is included in the outer face, near the portion of the boundary walk between the two vertices.

Suppose  $\sigma_1, \sigma_2$  are two disjoint strings in  $\Sigma$ . For  $i = 1, 2$ , let  $a_i, b_i$  be the ends of  $\sigma_i$ . Since  $\sigma_1$  and  $\sigma_2$  do not intersect inside  $\mathcal{O}$ , their ends do not alternate as we traverse  $\mathcal{O}$  in counterclockwise order. We may assume, by relabeling if necessary, that the ends occur in the order  $a_1, b_1, b_2, a_2$ .

We extend the  $a_i$ -ends of  $\sigma_1$  and  $\sigma_2$  so that they meet in a point  $p$  in the outer face. We do this extension so that the two added segments are in the outer face, and, more importantly, so that the interior face of the simple closed curve bounded by the added segments and the  $a_2a_1$ -arc in  $\mathcal{O}$  not containing  $\{b_1, b_2\}$ , does not include the inner face of  $\mathcal{O}$ . In Figure 2.8 we show the right and wrong way to extend, respectively.

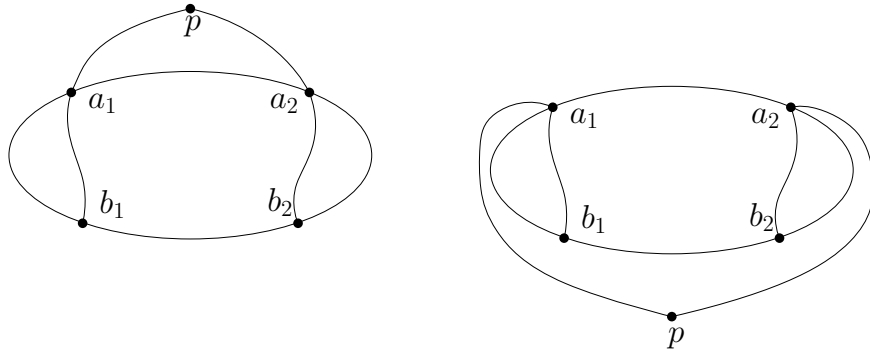


Figure 2.8: The right and wrong way to extend in the Exterior-Meeting Step.

We denote the new set of strings obtained as above by  $\Sigma'$ . To show that  $\Sigma'$  has no obstruction, we consider a cycle  $C$  in  $G(\Sigma')$ . If  $C$  does not contain  $p$ , then  $C$  is a cycle in  $G(\Sigma)$ , and so is not an obstruction in  $G(\Sigma')$ . Now suppose that  $p$  is in  $C$ .

The idea is to find three rainbows in  $C$ . To get the first one, we consider the path  $P_1$  obtained by traversing  $C$ , starting at  $p$ , continuing along the path induced by  $\sigma_1$ , and stopping just before we reach a first vertex not in  $\sigma_1$ . Let  $c_1$  be the last vertex in  $P_1$ , and let  $d_1$  be the neighbour of  $c_1$  in  $C$  that is not in  $P_1$ .

**Claim 1.** *The cycle  $C$  has a rainbow included in the disk  $\Delta_1$  bounded by  $\sigma_1$  and the  $a_1b_1$ -arc of  $\mathcal{O}$  not containing  $a_2$ .*

*Proof.* The vertex  $d_1$  is in one of the two bounded faces of  $\mathcal{O} \cup \sigma_1$ . Suppose that  $d_1$  is in the face  $F$  that is bounded by  $\sigma_1$  and the  $a_1b_1$ -arc of  $\mathcal{O}$  containing  $a_2$  and  $b_2$ . The rotation at  $c_1$  inside  $C$  does not include two edges in the same string  $\sigma$ , as otherwise  $\sigma$  and  $\sigma_1$  tangentially intersect at  $c_1$ . Therefore, when  $d_1 \in F$ ,  $c_1$  is a rainbow of  $C$  in  $\Delta_1$ .

Now suppose that  $d_1$  is in  $\Delta_1$ . Let  $P'_1$  be the path of  $C$  starting at  $c_1$  and the edge  $c_1d_1$ , and ending at the first vertex we encounter that is in  $\sigma_1$ . The cycle  $C'$  enclosed by  $P'_1$  and  $\sigma_1$  is not an obstruction, so it has at least three rainbows. The vertices in  $C' - V(P'_1)$  are reflecting inside  $C'$  because their rotations inside  $C'$  contain two edges in  $\sigma$ . Hence at least one internal vertex of  $P'_1$  is a rainbow in  $C'$ . This vertex is also a rainbow in  $C$ , and is included in  $\Delta_1$ .  $\square$

Considering  $\sigma_2$  instead of  $\sigma_1$ , Claim 1 yields a second rainbow in  $C$  inside an analogous disk  $\Delta_2$ . The third rainbow is  $p$ , showing that  $C$  is not an obstruction.  $\square$

Since the Disentangling Step, Face-Escaping Step and Exterior-Meeting Step can be performed without creating new obstructions, either: one of these steps can be performed to increase the number of pairwise intersecting strings in  $\Sigma$ ; or the strings in  $\Sigma$  are pairwise intersecting and all of them have their ends in the outer face, which implies that  $\Sigma$  can be extended to an arrangement of pseudolines.  $\square$

## 2.3 Proof of Lemmas 2.2.4 and 2.2.5

We deferred the proofs of Lemmas 2.2.4 and 2.2.5, both essential in the proof of Theorem 2.1.1, to this section.

Our next observation follows immediately from the definition of rainbow, and it will be repeatedly used in the next proofs.

**Useful Fact.** *Let  $\Sigma$  be set of strings in general position. Let  $v$  be a vertex that is in both the cycles  $C$  and  $C'$  of  $G(\Sigma)$  such that the rotation at  $v$  inside  $C$  includes the rotation at  $v$  inside  $C'$ . If  $v$  is a rainbow in  $C$ , then  $v$  is a rainbow in  $C'$ .*

Recall that a *near-obstruction* at  $v$  is a cycle  $C$  (in the underlying graph of a set of strings) that has precisely three rainbows, one of which is  $v$ . In Figure 2.9, we depict (up to symmetries) how two near-obstructions may intersect at  $v$ . In each of the nine diagrams,  $v$  is represented as a black dot, while the interiors of the near-obstructions are represented as dotted and dashed lines. In our next lemma, we will consider two near-obstructions at

$v$  that intersect only as in the last three diagrams, where every small open disk centered at  $v$  is included in the union of the disks bounded by the two near-obstructions. In the statement, an equivalent description is given in terms of the local rotation at  $v$ .

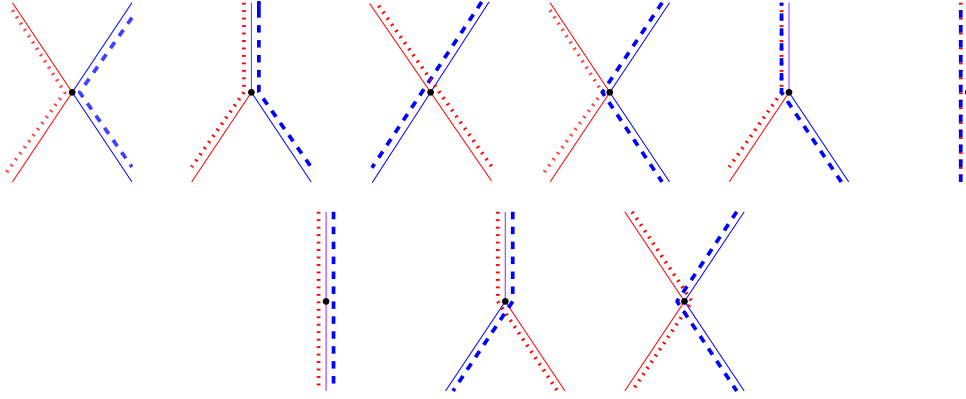


Figure 2.9: Two near obstructions at  $v$ .

**Lemma 2.2.4.** *Let  $\Sigma$  be a set of strings in general position. Suppose that  $C_1$  and  $C_2$  are cycles in  $G(\Sigma)$  that are near-obstructions at  $v$ , so that the rotation at  $v$  inside  $C_1$  includes (as a sublist) the rotation at  $v$  outside  $C_2$ , and that the rotation at  $v$  inside  $C_2$  includes the rotation at  $v$  outside  $C_1$ . Then  $G(\Sigma)$  has an obstruction.*

*Proof.* In order to obtain a contradiction, suppose that  $G(\Sigma)$  has no obstructions and that it contains such cycles  $C_1, C_2$ . The conditions on the rotation at  $v$  imply that every edge incident with  $v$  is in the interior of either  $C_1$  or  $C_2$ . Thus,  $v$  is not incident with the outer face of  $C_1 \cup C_2$ .

Our next goal is to show that  $C_1 \cap C_2$  has at least two vertices. If  $e$  is an edge of  $C_1$  incident with  $v$ , then either  $e$  is an edge of  $C_2$  or  $e$  is inside  $C_2$ . In the former case  $|V(C_1) \cap V(C_2)| \geq 2$ , thus we may assume that both edges of  $C_1$  incident with  $v$  are inside  $C_2$ . If  $v$  is the only vertex in  $V(C_1) \cap V(C_2)$ , then  $C_1 - v$  is in the interior of  $C_2$ , and hence the edges of  $C_2$  incident with  $v$  are not in the rotation at  $v$  inside  $C_1$ , a contradiction. Thus  $|V(C_1) \cap V(C_2)| \geq 2$ . It follows that  $C_1 \cup C_2$  is 2-connected; in particular, its outer face is bounded by a cycle  $C_{out}$ .

The Useful Fact applied to  $C = C_{out}$  and to each  $C' \in \{C_1, C_2\}$ , shows that every vertex that is a rainbow in  $C_{out}$  is also a rainbow in each of the cycles in  $\{C_1, C_2\}$  containing it. By assumption,  $C_{out}$  is not an obstruction, so it has at least three rainbows. The preceding two sentences imply that we may choose the labelling such that two of them, say  $p$  and  $q$ ,



are also rainbows in  $C_1$ . Neither  $p$  nor  $q$  is  $v$  and  $C_1$  is a near-obstruction. Thus,  $p$  and  $q$  are the only rainbows of  $C_{out}$  that are in  $C_1$ .

Since  $v \notin V(C_{out})$ ,  $C_1$  has a subpath  $P_v$  containing  $v$  in which only the ends of  $P_v$  are in  $C_{out}$ . Since  $v$  is not in the outer face of  $C_{out}$ ,  $P_v$  is included in the inner face of  $C_{out}$ . We let  $u$  and  $w$  be the ends of  $P_v$ , and let  $Q_{out}^1, Q_{out}^2$  be the  $uw$ -paths of  $C_{out}$ . The cycle  $C_1$  is inside one of the two disks bounded by  $P_v$  and one of  $Q_{out}^1$  and  $Q_{out}^2$ . By symmetry, we may assume that  $C_1$  is included in the disk bounded by  $Q_{out}^1 \cup P_v$ . In this case  $Q_{out}^2$  is a subpath of  $C_2$ .

Our desired contradiction will be obtained by finding three rainbows in  $C_2$  distinct from  $v$ . The first is relatively easy to find: if  $C_1 - (P_v)$  is the  $uw$  path in  $C_1$  distinct from  $P_v$ , we consider the cycle  $(C_1 - (P_v)) \cup Q_{out}^2$ . The disk bounded by  $(C_1 - (P_v)) \cup Q_{out}^2$  contains the one bounded by  $C_1$ . Then the Useful Fact applied to  $C = (C_1 - (P_v)) \cup Q_{out}^2$  and  $C' = C_1$ , implies that each vertex in  $C_1 - (P_v)$  that is rainbow in  $(C_1 - (P_v)) \cup Q_{out}^2$  is also rainbow in  $C_1$ . Since  $C_1$  has at most two rainbows in  $C_1 - (P_v)$ , namely  $p$  and  $q$ ,  $(C_1 - (P_v)) \cup Q_{out}^2$  must have a third rainbow  $r_1$  in the interior of  $Q_{out}^2$ . The interiors of the disks bounded by  $C_2$  and  $(C_1 - (P_v)) \cup Q_{out}^2$  are on the same side of  $Q_{out}^2$ ; thus  $r_1$  is a rainbow for  $C_2$ .

To find another rainbow in  $C_2$ , consider the edge  $e_u$  of  $C_2$  incident to  $u$  and not in  $Q_{out}^2$ . We claim that either  $u$  is a rainbow in  $C_2$  or that  $e_u$  is not included in the closed disk bounded by  $P_v \cup Q_{out}^2$ . Looking for a contradiction, suppose that  $u$  is reflecting in  $C_2$  and that  $e_u$  is included in the disk. Then we can find two edges in the rotation at  $u$ , included in the disk bounded by  $P_v \cup Q_{out}^2$ , that belong to the same string  $\sigma$ . The vertex  $u$  is not reflecting in  $C_1$ , as else, we would find another pair of edges in the rotation at  $u$  inside  $Q_{out}^1 \cup P_v$ , and included in a different string  $\sigma'$ ; in this case,  $\sigma$  and  $\sigma'$  tangentially intersect at  $u$ , a contradiction. Therefore  $u$  is a rainbow in  $C_1$ , so  $u$  is one of  $p$  and  $q$ . This implies that  $u$  is a rainbow in  $C_{out}$ , and hence, a rainbow in  $C_2$ , a contradiction.

If  $u$  is a rainbow in  $C_2$ , then this is the desired second one. Otherwise, the preceding paragraph shows that  $e_u$  is not in the closed disk bounded by  $P_v \cup Q_{out}^2$ . In this latter case,  $e_u$  is in a path  $P_u$  that starts at  $u$ , ends at  $u'$  in  $P_v$  and is otherwise disjoint from  $P_v$ .

Note that  $u' \neq w$ , as otherwise  $C_2 = P_u \cup Q_{out}^2$  and we have the contradiction that  $v$  is not in  $C_2$ . Let  $C_u$  be the cycle consisting of  $P_u$  and the  $uu'$ -subpath  $uP_vu'$  of  $P_v$ .

**Claim 1.** *If  $P_u$  does not have a rainbow of  $C_u$  in its interior, then:*

- (a)  $C_u$  and  $C_2$  are near-obstructions at  $v$  satisfying the conditions in Lemma 2.2.4; and
- (b) the closed disk bounded by the outer cycle of  $C_u \cup C_2$  contains fewer vertices than the disk bounded by  $C_{out}$ .

*Proof.* Suppose that all the rainbows of  $C_u$  are located in  $uP_vu'$ . Since  $C_u$  is not an obstruction, at least one of them is an interior vertex of  $uP_vu'$ . Each vertex in the interior of  $uP_vu'$  that is a rainbow in  $C_u$ , is also a rainbow in  $C_1$ . As  $v$  is the only vertex in the interior of  $P_v$  that is a rainbow in  $C_1$ ,  $v$  is the only rainbow of  $C_u$  that is in the interior of  $uP_vu'$ . Since  $C_u$  is not an obstruction,  $u$ ,  $u'$  and  $v$  are the only rainbows of  $C_u$ , and  $C_u$  is a near-obstruction at  $v$ . The rotation at  $v$  inside  $C_u$  is the same as inside  $C_1$ , so  $C_u$  and  $C_2$  satisfy the conditions in Lemma 2.2.4.

Let  $C'_{out}$  be the outer cycle of  $C_u \cup C_2$ . Since  $C_u \cup C_2 \subseteq C_1 \cup C_2$ , the exterior of  $C_{out}$  is included in the exterior of  $C'_{out}$ . This shows that the disk bounded by  $C_{out}$  includes the disk bounded by  $C'_{out}$ .

If both  $p$  and  $q$  are in  $C_2$ , then  $p$ ,  $q$  and  $r_1$  are rainbows in  $C_2$ , and also distinct from  $v$ , contradicting that  $C_2$  is a near-obstruction for  $v$ . Thus, we may assume  $p \notin C_2$ . Then  $p$  is not in  $P_u \subseteq C_2$  and, since  $p$  is not an interior vertex of  $P_v$ ,  $p \notin V(C_u)$ . Since  $p$  is in  $C_{out}$ , and  $p$  is not in  $C_u \cup C_2$ ,  $p$  is in the outer face of  $C_u \cup C_2$ . Then  $p$  is in the disk bounded by  $C_{out}$  but not by  $C'_{out}$ , as required.  $\square$

The proof of the existence of the additional two rainbows in  $C_2$  is by induction on the number of vertices in the closed disk bounded by  $C_{out}$ . If  $u$  is not a rainbow in  $C_2$  and  $P_u$  does not have a rainbow of  $C_2$  in its interior, then Claim 1 implies  $C_u$  and  $C_2$  make a smaller instance and we are done. Thus, we may assume one of them yields the next additional rainbow.

In the same way, either the induction applies or the last rainbow comes by considering the edge of  $C_2 - Q_{out}^2$  incident with  $w$ . It follows that  $v$ ,  $r_1$ , and these two other vertices are four different rainbows in  $C_2$ , contradicting the fact that  $C_2$  is a near-obstruction.  $\square$

Although the statements and proofs of Lemmas 2.2.4 and 2.2.5 are similar, some subtle differences make it hard to find a statement encapsulating both results. For instance, Lemma 2.2.5 assumes that  $G(\Sigma)$  has obstructions, while finding an obstruction is the conclusion of Lemma 2.2.4. We sketch the proof of Lemma 2.2.5, emphasizing such differences. It would be interesting to find a common theory behind these two lemmas.

**Lemma 2.2.5.** *Let  $\Sigma$  be a set of strings in general position. Let  $C_1$  and  $C_2$  be obstructions in  $G(\Sigma)$  with  $e \in E(C_1) \cap E(C_2)$ . If  $C_1$  and  $C_2$  include distinct sides of  $e$  in their interior faces, then  $G(\Sigma)$  has an obstruction not including  $e$ .*

*Sketch of the proof.* We start assuming that such cycles exist and that every obstruction includes  $e$ .

By assumption,  $C_1 \cap C_2$  has at least two vertices and, therefore,  $C_1 \cup C_2$  is 2-connected. Thus, its outer face is bounded by a cycle  $C_{out}$ .

The Useful Fact shows that every rainbow in  $C_{out}$  is a rainbow in each of the cycles  $C_1$  and  $C_2$  containing it.

Since  $C_1$  and  $C_2$  include different sides of  $e$ , it follows that  $e$  is not in  $C_{out}$ . Therefore  $C_{out}$  is not an obstruction. Thus,  $C_{out}$  has at least three rainbows, and by our previous observation, we may choose the labelling such that two of them, say  $p$  and  $q$ , are also rainbows in  $C_1$ . Because  $C_1$  is an obstruction,  $p$  and  $q$  are the only rainbows in  $C_1$ .

Then  $C_1$  has a subpath  $P_e$  of containing  $e$  and in which only the ends  $u$  and  $w$  of  $P_e$  are in  $C_{out}$ . Let  $Q_{out}^1$  and  $Q_{out}^2$  be the  $uw$ -paths of  $C_{out}$ . We may assume that  $C_1$  is drawn in the disk bounded by  $Q_{out}^1 \cup P_e$ .

Let  $C_1 - (P_e)$  be  $uw$ -path in  $C_1$  that is not  $P_e$ . Note that  $p$  and  $q$  are the only vertices in  $C_1 - (P_e)$  that are rainbows in  $(C_1 - (P_e)) \cup Q_{out}^2$ . Since  $(C_1 - (P_e)) \cup Q_{out}^2$  is not an obstruction, the interior  $Q_{out}^2$  has a vertex  $r_1$  that is a rainbow of  $(C_1 - (P_e)) \cup Q_{out}^2$ . This vertex  $r_1$  is also a rainbow of  $C_2$ .

Let  $e_u$  be the edge incident to  $u$  in  $C_2$  that is not in  $Q_{out}^2$ . As we did in Lemma 2.2.4, we can show that either  $u$  is a rainbow in  $C_2$  or that  $e_u$  is not included in the disk bounded by  $P_e \cup Q_{out}^2$ . We assume the latter situation, as in the former we found our desired second rainbow in  $C_2$ .

Let  $P_u$  be the subpath of  $C_2$  starting at  $u$ , continuing on  $e_u$ , and ending on the first vertex  $u' \in V(P_e) \cap V(C_2)$  distinct from  $u$ . Note that  $u' \neq w$ , as otherwise  $C_2 = P_u \cup Q_{out}^2$  and we have the contradiction that  $e$  is not in  $C_2$ . Let  $C_u$  be the cycle consisting of  $P_u$  and the  $uu'$ -subpath  $uP_eu'$  of  $P_e$ .

We claim that either  $P_u$  has an interior vertex that is a rainbow in  $C_2$  or that there is a pair of cycles  $C'_1$  and  $C'_2$  satisfying the conditions in Lemma 2.2.5, but with fewer vertices in the closed disk bounded by the outer cycle of  $C'_1 \cup C'_2$  than in the disk bounded by  $C_{out}$ .

Suppose that none of the interior vertices in  $P_u$  is a rainbow in  $C_2$ . Because the interior of  $P_e$  has no vertices that are rainbows in  $C_1$  (as  $p$  and  $q$  are the only rainbows of  $C_1$ ), the interior of  $uP_eu'$  has no vertices that are rainbows in  $C_u$ . Therefore  $C_u$  is an obstruction, and  $C'_1 = C_u$  and  $C'_2 = C_2$  is a pair of obstructions including both sides of  $e$ . As  $C_u \cup C_2 \subseteq C_1 \cup C_2$ , the closed disk bounded by the  $C_{out}$  contains the closed disk bounded by the outer cycle of  $C_u \cup C_2$ . Not both of  $p$  and  $q$  are in the outer cycle of  $C_u \cup C_2$ , as both  $p$  and  $q$  would be part of  $C_2$ , concluding that  $C_2$  has three rainbows  $p$ ,  $q$  and  $r_1$ , and contradicting that  $C_2$  is an obstruction.

From the previous paragraph, either  $C_u$  and  $C_2$  is a smaller instance, and we are done by induction on the number of vertices in the closed disk bounded by  $C_{out}$ , or we found our second rainbow of  $C_2$  in the interior of  $P_u$ .

In the same way, either the induction applies or the last rainbow comes by considering an edge of  $C_2 - Q_{out}^2$  incident with  $w$ . It follows that  $r_1$ , and these other two vertices are three different rainbows in  $C_2$ , contradicting that  $C_2$  is an obstruction.  $\square$

## 2.4 Finding obstructions in polynomial time

In this section we describe a polynomial-time algorithm that determines whether a set of strings has an obstruction. We will assume that our input is the underlying plane graph  $G(\Sigma)$  of a set  $\Sigma$  of simple strings in general position, and that every string in  $\Sigma$  is identified as a path in  $G(\Sigma)$  (see notation below).

The key idea behind the algorithm is simple: either find an obstruction in the outer boundary of  $G(\Sigma)$  or find a vertex in the outer boundary whose removal reduces our problem into a smaller instance.

We start by describing the vertex removal operation. Suppose that  $x$  is a vertex of  $G(\Sigma)$  incident to the outer face of  $G(\Sigma)$ . For each  $\sigma \in \Sigma$ , we consider the path  $P_\sigma$  of  $G(\Sigma)$  representing  $\sigma$ . Let  $P_\sigma - x$  be the plane graph obtained from  $P_\sigma$  by removing  $x$  and the edges of  $P_\sigma$  incident to  $x$  (if  $x \notin P_\sigma$ , then  $P_\sigma - x = P_\sigma$ ). Each component of  $P_\sigma - x$  is either a vertex that represents an end of  $\sigma$ , or a string. Let  $S_{\sigma,x}$  be the set of string components of  $P_\sigma - x$  and let  $\Sigma - x = \bigcup_{\sigma \in \Sigma} S_{\sigma,x}$ . Note that  $G(\Sigma - x)$  can be obtained from  $G(\Sigma)$  by removing  $x$  and the edges incident to  $x$ , and then suppressing the degree-2 vertices whose incident edges belong to the same string in  $\Sigma$ , as well as removing remaining degree-0 vertices (Figure 2.10 illustrates this process).

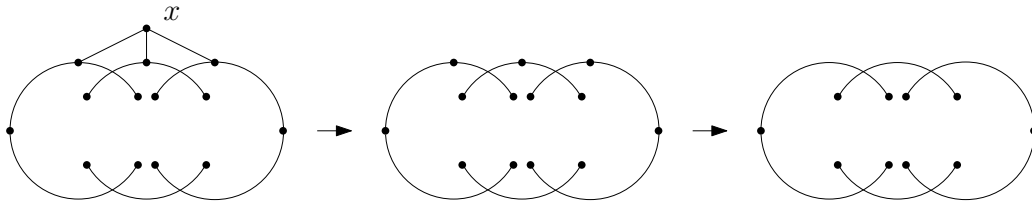


Figure 2.10: From  $\Sigma$  to  $\Sigma - x$ .

The next lemma is the key property used in the algorithm.

**Lemma 2.4.1.** *Let  $\Sigma$  be a set of simple strings in general position and let  $x$  be a vertex incident with the outer face. Then there is a 1 – 1 correspondence between the obstructions in  $G(\Sigma)$  not containing  $x$  and the obstructions of  $G(\Sigma - x)$ . Moreover, corresponding obstructions are the same simple closed curve.*

*Proof.* In general, there is a natural correspondence between cycles in  $G(\Sigma)$  not containing  $x$  and cycles in  $G(\Sigma - x)$ : if  $C$  is a cycle in  $G(\Sigma)$  not containing  $x$ , then every edge of  $C$  is not incident with  $x$ , and hence every edge is part of a string in  $\Sigma - x$ . Thus, there is a cycle  $C'$  in  $G(\Sigma - x)$  that represents the same simple closed curve as  $C$ . Conversely, each cycle  $C'$  in  $G(\Sigma - x)$  is a simple closed curve in  $\bigcup(\Sigma - x) \subseteq \bigcup \Sigma$ , and hence, there is a cycle  $C$  in  $G(\Sigma)$  representing the same simple closed curve as  $C'$ .

To complete the proof it is enough to show that any two cycles  $C, C'$  that correspond as above have the same rainbows. Since  $G(\Sigma - x)$  is obtained from suppressing and removing vertices in a subgraph of  $G(\Sigma)$ ,  $V(C') \subseteq V(C)$ . Thus,  $V(C) \setminus V(C')$  consists of suppressed and removed vertices in the process of converting  $G(\Sigma)$  into  $G(\Sigma - x)$ . Since  $x \notin V(C)$ , if  $v \in V(C)$  is suppressed, then the two edges of  $C$  incident to  $v$  belong to the same string in  $\Sigma$ . Therefore, none of the vertices in  $V(C) \setminus V(C')$  is a rainbow in  $C$ .

Every rainbow in  $C$  is also a rainbow in  $C'$  because every two edges of  $G(\Sigma - x)$  that are included in distinct strings of  $\Sigma$  are also included in distinct strings in  $\Sigma - x$ .

Conversely, suppose that  $v \in V(C) \cap V(C')$  is reflecting in  $C$ . Let  $\sigma \in \Sigma$  be a string including two edges of  $G(\Sigma)$  in the rotation at  $v$  inside  $C$ . Since  $x$  is drawn in the exterior of  $C$ , these two edges are part of the same string in  $\Sigma - x$ , and hence  $v$  is reflecting inside  $C'$ .

Therefore every rainbow of  $C'$  is a rainbow of  $C$ , and thus,  $C$  and  $C'$  have the same rainbows. □

A vertex in  $G(\Sigma)$  is an *outer-rainbow* if it is in the outer boundary and all the edges in its rotation belong to different strings. Note that every outer-rainbow is a rainbow for all the cycles in  $G(\Sigma)$  that contain it.

An *outer cycle* is a cycle of  $G(\Sigma)$  that has all its edges incident to the outer face of  $G(\Sigma)$ . For any graph  $G(\Sigma)$ , a *block* of  $G(\Sigma)$  is a maximal connected subgraph of  $G(\Sigma)$  with no cut-vertex. If  $G(\Sigma)$  is connected with at least two vertices, then each block is either an edge or is 2-connected. In the latter case, the outer face of the block is bounded by a cycle of the block.

We find obstructions by solving an auxiliary problem: finding obstructions including one or two fixed outer-rainbows. The next subroutine (Algorithm 1) describes how to find an obstruction containing two fixed outer-rainbows. Below we discuss its correctness.

---

**Algorithm 1:** Finding obstructions through two fixed outer-rainbows.

---

**Data:**  $G(\Sigma)$  and two outer-rainbows  $x$  and  $y$ .

**Result:** Either an obstruction containing  $x$  and  $y$  or that no such obstruction exists.

```

1 repeat
2   if there is no cycle containing  $x$  and  $y$  then
3     | return  $G(\Sigma)$  has no obstruction containing  $x$  and  $y$ ;
4   end
5   Find the outer cycle  $C$  containing  $x$  and  $y$ ;
6   while  $C$  is not the outer boundary of  $G(\Sigma)$  do
7     | Pick  $w \in V(G(\Sigma)) \setminus V(C)$  incident with the outer face;
8     |  $\Sigma \leftarrow \Sigma - w$ 
9   end
10  if  $C$  has a rainbow  $z \notin \{x, y\}$  in  $G(\Sigma)$  then
11    |  $\Sigma \leftarrow \Sigma - z$ ;
12  else
13    | return  $C$ ;
14  end
15 until  $V(G(\Sigma)) = \{x, y\}$ ;
16 return  $G(\Sigma)$  has no obstruction containing  $x$  and  $y$ .

```

---

To see that Algorithm 1 is correct, observe that when Step 2 does not apply, then Step 5 can be performed: if there is a cycle containing  $x$  and  $y$ , then, as  $x$  and  $y$  are incident to the outer face of  $G(\Sigma)$ , the outer boundary of the block containing  $x$  and  $y$  is an outer cycle  $C$  containing  $x$  and  $y$ . Every obstruction  $\mathcal{C}$  through  $x$  and  $y$  is drawn in the closed disk bounded by  $C$ . Lemma 2.4.1 guarantees that if we remove a vertex in the outer boundary that is not in  $C$  (Step 7) and we update  $\Sigma$  (Step 8), then  $\mathcal{C}$  (or more precisely, the cycle in the new  $G(\Sigma)$  that is the same simple closed curve as  $\mathcal{C}$ ) is an obstruction through  $x$  and  $y$ .

In Step 10, if  $x$  and  $y$  are the only rainbows of  $C$ , then  $C$  is an obstruction returned in Step 13. Else,  $C$  has a rainbow  $z \notin \{x, y\}$ . Any obstruction  $\mathcal{C}$  through  $x$  and  $y$  does not contain  $z$ , and hence removing  $z$  and updating  $\Sigma$  (Step 11) does not change the fact that  $\mathcal{C}$  is an obstruction in the new  $G(\Sigma)$ . This algorithm terminates as the number of vertices

in  $G(\Sigma)$  is always decreasing.

We now turn to Algorithm 2, used as subroutine in the main algorithm. Its correctness again easily follows from Lemma 2.4.1.

---

**Algorithm 2:** Finding obstructions through a fixed outer-rainbow.

---

**Data:**  $G(\Sigma)$  and an outer-rainbow vertex  $x$ .

**Result:** Either an obstruction containing  $x$  or that no such obstruction exists.

```

1 repeat
2   if there is no cycle containing  $x$  then
3     | return  $G(\Sigma)$  has no obstruction containing  $x$ ;
4   end
5   Find an outer cycle  $C$  containing  $x$ ;
6   if  $C$  has an outer-rainbow  $y \neq x$  then
7     | Run Algorithm 1 on  $(G(\Sigma), x, y)$ ;
8     | if  $G(\Sigma)$  has an obstruction  $D$  including  $x$  and  $y$ , then
9       | | return  $D$ ;
10    | end
11    |  $\Sigma \leftarrow \Sigma - y$ ;
12  else
13    | return  $C$ ;
14  end
15 until  $V(G(\Sigma)) = \{x\}$ ;
16 return  $G(\Sigma)$  has no obstruction containing  $x$ .
```

---

Finally we present the algorithm to find obstructions, whose correctness also relies on Lemma 2.4.1.

---

**Algorithm 3:** Finding obstructions.

---

**Data:**  $G(\Sigma)$ .

**Result:** Either finds an obstruction or that no such obstruction exists.

```
1 repeat
2   if  $G(\Sigma)$  has no cycles then
3     | return  $G(\Sigma)$  has no obstructions;
4   end
5   Find an outer cycle  $C$ ;
6   if  $C$  has no rainbows then
7     | return  $C$ ;
8   end
9   Pick a rainbow  $x$  in  $C$  ( $x$  is outer-rainbow in  $G(\Sigma)$ );
10  Run Algorithm 2 on  $(G(\Sigma), x)$ ;
11  if  $G(\Sigma)$  has an obstruction  $D$  including  $x$  then
12    | return  $D$ ;
13  end
14   $\Sigma \leftarrow \Sigma - x$ ;
15 until  $G(\Sigma) = \emptyset$ ;
16 return  $G(\Sigma)$  has no obstructions.
```

---

## 2.5 Pseudolinear drawings of $K_n$

In this section we present a simple proof of a characterization of pseudolinear drawings of complete graphs (Theorem 2.5.1), equivalent to the ones given in [5] and [8]. One of the equivalences, Corollary 2.5.2, is shown at the end of this section. The other equivalence (Lemma 4.4.8) is deferred to Subsection 4.4.2 of Chapter 4, where it plays an important role in the study of pseudospherical drawings of  $K_n$ .

**Theorem 2.5.1.** *A good drawing of a complete graph is pseudolinear if and only if it does not include the  $B$  configuration (see Figure 2.1).*

*Proof.* The unique cycle in a  $B$  configuration is an obstruction, so, by Theorem 2.1.1, no pseudolinear drawing of  $K_n$  can include it. Conversely, suppose that  $D$  is a good drawing of  $K_n$  that is not pseudolinear. Let  $\Sigma = \{D[e] : e \in E(K_n)\}$  be the set of edge-arcs, and



let  $G(\Sigma)$  be its underlying plane graph. In order to avoid confusion between vertices and edges of  $K_n$  and  $G(\Sigma)$ , vertices in  $G(\Sigma)$  are called *points*, and edges of  $G(\Sigma)$  are *segments*. Because  $D$  is good, each point is either in  $V(K_n)$  or a crossing.

For every cycle  $C$  in  $G(\Sigma)$ , we let  $\delta(C)$  be the set of points in  $C$  for which their two incident segments in  $D$  belong to distinct edges in  $\Sigma$ . Theorem 2.1.1 implies that  $G(\Sigma)$  has an obstruction  $C$ . We choose our obstruction  $C$  so that  $|\delta(C)|$  is as small as possible.

Since  $D$  is good,  $|\delta(C)| \geq 3$  and, because  $C$  is an obstruction, at most two vertices in  $\delta(C)$  are rainbows in  $C$ . Consider a point  $x \in \delta(C)$  that is reflecting inside  $C$ . Note that  $x$  is a crossing. Let  $\sigma_1$  and  $\sigma_2$  be the two edge-arcs in  $\Sigma$  crossed at  $x$ . We traverse  $\sigma_1$ , starting at  $x$ , continuing on the segment of  $\sigma_1$  included in the interior of  $C$ , until an end  $a_1 \in V(K_n)$  of  $\sigma_1$  is reached. Likewise we define  $a_2$  for  $\sigma_2$ . Henceforth, we refer to  $a_1$  and  $a_2$  as the *internal vertices* corresponding to the crossing  $x$ . The following claim explains why we call them “internal”.

**Claim 1.** *Let  $x \in \delta(C)$  be a point reflecting inside  $C$ . Then the two internal vertices corresponding to  $x$  are in the interior of  $C$ .*

*Proof.* Let  $a_1$  be an internal vertex corresponding to  $x$ , and suppose  $\sigma_1$  is the edge-arc including both  $x$  and  $a_1$ . Let  $\sigma'_1$  be the substring of  $\sigma_1$  having  $x$  and  $a_1$  as endpoints. Applying Observation 2.2.1 to our obstruction  $C$ , with  $\sigma = \sigma_1$  and  $\sigma' = \sigma'_1$ , we obtain that  $\sigma'_1 \cap C = \{x\}$ . Since points of  $\sigma'_1$  near  $x$  are in the interior face of  $C$ ,  $\sigma'_1 \setminus \{x\}$  is included in the interior of  $C$ . In particular,  $a_1$  is in such a face.  $\square$

Now we look at the points in  $\delta(C)$  that are not reflecting inside  $C$ . If  $x$  is one of them, then  $x$  is a vertex or a crossing. Suppose that  $x$  is a crossing. Let  $\sigma_1, \sigma_2$  be the edge-arcs crossing at  $x$ . Because  $x$  is not reflecting inside  $C$ , one of the two segments at  $x$  included in  $\sigma_1$  is in the outer face of  $C$ . We traverse  $\sigma_1$ , starting in  $x$ , and continuing in the outer face until we reach an end  $b_1$  of  $\sigma_1$ . Likewise we define  $b_2$  for  $\sigma_2$ . These vertices  $b_1, b_2$  are the *external vertices* corresponding to the crossing  $x$ .

**Claim 2.** *Let  $x$  be a crossing in  $\delta(C)$  that is not reflecting inside  $C$ , and let  $\sigma$  be an edge-arc including  $x$  and an external vertex  $b$  of  $x$ . If  $\sigma'$  is the substring of  $\sigma$  connecting  $x$  to  $b$ , then  $\sigma' \setminus \{x\}$  is included in the outer face of  $C$ .*

*Proof.* Applying Observation 2.2.1 to  $C$ ,  $\sigma_1$ , and  $\sigma'$ , we see that  $\sigma' \cap C = \{x\}$ . Since the points of  $\sigma'_1$  near  $x$  are in the outer face of  $C$ ,  $\sigma'_1 \setminus \{x\}$  is included in the outer face of  $C$ .  $\square$

It is convenient, in the case when  $x$  is a vertex of  $K_n$ , to let  $x$  be its own external vertex.

Henceforth we refer to the vertices of  $K_n$  that are internal to some crossing in  $C$  as the *internal vertices* of  $C$ , and likewise, the *external vertices* of  $C$  are the vertices of  $K_n$  that are external to some crossing or to a vertex in  $C$ .

**Claim 3.** *Every segment in  $C$  is included in an edge-arc whose ends are either internal or external vertices of  $C$ .*

*Proof.* Any segment  $s$  of  $C$  is contained in a subpath  $P$  of  $C$  whose ends are in  $\delta(C)$  but is otherwise disjoint from  $\delta(C)$ . This path  $P$  is part of an edge-arc  $\sigma \in \Sigma$ . Let  $a \in V(K_n)$  be one of the ends of  $\sigma$ , and suppose that  $x$  is the first end of  $P$  that we encounter when we traverse  $\sigma$  from  $a$  to the other end of  $\sigma$ . If  $\sigma$  is reflecting at  $x$ , then  $a$  is internal. If  $\sigma$  is not reflecting at  $x$ , then  $a$  is external. Likewise, the other end of  $\sigma$  is internal or external.  $\square$

Suppose that  $K_n$  has a vertex  $y$  that is neither external nor internal to  $C$ . Then, by our previous claim, the underlying plane graph of  $D[K_n - y]$  contains a cycle whose drawing is  $D[C]$  and is an obstruction. Thus,  $D[K_n - y]$  is not pseudolinear, and applying induction on  $n$ , we obtain that  $D[K_n - y]$  has a  $B$  configuration. Henceforth we assume that all the vertices of  $K_n$  are either internal or external to  $C$ .

**Claim 4.** *Either the outer face of  $D$  is bounded by a cycle of  $K_n$  or  $D$  has a  $B$  configuration.*

*Proof.* Suppose that the outer face of  $D$  is not bounded by a cycle of  $K_n$ . Then the outer face is incident to a crossing  $\times$  between two edge-arcs  $\sigma_1$  and  $\sigma_2$ . Let  $K$  be the crossing  $K_4$  induced by the ends of  $\sigma_1$  and  $\sigma_2$ . The drawing  $D[K]$  has exactly five faces, four of them incident to  $\times$ . Exactly one of the faces incident to  $\times$  includes the outer face of  $D$ . Such a face of  $D[K]$  is bounded by portions of  $\sigma_1$ ,  $\sigma_2$ , and an edge  $e$  of  $K_n$  connecting an end of  $\sigma_1$  to an end of  $\sigma_2$ . The drawing induced by  $\sigma_1$ ,  $\sigma_2$  and  $D[e]$  is a  $B$  configuration.  $\square$

Claims 1 and 4 imply that the outer cycle of  $D$  consists of only external vertices of  $C$ . Every external vertex either is associated with a crossing that is not reflecting inside  $C$ , or is itself a vertex of  $K_n$  in  $C$ . Because  $C$  has at most two points not reflecting inside  $C$ , and each of them has at most two external vertices, there are at most four points in the outer cycle of  $D$ . Thus the outercycle is a 3- or 4-cycle of  $K_n$ .

As  $C$  has at least three external vertices (in the outer cycle),  $\delta(C)$  has precisely two points  $p$  and  $q$  not reflecting inside  $C$ . The outer cycle of  $D$  has an edge  $uv$ , where  $u$  is external to  $p$  and  $v$  is external to  $q$  (possibly  $u = p$  or  $q = v$ ).

Consider the  $pq$ -path  $P$  in  $G(\Sigma)$ , starting at  $p$ , continuing on the edge-arc connecting  $p$  to  $u$ , then following the edge  $uv$  until we reach  $v$ , and ending by following the edge-arc connecting  $v$  to  $q$ . We finish our proof by considering two cases, depending on whether  $uv$  is a segment of  $C$ .

**Case.**  $uv$  is not a segment of  $C$ .

In this case, there exists a point  $w \in D[uv] \setminus D[C]$ . As  $D[uv]$  is part of the outer cycle, it contains neither crossings nor vertices in its interior, so the arcs in  $D[uv]$  connecting  $w$  to the ends  $u$  and  $v$  are internally disjoint from  $C$ . From Claim 2, it follows that the  $pu$ - and the  $qv$ -subpaths of  $P$  are internally disjoint from  $C$ . Thus  $P$  is an arc connecting  $p$  and  $q$  in the outer face of  $C$ .

Consider the cycle  $C'$  obtained from the union of  $P$  and the  $pq$ -path of  $C$  that lies in the outer face of  $D[C \cup P]$ .

We will show that  $C'$  is an obstruction by showing that  $u$  and  $v$  are the only rainbows of  $C'$ . If  $p \neq u$ , then the edge-arc  $\sigma$  connecting  $p$  and  $u$  shows that every point in  $(P - u) \cap \sigma$  is reflecting inside  $C'$ . Analogously, if  $q \neq v$ , the points distinct from  $v$  in the edge-arc connecting  $q$  and  $v$ , are reflecting inside  $C'$ . Thus the internal points in  $P$ , with the exception of  $u$  and  $v$ , are not reflecting. The same holds for the points in  $C' - P$ , as these points are not reflecting inside  $C$  (recall that  $p$  and  $q$  are the only rainbows of  $C$ ). Thus  $u$  and  $v$  are the only rainbows of  $C'$ .

Note that all the segments of  $C'$  are included in edges whose ends are  $u$ ,  $v$  or interior points of  $C$ . So if  $y$  is a vertex in the outercycle of  $D$  distinct from  $u$  and  $v$ ,  $D[K_n - y]$  also includes  $D[C']$  as an obstruction, implying that  $D[K_n - y]$  is not pseudolinear. Again, by induction on  $n$ , we obtain that  $K_n - y$  has a  $B$  configuration.

**Case.**  $uv$  is a segment of  $C$ .

In this case, as  $u, v$  are vertices of  $K_n$  in  $C$ , they are rainbows of  $C$ . Since  $p$  and  $q$  are the only rainbows,  $p = u, q = v$ , and  $D[uv]$  is a segment of  $C$ . Then, all the segments of  $C$  are included in edge-arcs whose ends are  $u, v$  or interior points of  $C$ . Again, remove a vertex in the outer cycle of  $D$  distinct from  $u$  and  $v$  to obtain a non-pseudolinear drawing of  $K_{n-1}$  in which, by induction, we find a  $B$  configuration.  $\square$

We conclude this section by showing a slightly different way to characterize pseudolinear drawings in terms of a four points property. This local property plays a central role in Chapter 4.

**Corollary 2.5.2.** *A good drawing  $D$  of  $K_n$  in the plane is pseudolinear if and only if, for every 3-cycle  $T$  and for every vertex  $v$  drawn in the bounded face of  $D[T]$ , the three edges connecting  $v$  to the vertices of  $T$  are contained in the disk bounded by  $D[T]$ .*

*Proof.* Suppose first that  $D$  is not a pseudolinear drawing of  $K_n$ . From Theorem 2.5.1 we know that there exists a path  $P = (x, y, z, w)$  that is drawn in  $D$  as a  $B$  configuration (Figure 2.1). Since  $D[\{x, y, z, w\}]$  is a crossing  $K_4$  in which  $xy$  crosses  $zw$ , the edge  $xz$  is not crossed in  $D[\{x, y, z, w\}]$ , and hence it connects  $x$  to  $z$  through the bounded face of  $D[P]$ . In this case  $(x, y, z, x)$  is a 3-cycle, containing  $w$  in its interior face, and such that  $D[wz]$  is not contained in the disk bounded by this 3-cycle.

Conversely, suppose that there exist a 3-cycle  $T = (x, y, z, x)$  and a vertex  $w$  for which at least one of the edges joining  $w$  to a vertex of  $T$ , say  $D[wz]$ , is not contained in the disk bounded by  $D[T]$ . The edge  $wz$  crosses at least one edge of  $T$ , and because  $D$  is good, it only crosses  $xy$ . Exactly one of  $x$  and  $y$  is not incident with the outer face  $F$  of  $D[T + wz]$ . Remove from  $D[T + wz]$  the edge connecting  $z$  to the one of  $x$  and  $y$  not incident with  $F$  to obtain a  $B$  configuration. This shows that  $D$  is not pseudolinear.  $\square$

## 2.6 Concluding remarks

In our initial attempts to formulate Theorem 2.1.1, we intended to characterize non-pseudolinear good drawings of graphs by means of having at least one of the configurations in Figure 2.3 as a subdrawing. We obtain this as an easy consequence of Theorem 2.1.1. We sketch its proof.

**Theorem 2.1.2.** *Let  $D$  be a non-pseudolinear good drawing of a graph  $H$ . Then there is a subset  $S$  of edge-arcs in  $\{D[e] : e \in E(H)\}$ , such that each  $\sigma \in S$  has a substring  $\sigma' \subseteq \sigma$  for which  $\bigcup_{\sigma \in S} \sigma'$  is one of the drawings in Figure 2.3.*

*Proof.* Take  $C$  an obstruction of the underlying plane graph associated to  $D$ . We choose  $C$  so that  $|\delta(C)|$  is as small as possible. Decompose  $C$  into a cyclic sequence of paths  $P_0, \dots, P_m$ , where  $P_i$  connects two points in  $\delta(C)$  and it is otherwise disjoint from  $\delta(C)$ . By using Observation 2.2.1, one can show that  $P_0, \dots, P_m$  belong to distinct edge-arcs  $\sigma_0, \dots, \sigma_m$ , respectively. For each  $P_i$ , we consider the string  $\sigma'_i$ , obtained by slightly extending the ends of  $P_i$  that are reflecting in  $C$ ; we extend them along  $\sigma_i$ .

Let  $x \in \delta(C)$  be an end shared by  $P_{i-1}$  and  $P_i$ . If  $x$  is reflecting in  $C$ , then  $x$  is a crossing between  $\sigma_{i-1}$  and  $\sigma_i$ . Moreover, the arcs added to  $P_{i-1}$  and  $P_i$  at  $x$  to obtain  $\sigma'_{i-1}$

and  $\sigma'_i$  are in the interior of  $C$ . If  $x$  is a rainbow in  $C$ , then  $P_i$  and  $P_{i-1}$  are not extended at  $x$ , and  $x$  acts as one of the black dots in Figure 2.3. The rest of the points in  $\delta(C)$  are crossings in  $\bigcup_{i=0}^m \sigma'_i$  facing the interior of  $C$ . Since  $C$  has at most two rainbows,  $\bigcup_{i=0}^m \sigma'_i$  is one in Figure 2.3.  $\square$

There are pseudolinear drawings that are not stretchable. For instance, consider the Non-Pappus configuration in Figure 2.11. Nevertheless, as an immediate consequence of Thomassen's main result in [29], pseudolinear and stretchable drawings are equivalent, under the assumption that every edge is crossed at most once.

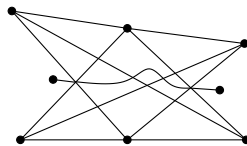


Figure 2.11: Non-Pappus configuration.

**Corollary 2.6.1.** *A drawing of a graph in which every edge is crossed at most once is stretchable if and only if it is pseudolinear.*

*Proof.* Let  $D$  be a drawing of a graph in which every edge is crossed at most once. If  $D$  is stretchable then clearly it is pseudolinear. To show the converse, suppose that  $D$  is pseudolinear. Then  $D$  does not contain any obstruction, and in particular, neither of the  $B$  and  $W$  configurations in Figure 2.1 occur in  $D$ . In [29], it was shown that not containing the  $B$  and  $W$  configurations is equivalent to being rectilinear.  $\square$

One can construct more general examples of pseudolinear drawings that are not stretchable by considering non-stretchable arrangements of pseudolines. However, such examples seem to inevitably have edges crossing several times. This leads to two natural questions.

**Question 1.** *Is it true that if  $D$  is a pseudolinear drawing in which every edge is crossed at most twice, then  $D$  is stretchable?*

**Question 2.** *Is it true that if  $D$  is a pseudolinear drawing in which all the crossings involve a fixed edge, then  $D$  is stretchable?*

# Chapter 3

## Straight Line Drawings of Graphs with Crossings

### 3.1 Introduction

In 1993, Bienstock and Dean showed that every graph  $G$  with  $\text{cr}(G) \leq 3$ , has  $\text{cr}(G) = \overline{\text{cr}}(G)$  [13]. A key idea used in their proof was that, under certain circumstances, one can modify an optimal drawing of a graph  $G$  (a drawing with fewest crossing pairs of edges as possible) to obtain a rectilinear drawing of  $G$  with the same number of crossings. This raises a natural question.

**Question 3.** *Given a drawing  $D$  of a graph  $G$ , can we find a rectilinear drawing  $\overline{D}$  of  $G$  with the same crossing pairs of edges as  $D$ ?*

For example, consider an  $n$ -cycle  $C$  drawn as a simple closed curve, with some of its chords drawn in the bounded side of  $C$ . Although, there are non-stretchable drawings of this type produced by considering a non-stretchable arrangement of pseudolines, as in Figure 3.1a, drawing  $C$  as a convex polygon in the plane induces a rectilinear drawing with the same pairs of crossing edges (Figure 3.1b).

This problem becomes more interesting when we let some chords of the  $n$ -cycle  $C$  be drawn in the outer face of  $C$ . We add the constraint that chords in the outer face are not crossing. In this case, it is less clear whether there exists a rectilinear drawing of the same graph with the chords drawn on the same sides of  $C$ . For instance, the 5-cycle with chords of Figure 3.2a is not pseudolinear. To obtain a rectilinear version of such a drawing, one would need to choose a different face to be the unbounded face, as in Figure 3.2b.

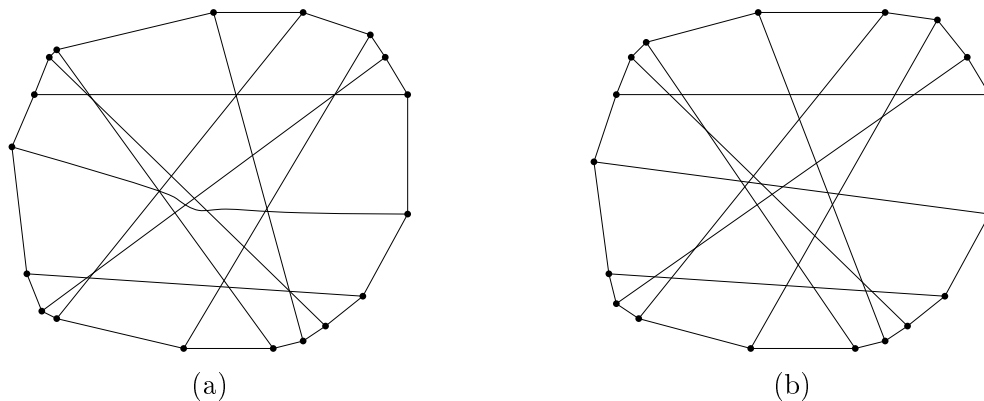


Figure 3.1: A non-stretchable and a rectilinear drawing with the same pair of crossing edges.

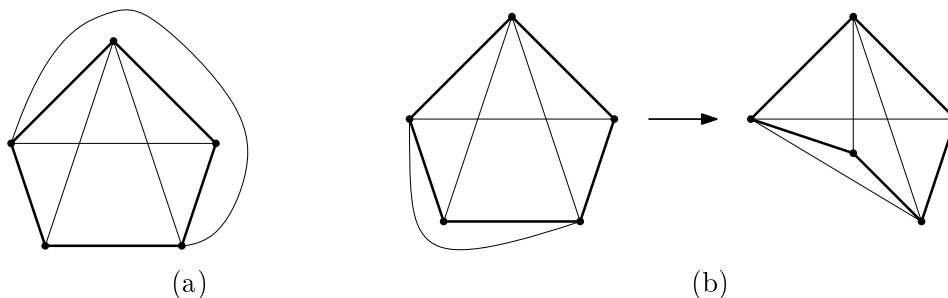


Figure 3.2

Our interest in this class of drawings (cycles with chords drawn in distinct sides of the cycle) came from realizing that many of the cases considered in the proof of Bienstock and Dean’s Theorem can be argued by applying Tutte’s Spring Theorem (Theorem 3.3.1) and by solving a more restricted version of Question 3 (explained below) for small instances of the drawings of cycles and chords. The main result in this chapter answers this restricted version of Question 3 for an extension of this class of drawings. To state this result, we need some terminology used throughout this chapter.

Let  $D$  be a drawing of a graph  $G$ . The subgraph induced by the edges of  $G$  not crossed in  $D$  is denoted as  $\text{pl}_D(G)$ , or more simply as  $\text{pl}(G)$ , if the drawing  $D$  is clear from the context. For a cycle  $C$  of  $\text{pl}(G)$ , let:

- $\text{Int}_D(C)$  be the subgraph induced by the vertices and edges of  $G$  drawn in the closed disk bounded by  $D[C]$ ;
- $\text{int}_D(C)$  be the subgraph induced by the vertices and edges of  $G$  drawn in the open disk bounded by  $D[C]$ ;
- $\text{Ext}_D(C)$  be the subgraph induced by the vertices and edges of  $G$  drawn in the closure of the outer face of  $D[C]$ ; and
- $\text{ext}_D(C)$  be the subgraph induced by the vertices and edges of  $G$  drawn in the outer face of  $D[C]$ .

The *chords* of  $C$  are the edges in  $E(G) \setminus E(C)$  between two vertices in  $V(C)$ . A chord of  $C$  is *external* if it belongs to  $\text{Ext}_D(C)$ , and *internal* if it belongs to  $\text{Int}_D(C)$ . A cycle of  $G$  is *facial* if it bounds a face of  $D$ .

A drawing  $D$  of a graph  $G$  is *bundled* if it is good, and every crossing in  $D$  is the crossing between two internal chords of a facial cycle of  $\text{pl}_D(G)$ . Every bundled drawing can be obtained from a planar embedding by selecting a set of facial cycles, and then adding, for each selected cycle, internal chords such that each of them cross at least once (they must cross in order not to be part of  $\text{pl}(G)$ ).

In a bundled drawing  $D$  of  $G$ , a *special cycle*  $S$  is a facial cycle of  $\text{pl}(G)$  bounding an inner-face and containing crossings in its interior. Every cycle bounding a face of  $D[\text{pl}(G)]$  is either facial or special. In Section 3.3, Figure 3.4a depicts a bundled drawing with the special cycles bounding coloured regions.

With the necessary terminology in hand, now we phrase the restricted version of Question 3: *Given a bundled drawing  $D$  of a graph  $G$ , is there a rectilinear drawing  $\overline{D}$  of  $G$  satisfying the following conditions?*

- (i)  $\text{pl}_D(G) = \text{pl}_{\overline{D}}(G)$ ;
- (ii)  $D[\text{pl}_D(G)]$  and  $\overline{D}[\text{pl}_{\overline{D}}(G)]$  are homeomorphic; and
- (iii) for every special cycle  $S$  of  $D$ ,  $\text{Int}_D(S)$  and  $\text{Int}_{\overline{D}}(S)$  are the same graph.

These three conditions strengthen the requirement of  $D$  and  $\overline{D}$  having the same pairs of crossing edges. For the drawing in Figure 3.2a, the answer to this question is negative, while for the one in Figure 3.2b, the answer is positive.



One would expect that, as  $D$  and  $\overline{D}$  satisfying (i)-(iii) are not very different from each other, a necessary condition to guarantee the existence of  $\overline{D}$  is that  $D$  is pseudolinear. Our main result in this chapter shows that pseudolinearity is a necessary and sufficient condition when the special cycles are well-behaved.

**Theorem 3.1.1.** *Let  $D$  be a bundled drawing of a graph  $G$ . Suppose that for every two distinct special cycles  $S_1$  and  $S_2$ , the intersection between  $S_1$  and  $S_2$  is either empty; a vertex; or an edge. Then  $D$  is pseudolinear if and only if there exists a rectilinear drawing  $\overline{D}$  of  $G$  such that:*

- (i)  $pl_D(G) = pl_{\overline{D}}(G)$ ;
- (ii)  $D[pl_D(G)]$  and  $\overline{D}[pl_{\overline{D}}(G)]$  are homeomorphic; and
- (iii) for every special cycle  $S$  of  $D$ ,  $Int_D(S)$  and  $Int_{\overline{D}}(S)$  are the same graph.

The results used in the proof Theorem 3.1.1 are presented in different sections of this chapter. In Section 3.2 we show that non-pseudolinear bundled drawings have a specific type of obstructions. In Section 3.3 we study a family of bundled drawings satisfying additional connectivity conditions; we refer to them as *crossing-webs*. We also introduce the notion of *elastic crossing-web*, that can be informally described as a drawing satisfying a generalized version of Tutte’s Spring Theorem. We conclude Section 3.3 by showing a family of elastic crossing-webs. This family is part of the basis of the induction in the proof of Theorem 3.4.4, the main result of Section 3.4.

In Section 3.4 we show that a *full crossing-web* is elastic if and only if is pseudolinear. In Section 3.5 we conclude the proof of Theorem 3.1.1 by showing that every pseudolinear bundled drawing can be extended into a pseudolinear full crossing-web.

This section is based on author’s original ideas, but it has been somewhat influenced by what will a more general work in collaboration with Richter and Thomassen.

## 3.2 Obstructions in bundled drawings

In this section we characterize non-pseudolinear bundled drawings of graphs. This characterization has the benefit of depending only on how special cycles and their chords are drawn. This will be used in the proofs of Observation 3.3.3 and Theorem 3.3.5.

Given a good drawing  $D$  of a graph  $G$ , let  $H(D)$  be its underlying plane graph obtained from  $D$ , by replacing crossings with degree 4 vertices. In Chapter 2, we showed that  $D$  is

pseudolinear if and only if  $H(D)$  has no obstructions (obstructions are defined in Section 2.2). A *generalized B* is an obstruction  $C$  that contains an entire edge  $D[xy]$  of  $G$  as a subpath of  $C$ . The obstructions in the figures in the first row of the third box in Figure 2.3 on page 7 are generalized Bs.

In Lemma 3.2.2 we will show that a bundled drawing, satisfying the conditions in Theorem 3.1.1, is not pseudolinear if and only if contains a generalized B. In fact, with the purpose of using this characterization in the next sections, we show that a more specific condition holds; this condition is phrased in Definition 3.2.1 and depicted in Figure 3.3.

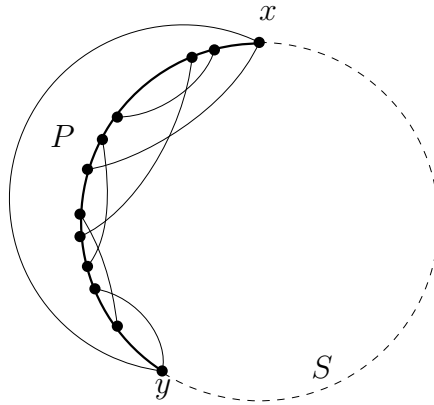


Figure 3.3: An external chord  $xy$  inducing a generalized  $B$  in  $S$ .

**Definition 3.2.1.** *Let  $D$  be a bundled drawing of a graph  $G$  and let  $S$  be a special cycle. Suppose that  $xy$  is an external chord of  $S$  and let  $P$  be the  $xy$ -path of  $S$  such that  $P$  is incident with the two inner faces of  $D[S + xy]$ . Then  $xy$  induces a generalized  $B$ , if, in the subdrawing  $D'$  of  $D$  obtained from  $D[P + xy]$  by adding the internal chords of  $S$  with both ends in  $P$ ,  $x$  and  $y$  are the only vertices of  $P$  in the outer boundary of  $D'$ .*

Observe that, if  $xy$  induces a generalized  $B$ , then the outer cycle of  $D'$  is a generalized  $B$  including the edge  $D[xy]$ .

**Lemma 3.2.2.** *Let  $D$  be a bundled drawing of a graph  $G$ , such that for every two distinct special cycles  $S_1$  and  $S_2$ , their intersection is either empty, a vertex or an edge. Then  $D$  is not pseudolinear if and only there exists a special cycle  $S$  and an outer chord of  $S$  inducing a generalized  $B$ .*

*Proof.* From the remark preceding Lemma 3.2.2, it is clear that if an outer chord of some special cycle induces a generalized  $B$ , then  $D$  is not pseudolinear.

Conversely, suppose that  $D$  is not pseudolinear. Let  $H(D)$  be the underlying plane graph of  $D$  (to avoid confusion, we call  $V(H(D))$  and  $E(H(D))$  the *points* and *segments* of  $H(D)$ ). Let  $C$  be an obstruction of  $H(D)$  minimizing  $|\delta(C)|$  (recall from Subsection 2.2.1 that  $\delta(C)$  is the set of points in  $V(C)$  for which their two incident segments in  $C$  are included in distinct edge-arcs of  $E(G)$ ).

Every point in  $C$  is either a vertex of  $G$  or a crossing in  $D$ . As every element in  $V(C) \cap V(G)$  is a rainbow for  $C$ ,  $|V(C) \cap V(G)| \leq 2$ .

**Claim 1.**  $|V(C) \cap V(G)| = 2$ .

*Proof.* By way of contradiction, suppose that  $|V(C) \cap V(G)| \leq 1$ . Then, for every pair of crossings that are points in  $V(C)$ , there is a path in  $C$  avoiding  $V(G)$ . Thus, every edge of  $E(G)$  involved in a crossing in  $V(C)$  is an inner chord of the same special cycle  $S$ . This in particular implies that the inner face  $F$  of  $C$  is part of the inner face of  $D[S]$ .

Let  $\times$  be a point in  $\delta(C)$  reflecting inside  $C$  ( $\times$  is a crossing in  $D$  because elements in  $V(G)$  are rainbows). The minimality of  $|\delta(C)|$  and Observation 2.2.1 imply that at least two ends of the edges crossed at  $\times$  are drawn in  $F$ . However, these two vertices cannot be drawn in the inner face of the special cycle  $D[S]$ , a contradiction.  $\square$

Thus, we may assume that there are two vertices  $x, y \in V(C) \cap V(G)$ . Since  $C$  is an obstruction and  $x$  and  $y$  are rainbows for  $C$ , all the points in  $V(C) \setminus \{x, y\}$  are reflecting inside  $C$ . Let  $P_1$  and  $P_2$  be the  $xy$ -paths of  $C$ . Since  $G$  is simple, at least one of these paths, say  $P_1$ , contains a point in  $\delta(C)$  distinct from  $x$  and  $y$ . The edges of  $E(G)$  involved in a crossing in  $V(P_1)$  are inner chords of the same special cycle  $S_1$ . Since  $x$  and  $y$  are ends of two of these edges,  $\{x, y\} \subseteq V(S_1)$ .

**Claim 2.**  $P_2$  is the edge  $D[xy]$ .

*Proof.* In the alternative, there exist points of  $V(P_2) \setminus \{x, y\}$  that are in  $\delta(C)$ . The edges of  $E(G)$  involved in crossings of  $V(P_2)$  are inner chords of the same special cycle  $S_2$ , and consequently  $\{x, y\} \subseteq V(S_2)$ .

Suppose that  $S_1 = S_2$ . Then, if we consider an element  $\times \in \delta(C)$ , distinct from  $x$  and  $y$ , Observation 2.2.1 implies that at least two of the four ends of the pair of edges crossing at  $\times$  are drawn in the inner face  $F$  bounded by  $C$ . However, because  $S_1 = S_2$ ,  $F$  is part of the inner face of  $D[S_1]$ , and consequently, there are vertices of  $G$  drawn in the interior of  $D[S_1]$ , a contradiction. Thus, we may assume that  $S_1 \neq S_2$ .

Since  $S_1$  and  $S_2$  have  $x$  and  $y$  in common, the assumption of how the special cycles intersect implies that  $xy \in E(S_1) \cap E(S_2)$ . In this case,  $xy$  is not crossed in  $D$ , and hence

it is drawn as an  $xy$ -arc in one of the two faces of  $C$ . Regardless of how  $xy$  is drawn in  $D$ , either  $P_1 \cup D[xy]$  or  $P_2 \cup D[xy]$  is an obstruction  $C'$ . Moreover, since  $D[xy]$  neither contains crossings nor vertices of  $G$  in its interior,  $|\delta(C')| < |\delta(C)|$ , contradicting the minimality of  $|\delta(C)|$ . This last contradiction implies that  $P_2$  is the edge  $D[xy]$ .  $\square$

**Claim 3.** *The edge  $xy$  is an outer chord of  $S_1$ .*

*Proof.* Suppose not. Since  $x$  and  $y$  are vertices of  $S_1$ ,  $xy$  is either an inner chord of  $S_1$  or  $xy \in E(S_1)$ . Either of these possibilities, and the fact that  $P_2 = D[xy]$ , imply that the inner face  $F$  bounded by  $C$  is included in the inner face of  $D[S_1]$ . Now, the existence of crossings in  $\delta(C)$  as interior points of  $P_1$  and Observation 2.2.1 imply the existence of vertices of  $G$  drawn in  $F$ . Then there are vertices of  $G$  drawn in the inner face of  $D[S_1]$ , a contradiction.  $\square$

Finally, to see that  $xy$  induces a generalized  $B$ , consider the  $xy$ -path  $P$  of  $S_1$  that is incident with the two inner faces of  $D[S_1 + xy]$ . Since  $P_1$  is an  $xy$ -arc drawn in the inner face of  $D[S_1]$ ,  $D[P]$  is in the inner face of  $D[xy] \cup P_1$ .

From Observation 2.2.1 it follows that every edge  $e \in E(G)$  that has a subsegment in  $P_1$  has its ends in  $V(P)$ . Thus, every such  $e$  is an inner chord of  $S_1$  with both ends in  $P$ . The simple closed curve  $C$  shows that  $x$  and  $y$  are the only vertices of  $V(P)$  incident with the outer face of the subdrawing induced by  $P + xy$  and the inner chords of  $S_1$  with both ends in  $P$ . Thus  $xy$  induces a generalized  $B$ , as desired.  $\square$

### 3.3 Crossing-webs

In this section we introduce crossing-webs, which are bundled drawings satisfying additional connectivity conditions. In Lemma 3.5.1 of Section 3.5 we will show that every bundled drawing can be included as part of a crossing-web (with the further condition of preserving pseudolinearity). We also introduce the notion of *elastic* crossing-webs, and characterize in Theorem 3.3.5, when a *basic* crossing-web is elastic.

A *crossing-web* is a triple  $\mathcal{W} = (G, D, \mathcal{O})$ , where  $D$  is a drawing of a graph  $G$  such that  $\text{pl}(G)$  is spanning and 2-connected; and the outer face of  $D$  is bounded by the cycle  $\mathcal{O}$  of  $\text{pl}(G)$ . We remark that in a crossing-web,  $D$  is a bundled drawing of  $G$ . We will refer to the special cycles of  $D$  also as the *special cycles* of  $\mathcal{W}$  (Figure 3.4a exhibits a crossing-web with its special cycles bounding distinct coloured regions).

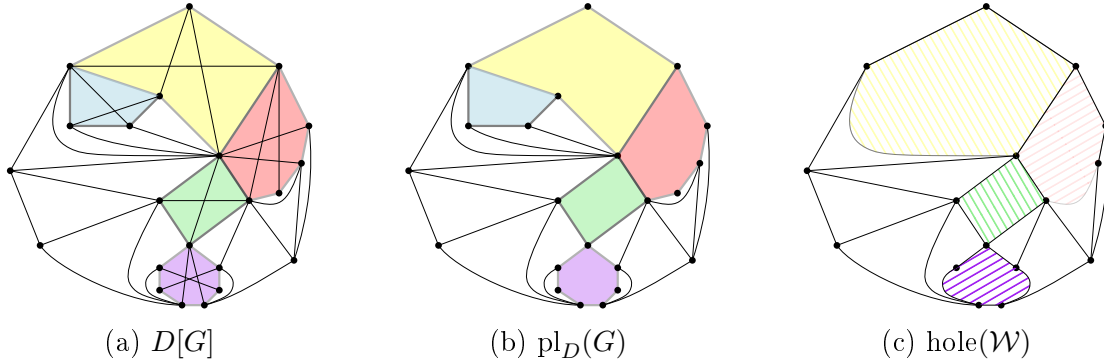


Figure 3.4: A crossing-web  $\mathcal{W} = (G, D, \mathcal{O})$

Let  $\phi$  be a rectilinear embedding of a cycle  $C$ . A *corner* is a vertex of  $C$  with angle in  $\phi[C]$  distinct from  $180^\circ$ . A *facet* is a path  $P$  of  $C$  joining two consecutive corners. We also refer to  $\phi[P]$  as a *facet* of  $\phi[C]$ . In case all the angles in  $\phi[C]$  are at most  $180^\circ$ , then  $\phi$  is a *weakly convex* embedding of  $C$ , and if all the angles are less than  $180^\circ$ , then  $\phi$  is a *strictly convex* embedding of  $C$ .

There are extensions to Fary's Theorem that improve the conditions imposed on the planar embeddings. Perhaps the most celebrated one is the following result by Tutte [30]:

**Theorem 3.3.1** (Tutte's Spring Theorem). *Let  $D$  be an embedding of a 3-connected planar graph with outer face  $\mathcal{O}$ . Then, for every strictly convex embedding  $\phi$  of  $\mathcal{O}$ , there exists a rectilinear embedding  $\bar{D}$  of  $G$  such that:*

- $\phi[\mathcal{O}] = \bar{D}[\mathcal{O}]$ ;
- $D$  is homeomorphic to  $\bar{D}$ ; and
- the boundary of every inner face is strictly convex in  $\bar{D}$ .

This theorem serves as our inspiration for the following definition:

**Definition 3.3.2.** *A crossing-web  $\mathcal{W} = (G, D, \mathcal{O})$  is elastic if, for every strictly convex embedding  $\phi$  of  $\mathcal{O}$ , there exists a rectilinear embedding  $\bar{D}$  of  $G$  such that:*

- (i)  $\phi[\mathcal{O}] = \bar{D}[\mathcal{O}]$ ;
- (ii)  $pl_D(G) = pl_{\bar{D}}(G)$  and  $\bar{D}[pl(G)]$  is homeomorphic to  $D[pl(G)]$ ;

- (iii) for every special cycle  $S$  in  $\mathcal{W}$ ,  $\text{Int}_D(S)$  and  $\text{Int}_{\overline{D}}(S)$  are the same graph; and
- (iv) every face of  $D$  bounded by a cycle of  $G$  is strictly convex in  $\overline{D}$ .

The natural example of an elastic crossing web  $(G, D, \mathcal{O})$  arises from letting  $D$  be an embedding of a 3-connected planar graph  $G$ . The next observation shows that pseudolinearity is a necessary condition for a crossing-web to be elastic.

**Observation 3.3.3.** *If  $(G, D, \mathcal{O})$  is an elastic crossing-web, then  $D$  is pseudolinear.*

*Proof.* Suppose that  $\mathcal{W}$  is elastic. Then, for an arbitrary strictly convex embedding  $\phi$  of  $\mathcal{O}$ , there exists a rectilinear drawing  $\overline{D}$  of  $G$  satisfying conditions (i)-(iv) in Definition 3.3.2. Conditions (ii) and (iii) imply that  $D$  and  $\overline{D}$  have the same special cycles, and each of them has the same external and internal chords. Moreover, these conditions also imply that an outer chord of a special cycle  $S$  induces a generalized  $B$  in  $D$  if and only if it induces a generalized  $B$  in  $\overline{D}$ . Because  $\overline{D}$  is pseudolinear, Lemma 3.2.2 implies that  $D$  is pseudolinear.  $\square$

If  $S$  is a special cycle of a crossing-web  $(G, D, \mathcal{O})$ , then  $\text{ec}(S)$  is the graph obtained from  $S$  by adding the external chords of  $S$  in  $D[G]$ . The outer cycle of  $D[\text{ec}(S)]$  is denoted as  $\mathcal{O}_S$ .

**Definition 3.3.4.** *A crossing-web  $\mathcal{W} = (G, D, \mathcal{O})$  is basic if*

- $\mathcal{W}$  has exactly one special cycle  $S$ ;
- $\mathcal{O} = \mathcal{O}_S$ ; and
- for every facial cycle  $C$  of  $D[\text{ec}(S)]$  distinct from  $S$  and  $\mathcal{O}$ ,  $\text{Int}_D(C)$  is a 3-connected planar graph embedded in the disk bounded by  $D[C]$ .

Next we show that pseudolinearity is also a sufficient condition for a basic crossing-web to be elastic.

**Theorem 3.3.5.** *A basic crossing-web  $(G, D, \mathcal{O})$  is elastic if and only if  $D$  is pseudolinear.*

For proving Theorem 3.3.5 we need two results. The first is an extension of Theorem 3.3.1.

**Theorem 3.3.6.** [28] *Let  $D$  be an embedding of 2-connected graph  $G$  in the plane with outer cycle  $\mathcal{O}$ . Suppose that  $\phi$  is a weakly convex embedding of  $\mathcal{O}$ . If, for every 2-cut  $\{x, y\}$ ,  $\{x, y\} \subseteq V(\mathcal{O})$  and  $x$  and  $y$  are not in the same facet, then there exists a rectilinear embedding  $\overline{D}$  of  $G$  such that:*

- $\phi[\mathcal{O}] = \overline{D}[\mathcal{O}]$ ;
- $\overline{D}$  is homeomorphic to  $D$ ; and
- every inner face of  $\overline{D}$  is bounded by a strictly convex cycle.

*In particular, if  $\phi$  is a strictly convex embedding of  $\mathcal{O}$ , such  $\overline{D}$  exists if and only if for every 2-cut  $\{x, y\}$ ,  $\{x, y\} \subseteq V(\mathcal{O})$  and  $xy \notin E(\mathcal{O})$ .*

The second result that we need is an observation about slightly modifying a rectilinear drawing of a graph. We sketch its proof.

**Observation 3.3.7.** *Let  $G$  be a graph and let  $\rho : V(G) \rightarrow \mathbb{R}^2$  be a map inducing a rectilinear good drawing of  $G$ . Then, there exists  $\epsilon > 0$  such that, if  $\rho' : V(G) \rightarrow \mathbb{R}^2$  is a map with  $\|\rho(v) - \rho'(v)\| < \epsilon$  for all  $v \in V(G)$ , then  $\rho'$  induces a rectilinear drawing of  $G$  such that:*

- (i)  $\rho[G]$  and  $\rho'[G]$  are homeomorphic;
- (ii) every cycle bounding a face of  $\rho[G]$  that is strictly convex in  $\rho$  is strictly convex in  $\rho'$ .

*Proof's sketch.* Suppose that, for some  $\epsilon > 0$  to be chosen later,  $\rho' : V(G) \rightarrow \mathbb{R}^2$  is such that  $\|\rho(v) - \rho'(v)\| < \epsilon$  for all  $v \in V(G)$ .

For  $v \in V(G)$ , let  $(v_1, v_2, \dots, v_r)$  be the cyclic counterclock-wise order in which  $v$  is joined to its neighbours  $v_1, \dots, v_r$  in  $\rho[G]$ . This sequence  $(v_1, v_2, \dots, v_r)$  is the *rotation* at  $v$  in  $\rho[G]$ .

- For each  $v \in V(G)$ , there is a sufficiently small  $\epsilon > 0$  for which the rotation at  $v$  in  $\rho[G]$  is the same as in  $\rho'[G]$ .
- For each pair  $(e, f) \in E(G) \times E(G)$  with  $e \neq f$ , there is a sufficiently small  $\epsilon > 0$  such that  $\rho[e]$  and  $\rho[f]$  are either crossing, incident or disjoint if and only if the same holds for  $\rho'[e]$  and  $\rho'[f]$ . Moreover, if  $e$  and  $f$  are crossing at  $\times$ , and  $H_{e,f}, H'_{e,f}$  are the underlying plane graphs of  $\rho[e \cup f]$  and  $\rho'[e \cup f]$ , respectively, then there is a sufficiently small  $\epsilon$  such that the rotations at  $\times$  in  $H_{e,f}$  and  $H'_{e,f}$  are the same.

- For every edge  $xy \in E(G)$ , and every pair  $\times_1$  and  $\times_2$  of crossings occurring in this linear order in  $D[xy]$  (when  $xy$  is oriented from  $x$  to  $y$ ), there is a sufficiently small  $\epsilon$ , such that the corresponding crossings  $\times'_1$  and  $\times'_2$  occur in the same order in  $\rho'[xy]$ .
- For every cycle  $C$  such that  $\rho[C]$  is strictly convex, there is a sufficiently small  $\epsilon$  guaranteeing that  $\rho'[C]$  is also strictly convex.

There are finitely many elements considered in each of the previous items. Thus, for a sufficiently small choice of  $\epsilon > 0$ ,  $\rho'$  satisfy all of the previous items. This, in particular implies the conclusion in Observation 3.3.7.  $\square$

In the proof of Theorem 3.3.5, Observation 3.3.7 is used every time we slightly move some points in a given rectilinear drawing, so we will not make an explicit reference to this observation.

*Proof of Theorem 3.3.5.* Let  $\mathcal{W} = (G, D, \mathcal{O})$  be a basic crossing-web with a unique special cycle  $S$ . Observation 3.3.3 shows that if  $\mathcal{W}$  is elastic, then  $D$  is pseudolinear.

Conversely suppose that  $D$  is pseudolinear. Let  $\phi$  be a strictly convex embedding of  $\mathcal{O}$ .

An *intermediate cycle* is a cycle  $C$  of  $\text{ec}(S)$  such that  $D[S]$  is drawn in the disk bounded by  $D[C]$ . A *partial drawing* with respect to  $C$ , is a function  $\rho$  that assigns to each vertex in  $V(\text{Ext}_D(C)) \cup V(S)$  a point in the plane with the following properties:

- (i) For every  $v \in V(\mathcal{O})$ ,  $\rho(v) = \phi(v)$ ;
- (ii) the map  $\rho$  restricted to  $V(\text{Ext}_D(C))$  induces a rectilinear embedding homeomorphic to  $D[\text{Ext}_D(C)]$ , in which the boundary cycle of every inner face, with the exception of  $C$ , is strictly convex;
- (iii) no two edges of  $\rho[C]$  are collinear;
- (iv) if  $xy \in E(C) \setminus E(S)$  and  $P$  is the  $xy$ -path of  $S$  incident with the two inner faces of  $D[S + xy]$ , then  $\rho[P]$  is embedded as a line segment connecting  $\rho(x)$  and  $\rho(y)$ ; and
- (v) if  $uv$  is an internal chord of  $S$  in  $D$  such that  $\rho(u)$  and  $\rho(v)$  are not drawn in the same facet of  $\rho[S]$ , then the open line segment joining  $\rho(u)$  to  $\rho(v)$  is included in the inner face of  $\rho[S]$ .



For example, there is a partial embedding  $\rho_0 : V(S) \rightarrow \mathbb{R}^2$  with respect to the intermediate cycle  $\mathcal{O}$ , obtained from letting  $\phi_0(v) = \phi(v)$  for every  $v \in V(\mathcal{O})$ ; and, for every edge  $xy \in E(\mathcal{O}) \setminus E(S)$ , the path  $P$  as in Item (iv) is embedded in  $\rho_0$  as a line segment connecting  $\phi(x)$  to  $\phi(y)$ . Note that  $\rho_0$  is not necessarily a drawing: for every  $xy \in E(\mathcal{O}) \setminus E(S)$ , the internal vertices of  $P$  are drawn over the edge  $\phi[xy]$ .

It is useful to think a partial drawing as a rectilinear embedding  $\rho$  of  $\text{Ext}_D(C)$ , in which  $\rho[C]$  induces a rectilinear drawing of  $\rho[S]$  (where the edges of  $C$  determine the facets of  $S$ ), and  $\rho[S]$  satisfies the visibility constraints for the inner chords of  $S$  given in (v).

Let  $C$  be an intermediate cycle for which there exists a partial drawing  $\rho$  with respect to  $C$ . We choose  $C$  such that  $|V(\text{Ext}_D(C))|$  is maximum. If  $C = S$ , then  $\rho$  is our desired rectilinear embedding

By way of contradiction suppose that  $C \neq S$ . Then  $C$  has an edge  $xy \in E(C) \setminus E(S)$ . For this edge  $xy$ , let  $P$  be the  $xy$ -path as in Item (iv).

**Claim 1.** *There exists an interior vertex  $z$  of  $P$  such that every internal chord  $uv$  of  $S$  with  $u, v \in V(P)$ , either both  $u$  and  $v$  are in the  $xz$ -path of  $P$ , or they are in the  $zy$ -path of  $P$ .*

*Proof.* Suppose not. Consider the subdrawing  $D'$  of  $D$  obtained from  $D[P + xy]$  by adding all the internal chords of  $S$  that have both ends in  $P$ . By assumption,  $x$  and  $y$  are the only vertices in the outer face of  $D'$ . Thus  $xy$  induces a generalized  $B$ , contradicting that  $D$  is pseudolinear (Lemma 3.2.2).  $\square$

Seeking a contradiction, our goal is to find an intermediate cycle  $C'$  and a partial drawing  $\rho'$  with respect to  $C'$ , for which  $\text{Ext}_D(C)$  is a proper subgraph of  $\text{Ext}_D(C')$ .

To find  $C'$ , consider the subgraph  $H_P$  of  $\text{ec}(S)$ , obtained from  $P$  by adding all the external chords of  $S$  with both ends in  $P$ . Let  $E_{xy}$  be the edges in  $H_P$  whose ends are not both in  $P[x, z]$  and not both in  $P[z, y]$ . Note that  $z$  and  $xy$  are incident with the same inner face  $F_{xy}$  of  $D[H_P - (E_{xy} \setminus \{xy\})]$ . Let  $P'$  be the  $xy$ -path in the boundary of  $F_{xy}$  distinct from the edge  $xy$ , and let  $C'$  be the cycle obtained from replacing  $xy$  in  $C$  by  $P'$ .

We start describing our first approximation to  $\rho'$ : For each vertex  $v$  in  $\text{Ext}_D(C)$ , let  $\rho_1(v) = \rho(v)$ . Note that vertices in  $P$  are located in  $\rho$  as points in the line segment joining  $\rho(x)$  to  $\rho(y)$ . Let  $\rho_1(z)$  be a point near  $\rho(z)$ , drawn in the inner face bounded by  $\rho[C]$ . Locate in  $\rho_1$  the vertices of  $P$  in such a way that the  $xz$ -path of  $P$  is embedded as a line segment joining  $\rho_1(x)$  to  $\rho_1(z)$ , and the  $zy$ -path of  $P$  is embedded as a line segment joining  $\rho_1(z)$  to  $\rho_1(y)$ . It is important to keep the images of vertices in  $P - \{x, y\}$  under  $\rho_1$  near their images under  $\rho$ .

As consequence of Claim 1, we can pick  $\rho_1(z)$  close enough  $\rho(z)$ , so that  $\rho_1$  satisfies the visibility requirements for the inner chords in Item (v). The map  $\rho_1$  is not necessarily a partial embedding: for instance,  $\rho_1$  may not satisfy Item (iii) with respect to  $C'$ , as some edges of  $P'$  might be drawn as part of  $xz$ -facet or as part of  $zy$ -facet of  $\rho_1[C']$ . Moreover, we have not defined where the vertices of  $\text{int}_D(P' + xy)$  are located. To fix this last issue, we need the following observation.

**Claim 2.** *The map  $\rho_1$  induces a rectilinear embedding of  $P' + E_{xy}$  in which the inner faces are bounded by weakly convex polygons.*

*Proof.* Let  $C_{xy} = P' + xy$ . By definition,  $\rho_1[C_{xy}]$  is embedded as a (weakly convex) triangle.

To show that  $\rho_1$  induces an embedding of  $P' + E_{xy}$ , first observe that every edge  $E_{xy}$  is drawn in  $\rho_1$  as a line segment connecting a point in the  $xz$ -facet of  $\rho_1[C_{xy}]$  to a point in the  $zy$ -facet of  $\rho_1[C_{xy}]$ . Thus, two edges  $x_1y_1, x_2y_2 \in E_{xy}$  cross in  $\rho_1$ , if and only if, the vertices  $x_1, x_2, y_1, y_2$  occur in this cyclic order as we follow some orientation of  $C_{xy}$ . However, this same condition would imply that  $x_1y_1, x_2y_2$  cross in  $D[P' + E_{xy}]$ , and  $D[P' + E_{xy}]$  is an embedding. Thus  $\rho_1$  induces a rectilinear embedding of  $P' + E_{xy}$ .

The way edges in  $E_{xy}$  are drawn in  $\rho_1$  induces a linear order on  $E_{xy}$ : an edge  $x_1y_1$  precedes  $x_2y_2$  if and only if the the triangle with corners  $\rho_1(x_1), \rho_1(z)$  and  $\rho_1(y_1)$  contains  $\rho_1[x_2y_2]$  as inner chord. In this case, each face of  $\rho_1[P' + E_{xy}]$  is either bounded by a weakly convex 4-gon (with two consecutive edges of  $E_{xy}$  its boundary), or bounded by a weakly convex triangle (with the last edge of  $E_{xy}$  and  $z$  in its boundary).  $\square$

Thus,  $\rho_1$  induces a rectilinear drawing of the graph obtained from the union of  $\text{Ext}_D(C)$  and  $P' + E_{xy}$ . To find a rectilinear drawing of  $\text{Ext}_D(C')$ , it remains to determine where the vertices of  $G$  drawn in the inner faces of  $D[P' + E_{xy}]$  are located under  $\rho_1$ .

For every inner face  $F$  of  $D[P' + E_{xy}]$ , let  $C_F$  be its boundary cycle. Since  $\mathcal{W}$  is basic,  $D[\text{Int}_D(C_F)]$  is a 3-connected planar graph embedded in the interior for  $D[C_F]$ . Apply Theorem 3.3.6 to extend  $\rho_1[C_F]$  to a rectilinear embedding of  $\text{Int}_D(C_F)$  in which every inner face is strictly convex. Such an embedding is how  $\rho_1$  is defined in its restriction to  $\text{Int}_D[C_F]$ .

At this stage,  $\rho_1$  satisfies the conditions for being a partial embedding with the exception of (iii), as either

- the edges in the  $xz$ -facet  $P'_{xz}$  of  $\rho_1[C']$  are in the same line; or
- the edges in the  $zy$ -facet of  $P'_{zy}$  of  $\rho_1[C']$  are in the same line.

To resolve this issue, we slightly push the interior vertices of each  $P'_{xz}$  and  $P'_{zy}$  towards the outer face of  $\rho_1(C')$ . To retain visibility of the inner chords imposed by (v), we push them so that the cycles  $P'_{xz} + xz$  and  $P'_{zy} + zy$  form strictly convex polygons. Let  $\rho'$  be the new function obtained from  $\rho_1$  after pushing these vertices; extend  $\rho'$  to all vertices in  $S$  in a way that, if  $uv \in E(P') \setminus E(S)$  and  $Q$  is the  $uv$ -path of  $S$  incident with the inner faces of  $D[S + uv]$ , then  $\rho'[Q]$  is drawn as line segment connecting  $\rho'(u)$  to  $\rho'(v)$ .

Now  $\rho'$  is a partial drawing of  $G$  with respect to  $C'$ , and  $|V(\text{Ext}_D(C'))| > |V(\text{Ext}_D(C))|$ . This contradicts our choice of  $C$ . Thus, there exists a partial embedding in which  $S$  is the intermediate cycle, showing that  $\mathcal{W}$  is elastic.  $\square$

### 3.4 Full crossing-webs

In this section we characterize when a full crossing-web is elastic. This is the main ingredient used in the proof of Theorem 3.1.1.

**Definition 3.4.1.** *A crossing-web  $\mathcal{W} = (G, D, \mathcal{O})$  is full if*

- every cycle of  $G$  bounding an inner face of  $D$  is a 3-cycle; and
- for every two distinct special cycles  $S_1$  and  $S_2$  in  $\mathcal{W}$ , the intersection between  $S_1$  and  $S_2$  is either empty, a vertex, or an edge.

Let  $\mathcal{W} = (G, D, \mathcal{O})$  be a crossing-web. A special cycle  $S$  is *maximal* if no other special cycle  $S'$  contains  $S$  in  $\text{Int}_D(\mathcal{O}_{S'})$  (recall that  $\mathcal{O}_S$  is the outer cycle of the subgraph of  $G$  induced by  $S$  and its external chords). The special cycle bounding the yellow region in Figure 3.4a is maximal, while the one bounding the blue region is not. We denote  $\text{hole}(\mathcal{W})$  as the plane graph obtained from  $\text{pl}(G)$  by removing, for each maximal special cycle  $S$  in  $\mathcal{W}$ , the vertices in  $\text{int}(\mathcal{O}_S)$  (see Figure 3.4c).

**Lemma 3.4.2.** *Let  $\mathcal{W}$  be a full crossing-web. Then*

- (i)  *$\text{hole}(\mathcal{W})$  is 2-connected; and*
- (ii) *if  $\{x, y\}$  is a 2-cut of  $G$ , then  $\{x, y\} \subseteq V(\mathcal{O})$  and  $xy \notin E(\mathcal{O})$ .*

*Proof.* We start by showing the following general result.

**Claim 1.** *In a 2-connected plane graph, if we remove all vertices drawn the interior of a cycle, then the resulting graph is 2-connected.*

*Proof.* Let  $C$  be a cycle of a 2-connected plane graph  $H$ . As  $G$  is connected, for every vertex in  $\text{Ext}(C)$ , there is a path in  $\text{Ext}(C)$  connecting such a vertex to some vertex in  $C$ . Thus  $\text{Ext}(C)$  is connected. Every face of  $\text{Ext}(C)$  is bounded by a cycle because it is either a face of  $H$ , or is bounded by  $C$ . Therefore  $\text{Ext}(C)$  is 2-connected.  $\square$

Item (i) follows from repeatedly applying Claim 1.

Let  $\{x, y\}$  be a 2-cut of  $\text{hole}(\mathcal{W})$ . Since  $D[\text{hole}(\mathcal{W})]$  is a planar embedding, there exists a simple closed curve  $\gamma$  in the plane intersecting  $D[\text{hole}(\mathcal{W})]$  at  $\{D[x], D[y]\}$ , and such that there is a vertex of  $\text{hole}(\mathcal{W})$  drawn on each side of  $\gamma$ .

Since  $\text{hole}(\mathcal{W})$  is 2-connected, there exists a cycle  $C_\gamma$  containing vertices drawn on distinct sides of  $\gamma$ . Such  $C_\gamma$  necessarily contains  $x$  and  $y$ , but  $xy \notin E(C_\gamma)$ . The existence of  $C_\gamma$  shows that (a) the two  $xy$ -arcs of  $\gamma$  are drawn in distinct faces  $F_1$  and  $F_2$  of  $D[\text{hole}(\mathcal{W})]$  incident with  $x$  and  $y$ ; and (b) if  $xy \in E(G)$ , then  $D[xy]$  is drawn in exactly one side of  $D[C_\gamma]$ .

**Claim 2.** *At least one of  $F_1$  and  $F_2$  is the outer face of  $D[\text{hole}(\mathcal{W})]$ .*

*Proof.* Suppose that both  $F_1$  and  $F_2$  are inner faces. Each of  $F_1$  and  $F_2$  is either a face of  $D$  bounded by a 3-cycle of  $G$  or, for some maximal special cycle  $S$ , is a face bounded by  $\mathcal{O}_S$ .

Suppose that  $xy \in E(G)$ . From (b) and the fact that  $F_1$  and  $F_2$  are symmetric, we may assume that  $D[xy]$  is drawn in the side of  $D[C_\gamma]$  containing  $F_2$ . In this case,  $F_1$  is not bounded by a 3-cycle of  $G$ , and hence there is a special cycle  $S_1$ , for which  $\mathcal{O}_{S_1}$  is the boundary of  $F_1$ . However,  $xy$  is an external chord of  $S_1$  drawn in the exterior of  $\mathcal{O}_{S_1}$ . This contradicts that  $\mathcal{O}_{S_1}$  is the outer cycle of  $\text{ec}(S_1)$ .

Therefore  $xy \notin E(G)$ . This implies that, for  $i = 1, 2$ , there exists a special cycle  $S_i$  such that  $F_i$  is bounded by  $\mathcal{O}_{S_i}$ . However,  $\{x, y\} \subseteq V(S_1) \cap V(S_2)$  and  $xy \notin E(G)$ , a contradiction.  $\square$

We may assume that  $F_1$  is the outer face of  $D[\text{hole}(\mathcal{W})]$ . This guarantees that  $\{x, y\} \subseteq V(\mathcal{O})$ , so we only need to show that  $xy \notin E(\mathcal{O})$ .

Suppose that  $xy \in E(\mathcal{O})$ . In this case,  $xy$  is incident with the outer face  $F_1$ , and from (a) and (b) it follows that  $xy$  is not in the boundary of  $F_2$ . Then  $F_2$  is not a face of  $D$

bounded by a 3-cycle of  $G$ . In this case, there is a maximal special cycle  $S_2$  with  $\mathcal{O}_{S_2}$  bounding  $F_2$ . However,  $xy$  is an external chord of  $S_2$  drawn in the outer face of  $\mathcal{O}_{S_2}$ , our final contradiction. Thus,  $xy \notin E(\mathcal{O})$ .  $\square$

Let  $\mathcal{W} = (G, D, \mathcal{O})$  be a crossing-web and let  $C$  be a cycle in  $\text{pl}_D(G)$ . Then  $(\text{Int}_D(C), D[\text{Int}_D(C)], C)$  is also a crossing-web that we denote as  $\mathcal{W}|_C$ .

**Lemma 3.4.3.** *Let  $\mathcal{W} = (G, D, \mathcal{O})$  be a full crossing-web that has a special cycle  $S_0$  for which  $\mathcal{O} = \mathcal{O}_{S_0}$ . Let  $\mathcal{W}'$  be a crossing-web obtained from  $\mathcal{W}$  by replacing, for each facial cycle  $C$  of  $\text{ec}(S_0)$  distinct from  $S_0$  and  $\mathcal{O}$ ,  $\mathcal{W}|_C$  by  $\text{hole}(\mathcal{W}|_C)$ . Then  $\mathcal{W}'$  is basic.*

*Proof.* Let  $C$  be a facial cycle of  $\text{ec}(S_0)$  distinct from  $S_0$  and  $\mathcal{O}$ . It suffices to show that  $\text{hole}(\mathcal{W}|_C)$  is 3-connected.

Let  $\mathcal{W}|_C = (G', D', C)$ . Since  $\mathcal{W}|_C$  is full, Lemma 3.4.2(i) implies that  $\text{hole}(\mathcal{W}|_C)$  is 2-connected.

Suppose that  $\{x, y\}$  is a 2-cut of  $G'$ . From Item (ii) of Lemma 3.4.2 we know that  $\{x, y\} \subseteq V(C)$  and  $xy \notin E(C)$ . Since  $C$  is a cycle of  $\text{ec}(S_0)$ ,  $\{x, y\} \subseteq V(S_0)$ . Because  $x$  and  $y$  are incident with the facial cycle  $C$  of  $\text{ec}(S_0)$  and because  $xy \notin E(C)$ , we see that  $xy \notin E(\text{ec}(S_0))$ .

Since  $\{x, y\}$  is a 2-cut in  $G'$ , there exists a face  $F$  in  $D'$  distinct from the outer face of  $D'$  that is incident with  $x$  and  $y$ . As  $xy$  is not in the boundary of  $F$  (because  $xy \notin E(\text{ec}(S_0))$ ),  $F$  is not bounded by a 3-cycle of  $G$ . Therefore, there exists a special cycle  $S'$  such that  $\mathcal{O}_{S'}$  is the boundary of  $F$ . However, this implies that  $S_0$  and  $S'$  have  $x$  and  $y$  in common, but  $xy \notin E(S_0) \cap E(S')$ , a contradiction.  $\square$

Now we are ready to characterize elastic full crossing-webs.

**Theorem 3.4.4.** *A full crossing-web  $\mathcal{W} = (G, D, \mathcal{O})$  is elastic if and only if  $D$  is pseudolinear.*

*Proof.* From Observation 3.3.3 it follows that  $(G, D, \mathcal{O})$  is elastic when  $D$  is pseudolinear. Conversely we assume that  $D$  is pseudolinear.

We will show that  $\mathcal{W}$  is elastic by induction on  $|V(G)|$ , where the statement clearly holds for  $|V(G)| \leq 3$ .

Let  $\phi$  be a strictly convex embedding of  $\mathcal{O}$ . From Lemma 3.4.2, we know that  $\text{hole}(\mathcal{W})$  is 2-connected and, for every 2-cut  $\{x, y\}$ ,  $\{x, y\} \subseteq V(\mathcal{O})$  and  $xy \notin E(\mathcal{O})$ . Thus, the ‘‘in particular’’ part of Theorem 3.3.6 shows that there exists an rectilinear embedding  $\theta$  of  $\text{hole}(\mathcal{W})$  such that

- $\theta[\mathcal{O}] = \phi[\mathcal{O}]$ ;
- $\theta[\text{hole}(\mathcal{W})]$  is homomorphic to  $D[\text{hole}(\mathcal{W})]$
- every facial cycle  $C$  of  $\theta$  is strictly convex.

Let  $\mathcal{S}_{max}$  be the set of maximal special cycles in  $\mathcal{W}$ .

**Case.**  $\text{hole}(\mathcal{W}) \neq \mathcal{O}$ .

In this case, for each special cycle  $S \in \mathcal{S}_{max}$ ,  $\mathcal{O}_S \neq \mathcal{O}$ . Thus, for each  $S \in \mathcal{S}_{max}$ ,  $\mathcal{W}|_{\mathcal{O}_S}$  is a full crossing-web with fewer vertices than  $\mathcal{W}$ . The induction hypothesis implies that  $\mathcal{W}|_{\mathcal{O}_S}$  is elastic; thus, for each  $S \in \mathcal{S}_{max}$ , there is a rectilinear drawing  $\overline{D}_S$  of  $\text{Int}_D(\mathcal{O}_S)$  satisfying Conditions (i)-(iv) in Definition 3.3.2 with respect to  $\mathcal{W}|_{\mathcal{O}_S}$  and  $\theta[\mathcal{O}_S]$ . Let  $\overline{D}$  be the extension of  $\theta$ , obtained from letting, for each  $S \in \mathcal{S}_{max}$  and for each vertex  $v$  in  $\text{Int}_D(\mathcal{O}_S)$ ,  $\overline{D}[v] = \overline{D}_S[v]$ . Then  $\overline{D}$  satisfies Conditions (i)-(iv) in Def. 3.3.2 with respect to  $\mathcal{W}$  and  $\phi$ .

**Case.**  $\text{hole}(\mathcal{W}) = \mathcal{O}$ .

In this case, there exists a special cycle  $S_0$  for which  $\mathcal{O}_{S_0} = \mathcal{O}$ . Every other special cycle  $S$  is drawn in the closure of a face of  $\text{ec}(S_0)$ . Consider the crossing-web  $\mathcal{W}' = (G', D', \mathcal{O})$  obtained from  $\mathcal{W}$  by replacing, for each facial cycle  $C$  of  $\text{ec}(S_0)$  distinct from  $S_0$  and  $\mathcal{O}$ ,  $\mathcal{W}|_C$  by  $\text{hole}(\mathcal{W}|_C)$ . Lemma 3.4.3 shows that  $\mathcal{W}'$  is basic and Theorem 3.3.5 implies that  $\mathcal{W}'$  is elastic. Let  $\overline{D}'$  be a rectilinear drawing satisfying Conditions (i)-(iv) in Definition 3.3.2 with respect to  $\mathcal{W}'$  and  $\phi$ .

Now, for each special cycle  $S$  of  $\mathcal{W}$  for which  $\mathcal{O}_S$  bounds a face of  $\overline{D}'$ ,  $\mathcal{W}|_{\mathcal{O}_S}$  is a full crossing-web, and, since it has fewer vertices than  $\mathcal{W}$ , is elastic. Since  $\overline{D}'[\mathcal{O}_S]$  is a convex drawing of  $\mathcal{O}_S$ , we can extend this to a rectilinear drawing of  $\text{Int}_D(\mathcal{O}_S)$  satisfying conditions (i)-(iv) in Definition 3.3.2. If we do this for every  $S$  in which  $\mathcal{O}_S$  bounds a face of  $\overline{D}'$ , then we obtain a rectilinear drawing  $\overline{D}$  of  $G$  satisfying conditions (i)-(iv) in Def. 3.3.2.  $\square$

## 3.5 Proof of Theorem 3.1.1

In this section we conclude the proof of Theorem 3.1.1 by showing that every pseudolinear bundled drawing with well-behaved special cycles is part of a pseudolinear full crossing-web.

**Lemma 3.5.1.** *Let  $D$  be a bundled drawing of a graph  $G$ . Suppose that, for every two distinct special cycles  $S_1$  and  $S_2$ , the intersection between  $S_1$  and  $S_2$  is either empty, a vertex or an edge. Then there exists a full crossing-web  $(G', D', \mathcal{O})$  such that  $G \subseteq G'$ ,  $D \subseteq D'$ , and for which  $D$  is pseudolinear if and only if  $D'$  is pseudolinear.*

*Proof.* Let  $D'$  be a drawing obtained by adding a cycle  $\mathcal{O}$  in the outer face of  $D$ , such that  $D$  is in the inner face of  $D'[\mathcal{O}]$ . As long as  $(G', D', \mathcal{O})$  is not a full crossing-web, apply one of the following steps:

1. If  $\text{pl}_{D'}(G')$  is not connected, then there exist two vertices  $u$  and  $v$  of  $G'$  in different components of  $\text{pl}_{D'}(G')$ , and incident with the same face  $F$  in  $D'[\text{pl}_{D'}(G')]$ . Add an arc connecting  $D[u]$  to  $D[v]$  through  $F$ , and extend  $G'$  and  $D'$  to include this new edge.
2. Suppose that  $\text{pl}_{D'}(G')$  is connected and that there is a face  $F$  of  $\text{pl}_{D'}(G')$  whose boundary is neither a special cycle nor a 3-cycle. Suppose that  $(w_0, w_1, \dots, w_{m-1}, w_0)$  is the boundary walk of  $F$ . Then add a cycle  $(v_0, v_1, \dots, v_{m-1}, v_0)$  drawn as a simple closed curve in  $F$ , and for  $i = 0, 1, \dots, m-1$ , connect  $v_i$  to  $w_i$  and to  $w_{i+1}$  (where the indices are read mod  $m$ ). Extend  $G'$  and  $D'$  to include the new vertices and edges.

Step 1 reduces the number of components of  $\text{pl}_{D'}(G')$ , while Step 2 reduces the number of inner faces of  $\text{pl}_{D'}(G')$  whose boundary is neither a special cycle nor a 3-cycle. When Steps 1 and 2 do not apply,  $\text{pl}_{D'}(G)$  is connected and the boundary of every inner face of  $\text{pl}_{D'}(G')$  is either a special cycle or a 3-cycle. Moreover, as every face of  $\text{pl}_{D'}(G')$  is bounded by a cycle,  $\text{pl}_{D'}(G')$  is 2-connected. Thus  $(G', D', \mathcal{O})$  is a crossing-web with  $G \subseteq G'$  and  $D \subseteq D'$ .

Steps 1 and 2 do not change the set of special cycles nor the way external and internal chords are drawn for a fixed special cycle. Thus,  $D$  has an external chord  $xy$  of some special cycle  $S$  inducing a generalized  $B$  if and only if  $xy$  induces a generalized  $B$  in  $D'$ . From Lemma 3.2.2, this is equivalent to the statement:  $D$  is pseudolinear if and only if  $D'$  is pseudolinear.  $\square$

*Proof of Theorem 3.1.1.* Let  $D$  be a nice drawing of a graph satisfying the conditions in Theorem 3.1.1.

Suppose that there is a rectilinear drawing  $\overline{D}$  satisfying Conditions (i)-(iii) of Theorem 3.1.1. Then  $D$  and  $\overline{D}$  has the same special cycles, and each of them have the same external and internal chords. Moreover, (i)-(iii) imply that an outer chord of a special cycle  $S$

induces a generalized  $B$  in  $D$  if and only if induces a generalized  $B$  in  $\overline{D}$ . Because  $\overline{D}$  is pseudolinear, Lemma 3.2.2 shows that  $D$  is pseudolinear.

Suppose now that  $D$  is pseudolinear. Let  $\mathcal{W} = (G', D', \mathcal{O})$  be a full crossing-web as in Lemma 3.5.1. Then, since  $D'$  is pseudolinear,  $\mathcal{W}$  is elastic. Thus, for some arbitrary strictly convex embedding  $\phi$  of  $\mathcal{O}$ , there exists a rectilinear drawing  $\overline{D}$  of  $G'$  satisfying (i)-(iv) in Def. 3.3.2. The restriction  $\overline{D}[G]$  satisfies (i)-(iii) of Theorem 3.1.1.  $\square$

### 3.6 Concluding remarks

The results in this chapter aimed to investigate Question 3 for bundled drawings. The motivation was to develop new tools for proving results such as Bienstock and Dean's Theorem [13].

In Theorem 3.1.1 we found sufficient conditions for a drawing  $D$  implying the existence of a rectilinear drawing  $\overline{D}$  having the same pairs of crossing edges. For this result, we assumed that  $D$  is bundled and that every pair of special cycles is either disjoint or that they intersect in a vertex or an edge. After some investigation, we believe that these conditions can be weakened. For instance:

**Question 4.** *Does the conclusion of Theorem 3.1.1 holds for every bundled drawing?*

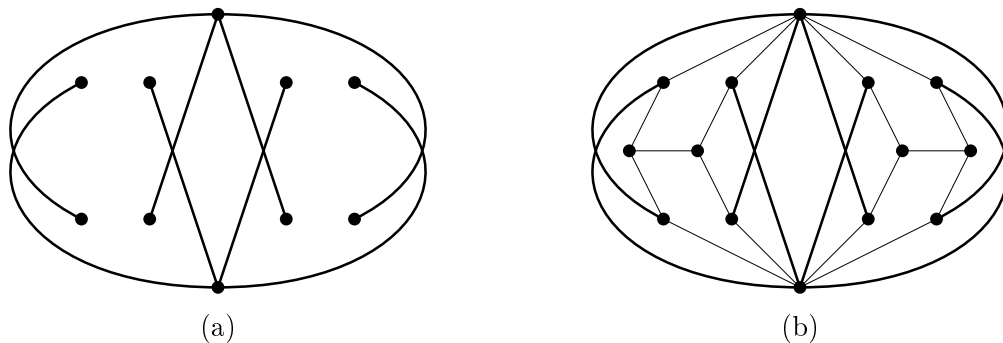


Figure 3.5: Steps to construct a graph with  $cr(G) = 4$  and with  $\overline{cr}(G) > 4$ .

In [13], Bienstock and Dean showed the existence of graphs with crossing number 4 and arbitrarily large rectilinear crossing number. Hernández-Vélez, Leños and Salazar [18] observed that these examples are obtained from a specific non-pseudolinear drawing, such as the one in Figure 3.5a, by adding a planar subdrawing, and replacing uncrossed edges



by edge-disjoint paths (in Figure 3.5b, the thin edges are replaced by edge disjoint paths). This standard technique has the purpose of making some edges undesirable to cross in an optimal drawing of the graph. The authors of [18] are inclined to believe that this is only way to build these examples, and they conjectured the following.

**Conjecture 3.6.1.** *There is a function  $f$  such that for every 3-connected graph  $G$ ,  $\overline{cr}(G) \leq f(cr(G))$ .*

The next question can be taken as a initial approach to this conjecture.

**Question 5.** *Is it true that every 3-connected graph with  $cr(G) = 4$ , has  $\overline{cr}(G) = cr(G)$ ? More specifically: can we find a set of minimal subdrawings  $\mathcal{F}$  such that every optimal drawing of graph with  $cr(G) = 4$  and  $\overline{cr}(G) > 4$  has an element of  $\mathcal{F}$  as a subdrawing?*

Let  $\tilde{cr}(G)$  denote the *pseudolinear crossing number* of  $G$ . In Schaefer's survey, *The graph crossing number and its variants* [26], he asks the question of whether there is a function  $f$  such that  $\overline{cr}(G) \leq f(\tilde{cr}(G))$ . Hernández-Vélez, Leaños and Salazar showed that a separation between  $\overline{cr}$  and  $\tilde{cr}$  exists, as there are arbitrarily large 3-connected graphs  $G$  with  $\overline{cr}(G) \geq (145/144)\tilde{cr}(G)$ . This contrasts with Theorem 3.1.1, where the existence of a pseudolinear drawing implies the existence of a similar rectilinear drawing with the same set of crossing pair of edges. There are instances of this problem where there is no clear separation between rectilinear and pseudolinear crossing numbers; for example, Balogh et al. conjectured in [9] that  $\overline{cr}(K_n) = \tilde{cr}(K_n)$ .

**Question 6.** *Can we identify features of a pseudolinear drawing  $D$  implying the existence of a rectilinear drawing  $\overline{D}$  of the same graph with  $cr(\overline{D}) \leq cr(D)$  (or with  $cr(\overline{D}) \leq f(cr(D))$  for some fixed function  $f$ )?*

The concept of elasticity plays a crucial role in the proof of Theorem 3.1.1. Elasticity is the ability of transforming a given drawing into a similar rectilinear drawing by extending any given strictly convex outer boundary and preserving pairs of crossing edges. Theorem 3.3.5 can be considered as an extension Tutte's Spring Theorem allowing some crossings. It is desirable to find extensions of Theorem 3.3.5.

# Chapter 4

## Pseudospherical Drawings of $K_n$

### 4.1 Introduction

Apart from rectilinear drawings, is there any other interesting class of geometric drawings that deserve being studied? To answer this, we recall the Harary-Hill Conjecture, stating that the crossing number of the complete graph  $K_n$  is equal to:

$$H(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor. \quad (4.1)$$

The British artist, Anthony Hill, is acknowledged for discovering a family of drawings of  $K_n$  having exactly  $H(n)$  crossings. These drawings are beautifully described in the historical survey by Beineke and Wilson, *The Early History of the Brick Factory Problem* [12]:

Label the vertices  $1, 2, \dots, n$ , and arrange the odd numbered ones equally around the inner of two concentric circles and the even ones around the outer circle. Then join all pairs of odd vertices inside the inner circle, join all pairs of even vertices outside the outer circle, and join even vertices to odd ones in the region between the circles.

The drawings described above, cannot be achieved by using shortest-arcs in the plane (i.e. straight line segments), but can be achieved by using shortest arcs in the sphere.

In general, if  $\mathcal{S}$  is the unit sphere in  $\mathbb{R}^3$ , then a *spherical drawing* of a graph  $G$  is one in which the vertices of  $G$  are represented as distinct points in  $\mathcal{S}$ , and every edge is a shortest-arc connecting its corresponding ends.

A shortest arc connecting two points  $p$  and  $q$  in  $\mathcal{S}$  is obtained as follows. Consider a plane through  $p$ ,  $q$  and the centre of  $\mathcal{S}$ . The intersection of this plane with  $\mathcal{S}$  is a *great circle* through  $p$  and  $q$ . The shortest  $pq$ -arc in the great circle is the shortest arc between  $p$  and  $q$  in the sphere. If  $p$  and  $q$  are not *antipodal*, that is, when they are not in the same line through the origin, then the great circle through  $p$  and  $q$  is unique, and so is the shortest  $pq$ -arc.

Although the Harary-Hill Conjecture is known to be true for certain classes of drawings of  $K_n$ , it is yet unknown that spherical drawings have at least  $H(n)$  crossings. In [23], Moon added more mystery to this problem by showing that the crossing number of a random spherical drawing of  $K_n$  is  $\frac{1}{4} \binom{n}{2} \binom{n-1}{2} \binom{n-2}{2} \binom{n-3}{2}$ .

In the proofs of [2] and [19] showing that rectilinear drawings of  $K_n$  have at least  $H(n)$  crossings, a crucial point was to relate the number of crossings in a given drawing to the separation properties of the  $\binom{n}{2}$  lines extending the edges. Understanding these separation properties, but for the curves extending the edges in spherical drawings, serve as our motivation for studying arrangement of pseudocircles.

An *arrangement of pseudocircles* is a set of simple closed curves in the sphere in which every two curves intersect at most twice, and every intersection is a crossing. If  $\gamma$  is a simple closed curve, then a *side* of  $\gamma$  is one of the two disks in  $\mathcal{S}$  bounded by  $\gamma$ . In spherical drawings, the great circles extending the edge-arcs form an arrangement of pseudocircles.

With the aim of finding a combinatorial extension of spherical drawings analogous to how pseudolinear drawings extend rectilinear drawings, there have been two significant questions under active consideration by the graph drawing community in recent years:

- (Q1) Do the edges of every good drawing of  $K_n$  in the sphere extend to an arrangement of pseudocircles?
- (Q2) If the edges of a drawing of  $K_n$  extend to an arrangement of pseudocircles, is there an extending arrangement in which any two pseudocircles intersect exactly twice?

In Subsection 4.2, we provide a negative answer to these questions by showing that:

1. the drawing of  $K_{10}$  in Figure 4.1 does not have any extension to an arrangement of pseudocircles;

2. the drawing of  $K_9$  in Figure 4.2 has an extension to an arrangement of pseudocircles, but there is no extension in which every two pseudocircles cross twice.

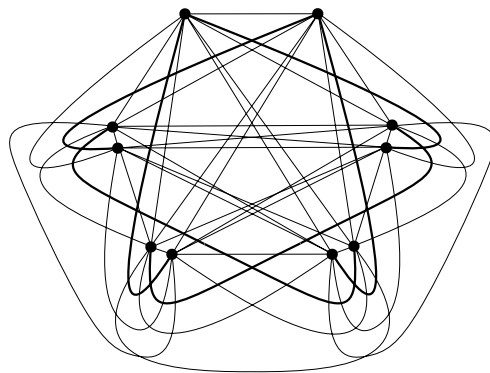


Figure 4.1: A drawing of  $K_{10}$  whose edges cannot be extended to an arrangement of pseudocircles.

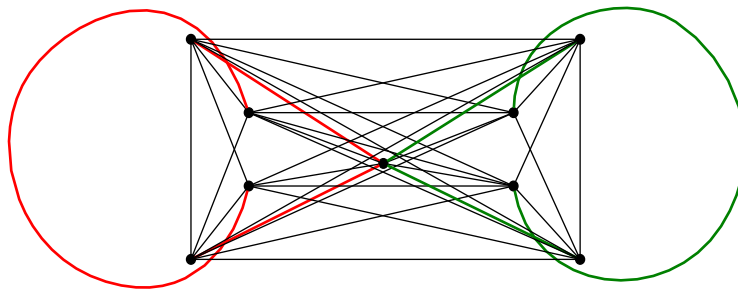


Figure 4.2: A drawing of  $K_9$  whose edges can be extended to an arrangement of pseudocircles, but there is no extension in which the pseudocircles are pairwise intersecting.

In an attempt to describe a combinatorial generalization of spherical drawings, we introduce the following as the definition of pseudospherical drawings.

**Definition 4.1.1.** A drawing  $D$  of  $K_n$  in the sphere  $\mathcal{S}$  is pseudospherical if there exists a family  $\{\gamma_e : e \in E(K_n)\}$  of simple closed curves in  $\mathcal{S}$  such that

(PS1)  $D[e] \subseteq \gamma_e$  and no vertex other than the ends of  $e$  is contained in  $\gamma_e$ ;

(PS2) for every extending curve  $\gamma_e$ , if two vertices  $x$  and  $y$  are drawn on the same side of  $\gamma_e$ , then  $D[xy]$  is completely drawn on that side of  $\gamma_e$ ; and

(PS3) if  $\gamma_e$  and  $\gamma_f$  are curves extending the edges  $e$  and  $f$ , respectively, then  $|\gamma_e \cap \gamma_f| \leq 2$ , and every intersection point in  $\gamma_e \cap \gamma_f$  is a crossing.

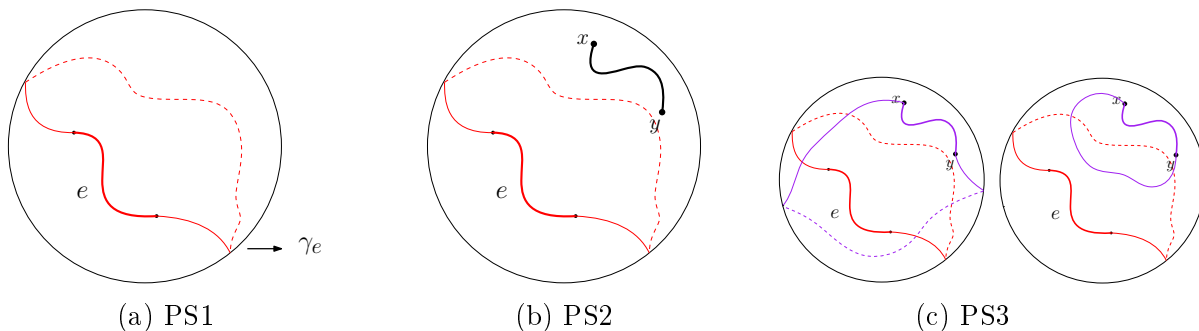


Figure 4.3: Pseudospherical drawings.

Throughout this chapter, we will see how this definition captures many of the essential features of spherical drawings.

Drawings for which there exists an extending set of curves only satisfying (PS1) and (PS3) are the ones considered in questions (Q1) and (Q3). Property (PS2) is an analog to the geometric fact that, if two points are drawn in the same hemisphere of  $\mathcal{S}$ , then the shortest-arc connecting them is also contained in that hemisphere. Adding (PS2) to our list of properties has a big impact on the structure of pseudospherical drawings that brings them closer to spherical.

As our major result for this chapter, we will show that the extending arrangement of pseudospheres can be chosen to resemble a set of great circles; this result answers (Q2) in a positive manner when (PS2) is part of our assumptions.

**Theorem 4.1.2.** *Let  $D$  be a pseudospherical drawing of  $K_n$ . Then there exists a set of simple closed curves satisfying (PS1), (PS2), and*

(PS3') if  $\gamma_e$  and  $\gamma_f$  are curves extending the edges  $e$  and  $f$ , respectively, then  $|\gamma_e \cap \gamma_f| = 2$ , and every intersection point in  $\gamma_e \cap \gamma_f$  is a crossing.

The difference between PS3 and PS3' is that in that in PS3' every two pseudocircles intersect exactly twice.

Questions (Q1) and (Q2) are answered in Section 4.2. In Section 4.3 we prove a technical theorem about arrangements of pseudocircles that will be used in the proof of Theorem

4.1.2. This technical theorem was the inspiration for the gadgets used to answer (Q1) and (Q2).

In Section 4.4 we introduce a class of drawings equivalent to pseudospherical drawings: hereditarily-convex drawings of  $K_n$ . The advantage of this definition is that it focuses only on properties of the drawing rather than in the pseudocircular extensions. Moreover, in [7] it was shown the existence of a  $O(n^6)$  algorithm to recognize whether a drawing of  $K_n$  is hereditarily-convex [7].

Section 4.4 is divided into four subsections. In Subsection 4.4.1 we show that pseudospherical drawings are hereditarily-convex. In Subsections 4.4.2, 4.4.3 and 4.4.4 we show that hereditarily-convex drawings are pseudospherical, and simultaneously give a structural description of hereditarily-convex drawings. This description is crucial in Section 4.5, where we present the proof of Theorem 4.1.2. Finally, in Section 4.6 we give some concluding remarks an open questions related to this chapter.

The present chapter is a collaborative work with Bruce Richter and Matthew Sunohara.

## 4.2 Solutions to Q1 and Q2

In this section we show that: (1) the drawing in Figure 4.1 cannot be extended to an arrangement of pseudocircles; and (2) the drawing in Figure 4.2 can be extended to an arrangement of pseudocircles, but there is no extension in which the pseudocircles are pairwise intersecting.

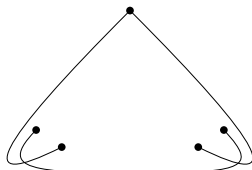


Figure 4.4: The drawing  $D_1$ .

The crucial point is to use the drawing in Figure 4.4, denoted as  $D_1$ , as a gadget forcing some pseudocircles be drawn in a specific region. We label the face of  $D_1$  incident with all of the five vertices as the *interior face*, while the face incident only with the degree 2 vertex is the *exterior face*. Denote the edge incident with two degree 1 vertices as  $e_1$ , while the other two edges are  $e_0$  and  $e_2$ .

Consider an arrangement  $\{\gamma_0, \gamma_1, \gamma_2\}$  of pseudocircles extending the edges of  $D_1$ , where  $\gamma_i$  extends  $e_i$ , for  $i = 0, 1, 2$ .

**Claim 1.** *Either  $\gamma_1$  is included in the closure of the interior face, or  $\gamma_0$  and  $\gamma_2$  are contained in the closure of the interior face.*

*Proof.* Suppose that  $\gamma_1$  is not included in the closure of the interior face. Because  $\gamma_1$  has points in the exterior face of  $D_1$  and because  $\gamma_1$  intersects each of  $\gamma_0$  and  $\gamma_2$  at the crossings between the pairs  $(e_1, e_0)$ ,  $(e_1, e_2)$ ,  $\gamma_1 \setminus e_1$  crosses each of  $e_0$  and  $e_2$  exactly once. This determines, up to homeomorphism, how  $\gamma_1$  is drawn in the sphere (see Figure 4.5). In particular, this determines all crossings between  $\gamma_1$  and each of  $\gamma_0$  and  $\gamma_2$ .

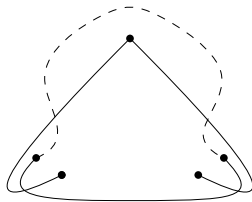


Figure 4.5: The only way  $\gamma_1$  can be drawn if it has points in the exterior face.

If  $\gamma_0$  has points in the exterior face, then, as we follow  $\gamma_0 \setminus e_0$  from the degree 1 end of  $e_0$ ,  $\gamma_0 \setminus e_0$  crosses into the exterior face. This crossing occurs in the interior of  $e_2$ , and together with the degree 2 vertex of  $D_1$ , they account for the intersection points between  $\gamma_0$  and  $\gamma_2$ . However, the ends of  $e_2$  are on distinct sides of a simple closed curve included in  $\gamma_0 \cup \gamma_2 \cup e_2$ , so the arc  $\gamma_2 \setminus e_2$  connects these ends without intersecting this closed curve, a contradiction.  $\square$

We overlap two copies of  $D_1$  to obtain the drawing  $D_2$  shown in Figure 4.6.

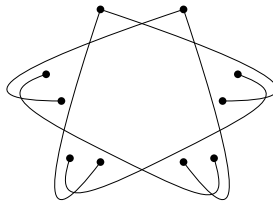


Figure 4.6: The drawing  $D_2$ .

**Claim 2.** *There is no arrangement of pseudocircles extending the edges of  $D_2$ .*

*Proof.* Claim 1 yields four possibilities of how the pseudocircles are drawn in the closure of the interior faces of their corresponding copies of  $D_1$ . It is routine to verify that in each of these four cases, there is a pair of pseudocircles that intersect four times (Figure 4.7).  $\square$

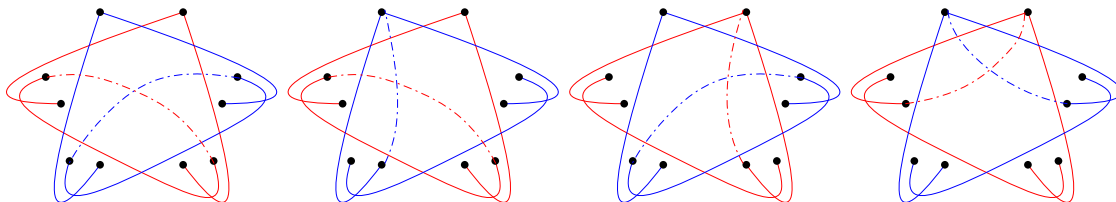


Figure 4.7: The four cases in Claim 2.

The drawing of  $K_{10}$  in Figure 4.1, contains  $D_2$ , and hence no arrangement of pseudocircles extends its edges.

The drawing of  $K_9$  in Figure 4.2 contains two copies of  $D_1$  where their interior faces are disjoint. This observation and Claim 1 imply that in any extension of this drawing to an arrangement of pseudocircles, there is a pair of disjoint pseudocircles.

Finally, to show that the drawing in Figure 4.2 has an extension to an arrangement of pseudocircles, note that the edges of this drawing, with the exception of two that are drawn as circular arcs, are straight line segments. Extend each straight line segment into a line, and each of the circular arcs into a circle. Map the extending curves from the plane to the sphere by using a stereographic projection; lines are mapped into pairwise intersecting great circles. If a line in the plane crosses one of the two circular extensions, then it crosses each circle exactly twice. Thus, all the extensions intersect at most twice.

### 4.3 Arcs in Arrangements of Pseudocircles

In this section we prove a technical result (Theorem 4.3.3) that restricts how crossings occur along an arc contained in the point set of an arrangement of pseudocircles. This result is an important piece in the proof of Theorem 4.5.1.

Let  $\Gamma$  be an arrangement of pseudocircles. We use  $P(\Gamma)$  to denote the point set  $\bigcup_{\gamma \in \Gamma} \gamma$ . We are interested in describing properties of the arcs of  $P(\Gamma)$ , that is, of homeomorphisms of  $[0, 1]$  in  $P(\Gamma)$ . In Figure 4.8 we depict an arc of an arrangement of pseudocircles. We introduce some standard notation helpful for referring to the features of any such arc.



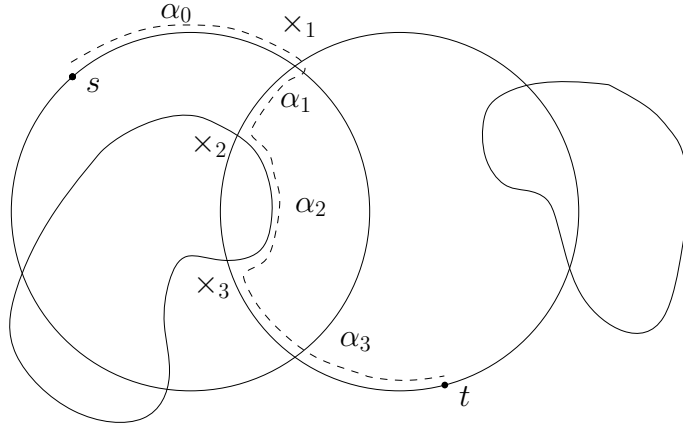


Figure 4.8: An arc in  $P(\Gamma)$

**Definition 4.3.1.** Let  $\Gamma$  be an arrangement of arcs, let  $A$  be an arc of  $P(\Gamma)$ , and suppose  $s$  and  $t$  are the ends of  $A$ . The decomposition of  $A$  in  $\Gamma$  is the unique sequence  $\alpha_0, \alpha_1, \dots, \alpha_m$  of subarcs of  $A$  such that

- (i)  $s$  is an end of  $\alpha_0$ ,  $t$  is an end of  $\alpha_m$ ;
- (ii) for each  $i = 0, 1, \dots, m$ , there is  $\gamma_i \in \Gamma$  for which  $\alpha_i \subseteq \gamma_i$ ;
- (iii) for  $i = 1, \dots, m$ , the curves  $\gamma_{i-1}$  and  $\gamma_i$  are distinct, and  $\alpha_{i-1} \cap \alpha_i$  is a crossing between  $\gamma_{i-1}$  and  $\gamma_i$ ;
- (iv) for  $i = 1, 2, \dots, m$ , the crossing  $\alpha_{i-1} \cap \alpha_i$  is denoted as  $\times_i$ ;
- (v) for convenience, we let  $\times_0 = s$  and  $\times_{m+1} = t$ ; and
- (v) the weight of  $A$  is  $m$ .

Consider an arc  $A$  with a decomposition as in Definition 4.3.1, and suppose  $A$  is oriented from  $s$  to  $t$ . This orientation defines the sides of  $A$ , that is, the points near  $A$  that are either on the left side or the right side. For  $i = 1, 2, \dots, m$ , since  $\gamma_{i-1}$  and  $\gamma_i$  intersect at  $\times_i$ , the ends of  $\gamma_{i-1} \setminus \alpha_{i-1}$  and  $\gamma_i \setminus \alpha_i$  near  $\times_i$  are drawn on the same side of  $A$ ; we denote this as *the side  $\times_i$  is facing*. In Figure 4.8,  $\times_1$  is facing the left, while  $\times_2$  and  $\times_3$  are facing the right of  $A$ .

**Definition 4.3.2.** Let  $\Gamma$  be an arrangement of pseudocircles, and let  $A$  be an arc with decomposition  $\alpha_0 \alpha_1 \cdots \alpha_m$ . For  $i = 0, 1, \dots, m$ :

- (i) the extensions  $\alpha_i^-$  and  $\alpha_i^+$  of  $\alpha_i$  are the components of  $\gamma_i \setminus A$  incident to  $\times_{i-1}$  and  $\times_i$ , respectively (the extensions of  $\alpha_1$  and  $\alpha_3$  are presented in Figure 4.9);
- (ii) for  $\varepsilon \in \{-, +\}$ ,  $\alpha_i^\varepsilon$  is coherent if the segments of  $\alpha_i^\varepsilon$  near its ends are on the same side of  $A$  (in Figure 4.9,  $\alpha_1^-$  is the only non-coherent extension; the other three are coherent);
- (iii) the extensions  $\alpha_0^-$  and  $\alpha_m^+$  are not coherent;
- (iv) for  $\varepsilon \in \{-, +\}$ ,  $y_i^\varepsilon$  is the end of  $\alpha_i^\varepsilon$  distinct from  $\times_i$  when  $\varepsilon = -$ , and distinct from  $\times_{i+1}$  when  $\varepsilon = +$ ; and
- (v)  $\alpha_i$  is coherent if  $\alpha_i^- \neq \alpha_i^+$ , and at least one of its extensions is coherent (in Figure 4.8,  $\alpha_2^-$  and  $\alpha_2^+$  are coherent, but  $\alpha_2$  is not as  $\alpha_2^- = \alpha_2^+$ );

If for every  $i \in \{0 \dots, m\}$ ,  $\alpha_i$  is coherent, then  $A$  is coherent.

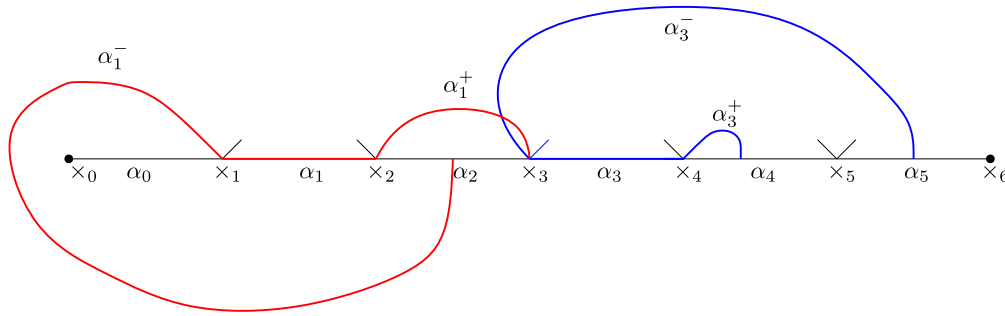


Figure 4.9: An arc in  $P(\Gamma)$  and some extensions

The next is a surprisingly technical result about arrangements of pseudocircles. Although, its proof is not particularly difficult, it plays a central role on the proof of Theorem 4.5.1. This result was also the inspiration for the gadgets in Figures 4.4, 4.6 used to answer (Q1) and (Q2).

**Theorem 4.3.3.** *Let  $\Gamma$  be an arrangement of pseudocircles in the sphere. Suppose that  $A$  is a coherent arc of  $P(\Gamma)$  with decomposition  $\alpha_0\alpha_1 \cdots \alpha_m$ . Then not all of  $\times_1, \times_2, \dots, \times_m$  face the same side of  $A$ .*

*Proof.* We proceed by induction on the weight  $m$  of  $A$ . When  $m = 0$ , none of the extensions of  $\alpha_0$  is coherent (Definition 4.3.2(iii)); thus the base case vacuously holds. Let us assume that  $m \geq 1$ .

It is convenient to consider the linear order  $\prec$  over the point set  $A$ , where  $p \prec q$  if  $p$  occurs first when we traverse  $A$  from  $s$  to  $t$ . We write  $p \preceq q$  if either  $p \prec q$  or  $p = q$ . Recall that, for  $j = 0, 1, \dots, m$ ,  $y_j^-$  is the end of  $\alpha_j^-$  distinct from  $\times_j$ , and  $y_j^+$  is the end of  $\alpha_j^+$  distinct from  $\times_{j+1}$ . The extension  $\alpha_j^+$  is *forwards* if  $\times_{j+1} \prec y_j^+$  and otherwise is *reversed*. Likewise, an extension  $\alpha_j^-$  is *forwards* if  $y_j^- \prec \times_j$  and otherwise is *reversed*. Note that ‘forwards’ and ‘reversed’ does not depend on whether we orient  $A$  from  $s$  to  $t$  or  $t$  to  $s$ .

In Claim 2 we rule out the existence of reversed coherent extensions; but first we need to understand more about forward extensions.

**Claim 1.** *Let  $\alpha_j^+$  be a forward coherent extension of  $\alpha_j$  with  $\times_{j+2} \prec y_j^+$ . Suppose that for every  $k > j$ ,  $\alpha_k^-$  is not a reversed coherent extension. Then*

- (i) *for every  $\times_i$  with  $\times_{j+1} \prec \times_i \prec y_j^+$ ,  $\alpha_i^- \cap \alpha_j^+ = \emptyset$ ; and*
- (ii)  *$\alpha_{j+1}^+ \cap \alpha_j^+ \neq \emptyset$ .*

*Likewise, let  $\alpha_j^-$  be a forward coherent extension of  $\alpha_j$  with  $y_j^- \prec \times_{j-1}$ . Suppose that, for every  $k < j$ ,  $\alpha_k^+$  is not a reversed coherent extension. Then*

- (iii) *for every  $\times_i$  with  $y_i^- \prec \times_i \prec \times_j$ ,  $\alpha_i^+ \cap \alpha_j^- = \emptyset$ ; and*
- (iv)  *$\alpha_{j-1}^- \cap \alpha_j^- \neq \emptyset$ .*

*Proof.* We show (i) and (ii), as (iii) and (iv) are the same, but for the traversal of  $A$  in reverse direction.

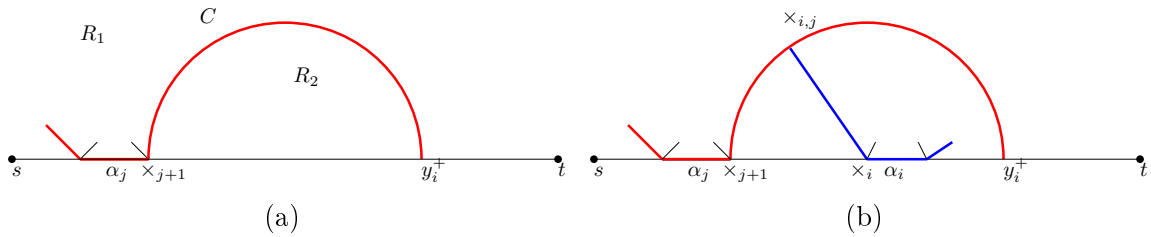


Figure 4.10: Extensions in Claim 1.

To show (i), suppose that for some  $\times_i$  with  $\times_{j+1} \prec \times_i \prec y_i^+$ ,  $\alpha_i^- \cap \alpha_j^+ \neq \emptyset$ . Let  $C$  be the simple closed curve obtained from the union of the two  $\times_{j+1}y_j^+$ -arcs in  $A \cup \alpha_j^+$ . This curve  $C$  defines two closed regions  $R_1$  and  $R_2$ ; one of these regions, say  $R_1$ , contains  $s$  and  $t$  (see Figure 4.10a).

Let  $\times_{j,i}$  be the first point we encounter in  $\alpha_j^+$ , when we traverse  $\alpha_i^-$  from  $\times_i$  to  $y_i^-$ . Since  $\times_i$  faces the same side of  $A$  as the ends of  $\alpha_j^+$ , the  $\times_i \times_{j,i}$ -arc of  $\alpha_i^-$  is included in  $R_2$  (Figure 4.10b). Moreover, since  $\alpha_i^-$  is not reversed,  $\times_{j,i} \neq y_j^+$ .

Let  $\alpha_j^*$  be the arc obtained by following  $\alpha_j$  from  $\times_j$  to  $\times_{j+1}$ , and then continuing on  $\alpha_j^+$  until we reach  $\times_{j,i}$ . Similarly, we let  $\alpha_i^*$  be the arc obtained by following  $\alpha_i$  from  $\times_{i+1}$  to  $\times_i$  and continuing on  $\alpha_i^-$  until we reach  $\times_{j,i}$ .

Let  $A'$  be the arc obtained from  $A$  by replacing its  $\times_j \times_{i+1}$ -subarc by  $\alpha_j^* \cup \alpha_i^*$ . Note that  $A'$  decomposes into  $\alpha_0 \alpha_1 \cdots \alpha_{j-1} \alpha_j^* \alpha_i^* \alpha_{i+1} \cdots \alpha_m$ , and it has smaller weight than  $A$ .

The crossings of  $A'$  as in Definition 4.3.1(iii) are given by the sequence  $\times_0, \times_1, \dots, \times_j, \times_{j,i}, \times_{i+1}, \dots, \times_{m+1}$ . Note that  $\times_{i,j}$  faces the same side as the rest of the crossings in  $A'$ , since the points of the extension  $(\alpha_j^*)^+$  near  $\times_{i,j}$ , are precisely the points in  $\alpha_j^+ \setminus \alpha_j^*$  near  $\times_{i,j}$ .

Let  $\alpha'$  be any segment of  $A'$ . To show  $\alpha'$  is coherent, note that  $\alpha'$  contains a segment  $\alpha_k$  of  $A$ . Let  $\alpha_k^\varepsilon$  be a coherent extension of  $\alpha_k$ . The extension  $(\alpha')^\varepsilon$  is obtained by following  $\alpha_k^\varepsilon$  from its end in  $\alpha_k$  (or from  $\times_{j,i}$  if  $\alpha_k^\varepsilon \in \{\alpha_j^+, \alpha_i^-\}$ ) and stopping at the first point in  $A'$  that we encounter. An encounter with  $A'$  is guaranteed: either  $y_k^\varepsilon \in A \cap A'$ , or  $y_k^\varepsilon \in A \setminus A'$ , and since the ends of  $A' \setminus A$  are on the same side of  $A$  as the ends of  $\alpha_k^\varepsilon$ ,  $\alpha_k^\varepsilon$  intersects  $A' \setminus A$  when  $y_k^\varepsilon \in A \setminus A'$ . Moreover, in any of the two cases, the ends of  $(\alpha')^\varepsilon$  are on the same side of  $A'$ , and hence  $(\alpha')^\varepsilon$  is coherent.

To complete the proof that  $\alpha'$  is coherent we need to show that  $(\alpha')^- \neq (\alpha')^+$ . First, observe that if both  $(\alpha')^-$  and  $(\alpha')^+$  are coherent, then  $(\alpha')^- \subseteq \alpha_k^-$  and  $(\alpha')^+ \subseteq \alpha_k^+$ . Since  $\alpha_k^- \neq \alpha_k^+$ ,  $(\alpha')^- \neq (\alpha')^+$ . Thus  $A'$  is a coherent arc of smaller weight than  $A$ , and with all its crossings facing the same side, contradicting the induction hypothesis. Then (i) holds.

For (ii), consider the  $\times_{j+1} y_j^+$ -arc  $A''$  of  $A$ . Let  $\times_{j+2}, \times_{j+3}, \dots, \times_\ell$  be the sequence of crossings of  $A$  in the interior of  $A''$ . Then  $A''$  decomposes into  $\alpha_{j+1} \alpha_{j+2} \cdots \alpha_{\ell-1} (\alpha_\ell \cap A'')$ . From (i) we know that, for  $i = j+2, \dots, \ell$ ,  $\alpha_i^-$  is coherent in  $A''$ , and hence all of  $\alpha_{j+2}, \dots, \alpha_\ell$  are coherent in  $A''$ . Since  $A''$  has smaller weight than  $A$ ,  $A''$  is not coherent. Then  $\alpha_{j+1}$  is not coherent in  $A''$ , and since  $\alpha_{j+1}^+$  intersects  $\alpha_j^+ \cup A'$  at least twice,  $\alpha_{j+1}^+ \cap \alpha_j^+ \neq \emptyset$ .  $\square$

**Claim 2.** *A has no reversed coherent extensions.*

*Proof.* By way of contradiction, suppose that  $A$  has a reversed coherent extension. By possibly reorienting  $A$ , we may assume that  $\alpha_j^+$  is a reversed coherent extension for some  $j \in \{1, \dots, m\}$ , and that  $j$  is the smallest index with that property (note that  $\alpha_0^+$  is coherent but not reversed).

Let  $C$  be the simple closed curve bounded in  $A \cup \alpha_j^+$ . The ends of  $\alpha_j^+$  face the same side of  $A$  than  $\times_j$ , thus, if we follow the extension  $\alpha_{j-1}^+$  from  $\times_j$  to  $y_{j-1}^+$ , then we either encounter a point in  $\alpha_j^+$  or in  $C \cap A$ . As  $\alpha_{j-1}^+$  intersects  $C$  at least twice and is not reversed,  $\alpha_{j-1}^+$  intersects  $\alpha_j \cup \alpha_j^+$  in a point  $\times$  distinct from  $\times_j$  and  $y_j^+$ . Thus  $\gamma_{j-1} \cap \gamma_j = \{\times_j, \times\}$ , and hence  $y_j^+ \notin \alpha_{j-1}$  (note that  $y_j^+ \neq \times_j$  because  $\alpha_j^- \neq \alpha_j^+$ ).

The ends of  $\alpha_j^+$  face the same side of  $A$  than  $\times_{j-1}$ , thus  $\alpha_j^-$  intersects  $C$  twice. Since  $\alpha_j^-$  does not intersect the interior of  $\alpha_j \cup \alpha_j^+$ ,  $\alpha_j^- \cap C = \{y_j^-, \times_j\}$ . Because  $\gamma_{j-1} \cap \gamma_j = \{\times_j, \times\}$ ,  $y_j^- \notin \alpha_{j-1}$ , and hence  $y_j^+ \preceq y_j^- \prec \times_{j-1}$ . Thus  $\alpha_j^-$  is a forward coherent extension of  $\alpha_j$  with  $y_j^- \prec \times_{j-1}$ . The minimality of  $j$  implies that for every  $k < j$ ,  $\alpha_k^+$  is not a reversed coherent extension. From Claim 1 (iv), we know that  $\alpha_{j-1}^-$  and  $\alpha_j^-$  have non-empty intersection, and since  $\alpha_{j-1}^- \neq \alpha_{j-1}^+$ , they intersect in point distinct from  $\times_j$  and  $\times$ . This contradicts that  $|\gamma_{j-1} \cap \gamma_j| = 2$ .  $\square$

Our next goal is to show in Claim 4 that forward coherent extensions are “small jumps”. But first we need to show the following.

**Claim 3.** *Let  $k \in \{1, 2, \dots, m\}$ . Suppose  $\alpha_{k-1}^-$  is not a coherent extension and that  $\gamma_k \setminus \alpha_k^-$  has an intersection with  $\gamma_{k-1}$ . Then  $\alpha_k^-$  is not coherent. Likewise, if  $\alpha_k^+$  is not coherent and  $\gamma_{k-1} \setminus \alpha_{k-1}^+$  has an intersection with  $\gamma_k$ , then  $\alpha_{k-1}^+$  is not coherent.*

*Proof.* We prove the first statement; the second is the same, but for the traversal of  $A$  in the reverse direction.

Let  $\times$  be an intersection between  $\gamma_k \setminus \alpha_k^-$  and  $\gamma_{k-1}$ . Since  $\times_k \in \alpha_k^-$  is also an intersection between  $\gamma_{k-1}$  and  $\gamma_k$ ,  $\gamma_{k-1} \cap \gamma_k = \{\times, \times_k\}$ .

Consider a simple closed curve  $C$  obtained from the union of  $\alpha_{k-1}^-$  and the subarc of  $A$  connecting the ends of  $\alpha_{k-1}^-$ . Let  $p$  be a point of  $\alpha_k^-$  near  $\times_k$ , and let  $A'$  be the  $\times_0 \times_{k-1}$ -arc of  $A$ . Consider an arc  $\delta$ , starting on  $p$ , following  $A$  alongside  $\alpha_{k-1}$ , crossing  $\alpha_{k-1}^-$ , and continuing beside  $A'$  until we reach a point near  $\times_0$ . Since  $\alpha_{k-1}^-$  is not coherent,  $\delta$  crosses  $C$  exactly once, and hence  $p$  is on a side of  $C$ , different from the one containing the points near  $A'$  on the side of  $A$  faced by the crossings in its decomposition.

By way of contradiction, suppose that  $\alpha_k^-$  is coherent. Claim 2 implies that  $y_k^- \prec \times_k$ . Since  $\alpha_k^-$  does not intersect  $\gamma_{k-1} \setminus \{\times_k\}$ ,  $\alpha_k^-$  is disjoint from  $C$ . This and the last observation in the previous paragraph together imply that  $y_k^- \in \alpha_{k-1} \setminus \{\times_k\}$ . Then  $y_k^-$ ,  $\times_k$  and  $\times$  are three different points in  $\gamma_{k-1} \cap \gamma_k$ , a contradiction.  $\square$

In the final step we get an accurate description of all the coherent extensions.

**Claim 4.** For  $j \in \{0, 1, \dots, m-1\}$ , if  $\alpha_j^+$  is a coherent extension, then  $y_j^+ \in \alpha_{j+1}$ . Likewise, for  $j \in \{1, \dots, m\}$ , if  $\alpha_j^-$  is a coherent extension, then  $y_j^- \in \alpha_{j-1}$ .

*Proof.* We prove the first statement; the second is the same, but for the traversal of  $A$  in the reverse direction.

Suppose by the way of contradiction that the first statement is false. Let  $j$  be the smallest index for which  $\alpha_j^+$  is coherent and  $y_j^+ \notin \alpha_{j+1}$ .

We claim that, for  $k = 0, 1, \dots, j$ ,  $\alpha_k^-$  is not coherent. The proof is by induction on  $k$ , where the case  $k = 0$  holds as  $\alpha_0^-$  is not coherent (see Definition 4.3.2 (iii)). Now, inductively suppose that  $\alpha_{k-1}^-$  is not coherent for  $k \in \{0, \dots, j\}$ . Since  $\alpha_{k-1}^+$  is coherent, the choice of  $j$  implies that  $y_{k-1}^+ \in \alpha_k$ . From this we know that  $\gamma_k \setminus \alpha_k^-$  and  $\gamma_{k-1}$  intersect at  $y_{k-1}^+$ . Claim 3 implies that  $\alpha_k^-$  is not coherent, as desired. Now we know that  $\alpha_j^-$  is not coherent.

Claim 2 and the fact that  $\alpha_j^+$  is coherent imply that  $y_j^+ \in \alpha_\ell \setminus \{\times_\ell\}$ , for some  $\ell \geq j+2$ . By applying Claims 1 and 2, we obtain that (i)  $\alpha_{j+2}^-$  is disjoint from  $\alpha_j^+$ ; and (ii)  $\alpha_{j+1}^+ \cap \alpha_j^+ \neq \emptyset$ .

From (i) and the fact that  $\alpha_{j+2}^-$  is not reversed, we know that  $y_{j+2}^- \in \alpha_{j+2}$ , and hence  $\gamma_{j+1} \cap \gamma_{j+2} = \{y_{j+2}^-, \times_{j+2}\}$ .

Observation (ii) and Claim 3 imply that  $\alpha_{j+1}^-$  is not coherent, and hence  $\alpha_{j+1}^+$  is coherent. From Claim 2 and the fact that  $\gamma_{j+1} \cap \gamma_{j+2} = \{y_{j+2}^-, \times_{j+2}\}$  it follows that, for some  $r \geq j+3$ ,  $y_{j+1}^+ \in \alpha_r \setminus \{\times_r\}$ . Since  $\times_{j+2} \prec \times_{j+3} \prec y_{j+1}^+$ , Claim 1 implies that  $\alpha_{j+2}^+ \cap \alpha_{j+1}^- \neq \emptyset$ . However, this contradicts that  $\gamma_{j+1} \cap \gamma_{j+2} = \{y_{j+2}^-, \times_{j+2}\}$ ; therefore Claim 4 holds.  $\square$

Definition 4.3.2 (iii) tells us that  $\alpha_0^-$  is not coherent, and hence  $\alpha_0^+$  is coherent. Likewise,  $\alpha_m^-$  is coherent. Then there exists  $j \in \{0, \dots, m-1\}$  for which  $\alpha_j^+$  and  $\alpha_{j+1}^-$  are coherent. Claim 4 implies that  $y_j^+ \in \alpha_{j+1}$  and  $y_{j+1}^- \in \alpha_j$ . But this means that  $\gamma_j$  and  $\gamma_{j+1}$  intersect in at least three points, the final contradiction.  $\square$

## 4.4 Hereditarily-convex drawings

The aim of the next four subsections is to show that pseudospherical drawings of  $K_n$  are characterized by a simple local property of the drawing known as *convexity*.

### 4.4.1 Pseudospherical drawings are hereditarily-convex.

In this subsection we introduce the notion of convexity and show that pseudospherical drawings of  $K_n$  are hereditarily-convex.

**Definition 4.4.1.** *Let  $D$  be a drawing of  $K_n$  in the sphere.*

- *Let  $T$  be a 3-cycle of  $K_n$ . A convex side of  $T$  is a disk  $\Delta$  bounded by  $D[T]$  such that, for every vertex  $v$  drawn inside  $\Delta$ , the three edges connecting  $v$  to the vertices of  $T$  are drawn in  $\Delta$ .*
- *A choice of convex sides is a set  $\{\Delta_T : T \text{ is a 3-cycle of } K_n\}$  such that each  $\Delta_T$  is a convex side of  $T$ .*
- *The drawing  $D$  is convex if it has a choice of convex sides.*
- *The drawing  $D$  is hereditarily-convex or simply h-convex, if there is a choice of convex sides  $\{\Delta_T : T \text{ is a 3-cycle}\}$  such that, if  $T, T'$  are 3-cycles for which  $D[T'] \subseteq \Delta_T$ , then  $\Delta_{T'} \subseteq \Delta_T$ .*
- *The drawing  $D$  is face-convex or simply f-convex, if there is a face  $F$  such that, if for every 3-cycle  $T$  we let  $\Delta_T$  be its side for which  $\Delta_T \cap F = \emptyset$ , then  $\{\Delta_T : T \text{ is a 3-cycle}\}$  is a choice of convex sides.*

It is immediate that face-convex drawings are hereditarily-convex. In connection with Chapter 2, we will see in Subsection 4.4.2 that face-convex drawings are equivalent to pseudolinear drawings of  $K_n$  in the plane. Convex drawings of  $K_n$  have received recent attention from the community: during the Crossing Numbers Workshop 2017 in Osnabrück, Aichholzer found, using computer assistance, that, for  $n \leq 12$ , every optimal drawing of  $K_n$  is convex. We refer to [7] for a more detailed treatment on convex drawings of  $K_n$ . The aim of this section is to show the following:

**Theorem 4.4.2.** *A drawing of  $K_n$  is pseudospherical if and only if it is hereditarily-convex.*

Is it worth noting that in [7] it was shown that h-convex drawings are those from obtained from excluding two drawings of  $K_5$  and one drawing of  $K_6$ . This observation and Theorem 4.4.2 imply the existence of an  $O(n^6)$  algorithm for testing whether a drawing is pseudospherical.

In this subsection we prove the easy direction of Theorem 4.4.2: pseudospherical drawings are hereditarily-convex (Lemma 4.4.4). In Subsection 4.4.4 we complete the proof by showing that hereditarily-convex drawings are pseudospherical.

We start with a basic observation.

**Lemma 4.4.3.** *In a pseudospherical drawing of  $K_n$ , if  $e$  and  $f$  are distinct edges and  $\gamma_e$  is the pseudocircle extending  $e$ , then  $\gamma_e$  intersects  $f$  at most once.*

*Proof.* Suppose that  $e, f$  are edges for which  $\gamma_e$  intersects  $f$  at least twice. Since  $\gamma_f$  intersects  $\gamma_e$  in at most two points,  $\gamma_e$  intersects  $f$  exactly twice. Then the ends of  $f$  are in the same side of  $\gamma_e$ , but  $f$  is not contained in that side, contradicting (PS2).  $\square$

**Lemma 4.4.4.** *Let  $D$  be a pseudospherical drawing of  $K_n$ . Then every 3-cycle  $T$  has a side  $\Delta_T$  whose interior is disjoint from the three pseudocircles extending the edges of  $T$ . Moreover,  $\{\Delta_T : T \text{ is a 3-cycle}\}$  is a choice of convex sides witnessing that  $D$  is h-convex.*

*Proof.* Let  $e$  be an edge of  $T$ . The pseudocircle  $\gamma_e$  extending  $e$  intersects the other two edges of  $T$  at the ends of  $e$ . Lemma 4.4.3 implies that  $\gamma_e \setminus D[e]$  is drawn in the interior of a side  $\Delta'_T$  of  $D[T]$ .

For any edge  $f$  of  $T$  distinct from  $e$ ,  $\gamma_f$  intersects  $\gamma_e$  at the end that  $e$  and  $f$  have in common. Since this intersection is a crossing, the points of  $\gamma_f \setminus D[f]$  near this crossing are in the interior of  $\Delta'_T$ , and so is  $\gamma_f \setminus D[f]$ . Then the other side  $\Delta_T$  of  $D[T]$  has its interior disjoint from the three pseudocircles extending the edges of  $T$ .

To show that  $\Delta_T$  is convex, consider a vertex  $x$  drawn in the interior of  $\Delta_T$ . Note that  $\Delta_T$  is the intersection of three disks bounded by pseudocircles, each of them going through one of the edges of  $T$ . This and (PS2) imply that any edge having both ends in  $\Delta_T$  is drawn in  $\Delta_T$ . In particular, this holds for the edges connecting  $x$  to the three vertices in  $T$ , and hence  $\Delta_T$  is convex.

Finally, in order to show that  $\{\Delta_T : T \text{ is a 3-cycle}\}$  witness that  $D$  is h-convex, we consider a pair of 3-cycles  $T_1, T_2$  such that  $D[T_2] \subseteq \Delta_{T_1}$ .

**Claim 1.** *If  $T_1$  and  $T_2$  have at least one vertex in common, then  $\Delta_{T_2} \subseteq \Delta_{T_1}$ .*

*Proof.* Let  $u \in V(T_1) \cap V(T_2)$  and let  $e$  be an edge of  $T_1$  incident with  $u$ . If  $T_1 = T_2$ , then the claim trivially holds, so we may assume that  $T_2$  has a vertex  $v$  not in  $T_1$ . The choice of  $\Delta_{T_1}$  and the hypothesis on  $T_2$  imply that  $\gamma_e$  and  $D[T_2]$  are drawn in distinct sides of  $D[T_1]$ . Since  $D[u]$  is a crossing between  $\gamma_e$  and  $\gamma_{uv}$  (PS3), there are points of  $\gamma_{uv}$  on both sides of  $\gamma_e$ . Consequently,  $\gamma_{uv}$  is drawn on the side of  $D[T_2]$  not contained in  $\Delta_{T_1}$ ; the same holds for any pseudocircle that extends an edge of  $T_2$ . Thus  $\Delta_{T_2} \subseteq \Delta_{T_1}$ .  $\square$



From Claim 1, we may assume that  $T_1$  and  $T_2$  have no vertices in common. Let  $u \in V(T_2) \setminus V(T_1)$  and let  $V(T_1) = \{x, y, z\}$ . Since  $\Delta_{T_1}$  is convex and  $u$  is drawn in the interior of  $\Delta_{T_1}$ , the three edges connecting  $u$  to  $x$ ,  $y$  and  $z$  are drawn in  $\Delta_{T_1}$ . If one of these three edges, say  $ux$ , intersects the edge in  $T_2 - u$ , then the side of  $T_2$  containing  $x$  is not convex. Because  $\Delta_{T_2}$  is convex,  $\Delta_{T_2}$  is the side of  $T_2$  included in  $\Delta_{T_1}$ . Thus, we may assume that none of  $ux$ ,  $uy$  and  $uz$  cross the edge in  $T_2 - u$ . This implies that the two vertices of  $T_2 - u$  are drawn in one of the three faces of  $D[T_1 + u]$  included in  $\Delta_{T_1}$ . By symmetry, we may assume that the side  $\Delta$  of  $T'_1 = (x, y, u)$  included in  $\Delta_{T_1}$  is the one in which  $T_2$  is drawn. Claim 1 applied to  $T_1$  and  $T'_1$  implies that  $\Delta_{T'_1} = \Delta$ , and applied to  $T'_1$  and  $T_2$  implies that  $\Delta_{T_2} \subseteq \Delta_{T'_1}$ . Then  $\Delta_{T_2} \subseteq \Delta_{T_1}$ , and  $D$  is h-convex.  $\square$

## 4.4.2 Face-convex drawings

A drawing  $D$  of  $K_n$  is face-convex if there is a face  $F$  such that, for every 3-cycle  $T$ , the side of  $T$  disjoint from  $F$  is convex. In this subsection we explain why face-convex drawings are equivalent to pseudolinear drawings of  $K_n$  in the plane. We will see in the next section that face-convex drawings are the basic pieces in the description of the structure of an h-convex drawing. In Lemma 4.4.8 and Corollary 4.4.9 we investigate some properties of face-convex drawings that we need in the next section.

A drawing  $D$  in the sphere  $\mathcal{S}$  is *equivalent* to a drawing  $D'$  in the plane  $\mathbb{R}^2$  if there exists a point  $p$  in a face of  $D$  and an homeomorphism  $f : \mathcal{S} \setminus \{p\} \rightarrow \mathbb{R}^2$  such that the image  $f(D)$  is homeomorphic to  $D'$ . In Section 2.5 of Chapter 2, we characterized pseudolinear drawings of  $K_n$  in the plane in terms of forbidding the drawing of a path of length 3 known as the B configuration. From this characterization we obtained the following.

**Corollary 2.5.2.** *A good drawing  $D$  of  $K_n$  in the plane is pseudolinear if and only if, for every 3-cycle  $T$  and for every vertex  $v$  drawn in the bounded face of  $D[T]$ , the three edges connecting  $v$  to the vertices of  $T$  are contained in the disk bounded by  $D[T]$ .*

From this corollary it immediately follows the main result of [8].

**Theorem 4.4.5.** [8] *A drawing of  $K_n$  is face-convex if and only if it is equivalent to a pseudolinear drawing of  $K_n$  in the plane.*

In the remainder of this section we prove Lemma 4.4.8, an alternative characterization of pseudolinear drawings of  $K_n$  due to Aichholzer et al. [5].

**Notation 4.4.6.** Let  $D$  be a drawing of  $K_n$  and let  $J$  be a crossing  $K_4$  of  $D$ . If  $C_J$  denotes the 4-cycle consisting of the uncrossed edges in  $D[J]$ , then the crossing side of  $J$  is the disk bounded by  $D[C_J]$  that contains the crossing of  $D[J]$ .

Our next observation immediately follows from the definition of convex drawing.

**Observation 4.4.7.** Let  $D$  is a convex drawing of  $K_n$  and let  $J$  be a crossing  $K_4$  in  $D$ . Then each 3-cycle in  $J$  has a unique convex side: the one included in the crossing side of  $J$ .

**Lemma 4.4.8.** Let  $D$  be a drawing of  $K_n$  in the sphere, and let  $F$  be a face of  $D$ . Then the following are equivalent:

- (i)  $F$  witnesses face-convexity; and
- (ii) the crossing side of every crossing  $K_4$  is disjoint from  $F$ .

*Proof.* Suppose that (i) holds. By way of contradiction, suppose that  $J$  is a crossing  $K_4$  for which its crossing side is not disjoint from  $F$ . Then  $F$  is included in a face  $F'$  of  $D[J]$  incident with the crossing of  $J$ . The face  $F'$  is bounded by two segments of the edges crossing in  $J$  and by an uncrossed edge  $e$ . Let  $T$  be any 3-cycle of  $J$  having  $e$  as an edge. From Observation 4.4.7 it follows that the unique convex side of  $T$  is the one containing  $F$ . However, this contradicts that  $F$  witnesses face-convexity.

Conversely, suppose that (ii) holds. Let  $T$  be a 3-cycle and let  $\Delta_T$  be the side of  $T$  disjoint from  $F$ . If  $\Delta_T$  is not convex, then there exists a vertex  $u$  drawn in the interior of  $\Delta_T$ , and a vertex  $v \in V(T)$  for which  $D[uv] \not\subseteq \Delta_T$ . Thus,  $uv$  crosses at least one edge in  $T$ , and since  $D$  is a good drawing, it crosses only the edge in  $T - v$ . Then  $T + u$  induces a crossing  $K_4$ . Moreover, since  $uv$  and the edge in  $T - v$  are the only crossing edges in  $T + u$ , the edges connecting  $u$  to the vertices in  $T - v$  are included in  $\Delta_T$ . This shows that the face of  $D[T + u]$  bounded by a 4-cycle is included in  $\Delta_T$ , and hence the crossing side of  $T + u$  is not disjoint from  $F$ , contradicting (ii). Thus  $\Delta_T$  is convex for every  $T$ , and hence  $F$  witnesses face-convexity.  $\square$

**Corollary 4.4.9.** Let  $n$  be an integer with  $n \geq 3$ . If  $F$  is a face of a drawing  $D$  of  $K_n$  witnessing that  $D$  is  $f$ -convex, then  $F$  is bounded by a cycle of  $K_n$ .

*Proof.* Consider the underlying plane graph  $G$  of  $D$  obtained by replacing each crossing of  $D$  by a degree 4 vertex. From [24, Lemma 5] it follows that  $G$  is 2-connected (in fact, the

authors showed that, for  $n \geq 5$ ,  $G$  is 3-connected). Consequently  $F$  is bounded by a cycle of  $G$ .

If the boundary cycle of  $F$  is not a cycle of  $K_n$ , then there is at least one crossing  $\times$  of  $D$  incident with  $F$ . Let  $J$  be the crossing  $K_4$  induced by the ends of the pair of edges crossing at  $\times$ . Since  $\times$  is incident with  $F$ , one of the four faces of  $D[J]$  incident with  $\times$  contains  $F$ . Then the crossing side of  $J$  is not disjoint from  $F$ , and this contradicts Lemma 4.4.8.  $\square$

### 4.4.3 The face-convex sides of an hereditarily-convex drawing

In this section we describe the structure of hereditarily-convex drawings needed to complete the proof of Theorem 4.4.2. We will see that, in any hereditarily-convex drawing of  $K_n$ , every edge induces a natural decomposition into two face-convex (or pseudolinear) drawings. This decomposition generalizes the fact that, in a spherical drawing, if we consider a great circle  $\gamma$  extending an edge, then each of the two drawings induced by the vertices on one side of  $\gamma$  is equivalent to a rectilinear drawing in the plane.

**Notation 4.4.10.** *Let  $D$  be a drawing of  $K_n$  with a choice of convex sides  $\{\Delta_T : T \text{ is a 3-cycle}\}$  witnessing  $h$ -convexity. For each  $e \in E(K_n)$ , let  $\vec{e}$  be an arbitrary orientation of  $e$ .*

- *The set  $\Sigma_e^1$  consists on the vertices  $v$  in  $K_n$  not incident with  $e$ , for which the 3-cycle  $T$  containing  $v$  and  $e$  has its chosen convex side  $\Delta_T$  to the left of  $\vec{e}$ . The set  $\Sigma_e^2$  consists on the vertices  $v$  not incident with  $e$ , for which the 3-cycle  $T$  containing  $v$  and  $e$  has  $\Delta_T$  to the right of  $\vec{e}$ .*
- *For  $i = 1, 2$ , we let  $D_e^i$  be the subdrawing of  $D$  induced by the vertices in  $\Sigma_e^i$  and  $e$ .*

We should make few points about this notation. First, the orientation that we choose for each edge is irrelevant to us; it only helps us distinguish the sides of  $e$ . Second, we will assume that every choice of convex sides witnessing  $h$ -convexity also comes with an arbitrary orientation of the edges of  $K_n$ , so that the 1- and 2-sides of  $e$  are predetermined, as well as  $\Sigma_e^1$  and  $\Sigma_e^2$ .

The next result will be used in the proofs of 4.4.14 and 4.4.16.

**Lemma 4.4.11.** *Let  $D$  be an  $h$ -convex drawing of  $K_n$  with respect to a given choice of convex sides. Suppose that  $J$  is a crossing  $K_4$  in  $D$ , and let  $v \in V(K_n) \setminus V(J)$  be a vertex drawn in the crossing side of  $J$ . Then*

- (i) the four edges connecting  $v$  to the vertices of  $J$  are drawn in the crossing side of  $J$ ;  
and
- (ii) these four edges partition the crossing side of  $J$  into four regions; each of them is the chosen convex side of their boundary.

*Proof.* Suppose that  $x, y, z, w$  are the vertices of  $J$ , and we may choose these labels so that  $xz$  crosses  $yw$ . Without loss of generality, suppose that  $v$  is in the face bounded by  $D[xy]$  and by some subsegments of  $D[xz]$  and  $D[yw]$ .

Suppose that  $\{\Delta_T : T \text{ is a 3-cycle}\}$  is the given choice of convex sides. Observation 4.4.7 implies that  $\Delta_{xyz}$  and  $\Delta_{xyw}$  are included in the crossing side of  $J$ , and consequently, so are the edges connecting  $v$  to the vertices of  $J$ . Then (i) holds.

To show (ii), consider any of the four closed regions of the crossing side of  $J$ , bounded by an edge  $e \in E(J)$  and by the edges connecting  $v$  to the ends of  $e$ . If  $e \neq zw$  then this region is contained in either  $\Delta_{xyz}$  or  $\Delta_{xyw}$ , and hence is convex. Otherwise, this region is bounded by the 3-cycle  $(v, z, w)$ , and is contained in the crossing side of the  $K_4$  induced by  $v, x, z$  and  $w$ . Now the result follows from Observation 4.4.7.  $\square$

**Lemma 4.4.12.** *Let  $D$  be an  $h$ -convex drawing of  $K_n$  with respect to a given choice of convex sides. Let  $e, f \in E(K_n)$ . If  $f$  crosses  $e$ , then the ends of  $f$  are in distinct ones of  $\Sigma_e^1$  and  $\Sigma_e^2$ . Consequently, for  $i = 1, 2$ ,  $e$  is not crossed in  $D_e^i$ .*

*Proof.* Let  $J$  be the crossing  $K_4$  containing the four ends of  $e$  and  $f$ . If  $T_1$  and  $T_2$  are the 3-cycles of  $J$  containing  $e$ , then the disks bounded by  $T_1$  and  $T_2$  included in the crossing side of  $J$  contain distinct sides of  $e$ . These disks are the chosen convex sides by Observation 4.4.7, and hence Lemma 4.4.12 holds.  $\square$

Our previous lemma shows that, in any  $h$ -convex drawing  $D$  of  $K_n$ , every edge  $e$  is incident to two faces of  $D_e^i$ ; each of them includes one side of  $e$ .

**Notation 4.4.13.** *Let  $D$  be an  $h$ -convex drawing of  $K_n$  with respect to a given choice of convex sides. For  $e \in E(K_n)$  and for  $i \in \{1, 2\}$ , we let  $F_e^i$  be the face of  $D_e^i$  incident with  $e$  that includes the  $(3 - i)$ -side of  $e$ .*

**Lemma 4.4.14.** *Let  $D$  be an  $h$ -convex drawing of  $K_n$  with respect to a choice  $\{\Delta_T : T \text{ is a 3-cycle}\}$  of convex sides, and let  $e \in E(K_n)$ . If, for  $i \in \{1, 2\}$ ,  $|\Sigma_e^i| \geq 1$ , then every 3-cycle  $T$  in  $D_e^i$  has  $\Delta_T \cap F_e^i = \emptyset$ . In particular  $D_e^i$  is  $f$ -convex.*

*Proof.* Let  $J$  be any crossing  $K_4$  in  $D_e^i$  and suppose  $F_J$  is the face of  $D[J]$  bounded by a 4-cycle. By Lemma 4.4.8, it is enough to show that  $F_e^i \subseteq F_J$ .

Lemma 4.4.12 implies that  $e$  does not cross any edge in  $J$ . In particular, if  $e \in E(J)$ , then  $e$  is part of the 4-cycle bounding  $F_J$ . Moreover, Observation 4.4.7 tells us that, if  $e \in E(J)$ , then the 3-cycles of  $J$  through  $e$  have their chosen convex sides disjoint from  $F_J$ ; in this case  $F_e^i \subseteq F_J$ , as desired.

Now suppose that  $e \notin E(J)$ . In this case,  $e$  has an end  $u \notin V(J)$ , drawn in a face of  $D[J]$ . If  $D[u] \in F_J$ , then, since  $e$  does not cross any edge in  $J$ ,  $e$  is drawn in the closure of  $F_J$ . Because  $F$  is incident with  $e$  in  $D_e^i$ ,  $F_e^i \subseteq F_J$ .

By way of contradiction suppose that  $u$  is drawn in a face  $F'$  of  $D[J]$  distinct from  $F_J$ .

Lemma 4.4.11 implies that the four edges from  $u$  to the vertices in  $J$  are drawn inside the crossing side of  $J$ . These four edges partition the crossing side of  $J$  into four closed regions. Each of these four regions is the convex side of the 3-cycle bounding the region. If  $e$  is one of the four edges connecting  $u$  to a vertex in  $J$ , then two of the 3-cycles bounding these four regions have their convex sides containing distinct sides of  $e$ . This contradicts that all the vertices of  $J$  are in  $D_e^i$ . Thus, we may assume that  $e$  is none of the four edges connecting  $u$  to a vertex in  $J$ .

Let  $v$  be the end of  $e$  distinct from  $u$ . Since  $e$  does not cross any edge of  $J$ ,  $v$  is located in one of the four previously described regions; call it  $\Delta$ . Let  $T$  be the 3-cycle of  $J + u$  bounding  $\Delta$ . By Lemma 4.4.11,  $\Delta$  is convex, and hence  $e$  and the other two edges connecting  $v$  to the vertices in  $T - u$  are drawn in  $\Delta$ . Now, as  $D$  is h-convex, the 3-cycles in  $T + v$  containing  $e$  have their chosen convex sides including distinct sides of  $e$ , a contradiction.  $\square$

Recall that, in every f-convex drawing of  $K_n$ , a face witnessing f-convexity in  $D$  is bounded by a cycle of  $K_n$  (Corollary 4.4.9). With this and Lemma 4.4.14 in hand, we introduce the following.

**Notation 4.4.15.** *Let  $D$  be an h-convex drawing with respect to a given choice of convex sides. Let  $e \in E(K_n)$ . For  $i = 1, 2$ ,  $C_e^i$  denotes the cycle bounding  $F_e^i$ . We let  $\Delta_e^i$  be the disk bounded by  $D[C_e^i]$  disjoint from  $F_e^i$ .*

The remainder of this subsection is devoted to refining our structural description of h-convex drawings, by showing that for every edge  $e$ ,  $\Delta_e^1 \cap \Delta_e^2 = D[e]$  (Lemma 4.4.18). The proof of this lemma is achieved in a sequence of three small steps.

Our first step is to show that each disk  $\Delta_e^i$  does not contain vertices from  $\Sigma_e^{(3-i)}$ .

**Lemma 4.4.16.** *Let  $D$  be an  $h$ -convex drawing of  $K_n$  with respect to a given choice of convex sides. Let  $e \in E(K_n)$ . For  $i = 1, 2$ , if  $x \in \Sigma_e^{3-i}$ , then  $D[x] \notin \Delta_e^i$ .*

*Proof.* By way of contradiction, suppose that, for some  $i = 1, 2$ , there exists  $x \in \Sigma_e^{3-i}$  with  $D[x] \in \Delta_e^i$ .

First, suppose that  $C_e^i$  is a 3-cycle. Then  $x$  is drawn in the convex side  $\Delta_e^i$ . Thus, the two edges connecting  $x$  to the ends of  $e$  are also drawn in  $\Delta_e^i$ . Since  $D$  is  $h$ -convex, the convex side of the 3-cycle containing  $e$  and  $x$  is the one included in  $\Delta_e^i$ . This implies that  $x \in \Sigma_e^i$ , a contradiction.

In the alternative, suppose that  $C_e^i$  has length at least 4. For any edge  $f$  of  $C_e^i$  not incident with  $e$ , the  $K_4$  induced by the ends of  $e$  and  $f$  is drawn in  $\Delta_e^i$ , and hence is crossing. From Lemmas 4.4.14 and 4.4.8 it follows that every crossing  $K_4$  obtained in this way has its crossing side included in  $\Delta_e^i$ . Moreover,  $\Delta_e^i$  is the union of the crossing sides of the  $K_4$ s obtained in this way, thus at least one of them, say  $J$ , contains  $D[x]$  in its crossing side.

Since  $D[e]$  is part of the boundary of the crossing side of  $J$ , Lemma 4.4.11 implies that the 3-cycle containing  $x$  and  $e$ , as well as its chosen convex side, is included in  $\Delta_e^i$ . Therefore  $x \in \Sigma_e^i$ , a contradiction.  $\square$

In our second step, we show that edges belonging to distinct ones of  $D_e^1$  and  $D_e^2$  do not cross.

**Lemma 4.4.17.** *Let  $D$  be an  $h$ -convex drawing of  $K_n$  with a given choice of convex sides. Let  $e \in E(K_n)$ . Then no edge of  $D_e^1$  crosses any edge of  $D_e^2$ .*

*Proof.* We start with the following observation.

**Claim 1.** *If an edge  $f_1$  of  $D_e^1$  crosses an edge  $f_2$  of  $D_e^2$ , then at least one of  $f_1$  and  $f_2$  has no ends in common with  $e$ .*

*Proof.* Suppose that each of  $f_1$  and  $f_2$  has an end in common with  $e$ . The ends of this pair of edges induce a crossing  $K_4$  for which  $e$  is part the 4-cycle bounding a face. Observation 4.4.7 shows that the ends of  $f_1$  and  $f_2$  that are not in  $e$  are in the same one of  $\Sigma_e^1$  and  $\Sigma_e^2$ . Thus  $f_1$  and  $f_2$  are in the same of  $D_e^1$  and  $D_e^2$ , a contradiction.  $\square$

By way of contradiction, suppose that  $x_1y_1$  is an edge of  $D_e^1$  crossing an edge  $x_2y_2$  of  $D_e^2$ . Claim 1 implies that at least one of these edges, say  $x_1y_1$ , is not incident with  $e$ . Let  $J_1$  be the  $K_4$  induced by the ends of  $x_1y_1$  and  $e$ .

First, suppose that one of the ends of  $x_2y_2$ , say  $y_2$ , is an end of  $e$ . As  $x_1y_1$  crosses  $x_2y_2$ , Lemma 4.4.12 shows that  $x_2y_2 \neq e$ , and hence  $x_2 \in \Sigma_e^2$ . Claim 1 implies that  $x_1y_1$  is the only edge of  $J_1$  that crosses  $x_2y_2$ . Lemma 4.4.16 implies that  $D[x_2] \notin \Delta_e^1$ . Thus, when we traverse  $D[x_2y_2]$  from  $x_2$  to  $y_2$ , the first point of  $D[J_1]$  that we encounter is the crossing  $\times$  between  $x_2y_2$  and  $x_1y_1$ . This implies that  $x_1y_1$  is incident with the face of  $D[J_1]$  containing  $x_2$ . Hence  $J_1$  is a crossing  $K_4$  where  $x_1y_1$  and  $e$  are in the 4-cycle of  $D[J_1]$  bounding a face containing  $D[x_2]$ .

If we continue traversing  $x_2y_2$  across  $\times$ , then we enter into a face  $F$  of  $D[J_1]$  included in the crossing side of  $J_1$ . When we follow  $x_2y_2$  until we reach  $y_2$ ,  $x_2y_2$  must cross another edge that has a segment in the boundary of  $F$ . However, the existence of a second crossing contradicts Claim 1.

Now we assume that  $e$  and  $x_2y_2$  have no ends in common. Let  $J_2$  be the  $K_4$  induced by the ends of  $e$  and  $x_2y_2$ . From what we just showed, we may assume that the only crossing between an edge in  $J_1$  and an edge in  $J_2$  is between  $x_1y_1$  and  $x_2y_2$ . Since  $D[x_2] \notin \Delta_e^1$ , when we traverse  $D[x_2y_2]$  from  $x_2$  to  $y_2$ , the first point we encounter is the crossing  $\times$  between  $x_2y_2$  and  $x_1y_1$ . Again, this implies that  $D[J_1]$  is a crossing  $K_4$ , having  $x_1y_1$  and  $e$  in the 4-cycle bounding a face  $F_J$  containing  $D[x_1]$ .

Since the crossing side of  $J_1$  is included in  $\Delta_e^1$ , and  $D[y_2] \notin \Delta_e^1$ ,  $D[y_2]$  is also drawn in  $F_J$ . As we continue traversing  $x_1y_2$  across  $\times$ , we enter into a face  $F$  of  $D[J_1]$  included in the crossing side of  $J_1$ . Since  $D[x_2y_2]$  crosses an even number of times the boundary of  $F$ ,  $x_2y_2$  crosses some other edge of  $J_1$ , a contradiction.  $\square$

We reached our third and final step towards understanding the structure of h-convex drawings.

**Lemma 4.4.18.** *Let  $D$  be an h-convex drawing with a given choice of convex sides, and let  $e \in E(K_n)$ . If  $\Delta_e^1$  and  $\Delta_e^2$  are the disks defined as in Notation 4.4.15, then  $\Delta_e^1 \cap \Delta_e^2 = D[e]$ .*

*Proof.* Let  $u$  and  $v$  be the ends of  $e$ . Since  $\Sigma_e^1 \cap \Sigma_e^2 = \emptyset$ ,  $u$  and  $v$  are the only vertices that  $C_e^1$  and  $C_e^2$  have in common. Then  $D[C_e^1 \cup C_e^2]$  is the drawing of three internally-disjoint  $uv$ -paths. Moreover, Lemmas 4.4.12 and 4.4.17 imply that  $D[C_e^1 \cup C_e^2]$  is drawn as three internally disjoint  $uv$ -arcs in the sphere.

As the open  $uv$ -arc  $D[C_e^1] \setminus D[e]$  contains at least one vertex of  $\Sigma_e^1$ , Lemma 4.4.16 guarantees that such an arc is not drawn in  $\Delta_e^2$ . Likewise,  $D[C_e^2] \setminus D[e]$  is not drawn in  $\Delta_e^1$ . So each of  $\Delta_e^1$  and  $\Delta_e^2$  is the closure of a face in  $D[C_e^1 \cup C_e^2]$ , and these disks only intersect at  $D[e]$ .  $\square$

#### 4.4.4 Hereditarily-convex drawings are pseudospherical

In this subsection we conclude the proof of Theorem 4.4.2 (pseudospherical and h-convex drawings are equivalent), and simultaneously pave our way towards the proof of Theorem 4.1.2 (pseudospherical drawings have an extension satisfying (PS3')). We assume that the reader is familiar with Notation 4.4.10-4.4.15 given in the previous subsection.

Our aim in this subsection is to extend the edges of an hereditarily-convex drawing  $D$  of  $K_n$  to an arrangement of pseudocircles. Lemmas 4.4.14 and 4.4.18 describe how every edge  $e$  induces a partition into two f-convex drawings that intersect in  $D[e]$ . This partition suggests that, if  $\gamma_e$  is a pseudocircle extending  $e$ , then  $\Delta_e^1$  and  $\Delta_e^2$  are on distinct sides of  $\gamma_e$ . This condition is the same as assuming that  $\gamma_e \setminus D[e]$  is an arc connecting the ends of  $e$  in  $\mathcal{S} \setminus (\Delta_e^1 \cup \Delta_e^2)$ . This last condition is what we define as the basic property that any “feasible” pseudocircle extending  $e$  must satisfy. We generalize this notion of “feasibility” to allow multiple pseudocircles extending a given subset of edges in  $K_n$ .

**Notation 4.4.19.** *Let  $D$  be an h-convex drawing of  $K_n$  with a given choice of convex sides and let  $e \in E(K_n)$ . We denote as  $F_e$  the face of  $D_e^1 \cup D_e^2$  bounded by  $(C_e^1 - e) \cup (C_e^2 - e)$ .*

Notice that  $F_e = F_e^1 \cap F_e^2 = \mathcal{S} \setminus (\Delta_e^1 \cup \Delta_e^2)$ .

**Definition 4.4.20.** *Let  $D$  be an h-convex drawing of  $K_n$  in the sphere, and let  $J \subseteq E(K_n)$ . A set of pseudocircles extending  $J$  is a set  $\{\gamma_f : f \in J\}$  of simple closed curves satisfying (PS1)-(PS3) and such that, for each  $f \in J$ :*

- (i)  $\gamma_f \setminus D[f]$  is an open arc connecting the ends of  $f$  in  $F_f$ ; and
- (ii) for each  $e \in E(K_n)$ , the intersection of  $\gamma_f$  with  $D[e]$  is either empty; a crossing with the interior of  $D[e]$ ; a single vertex that is a common end of  $e$  and  $f$ ; or  $D[e]$  and  $f = e$ .

The next remark follows from the fact that, for each edge  $f \in J$ ,  $\Delta_f^1$  and  $\Delta_f^2$  are contained in distinct sides of  $\gamma_f$ .

**Remark 4.4.21.** *If  $\gamma_f$  is an element of a set of pseudocircles extending  $J$ , and  $\gamma_f$  crosses an edge, then the ends of such an edge are in distinct ones of  $\Sigma_f^1$  and  $\Sigma_f^2$ . Moreover, if  $e_1$  and  $e_2$  are edges crossed by  $\gamma_f$ , and  $x_1, x_2$  are ends of  $e_1$  and  $e_2$ , respectively, then  $x_1$  and  $x_2$  belong to the same one of  $\Sigma_f^1$  and  $\Sigma_f^2$  if and only if they are on the same side of  $\gamma_f$ .*



We aim to show that, if  $J \subseteq E(K_n)$  and  $e \in E(K_n) \setminus J$ , then every set  $\Gamma_J$  of pseudocircles extending  $J$  can be enlarged to one extending  $J \cup \{e\}$ . For this, we consider a pseudocircle  $\gamma_e$  extending  $e$ , obtained as the union of  $D[e]$  and an arc joining the ends of  $e$  in  $F_e$ , very near to  $D[C_e^1 - e]$ . In Lemmas 4.4.23 and 4.4.24 we will show that such  $\gamma_e$  can be chosen such that  $\Gamma_J \cup \{\gamma_e\}$  extends  $J \cup \{e\}$ .

Our next observation will be constantly used in the proofs of Lemmas 4.4.23 and 4.4.24.

**Lemma 4.4.22.** *Let  $D$  be an  $h$ -convex drawing with a given choice of convex sides. Suppose that  $e$  and  $f$  are distinct edges of  $E(K_n)$ . Then there are no four vertices  $x_1, x_2, y_1, y_2$  occurring in this cyclic order around  $C_e^1$ , and such that  $x_i$  and  $y_i$  are drawn in  $D_f^i$ , for  $i = 1, 2$ .*

*Proof.* If such four vertices exists occur in this cyclic order, then the edges  $x_1y_1$  and  $x_2y_2$  are drawn in  $\Delta_e^1$ , and hence they cross. However, the edge  $x_1y_1$  is in  $D_f^1$ , while  $x_2y_2$  is in  $D_f^2$ , contradicting Lemma 4.4.12.  $\square$

**Lemma 4.4.23.** *Let  $D$  be an  $h$ -convex drawing of  $K_n$  with a given choice of convex sides and let  $J \subseteq E(K_n)$ . Suppose that  $\{\gamma_f : f \in J\}$  is a set of pseudocircles extending  $J$ . Let  $e, f$  be distinct edges of  $K_n$  with  $e \notin J$  and  $f \in J$ . Then there exists a simple closed curve  $\gamma_e$  with  $D[e] \subseteq \gamma_e$  and  $\gamma_e \setminus D[e]$  drawn in  $F_e$ , sufficiently near  $D[C_e^1 - e]$ , such that  $|\gamma_f \cap \gamma_e| \leq 2$ .*

*Proof.* As  $\gamma_e$  is constructed by considering a curve in  $F_e$  near  $D[C_e^1 - e]$ , we note that any forced crossing between  $\gamma_e \setminus D[e]$  and  $\gamma_f$  arises from any short segment  $\sigma$  of  $\gamma_f$ , with one end  $z \in D[C_e^1]$ , and the rest of  $\sigma \setminus z$  drawn in  $F_e$ .

Having the previous paragraph under consideration, let  $Z = \gamma_f \cap (D[C_e^1] \setminus D[e])$ , and for each  $z \in Z$ , consider a small open arc  $\alpha$  of  $\gamma_f$  centered at  $z$ . For  $z \in Z$ , define the contribution  $c(z)$  of  $z$  to be the number of components of  $\alpha \setminus z$  drawn in  $F_e$ . In such case,  $c(z)$  is either 0, 1 or 2, and it counts the forced crossings near  $D[z]$ .

Thus, by letting  $c(\gamma_f) = \sum_{z \in Z} c(z)$ , we can choose  $\gamma_e$  sufficiently near  $C_e^1$  such that

$$|\gamma_e \cap \gamma_f| = c(\gamma_f) + |\gamma_f \cap D[e]|.$$

Since  $|\gamma_e \cap \gamma_f|$  is even and  $|\gamma_f \cap D[e]| \leq 1$ , it suffices to show that  $c(\gamma_f) \leq 2$ .

**Claim 1.** *If for some  $z \in Z$ ,  $c(z) = 2$ , then  $c(\gamma_f) \leq 2$ .*

*Proof.* By assumption, if  $\gamma_f$  intersects the interior of an edge, then it crosses that edge. Because  $\gamma_f$  does not cross from  $F_e$  into  $\Delta_e^1$  at  $z$ ,  $z$  must be a vertex of  $C_e^1$ . Moreover, since  $\gamma_f$  satisfies Item (ii) in Definition 4.4.20,  $z$  is an end of  $f$ .

Because  $c(z) = 2$ , the points of  $D[f]$  near  $z$  are drawn in  $F_e$ . As every edge that has both ends in  $D_e^1$  is drawn in  $\Delta_e^1$ , the end of  $f$  distinct from  $z$  is in  $\Sigma_e^2$ .

Suppose that there exists  $z' \in Z \setminus \{z\}$  with  $c(z') > 0$ . From what we just observed,  $z'$  is not an end of  $f$ , and hence is a crossing between  $\gamma_f$  and an edge  $e'$  of  $C_e^1 - e$ . Traverse  $\gamma_f$ , starting at  $z'$ , continuing in the interior of  $\Delta_e^1$ , until we reach a point  $z''$  in  $D[C_e^1]$ . Since  $c(z) = 2$ ,  $z'' \notin \{z, z'\}$ . Thus  $z''$  is also a crossing between  $\gamma_f$  and an edge  $e''$  in  $C_e^1 - e'$ .

Let  $x_1$  and  $y_1$  be ends of  $e'$  and  $e''$ , respectively, and we choose them such that they are in the same component of  $D[C_e^1] \setminus \{z', z''\}$  containing  $z$ . Let  $x_2$  be the end of  $e'$  distinct from  $x_1$ . As  $\gamma_f$  crosses each of  $e'$  and  $e''$  exactly once, Remark 4.4.21 shows that  $x_1$  and  $y_1$  are on the same one of  $\Sigma_f^1$  and  $\Sigma_f^2$ . By reorienting  $f$ , if necessary, we may assume  $x_1, y_1 \in \Sigma_f^1$ . In this case  $x_2 \in \Sigma_f^2$ . By letting  $y_2 = z$ , we obtain vertices  $x_1, x_2, y_1, y_2$  occurring in this cyclic order in  $C_e^1$ , and such that  $x_i, y_i$  belong to  $D_e^i$ , for  $i = 1, 2$ . This contradicts Lemma 4.4.22.  $\square$

**Claim 2.** *If  $f \in E(C_e^1)$ , then  $c(\gamma_f) \leq 2$ .*

*Proof.* As  $D[f] \subseteq \gamma_f$ , each end of  $f$  has a contribution to  $c$  of at most 1. Suppose that there exists  $z \in Z$ , that is not an end of  $f$ , with  $c(z) > 0$ . Then, from Claim 1, we may assume that  $c(z) = 1$ . Note that  $z$  is a crossing between  $\gamma_f$  and an edge  $e'$  of  $C_e^1 - e$  not incident with  $f$ . As the ends of  $e'$  belong to distinct ones of  $\Sigma_e^1$  and  $\Sigma_e^2$ , and the ends of  $f$  belong to both  $D_f^1$  and  $D_f^2$ , we can label the ends of  $e'$  and  $f$  using labels  $x_1, x_2, y_1, y_2$  such that they occur in this cyclic order in  $C_e^1$ , and such that  $x_i, y_i$  are drawn in  $D_e^i$ , for  $i = 1, 2$ . This contradicts Lemma 4.4.22.  $\square$

From the previous two claims, we may assume that  $f \notin E(C_e^1)$  and that each element in  $Z$  with positive contribution has contribution 1.

Let  $z \in Z$  with  $c(z) = 1$ . Follow  $\gamma_f$  from  $F_e$  into  $\Delta_e^1$  across  $z$ , and continue along  $\gamma_z$  inside  $\Delta_e^1$ , until we reach a point  $z'$  just before we cross into  $F_e$ . Since  $c(z') = 1$ , this shows that the elements in  $Z$  with positive contribution are paired in such a way that a pair corresponds to the ends of an arc-component of  $\gamma_f \cap \Delta_e^1$ .

By way of contradiction, suppose that  $c(\gamma_f) > 2$ . It follows that there are four vertices  $z_1, z'_1, z_2, z'_2$  with contribution 1, and such that, for  $i = 1, 2$ ,  $z_i$  and  $z'_i$  are ends of the same arc-component  $\alpha_i$  of  $\gamma_f \cap \Delta_e^1$ .

**Claim 3.** *At least one of  $\alpha_1$  and  $\alpha_2$  is disjoint from the ends of  $f$ .*

*Proof.* If both ends of  $f$  are in  $\Delta_e^1$ , then  $D[f] \subseteq \Delta_e^1$ , and hence they belong to the same arc-component of  $\gamma_f \cap \Delta_e^1$ . In this case the claim holds.

In the alternative, at most one end of  $f$  is part of  $\Delta_e^1$ . Since  $\alpha_1$  and  $\alpha_2$  are disjoint subsets of  $\Delta_e^1$ , at least one of them is disjoint from the ends of  $f$ .  $\square$

By symmetry, we may assume that the ends of  $f$  are not in  $\alpha_1$ , and hence  $z_1$  and  $z'_1$  are crossings between  $\gamma_f$  and distinct edges  $e_1, e'_1 \in E(C_e^1)$ .

Let  $x_1$  and  $y_1$  be ends of  $e_1, e'_1$ , respectively, and chosen such that they are in the same component of  $D[C_e^1] \setminus \{z_1, z'_1\}$  containing  $\{z_2, z'_2\}$ . Let  $x_2$  be the end of  $e_1$  distinct from  $x_1$ . In this case,  $x_1$  and  $y_1$  belong to the same one  $\Sigma_f^1$  and  $\Sigma_f^2$ . By reorienting  $f$ , if necessary, we may assume that  $x_1, y_1 \in \Sigma_f^1$ . Because  $\gamma_f$  crosses  $e_1, x_2 \in \Sigma_f^2$ .

If one of  $z_2$  and  $z'_2$  is an end of  $f$ , then we label such an end of  $f$  as  $y_2$ . In the alternative, both  $z_2, z'_2$  are crossings. If  $e_2$  is the edge of  $C_e^1$  that crosses  $\gamma_f$  at  $z_2$ , then one of the two ends of  $e_2$  is in  $\Sigma_f^2$ ; in this case, we label such an end as  $y_2$ .

In any case, we obtained four vertices  $x_1, x_2, y_1, y_2$  occurring in this cyclic order in  $C_e^1$ , and such that  $x_i, y_i$  are drawn in  $D_e^i$ , for  $i = 1, 2$ , contradicting Lemma 4.4.22. This shows that  $c(\gamma_2) \leq 2$ , as desired.  $\square$

The next Lemma guarantees that, in the process of enlarging a set of pseudocircles extending some edges of  $K_n$ , the added pseudocircle satisfies (ii) in Definition 4.4.20.

**Lemma 4.4.24.** *Let  $D$  be an  $h$ -convex drawing of  $K_n$  with a given choice of convex sides, and let  $e$  and  $e'$  be distinct edges of  $K_n$ . Then there exists a simple closed curve  $\gamma_e$  with  $D[e] \subseteq \gamma_e$  and  $\gamma_e \setminus D[e]$  drawn in  $F_e$ , sufficiently near  $D[C_e^1 - e]$ , such that the intersection between  $\gamma_e$  and  $D[e']$  is either empty; a crossing with the interior of  $D[e']$ ; or a single vertex that is a common end of  $e$  and  $e'$ .*

*Proof.* If  $e$  and  $e'$  have an end in common, then  $D[e'] \subseteq \Delta_e^1 \cup \Delta_e^2$ , and hence the intersection of any curve  $\gamma_e$ , with  $\gamma_e \setminus D[e] \subseteq F_e$ , is  $D[e] \cap D[e']$ . In this case, Lemma 4.4.24 holds.

Henceforth we assume that  $e$  and  $e'$  have no ends in common. For this case, the proof is similar to the proof of Lemma 4.4.23: let  $Z = D[e'] \cap D[C_e^1]$ . For each  $z \in Z$ , we consider a short arc  $\alpha$  of  $D[e']$  centered at  $z$ ; in case  $z$  is an end of  $D[e']$ , we instead let  $\alpha$  be a short segment of  $D[e']$  starting at  $z$ . In any case, the contribution  $c(z)$  of  $z$  is the number of components  $\alpha \setminus z$  drawn in  $F_e$ .

Observe that  $c(z)$  is the number of forced crossings near  $z$ , between  $D[e']$  and a pseudocircle extending  $D[e]$  drawn near  $D[C_e^1]$ . Since every two edges are either crossing, have an edge in common or are disjoint,  $c(z) \in \{0, 1\}$  for all  $z \in Z$ .

Thus, if we let  $c(e') = \sum_{z \in Z} c(z)$ , and we choose  $\gamma_e$  near enough  $D[C_e^1]$ , then

$$|\gamma_e \cap D[e']| = c(e').$$

It is enough to show that  $c(e') \leq 2$ .

If  $e'$  has both ends in the same one of  $\Sigma_e^1$  and  $\Sigma_e^2$ , then  $D[e'] \subseteq \Delta_e^1 \cup \Delta_e^1$ , and consequently,  $c(e') = 0$ . Thus, we may assume that  $e'$  has ends on each of  $\Sigma_e^1$  and  $\Sigma_e^2$ .

By way of contradiction, suppose that  $c(e') > 2$ . Let  $z, z', z'' \in Z$  such that  $c(z) = c(z') = c(z'') = 1$ . At least two of these points in  $Z$ , say  $z$  and  $z'$ , are not ends of  $e'$ . In this case, each of  $z, z'$  is a crossing between  $D[e']$  and distinct edges  $f, f' \in E(C_e^1)$ . Let  $x_1$  and  $y_1$  be ends of  $f$  and  $f'$ , respectively. We choose  $x_1$  and  $y_1$  to be in the same component of  $C_e^1 - f - f'$  containing  $z''$ . In this case,  $x_1$  and  $y_1$  belong to the same one of  $\Sigma_{e'}^1$  and  $\Sigma_{e'}^2$ . Let  $x_2$  be the end of  $f$  distinct from  $x_1$ . By reorienting  $e'$ , if necessary, we may assume that  $x_1$  and  $y_1$  are in  $\Sigma_{e'}^1$ , while  $x_2 \in \Sigma_{e'}^2$ .

Either  $z''$  is an end of  $e'$ , and then we let  $y_2 = z''$ , or  $z''$  is a crossing between  $e'$  and an edge  $f''$  of  $C_e^1$ , and then we choose  $y_2$  to be the end of  $f''$  in  $\Sigma_{e'}^2$ . In any case, the vertices  $x_1, x_2, y_1, y_2$  occur in this cyclic order in  $C_e^1$ , and such that  $x_i, y_i$  belong to  $D_e^i$ , for  $i = 1, 2$ . This contradiction to Lemma 4.4.22 shows that  $c(e') \leq 2$ , as desired.  $\square$

The next result follows immediately from Lemma 4.4.23 and 4.4.24.

**Lemma 4.4.25.** *Let  $D$  be an  $h$ -convex drawing with a given choice of convex sides, and let  $J \subseteq E(K_n)$ . Suppose that  $\{\gamma_f : f \in J\}$  is a set of pseudocircles extending  $J$ . Then, for every  $e \in E(K_n) \setminus J$ , there exists a simple closed curve  $\gamma_e$ , sufficiently near  $D[C_e^1]$ , such that  $\{\gamma_f : f \in J \cup \{e\}\}$  is a set of pseudocircles extending  $J \cup \{e\}$ .  $\square$*

The following Corollary completes the proof Theorem 4.4.2, stating that  $h$ -convex drawings are pseudospherical, and it is a straightforward application of Lemma 4.4.25.

**Corollary 4.4.26.** *Every  $h$ -convex drawing of  $K_n$  is pseudospherical.  $\square$*

## 4.5 Finding curves that also satisfy (PS3')

In this section we conclude the proof of Theorem 4.1.2. For this, we will use the fact that pseudospherical drawings are  $h$ -convex and we will prove the following.

**Theorem 4.5.1.** *Every h-convex drawing of  $K_n$  has a set of pseudocircles extending  $E(K_n)$  and satisfying (PS3').*

In Subsection 4.4.2 we saw that every f-convex drawing  $D$  of  $K_n$  is equivalent, via a stereographic projection  $\phi$ , to a pseudolinear drawing  $D'$  in the plane. Then, by considering the inverse image  $\phi^{-1}$  of each pseudoline in an arrangement extending the edges in  $D'$ , and adding to each of these images the point of the sphere from which we are projecting in  $\phi$ , we get an arrangement of pseudocircles extending  $E(K_n)$  and satisfying (PS3'). Thus, in the proof of Theorem 4.5.1, we will assume that the drawing we are considering is not f-convex.

The next lemma, the first ingredient we need to show Theorem 4.5.1, tells us more about a set of extending pseudocircles in an h-convex drawing that is not f-convex.

**Lemma 4.5.2.** *Let  $D$  be an h-convex drawing and let  $J \subsetneq E(K_n)$ . Suppose that  $\{\gamma_f : f \in J\}$  is a set of pseudocircles extending  $J$  and that  $e \in E(K_n) \setminus J$ . Then either  $D$  is f-convex, or for every  $f \in J$ ,  $\gamma_f \cap F_e \neq \emptyset$ .*

*Proof.* Suppose that for some  $f \in J$ ,  $\gamma_f \cap F_e = \emptyset$ . This implies that  $\gamma_f \subseteq \Delta_e^1 \cup \Delta_e^2$ . From Lemma 4.4.18 we know that  $\Delta_e^1 \cap \Delta_e^2 = D[e]$ , and since  $\gamma_f$  crosses  $D[e]$  at most once,  $\gamma_f \subseteq \Delta_e^\ell$  for some  $\ell \in \{1, 2\}$ . In particular, (i)  $f$  has both ends in  $\Delta_e^\ell$ ; and (ii) every vertex of  $C_e^\ell$  is in the same one of  $\Delta_f^1$  and  $\Delta_f^2$ .

**Claim 1.**  *$f$  is an edge of  $C_e^\ell$ .*

*Proof.* If an edge  $xy$  crosses  $f$ , then  $x, y$ , are in different ones of  $\Sigma_f^1$  and  $\Sigma_f^2$ . Thus, (ii) implies that no edge joining two vertices in  $C_e^\ell$  crosses  $f$ . In particular, if we let  $z$  be an end of  $e$ , and let  $xy$  run through the edges of  $C_e^\ell - z$ , then, as the chosen convex sides of the 3-cycles  $xyz$  cover  $\Delta_e^\ell$ , it follows that one of these contains  $D[f]$ . Let  $xyz$  be such a cycle whose chosen convex side includes  $D[f]$ .

By way of contradiction suppose that  $f$  has an end  $u$  distinct from  $x, y$  and  $z$ . Since  $u$  is drawn in the chosen convex side  $\Delta_{xyz}$  of  $xyz$ , so also are the edges  $ux, uy, uz$ . By h-convexity, the chosen convex sides of the triangles  $uxy, uxz, uyz$  are included in  $\Delta_{xyz}$ .

Let  $v$  be the end of  $f$  distinct from  $u$ . If  $v \in \{x, y, z\}$ , then the two vertices of  $\{x, y, z\} \setminus \{v\}$  are on distinct ones of  $\Sigma_f^1$  and  $\Sigma_f^2$ , contradicting (ii). Thus,  $v$  is in the chosen convex side bounded by one of  $uxy, uxz, uyz$ , and the two of  $\{x, y, z\}$  in such boundary are on distinct ones of  $\Sigma_f^1$  and  $\Sigma_f^2$ , a contradiction. Thus, both ends of  $f$  are in  $C_e^\ell$ .

If  $f \notin E(C_e^\ell)$ , then  $f$  is a chord of  $C_e^\ell$  drawn in  $\Delta_e^\ell$ . Then there is an edge  $xy$  with ends in  $C_e^\ell$  that crosses  $f$ . In such case,  $x$  and  $y$  are on distinct ones of  $\Sigma_f^1$  and  $\Sigma_f^2$ , a contradiction.  $\square$

Lemma 4.4.14 shows that  $D_e^\ell$  is f-convex. Since  $f$  is an edge in  $C_e^\ell$ , and  $C_e^\ell$  is the boundary of the face  $F_e^\ell$  witnessing f-convexity in  $D_e^\ell$ , all the vertices of  $D_f^\ell$  not incident with  $f$  are in  $\Sigma_f^k$ , for some  $k \in \{1, 2\}$ . As  $\gamma_f \subseteq \Delta_e^\ell$ , every vertex of  $\Delta_e^{3-\ell}$  is in the same  $\Sigma_f^k$  as the ends of  $e$  not incident with  $f$ . Then all the vertices of  $K_n$  are in  $D_f^k$ , and hence  $D = D_f^k$ . It follows from Lemma 4.4.14 that  $D$  is f-convex.  $\square$

We are ready to prove Theorem 4.5.1.

*Proof of Theorem 4.5.1.* Let  $D$  be an h-convex drawing of  $K_n$ . From the discussion preceding Lemma 4.5.2, we may assume that  $D$  is not f-convex.

Let  $J$  be a set of  $E(K_n)$  for which there exists a set  $\{\gamma_f : f \in J\}$  of pairwise intersecting pseudocircles extending  $J$ . From Lemma 4.4.24, we may choose  $J$  to have at least one edge.

If  $J = E(K_n)$ , then  $J$  is our desired set of pseudocircles. Otherwise, let  $e \in E(K_n) \setminus J$ . Our aim is to find a pseudocircle  $\gamma_e$  extending  $e$ , and such that  $\{\gamma_f : f \in J\} \cup \{\gamma_e\}$  is a set of pairwise intersecting pseudocircles extending  $J \cup \{e\}$ .

Let  $M$  be the set of edges  $g \in E(K_n) \setminus (J \cup \{e\})$  such that  $D[g] \cap F_e \neq \emptyset$ . We repeatedly apply Lemma 4.4.25 to the edges  $g$  in  $M$  to obtain pseudocircles  $\delta_g$ . This yields a set  $\Gamma = \{\gamma_f : f \in J\} \cup \{\delta_g : g \in M\}$  of pseudocircles extending  $J \cup M$ .

For our initial approximation to  $\gamma_e$ , we consider the simple closed curve  $\gamma_e^0$  extending  $e$ , obtained from applying Lemma 4.4.25 to  $\Gamma$  and  $e$ . We let  $I_0$  be the set of edges in  $J \cup M$ , for which their corresponding curves in  $\Gamma$  intersect  $\gamma_e^0$ . We remark that  $M \subseteq I_0$ : every edge  $g \in M$  cannot have both ends in the same one of  $\Delta_e^1$  and  $\Delta_e^2$ . Thus, if we follow  $D[g]$  from a point in  $F_e$  towards the end of  $D[g]$  in  $\Delta_e^1$ ,  $D[g]$  crosses  $D[C_e^1]$ . As  $\gamma_e^0$  is chosen to be near  $D[C_e^1]$ ,  $\delta_g$  crosses  $\gamma_e^0$ .

A *feasible extension for  $e$*  is a simple closed curve  $\gamma$  including  $D[e]$ , such that  $\Gamma \cup \{\gamma\}$  is a set of pseudocircles extending  $J \cup M \cup \{e\}$  in  $D$ . For every feasible extension  $\gamma$ , the arc  $\gamma \setminus D[e]$  partitions  $F_e$  into two open regions; we refer to the region bounded by  $(\gamma \setminus D[e]) \cup D[C_e^2 - e]$  as  $\Theta_\gamma$  (see Figure 4.11a). We use  $cr(\Theta_\gamma)$  to denote the number of crossings of  $\Gamma$  inside  $\Theta_\gamma$ .

Informally speaking,  $cr(\Theta_\gamma)$  measures how close is  $\gamma$  to  $C_e^2$ . The idea of the proof is to start with the feasible extension  $\gamma = \gamma_e^0$  (that indeed is far from  $C_e^2$ ); then we slide  $\gamma$

towards  $C_e^2$ . At each step, either  $\gamma$  intersects more curves in  $\Gamma$  or  $cr(\Theta_\gamma)$  decreases. The key is to show that if there is a moment where  $\gamma$  cannot be pushed closer to  $C_e^2$  it is because our current curve intersects all the curves in  $\Gamma$ .

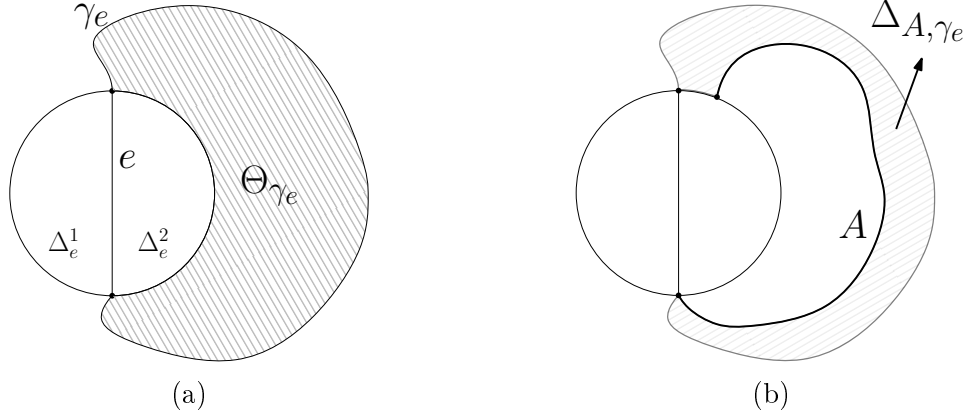


Figure 4.11: Definitions in the proof of Theorem 4.5.1.

We formalize this by considering a critical curve: among all the feasible extensions of  $e$ , we pick  $\gamma_e$  such that:

- (i) the set  $I \subseteq J \cup M$  of edges whose corresponding curves in  $\Gamma$  intersect  $\gamma_e$  is such that  $I_0 \subseteq I$ ;
- (ii) subject to (i),  $I$  is as large as possible; and
- (iii) subject to (i) and (ii),  $cr(\Theta_{\gamma_e})$  is as small as possible.

We aim to show that  $J \subseteq I$ . By way of contradiction suppose that  $J \setminus I$  is not empty.

**Claim 1.** *Let  $f \in J \setminus I$ . Then  $\gamma_f \cap \Theta_{\gamma_e} \neq \emptyset$ .*

*Proof.* Since  $D$  is not  $f$ -convex, Lemma 4.5.2 implies that  $\gamma_f \cap F_e \neq \emptyset$ . Thus, if  $\gamma_f$  is disjoint from  $\Theta_{\gamma_e}$ , then  $\gamma_f$  has points in the open subregion of  $F_e$  bounded by  $D[C_e^1 - e] \cup (\gamma_e \setminus D[e])$ . Since the ends of  $f$  are in  $\Delta_e^1 \cup \Delta_e^2$ ,  $\gamma_f$  intersects the boundary of this open subregion. Then  $\gamma_f$  either intersects  $D[C_e^1 - e]$  or  $\gamma_e$ . Any curve in  $\Gamma$  that crosses from  $F_e$  into  $D[C_e^1]$  intersects  $\delta_e^0$ , and hence is in  $I_0$ . Thus, in either case of  $\gamma_f$  intersecting  $D[C_e^1 - e] \cup (\gamma_e \setminus D[e])$ , condition (i) implies that  $\gamma_f$  intersects  $\gamma_e$ , contradicting that  $f \in J \setminus I$ .  $\square$

We claim that, for every  $f \in J \setminus I$ , there is no face  $F$  of  $\bigcup \Gamma$  for which both  $\gamma_e \setminus D[e]$  and  $\gamma_f$  have arcs incident with  $F$ . In the alternative, we can apply a Reidemeister Type II move to shift a portion of  $\gamma_e \setminus D[e]$  across  $\gamma_f$ , contradicting our choice of  $\gamma_e$  (see (ii)).

Therefore, if we let  $\Gamma_I = \Gamma \setminus \{\gamma_f : f \in J \setminus I\}$ , then there exists an arc  $A$  in  $\bigcup \Gamma_I$  with ends in the closed arc  $D[C_e^2 - e]$  and otherwise contained in  $\Theta_{\gamma_e}$ , separating  $\gamma_e$  from  $(\bigcup(\Gamma \setminus \Gamma_I)) \cap \Theta_{\gamma_e}$ .

Let  $\Delta_{A, \gamma_e}$  be the closure of the component of  $\Theta_{\gamma_e} \setminus A$  incident with  $\gamma_e \setminus D[e]$  (see Figure 4.11b). We choose  $A$  so that  $\Delta_{A, \gamma_e}$  is minimal.

Our goal now is to show that we can slide  $\gamma_e$  towards  $C_e^2$  without having a curve in  $\Gamma_i$  disjoint from the new  $\gamma_e$ . For this, we start by describing how the arcs in  $\Gamma_I$  intersect  $\Delta_{A, \gamma_e}$ .

**Claim 2.** *Let  $\gamma \in \Gamma_I$ . Then every arc-component of  $\gamma \cap \Delta_{A, \gamma_e}$  has an end in the interior of  $\gamma_e \setminus D[e]$  and an end not in the interior of  $\gamma_e \setminus D[e]$ .*

*Proof.* Let  $\beta$  be an arc of  $\gamma \cap \Delta_{A, \gamma_e}$ . If  $\beta$  has both ends not in the interior of  $\gamma_e \setminus D[e]$ , then there is an arc  $A'$  in  $\bigcup \Gamma_I$ , with  $\Delta_{A', \gamma_e}$  properly contained in  $\Delta_{A, \gamma_e}$ , contradicting the choice of  $A$ .

Suppose now that  $\beta$  has both ends in the interior of  $\gamma_e \setminus D[e]$ . Clearly  $\gamma \cap \gamma_e$  is not empty, so, for some  $f \in I$ ,  $\gamma = \gamma_f$ . Since  $|\gamma \cap \gamma_e| = 2$ ,  $\gamma \setminus \beta$  is contained in the side of  $\gamma_e$  that contains  $\Delta_e^1$ . Therefore  $f \notin M$ .

Let  $f' \in J \setminus I$ . Since  $\gamma_e \cap \gamma_{f'} = \emptyset$ , Claim 1 implies that  $\gamma_{f'}$  is entirely drawn in the side of  $\gamma_e$  containing  $\Delta_e^2$ . Moreover,  $\gamma_{f'}$  does not intersect  $\beta$ , as the arc  $A$  separates  $\gamma_e \setminus D[e]$  from  $\gamma_{f'} \cap \Theta_{\gamma_e}$ . This implies that  $\gamma_f \cap \gamma_{f'} = \emptyset$ , contradicting that the arcs in  $\{\gamma_f : f \in J\}$  are pairwise intersecting.  $\square$

We return to the context of Subsection 4.3, and consider a decomposition of  $A$  with respect to the arrangement  $\Gamma$ .

**Claim 3.** *At least one crossing in the decomposition of  $A$  faces the side of  $A$  included in  $\Delta_{A, \gamma_e}$ .*

*Proof.* By way of contradiction suppose that all the crossings in  $A$  face the side of  $A$  disjoint from  $\Delta_{A, \gamma_e}$ . We will arrive to a contradiction by showing that  $A$  is coherent.

Let  $\alpha$  be an arc in the decomposition of  $A$ . Since  $A$  is an arc in  $\bigcup \Gamma_I$ ,  $\alpha$  is included in some curve  $\gamma \in \Gamma$  such that  $\gamma \cap \gamma_e \neq \emptyset$ .



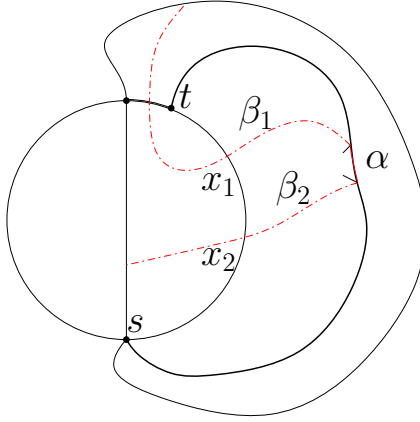


Figure 4.12: Illustrating some points in Claim 3.

Let  $\beta_1$  and  $\beta_2$  be the two disjoint arcs of  $\gamma \setminus \alpha$ , connecting an end of  $\alpha$  to a point in  $\gamma_e$ , and having no internal points in  $\gamma_e$  (Figure 4.12).

Let  $s$  and  $t$  be the ends of  $A$ , and let  $B$  be the  $st$ -arc of  $D[C_e^2 - e]$ . For  $i = 1, 2$ , let  $a_i$  be the end of  $\beta_i$  in  $\alpha$ . In case  $a_i \notin \{s, t\}$ , when we traverse  $\beta_i$  from  $a_i$  to its end in  $\gamma_e$ , we have a first encounter with  $A \cup B$  at a point that we denote as  $x_i$  (for this part, we are using the fact that the crossings in the decomposition of  $A$  are facing towards the side of  $A$  disjoint from  $\Delta_{A, \gamma_e}$ ). If  $a_i \in \{s, t\}$ , we simply let  $x_i = a_i$ .

If, for some  $i = 1, 2$ ,  $x_i$  is in the interior of  $A$ , then the  $a_i x_i$ -arc of  $\beta_i$  is a coherent extension of  $\alpha$ . Thus, as our goal is to show that  $\alpha$  is coherent, we may assume that  $x_1$  and  $x_2$  are in  $B$ .

Let  $Z = \gamma \cap D[C_e^2]$ . For each  $z \in Z$ , consider a short arc  $\sigma$  of  $\gamma$  centered at  $z$ . Let  $c(z)$  be the number of components of  $\sigma \setminus z$  drawn in  $F_e$ . By reversing the roles of  $\Sigma_e^1$  and  $\Sigma_e^2$  in Lemma 4.4.24, we know that there is a simple closed curve  $\delta_e$  extending  $e$ , with  $\delta_e \setminus D[e]$  drawn in  $F_e$  near  $D[C_e^2 - e]$ , and such that  $|\delta_e \cap \gamma| \leq 2$ . This implies that  $\sum_{z \in Z} c(z) \leq 2$ .

Since  $\gamma$  intersects  $D[e]$  at most once, at least one of  $\beta_1$  and  $\beta_2$ , say  $\beta_1$ , has one end in  $\gamma_e \setminus D[e]$ . Each of  $x_1$  and  $x_2$  is in  $Z$ ; furthermore, for  $i = 1, 2$ , there are points of  $\gamma \cap F_e$  near  $x_i$ , and thus  $c(x_i) > 0$ . If we traverse  $\beta_1$  from  $x_1$  towards its end in  $\gamma_e \setminus D[e]$ , there is first point  $x'_1$  from which  $\beta_1$  crosses from  $C_e^1$  into  $F_e$ . Such a point has  $c(x'_1) > 0$ . Either  $x'_1 \neq x_1$ , and there are three points in  $Z$  having positive contribution  $c$ , or  $x'_1 = x_1$ , and  $c(x_1) = 2$ . In any case,  $\sum_{z \in Z} c(z) > 2$ , contradicting the conclusion of the previous paragraph.

Thus, each arc  $\alpha$  has a coherent extension, and hence  $A$  is coherent. However, Theorem

4.3.3 implies that not all crossings in the decomposition of  $A$  face the same side of  $A$ . This final contradiction shows that at least one of the crossings faces the side of  $A$  included in  $\Delta_{A,\gamma_e}$ .  $\square$

Recall that  $\Gamma_I$  is the set of curves in  $\Gamma$  that intersect  $\gamma_e$ . From what we just showed, we may assume that there exists distinct curves  $\gamma_1, \gamma_2$  in  $\Gamma_I$ , crossing at  $\times_{1,2}$ , where  $\times_{1,2}$  is a crossing in the interior of  $A$  facing the side of  $A$  included in  $\Delta_{A,\gamma_e}$ . For  $i = 1, 2$ , let  $\beta_i$  be the arc of  $\gamma_i \cap \Delta_{A,\gamma_e}$  having  $\times_{1,2}$  as an end.

From Claim 2 we know that, for  $i = 1, 2$ , the end  $a_i$  of  $\beta_i$  distinct from  $\times_{1,2}$  is in the interior of  $\gamma_e \setminus D[e]$ .

Since  $\beta_1 \subseteq \gamma_1$  and  $\beta_2 \subseteq \gamma_2$ ,  $|\beta_1 \cap \beta_2| \leq 2$ . If  $\beta_1$  and  $\beta_2$  cross twice, we denote their crossing distinct from  $\times_{1,2}$  as  $\times_{1,2}^*$ ; otherwise, let  $\times_{1,2}^* = \times_{1,2}$ .

Consider the simple closed curve  $\theta$  obtained by traversing  $\beta_1$  from  $\times_{1,2}^*$  to  $a_1$ , then following  $\gamma_e$  until we reach  $a_2$ , and returning back to  $\times_{1,2}^*$  by following  $\beta_2$ . We claim that none of the curves in  $\Gamma$  cross the interior of each of  $\theta \cap \beta_1$  and  $\theta \cap \beta_2$ .

Suppose that some curve  $\gamma_3 \in \Gamma$  crosses the interior of  $\theta \cap \beta_1$  at  $\times_{1,3}$ . Let  $\beta_3$  be the arc-component of  $\gamma_3 \cap \Delta_{A,\gamma_e}$  containing  $\times_{1,3}$ . Claim 2 implies that  $\beta_3$  has a subarc  $\beta_3'$  connecting  $\times_{1,3}$  to an end of  $\beta_3$  not in the interior of  $\gamma_e \setminus D[e]$ . We may assume that  $\beta_3'$  has no further intersection with  $\beta_1$ .

The set  $A \cup \beta_3' \cup \beta_1$  contains an arc  $A'$  such that  $\Delta_{A',\gamma_e}$  is properly contained in  $\Delta_{A,\gamma_e}$ , contradicting the choice of  $A$ . Thus no arc in  $\Gamma$  intersects the interior of  $\theta \cap \beta_1$ , and likewise no arc in  $\Gamma$  intersects the interior of  $\theta \cap \beta_2$ . Moreover, no arc in  $\Gamma$  intersects  $\theta \cap \gamma_e$  as consequence of Claim 2.

From the previous discussion we conclude that no arc in  $\Gamma$  has a point in the side of  $\theta$  contained in  $\Delta_{A,\gamma_e}$ . Therefore we can perform a Redeimeister III move, shifting the portion of  $\gamma_e$  in  $\theta$  across  $\times_{1,2}^*$ . The obtained pseudocircle  $\gamma_e'$  extending  $D[e]$  intersects the same curves in  $\Gamma$  as  $\gamma_e$ , and it has  $cr(\Theta_{\gamma_e'}) < cr(\Theta_{\gamma_e})$ , contradicting the choice of  $\gamma_e$ .

This last contradiction shows that a feasible extension  $\gamma_e$  of  $e$  satisfying (i), (ii) and (iii), intersects all the curves in  $\{\gamma_f : f \in J\}$ . Thus, there is a set of pairwise intersecting pseudocircles extending  $J \cup \{e\}$ . The result now follows from applying an easy induction.  $\square$

## 4.6 Concluding remarks

In this section we present some observations and questions related to pseudospherical drawings that can be interesting topics for future research.

A classic result of Barány and Füredi [10], states that given  $n$  points in general position in the plane, there are at least  $n^2 + o(n^2)$  triangles formed by taking three of these  $n$  points, that contain none of the other  $n - 3$  points in their interior. This is the problem of counting the number of *empty triangles* in a configuration of points in the plane. There are configurations with as few as approximately  $1.6n^2 + o(n^2)$  empty triangles [11], and it is an open problem to find the right coefficient for  $n^2$ .

In a more general setting, given a good drawing of  $K_n$ , an empty triangle is a 3-cycle that contains no vertices in the interior of one of its sides. Harborth found a family of good drawings of  $K_n$  with at most  $2n - 4$  empty triangles [17], and Aichholzer et al. showed that every good drawing of  $K_n$  has at least  $n$  empty triangles [3]. In contrast, the Barány and Füredi's result can be extended to pseudolinear drawings of  $K_n$  [8], so pseudolinear drawings of  $K_n$  also have least  $n^2 + o(n^2)$  empty triangles. The next is a lower bound for the number of empty triangles in a pseudospherical drawing of  $K_n$ . Its proof uses the fact that in every pseudospherical drawing of  $K_n$ , each edge partitions the drawing into two pseudolinear drawings (Lemma 4.4.14).

**Theorem 4.6.1.** *If  $D$  is a pseudospherical drawing of  $K_n$  then  $D$  has at least  $\frac{3}{4}n^2 + o(n^2)$  empty triangles.*

*Proof.* If  $D$  is f-convex, then it has  $n^2 + o(n^2)$  empty triangles. Thus, we may assume  $D$  is not f-convex. Since  $D$  is h-convex, we consider an h-convex choice  $\{\Delta_T : T \text{ is a 3-cycle}\}$  of convex sides.

**Claim 1.** *Every edge of  $K_n$  is in at least two empty triangles.*

*Proof.* Let  $f \in E(K_n)$ . For  $i = 1, 2$ , consider  $D_f^i$  the drawing induced by the vertices drawn in  $\Delta_f^i$  (see Notation 4.4.10-4.4.15). For  $i = 1, 2$ , let  $v_i \in \Sigma_f^i$  and let  $T_{v_i}$  be the 3-cycle containing  $v_i$  and  $f$ . Since  $\Delta_{T_{v_i}}$  is convex, every vertex  $v'_i$  drawn in the interior of  $\Delta_{T_{v_i}}$ , has its  $\Delta_{T_{v'_i}}$  properly included in  $\Delta_{T_{v_i}}$ , but  $\Delta_{T_{v'_i}}$  has fewer vertices than  $\Delta_{T_{v_i}}$  in its interior. Thus, by choosing vertices  $v_1 \in \Sigma_f^1$  and  $v_2 \in \Sigma_f^2$  minimizing the numbers of vertices in the interior of  $\Delta_{T_{v_1}}$  and  $\Delta_{T_{v_2}}$ , we find our two desired empty triangles  $T_{v_1}$  and  $T_{v_2}$ .  $\square$

Let  $e$  be an edge of  $K_n$ . Suppose that there are  $k$  of the vertices of  $K_n$  in  $\Delta_e^1$  and  $n - k + 2$  vertices in  $\Delta_e^2$ . Then, because the drawings  $D_e^1$  and  $D_e^2$  are f-convex (Lemma 4.4.14), each of these drawings exhibit at least  $k^2 + o(n^2)$  and  $(n - k + 2)^2 + o(n^2)$  empty triangles.

Consider the pairs  $(f, T)$ , where  $f$  is an edge with ends on distinct ones of  $\Sigma_e^1$  and  $\Sigma_e^2$ , and  $T$  is an empty triangle containing  $f$ . Claim 1 implies that each edge with one end in  $\Sigma_e^1$  and the other in  $\Sigma_e^2$  is in at least two of these pairs. Thus, there are at least  $2(k - 2)(n - k - 2)$  of such pairs. Moreover, at most two of such pairs contain a fixed triangle  $T$ . This counting gives  $(k - 2)(n - k - 2)$  additional empty triangles.

In total, there are at least  $k^2 + (n - k + 2)^2 + (k - 2)(n - k - 2) + o(n^2)$  empty triangles. By optimizing  $k$ , it follows that this number is at least  $\frac{3}{4}n^2 + o(n^2)$ .  $\square$

So naturally we have the following question:

**Question 7.** *Can we improve the coefficient  $\frac{3}{4}$  in Theorem 4.6.1?*

In [25], Rafla stated the following beautiful conjecture:

**Conjecture 4.6.2.** *Every good drawing of  $K_n$  in the sphere has a Hamilton cycle drawn without self-crossings.*

There is a folklore proof for Rafla's Conjecture for rectilinear drawings (that can be easily adapted for pseudolinear drawings). We extend this result by showing that Rafla's conjecture holds for pseudospherical drawings.

**Theorem 4.6.3.** *Every pseudospherical drawing of  $K_n$  has a Hamilton cycle drawn without self-crossings.*

*Proof.* Suppose that  $D$  is a pseudospherical drawing of  $K_n$ , and let  $\Gamma$  be an arrangement of pseudocircles extending the edges of  $K_n$ . For an edge  $e \in E(K_n)$ , define its weight  $w(e)$  to be the number of pseudocircles in  $\Gamma$  crossing  $e$ .

Let  $H$  be a Hamilton cycle minimizing  $\sum_{e \in E(H)} w(e)$ . Suppose by way of contradiction that  $H$  is self-crossed. Let  $xz$  and  $yw$  be edges of  $H$  crossing at  $\times$ , and denote the crossing  $K_4$  containing these edges as  $J$ . The graph  $H - xz - yw$  consists of two vertex-disjoint paths connecting two vertices of  $J$ , that by symmetry, we may assume that one of the paths connects  $x$  to  $y$ , while the other path connects  $z$  to  $w$ .

Consider the Hamilton cycle  $H' = (H - xz - yw) + xw + yz$ . Let  $\alpha$  be the arc obtained by following  $D[xz]$  from  $x$  to  $\times$ , and continuing along  $D[yw]$  from  $\times$  to  $w$ . Since

every pseudocircle in  $\Gamma$  intersects the simple closed curve  $D[xw] \cup \alpha$  an even number of times, and none of them can intersect  $xw$  more than once, every curve in  $\Gamma$  crossing  $xw$ , crosses at least one of  $xz$  and  $yw$ . Likewise, every curve in  $\Gamma$  crossing  $yz$  crosses at least one of  $xz$  and  $yw$ . Moreover, because  $xz$  crosses  $yw$ , but  $xw$  does not cross  $yz$ ,  $\sum_{e \in E(H')} w(e) \leq \sum_{e \in E(H)} w(e) - 2$ , a contradiction.  $\square$

It is interesting to note that the fact that  $\Gamma$  satisfies (PS3) is not used in the previous proof.

Our last question is about extending the notion of pseudospherical to graphs that are not necessarily  $K_n$ . In particular:

**Question 8.** *Can we characterize drawings of graphs whose edges can be extended to a set of simple closed curves satisfying (PS1), (PS2) and (PS3)? Can we find their minimal obstructions?*

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