Contributions to the Study of General Relativistic Shear-Free Perfect Fluids

An Approach Involving Cartan's Equivalence Method, Differential Forms and Symbolic Computation

by

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Abstract

It has been conjectured that general relativistic shear-free perfect fluids with a barotropic equation of state, and such that the energy density, μ, and the pressure, p, satisfy $\mu + p \neq 0$, cannot simultaneously be rotating and expanding (or contracting). A survey of the known results about this conjecture is included herein. We show that the conjecture holds true under either of the following supplementary conditions: 1) the Weyl tensor is purely magnetic with respect to the flow velocity vector or 2) $dp/d\mu = -1/3$.

Any hypersurface-homogeneous shear-free perfect fluid which is not space-time homogeneous and whose acceleration vector is not parallel to the vorticity vector belongs to one of three invariantly defined classes, labelled A, B and C. It is found that the Petrov types which are allowed in each class are as follows: for class A, type I only; for class B, types I, II and III; and for class C, types I, D, II and N.

Two-dimensional pseudo-Riemannian space-times are classified in a manner similar to that of the Karlhede classification of four-dimensional general-relativistic space-times.

In an appendix, the `forms` differential forms package for the Maple program is described.

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Katka et Mélanie

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Chapter 1

Introduction

If I have seen farther it is by standing on the shoulders of giants.

Sir Isaac Newton

IN THE process of finding physically meaningful solutions to Einstein's field equations of general relativity, one is often confronted with the possibility that an apparently new metric describes the same spacetime as that given by an already known one. The problem is compounded by the fact that the physical properties of a given metric are unchanged by a coordinate transformation. The detection of the equivalence of two metrics is then a very difficult problem, even if one confines oneself to local considerations. One way to attack the problem of equivalence is to compute, from each metric, a set of invariants. If the invariants from the first metric are not equivalent to the invariants from the second metric, then the two metrics cannot describe the same spacetime. For example, if the Riemann tensor vanishes for one of the metrics, but not for the other, then the two metrics cannot be equivalent. The problem associated with the equivalence of metrics is therefore reduced to that of finding the equivalence of invariants constructed from the metrics. Even if the question of the equivalence of two particular metrics cannot be completely settled, partial information can be gained from a study of the invariants. Metrics can thus be classified. For example, the Petrov classification of the Weyl tensor and the Segre and Plebanskí classifications of the Ricci tensors are classification schemes based on constructing invariants from the Riemann tensor. Another set of invariants that can be derived from a particular metric is its symmetry group. It may seem that classifications based on invariants such as the Riemann tensor and classifications according to symmetry groups have little to do with each other. However, a deeper examination reveals an amazing interplay between the two approaches; they are two facets of a very powerful theory. Indeed, they both can be found using the method of equivalence of Cartan, which is a systematic method of finding invariants. In particular, when applied to the study of the equivalence of metrics, Cartan's method uncovers the results that the relevant invariants for the orthogonal group of transformation are the Riemann tensor and its derivatives. Similarly, the invariants for the conformal group of transformations are found to be the Weyl tensor, a tensor which reduces to the Cotton-York tensor in the three dimensional case, and their derivatives. Cartan's method also uncovers the various symmetry groups of the metrics.

In chapter 2, manifolds with pseudo-Riemannian real analytic metrics are studied using the method of Cartan. In the context of general relativity, however, many metrics can describe the same physical spacetime. Ignoring discrete transformations, this internal indeterminacy is encoded in a group of transformations, the special orthogonal group. The method of equivalence transforms the study of the original manifold to a new manifold that includes the group of indeterminacy as well as the original manifold. The invariants given by the method are quantities which are defined on the enlarged manifold. We show the well-known result that the invariants associated with the pseudo-Riemannian metrics (using the special orthogonal group) are the Riemann tensor (on the enlarged manifold) and its derivatives. We then show how these invariants of the enlarged manifold can be computed by lifting calculations done on the original manifold, which are nothing more than the classical calculations. We then do similar calculations when the group of transformations is the conformal group, a group that is larger than the internal group of indeterminacy. This new equivalence problem has as invariants (defined on the enlarged manifold) the Weyl tensor, a tensor which reduces to the Cotton-York in the three dimensional case and their derivatives. In the process, we uncover a set of one-forms that contain the information of the Ricci tensor. It is not clear whether any meaning can be given to the particular combinations of Ricci tensor components that appear in these one-forms. The calculations in chapter 2 are illustrated throughout with explicit calculations for the situation of real analytic two-dimensional pseudo-Riemannian metrics. We also give a classification, which appears to be new, of manifolds that possess such metrics. This classification is a similar to the Karlhede classification, which is a modification to the method of Cartan that is better suited for the space-times of general relativity. We also illustrate the calculation of the invariants for the situation of conformally flat metrics.

In chapter 3, we concentrate on the four-dimensional spacetimes of general relativity. We show how the structure equations of such manifolds can be obtained using differential forms. This approach is dual to the method of orthonormal tetrads. The structure equations involve functions, the kinematic quantities, of which we present two similar invariant constructions. The differential forms method, or the orthonormal tetrad method, have the advantage over coordinate methods that the equations of general relativity become first order differential equations, instead of equations of second order. There is a price to be paid, however. The number of equations is larger, since the set of equations that do not appear with the coordinate methods is the set of Jacobi identities, which are obtained by differentiating the structure equations. We then give expressions for the Riemann, Ricci and Weyl tensors in terms of the kinematic quantities. The Einstein field equations are given, then specialized to the case of a perfect fluid. Since we shall be interested in a fluid with a barotropic equation of state, the field equations introduce a single function, the energy density, in addition to the aforementioned kinematic quantities. The condition that two applications of the exterior derivative to a function must vanish gives integrability conditions. With the integrability conditions of the energy density, the basic equations are then all described.

We also present an invariant determination of an orthonormal tetrad that is well suited to the study of rotating perfect fluids. This choice implies the vanishing of many kinematic quantities, thereby simplifying our equations. This tetrad will be used in chapter 5.

In chapter 4, the Petrov classification of the Weyl tensor is presented in a manner that is slightly different, yet fully equivalent, to the usual method in general relativity. The approach presented herein focuses on the eigenvalues of a three by three complex matrix and on the dimensions of their corresponding eigenspaces. We also present a set of equations to convert between the Newman-Penrose components of the Weyl tensor, the components we presented in chapter 3 and the aforementioned three by three matrix. It is felt that this chapter clarifies the various interconnections between the different approaches to the Petrov classification.

In chapter 5, we turn our attention to general relativistic shear-free perfect fluids with a barotropic equation of state. It has been conjectured that such a fluid cannot be both rotating and expanding (or contracting). The first result showing a special case of the conjecture dates back to 1950. There are no known general relativistic counter-examples; however there are some in Newtonian gravity. Various special cases of the conjecture have been proved over the years, though as yet, its validity in the general case has still not been established. In the first part of chapter 5, we present a detailed history of the various partial results. We identify various properties that we feel were critical to the success. We also identify as a recurring theme the computation of torsion, which enables one to focus on the integrability conditions that are of lower order than is expected at any particular stage of a proof. In the second part of chapter 5, we establish the veracity of the conjecture for the special case when the Weyl tensor is purely magnetic with respect to the fluid flow. In the last part of the chapter, we show that the conjecture also holds for the case of a perfect fluid with a barotropic equation of state such that the derivative of the pressure with respect to the energy density is equal to $-1/3$. Such fluids include the coasting universes of inflation theory.

Should the shear-free conjecture hold, then the possible spacetimes that satisfy the hypotheses of the conjecture can be classified into two broad classes according to whether they are expanding (or contracting) or not. If their rate of expansion is not zero, then the shear-free conjecture would force them to be irrotational. This situation is well understood, all such spacetimes having been classified and examined by Collins and Wainwright (1983). If, however, the fluid has zero expansion, not all spacetimes have been identified. There are partial results in the literature. It is the subject of chapter 6 to find the Petrov types of a subclass of the expansion-free shear-free rotating spacetimes that has been previously identified. These spacetimes are hypersurface-homogeneous without being fully homogeneous. Also, their vorticity vector is linearly independent of their acceleration vector. The spacetimes we consider are divided into three cases, the simplest of which has already appeared in the literature in a study of rotating spacetimes with a Killing vector parallel to the axis of rotation. The determination of the Petrov type for each of the three cases is for the most part fairly straightforward. There are, however, two Petrov types in one of the cases that are surprisingly difficult to rule out. The question arises of showing that a particular set of polynomials has no solutions. In theory, doing so is simple: variables are eliminated one by one until a contradiction results that a non-zero integer is equal to zero. In practice, the expressions become so large that even being able to finish the computation is a difficult endeavour. The order in which the calculations are done is critical. Even so, we had to use various transformations to reduce the expression sizes. A further complication arises from the fact that at one point, a particular polynomial factorizes. The manner in which it does so precludes the use of certain evaluation techniques from the starting point. One must first use more straightforward methods in order to identify the factors of this polynomial. Once this is done, the evaluation techniques can be used to reduce the expression sizes. In spite of the various practical obstacles, it was found possible to complete the classification task. Various symbolic computation tools were considered, and tried, in order to resolve the problem of the presence of solutions to the set of polynomials. One theoretical development which initially appeared to be promising was the Gröbner bases method due to Buchberger (1985) for which the `grobner` package of Maple seemed especially useful. Unfortunately, it could not handle the polynomials which arose in the present problem. The computations could not finish, for lack of time. In retrospect, this is not surprising, considering the number of mathematical tools that were in the end used in order to complete the problem in a step by step manner.

Finally, we present in appendix A a differential forms package for the Maple symbolic computation program. The `forms` package implements the basic operations on differential forms and vectors. It also implements higher level functions such as tools to solve for unknown differential forms, to test whether a particular differential form is an element of a given differential ideal, to implement an inner product between differential forms and to compute operations such as the Hodge star of a differential form. We considered the use of the `difforms` package provided with Maple. It soon was apparent that `difforms` was not adequate for our needs[1] and that it would be faster to implement a new differential forms package than to modify the existing one. The package `forms` of appendix A was used as the main computational tool for chapter 5.

We make use of the following conventions, unless indicated otherwise. Indices are raised and lowered with a metric tensor whose signature is $(-+++)$. We use geometric units in which $8\pi G = c = 1$, where G is the Newtonian gravitational constant and c is the velocity of light in vacuum. The Riemann tensor, ${R^i}_{jk\ell}$, is defined by ${v^i}_{;\ell;k} - {v^i}_{;k;\ell} = {R^i}_{jk\ell} v^j$ for any C^2 vector field $\vec{v}$, with the semicolon denoting covariant differentiation. The Ricci tensor, R_{ij}, is defined by the contraction $R_{ij} = {R^k}_{ikj}$, and the Ricci scalar, R, by the contraction $R = {R^i}_i$.

[1] In particular, `difforms` does not handle vectors which are needed for the Lie derivative and for the interior product of a vector and a differential form.

Chapter 2

Applications of the Equivalence Method

Un bon livre devrait toujours former un véritable lien entre celui qui l'écrit et celui qui le lit. *Laure Conan*

IN THIS chapter, the equivalence method of Cartan is used to study the equivalence of pseudo-Riemannian real analytic metrics. The approach of Cartan involves the transformation of the problem of equivalence on a given manifold to a problem of equivalence on a new manifold, consisting of the original manifold augmented by a group of transformations.

We first look at the equivalence of metrics under the action of the orthogonal group. The application of the theory of Cartan shows that the geometric objects which allow a decision of whether two metrics are equivalent under this group are the Riemann tensor and its covariant derivatives up to an order determined by the method. These geometric objects are defined on the enlarged space. We show that the appropriate calculations need not be done solely on the enlarged space, but the main portion can be done on the original manifold. We then look at the equivalence of two metrics under the conformal group of transformations. We show that some of the invariant functions given by the method are the Weyl tensor components that are defined on the enlarged manifold. The other invariants functions are given by a tensor, which reduces to the Cotton-York tensor in the three-dimensional case. We then compute explicitly the various geometric objects, given by the method of Cartan, for the case of conformally flat metrics. In that case, all invariants vanish when the dimension of the metrics is greater than two.

Throughout our development, we illustrate the method by applying it to the two-dimensional pseudo-Riemannian real analytic metrics. We demonstrate the well-known result that all of these spaces are conformally equivalent. We then investigate the equivalence problem under the orthogonal group. The Riemann tensor, which in this case is a scalar, is obtained. A classification is provided of the real analytic two-dimensional pseudo-Riemannian metrics. This classification appears to be new. It involves the various groups of symmetry of those metrics, but distinguishes two classes of metrics without symmetry. This example illustrates the program of classification of spacetime metrics undertaken by a number of authors such as Karlhede (1980a), Karlhede (1980b), Karlhede and Lindström (1982), Karlhede and MacCallum (1982), Bradley and Karlhede (1990), Collins, d'Inverno and Vickers (1990), Joly and MacCallum (1990), Åman *et al.* (1991), Koutras (1992) and Collins *et al.* (1993). MacCallum (1991) gives a nice review of the progress in the classification of exact solutions of general relativity and of the computer programs involved in that classification. An interesting new development, which can be found in Paiva *et al.* (1993), is the use of the techniques involved in the Karlhede classification in order to find limits of spacetimes in a coordinate-free approach.

We shall often resort to the *Cartan Lemma* (Cartan, 1945). The statement of this lemma is as follows:

Lemma 1 (Cartan) *Let $\omega^1, \ldots, \omega^p$ be p one-forms which are linearly independent pointwise on an n-dimensional manifold M, with $p \leq n$. Let $\eta_1, \ldots, \eta_p$ be p one-forms on M satisfying*

$$\eta_i \wedge \omega^i = 0.$$

Then there exist C^∞ functions A_{ij}, with $A_{ij} = A_{ji}$, such that

$$\eta_i = A_{ij}\omega^j \quad (i = 1, \ldots, p).$$

Here, and throughout this work, we use Einstein's summation convention. The proof[1] of this lemma is as follows. Since $\omega^1, \ldots, \omega^p$ are all independent, they form part of a basis over M. This basis is formed by adjoining $p - n$ independent one-forms $\xi^1, \ldots, \xi^{p-n}$. Since for each i ($1 \leq i \leq p$), the one-form η_i is defined over M, it can be expanded in this basis; therefore, we obtain $\eta_i = A_{ij}\omega^j + B_{ij}\xi^j$, where A_{ij} and B_{ij} are functions. The condition on η_i translates into $A_{ij}\omega^j \wedge \omega^i + B_{ij}\xi^j \wedge \omega^i = 0$. Since the ξ^j are all independent of the ω^i, and they are all independent pairwise with each other, then the coefficients of $\xi^j \wedge \omega^i$ must all vanish, *i.e.* $B_{ij} = 0$ for all i and for all j. We are left with $(A_{ij} - A_{ji})\omega^{|i} \wedge \omega^{j|} = 0$, where $|i, j|$ indicates that $i \leq j$. Since $\omega^{|i} \wedge \omega^{j|}$ are all independent of each other, their coefficients must vanish, *i.e.* $A_{ij} = A_{ji}$. ▮

We note that the method of proof allows us to generalize the Cartan lemma to conclude that a set of p differential forms, η_i of degree q satisfying

$$\eta_i \wedge \omega^i = 0,$$

[1] A similar proof of this lemma is found in Appendix A.

must also satisfy

$$\eta_i = \xi_{ij} \wedge \omega^j$$

for some differential forms ξ_{ij} of degree $q-1$ that obey

$$\xi_{ij} \wedge \omega^j \wedge \omega^i = 0.$$

The proof is very similar to that of the standard Cartan lemma. We shall not introduce a new name for this generalization; the context being clear as to which version of the lemma is being used. Related to this generalization is the Cartan-Poincaré lemma, which appears in section VIII.2 of Bryant *et al.* (1991).[2]

2.1 Equivalence under the orthonormal group

The purpose of this section is to present a group invariant approach to defining and calculating the Riemann tensor. This approach is based on that of Cartan as expounded in Gardner (1989). We generalize the work therein by allowing for a metric of any signature. We also show explicitly how the calculations on the enlarged manifold can be done by lifting calculations on the original manifold. The theory is illustrated by performing the appropriate calculations for two-dimensional real analytic pseudo-Riemannian metrics, which will be referred to as 1+1 metrics.

A spacetime, in general relativity, is a four-dimensional manifold possessing a Lorentzian metric with signature $-+++$. In the tangent space of each point, therefore, the metric is simply the Minkowski metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. The metric may always be written as $\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a \otimes \mathrm{d}x^b$, whether one is dealing with a flat geometry in general coordinates, or a non-flat spacetime. This metric,

[2] I am grateful to R. Gardner for pointing out this lemma.

since it is not degenerate by hypothesis, can then be diagonalized as $\mathrm{d}s^2 = -(\sigma^0)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2$.

In general, we shall consider non-degenerate metrics of arbitrary dimension and signature; they may therefore be expressed as $\mathrm{d}s^2 = \sum_a \eta_{aa}(\sigma^a)^2$, where η_{aa} is the diagonal signature matrix. We shall give greater details of the computations in the case of the 1+1 metrics. Even though some features of the calculations are absent for metrics of such a small dimension, they still provide a useful model to keep in mind because the calculations are comparatively simple, and yet many features of higher-dimensional problems are indeed present.

The choice of diagonalization is not unique however. If we define $\bar{\omega} = S\sigma$, then $\bar{\omega}$ is also an acceptable choice for the diagonalization, provided that $\bar{\omega}^t\eta\bar{\omega} = \sigma^t\eta\sigma$. This implies that $\sigma^t S^t \eta S\sigma = \sigma^t\eta\sigma$ for all σ. Therefore S must obey the restriction that $S^t\eta S = \eta$. This is the definition of the statement that, ignoring reflections, S belongs to the group $\mathrm{SO}(p, q, \mathbf{R})$, where p is the number of plus signs in the signature and q is the number of minus signs. For spacetimes, the group is $\mathrm{SO}(3, 1, \mathbf{R})$. Because of its importance, this group is also referred to as the Lorentz group. For 1+1 spacetimes, the group is $\mathrm{SO}(1, 1, \mathbf{R})$. We shall restrict ourselves to real analytic transformations.

We now construct a differentiable manifold from the original space-time and the aforementioned group of transformations. This transforms the problem of equivalence over the space U to a question of equivalence over the space $U \times G$, where G is the group of which S is a member. In some sense, we are thus simultaneously considering all possible choices of S. The steps of considering orthogonal frames and of lifting the problem to a space that includes the group of allowed transformations form the essence of Cartan's equivalence method.

We first consider some calculations for 1+1 metrics, in order to illustrate the steps of the general case. The group $\mathrm{SO}(1,1,\mathbf{R})$ is easily parametrized; therefore, we can explicitly give part[3] of the basis to the space of differential forms of elements of $U \times G$. If we parametrize G by α, then we can define

$$\bar{\omega}^0 = (\cosh\alpha)\sigma^0 + (\sinh\alpha)\sigma^1$$

and

$$\bar{\omega}^1 = (\sinh\alpha)\sigma^0 + (\cosh\alpha)\sigma^1,$$

since $-(\bar{\omega}^0)^2 + (\bar{\omega}^1)^2 = -(\sigma^0)^2 + (\sigma^1)^2$. The cobasis elements σ are defined over U and the cobasis elements $\bar{\omega}$ are defined over $U \times G$, where G =SO(1,1,**R**). If we rewrite this in terms of matrices, then $\bar{\omega} = S\sigma$, where

$$S = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}.$$

We must find the variation of the frames in a small neighbourhood. We start with the structure equations over U, given as the exterior derivatives of the elements of the original cobasis σ in terms of themselves. We then look at the implications for the $U \times G$ space. For the 1+1 case, then, we therefore start with

$$\mathrm{d}\sigma^0 = F_1\sigma^0 \wedge \sigma^1$$

and

$$\mathrm{d}\sigma^1 = F_2\sigma^0 \wedge \sigma^1.$$

The structure equations of $U \times G$ are found by the following calculations:

$$\mathrm{d}\begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \end{pmatrix} = \begin{pmatrix} \sinh\alpha & \cosh\alpha \\ \cosh\alpha & \sinh\alpha \end{pmatrix} \mathrm{d}\alpha \wedge \begin{pmatrix} \sigma^0 \\ \sigma^1 \end{pmatrix} +$$

[3]Since $U \times G$ is 3-dimensional, $\bar{\omega}^0$ and $\bar{\omega}^1$ cannot form a full basis.

$$\begin{aligned} &\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} \mathrm{d}\sigma^0 \\ \mathrm{d}\sigma^1 \end{pmatrix} \\ = &\begin{pmatrix} \sinh\alpha & \cosh\alpha \\ \cosh\alpha & \sinh\alpha \end{pmatrix} \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \mathrm{d}\alpha \wedge \begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \end{pmatrix} \\ &+ \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} F_1\bar{\omega}^0 \wedge \bar{\omega}^1 \\ F_2\bar{\omega}^0 \wedge \bar{\omega}^1 \end{pmatrix}, \end{aligned}$$

where the equality $\bar{\omega}^0 \wedge \bar{\omega}^1 = \sigma^0 \wedge \sigma^1$ was used in order to express the results in terms of forms over $U \times G$ rather than over U. The structure equations over $U \times G$ are therefore

$$\mathrm{d}\begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d}\alpha \\ \mathrm{d}\alpha & 0 \end{pmatrix} \wedge \begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \end{pmatrix} + \begin{pmatrix} \cosh\alpha\, F_1 + \sinh\alpha\, F_2 \\ \sinh\alpha\, F_1 + \cosh\alpha\, F_2 \end{pmatrix} \bar{\omega}^0 \wedge \bar{\omega}^1.$$

For metrics of any dimension, the corresponding structure equations are given by

$$\mathrm{d}\bar{\omega} = \mathrm{d}S \wedge \sigma + S\mathrm{d}\sigma,$$

which is, when expressed over $U \times G$,

$$\mathrm{d}\bar{\omega} = (\mathrm{d}SS^{-1} + \vartheta(U,S)) \wedge \bar{\omega}, \tag{2.1}$$

where the terms $\vartheta(U,S)$ are linear in $\bar{\omega}$. Differentiating $S^t\eta S = \eta$ gives the following defining relations for the Lie algebra $\mathrm{so}(p,q,\mathbf{R})$ corresponding to the Lie group $\mathrm{SO}(p,q,\mathbf{R})$:

$$\mathrm{d}(S^t)\eta S + S^t\eta\mathrm{d}S = 0.$$

In order to use these relations together with the $U \times G$ structure equations, we obtain the following equivalent expression by multiplication on the left by $(S^{-1})^t$ and on the right by S^{-1} :

$$(\mathrm{d}SS^{-1})^t\eta + \eta(\mathrm{d}SS^{-1}) = 0. \tag{2.2}$$

This exhibits the rôle of η: if we use it to raise and lower indices, the above line states that $\mathrm{d}SS^{-1}$, with indices lowered, is antisymmetric .

On the 1+1 space, if we define

$$\Pi = \mathrm{d}\alpha + (F_1 \cosh\alpha + F_2 \sinh\alpha)\bar{\omega}^0 - (F_1 \sinh\alpha + F_2 \cosh\alpha)\bar{\omega}^1 \tag{2.3}$$

then the structure equations on $U \times G$ for the $1+1$ metrics can be rewritten as

$$\mathrm{d}\begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \end{pmatrix} = \begin{pmatrix} 0 & \Pi \\ \Pi & 0 \end{pmatrix} \wedge \begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \end{pmatrix}. \tag{2.4}$$

The matrix

$$\begin{pmatrix} 0 & \Pi \\ \Pi & 0 \end{pmatrix}$$

is antisymmetric when the first index is lowered. This indicates that it is an element of so(1,1,$\mathbf{R}$). The idea behind the definition of Π is to gather, as much as possible, quantities that can be changed by the group parameter, α.

We observe that, for the 1+1 metrics, there are no longer any terms that are explicitly quadratic in $\bar{\omega}$. For future reference, such terms will be referred to as torsion terms, or as the torsion. The requirement that the torsion vanish here, or equivalently that the torsion be completely absorbed, determines Π uniquely. This statement is rarely true in the application of the method of equivalence.

For general metrics, we can always write, using an index-free notation, the structure equations as

$$\mathrm{d}\bar{\omega} = \Delta \wedge \bar{\omega}, \tag{2.5}$$

where we recall that $\bar{\omega} = S\sigma$. The matrix Δ is an n by n matrix of one-forms. The matrix Δ can be split, non-uniquely, into a part that is independent of derivatives of group parameters and a part that does contain derivatives of group parameters.

In the present paragraph, we show that we can find, using Δ, a uniquely defined matrix, δ, belonging to $so(p,q,\mathbf{R})$ and such that

$$\mathrm{d}\bar{\omega} = \delta \wedge \bar{\omega}.$$

In order that the structure equations (2.5) be identical with (2.1), the matrix Δ must obey the condition

$$(\mathrm{d}SS^{-1} - \Delta + \vartheta(U,S)) \wedge \bar{\omega} = 0.$$

Therefore, we obtain, by using the Cartan Lemma, that

$$\Delta - \mathrm{d}SS^{-1} \equiv 0 \mod \text{base},$$

where by "mod base" we mean that the given congruence holds up to a linear combination of the basis $\bar{\omega}$. From this we can infer that

$$(\Delta - \mathrm{d}SS^{-1})^t \eta + \eta(\Delta - \mathrm{d}SS^{-1}) \equiv 0 \mod \text{base}.$$

Taking into account (2.2), this last congruence simplifies to

$$\Delta^t \eta + \eta \Delta \equiv 0 \mod \text{base}.$$

We thus conclude that there are no derivatives of group parameters in $\Delta^t\eta + \eta\Delta$. Because of that fact, these components of Δ are called the principal components of first order[4] (Gardner, 1989). The equivalence method approach then suggests that we perform the expansion $\Delta = \delta + \Psi$, where

$$\eta\Psi = 1/2(\Delta^t\eta + \eta\Delta)$$

[4]The order refers to the number of times this step of identifying terms independent of group derivatives in the matrix Δ is reached in the method of equivalence; for details, see Gardner (1989)

and

$$\eta\delta = 1/2(\eta\Delta - \Delta^t\eta),$$

to get the structure equations re-expressed as

$$\mathrm{d}\bar{\omega} = \delta \wedge \bar{\omega} + \Psi \wedge \bar{\omega}.$$

The functions Ψ do not contain derivatives of group parameters, and so they are expressible in the basis ω. We therefore have

$$(\Psi)^i{}_j = \Psi^i{}_{jk}\bar{\omega}^k,$$

for some functions $\Psi^i{}_{jk}$. Without loss of generality, we can antisymmetrize $\Psi^i{}_{jk}$ on the lower two indices, since we do not thereby modify the structure equations. We notice that $\delta^t\eta + \eta\delta = 0$, and so δ satisfies the Lie algebra relations of $\mathrm{so}(p, q, \mathbf{R})$. We try to eliminate as many of the functions Ψ as possible, by modifying δ,without changing its Lie algebra structure. This step is the absorption of torsion. Let Π be an $n \times n$ matrix of one-forms expressed in the $\bar{\omega}$ basis. Each entry therefore has n terms. We consider the coefficients in these terms to be the unknowns in the system of linear equations $\Pi \wedge \bar{\omega} = \Psi \wedge \bar{\omega}$, and we add the restriction that Π must obey the condition

$$\Pi^t\eta + \eta\Pi = 0. \qquad (2.6)$$

Note that there are $\frac{n^2(n-1)}{2}$ equations with $\frac{n^2(n-1)}{2}$ unknowns. We perform the expansion

$$(\Pi)^i{}_j = \Pi^i{}_{jk}\bar{\omega}^k,$$

for some functions $\Pi^i{}_{jk}$. After lowering the indices with η, the linear equations to be satisfied are

$$\Pi_{ijk}\bar{\omega}^k \wedge \bar{\omega}^j = \Psi_{ijk}\bar{\omega}^k \wedge \bar{\omega}^j.$$

The coefficients of the independent terms therefore obey

$$\Pi_{ijk} - \Pi_{ikj} = \Psi_{ijk} - \Psi_{ikj}.$$

The Lie algebra condition (2.6) is

$$\Pi_{ijk}\bar{\omega}^k + \Pi_{jik}\bar{\omega}^k = 0,$$

so

$$\Pi_{ijk} + \Pi_{jik} = 0.$$

Together, these imply that

$$\begin{aligned}
\Pi_{ijk} &= -\Pi_{jik} = -(\Pi_{jki} + \Psi_{jik} - \Psi_{jki}) \\
&= \Pi_{kji} - \Psi_{jik} + \Psi_{jki} \\
&= (\Pi_{kij} + \Psi_{kji} - \Psi_{kij}) - \Psi_{jik} + \Psi_{jki} \\
&= -\Pi_{ikj} + \Psi_{kji} - \Psi_{kij} - \Psi_{jik} + \Psi_{jki} \\
&= -(\Pi_{ijk} + \Psi_{ikj} - \Psi_{ijk}) + \Psi_{kji} - \Psi_{kij} - \Psi_{jik} + \Psi_{jki}.
\end{aligned}$$

This can be simplified due to the antisymmetry $\Psi_{ijk} = -\Psi_{ikj}$. Therefore, the unknowns Π_{ijk} are solved in terms of the torsion coefficients as

$$\Pi_{ijk} = \Psi_{ijk} - \Psi_{jik} - \Psi_{kij}.$$

The torsion can thus be eliminated by defining $\varphi := \delta + \Pi$, to get $\mathrm{d}\bar{\omega} = \varphi \wedge \bar{\omega}$ with $\varphi^t\eta + \eta\varphi = 0$. This determines φ uniquely.

It is rarely the case that all torsion can be made to vanish. Usually only some torsion terms can be set to zero. This being the case, the next step in the equivalence method would be to try to use the group G to normalize some of the remaining torsion terms to particular values. For example, if the group acts by multiplication on some torsion terms, then a number of these could be normalized to 1. Requiring that the normalization be preserved restricts the group G to one of its subgroups.

At this stage, we have that φ and $\bar{\omega}$ are invariants on $U \times G$. Therefore the group of freedom on this structure consists solely of the identity. When this is the case, we say that we have an e-structure. The theory of the equivalence of e-structures now enables us to state that the fundamental invariants of the problem are given by the functions involved in the structure equations of the e-structure. These functions, γ, are invariants, in the sense that if Φ is the transformation that takes U to V, then $\gamma|_U = \gamma|_V \circ \Phi$. We shall first find these invariants for the 1+1 spacetimes, then we shall do so for general spacetimes. We define F_s to be the set consisting of the invariants and their covariant derivatives up to order $s-1$. We consider F_s to be lexicographically ordered. The **rank** k_s of F_s at a point p is the rank of the span of $\mathrm{d}(F_s)$ at p. The **order** of F_s at p is the smallest j for which $k_j = k_{j+1}$. An e-structure is said to have **regular rank** ρ at p if the rank of the F_s of the e-structure is ρ in a neighbourhood of p. We point out that the rank and the order of an e-structure are invariant quantities. The theory (Gardner, 1989) allows us to state that if the rank of a regular n-dimensional e-structure is ρ, then the e-structure admits an $(n-\rho)$-dimensional symmetry group.

For 1+1 spacetimes, we proceed as follows. From the exterior derivative of the 1+1 structure equations (2.4), we get

$$0 = \mathrm{d}^2 \begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d}\Pi \\ \mathrm{d}\Pi & 0 \end{pmatrix} \wedge \begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \end{pmatrix} - \begin{pmatrix} 0 & \Pi \\ \Pi & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & \Pi \\ \Pi & 0 \end{pmatrix} \wedge \begin{pmatrix} \bar{\omega}^0 \\ \bar{\omega}^1 \end{pmatrix}.$$

One of the relations we thereby obtain is

$$0 = \mathrm{d}\Pi \wedge \bar{\omega}^1.$$

This implies that

$$\mathrm{d}\Pi = \tau \wedge \bar{\omega}^1, \tag{2.7}$$

where τ is a 1-form on $U \times G$. The other relation we obtain is

$$0 = \mathrm{d}\Pi \wedge \bar{\omega}^0 = \tau \wedge \bar{\omega}^1 \wedge \bar{\omega}^0,$$

where we have used (2.7). The Cartan lemma then states that τ is a linear combination of $\bar{\omega}^0$ and $\bar{\omega}^1$, *i.e.*

$$\tau = R\,\bar{\omega}^0 + R'\,\bar{\omega}^1.$$

Again using (2.7), the derivative of the connection form Π is therefore

$$\mathrm{d}\Pi = R\,\bar{\omega}^0 \wedge \bar{\omega}^1. \tag{2.8}$$

The function R is the required invariant function. It is just the lifted Riemann tensor component ${R^0}_{101}$.

We now proceed to obtain R explicitly in terms of the functions F_1 and F_2. If we expand the derivatives of F_1 and F_2 in the σ basis (since F_1 and F_2 are defined on U), we get

$$\mathrm{d}F_1 = F_{1|\sigma^0}\sigma^0 + F_{1|\sigma^1}\sigma^1$$

and a similar expression for F_2. These expressions can be used as definitions for $F_{1|\sigma^0}$, $F_{1|\sigma^1}$, $F_{2|\sigma^0}$ and $F_{2|\sigma^1}$. We differentiate equation (2.3) and hence obtain

$$\mathrm{d}\Pi = [-F_{1|\sigma^1} - F_{2|\sigma^0} + (F_1)^2 - (F_2)^2]\bar{\omega}^0 \wedge \bar{\omega}^1,$$

after converting the result into the $\bar{\omega}$ basis. (In this case, $\bar{\omega}^0 \wedge \bar{\omega}^1$ is just $\sigma^0 \wedge \sigma^1$, but this is rarely true.) Comparison with (2.8) provides us with the result that $R = -F_{1|\sigma^1} - F_{2|\sigma^0} + (F_1)^2 - (F_2)^2$, as required.

We now turn to spacetimes of any dimension. After the absorption of torsion, the structure equations are

$$\mathrm{d}\bar{\omega} = \varphi \wedge \bar{\omega},$$

where φ is uniquely determined and obeys the condition

$$\varphi^t \eta + \eta \varphi = 0.$$

The exterior derivative of the structure equations is

$$0 = \mathrm{d}^2 \bar{\omega} = (\mathrm{d}\varphi - \varphi \wedge \varphi) \wedge \bar{\omega}.$$

The quantity in parentheses contains the information about the curvature of the spacetime. This justifies the definition

$$\Theta := \mathrm{d}\varphi - \varphi \wedge \varphi, \tag{2.9}$$

where this curvature two-form is constrained by

$$0 = \Theta \wedge \bar{\omega}. \tag{2.10}$$

We note that the definition of Θ forces it to obey $\Theta^t \eta + \eta \Theta = 0$. By the Cartan lemma, the constraint (2.10) on Θ implies that it can be expanded in the basis $\bar{\omega}$, the coefficients being one-forms:

$$\Theta^i{}_j = \psi^i{}_{jk} \wedge \bar{\omega}^k.$$

The one-forms $\psi^i{}_{jk}$ are not arbitrary since they must satisfy the constraint (2.10) on Θ. This produces the following equivalence:

$$0 = \Theta^i{}_j \wedge \bar{\omega}^j \Leftrightarrow \psi^i{}_{jk} \wedge \bar{\omega}^k \wedge \bar{\omega}^j = 0.$$

A cobasis for a space of dimension n is $\bar{\omega}^1, \bar{\omega}^2, ..., \bar{\omega}^n$. If we multiply the constraint on ψ with all possible combinations of $n-2$ cobasis forms, we obtain the relations (noting that most terms in the sum disappear)

$$({\psi^i}_{jk} - {\psi^i}_{kj}) \wedge \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \cdots \wedge \bar{\omega}^n = 0,$$

and therefore ψ is symmetric in the two lower indices, up to linear combinations of the cobasis, *i.e.*

$${\psi^i}_{jk} \equiv {\psi^i}_{kj} \mod \text{base}.$$

Similarly, the antisymmetry of Θ with its indices lowered translates into the following antisymmetry of ψ:

$$\Theta_{ij} = -\Theta_{ji} \Leftrightarrow (\psi_{ijk} + \psi_{jik}) \wedge \bar{\omega}^k = 0,$$

where the indices are lowered (and raised) using η. Multiplying this constraint with all possible combinations of $n-1$ cobasis forms, we obtain

$$(\psi_{ijk} + \psi_{jik}) \wedge \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \cdots \wedge \bar{\omega}^n = 0.$$

We can therefore conclude that the following congruences hold:

$$\psi_{ijk} \equiv \psi_{ikj} \equiv -\psi_{jik} \equiv 0 \mod \text{base},$$

which imply that

$${\psi^i}_{jk} \equiv 0 \mod \text{base}.$$

This shows that ${\psi^i}_{jk}$ can be expanded in the cobasis as follows:

$$\psi^i_{jk} = \frac{1}{2} {S^i}_{jk\ell} \bar{\omega}^\ell,$$

for some functions ${S^i}_{jk\ell}$. This demonstrates that ψ, and hence Θ, does not contain derivatives of the group parameters. It therefore follows that

$${\Theta^i}_j = \frac{1}{2} {S^i}_{jk\ell} \bar{\omega}^\ell \wedge \bar{\omega}^k.$$

The coefficients $S^i{}_{jk\ell}$ are the fundamental invariants of the problem.

When we take the exterior derivative of Θ, as given by its definition (2.9), we obtain

$$\begin{aligned} \mathrm{d}\Theta &= -\mathrm{d}\varphi \wedge \varphi + \varphi \wedge \mathrm{d}\varphi \\ &= -\Theta \wedge \varphi - \varphi \wedge \varphi \wedge \varphi + \varphi \wedge \Theta + \varphi \wedge \varphi \wedge \varphi \\ &= -\Theta \wedge \varphi + \varphi \wedge \Theta. \end{aligned}$$

This calculation simply yields the Bianchi identities on $U \times G$.

The structure equations on $U \times SO(p, q, \mathbf{R})$ can be summarized as follows:

$$\mathrm{d}\bar{\omega} = \varphi \wedge \bar{\omega}$$

and

$$\mathrm{d}\varphi = \varphi \wedge \varphi + \Theta.$$

With indices, these become

$$\mathrm{d}\bar{\omega}^i = \varphi^i{}_j \wedge \bar{\omega}^j$$

and

$$\mathrm{d}\varphi^i{}_j = \varphi^i{}_k \wedge \varphi^k{}_j + \frac{1}{2} S^i{}_{jk\ell} \bar{\omega}^\ell \wedge \bar{\omega}^k,$$

respectively.

So far, the calculations have been made on $U \times G$. This is more complicated than calculating on U. Furthermore, the "classical" results do not involve the group G. We therefore need to find the contribution of G, in order to recover the classical approach.

We define a left-action on G by multiplication on the left by a constant:

$$\begin{aligned} L_C \ &: \ G \to G \\ & \quad\ S \mapsto CS. \end{aligned}$$

This action on G induces an action (a pull-back) on the cobasis over $U \times G$:

$$L_C^* \bar{\omega} = L_C^*(S\sigma) = CS\sigma = C\bar{\omega}.$$

We can determine the induced action on the connection forms φ since pull-backs commute with exterior differentiation, and since the pull-back of an exterior product is the exterior product of the pull-back. The sequence of equalities

$$L_C^* \mathrm{d}\bar{\omega} = \mathrm{d}(L_C^*\bar{\omega}) = \mathrm{d}(C\bar{\omega}) = C\mathrm{d}\bar{\omega} = L_C^*(\varphi \wedge \bar{\omega}) = (L_C^*\varphi) \wedge (L_C^*\bar{\omega})$$

leads to

$$\varphi \wedge \bar{\omega} = [C^{-1}(L_C^*\varphi)C] \wedge \bar{\omega}.$$

We then conclude that

$$L_C^*\varphi = C\varphi C^{-1},$$

after invoking the uniqueness of φ. This type of action is called an adjoint action. Also, by the uniqueness of φ, $C^{-1}(L_C^*\varphi)C$ has the same index symmetries as φ. The induced action on $\varphi = \mathrm{d}SS^{-1} + \vartheta_U(u, S)$ leads to

$$L_C^*\varphi = \mathrm{d}(CS)(CS)^{-1} + L_C^*\vartheta_U(u, CS),$$

where, as can be expected, $L_C^*\vartheta_U(u, CS)$ means $(L_C^*\vartheta_U)|_{(u,CS)}$. Therefore, the action on ϑ_U obeys

$$L_C^*\vartheta_U(u, CS) = C\vartheta_U(u, S)C^{-1}.$$

Pointwise, we can make the choice of $C = S^{-1}$, provided, it seems, that we do not differentiate the results; we shall show in the next paragraph that, actually, we can perform the differentiation. We thus obtain the equivalent connection forms on U. With the definition

$$\vartheta_U(u) := L_{S^{-1}}^*\vartheta_U(u, S^{-1}S) = S^{-1}\vartheta_U(u, S)S,$$

the connection forms on U and those on $U \times G$ are related by

$$\vartheta_U(u, S) = S\vartheta_U(u)S^{-1}.$$

Similarly, the action on Θ obeys

$$\begin{aligned} L_C^*\Theta(u, CS) &= L_C^*(\mathrm{d}\varphi - \varphi \wedge \varphi) \\ &= \mathrm{d}(C\varphi C^{-1}) - C\varphi C^{-1} \wedge C\varphi C^{-1} \\ &= C(\mathrm{d}\varphi - \varphi \wedge \varphi)C^{-1} \\ &= C\Theta(u, S)C^{-1}. \end{aligned}$$

We can therefore define

$$\Theta(u) := L_{S^{-1}}^*\Theta(u, e) = S^{-1}\Theta(u, S)S,$$

which leads to

$$\Theta(u, S) = S\Theta(u)S^{-1}.$$

We now explicitly[5] show that we can indeed differentiate on U and obtain the appropriate quantities, without first going to $U \times G$ and then choosing a particular value of S. This is of value, since differentiating on U is easier than on $U \times G$. Once we know the result on U, it is easy to lift the result to $U \times G$. We are then able to apply the results of the method of equivalence.

We start by showing that we can compute $\Theta(u)$ by staying on U. For

$$\begin{aligned} &\mathrm{d}\vartheta_U(u) - \vartheta_U(u) \wedge \vartheta_U(u) \\ &= \mathrm{d}[S^{-1}\vartheta_U(u, S)S] - S^{-1}\vartheta_U(u, S) \wedge \vartheta_U(u, S)S \\ &= \mathrm{d}[S^{-1}\varphi S - S^{-1}\mathrm{d}S] - S^{-1}[\varphi - \mathrm{d}SS^{-1}] \wedge [\varphi - \mathrm{d}SS^{-1}]S \end{aligned}$$

[5] See page 27 ff. for some comments on the calculations on U.

$$\begin{aligned}
&= -S^{-1}\mathrm{d}SS^{-1}\wedge\varphi S + S^{-1}\mathrm{d}(\varphi)S - S^{-1}\varphi\wedge\mathrm{d}S + S^{-1}\mathrm{d}SS^{-1}\wedge\mathrm{d}S \\
&\quad -S^{-1}\varphi\wedge\varphi S + S^{-1}\mathrm{d}SS^{-1}\wedge\varphi S + S^{-1}\varphi\wedge\mathrm{d}S - S^{-1}\mathrm{d}SS^{-1}\wedge\mathrm{d}S \\
&= S^{-1}\mathrm{d}(\varphi)S - S^{-1}\varphi\wedge\varphi S \\
&= S^{-1}(\Theta(u,S))S \\
&= S^{-1}(S\Theta(u)S^{-1})S \\
&= \Theta(u),
\end{aligned}$$

where the use of the relation $\mathrm{d}(S^{-1}) = -S^{-1}\mathrm{d}SS^{-1}$ has been made. We caution that one needs to be careful with the signs of the exterior derivative and with the ordering of the various quantities, since matrices do not, in general, commute.

We now compute the Bianchi identities on U:

$$\begin{aligned}
\mathrm{d}\Theta(u) &= \mathrm{d}(S^{-1}\Theta(u,S)S) \\
&= -S^{-1}\mathrm{d}SS^{-1}\wedge\Theta(u,S)S + S^{-1}\mathrm{d}(\Theta(u,S))S + S^{-1}\Theta(U,S)\wedge\mathrm{d}S \\
&= -S^{-1}\mathrm{d}S\wedge\Theta(u) + S^{-1}(-\Theta(u,S)\wedge\varphi \\
&\quad +\varphi\wedge\Theta(u,S))S + \Theta(u)\wedge S^{-1}\mathrm{d}S \\
&= -S^{-1}\mathrm{d}S\wedge\Theta(u) - S^{-1}\Theta(u,S)\wedge(\mathrm{d}SS^{-1} + \vartheta(u,S)S) \\
&\quad +S^{-1}(\mathrm{d}SS^{-1} + \vartheta(u,S))\wedge\Theta(u,S)S + \Theta(u)\wedge S^{-1}\mathrm{d}S \\
&= \varphi(u)\wedge\Theta(u) - \Theta(u)\wedge\varphi(u).
\end{aligned}$$

These are the same equations as on $U \times G$. Further differentiation does not give anything new.

Finally, we show that we can get $\vartheta(u)$ from the cobasis on U:

$$\begin{aligned}
\mathrm{d}\sigma &= \mathrm{d}(S^{-1}\bar{\omega}) = \mathrm{d}(S^{-1})\wedge\bar{\omega} + S^{-1}\mathrm{d}\bar{\omega} \\
&= -S^{-1}\mathrm{d}SS^{-1}\wedge\bar{\omega} + S^{-1}\varphi\wedge\bar{\omega}
\end{aligned}$$

$$\begin{aligned} &= -S^{-1}\mathrm{d}SS^{-1} \wedge S\sigma + S^{-1}\mathrm{d}SS^{-1} \wedge S\sigma + S^{-1}\vartheta_U(u,S) \wedge S\sigma \\ &= S^{-1}(S\vartheta_U(u)S^{-1})S \wedge \sigma \\ &= \vartheta_U(u) \wedge \sigma. \end{aligned}$$

In summary, we can calculate ϑ_U and $\Theta(u)$ and the Bianchi identities without involving G at all. This is exactly the classical calculation, as can be found, for example, in Misner, Thorne and Wheeler (1973). The calculations for the equivalence method, which require the space $U \times G$, can therefore be done by first computing on U, then lifting to $U \times G$ by change of basis and multiplication by matrices, without any further differentiations.

This enables us to identify $\vartheta_U(u)$ as the connection one-forms, as found for example in Misner, Thorne and Wheeler (1973), and $\Theta(u)$ as the Riemann curvature two-forms. Expanding them in the cobasis over U, we get

$$\vartheta(u)]^i{}_j = \Gamma^i{}_{jk}\,\sigma^k$$

and

$$\Theta(u)]^i{}_j = \frac{1}{2}R^i{}_{jk\ell}\,\sigma^\ell \wedge \sigma^k,$$

where $\Gamma^i{}_{jk}$ are the Christoffel symbols and $R^i{}_{jkl}$ are the Riemann tensor components.

The group $SO(p,q,\mathbf{R})$ is the natural group to use in the study of pseudo-Riemannian manifolds. Furthermore, it is quite natural to use connections that render the structure equations to be torsion-free. These natural requirements can be seen as follows. The exterior derivative operation can be extended to vector-valued objects. There are more details given in Misner *et al.* (1973). Given $\{\vec{e}_a\}$ a vector basis, define d to be a differentiation such that it is equal to the ordinary exterior derivative when applied to functions and differential forms, and such that

$\mathrm{d}\vec{e}_a = \vec{e}_b \sigma^b{}_a$. On vectors with scalar coefficients, this derivative is then a definition for covariant differentiation of a vector. This is readily extended to tensor products of vectors with scalar coefficients by using the product rule. The expressions $\sigma^b{}_a$ are referred to as a connection. Let $\{\vec{e}_a\}$ be chosen dual to the cobasis σ_b; therefore, it satisfies the bilinear pairing $< \vec{e}_a, \sigma^b >= \delta^b_a$. We need a well defined relation between the derivative of $\vec{e}_a$ and that of σ_b. This is obtained by requiring the vanishing of the derivative the invariantly-defined vector-valued one-form $\vec{e}_a \otimes \sigma^a := \mathcal{P}$. We thus require

$$\vec{e}_c \otimes \sigma^c{}_a \wedge \sigma^a + \vec{e}_a \otimes \mathrm{d}\sigma^a = 0,$$

whence

$$\mathrm{d}\sigma^b = -\sigma^b{}_c \wedge \sigma^c.$$

We are thus led to torsion-free space-times. The connection is not uniquely specified. One natural invariant requirement is that it be chosen so that covariant differentiation be compatible with the metric; in other words, that the covariant derivative of the metric vanish. The (dual of) the metric is given by

$$\mathbf{g} = \sum_a \eta^{aa} \vec{e}_a \otimes \vec{e}_a.$$

Its covariant derivative, which we require to vanish, is given by

$$0 = \mathrm{d}\mathbf{g} = \eta^{aa} \vec{e}_c \sigma^c{}_a \otimes \vec{e}_a + \eta^{aa} \vec{e}_a \otimes \vec{e}_c \sigma^c{}_a,$$

which is equivalent to

$$0 = \vec{e}_c \sigma^{ca} \otimes \vec{e}_a + \vec{e}_a \otimes \vec{e}_c \sigma^{ca}.$$

It follows then that $\sigma^{ac} + \sigma^{ca} = 0$ or, equivalently, $\sigma_{ac} + \sigma_{ca} = 0$. These relations are exactly the defining relations of the Lie algebra $so(p, q, \mathbf{R})$. From previous results in the present chapter, it is clear that the connection is now uniquely determined. The fact that the torsion-free connection is that choice of connection which is

$SO(p, q, \mathbf{R})$-invariant is exactly[6] the reason why the equivalence calculations on $U \times G$ can be done first on U.

Note that the theory of the method of equivalence confirms the classical theorem that an n-dimensional Riemannian metric is determined up to isometries by prescribing the Christoffel symbols, the Riemannian curvature tensor and its derivatives up to order $n+1+n(n-1)/2$. The precise statement of this theorem contains conditions, on an e-structure, of regularity, equal order, equal rank, and preservation of dependency. We refer to Gardner (1989) for the precise specification of these conditions. We shall illustrate some of these points when we classify the 1+1 metrics. We further remark that the order stated in the theorem is one more than the dimension of $U \times G$. The stated number of differentiations is an upper bound. Usually much less than this is needed to determine the equivalence of two metrics, whether or not symmetries are involved.

Since the whole problem of equivalence on $U \times G$ can be completely solved by reducing to a computation on $U \times \{e\} \cong U$, and then multiplying by appropriate matrices, we might as well choose the representation of $U \times \{e\}$ in such a fashion as to simplify the computations. This provides a geometric justification for the usual practice of rotating an orthonormal tetrad so that one eliminates as many kinematic quantities as possible on $U \times \{e\}$, since they are invariantly defined on $U \times G$.

A standard procedure for classifying metrics involves using an eigenvalue approach on the Weyl and the Ricci tensors. This approach reduces the group G to one of its subgroups by choosing invariantly defined frames based on quantities appearing in the Riemann tensor. As Bradley and Karlhede (1990) remarked, it

[6] I am grateful to M.A.H. MacCallum for pointing this out.

is difficult to use the Christoffel symbols directly on U to carry out the appropriate reduction, since they are not tensorial in nature. However, on $U \times G$, the corresponding objects, φ, are tensorial. This gives a further justification for the approach used in the orthonormal tetrad techniques, where a frame can often be fixed by requiring that certain combinations of Christoffel symbols be made to vanish. Since this allows us more possibilities to reduce the group G than by solely using the Riemann tensor, the number of derivatives required for a classification can be reduced. For four-dimensional spacetimes, it has been shown that there is an upper bound of seven derivatives of the Riemann tensor. A summary of the relevant results can be found in Collins *et al* (1993). It appears likely that the upper bound will be reduced to six; the only situations where that bound of six has not been proved are the non-vacuum type-N metrics and a class of conformally flat metrics. In Collins, d'Inverno and Vickers (1990), the question was posed as to whether one needs to proceed beyond the third derivative. Since then, Koutras (1992) has answered that query by exhibiting a spacetime that requires four derivatives for its classification. So far, this is the highest number of differentiations that has been required for classifying a spacetime. In short the maximum number of necessary differentiations is at least four, no more than seven and very possibly no more than six.

2.2 Equivalence under the conformal group

In this section, we study the equivalence of metrics under the conformal group of transformations $CO(p, q, \mathbf{R}) = \{\lambda S | \lambda \in \mathbf{R}^*, S \in SO(p, q, \mathbf{R})\}$, where $\mathbf{R}^*$ represents the non-zero real numbers. We shall show that the geometric object allowing us to classify metrics under the conformal group is the Weyl tensor. Unlike the situation of the classification under the orthonormal group, a single lift is not sufficient to solve the problem. This section follows Gardner (1989), who stops after providing the structure equation on the twice-lifted space with a positive-definite metric. In the present work, metrics of arbitrary signature are allowed. We compute the action of the conformal group on the Weyl tensor. Some special cases of the classification are briefly addressed. We discuss the significance of particular one-forms that were introduced during the calculation; these contain the Ricci part of the Riemann tensor. Using the method of calculation discussed in this section, we then show explicitly the well-known result that, for conformally flat metrics, the Weyl tensor vanishes.

Let there be a non-degenerate metric given by $\mathrm{d}s^2 = \sigma^t \eta \sigma$, where

$$\eta = \mathrm{diag}(\underbrace{-1, -1, \cdots, -1}_{q}, \underbrace{1, 1, \cdots, 1}_{p}).$$

This metric will be used to raise and lower indices.

The one-forms σ give a coframe for the cotangent space to the base manifold U. We lift the problem to the space $U \times G$, where G is, in this case, the conformal group. We therefore look at the lifted coframe $\omega = \lambda S \sigma$, where $S^t \eta S = \eta$, $S \in SO(p, q, \mathbf{R})$ and $\lambda \in \mathbf{R}^*$. The structure equations, which are obtained by differentiating ω, contain terms that are linear and quadratic in ω. Therefore, they can be expressed as $\mathrm{d}\omega = \Delta \wedge \omega$, where Δ is a particular matrix of one-forms.

We now proceed to determine which entries in Δ do not contain derivatives of group parameters. The defining relations of the orthogonal group $SO(p, q, \mathbf{R})$ are $S^t \eta S = \eta$. Taking the exterior derivative of these relations gives the defining expressions of the corresponding Lie algebra $so(p, q, \mathbf{R})$, that is, $\eta \mathrm{d}SS^{-1} + (\mathrm{d}SS^{-1})^t \eta = 0$. This implies that the defining relations of the Lie algebra associated with the conformal group $CO(p,q,\mathbf{R})$ obey the condition

$$\eta\, \mathrm{d}(\lambda S)(\lambda S)^{-1} + (\mathrm{d}(\lambda S)(\lambda S)^{-1})^t \eta = 2\mathrm{d}\lambda\, \lambda^{-1}\, \eta.$$

As a consequence of this, since the structure equations are of the form

$$\mathrm{d}\omega = \mathrm{d}(\lambda S)(\lambda S)^{-1} \wedge \omega + \text{terms quadratic in } \omega,$$

there are no derivatives of the group parameters in the combination

$$\eta\Delta + \Delta^t \eta - \frac{2}{n}(\mathrm{trace}\,\Delta)\eta$$

of entries of the matrix Δ. We say that this combination gives the principal components of the first order for the present equivalence problem. They are linear in ω, and so the corresponding parts of the structure equations are quadratic in ω. Consequently, the principal components of first order yield the torsion. The torsion is not necessarily unique for a given problem; by varying the derivatives of the group, the torsion can change, and sometimes can even be made to vanish. The other components of Δ can split into a diagonal part and an antisymmetric part (once indices are lowered). To summarize, the structure equations can be written as

$$\mathrm{d}\omega = (\tilde{\phi} + \tilde{\alpha} I_n) \wedge \omega + \text{torsion},$$

where $\tilde{\phi}$ is antisymmetric with indices lowered, *i.e.* it obeys $\eta\tilde{\phi} + \tilde{\phi}^t\eta = 0$, and where I_n is the n-dimensional identity matrix.

From the equivalence problem under $SO(p,q,\mathbf{R})$, we know that all the torsion can be absorbed into $\tilde{\phi}$. In that situation, the absorption was unique. This is not so in the present situation, since there are more independent group parameters than needed to do the absorption. We can still vary $\tilde{\alpha}$. This produces torsion terms, which can be absorbed into $\tilde{\phi}$. Performing that absorption, we conclude that the structure equations can be written as

$$\mathrm{d}w = (\phi + \alpha\, I_n) \wedge \omega, \tag{2.11}$$

where ϕ is antisymmetric with indices lowered, *i.e.* $\eta\phi + \phi^t\eta = 0$, where I_n is the n-dimensional identity matrix, and where ϕ and α contain group derivatives. There is no longer any torsion, and so there is no permanent torsion. Unlike the situation in the previous section, ϕ and α are not uniquely determined. Therefore there is still some freedom left after making the torsion vanish.

Sufficient conditions for this system to be integrable are provided by the Cartan-Kähler theorem, which is a geometric generalization of the Cauchy-Kowalewski theorem. We refer the reader to Bryant *et al.* (1991) for the statement and proof of this difficult theorem. However, for a problem such as the one we are dealing with, the theorem applies whenever the exterior differential system is real analytic and satisfies the condition of being in involution. This notion of involution is not that of Frobenius theory. Fortunately, Cartan has provided a simple test which can even be used as the definition of involution. For the situation we are considering, Cartan's test is as follows (for further details, see Gardner (1989)). We start with the matrix $\phi + \alpha I_n \mod \text{base}$. We construct a set Σ_1 as follows. We first let Σ_1 be the empty set. Then we perform the following step as many times as possible: add to Σ_1 an element of the matrix, noting the row from which it came, provided that the chosen element is independent of elements already in Σ_1 and provided it did not come from a row already used. When there are many ways to construct Σ_1, we choose one way amongst those that maximize the cardinality of Σ_1. We then construct Σ_i, with $i \geq 2$, in a similar fashion using the matrix $\phi + \alpha I_n \mod(\text{base} \cup \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_{i-1})$. The **$i$th Cartan character** is defined to be the cardinality of Σ_i. The **Cartan**

character is defined to be the integer $\sigma = \sum_i i\sigma_i$. The system is said to be **in involution** if its Cartan character is equal to the degree of freedom in it.

For the case under consideration, we can construct Σ_1 using the first column of $\phi + \alpha I_n$. The elements are all independent, and therefore $\sigma_1 = n$. Note that $(\phi + \alpha I_n \mod \text{base})$ is an $(n-1) \times (n-1)$ antisymmetric matrix. The next Cartan characters are then $\sigma_j = n - j$, for $j = 2, \ldots, n$. The Cartan character is the sum $\sigma = \sum_j j\,\sigma_j = \frac{n(n+1)(n-1)}{6} + 1$. The system is in involution if this number is equal to the degree of freedom in ϕ and α. In order to find this degree of freedom, we suppose that we can find ϕ' and α' also satisfying

$$\mathrm{d}w = (\phi' + \alpha'\, I_n) \wedge \omega, \tag{2.12}$$

where $\eta\phi' + \phi'^t\eta = 0$. By subtracting (2.11), we must have

$$((\phi' - \phi) + (\alpha' - \alpha)\, I_n) \wedge \omega = 0.$$

Putting in the indices, this is just

$$\Big((\phi' - \phi)^i{}_j + (\alpha' - \alpha)\delta^i{}_j\Big) \wedge \omega^j = 0. \tag{2.13}$$

Using the Cartan lemma, we deduce that

$$(\phi' - \phi)^i{}_j + (\alpha' - \alpha)\delta^i{}_j = A^i{}_{jk}\,\omega^k,$$

with $A^i{}_{jk} = A^i{}_{kj}$. Now, taking the trace, we obtain $\alpha' - \alpha = \frac{1}{n} A^i{}_{ik}\,\omega^k$. By simple renaming of $A^i{}_{ik}/n$ by A_k, we find that $\alpha' = \alpha + A_i\,\omega^i$. Substitution into (2.13), and making use of the fact that $(\phi' - \phi)$ is antisymmetric with indices lowered, yields that $(\phi' - \phi)$ is now uniquely determined. Explicitly, we obtain $A^i{}_{jk} = A_k\delta^i{}_j + A_j\delta^i{}_k - A_\ell\eta^{\ell i}\eta_{jk}$; therefore, $(\phi')^i{}_j = \phi^i{}_j + (A_j\delta^i{}_k - A_\ell\eta^{i\ell}\eta_{jk})\omega^k$. The degree of freedom is then n, which is the number of functions A_i. In summary, we have involution if and only if

$$\frac{n(n+1)(n-1)}{6} + 1 = n.$$

The solutions to this are $n = 1, 2, -3$. We therefore have involution if and only if $n = 1$ or 2, since the solution $n = -3$ is, of course, extraneous. This is saying that all real analytic one-dimensional metrics are equivalent under conformal transformations, and that all real analytic two-dimensional metrics (with the same signature) are equivalent to each other under the action of the conformal group.

In the other cases, that is, when $n > 2$, the system is not involutive. We are now faced with another equivalence problem, where the group of indeterminacy is now the n-dimensional group $G^{(1)}$ of the functions A_i. We therefore lift the equivalence problem on $U \times G$ to an equivalence problem on $U \times G \times G^{(1)}$. There is a gain, since $\dim G^{(1)} < \dim G$. Because $G^{(1)}$ is defined to be the group that preserves the relation $\mathrm{d}w = (\phi + \alpha I_n) \wedge \omega$, the lift from $U \times G$ to $U \times G \times G^{(1)}$ also satisfies the same equation. We therefore keep the same notation; but now, ω, α and ϕ indicate forms over $U \times G \times G^{(1)}$. We have computed the derivative of ω on $U \times G$; we showed that it can be made torsion-free on that space. The expression for the derivative of ω on $U \times G \times G^{(1)}$ is, of course, the same as the one on $U \times G$. However, for the purpose of the equivalence problem on the lifted space $U \times G \times G^{(1)}$, the derivative contains only torsion terms, since $(\phi + \alpha I_n) \wedge \omega$ does not contain derivatives of elements of $G^{(1)}$. We now require the derivatives of α and ϕ. To compute them, we first take the exterior derivative of $\mathrm{d}\omega$, to obtain

$$0 = \mathrm{d}^2\omega = [(\mathrm{d}\phi - \phi \wedge \phi) + \mathrm{d}\alpha\, I_n] \wedge \omega. \tag{2.14}$$

Let us define

$$\Theta := \mathrm{d}\phi - \phi \wedge \phi + \mathrm{d}\alpha\, I_n.$$

The trace-free part of Θ is antisymmetric when its indices are lowered, and so obeys the relation

$$\eta\Theta + \Theta^t\eta - \frac{2}{n}\mathrm{trace}\,\Theta\,\eta = 0. \tag{2.15}$$

Also, Θ obeys the condition that

$$\Theta \wedge \omega = 0,$$

which, by the Cartan lemma, implies that

$$\Theta = \psi \wedge \omega.$$

This is, using indices,

$$\Theta^i{}_j = \psi^i{}_{jk} \wedge \omega^k.$$

The condition (2.15) on Θ then gives

$$\psi_{ijk} \wedge \omega^k + \psi_{jik} \wedge \omega^k - \frac{2}{n}\,\psi^\ell{}_{\ell k} \wedge \omega^k\,\eta_{ij} = 0.$$

If we multiply this equation with all possible combinations of $(n-1)$ cobasis elements ω, we obtain, by application of the Cartan lemma, that

$$\psi_{ijk} \equiv -\psi_{jik} + 2\epsilon_k\,\eta_{ij} \bmod \text{base},$$

where we define $\epsilon_k := \frac{1}{n}\psi^\ell{}_{\ell k}$. In terms of ψ, the integrability condition (2.14) becomes

$$\psi^i{}_{jk} \wedge \omega^k \wedge \omega^j = 0.$$

If we multiply this relation with all possible combinations of $(n-2)$ cobasis elements ω, we find, after lowering the i index, that ψ also obeys

$$\psi_{ijk} \equiv \psi_{ikj} \bmod \text{base}.$$

We can then solve for ψ_{ijk}. The solution is

$$\psi_{ijk} = \epsilon_j \eta_{ki} - \epsilon_i \eta_{jk} + \epsilon_k \eta_{ij} + A_{ijk\ell}\, \omega^\ell,$$

where $A_{ijk\ell}$ are functions. By back substitution, we obtain that

$$\Theta_{ij} = \epsilon_j \wedge \omega_i - \epsilon_i \wedge \omega_j + \eta_{ij} \epsilon_k \wedge \omega^k + A_{ijk\ell}\, \omega^\ell \wedge \omega^k.$$

Without loss of generality, we can assume that $A_{ijk\ell}$ is antisymmetric in the last two indices, *i.e.*

$$A_{ijk\ell} + A_{ij\ell k} = 0,$$

since the symmetric part is cancelled when the antisymmetry in $\omega^\ell \wedge \omega^k$ is taken into account. This entails that there are at most $n^3(n-1)/2$ independent functions $a_{ijk\ell}$ for an n-dimensional manifold U. Because $\Theta_{ij} = -\Theta_{ji}$, it follows that $A_{ijk\ell}$ is antisymmetric in the first two indices, *i.e.*

$$A_{ijk\ell} + A_{jik\ell} = 0.$$

This reduces the number of independent components of A to $n^2(n-1)^2/4$. Furthermore, the requirement that $\Theta_{ij} \wedge \omega^k = 0$ imposes the condition

$$A_{i[jk\ell]} = 0.$$

In these equations, there are n possibilities for the index i and $n(n-1)(n-2)/6$ possibilities for the other three indices. The number of independent entries in A is therefore $n^2(n^2-1)/12$. The derivatives of elements of $G^{(1)}$ appear solely in the various terms ϵ_i. This allows us to give the structure equations on $U \times G \times G^{(1)}$ as

$$\mathrm{d}\begin{pmatrix} \alpha \\ \phi \\ \omega \end{pmatrix} = \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & 0 & \delta(\epsilon) \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \alpha \\ \phi \\ \omega \end{pmatrix} + \begin{pmatrix} 0 \\ \phi \wedge \phi + \mathcal{A} \\ (\phi + \alpha I_n) \wedge \omega \end{pmatrix},$$

where

$$[\delta(\epsilon) \wedge \omega]^i{}_j = (\epsilon_j \delta^i{}_k - \epsilon_\ell \eta^{\ell i} \eta_{kj}) \wedge \omega^k$$

and

$$\mathcal{A} = A^i{}_{jk\ell} \omega^\ell \wedge \omega^k.$$

The last term in these structure equations is the torsion.

We now proceed to eliminate as many torsion terms as possible, using the $G^{(1)}$ freedom in ϵ. This proceeds as follows. The functions ϵ and $A_{ijk\ell}$ are not uniquely defined. Suppose that $\bar{\epsilon}_i$ and $\bar{A}_{ijk\ell}$ also satisfy:

$$\Theta_{ij} = \bar{\epsilon}_j \wedge \omega_i - \bar{\epsilon}_i \wedge \omega_j + \eta_{ij} \bar{\epsilon}_k \wedge \omega^k + \bar{A}_{ijk\ell}\, \omega^\ell \wedge \omega^k.$$

By subtraction of these two equalities on Θ_{ij}, it follows that

$$\Big((\bar{\epsilon}_j - \epsilon_j)\eta_{ik} - (\bar{\epsilon}_i - \epsilon_i)\eta_{jk} + \eta_{ij}(\bar{\epsilon}_k - \epsilon_k) + (\bar{A}_{ijk\ell} - A_{ijk\ell})\, \omega^\ell \Big) \wedge \omega^k = 0.$$

If we multiply this last expression by all possible ω, with the exception of ω^i, all the terms are eliminated except the one with η_{ij}. By application of the Cartan lemma, we can conclude that

$$\bar{\epsilon}_k - \epsilon_k = B_{km} \omega^m,$$

where the B_{km} are functions. These functions represent a certain amount of freedom that can be used for eventual removal of torsion. They are not, however, totally arbitrary, since they must obey the condition

$$\Big((B_{j\ell} \eta_{ik} - B_{i\ell} \eta_{jk} + B_{k\ell} \eta_{ij} + (\bar{A}_{ijk\ell} - A_{ijk\ell}) \Big)\, \omega^\ell \wedge \omega^k = 0.$$

If we exchange i and j and add to the original expression, we obtain

$$B_{k\ell}\, \omega^\ell \wedge \omega^k = 0,$$

from which one concludes that $B_{k\ell}$ is symmetric. We now proceed to use the arbitrariness left in $B_{k\ell}$. Now $B_{j\ell}$ obeys

$$B_{j\ell}\eta_{ik}-B_{jk}\eta_{i\ell}-B_{i\ell}\eta_{jk}+B_{ik}\eta_{j\ell}+(B_{k\ell}-B_{\ell k})\eta_{ij}+(\bar{A}_{ijk\ell}-A_{ijk\ell})-(\bar{A}_{ij\ell k}-A_{ij\ell k})=0,$$

which simplifies to

$$B_{j\ell}\eta_{ik}-B_{jk}\eta_{i\ell}-B_{i\ell}\eta_{jk}+B_{ik}\eta_{j\ell}+2\,(\bar{A}_{ijk\ell}-A_{ijk\ell})=0.$$

If we raise the index i and take the trace on k and i, we obtain

$$(n-2)\,B_{j\ell}+B\eta_{j\ell}+2\,(\bar{A}^i{}_{ji\ell}-A^i{}_{ji\ell})=0,$$

where $B := B^i{}_i$. If we now raise j and take the trace, we obtain

$$(n-1)B+\bar{A}^{ij}{}_{ij}-A^{ij}{}_{ij}=0.$$

We can use the freedom in B to set

$$B=\frac{A^{ij}{}_{ij}}{n-1}.$$

The remaining freedom in B_{ij} is used to set

$$B_{j\ell}=\frac{-B\eta_{j\ell}+2A^i{}_{ji\ell}}{n-2},$$

which is consistent with the definition of B. In this manner, the freedom in B_{ij} is used to set the trace of A to vanish, *i.e.* $\bar{A}^i{}_{ji\ell}=0$. The other $\bar{A}^i{}_{jkl}$ are then equal to their un-barred versions. When such a choice of $B_{j\ell}$ as described above has been made, we denote the resulting $\bar{A}^i{}_{jk\ell}$ by $\frac{1}{2}W^i{}_{jk\ell}$. At this juncture, we note that W possesses the algebraic symmetries of the Weyl tensor. There are $n(n+1)/2$ independent $B_{k\ell}$. This means that there are $n(n+1)(n+2)(n-3)/12$ independent entries in $W_{ijk\ell}$. This means that if U is a three dimensional manifold, W is always zero. We also note that now ϵ and W are uniquely defined. Therefore, the group $G^{(2)}$ of freedom consists solely of the identity element. We thus have a uniquely defined coframe, or e-structure, on $U \times G \times G^{(1)} \times G^{(2)}$ given by ω, ϕ, α and ϵ. In order to find the fundamental invariants for this equivalence problem, we first need to ascertain whether this e-structure is in involution. The part of the structure equations that involve derivatives of the group $G^{(1)}$ is:

$$\mathrm{d}\begin{pmatrix} \alpha \\ \phi_{ij} \end{pmatrix} = \begin{pmatrix} \epsilon_k \wedge \omega^k \\ \epsilon_j \wedge \omega_i - \epsilon_i \wedge \omega_j \end{pmatrix} + \cdots .$$

The right hand side has the form of a matrix $M(\epsilon)$ multiplied, using exterior multiplication, by ω. The Cartan character from $M(\epsilon)$ is easily seen to be non-zero. However, there is no degree of freedom in the definition of ϵ. It follows that the system is not in involution whence a prolongation step is needed. The prolongation is obtained by computing the value of $\mathrm{d}\epsilon$.

We already have expressions[7] for the exterior derivatives of ω, ϕ and α. The prolongation step will give the structure equations on $U \times G \times G^{(1)} \times G^{(2)}$. We obtain $\mathrm{d}\epsilon$ by examining the integrability condition of α, which is given by

$$\begin{aligned} 0 = \mathrm{d}^2\alpha = \mathrm{d}(\epsilon_k \wedge \omega^k) &= \mathrm{d}\epsilon_k \wedge \omega^k - \epsilon_\ell \wedge ({\phi^l}_k + \alpha {\delta^l}_k) \wedge \omega^k \\ &= \left(\mathrm{d}\epsilon_k - \epsilon_\ell \wedge {\phi^\ell}_k - \epsilon_k \wedge \alpha\right) \wedge \omega^k. \end{aligned}$$

It follows, by the Cartan lemma, that

$$\mathrm{d}\epsilon_k = \epsilon_\ell \wedge {\phi^\ell}_k + \epsilon_k \wedge \alpha + \zeta_{k\ell} \wedge \omega^\ell,$$

where the functions ζ are one-forms subject to the restrictionthat

$$\zeta_{k\ell} \wedge \omega^\ell \wedge \omega^k = 0. \tag{2.16}$$

[7]We have these derivatives on $U \times G \times G^{(1)}$, but because $G^{(2)}$ is defined to be the group the preserves the form of those derivatives we need not introduce new notation even though ω, ϕ, α and ϵ are now defined on the space $U \times G \times G^{(1)} \times G^{(2)}$.

Exterior multiplication of this last expression by all the other cobasis elements ω, enables us to conclude, through the Cartan lemma, that

$$\zeta_{k\ell} = H_{k\ell m}\omega^m, \tag{2.17}$$

where $H_{k\ell m}$ are functions. Back substitution reveals that

$$H_{[k\ell m]} = 0. \tag{2.18}$$

Since we are only interested in ζ in so far as it appears in the product

$$\zeta_{k\ell} \wedge \omega^\ell = H_{k\ell m}\,\omega^m \wedge \omega^\ell,$$

we can, without loss of generality require that G be antisymmetric in the last two indices, *i.e.* $H_{k\ell m} = -H_{km\ell}$.

The structure equations on $U \times G \times G^{(1)} \times G^{(2)}$ can be summarized as follows:

$$\begin{aligned}
\mathrm{d}\omega &= (\phi + \alpha I_n) \wedge \omega, \\
\mathrm{d}\phi &= \phi \wedge \phi - [\epsilon \wedge \omega] + \mathcal{W}, \\
\mathrm{d}\alpha &= \epsilon \wedge \omega \\
&\text{and} \\
\mathrm{d}\epsilon &= \epsilon \wedge \phi + \epsilon \wedge \alpha I_n + \mathcal{H}.
\end{aligned}$$

With indices, these become

$$\begin{aligned}
\mathrm{d}\omega^i &= ({\phi^i}_j + \alpha\,{\delta^i}_j) \wedge \omega^j, \\
\mathrm{d}\phi_{ij} &= \phi_{ik} \wedge {\phi^k}_j + \epsilon_j \wedge \omega_i - \epsilon_i \wedge \omega_j + \frac{1}{2} W_{ijk\ell}\,\omega^\ell \wedge \omega^k, && (2.19) \\
\mathrm{d}\alpha &= \epsilon_k \wedge \omega^k && (2.20) \\
&\text{and} \\
\mathrm{d}\epsilon_k &= \epsilon_\ell \wedge {\phi^\ell}_k + \epsilon_k \wedge \alpha + H_{k\ell m}\,\omega^m \wedge \omega^\ell, && (2.21)
\end{aligned}$$

where

$$H_{k\ell m} = -H_{km\ell},$$

$$H_{k\ell m}\omega^k \wedge \omega^\ell \wedge \omega^m = 0$$

and W has the index symmetries of the Weyl tensor.

The fundamental invariants of this equivalence problem are given by $W^i{}_{jk\ell}$, $H_{k\ell m}$ and their derivatives.

In this paragraph we exhibit the relation between $\mathcal{W}$ and the Weyl tensor. We first define a left action on the group G as follows:

$$\begin{aligned} L_C^{(1)} \ &: \ G \to G \\ &\quad (\lambda, S) \mapsto (C\lambda, S). \end{aligned}$$

The action induced on the cobasis ω is

$$L_C^{(1)*}\omega^i = C\omega^i.$$

This enables us to define ξ^i as follows:

$$L^{(1)*}{}_{\lambda^{-1}}\omega^i = \lambda^{-1}\omega^i =: \xi^i,$$

from which $\omega^i = \lambda\xi^i$. We can consider ξ^i as a quantity defined over the manifold $U \times SO(p, q, \mathbf{R})$, and so we can use the results of the previous section. The structure equations satisfied by ξ^i are then

$$\mathrm{d}\xi^i = \varphi^i{}_j \wedge \xi^j,$$

and those satisfied by $\varphi^i{}_j$ are

$$\mathrm{d}\varphi^i{}_j = \varphi^i{}_j \wedge \varphi^j{}_k + \frac{1}{2}S^i{}_{jk\ell}\omega^\ell \wedge \omega^k. \tag{2.22}$$

This enables us to compute the structure equations for the conformal space in terms of the Riemannian space:

$$\begin{aligned} \mathrm{d}\omega^i &= \mathrm{d}\lambda \wedge \xi^i + \lambda \varphi^i{}_j \wedge \xi^j \\ &= (\mathrm{d}\lambda\lambda^{-1}\delta^i{}_j + \varphi^i{}_j) \wedge \omega^j. \end{aligned}$$

We use the discussion that begins with equation (2.12) above to enable us to identify

$$\alpha = \mathrm{d}\lambda\lambda^{-1} + M_i\omega^i$$

and

$$\phi^i{}_j = \varphi^i{}_j + (M_j\delta^i{}_k - M_\ell\eta^{i\ell}\eta_{jk})\omega^k,$$

where M_i is a member of $G^{(1)}$. By differentiation of α, we obtain, after using equation (2.20) and Cartan's lemma,

$$\epsilon_i = \mathrm{d}M_i + M_j\phi^j{}_i + M_i\alpha + B_{ij}\omega^j, \tag{2.23}$$

with $B_{ij} = B_{ji}$. We can now define a left action on $G \times G^{(1)}$ by the following:

$$\begin{aligned} L^{(2)}_{C,K} &: G \times G^{(1)} \to G \times G^{(1)} \\ &(\lambda, S; M) \mapsto (C\lambda, S; K + M), \end{aligned}$$

where $K = (K_1, K_2, \ldots, K_n)$ and $M = (M_1, M_2, \ldots, M_n)$. It follows that

$$\begin{aligned} L^{(2)*}_{C,K}\omega^i &= C\omega^i, \\ L^{(2)*}_{C,K}\alpha &= \mathrm{d}\lambda\lambda^{-1} + (K_i + M_i)C\omega^i \end{aligned}$$

and

$$L^{(2)*}_{C,K}\phi^i{}_j = \varphi^i{}_j + [(K_j + M_j)\delta^i{}_k - (K_\ell + M_\ell)\eta^{i\ell}\eta_{jk}]C\omega^k.$$

We can recover ξ^i and $\varphi^i{}_j$ using this action, since

$$L^{(2)*}_{\lambda^{-1},-M}\omega^i = \lambda^{-1}\omega^i = \xi^i$$

and

$$L^{(2)*}_{\lambda^{-1},-M}\phi^i{}_j \;\; = \;\; \varphi^i{}_j.$$

We can also obtain the contribution of the conformal factor:

$$L^{(2)*}_{\lambda^{-1},-M}\alpha = \mathrm{d}\lambda\lambda^{-1}.$$

From equation (2.23), we obtain that the induced action on ϵ is as follows:

$$L^{(2)*}_{C,K}\epsilon_k = \mathrm{d}M_k + (K_i + M_i)(\phi^i{}_k + \alpha\delta^i{}_k) + L^{(2)*}_{C,K}B_{jk}C\omega^j,$$

where $B_{jk} = B_{kj}$. This implies that

$$L^{(2)*}_{\lambda^{-1},-M}\epsilon_k = \mathrm{d}M_k + B_{jk}|_{(1,0)}\xi^j,$$

where we have used the definition

$$B_{jk}|_{(1,0)} := L^{(2)*}_{\lambda^{-1},-M}B_{jk}.$$

On the one hand, we have

$$\begin{aligned}
L^{(2)*}_{C,K}\mathrm{d}\phi_{ij} &= \mathrm{d}(L^{(2)*}_{C,K}\phi_{ij}) \\
&= \mathrm{d}(\varphi^i{}_j + [(K_j + M_j)\delta^i{}_k - (K_\ell + M_\ell)\eta^{i\ell}\eta_{jk} \wedge C\omega^k \\
&= \mathrm{d}\varphi^i{}_j + \mathrm{d}M_j \wedge C\omega^i - \mathrm{d}M_\ell\eta^{i\ell}\eta_{jk}] \wedge C\omega^k) \\
&\qquad + [(K_j + M_j)\delta^i{}_k - (K_\ell + M_\ell)\eta^{i\ell}\eta_{jk}]C(\phi^k{}_m + \alpha\delta^k{}_m) \wedge \omega^m).
\end{aligned}$$

On the other hand, we have, using (2.19), that

$$\begin{aligned}
L^{(2)*}_{C,K}\mathrm{d}\phi_{ij} &= L^{(2)*}_{C,K}(\phi_{ik} \wedge \phi^k{}_j + \eta_{ik}\epsilon_j \wedge \omega^k - \eta_{jk}\epsilon_i \wedge \omega^k + W_{ijk\ell}\,\omega^k \wedge \omega^l) \\
&= \{\varphi_{ik} + [(K_k + M_k)\eta_{im} - (K_i + M_i)\eta_{km}] \wedge C\omega^m\} \wedge \\
&\quad \wedge\{\varphi^k{}_j + [(K_j + M_j)\delta^k{}_n - (K_\ell + M_\ell)\eta^{k\ell}\eta_{jn}] \wedge C\omega^n\} \\
&\quad +\eta_{ik}[\mathrm{d}M_j + (K_\ell + M_\ell)(\phi^\ell{}_j + \alpha\delta^\ell{}_j) + L^{(2)*}_{C,K}B_{jk}C\omega^m] \wedge C\omega^k \\
&\quad -\eta_{jk}[\mathrm{d}M_i + (K_\ell + M_\ell)(\phi^\ell{}_i + \alpha\delta^\ell{}_i) + L^{(2)*}_{C,K}B_{jk}C\omega^m] \wedge C\omega^k \\
&\quad +(\frac{1}{2}L^{(2)*}_{C,K}W^i{}_{jk\ell})C\omega^\ell \wedge C\omega^k.
\end{aligned}$$

If we let C be equal to λ^{-1} and K be equal to $-M$, then we obtain

$$\begin{aligned} & \mathrm{d}\varphi_{ij} + \mathrm{d}M_j \wedge \eta_{i\ell}\xi^\ell - \mathrm{d}M_i \wedge \eta_{jk}\xi^k \\ = \; & \varphi_{ik} \wedge \varphi_j^k + [\mathrm{d}M_j + B_{jk}|_{(1,0)}\xi^k] \wedge \eta_{i\ell}\xi^\ell - [\mathrm{d}M_i + B_{ik}|_{(1,0)}\xi^k] \wedge \eta_{j\ell}\xi^\ell \\ & + (\frac{1}{2}L^{(2)*}_{\lambda^{-1},-M} W_{ijk\ell})\xi^\ell \wedge \xi^k. \end{aligned}$$

We can replace $\mathrm{d}\varphi_{ij}$ using its value in (2.22). Doing so, we can solve for

$$(\frac{1}{2}L^{(2)*}_{\lambda^{-1},-M} W_{ijk\ell}) = S_{ijk\ell} - B_{ik}|_{(1,0)}\eta_{j\ell} + B_{i\ell}|_{(1,0)}\eta_{jk} + B_{jk}|_{(1,0)}\eta_{i\ell} - B_{j\ell}|_{(1,0)}\eta_{ik}.$$

If we raise i, let $k = i$ and then sum, we obtain

$$0 = S^i{}_{ji\ell} - B^i{}_i|_{(1,0)}\eta_{j\ell} + B_{j\ell}|_{(1,0)} + B_{j\ell}|_{(1,0)} - nB_{j\ell}|_{(1,0)},$$

where we have used the fact that $W^i_{ji\ell} = 0$. This enables us to isolate $B_{j\ell}|_{(1,0)}$ and so obtain

$$B_{j\ell}|_{(1,0)} = \frac{1}{n-2}[S^i{}_{ji\ell} - \frac{\eta_{j\ell}}{2(n-1)} S^{i\ell}{}_{i\ell}]. \tag{2.24}$$

Since $S^i{}_{jk\ell}$ is the Riemann tensor on $U \times SO(p,q,\mathbf{R})$, it follows that $B_{j\ell}|_{(1,0)}$ is isomorphic to the Ricci tensor on $U \times SO(p,q,\mathbf{R})$, and therefore the trace-free tensor $L^{(2)*}_{\lambda^{-1},-M} W_{ijk\ell}$ is the Weyl tensor on $U \times SO(p,q,\mathbf{R})$. The quantity $\mathcal{W}$ can now be identified as the matrix of two-forms representing the Weyl tensor on $U \times G \times G^{(1)}$.

From equation (2.23), we obtain by exterior differentiation the following

$$\mathrm{d}\epsilon_i = \mathrm{d}M_j \wedge (\phi^j)_i + M_j\,\mathrm{d}\phi^j{}_i + \mathrm{d}M_i \wedge \alpha + M_i\,\mathrm{d}\alpha + \mathrm{d}(B_{ij}\omega^j).$$

From this value for $\mathrm{d}\epsilon_i$, we deduce that

$$\begin{aligned} L^{(2)*}_{\lambda^{-1},-M}\epsilon_i \; = \; & \mathrm{d}M_j \wedge \varphi^j{}_i + \mathrm{d}M_i \wedge \mathrm{d}\lambda\,\lambda^{-1} + (\mathrm{d}B_{ij})\,|_{(1,0)} \wedge \omega^j \\ & + \left(B_{ij}|_{(1,0)}\right) \wedge \left(\varphi^j{}_k \wedge \xi^k + \mathrm{d}\lambda\,\lambda^{-1} \wedge \xi^j\right). \end{aligned} \tag{2.25}$$

From equation (2.21), one deduces that

$$\begin{aligned} L^{(2)*}_{\lambda^{-1},-M}\epsilon_i &= \left(\mathrm{d}M_j + B_{jk}|_{(1,0)}\xi^k\right)\wedge\varphi^j{}_i + \left(\mathrm{d}M_i + B_{ji}|_{(1,0)}\xi^j\right)\wedge\mathrm{d}\lambda\,\lambda^{-1} \quad (2.26)\\ &+ \left(L^{(2)*}_{(\lambda^{-1},-M)}H_{i\ell m}\right)\xi^m\wedge\xi^\ell. \end{aligned}$$

Comparing equations (2.25) and (2.26), we obtain the condition

$$\left(\mathrm{d}B_{ij}|_{(1,0)} - L^{(2)*}_{(\lambda^{-1},-M)}H_{ijm}\xi^m\right)\wedge\xi^j = 0. \quad (2.27)$$

We now proceed to compute the action of the group on $\mathcal{W}$, the Weyl tensor. This is done by taking the exterior derivative of $\mathrm{d}\alpha$ and $\mathrm{d}\phi$, and looking at what happens to $d\mathcal{W}$ modulo the cobasis ω.

Differentiation of (2.19) yields

$$\begin{aligned} 0 &= \left(\phi_{ip}\wedge\phi_{mk}\eta^{pm} + \epsilon_k\wedge\omega^p\eta_{ki} - \epsilon_i\wedge\omega^p\eta_{pk} + \frac{1}{2}W_{ikpm}\omega^p\wedge\omega^m\right)\wedge\phi_{\ell j}\eta^{k\ell}\\ &-\phi_{ik}\wedge\left(\phi_{\ell m}\wedge\phi_{pj}\eta^{mp} + \epsilon_j\wedge\omega^m\eta_{ml} - \epsilon_\ell\wedge\omega^p\eta_{pj} + \frac{1}{2}W_{\ell jnm}\omega^p\wedge\omega^m\right)\eta^{kl}\\ &+\left(\epsilon_\ell\wedge\phi^\ell{}_j + \epsilon_j\wedge\alpha + \zeta_{j\ell}\wedge\omega^\ell\right)\wedge\omega^k\eta_{ki}\\ &-\epsilon_j\wedge\left(\phi^k{}_\ell + \alpha\delta^k{}_\ell\right)\wedge\omega^\ell\eta_{ki}\\ &+\epsilon_i\wedge\left(\phi^k{}_\ell + \alpha\delta^k{}_\ell\right)\wedge\omega^\ell\eta_{kj} + \frac{1}{2}\mathrm{d}W_{ijk\ell}\wedge\omega^k\wedge\omega^\ell\\ &+\frac{1}{2}W_{ijk\ell}\left(\phi^k{}_m + \alpha\delta^k{}_m\right)\wedge\omega^m\wedge\omega^\ell - \frac{1}{2}W_{ijk\ell}\wedge\left(\phi^\ell{}_m + \alpha\delta^\ell{}_m\right)\wedge\omega^m, \end{aligned}$$

which simplifies to

$$\begin{aligned} 0 &= (\mathrm{d}W_{ijk\ell} + 2W_{ijk\ell}\alpha - W^m{}_{jk\ell}\phi_{im} \quad (2.28)\\ &\quad + W_{imk\ell}\phi^m{}_j + W_{ijm\ell}\phi^m{}_k + W_{ijkm}\phi^m{}_\ell\\ &\quad + 2\zeta_{j\ell}\eta_{ki} - 2\zeta_{i\ell}\eta_{kj})\wedge\omega^k\wedge\omega^\ell. \end{aligned}$$

Multiplying (2.28) with all possible exterior products of $(n-2)$ different ω, we can conclude that

$$0 \equiv \mathrm{d}W_{ijk\ell} + 2W_{ijk\ell}\alpha - W^m{}_{jk\ell}\phi_{im} \quad (2.29)$$

$$+ W_{imk\ell}\phi^m{}_j + W_{ijm\ell}\phi^m{}_k + W_{ijkm}\phi^m{}_\ell \quad \text{mod base.}$$

It follows that $\mathcal{W}$ transforms as a tensor under the $SO(p, q, \mathbf{R})$ group (through ϕ) and scales as λ^2 under stretching (through α). This means that

$$W^i{}_{jk\ell} = S^i{}_m (S^{-1})^r{}_j (S^{-1})^p{}_k (S^{-1})^q{}_\ell \lambda^{-2} \tilde{W}^m{}_{rqp}, \tag{2.30}$$

where $\tilde{W}$ is W evaluated at a fixed choice of the group parameters. Differentiating (2.30), we obtain

$$\begin{aligned} \mathrm{d}W^i{}_{jk\ell} \;\; \equiv \;\; & (\mathrm{d}SS^{-1})^i{}_m W^m{}_{jk\ell} - (\mathrm{d}SS^{-1})^m{}_j W^i{}_{mk\ell} - (\mathrm{d}SS^{-1})^m{}_k W^i{}_{jm\ell} \\ & - (\mathrm{d}SS^{-1})^m{}_\ell W^i{}_{jkm} - 2(\mathrm{d}\lambda\lambda^{-1}) W^i{}_{jk\ell} \quad \text{mod base.} \end{aligned}$$

This is equivalent to the congruence (2.29).

Various special cases are apparent. The first special case is if all $W^i{}_{jk\ell}$ vanish.[8] Since in that case $\mathcal{W} = 0$, we cannot use it to perform a group reduction in an invariant way. We shall now analyze this situation in more details. Equation (2.28) reduces to

$$2(\zeta_{j\ell}\eta_{ki} - \zeta_{i\ell}\eta_{kj}) \wedge \omega^k \wedge \omega^\ell.$$

Using equation (2.17) and remembering that H is symmetric in the last two indices, this is equivalent to

$$(\eta_{kj} H_{imp} - \eta_{ki} H_{jmn})\, \omega^k \wedge \omega^p \wedge \omega^m = 0. \tag{2.31}$$

Due to the antisymmetry in i and j, there are $n(n-1)/2$ (exterior) equations. The number of unknowns, H_{ijk} is $n^2(n-1)/2$. Consider the sets $\{i, j\}$ and $\{m, p\}$. If they are equal, the corresponding terms in equation (2.31) vanish either because of

[8] Of course, if the manifold U is three-dimensional, then $\mathcal{W}$ is always zero; therefore, this does not represent any restriction.

η (when $k \notin \{i, j\}$) or because of the exterior product (when $k \in \{m, p\} = \{i, j\}$). The terms corresponding to the situation when the intersection of $\{i, j\}$ and $\{m, p\}$ is empty imply that $H_{imp} = H_{jmp} = 0$. This is because η is diagonal and that all $\omega^k \wedge \omega^m \wedge \omega^p$, with $k \in \{i, j\}$, are all independent of the other possibilities. The value of $n(n-1)(n-2)/2$ unknowns is thus given. This leave $n(n-1)$ unknowns to be found. Note that the step just performed requires the dimension of U be at least four. The remaining situation is that when the intersection of $\{i, j\}$ and $\{m, p\}$ has one element. Without loss of generality, we can choose $m \in \{i, j\}$ and $p \notin \{i, j\}$. Taking into account the vanishing of the aforementioned H_{imp}, equation (2.31) reduces to

$$(\eta_{jj} H_{iip} + \eta_{ii} H_{jjp})\, \omega^n \wedge \omega^i \wedge \omega^j = 0 \quad \text{(No sum on } i, j). \tag{2.32}$$

Each of the $n(n-1)/2$ such exterior equation imply the vanishing of $n-2$ coefficients. There are therefore $n(n-1)(n-2)/2$ such equations which are homogeneous in H_{iip}. Note that $n(n-1)(n-2)/2 \geq n(n-1)$ for $n \geq 4$. The equality arises only when $n = 4$. From the equations implied by equation (2.32), consider the subset given by

$$\eta_{jj} H_{00p} + \eta_{00} H_{jjp} = 0, \quad j \neq 0$$

and

$$\eta_{11} H_{22p} + \eta_{22} H_{11p} = 0.$$

There are $n(n-1)$ such equation. The determinant of the matrix of coefficient is easily seen to be, up to a sign, $\left(2\eta_{00}^{(n-2)} \eta_{11} \eta_{22}\right)^n$; therefore, it does not vanish whence the only solution to H_{ijk} is the trivial solution. We then conclude that the dimension of U is greater than three, all the functions H_{ijk} vanish.

We now turn to the 3-dimensional case. In that situation $W_{ijk\ell}$ necessarily vanishes. There are a maximum of $n^2(n-1)/2 = 9$ components of H^i_{jk}, due to the antisymmetry in j and k. This maximum is immediately reduced to 8 because of the single constraint (2.18). Equation (2.28) reduces to

$$\eta_{ki} H_{j\ell m} \omega^m \wedge \omega^k \omega \wedge^\ell - \eta_{kj} H_{i\ell m} \omega^m \wedge \omega^k \wedge \omega^\ell = 0.$$

Since η is diagonal, this equation is equivalent to

$$\eta_{ii} H_{j\ell m} \omega^m \wedge \omega^i \omega \wedge^\ell - \eta_{jj} H_{i\ell m} \omega^m \wedge \omega^j \wedge \omega^\ell = 0 \quad \text{(No sum on } i, j\text{)},$$

which, in turn, is equivalent to

$$\eta_{ii} \eta_{jj} \left(H^j{}_{\ell m} \omega^m \wedge \omega^i \omega \wedge^\ell - H^i{}_{\ell m} \omega^m \wedge \omega^j \wedge \omega^\ell \right) = 0 \quad \text{(No sum on } i, j\text{)}, \tag{2.33}$$

The indices i, j are two of three possibles values of indices in a three dimensional space. Let the index r denote the third one. Since H is antisymmetric in the last two indices, the previous equation yields that the trace $H^k{}_{k\ell}$ vanishes. This reduces the number of components of $H^i{}_{jk}$ to 5. Let the quantity L be defined implicitly as follows:

$$H^i{}_{jk} \omega^j \wedge \omega^k \wedge \omega^s = L^{is} \omega^0 \wedge \omega^1 \wedge \omega^2. \tag{2.34}$$

Equation (2.33) gives

$$\eta_{ii} \eta_{jj} \left(L^{ij} - L^{ji} \right) \omega^0 \wedge \omega^1 \wedge \omega^2 = 0$$

whence L_{ij} is symmetric. Lowering i in equation (2.34), letting $s = i$ and taking the sum over i yields

$$H_{ijk} \omega^j \wedge \omega^k \omega^i = L_i{}^i \omega^0 \wedge \omega^1 \wedge \omega^2.$$

Since the left hand side of this equality vanishes, the quantity L must be trace-free. The fundamental invariants of this three-dimensional conformal equivalence problem are the five quantities L_{ij} and their covariant derivatives. Using equation (2.27), we find that

$$L_{is} = B_{ik;\ell} - B_{i\ell;k},$$

where (s, k, ℓ) is a cyclic permutation of $(0, 1, 2)$ and where the semi-colon denotes covariant differentation. Given equation (2.24), specialized to the three dimensional case (*i.e.* the case where $n = 3$), we can identify L_{is} as the Cotton-York, or Weyl-Schouten, tensor (Kramer et al., 1980).

If all the $W^i{}_{jk\ell}$ are constants, we have another special case. Since in that case varying the group does not produce any change in $\mathcal{W}$, then $\mathcal{W}$ cannot be used to perform a reduction of the group in an invariant way. Another way to see this is that, in this case, the rank of the $e-$structure on $U \times G \times G^{(1)}$ is zero. This implies that there is, for such a space, a group of symmetry with the same dimension as that of $U \times G \times G^{(1)}$; this is the maximal symmetry group possible. Therefore, there are no privileged directions; such directions would allow us to do a group reduction. It is important that $\mathcal{W}$ is defined on $U \times G \times G^{(1)}$. Constancy of $\mathcal{W}$ on U does not necessarily imply the constancy discussed here. The rest of Cartan's classification approach would involve consideration of the rank of $\mathrm{d}\mathcal{W}$, and of further derivatives.

In the process of this calculation, the forms ϵ were introduced. They contain the non-Weyl part, *i.e.* the Ricci part, of the Riemann tensor. The particular combination of Ricci tensor components appearing in ϵ is exactly the combination that is differentiated in the definition of the Cotton-York tensor, see Kramer *et al.* (1980), in the case of the three-dimensional manifolds. The forms ϵ do allow us to compute the Weyl two-forms directly from the Riemann two-forms without first exhibiting the Riemann tensor from the two-forms.

2.2.1 Conformally flat metrics

We now present an example which illustrates calculations involving the preceding theory. The starting point is a metric that is conformally equivalent to a flat metric.

We proceed to compute the Weyl tensor, and thereby demonstrate the well-known fact that it is zero. Suppose that

$$\omega = zZ\sigma$$

where σ is a $1 \times n$ array of exact differential forms $\sigma = \mathrm{d}x$, z is a non-zero real number and Z is a member of $SO(p, q, \mathbf{R})$. Then, by differentiation, we have that

$$\left(\mathrm{d}(zZ)(zZ)^{-1} - \phi - \alpha I\right) \wedge \omega = 0.$$

We define

$$H := \mathrm{d}(zZ)(zZ)^{-1} = \mathrm{d}zz^{-1}I + \mathrm{d}ZZ^{-1}.$$

We can split H into the trace part $\mathrm{d}zz^{-1}$ and trace-free part $\mathrm{d}ZZ^{-1}$. By application of the Cartan lemma, we have

$$\phi^i{}_j + \alpha\delta^i{}_j - H^i{}_j = C^i{}_{jk}\omega^k,$$

for some functions $C^i{}_{jk}$. Taking the trace, it follows that

$$\alpha = \mathrm{d}zz^{-1} + f_k\omega^k,$$

where we define $f_k := \frac{1}{n}C^i{}_{ik}$. Back substitution shows that

$$(\phi - \mathrm{d}ZZ^{-1})^i{}_j = C^i{}_{jk}\omega^k,$$

subject to

$$(f_k\delta^i{}_j + C^i{}_{jk})\,\omega^k \wedge \omega^j = 0.$$

The latter expression implies that

$$C^i{}_{kj} = C^i{}_{jk} - f_j\delta^i{}_k + f_k\delta^i{}_j.$$

Now, since Z is a member of $SO(p,q,\mathbf{R})$, then $(\mathrm{d}ZZ^{-1})_{ij} = -(\mathrm{d}ZZ^{-1})_{ji}$. Therefore,

$$C_{ijk} = -C_{jik}.$$

We can then solve for C_{ijk} and obtain

$$C_{ijk} = -f_i\eta_{jk} + f_j\eta_{ik}.$$

Therefore

$$(\phi - \mathrm{d}ZZ^{-1})_{ij} = -f_i\eta_{jk}\omega^k + f_j\eta_{ik}\omega^k.$$

Taking the exterior derivative of α, we get

$$\mathrm{d}\alpha = (\mathrm{d}f_k + f_\ell\phi^\ell{}_k + f_\ell\alpha\delta^\ell{}_k) \wedge \omega^k.$$

This enables us to compute, using (2.20), that

$$\epsilon_k = \mathrm{d}f_k + f_\ell\phi^\ell{}_k + f_\ell\alpha\delta^\ell{}_k + G_{k\ell}\omega^\ell, \tag{2.35}$$

for some functions $G_{k\ell}$ satisfying $G_{k\ell} = G_{\ell k}$. The non-diagonal connection forms are

$$\phi^i{}_j = (dZ\,Z^{-1})^i{}_j - \eta^{\ell i} f_\ell \eta_{jk}\omega^k + f_j\omega^i. \tag{2.36}$$

The exterior derivative of this last expression is

$$\begin{aligned} \mathrm{d}\phi^i{}_j \;&=\; (\mathrm{d}ZZ^{-1})^i{}_k \wedge (\mathrm{d}ZZ^{-1})^k{}_j - \eta^{\ell i}\mathrm{d}f_\ell\eta_{jk} \wedge \omega^k - \eta^{\ell i} f_\ell \eta_{jk}(\phi^k{}_\ell + \alpha\delta^k{}_\ell) \wedge \omega^\ell \\ &\quad + \mathrm{d}f_j \wedge \omega^i + f_j(\phi^i{}_\ell + \alpha\delta^i{}_\ell) \wedge \omega^\ell. \end{aligned}$$

Using (2.35) and (2.36), this becomes

$$\begin{aligned} \mathrm{d}\phi^i{}_j \;&=\; \phi^i{}_k \wedge \phi^k{}_j + \epsilon_j \wedge \omega^i - \eta^{im}\eta_{jk}\epsilon_m \wedge \omega^k \\ &\quad + \Big(-G_{j\ell}\delta^i{}_k - f_\ell f_j \delta^i{}_k + \eta_{j\ell}\eta^{mn} f_m f_n \delta^i{}_k + \eta^{im}\eta_{jk} G_{m\ell} \\ &\qquad - \eta^{im}\eta_{k\ell} f_m f_j + \eta^{im}\eta_{jk} f_m f_\ell\Big)\, \omega^\ell \wedge \omega^k. \end{aligned} \tag{2.37}$$

We now examine the last term. We first note that the term with $\eta_{k\ell}$ is symmetric in $k\ell$, and therefore vanishes when multiplied with $\omega^\ell \wedge \omega^k$ and summed over all possibilities. The coefficients of the independent two-forms $\omega^\ell \wedge \omega^k$ simplify to

$$\begin{aligned}&-G_{j\ell}\delta^i{}_k + G_{jk}\delta^i{}_\ell - f_\ell f_j \delta^i{}_k + f_k f_j \delta^i{}_\ell + \eta_{j\ell}\eta^{mn} f_m f_n \delta^i{}_k - \eta_{jk}\eta^{mn} f_m f_n \delta^i{}_\ell \\ &+\eta^{im}\eta_{jk}G_{m\ell} - \eta^{im}\eta_{j\ell}G_{mk} + \eta^{im}\eta_{jk} f_m f_\ell - \eta^{im}\eta_{j\ell} f_m f_k,\end{aligned}$$

which we define to be $J^i{}_{jk\ell}$. It follows that J is antisymmetric in $k\ell$, and also in ij, when the index i is lowered. Let $i = k$, then sum. Then raise j, let $j = \ell$, and sum. The result is

$$2(1-n)(G^i{}_i - \frac{n-2}{2} f^i f_i) = J^{ij}{}_{ij}.$$

It therefore follows that $J^{ij}{}_{ij}$ can be set to zero (without loss of generality) by letting

$$G^i{}_i = \frac{n-2}{2} f^i f_i. \tag{2.38}$$

With back substitution, it follows that we can set $J^i{}_{ji\ell}$ to zero by letting

$$G_{j\ell} = -f_j f_\ell + \frac{1}{2} f^m f_m \eta_{j\ell},$$

which is consistent with (2.38). Actually, by direct calculation, one can verify that not only the trace $J^i{}_{ji\ell}$ is translated to zero by the present choice of $G_{j\ell}$ but also every $J^i{}_{jk\ell}$ made to vanish. With these choices, we obtain

$$\begin{aligned}\omega^i &= zZ^i{}_j \mathrm{d}x^j \\ \phi_{ij} &= (\mathrm{d}ZZ^{-1})_{ij} - f_i\eta_{jk}\omega^k + f_j\eta_{ik}\omega^k \\ \alpha &= \mathrm{d}zz^{-1} + f_k\omega^k\end{aligned}$$

and

$$\epsilon_k = \mathrm{d}f_k + f_\ell\phi^\ell{}_k + f_\ell\alpha\delta^\ell{}_k - f_k f_\ell\omega^\ell + \frac{1}{2} f^m f_m \eta_{k\ell}\omega^\ell,$$

where $Z \in SO(p, q, \mathbf{R})$. Since $J^i{}_{jk\ell} = 0$, then equation (2.37) becomes

$$\mathrm{d}\phi^i{}_j = \phi^i{}_k \wedge \phi^k{}_j + \epsilon_j \wedge \omega^i - \eta^{im}\eta_{jk}\epsilon_m \wedge \omega^k.$$

Thus, using (2.19), we find that, with this choice of metric, the Weyl tensor is zero. Direct calculation shows that, for the manifolds we are investigating,

$$\mathrm{d}\epsilon_k = \epsilon_\ell \wedge \phi^\ell{}_k + \epsilon_k \wedge \alpha.$$

The invariants $H_{jk\ell}$ are then all equal to zero. This is compatible with the results of the preceding section, for manifolds U of dimension greater than three, that the functions H_{ijk} must vanish when the Weyl tensor does so.

In summary, for conformally flat metrics, all the fundamental invariants vanish. We can invoke the theory of the equivalence to conclude that all real analytic pseudo-Riemannian manifolds of dimension greater or equal to four such that their Weyl tensor vanishes are conformally equivalent. In particular, since flat metrics have their Weyl tensor equal to zero, all such aforementioned manifolds are conformally flat if and only if they have zero Weyl tensor. Similarly, all three-dimensional real analytic pseudo-Riemannian manifolds are conformally flat if and only if their Cotton-York tensor vanishes. Also, all real analytic pseudo-Riemannian manifolds of dimension one or two are conformally flat. These results are well known, see Kramer *et al.* (1980)

2.3 A classification of 1+1 metrics

In this section, we classify real analytic pseudo-Riemannian two-dimensional metrics using the method of equivalence of Cartan. Afterwards, we redo the classification with a slightly different point of view that emphasizes the physical aspects of the various cases. This second classification also illustrates the difference between the Karlhede classification and that based on the method of Cartan. We recall that on $U \times SO(1,1,\mathbf{R})$, the structure equations are (cf. (2.4) and (2.8)) the e-structure

$$\begin{aligned} \mathrm{d}\omega^0 &= \Pi \wedge \omega^1 \\ \mathrm{d}\omega^1 &= \Pi \wedge \omega^0 \end{aligned}$$

and

$$\mathrm{d}\Pi = R\,\omega^0 \wedge \omega^1.$$

Taking the exterior derivative of the last equation yields

$$0 = \mathrm{d}^2\Pi = \mathrm{d}R \wedge \omega^0 \wedge \omega^1.$$

By the Cartan lemma, this implies that

$$\mathrm{d}R = A\omega^0 + B\omega^1, \tag{2.39}$$

where A and B are functions.

The first case to consider is when the rank (as defined on page 19) of $\{\mathrm{d}R\}$ is zero. It follows that A and B are both zero, and that R is a constant. In that case, the derivative of R does not produce any new invariants, and so the rank of this e-structure is 0 and the order is 0. The dimension of $U \times SO(1,1,\mathbf{R})$ is 3. There is a three-dimensional group of symmetry for these metrics. The dimension of this group is obtained by subtracting the rank of the e-structure from the dimension of the space $U \times SO(1,1,\mathbf{R})$.

We now suppose that the rank of $\{\mathrm{d}R\}$ is one. The Riemann curvature R is an invariant function. Therefore, the order of the e-structure is at least one. It is exactly one if the derivative of R does not produce any new invariants. As a first step, we compute the derivatives of A and B from the integrability condition on R, thereby obtaining

$$0 = \mathrm{d}^2 R = \mathrm{d}A \wedge \omega^0 + \mathrm{d}B \wedge \omega^1 + A\Pi \wedge \omega^1 + B\Pi \wedge \omega^0.$$

We can then isolate $\mathrm{d}A$ and $\mathrm{d}B$, and obtain

$$\mathrm{d}A = -B\Pi + C\omega^0 + D\omega^1$$

and

$$\mathrm{d}B = -A\Pi + D\omega^0 + E\omega^1,$$

where C, D and E are functions. We remark that if $A = 0$, then $\mathrm{d}A = 0$ implies $B = 0$. Conversely, if $B = 0$, then $\mathrm{d}B = 0$ implies $A = 0$. Since, in the present situation, R cannot be constant, we must have that $A^2 + B^2 \neq 0$. If the order of the e-structure is one, then the fact that differentiating R does not produce new invariants means that the rank of $\{\mathrm{d}R, \mathrm{d}A, \mathrm{d}B\}$ is one. This requires that $\mathrm{d}R \wedge \mathrm{d}A = 0$, which is just

$$-BA\omega^0 \wedge \Pi + AD\omega^0 \wedge \omega^1 + B^2\Pi \wedge \omega^1 - BC\omega^0 \wedge \omega^1 = 0.$$

Since ω^0, ω^1 and Π are independent, this means that $B^2 = 0$, or $B = 0$. Similarly, $\mathrm{d}R \wedge \mathrm{d}B = 0$ implies that $A = 0$. Now, we have already observed that $A^2 + B^2 \neq 0$, and therefore the case of order one cannot happen. This result can also be obtained from a group consideration.[9] Suppose that the order is exactly one. That entails that the rank must be equal to one whence there is a two-dimensional isometry group. Also, there is a single invariant on $M \times G$. By the preceding this invariant can be taken to be R. In addition, on M, except at isolated points, the orbits of the isometry group must be two-dimensional; therefore, R must be constant on M and thus also on $M \times G$. This is a contradiction.

[9] I thank M.A.H. MacCallum for noting this line of argument.

We now turn to the situation when the order is at least two. This means that $\mathrm{d}R$ produces at least one more invariant. There are two situations, depending on whether the rank of $\{\mathrm{d}R, \mathrm{d}A, \mathrm{d}B\}$ is two or three. If this rank is three, then differentiations of A and B cannot produce any new invariants independent of R, A and B. Therefore the order of the e-structure is two and its rank is three. There is no symmetry in this structure since its rank is equal to the dimension of the space $U \times SO(1,1,\mathbf{R})$ on which it is defined.

When the rank of $\{\mathrm{d}R, \mathrm{d}A, \mathrm{d}B\}$ is two, there is a functional relationship between R, A and B. Their derivatives obey the relation $\mathrm{d}R \wedge \mathrm{d}A \wedge \mathrm{d}B = 0$. This is

$$[-(A^2 + B^2)D + AB(C+E)] \wedge \Pi \wedge \omega^0 \wedge \omega^1 = 0,$$

where use has been made of the integrability condition on R. In this situation, B cannot produce invariants that are not already given by R or A, and so we consider the integrability condition on A. From $\mathrm{d}^2 A = 0$, we obtain

$$0 = -\mathrm{d}B \wedge \Pi - B\mathrm{d}\Pi + \mathrm{d}C \wedge \omega^0 + C\mathrm{d}\omega^0 + \mathrm{d}D \wedge \omega^1 + D\mathrm{d}\omega^1.$$

We deduce that

$$\mathrm{d}C = -2D\Pi + H\omega^0 + I\omega^1$$

and

$$\mathrm{d}D = -(C+E)\Pi + (BR + I)\omega^0 + J\omega^1,$$

where H, I and J are functions. If the order of the e-structure is 2 then the rank of $\{\mathrm{d}R, \mathrm{d}A, \mathrm{d}B, \mathrm{d}C, \mathrm{d}D\}$ is equal to the rank of $\{\mathrm{d}R, \mathrm{d}A, \mathrm{d}B\}$, which is 2. Since B is functionally dependent on R and A, it suffices to require that $\mathrm{d}R \wedge \mathrm{d}A \wedge \mathrm{d}C = 0$ and $\mathrm{d}R \wedge \mathrm{d}A \wedge \mathrm{d}D = 0$. These conditions translate to

$$0 = \Big(BAI - 2D(AD - BD) - B^2 H\Big)\, \Pi \wedge \omega^0 \wedge \omega^1$$

and

$$0 = \Big(BAJ - (AD - BC)(C + E) - B^2(BR + I)\Big)\,\Pi \wedge \omega^0 \wedge \omega^1.$$

Since the order of the e-structure is 2, there is a one-parameter group of symmetries. If either of these last two conditions is not satisfied, then the rank of $\{\mathrm{d}R, \mathrm{d}A, \mathrm{d}B, \mathrm{d}C, \mathrm{d}D\}$ is 3. In this case the order is 3 and there is no group of symmetry.

We summarize these results in table 2.1.

order of the e-structure	*rank of the e-structure*	*symmetry*
1	0	3-dimensional group
1	1	this situation does not happen
2	2	1-dimensional group
2	3	no symmetry
3	3	no symmetry

Table 2.1: Classification of 1+1 metrics

We now examine the classification from a slightly different point of view in order to shed more light as to the physical significance of the various cases.[10] Equation (2.39) can be rewritten as

$$\mathrm{d}R = (A\cosh\alpha + B\sinh\alpha)\sigma^0 + (A\sinh\alpha + B\cosh\alpha)\sigma^1$$

If $\mathrm{d}R = 0$, we are in the situation with the 3-dimensional isometry group and so the only invariant of this problem, *viz.* R, is constant. Hence, we suppose $A^2 + B^2 \neq 0$.

[10] I am grateful to M.A.H. MacCallum for his remarks concerning the null versus non-null characterization of $\mathrm{d}R$.

If $|A/B| \neq 1$, in other words, when $\mathrm{d}R$ is non-null, we can make [11] $B = 0$. This fixes the group parameter α and hence we are no longer directly working with the e-structure[12] but with a normal-form-structure. Differentiating $\mathrm{d}R = A\omega^0$, we obtain

$$0 = \mathrm{d}^2R = \mathrm{d}A \wedge \omega^0 + A\Pi \wedge \omega^1,$$

whence, by the Cartan lemma,

$$\mathrm{d}A = C\omega^0 + D\omega^1$$

and

$$\Pi = (D/A)\omega^0 + E\omega^1.$$

The product

$$\mathrm{d}R \wedge \mathrm{d}A = AD\,\omega^0 \wedge \omega^1$$

vanishes if and only if $D = 0$, since $A = 0$ has already been excluded. We first suppose that $D \neq 0$. Both the order and the rank of the normal-form-structure are equal to two. The invariants of the problem are R and A. There is no isometry in this situation since the dimension[13] of $U \times G$ is two. We note that this situation corresponds to that of *order* $= 2$ and *rank* $= 3$ in the table 2.1. If $D = 0$, then the rank and the order of the normal-form-structure are 1. There is therefore a one-dimensional isometry group. The only invariant of the problem is R. We note that this situation corresponds to that of *order* $= 2$ and *rank* $= 2$ in the table 2.1.

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[11] If $|A/B| < 1$ then the discrete transformation $(\omega^0, \omega^1) \mapsto \sqrt{-1}(\omega^1, \omega^0)$ is needed to keep α real.

[12] Note that the e in e-structure refers to the group $G^{(1)}$ in $U \times G \times G^{(1)}$. We are reducing the group G to one of its subgroup G'. We are thus working with $U \times G' \times G^{(1)}$.

[13] Rotating the dyad so that $B = 0$ reduces the group of indeterminacy to a zero-dimensional group.

If $|A/B| = 1$, then $\mathrm{d}R$ is null. It follows that $B = \pm A$. We choose to consider $B = A$; the situation of $B = -A$ being analogous. We therefore have

$$\mathrm{d}R = A(\omega^0 + \omega^1).$$

Differentiation of $\mathrm{d}R$ yields

$$\mathrm{d}A = -A\Pi + C(\omega^0 + \omega^1),$$

after invoking the Cartan lemma. The rank relation

$$\mathrm{d}A \wedge \mathrm{d}R = -A^2\Pi \wedge (\omega^0 + \omega^1)$$

does not vanish since we have already considered the situation of a 3-dimensional isometry group. Differentiation of $\mathrm{d}A$ and the Cartan lemma imply that

$$\mathrm{d}C = -2C\Pi + D(\omega^0 + \omega^1) + \frac{1}{2}A(\omega^0 - \omega^1).$$

The rank test-quantity

$$\mathrm{d}A \wedge \mathrm{d}R \wedge \mathrm{d}C = -A^3 R\, \Pi \wedge \omega^0 \wedge \omega^1$$

cannot vanish. There cannot be any further independent invariant functions. Both the order and the rank of the present e-structure are equal to 3 whence there are no isometries.

2.4 Comments

It may now be seen that Cartan's method of equivalence leads naturally to the Riemann and Weyl tensor. It also unifies classifications of the metric based on the Riemann tensor, such as the Petrov classification of the Weyl tensor and the

Plebanskí and Segre classifications of the Ricci tensors, and those based on groups of symmetry of the metric (see, for example, Kramer et al. (1980), McIntosh et al. (1981) and Joly and MacCallum (1990)). The works by Karlhede (1980a), Karlhede (1980b), Karlhede and Lindström (1982), Karlhede and MacCallum (1982), Bradley and Karlhede (1990), Collins and al. (1990), Joly and MacCallum (1990), Åman et al. (1991), Koutras (1992) and others follow the method of equivalence of Cartan, with a modification, known as the Karlhede classification, to be better suited for the purpose of the study of spacetimes. The equivalence method allows the various covariant derivatives of the Riemann tensor to play a rôle in the classification. The last section uses the classification of two-dimensional metrics to illustrate the classification of higher dimensional metrics and shows the usefulness of finding normal forms[14] to reduce the number of derivatives needed. As a by-product of the classification with respect to the conformal group, we have found an efficient way of obtaining the Weyl curvature two-forms, given the Riemann curvature two-forms.

[14]This is the essence of the modification of Karlhede to the method of Cartan as applied to manifolds of general relativity.

Chapter 3

Orthonormal Frame Formalism

All men by nature desire to know.

Aristotle

IN THIS chapter, we focus on the geometry of U, where U is a four-dimensional Lorentzian manifold. Let the metric be given by

$$g = \mathrm{d}s^2 = -\bar{\omega}^0 \otimes \bar{\omega}^0 + \bar{\omega}^1 \otimes \bar{\omega}^1 + \bar{\omega}^2 \otimes \bar{\omega}^2 + \bar{\omega}^3 \otimes \bar{\omega}^3. \tag{3.1}$$

3.1 Structure equations

In this section, we describe the structure equations of a Lorentzian spacetime with an invariantly defined[1] unit timelike future-pointing vector. These structure equations enable us to define various kinematic quantities. We shall provide two methods of giving an interpretation to these kinematic quantities. The method we use is

[1] We shall concern ourselves with only local considerations. Not all spacetimes admit such a global unit timelike future-pointing vector field.

closely related to that of MacCallum (1973). Our approach uses differential forms, whereas MacCallum used the geometrical objects dual to one-forms, namely, vectors.

Let $\vec{e}_0$ be the invariantly defined (locally) unit timelike vector admitted by the spacetime under consideration. For a perfect fluid spacetime with $\mu + p \neq 0$, the vector $\vec{e}_0$ can be chosen in an invariant way as the unique future-pointing unit timelike eigenvector of the Ricci tensor(Ellis, 1971). This eigenvector is the velocity vector of the fluid flow. In a coordinate basis, $\vec{e}_0$ can be written as

$$\vec{e}_0 = u^i \frac{\partial}{\partial x^i}. \tag{3.2}$$

Let $\bar{\omega}^0$ be the one-form dual to $\vec{e}_0$. In a coordinate basis, $\bar{\omega}^0$ can be written as

$$\bar{\omega}^0 = -u_i \, dx^i. \tag{3.3}$$

The interior product of $\bar{\omega}^0$ and $\vec{e}_0$ satisfies

$$1 = \vec{e}_0 \rfloor \bar{\omega}^0 = -u^i u_i. \tag{3.4}$$

This is consistent with the unit timelike character of the velocity. We complete the orthonormal cobasis by choosing three covectors

$$\bar{\omega}^\alpha = A^\alpha{}_i \mathrm{d}x^i. \tag{3.5}$$

The corresponding vector basis elements are

$$\vec{e}_\alpha = B^i{}_\alpha \frac{\partial}{\partial x^i}.$$

The condition of orthonormality implies

$$\vec{e}_\alpha \rfloor \bar{\omega}^\beta = \delta^\beta{}_\alpha = B^i{}_\alpha A^\beta{}_i, \tag{3.6}$$

$$\vec{e}_0 \rfloor \bar{\omega}^\alpha = u^i A^\alpha{}_i = 0 \tag{3.7}$$

and

$$\vec{e}_\alpha \rfloor \bar{\omega}^0 = -B^i{}_\alpha u_i = 0. \tag{3.8}$$

The coordinate cobasis satisfies

$$\mathrm{d}x^i = u^i\bar{\omega}^0 + B^i{}_\alpha\bar{\omega}^\alpha.$$

This is easily verified by substitution into (3.3) and (3.5), followed by simplification using (3.4) and (3.8).

The fluid flow vector $\vec{u}$ is given by (3.2) in a coordinate basis and by $(+1)\,\vec{e}_0$ in the tetrad basis. The corresponding covector $\underline{u} = g(\vec{u})$, where $g(\vec{u})$ denotes the contraction of the metric (3.1) with the vector $\vec{u}$, is given by $u_i\,\mathrm{d}x^i$ in coordinates and by $(-1)\,\bar{\omega}^0$ in the tetrad basis.

We now proceed to calculate the various kinematic quantities. This is done by first computing the structure equations and then identifying their various components. The first structure equation is obtained by differentiating (3.3) to obtain

$$\mathrm{d}\bar{\omega}^0 = -\mathrm{d}u_i \wedge \mathrm{d}x^i. \tag{3.9}$$

Since $\mathrm{d}u_i$ can be expanded in the coordinate cobasis as follows:

$$\mathrm{d}u_i = u_{i|j}\mathrm{d}x^j = \vec{e}_a(u_i)\bar{\omega}^a,$$

equation (3.9) becomes

$$\begin{aligned} \mathrm{d}\bar{\omega}^0 &= -\mathrm{d}u_i \wedge \mathrm{d}x^i \\ &= -\mathrm{d}u_i \wedge (u^i\bar{\omega}^0 + B^i{}_\alpha\bar{\omega}^\alpha) \\ &= -\vec{e}_a(u_i)\,\bar{\omega}^a \wedge (u^i\bar{\omega}^0 + B^i{}_\alpha\bar{\omega}^\alpha). \end{aligned} \tag{3.10}$$

The acceleration, $\dot{\underline{u}} = \dot{u}_\alpha\bar{\omega}^\alpha$, of the $\vec{e}_0-$congruence must be perpendicular to the velocity, since the velocity has unit length. Therefore the acceleration does not have a $\bar{\omega}^0$ component; it is, however, equal to $\vec{u}\rfloor\mathrm{d}\underline{u} = -\vec{e}_0\rfloor\mathrm{d}\bar{\omega}^0$. Since

$$\vec{e}_0\rfloor\mathrm{d}\bar{\omega}^0 = u^i\vec{e}_\beta(u_i)\bar{\omega}^\beta - B^i{}_\alpha\vec{e}_0(u_i)\bar{\omega}^\alpha,$$

the acceleration components are given by

$$\dot{u}_\alpha = -\vec{e}_\alpha \rfloor \vec{e}_0 \rfloor \mathrm{d}\bar{\omega}^0 = -u^i \vec{e}_\alpha(u_i) + B^i{}_\alpha \vec{e}_0(u_i).$$

We next compute the part of the structure equation (3.10) that is independent of $\bar{\omega}^0$. First, we have

$$\vec{e}_\alpha \rfloor \mathrm{d}\bar{\omega}^0 = -\vec{e}_\alpha(u_i)(u^i \bar{\omega}^0 + B^i{}_\beta \bar{\omega}^\beta) + \vec{e}_j(u_i)\bar{\omega}^j B^i{}_\alpha,$$

where the sum over j omits $j = \alpha$. Then, we have

$$\vec{e}_\beta \rfloor \vec{e}_\alpha \rfloor \mathrm{d}\bar{\omega}^0 = -\vec{e}_\alpha(u_i) B^i{}_\beta + \vec{e}_\beta(u_i) B^i{}_\alpha.$$

These quantities are antisymmetric and perpendicular to $\vec{e}_0$, and so they can be grouped as the one-form $2\omega_\gamma \bar{\omega}^\gamma$, where (α, β, γ) is an even permutation of $(1, 2, 3)$. These kinematic quantities correspond to the (rate of) vorticity of the $\vec{e}_0$-congruence, as can be seen by noting that

$$\begin{aligned} \omega_1 \bar{\omega}^0 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3 + \omega_2 \bar{\omega}^0 \wedge \bar{\omega}^3 \wedge \bar{\omega}^1 + \omega_3 \bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^2 &= \\ &= \bar{\omega}^0 \wedge \mathrm{d}\bar{\omega}^0 \\ &= \underline{u} \wedge \mathrm{d}\underline{u}. \end{aligned}$$

The structure equation (3.10) can therefore be written as

$$\mathrm{d}\bar{\omega}^0 = -\dot{u}_\alpha \bar{\omega}^0 \wedge \bar{\omega}^\alpha + 2\omega_\gamma \bar{\omega}^{|\alpha} \wedge \bar{\omega}^{\beta|},$$

where by $|\alpha\beta|$ we indicate that $\alpha \leq \beta$.

To compute the remaining structure equations, we start by differentiating (3.5), which gives

$$\mathrm{d}\bar{\omega}^\alpha = \mathrm{d}A^\alpha_i \wedge \mathrm{d}x^i = \mathrm{d}A^\alpha_i \wedge (u^i \bar{\omega}^0 + B^i_\alpha \bar{\omega}^\alpha).$$

Differentiating (3.7) and (3.6), we obtain $\mathrm{d}A^{\alpha}{}_{i}\, u^{i} = -A^{\alpha}{}_{i}\mathrm{d}u^{i}$ and $\mathrm{d}A^{\beta}{}_{i}B^{i}{}_{\alpha} = -A^{\beta}{}_{i}\mathrm{d}B^{i}{}_{\alpha}$, from which we deduce that

$$\mathrm{d}\bar{\omega}^{\alpha} = -A^{\alpha}{}_{i}\vec{e}_{\beta}(u^{i})\bar{\omega}^{\beta} \wedge \bar{\omega}^{0} - A^{\alpha}{}_{i}\vec{e}_{j}(B^{i}{}_{\beta})\bar{\omega}^{j} \wedge \bar{\omega}^{\beta}.$$

We examine the part of these structure equations involving the $\vec{e}_0$−congruence. In order to do this, we first compute

$$\vec{e}_{\gamma}\rfloor\mathrm{d}\bar{\omega}^{\alpha} = -A^{\alpha}{}_{i}\vec{e}_{\gamma}(u^{i})\bar{\omega}^{0} - A^{\alpha}{}_{i}\vec{e}_{\gamma}(B^{i}{}_{\beta})\bar{\omega}^{\beta} + A^{\alpha}{}_{i}\vec{e}_{j}(B^{i}{}_{\gamma})\bar{\omega}^{j}.$$

From this, we obtain the required components, which are

$$\vec{e}_{0}\rfloor\vec{e}_{\gamma}\rfloor\mathrm{d}\bar{\omega}^{\alpha} = -A^{\alpha}{}_{i}\vec{e}_{\gamma}(u^{i}) + A^{\alpha}{}_{i}\vec{e}_{0}(B^{i}{}_{\gamma}). \tag{3.11}$$

This can be decomposed into a part that is symmetric in α and γ and into a part that is antisymmetric. The antisymmetric part is given by

$$\frac{1}{2}[-A^{\alpha}{}_{i}\vec{e}_{\gamma}(u^{i}) + A^{\gamma}{}_{i}\vec{e}_{\alpha}(u^{i}) + A^{\alpha}{}_{i}\vec{e}_{0}(B^{i}{}_{\gamma}) - A^{\gamma}{}_{i}\vec{e}_{0}(B^{i}{}_{\alpha})].$$

The first two terms in the square brackets are just $2\omega_{\beta}$ with the index raised, where (α, β, γ) is an even permutation of $(1, 2, 3)$. The last two terms can be grouped together to define the vector $\Omega^{\alpha}\vec{e}_{\alpha}$, where

$$\Omega^{\alpha} = +A^{\beta}_{i}\vec{e}_{0}(B^{i}_{\gamma}) - A^{\gamma}_{i}\vec{e}_{0}(B^{i}_{\beta}),$$

with (α, β, γ) an even permutation of $(1, 2, 3)$. Since we are using metric components in the orthonormal tetrad, we can lower the index on Ω, a space-like quantity, without changing its value. These terms correspond to the rotation of the $\vec{e}_{\alpha}$− axes with respect to a Fermi-Walker propagated tetrad. The symmetric part of (3.11), denoted by $\theta^{\alpha}{}_{\gamma}$ is given by

$$\frac{1}{2}[-A^{\alpha}{}_{i}\vec{e}_{\gamma}(u^{i}) - A^{\gamma}{}_{i}\vec{e}_{\alpha}(u^{i}) + A^{\alpha}{}_{i}\vec{e}_{0}(B^{i}{}_{\gamma}) + A^{\gamma}{}_{i}\vec{e}_{0}(B^{i}{}_{\alpha})].$$

Lowering the upper index of the space-like quantity $\theta^{\alpha}{}_{\gamma}$ does not change its value, since we are using the orthonormal basis. This quantity is the (rate of) expansion of the $\vec{e}_0$-congruence.

What has been obtained so far can be summarized by the equation

$$\mathrm{d}\bar{\omega}^{\beta} + (\theta^{\beta}{}_{\gamma} + \Omega_{\alpha})\bar{\omega}^0 \wedge \bar{\omega}^{\gamma} = -A^{\beta}{}_{i}\vec{e}_{\gamma}(B^{i}{}_{\alpha})\bar{\omega}^{\gamma} \wedge \bar{\omega}^{\alpha}.$$

We now wish to interpret the right-hand side of this expression. We choose (arbitrarily for now) one unit axis perpendicular to $\bar{\omega}^0$ and call it $\bar{\omega}^1$. The previous treatment, which was applied to $\bar{\omega}^0$, can act as a guide to the situation with $\bar{\omega}^1$. We temporarily ignore the terms that involve $\bar{\omega}^0$, since they already have been interpreted. The following, therefore, can be thought of as involving appropriate projections onto the space perpendicular to the $\vec{e}_0$-congruence. Accordingly, we look at the terms involving $\bar{\omega}^1$ in the structure equation for $\bar{\omega}^1$. The expression

$$\vec{e}_1\rfloor - A^{1}{}_{i}\vec{e}_{\gamma}(B^{i}{}_{\alpha})\bar{\omega}^{\gamma} \wedge \bar{\omega}^{\alpha} = -A^{1}{}_{i}\vec{e}_1(B^{i}{}_{A})\bar{\omega}^{A} + A^{1}{}_{i}\vec{e}_A(B^{i}{}_{1})\bar{\omega}^{A}$$

has components

$$\vec{e}_A\rfloor\vec{e}_1\rfloor - A^{1}{}_{i}\vec{e}_{\gamma}(B^{i}{}_{\alpha})\bar{\omega}^{\gamma} \wedge \bar{\omega}^{\alpha} = -A^{1}{}_{i}\vec{e}_1(B^{i}{}_{A}) + A^{1}{}_{i}\vec{e}_A(B^{i}{}_{1}) =: d_A,$$

which represent the spatial part of the acceleration of the $\vec{e}_1$-congruence. The spatial part of the vorticity of this congruence is given by

$$\vec{e}_3\rfloor\vec{e}_2\rfloor - A^{1}{}_{i}\vec{e}_{\gamma}(B^{i}{}_{\alpha})\bar{\omega}^{\gamma} \wedge \bar{\omega}^{\alpha} = -A^{1}{}_{i}\vec{e}_2(B^{i}{}_{3}) + A^{1}{}_{i}\vec{e}_3(B^{i}{}_{2}) =: -n.$$

The other components of the structure equations involving $\bar{\omega}^1$ have coefficients given by

$$\vec{e}_B\rfloor\vec{e}_1\rfloor - A^{A}_{i}\vec{e}_{\gamma}(B^{i}{}_{\alpha})\bar{\omega}^{\gamma} \wedge \bar{\omega}^{\alpha} = -A^{A}_{i}\vec{e}_1(B^{i}{}_{B}) + A^{A}_{i}\vec{e}_B(B^{i}{}_{1}).$$

As before, this expression can be decomposed into a symmetric part and an anti-symmetric part. The symmetric part, which is

$$\frac{1}{2}[-A^A{}_i\vec{e}_1(B^i{}_B) - A^B{}_i\vec{e}_1(B^i{}_A) + A^A{}_i\vec{e}_B(B^i{}_1) + A^B{}_i\vec{e}_B(A^i{}_1)] =: \frac{1}{2}\hat{\theta}^A_B,$$

measures the spatial component of the expansion rate of the $\vec{e}_1$-congruence. The antisymmetric part, which is given by

$$\frac{1}{2}[-A^A{}_i\vec{e}_1(B^i{}_B) + A^B{}_i\vec{e}_1(A^i{}_B) + A^A{}_i\vec{e}_B(B^i{}_1) - A^B{}_i\vec{e}_B(A^i{}_1)] =: \frac{1}{2}\hat{\Omega},$$

measures the spatial component of the angular velocity of the dyad $\{\vec{e}_2, \vec{e}_3\}$ along the $\vec{e}_1$-congruence.

The only components of the structure equations that are left to interpret are those independent of both $\bar{\omega}^0$ and $\bar{\omega}^1$. They are given by

$$\vec{e}_B\rfloor\vec{e}_A\rfloor - A^A{}_i\vec{e}_\gamma(B^i{}_\alpha)\bar{\omega}^\gamma \wedge \bar{\omega}^\alpha = -A^A{}_i\vec{e}_A(B^i{}_B) + A^A{}_i\vec{e}_B(B^i{}_A).$$

There are only two such terms; the first is

$$\vec{e}_2\rfloor\vec{e}_3\rfloor - A^3{}_i\vec{e}_\gamma(B^i{}_\alpha)\bar{\omega}^\gamma \wedge \bar{\omega}^\alpha = -A^3{}_i\vec{e}_3(B^i{}_2) + A^3{}_i\vec{e}_B(2^i_3) =: -A_2,$$

and the second is

$$\vec{e}_3\rfloor\vec{e}_2\rfloor - A^2{}_i\vec{e}_\gamma(B^i{}_\alpha)\bar{\omega}^\gamma \wedge \bar{\omega}^\alpha = -A^2{}_i\vec{e}_2(B^i{}_3) + A^2{}_i\vec{e}_3(B^i{}_2) =: -A_3.$$

The quantity A_2 measures the projection of the acceleration of the $\vec{e}_2$-congruence and A_3, the expansion of the $\vec{e}_2$-congruence.

Ellis (1971) gives a very clear introduction to the kinematic quantities, $\dot{u}_\alpha, \theta_{\alpha\beta}$ and ω_α, associated with the $\vec{e}_0$-congruence. The interpretation of the quantities associated with the $\vec{e}_1$-congruence, namely $d_A, n, \hat{\Omega}, \hat{\theta}_{AB}$ parallels the similar interpretation of the $\vec{e}_0$-congruence quantities, namely $\dot{u}_\alpha, \omega_\alpha, \Omega_\alpha, \theta_{\alpha\beta}$. There is also a parallel with the quantities associated with the $\vec{e}_2$-congruence, namely A_2 and A_3. The choices of sign in the above definitions of the kinematic quantities have been made in accordance with those of White and Collins (1984), who first[2] defined $d_A, A_A, \hat{\theta}_{AB}, \hat{\Omega}$ and n.

To summarize this section, the structure equations can be written as follows:

$$\mathrm{d}\bar{\omega}^0 = -\dot{u}_\alpha\,\bar{\omega}^0\wedge\bar{\omega}^\alpha + 2\omega_\gamma\,\bar{\omega}^{|\alpha}\wedge\bar{\omega}^{\beta|}, \tag{3.12}$$

$$\begin{aligned}\mathrm{d}\bar{\omega}^1 = {}& \theta_{11}\bar{\omega}^0\wedge\bar{\omega}^1 + (\theta_{12}+\omega_3+\Omega_3)\bar{\omega}^0\wedge\bar{\omega}^2 + (\theta_{13}-\omega_2-\Omega_2)\bar{\omega}^0\wedge\bar{\omega}^3 \\ & + d_2\,\bar{\omega}^1\wedge\bar{\omega}^2 - n\,\bar{\omega}^2\wedge\bar{\omega}^3 - d_3\,\bar{\omega}^3\wedge\bar{\omega}^1,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\mathrm{d}\bar{\omega}^2 = {}& (\theta_{12}-\omega_3-\Omega_3)\bar{\omega}^0\wedge\bar{\omega}^1 + \theta_{22}\,\bar{\omega}^0\wedge\bar{\omega}^2 + (\theta_{23}+\omega_1+\Omega_1)\bar{\omega}^0\wedge\bar{\omega}^3 \\ & + \hat{\theta}_{22}\bar{\omega}^1\wedge\bar{\omega}^2 - A_3\,\bar{\omega}^2\wedge\bar{\omega}^3 + (-\hat{\Omega}-\hat{\theta}_{23})\,\bar{\omega}^3\wedge\bar{\omega}^1\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\mathrm{d}\bar{\omega}^3 = {}& (\theta_{13}+\omega_2+\Omega_2)\bar{\omega}^0\wedge\bar{\omega}^1 + (\theta_{23}-\omega_1-\Omega_1)\bar{\omega}^0\wedge\bar{\omega}^2 + \theta_{33}\,\bar{\omega}^0\wedge\bar{\omega}^3 \\ & + (\hat{\theta}_{23}-\hat{\Omega})\bar{\omega}^1\wedge\bar{\omega}^2 + A_2\,\bar{\omega}^2\wedge\bar{\omega}^3 - \hat{\theta}_{33}\,\bar{\omega}^3\wedge\bar{\omega}^1,\end{aligned} \tag{3.15}$$

where $(\alpha\beta\gamma)$ is an even permutation of (123).

The following is an alternative characterization of the various kinematic quantities. The vector $\vec{e}_0$ is invariantly defined, and so the Lie derivative along $\vec{e}_0$ of the metric is also an invariantly defined quantity. The Lie derivative along $\vec{e}_0$ of the one-forms $\bar{\omega}^a$ is given by

$$\begin{aligned}\mathcal{L}_{\vec{e}_0}\bar{\omega}^0 &= \mathrm{d}(\vec{e}_0\rfloor\bar{\omega}^0) + \vec{e}_0\rfloor\mathrm{d}\bar{\omega}^0 = -\dot{u}_1\bar{\omega}^1 - \dot{u}_2\bar{\omega}^2 - \dot{u}_3\bar{\omega}^3, \\ \mathcal{L}_{\vec{e}_0}\bar{\omega}^1 &= \theta_{11}\bar{\omega}^1 + (\theta_{12}+\omega_3+\Omega_3)\bar{\omega}^2 + (\theta_{13}-\omega_2-\Omega_2)\bar{\omega}^3, \\ \mathcal{L}_{\vec{e}_0}\bar{\omega}^2 &= (\theta_{12}-\omega_3-\Omega_3)\bar{\omega}^1 + \theta_{22}\bar{\omega}^2 + (\theta_{23}+\omega_1+\Omega_1)\bar{\omega}^3\end{aligned}$$

[2]Similar, although not identical, quantities were previously defined by Greenberg (1970) and by Harness (1982)

and

$$\mathcal{L}_{\vec{e}_0}\bar{\omega}^3 = (\theta_{13}+\omega_2+\Omega_2)\bar{\omega}^1 + (\theta_{23}-\omega_1-\Omega_1)\bar{\omega}^2 + \theta_{33}\bar{\omega}^3.$$

The Lie derivative of the metric is

$$\mathcal{L}_{\vec{e}_0}g = \dot{u}_\alpha(\bar{\omega}^0\otimes\bar{\omega}^\alpha + \bar{\omega}^\alpha\otimes\bar{\omega}^0) + 2\theta_{\alpha\beta}\bar{\omega}^\alpha\otimes\bar{\omega}^\beta. \tag{3.16}$$

We first note that $-\mathcal{L}_{\vec{e}_0}\bar{\omega}^0$ is invariantly defined. It measures the change in length along the fluid flow direction as the flow is followed. It measures acceleration since the fluid flow vector has unit length. The last term of (3.16) measures changes of spatial length as the fluid flow is followed. The expansion tensor is therefore given by

$$\begin{aligned}\frac{1}{2}(\bar{\omega}^0\otimes\bar{\omega}^0)\wedge(\mathcal{L}_{\vec{e}_0}g) &:= \frac{1}{2}(\mathcal{L}_{\vec{e}_0}g)_{ab}(\bar{\omega}^0\wedge\bar{\omega}^a)\otimes(\bar{\omega}^0\wedge\bar{\omega}^b)\\ &= \theta_{\alpha\beta}(\bar{\omega}^0\wedge\bar{\omega}^\alpha)\otimes(\bar{\omega}^0\wedge\bar{\omega}^\beta).\end{aligned}$$

The expansion scalar, θ, is found by considering the propagation of the volume form, as follows:

$$\theta\bar{\omega}^0\wedge\bar{\omega}^1\wedge\bar{\omega}^2\wedge\bar{\omega}^3 = \mathcal{L}_{\vec{e}_0}\bar{\omega}^0\wedge\bar{\omega}^1\wedge\bar{\omega}^2\wedge\bar{\omega}^3 = (\theta_{11}+\theta_{22}+\theta_{33})\bar{\omega}^0\wedge\bar{\omega}^1\wedge\bar{\omega}^2\wedge\bar{\omega}^3.$$

The following is also an invariant quantity:

$$\bar{\omega}^0\wedge \mathrm{d}\bar{\omega}^0 = 2\omega_3\bar{\omega}^0\wedge\bar{\omega}^1\wedge\bar{\omega}^2 + 2\omega_2\bar{\omega}^0\wedge\bar{\omega}^3\wedge\bar{\omega}^1 + 2\omega_1\bar{\omega}^0\wedge\bar{\omega}^2\wedge\bar{\omega}^3.$$

It does not involve any change of length as seen by an observer travelling with the flow, since otherwise it would appear in (3.16). Hence it represents the rate of rotation of the fluid flow. The vorticity vector (with index lowered) can then be found by

$$\omega_\alpha\bar{\omega}^\alpha = *\frac{1}{2}\bar{\omega}^0\wedge \mathrm{d}\bar{\omega}^0,$$

where $*$ is the Hodge[3] star operator. This operator is a linear operator that obeys

$$\begin{aligned} *\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^2 &= \bar{\omega}^3, \\ *\bar{\omega}^0 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3 &= \bar{\omega}^1, \\ *\bar{\omega}^0 \wedge \bar{\omega}^3 \wedge \bar{\omega}^1 &= \bar{\omega}^2 \end{aligned}$$

and

$$*\bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3 = \bar{\omega}^0.$$

The spatial triad rotation, $\Omega_a \bar{\omega}^a$, is given by

$$\Omega_a \bar{\omega}^a = - * \frac{1}{2} \bar{\omega}^0 \wedge \left(\sum_b (\mathrm{d}\bar{\omega}^b) \wedge \bar{\omega}^b \right).$$

The invariant definition of $\bar{\omega}^0$ thus implies an invariant characterization of $\dot{u}_\alpha \bar{\omega}^\alpha$, $\omega_\alpha \bar{\omega}^\alpha$ and $\theta_{\alpha\beta} \bar{\omega}^\alpha \otimes \bar{\omega}^\beta$. At this point, the group of indeterminacy is $SO(3, 0, \mathbf{R})$, representing the possible rotations of the $1-2-3$ triad. Using the aforementioned quantities, it may be possible to define uniquely the direction of $\vec{e}_1$. For example, the acceleration vector, the vorticity vector or the triad rotation vector, if they do not vanish, can each be chosen as this invariant direction. Another choice of invariant direction can usually be made by examining the eigenvectors of the expansion tensor, by choosing the eigenvector with the smallest eigenvalue, if the eigenvalues are all different, or by choosing the eigenvector corresponding to the non-repeated eigenvalue, if two eigenvalues are equal. The only situation when we cannot find an invariant direction using the acceleration vector, the vorticity vector, the triad rotation vector or the expansion tensor is when the acceleration, vorticity, and triad rotation vectors all vanish, and, at the same time, the expansion tensor has three equal eigenvalues.

[3]See also page 93.

We shall suppose that we can invariantly find $\bar{\omega}^1$. The expression $\bar{\omega}^0 \wedge \mathcal{L}_{\vec{e}_1}\bar{\omega}^1$ is then an invariant quantity. It corresponds to the acceleration of the $\vec{e}_1$-congruence, projected into the $1-2-3$ triad. We can therefore find d_2 and d_3 by computing

$$\bar{\omega}^0 \wedge \mathcal{L}_{\vec{e}_1}\bar{\omega}^1 = d_2\bar{\omega}^0 \wedge \bar{\omega}^2 + d_3\bar{\omega}^0 \wedge \bar{\omega}^3.$$

The (projected) tensor, $\hat{\theta}_{AB}$, corresponding to the expansion of the $\vec{e}_1$-congruence is computed as follows:

$$\frac{1}{2}\left((\bar{\omega}^0 \wedge \bar{\omega}^1) \otimes (\bar{\omega}^0 \wedge \bar{\omega}^1)\right) \wedge \mathcal{L}_{\vec{e}_1} g = \hat{\theta}_{AB}(\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^A) \otimes (\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^B).$$

The (projected) vorticity, n, of the $\vec{e}_1$-congruence obeys

$$-\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \mathrm{d}\bar{\omega}^1 = n\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3.$$

The (projected) rotation of the 2–3 dyad with respect to a Fermi-Walker propagated $\vec{e}_1$-congruence is given by

$$-\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \left(\sum_\alpha (\mathrm{d}\bar{\omega}^\alpha) \wedge \bar{\omega}^\alpha\right) = (n + 2\hat{\Omega})\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3.$$

Given an invariantly defined $\vec{e}_0$, an invariant definition of $\vec{e}_1$ thus enables us to obtain an invariant characterization of $d_A\bar{\omega}^A, \hat{\theta}_{AB}\bar{\omega}^A \otimes \bar{\omega}^B, n$ and $\hat{\Omega}$. The remaining indeterminacy is $SO(1, 0, \mathbf{R})$, representing the rotations of the 2–3 dyad.

The acceleration of the $\vec{e}_2$-congruence, projected in the 2–3 space, is given by $A_3\bar{\omega}^3$, and is computed using

$$-\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \mathcal{L}_{\vec{e}_2}\bar{\omega}^2 = A_3\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^3.$$

The tensor $A_2\bar{\omega}^2$ corresponding to (projection of) the expansion of the $\vec{e}_2$-congruence is computed as follows:

$$\frac{1}{2}\left((\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^2) \otimes (\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^2)\right) \wedge \mathcal{L}_{\vec{e}_2} g =$$
$$A_2(\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3) \otimes (\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3).$$

3.2 Jacobi identities

The exterior derivative of the structure equations (3.12)–(3.15) provides constraints on the first order derivatives of the kinematic quantities. They take the form of four equations in the six-dimensional space with basis $\bar\omega^0\wedge\bar\omega^1$, $\bar\omega^0\wedge\bar\omega^2$, $\bar\omega^0\wedge\bar\omega^3$, $\bar\omega^1\wedge\bar\omega^2$, $\bar\omega^2\wedge\bar\omega^3$ and $\bar\omega^3\wedge\bar\omega^1$. There are therefore 24 coefficients that must vanish, although not all of them are independent. The equations represent the Jacobi identities of the Lie algebra generated by $\vec{e}_0, \vec{e}_1, \vec{e}_2$ and $\vec{e}_3$. These identities are equivalent to

$$
\begin{aligned}
\partial_1 A_3 &= -\partial_2\hat\Omega + 2\theta_{22}\omega_2 + \hat\Omega d_2 - A_3\hat\theta_{33} + \partial_3\hat\theta_{22} - 2\omega_1\Omega_3 + 2\omega_1\theta_{12} - \partial_2\hat\theta_{23} \\
&\quad +2\omega_3\Omega_1 - \hat\Omega A_2 + 2\omega_3\theta_{23} + \hat\theta_{23}d_2 - \hat\theta_{23}A_2 - \hat\theta_{22}d_3, \\
\partial_0 A_2 &= -\partial_3\theta_{23} + \partial_2\theta_{33} - n\omega_2 - n\Omega_2 - n\theta_{13} - \Omega_2\hat\theta_{23} + \Omega_2\hat\Omega + \theta_{13}\hat\theta_{23} - \theta_{13}\hat\Omega \\
&\quad -\omega_2\hat\theta_{23} + \omega_2\hat\Omega + \partial_3\omega_1 + \partial_3\Omega_1 - A_2\theta_{22} - \hat\theta_{33}\Omega_3 - \hat\theta_{33}\omega_3 \\
&\quad -\hat\theta_{33}\theta_{12} + \Omega_1 A_3 - \theta_{23}\dot u_3 + \omega_1\dot u_3 + \Omega_1\dot u_3 + \theta_{33}\dot u_2 + \omega_1 A_3 - \theta_{23}A_3, \\
\partial_0 d_3 &= -\Omega_2\hat\theta_{33} + \theta_{13}\hat\theta_{33} - n\Omega_3 + \theta_{13}\dot u_1 - \omega_2\dot u_1 + \omega_3\hat\Omega + \Omega_3\hat\theta_{23} - \partial_3\theta_{11} \\
&\quad +\partial_1\theta_{13} - \partial_1\omega_2 - \partial_1\Omega_2 - d_2\Omega_1 + \Omega_3\hat\Omega + \theta_{12}\hat\theta_{23} - \Omega_2\dot u_1 \\
&\quad +n\theta_{12} - \theta_{11}\dot u_3 + \theta_{12}\hat\Omega - n\omega_3 - d_3\theta_{33} - d_2\omega_1 \\
&\quad -d_2\theta_{23} + \omega_3\hat\theta_{23} - \omega_2\hat\theta_{33}, \\
\partial_0 d_2 &= -n\omega_2 - n\Omega_2 - n\theta_{13} - \Omega_2\hat\theta_{23} + \Omega_2\hat\Omega + \theta_{13}\hat\theta_{23} - \theta_{13}\hat\Omega - \omega_2\hat\theta_{23} + \theta_{12}\dot u_1 \\
&\quad +\omega_2\hat\Omega + \theta_{12}\hat\theta_{22} + \omega_3\hat\theta_{22} - \partial_2\theta_{11} + \partial_1\theta_{12} + \partial_1\omega_3 + \partial_1\Omega_3 \\
&\quad +d_3\Omega_1 + \Omega_3\hat\theta_{22} + \Omega_3\dot u_1 + \omega_3\dot u_1 - \theta_{11}\dot u_2 - d_3\theta_{23} + d_3\omega_1 - d_2\theta_{22}, \\
\partial_0 A_3 &= -\partial_2\theta_{23} - n\Omega_3 + \omega_3\hat\Omega + \Omega_3\hat\theta_{23} + \partial_3\theta_{22} - \partial_2\omega_1 - \partial_2\Omega_1 + \hat\theta_{22}\omega_2 \\
&\quad +\Omega_3\hat\Omega + \theta_{12}\hat\theta_{23} + n\theta_{12} + \theta_{12}\hat\Omega - \Omega_1\dot u_2 - \omega_1\dot u_2 - \theta_{23}\dot u_2 \\
&\quad -\Omega_1 A_2 - \omega_1 A_2 - \theta_{23}A_2 + \theta_{22}\dot u_3 + \hat\theta_{22}\Omega_2 + \omega_3\hat\theta_{23} \\
&\quad -\hat\theta_{22}\theta_{13} - A_3\theta_{33} - n\omega_3,
\end{aligned}
$$

$$\begin{aligned}
\partial_0 n &= \partial_3\omega_3 + \partial_2\omega_2 - \partial_2\theta_{13} + \omega_3\dot{u}_3 + d_2\Omega_2 + \theta_{12}\dot{u}_3 + \omega_3 A_3 + \theta_{12}A_3 + \partial_3\Omega_3 \\
&+\partial_2\Omega_2 - \theta_{13}\dot{u}_2 + \Omega_2 A_2 + \Omega_3 A_3 + \partial_3\theta_{12} + \Omega_2\dot{u}_2 + \Omega_3\dot{u}_3 \\
&-n\theta_{22} + d_3\Omega_3 + \theta_{11}n + \omega_2 A_2 - \theta_{13}A_2 + \omega_2\dot{u}_2 + d_3\omega_3 \\
&-d_2\theta_{13} + d_2\omega_2 - n\theta_{33} + d_3\theta_{12},
\end{aligned}$$

$$\begin{aligned}
\partial_0\hat{\theta}_{33} &= A_2\omega_3 - 2\omega_1\hat{\theta}_{23} - \Omega_2\dot{u}_3 - \partial_3\omega_2 + 2\theta_{23}\hat{\Omega} - A_2\theta_{12} + \Omega_2 d_3 - \omega_2\dot{u}_3 \qquad (3.17) \\
&-\hat{\theta}_{33}\theta_{11} + \partial_1\theta_{33} - \theta_{13}\dot{u}_3 + \theta_{33}\dot{u}_1 - 2\Omega_1\hat{\theta}_{23} + \theta_{13}d_3 + A_2\Omega_3 \\
&-\partial_3\theta_{13} + \omega_2 d_3 - \partial_3\Omega_2,
\end{aligned}$$

$$\begin{aligned}
\partial_0\omega_1 &= -\frac{1}{2}\dot{u}_3 A_2 + \Omega_3\omega_2 - \Omega_2\omega_3 + \theta_{12}\omega_2 - \omega_1\theta_{22} + \frac{1}{2}\dot{u}_1 n - \frac{1}{2}\partial_2\dot{u}_3 - \omega_1\theta_{33} \\
&+\theta_{13}\omega_3 + \frac{1}{2}\dot{u}_2 A_3 + \frac{1}{2}\partial_3\dot{u}_2,
\end{aligned}$$

$$\begin{aligned}
\partial_0\omega_2 &= \frac{1}{2}\dot{u}_1 d_3 + \omega_3\theta_{23} + \frac{1}{2}\dot{u}_2\hat{\Omega} + \frac{1}{2}\partial_1\dot{u}_3 + \omega_3\Omega_1 + \frac{1}{2}\dot{u}_3\hat{\theta}_{33} \\
&-\omega_2\theta_{11} + \frac{1}{2}\dot{u}_2\hat{\theta}_{23} - \omega_2\theta_{33} + \omega_1\theta_{12} - \frac{1}{2}\partial_3\dot{u}_1 - \omega_1\Omega_3, \qquad (3.18)
\end{aligned}$$

$$\begin{aligned}
\partial_0\omega_3 &= -\frac{1}{2}\dot{u}_2\hat{\theta}_{22} - \omega_3\theta_{22} + \omega_2\theta_{23} - \frac{1}{2}\partial_1\dot{u}_2 + \frac{1}{2}\dot{u}_3\hat{\Omega} - \frac{1}{2}\dot{u}_3\hat{\theta}_{23} - \omega_3\theta_{11} - \frac{1}{2}\dot{u}_1 d_2 \\
&+\frac{1}{2}\partial_2\dot{u}_1 + \omega_1\Omega_2 - \omega_2\Omega_1 + \omega_1\theta_{13}, \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
\partial_0\hat{\theta}_{23} &= \frac{1}{2}\partial_3\omega_3 - \frac{1}{2}\partial_2\omega_2 - \frac{1}{2}\partial_2\theta_{13} + \frac{1}{2}\omega_3\dot{u}_3 + \frac{1}{2}d_2\Omega_2 - \frac{1}{2}\theta_{12}\dot{u}_3 + \partial_1\theta_{23} \qquad (3.20) \\
&-\frac{1}{2}\omega_3 A_3 + \frac{1}{2}\theta_{12}A_3 + \frac{1}{2}\partial_3\Omega_3 - \frac{1}{2}\partial_2\Omega_2 - \frac{1}{2}\theta_{13}\dot{u}_2 + \frac{1}{2}\Omega_2 A_2 - \frac{1}{2}\Omega_3 A_3 \\
&-\frac{1}{2}\partial_3\theta_{12} - \frac{1}{2}\Omega_2\dot{u}_2 + \hat{\theta}_{33}\Omega_1 - \Omega_1\hat{\theta}_{22} + \frac{1}{2}\Omega_3\dot{u}_3 + \theta_{23}\dot{u}_1 - \omega_1\hat{\theta}_{22} \\
&-\frac{1}{2}d_3\Omega_3 + \frac{1}{2}\omega_2 A_2 + \frac{1}{2}\theta_{13}A_2 - \frac{1}{2}\omega_2\dot{u}_2 - \hat{\theta}_{23}\theta_{11} - \frac{1}{2}d_3\omega_3 + \frac{1}{2}d_2\theta_{13} \\
&+\frac{1}{2}d_2\omega_2 + \hat{\Omega}\theta_{22} - \theta_{33}\hat{\Omega} + \omega_1\hat{\theta}_{33} + \frac{1}{2}d_3\theta_{12},
\end{aligned}$$

$$\begin{aligned}
\partial_0\hat{\Omega} &= \partial_1\Omega_1 - \frac{1}{2}\partial_3\omega_3 - \frac{1}{2}\partial_2\omega_2 + \Omega_1\dot{u}_1 + \frac{1}{2}\partial_2\theta_{13} + \frac{3}{2}\omega_3\dot{u}_3 - \frac{1}{2}d_2\Omega_2 \\
&-\frac{1}{2}\theta_{12}\dot{u}_3 - \hat{\Omega}\theta_{11} - \frac{3}{2}\omega_3 A_3 + \frac{1}{2}\theta_{12}A_3 + \frac{1}{2}\partial_3\Omega_3 + \frac{1}{2}\partial_2\Omega_2 + \frac{1}{2}\theta_{13}\dot{u}_2 \\
&-\frac{1}{2}\Omega_2 A_2 - \frac{1}{2}\Omega_3 A_3 - \frac{1}{2}\partial_3\theta_{12} + \frac{1}{2}\Omega_2\dot{u}_2 + \frac{1}{2}\Omega_3\dot{u}_3 + 2\dot{u}_1\omega_1 - \omega_1\hat{\theta}_{22}
\end{aligned}$$

$$\begin{aligned}
& -\theta_{33}\hat{\theta}_{23} - \theta_{23}\hat{\theta}_{22} - \frac{1}{2}d_3\Omega_3 - \frac{3}{2}\omega_2 A_2 - \frac{1}{2}\theta_{13}A_2 + \frac{3}{2}\omega_2\dot{u}_2 + \hat{\theta}_{23}\theta_{22} \\
& +\frac{1}{2}d_3\omega_3 - \frac{1}{2}d_2\theta_{13} + \frac{1}{2}d_2\omega_2 + \hat{\theta}_{33}\theta_{23} - \omega_1\hat{\theta}_{33} + \frac{1}{2}d_3\theta_{12}, \\
\partial_0\hat{\theta}_{22} &= \theta_{22}\dot{u}_1 + \theta_{12}d_2 + \partial_1\theta_{22} - A_3\theta_{13} - \Omega_3 d_2 + \Omega_3\dot{u}_2 - 2\theta_{23}\hat{\Omega} + \omega_3\dot{u}_2 \qquad (3.21) \\
& -\theta_{12}\dot{u}_2 - \hat{\theta}_{22}\theta_{11} - A_3\omega_2 + 2\omega_1\hat{\theta}_{23} + 2\Omega_1\hat{\theta}_{23} + \partial_2\Omega_3 - A_3\Omega_2 \\
& -\partial_2\theta_{12} + \partial_2\omega_3 - \omega_3 d_2, \\
\partial_3\hat{\Omega} &= 2\theta_{33}\omega_3 - \partial_2\hat{\theta}_{33} + 2\omega_1\Omega_2 - \hat{\Omega}A_3 + \hat{\theta}_{33}d_2 + 2\omega_2\theta_{23} + \partial_1 A_2 \\
& -2\omega_2\Omega_1 + \hat{\theta}_{23}A_3 + A_2\hat{\theta}_{22} + \partial_3\hat{\theta}_{23} + \hat{\Omega}d_3 - \hat{\theta}_{23}d_3 + 2\omega_1\theta_{13}, \\
\partial_1\omega_1 &= -\omega_1\hat{\theta}_{33} - \partial_2\omega_2 + d_2\omega_2 + \omega_2\dot{u}_2 + \dot{u}_1\omega_1 + \omega_3\dot{u}_3 + d_3\omega_3 - \omega_2 A_2 \\
& -\partial_3\omega_3 - \omega_3 A_3 - \omega_1\hat{\theta}_{22}
\end{aligned}$$

and

$$\begin{aligned}
\partial_3 d_2 &= 2\Omega_2\omega_3 - 2\theta_{11}\omega_1 - 2\theta_{12}\omega_2 + \partial_1 n - 2\theta_{13}\omega_3 - d_2 A_3 + n\hat{\theta}_{22} - 2\Omega_3\omega_2 \\
& +d_3 A_2 + n\hat{\theta}_{33} + \partial_2 d_3.
\end{aligned}$$

3.3 Connection

The connection, φ, is the unique matrix of one-forms that satisfies

$$\mathrm{d}\bar{\omega}^i = \varphi^i{}_j \wedge \bar{\omega}^j$$

and

$$\varphi_{ij} + \varphi_{ji} = 0,$$

where $\varphi_{ij} = \varphi^k{}_j\eta_{ki}$, and η is the signature matrix $\eta = \mathrm{diag}(-1, 1, 1, 1)$. Solving for $\varphi^i{}_j$, we obtain

$$\varphi^0{}_1 = -\dot{u}_1\bar{\omega}^0 - \theta_{11}\bar{\omega}^1 + (-\omega_3 - \theta_{12})\bar{\omega}^2 + (\omega_2 - \theta_{13})\bar{\omega}^3,$$

$$
\begin{aligned}
\varphi^0{}_2 &= -\dot{u}_2\bar{\omega}^0 + (\omega_3 - \theta_{12})\bar{\omega}^1 - \theta_{22}\bar{\omega}^2 - (\theta_{23} + \omega_1)\bar{\omega}^3,\\
\varphi^0{}_3 &= -\dot{u}_3\bar{\omega}^0 - (\omega_2 + \theta_{13})\bar{\omega}^1 + (-\theta_{23} + \omega_1)\bar{\omega}^2 - \theta_{33}\bar{\omega}^3,\\
\varphi^1{}_2 &= \Omega_3\bar{\omega}^0 + d_2\bar{\omega}^1 + \hat{\theta}_{22}\bar{\omega}^2 + (\hat{\theta}_{23} + \frac{1}{2}n)\bar{\omega}^3,\\
\varphi^2{}_3 &= \Omega_1\bar{\omega}^0 + (\hat{\Omega} - \frac{1}{2}n)\bar{\omega}^1 - A_3\bar{\omega}^2 + A_2\bar{\omega}^3
\end{aligned}
$$

and

$$
\varphi^3{}_1 = \Omega_2\bar{\omega}^0 - d_3\bar{\omega}^1 + (\hat{\theta}_{23} + \frac{1}{2}n)\bar{\omega}^2 - \hat{\theta}_{33}\bar{\omega}^3.
$$

3.4 Riemann, Ricci and Weyl tensors

The Riemann curvature two-forms are given by

$$
\Theta^a{}_b = \mathrm{d}\varphi^a{}_b + \varphi^a{}_c \wedge \varphi^c_b. \tag{3.22}
$$

Explicitly, the various curvature two-forms are

$$
\begin{aligned}
\Theta^0{}_1 &= R^0{}_{101}\bar{\omega}^0 \wedge \bar{\omega}^1 + R^0{}_{102}\bar{\omega}^0 \wedge \bar{\omega}^2 + R^0{}_{103}\bar{\omega}^0 \wedge \bar{\omega}^3\\
&\quad + R^0{}_{112}\bar{\omega}^1 \wedge \bar{\omega}^2 + R^0{}_{123}\bar{\omega}^2 \wedge \bar{\omega}^3 + R^0{}_{131}\bar{\omega}^3 \wedge \bar{\omega}^1,\\
\Theta^0{}_2 &= R^0{}_{102}\bar{\omega}^0 \wedge \bar{\omega}^1 + R^0{}_{202}\bar{\omega}^0 \wedge \bar{\omega}^2 + R^0{}_{203}\bar{\omega}^0 \wedge \bar{\omega}^3\\
&\quad + R^0{}_{212}\bar{\omega}^1 \wedge \bar{\omega}^2 + R^0{}_{223}\bar{\omega}^2 \wedge \bar{\omega}^3 - R^0{}_{231}\bar{\omega}^1 \wedge \bar{\omega}^3,\\
\Theta^0{}_3 &= R^0{}_{103}\bar{\omega}^0 \wedge \bar{\omega}^1 + R^0{}_{203}\bar{\omega}^0 \wedge \bar{\omega}^2 + R^0{}_{303}\bar{\omega}^0 \wedge \bar{\omega}^3\\
&\quad + (-R^0{}_{123} - R^0{}_{231})\bar{\omega}^1 \wedge \bar{\omega}^2 + R^0{}_{323}\bar{\omega}^2 \wedge \bar{\omega}^3 + R^0{}_{331}\bar{\omega}^3 \wedge \bar{\omega}^1,\\
\Theta^1{}_2 &= -R^0{}_{112}\bar{\omega}^0 \wedge \bar{\omega}^1 - R^0{}_{212}\bar{\omega}^0 \wedge \bar{\omega}^2 + (R^0{}_{123} + R^0{}_{231})\bar{\omega}^0 \wedge \bar{\omega}^3\\
&\quad + R^1{}_{212}\bar{\omega}^1 \wedge \bar{\omega}^2 + R^1{}_{223}\bar{\omega}^2 \wedge \bar{\omega}^3 + R^1{}_{231}\bar{\omega}^1 \wedge \bar{\omega}^3,\\
\Theta^2{}_3 &= -R^0{}_{123}\bar{\omega}^0 \wedge \bar{\omega}^1 - R^0{}_{223}\bar{\omega}^0 \wedge \bar{\omega}^2 - R^0{}_{323}\bar{\omega}^0 \wedge \bar{\omega}^3\\
&\quad + R^1{}_{223}\bar{\omega}^1 \wedge \bar{\omega}^2 + R^2{}_{323}\bar{\omega}^2 \wedge \bar{\omega}^3 + R^2{}_{331}\bar{\omega}^3 \wedge \bar{\omega}^1
\end{aligned}
$$

and

$$\begin{aligned}\Theta^3{}_1 &= -R^0{}_{131}\bar{\omega}^0\wedge\bar{\omega}^1 - R^0{}_{231}\bar{\omega}^0\wedge\bar{\omega}^2 - R^0{}_{331}\bar{\omega}^0\wedge\bar{\omega}^3\\ &\quad +R^1{}_{231}\bar{\omega}^1\wedge\bar{\omega}^2 + R^2{}_{331}\bar{\omega}^2\wedge\bar{\omega}^3 + R^3{}_{131}\bar{\omega}^3\wedge\bar{\omega}^1.\end{aligned}$$

The twenty quantities $R^0{}_{101}$, $R^0{}_{102}$, $R^0{}_{103}$, $R^0{}_{112}$, $R^0{}_{123}$, $R^0{}_{131}$, $R^0{}_{202}$, $R^0{}_{203}$, $R^0{}_{212}$, $R^0{}_{223}$, $R^0{}_{231}$, $R^0{}_{303}$, $R^0{}_{323}$, $R^0{}_{331}$, $R^1{}_{212}$, $R^1{}_{223}$, $R^1{}_{231}$, $R^2{}_{323}$, $R^2{}_{331}$, and $R^3{}_{131}$ are given by:

$$\begin{aligned}
R^0{}_{101} &= -\theta_{11}^2 + 2\Omega_3\theta_{12} - \dot{u}_3 d_3 + \omega_3^2 + \partial_1\dot{u}_1 - 2\Omega_2\theta_{13} - \partial_0\theta_{11}\\
&\quad -\theta_{13}^2 + \dot{u}_1^2 + \omega_2^2 - \theta_{12}^2 - \dot{u}_2 d_2,\\
R^0{}_{102} &= \frac{1}{2}\partial_1\dot{u}_2 - \frac{1}{2}\dot{u}_3\hat{\Omega} + \frac{1}{2}\dot{u}_1 d_2 + \frac{1}{2}\dot{u}_3 n - \theta_{13}\theta_{23} + \theta_{13}\Omega_1 + \theta_{22}\Omega_3 + \dot{u}_1\dot{u}_2\\
&\quad -\theta_{11}\theta_{12} - \omega_2\omega_1 - \Omega_2\theta_{23} - \frac{1}{2}\dot{u}_3\hat{\theta}_{23} - \theta_{12}\theta_{22} - \frac{1}{2}\dot{u}_2\hat{\theta}_{22} - \partial_0\theta_{12}\\
&\quad -\theta_{11}\Omega_3 + \frac{1}{2}\partial_2\dot{u}_1,\\
R^0{}_{103} &= \frac{1}{2}\dot{u}_2\hat{\Omega} + \frac{1}{2}\partial_1\dot{u}_3 + \frac{1}{2}\dot{u}_1 d_3 - \omega_3\omega_1 - \partial_0\theta_{13} - \theta_{11}\theta_{13} - \frac{1}{2}\dot{u}_2\hat{\theta}_{23}\\
&\quad +\dot{u}_1\dot{u}_3 + \frac{1}{2}\partial_3\dot{u}_1 - \theta_{12}\Omega_1 + \theta_{11}\Omega_2 + \Omega_3\theta_{23} - \theta_{13}\theta_{33} - \theta_{33}\Omega_2\\
&\quad -\frac{1}{2}\dot{u}_2 n - \theta_{12}\theta_{23} - \frac{1}{2}\dot{u}_3\hat{\theta}_{33},\\
R^0{}_{112} &= \partial_2\theta_{11} - 2\theta_{12}\hat{\theta}_{22} - \partial_1\theta_{12} - \theta_{11}d_2 - \partial_1\omega_3 - 2\omega_3\dot{u}_1 - \omega_2\hat{\Omega}\\
&\quad +d_2\theta_{22} - d_3\omega_1 + d_3\theta_{23} - 2\theta_{13}\hat{\theta}_{23} + \frac{1}{2}n\theta_{13} + \frac{1}{2}n\omega_2 + \theta_{13}\hat{\Omega},\\
R^0{}_{123} &= \theta_{23}\hat{\theta}_{22} + \omega_3 A_3 + \partial_2\omega_2 + \theta_{11}n + \omega_1\hat{\theta}_{33} + \theta_{33}\hat{\theta}_{23} + \omega_2 A_2 - \theta_{13}A_2\\
&\quad -\frac{1}{2}n\theta_{33} - \partial_2\theta_{13} - \hat{\theta}_{23}\theta_{22} + \omega_1\hat{\theta}_{22} + \theta_{12}A_3 + \partial_3\omega_3 - \frac{1}{2}n\theta_{22} +\\
&\quad \partial_3\theta_{12} - \hat{\theta}_{33}\theta_{23} - 2\dot{u}_1\omega_1,\\
R^0{}_{131} &= \omega_3\hat{\Omega} + \theta_{11}d_3 - d_2\theta_{23} + 2\theta_{13}\hat{\theta}_{33} - d_3\theta_{33} + \partial_1\theta_{13} - 2\omega_2\dot{u}_1\\
&\quad +\theta_{12}\hat{\Omega} - d_2\omega_1 - \frac{1}{2}n\omega_3 + \frac{1}{2}n\theta_{12} + 2\theta_{12}\hat{\theta}_{23} - \partial_1\omega_2 - \partial_3\theta_{11},\\
R^0{}_{202} &= -\theta_{22}^2 - \theta_{23}^2 - \partial_0\theta_{22} - \theta_{12}^2 + \dot{u}_2^2 + \omega_1^2 + \omega_3^2\\
&\quad -2\Omega_3\theta_{12} + \dot{u}_1\hat{\theta}_{22} + \partial_2\dot{u}_2 + 2\Omega_1\theta_{23} + \dot{u}_3 A_3,
\end{aligned}$$

$$
\begin{aligned}
R^0{}_{203} &= -\theta_{22}\Omega_1 + \frac{1}{2}\partial_3\dot{u}_2 + \dot{u}_3\dot{u}_2 - \frac{1}{2}\dot{u}_2 A_3 + \theta_{33}\Omega_1 + \Omega_2\theta_{12} - \theta_{13}\theta_{12} - \frac{1}{2}\dot{u}_3 A_2 \\
&\quad -\theta_{13}\Omega_3 - \theta_{23}\theta_{22} + \frac{1}{2}\partial_2\dot{u}_3 - \omega_2\omega_3 + \dot{u}_1\hat{\theta}_{23} - \theta_{33}\theta_{23} - \partial_0\theta_{23}, \\
R^0{}_{212} &= -\theta_{23}\hat{\theta}_{23} + \frac{1}{2}\omega_1 n - 2\theta_{12}d_2 - \omega_1\hat{\theta}_{23} + \partial_2\theta_{12} - \partial_1\theta_{22} - \frac{1}{2}\theta_{23}n \\
&\quad +\hat{\theta}_{22}\theta_{11} - \partial_2\omega_3 + A_3\theta_{13} - \theta_{22}\hat{\theta}_{22} - 2\omega_3\dot{u}_2 + A_3\omega_2 + 2\theta_{23}\hat{\Omega}, \\
R^0{}_{223} &= -\partial_2\omega_1 + \hat{\theta}_{22}\omega_2 - \partial_2\theta_{23} + \partial_3\theta_{22} - \hat{\theta}_{22}\theta_{13} - \frac{1}{2}n\omega_3 - A_3\theta_{33} \\
&\quad +A_3\theta_{22} + \omega_3\hat{\theta}_{23} + \theta_{12}\hat{\theta}_{23} - 2\omega_1\dot{u}_2 - 2\theta_{23}A_2 + \frac{3}{2}n\theta_{12}, \\
R^0{}_{231} &= \frac{1}{2}n\theta_{33} + \theta_{13}A_2 - \omega_2\dot{u}_2 + \partial_1\theta_{23} - \omega_3 A_3 - \partial_3\theta_{12} - \partial_2\omega_2 \\
&\quad +\omega_3\dot{u}_3 + d_2\theta_{13} - \frac{1}{2}\theta_{11}n + \hat{\Omega}\theta_{22} + \hat{\theta}_{33}\theta_{23} + \hat{\theta}_{23}\theta_{22} - \hat{\theta}_{23}\theta_{11} \\
&\quad +d_3\theta_{12} - \theta_{33}\hat{\Omega} + \dot{u}_1\omega_1 - \omega_1\hat{\theta}_{22}, \\
R^0{}_{303} &= -\theta_{33}^2 + \omega_2^2 + \dot{u}_2 A_2 - \theta_{23}^2 + \dot{u}_3^2 - 2\Omega_1\theta_{23} + 2\Omega_2\theta_{13} + \partial_3\dot{u}_3 \\
&\quad +\dot{u}_1\hat{\theta}_{33} + \omega_1^2 - \theta_{13}^2 - \partial_0\theta_{33}, \\
R^0{}_{323} &= -\theta_{13}\hat{\theta}_{23} - \partial_3\omega_1 - 2\omega_1\dot{u}_3 - \partial_2\theta_{33} + A_2\theta_{22} + \hat{\theta}_{33}\theta_{12} + \frac{3}{2}n\theta_{13} \\
&\quad +\omega_2\hat{\theta}_{23} + \frac{1}{2}n\omega_2 + 2\theta_{23}A_3 - \theta_{33}A_2 + \hat{\theta}_{33}\omega_3 + \partial_3\theta_{23}, \\
R^0{}_{331} &= -A_2\theta_{12} - \frac{1}{2}\theta_{23}n - \omega_1\hat{\theta}_{23} + \theta_{23}\hat{\theta}_{23} - \partial_3\omega_2 + 2\theta_{23}\hat{\Omega} - 2\omega_2\dot{u}_3 \\
&\quad +\theta_{33}\hat{\theta}_{33} + A_2\omega_3 - \partial_3\theta_{13} - \frac{1}{2}\omega_1 n - \hat{\theta}_{33}\theta_{11} + 2\theta_{13}d_3 + \partial_1\theta_{33}, \\
R^1{}_{212} &= -2\hat{\Omega}\hat{\theta}_{23} + n\hat{\theta}_{23} + 2\Omega_3\omega_3 + \hat{\theta}_{22}^2 - A_3 d_3 + \hat{\theta}_{23}^2 - \partial_2 d_2 \\
&\quad -\frac{1}{4}n^2 - \theta_{11}\theta_{22} - \omega_3^2 + d_2^2 + \partial_1\hat{\theta}_{22} + \theta_{12}^2, \\
R^1{}_{223} &= -\omega_3\omega_1 + \frac{1}{2}\partial_2 n + \partial_2\hat{\theta}_{23} + 2\omega_1\Omega_3 + A_3\hat{\theta}_{33} + 2\hat{\theta}_{23}A_2 + \theta_{22}\theta_{13} \\
&\quad -\partial_3\hat{\theta}_{22} - \omega_3\theta_{23} - \theta_{22}\omega_2 - n d_2 - A_3\hat{\theta}_{22} - \theta_{12}\theta_{23} - \omega_1\theta_{12}, \\
R^1{}_{231} &= -\theta_{13}\omega_3 + n\hat{\theta}_{22} - d_2 A_3 - \hat{\theta}_{23}\hat{\theta}_{22} + \frac{1}{2}\partial_1 n - \hat{\theta}_{33}\hat{\theta}_{23} - \partial_1\hat{\theta}_{23} \\
&\quad -d_3 d_2 + \hat{\theta}_{33}\hat{\Omega} + \theta_{11}\theta_{23} - \theta_{11}\omega_1 + 2\Omega_2\omega_3 - \hat{\theta}_{22}\hat{\Omega} - \theta_{12}\omega_2 \\
&\quad -\theta_{13}\theta_{12} - \omega_2\omega_3 + \partial_2 d_3,
\end{aligned}
$$

$$
\begin{aligned}
R^2{}_{323} &= \partial_2 A_2 - \hat{\theta}_{23}^2 + A_2^2 + \frac{3}{4}n^2 + 2\Omega_1\omega_1 + A_3^2 + \hat{\theta}_{22}\hat{\theta}_{33} - \omega_1^2 \\
&\quad +\theta_{23}^2 - \theta_{22}\theta_{33} + \partial_3 A_3 - \hat{\Omega}n, \\
R^2{}_{331} &= -\omega_2\omega_1 + \omega_1\theta_{13} + \theta_{33}\theta_{12} + \partial_3\hat{\theta}_{23} + 2\omega_1\Omega_2 + 2\hat{\theta}_{23}A_3 + nd_3 \\
&\quad +\omega_2\theta_{23} + A_2\hat{\theta}_{22} - \frac{1}{2}\partial_3 n - \theta_{13}\theta_{23} - A_2\hat{\theta}_{33} - \partial_2\hat{\theta}_{33} + \theta_{33}\omega_3
\end{aligned}
$$

and

$$
\begin{aligned}
R^3{}_{131} &= -\theta_{11}\theta_{33} + \partial_1\hat{\theta}_{33} - \partial_3 d_3 + 2\hat{\Omega}\hat{\theta}_{23} - \omega_2^2 - n\hat{\theta}_{23} + d_3^2 \\
&\quad +\hat{\theta}_{23}^2 + \theta_{13}^2 + \hat{\theta}_{33}^2 + 2\Omega_2\omega_2 - \frac{1}{4}n^2 - d_2A_2.
\end{aligned}
$$

The Ricci tensor, which is a 4×4 symmetric tensor, is formed by contracting the Riemann tensor, *i.e.* $R_{ab} = R^i{}_{aib}$. The components of the Ricci tensor are thus obtained by calculating

$$
\begin{aligned}
R_{00} &= -R^0{}_{101} - R^0{}_{202} - R^0{}_{303}, \\
R_{11} &= R^0{}_{101} + R^1{}_{212} + R^3{}_{131}, \\
R_{22} &= R^0{}_{202} + R^1{}_{212} + R^2{}_{323}, \\
R_{33} &= R^0{}_{303} + R^3{}_{131} + R^2{}_{323}, \\
R_{01} &= -R^0{}_{212} + R^0{}_{331}, \\
R_{02} &= R^0{}_{112} - R^0{}_{323}, \\
R_{03} &= -R^0{}_{131} + R^0{}_{223}, \\
R_{12} &= R^0{}_{102} - R^2{}_{331}, \\
R_{13} &= R^0{}_{103} - R^1{}_{223}
\end{aligned}
$$

and

$$
R_{23} = R^0{}_{203} - R^1{}_{231}.
$$

They are therefore given by

$$
R_{00} = \partial_0\theta_{33} - \dot{u}_1\hat{\theta}_{22} - \dot{u}_1\hat{\theta}_{33} - \dot{u}_2A_2 - \partial_1\dot{u}_1 - \partial_2\dot{u}_2 - \partial_3\dot{u}_3 + \partial_0\theta_{11} + \partial_0\theta_{22}
$$

$$
\begin{aligned}
& -\dot{u}_1^2+\theta_{11}^2-\dot{u}_3^2+\theta_{33}^2-2\omega_1^2+2\theta_{23}^2+2\theta_{13}^2-2\omega_2^2 \\
& +\theta_{22}^2-\dot{u}_2^2-2\omega_3^2+2\theta_{12}^2+\dot{u}_2 d_2+\dot{u}_3 d_3-\dot{u}_3 A_3, \\
R_{01} \quad = \quad & \partial_1\theta_{22}+\partial_1\theta_{33}+2\theta_{23}\hat{\theta}_{23}+\theta_{22}\hat{\theta}_{22}-\partial_2\theta_{12}-\partial_3\theta_{13}-\partial_3\omega_2+\partial_2\omega_3 \\
& +\theta_{33}\hat{\theta}_{33}-\omega_1 n-A_3\theta_{13}-A_3\omega_2+2\omega_3\dot{u}_2-\hat{\theta}_{22}\theta_{11} \\
& +2\theta_{12}d_2-\hat{\theta}_{33}\theta_{11}-A_2\theta_{12}+A_2\omega_3+2\theta_{13}d_3-2\omega_2\dot{u}_3, \\
R_{02} \quad = \quad & -\partial_3\theta_{23}+\partial_2\theta_{33}-n\theta_{13}-\theta_{13}\hat{\theta}_{23}+\theta_{13}\hat{\Omega}-\omega_2\hat{\theta}_{23}-\omega_2\hat{\Omega} \\
& -2\theta_{12}\hat{\theta}_{22}+\partial_2\theta_{11}-\partial_1\theta_{12}+\partial_3\omega_1-\partial_1\omega_3+\theta_{33}A_2-2\omega_3\dot{u}_1 \\
& -\theta_{11}d_2-\hat{\theta}_{33}\omega_3-\hat{\theta}_{33}\theta_{12}-A_2\theta_{22}+2\omega_1\dot{u}_3-2\theta_{23}A_3 \\
& +d_3\theta_{23}-d_3\omega_1+d_2\theta_{22}, \\
R_{03} \quad = \quad & -2\theta_{13}\hat{\theta}_{33}-\partial_2\theta_{23}+2\omega_2\dot{u}_1+A_3\theta_{22}-\omega_3\hat{\Omega}+\partial_3\theta_{11}-\partial_1\theta_{13}+\partial_3\theta_{22} \\
& -\partial_2\omega_1+\partial_1\omega_2-\theta_{12}\hat{\theta}_{23}+n\theta_{12}-\theta_{11}d_3-\theta_{12}\hat{\Omega}-2\omega_1\dot{u}_2 \\
& -2\theta_{23}A_2+\hat{\theta}_{22}\omega_2-\hat{\theta}_{22}\theta_{13}-A_3\theta_{33}+d_3\theta_{33}+d_2\omega_1+d_2\theta_{23}+\omega_3\hat{\theta}_{23}, \\
R_{11} \quad = \quad & d_2^2+\hat{\theta}_{22}^2-d_2A_2-\theta_{11}\theta_{33}+\partial_1\hat{\theta}_{22}+\partial_1\dot{u}_1-\partial_0\theta_{11}+\partial_1\hat{\theta}_{33}-\partial_2 d_2 \\
& -\partial_3 d_3+2\Omega_2\omega_2+\dot{u}_1^2-\theta_{11}^2+d_3^2+\hat{\theta}_{33}^2+2\Omega_3\theta_{12}+2\hat{\theta}_{23}^2-\frac{1}{2}n^2 \\
& -2\Omega_2\theta_{13}-\dot{u}_2 d_2-\dot{u}_3 d_3-A_3 d_3-\theta_{11}\theta_{22}+2\Omega_3\omega_3, \\
R_{12} \quad = \quad & -\frac{1}{2}\dot{u}_3\hat{\Omega}-\theta_{11}\Omega_3+\frac{1}{2}\dot{u}_1 d_2+\frac{1}{2}\dot{u}_3 n-\omega_2\theta_{23}-\theta_{12}\theta_{22}+\theta_{22}\Omega_3-\Omega_2\theta_{23} \\
& -\omega_1\theta_{13}-\theta_{11}\theta_{12}-2\omega_1\Omega_2-\partial_3\hat{\theta}_{23}+\frac{1}{2}\partial_2\dot{u}_1-\partial_0\theta_{12}+\frac{1}{2}\partial_3 n \\
& +\frac{1}{2}\partial_1\dot{u}_2+\partial_2\hat{\theta}_{33}-A_2\hat{\theta}_{22}-\frac{1}{2}\dot{u}_2\hat{\theta}_{22}-\frac{1}{2}\dot{u}_3\hat{\theta}_{23}+A_2\hat{\theta}_{33}-nd_3 \\
& -\theta_{33}\omega_3-\theta_{33}\theta_{12}-2\hat{\theta}_{23}A_3+\theta_{13}\Omega_1+\dot{u}_1\dot{u}_2, \\
R_{13} \quad = \quad & \frac{1}{2}\partial_1\dot{u}_3-\partial_2\hat{\theta}_{23}-\frac{1}{2}\partial_2 n+\frac{1}{2}\partial_3\dot{u}_1+\partial_3\hat{\theta}_{22}-\partial_0\theta_{13}+\frac{1}{2}\dot{u}_1 d_3+\frac{1}{2}\dot{u}_2\hat{\Omega} \\
& -A_3\hat{\theta}_{33}+\theta_{11}\Omega_2-\theta_{11}\theta_{13}-\theta_{13}\theta_{33}+\dot{u}_1\dot{u}_3-\frac{1}{2}\dot{u}_2\hat{\theta}_{23}-\theta_{33}\Omega_2 \\
& -\frac{1}{2}\dot{u}_3\hat{\theta}_{33}-2\omega_1\Omega_3-\theta_{12}\Omega_1-\frac{1}{2}\dot{u}_2 n+\omega_3\theta_{23}+\omega_1\theta_{12}+A_3\hat{\theta}_{22} \\
& +nd_2+\theta_{22}\omega_2-2\hat{\theta}_{23}A_2-\theta_{22}\theta_{13}+\Omega_3\theta_{23},
\end{aligned}
$$

$$
\begin{aligned}
R_{22} &= \dot{u}_2^2 + A_2^2 + \partial_1\hat{\theta}_{22} - \partial_0\theta_{22} - \theta_{22}^2 - \partial_2 d_2 + \partial_2 A_2 + \partial_3 A_3 + \partial_2\dot{u}_2 \\
&\quad + \dot{u}_3 A_3 + 2\Omega_1\theta_{23} - A_3 d_3 - \theta_{11}\theta_{22} - 2\hat{\Omega}\hat{\theta}_{23} + n\hat{\theta}_{23} + 2\Omega_3\omega_3 \\
&\quad + \hat{\theta}_{22}^2 + \frac{1}{2}n^2 - \theta_{22}\theta_{33} - \hat{\Omega}n + \hat{\theta}_{22}\hat{\theta}_{33} + A_3^2 + \dot{u}_1\hat{\theta}_{22} \\
&\quad - 2\Omega_3\theta_{12} + d_2^2 + 2\Omega_1\omega_1, \\
R_{23} &= -\partial_0\theta_{23} - \partial_2 d_3 - \frac{1}{2}\partial_1 n - \theta_{33}\theta_{23} + \frac{1}{2}\partial_3\dot{u}_2 + \frac{1}{2}\partial_2\dot{u}_3 + \partial_1\hat{\theta}_{23} \\
&\quad - \frac{1}{2}\dot{u}_3 A_2 - \theta_{13}\Omega_3 + d_2 A_3 - n\hat{\theta}_{22} + \theta_{13}\omega_3 + \theta_{12}\omega_2 - \frac{1}{2}\dot{u}_2 A_3 \\
&\quad + \hat{\theta}_{23}\hat{\theta}_{22} + \theta_{11}\omega_1 + \hat{\theta}_{33}\hat{\theta}_{23} - \hat{\theta}_{33}\hat{\Omega} + \hat{\theta}_{22}\hat{\Omega} - \theta_{11}\theta_{23} + d_3 d_2 \\
&\quad + \dot{u}_1\hat{\theta}_{23} + \Omega_2\theta_{12} - \theta_{23}\theta_{22} + \theta_{33}\Omega_1 + \dot{u}_3\dot{u}_2 - \theta_{22}\Omega_1 - 2\Omega_2\omega_3
\end{aligned}
$$

and

$$
\begin{aligned}
R_{33} &= A_2^2 + \partial_1\hat{\theta}_{33} - \partial_0\theta_{33} - \partial_3 d_3 + \partial_2 A_2 + \partial_3 A_3 + \partial_3\dot{u}_3 - 2\Omega_1\theta_{23} + 2\Omega_2\omega_2 \\
&\quad - d_2 A_2 - \theta_{11}\theta_{33} + d_3^2 - \theta_{33}^2 + \hat{\theta}_{33}^2 + \dot{u}_1\hat{\theta}_{33} + \dot{u}_2 A_2 \\
&\quad + 2\hat{\Omega}\hat{\theta}_{23} - n\hat{\theta}_{23} + \dot{u}_3^2 + 2\Omega_2\theta_{13} + \frac{1}{2}n^2 - \theta_{22}\theta_{33} - \hat{\Omega}n \\
&\quad + \hat{\theta}_{22}\hat{\theta}_{33} + A_3^2 + 2\Omega_1\omega_1.
\end{aligned}
$$

The ϵ_i that absorb the Ricci tensor components from the Riemann curvature two-forms are given by $\epsilon_i = B_{ij}\bar{\omega}^j$, where $B_{ij} = B_{ji}$ and where

$$
\begin{aligned}
B_{00} &= -\frac{1}{12}R_{33} - \frac{5}{12}R_{00} - \frac{1}{12}R_{22} - \frac{1}{12}R_{11}, \\
B_{01} &= -\frac{1}{2}R_{01}, \\
B_{02} &= -\frac{1}{2}R_{02}, \\
B_{03} &= -\frac{1}{2}R_{03}, \\
B_{11} &= -\frac{5}{12}R_{11} + \frac{1}{12}R_{33} - \frac{1}{12}R_{00} + \frac{1}{12}R_{22}, \\
B_{12} &= -\frac{1}{2}R_{12}, \\
B_{13} &= -\frac{1}{2}R_{13},
\end{aligned}
$$

$$B_{22} = -\frac{5}{12}R_{22} + \frac{1}{12}R_{33} - \frac{1}{12}R_{00} + \frac{1}{12}R_{11},$$
$$B_{23} = -\frac{1}{2}R_{23}$$

and

$$B_{33} = -\frac{1}{12}R_{00} - \frac{5}{12}R_{33} + \frac{1}{12}R_{22} + \frac{1}{12}R_{11}.$$

The Weyl curvature two-forms are given by

$$\begin{aligned}
\mathcal{W}^0{}_1 &= W^0{}_{101}\bar{\omega}^0 \wedge \bar{\omega}^1 + W^0{}_{102}\bar{\omega}^0 \wedge \bar{\omega}^2 + W^0{}_{103}\bar{\omega}^0 \wedge \bar{\omega}^3 \\
&\quad + W^0{}_{112}\bar{\omega}^1 \wedge \bar{\omega}^2 + W^0{}_{123}\bar{\omega}^2 \wedge \bar{\omega}^3 + W^0{}_{131}\bar{\omega}^3 \wedge \bar{\omega}^1, \\
\mathcal{W}^0{}_2 &= W^0{}_{102}\bar{\omega}^0 \wedge \bar{\omega}^1 + W^0{}_{202}\bar{\omega}^0 \wedge \bar{\omega}^2 + W^0{}_{203}\bar{\omega}^0 \wedge \bar{\omega}^3 \\
&\quad + W^0{}_{212}\bar{\omega}^1 \wedge \bar{\omega}^2 + W^0{}_{131}\bar{\omega}^2 \wedge \bar{\omega}^3 - W^0{}_{231}\bar{\omega}^1 \wedge \bar{\omega}^3, \\
\mathcal{W}^0{}_3 &= W^0{}_{103}\bar{\omega}^0 \wedge \bar{\omega}^1 + W^0{}_{203}\bar{\omega}^0 \wedge \bar{\omega}^2 - (W^0{}_{101} + W^0{}_{202})\bar{\omega}^0 \wedge \bar{\omega}^3 \\
&\quad + (-W^0{}_{123} - W^0{}_{231})\bar{\omega}^1 \wedge \bar{\omega}^2 + W^0{}_{112}\bar{\omega}^2 \wedge \bar{\omega}^3 + W^0{}_{212}\bar{\omega}^3 \wedge \bar{\omega}^1, \\
\mathcal{W}^1{}_2 &= -W^0{}_{112}\bar{\omega}^0 \wedge \bar{\omega}^1 - W^0{}_{212}\bar{\omega}^0 \wedge \bar{\omega}^2 + (W^0{}_{123} + W^0{}_{231})\bar{\omega}^0 \wedge \bar{\omega}^3 \\
&\quad - (W^0{}_{101} + W^0{}_{202})\bar{\omega}^1 \wedge \bar{\omega}^2 + W^0{}_{103}\bar{\omega}^2 \wedge \bar{\omega}^3 + W^0{}_{203}\bar{\omega}^1 \wedge \bar{\omega}^3, \\
\mathcal{W}^2{}_3 &= -W^0{}_{123}\bar{\omega}^0 \wedge \bar{\omega}^1 - W^0{}_{131}\bar{\omega}^0 \wedge \bar{\omega}^2 - W^0{}_{112}\bar{\omega}^0 \wedge \bar{\omega}^3 \\
&\quad + W^0{}_{103}\bar{\omega}^1 \wedge \bar{\omega}^2 + W^0{}_{101}\bar{\omega}^2 \wedge \bar{\omega}^3 + W^0{}_{102}\bar{\omega}^3 \wedge \bar{\omega}^1
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{W}^3{}_1 &= -W^0{}_{131}\bar{\omega}^0 \wedge \bar{\omega}^1 - W^0{}_{231}\bar{\omega}^0 \wedge \bar{\omega}^2 - W^0{}_{212}\bar{\omega}^0 \wedge \bar{\omega}^3 \\
&\quad + W^0{}_{203}\bar{\omega}^1 \wedge \bar{\omega}^2 + W^0{}_{102}\bar{\omega}^2 \wedge \bar{\omega}^3 + W^0{}_{202}\bar{\omega}^3 \wedge \bar{\omega}^1,
\end{aligned}$$

where

$$\begin{aligned}
W^0{}_{101} &= -\frac{1}{6}\partial_1\hat{\theta}_{22} - \frac{1}{3}\partial_0\theta_{11} - \frac{1}{6}\partial_1\hat{\theta}_{33} + \frac{1}{6}\partial_0\theta_{22} + \frac{1}{6}\partial_0\theta_{33} + \frac{1}{3}\partial_1\dot{u}_1 + \frac{1}{6}\partial_2 d_2 \\
&\quad + \frac{1}{6}\partial_3 d_3 + \frac{1}{3}\partial_2 A_2 + \frac{1}{3}\partial_3 A_3 - \frac{1}{6}\partial_2\dot{u}_2 - \frac{1}{6}\partial_3\dot{u}_3 + \frac{1}{3}A_2^2 + \frac{1}{3}A_3^2 \\
&\quad + \frac{1}{6}\theta_{33}^2 - \frac{1}{6}\dot{u}_3^2 - \frac{1}{3}\theta_{11}^2 + \frac{1}{3}\dot{u}_1^2 - \frac{1}{6}\hat{\theta}_{22}^2 - \frac{1}{6}d_2^2 - \frac{1}{6}\dot{u}_3 A_3
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{3}\Omega_3\omega_3+\frac{1}{6}\theta_{11}\theta_{22}+\frac{1}{6}A_3d_3-\frac{1}{3}\dot{u}_3d_3-\frac{1}{3}\dot{u}_2d_2-\frac{1}{3}\hat{\Omega}n-\frac{1}{3}\Omega_2\omega_2\\
& -\frac{1}{6}\dot{u}_2A_2-\frac{1}{6}\dot{u}_1\hat{\theta}_{33}+\frac{1}{3}n^2+\frac{2}{3}\Omega_1\omega_1-\frac{1}{3}\theta_{22}\theta_{33}+\frac{1}{3}\hat{\theta}_{22}\hat{\theta}_{33}+\frac{1}{6}\theta_{11}\theta_{33}\\
& +\frac{1}{6}d_2A_2-\frac{1}{6}\dot{u}_1\hat{\theta}_{22}-\Omega_2\theta_{13}+\frac{1}{3}\omega_3^2-\frac{1}{3}\theta_{12}^2-\frac{1}{6}\hat{\theta}_{33}^2-\frac{1}{3}\theta_{13}^2\\
& +\frac{1}{3}\omega_2^2+\frac{1}{6}\theta_{22}^2-\frac{1}{6}\dot{u}_2^2-\frac{2}{3}\hat{\theta}_{23}^2+\Omega_3\theta_{12}-\frac{2}{3}\omega_1^{2}+\frac{2}{3}\theta_{23}^2-\frac{1}{6}d_3^2,\\
W^0{}_{102} \;=\; & -\partial_0\omega_3+\frac{1}{2}\partial_3\hat{\theta}_{23}+\frac{3}{4}\partial_2\dot{u}_1-\frac{1}{2}\partial_0\theta_{12}-\frac{1}{4}\partial_3n-\frac{1}{4}\partial_1\dot{u}_2-\frac{1}{2}\partial_2\hat{\theta}_{33}\\
& -\omega_3\theta_{22}-\omega_3\theta_{11}-\omega_2\Omega_1-\frac{1}{2}\theta_{11}\theta_{12}-\frac{1}{4}\dot{u}_1d_2-\frac{3}{4}\dot{u}_3\hat{\theta}_{23}+\frac{1}{4}\dot{u}_3\hat{\Omega}\\
& +\frac{3}{2}\omega_2\theta_{23}+2\omega_1\Omega_2+\frac{3}{2}\omega_1\theta_{13}-\omega_2\omega_1-\theta_{13}\theta_{23}+\frac{1}{2}\theta_{33}\omega_3+\frac{1}{2}A_2\hat{\theta}_{22}\\
& +\hat{\theta}_{23}A_3-\frac{1}{2}\theta_{12}\theta_{22}+\frac{1}{4}\dot{u}_3n-\frac{1}{2}\theta_{11}\Omega_3+\frac{1}{2}\dot{u}_1\dot{u}_2+\frac{1}{2}\theta_{13}\Omega_1-\frac{3}{4}\dot{u}_2\hat{\theta}_{22}\\
& +\frac{1}{2}\theta_{33}\theta_{12}-\frac{1}{2}A_2\hat{\theta}_{33}+\frac{1}{2}nd_3+\frac{1}{2}\theta_{22}\Omega_3-\frac{1}{2}\Omega_2\theta_{23},\\
W^0{}_{103} \;=\; & -\frac{1}{4}\partial_1\dot{u}_3+\frac{1}{2}\partial_2\hat{\theta}_{23}+\partial_0\omega_2+\frac{1}{4}\partial_2n+\frac{3}{4}\partial_3\dot{u}_1-\frac{1}{2}\partial_3\hat{\theta}_{22}-\frac{1}{2}\partial_0\theta_{13}\\
& +\hat{\theta}_{23}A_2+\omega_2\theta_{11}+\omega_2\theta_{33}-\frac{1}{2}\theta_{12}\Omega_1-\frac{1}{2}\theta_{22}\omega_2+\frac{1}{2}A_3\hat{\theta}_{33}-\frac{1}{4}\dot{u}_2\hat{\Omega}\\
& -\frac{3}{4}\dot{u}_3\hat{\theta}_{33}-\omega_3\Omega_1-\frac{3}{2}\omega_3\theta_{23}-\frac{1}{4}\dot{u}_1d_3+2\omega_1\Omega_3-\frac{3}{2}\omega_1\theta_{12}+\frac{1}{2}\dot{u}_1\dot{u}_3\\
& -\omega_3\omega_1-\frac{1}{2}\theta_{33}\Omega_2-\frac{1}{2}\theta_{11}\theta_{13}+\frac{1}{2}\theta_{22}\theta_{13}-\theta_{12}\theta_{23}-\frac{3}{4}\dot{u}_2\hat{\theta}_{23}-\frac{1}{2}\theta_{13}\theta_{33}\\
& +\frac{1}{2}\theta_{11}\Omega_2+\frac{1}{2}\Omega_3\theta_{23}-\frac{1}{4}\dot{u}_2n-\frac{1}{2}A_3\hat{\theta}_{22}-\frac{1}{2}nd_2,\\
W^0{}_{112} \;=\; & -\omega_1\dot{u}_3+\frac{1}{2}A_2\theta_{22}+\theta_{23}A_3+\frac{1}{2}d_3\theta_{23}-\frac{1}{2}\partial_2\theta_{33}-\frac{1}{2}\omega_2\hat{\Omega}+\frac{1}{2}\hat{\theta}_{33}\theta_{12}\\
& -\frac{1}{2}\partial_1\theta_{12}+n\theta_{13}+\frac{1}{2}d_2\theta_{22}+\frac{1}{2}\partial_2\theta_{11}+\frac{1}{2}\hat{\theta}_{33}\omega_3+\frac{1}{2}\omega_2\hat{\theta}_{23}-\frac{1}{2}d_3\omega_1\\
& -\frac{1}{2}\theta_{11}d_2+\frac{1}{2}\partial_3\theta_{23}-\frac{1}{2}\theta_{33}A_2-\frac{1}{2}\partial_3\omega_1+\frac{1}{2}\theta_{13}\hat{\Omega}-\omega_3\dot{u}_1+\frac{1}{2}n\omega_2\\
& -\theta_{12}\hat{\theta}_{22}-\frac{3}{2}\theta_{13}\hat{\theta}_{23}-\frac{1}{2}\partial_1\omega_3,\\
W^0{}_{123} \;=\; & \theta_{23}\hat{\theta}_{22}+\omega_3A_3+\partial_2\omega_2+\theta_{11}n+\omega_1\hat{\theta}_{33}+\theta_{33}\hat{\theta}_{23}+\omega_2A_2\\
& -\theta_{13}A_2-\frac{1}{2}n\theta_{33}-\partial_2\theta_{13}-\hat{\theta}_{23}\theta_{22}+\omega_1\hat{\theta}_{22}+\theta_{12}A_3+\partial_3\omega_3
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{2}n\theta_{22} + \partial_3\theta_{12} - \hat{\theta}_{33}\theta_{23} - 2\dot{u}_1\omega_1, \\
{W^0}_{131} \;=\; & -(\omega_2\dot{u}_1 + \frac{1}{2}n\omega_3 + \frac{1}{2}A_3\theta_{33} + \frac{1}{2}\partial_1\omega_2 - \frac{1}{2}\partial_1\theta_{13} + \frac{1}{2}\partial_2\omega_1 - n\theta_{12} \\
& -\frac{1}{2}\partial_3\theta_{22} + \theta_{23}A_2 + \frac{1}{2}d_2\theta_{23} - \frac{1}{2}\hat{\theta}_{22}\omega_2 + \frac{1}{2}d_2\omega_1 + \frac{1}{2}d_3\theta_{33} + \omega_1\dot{u}_2 \\
& -\theta_{13}\hat{\theta}_{33} - \frac{1}{2}\omega_3\hat{\theta}_{23} + \frac{1}{2}\partial_3\theta_{11} - \frac{1}{2}\omega_3\hat{\Omega} - \frac{3}{2}\theta_{12}\hat{\theta}_{23} + \frac{1}{2}\hat{\theta}_{22}\theta_{13} - \frac{1}{2}\theta_{12}\hat{\Omega} \\
& -\frac{1}{2}\theta_{11}d_3 - \frac{1}{2}A_3\theta_{22} + \frac{1}{2}\partial_2\theta_{23}), \\
{W^0}_{202} \;=\; & -\frac{1}{6}\partial_1\hat{\theta}_{22} + \frac{1}{6}\partial_0\theta_{11} + \frac{1}{3}\partial_1\hat{\theta}_{33} - \frac{1}{3}\partial_0\theta_{22} + \frac{1}{6}\partial_0\theta_{33} - \frac{1}{6}\partial_1\dot{u}_1 + \frac{1}{6}\partial_2 d_2 \\
& -\frac{1}{3}\partial_3 d_3 - \frac{1}{6}\partial_2 A_2 - \frac{1}{6}\partial_3 A_3 + \frac{1}{3}\partial_2\dot{u}_2 - \frac{1}{6}\partial_3\dot{u}_3 - \frac{1}{6}A_2^2 - \frac{1}{6}A_3^2 \\
& +\frac{1}{6}\theta_{33}^2 - \frac{1}{6}\dot{u}_3^2 + \frac{1}{6}\theta_{11}^2 - \frac{1}{6}\dot{u}_1^2 - \frac{1}{6}\hat{\theta}_{22}^2 - \frac{1}{6}d_2^2 + \frac{1}{3}\dot{u}_3 A_3 \\
& -\frac{1}{3}\Omega_3\omega_3 + \frac{1}{6}\theta_{11}\theta_{22} + \frac{1}{6}A_3 d_3 + \frac{1}{6}\dot{u}_3 d_3 + \frac{1}{6}\dot{u}_2 d_2 + \frac{1}{6}\hat{\Omega}n + \frac{2}{3}\Omega_2\omega_2 \\
& -\frac{1}{2}n\hat{\theta}_{23} + \Omega_1\theta_{23} - \frac{1}{6}\dot{u}_2 A_2 - \frac{1}{6}\dot{u}_1\hat{\theta}_{33} - \frac{1}{6}n^2 + \hat{\Omega}\hat{\theta}_{23} - \frac{1}{3}\Omega_1\omega_1 \\
& +\frac{1}{6}\theta_{22}\theta_{33} - \frac{1}{6}\hat{\theta}_{22}\hat{\theta}_{33} - \frac{1}{3}\theta_{11}\theta_{33} - \frac{1}{3}d_2 A_2 + \frac{1}{3}\dot{u}_1\hat{\theta}_{22} + \frac{1}{3}\omega_3^2 - \frac{1}{3}\theta_{12}^2 \\
& +\frac{1}{3}\hat{\theta}_{33}^2 + \frac{2}{3}\theta_{13}^2 - \frac{2}{3}\omega_2^2 - \frac{1}{3}\theta_{22}^2 + \frac{1}{3}\dot{u}_2^2 + \frac{1}{3}\hat{\theta}_{23}^2 - \Omega_3\theta_{12} \\
& +\frac{1}{3}\omega_1^2 - \frac{1}{3}\theta_{23}^2 + \frac{1}{3}d_3^2, \\
{W^0}_{203} \;=\; & -\partial_0\omega_1 - \omega_2\omega_3 - \theta_{13}\theta_{12} + \frac{1}{2}n\hat{\theta}_{22} - \frac{3}{4}\dot{u}_3 A_2 + \frac{1}{2}\Omega_2\theta_{12} + \frac{1}{2}\dot{u}_3\dot{u}_2 \\
& +\frac{1}{2}\theta_{11}\theta_{23} - \frac{1}{2}\hat{\theta}_{33}\hat{\theta}_{23} + \frac{1}{4}\dot{u}_2 A_3 - \frac{1}{2}\theta_{11}\omega_1 + \frac{1}{2}\theta_{33}\Omega_1 - \frac{1}{2}\theta_{33}\theta_{23} - \frac{1}{2}\theta_{23}\theta_{22} \\
& -\frac{1}{2}\partial_0\theta_{23} + \frac{1}{2}\partial_2 d_3 + \frac{1}{4}\partial_1 n + \frac{3}{4}\partial_3\dot{u}_2 - \frac{1}{4}\partial_2\dot{u}_3 - \frac{1}{2}\partial_1\hat{\theta}_{23} + \frac{1}{2}\theta_{13}\omega_3 \\
& +\frac{1}{2}\theta_{12}\omega_2 - \frac{1}{2}d_3 d_2 - \frac{1}{2}\hat{\theta}_{23}\hat{\theta}_{22} - \frac{1}{2}d_2 A_3 - \frac{1}{2}\theta_{13}\Omega_3 - \frac{1}{2}\hat{\theta}_{22}\hat{\Omega} \\
& +\frac{1}{2}\dot{u}_1\hat{\theta}_{23} - \frac{1}{2}\theta_{22}\Omega_1 - \omega_1\theta_{33} + \frac{1}{2}\dot{u}_1 n - \omega_1\theta_{22} + \Omega_3\omega_2 + \frac{1}{2}\hat{\theta}_{33}\hat{\Omega}, \\
{W^0}_{212} \;=\; & \frac{1}{2}\hat{\theta}_{22}\theta_{11} - \frac{1}{2}\hat{\theta}_{33}\theta_{11} + \theta_{13}d_3 - \frac{1}{2}\partial_1\theta_{22} + 2\theta_{23}\hat{\Omega} - \frac{1}{2}A_2\theta_{12} + \frac{1}{2}A_3\theta_{13} \\
& +\frac{1}{2}A_2\omega_3 - \theta_{12}d_2 + \frac{1}{2}\partial_2\theta_{12} - \frac{1}{2}\theta_{23}n - \omega_3\dot{u}_2 - \frac{1}{2}\theta_{22}\hat{\theta}_{22} - \frac{1}{2}\partial_2\omega_3
\end{aligned}
$$

$$+\frac{1}{2}\partial_1\theta_{33} - \frac{1}{2}\partial_3\omega_2 - \omega_2\dot{u}_3 - \omega_1\hat{\theta}_{23} + \frac{1}{2}\theta_{33}\hat{\theta}_{33} - \frac{1}{2}\partial_3\theta_{13} + \frac{1}{2}A_3\omega_2$$

and

$$\begin{aligned} W^0{}_{231} &= -(-\hat{\Omega}\theta_{22} + \theta_{33}\hat{\Omega} - \omega_1\hat{\theta}_{33} - \partial_1\theta_{23} - \hat{\theta}_{33}\theta_{23} + \frac{1}{2}\theta_{11}n - \partial_3\omega_3 - \omega_2 A_2 \\ &\quad +\partial_3\theta_{12} + d_2\omega_2 - d_3\theta_{12} - \partial_1\omega_1 - \theta_{13}A_2 - d_2\theta_{13} + 2\omega_2\dot{u}_2 + \hat{\theta}_{23}\theta_{11} \\ &\quad +d_3\omega_3 - \frac{1}{2}n\theta_{33} - \hat{\theta}_{23}\theta_{22}). \end{aligned}$$

These equalities are equivalent to the following ones, taking into account the Jacobi identities:,

$$\begin{aligned} W^0{}_{101} &= R^0{}_{101} + \frac{1}{6}(2R_{00} - 2R_{11} + R_{22} + R_{33}), \\ W^0{}_{102} &= R^0{}_{102} - \frac{1}{2}R_{12}, \\ W^0{}_{103} &= R^0{}_{103} - \frac{1}{2}R_{13}, \\ W^0{}_{112} &= R^0{}_{112} - \frac{1}{2}R_{02}, \\ W^0{}_{123} &= R^0{}_{123}, \\ W^0{}_{131} &= R^0{}_{131} + \frac{1}{2}R_{03}, \\ W^0{}_{202} &= R^0{}_{202} + \frac{1}{6}(2R_{00} + R_{11} - 2R_{22} + R_{33}), \\ W^0{}_{203} &= R^0{}_{203} - \frac{1}{2}R_{23} \end{aligned}$$

and

$$W^0{}_{231} = R^0{}_{231}.$$

3.5 Einstein field equations

For a perfect fluid, with μ being the energy density and p being the pressure, the energy-momentum tensor is given by

$$T_{ab}\bar{\omega}^a \otimes \bar{\omega}^b = \mu\bar{\omega}^0 \otimes \bar{\omega}^0 + p\sum_{\alpha} \bar{\omega}^\alpha \otimes \bar{\omega}^\alpha. \tag{3.23}$$

With the sign convention used in (3.22) for the Riemann curvature tensor, the Einstein field equations are given by

$$R_{ab} - \frac{R}{2}g_{ab} - \Lambda g_{ab} = T_{ab}.$$

An equivalent expression is given by

$$R_{ab} = -T_{ab} + \frac{T}{2}g_{ab} - \Lambda g_{ab},$$

where $T = T^a{}_a$ is the trace of the energy-momentum tensor. It follows then that the Einstein field equations, for a perfect fluid, are given by

$$R_{00} = -\tfrac{\mu}{2} - \tfrac{3p}{2} + \Lambda,$$

$$R_{11} = R_{22} = R_{33} = -\tfrac{\mu}{2} + \tfrac{p}{2} - \Lambda,$$

and

$$R_{01} = R_{02} = R_{03} = R_{12} = R_{13} = R_{23} = 0.$$

3.6 Integrability conditions on the energy density

Closely related to the Jacobi identities are the integrability conditions on the energy density, μ. They are determined by taking all the commutation relations on μ. This is easily computed using differential forms, by making use of the identity:

$$\mathrm{d}^2\mu = 0.$$

We notice that, since p is a function of μ, the preceding implies

$$0 = \mathrm{d}\left(\frac{\mathrm{d}p}{\mu + p}\right). \tag{3.24}$$

Using the contracted Bianchi identities, we obtain

$$0 = \mathrm{d}\left(p'\theta\bar{\omega}^0 + \dot{u}_\alpha\bar{\omega}^\alpha\right).$$

The evaluation of this equation implies that a particular two-form must vanish. Therefore, the six components of this two-form must also vanish, and so

$$\begin{aligned}
\partial_0\dot{u}_1 &= \frac{1}{p'}\left(-\dot{u}_1\theta p''(\mu+p) + \dot{u}_2\omega_3 p' - \dot{u}_3\theta_{13}p' + p'^2\partial_1\theta_{11} - p'\theta\dot{u}_1 + p'^2\theta\dot{u}_1\right.\\
&\quad \left. -\dot{u}_3\Omega_2 p' - \dot{u}_3\omega_2 p' + \dot{u}_2\Omega_3 p' - \dot{u}_2\theta_{12}p'\right), \qquad (3.25)\\
\partial_0\dot{u}_2 &= \frac{1}{p'}\left(-\dot{u}_2\theta p''(\mu+p) + p'^2\partial_2\theta - \dot{u}_1\theta_{12}p' - \dot{u}_1\omega_3 p' + \dot{u}_2 p'^2\theta - \dot{u}_2 p'\theta_{22}\right.\\
&\quad \left. -\dot{u}_3\theta_{23}p' - \dot{u}_1\Omega_3 p' + \dot{u}_3\Omega_1 p' + \dot{u}_3\omega_1 p'\right)\\
\partial_0\dot{u}_3 &= \frac{1}{p'}\left(-\dot{u}_3\theta p''(\mu+p) - \dot{u}_1\theta_{13}p' + \dot{u}_1\omega_2 p' - \dot{u}_2\Omega_1 p' - \dot{u}_2\theta_{23}p' + \dot{u}_1\Omega_2 p'\right.\\
&\quad \left. +\dot{u}_3 p'^2\theta - \dot{u}_3 p'\theta_{33} + p'^2\partial_3\theta - \dot{u}_2\omega_1 p'\right),\\
\partial_1\dot{u}_2 &= \partial_2\dot{u}_1 - 2\omega_3\theta p' + \dot{u}_3\hat{\Omega} - \dot{u}_2\hat{\theta}_{22} - \dot{u}_3\hat{\theta}_{23} - \dot{u}_1 d_2, \qquad (3.26)\\
\partial_2\dot{u}_3 &= -2\omega_1\theta p' - \dot{u}_3 A_2 + \dot{u}_2 A_3 + \partial_3\dot{u}_2 + \dot{u}_1 n
\end{aligned}$$

and

$$\partial_3\dot{u}_1 = -2\omega_2\theta p' + \dot{u}_2\hat{\theta}_{23} + \partial_1\dot{u}_3 + \dot{u}_1 d_3 + \dot{u}_3\hat{\theta}_{33} + \dot{u}_2\hat{\Omega}. \tag{3.27}$$

We note that the quantity being differentiated in (3.24) is the negative of the differential of the function F of White and Collins (1984).

3.7 Tetrad determination

The Lorentzian metric (3.1) enables us to construct an orthonormal tetrad with axes $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ in the tangent space of each point of the spacetime. The tetrads are not uniquely determined. The group of freedom in their orientation, ignoring reflections, is the full Lorentz group SO(3,1,$\mathbf{R}$). We now require that the $\vec{e}_0-$axis of each tetrad be aligned with the unique future-pointing unit timelike eigenvector of the energy-momentum tensor (3.23) of a perfect fluid. The vector $\vec{e}_0$ is then the fluid flow velocity vector of the fluid. This restricts the possible tetrads. The indeterminacy in their definition is now isomorphic to SO(3,0,$\mathbf{R}$), corresponding to rotations of the spacelike triad $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. The structure equations are now those given in section 3.1.

We shall study perfect fluids that are shear-free; these are fluids where the expansion tensor of the fluid possesses the simple form

$$\theta_{\alpha\beta} = \frac{\theta}{3}\delta_{\alpha\beta}. \tag{3.28}$$

Since the fluids that are of particular interest to us are rotating fluids, we now choose the $\vec{e}_1-$axis in such a way that it is parallel to the vorticity vector of the fluid. This choice involves solely rotations of the triad $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Since equation (3.28) is invariant under such rotations, this choice of $\vec{e}_1$ does not impose any restrictions on the spacetime. We thus have that $\omega_2 = \omega_3 = 0$. The indeterminacy in the tetrad is now SO(1,0,$\mathbf{R}$), representing rotations of the dyad $(\vec{e}_2, \vec{e}_3)$, together with a possible reflection $\vec{e}_1 \mapsto -\vec{e}_1$ and a reflection in the $(2-3)$ space, $(\vec{e}_2, \vec{e}_3) \mapsto (-\vec{e}_2, \vec{e}_3)$. The Jacobi identity (3.18), the integrability condition (3.27), the shear-free condition and the condition that $\omega_2 = \omega_3 = 0 \neq \omega_1$ requires that Ω_3 be zero. Similarly, equation (3.19), equation (3.26), the shear-free condition and $\omega_2 = \omega_3 = 0$ imply that $\Omega_2 = 0$.

At this point, $\bar{\omega}^0$ and $\bar{\omega}^1$ are determined. Let α be a parameter representing the rotational freedom left in the determination of $\bar{\omega}^2$ and $\bar{\omega}^3$. Let $\bar{\omega}^{2\prime}$ and $\bar{\omega}^{3\prime}$ be another choice for these directions. The relation between $(\bar{\omega}^2, \bar{\omega}^3)$ and $(\bar{\omega}^{2\prime}, \bar{\omega}^{3\prime})$ is

$$\begin{pmatrix} \bar{\omega}^{2\prime} \\ \bar{\omega}^{3\prime} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \bar{\omega}^{2} \\ \bar{\omega}^{3} \end{pmatrix}.$$

It follows then that the structure equations for $\bar{\omega}^{2\prime}$ and $\bar{\omega}^{3\prime}$ in terms of the kinematic quantities associated with $\bar{\omega}^2$ and $\bar{\omega}^3$ are

$$\begin{aligned} \mathrm{d}\bar{\omega}^{2\prime} &= \mathrm{d}\alpha \wedge \bar{\omega}^{3\prime} + \frac{\theta}{3}\bar{\omega}^0 \wedge \bar{\omega}^{2\prime} + (\omega_1 + \Omega^1)\bar{\omega}^0 \wedge \bar{\omega}^{3\prime} \\ &\quad + (\hat{\theta}_{22}\cos^2\alpha + 2\hat{\theta}_{23}\cos\alpha\sin\alpha + \hat{\theta}_{33}\sin^2\alpha)\bar{\omega}^1 \wedge \bar{\omega}^{2\prime} \\ &\quad + (A_2\sin\alpha - A_3\cos\alpha)\bar{\omega}^{2\prime} \wedge \bar{\omega}^{3\prime} \\ &\quad + \left[(\hat{\theta}_{22} - \hat{\theta}_{33})\cos\alpha\sin\alpha - \hat{\Omega} + \hat{\theta}_{23}(\sin^2\alpha - \cos^2\alpha)\right]\bar{\omega}^{3\prime} \wedge \bar{\omega}^1 \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}\bar{\omega}^{3\prime} &= -\mathrm{d}\alpha \wedge \bar{\omega}^{2\prime} - (\omega_1 + \Omega^1)\bar{\omega}^0 \wedge \bar{\omega}^{2\prime} + \frac{\theta}{3}\bar{\omega}^0 \wedge \bar{\omega}^{3\prime} \\ &\quad + \left[(\hat{\theta}_{33} - \hat{\theta}_{22})\cos\alpha\sin\alpha - \hat{\Omega} + \hat{\theta}_{23}(\cos^2\alpha - \sin^2\alpha)\right]\bar{\omega}^1 \wedge \bar{\omega}^{2\prime} \\ &\quad + (A_2\cos\alpha + A_3\sin\alpha)\bar{\omega}^{2\prime} \wedge \bar{\omega}^{3\prime} \\ &\quad + (-\hat{\theta}_{22}\sin^2\alpha + 2\hat{\theta}_{23}\cos\alpha\sin\alpha - \hat{\theta}_{33}\cos^2\alpha)\bar{\omega}^{3\prime} \wedge \bar{\omega}^1. \end{aligned}$$

If we let ω_1' and Ω_1' be the kinematic quantities analogous to ω_1 and Ω_1, then

$$\begin{aligned} 2(\omega_1' + \Omega_1')\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^{2\prime} \wedge \bar{\omega}^{3\prime} &= \bar{\omega}^1 \wedge \bar{\omega}^{2\prime} \wedge \mathrm{d}\bar{\omega}^{2\prime} + \bar{\omega}^1 \wedge \bar{\omega}^{3\prime} \wedge \mathrm{d}\bar{\omega}^{3\prime} \\ &= 2\mathrm{d}\alpha \wedge \bar{\omega}^1 \wedge \bar{\omega}^{2\prime} \wedge \bar{\omega}^{3\prime} + 2(\omega_1 + \Omega_1)\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^{2\prime} \wedge \bar{\omega}^{3\prime}. \end{aligned}$$

We can therefore require that $\omega_1' + \Omega_1' = 0$, provided that we require that $\partial_0\alpha + (\omega_1 + \Omega_1) = 0$. This result that $\omega_1 + \Omega_1$ can be set to zero also holds in the situation when the fluid has non-vanishing shear. There is still freedom in the choice of α, provided that we maintain the constraint that $\partial_0\alpha = 0$. We compute the effect of the rotation on the quantity $\hat{\theta}_{23}$ as follows:

$$2\hat{\theta}_{23}'\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^{2\prime} \wedge \bar{\omega}^{3\prime} = -\mathrm{d}\bar{\omega}^{2\prime} \wedge \bar{\omega}^0 \wedge \bar{\omega}^{2\prime} + \mathrm{d}\bar{\omega}^{3\prime} \wedge \bar{\omega}^0 \wedge \bar{\omega}^{3\prime}$$
$$= \; 2\Big((\hat{\theta}_{33} - \hat{\theta}_{22})\sin\alpha\cos\alpha + \hat{\theta}_{23}(\cos^2\alpha - \sin^2\alpha)\Big)\,\bar{\omega}^0 \wedge \bar{\omega}^1 \wedge \bar{\omega}^{2\prime} \wedge \bar{\omega}^{3\prime}.$$

We can set $\hat{\theta}_{23}$ to be zero, by choosing α such that

$$(\hat{\theta}_{33} - \hat{\theta}_{22})\sin\alpha\cos\alpha + \hat{\theta}_{23}(\cos^2\alpha - \sin^2\alpha) = 0. \tag{3.29}$$

Of course, when $\hat{\theta}_{22} = \hat{\theta}_{23} = \hat{\theta}_{33} = 0$ no constraints are thereby imposed on α. Apart from this special situation, the tetrad $\{\vec{e}_a\}$ is then completely determined, up to possible reflections. This is allowed provided that equation (3.29) is propagated along the fluid flow without introducing new constraints. For the present situation, equations (3.17), (3.20) and (3.21) reduce to

$$\partial_0\hat{\theta}_{33} = -\frac{\theta}{3}\hat{\theta}_{33} + \frac{1}{3}\partial_1\theta + \frac{\theta}{3}\dot{u}_1,$$
$$\partial_0\hat{\theta}_{23} = -\frac{\theta}{3}\hat{\theta}_{23}$$

and

$$\partial_0\hat{\theta}_{22} = -\frac{\theta}{3}\hat{\theta}_{22} + \frac{1}{3}\partial_1\theta + \frac{\theta}{3}\dot{u}_1.$$

Using these expressions, the differentiation of equation (3.29) along $\vec{e}_0$ yields the identity $0 = 0$. We remark that if the fluid possesses shear, then this differentiation of (3.29) will, in general, introduce new constraints. We also note that, prior to setting $\omega_1 + \Omega_1 = 0$, we could have set $\hat{\theta}_{23} = 0$. Propagating this expression in the fluid flow direction would have forced $\omega_1 + \Omega_1$ to vanish, without loss of generality, except when $\hat{\theta}_{22} = \hat{\theta}_{33}$. The tetrad is now fixed, up to reflection of axes, except when $\hat{\theta}_{22} = \hat{\theta}_{33}$. When it is the case that $\hat{\theta}_{22} = \hat{\theta}_{33}$, remembering that $\hat{\theta}_{23} = 0$ was imposed, we can still set $\omega_1 + \Omega_1$ to zero, but there are no restrictions on α, *i.e.* there is the full freedom of rotation of the 2-3 dyad.

Chapter 4

The Petrov classification of the Weyl tensor

Que diable allait-il faire dans cette galère?

Molière

IN THIS chapter, we show how the Weyl tensor can be classified using results from linear algebra. We refer to Grossman (1984) for an elementary introduction to the concepts from linear algebra that we shall need. In particular, for a two-dimensional matrix with a double eigenvalue for which the associated eigenspace is one-dimensional, Grossman (1984) shows how to compute a vector that is linearly independent of this eigenspace. We use that example in Grossman (1984) as a guide for our calculations for the cases when the dimension of the eigenspace associated with a repeated eigenvalue is less than the multiplicity of the eigenvalue. The other calculations are from the present author. Hungerford (1974) is a more advanced reference about algebra in general and linear algebra in particular. Kramer et al. (1980) provide more information about the Petrov classification.

The fluid flow vector may be employed to split the Weyl tensor into two tensors: the electric part, denoted by E_{ab}, and the magnetic part, denoted by H_{ab}. More information about the electric and magnetic parts of the Weyl tensor can be found in Kramer et al. (1980) and in Ellis (1971). These tensors are symmetric and trace-free. Relative to a frame in which $\vec{e}_0$ is defined to be the fluid flow tangent vector, they satisfy $E_{0a} = H_{0a} = 0$ and obey

$$\begin{aligned}
\mathcal{W}^0{}_1 &= -E_{11}\bar{\omega}^0 \wedge \bar{\omega}^1 - E_{12}\bar{\omega}^0 \wedge \bar{\omega}^2 - E_{13}\bar{\omega}^0 \wedge \bar{\omega}^3 \\
&\quad -H_{11}\bar{\omega}^2 \wedge \bar{\omega}^3 - H_{12}\bar{\omega}^3 \wedge \bar{\omega}^1 - H_{13}\bar{\omega}^1 \wedge \bar{\omega}^2, \\
\mathcal{W}^0{}_2 &= -E_{12}\bar{\omega}^0 \wedge \bar{\omega}^1 - E_{22}\bar{\omega}^0 \wedge \bar{\omega}^2 - E_{23}\bar{\omega}^0 \wedge \bar{\omega}^3 \\
&\quad -H_{12}\bar{\omega}^2 \wedge \bar{\omega}^3 - H_{22}\bar{\omega}^3 \wedge \bar{\omega}^1 - H_{23}\bar{\omega}^1 \wedge \bar{\omega}^2
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{W}^0{}_3 &= -E_{13}\bar{\omega}^0 \wedge \bar{\omega}^1 - E_{23}\bar{\omega}^0 \wedge \bar{\omega}^2 - E_{33}\bar{\omega}^0 \wedge \bar{\omega}^3 \\
&\quad -H_{13}\bar{\omega}^2 \wedge \bar{\omega}^3 - H_{23}\bar{\omega}^3 \wedge \bar{\omega}^1 - H_{33}\bar{\omega}^1 \wedge \bar{\omega}^3.
\end{aligned}$$

Since both the electric part and the magnetic part of the Weyl tensor are trace-free, it follows that $E_{11} + E_{22} + E_{33} = 0$ and $H_{11} + H_{22} + H_{33} = 0$. This enables us to make the identifications:

$$\begin{aligned}
E_{11} &= -W^0{}_{101}, \\
E_{12} &= -W^0{}_{102}, \\
E_{13} &= -W^0{}_{103}, \\
E_{22} &= -W^0{}_{202}, \\
E_{23} &= -W^0{}_{203}, \\
E_{33} &= -E_{11} - E_{22}, \\
H_{11} &= -W^0{}_{123}, \\
H_{12} &= -W^0{}_{131},
\end{aligned}$$

$$\begin{aligned} H_{13} &= -W^0{}_{112}, \\ H_{23} &= -W^0{}_{212}, \\ H_{22} &= -W^0{}_{231} \\ &\text{and} \\ H_{33} &= -H_{11} - H_{22}. \end{aligned}$$

Introducing the definitions

$$\mathcal{W} = (\bar{\omega}^a \wedge \bar{\omega}^b) \otimes \mathcal{W}_{ab}, \tag{4.1}$$

and

$$(\vec{e}_a \otimes \vec{e}_b)\rfloor\left((\eta^c \wedge \eta^d) \otimes (\eta^e \wedge \eta^f)\right) := (\delta^c_a \eta^d - \delta^d_a \eta^c) \otimes (\delta^e_b \eta^f - \delta^f_b \eta^e),$$

with $\rfloor$ (the hook operator) extended by bilinearity, it then follows that the electric part of the Weyl tensor with respect to the fluid flow is obtained by

$$E_{ab}(\bar{\omega}^a \otimes \bar{\omega}^b) = (\bar{\omega}^0 \otimes \bar{\omega}^0)\rfloor\mathcal{W}, \tag{4.2}$$

and the magnetic part of the Weyl tensor with respect to the fluid flow is obtained by

$$H_{ab}(\bar{\omega}^a \otimes \bar{\omega}^b) = (\bar{\omega}^0 \otimes \bar{\omega}^0)\rfloor\left((\bar{\omega}^a \wedge \bar{\omega}^b) \otimes *\mathcal{W}_{ab}\right), \tag{4.3}$$

where the $*$ operator[1] is a linear operator that obeys

$$\begin{aligned} *(\bar{\omega}^0 \wedge \bar{\omega}^1) &= \bar{\omega}^2 \wedge \bar{\omega}^3, \\ *(\bar{\omega}^0 \wedge \bar{\omega}^2) &= \bar{\omega}^3 \wedge \bar{\omega}^1, \\ *(\bar{\omega}^0 \wedge \bar{\omega}^3) &= \bar{\omega}^1 \wedge \bar{\omega}^2, \\ *(\bar{\omega}^1 \wedge \bar{\omega}^2) &= \bar{\omega}^0 \wedge \bar{\omega}^3, \end{aligned}$$

[1]See also page 71.

$$*(\bar{\omega}^2 \wedge \bar{\omega}^3) = \bar{\omega}^0 \wedge \bar{\omega}^1$$

and

$$*(\bar{\omega}^3 \wedge \bar{\omega}^1) = \bar{\omega}^0 \wedge \bar{\omega}^2.$$

We define the null vectors $\vec{k}$, $\vec{\ell}$, $\vec{m}$ and $\vec{\bar{m}}$ in such a way that they satisfy:

$$\sqrt{2}\vec{k} = \vec{e}_0 + \vec{e}_3,$$

$$\sqrt{2}\vec{\ell} = \vec{e}_0 - \vec{e}_3,$$

$$\sqrt{2}\vec{m} = \vec{e}_1 - i\vec{e}_2$$

and

$$\sqrt{2}\vec{\bar{m}} = \vec{e}_1 + i\vec{e}_2.$$

This enables us to define the components of the Weyl tensor as follows:

$$\Psi_0 = (\vec{k} \wedge \vec{m}) \otimes (\vec{k} \wedge \vec{m}) \rfloor \mathcal{W},$$

$$\Psi_1 = (\vec{k} \wedge \vec{\ell}) \otimes (\vec{k} \wedge \vec{m}) \rfloor \mathcal{W},$$

$$\Psi_2 = (\vec{k} \wedge \vec{\ell}) \otimes (\vec{k} \wedge \vec{\ell} - \vec{m} \wedge \vec{\bar{m}}) \rfloor \mathcal{W},$$

$$\Psi_3 = -(\vec{k} \wedge \vec{\ell}) \otimes (\vec{\ell} \wedge \vec{\bar{m}}) \rfloor \mathcal{W}$$

and

$$\Psi_4 = (\vec{\ell} \wedge \vec{\bar{m}}) \otimes (\vec{\ell} \wedge \vec{\bar{m}}) \rfloor \mathcal{W},$$

where the exterior product of two vectors, denoted by $\wedge$ is an antisymmetric, associative and bilinear operation. In terms of the components of the electric and magnetic parts of the Weyl tensors, we obtain:

$$\Psi_0 = \frac{1}{2}(E_{11} - E_{22} + 2H_{12}) + \frac{i}{2}(H_{11} - H_{22} - 2E_{12}),$$

$$\Psi_1 = -\frac{1}{2}(E_{13} + H_{23}) + \frac{i}{2}(E_{23} - H_{13}),$$

$$\Psi_2 = \frac{1}{2}E_{33} + \frac{i}{2}H_{33},$$
$$\Psi_3 = \frac{1}{2}(E_{13} - H_{23}) + \frac{i}{2}(E_{23} + H_{13})$$

and

$$\Psi_4 = \frac{1}{2}(E_{11} - E_{22} - 2H_{12}) + \frac{i}{2}(H_{11} - H_{22} + 2E_{12}).$$

The inverse relations are

$$E_{11} = \frac{1}{4}(\Psi_0 + \Psi_4 - 2\Psi_2 - 2\bar{\Psi}_2),$$
$$E_{12} = \frac{i}{4}(\Psi_0 - \Psi_4 - \bar{\Psi}_0 + \bar{\Psi}_4),$$
$$E_{13} = \frac{1}{2}(\Psi_3 - \Psi_1 + \bar{\Psi}_3 - \bar{\Psi}_1),$$
$$E_{22} = -\frac{1}{4}(\Psi_0 + \Psi_4 + 2\Psi_2 + 2\bar{\Psi}_2),$$
$$E_{23} = \frac{i}{2}(\bar{\Psi}_1 + \bar{\Psi}_3 - \Psi_1 - \Psi_3),$$
$$H_{11} = \frac{i}{4}(2\Psi_2 - 2\bar{\Psi}_2 - \Psi_0 - \Psi_4),$$
$$H_{12} = \frac{1}{4}(\Psi_0 - \Psi_4 + \bar{\Psi}_0 - \bar{\Psi}_4),$$
$$H_{13} = \frac{i}{2}(\Psi_1 - \Psi_3 - \bar{\Psi}_1 + \bar{\Psi}_3),$$
$$H_{22} = \frac{i}{4}(\Psi_0 + \Psi_4 + 2\Psi_2 - 2\bar{\Psi}_2)$$

and

$$H_{23} = -\frac{1}{2}(\Psi_1 + \Psi_3 + \bar{\Psi}_1 + \bar{\Psi}_3).$$

All the information in the Weyl tensor can be regrouped in the matrix $Q = E+iH$, which is equivalent to

$$Q = \begin{pmatrix} \frac{1}{2}(\Psi_0 + \Psi_4 - 2\Psi_2) & \frac{i}{2}(\Psi_0 - \Psi_4) & \Psi_3 - \Psi_1 \\ \frac{i}{2}(\Psi_0 - \Psi_4) & -\frac{1}{2}(\Psi_0 + \Psi_4 + 2\Psi_2) & -i(\Psi_1 + \Psi_3) \\ \Psi_3 - \Psi_1 & -i(\Psi_1 + \Psi_3) & 2\Psi_2 \end{pmatrix}.$$

This is a symmetric, trace-free complex matrix. Since the trace of a square matrix, M, is equal to the sum of the eigenvalues of M,[2] it follows that the sum of the eigenvalues of Q vanishes.

The matrix Q can be classified according to its eigenvalues and eigenvectors. Let λ be an eigenvector of Q; therefore, λ satisfies the characteristic polynomial of Q :

$$\mathcal{K} = \det(Q - \lambda I_3) = -\lambda^3 + \lambda I - 2J = 0,$$

with I_3 being the three-dimensional identity matrix and the invariants I and J satisfying:

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3(\Psi_2)^2$$

and

$$J = \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix} = \Psi_0\Psi_2\Psi_4 + 2\Psi_1\Psi_2\Psi_3 - \Psi_4(\Psi_1)^2 - \Psi_0(\Psi_3)^2 - (\Psi_2)^3.$$

For an eigenvalue to be repeated there must be a common zero of $\mathcal{K}$ and $d\mathcal{K}/d\lambda$. Therefore λ is a repeated eigenvalue if and only if the resultant of $\mathcal{K}$ and $d\mathcal{K}/d\lambda$ with respect to λ is zero. We conclude, then, that there is a repeated eigenvalue if and only if I and J satisfy $I^3 = 27\ J^2$. We say that a spacetime is of Petrov type I if the eigenvalues are all different, or equivalently, if $I^3 \neq 27\ J^2$. Since all the eigenvalues are different, the minimal polynomial of Q for Petrov type I is equal to $(Q - \lambda_1 I_3)(Q - \lambda_2 I_3)(Q - \lambda_3 I_3) = 0$, where λ_1, λ_2 and λ_3 are the three different eigenvalues.

[2] This follows since trace(AB)=trace(BA) and a matrix M is similar to a diagonal matrix with the diagonal elements equal to the eigenvalues of M.

For Q to admit a triply repeated eigenvalue, $\mathcal{K}$, $d\mathcal{K}/d\lambda$ and $d^2\mathcal{K}/d\lambda^2$ must possess a common factor. Taking the pairwise resultants with respect to λ, and equating them to zero, it follows that the invariants I and J must both vanish. The repeated eigenvalue must therefore be zero. We look at the eigenspace belonging to the triple eigenvalue zero. This space must be at least one-dimensional, otherwise there would not be any eigenvectors, and so there would not be any eigenvalues. Suppose that the eigenspace is three-dimensional. Since the dimension of the eigenspace is the same as the space to which Q applies, then any vector is an eigenvector of Q. In particular, we must have

$$Q\begin{pmatrix}1\\0\\0\end{pmatrix} = Q\begin{pmatrix}0\\1\\0\end{pmatrix} = Q\begin{pmatrix}0\\0\\1\end{pmatrix} = 0.$$

It follows that the tensor Q must vanish, and so the spacetimes that belong to this class are the conformally flat spacetimes. They are said to belong to the class of spacetimes of Petrov type O.

If the eigenspace belonging to the triple eigenvalue zero is two-dimensional, we can choose two independent vectors $\vec{x_1}$ and $\vec{x_2}$ as a basis for this eigenspace. Choose a vector $\vec{w}$ independent of $\vec{x_1}$ and $\vec{x_2}$. Such a vector must exist, since the eigenspace is not three-dimensional. The vector $Q\vec{w}$ cannot be zero, or even proportional to $\vec{w}$, since $\vec{w}$ cannot be an eigenvector, and so can be expressed in terms of $\vec{w}$, $\vec{x_1}$ and $\vec{x_2}$:

$$Q\vec{w} = a\vec{w} + b_1\vec{x_1} + b_2\vec{x_2},$$

where $(b_1)^2 + (b_2)^2 \neq 0$. We want to show that $a = 0$. Suppose that $a \neq 0$, and therefore a is not an eigenvalue of Q. It follows that $B := (Q - a\, I_3)$ is invertible. Therefore

$$\vec{w} = b_1 B^{-1}\vec{x_1} + b_2 B^{-1}\vec{x_2}.$$

On the other hand, since $b_1\vec{x_1} + b_2\vec{x_2}$ is an eigenvector of Q, we have that

$$\begin{aligned} B^{-1}Q(b_1\vec{x_1} + b_2\vec{x_2}) &= B^{-1}\, 0 = 0 \\ &= (I_3 + a\, B^{-1})(b_1\vec{x_1} + b_2\vec{x_2}) \\ &= b_1\vec{x_1} + b_2\vec{x_2} + a\, B^{-1}(b_1\vec{x_1} + b_2\vec{x_2}) \\ &= b_1\vec{x_1} + b_2\vec{x_2} + a\, \vec{w}, \end{aligned}$$

which is a contradiction since this implies that $a = 0$. It follows then that, indeed, $a = 0$, and so $Q\vec{w}$ is an eigenvector of Q. Note that Q cannot be equal to zero, since the vector $\vec{w}$ is not an eigenvector of Q. Because every vector $\vec{x}$ can be expressed as a combination of $\vec{w}$, $\vec{x_1}$ and $\vec{x_2}$, it follows that, $\forall \vec{x}$, $(Q^2)\vec{x} = 0$. Therefore the minimal polynomial of Q for spacetimes belonging to this class is Q^2. Such spacetimes are said to be of Petrov type N.

Now suppose that the eigenspace belonging to the triple eigenvalue zero of Q is one-dimensional. Let $\vec{x}$ be a non-trivial eigenvector of Q. Every other eigenvector of Q must then be a multiple of $\vec{x}$. Let $\vec{y_1}$ and $\vec{y_2}$ be two vectors, independent of each other and of $\vec{x}$, and so $\vec{y_1}$ and $\vec{y_2}$ are not eigenvectors of Q. Since $\vec{x}$, $\vec{y_1}$ and $\vec{y_2}$ form a basis, $Q\vec{y_1}$ can be expressed as

$$Q\vec{y_1} = a\vec{x} + b_1\vec{y_1} + b_2\vec{y_2}.$$

Similarly, we obtain

$$Q\vec{y_2} = c\vec{x} + d_1\vec{y_1} + d_2\vec{y_2}.$$

By taking $\vec{z} := d_2\vec{y_1} - b_2\vec{y_2}$, we see that $Q\vec{z}$ does not have a component along $\vec{y_2}$. Since there is no loss of generality in taking $\vec{y_1}$ to be this vector $\vec{z}$, and finding an appropriate vector $\vec{y_2}$, we can assume that b_2 is equal to zero. For simplicity, we shall denote b_1 by b. We then have

$$(Q - b\, I_3)\vec{y_1} = a\vec{x}.$$

By the same argument as in the preceding paragraph, the quantity b must be an eigenvalue of Q, and so must equal zero. It follows that $Q\vec{y_1}$ is an eigenvector of Q. We now apply the matrix Q to the vector $\vec{y_2}$. From the result, we obtain

$$(Q - d_2\ I_3)\vec{y_2} = c\vec{x} + d_1\vec{y_1}.$$

Let $C := Q - d_2\ I_3$. We first suppose that d_2 is not an eigenvalue of Q, that is, we suppose that d_2 is not zero, whence C is invertible. It follows that

$$\vec{y_2} = cC^{-1}\vec{x} + d_1C^{-1}\vec{y_1}.$$

Since $\vec{x}$ is an eigenvector of Q, we have that $0 = C^{-1}Q\vec{x} = C^{-1}(C + d_2I_3)\vec{x} = \vec{x} + d_2C^{-1}\vec{x}$, whence $C^{-1}\vec{x}$ is a multiple of $\vec{x}$. The quantity $Q\vec{y_2}$ is an eigenvector of Q, so we must have $Q\vec{y_1} = e\vec{x}$, for some non-zero constant e. Because $Q = C + d_2I_3$, we obtain $eC^{-1}\vec{x} = \vec{y_1} + d_2C^{-1}\vec{y_1}$; therefore, $C^{-1}\vec{y_1}$ belongs to the space spanned by $\vec{x}$ and $\vec{y_1}$, and so also must $\vec{y_2}$. This is a contradiction; therefore d_2 must be an eigenvalue of Q, and so d_2 must be zero. We thus get that $Q\vec{y_2} = c\vec{x} + d_1\vec{y_1}$. The quantity d_1 cannot be equal to zero, since otherwise $Q(c\vec{y_1} - a\vec{y_2}) = 0$, and so $c\vec{y_1} - a\vec{y_2} = f\vec{x}$, for some f. This cannot be, since $\vec{y_1}$, $\vec{y_2}$ and $\vec{x}$ are linearly independent, whence, $(Q^2)\vec{y_2} = d_1\vec{x} \neq 0$, and so $Q^2 \neq 0$. However, every vector $\vec{w}$ is expressible as a linear combination of $\vec{y_1}$, $\vec{y_2}$ and $\vec{x}$, so it must follow that $\forall\vec{w}, (Q^3)\vec{w} = 0$. The minimal polynomial of Q is then Q^3. Spacetimes belonging to the present class are said to be of Petrov type III.

We now consider the situation of a double eigenvalue, λ. Since the sum of the eigenvalues must be zero, the non-repeated eigenvalue must be -2λ. We have already handled the situation of a triple eigenvalue, so we can impose the condition $\lambda \neq 0$. Suppose that the eigenspace of the double eigenvalue λ is two-dimensional. Let $\vec{x_1}$ and $\vec{x_2}$ be two eigenvectors of Q that form a basis of the eigenspace of λ.

Let $\vec{y}$ be an eigenvector that belongs to the eigenvalue -2λ. The vector $\vec{y}$ must be orthogonal to both $\vec{x_1}$ and $\vec{x_2}$, since it belongs to a different eigenvalue. Thus the vectors $\vec{y}$, $\vec{x_1}$ and $\vec{x_2}$ form a basis for the full space. Let $\vec{w}$ be any vector. There must exist numbers a_1, a_2 and b such that

$$\vec{w} = a_1\vec{x_1} + a_2\vec{x_2} + b\vec{y}.$$

Applying the $Q + 2\lambda\, I_3$ operator to $\vec{w}$ yields

$$\begin{aligned}(Q + 2\lambda\, I_3)\vec{w} &= a_1(Q - \lambda\, I_3)\vec{x_1} + a_2(Q - \lambda\, I_3)\vec{x_2} + b(Q + 2\lambda\, I_3)\vec{y} \\ &\quad +3a_1\lambda\vec{x_1} + 3a_2\lambda\vec{x_2} \\ &= 3a_1\lambda\vec{x_1} + 3a_2\lambda\vec{x_2},\end{aligned}$$

whence,

$$(Q - \lambda\, I_3)(Q + 2\lambda\, I_3)\vec{w} = 0.$$

Since $\vec{w}$ is arbitrary, the minimal polynomial of Q must be $(Q - \lambda\, I_3)(Q + 2\lambda\, I_3)$. Spacetimes that belong to the present class are said to be of Petrov type D.

Now suppose that the eigenspace of the double eigenvalue λ has dimension 1. Let $\vec{x}$ and $\vec{y}$ be non-trivial vectors belonging to the eigenspace of λ and -2λ respectively. The vectors $\vec{x}$ and $\vec{y}$ are orthogonal to each other, since they belong to different eigenvalues. Let $\vec{z}$ be a vector orthogonal to both $\vec{x}$ and $\vec{y}$. The vector $\vec{z}$ cannot be an eigenvector of Q. Hence there exist scalar functions a, b and c such that

$$Q\vec{z} = a\vec{x} + b\vec{y} + c\vec{z}.$$

The functions a and b cannot vanish at the same time, otherwise $\vec{z}$ would be an eigenvector. Define C to be equal to $Q - cI_3$. If c is not an eigenvalue, then C is invertible, and we obtain

$$z = C^{-1}(a\vec{x} + b\vec{y}).$$

We also have

$$\begin{aligned} 0 &= C^{-1}(Q - \lambda\ I_3)\vec{x} = C^{-1}[C + (c - \lambda)I_3]\vec{x} \\ &= x + C^{-1}(c - \lambda)\vec{x} \end{aligned}$$

and

$$\begin{aligned} 0 &= C^{-1}(Q + 2\lambda\ I_3)\vec{y} = C^{-1}[C + (c + 2\lambda)I_3]\vec{y} \\ &= y + C^{-1}(c + 2\lambda)\vec{y}. \end{aligned}$$

It follows that

$$(c - \lambda)(c + 2\lambda)\vec{z} = -(c - \lambda)b\vec{y} - a(c + 2\lambda)\vec{x}.$$

Now this is a contradiction, since c is assumed not to be an eigenvalue and $\vec{x}, \vec{y}$ and $\vec{z}$ are independent. Therefore C is not invertible and so c must be an eigenvalue. Suppose that $c = -2\lambda$, and so

$$(Q + 2\lambda\ I_3)\vec{z} = a\vec{x} + b\vec{y}.$$

Because a matrix must satisfy its characteristic equation, Q must satisfy

$$(Q - \lambda\ I_3)^2(Q + 2\lambda\ I_3) = 0;$$

however,

$$(Q - \lambda\ I_3)^2(Q + 2\lambda\ I_3)\vec{z} = b\lambda^2\vec{y}.$$

In this case, b must be zero. Since

$$Q(3\lambda\vec{z} - a\vec{x}) = 3\lambda(-2\lambda\vec{z} + a\vec{x}) - a\lambda\vec{x} = -2\lambda(3\vec{x} - a\vec{x}),$$

we have that $3\lambda\vec{z} - a\vec{x}$ is in the eigenspace of -2λ and so must be proportional to $\vec{y}$. This is a contradiction. We thus have $c = \lambda$, and so

$$(Q - \lambda\ I_3)\vec{z} = a\vec{x} + b\vec{y}.$$

This is compatible with the characteristic equation of Q, and so there are no further restrictions on a and b, provided that neither a nor b is zero. Let $\vec{w}$ be any vector. There are then functions c, d and e such that $\vec{w} = c\vec{x}+d\vec{y}+e\vec{z}$. We find the minimal polynomial of Q using the following computations:

$$(Q + 2\lambda\ I_3)\vec{w} = 3c\lambda\vec{x} + e(a\vec{x} + b\vec{y} + 3\lambda\vec{z}),$$

$$(Q - \lambda\ I_3)(Q + 2\lambda\ I_3)\vec{w} = -3eb\lambda\vec{y} + 3ea\lambda\vec{x} + 3eb\lambda\vec{y} = 3ea\lambda\vec{x}$$

and

$$(Q - \lambda\ I_3)^2(Q + 2\lambda\ I_3)\vec{w} = 0.$$

The minimal polynomial of Q is then $(Q-\lambda\ I_3)^2(Q+2\lambda\ I_3)$. Spacetimes belonging to this class are those of Petrov type II.

We can summarize the content of the present section into table 4.1 where the Petrov type is given by the most restrictive matrix that applies.[3]

[3] A similar table appears as table 4.1 of Kramer et al. (1980).

Petrov type	Matrix condition (use the most restrictive matrix condition that applies)	Dimension of eigenspace
I	$(Q - \lambda_1 \, I_3)(Q - \lambda_2 \, I_3)(Q - \lambda_3 \, I_3) = 0$ $\lambda_1, \lambda_2, \lambda_3$ all different	$< \lambda_1 >= 1$ $< \lambda_2 >= 1$ $< \lambda_3 >= 1$
D	$(Q + \frac{\lambda}{2} \, I_3)(Q - \lambda \, I_3) = 0$	$< -\frac{\lambda}{2} >= 2$ $< \lambda >= 1$
II	$(Q + \frac{\lambda}{2} \, I_3)^2(Q - \lambda \, I_3) = 0$	$< -\frac{\lambda}{2} >= 1$ $< \lambda >= 1$
N	$Q^2 = 0$	$< 0 >= 2$
III	$Q^3 = 0$	$< 0 >= 1$
O	$Q = 0$	$< 0 >= 3$
The expression $< \lambda >$ is defined to be the dimension of the eigenspace associated with the eigenvalue λ.		

Table 4.1: Petrov types.

Chapter 5

The shear-free conjecture

Only a life lived for others is a life worth while. *Albert Einstein*

THERE IS a growing body of evidence that the following conjecture, which we shall refer to as the **shear-free conjecture**, is true:

Conjecture 1 *A shear-free perfect fluid that obeys a barotropic equation of state,* $p = p(\mu)$, *such that* $\mu + p \neq 0$, *and satisfying the field equations of general relativity, is necessarily either irrotational or expansion-free, i.e.,* $\sigma = 0 \Rightarrow \omega\theta = 0$.

This conjecture appears to have first appeared in the literature in King (1974). King attributes it to Treciokas and Ellis (1971).

In the following pages, we present a historical account of the various results supporting the conjecture. Thereafter, the conjecture is proved for two special cases. The first case is that when the Weyl tensor is purely magnetic with respect to the fluid flow vector. The second case is that of a coasting[1] universe, *i.e.* one with the equation of state satisfying $dp/d\mu = -1/3$.

5.1 Historical survey

In this section, we review the basic results previously obtained with respect to the shear-free conjecture. Particular attention is paid to features that were critical to the success of the authors in establishing the veracity of the conjecture under various hypotheses. Collins (1986) gives a quite extensive discussion on shear-free fluids in general relativity. In particular, he provides a survey of the literature on the shear-free conjecture and its consequences.

The first result of which I am aware concerning the shear-free conjecture is contained in the work of Gödel (1950). Gödel considers spacetimes with a dust source, *i.e.* perfect fluids with vanishing pressure. He requires the spacetime to be spatially homogeneous and rotating, with non-constant energy density. Since the energy density varies, the space must be expanding. Therefore he requires that the product $\omega\theta$ be non-vanishing. Since the dust is rotating, the flow velocity cannot be orthogonal to the surfaces of homogeneity, *i.e.* the spacetime is tilted. Furthermore, Gödel requires that the isometry group must be compact. He shows that the group must be a three-parameter group that cannot be commutative, and therefore that it

> must be isomorphic (as a group of transformations) with the right (or left) translations of a 3-space of constant positive curvature, or with

[1] Matter-energy is between attracting and repulsing regimes as can be seen from Raychaudhuri's equation.

> these translations plus certain rotations by an angle π.
> (Gödel, 1950)

It follows then that the spacetime must be a tilted spatially homogeneous spacetime of Bianchi-Behr type IX. If in addition the metric induced in the 3-spaces of constant density is positive definite, or, equivalently (Gödel, 1950), if the spacetime contains no closed timelike lines, then the expansion tensor cannot be, at any instant of time, rotationally symmetric about the axis of rotation. This therefore requires that the spacetime exhibits shear. Gödel considers it very likely that

> there exist no rotating spatially homogeneous and expanding solutions whatsoever in which the ellipsoid of expansion is *permanently [Gödel's emphasis]* rotationally symmetric around ω.

Schücking (1957) generalizes the result of Gödel (1950) on the shear-free conjecture to general spatially homogeneous dust. Schücking remarks that shear-free models with simultaneous expansion and rotation would represent spacetimes that are intermediate between the isotropically expanding Friedmann models without rotation and the stationary rotating Gödel models without expansion. Schücking writes the line element for a spatially homogeneous spacetime with dust in comoving coordinates as follows (with the convention of Gödel (1950) that has Greek indices running over 0 to 3 and Latin indices running over 1 to 3):

$$ds^2 = (dx^0)^2 + 2g_{0i}(x^j)dx^0\, dx^i + g_{ik}(x^\mu)dx^i\, dx^k.$$

The velocity is given by

$$u^\mu = \frac{dx^\mu}{ds} = \delta^\mu_0.$$

The length scale R is defined as $R^3 = \sqrt{-g}$, where $g = \det|g_{\mu\nu}|$. The equation of continuity and the requirement of spatial homogeneity necessitate that R separates as

$$R(x^r) = S(x^0)W(x^j).$$

The (00) and the $(0i)$ Einstein field equations, since they must hold for arbitrary x^j, are then two differential equations for $S(x^0)$ which are incompatible with each other under the requirement that both the expansion and the vorticity be non-zero. Schücking (1957) mentions that models with both vorticity and expansion do exist in Newtonian cosmology, as long as one does not neglect the ambiguity[2] (which is characteristic of such models) of the boundary conditions. The result of Schücking (1957) was generalized by Ellis (1967) to general dust and by Banerji (1968) for perfect fluids with an equation of state $p = (\gamma - 1)\mu$, such that $\gamma \neq 10/9$.

Ellis (1967) studies general relativistic pressure-free matter. The scope of this work covers much more than the shear-free conjecture; however, we shall restrict ourselves to that aspect. Ellis proves the conjecture for shear-free dust, *i.e.* for fluids without pressure. An immediate consequence of requiring that the pressure vanish is that the acceleration must also vanish; this is proved using three of the four contracted Bianchi identities. The framework used is the orthonormal technique. Ellis proves the conjecture for shear-free dust by showing that a contradiction is reached after making the hypothesis that neither the expansion nor the vorticity vanishes. A sketch of the proof follows.

The $\vec{e}_0-$axis is chosen to be the fluid flow velocity. The $\vec{e}_1-$axis is chosen to

[2]The problem is that it is not possible to invariantly separate the inertial and the gravitational parts of the acceleration. For more details see Ellis (1971).

be parallel to the vorticity vector. The vectors $\vec{e}_2$ and $\vec{e}_3$ have the freedom of rotation through the angle Θ. The propagation of this angle along $\vec{e}_0$ is chosen such that $\omega_1 + \Omega_1 = 0$. The Jacobi identities imply that $\Omega_2 = \Omega_3 = 0$. For shear-free fluids, then, $\omega_\alpha + \Omega_\alpha = 0$, with the convention that Greek indices run over 1 to 3 and Latin indices run over 0 to 3. The propagation along $\vec{e}_1$ of Θ is chosen in such a way to set $\hat{\Omega} - \hat{\theta}_{23} = 0$. The Jacobi identities, some of the Einstein field equations and the remaining contracted Bianchi identity, which expresses conservation of energy, are then computed and used to find the four derivatives of the expansion, the $\vec{e}_0-$ and the $\vec{e}_1-$derivatives of the vorticity and the $\vec{e}_0-$derivatives of $\theta, \omega, d_2, d_3, n, A_2, A_3, \hat{\theta}_{22}, \hat{\theta}_{23}, \hat{\theta}_{33}$ and μ. The $[\vec{e}_0, \vec{e}_2]$ and $[\vec{e}_0, \vec{e}_3]$ commutation relations on ω are then used to find the propagation of $\partial_2\omega$ and $\partial_3\omega$ along $\vec{e}_0$, where we denote the $\vec{e}_i-$derivative by ∂_i. The propagation along $\vec{e}_0$ of the equations is then used exclusively as the tool to generate further equations. The three spatial derivatives of μ and various algebraic constraints are found. It is shown that $\hat{\theta}_{22} + \hat{\theta}_{33} = 0$ and $n = 0$. The propagations along $\vec{e}_0$ of $\partial_2\mu$ and of $\partial_3\mu$ yield two equations involving $\partial_2\omega$ and $\partial_3\omega$. Propagation of these yields two other such equations; from these last four equations, the relation $d_2\partial_3\omega - d_3\partial_2\omega = 0$ is deduced. The propagation along $\vec{e}_0$ of the (11), (22) and (33) field equations produces the required contradiction.

We note, as did Ellis, that the timelike $\vec{e}_0-$congruence is the principal feature of this paper. As White and Collins (1984) observed, the proof of Ellis also holds for the more general situation when the pressure is constant. Any non-zero constant pressure can be absorbed into the cosmological term Λ, with the appropriate adjustment of the definition of the energy density. White and Collins (1984) give a slightly different proof for this case, but in the same notation as that used in the present work.

As a note on the history of the conjecture, we mention that Ellis (1967) asks the question "under what more general[3] conditions does such a result[4] hold?"

Banerji (1968) considers shear-free rotating spatially homogeneous perfect fluid spacetimes with a gamma law equation of state $p = (\gamma - 1)\mu$, where $\gamma - 1$ is positive. He finds that the conjecture holds except possibly when $\gamma = 10/9$. The method of study is based on coordinates. Let the surfaces of homogeneity be labelled by t = constant. The metric is given by $ds^2 = dt^2 + 2g_{4i}dt\,dx^i + g_{ik}dx^i\,dx^k$. The function $G = \sqrt{-g}$ satisfies $\dot{G}/G = \theta/3$, where the dot $\dot{()}$ indicates differentiation along the fluid flow and Latin indices run from 1 to 3. For spatially homogeneous spacetimes, the function G separates as the product of a function, S, of x^4 alone and a function, W, which is independent of x^4. The vorticity must be of the form $\omega^2 = AS^{6\gamma+2}$, with A being a positive constant. The (00) equation and a particular combination of the (0α) field equations give, by integration, an algebraic relation on the function S. The requirement that $\theta\omega \neq 0$ then requires that S be equal to $\sqrt{-E}t$, where E is a negative constant. The requirement that the energy density not vanish then shows that the only values for γ are $\gamma = 1$ and $\gamma = 10/9$. The value $\gamma = 1$ corresponds to dust, for which Schücking (1957) has shown the veracity of the shear-free conjecture in the case of spatially homogeneous spacetimes. Banerji considers "not unlikely" that the case $p = \mu/9$ can also be ruled out, but does not give a proof for this situation.

Ellis (1971) mentions that for conformally flat spacetimes, the Bianchi identities require that perfect fluids must be shear-free, irrotational and geodesic. In other

[3]than the conditions of the dust-filled world with homogeneous space sections considered by Schücking.

[4]that $\theta \neq 0 = \sigma \Rightarrow \omega = 0$.

words, they must be the Friedmann-Robertson-Walker models. The result is also contained implicitly in Stephani (1967b) and (1967a) who investigated conformally flat solutions of the Einstein field equations for a perfect fluid or an electromagnetic field.

Treciokas and Ellis (1971) proves the conjecture for the case of a shear-free fluid with the equation of state $p = \mu/3$. The method of proof is coordinate-based. First, Treciokas and Ellis show that for a shear-free perfect fluid with a barotropic equation of state and with non-zero vorticity, local co-moving coordinates can be chosen so that the spacetime metric is

$$ds^2 = \frac{1}{w^2(x^a)}\left(f_{\alpha\beta}(x^\sigma)dx^\alpha\, dx^\beta - v^2(x^a)(dx^0 + x^2\, dx^3)^2\right),$$

with $v := w/r$, where

$$w := \exp\left(\int_{\mu_0}^{\mu} \frac{d\mu}{3(\mu+p)}\right)$$

and

$$r := \exp\left(\int_{p_0}^{p} \frac{dp}{\mu+p}\right).$$

The convention that Latin indices go from 0 to 3 and that Greek indices go from 1 to 3 is used. The only quantities appearing in the metric that depend on time (x^0) are $w(x^a)$ and $v(w)$. The authors define $W(x^a) := w_{,0}$ and $X_\alpha(x^a) := w_{,a} - a_\alpha W$, where $a_\nu(x^\sigma) := x^2\delta^3_\nu$. The expansion of the fluid vanishes if and only if W does. The X_a are related to the acceleration terms. We note that the exterior derivative of w is $W(dx^0 + x^2\, dx^3) + X_\alpha\, dx^\alpha$. The critical condition that $p = \mu/3$ translates into $v = 1$. There is then a precise correspondence with the spacetimes (within the class under consideration) that are conformal to a static spacetime. An outline of the proof of the conjecture in this case now follows.

The (00) field equation yields an expression for $\partial_0 W$; the (0ν) field equations give $\partial_0 X_\nu$ and the $(\mu\nu)$ ones give $\partial_\mu X_\nu$. The only expressions that contain derivatives of W are those for $\partial_\mu X_\nu$. They contain the term $\partial_0 W$. The (00), (0ν) and (23) field equations are differentiated with respect to the variable x^0. The resulting equations are denoted by $(00)_{,0}$, $(0\nu)_{,0}$ and $(23)_{,0}$, respectively, where $(ab)_{,0}$ denotes differentiation of the (ab) field equation with respect to x^0. The $(00)_{,0}$ equation yields an expression for $\partial_0\partial_0 W$. The $(0\nu)_{,0}$ equations are then put in the form of equations that are linear in X_ν, with coefficients in which the only dependence on x^0 appears in the function w. The determinant of these three equations, considering X_ν as the variables, is a polynomial in w with coefficients independent of x^0. By repeated differentiation with respect to x^0, one can conclude that this determinant can vanish if and only if all coefficients of the polynomial in w vanish. It is therefore of critical importance that w appears only in a polynomial fashion. The leading coefficient, $(8\mu_0/3)^3$, cannot vanish, and therefore neither can the determinant. One can then solve for the variables X_ν. They appear as the ratio of a polynomial in w of degree 5 by a polynomial in w of degree 6 (the aforementioned determinant). Then one solves for W from $(23)_{,0}$ and substitutes the result in (00). After multiplication by a suitable power of the determinant and by a suitable power of a particular time independent function, Treciokas and Ellis (1971) obtain that a certain polynomial in w, with coefficients independent of x^0, vanishes. The leading coefficient of this polynomial must then vanish, but this is a contradiction because it is equal to $5(\mu_0/3)^2(8\mu_0/3)^9$, a non-zero quantity.

We note that here as well the derivatives with respect to x^0 are extremely important. The crucial part of this proof is that the authors obtained the vanishing of quantities that are *polynomials* in the time-dependent variable. Unfortunately, this desirable feature does not appear to be generic, and so this method of proof is unlikely to apply to the full conjecture.

We also note that higher order derivatives are eliminated as soon as possible in favour of lower order derivatives. The highest order derivative appearing explicitly is $\partial_0\partial_0 W$. This corresponds to second order derivatives of the kinematic quantities.

Treciokas and Ellis (1971) also provide an outline of the proof of the conjecture for the situation when there exists a function β such that the acceleration potential r and its derivative along the fluid flow $\dot{r}$ are related by $\dot{r} = \beta(r)$. Except in the situation of dust, for which, anyway, Ellis (1967) established the truth of the conjecture, this case of Treciokas and Ellis (1971) is equivalent to the situation considered by Lang and Collins (1988). This work of Lang and Collins, which will be examined below, provides the first *full* published proof, as far as we are aware, for this situation.

Treciokas and Ellis (1971) mention that they would like to know the precise conditions for which the requirement of vanishing shear entails that the product $\omega\theta$ vanish. They conjecture that

> It is conceivably true for all perfect fluid solutions, or for all perfect [fluid] solutions with an equation of state of the form $p = p(\mu)$.

Treciokas and Ellis (1971) also mention that their result does not hold in the corresponding Newtonian theory. Furthermore, the condition of vanishing shear does not impose restrictions on Newtonian spacetimes, unlike in the relativistic theory. Treciokas and Ellis conjecture that the energy-momentum tensor will be that corresponding to a perfect fluid only if the shear vanishes. Collins (1987) uses this conjecture in a study on the uniqueness of the Friedmann-Robertson-Walker cosmological models.

King and Ellis (1973) generalize the work of Banerji (1968) by removing the conditions on the equation of state. They prove the conjecture for homogeneous cosmological models, provided the reasonable condition $\mu + p > 0$ holds. The technique used in this proof is the method of tetrads. Let $S(t)$ represent the surfaces of homogeneity. Let the vector $\vec{n}$ be the unique future-directed normal vector field determined by $S(t)$. If the vector $\vec{n}$ does not equal the fluid flow vector, then the model is said to be tilted. The orthogonal tetrad used by King and Ellis in the proof of the conjecture is a normalized fluid basis. The vector $\vec{e}_0$ is a future-pointing vector parallel to the fluid flow that has length r^{-1}, where

$$r(t) := \exp \int_{t_0}^{t} \frac{\mathrm{d}p/\mathrm{d}t}{\mu + p} \mathrm{d}t.$$

This factor is included in order to simplify the tetrad form of the conservation equations. The vector $\vec{e}_3$ is chosen to be in the 2-plane spanned by $\vec{u}$ and $\vec{n}$. The vectors $\vec{e}_1$ and $\vec{e}_2$ are unit vectors that span the 2-planes orthogonal to $\vec{n}$ and $\vec{e}_3$. The freedom of rotation in the definition of $\vec{e}_1$ and $\vec{e}_2$ is chosen so that the $\vec{e}_0 \rfloor \vec{e}_1 \rfloor \mathrm{d}\bar{\omega}^2$ connection coefficient vanishes. All the connection coefficients are functions of t only. King and Ellis note that the crux of the proof is that if the fluid does not possess shear, then the Jacobi identities and the renormalized tilt parameter $\lambda := r \tanh \beta$, where $\cosh \beta := -g(\vec{u}, \vec{n})$, can be integrated up to a quadrature, in terms of a length parameter ℓ, which has the same $t-$dependence as the function G of Banerji (1968), defined by $\dot{\ell}/\ell = \theta(t)/3$, where the dot $\dot{()}$ represents the covariant derivative along the fluid flow lines. Three cases arise (i) $\omega^2\omega^3 \neq 0$, (ii) $0 = \omega^3 \neq \omega^2$ and (iii) $0 = \omega^2 \neq \omega^3$. The assumption that $\omega\theta \neq 0$, together with the field equations, then yields a contradiction. King and Ellis (1973) describe the proof as "straightforward and tedious" and therefore do not give details beyond an outline but refer to King (1973). We have not consulted King (1973), especially since the work of Lang and Collins (1988), as discussed below, encompasses the present part of that of King and Ellis (1973). The work of White (1981) relaxes the condition $\mu + p > 0$ by showing that the conjecture is true for spatially homogeneous spacetimes under the more general condition $\mu + p \not\equiv 0$. Incidentally, as a historical note, Lang and Collins (1988) notice that the work considered by King and Ellis (1973) is a special case[5] of one of the situations considered by Treciokas and Ellis (1971); thereby, an alternative proof of the conjecture for the situation of King and Ellis (1973) could have been obtained.

King (1974) studies singularities of shear-free perfect fluids. Under certain conditions, such fluids cannot have matter singularities. As a consequence of his result, he considers very plausible the truth of the shear-free conjecture, attributed by him to Treciokas and Ellis (1971). King (1974) states the conjecture as follows:

> ... that either the expansion θ or the vorticity ω must vanish in a shear-free perfect fluid model, at least for $p = p(\mu)$ [such that]
>
> $$0 \leq \frac{\mathrm{d}p}{\mathrm{d}\mu} \leq \frac{1}{3}.$$

King (1974) thus provides the first allusion in the literature to the conjecture.

White and Collins (1984) show that the shear-free conjecture holds when the vorticity is parallel to the acceleration, including the degenerate case of geodesic flow. The method involves the use of the orthonormal tetrad technique in a proof by contradiction that first assumes that $\omega\theta \neq 0$. The $\vec{e}_0-$axis is chosen to be along the tangent to the flow, normalized so that the flow velocity is unit. The $\vec{e}_1-$axis is chosen to be in the common direction of the acceleration and of the vorticity.

The proof splits into two cases. The first case, when the flow is not geodesic, is the simpler of the two. We note that in the proof, the derivatives in the four

[5] Defined by $\dot{r} = \beta(r)$ in the notation of Treciokas and Ellis (1971).

directions of θ and of the acceleration have been isolated as early as possible. Commutation relations have been used on variables. No second order derivatives needed to be isolated. There was then a crucial propagation of various expressions along the fluid flow direction.

In the second case, the flow is geodesic. Apart from an integration constant, this is essentially the situation of dust considered by Ellis (1967). White and Collins (1984) provide a proof similar to that of Ellis (1967), but in their notation. This enables a more direct comparison with the non-geodesic case, and clarifies the rôle of the intrinsic geometrical quantities. It also enables the direct use of the intermediate results of White and Collins (1984) in the study of shear-free perfect fluids that is found in Collins and White (1984). As before, the proof uses commutation relations on the expansion, the energy density and the vorticity. As well, differentiation along the flow direction is still crucial; however, a new feature arises: second order derivatives are calculated (namely $\partial_0\partial_2\omega$ and $\partial_0\partial_3\omega$) and eventually eliminated. This is an indication that the geodesic case of the conjecture is more complex than the first case since second order derivatives are involved. That second order derivatives are eliminated (algebraically), yielding equations with only lower order derivatives, is a new feature that will recur in the proof of other situations.

In the work of Lang and Collins (1988), the rate of expansion is functionally related to the energy density. This is equivalent to requiring that the fluid obeys a type of homogeneity proposed by Bonnor and Ellis (1986), namely, the postulate of uniform thermal history (PUTH). This postulate is based on the assumption that similar thermodynamic histories imply similar dynamical histories. This requires that, for example, pressures and densities are not substantially affected by non-thermodynamic factors, such as gravitational waves. The postulate is expressed as follows:

> Both density, ρ and entropy per baryon S are uniform for the fundamental observers in the Universe.
> (Bonnor and Ellis, 1986).

The framework is that of an orthonormal tetrad aligned as follows. The vector $\vec{e}_0$ is aligned with the fluid flow, and is unit. The $\vec{e}_1-$axis is parallel to the vorticity vector. The other two axes are rotated such that the (projected) shear tensor of the $\vec{e}_1-$ congruence be diagonal ($\hat{\sigma}_{23} = 0$). The shear-free conjecture is proved by contradiction, supposing first that $\omega\theta(\mu{+}p) \neq 0$, then showing inconsistency. There are six different cases to be treated. The first case is when the energy density, μ, is constant. One of the contracted Bianchi identities gives immediately the required contradiction.

The second case is that when the pressure, p, is constant. This case is basically covered by Ellis (1967). As mentioned in White and Collins (1984), the constant can be "absorbed" into the cosmological constant followed by a reinterpretation of μ and p. If this is done, then the proof of Ellis (which was for vanishing p) carries through without changes. In this situation, the conjecture holds without any further restrictions on θ. White and Collins (1984) give a proof very similar to that of Ellis (1967).

The third case has the acceleration parallel to the vorticity. This has been treated by White and Collins (1984). At this point, we prove that requiring that, in a general setting, the acceleration be non-zero and parallel to the vorticity, *necessarily* implies that the expansion and the energy density are functionally related.

That this is the situation was not realized by Lang and Collins when they established their results. The proof is as follows. Since $\dot{u}_2 \equiv \dot{u}_3 \equiv 0$ then $\partial_2\mu = \partial_3\mu = 0$, by the Bianchi identities. Also $\partial_2\theta = \partial_3\theta = 0$ by the commutation relations on μ, given by equation (3.1) of White and Collins (1984). Therefore

$$\mathrm{d}\theta \wedge \mathrm{d}\mu = 0$$

if and only if

$$\partial_0\theta\partial_1\mu - \partial_1\theta\partial_0\mu = 0.$$

Now, $\partial_0\mu = -(\mu+p)\theta$ and $\partial_1\mu = -(\mu+p)\dot{u}/p'$ by the contracted Bianchi identities; $\partial_0\theta = (3/4)n^2$ as in equation (3.3) of White and Collins (1984); and $\partial_1\theta = (3/2)n\omega$ by the (01) field equation. Therefore

$$\begin{aligned} & \partial_0\theta\partial_1\mu - \partial_1\theta\partial_0\mu = \\ = \; & -(3/4)n^2(\mu+p)\dot{u}/p' + (3/2)n\omega(\mu+p)\theta \\ = \; & -(3/4)n(\mu+p)/p' \times 2\omega p'\theta + (3/2)n\omega(\mu+p)\theta \\ = \; & 0, \end{aligned}$$

where use has been made of equation (3.5) of White and Collins (1984), viz. $2\omega p'\theta = n\dot{u}$. The non-geodesic situation treated in White and Collins (1984) and by Collins and White (1984) is then a *proper* subcase of that covered by Lang and Collins, and therefore obeys PUTH.

After these first three cases, for which the proof of the conjecture is either immediate or has been done in previous work, Lang and Collins now turn to the main part of the proof. Four torsion expressions are computed. These expressions were not recognized as such by Lang and Collins (1988), but for much of the present work, torsion will be a useful notion. It may be explained in loose terms as follows.[6]

[6] More will be mentioned about the torsion after the discussion of the sixth case.

The torsion equations arise from particular combinations of commutation relations. Normally, commutation relations on algebraic quantities give second order derivatives. The torsions are the combinations that give derivatives of lower order than expected. In this case, the torsions would be expressions involving derivatives of at most first order. The four torsions just noted are even more special, since they do not involve derivatives at all, but only algebraic quantities.

We now examine the fourth, fifth and six cases of the proof. The fourth case corresponds to constant fluid expansion ($\theta' = 0$). In the proof, the operator $(1/\theta)\partial_0$ is used twice, where ∂_0 is the derivative along $\vec{e}_0$. The proof is completed by noting that the flow is necessarily geodesic. This case therefore reduces to one already treated.

The fifth case has the equation of state obeying $p' = 1/9$, excluding the situation covered in the fourth case. It is interesting to note that this rather peculiar equation of state also appeared as the one exceptional case in the work of Banerji (1968) that was not treated, although it appears here in a broader context. The operators $(1/\theta)\partial_0$ and ∂_0 are used. Also a further torsion equation, involving a first order derivative, is obtained. This equation enables the authors to solve for $\partial_2\dot{u}_3$ and then for $\partial_3\dot{u}_2$. With this, the commutation relation $[\vec{e}_0, \vec{e}_1]$ on $\dot{u}_2$ now becomes a torsion equation which leads to $\dot{u}_2\dot{u}_3 = 0$. The choice is made to set $\dot{u}_2 = 0$.[7] A further torsion equation was then evaluated, yielding a value for $\partial_1\dot{u}_3$. This enables the commutation relation $[\vec{e}_0, \vec{e}_1]$ on $\dot{u}_3$ to become a torsion equation, from which the conclusion that $\dot{u}_1$ vanishes is obtained. All the preceding results and the commutation relation $[\vec{e}_0, \vec{e}_2]$ on d_3 then produce a contradiction, namely that $\theta\omega$ should vanish.

[7] The other choice of $\dot{u}_3 = 0$ is completely symmetric, and so there is no loss of generality.

The sixth case is the general case where $(p' - 1/9)\theta' \neq 0$. The authors start by obtaining a few expressions involving only functions of μ. Then they derive a homogeneous system of three linear equations (with coefficients being functions of μ only) in $\dot{u}^2, \omega^2$ and $\mu + p$. The differential operations used are differentiation with respect to μ and differentiation along the flow vector. The trivial solution to this linear system is to be rejected, and so the determinant (a function of μ only) must vanish. This determinant takes the form of a bivariate polynomial in p' and G (a particular function of μ involving p''). The derivatives of G and of p' with respect to μ were previously calculated and are expressible in terms of G and p'. Therefore, by differentiating the above bivariate polynomial with respect to μ, another similar polynomial is obtained. In order that they have simultaneous solutions, their resultant with respect to G must also vanish. This resultant is a non-trivial univariate polynomial[8] in p'. Consequently, p' is a constant. This crucial step then leads one to the result that $\mu + p = 0$. This is the required contradiction.

In their remarks, Lang and Collins (1988) noted that 6 commutation relations were applied to 12 variables, leading to 72 equations. There were two combinations of those commutation relations that were purely algebraic. Normally, commutation relations on algebraic quantities give expressions with second order derivatives. There may be combinations involving lower order derivatives, and, as can be seen in the proof of Lang and Collins, such combinations were also used. We note they were also used in White and Collins (1984). No further justification was given to this procedure, other than that it works. It so happens that finding these combinations is a well defined procedure of the theory of exterior differential systems,[9] that of finding the *torsion*. We point out that whenever known relations are propagated,

[8]The resultant is of degree 60, has 53 terms and has some coefficients with over 40 digits!

[9]For more information about exterior differential systems, see Bryant et al. (1991).

new torsion expressions may appear. We also note that the cases where the proof is the most difficult are those where the acceleration is perpendicular to the vorticity.

From equation (4.19) of Ellis (1971) it is immediate that all non-rotating shear-free perfect fluids must necessarily have a vanishing magnetic part of the Weyl tensor. In an article by Collins (1984), it is shown that the converse does not necessarily hold, but that if the fluid is rotating, then the expansion must vanish (under the usual assumptions of $\mu+p \neq 0$ and a barotropic equation of state). First, for the case of geodesic flow, the situation is covered by White and Collins (1984) and by Ellis (1967); this therefore needs no further attention as far as the conjecture is concerned.

A sketch of the proof of the conjecture for non-geodesic flow with a purely electric Weyl tensor follows. The tetrad is chosen such that $\vec{e}_0$ is along the fluid flow and is unit. The $\vec{e}_1-$congruence is chosen to be parallel to the vorticity. It is assumed that the vorticity is non-zero. From the (full) Bianchi identities, it follows that the vorticity vector is an eigenvector[10] of the symmetric tensor E_{ab}, representing the electric part of the Weyl tensor. In the chosen frame, $E_{0a} = E_{12} = E_{13} = 0$ $(a = 0, 1, 2, 3)$ and therefore the $\vec{e}_0-$propagation of E_{11} simplifies to $\partial_0 E_{11}+\theta E_{11} = 0$. The eigenvalue corresponding to the vorticity vector is $-(1/3)(\mu+p)$; by the choice of tetrad, E_{11} must therefore be equal to this eigenvalue. The conclusion follows from the propagation of E_{11} and the contracted Bianchi identity $\partial_0\mu + (\mu + p)\theta = 0$.

Collins then proceeds to examine further the case when the vorticity does not vanish (and therefore, the expansion must vanish). We note that, in this situation,

[10]The fact that the vorticity vector is either zero or is an eigenvector of E_{ab} was independently noticed by Barnes (1984).

$\vec{e}_0$ is again distinguished, by being a Killing vector. The process of finding the torsion is again used (although not in any explicit way), as is the process of isolating the various derivatives of the acceleration vector components. Two classes appear according to whether or not the acceleration is parallel to the vorticity. In the first class, they are not parallel. The tetrad is rotated so that $\dot{u}_3 = 0$. It follows that $\vec{e}_3$ is a second Killing vector. The situation where the acceleration is perpendicular to the vorticity is again distinguished since, in that case, there is a third Killing vector, namely $\vec{e}_1$. Because there is a Killing vector parallel to the vorticity vector, such spacetimes belong to a class of models investigated by Krasiński (1978). This class will be studied in more detail as *case C* of chapter 6. In the second class, where the acceleration is parallel to the vorticity, the condition that the acceleration be also perpendicular to the vorticity (and so, that the acceleration vanish) again arises as a special case – the vorticity is constant, and so are the pressure and energy density. This is the Gödel solution, generalized to include pressure. This subclass is also distinguished in that there is a G_5 isometry group instead of a G_4.

Carminati (1987) proves the shear-free conjecture for the situation when the Weyl tensor is of type N. The actual result is stronger than that of the conjecture. The spacetimes under consideration are shown to have vanishing volume expansion and necessarily non-vanishing vorticity. The Newman-Penrose (NP) formalism is used for the calculations. The null tetrad $\{\vec{\ell}, \vec{n}, \vec{m}, \vec{\bar{m}}\}$ is chosen in the following manner. The vector $\vec{\ell}$ is chosen to be the repeated principal null direction of the Weyl tensor. By a rotation that leaves $\vec{\ell}$ fixed, $\vec{n}$ is made to lie in the two-space spanned by $\vec{\ell}$ and the fluid velocity vector $\vec{u}$. Then $\vec{\ell}$ and $\vec{n}$ are rescaled so that $\vec{u} = (1/\sqrt{2})(\vec{\ell} + \vec{n})$. The freedom left in the choice of the tetrad is a multiplication of the vector $\vec{m}$ by a complex number with unit modulus. Imposing this choice of tetrad, the shear-free condition, the barotropic equation of state and the condition that the spacetime be of Petrov type N in the Bianchi identities and the NP equations readily leads to the result that the repeated null congruence of the Weyl tensor is non-geodesic and that the fluid is necessarily rotating. The assumption is made that the fluid has non-zero expansion. Three subcases arise, each of which leads to a contradiction. The first subcase has $dp/d\mu \not\equiv 0$ and $1 + 3\,dp/d\mu \not\equiv 0$. After some calculations, a contradiction is reached. Derivatives of Weyl tensor components were used. The second subcase has $dp/d\mu = 0$. This case is quickly shown to be impossible. The remaining rotational freedom of the tetrad is then used to impose on the NP quantities α and β the restriction that $\bar{\alpha} + \beta = \alpha + \bar{\beta}$. This is a condition on a component of the acceleration divided by $dp/d\mu$. The third subcase has $1 + 3\,dp/d\mu = 0$. This case is shown to be impossible after some calculations. The techniques of calculations are similar to that used in the orthonormal tetrad approach, except that the Bianchi identities are used explicitly. The Weyl tensor and Ricci tensor components also appear explicitly, instead of being expressed in terms of the equivalent of the kinematic quantities and their derivatives. Commutation relations on the energy density are used. The various derivatives are applied to propagate algebraic relations. The highest order derivative appears as the first derivative of the Weyl tensor components; therefore, second order derivatives of the kinematic quantities are potentially involved. The result proved is actually even stronger than showing that the expansion vanishes, which is equivalent to asking that the NP quantities[11] ρ and μ satisfy $\bar{\rho} - \mu = 0$. Carminati (1987) shows that both ρ and μ vanish. The extra conditions can be interpreted as constraints on the kinematic quantities of the $\vec{v}-$congruence, where $\vec{v}$ is defined as the unit vector orthogonal to the fluid flow vector, and lying in the two-space spanned by $\vec{\ell}$ and $\vec{n}$.

[11] The μ used here is the NP quantity. It should not be confused with the energy density which is denoted by μ everywhere else in the present work.

It is of interest to note that the fluid in the spacetimes under consideration must have non-zero acceleration and vorticity, and that the acceleration is orthogonal to the vorticity. Carminati (1987) suggests that an avenue for further exploration is to consider fluids where the acceleration is perpendicular to the vorticity, regardless of the Petrov type. This would complement the results of White and Collins (1984), and is very closely related to the spacetimes explored by Krasiński (1978).

In a later article, Carminati (1988) showed that perfect fluid spacetimes of Petrov type N, for which he had proved that the conjecture holds, are stationary, possess a three-parameter abelian group of local isometries acting simply transitively on time-like hypersurfaces and possess one Killing vector parallel to the flow velocity and another parallel to the vorticity vector. The presence of this last Killing vector entails that spacetimes of Petrov type N must belong to the class of spacetimes studied by Krasiński (1978), and so must belong to our *case C* of chapter 6. Our result that there are no spacetimes within the scope of chapter 6 of Petrov type N that belong to either our *case A* or our *case B* is compatible with the result of Carminati (1988).

Carminati (1990) proves the conjecture for a subcase of the Petrov type III spacetimes. The framework for the proof is the Newman-Penrose formalism, which uses null tetrads. The tetrad is initially chosen as in Carminati (1987). The cases when the pressure is constant, and when it is equal (up to an additive constant) to a third of the energy density have already been solved. The conjecture is then proved for the so-called "aligned" cases. The first aligned case is defined to be that arising when the acceleration vector lies in the two-space spanned by $\vec{m}$ and $\bar{\vec{m}}$. From the $[\delta, \bar{\delta}]$ commutation relation on the energy density, two classes emerge. The first class is further divided into two subclasses, according to whether or not the vorticity vector has a component along the vector $\vec{\ell}-\vec{n}$, *i.e.* depending on whether or not the vorticity vector lies in the two-space spanned by $\vec{m}$ and $\bar{\vec{m}}$. The second class necessarily does not have such a vorticity component. The second aligned case is when the fluid velocity vector lies in the two-spaces spanned by the principal null directions of the Weyl tensor. There are three subcases to be considered.

We note that the $[\delta, \Delta]$ commutation relation was applied to the Ψ_3 Weyl tensor component. Therefore this proof possibly entails the computation of third order derivatives of kinematic quantities. However, both the $\delta-$ and the $\Delta-$derivatives of Ψ_3 were obtained in terms of the kinematic quantities and the energy density. The result of the commutation relation is an algebraic restriction. The highest order derivatives that appear explicitly in this work arise from the first derivatives of Weyl tensor components. These involve second order derivatives of the kinematic quantities. The situation when $\mathrm{d}p/\mathrm{d}\mu = -1/3$ arises as a special case in various places in the proof.

A spacetime admits a *conformal Killing vector*, $\vec{\xi}$ if

$$\mathcal{L}_{\vec{\xi}}g_{ab} = 2\psi\, g_{ab},$$

where $\mathcal{L}_{\vec{\xi}}$ is the Lie derivative along $\vec{\xi}$. The function $\psi(x^a)$ is called the *conformal factor*. If the second covariant derivatives of ψ do not vanish, then $\vec{\xi}$ is called a *proper* conformal Killing vector. If the second covariant derivatives of ψ do vanish, but the first do not, then $\vec{\xi}$ is called a *special* conformal Killing vector. If ψ is a non-zero constant then $\vec{\xi}$ is a *homothetic* vector, whereas if ψ is zero, then $\vec{\xi}$ is a *Killing* vector. Coley (1991) has shown that if there exists a conformal Killing vector parallel to the velocity four-vector, then the shear is necessarily zero. If the vector is a proper conformal Killing vector, then the expansion is non-zero but the vorticity vanishes. The same conclusion holds if the vector is a homothetic vector, whereas if the vector is a Killing vector, then the expansion must vanish. In all cases, the conjecture holds. Coley (1991) also gives the necessary changes to extend the proof of Treciokas and Ellis (1971) to cover situation when the equation of state is $p = \mu/3 + K$ for any constant K. The original proof of Treciokas and Ellis (1971) requires K to be zero.

In summary, the shear-free conjecture is known to hold in the following situations:

1. Spatially homogeneous dust of Bianchi type IX (Gödel, 1950)
2. Spatially homogeneous dust (Schücking, 1957). This generalizes 1.
3. All dust (Ellis, 1967). This generalizes 2. The validity of this result actually holds for constant pressure (White and Collins, 1984).
4. Spatially homogeneous spacetimes with equation of state $p = (\gamma - 1)\mu$, $\gamma \neq 10/9$ (Banerji, 1968). This generalizes 2.
5. Conformally flat spacetimes, *i.e.* spacetimes of Petrov type O (Ellis, 1971).
6. Perfect fluid with $p = \mu/3$ (this includes a relativistic gas) and claim of a proof for PUTH (Treciokas and Ellis, 1971).
7. All spatially homogeneous spacetimes with $\mu + p > 0$ (King and Ellis, 1973). This generalizes 4.
8. Perfect fluids with acceleration parallel to the vorticity and with $\mu + p \neq 0$; this includes the case of constant pressure (White and Collins, 1984). This generalizes 3.

9. Perfect fluids that obey PUTH and with $\mu + p \neq 0$ (Lang and Collins, 1988). This generalizes 7 and has as a proper subcase the non-geodesic portion of 8.

10. Perfect fluids with a Weyl tensor which is purely electric with respect to the fluid (Collins, 1984).

11. Petrov type N spacetimes (Carminati, 1987).

12. "Aligned" Petrov type III spacetimes (Carminati, 1990).[12]

13. Fluids with a conformal Killing vector parallel to the velocity, together with the extension of 6 to cover $p = \mu/3 + \text{constant}$(Coley, 1991).

14. In this work, we show that the conjecture also holds for perfect fluids with a Weyl tensor which is purely magnetic with respect to the fluid.

15. Also in this work, we show that the conjecture holds for coasting universes, *i.e.* universes that obey $p = -\mu/3 + \text{constant}$.

5.2 Shear-free conjecture for spaces with a purely magnetic Weyl tensor

In this section, we shall examine the spacetimes that not only satisfy the hypotheses of the shear-free conjecture, but also satisfy the extra constraint that the electric part of the Weyl tensor with respect to the fluid flow vector vanishes. We shall prove that for such fluids, the shear-free conjecture is valid. The proof presented

[12] Carminati has recently informed us that he has extended this result to all Petrov type III spacetimes.

hereinafter is by contradiction, first assuming that neither the vorticity nor the expansion vanishes.

For a perfect fluid with an equation of state that satisfies $p' = 0$, it is already known that the conjecture holds, regardless of further conditions on the Weyl tensor. The validity of the conjecture was shown for case $p = 0$ by Ellis (1967). White and Collins (1984) showed that with a small modification, the proof of Ellis is valid for the pressure equal to *any* constant value. Treciokas and Ellis (1971) have proved the conjecture for the case when $p = \mu/3$. Coley (1991) has extended this result to $p = \mu/3 + K$, with K being *any* constant. As a result of the foregoing discussion, we can therefore assume throughout the remainder of this chapter that the equation of state is such that $p'(3p' - 1) \neq 0$.

For shear-free perfect fluids, with the $\vec{e}_0$−axis along the fluid flow velocity, the $\vec{e}_1$-axis along the vorticity vector, and the $\vec{e}_2$-axis and $\vec{e}_3$-axis such that $\hat{\theta}_{23}$ can be set to zero, the Riemann curvature two-forms are:

$$
\begin{aligned}
\mathbf{R}^0{}_1 \;=\;& \left(\dot{u}_2\, d_2 + \dot{u}_3\, d_3 + \frac{\partial_0\theta}{3} - \partial_1\dot{u}_1 - \dot{u}_1^2 + \frac{\theta^2}{9}\right)(\eta^0 \wedge \eta^1) \\
&+ \left(\dot{u}_2\, \hat{\theta}_{22} - \dot{u}_2\, \dot{u}_1 - \frac{\dot{u}_3\, n}{2} - \partial_2\dot{u}_1\right)(\eta^0 \wedge \eta^2) \\
&+ \left(\frac{\dot{u}_2\, n}{2} - \partial_3\dot{u}_1 + \dot{u}_3\, \hat{\theta}_{33} - \dot{u}_3\, \dot{u}_1\right)(\eta^0 \wedge \eta^3) \\
&+ \left(d_3\, \omega - \frac{\partial_2\theta}{3}\right)(\eta^1 \wedge \eta^2) \\
&+ \left(2\, \dot{u}_1\, \omega - \omega\hat{\theta}_{22} - \omega\hat{\theta}_{33}\right)(\eta^2 \wedge \eta^3) \\
&+ \left(-\frac{\partial_3\theta}{3} - d_2\, \omega\right)(\eta^1 \wedge \eta^3), \\
\mathbf{R}^0{}_2 \;=\;& \left(\dot{u}_3\, \hat{\Omega} - \dot{u}_1\, d_2 - \frac{\dot{u}_3\, n}{2} - \partial_1\dot{u}_2 - \dot{u}_2\, \dot{u}_1\right)(\eta^0 \wedge \eta^1) \\
&+ \left(\frac{\theta^2}{9} - \partial_2\dot{u}_2 - \dot{u}_3\, A_3 + \frac{\partial_0\theta}{3} - \dot{u}_2^2 - \dot{u}_1\, \hat{\theta}_{22} - \omega^2\right)(\eta^0 \wedge \eta^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{2\,\omega\theta}{3}-\dot{u}_3\,\dot{u}_2-\frac{\dot{u}_1\,n}{2}+\dot{u}_3\,A_2+\partial_0\omega-\partial_3\dot{u}_2\right)(\eta^0\wedge\eta^3)\\
&+\left(\frac{\partial_1\theta}{3}-\frac{\omega n}{2}\right)(\eta^1\wedge\eta^2)\\
&+\left(2\,\dot{u}_2\,\omega-\frac{\partial_3\theta}{3}+\partial_2\omega\right)(\eta^2\wedge\eta^3)\\
&+\left(\partial_1\omega+\omega\hat{\theta}_{33}\right)(\eta^1\wedge\eta^3),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{R}^0{}_3 \;=\;& \left(-\dot{u}_3\,\dot{u}_1-\partial_1\dot{u}_3-\dot{u}_2\,\hat{\Omega}+\frac{\dot{u}_2\,n}{2}-\dot{u}_1\,d_3\right)(\eta^0\wedge\eta^1)\\
&+\left(-\partial_2\dot{u}_3-\dot{u}_3\,\dot{u}_2+\frac{\dot{u}_1\,n}{2}-\frac{2\,\omega\theta}{3}-\partial_0\omega+\dot{u}_2\,A_3\right)(\eta^0\wedge\eta^2)\\
&+\left(\frac{\partial_0\theta}{3}+\frac{\theta^2}{9}-\omega^2-\partial_3\dot{u}_3-\dot{u}_3^2-\dot{u}_2\,A_2-\dot{u}_1\,\hat{\theta}_{33}\right)(\eta^0\wedge\eta^3)\\
&+\left(-\omega\hat{\theta}_{22}-\partial_1\omega\right)(\eta^1\wedge\eta^2)\\
&+\left(\partial_3\omega+\frac{\partial_2\theta}{3}+2\,\dot{u}_3\,\omega\right)(\eta^2\wedge\eta^3)\\
&+\left(\frac{\partial_1\theta}{3}-\frac{\omega n}{2}\right)(\eta^1\wedge\eta^3),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{R}^1{}_2 \;=\;& \left(-\partial_0 d_2-d_3\,\omega-\frac{\theta\,\dot{u}_2}{3}-\frac{\theta\,d_2}{3}\right)(\eta^0\wedge\eta^1)\\
&+\left(\frac{\omega n}{2}-\partial_0\hat{\theta}_{22}-\frac{\theta\,\hat{\theta}_{22}}{3}+\frac{\theta\,\dot{u}_1}{3}\right)(\eta^0\wedge\eta^2)\\
&+\left(\dot{u}_1\,\omega-\omega\hat{\theta}_{33}-\frac{\partial_0 n}{2}-\frac{\theta\,n}{6}\right)(\eta^0\wedge\eta^3)\\
&+\left(\frac{\theta^2}{9}-{d_2}^2+\partial_2 d_2+\frac{n^2}{4}+d_3\,A_3-\partial_1\hat{\theta}_{22}-\hat{\theta}_{22}^2\right)(\eta^1\wedge\eta^2)\\
&+\left(\partial_3\hat{\theta}_{22}-\frac{\partial_2 n}{2}+d_2\,n-A_3\,\hat{\theta}_{33}+A_3\,\hat{\theta}_{22}\right)(\eta^2\wedge\eta^3)\\
&+\left(\hat{\theta}_{33}\,\hat{\Omega}+\partial_3 d_2-\frac{\partial_1 n}{2}+\frac{\omega\theta}{3}-d_2\,d_3-d_3\,A_2-n\hat{\theta}_{33}-\hat{\theta}_{22}\,\hat{\Omega}\right)(\eta^1\wedge\eta^3),
\end{aligned}
$$

$$
\mathbf{R}^2{}_3 \;=\; \left(\frac{\partial_0 n}{2}-\dot{u}_1\,\omega-\frac{\theta\,\hat{\Omega}}{3}-\partial_0\hat{\Omega}+\frac{\theta\,n}{6}-\partial_1\omega\right)(\eta^0\wedge\eta^1)
$$

$$
\begin{aligned}
&+\left(\frac{\theta\,A_3}{3}-\partial_2\omega+\partial_0 A_3-2\,\dot{u}_2\,\omega-\frac{\theta\,\dot{u}_3}{3}\right)(\eta^0\wedge\eta^2)\\
&+\left(\frac{\theta\,\dot{u}_2}{3}-\partial_0 A_2-\partial_3\omega-\frac{\theta\,A_2}{3}-2\,\dot{u}_3\,\omega\right)(\eta^0\wedge\eta^3)\\
&+\left(\hat{\Omega}\,A_2-\hat{\Omega}\,d_2+\hat{\theta}_{22}\,d_3+d_2\,n+\partial_1 A_3-\frac{\partial_2 n}{2}+\partial_2\hat{\Omega}+A_3\,\hat{\theta}_{22}\right)(\eta^1\wedge\eta^2)\\
&+\left(3\,\omega^2+\hat{\Omega}\,n-\partial_3 A_3-\partial_2 A_2+\frac{\theta^2}{9}\right.\\
&\qquad\left.-\hat{\theta}_{22}\,\hat{\theta}_{33}-\frac{3\,n^2}{4}-{A_3}^2-{A_2}^2\right)(\eta^2\wedge\eta^3)\\
&+\left(\partial_3\hat{\Omega}-\hat{\theta}_{33}\,d_2-\frac{\partial_3 n}{2}+\hat{\Omega}\,A_3-\partial_1 A_2+nd_3-\hat{\Omega}\,d_3-A_2\,\hat{\theta}_{33}\right)(\eta^1\wedge\eta^3)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{R}^1{}_3 \;=\;& \left(d_2\,\omega-\frac{\theta\,d_3}{3}-\frac{\theta\,\dot{u}_3}{3}-\partial_0 d_3\right)(\eta^0\wedge\eta^1)\\
&+\left(\omega\hat{\theta}_{22}-\dot{u}_1\,\omega+\frac{\theta\,n}{6}+\frac{\partial_0 n}{2}\right)(\eta^0\wedge\eta^2)\\
&+\left(\frac{\omega n}{2}+\frac{\theta\,\dot{u}_1}{3}-\partial_0\hat{\theta}_{33}-\frac{\theta\,\hat{\theta}_{33}}{3}\right)(\eta^0\wedge\eta^3)\\
&+\left(\partial_2 d_3-\hat{\theta}_{22}\,\hat{\Omega}+\frac{\partial_1 n}{2}+\hat{\theta}_{33}\,\hat{\Omega}-\frac{\omega\theta}{3}+n\hat{\theta}_{22}-d_2\,d_3-d_2\,A_3\right)(\eta^1\wedge\eta^2)\\
&+\left(nd_3-\frac{\partial_3 n}{2}+A_2\,\hat{\theta}_{22}-\partial_2\hat{\theta}_{33}-A_2\,\hat{\theta}_{33}\right)(\eta^2\wedge\eta^3)\\
&+\left(\frac{\theta^2}{9}+d_2\,A_2-{d_3}^2+\frac{n^2}{4}-\partial_1\hat{\theta}_{33}-\hat{\theta}_{33}^2+\partial_3 d_3\right)(\eta^1\wedge\eta^3).
\end{aligned}
$$

Specializing the results of chapter 3, we find that the Einstein field equations, the Jacobi identities, the commutation relations on the acceleration potential and the contracted Bianchi identities are equivalent to the following thirty-three equations:

$$
\partial_0\dot{u}_1 = \frac{6\,\theta\,\dot{u}_1\,{p'}^2-6\,\dot{u}_1\,p''\,\theta\,p-2\,\theta\,\dot{u}_1\,p'-6\,\dot{u}_1\,p''\,\theta\,\mu+9\,{p'}^2\omega n}{6\,p'}, \tag{5.1}
$$

$$
\partial_0\dot{u}_2 = -\frac{3\,\theta\,p''\,\dot{u}_2\,p+\theta\,\dot{u}_2\,p'+3\,\dot{u}_2\,\mu\,p''\,\theta-3\,\partial_2\theta\,{p'}^2-3\,\theta\,\dot{u}_2\,{p'}^2}{3\,p'},
$$

$$
\begin{aligned}
\partial_0\dot{u}_3 &= -\frac{3\,\dot{u}_3\,p''\,\theta\,p - 3\,\theta\,\dot{u}_3\,p'^2 + \theta\,\dot{u}_3\,p' - 3\,\partial_3\theta\,p'^2 + 3\,\dot{u}_3\,\mu\,p''\,\theta}{3\,p'},\\
\partial_0\omega &= p'\,\omega\theta - \frac{2\,\omega\theta}{3},\\
\partial_0 n &= -\frac{\theta\,n}{3},\\
\partial_0 d_2 &= -\frac{\partial_2\theta}{3} - \frac{\theta\,d_2}{3} - \frac{\theta\,\dot{u}_2}{3},\\
\partial_0 d_3 &= -\frac{\partial_3\theta}{3} - \frac{\theta\,\dot{u}_3}{3} - \frac{\theta\,d_3}{3},\\
\partial_0 A_2 &= \frac{\theta\,\dot{u}_2}{3} - \frac{\theta\,A_2}{3} + \frac{\partial_2\theta}{3},\\
\partial_0 A_3 &= \frac{\partial_3\theta}{3} + \frac{\theta\,\dot{u}_3}{3} - \frac{\theta\,A_3}{3},\\
\partial_0\hat{\Omega} &= -\frac{\theta\,\hat{\Omega}}{3},\\
\partial_0\hat{\theta}_{22} &= \frac{\theta\,\dot{u}_1}{3} - \frac{\theta\,\hat{\theta}_{22}}{3} + \frac{\omega n}{2},\\
\partial_0\hat{\theta}_{33} &= \frac{\theta\,\dot{u}_1}{3} - \frac{\theta\,\hat{\theta}_{33}}{3} + \frac{\omega n}{2},\\
\partial_0\mu &= -\theta\,\mu - \theta\,p,\\
\partial_1\dot{u}_1 &= \dot{u}_2\,d_2 + \dot{u}_3\,d_3 + \partial_0\theta + \frac{3\,p}{2} - \dot{u}_1^2 + \frac{\theta^2}{3} - \partial_2\dot{u}_2 - \dot{u}_3\,A_3 - \dot{u}_2^2 - \dot{u}_1\,\hat{\theta}_{22}\\
&\quad -2\,\omega^2 - \partial_3\dot{u}_3 - \dot{u}_3^2 - \dot{u}_2\,A_2 - \dot{u}_1\,\hat{\theta}_{33} - \Lambda + \frac{\mu}{2},\\
\partial_1\omega &= \dot{u}_1\,\omega - \omega\hat{\theta}_{22} - \omega\hat{\theta}_{33},\\
\partial_1\theta &= \frac{3\,\omega n}{2},\\
\partial_1 n &= \frac{2\,\omega\theta}{3} - 2\,\dot{u}_3\,\dot{u}_2 - \dot{u}_1\,n + 2\,\dot{u}_3\,A_2 - 2\,\partial_3\dot{u}_2 + 2\,\hat{\theta}_{33}\,\hat{\Omega} + 2\,\partial_3 d_2\\
&\quad -2\,\hat{\theta}_{22}\,\hat{\Omega} - 2\,d_2\,d_3 - 2\,d_3\,A_2 - 2\,n\hat{\theta}_{33} + 2\,p'\,\omega\theta,\\
\partial_1 A_2 &= \partial_2\hat{\theta}_{33} + \partial_3\hat{\Omega} - \hat{\theta}_{33}\,d_2 - \hat{\Omega}\,d_3 + \hat{\Omega}\,A_3 - A_2\,\hat{\theta}_{22},\\
\partial_1 A_3 &= \hat{\Omega}\,d_2 - \hat{\theta}_{22}\,d_3 - \partial_2\hat{\Omega} - A_3\,\hat{\theta}_{33} + \partial_3\hat{\theta}_{22} - \hat{\Omega}\,A_2,\\
\partial_1\mu &= -\frac{(\mu + p)\,\dot{u}_1}{p'},\\
\partial_2\dot{u}_1 &= \dot{u}_1\,d_2 + \dot{u}_2\,\hat{\theta}_{22} - \dot{u}_3\,\hat{\Omega} + \partial_1\dot{u}_2, \qquad (5.2)
\end{aligned}
$$

$$
\begin{aligned}
\partial_2\dot{u}_3 &= \dot{u}_1\, n - \dot{u}_3\, A_2 + \partial_3\dot{u}_2 - 2\, p'\, \omega\theta + \dot{u}_2\, A_3, \\
\partial_2\omega &= \frac{2\,\partial_3\theta}{3} + d_2\,\omega - 2\,\dot{u}_2\,\omega, \\
\partial_2 n &= 2\,\partial_1\dot{u}_3 - \dot{u}_2\, n + 2\,\dot{u}_2\,\hat{\Omega} + 2\,\dot{u}_1\, d_3 + 2\,\dot{u}_3\,\dot{u}_1 + 2\,\partial_3\hat{\theta}_{22} \\
&\quad +2\, d_2\, n - 2\, A_3\,\hat{\theta}_{33} + 2\, A_3\,\hat{\theta}_{22} \\
\partial_2 d_2 &= \frac{p}{2} - \partial_3\dot{u}_3 - \frac{n^2}{4} + \frac{\partial_0\theta}{3} - \dot{u}_3^2 + \partial_1\hat{\theta}_{22} + \frac{\mu}{2} - \omega^2 - d_3\, A_3 \\
&\quad -\dot{u}_1\,\hat{\theta}_{33} - \dot{u}_2\, A_2 + {d_2}^2 + \hat{\theta}_{22}^2, \\
\partial_2 d_3 &= 2\,\dot{u}_3\,\dot{u}_2 + \dot{u}_1\, n - 2\,\dot{u}_3\, A_2 + 2\,\partial_3\dot{u}_2 - 2\,\hat{\theta}_{33}\,\hat{\Omega} - \partial_3 d_2 + 2\,\hat{\theta}_{22}\,\hat{\Omega} \\
&\quad +2\, d_2\, d_3 + d_3\, A_2 + n\hat{\theta}_{33} - 2\, p'\,\omega\theta - n\hat{\theta}_{22} + d_2\, A_3, \\
\partial_2 A_2 &= -\partial_3\dot{u}_3 - \partial_2\dot{u}_2 + p - \frac{3\, n^2}{4} + \frac{2\,\partial_0\theta}{3} + \frac{\theta^2}{3} - \dot{u}_2\, A_2 - \dot{u}_3^2 - \partial_3 A_3 \\
&\quad -\Lambda + \omega^2 - \dot{u}_3\, A_3 - \dot{u}_1\,\hat{\theta}_{22} + \hat{\Omega}\, n - \hat{\theta}_{22}\,\hat{\theta}_{33} - {A_3}^2 - {A_2}^2 - \dot{u}_2^2 - \dot{u}_1\,\hat{\theta}_{33}, \\
\partial_2\mu &= -\frac{(\mu + p)\,\dot{u}_2}{p'}, \\
\partial_3\dot{u}_1 &= \partial_1\dot{u}_3 + \dot{u}_2\,\hat{\Omega} + \dot{u}_1\, d_3 + \dot{u}_3\,\hat{\theta}_{33}, \qquad (5.3) \\
\partial_3 n &= -2\,\dot{u}_2\,\dot{u}_1 - \dot{u}_3\, n - 2\,\dot{u}_1\, d_2 + 2\,\dot{u}_3\,\hat{\Omega} - 2\,\partial_1\dot{u}_2 + 2\, n d_3 \\
&\quad +2\, A_2\,\hat{\theta}_{22} - 2\,\partial_2\hat{\theta}_{33} - 2\, A_2\,\hat{\theta}_{33}, \\
\partial_3 d_3 &= \frac{p}{2} - \partial_2\dot{u}_2 - \frac{n^2}{4} + \frac{\partial_0\theta}{3} + \partial_1\hat{\theta}_{33} + \frac{\mu}{2} - \omega^2 - \dot{u}_3\, A_3 \\
&\quad -\dot{u}_1\,\hat{\theta}_{22} + \hat{\theta}_{33}^2 + {d_3}^2 - \dot{u}_2^2 - d_2\, A_2, \\
\partial_3\omega &= d_3\,\omega - \frac{2\,\partial_2\theta}{3} - 2\,\dot{u}_3\,\omega
\end{aligned}
$$

and

$$
\partial_3\mu = -\frac{(\mu + p)\,\dot{u}_3}{p'}.
$$

The requirement that the electric part of the Weyl tensor vanish is equivalent to the following equations:

$$
\begin{aligned}
\partial_0\theta &= 3\,\dot{u}_3^2 - \frac{\theta^2}{3} + 3\,\omega^2 + 3\,\partial_3\dot{u}_3 - \frac{3\, p}{2} + 3\,\dot{u}_2\, A_2 + 3\,\dot{u}_1\,\hat{\theta}_{33} - \frac{\mu}{2} + \Lambda, \\
\partial_1\dot{u}_2 &= \dot{u}_3\,\hat{\Omega} - \dot{u}_1\, d_2 - \frac{\dot{u}_3\, n}{2} - \dot{u}_2\,\dot{u}_1, \qquad (5.4)
\end{aligned}
$$

$$\begin{aligned}\partial_1\dot{u}_3 &= -\dot{u}_3\,\dot{u}_1 - \dot{u}_2\,\hat{\Omega} + \frac{\dot{u}_2\,n}{2} - \dot{u}_1\,d_3, \\ \partial_2\dot{u}_2 &= \dot{u}_3^2 - \dot{u}_3\,A_3 - \dot{u}_2^2 - \dot{u}_1\,\hat{\theta}_{22} + \partial_3\dot{u}_3 + \dot{u}_2\,A_2 + \dot{u}_1\,\hat{\theta}_{33}\end{aligned} \tag{5.5}$$

and

$$\partial_3\dot{u}_2 = \dot{u}_3\,A_2 - \frac{\dot{u}_1\,n}{2} + p'\,\omega\theta - \dot{u}_3\,\dot{u}_2.$$

The contact form representing the derivative of $\dot{u}_2$ is therefore

$$\begin{aligned}\mathbf{d}\dot{u}_2 &= -\frac{\left(3\,\dot{u}_2\,p''\,\theta\,p + 3\,\dot{u}_2\,\mu\,p''\,\theta - 3\,\partial_2\theta\,p'^2 - 3\,\theta\,\dot{u}_2\,p'^2 + \theta\,\dot{u}_2\,p'\right)\eta^0}{3\,p'} \\ &+ \left(\dot{u}_3\,\hat{\Omega} - \dot{u}_1\,d_2 - \frac{\dot{u}_3\,n}{2} - \dot{u}_2\,\dot{u}_1\right)\eta^1 \\ &+ \left(\dot{u}_3^2 - \dot{u}_3\,A_3 - \dot{u}_2^2 - \dot{u}_1\,\hat{\theta}_{22} + \partial_3\dot{u}_3 + \dot{u}_2\,A_2 + \dot{u}_1\,\hat{\theta}_{33}\right)\eta^2 \\ &+ \left(-\dot{u}_2\,\dot{u}_3 - \frac{\dot{u}_1\,n}{2} + \dot{u}_3\,A_2 + p'\,\omega\theta\right)\eta^3\end{aligned} \tag{5.6}$$

and the contact form representing the derivative of $\dot{u}_3$ is

$$\begin{aligned}\mathbf{d}\dot{u}_3 &= \left(\frac{-6\,\dot{u}_3\,p''\,\theta\,p - 2\,p'\,\theta\,\dot{u}_3 + 6\,\theta\,\dot{u}_3\,p'^2 - 6\,\dot{u}_3\,\mu\,p''\,\theta + 6\,\partial_3\theta\,p'^2}{6\,p'}\right)\eta^0 \\ &+ \left(-\,\dot{u}_1\,\dot{u}_3 -\,\dot{u}_2\,\hat{\Omega} + \frac{1}{2}\,\dot{u}_2\,n -\,\dot{u}_1\,d_3\right)\eta^1 \\ &+ \left(-\,\dot{u}_2\,\dot{u}_3 +\,\dot{u}_2\,A_3 -\,p'\omega\theta + \frac{1}{2}\,\dot{u}_1\,n\right)\eta^2 \\ &+ \partial_3\dot{u}_3\,\eta^3.\end{aligned} \tag{5.7}$$

Adding the exterior derivative of (5.6) multiplied by $\eta^0\wedge\eta^3$, to the exterior derivative of (5.7) multiplied by $\eta^0 \wedge \eta^2$, taking into account all the previous information, we obtain

$$\left(\hat{\theta}_{22} - \hat{\theta}_{33}\right)\omega^2(\eta^0 \wedge \eta^1 \wedge \eta^2 \wedge \eta^3) = 0.$$

The operation just performed is equivalent to adding together the $[\vec{e}_1, \vec{e}_2]$ commutator applied to $\dot{u}_2$ and the $[\vec{e}_3, \vec{e}_1]$ commutator on $\dot{u}_3$. This particular combination ensures that no second derivatives appear. Such an operation is called 'finding the non-absorbable torsion'. Now since the vorticity, ω, does not vanish by hypothesis, we conclude that

$$\hat{\theta}_{22} = \hat{\theta}_{33}. \tag{5.8}$$

Evaluation of the following seven torsion expressions:

$$[\vec{e}_1, \vec{e}_2]n + 2[\vec{e}_2, \vec{e}_3]d_2 - 2[\vec{e}_1, \vec{e}_3]\hat{\theta}_{22},$$
$$[\vec{e}_0, \vec{e}_1]A_3 - [\vec{e}_0, \vec{e}_3]\hat{\theta}_{22} + [\vec{e}_0, \vec{e}_2]\hat{\Omega} + \tfrac{1}{2}[\vec{e}_1, \vec{e}_2]\omega,$$
$$[\vec{e}_0, \vec{e}_1]A_2 - [\vec{e}_0, \vec{e}_3]\hat{\Omega} - [\vec{e}_0, \vec{e}_2]\hat{\theta}_{22} - \tfrac{1}{2}[\vec{e}_1, \vec{e}_3]\omega,$$
$$[\vec{e}_2, \vec{e}_3]d_3 + \tfrac{1}{2}[\vec{e}_1, \vec{e}_3]n + [\vec{e}_1, \vec{e}_2]\hat{\theta}_{22},$$
$$[\vec{e}_0, \vec{e}_2]n - 2[\vec{e}_0, \vec{e}_2]\hat{\theta}_{22},$$

and

$$[\vec{e}_0, \vec{e}_2]A_2 + [\vec{e}_0, \vec{e}_3]A_3 - \tfrac{1}{2}[\vec{e}_2, \vec{e}_3]\omega$$

provides relations equivalent to the following equalities:

$$\begin{aligned}
\partial_2\theta &= \frac{9\,\omega^2 p'\, d_3 + \mu\,\dot{u}_3 + p\dot{u}_3}{3\,p'\,\omega}, \qquad (5.9)\\
\partial_3\theta &= -\frac{9\,\omega^2 p'\, d_2 + p\dot{u}_2 + \mu\,\dot{u}_2}{3\,p'\,\omega},\\
\partial_1 d_2 &= 5\,\hat{\theta}_{22}\, d_2 - 5\,\dot{u}_1\, d_2 - 3\,\dot{u}_1\,\dot{u}_2 + 3\,\hat{\theta}_{22}\,\dot{u}_2 + d_3\,\hat{\Omega} + \frac{1}{2}\,n\dot{u}_3\\
&\quad + \frac{8\,\hat{\theta}_{22}\, p\dot{u}_2 + 4\,n\mu\,\dot{u}_3 + 8\,\hat{\theta}_{22}\,\mu\,\dot{u}_2 - 4\,\dot{u}_1\, p\dot{u}_2 - 4\,\dot{u}_1\,\mu\,\dot{u}_2 + 4\,np\dot{u}_3}{18\,\omega^2 p'},\\
\partial_1 d_3 &= -\Bigg(d_2\,\hat{\Omega} - 5\,\hat{\theta}_{22}\, d_3 + 5\,\dot{u}_1\, d_3 + \frac{1}{2}\,n\dot{u}_2 + 3\dot{u}_1\,\dot{u}_3 - 3\,\hat{\theta}_{22}\,\dot{u}_3\\
&\quad + \frac{-8\,\hat{\theta}_{22}\,\mu\,\dot{u}_3 + 4\,np\dot{u}_2 + 4\,n\mu\,\dot{u}_2 - 8\,\hat{\theta}_{22}\, p\dot{u}_3 + 4\,\dot{u}_1\,\mu\,\dot{u}_3 + 4\,\dot{u}_1\, p\dot{u}_3}{18\,\omega^2 p'}\Bigg),\\
\partial_3 d_2 &= d_3\, A_2 - 3p'\theta\omega - \frac{3}{2}\,\dot{u}_1\, n + 2\,\hat{\theta}_{22}\, n + \frac{2}{3}\omega\,\theta + d_3\, d_2\\
&\quad + \frac{2\,d_2\,\mu\,\dot{u}_3 - 2\,d_3\, p\dot{u}_2 - 2\,d_3\,\mu\,\dot{u}_2 + 2\,d_2\, p\dot{u}_3}{18\,\omega^2 p'},\\
\partial_2\hat{\theta}_{22} &= -\Bigg(\frac{1}{2}\,n\dot{u}_3 + \dot{u}_1\, d_2 - \hat{\theta}_{22}\, d_2
\end{aligned}$$

$$+\frac{2\,\dot u_1\,\mu\,\dot u_2+2\,\dot u_1\,p\dot u_2+3\,n\mu\,\dot u_3-2\,\hat\theta_{22}\,p\dot u_2-2\,\hat\theta_{22}\,\mu\,\dot u_2+3\,np\dot u_3}{18\,\omega^2p'}\Bigg)$$

and

$$\begin{aligned}\partial_3\hat\theta_{22} &= \frac{1}{2}n\dot u_2-\dot u_1\,d_3+\hat\theta_{22}\,d_3\\ &+\frac{-2\,\dot u_1\,\mu\,\dot u_3+3\,np\dot u_2+3\,n\mu\,\dot u_2+2\,\hat\theta_{22}\,p\dot u_3-2\,\dot u_1\,p\dot u_3+2\,\hat\theta_{22}\,\mu\,\dot u_3}{18\,\omega^2p'}.\end{aligned}$$

Evaluation of the following five combinations of commutation relations:

$$[\vec e_0,\vec e_1]\theta+3[\vec e_1,\vec e_2]\dot u_2,$$
$$[\vec e_0,\vec e_2]\theta-3[\vec e_1,\vec e_2]\dot u_1+\tfrac{3}{2}[\vec e_0,\vec e_3]\omega,$$
$$[\vec e_0,\vec e_3]\theta-3[\vec e_1,\vec e_3]\dot u_1-\tfrac{3}{2}[\vec e_0,\vec e_2]\omega,$$
$$[\vec e_2,\vec e_3]\dot u_1$$

and

$$[\vec e_0,\vec e_1]\dot u_2+p'[\vec e_1,\vec e_2]\theta$$

yields the following equalities:

$$\begin{aligned}p''' &= \Big(12\,p'\,\dot u_1\,\mu\,p''\,p\dot u_3+54\,{p'}^2\dot u_1\,p''\,p\omega^2d_3+18\,{p'}^3\dot u_1\,\omega^2d_3\\ &+2\,{p'}^2\dot u_1\,\mu\,\dot u_3+2\,{p'}^2\dot u_1\,p\dot u_3+54\,{p'}^2\dot u_1\,\mu\,p''\,\omega^2d_3\\ &-54\,\dot u_1\,{p'}^4\omega^2d_3-6\,\dot u_1\,{p'}^3\mu\,\dot u_3-6\,\dot u_1\,{p'}^3p\dot u_3\\ &+6\,p'\,\dot u_1\,\mu^2p''\,\dot u_3+6\,p'\,\dot u_1\,p''\,p^2\dot u_3+12\,\omega\dot u_1\,{p'}^3\theta\,\dot u_2-36\,\omega\dot u_1\,{p'}^4\theta\,\dot u_2\\ &+18\,\omega\dot u_2\,{p''}^2\theta\,\mu^2\dot u_1+36\,\omega\dot u_2\,{p''}^2\theta\,p\mu\,\dot u_1-18\,\omega\dot u_2\,p''\,\theta\,p'\,\mu\,\dot u_1\\ &+18\,\omega\dot u_2\,{p''}^2\theta\,p^2\dot u_1+9\,\dot u_2\,{p'}^3\omega^2n+36\,\omega\dot u_1\,{p'}^2\dot u_2\,p''\,\theta\,\mu\\ &-18\,\omega\dot u_2\,p''\,\theta\,p'\,p\dot u_1+36\,\omega p''\,\theta\,p{p'}^2\dot u_2\,\dot u_1+27\,\dot u_2\,p''\,p{p'}^2\omega^2n\\ &+27\,\dot u_2\,p''\,\mu\,{p'}^2\omega^2n-27\,{p'}^4\omega^2n\dot u_2\Big)\Big/\Big(18\,\omega\dot u_2\,\theta\,p'\,\dot u_1\,(p+\mu)^2\Big),\\ \hat\theta_{22} &= \frac{\dot u_1\,(p+\mu+18\,\omega^2p')}{18\,\omega^2p'},\\ n &= \frac{\theta\,\dot u_1\,\Big(18\,\omega^2p''\,\mu-18\,{p'}^2\omega^2+6\,\omega^2p'+p'\,\mu+18\,\omega^2p''\,p+p'\,p\Big)}{9\,\omega^3p'\,(3\,p'-1)},\end{aligned}$$

$$
\begin{aligned}
d_2 \;=\; & -\Big(27\,\theta\,\omega^2 p''\,p\dot{u}_3 - 27\,\dot{u}_3\,{p'}^2\omega^2\theta - 4\,wp\dot{u}_2 + 2\,\theta\,p'\,p\dot{u}_3 - 4\,\omega\mu\,\dot{u}_2 \\
& +27\,\theta\,\omega^2 p''\,\mu\,\dot{u}_3 + 9\,\omega^2 p'\,\theta\,\dot{u}_3 + 9\,\omega p'\,p\dot{u}_2 \\
& +9\,\omega p'\,\mu\,\dot{u}_2 + 2\,\theta\,p'\,\mu\,\dot{u}_3)/\Big(27\,\omega^3 p'\,\left(3\,p'-1\right)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
d_3 \;=\; & -\Big(27\,\dot{u}_2\,{p'}^2\omega^2\theta - 9\,\omega^2 p'\,\theta\,\dot{u}_2 - 27\,\theta\,\omega^2 p''\,\mu\,\dot{u}_2 - 27\,\theta\,\omega^2 p''\,p\dot{u}_2 \\
& +9\,\omega\mu\,p'\,\dot{u}_3 - 2\,\theta\,p'\,p\dot{u}_2 - 2\,\theta\,p'\,\mu\,\dot{u}_2 - 4\,wp\dot{u}_3 \\
& +9\,\omega p'\,p\dot{u}_3 - 4\,\omega\mu\,\dot{u}_3)/\Big(27\,\omega^3 p'\,\left(3\,p'-1\right)\Big)\,,
\end{aligned}
$$

where the assumption that $\dot{u}_1\dot{u}_2 \neq 0$ has been made. We recall also we can assume that the equation of state satisfies $p'(3p'-1) \neq 0$. The $[\vec{e}_0, \vec{e}_3]\dot{u}_1$ commutation relation then provides

$$p'' = \frac{p'\,\left(3\,p'-1\right)}{3(\mu+p)}. \tag{5.10}$$

The torsion expression

$$[\vec{e}_0, \vec{e}_2]\dot{u}_2 - [\vec{e}_0, \vec{e}_1]\dot{u}_1 + \frac{3p'}{2}[\vec{e}_0, \vec{e}_2]d_2 - \frac{3p'}{2}[\vec{e}_0, \vec{e}_3]d_3 - \frac{3p'}{4}[\vec{e}_2, \vec{e}_3]\omega$$

implies that

$$p' = -\frac{1}{6}.$$

This is inconsistent with equation (5.10). We must therefore have that $\dot{u}_1\dot{u}_2 = 0$.

We now consider the case that $\dot{u}_1$ does not vanish. It follows then that $\dot{u}_2$ must be equal to zero. Differentiation of $\dot{u}_2 = 0$ along $\vec{e}_0$ yields that $\partial_2\theta = 0$, which is equivalent to

$$9\,\omega^2 p'\,d_3 + \dot{u}_3(\mu+p) = 0 \tag{5.11}$$

by the relation (5.9). Subtracting the $[\vec{e}_1, \vec{e}_3]\dot{u}_1$ commutation relation from the $[\vec{e}_2, \vec{e}_3]\dot{u}_2$ commutation relation is equivalent to

$$-3(1-3p')d_3\omega^2 + (-\frac{4}{9p'}+1)\dot{u}_3(\mu+p) = 0. \tag{5.12}$$

Elimination of d_3 between equations (5.11) and (5.12) then requires that

$$\dot{u}_3\omega^2 p'(\mu + p) = 0.$$

from which it follows that $\dot{u}_3 = 0$. Since $\dot{u}_2$ also vanishes, we are therefore in the situation when the acceleration is parallel to the vorticity. The shear-free conjecture was proved for that situation by White and Collins (1984).

Henceforth, we assume that $\dot{u}_1$ is zero. Differentiation of $\dot{u}_1 = 0$ along $\vec{e}_0$, as given by equation (5.1), implies that $n = 0$. Furthermore, differentiation along $\vec{e}_2$, as given by equation (5.2), together with the Weyl tensor constraint (5.4) implies that $\dot{u}_2\hat{\theta}_{22} = 0$, and differentiation along $\vec{e}_3$, as given by equation (5.3), together with the Weyl tensor constraint (5.5) and with the equality (5.8) implies that $\dot{u}_3\hat{\theta}_{22} = 0$. If $\hat{\theta}_{22} \neq 0$, then both $\dot{u}_2$ and $\dot{u}_3$ are equal to zero; therefore, the situation is that for which White and Collins (1984) proved that the shear-free conjecture holds. Hence, we assume that $\dot{u}_1 = \hat{\theta}_{22} = 0$.

The following five torsion expressions:

$$2[\vec{e}_0, \vec{e}_3]\hat{\Omega} - 2[\vec{e}_0, \vec{e}_1]A_2 + [\vec{e}_1, \vec{e}_3]\omega,$$

$$3[\vec{e}_1, \vec{e}_2]\omega - 2[\vec{e}_1, \vec{e}_3]\theta,$$

$$\begin{aligned}\Big(&-3(2\dot{u}_2^2 + \dot{u}_3^2){p'}^2[\vec{e}_0, \vec{e}_2]d_2 - 6\dot{u}_2\dot{u}_3{p'}^2[\vec{e}_0, \vec{e}_3]d_2 - 3(\dot{u}_2^2 + 2\dot{u}_3^2){p'}^2[\vec{e}_0, \vec{e}_3]d_3\\ &-3(\dot{u}_2^2 + \dot{u}_3^2){p'}^2[\vec{e}_2, \vec{e}_3]\omega + \dot{u}_3^2 p'[\vec{e}_0, \vec{e}_2]\dot{u}_2 - 2\dot{u}_2\dot{u}_3 p'[\vec{e}_0, \vec{e}_3]\dot{u}_2\\ &+\dot{u}_2^2 p'[\vec{e}_0, \vec{e}_3]\dot{u}_3\big) \,/\, (\theta\omega^2(\dot{u}_2^2 + \dot{u}_3^2))\,,\end{aligned}$$

$$p'[\vec{e}_2, \vec{e}_3]d_2$$

and

$$p'[\vec{e}_2, \vec{e}_3]d_3$$

imply, respectively,

$$\partial_1 d_2 \;=\; \hat{\Omega} d_3,$$

$$
\begin{aligned}
\partial_1 d_3 &= -\hat{\Omega} d_2, \\
p'' &= -\frac{2}{3}\frac{p'(27p'^2 - 6p' - 1)}{\mu + p}, \\
\partial_2 \theta &= \frac{1}{3}\frac{\dot{u}_3(\mu+p) + 9p'd_3\omega^2}{p'\omega}
\end{aligned}
\tag{5.13}
$$

and

$$
\partial_3 \theta = -\frac{1}{3}\frac{\dot{u}_2(\mu+p) + 9p'd_2\omega^2}{p'\omega}.
$$

Differentiation of p'' with respect to μ gives

$$
p''' = \frac{2}{9}\,\frac{p'(27p'^2 - 6p' - 1)(162p'^2 - 21p' + 1)}{(\mu+p)^2}.
$$

The combination of commutation relations $[\vec{e}_0, \vec{e}_2]\dot{u}_2 + 3p'[\vec{e}_0, \vec{e}_2]d_2$ implies that

$$
\begin{aligned}
&(\mu+p)(6\dot{u}_2\dot{u}_3 p' + 2d_3\dot{u}_2 p' - 2d_2\dot{u}_3 p' - 36p'^2\dot{u}_2\dot{u}_3 + 2\dot{u}_2\dot{u}_3) - 18\theta p'^2\omega^3 \\
&-324p'^3\dot{u}_2\omega^2 d_3 + 54p'^2\dot{u}_2\omega^2 d_3 + 18\dot{u}_2\omega^2 d_3 p' + 54p'^2\theta\dot{u}_2^2\omega + 54p'^3\theta\omega^3 \\
&-1944\dot{u}_2^2 p'^4\theta\omega + 486\dot{u}_2^2 p'^3\theta\omega = 0;
\end{aligned}
\tag{5.14}
$$

that of $[\vec{e}_0, \vec{e}_3]\dot{u}_3 + 3p'[\vec{e}_0, \vec{e}_3]d_3$ results in

$$
\begin{aligned}
&(\mu+p)(-2d_2\dot{u}_3 p' - 2\dot{u}_2\dot{u}_3 + 36p'^2\dot{u}_2\dot{u}_3 - 6\dot{u}_2\dot{u}_3 p' + 2d_3\dot{u}_2 p') - 54p'^2\dot{u}_3\omega^2 d_2 \\
&+486\dot{u}_3^2 p'^3\theta\omega + 54p'^2\theta\dot{u}_3^2\omega - 1944\dot{u}_3^2 p'^4\theta\omega \\
&-18\theta p'^2\omega^3 + 54p'^3\theta\omega^3 - 18\dot{u}_3\omega^2 d_2 p' + 324p'^3\omega^2\dot{u}_3 d_2 = 0,
\end{aligned}
\tag{5.15}
$$

and that of $2[\vec{e}_0, \vec{e}_2]d_2 + 2[\vec{e}_0, \vec{e}_3]d_3 + [\vec{e}_2, \vec{e}_3]\omega$ gives

$$
(\mu+p)(-2d_2\dot{u}_3 + 2d_3\dot{u}_2) - 18\theta\omega^3 p' + 54\theta p'^2\omega^3 = 0. \tag{5.16}
$$

Dividing the difference of equation (5.14) and p' times equation (5.16) by $3p' - 1$ yields

$$
-2\dot{u}_2\left(324\dot{u}_2 p'^3\theta\omega + 54d_3 p'^2\omega^2 + 27\theta p'^2\dot{u}_2\omega + (6p'+1)\dot{u}_3(\mu+p) + 9d_3p'\omega^2\right) = 0 \tag{5.17}
$$

Subtracting p' times equation (5.16) from equation (5.15) and dividing the result by $3p'-1$ yields

$$2\dot{u}_3\left(-324\dot{u}_3{p'}^3\theta\omega+54d_2{p'}^2\omega^2-27\theta{p'}^2\dot{u}_3\omega+(6p'+1)\dot{u}_2(\mu+p)+9d_2p'\omega^2\right)=0. \tag{5.18}$$

Now, p' cannot be equal to $-1/6$, as can be seen by substitution into equation (5.13). This enables us to divide the difference of $\dot{u}_3^2$ times equation (5.17) and $\dot{u}_2^2$ times equation (5.18) by the product $\dot{u}_2\dot{u}_3(6p'+1)$. Doing so gives the relation

$$-18\dot{u}_3\omega^2d_3p'-18\dot{u}_2\omega^2d_2p'-2\dot{u}_3^2\mu-2\dot{u}_3^2p-2\dot{u}_2^2p-2\dot{u}_2^2\mu=0. \tag{5.19}$$

The combination of commutation relations

$$3\dot{u}_3[\vec{e}_0,\vec{e}_2]\omega-2\dot{u}_3[\vec{e}_0,\vec{e}_3]\theta-3\dot{u}_2[\vec{e}_0,\vec{e}_3]\omega-2\dot{u}_2[\vec{e}_0,\vec{e}_2]\theta$$

is equivalent to

$$-\omega\left((-27p'+81{p'}^2)\omega^2(\dot{u}_2d_2+\dot{u}_3d_3)+(9p'-4)(\dot{u}_2^2+\dot{u}_3^2)(\mu+p)\right)=0. \tag{5.20}$$

Subtracting $\omega(9p'-4)$ times equation (5.19) from twice equation (5.20) yields

$$-18\omega^3p'(\dot{u}_3d_3+\dot{u}_2d_2)=0,$$

from which we deduce that $\dot{u}_2d_2+\dot{u}_3d_3=0$. Equation (5.19) then simplifies and becomes:

$$-2(\dot{u}_3^2+\dot{u}_2^2)(\mu+p)=0,$$

i.e. $\dot{u}_2=\dot{u}_3=0$, and so the acceleration is parallel to the vorticity. By White and Collins (1984), the validity of the conjecture holds in this case also.

5.3 Perfect fluids with an equation of state that obeys $dp/d\mu = -\frac{1}{3}$

We now prove the conjecture for the special situation of a general relativistic perfect fluid with a barotropic equation of state that satisfies $dp/d\mu = -1/3$. We show that the requirement that neither the vorticity, ω, nor the expansion, θ, vanish leads to a contradiction. While this equation is admittedly rather unphysical in the context of standard general relativity, it does represent an interesting limiting case for which the validity of the shear-free conjecture has heretofore not been established, as far as we are aware. Some further discussion of the physical relevance of this equation of state will be provided at the end of the present section.

We use an orthonormal tetrad with the $\vec{e}_0-$axis along the fluid flow velocity, the $\vec{e}_1-$axis along the vorticity vector, and the $\vec{e}_2-$axis and $\vec{e}_3-$axis such that $\hat{\theta}_{23}$ is set to zero. The Einstein field equations, the Jacobi identities, the commutation relations on the acceleration potential and the contracted Bianchi identities are obtained by setting $p' = -1/3$ in the thirty-three equalities beginning with equation (5.1), where the prime ($'$) denotes differentation with respect to the energy density, μ.

The combinations of commutation relations

$$-2[\vec{e}_0,\vec{e}_3]d_2+[\vec{e}_0,\vec{e}_1]n+2[\vec{e}_0,\vec{e}_3]\dot{u}_2,$$
$$[\vec{e}_0,\vec{e}_2]n-2[\vec{e}_0,\vec{e}_3]\dot{u}_1-2[\vec{e}_0,\vec{e}_3]\hat{\theta}_{22},$$
$$[\vec{e}_0,\vec{e}_3]n+2[\vec{e}_0,\vec{e}_2]\dot{u}_1+2[\vec{e}_0,\vec{e}_2]\hat{\theta}_{33},$$
$$[\vec{e}_0,\vec{e}_2]\dot{u}_2+[\vec{e}_0,\vec{e}_3]\dot{u}_3-2[\vec{e}_0,\vec{e}_2]\dot{u}_1+3[\vec{e}_0,\vec{e}_2]A_2+3[\vec{e}_0,\vec{e}_3]A_3-[\vec{e}_2,\vec{e}_3]\omega,$$
$$[\vec{e}_0,\vec{e}_1]\dot{u}_1+[\vec{e}_0,\vec{e}_2]\dot{u}_2-2[\vec{e}_0,\vec{e}_3]\dot{u}_3+3[\vec{e}_0,\vec{e}_1]\hat{\theta}_{22}-3[\vec{e}_0,\vec{e}_2]d_2-[\vec{e}_2,\vec{e}_3]\omega$$

and

$$[\vec{e}_0,\vec{e}_1]\dot{u}_1-2[\vec{e}_0,\vec{e}_2]\dot{u}_2+[\vec{e}_0,\vec{e}_3]\dot{u}_3+3[\vec{e}_0,\vec{e}_1]\hat{\theta}_{33}-3[\vec{e}_0,\vec{e}_3]d_3-[\vec{e}_2,\vec{e}_3]\omega$$

are equivalent to the following equalities:

$$0 = (8/3)\theta\dot{u}_3\dot{u}_2+(4/3)\dot{u}_3\partial_2\theta+(4/3)\partial_3\theta\dot{u}_2, \quad (5.21)$$
$$0 = -(4/3)\dot{u}_1\partial_3\theta-(8/3)\theta\dot{u}_3\dot{u}_1-2\omega n\dot{u}_3, \quad (5.22)$$
$$0 = (4/3)\dot{u}_1\partial_2\theta+(8/3)\theta\dot{u}_1\dot{u}_2+2n\dot{u}_2\omega, \quad (5.23)$$
$$0 = (4/3)\theta\dot{u}_2^2+(4/3)\dot{u}_2\partial_2\theta-(16/3)\omega^2\theta-(8/3)\theta\dot{u}_1^2 +(4/3)\theta\dot{u}_3^2-4\dot{u}_1\omega n+(4/3)\partial_3\theta\dot{u}_3, \quad (5.24)$$
$$0 = (4/3)\theta\dot{u}_2^2+(4/3)\dot{u}_2\partial_2\theta+(8/3)\omega^2\theta+(4/3)\theta\dot{u}_1^2 -(8/3)\theta\dot{u}_3^2+2\dot{u}_1\omega n-(8/3)\partial_3\theta\dot{u}_3 \quad (5.25)$$

and

$$0 = -(8/3)\theta\dot{u}_2^2-(8/3)\dot{u}_2\partial_2\theta+(8/3)\omega^2\theta+(4/3)\theta\dot{u}_1^2 +(4/3)\theta\dot{u}_3^2+2\dot{u}_1\omega n+(4/3)\partial_3\theta\dot{u}_3. \quad (5.26)$$

We compute the resultant with respect to $\partial_3\theta$ of equation (5.21) and equation (5.22). We then eliminate $\partial_2\theta$ from the result, using the resultant with equation (5.23). We thus obtain

$$\dot{u}_3\dot{u}_2\dot{u}_1(3\omega n+2\theta\dot{u}_1)=0. \quad (5.27)$$

Similarly, we compute the resultant with respect to $\partial_3\theta$ of equation (5.22) and equation (5.24). We then eliminate $\partial_2\theta$ from the result, using the resultant with equation (5.23). Thus, we get

$$\begin{aligned} &2\dot{u}_2^2\dot{u}_1^2\theta + 8\dot{u}_1^2\omega^2\theta + 4\theta\dot{u}_1^4 + 2\theta\dot{u}_3^2\dot{u}_1^2 + 6\dot{u}_1^3\omega n \\ &\quad +3\dot{u}_1\omega n\dot{u}_3^2 + 3\dot{u}_2^2\dot{u}_1 n\omega = 0. \end{aligned} \tag{5.28}$$

Elimination of n between equation (5.27) and equation (5.28) using the resultant yields

$$\dot{u}_3\dot{u}_2\dot{u}_1^3\omega^3\theta = 0, \tag{5.29}$$

whereby $\dot{u}_1\dot{u}_2\dot{u}_3 = 0$. Adding twice equation (5.24) to equation (5.25) results in

$$2\theta\dot{u}_2^2 + 2\dot{u}_2\partial_2\theta - 4\omega^2\theta - 2\theta\dot{u}_1^2 - 3\dot{u}_1\omega n = 0, \tag{5.30}$$

whereas subtraction of equation (5.24) from equation (5.25) yields

$$4\omega^2\theta + 2\theta\dot{u}_1^2 - 2\theta\dot{u}_3^2 + 3\dot{u}_1\omega n - 2\partial_3\theta\dot{u}_3 = 0. \tag{5.31}$$

We eliminate $\partial_2\theta$ between equations (5.30) and (5.23), and eliminate $\partial_3\theta$ between equations (5.31) and (5.22) to obtain

$$2\dot{u}_1\theta\dot{u}_2^2 + 4\dot{u}_1\omega^2\theta + 2\theta\dot{u}_1^3 + 3\dot{u}_1^2\omega n + 3\dot{u}_2^2 n\omega = 0 \tag{5.32}$$

and

$$4\dot{u}_1\omega^2\theta + (16/3)\theta\dot{u}_1^3 + 2\theta\dot{u}_3^2\dot{u}_1 + 3\dot{u}_1^2\omega n + 3\omega n\dot{u}_3^2 = 0, \tag{5.33}$$

respectively.

We now look at the three cases implied by equation (5.29). The first case has $\dot{u}_3 = 0$. Equations (5.21) and (5.22) show that if $\partial_3\theta$ is not equal to zero, then the flow is geodesic. However, this is not compatible with the requirement that $p' = -1/3$, since geodesic flow implies that $p' = 0$. It follows therefore that $\partial_3\theta = 0$.

The sum of the resultant of equations (5.23) and (5.24), with respect to $\partial_2\theta$, and twice the resultant of equations (5.23) and (5.25), with respect to $\partial_2\theta$, reduces to

$$\dot{u}_2^2(3\omega n + 2\theta\dot{u}_1) = 0. \tag{5.34}$$

The situation of $\dot{u}_2 = \dot{u}_3 = 0$ was covered by White and Collins (1984), who showed that the shear-free conjecture holds in this case. We can thus suppose that $\dot{u}_2 \neq 0$. The resultant of equations (5.23) and (5.25) with respect to $\partial_2\theta$ subtracted from the resultant of equations (5.23) and (5.24) with respect to $\partial_2\theta$ simplifies to

$$\dot{u}_1(2\theta\dot{u}_1^2 + 3\dot{u}_1\omega n + 4\omega^2\theta) = 0. \tag{5.35}$$

Eliminating n between equations (5.34) and (5.35) yields

$$\dot{u}_2^2\dot{u}_1\omega^2\theta = 0,$$

whence $\dot{u}_1 = 0$. Propagation of $\dot{u}_1 = 0$ along the fluid flow implies the vanishing of n. Equation (5.31) then gives that $\omega\theta = 0$ and so the shear-free conjecture holds.

The second case implied by equation (5.29) has $\dot{u}_2 = 0 \neq \dot{u}_3$. Since our choice of tetrad and the structure equations (3.12) (to 3.15) are invariant under the discrete symmetry $\vec{e}_2 \mapsto \vec{e}_3, \vec{e}_3 \mapsto -\vec{e}_2$, so also are our equations. In particular this implies that $\dot{u}_2 = 0 \neq \dot{u}_3$ is equivalent to the situation of $\dot{u}_3 = 0 \neq \dot{u}_2$ which we treated in the preceding paragraph. Thus, the shear-free conjecture holds for the present case as well.

The third, and last, case implied by equation (5.29) has $\dot{u}_1 = 0 \neq \dot{u}_2\dot{u}_3$. Propagation of $\dot{u}_1 = 0$ along the fluid flow, given by equation (3.25), entails that $n = 0$. The resultant of equations (5.21) and (5.30) with respect to $\partial_2\theta$ simplifies to

$$\theta\dot{u}_3\dot{u}_2^2 + 2\dot{u}_3\omega^2\theta + \partial_3\theta\dot{u}_2^2 = 0. \tag{5.36}$$

The resultant of equations (5.31) and (5.36) with respect to $\partial_3\theta$ becomes

$$\omega^2\theta(\dot{u}_3^2 + \dot{u}_2^2) = 0,$$

which is a contradiction. The shear-free conjecture therefore holds in this third case as well.

The situation of $p' = -1/3$ includes spacetimes that obey a gamma-law of state $p = (\gamma - 1)\mu$ with $\gamma = 2/3$. These spacetimes are generally regarded as non-physical since γ is usually restricted to lie between 1 and 2. Other conditions which are frequently imposed on the equation of state are $\mu + p > 0$ and $\mu + 3p > 0$ (see Ellis (1971) for more details). The case where $\gamma = 2/3$ is then a limiting case of the second condition. There are further spacetimes where $\gamma = 2/3$ is a limiting value. Raychaudhuri's equation, which is the (00) Einstein field equation, is given by Ellis (1971) as being

$$3\ddot{\ell}/\ell = 2(\omega^2 - \sigma^2) + \dot{u}^a_{;a} - \frac{1}{2}(\mu + 3p) + \Lambda,$$

where ℓ is a length scale obeying $\dot{\ell}/\ell = \theta/3$. From this equation, it is readily apparent that matter-energy is in some sense attractive when $\mu + 3p > 0$ and repulsive when $\mu + 3p > 0$. The limiting situation, when $\mu + 3p = 0$ reduces to $\gamma = 2/3$ for a gamma-law of state. To clarify further the rôle of $\mu + 3p$, we shall discuss Raychaudhuri's equation in situations of especial physical interest. If we consider the situation of a static star model filled with a perfect fluid (and the cosmological constant taken to be zero), then Raychaudhuri's equation, which is the (00) Einstein field equation, reduces to

$$\dot{u}^a_{;a} = (1/2)(\mu + 3p),$$

as given by Ellis (1971). For the Friedman-Robertson-Walker solutions, Raychaudhuri's equation becomes, as given by Ellis (1973),

$$3\frac{\ddot{R}}{R} + \frac{1}{2}(\mu + 3p) - \Lambda = 0,$$

with $3\dot{R}/R$ being the expansion θ. When the cosmological constant is zero, $\gamma = 2/3$ again represents a special situation, being a critical value that separates accelerating universes from decelerating universes. In the Einstein static solution, which is a Friedmann-Robertson-Walker model with $\theta = 0$, the cosmological constant obeys $\Lambda = (1/2)(\mu + 3p)$, and therefore changes sign at $\gamma = 2/3$. The value $\gamma = 2/3$ is also a limiting case of Gödel's universe, generalized to include pressure (Ellis, 1973), since such spacetimes obey

$$\begin{aligned} &2\omega^2 + \Lambda = \tfrac{1}{2}(\mu + 3p) \\ &\tfrac{1}{2}(\mu - p) = -\Lambda. \end{aligned} \tag{5.37}$$

Spacetimes with $p' = -1/3$ are a genuine special case of the shear-free conjecture. This can be seen, for example, by computing the combination of commutation relations

$$-[\vec{e}_2, \vec{e}_3]\theta + 3[\vec{e}_0, \vec{e}_2]\dot{u}_3 - 3[\vec{e}_0, \vec{e}_3]\dot{u}_2,$$

which gives

$$(1 + 3p')\left((3/2)\omega n^2 + \partial_3\partial_2\theta - \partial_2\partial_3\theta + \partial_2\theta A_3 - \partial_3\theta A_2 - 2\partial_0\theta\omega\right) = 0. \tag{5.38}$$

We note that when $p' = -1/3$, equation (5.38) becomes a trivial torsion equation. Other non-torsion expressions become non-trivial torsion expressions when $p' = -1/3$. An example of this situation is given by the combination of commutation relations

$$-2[\vec{e}_0, \vec{e}_3]d_2 + [\vec{e}_0, \vec{e}_1]n + 2[\vec{e}_1, \vec{e}_3]\dot{u}_2,$$

which is

$$\begin{aligned}&(1+3p')\Big(-(1/3)\partial_3\partial_2\theta+(2/3)\partial_0\theta\omega-(2/3)\partial_3\dot{u}_2\theta+(2/3)\partial_3\theta A_2-(1/2)\omega n^2\\&\quad+(2/3)A_2\dot{u}_3\theta-(1/3)\theta\dot{u}_1 n+(2/3)p'\theta^2\omega\Big)\\&\quad+(2/3)\dot{u}_3\theta\dot{u}_2-6p'\theta\dot{u}_2\dot{u}_3-4p'\partial_3\theta\dot{u}_2-4p'\dot{u}_3\partial_2\theta\\&\quad+\frac{p''(\mu+p)}{p'}\Big(2\dot{u}_3\partial_2\theta+2\theta\partial_3\dot{u}_2+2\dot{u}_2\partial_3\theta-2\frac{\dot{u}_3\dot{u}_2\theta}{p'}+6\dot{u}_2\dot{u}_3\theta\\&\quad+n\dot{u}_1\theta-2A_2\dot{u}_3\theta-2\omega\theta^2p'\Big)\\&\quad+\left(\frac{p''^2(\mu+p)^2}{p'^3}-\frac{p'''(\mu+p)^2}{p'^2}\right)(2\dot{u}_2\dot{u}_3\theta)=0.\end{aligned}\tag{5.39}$$

Equation (5.39) becomes a torsion equation when $p'=-1/3$, and reduces to equation (5.21). There is thus a substantial reduction in computational work.

It is of interest to note that $p'=-1/3$ was obtained as an intermediate result in parts of previous proofs of the conjecture. For example, it appears in White and Collins (1984), in Carminati (1987) and in Carminati (1990),

In three of the cases[13] discussed by Collins and White (1984), the matter necessarily obeys the equation of state $\mu+3p-2\Lambda=0$. Collins and White (1984) mention that this equation of state is physically unreasonable, but point out that such an equation of state, with $\mu+3p=constant$, occurs for a class of solutions due to Wahlquist (1968), of which a limiting case, with $\mu+3p=0$, is due to Vaidya (1977). These solutions are of Petrov type D with a shear-free, expansion-free, rotating and accelerating fluid flow. They admit an abelian G_2 isometry group acting on timelike orbits.

While $\gamma=2/3$ may be unphysical in the context of standard general relativistic cosmology, it is certainly not so in the context of inflationary cosmology. Ellis (1990)

[13]Labelled by IIAAii, IIIAAi and IIIAAii by Collins and White (1984)

mentions that, for Friedmann-Robertson-Walker models, the value $\gamma = 2/3$ is a critical one which separates decelerating models from accelerating models. Universes with $\gamma = 2/3$ are called *coasting* universes. Accelerating models, called *inflationary* models, violate the usual inequalities on the energy. If the cosmological constant is positive, a non-interacting mixture of matter, radiation and the cosmological constant would evolve from a radiation-dominated universe ($\gamma \approx 4/3$) to a matter-dominated universe ($\gamma \approx 1$), then asymptotically to a universe dominated by the cosmological constant ($\gamma \to 0$). There will therefore be a point when the critical value of $\gamma = 2/3$ is attained. Coasting universes can be obtained in terms of a scalar field solution, but not by any known simple matter. In particular, there exists a coasting generalized version of the Milne universe. The classical Milne universe is empty; however, the scalar field allows the generalized version to be non-empty. Coasting universes solve, in a weak sense, the horizon problem, which relates to the following question: why do two widely separated regions of the sky have similar background radiation when not enough time, classically, has elapsed for these regions to be causally related? The coasting universes allow for the possibility of a mechanism that would ensure that all such regions be indeed causally related, but do not guarantee in general the existence of such mechanisms (which is why it is only in a weak sense that coasting universes solve the horizon problem).

Chapter 6

Rotating non-expanding shear-free hypersurface-homogeneous spacetimes

On rencontre sa destinée souvent par des chemins qu'on prend pour l'éviter.

Jean de la Fontaine

WE consider a perfect-fluid shear-free spacetime that is rotating but not expanding. The particular class of spacetimes we shall examine was first described by Collins (1988); however, we provide a different characterization. The vector $\vec{e}_0$ is chosen to be the normalized velocity vector which can be taken as the unique unit time-like future-pointing eigenvector of the Ricci tensor, provided that the energy density, μ, and the pressure, p, are such that $\mu + p \neq 0$. Suppose that the spacetime admits a unique exact unit space-like covector that is annihilated by $\vec{e}_0$. Let $\vec{e}_1$ be the vector that corresponds to this covector via the metric. The above conditions require the vanishing of the kinematic quantities $\theta_{\alpha\beta}$, $\omega_2+\Omega_2$, $\omega_3+\Omega_3$, d_2, d_3 and n, which appear in equations (3.12) to (3.15). We rotate the $\vec{e}_2-$ and $\vec{e}_3-$axes by an angle Θ as follows:

$$\vec{e}_2 \mapsto \cos\Theta\vec{e}_2+\sin\Theta\vec{e}_3$$

and

$$\vec{e}_3 \mapsto -\sin\Theta\vec{e}_2+\cos\Theta\vec{e}_3.$$

This rotation is used so that ω_3 is set to zero at a point. We are then free to make ω_3 vanish on a hypersurface transverse to the fluid flow. Propagation of ω_3 along $\vec{e}_0$, given by the Jacobi identity 3.19 simplified using equation (3.26), shows that ω_3 is then zero everywhere provided that $\omega_2(\omega_1+\Omega_1)$ vanishes. Now this is easily ensured, since under the aforementioned rotation, $\omega_1+\Omega_1$ transforms by the formula

$$\omega_1+\Omega_1 \mapsto \omega_1+\Omega_1+\partial_0\Theta.$$

By choosing the rotation so that $\partial_0\Theta=-(\omega_1+\Omega_1)$, we can ensure that $\omega_1+\Omega_1$ is zero and thus also the same applies to ω_3.

The structure equations now obey:

$$\begin{aligned} \mathrm{d}\bar{\omega}^0 &= -\dot{u}_1\bar{\omega}^0\wedge\bar{\omega}^1-\dot{u}_2\bar{\omega}^0\wedge\bar{\omega}^2-\dot{u}_3\bar{\omega}^0\wedge\bar{\omega}^3 \\ &\quad +2\omega_1\bar{\omega}^2\wedge\bar{\omega}^3+2\omega_2\bar{\omega}^3\wedge\bar{\omega}^1, \\ \mathrm{d}\bar{\omega}^1 &= 0, \\ \mathrm{d}\bar{\omega}^2 &= \hat{\theta}_{22}\bar{\omega}^1\wedge\bar{\omega}^2-A_3\bar{\omega}^2\wedge\bar{\omega}^3+(-\hat{\Omega}-\hat{\theta}_{23})\bar{\omega}^3\wedge\bar{\omega}^1 \end{aligned}$$

and

$$\mathrm{d}\bar{\omega}^3 = (\hat{\theta}_{23}-\hat{\Omega})\bar{\omega}^1\wedge\bar{\omega}^2+A_2\bar{\omega}^2\wedge\bar{\omega}^3-\hat{\theta}_{33}\bar{\omega}^3\wedge\bar{\omega}^1.$$

Since $\bar{\omega}^1$ is exact, it defines (locally) a coordinate function, x. We require that all the kinematic quantities, the pressure, p, and the energy density, μ, be non-constant functions of x only. Because of this, $\partial_2 p = \partial_3 p = 0$, and therefore $\dot{u}_2 = \dot{u}_3 = 0$. Since p is a non-constant function of x, the acceleration does not vanish, and so $\dot{u}_1$ is not equal to zero. The (01) field equation simplifies to $A_3\omega_2 = 0$; the (02) field equation to $\omega_2(\hat{\theta}_{23}+\hat{\Omega}) = 0$; the (12) field equation to $A_2(\hat{\theta}_{22}-\hat{\theta}_{33})+2A_3\hat{\theta}_{23}+2\omega_1\Omega_2 = 0$; and the (13) field equation to $A_3(\hat{\theta}_{22} - \hat{\theta}_{33}) - 2A_2\hat{\theta}_{23} = 0$. If $\omega_2 = 0$ then the vorticity and the acceleration are parallel, in which case, the situation has been studied by Collins and White (1984). The relevant situation here is case III of Collins and White (1984), since we require shear-free non-expanding rotating fluids. If instead we require that $\omega_2 \neq 0$, then we have $A_3 = 0$ and $\hat{\Omega} = -\hat{\theta}_{23}$. We are now in the situation studied by Collins (1988), in which the spacetime is hypersurface-homogeneous (the orbits of the isometry group being given by $\{x = \text{constant}\}$), and we shall be concerned with this in the remainder of this chapter.

For ease of comparison, since Collins (1988) uses the notation of MacCallum (1973), we shall make use of the following quantities:

$$
\begin{aligned}
n_{23} &:= (\hat{\theta}_{33} - \hat{\theta}_{22})/2, \\
a_1 &:= -(\hat{\theta}_{33} + \hat{\theta}_{22})/2, \\
a_2 &:= -A_2/2 \\
&\text{and} \\
n_{33} &:= -2\hat{\theta}_{23}.
\end{aligned}
$$

The inverse relations are:

$$
\begin{aligned}
\hat{\theta}_{22} &:= -(a_1 + n_{23}), \\
\hat{\theta}_{33} &:= n_{23} - a_1,
\end{aligned}
$$

$$A_2 := -2a_2$$

and

$$\hat{\theta}_{23} := -n_{33}/2.$$

The structure equations are therefore given by:

$$\begin{aligned}
\mathrm{d}\bar{\omega}^0 &= -\dot{u}_1\bar{\omega}^0 \wedge \bar{\omega}^1 + 2\omega_1\bar{\omega}^2 \wedge \bar{\omega}^3 + 2\omega_2\bar{\omega}^3 \wedge \bar{\omega}^1, \\
\mathrm{d}\bar{\omega}^1 &= 0, \\
\mathrm{d}\bar{\omega}^2 &= -(a_1 + n_{23})\bar{\omega}^1 \wedge \bar{\omega}^2 \\
&\text{and} \\
\mathrm{d}\bar{\omega}^3 &= -n_{33}\bar{\omega}^1 \wedge \bar{\omega}^2 - 2a_2\bar{\omega}^2 \wedge \bar{\omega}^3 + (a_1 - n_{23})\bar{\omega}^3 \wedge \bar{\omega}^1.
\end{aligned}$$

We note that the tetrad is now uniquely determined. The (13) field equation now simplifies to

$$a_2 n_{33} = 0, \tag{6.1}$$

whereas the (12) field equation simplifies to

$$2a_2 n_{23} - \omega_1\omega_2 = 0. \tag{6.2}$$

One combination of the Einstein field equations gives the constraint

$$4\omega_1^2 - 4\omega_2^2 + 8\dot{u}_1 a_1 + 4(p - \Lambda) - 4a_1^2 - 16a_2^2 + 4n_{23}^2 + n_{33}^2 = 0. \tag{6.3}$$

The remaining Einstein field equations, Jacobi identities and contracted Bianchi identities give the propagation along $\vec{e}_1$ of the quantities as follows:

$$\begin{aligned}
\partial_1\omega_1 &= \dot{u}_1\omega_1 + 2\omega_1 a_1 + 2\omega_2 a_2, \\
\partial_1\omega_2 &= \omega_2(-2\dot{u}_1 + n_{23} + a_1), \\
\partial_1\dot{u}_1 &= -\Lambda + (3/2)p + (1/2)\mu - 2\omega_1^2 - 2\omega_2^2 - \dot{u}_1^2 + 2\dot{u}_1 a_1,
\end{aligned}$$

$$\begin{aligned}
\partial_1 a_1 &= (1/2)p + (1/2)\mu - \omega_1^2 - 2\omega_2^2 + \dot{u}_1 a_1 + n_{23}^2 + a_1^2 + (1/4)n_{33}^2, \\
\partial_1 a_2 &= a_2(n_{23} + a_1), \\
\partial_1 n_{23} &= -\dot{u}_1 n_{23} + 2a_1 n_{23} + (1/2)n_{33}^2 + \omega_2^2, \\
\partial_1 n_{33} &= n_{33}(-\dot{u}_1 + 2a_1 - 2n_{23}), \\
\partial_1 p &= -\dot{u}_1(\mu + p)
\end{aligned}$$

and

$$\partial_1 \Lambda = 0.$$

The quantity Λ is the cosmological constant. Therefore, the only quantity for which there is not a propagation equation is the energy density μ. These equations reproduce the results of Collins (1988).

As noted by Collins (1988), the quantity ω_1 vanishes if and only if the quantity a_2 does. The proof is as follows. Suppose that $\omega_1 = 0$. Propagation of ω_1 entails that $\omega_2 a_2 = 0$. Since we are operating under the assumption that $\omega_2 \neq 0$, then $a_2 = 0$. Conversely, if we assume that $a_2 = 0$, equation (6.2) implies that $\omega_1 = 0$. Therefore requiring that $a_2 = 0$ is equivalent to requiring that the vorticity be orthogonal to the acceleration for the spacetimes under consideration. We note that, since ω_2 does not vanish, the quantity n_{23} cannot vanish. If n_{23} did vanish, the propagation of n_{23} would imply that n_{33} and ω_2 both vanish.

Because of equation (6.1), there are three cases to be considered. The first case, which we shall refer to as *case A*, has $n_{33} = 0 \neq a_2$. Since $a_2 \neq 0$, it follows that $\omega_1 \neq 0$. Therefore, *case A* has $n_{33} = 0 \neq \omega_1 \omega_2 a_2 n_{23} \dot{u}_1$. Collins (1988) has identified that spacetimes belonging to *case A* admit a G_3 isometry group of Bianchi-Behr type VI_h with $h = -1$ (*i.e.* Bianchi type III). Also, there is a Killing vector which is not parallel to the fluid velocity vector and orthogonal to the vorticity vector.

The second case resulting from equation (6.1), *case B,* has $a_2 = 0 \neq n_{33}$. By the discussion above, requiring $a_2 = 0$ is equivalent to requiring $\omega_1 = 0$. Therefore, *case B* has the constraints $a_2 = \omega_1 = 0 \neq n_{33}n_{23}\omega_2\dot{u}_1$. Collins (1988) has found that spacetimes in case B admit a G_3 isometry group of Bianchi type I and that there is a Killing vector which is independent of the fluid velocity vector and orthogonal to the vorticity vector.

The third case, *case C,* has $n_{33} = a_2 = 0$. By the preceding discussion, *case C* has the constraints $n_{33} = a_2 = \omega_1 = 0 \neq \omega_2 n_{23}\dot{u}_1$. Collins (1988) has identified that the spacetimes which belong to *case C* admit a G_3 isometry group of Bianchi type I. They have a Killing vector which is independent of the velocity vector and orthogonal to the vorticity. Furthermore, *case C* is the only case where there is an additional Killing vector which is parallel to the vorticity; this is equivalent for the spacetimes under consideration to having a Killing vector which is independent of the velocity vector and which lies in the 2-surfaces spanned by the velocity vector and the vorticity vector. Spacetimes belonging to *case C* coincide with the spacetimes studied by Krasiński (1978).

We now wish to further the study of those spacetimes started by Collins (1988). We shall be interested in finding which Petrov types of the Weyl tensor are allowed in each of the three cases identified above. More information about the Petrov classification can be found in chapter 4. The Weyl tensor can be decomposed into two matrices with the help of the velocity vector, $\vec{e}_0$. The electric part of the Weyl tensor, with respect to $\vec{e}_0$, is given by the (real) 3×3 trace-free symmetric matrix $E_{\alpha\beta}$ where the entries satisfy:

$$\begin{aligned}
E_{11} &= -(2/3)\Lambda + p + (1/3)\mu - 2\omega_1^2 - \omega_2^2 + 2\dot{u}_1 a_1, \\
E_{12} &= E_{21} = -\omega_2\omega_1, \\
E_{13} &= E_{31} = 0,
\end{aligned}$$

$$E_{22} = -\dot{u}_1 a_1 + \omega_1^2 - \dot{u}_1 n_{23} + (1/3)\Lambda - (1/2)p - (1/6)\mu,$$
$$E_{23} = E_{32} = -(1/2)\dot{u}_1 n_{33}$$

and

$$E_{33} = -(E_{11} + E_{22}) = (1/3)\Lambda - (1/2)p - (1/6)\mu + \omega_1^2 + \omega_2^2 - \dot{u}_1 a_1 + \dot{u}_1 n_{23}.$$

The magnetic part of the Weyl tensor, with respect to $\vec{e}_0$, is[1] also a (real) 3×3 trace-free symmetric matrix $H_{\alpha\beta}$ with entries given by:

$$H_{11} = 2\dot{u}_1\omega_1 + 2\omega_2 a_2 + 2\omega_1 a_1,$$
$$H_{12} = H_{21} = \omega_2(n_{23} + a_1),$$
$$H_{13} = H_{31} = (1/2)\omega_2 n_{33},$$
$$H_{22} = -\dot{u}_1\omega_1 - \omega_1(a_1 + n_{23}),$$
$$H_{23} = H_{32} = -(1/2)n_{33}\omega_1$$

and

$$H_{33} = -(H_{11} + H_{22}) = -\dot{u}_1\omega_1 - \omega_1 a_1 - 2\omega_2 a_2 + \omega_1 n_{23}.$$

Some properties of spacetimes with $E_{ab} = 0$ as well as for spacetimes with $H_{ab} = 0$ can be found in chapter 5.

We form the complex matrix $Q_{\alpha\beta} := E_{\alpha\beta} + iH_{\alpha\beta}$. The Petrov type can be found by looking at the elementary divisors and multiplicities of the eigenvalues of Q (Kramer et al., 1980).[2] We shall follow the matrix criteria given in Kramer et al. (1980) to determine the allowed Petrov types for each of the three cases identified above, *i.e.* for *case A*: $n_{33} = 0, a_2 \neq 0$; $n_{23}\omega_1\omega_2\dot{u}_1 \neq 0.$, *case B*: $n_{33} \neq 0, a_2 = 0$; $\omega_1 = 0, n_{23}\omega_2\dot{u}_1 \neq 0.$ and *case C*: $n_{33} = 0, a_2 = 0; \omega_1 = 0, n_{23}\omega_2\dot{u}_1 \neq 0.$

These cases can be regrouped in the specialization diagram given in table (6.1)

[1] Strictly speaking, H_{ab} is a tensor which is isomorphic to the 3×3 matrix given above.

[2] See also chapter 4.

that appears on page 154.

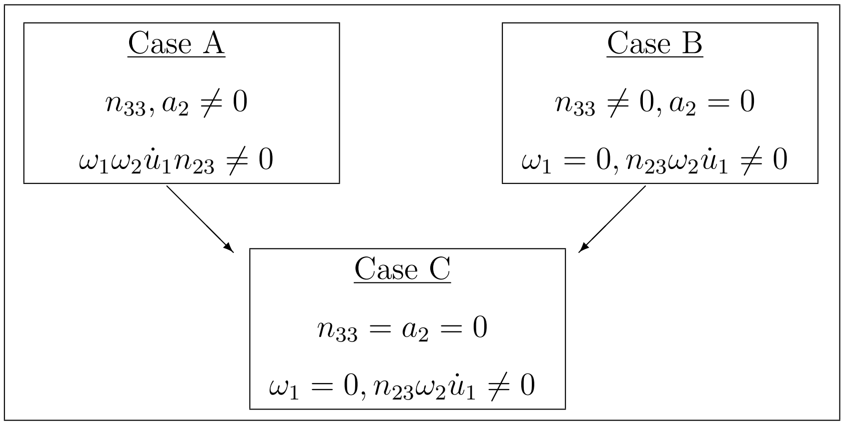

Table 6.1: Specialization diagram

6.1 Case A: $n_{33} = 0, a_2 \neq 0$; $n_{23}\omega_1\omega_2\dot{u}_1 \neq 0$.

The propagation equations for *case A* are

$$
\begin{aligned}
\partial_1 a_1 &= (1/2)p + (1/2)\mu - \omega_1^2 - 2\omega_2^2 + \dot{u}_1 a_1 + n_{23}^2 + a_1^2, \\
\partial_1 a_2 &= a_2(n_{23} + a_1), \\
\partial_1 \dot{u}_1 &= -\Lambda + (3/2)p + (1/2)\mu - 2\omega_1^2 - 2\omega_2^2 - \dot{u}_1^2 + 2\dot{u}_1 a_1 \\
\partial_1 \omega_1 &= \dot{u}_1\omega_1 + 2\omega_1 a_1 + 2\omega_2 a_2, \\
\partial_1 \omega_2 &= \omega_2(-2\dot{u}_1 + n_{23} + a_1), \\
\partial_1 n_{23} &= -\Lambda + p + 2\dot{u}_1 a_1 - \dot{u}_1 n_{23} + \omega_1^2 + 2a_1 n_{23} - a_1^2 - 4a_2^2 + n_{23}^2, \\
\partial_1 p &= -\dot{u}_1(\mu + p) \\
&\text{and} \\
\partial_1 \Lambda &= 0.
\end{aligned}
$$

There are also the further two constraints:

$$\Lambda = -a_1^2 + p + 2\dot{u}_1 a_1 + \omega_1^2 - 4a_2^2 - \omega_2^2 + n_{23}^2 \tag{6.4}$$

and

$$\omega_2\omega_1 - 2a_2 n_{23} = 0. \tag{6.5}$$

The matrix Q is

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{pmatrix},$$

where

$$\begin{aligned}
Q_{11} &= (2/3)a_1^2 + (1/3)p + (2/3)\dot{u}_1 a_1 - (8/3)\omega_1^2 + (8/3)a_2^2 - (1/3)\omega_2^2 \\
&\quad -(2/3)n_{23}^2 + (1/3)\mu + i(2\dot{u}_1\omega_1 + 2\omega_2 a_2 + 2\omega_1 a_1), \\
Q_{12} &= -\omega_2\omega_1 + i(\omega_2 n_{23} + \omega_2 a_1), \\
Q_{22} &= -(1/3)\dot{u}_1 a_1 + (4/3)\omega_1^2 - \dot{u}_1 n_{23} - (1/3)a_1^2 - (1/6)p - (4/3)a_2^2 \\
&\quad -(1/3)\omega_2^2 + (1/3)n_{23}^2 - (1/6)\mu + i(\dot{u}_1\omega_1 + \omega_1 a_1 + \omega_1 n_{23}),
\end{aligned}$$

and

$$\begin{aligned}
Q_{33} &= -(Q_{11} + Q_{22}) \\
&= -(1/3)a_1^2 - (1/6)p - (1/3)\dot{u}_1 a_1 + (4/3)\omega_1^2 - (4/3)a_2^2 + (2/3)\omega_2^2 \\
&\quad +(1/3)n_{23}^2 - (1/6)\mu + \dot{u}_1 n_{23} + i(-\dot{u}_1\omega_1 - \omega_1 a_1 - 2\omega_2 a_2 + \omega_1 n_{23}).
\end{aligned}$$

We immediately find that there are no spacetimes of Petrov type O, since the real part of Q_{12} does not vanish, and so the matrix Q cannot vanish.

In order that the Petrov type be N, the matrix Q must satisfy $Q^2 = 0$ with $Q \neq 0$. Therefore, Q_{33} must vanish. The real part of Q_{33} provides an expression for the energy density:

$$\mu = -2\dot{u}_1 a_1 + 6\dot{u}_1 n_{23} - p + 8\omega_1^2 + 4\omega_2^2 + 2n_{23}^2 - 2a_1^2 - 8a_2^2.$$

This is then used to reexpress Q^2 without μ. The constraint (6.5) gives an expression for

$$a_2 = \frac{\omega_1\omega_2}{2n_{23}}.$$

The imaginary part of $(Q^2)_{12}$, which is

$$-\omega_1^2\omega_2(a_1 n_{23} - n_{23}^2 + \dot{u}_1 n_{23} + \omega_2^2) = 0,$$

yields an expression for $\dot{u}_1$:

$$\dot{u}_1 = -\frac{a_1 n_{23} - n_{23}^2 + \omega_2^2}{n_{23}}.$$

The imaginary part of $(Q^2)_{22}$ is

$$2\omega_1 n_{23}(\omega_2 - 2n_{23})(\omega_2 + 2n_{23})(a_1 n_{23} - n_{23}^2 + \omega_2^2) = 0.$$

Since $\dot{u}_1$ is constrained to be non-zero, it follows that $\omega_2 = \pm 2n_{23}$. For both situations, using the expressions just obtained for $\dot{u}_1, \omega_2$ and a_2, we find from the expression for μ that $\mu + p = 0$. There are therefore no type N solutions.

For Petrov type III, the matrix condition is $Q^3 = 0$ with $Q^2 \neq 0$. In type III, all three eigenvalues must be equal to zero. Since Q is trace-free and symmetric, and since the vector $(0, 0, 1)$ is an eigenvector of Q, it follows that Q must be of the form

$$\begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is then $L(L^2 - A^2 - B^2) = 0$. For $L = 0$ to be a triple root, it follows that $A^2 + B^2 = 0$, and so $A = \pm Bi$. But this implies that Q^2 must be zero. Therefore there are no type III spacetimes in *case A*.

For Petrov types II and D, there is a (non-zero) double eigenvalue. From the structure of the matrix Q, it is immediate that Q_{33} is an eigenvalue, with associated eigenvector $(0, 0, 1)$. Since the trace of a matrix is equal to the sum of its eigenvalues, there are two cases to consider for the present situation, depending on whether or not Q_{33} is the repeated eigenvalue.

We first consider the situation when Q_{33} is the double eigenvalue. The matrix $Q - Q_{33}\, I_3$, with I_3 denoting the three-dimensional identity matrix, is given by

$$Q = \begin{pmatrix} 2E_{11} + E_{22} + i(2H_{11} + H_{22}) & E_{12} + iH_{12} & 0 \\ E_{12} + iH_{12} & 2E_{22} + E_{11} + i(2H_{22} + H_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One of the possible eigenvectors belonging to the eigenvalue Q_{33} is $(0, 0, 1)$. There will be another such eigenvector, linearly independent of $(0, 0, 1)$ if and only if the determinant

$$\begin{vmatrix} 2E_{11} + E_{22} + i(2H_{11} + H_{22}) & E_{12} + iH_{12} \\ E_{12} + iH_{12} & 2E_{22} + E_{11} + i(2H_{22} + H_{11}) \end{vmatrix}$$

vanishes. If this last determinant does vanish, then the Petrov type is D, otherwise, the Petrov type is II. On the other hand, the quantity $-2Q_{33}$ is also an eigenvalue, which entails that the determinant of $Q + 2Q_{33}\, I_3$ must vanish, *i.e.*

$$\begin{vmatrix} -E_{11} - 2E_{22} + i(-H_{11} - 2H_{22}) & E_{12} + iH_{12} & 0 \\ E_{12} + iH_{12} & -E_{22} - 2E_{11} + i(-H_{22} - 2H_{11}) & 0 \\ 0 & 0 & Q_{33} \end{vmatrix} = 0.$$

This is precisely the condition that the Petrov type be D, since $Q_{33} \neq 0$. Since the determinant is a complex valued quantity, its vanishing actually represents two conditions. The vanishing of the real part of the determinant gives the first condition:

$$\begin{aligned} & -(\mu+p)(\dot{u}_1 n_{23} + \frac{1}{2}\omega_2^2) - \dot{u}_1 a_1 \omega_2^2 - 2\dot{u}_1^2 a_1 n_{23} - 2a_1^2 \dot{u}_1 n_{23} \\ & +2\omega_2^2 n_{23} a_1 + 14\omega_1^2 \dot{u}_1 n_{23} - 8a_2^2 \dot{u}_1 n_{23} + 3\omega_2^2 \dot{u}_1 n_{23} + 10\omega_2 a_2 \omega_1 n_{23} \\ & -6\omega_1 a_1 \omega_2 a_2 + 6\omega_1^2 a_1 n_{23} + \omega_2^4 - 12a_2^2\omega_2^2 + 2n_{23}^3 \dot{u}_1 \\ & +3\omega_1^2\omega_2^2 - 2\omega_1^2 n_{23}^2 + 2n_{23}^2\omega_2^2 - 6\dot{u}_1\omega_1\omega_2 a_2 + 2\dot{u}_1^2 n_{23}^2 = 0. \quad (6.6) \end{aligned}$$

The second condition is attained by requiring that the imaginary part of the determinant be equal to zero:

$$\begin{aligned} & (\mu+p)(\omega_2 a_2 - \omega_1 n_{23}) - 8a_2^2\omega_1 n_{23} - 2a_1^2\omega_1 n_{23} + 2a_1^2\omega_2 a_2 \\ & +4\dot{u}_1 n_{23}^2\omega_1 - 3\dot{u}_1\omega_1\omega_2^2 - \omega_1 a_1 \omega_2^2 + 5\omega_2^2\omega_1 n_{23} - 6\dot{u}_1^2\omega_1 n_{23} \\ & -10\omega_2 a_2 \dot{u}_1 n_{23} + 2\dot{u}_1 a_1 \omega_2 a_2 - 2n_{23}^2\omega_2 a_2 + 8\omega_1^3 n_{23} - 8\omega_1^2\omega_2 a_2 \\ & -8\dot{u}_1 a_1 \omega_1 n_{23} + 8a_2^3\omega_2 - 6\omega_2^3 a_2 + 2n_{23}^3\omega_1 = 0. \quad (6.7) \end{aligned}$$

We eliminate $\mu + p$ between equations (6.6) and (6.7) to obtain:

$$\begin{aligned} & -2\dot{u}_1^2 n_{23}^3\omega_1 - a_1^2\omega_2^3 a_2 + 6\omega_1^3 a_1 n_{23}^2 + (3/2)\dot{u}_1\omega_1\omega_2^4 + (1/2)\omega_1 a_1 \omega_2^4 \\ & -(3/2)\omega_2^4\omega_1 n_{23} + 6\dot{u}_1^3\omega_1 n_{23}^2 - n_{23}^2\omega_2^3 a_2 + 6\omega_1^3 n_{23}^2 \dot{u}_1 - \omega_1^3 n_{23}\omega_2^2 \\ & +\omega_1^2\omega_2^3 a_2 + n_{23}^3\omega_1\omega_2^2 + 8a_2^3\omega_2^3 + 2\omega_2^5 a_2 - 2\omega_1^3 n_{23}^3 - 18a_2^2\omega_1 n_{23}\omega_2^2 \\ & +6\dot{u}_1\omega_1\omega_2^2 a_2^2 - 12\omega_1^2 a_1 \omega_2 a_2 n_{23} + 6\omega_1 a_1 \omega_2^2 a_2^2 + 6\dot{u}_1^2 a_1 \omega_1 n_{23}^2 \\ & -2\omega_2^3 n_{23} a_1 a_2 + 2\omega_2^2 n_{23}^2 a_1 \omega_1 + 12\omega_2 a_2 \omega_1^2 n_{23}^2 + a_1^2\omega_1 n_{23}\omega_2^2 \\ & -4\dot{u}_1 n_{23}^2\omega_1\omega_2^2 + 6\dot{u}_1^2\omega_1\omega_2^2 n_{23} + 4\omega_1 a_1 \omega_2^2 \dot{u}_1 n_{23} + 8\omega_2 a_2 \dot{u}_1^2 n_{23}^2 \\ & +8\omega_2^3 a_2 \dot{u}_1 n_{23} - 12\omega_1^2\omega_2 a_2 \dot{u}_1 n_{23} = 0. \quad (6.8) \end{aligned}$$

Equation (6.8) is then differentiated three times. Each time, equation (6.4) is used to eliminate Λ, and then equation (6.7) is used to eliminate $\mu + p$. The three equations thus obtained have 109, 291 and 648 terms respectively. Since the exact expressions are not very illuminating in themselves, they, as well as other long equations, will be omitted from the present text. Sufficient details, however, will be provided so that any omitted equation can be calculated.[3] The main problem to control is that the intermediate calculations become quite large. The order in which the operations are performed and the various projections that are used turn out to be critical in being able to complete the calculations. The steps are as follows. Factor every polynomials that are obtained. Each factor corresponds to a branch in the calculations. The main reason for keeping the polynomials factor-free is to keep their sizes down. Denote equation (6.5) by `T1`; equation (6.8) by `T2`; and the three successive derivatives of equation (6.8) by `T3, T4` and `T5`. Equations (`T1-T5`) are polynomial equations that are homogeneous. We set $n_{23} = 1$ in equations (`T1-T5`), thereby breaking the homogeneity of the equations. This is equivalent to replacing each variable by itself divided by n_{23}. We therefore are working in a projective space. This reduces the size of the equations that are to come, since we obtain real numbers where polynomials in n_{23} would have appeared. The projective forms of equations are labelled by `T1a-T5a`. Equation (`T1a`) is used to eliminate a_2 from the other equations, using the resultant. The variable a_2 has been chosen since it appears as the variable of lowest degree.

Since computing a resultant entails computing a determinant of a matrix[4] with

[3]The use of a symbolic calculator proves to be essential.

[4]This is the Sylvester matrix.

dimension[5] twice[6] the degree[7] of the variable which is to be eliminated, it is important to keep the degrees as low as possible. If there are several variables to be eliminated, the first tendency might be to start by eliminating the higher degree variables. That it is actually better to start with the lower degree variables is easily seen by thinking about three bivariate equations, linear in one variable, but of degree ten, say, in the second variable. If one eliminates the linear variable, one would get two equations of at most twentieth degree in the second variable. The numerical coefficients are of the order of magnitude of the product of the largest coefficient in each of the polynomials. The determinant of a matrix of dimension 40 would be computed. On the other hand, starting with elimination of the higher degree variable, one would compute the determinant of two matrices of dimension 20, with terms linear in the remaining variable. This would yield two polynomials whose potential degree is 20. The numerical coefficients are of potential order of magnitude of the product raised to the twentieth power of the largest coefficient in each of the polynomials. As in the first approach, the determinant of a matrix of order 40 would need to be computed. The big difference is that the numerical coefficients are bigger in the second approach. This effect is magnified the more variables there are.[8]

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[5]The dimension of a square matrix is defined to be the number of rows (or columns) of the matrix.

[6]There is another method of computing the resultant. It involves computing the determinant of a Bezout matrix which has dimension equal to the maximum degree of the polynomials. Its entries are, however, more complicated than in the Sylvester matrix. In either case, the point that the needed expressions cannot be computed in the straightforward way still holds.

[7]This is for polynomials in which the degree of the unknown is the same. The exact dimension of the matrix for two polynomials is equal to the sum of their degrees.

[8]As an example of this effect, let us suppose that we are given the three polynomial equations $tx^4+(t+1)x+3=0$, $(t+2)x^4+(t+2)x^2+4=0$ and $x^4+t+3=0$. Eliminating t first, followed

Equation T11 is obtained from equations T1a and T2a by taking the resultant with respect to a_2 followed by a division by ω_1. Equation T12 is obtained from T1a and T3a, with a division by ω_1^2. Equation T13 is obtained from T1a and T4a, followed by a division by ω_1^3. Lastly, equation T14 is obtained from equations T1a and T5a followed by a division by ω_1^4. Since ω_1 and ω_2 only appear with even degree in equations T11 to T14, it is worthwile to replace ω_1^2 and ω_2^2 by new variables, W_1 and W_2, respectively. Now variable W_1 is the variable of least degree in T11-T14. We then use equation T11 to eliminate W_1 from the other equations. Equation T21 is obtained from equations T11 and T12, together with a division by $(W_2-2)^2$. We shall consider later the situation when $W_2 - 2 = 0$, which is equivalent to $\omega_2^2 - 2n_{23}^2 = 0$, but for now, we assume that this factor does not vanish. Equation T22 is obtained from equations T11 and T13, and a division by $(W_2 - 2)^2 = 0$. Equation T23 is obtained from equations T11 and T14, and a division by $(W_2 - 2)^3$. It is important that these factors of $W_2 - 2$ be removed, otherwise resultants with respect to W_2 would be zero, indicating the presence of W_2-2 as a common factor, but not telling us any information about other possible common factors involving variables other than W_2. Next, the resultant T31 of T21 and T22 with respect to a_1 is calculated. It has

$$(2\dot{u}_1 + W_2)^4(3\dot{u}_1 + 2W_2 - 2)^{10}(W_2 - 4)^{20}(W_2 - 2)^{11}W_2^6\dot{u}_1^6 \tag{6.9}$$

as factors. We remove from the resultant these factors, whose possible vanishing we shall consider later, and denote the result by T31a. The next step would be to compute the resultant of T21 and T23 with respect to a_1. This, however, is a lengthy calculation. It is not clear that it can be carried out, and the step following

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by x gives -72145632. This is close to $(3 \times 4)^{7.2}$. Doing the elimination in the opposite ordering gives 60871023332170267428043092623336. This is about $(3 \times 4)^{28.5}$. That these numbers do not equal to zero indicates that there are no common zeros to the polynomials.

the elimination of a_1 certainly could not be computed directly.[9] A small prime number is chosen; the value 19 is adequate.[10] We replace W_2 by this small prime in `T21, T23` and `T31a`. Then, the resultant of the modified `T21` and `T23` with respect to a_1 is computed, and the result is labelled by `T32`. Then `T41` is computed by taking the resultant with respect to a_1 of `T32` and the modified `T31a`. If we had not removed the factors given by (6.9) from `T31`, we would have found that `T41` is zero. Therefore, at least some of the factors of (6.9) are common to the two resultants `T31` and `T32`. Since these factors needed to be identified, it was not possible to set W_2 to be the chosen prime from the outset.[11] Having removed the factors (6.9) from

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[9] We may consider an estimate of the magnitude of the calculation, as follows. Equation `T31` is already of degree 35 in $\dot{u}_1$ and of degree 75 in W_2, implying that equation `T31a` is of degree 15 in $\dot{u}_1$ and of degree 24 in W_2. Equation `T32` has a higher degree than `T31`. Even if the factors (6.9) are divisors of equation `T32`, the corresponding equation `T32a` would be of at least degree 15 in $\dot{u}_1$ and of degree 24 in W_2. Eliminating $\dot{u}_1$ between `T31a` and `T32a` involves finding the determinant of a matrix of dimension 30 with entries being polynomials in W_2 with degree of the order of 24. The result would be a polynomial in W_2 with degree of the order of 24×30. The numerical coefficients in `T31` are of the order of 10^{40} to 10^{80}. The polynomial in W_2 would then have coefficients of the order of $10^{40\times 30}$. Roughly, we then have 700 terms with coefficients of 1200 digits. This is 0.8 megabytes just to give the coefficients. In terms of time, it took about 7000 seconds to compute equation `T31` on the machine `jeeves.uwaterloo.ca` which is a DECsystem 5500 running Ultrix 4.2a and is about 30 times faster than a VAX780. Calculation of `T32` would take even longer. It is clear that the resultant between `T31a` and `T32` should not be attempted, since the required time behaves as the cube of the dimension of the matrix whose determinant we compute; and this is assuming the fortuitous case that the coefficients do not increase in magnitude, an assumption we already know does not hold.

[10] It is not required that the number be a prime number. For more information, see the following footnote.

[11] Actually, given a bound on the degree of relevant polynomials, it is possible to do the evaluation at enough prime values to be able to find the actual factors. For our purposes, such a

T31, the value of T41 is not zero, but rather an integer comprising 5304 digits.[12] This value is the value of the full resultant T41 when evaluated at W_2 equal to the chosen prime 19. Since the number obtained is not zero, we know that the full value of the resultant T41 is a polynomial in W_2. Equating this polynomial to zero, we can conclude that W_2 has to be a constant. In terms of the original variables, we can then conclude that w_2 is proportional to n_{23}. The constant of proportionality cannot be zero, and has to be finite, since the product $\omega_2 n_{23}$ cannot be zero.

Taking into account the various common factors already identified, the present situation therefore subdivides into 3 cases. The first subcase has $\omega_2 = An_{23}$, with A a non-zero constant. The second subcase has $\omega_2^2 = -2\dot{u}_1 n_{23}$ and the third subcase has $3\dot{u}_1 n_{23} + 2\omega_2^2 - 2n_{23}^2 = 0$.

The first subcase has

$$\omega_2 - An_{23} = 0. \tag{6.10}$$

Equation (6.10) is used to eliminate n_{23}. Equation (6.5) becomes

$$2a_2 - \omega_1 A = 0. \tag{6.11}$$

Equation (6.11) is used to eliminate ω_1. Differentiation of equation (6.10) gives

$$-A\dot{u}_1 - Aa_1 - A^2\omega_2 + \omega_2 = 0. \tag{6.12}$$

Equation (6.12) is used to eliminate a_1. Equation (6.8) becomes

$$a_2\omega_2^2(8A^4\omega_2\dot{u}_1^2 + 8A^4\omega_2^3 + 32a_2^2A^2\omega_2 - 56a_2^2A^4\omega_2 - 14A^6\omega_2^3$$

calculation turned out to be unnecessary. The reason for choosing prime numbers is that it is then easier to compute the value of the actual factors.

[12]This value of 5304 is of the same order of magnitude as the 1200 we arrived at in the previous footnote. We may regard this as illustrating that our method of estimating such values is reasonably accurate.

$$+28a_2^2A^6\omega_2 - 8A^5\omega_2^2\dot{u}_1 - 4A^8\omega_2 a_2^2 + 7A^8\omega_2^3 - A^{10}\omega_2^3 + 2A^6\dot{u}_1^2\omega_2$$
$$-2A^9\dot{u}_1\omega_2^2 - A^8\dot{u}_1^2\omega_2 + 10A^7\dot{u}_1\omega_2^2) = 0. \quad (6.13)$$

Equation (6.13) is used to eliminate $\dot{u}_1$. Equation (T3) then becomes

$$-a_2^7A^{22}\omega_2^{15}(A-2)^7(A+2)^7(A^2-2)^2(72\omega_2^2A^{10}a_2^2 + 48\omega_2^2A^8a_2^2$$
$$-480A^6\omega_2^2a_2^2 + 128\omega_2^4 - 384A^2\omega_2^4 + 30\omega_2^4A^8 - 404A^6\omega_2^4 + 9\omega_2^4A^{10}$$
$$+648A^4\omega_2^4 + 144a_2^4A^{10} + 768A^2a_2^2\omega_2^2 - 192A^4\omega_2^2a_2^2$$
$$+1152A^4a_2^4 - 576A^6a_2^4 - 288A^8a_2^4)(4a_2^2 + \omega_2^2A^2)^2 = 0.$$

It follows then that either ω_2 is proportional to a_2 or A is equal to $2, -2, \sqrt{2}$ or $-\sqrt{2}$. Differentiating $\omega_2 - Ba_2 = 0$, with B a non-zero constant, one gets $-2\dot{u}_1 Ba_2 = 0$, a contradiction. If $\omega_2 = 2n_{23}$ or $\omega_2 = -2n_{23}$, then differentiation of equation (6.12) shows that $\mu + p = 0$, a contradiction. If $\omega_2^2 = 2n_{23}^2$, then differentiation of equation (6.12) gives that $\mu + p = 4(a_2^2 + n_{23}^2)$. Differentiation of $\omega_2^2 - 2n_{23}^2 = 0$ implies that $n_{23} + a_1 + \dot{u}_1 = 0$. Equation (6.7) then gives $4\sqrt{2}a_2n_{23}^2(n_{23} + a_1) = 0$, whence $n_{23} + a_1 = 0$. This in turn implies that $\dot{u}_1 = 0$, a contradiction. There are therefore no spacetimes that belong to the first subcase.

The second subcase has

$$\omega_2^2 + 2\dot{u}_1 n_{23} = 0. \quad (6.14)$$

Differentiation of equation (6.14) gives

$$-2\dot{u}_1\omega_2^2 + 2a_1\omega_2^2 + 2n_{23}a_1^2 - 6n_{23}\omega_1^2 + 8a_2^2n_{23} - 2n_{23}^3 + n_{23}(\mu+p) - 4n_{23}\dot{u}_1^2 + 4n_{23}\dot{u}_1a_1 = 0. \quad (6.15)$$

The variable $\dot{u}_1$ is eliminated between equations (6.14) and (6.15), and factors of n_{23} are removed from the result. This gives

$$-2n_{23}^2 + 2a_1^2 - 6\omega_1^2 + 8a_2^2 + \mu + p = 0. \quad (6.16)$$

We use equation (6.16) to remove μ, equation (6.14) to remove $\dot{u}_1$ and equation (6.5) to eliminate ω_2 from equations (6.6), (6.7) and T3. We thus obtain

$$\begin{aligned} 8n_{23}^3\omega_1(-\omega_1^4 n_{23} + 3\omega_1^4 a_1 + 2a_2^2 n_{23}\omega_1^2 - 6a_2^2\omega_1^2 a_1 + 2a_2^2 n_{23}^3 & & \\ +4a_2^2 n_{23}^2 a_1 + 2a_2^2 n_{23} a_1^2 - 4a_2^4 n_{23}) &= 0, \end{aligned} \tag{6.17}$$

$$8n_{23}^3(\omega_1^6 - 2a_2^2\omega_1^4 + 6a_2^2 n_{23}^2\omega_1^2 + 6a_2^2 n_{23}\omega_1^2 a_1 - 4a_2^4 n_{23}^2 - 4a_2^4 n_{23} a_1) = 0. \tag{6.18}$$

and the equation that arises from T3. The resultant of equations (6.17) and (6.18) with respect to a_1 is

$$\begin{aligned} -8192\omega_1^1 2 + 38912a_2^2\omega_1^{10} - 73728a_2^2 n_{23}^2\omega_1^8 + 245760a_2^4 n_{23}^2\omega_1^6 & \\ -57344a_2^4\omega_1^8 - 303104a_2^6 n_{23}^2\omega_1^4 + 24576a_2^6\omega_1^6 & \\ +163840a_2^8 n_{23}^2\omega_1^2 - 32768a_2^{10} n_{23}^2 &= 0, \end{aligned} \tag{6.19}$$

after division by $n_{23}^{10}\omega_1 a_2^2$. The resultant of the transformed T3 and equation (6.17) with respect to a_1 becomes

$$\begin{aligned} -21\omega_1^{22} + 166a_2^2\omega_1^{20} - 510\omega_1^{18}a_2^4 - 324\omega_1^{18}a_2^2 n_{23}^2 + 2100\omega_1^{16}a_2^4 n_{23}^2 & \\ +756\omega_1^{16}a_2^6 - 1296\omega_1^{14}a_2^4 n_{23}^4 - 5132\omega_1^{14}a_2^6 n_{23}^2 - 536\omega_1^{14}a_2^8 & \\ +144\omega_1^{12}a_2^{10} + 8640\omega_1^{12}a_2^6 n_{23}^4 + 5560\omega_1^{12}a_2^8 n_{23}^2 - 1696\omega_1^{10}a_2^{10} n_{23}^2 & \\ -24624\omega_1^{10}a_2^8 n_{23}^4 + 38688\omega_1^8 a_2^{10} n_{23}^4 - 1728\omega_1^8 a_2^{12} n_{23}^2 + 1600\omega_1^6 a_2^{14} n_{23}^2 & \\ -36064\omega_1^6 a_2^{12} n_{23}^4 - 384\omega_1^4 a_2^{16} n_{23}^2 + 19904\omega_1^4 a_2^{14} n_{23}^4 & \\ -6016a_2^{16} n_{23}^4\omega_1^2 + 768a_2^{18} n_{23}^4 &= 0, \end{aligned}$$

after division by $n_{23}^{17}\omega_1 a_2^2(\omega_1^2 - 2a_2^2)$. The resultant of equations (6.19) and (6.20) with respect to n_{23}^2 is

$$\begin{aligned} -67108864a_2^4\omega_1^{10}(3\omega_1^{10} - 88\omega_1^8 a_2^2 + 371\omega_1^6 a_2^4 - 534\omega_1^4 a_2^6 + 324\omega_1^2 a_2^8 - 72a_2^{10}) & \\ (\omega_1 - a_2)^3(\omega_1 + a_2)^3(3\omega_1^2 - 2a_2^2)^3(\omega_1^2 - 2a_2^2)^3 &= 0. \end{aligned}$$

We thus conclude that ω_1 is proportional to a_2 and so we set

$$\omega_1 = Ba_2, \tag{6.20}$$

with B being a non-zero constant. Differentiation of equation (6.20) gives

$$\dot{u}_1 Ba_2 + Ba_2 a_1 + 2\omega_2 a_2 - Ba_2 n_{23} = 0,$$

which is equivalent to

$$-2B^2 a_1 + 2B^2 n_{23} - 4n_{23} = 0, \tag{6.21}$$

after elimination of a_2 with equation (6.20) and of ω_2 with equation (6.5). The derivative of (6.21) is equivalent to

$$-B^4 a_2^2 - \omega_2^2 + 2B^2 a_2^2 + B^2 \omega_2^2 = 0. \tag{6.22}$$

Differentiation of (6.22) leads to

$$16B^4 a_2^4 \dot{u}_1^2 (B-1)^2 (B+1)^2 (B^2-2)^2 = 0,$$

which shows that B must be equal to $1, -1, \sqrt{2}$ or $-\sqrt{2}$. Substitution of these four values into equation (6.22) leads to contradictions in all cases. There are therefore no spacetimes that belong to the second subcase.

The third subcase has

$$3\dot{u}_1 n_{23} + 2\omega_2^2 - 2n_{23}^2 = 0,$$

which we shall refer to as being equation `P1`. We shall refer to equation (6.5) as equation `P2`, equation (6.6) as `P3`, equation (6.7) as `P4`, the derivative of `P1` as `P5` and `T3` as `P6`. We compute `P13` as the resultant of `P1` and `P3` with respect to $\dot{u}_1$. Similarly we compute `P14, P15` and `P16` as the resultants with `P1` of `P4, P5` and `P6` with respect to $\dot{u}_1$. Then `P123, P124, P125` and `P126` are obtained as the resultants of `P2` with, respectively, `P13, P14, P15` and `P16` with respect to ω_1. Then, `P1235` is obtained from the resultant of `P123` and `P125` with respect to μ. Also, `P1245` is obtained from the resultant of `P124` and `P125` with respect to μ. Lastly, `P12456` is obtained by taking the resultant of `P126` and `P1245` with respect to a_1. Whenever they appear, we shall remove common factors of powers of n_{23}, a_2 and ω_2. We let ω_2 be equal to the prime number [13] 17 and take the resultant modulo 7 with respect to n_{23} of `P12345` and `P12456`. The answer is

$$\begin{aligned} 5(2a_2^{40} + 4a_2^{32} + 2a_2^{34} + a_2^{36} + 4a_2^{38} + 2a_2^{28} + 5a_2^{24} + a_2^{26} + a_2^{30} + 3a_2^{20} \\ +6a_2^{22} + 6a_2^{66} + 6a_2^{68} + 6a_2^{64} + 4a_2^{58} + 2a_2^{60} + 2a_2^{62} + 2a_2^{50} + 5a_2^{52} \\ +3a_2^{54} + a_2^{56} + a_2^{42} + 4a_2^{44} + 2a_2^{46})^2 \equiv 0 \bmod 7. \end{aligned}$$

This shows that ω_2 is proportional to a_2. Differentiation of $\omega_2 - Aa_2 = 0$, where A is a non-zero constant, shows that $-2\dot{u}_1 A a_2 = 0$. This is a contradiction. We can then conclude that there are no *case A* spacetimes of Petrov type D.

We now consider the case when Q_{33} is the non-repeated eigenvalue of Q. Since the eigenvalues of Q must sum to zero, the repeated eigenvalue is $-(1/2)Q_{33}$. The matrix $Q + (1/2)Q_{33}\, I_3$, which is

$$\begin{pmatrix} (1/2)[E_{11} - E_{22} + i(H_{11} - H_{22})] & E_{12} + iH_{12} & 0 \\ E_{12} + iH_{12} & -(1/2)[E_{11} - E_{22} + i(H_{11} - H_{22})] & 0 \\ 0 & 0 & (3/2)Q_{33} \end{pmatrix},$$

implies that the eigenvectors associated with $-(1/2)Q_{33}$ are orthogonal to $(0, 0, 1)$. The requirement that the eigenspace of $-(1/2)Q_{33}$ be two-dimensional requires that

[13] The numbers 7 and 17 are arbitrary. They were chosen because they were small and because we could obtain the results we sought. Most positive integers would have been appropriate.

the submatrix

$$\begin{pmatrix} (1/2)[E_{11}-E_{22}+i(H_{11}-H_{22})] & E_{12}+iH_{12} \\ E_{12}+iH_{12} & -(1/2)[E_{11}-E_{22}+i(H_{11}-H_{22})] \end{pmatrix} \quad (6.23)$$

be scalar, *i.e.* a multiple of the identity matrix. However, if the matrix (6.23) is scalar, then E_{12}, which is $-\omega_1\omega_2$, vanishes. This is a contradiction, whence the Petrov type must be II. Since $-Q_{33}/2$ is an eigenvalue, the determinant of the matrix (6.23) must be zero. This determinant factors as:

$$\begin{aligned}(1/2)a_1^2+(1/4)p+(1/2)\dot{u}_1a_1-2\omega_1^2+2a_2^2-(1/2)n_{23}^2+(1/4)\mu \\ +(3/2)i\dot{u}_1\omega_1+i\omega_2a_2+(3/2)i\omega_1a_1+(1/2)\dot{u}_1n_{23}+(1/2)i\omega_1n_{23} \\ +i(-\omega_2\omega_1+i\omega_2n_{23}+i\omega_2a_1)\end{aligned} \qquad (6.24)$$

times

$$\begin{aligned}(1/2)a_1^2+(1/4)p+(1/2)\dot{u}_1a_1-2\omega_1^2+2a_2^2-(1/2)n_{23}^2+(1/4)\mu \\ +(3/2)i\dot{u}_1\omega_1+i\omega_2a_2+(3/2)i\omega_1a_1+(1/2)\dot{u}_1n_{23}+(1/2)i\omega_1n_{23} \\ -i(-\omega_2\omega_1+i\omega_2n_{23}+i\omega_2a_1).\end{aligned} \qquad (6.25)$$

We first suppose that the first factor (6.24) is equal to zero. The vanishing of the real part of (6.24) gives a value for μ :

$$\begin{aligned}(1/2)\dot{u}_1a_1+(1/2)\dot{u}_1n_{23}-\omega_2n_{23}-\omega_2a_1 \\ +(1/4)\mu+(1/4)p-2\omega_1^2-(1/2)n_{23}^2+(1/2)a_1^2+2a_2^2 &= 0.\end{aligned} \qquad (6.26)$$

The vanishing of the imaginary part of (6.24) gives

$$(3/2)\dot{u}_1\omega_1+\omega_2a_2+(3/2)\omega_1a_1-\omega_2\omega_1+(1/2)\omega_1n_{23}=0. \qquad (6.27)$$

The derivative of equation (6.27) is equivalent to

$$\begin{aligned}\omega_1(\omega_1+a_2)(9\omega_1^3a_2^2+\omega_2^2\omega_1^3+2\omega_2^2a_2\omega_1^2 \\ -3\dot{u}_1\omega_1^2\omega_2a_2-9\omega_1^2a_2^3-3\dot{u}_1\omega_1\omega_2a_2^2-2\omega_2^2a_2^2\omega_1-3a_2^3\omega_2^2) &= 0,\end{aligned} \qquad (6.28)$$

where equation (6.4) is used to eliminate Λ, equation (6.26) is used to eliminate μ, equation (6.27) is used to eliminate a_1 and equation (6.5) is used to eliminate n_{23}. There are therefore two possibilities, according to whether or not $\omega_1 + a_2 = 0$. If $\omega_1 + a_2$ is indeed equal to zero, equation (6.5) gives that $\omega_2 + 2n_{23} = 0$. Equation (6.27) gives that $a_1 + \dot{u}_1 + 3n_{23} = 0$. In turn, equation (6.26) gives $\mu + p = 0$, a contradiction. Therefore $\omega_1 + a_2 \neq 0$. Equation (6.28), divided by $\omega_1(\omega_1 + a_2)$, will be used to eliminate $\dot{u}_1$. Furthermore, equation (6.28) is differentiated. The result is equivalent to

$$\begin{aligned} -\frac{19683}{32}\omega_1^6 a_2^4 + \frac{6561}{32}\omega_1^6 a_2^2\omega_2^2 + \frac{729}{32}\omega_1^6\omega_2^4 + \frac{243}{8}\omega_1^5\omega_2^4 a_2 \\ +\frac{19683}{16}\omega_1^5 a_2^5 - \frac{2187}{8}\omega_1^5 a_2^3\omega_2^2 - \frac{2673}{32}\omega_1^4\omega_2^4 a_2^2 - \frac{24057}{32}\omega_1^4 a_2^4\omega_2^2 \\ -\frac{19683}{32}\omega_1^4 a_2^6 + \frac{2187}{16}\omega_1^3 a_2^5\omega_2^2 - \frac{729}{4}\omega_1^3 a_2^3\omega_2^4 + \frac{6561}{16}\omega_1^2 a_2^6\omega_2^2 \\ -\frac{2673}{32}\omega_1^2 a_2^4\omega_2^4 + \frac{243}{8}a_2^5\omega_2^4\omega_1 + \frac{729}{32}a_2^5\omega_2^4 \quad = \quad 0. \end{aligned} \tag{6.29}$$

Equation (6.29) is used to eliminate ω_1. The result of differentiating equation (6.29) implies that

$$a_2^{78}\omega_2^{32}(343\omega_2^4 + 234a_2^2\omega_2^2 - 81a_2^4)(\omega_2^2 + 4a_2^2)^6 = 0. \tag{6.30}$$

This implies that ω_2 is proportional to a_2. Propagation of this proportionality relation yields a contradiction.

If the factor (6.24) is not equal to zero, then, for the spacetime to be of Petrov type II, the factor (6.25) must be zero. The same steps as in the preceding paragraph are followed, replacing the factor (6.24) by the factor (6.25). Two cases appear, according as ω_1+a_2 vanishes or not. If ω_1+a_2 does vanish, a contradiction is reached in the same manner as that above. If $\omega_1 + a_2$ is not zero, the same steps as in the preceding paragraph lead to exactly the same equation (6.30) that was obtained in the first subcase. It follows then that ω_2 is proportional to a_2. Propagation of that proportionality relation leads to a contradiction. There are therefore no Petrov type II solutions in *case A*.

If there are spacetimes in *case A,* they must be of Petrov type I.

6.2 Case B: $n_{33} \neq 0, a_2 = 0;\ \omega_1 = 0, n_{23}\omega_2\dot{u}_1 \ \neq 0.$

For *case B*, the propagation equations are

$$\begin{aligned}
\partial_1 a_1 &= (1/2)p + (1/2)\mu - 2\omega_2^2 + \dot{u}_1 a_1 + n_{23}^2 + a_1^2 + (1/4)n_{33}^2, \\
\partial_1 \dot{u}_1 &= -\Lambda + (3/2)p + (1/2)\mu - 2\omega_2^2 - \dot{u}_1^2 + 2\dot{u}_1 a_1, \\
\partial_1 \omega_2 &= -2\dot{u}_1\omega_2 + \omega_2 n_{23} + \omega_2 a_1, \\
\partial_1 n_{23} &= -\Lambda + p + 2\dot{u}_1 a_1 - \dot{u}_1 n_{23} + 2a_1 n_{23} - a_1^2 + (3/4)n_{33}^2 + n_{23}^2, \\
\partial_1 n_{33} &= -\dot{u}_1 n_{33} + 2n_{33}a_1 - 2n_{33}n_{23}, \\
\partial_1 p &= -\dot{u}_1(\mu + p) \\
&\text{and} \\
\partial_1 \Lambda &= 0.
\end{aligned}$$

The cosmological constant, Λ satisfies

$$\Lambda = (1/4)n_{33}^2 + p + 2\dot{u}_1 a_1 - a_1^2 - \omega_2^2 + n_{23}^2.$$

The matrix Q is of the form

$$\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{33} & Q_{33} \end{pmatrix},$$

where

$$
\begin{aligned}
Q_{11} &= -(1/6)n_{33}^2 + (1/3)p + (2/3)\dot{u}_1 a_1 + (2/3)a_1^2 - (1/3)\omega_2^2 \\
&\quad -(2/3)n_{23}^2 + (1/3)\mu, \\
Q_{12} &= i\omega_2(n_{23} + a_1), \\
Q_{13} &= (1/2)i\omega_2 n_{33}, \\
Q_{22} &= -(1/3)\dot{u}_1 a_1 - \dot{u}_1 n_{23} + (1/12)n_{33}^2 - (1/6)p - (1/3)a_1^2 \\
&\quad -(1/3)\omega_2^2 + (1/3)n_{23}^2 - (1/6)\mu, \\
Q_{23} &= -(1/2)\dot{u}_1 n_{33}
\end{aligned}
$$

and

$$
\begin{aligned}
Q_{33} &= -(Q_{11} + Q_{22}) \\
&= (1/12)n_{33}^2 - (1/6)p - (1/3)\dot{u}_1 a_1 - (1/3)a_1^2 + (2/3)\omega_2^2 \\
&\quad +(1/3)n_{23}^2 - (1/6)\mu + \dot{u}_1 n_{23}.
\end{aligned}
$$

The possibility of a Petrov type O spacetime is rejected because that would require that the matrix Q vanish. This cannot be so since the imaginary part of Q_{13} is necessarily nonzero.

A Petrov type N spacetime requires that $Q^2 = 0$ with $Q \neq 0$. The expression in $(Q^2)_{13} = 0$ is

$$(1/24)i\omega_2 n_{33}(4\omega_2^2 - n_{33}^2 - 4n_{23}^2 + 2p - 8\dot{u}_1 a_1 + 4a_1^2 + 2\mu) = 0,$$

from which μ is isolated:

$$\mu = (1/2)n_{33}^2 - p + 4\dot{u}_1 a_1 - 2a_1^2 - 2\omega_2^2 + 2n_{23}^2. \tag{6.31}$$

The expression $(Q^2)_{23} = 0$ becomes, after division by n_{33}

$$-(1/2)\omega_2^2 n_{23} - (1/2)\omega_2^2 a_1 + \dot{u}_1^2 a_1 - (1/2)\dot{u}_1\omega_2^2 = 0. \tag{6.32}$$

The resultant, with respect to a_1 of equation (6.32) with the equation $(Q^2)_{11} = 0$ is, after division by ω_2^2, equal to

$$\begin{aligned} 4\omega_2^6 + 16\dot{u}_1 n_{23}\omega_2^4 - 4\omega_2^4\dot{u}_1^2 - \omega_2^4 n_{33}^2 - 16\dot{u}_1^3 n_{23}\omega_2^2 + 16\dot{u}_1^2 n_{23}^2\omega_2^2 & \\ +4\omega_2^2 n_{33}^2\dot{u}_1^2 - 16\dot{u}_1^4 n_{23}^2 - 4\dot{u}_1^4 n_{33}^2 &= 0. \end{aligned}$$

Multiplication of equation (6.31) by $8[\dot{u}_1^2 - (1/2)\omega_2^2]^2$ yields, after taking into account equations (6.32) and (6.33), that

$$8(\mu + p)[\dot{u}_1^2 - (1/2)\omega_2^2]^2 = 0,$$

whence

$$\omega_2^2 = 2\dot{u}_1^2.$$

Substitution into equation (6.32) reveals that $n_{23} + \dot{u}_1 = 0$. Differentiation of $n_{23} + \dot{u}_1 = 0$ then yields $n_{33}^2 - 4\dot{u}_1^2 = 0$. Propagation of $n_{33}^2 - 4\dot{u}_1^2 = 0$ implies that $\dot{u}_1 = 0$. This is a contradiction in *case B,* since $n_{33} \neq 0$. There are therefore no spacetimes in *case B* that belong to Petrov type N.

The matrix condition for a spacetime to belong to Petrov type III is $Q^3 = 0$ with $Q^2 \neq 0$. A direct calculation shows that there are only two independent components in Q^3. The entry $(Q^3)_{13} = 0$, which is

$$\begin{aligned} -4(\mu + p)^2 + (-16\dot{u}_1 a_1 + 4n_{33}^2 - 16a_1^2 + 8\omega_2^2 + 16n_{23}^2)(\mu + p) & \\ +96\omega_2 n_{23} a_1 + 8n_{33}\dot{u}_1 a_1 + 16\dot{u}_1 a_1\omega_2 + 32\dot{u}_1 a_1 n_{23}^2 - n_{33}^4 - 16n_{23}^4 & \\ -16a_1^4 - 16\omega_2^4 + 8\omega_2^2 n_{33}^2 + 8n_{33}^2 a_1^2 - 8n_{33}^2 n_{23}^2 - 16\dot{u}_1^2 a_1^2 & \\ -32\dot{u}_1 a_1^3 + 64a_1^2\omega_2^2 + 32a_1^2 n_{23}^2 + 32\omega_2^2 n_{23}^2 - 48\dot{u}_1 n_{23}\omega_2^2 & \\ -12\dot{u}_1^2 n_{33}^2 - 48\dot{u}_1^2 n_{23}^2 &= 0, \quad (6.33) \end{aligned}$$

can provide a value for the energy density, since the coefficient of the highest power of μ therein cannot vanish. We take the resultant of $(Q^3)_{11} = 0$ and equation (6.33) with respect to μ. Differentiation of the result does not yield anything new. There can be solutions of Petrov type III, but some constraints must be met.

For spacetimes to be of Petrov type D, their Weyl tensor must be such that the matrix equation $M := (Q + \lambda/2\ I_3)(Q - \lambda\ I_3) = 0$ must be satisfied. The condition $M_{13} = 0$, which is

$$1/24 i\omega_2 n_{33}\left(4\omega_2^2 - n_{33}^2 - 4n_{23}^2 - 8\dot{u}_1 a_1 + 4a_1^2 + 2(\mu + p) - 6\lambda\right) = 0,$$

produces a value for λ, viz.,

$$\lambda = -(1/6)n_{33}^2 - (4/3)\dot{u}_1 a_1 + (2/3)a_1^2 + (2/3)\omega_2^2 - (2/3)n_{23}^2 + (1/3)(\mu + p).$$

Then, the equation $M_{12} = 0$ reduces to

$$-(1/4)\omega_2 i(4\omega_2^2 n_{23} + 4\omega_2^2 a_1 + n_{33}^2\dot{u}_1 - 4\dot{u}_1 a_1^2 + 4n_{23}^2\dot{u}_1) = 0,$$

which will be used to eliminate n_{33}. We deduce that $n_{23} + \dot{u}_1 = 0$ from equation $M_{22} = 0$, which is

$$(\dot{u}_1/2)(n_{23} + a_1)(\mu + p) = 0.$$

The condition $M_{11} = 0$ now simplifies to

$$-(\dot{u}_1 n_{23} + (1/2)\omega_2^2)(\mu + p) = 0.$$

Differentiation of $a_1 + n_{23} = 0$ leads to $\mu + p = 0$, a contradiction. There can therefore not be any *case B* solutions that are of Petrov type D.

Petrov type II spacetimes have a Weyl tensor that obeys the matrix condition $N := (Q + \lambda/2\ I_3)^2(Q - \lambda\ I_3) = 0$, yet do not satisfy the condition for Petrov type D. The equation given by $N_{13} = 0$, *i.e.*

$$(1/96)\omega_2 n_{33} i[-36\lambda^2 - 96\omega_2^2 n_{23} a_1 - 8n_{33}^2\dot{u}_1 a_1 - 16\dot{u}_1 a_1\omega_2^2$$

$$-32\dot{u}_1 a_1 n_{23}^2 + n_{33}^4 + 16n_{23}^4 + 16a_1^4 + 16\omega_2^4 - 8\omega_2^2 n_{33}^2 - 8n_{33}^2 a_1^2$$
$$+8n_{33}^2 n_{23}^2 + 16\dot{u}_1^2 a_1^2 + 32\dot{u}_1 a_1^2 - 64a_1^2\omega_2^2 - 32a_1^2 n_{23}^2 - 32\omega_2^2 n_{23}^2$$
$$+48\dot{u}_1 n_{23}\omega_2^2 + 12\dot{u}_1^2 n_{33}^2 + 48\dot{u}_1^2 n_{23}^2(-16n_{23}^2 - 8\omega_2^2 + 16a_1^2$$
$$-4n_{33}^2 + 16\dot{u}_1 a_1)(\mu + p) + 4(\mu + p)^2] \quad = \quad 0, \quad (6.34)$$

provides an expression for λ^2. Multiplying this expression by λ gives the value of λ^3. Substitution of the equalities for λ^2 and λ^3 into the matrix N results in a diagonal matrix. It turns out that this matrix is a scalar matrix; in other words, the three non-trivial entries are actually equal, and so N is now proportional to the identity matrix. The resultant of this non-trivial entry of N and equation (6.34) with respect to λ yields an equation with 923 terms. This equation can be considered a definition for the energy density, μ, except when all the coefficients of the various powers of μ vanish or when there are no real-valued solutions for μ. We now turn our attention to the situation when it is indeed the case that this equation of 923 terms has its coefficients of the various powers of μ vanishing. The highest power of μ is 4. We require the vanishing of the corresponding coefficient, viz.

$$-(1/12)\dot{u}_1^2 n_{33}^2 - (1/3)\dot{u}_1^2 n_{23}^2 - (1/12)\omega_2^4 - (1/3)\dot{u}_1 n_{23}\omega_2^2 = 0. \quad (6.35)$$

We shall use equation (6.35) to eliminate n_{33}. The derivative of equation (6.35) becomes

$$\frac{\omega_2^2(2\dot{u}_1 n_{23} + \omega_2^2)[\omega_2^4 + 4\dot{u}_1 n_{23}\omega_2^2 + 4\dot{u}_1^2 a_1^2 + 2(\mu + p)\dot{u}_1^2]}{24\dot{u}_1^3} = 0 \quad (6.36)$$

The term $2\dot{u}_1 n_{23} + \omega_2^2$ cannot vanish, otherwise equation (6.35) would imply that $-(\dot{u}_1^2 n_{33}^2)/12 = 0$, a contradiction. Equations (6.35) and (6.36) determine a value for μ, viz.

$$\mu = -p - 2a_1^2 + (1/2)n_{33}^2 + 2n_{23}^2.$$

We now return to the equation with 923 terms. The vanishing of the coefficient of μ^3 therein simplifies to

$$\omega_2^2(-2\dot{u}_1 a_1 + \omega_2^2)^2(2\dot{u}_1 n_{23} + \omega_2^2) = 0,$$

after making use of equation (6.35) to eliminate n_{33}. Since we have already ruled out the possibility that $2\dot{u}_1 n_{23} + \omega_2^2 = 0$, we must have that $\omega_2^2 = 2\dot{u}_1 a_1$. Differentiating $\omega_2^2 - 2\dot{u}_1 a_1 = 0$ implies that $n_{23} + a_1 = 0$. Equation (6.35) now reduces to $-(1/12)\dot{u}_1^2 n_{33}^2 = 0$, a contradiction. There can therefore be solutions of Petrov type II, provided that an expression with 923 terms (mentioned above) yields a value for μ.

In summary, there are no solutions in *case B* that are of Petrov types D, N or O. If there are spacetimes in *case B*, they must be of Petrov types I, II or III.

6.3 Case C: $n_{33} = 0, a_2 = 0; \omega_1 = 0, n_{23}\omega_2\dot{u}_1 \neq 0.$

For this situation, the propagation equations reduce to

$$\begin{aligned}
\partial_1 a_1 &= (1/2)(p+\mu) - 2\omega_2^2 + \dot{u}_1 a_1 + n_{23}^2 + a_1^2,\\
\partial_1 \dot{u}_1 &= -\Lambda + (3/2)p + (1/2)\mu - 2\omega_2^2 - \dot{u}_1^2 + 2\dot{u}_1 a_1,\\
\partial_1 \omega_2 &= -2\dot{u}_1\omega_2 + \omega_2 n_{23} + \omega_2 a_1,\\
\partial_1 n_{23} &= -\Lambda + p + 2\dot{u}_1 a_1 - \dot{u}_1 n_{23} + 2a_1 n_{23} - a_1^2 + n_{23}^2,\\
\partial_1 p &= -\dot{u}_1(\mu + p)
\end{aligned}$$

and

$$\partial_1 \Lambda = 0.$$

Equation (6.3) can be used to solve for Λ, giving

$$\Lambda = -\omega_2^2 + p + 2\dot{u}_1 a_1 - a_1^2 + n_{23}^2.$$

The matrix Q is

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{pmatrix},$$

where

$$\begin{aligned} Q_{11} &= -(1/3)\omega_2^2 + (1/3)(p+\mu) + (2/3)\dot{u}_1 a_1 + (2/3)a_1^2 - (2/3)n_{23}^2, \\ Q_{12} &= i\omega_2(n_{23} + a_1), \\ Q_{22} &= -(1/3)\dot{u}_1 a_1 - \dot{u}_1 n_{23} - (1/3)\omega_2^2 - (1/3)a_1^2 + (1/3)n_{23}^2 - (1/6)(p+\mu) \end{aligned}$$

and

$$\begin{aligned} Q_{33} &= -(Q_{11} + Q_{22}) \\ &= (2/3)\omega_2^2 - (1/3)\dot{u}_1 a_1 - (1/3)a_1^2 + (1/3)n_{23}^2 \\ &\quad -(1/6)(\mu + p) + \dot{u}_1 n_{23}. \end{aligned} \tag{6.37}$$

For Petrov type O, the matrix condition is that Q be equal to zero. Since ω_2 does not vanish, we must have $n_{23} + a_1 = 0$. The quantity $Q_{11} - Q_{22}$, which equals $(1/2)(\mu + p)$, must also be zero, since Q vanishes; however, this is a contradiction. There are therefore no Petrov type O solutions in *case C*.

There are also no spacetimes of Petrov type III since the vector $(0, 0, 1)$ is a non-null eigenvector of Q. The proof that there are no Petrov type III spacetimes in *case C* is identical to that presented for *case A,* and therefore is omitted here.

In order that a spacetime be of Petrov type N, the matrix Q must satisfy $Q^2 = 0$ with $Q \neq 0$. The entry $(Q^2)_{11} = 0$ can be used to find a value for μ:

$$\mu = 4\omega_2^2 - p - 2\dot{u}_1 a_1 - 2a_1^2 + 2n_{23}^2 + 6\dot{u}_1 n_{23}. \tag{6.38}$$

The only remaining independent entry in $Q^2 = 0$ is given by

$$(\omega_2 a_1 + \omega_2 n_{23} + \omega_2^2 + 2\dot{u}_1 n_{23})(-\omega_2 a_1 - \omega_2 n_{23} + \omega_2^2 + 2\dot{u}_1 n_{23}) = 0. \tag{6.39}$$

The propagation of this equation does not yield any new restrictions. Therefore there can be type N spacetimes in *case C*.

For Petrov type II or type D there is a non-zero repeated eigenvalue. Because of the structure of Q, one of the eigenvalues is E_{33}. The vector $(0,0,1)$ is one eigenvector associated with the eigenvalue E_{33}. Since Q is trace-free, the sum of the eigenvalues must be zero. Therefore, there are two cases to consider, depending on whether or not E_{33} is the repeated eigenvalue.

Suppose that the repeated value is indeed E_{33}.The matrix $Q - E_{33}\ I_3$ is

$$\begin{pmatrix} 2E_{11}+E_{22} & iH_{12} & 0 \\ iH_{12} & E_{11}+2E_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the vector $(0,0,1)$ is an eigenvector belonging to the eigenvalue E_{33}, the dimension of the eigenspace of Q associated with E_{33} is either two or one according as $(2E_{11}+E_{22})(E_{11}+2E_{22})+H_{12}^2$ vanishes or not, whence the Petrov type is D or II, respectively. However, $-2E_{33}$ is the non-repeated eigenvalue, and so the matrix $Q+2E_{33}\ I_3$, which is

$$\begin{pmatrix} -E_{11}-2E_{22} & iH_{12} & 0 \\ iH_{12} & -2E_{11}-E_{22} & 0 \\ 0 & 0 & 3E_{33} \end{pmatrix},$$

must be singular. The expression $(E_{11}+2E_{22})(2E_{11}+E_{22})+H_{12}^2$ must then vanish, whence the Petrov type must be D. Explicitly, the equation

$$(E_{11}+2E_{22})(2E_{11}+E_{22})+H_{12}^2 = 0$$

is given by

$$\begin{aligned} &-(\dot{u}_1 n_{23} + (1/2)\omega_2^2)(\mu+p) + \omega_2^4 + \omega_2^2(3\dot{u}_1 n_{23} - \dot{u}_1 a_1 + 2n_{23}^2 + 2n_{23}a_1) \\ &\quad +\dot{u}_1 n_{23}(-2\dot{u}_1 a_1 - 2a_1^2 + 2n_{23}^2 + 2\dot{u}_1 n_{23}) = 0. \end{aligned} \tag{6.40}$$

This gives a definition for μ, unless its coefficient vanishes, *i.e.* unless $2\dot{u}_1 n_{23} + \omega_2^2 = 0$. Supposing that, indeed, $2\dot{u}_1 n_{23} + \omega_2^2 = 0$, then equation (6.40) implies that $n_{23} + a_1 = 0$, and the requirement that this is propagated now shows that $\mu + p = 0$, which is a contradiction. There can therefore be *case C* spacetimes of Petrov type D; however, some constraints need to be satisfied.

Now suppose that the repeated eigenvalue is not E_{33}; it must then be $-E_{33}/2$. Therefore, the matrix $Q + (E_{33}/2)I_3$ is

$$\begin{pmatrix} (E_{11} - E_{22})/2 & iH_{12} & 0 \\ iH_{12} & (E_{22} - E_{11})/2 & 0 \\ 0 & 0 & 3E_{33}/2 \end{pmatrix}$$

Since $-E_{33}/2$ is an eigenvalue, the determinant of this matrix must be zero, forcing $4H_{12}^2 - (E_{11} - E_{22})^2 = 0$. This is equivalent to

$$\begin{aligned}(p + \mu - 4\omega_2 a_1 + 2\dot{u}_1 a_1 + 2a_1^2 - 4\omega_2 n_{23} + 2\dot{u}_1 n_{23} - 2n_{23}^2)\times \\ \times(p + \mu + 4\omega_2 a_1 + 2\dot{u}_1 a_1 + 2a_1^2 + 4\omega_2 n_{23} + 2\dot{u}_1 n_{23} - 2n_{23}^2) = 0,\end{aligned} \tag{6.41}$$

which gives two possible values for μ. If $H_{12} = 0$ and $E_{11} = E_{22}$ then the dimension of the eigenspace associated with $-E_{33}/2$ is two, whence the Petrov type is D; otherwise, the dimension is one, whence the Petrov type is II. If the Petrov type is D, then the condition $H_{12} = 0$ implies that $n_{23} + a_1 = 0$, and the condition $E_{11} = E_{22}$ necessitates that $(1/2)(\mu + p) + \dot{u}_1(n_{23} + a_1) + (a_1 + n_{23})(a_1 - n_{23}) = 0$. Together, these two conditions imply that $\mu + p = 0$, which is a contradiction, and thus the spacetimes must be of Petrov type II.

In summary, spacetimes of Petrov type III and O are not allowed in *case C*. The other Petrov types are allowed but under the presence of certain constraints.

6.4 Summary

The only spacetimes allowed in *case A* must be of Petrov type I. Spacetimes that belong to *case B* cannot belong to Petrov types O, N, or D. There can be solutions of type I. There can also be solutions of Petrov type III, but some constraints have to be met. There can also be solutions of type II, provided that a particular equation of 923 terms contains terms involving μ. If the coefficients of μ all vanish in that particular equation, then there are no solutions. We note that since the Petrov types O and D are ruled out, spacetimes with a purely electric Weyl tensor that belong to either of class A or of class B must be of Petrov type I (see Kramer et al. (1980) who mention the fact that if the matrix Q is real, the only allowed Petrov types are O, D and I).

For spacetimes that belong to *case C*, there are no solutions of Petrov types O and III. There can be solutions of Petrov type N. In these spacetimes, the fluid has the energy given by equation (6.38) and the solutions are subject to the constraint (6.39). There can also be Petrov type D solutions. They have Q_{33}, given by equation (6.37), as a double eigenvalue. The energy density is given implicitly by equation (6.40) and the quantity $2\dot{u}_1 n_{23} + \omega_2^2$ cannot vanish. Furthermore, there can be Petrov type II solutions. They have Q_{33} as the non-repeated eigenvalue. The energy density must satisfy equation (6.41). The quantities $\omega_2(n_{23} + a_1)$ and $(1/6)(p + \mu) + \dot{u}_1(a_1 + n_{23}) + a_2^2 - n_{23}^2$ cannot both vanish on an open set. There can also be solutions of type I.

The results we have obtained for Petrov type N are compatible with those obtained by Carminati (1988), who showed that Petrov type N shear-free perfect fluids with a barotropic equation of state must belong to the class studied by Krasiński (1978), and therefore must belong to our *case C*.

There are no spacetimes within the class we are studying that are conformally flat, *i.e.* of Petrov type O. This, of course, is compatible with Ellis (1971) who attributes to Trümper[14] the result that conformally flat spacetimes with a barotropic equation of state must be shear-free, geodesic and irrotational and so must belong to the Friedmann-Robertson-Walker models.

Kramer et al. (1980) mention that they were not aware of the existence of any perfect fluid solutions of Petrov type III. A superficial search of the literature did not reveal any solutions other than the work of Allnutt (1981) which uncovered a perfect fluid of Petrov type III that possesses non-zero shear. Carminati (1990) mentions the article of Allnutt and adds that, as far as he is aware, there are no known shear-free perfect fluid solutions of Petrov type III. We have demonstrated the possible existence of such spacetimes in our *case B*, although they are subject to rather complicated (yet readily accessible) constraints.

We have summarized the previous results in table (6.2) appearing on page 181.

[14]Ellis does not give an exact reference.

Petrov Type	*Case A*	*Case B*	*Case C*
I	Allowed	Allowed	Allowed
D	Disallowed	Disallowed	Allowed
II	Disallowed	Allowed	Allowed
N	Disallowed	Disallowed	Allowed
III	Disallowed	Allowed	Disallowed
O	Disallowed	Disallowed	Disallowed

Table 6.2: Allowed Petrov Types

Appendix A

The `forms` Maple package for differential forms

De la discussion jaillit la lumière.

Proverbe français

THE Maple package `forms` is a collection of programs for calculations involving differential forms and their dual vectors. Maple V or Maple V release 2 is required in order to use it. A standard reference about Maple is Char *et al.* (1991).

The following functions are provided:

`adjoint_d`: compute the adjoint differential, or coderivative of a form.

`cartan_lemma`: solve for unknown forms

`cauchy_char`: compute the Cauchy characteristic of a differential ideal

`d`: compute the exterior derivative

`derived_ideal`: compute the derived ideal of a differential ideal `express_base`: express a form over a basis

`form_coeffs`: find the coefficients of forms

`form_to_vec`: take a basic form to a basic vector[1]

`form_part`: find the non scalar part of a term

`hodge_star`: apply on a form the hodge star operator with respect to an inner product

`hook`: compute the interior product of a form by a vector

`in_ideal`: verify if a form belongs to given differential ideal

`inner_product`: compute an inner product between two forms

`item_map`: apply an operation to elements of nested structures

`laplace_beltrami`: apply a generalized Laplacian to a form

`lie`: compute the lie derivative of a form

`linear_divisors`: compute the linear divisors of a form

`linear_solve`: solve linear equations; extends `solve(..., linear)`

`mod_ideal`: finds a representative of a form modulo a differential ideal

`scalar_part`: find the coefficient of a basic form

`standard_form`: regroup forms according to basic forms

[1]A *basic* form is a `nform`, a `form` or a `dform`. A *basic* vector is a `nvector`, a `vector` or a `dvector`.

`subs_form`: substitute forms in other forms; extends `subs()`

`vec_scalar_part`: find the coefficient of a multivector term

`vec_subs`: substitute multivectors. extends `subs()`

`vec_to_form`: take a basic vector to a basic form

`vec_wedge`: compute the exterior multiplication of vectors

`vector_part`: find the multivector part of a term

`wdegree`: find the degree of a form

`wedge`: compute the exterior multiplication of forms

In order to use the `forms` package, it must first be loaded in Maple via the `with()` facility.

```
> with(forms):
```

In the Maple examples below, it is useful the remember that the ordering of terms in a sum, of factors in a product and elements in a set are session dependent. The output of each example may thus be different from that shown in the present document.

Let V be a real vector space of dimension n, and V^* its dual space. Elements of V are called vectors; those of V^*, covectors or 1-forms.

A.1 Differential forms

Any non-compound Maple expression is a differential form of degree 0. We shall refer to such forms as 0-forms. Compound Maple expressions are quantities like sets, lists, expression sequences and so on. From 0-forms, one can get differential forms of degree 1, or 1-forms, by using the exterior derivative operation `d()`. For example,

```
> F1:=x*y+3*z;
                              F1 := x y + 3 z
```

is a 0-form.

```
> dF1:=d(F1);
            dF1 := dform(1, x) y + x dform(1, y) + 3 dform(1, z)
```

As one can see, the exterior derivative operates on 0-forms as a differential operator and produces a 1-form. The notation `dform(1,x)` represents a closed 1-form with name `x`. This name is used to distinguish between various differential forms and should be either a Maple name or a Maple indexed[2] name. By definition, closed forms are differential forms whose exterior derivative is zero. One can use those `dform` expressions to build other 1-forms.

```
> F2:=x*d(y)+t*d(z);
                   F2 := x dform(1, y) + t dform(1, z)
```

One can use the standard addition of Maple to add differential forms together.

```
> dF1+3*F2;
     dform(1, x) y + 4 x dform(1, y) + 3 dform(1, z) + 3 t dform(1, z)
```

Like terms are combined using the `standard_form()` operation.

```
> standard_form(dF1+3*F2);

          dform(1, x) y + (3 + 3 t) dform(1, z) + 4 x dform(1, y)
```

[2]There is no extra support for forms with indexed names. Further development of the `forms` package could involve index symmetries and also Einstein's summation convention.

The coefficient multiplying a basic form can be obtained with the `scalar_part()` function. The basic form itself is obtained using the `form_part()` function.

```
> scalar_part(3*x*d(y));
                                   3 x
> form_part(3*x*d(y));
                                dform(1, y)
```

Two differential forms can be multiplied together. However, since the multiplication of differential forms is not necessarily commutative, the multiplication provided by Maple cannot[3] be used. The appropriate multiplication, the exterior multiplication, is obtained through the `wedge()` operation.[4]

```
> standard_form(wedge(dF1,F2));
              y x wedge(dform(1, x), dform(1, y))
                   + (x t - 3 x) wedge(dform(1, y), dform(1, z))
                   + y t wedge(dform(1, x), dform(1, z))
```

The notation `wedge(dform(1,y), dform(1,z))` means that the differential forms that are arguments to the `wedge()` function are multiplied together using exterior multiplication. The `wedge()` operation is distributive. Scalar functions (0-forms) move out of `form()`. The ordering within the square brackets is unique during a

[3]Even if one could "overload" the $*$ operator of Maple, it is arguably better to have a different notation for each type of multiplication. For an example to ponder about, consider the design of a system that could handle tensor multiplication of exterior multiplications of arrays of quaternion-valued differential forms. The `Gauss` package for Maple, see Gruntz *et al.* (1993), is a suitable environment for such a system.

[4]Unfortunately, this associative operator cannot be made into an infix operator if it has more than two arguments without the appearance of extraneous parentheses. However doing `` alias(`&^`=wedge); `` will make give an infix operator for the exterior product of two differential forms.

Maple session, but can change from one session to another.

```
> standard_form(wedge(F2,dF1));
             - y x wedge(dform(1, x), dform(1, y))
                  + (3 x - x t) wedge(dform(1, y), dform(1, z))
                  - y t wedge(dform(1, x), dform(1, z))
```

The `wedge()` and the `d()` operations are then appropriate tools to construct differential forms of various degrees.

```
> F3:=standard_form(wedge(wedge(F2,dF1),d(x)));
       F3 := (3 x - x t) wedge(dform(1, x), dform(1, y), dform(1, z))
```

The degree of a differential form is obtained by `wdegree()`. Note that for the answer of `wdegree()` to be valid, each term of its argument must be of the same form-degree.[5]

```
> wdegree(F3);
                                   3
```

The set of p-forms, or forms of degree p, is denoted by $\bigwedge^p(V^*)$. The exterior algebra of V^* is the graded algebra

$$\bigwedge(V^*) := \bigwedge^0(V^*) \oplus \bigwedge^1(V^*) \oplus \cdots \oplus \bigwedge^n(V^*),$$

where $\bigwedge^0(V^*)$ is the set of real (complex) valued functions and $\bigwedge^1(V^*)$ is the covector space V^*. Exterior multiplication is associative and distributive, but not commutative. It satisfies the relation

$$\alpha \wedge \beta = (-1)^{(pq)} \beta \wedge \alpha, \quad \alpha \in \bigwedge^p(V^*), \beta \in \bigwedge^q(V^*).$$

[5]There is a question of efficiency behind this design. Assuming that the argument of `wdegree()` is homogeneous in degree allows for a constant time calculation. Without that assumption, every term would need to be checked, therefore checking the form-degree would be an operation with a cost linear in the number of input terms. A test for checking degree-homogeneity of `fm1` is `evalb(nops(map(wdegree, convert(fm1,set)))=1)`. The `forms` package can otherwise handle forms of non-homogeneous degree; in particular, exterior multiplication is handled correctly.

The operation `subs_form()` is used to substitute forms into other forms. The function `subs()` of Maple is not adequate, since it does not preserve the canonical forms that the `forms` package uses. Since the first step that `subs_form()` performs is to use Maple's `subs()` command, the rules governing the use of `subs()` also apply here.

```
> standard_form(subs_form(d(z)=3*x*d(t)+d(u), F3));
              2
      - 3 x  (- 3 + t) wedge(dform(1, t), dform(1, x), dform(1, y))

           - x (- 3 + t) wedge(dform(1, u), dform(1, x), dform(1, y))
```

The various coefficients of a differential form are obtained with the function `form_coeffs()`.[6]

```
> form_coeffs(dF1,{d(x),d(y),d(z)});
                                  x, y, 3
```

Note that the form which is passed as a first argument to `form_coeffs()` must be constructible from the elements in the (optional) second argument, otherwise an error will be reported. This is quite useful because normally the results of `form_coeffs()` are only useful if the elements of the second argument are independent. For example, if one knows that `z` is a function of `x` and `y`, then the derivative of `z` would be expressible in terms of `d(x)` and `d(y)`.

```
> dz:=d(z(x,y));
               /  d         \                /  d         \
       dz := |---- z(x, y)| dform(1, x) + |---- z(x, y)| dform(1, y)
               \ dx         /                \ dy         /
```

[6]Note that the order can vary. An expression sequence is returned to be consistent with the `coeffs()` function of Maple.

The coefficients[7] in dF1 would then be

```
> form_coeffs(dF1,{d(x),d(y)});
Error, (in form_coeffs) Non basis form(s) present:, {dform(1,z)}
```

The error message signals that we assumed that the form dF1 can be constructed solely with d(x) and d(y). We first have to express dz in terms of d(x) and d(y), and substitute the result into dF1.

```
> dF1_a:=subs_form(d(z)=dz,dF1);
                                                   /  d         \
   dF1_a := dform(1, x) y + x dform(1, y) + 3 |---- z(x, y)| dform(1, x)
                                                   \ dx         /

                  /  d         \
         + 3 |---- z(x, y)| dform(1, y)
                  \ dy         /
```

One can then find the coefficients which were being sought.

```
> form_coeffs(dF1_a,{d(x),d(y)});
                        /  d         \        /  d         \
                  x + 3 |---- z(x, y)|, y + 3 |---- z(x, y)|
                        \ dy         /        \ dx         /
```

Sometimes it is useful to express a one-form with respect to a basis. This may happen, for example, when one wants to express the derivative of a function (i.e. the contact equation). Very often, one needs to invent new names for the various coefficients. The function express_base() was written to simplify this. It takes a form and a basis, and returns an equality where the left-hand side is the form, and the right hand-side is the expanded version of it. An optional third argument gives a method for constructing the names for the coefficients.

[7]Note that some releases of Maple would have returned D[1](z)(x,y) as the form of the coefficients of dz.

As an example, we can expand `d(z)` in terms of `d(x)` and `d(y)` using the `express_base`() facility.

```
> substitutions:={express_base(d(z),{d(x),d(y)})};
     substitutions := {dform(1, z) = z_x dform(1, x) + z_y dform(1, y)}



> dF1_b:=subs_form(substitutions,dF1);
dF1_b := dform(1, x) y + x dform(1, y) + 3 z_x dform(1, x) + 3 z_y dform(1, y)

> form_coeffs(dF1_b,{d(x),d(y)});
                              y + 3 z_x, x + 3 z_y
```

Of course, the expression for `d(z)` can be constructed using the facilities of Maple:

```
> substitutions_2:={d(z)=zx*d(x)+zy*d(y)};
     substitutions_2 := {dform(1, z) = zx dform(1, x) + zy dform(1, y)}

> dF1_c:=standard_form(subs_form(substitutions_2,dF1));
          dF1_c := (x + 3 zy) dform(1, y) + (y + 3 zx) dform(1, x)
```

The optional third argument to `express_base`() is a function that will be called with three arguments: a name, a base element and a number. It should return a name constructed with this information. For example `proc(name,base_element, ind)` could return on `(F,dform(1,x),3)` something like `F_x` or `F3`.

```
> substitutions_3:=express_base(d(z),[d(x),d(y)],<name[ind]|name,base,ind>);
    substitutions_3 := dform(1, z) = z[1] dform(1, x) + z[2] dform(1, y)

> dF1_d:=standard_form(subs_form(substitutions_3,dF1));
        dF1_d := (x + 3 z[2]) dform(1, y) + (y + 3 z[1]) dform(1, x)
```

The differential forms seen so far have been constructed with the exterior derivative, `d`(), of functions and forms and with the exterior multiplication, `wedge`, of forms. It is quite useful to be able to use differential forms without necessarily having to construct them out of scalar functions. The notation `nform(`*degree*`,` *name*`)` is used to specify a differential form.

```
> F4:=nform(3,w);
                              F4 := nform(3, w)
> dF4:=d(F4);
                              dF4 := dform(4, w)
> d2dF4:=d(dF4);
                                  d2dF4 := 0
> wedge(F4,F2), wedge(F2,F4);
x wedge(nform(3, w), dform(1, y)) + t wedge(nform(3, w), dform(1, z)),
    - x wedge(nform(3, w), dform(1, y)) - t wedge(nform(3, w), dform(1, z))
```

A.2 Vectors and multivectors

A vector is an object which is dual to a one-form. In the package **forms**, basic vectors are **nvector(***name1***)**, which is dual to **form(1,** *name1***)**, and **dvector(***name1***)**, which is dual to **dform(1,***name1***)**. Vectors are formed by linear combinations (over the maple expressions) of basic vectors.

```
> V1:=dvector(x)+3*z*dvector(y)-u*dvector(z);
                 V1 := dvector(x) + 3 z dvector(y) - u dvector(z)
> V2:=-dvector(x)+2*x*dvector(y)+3*u*dvector(z);
               V2 := - dvector(x) + 2 x dvector(y) + 3 u dvector(z)
> V3:=nvector(w1);
                              V3 := nvector(w1)
> V4:=expand(V1+3*x*V2);
      V4 := dvector(x) + 3 z dvector(y) - u dvector(z) - 3 x dvector(x)
                2
           + 6 x  dvector(y) + 9 x u dvector(z)
```

Vectors can be multiplied together with **vec_wedge**(). The notation for the **vec_wedge** of basic vectors is **vector**([*list of basis vectors*]).

```
> vec_wedge(V1,V2);
```

```
2 x vec_wedge(dvector(x), dvector(y))
     + 2 u vec_wedge(dvector(x), dvector(z))
     + 3 z vec_wedge(dvector(x), dvector(y))
     + 9 z u vec_wedge(dvector(y), dvector(z))
     + 2 u x vec_wedge(dvector(y), dvector(z))
```

The set of p-vectors, formed by the sum of terms that are the exterior products of p vectors, is denoted by $\bigwedge^p(V)$. The exterior algebra of V^* is the graded algebra

$$\bigwedge(V) := \bigwedge^0(V) \oplus \bigwedge^1(V) \oplus \cdots \oplus \bigwedge^n(V),$$

where $\bigwedge^0(V)$ is the real (complex) valued functions and $\bigwedge^1(V)$ is the vector field V. The exterior multiplication is associative and distributive, but not commutative. It satisfies the relation

$$\xi \wedge \eta = (-1)^{(pq)} \eta \wedge \xi, \quad \xi \in \bigwedge^p(V), \eta \in \bigwedge^q(V).$$

Two functions[8] help in the construction of vectors, namely, `form_to_vec`() and `vec_to_form`(). The function `form_to_vec`() takes a basic form, and returns the corresponding basic vector. The function `vec_to_form`() does the opposite.

```
> form_to_vec(nform(1,w1));
                          nvector(w1)

> form_to_vec(d(x));
                           dvector(x)

> vec_to_form(dvector(y));
                          dform(1, y)

> vec_to_form(nvector(w2));
```

[8]These functions are used for formal manipulations. They are not intended to mathematically convert between forms and vectors via a pairing such as $< \vec{e}_a, \eta^b >= \delta_a^b$. For such a conversion, the function `hook`(), in conjunction with `solve()`, is more suitable.

```
nform(1, w2)
```

One can find the scalar function multiplying a basic vector with the function `vec_scalar_part()`. The basic vector is returned with the function `vector_part()`.

```
> vec_scalar_part(3*x*nvector(w1));
                                    3 x

> vector_part(3*x*nvector(w1));
                                nvector(w1)
```

For the same reason why one should not do substitutions in forms using the Maple `subs()` function, but rather with the `forms`-package `subs_form()`, the same situation holds for vectors. The function `vec_subs()` is provided to do the work.

```
> vec_subs(dvector(z)=y*dvector(x)+x*dvector(y), vec_wedge(V1,V2));
          2 x %1 + 2 u x %1 + 3 z %1 - 9 z u y %1 - 2 u x y %1
%1 :=              vec_wedge(dvector(x), dvector(y))
```

Let v_i be elements of V and w^j be elements of V^*. One can define a pairing $< v_i, w^j >$ which is linear in each argument and is a real (or complex) number. If v_i is chosen to be dual to w^i, then $< v_i, w^j >$ is equal to δ_i^j. This pairing is extended to elements of $\bigwedge^p(V)$ and $\bigwedge^p(V^*)$ as follows: Let $\xi := v_1 \wedge \cdots \wedge v_p$ and $\alpha := w^1 \wedge \cdots \wedge w^p$. The pairing $< \xi, \alpha >$ is defined to be the determinant of the matrix $M_i^j :=< v_i, w^j >$. The definition of this pairing is then extended using linearity in both arguments. If the degree of the multivector is higher than the degree of the form, the pairing is defined to be zero.

Given $\xi \in V$, the interior product $(\xi \rfloor)$ of ξ with a p-form, giving a $(p-1)$-form, is defined implicitly as follows:

$$< \eta, \xi \rfloor \alpha >=< \xi \wedge \eta, \alpha > \quad \forall \eta \in \bigwedge^p(V), \alpha \in \bigwedge^p(V^*)$$

The function `hook()` is the implementation in the `forms` package of the inner product.

```
> F5:=a*wedge(d(x),d(y))+b*wedge(d(y),d(z))+c*wedge(d(z),d(x));
              F5 := a wedge(dform(1, x), dform(1, y))
                   + b wedge(dform(1, y), dform(1, z))
                   - c wedge(dform(1, x), dform(1, z))

> V5:=form_to_vec(d(x));
                          V5 := dvector(x)

> V6:=form_to_vec(d(y));
                          V6 := dvector(y)

> hook(V5,F5);
                   a dform(1, y) - c dform(1, z)

> hook(vec_wedge(V5,V6),F5);
                                 a
```

A.3 Higher level functions

The higher level functions are functions that build upon the basic differential exterior algebra functions we have seen so far. Bryant et al. (1991) provide more information about the concepts involve.

Suppose that $\texttt{nform}(p, \alpha)$ is a p-form. The space of linear divisors of $\texttt{nform}(p, \alpha)$ is the set of one-forms whose exterior product with $\texttt{nform}(p, \alpha)$ vanish. This space is calculated with the function `linear_divisors()`.

```
> linear_divisors(wedge(d(x), d(y)), DIV);
    DIV[1] dform(1, x) + DIV[2] dform(1, y), {0}, {DIV[1], DIV[2]}
> F6:=wedge( d(x)+3*d(y), wedge ( d(u), d(t) ) + wedge( d(x), d(z) ) );
```

```
        F6 := - wedge(dform(1, x), dform(1, t), dform(1, u))
             - 3 wedge(dform(1, y), dform(1, t), dform(1, u))
             - 3 wedge(dform(1, x), dform(1, y), dform(1, z))

> linear_divisors(F6, divisor);
  1/3 divisor[1] dform(1, x) + divisor[1] dform(1, y), {}, {divisor[1]}

> linear_divisors(a*wedge(d(x),d(y))+b*wedge(d(t),d(z)), DIV);
                              0, {}, {}
```

The second argument to `linear_divisors`() is a name which will be used in constructing the arbitrary parameters in the answer. The function `linear_divisors`() returns a sequence of three expressions. The answer is given by the first expression parametrized by all possible values of the parameters given in the third expression. The second expression is the set of relations, if any, that must be equal to zero for the answer to be valid. Note that all the basic forms appearing in the first argument of `linear_divisors`() are assumed to be independent.

A subring $\mathcal{I} \subset \bigwedge(V^*)$ is called an ideal if every element α of $\mathcal{I}$ is of homogeneous degree and if $\alpha \in \mathcal{I}$ implies that $\alpha \wedge \beta \in \mathcal{I}$ for all $\beta \in \bigwedge(V^*)$. Furthermore, $\mathcal{I}$ is called a *differential* ideal if $\mathcal{I}$ is closed under exterior differentiation, *i.e.* if the exterior derivative of every element of $\mathcal{I}$ belongs to $\mathcal{I}$. For the `forms` package, differential ideals are represented by a set of differential forms that will be used as generators for the ideal.

```
> Ideal1:={d(y)-p*d(x)};
               Ideal1 := {dform(1, y) - p dform(1, x)}

> Ideal2:=Ideal1 union map(d, Ideal1);
  Ideal2 :=
      {wedge(dform(1, x), dform(1, p)), dform(1, y) - p dform(1, x)}

> Ideal3:=Ideal2 union {d(H(x,y,p))};
```

```
Ideal3 := {wedge(dform(1, x), dform(1, p)),

    dform(1, y) - p dform(1, x),

    /  d           \                /  d           \
    |---- H(x, y, p)| dform(1, x) + |---- H(x, y, p)| dform(1, y)
    \ dx           /                \ dy           /

           /  d           \
         + |---- H(x, y, p)| dform(1, p)                         }
           \ dp           /
```

Given an ideal $\mathcal{I}$, the Cauchy characteristic space of $\mathcal{I}$ is the set of vectors whose interior product with all the members of $\mathcal{I}$ is itself a member of $\mathcal{I}$. This space is calculated with the function `cauchy_char()`.

```
> cauchy_char(Ideal1, CC);
           CC[1] dvector(x)
           ---------------- + CC[1] dvector(y), {}, {CC[1]}
                   p


> cauchy_char(Ideal2, CC);
                                    0, {}, {}

> cauchy_char(Ideal3,CC);
           /  d           \
     CC[1] |---- H(x, y, p)| p dvector(y)
           \ dp           /
 - -------------------------------------- + CC[1] dvector(p)
   /  d           \   /  d           \
   |---- H(x, y, p)| + |---- H(x, y, p)| p
   \ dx           /   \ dy           /
```

```
            /  d             \
     CC[1] |---- H(x, y, p)| dvector(x)
            \ dp             /
- --------------------------------------,             {}, {CC[1]}
   /  d            \   /  d            \
   |---- H(x, y, p)| + |---- H(x, y, p)| p
   \ dx            /   \ dy            /
```

The interpretation of the answer and of the second argument of the function `cauchy_char()` is similar to what was described for the linear divisors.[9]

The retracting subspace of the differential ideal $\mathcal{I}$ is the annihilator of the Cauchy characteristic space of $\mathcal{I}$ (*i.e.* all the differential one-forms whose interior products by members of the Cauchy characteristic space of $\mathcal{I}$ vanish). This space is calculated with the function `retraction()`.

```
> retraction(Ideal1,RR);
               RR[1] dform(1, y)
             - ----------------- + RR[1] dform(1, x), {}, {RR[1]}
                       p

> retraction(Ideal2,RR);
                                 0, {0}, {}


> retraction(Ideal3,RR);
      RR[2] dform(1, y) + RR[1] dform(1, p) +
```

[9]In particular, the answer to `cauchy_char`(Ideal1) may vary by overall factors which could be included in `CC[1]`. It is not clear which is the best strategy as to which factors should be absorbed. Part of the problem is to avoid removing factors which could become zero.

```
/         /  d            \                        \
|         |---- H(x, y, p)| p RR[2]                |
|         \ dp            /                        |
|- ---------------------------------------- + RR[1]|
|  /  d            \   /  d            \           |
|  |---- H(x, y, p)| + |---- H(x, y, p)| p         |
\  \ dx            /   \ dy            /           /

//  d            \   /  d            \  \
||---- H(x, y, p)| + |---- H(x, y, p)| p| dform(1, x)
\\ dx            /   \ dy            /  /

   / /  d            \
  /  |---- H(x, y, p)|,                              {},
 /   \ dp            /

{RR[2], RR[1]}
```

```
> map(simplify,standard_form("[1]));
        RR[2] dform(1, y) + RR[1] dform(1, p) - dform(1, x) (

          /  d            \                  /  d            \
          |---- H(x, y, p)| p RR[2] - RR[1] |---- H(x, y, p)| p
          \ dp            /                  \ dy            /

                  /  d            \    / /  d            \
           - RR[1] |---- H(x, y, p)|)  /  |---- H(x, y, p)|
                  \ dx            /  /   \ dp            /
```

The function in_ideal() tests whether a particular differential form is a member of a given differential ideal.

```
> in_ideal(d(x), Ideal1);
                                false
```

```
> in_ideal(wedge(d(x),d(y)), Ideal1);
                              true
```

The (first-)derived system of an ideal $\mathcal{I}$ is the set of elements of $\mathcal{I}$ whose exterior derivative is also a member of $\mathcal{I}$. This is calculated by the function `derived_ideal()`.

```
> Ideal4:={d(y)-p*d(x), d(p)-q*d(x)};
  Ideal4 := {dform(1, y) - p dform(1, x), dform(1, p) - q dform(1, x)}

> derived_ideal(Ideal4);
                 {dform(1, y) - p dform(1, x)}, {}

> derived_ideal( derived_ideal(Ideal4)[1] );
                             {}, {}
```

The `derived_ideal()` function has an optional second argument that is used to give to `derived_ideal()` the expressions for the various derivatives.

```
> derived_ideal({nform(1,a),nform(1,b)},
>  {d(nform(1,a))=wedge(nform(1,a), nform(1,b) ),
>  d(nform(1,b))=wedge( nform(1,a), nform(1,c))} );

                 {nform(1, b), nform(1, a)}, {0}

> derived_ideal({nform(1,a),nform(1,b)},
> {d(nform(1,a))=wedge(nform(1,a), nform(1,b) ),
>  d(nform(1,b))=wedge( nform(1,c), nform(1,e))} );

                       {nform(1, a)}, {}
```

The answer consists of a sequence of two sets: the derived ideal and the set of quantities that have been assumed to vanish.

A very useful result in exterior differential algebra is the following. Let M be an $n-$dimensional manifold. Let $\{\omega^i\}$ be a set of p independent one-forms, where $p<n$. (The independence condition is determined by requiring that the exterior product

of all p of these 1-forms gives a non-zero result.) Suppose that we have a set of p one-forms $\{\theta^i\}$ over that same manifold M satisfying $\sum_{i=1}^{p} \theta^i \wedge \omega^i = 0$. Then Cartan's lemma states that there are $p(p+1)/2$ functions A_{ij}, with $A_{ij} = A_{ji}$, such that $\theta_i = A_{ij}\omega^j$. The method of proof [10] is to complete the set of functions ω^i to a basis of T^*M by adjoining $(n-p)$ one-forms α^a. Since the one-forms $\theta^i \in T^*M$, they can be expanded uniquely in this constructed basis: $\theta^i = A_{ij}\omega^j + B_{ia}\alpha^a$. We substitute this in the condition on θ^i, to obtain $A_{ij}\omega^j \wedge \omega^i + B_{ia}\alpha^a \wedge \omega^i = 0$. Since the functions ω^i and the α^a are all pairwise independent, it follows that $A_{ij} - A_{ji} = B_{ia} = 0$.

The proof is instructive, since it allows us to extend the lemma. Suppose we have a set of exterior algebraic equalities involving the one-forms of a basis of T^*M and other one-forms that are taken as unknowns but members of T^*M. We can expand these unknown one-forms with respect to the basis, with the various coefficients left as unknown functions. These expansions are substituted in the given equalities. We then put the result in standard order and equate to zero all the coefficients of the basic forms. We then solve for as many unknown functions as possible. The relations that we are left with, not involving the unknown functions, cannot be made to vanish. They determine quantities known by the collective term of the non-absorbable torsion. (For systems satisfying the hypothesis of the Cartan lemma, all the the torsion can be absorbed).

Now, substituting the solved functions into the unknown one-forms gives us the answer we seek. We may have some functions that are still undetermined (in the standard Cartan lemma, these are the coefficients of the symmetric $p \times p$ matrix A_{ij}). Depending on the problem that is being solved, these parameters may have an interpretation (for example in the method of equivalence, they may represent the

[10] There is a similar proof on page 10. The present proof is included in order that this appendix be self-contained.

parameters of the subgroup involved in the prolongation step of the algorithm). The torsion is obtained by substituting the solved one-forms into the original problem and simplifying.

```
> cartan_lemma( wedge(nform(1,F[1]),d(x))+wedge(nform(1,F[2]),d(y)),
>      {d(x),d(y)}, P);
      [{nform(1, F[1]) = P[2] dform(1, y) + P[3] dform(1, x),
           nform(1, F[2]) = P[1] dform(1, y) + P[2] dform(1, x)},
          {P[3], P[1], P[2]}]

> F7:=wedge(nform(1,G), d(x)) + wedge(d(y),3*d(z));
F7 := wedge(nform(1, G), dform(1, x)) - 3 wedge(dform(1, z), dform(1, y))

> ans:=cartan_lemma( F7, {d(x), d(y), d(z) }, P);
            ans := [{nform(1, G) = P[1] dform(1, x)}, {P[1]}]

> torsion:=subs_form(ans[1], F7);
             torsion := - 3 wedge(dform(1, z), dform(1, y))


> cartan_lemma( {F7, wedge(d(z),d(x))}, {d(x), d(y)}, P);
 [{nform(1, G) = - 3 P[2] dform(1, y) + P[1] dform(1, x),
      dform(1, z) = P[2] dform(1, x)},                   {P[1], P[2]}]
```

Given a set of differential forms, one can construct an ideal $\mathcal{I}$ using these differential forms as generators with the multiplication operator being the exterior product. The mod_ideal() function of a differential form ω finds a representative for the equivalence[11] class $\omega + \mathcal{I}$.

```
> mod_ideal(d(p),Ideal4);
                                  q dform(1, x)

> mod_ideal(wedge(d(p),d(y)),Ideal4);
```

[11]The actual representative can change from a Maple session to another. However, if ω is in the ideal, then the result of mod_ideal() is guaranteed to be 0.

```
                                    0

> mod_ideal(d(x),Ideal4);
                                dform(1, x)

> mod_ideal(d(y),Ideal4);
                               p dform(1, x)

> mod_ideal(wedge(d(p),d(z)),Ideal4);
                    - q wedge(dform(1, z), dform(1, x))
```

Let L be an n-dimensional space of differential one-forms with an inner product:

$$g : L \times L \to \mathbf{R}$$

We can extend this inner product to an inner product over the exterior algebra on L

$$g : \bigwedge L \; \times \; \bigwedge L \to \mathbf{R}$$

as follows. First, if the two arguments of the inner product have different wedge degree, then the answer is zero. Second, since the inner product is linear in each argument, we need only consider simple p-forms. Let α and β be expanded in one-forms as $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^p$, and $\beta = \beta^1 \wedge \cdots \wedge \beta^p$. Then

$$g(\alpha, \beta) = \det\Big(g(\alpha^i, \beta^j)\Big)$$

The function `inner_product()` calculates the inner product between two differential forms given an orthonormal basis, and a signature list[12], which defaults to

[12] The signature list gives the diagonal of the inner product between all the elements of the orthonormal basis – this is not always positive if we allow pseudo-Riemannian bases such as the ones appearing in relativity.

all begin equal to one.

```
> inner_product(d(x)+3*d(y), -d(x)+4*d(y), [d(x),d(y)]);
                                        11

> inner_product(d(x)+3*d(y), -d(x)+4*d(y), [d(x),d(y)],[-1,1]);
                                        13

> inner_product( wedge(d(x),d(y)), wedge(d(x),d(z)), [d(x), d(y), d(z)]);
                                        0

> inner_product( wedge(d(x),d(y)), wedge(d(x),d(y)), [d(x), d(y), d(z)]);
                                        1
```

Given L, a differential forms space (of dimension n) with an inner product g, and given an orientation on L, we can define an operator $*$ taking p-forms into $(n-p)$-forms. This operator is called the (Hodge) star operator. Let σ be the volume form on L.

Let α be a p-form Then $*\alpha$ is the unique $(n-p)$-form that satisfies

$$\alpha \wedge \beta = g(*\alpha, \beta)\sigma$$

for all $(n-p)$-forms β.

The function `hodge_star`() calculates this operation. It takes as arguments the differential form operated upon, an orthonormal basis and (optional) a signature list.

```
> hodge_star(d(x),[d(x),d(y),d(z)]);
                       - wedge(dform(1, z), dform(1, y))

> hodge_star(wedge(d(y),d(z)),[d(x),d(y),d(z)]);
                                    dform(1, x)

> hodge_star(d(x),[d(x),d(y),d(z)],[-1,1,1]);
```

```
                    - wedge(dform(1, z), dform(1, y))

> hodge_star(wedge(d(y),d(z)),[d(x),d(y),d(z)],[-1,1,1]);
                                 - dform(1, x)

> hodge_star(d(x)+2*d(y),[d(x),d(y)]);
                          dform(1, y) - 2 dform(1, x)
```

When we have a space on which the Hodge star operator can be defined, then from the exterior derivative, one can construct another differential operator δ taking a p-form to a $(p-1)$-form as follows:

$$\delta\alpha = (-1)^{(np+n+1)} * \mathtt{d} * \alpha.$$

The name `adjoint_d` comes from the following property. If α is a p-form, and β is a $p+1$-form, and g is the inner product on the space then,

$$g(\mathtt{d}\alpha, \beta) = g(\alpha, \delta\beta).$$

This operator is also known as the co-differential.

```
> adjoint_d(y*d(x),[d(x),d(y),d(z)]);
                                    0

> adjoint_d((x*y)*d(x),[d(x),d(y),d(z)]);
                                   - y
```

We now have all the ingredients to define an operator Δ that generalizes the Laplacian operator on functions (actually, minus one times the Laplacian operator). It is defined as

$$\Delta := \mathtt{d} \circ \delta + \delta \circ \mathtt{d}.$$

This operator is known as the Laplace-Beltrami operator. It also is known as the harmonic operator. The function `laplace_beltrami()` implements this operator.

It takes as arguments the differential form on which the operator is applied, the orthonormal basis and a contact set.

The contact set is there for the following reason. Between the application of the second differentiation in each term of the Laplace-Beltrami operator, one has to take into account the expansion of the first differentiation in the space $\bigwedge L$, otherwise, the star operator cannot be applied.

```
> basis:=[d(x),d(y),d(z)]:
> contact:={express_base(d(f),[d(x),d(y),d(z)])}:
> contact:=contact union map(express_base, {d(f_x),d(f_y),d(f_z)},basis);

contact := {dform(1, f) = f_x dform(1, x) + f_y dform(1, y) + f_z dform(1, z),

    dform(1, f_x) = f_x_x dform(1, x) + f_x_y dform(1, y) + f_x_z dform(1, z),

    dform(1, f_y) = f_y_x dform(1, x) + f_y_y dform(1, y) + f_y_z dform(1, z),

    dform(1, f_z) = f_z_x dform(1, x) + f_z_y dform(1, y) + f_z_z dform(1, z)}

> laplace_beltrami(f,basis,contact);

                         - f_x_x - f_y_y - f_z_z
```

The Lie derivative of a differential form with respect to a vector is obtained using the `lie()` operation. The first argument is the vector in the direction of which the derivative is applied. The second argument is the differential form to be differentiated. An optional argument is used to specify the exterior derivatives of the various quantities.

```
> f1:=x*d(y)+y^2*d(z);
                                              2
                      f1 := x dform(1, y) + y  dform(1, z)

> lie(dvector(y),f1);
```

```
                                2 y dform(1, z)

> alias(a=nform(1,a_),b=nform(1,b_), Avec=nvector(a_));
                                 I, a, b, Avec

> lie(Avec,a+3*b,{d(a)=7*wedge(a,b),d(b)=9*wedge(a,b)});
                                     34 b
```

A.4 Utility functions

It is often the case that a function's natural argument is a single item (as opposed to a matrix, equality, set, list, etc.). If we apply that function to a composite object, such as a set, the natural thing to do would be to apply the function to each individual member of the composite object. For example, taking the exterior derivative of a matrix is just the matrix of exterior derivatives applied to each member of the matrix.

Maple provides an operation to do this: `map()`. Unfortunately, this works only at a depth of one level. The function `item_map`() generalizes `map()` to work to any desired depth. The first argument of `item_map`() is a function. The second argument is a list containing all the other arguments to the function. The third argument specifies which "slot" needs to be expanded (by default, the first slot is the one that is expanded). The fourth argument (optional) is a set of types over which `item_map`() is recursively invoked, and the last argument specifies the depth of recursion (default is infinite).

```
> item_map(d,[ { [ax=bx+cx], [ [d(cx) = ex*d(fx)]] } ] );
             {[dform(1, ax) = dform(1, bx) + dform(1, cx)],

                [[0 = - wedge(dform(1, fx), dform(1, ex))]]}
```

```
> item_map(fn,[ { [ax=bx+cx], [ [d(cx) = ex*d(fx)]] } ],1, {list,set,`=`},2 );
          {[fn(ax = bx + cx)], [fn([dform(1, cx) = ex dform(1, fx)])]}

> item_map(fn,[ { [ax=bx+cx], [ [d(cx) = ex*d(fx)]] } ],1, {list,set,`=`},3 );

        {[[fn(dform(1, cx) = ex dform(1, fx))]], [fn(ax) = fn(bx + cx)]}

> item_map(fn,[ { [ax=bx+cx], [ [d(cx) = ex*d(fx)]] } ],1, {list,set},3 );

          {[fn(ax = bx + cx)], [[fn(dform(1, cx) = ex dform(1, fx))]]}
```

The function `linear_solve()` is an extension to the Maple function `solve(..., linear)`. It returns the set of expressions that have been assumed to be equal to zero in order that the solution set be valid.

```
> solve({x-a,x-b},{x});
# Note NULL result. This indicates no solution
> linear_solve({x-a,x-b},{x});
                               [{x = a}, {a - b}]
# This is interpreted as : the solution is x=a, provided a-b=0.
```

A.5 Points to keep in mind

While forms used with the `forms` package can be inhomogeneous in degree, it is important important to realize that some of the functions require homogeneity. For example, the `wdegree()` function will return the degree of only one of the terms and will assume that all the other terms will have the same degree. Functions such as addition, `d()`, `wedge()`, `subs_form()` will work with inhomogeneous forms. Any functions that are described in the higher level functions section must be assumed to require homogeneous forms.

It is also recommended that the exterior derivative be used to construct expressions involving `dform`. The `wedge()` operator is to be used to multiply forms together.

Giving a set of independent basic forms to `form_coeffs`() will detect the cases when a dependent form is present in the first argument. The second argument is optional, and its omission will cause `form_coeffs`() to assume that every basic form is independent. If this is not the case, then too many coefficients will be returned.

A.6 Making `forms` laconic

The package has been designed to be rather verbose. The main reasons are to avoid clashes with other Maple names, to avoid obtuse abbreviations and to avoid ambiguity. Since Maple provides an aliasing facility, it is easy to replace long expressions with shorter ones. Here are a few hints to use Maple's alias() function effectively.

The normal syntax is `alias(short=long)` where `long` is a long expression, and `short` is a name that will be used to abbreviate `long`. For example, `alias(alpha=nform(2,alpha_))` can be used to define a two-form with name `alpha`. It is recommended that different names be used on the two sides of the equality in the `alias()` expression. This is why an underscore was appended to `alpha`. The reason for this recommendation is because the expression `op(2,alpha)` returns `alpha_`. If the "internal" name had been `alpha,` then `op(2,alpha)` could not be visually distinguished from `alpha`. It is important to note that `long` will not be evaluated, and cannot itself use abbreviations. Therefore, in order to give an alias for the derivative of `alpha`, use

```
> alias(alpha=nform(2,alpha_)):
> eval(subs(dalpha_=d(alpha), 'alias(dalpha=dalpha_)')):
```

The `alias()` statements must come in the order shown.

The expression `alias(V=wedge);` can be used to shorten input of data. A better solution to shorten the output is to use the neutral operators of Maple, together with functions to transform expressions. This is done by making the definitions

```
> shorten:=proc(item)
>  eval(subs('wedge'=proc() &^(args) end, item))
> end:

> lengthen:=proc(item)
>  eval(subs(`&^`=wedge, item))
> end:
```

An example showing the use of the preceding definitions is

```
> eval(subs(_dx=d(x), _dy=d(y), _dt=d(t), 'alias(dx=_dx, dy=_dy, dt=_dt)')):
> A:=t*wedge(dx,dy)+x*wedge(dx,dt);
                 A := - t wedge(dy, dx) - x wedge(dt, dx)


> shorten(A);
                       - t (dy &^ dx) - x (dt &^ dx)


> lengthen(");
                    - t wedge(dy, dx) - x wedge(dt, dx)


> shorten(d("));
                              &^(dy, dt, dx)
```

Note that the `forms` package does not use the `&^` operator. It is therefore necessary to use `lengthen()` before applying any `forms` operation to expressions involving `&^`.

A.7 Extensibility

Various functions have facilities to extend their domain of definition. The exterior derivative function, `d`(), allows for the following. If the function `‘forms/d/alpha‘` exists, then `d(alpha(args))` will be the result to the call `‘forms/d/alpha‘(args)`. If the function `‘forms/d2/alpha‘` exists, then `d(alpha(args))` will result in a call to `‘forms/d2/alpha‘(alpha(args),fm)`. The function `‘forms/d2/alpha‘` must give the result of the differentiation of `‘alpha(args)‘` wedged with the form `‘fm‘`.

Likewise, the function `lie`() applied to a function `fn()` will call the function `‘forms/lie/f‘`, if it exists, with arguments: the direction vector followed by the original arguments to the function `f` and then followed by the structure equations that were passed as third argument to the `lie`() function.

A.8 Vector-valued differential forms

Vector-valued differential forms are necessary for moving frame calculations. Under the operation of `d`(), the vector parts of a differential form are assumed to behave as a scalar. Their exterior derivatives multiply the form parts on the left. The derivative of a vector, say `nvector(A)`, is given a name suitable for substitution via `subs_form`(). This substitution must take place prior to a further differentiation.

```
> d(nvector(AA));
                              nform(1, D_nvector(AA))

> d(nvector(AA)*nform(1,WW));
wedge(nform(1, D_nvector(AA)), nform(1, WW)) + nvector(AA) dform(2, WW)
```

A.9 Further information

An advanced study of differential forms can be found in Bryant et al. (1991). An excellent reference is Flanders (1963). Gardner (1989) applies differential forms to the problem of equivalence. Exterior differential systems are the subject of Cartan (1945).

Bibliography

Allnutt, J. A. (1981). A Petrov type-III perfect fluid solution of Einstein's equations. *Gen. Rel. Grav.*, 13:1017–1020.

Åman, J. E., d'Inverno, R. A., Joly, G. C., and MacCallum, M. A. H. (1991). Quartic equations and classification of Riemann tensors in general relativity. *Gen. Rel. Grav.*, 23:1023–1055.

Banerji, S. (1968). Homogeneous cosmological models without shear. *Progress of Theoretical Physics*, 39:365–371.

Barnes, A. (1984). Shear-free flows of a perfect fluid. In Bonner, W. B., Islam, J. N., and MacCallum, M. A. H., editors, *Classical General Relativity*. Cambridge University Press, Cambridge. Conference on Classical General Relativity. 1983: City University, London.

Bonnor, W. B. and Ellis, G. F. R. (1986). Observational homogeneity of the universe. *Monthly Notices of the Royal Astronomical Society*, 218:605–614.

Bradley, M. and Karlhede, A. (1990). On the curvature description of gravitational fields. *Class. Quantum Grav.*, 7:449 – 463.

Bryant, R. L., Chern, S. S., Gardner, R. B., Goldschmidt, H. L., and Griffiths, P. A. (1991). *Exterior Differential Systems.* Springer-Verlag, New York.

Buchberger, B. (1985). Gröbner bases: An algorithmic method in polynomial ideal theory. In Bose, N. K., editor, *Multidimensional Systems Theory*, chapter 6, pages 184–232. D. Reidel Publishing Company.

Carminati, J. (1987). Shear-free perfect fluids in general relativity. I. Petrov type N Weyl tensor. *J. Math. Phys.*, 28:1848–1853.

Carminati, J. (1988). Type N, shear-free, perfect fluid space-times with a barotropic equation of state. *Gen. Rel. Grav.*, 20:1239–1248.

Carminati, J. (1990). Shear-free perfect fluids in general relativity. II. Aligned, Petrov type III space-times. *J. Math. Phys.*, 31:2434–2440.

Cartan, E. (1945). *Les systèmes différentiels extérieurs et leurs applications géométriques.* Hermann, Paris. Republished in 1971.

Char, B. W., Geddes, K. O., Gonnet, G. H., Leong, B. L., Monagan, M. B., and Watt, S. M. (1991). *Maple Language Reference Manual.* Springer-Verlag, New York.

Coley, A. A. (1991). Fluid spacetimes admitting a conformal Killing vector parallel to the velocity vector. *Class. Quantum Grav.*, 8:955–968.

Collins, C. B. (1984). Shear-free perfect fluids with zero magnetic Weyl tensor. *J. Math. Phys.*, 25:995–1000.

Collins, C. B. (1986). Shear-free fluids in general relativity. *Can. J. Phys.*, 64:191–199.

Collins, C. B. (1987). How unique are the Friedmann-Robertson-Walker models of the universe? *Gen. Rel. Grav.*, 19:493–497.

Collins, C. B. (1988). Homogeneous and hypersurface-homogeneous shear-free perfect fluids in general relativity. *Gen. Rel. Grav.*, 20:847–855.

Collins, C. B. and Wainwright, J. (1983). Role of shear in general-relativistic cosmological and stellar models. *Phys. Rev. D*, 27:1209–1218.

Collins, C. B. and White, A. J. (1984). A class of shear-free perfect fluids in general relativity. II. *J. Math. Phys.*, 25:1460–1472.

Collins, J. M. and d'Inverno, R. A. (1993). The Karlhede classification of type-D non-vacuum spacetimes. *Class. Quantum Grav.*, 10:343–351.

Collins, J. M., d'Inverno, R. A., and Vickers, J. A. (1990). The Karlhede classification of type D vacuum spacetimes. *Class. Quantum Grav.*, 7:2005–2015.

Ellis, G. F. R. (1967). Dynamics of pressure-free matter in general relativity. *J. Math. Phys.*, 8:1171–1194.

Ellis, G. F. R. (1971). Relativistic cosmology. In Sachs, R. K., editor, *Proceedings of the International School of Physics "Enrico Fermi", Course XLVII, 1969*, pages 104–182. Academic Press, London.

Ellis, G. F. R. (1973). Relativistic cosmology. In Schatzman, E., editor, *Lectures at the International Summer School of Physics, Cargèse, Corsica (1971)*, pages 1–60. Gordon and Breach, New York.

Ellis, G. F. R. (1990). Standard and inflationary cosmologies. Lecture given at the Summer Research Institute GRAVITATION, Banff Centre, Banff, Alberta, Canada.

Flanders, H. (1963). *Differential Forms with Applications to the Physical Sciences*. Academic Press, New York. Republished by Dover, Mineola, New York, 1989.

Gardner, R. B. (1989). *The Method of Equivalence and Its Applications*. Society for Industrial and Applied Mathematics, Philadelphia.

Gödel, K. (1950). Rotating universes in general relativity theory. *International Congress of Mathematicians*, 1:175–181.

Greenberg, P. J. (1970). The general theory of space-like congruences with an application to vorticity in relativistic hydrodynamics. *J. Math. Anal. Appl.*, 30:128 – 143.

Grossman, S. I. (1984). *Elementary Linear Algebra*. Wadsworth Publishing Company, Belmont, California, second edition.

Gruntz, D. and Monagan, M. B. (1993). Introduction to gauss. *MapleTech*, 9:23–35. Birkäuser, Boston.

Harness, R. S. (1982). Space-times homogeneous on a time-like hypersurface. *J. Phys. A*, 15:135–151.

Hungerford, T. W. (1974). *Algebra*. Springer-Verlag, New York.

Joly, G. C. and MacCallum, M. A. H. (1990). Computer-aided classification of the Ricci tensor in general relativity. *Class. Quantum Grav.*, 7:541–556.

Karlhede, A. (1980a). On a coordinate-invariant description of Riemannian manifolds. *Gen. Rel. Grav.*, 12:963–970.

Karlhede, A. (1980b). A review of the geometrical equivalence of metrics in general relativity. *Gen. Rel. Grav.*, 12:693–707.

Karlhede, A. and Lindström, U. (1982). Finding space-time geometries without using a metric. *Gen. Rel. Grav.*, 15:597–610.

Karlhede, A. and MacCallum, M. A. H. (1982). On determining the isometry group of a Riemannian space. *Gen. Rel. Grav.*, 14:673–982.

King, A. R. (1973). *Cosmological models and their singularities*. PhD thesis, University of Cambridge.

King, A. R. (1974). Generalized shear-free singularities. *Gen. Rel. Grav.*, 5:371–377.

King, A. R. and Ellis, G. F. R. (1973). Tilted homogeneous cosmological models. *Commun. math. Phys.*, 31:209–242.

Koutras, A. (1992). A spacetime for which the Karlhede invariant classification requires the fourth covariant derivative of the Riemann tensor. *Class. Quantum Grav.*, 9:L143–L145.

Kramer, D., Stephani, H., MacCallum, M., and Herlt, E. (1980). *Exact Solutions of Einstein's Field Equations*. Cambridge University Press, Cambridge.

Krasiński, A. (1978). All flow-stationary cylindrically symmetric solutions of the Einstein field equations for a rotating isentropic perfect fluid. *Rep. Math. Phys.*, 14:225–236.

Lang, J. M. and Collins, C. B. (1988). Observationally homogeneous shear-free perfect fluids. *Gen. Rel. Grav.*, 20:683–710.

MacCallum, M. A. H. (1973). Cosmological models from a geometric point of view. In Schatzman, E., editor, *Lectures at the International Summer School* *of Physics, Cargèse, Corsica (1971)*, pages 61–174. Gordon and Breach, New York.

MacCallum, M. A. H. (1991). Computer-aided classification of exact solutions in general relativity. In Cianci, R., de Ritis, R., Francaviglia, M., Marmo, G., Rubano, C., and Scudellaro, P., editors, *Ninth Italian conference on General Relativity and Gravitational Physics, Capri (Napoli) 1990*, pages 318–337. World Scientific, London.

McIntosh, C. B. G., Foyster, J. M., and Lun, A. W. C. (1981). The classification of the Ricci and Plebanski tensors in general relativity using Newman-Penrose formalism. *J. Math. Phys.*, 22:2620–2623.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*. W.H. Freeman and Company, New York.

Paiva, F. M., Rebouças, M. J., and MacCallum, M. A. H. (1993). On limits of spacetimes – a coordinate-free approach. *Class. Quantum Grav.*, 10:1165–1178.

Schücking, E. (1957). Homogene scherungsfreie weltmodelle in der relativistischen kosmologie. *Naturwissenschaften*, 19:507.

Stephani, H. (1967a). Konform flach gravitationsfelder. *Commun. math. Phys.*, 5:337–342.

Stephani, H. (1967b). Über lösungen der Einsteinschen feldgleichungen, die sich in einen fünfdimensionalen flachen raum einbetten lassen. *Commun. math. Phys.*, 4:137 – 142.

Treciokas, R. and Ellis, G. F. R. (1971). Isotropic solutions of the Einstein-Boltzmann equations. *Commun. math. Phys.*, 23:1–22.

Vaidya, P. C. (1977). The Kerr metric in cosmological background. *Pramana*, 8:512–217.

Wahlquist, H. D. (1968). Interior solution for a finite rotating body of perfect fluid. *Phys. Rev.*, 172:1291–1296.

White, A. J. (1981). Shear-free perfect fluids in general relativity. Master's thesis, University of Waterloo.

White, A. J. and Collins, C. B. (1984). A class of shear-free perfect fluids in general relativity. I. *J. Math. Phys.*, 25:332–337.

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