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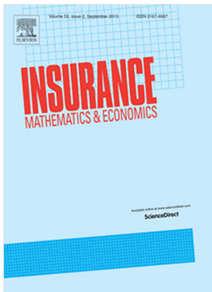

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Expected Utility of the Drawdown-Based Regime-Switching Risk Model with State-Dependent Termination

David Landriault* Bin Li* Shu Li†

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Abstract

In this paper, we model an entity's surplus process X using the drawdown-based regime-switching (DBRS) dynamics proposed in [9]. We introduce the state-dependent termination time to the model, and provide rationale for its introduction in insurance contexts. By examining some related potential measures, we first derive an explicit expression for the expected terminal utility of the entity in the DBRS model with Brownian motion dynamics. The analysis is later generalized to time-homogeneous Markov framework, where the spectrally negative Lévy model is also discussed as a special example. Our results show that, even considering the impact of the termination risk, the DBRS strategy can still outperform its counterparts in either single regime strategy. This study shows that the DBRS model is not myopic, as it not only helps to recover from significant losses, but also may improve the insurer's overall welfare.

1 Introduction

The development of effective risk management mechanisms to help control and mitigate the underlying risks of a given surplus/value process is of paramount importance to insurers. This explains the vast interest this research topic has received within the actuarial science community over the years (see, e.g., [1] and references therein). Broadly speaking, an insurer aims to maintain a steady and healthy growth rate for its underlying business while simultaneously controlling the risk of extreme losses which may adversely affect its business operations. This *growth/risk* trade-off is a delicate one for the insurer to balance and hence, modern risk metrics are constantly developed and utilized to provide informative and timely guidance to the insurer.

With this broad context in mind and inspired from an application in the fund management industry, a drawdown-based regime-switching (DBRS) insurance risk model was proposed by [9] in which drawdown is used as a dynamic risk metric to measure the magnitude of the drop of insurance surplus from its maximum. As discussed in [9], in comparison to the traditional risk metrics which rely on a fixed threshold level to assess the solvency risk (such as the ruin probability), drawdown follows more closely the dynamic changes of insurance surplus over time and hence, can be used to provide timely warning to decision makers on solvency matters. This application of drawdown in insurance risk modelling is consistent with the common use of drawdown in the fund management industry (e.g., [16]), which has been motivated from its close relationship with fund redemption. Other practical and theoretical studies of drawdown can be found in [2], [10] - [14], [17] - [18], and references therein.

In [9], a drawdown metric is used to characterize periods of extreme insurance losses. Given that an insurer's reinsurance, investment, and other business strategies will likely have to be modified to resolve an

*Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada
†Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA (shuli@illinois.edu)

episode of financial distress, the DBRS model allows the insurance surplus to experience *dynamic changes* on a regime-switching basis. More precisely, the dynamics of the surplus process changes according to the alternating occurrence of the following two events: (1) the surplus process experiences a large drawdown over a pre-determined size $a > 0$ and (2) the surplus process recovers its previous maximum. The former event is used as a trigger to initiate a period of financial distress while the latter is used to reset the surplus process dynamic to its "normal" non-distressed behavior (i.e., to end the period of financial distress). Reasons to consider the DBRS risk process are multifold; interested readers are referred to [9] where a detailed account can be found of how the DBRS strategy can help characterize an insurer's business cycle.

Note that a key design feature of the DBRS model is that the surplus dynamics under the distressed regime only remain effective until the financial distress is resolved. One may consider a situation where the insurer is subject to additional financing/liquidity constraints under the distressed regime which may lead to a suboptimal business strategy for the insurer. Other business related strategies of the insurer (such as those related to its capital structure, investment policy, dividend policy, and others) may have to be adjusted when the insurer is going through a distressed period. For instance, the insurer may revise its pricing practices to better reflect the existing business environment which may, in turn, have an immediate impact on the insurer's policyholders demographic (via a change in its policyholders' retention rate and/or its ability to attract new policyholders). In light of the significance of the retention (surrender) risk to an insurer's profitability, we aim to provide a more comprehensive assessment of the benefits of the DBRS strategy. As the main implication of this paper, the present analysis shows that the DBRS model is *not myopic*, as it not only helps to recover from significant losses, but also may *improve the insurer's overall welfare*.

In terms of mathematical formulation (which will be described formally in Section 2), the surplus process X is assumed to exhibit the DBRS dynamics (with two distinct underlying processes X^1 and X^2) as introduced in [9]. As a novel extension, we further introduce a *state-dependent termination time* (which we shall denote by ξ) with different killing rates when the DBRS process X operates under dynamics X^1 or X^2. In addition to the state-dependent termination time, it is also natural to set a termination whenever one of the following two events occurs: the surplus drops below level 0 (denoted by T_0^-) or the surplus reaches a target level b (denoted by T_b^+). To evaluate the insurer's overall welfare, a general utility function $U(\cdot)$ is imposed, and the main problem is to calculate the benchmark expected terminal utility (ETU), i.e., for $q \geq 0$,

$$V_a(u) = \mathbb{E}\left[e^{-q\kappa} U\left(X_\kappa\right) \middle| X_0 = u \right], \tag{1.1}$$

where $\kappa := \xi \wedge T_0^- \wedge T_b^+$. Note that, in (1.1), $a > 0$ is the pre-determined drawdown level to trigger the dynamic changes.

Analytic expressions of the value function V_a will first be given when X^1 and X^2 are Brownian motions (primarily, for ease of presentation). We later extend the analysis to the case where X^1 and X^2 are two *general time-homogeneous Markov processes*. The key is to analyze a few exit densities, which are shown to be the unique solution to a class integro-differential equations. From the practical side, we are able to show that, even when the state-dependent termination rate is included, the DBRS model may improve the overall welfare of the insurer (quantified by the value function V_a), i.e.,

$$V_a(u) > \max\left\{V^1(u), V^2(u)\right\}, \tag{1.2}$$

for some $a > 0$, where $V^1(u)$ and $V^2(u)$ are defined as the ETU (1.1) under the single regime processes $X \equiv X^1$ and $X \equiv X^2$, respectively. In other words, under certain model setups, insurers are better off by adopting the DBRS dynamic changing strategy instead of sticking to either of the selected underlying models X^1 or X^2.

Although the focus of this paper is put on the application of the DBRS changing dynamics in the insurance surplus context, the mathematical model and its analysis may also be applied more broadly. Indeed, numerous entities (e.g., investors and financial institutions) are sensitive to large drops in wealth, the occurrence of which may lead to adjustments to the entity's overall business operations for some time. The DBRS model provides a natural mathematical framework to quantitatively assess the impact of such adjustments on an entity's strategic goals. For instance, in the variable annuity (VA) context, a state-dependent fee structure was recently proposed by [3] to reduce the surrender risk of VA buyers. Mathematically, the underlying account value process follows a so-called refracted-type process (see, e.g., [8]) for which a constant level is used to trigger dynamic changes. Alternatively, one may consider a given drawdown level as the trigger for these dynamic changes which may be more effective for risk management purposes, especially in the context of VAs with ratchet-type features.

The rest of the paper is organized as follows: in Section 2, we provide a detailed mathematical description of the DBRS process X and further discuss the primary quantity of interest V_a introduced in (1.1). In Section 3, we first consider the DBRS process when both surplus dynamics are governed by Brownian motions. An explicit expression for the expected present value of the insurer's terminal surplus is obtained. Numerical examples are considered in Section 3.2. In particular, by considering a utility function of some basic form, we provide a sufficient condition for (1.2) to hold for some $a > 0$. The more general case of time-homogeneous Markov processes will be tackled in Section 4. Further details will be given under the special case of spectrally negative Lévy processes in Section 5.

2 Problem formulation

Mathematically speaking, let $X = \{X_t\}_{t\geq 0}$ be the surplus process defined on a filtered probability space $(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions of completeness and right continuity. The drawdown process $Y = \{Y_t\}_{t\geq 0}$ of X is defined as

$$Y_t = M_t - X_t,$$

where $M_t = \sup_{0\leq s\leq t} X_s$ is the running maximum of X at time t. Let $T_x^{+(-)} = \inf\{t \geq 0 : X_t > (<)\ x\}$ be the first passage times of X for level $x \in \mathbb{R}$. We define the first drawdown time of X (i.e., the first passage time of Y) for a fixed level $a > 0$ as

$$\tau_a = \inf\{t \geq 0 : Y_t > a\}.$$

The dynamics of the DBRS process X of interest in this paper is assumed to follow

$$\mathrm{d}X_t = \begin{cases} \mathrm{d}X_t^1, & \text{if } Q_t = 1, \\ \mathrm{d}X_t^2, & \text{if } Q_t = 2, \end{cases} \tag{2.1}$$

with initial surplus $X_0 = u \geq 0$, where X^1 and X^2 are two given processes,

$$Q_t = \begin{cases} 1, & \text{if } \sup_{l_t\leq s\leq t} Y_s < a, \\ 2, & \text{if } \sup_{l_t\leq s\leq t} Y_s \geq a, \end{cases}$$

with $Q_0 = 1$ and $l_t = \sup\{s \le t : Y_s = 0\}$ is the last time the process X is at its running maximum prior to or at time t. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$. The sequence of regime switching times $\{\eta^{(i)}\}_{i\in\mathbb{N}}$ of $\{Q_t\}_{t\ge 0}$ is defined recursively as

$$\eta^{(0)} = 0, \quad \eta^{(i)} = \inf\left\{t \ge \eta^{(i-1)} : Q_t \ne Q_{\eta^{(i-1)}}\right\} \text{ for } i \in \mathbb{N}^+,$$

Note that the DBRS process (2.1) is proposed by [9] where X^1 and X^2 are assumed to be two spectrally negative Lévy processes (SNLPs). In this paper, X^1 and X^2 will be generalized to time-homogeneous Markov processes. The special cases where X^1 and X^2 are either Brownian motions or SNLPs will also be given special consideration.

For the DBRS process X, we further incorporate a *state-dependent termination time* ξ defined as follows. Let $\{e^{(i)}\}_{i\in\mathbb{N}^+}$ be a sequence of independent random variables such that

$$e^{(i)} \stackrel{d}{=} \begin{cases} e_1, & \text{if } i \text{ is odd,} \\ e_2, & \text{if } i \text{ is even,} \end{cases}$$

where e_k is an exponential random variable with mean $1/\lambda_k > 0$ for $k = 1, 2$. Then

$$\xi = \inf_{i\in\mathbb{N}}\left\{\eta^{(i)} + e^{(i+1)} : \eta^{(i)} + e^{(i+1)} < \eta^{(i+1)}\right\}.$$

Intuitively, the (instantaneous) termination rate of ξ in regime k (i.e., $Q = k$) is equal to λ_k for $k = 1, 2$. Or equivalently, the termination rate of ξ at time t is equal to λ_{Q_t} for all $t \ge 0$. The definition of ξ also implies the conditional distribution of ξ is of the form

$$\mathbb{P}\left(\xi > t \middle| \mathcal{F}_t\right) = \mathbb{P}\left(\xi > t \middle| \{Q_s\}_{0\le s\le t}\right) = e^{-\lambda_1\theta^1(t) - \lambda_2\theta^2(t)}, \quad t \ge 0, \tag{2.2}$$

where

$$\theta^k(t) = \int_0^t 1_{\{Q_s = k\}} \mathrm{d}s, \quad t \ge 0,$$

represents the occupation time of Q in regime k ($k = 1, 2$) up to time t. Clearly, $\theta^1(t) + \theta^2(t) = t$ as Q has only two state regimes. Figure 1 depicts the regime switching of the underlying dynamics and the termination rate of our model.

To evaluate the overall performance of the DBRS model X with state-dependent termination, we consider an entity whose terminal surplus is assessed at the earliest time among the following three events: (1) X reaches a target level b ($b > u$); (2) X drops below level 0; (3) X is "killed" at time ξ. Formally speaking, consider a general monotone increasing (nondecreasing) utility function U, our objective is to evaluate the entity's expected terminal utility (ETU) under the DBRS model (2.1), that is,

$$V_a(u) = \mathbb{E}_u\left[e^{-q\kappa} U\left(X_\kappa\right)\right],$$

where $q \ge 0$ is the discount rate and $\kappa = \xi \wedge T_0^- \wedge T_b^+$ is the terminal time. Here and thereafter, we denote $\mathbb{E}_u$ as the law of X given that $X_0 = u$, and $\mathbb{P}_u$ as the corresponding conditional probability.

Throughout the paper, we confine ourselves to the most interesting case where the initial surplus $u \in [a, b]$. This is because the case $u < a$ is not very practical and can also be easily obtained from the case $u \in [a, b]$. Moreover, we often make use of the expectation $\mathbb{E}_u^k[\cdot]$ ($k = 1, 2$) to denote that all processes and stopping times under the (conditional) expectation are those related to process X^k only (given that $X_0^k = u$). For instance, $\mathbb{E}_u^1[e^{-qT_b^+} 1_{\{T_b^+ < T_0^-, X_{T_b^+} \in \mathrm{d}z\}}] := \mathbb{E}_u[e^{-qT_b^{1,+}} 1_{\{T_b^{1,+} < T_0^{1,-}, X^1_{T_b^{1,+}} \in \mathrm{d}z\}}]$, where $T_x^{k,+(-)} = \inf\left\{t \ge 0 : X_t^k > (<)\ x\right\}$ are denoted as the first passage times of X^k ($k = 1, 2$).

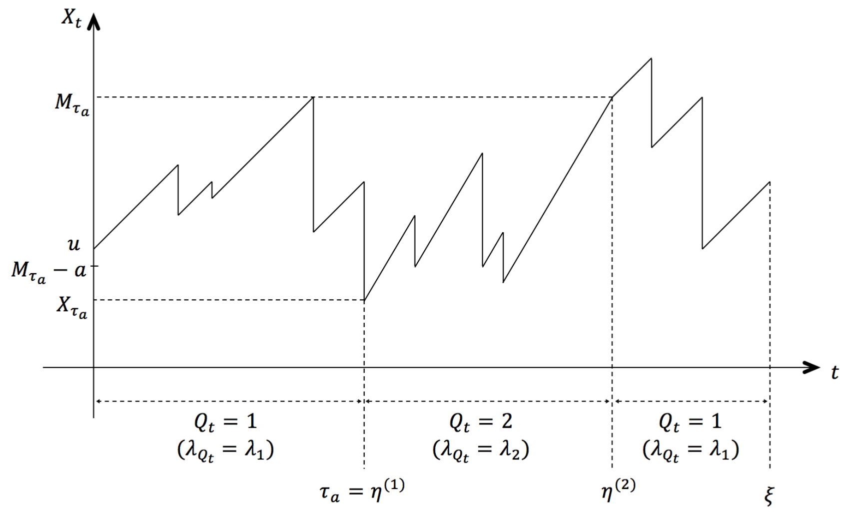

Figure 1: A sample path of the DBRS process X with state-dependent termination rates

3 DBRS with Brownian motion dynamics

In this section, we confine the underlying processes X^1 and X^2 to be two Brownian motions. As we will see, the analysis of the value function V_a is considerably simpler under this model assumption (in comparison to the general time-homogenous Markov process setting in Section 4).

3.1 Analysis of the ETU V_a

Let X^k $(k = 1, 2)$ be a Brownian motion with Laplace exponent $\psi_k(s) := \log \mathbb{E}\left[e^{sX_1^k}\right] = c_k s + \frac{1}{2}\sigma_k^2 s^2$ for $s \geq 0$, where $c_k \in \mathbb{R}$ is the drift and $\sigma_k > 0$ is the volatility of X^k. It is well-known that first passage and drawdown problems pertaining to the Brownian motion X^k heavily rely on the first and second q-scale functions, which are defined as

$$W_k^{(q)}(x) = \frac{1}{\sqrt{c_k^2 + 2q\sigma_k^2}} \left(e^{\rho_k x} - e^{-R_k x}\right) 1_{\{x \geq 0\}}, \tag{3.1}$$

and

$$Z_k^{(q)}(x) = 1 + q \int_0^x W_k^{(q)}(y)dy = \frac{q}{\sqrt{c_k^2 + 2q\sigma_k^2}} \left(\frac{1}{\rho_k} e^{\rho_k x} + \frac{1}{R_k} e^{-R_k x}\right), \quad x \geq 0,$$

where $\rho_k = \left(\sqrt{c_k^2 + 2q\sigma_k^2} - c_k\right)/\sigma_k^2$ and $R_k = \left(\sqrt{c_k^2 + 2q\sigma_k^2} + c_k\right)/\sigma_k^2$. More details on scale functions within the more general class of SNLPs can be found in e.g., [4]–[6] or later in Section 5.

The following lemma summarizes some of the important preliminary results, including the two-sided exit probability (3.2) (e.g., Equation 8.11 of [7]), the potential measure (3.3) (e.g., Theorem 8.7 of [7]), and the joint law of (τ_a, M_{τ_a}) (3.4) (e.g., Theorem 2.1 of [9]).

Lemma 3.1 *For $k=1,2$, $u\in[0,b]$ and $q\geq 0$, we have*

$$\mathbb{E}_u^k\left[e^{-qT_b^+}1_{\{T_b^+<T_0^-\}}\right]=\frac{W_k^{(q)}(u)}{W_k^{(q)}(b)},\tag{3.2}$$

$$\mathbb{E}_u^k\left[e^{-qe_k}1_{\{X_{e_k}\in\mathrm{d}y,e_k<T_b^+\wedge T_0^-\}}\right]=\lambda_k\left(\frac{W_k^{(q_k)}(u)\,W_k^{(q_k)}(b-y)}{W_k^{(q_k)}(b)}-W_k^{(q_k)}(u-y)\right)\mathrm{d}y,\quad y\in(0,b),\tag{3.3}$$

$$\mathbb{E}^k\left[e^{-q\tau_a}1_{\{M_{\tau_a}\in\mathrm{d}z\}}\right]=\frac{\sigma_k^2}{2}\left(\frac{\left(W_k^{(q)\prime}(a)\right)^2}{W_k^{(q)}(a)}-W_k^{(q)\prime\prime}(a)\right)e^{-\frac{W_k^{(q)\prime}(a)}{W_k^{(q)}(a)}z}\mathrm{d}z,\quad z>0,\tag{3.4}$$

where $q_k=q+\lambda_k$.

We also recall some results on first passage times obtained under the DBRS risk process of [9] which will be of great help in the analysis of (1.1). It is worth pointing out that these results were derived when more generally X^1 and X^2 are SNLPs, but we state here their simplified representations when X^1 and X^2 are Brownian motions.

Proposition 3.1 *For $u\in[a,b]$ and $s,q\geq 0$, the generalized two-sided exit identities are given by*

$$\mathbb{E}_u\left[e^{-s\theta^1(T_b^+)-q\theta^2(T_b^+)}1_{\{T_b^+<T_0^-\}}\right]=e^{-\int_u^b C_{s,q}(w)\mathrm{d}w},\tag{3.5}$$

and

$$\mathbb{E}_u\left[e^{-s\theta^1(T_0^-)-q\theta^2(T_0^-)}1_{\{T_0^-<T_b^+\}}\right]=\int_u^b e^{-\int_u^z C_{s,q}(w)\mathrm{d}w}D_{s,q}(z)\mathrm{d}y,\tag{3.6}$$

where

$$C_{s,q}(z)=\frac{W_1^{(s)\prime}(a)}{W_1^{(s)}(a)}\left\{1-\frac{\sigma_1^2}{2}\frac{W_2^{(q)}(z-a)}{W_2^{(q)}(z)}\left(W_1^{(s)\prime}(a)-W_1^{(s)\prime\prime}(a)\frac{W_1^{(s)}(a)}{W_1^{(s)\prime}(a)}\right)\right\},\tag{3.7}$$

and

$$D_{s,q}(z)=\frac{\sigma_1^2}{2}\left(\frac{\left(W_1^{(s)\prime}(a)\right)^2}{W_1^{(s)}(a)}-W_1^{(s)\prime\prime}(a)\right)\left(Z_2^{(q)}(z-a)-Z_2^{(q)}(z)\frac{W_2^{(q)}(z-a)}{W_2^{(q)}(z)}\right).\tag{3.8}$$

Proof: See Theorems 3.1 and 3.2 of [9].

To obtain an expression for V_a, we first consider separately the contributions of all three causes of the termination time κ, namely T_b^+, T_0^- and ξ. From the continuity of the sample paths of Brownian motions, we have

$$V_a(u)=U(b)\mathbb{E}_u\left[e^{-qT_b^+}1_{\{T_b^+<T_0^-\wedge\xi\}}\right]+U(0)\mathbb{E}_u\left[e^{-qT_0^-}1_{\{T_0^-<T_b^+\wedge\xi\}}\right]+\int_0^b U(y)F^{(q)}(u,\mathrm{d}y),\tag{3.9}$$

where we define

$$F^{(q)}(u,\mathrm{d}y)=\mathbb{E}_u\left[e^{-q\xi}1_{\{X_\xi\in\mathrm{d}y,\xi<T_b^+\wedge T_0^-\}}\right],\quad y\in(0,b).\tag{3.10}$$

Note that $F^{(q)}(u,\mathrm{d}y)/q$ can be interpreted as the q-potential measure of the DBRS model with state-dependent killing before exiting the interval $[0,b]$. Also, we denote by $f^{(q)}$ the density associated to the measure $F^{(q)}$, i.e.

$$f^{(q)}(u,y)\,\mathrm{d}y=F^{(q)}(u,\mathrm{d}y).$$

In the following proposition, an explicit expression for $f^{(q)}$ is given.

Proposition 3.2 *For $q \geq 0$, $u \in [a,b]$, and $y \in (0,b)$, we have*

$$f^{(q)}(u,y) = \int_u^b e^{-\int_u^z C_{q_1,q_2}(w)\mathrm{d}w} g_{q_1,q_2}(z,y)\mathrm{d}z, \tag{3.11}$$

where

$$\begin{aligned} g_{q_1,q_2}(z,y) = {} & \lambda_1 \left(W_1^{(q_1)\prime}(z-y) - W_1^{(q_1)}(z-y) \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} \right) 1_{\{z-a<y<z\}} \\ & + \lambda_2 W_2^{(q_2)}(z-y)\left(C_{q_1,q_2}(z-y) - C_{q_1,q_2}(z)\right) 1_{\{0<y<z\}}. \end{aligned} \tag{3.12}$$

Proof. We first use an infinitesimal argument to derive an ordinary differential equation (ODE) for $f^{(q)}$. We proceed by considering a mid-step target level $u+\varepsilon < b$ for some small $\varepsilon > 0$ (starting from an initial surplus u). Hence, by conditioning on whether or not $T_{u+\varepsilon}^+$ occurs before ξ and using the strong Markov property of the DBRS process X at new running maxima, (3.10) becomes

$$\begin{aligned} F^{(q)}(u,\mathrm{d}y) &= \mathbb{E}_u\left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, T_{u+\varepsilon}^+ < \xi < T_b^+ \wedge T_0^-\}}\right] + \mathbb{E}_u\left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \xi < T_{u+\varepsilon}^+ \wedge T_0^-\}}\right] \\ &= \mathbb{E}_u\left[e^{-qT_{u+\varepsilon}^+} 1_{\{T_{u+\varepsilon}^+ < \xi \wedge T_0^-\}}\right] F^{(q)}(u+\varepsilon,\mathrm{d}y) + \mathbb{E}_u\left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \xi < T_{u+\varepsilon}^+ \wedge T_0^-\}}\right]. \end{aligned} \tag{3.13}$$

For the first term on the right-hand side of (3.13), using (2.2) and the identity $\theta^1(t) + \theta^2(t) = t$, we have

$$\begin{aligned} \mathbb{E}_u\left[e^{-qT_{u+\varepsilon}^+} 1_{\{T_{u+\varepsilon}^+ < \xi \wedge T_0^-\}}\right] &= \mathbb{E}_u\left[\mathbb{E}\left[e^{-qT_{u+\varepsilon}^+} 1_{\{T_{u+\varepsilon}^+ < \xi \wedge T_0^-\}} \middle| \mathcal{F}_{T_{u+\varepsilon}^+}\right]\right] \\ &= \mathbb{E}_u\left[\mathbb{E}\left[e^{-qT_{u+\varepsilon}^+} e^{-\lambda_1\theta^1(T_{u+\varepsilon}^+) - \lambda_2\theta^2(T_{u+\varepsilon}^+)} 1_{\{T_{u+\varepsilon}^+ < T_0^-\}} \middle| \mathcal{F}_{T_{u+\varepsilon}^+}\right]\right] \\ &= \mathbb{E}_u\left[\mathbb{E}\left[e^{-q_1\theta^1(T_{u+\varepsilon}^+) - q_2\theta^2(T_{u+\varepsilon}^+)} 1_{\{T_{u+\varepsilon}^+ < T_0^-\}} \middle| \mathcal{F}_{T_{u+\varepsilon}^+}\right]\right] \\ &= \mathbb{E}_u\left[e^{-q_1\theta^1(T_{u+\varepsilon}^+) - q_2\theta^2(T_{u+\varepsilon}^+)} 1_{\{T_{u+\varepsilon}^+ < T_0^-\}}\right], \end{aligned} \tag{3.14}$$

which is known from (3.5) to be given by

$$\mathbb{E}_u\left[e^{-qT_{u+\varepsilon}^+} 1_{\{T_{u+\varepsilon}^+ < \xi \wedge T_0^-\}}\right] = e^{-\int_u^{u+\varepsilon} C_{q_1,q_2}(w)\mathrm{d}w}. \tag{3.15}$$

As for the second term on the right-hand side of (3.13), we condition on whether τ_a occurs before or after the state-dependent termination time ξ. Note that the dynamics of X will experience a change from X^1 to X^2 at the moment of τ_a. Then,

$$\begin{aligned} & \mathbb{E}_u\left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \xi < T_{u+\varepsilon}^+ \wedge T_0^-\}}\right] \\ & = \mathbb{E}_u\left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \xi < \tau_a \wedge T_{u+\varepsilon}^+ \wedge T_0^-\}}\right] + \mathbb{E}_u\left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \tau_a < \xi < T_{u+\varepsilon}^+ \wedge T_0^-\}}\right]. \end{aligned} \tag{3.16}$$

For the first term on the right-hand side of (3.16), since $\xi < \tau_a$, we have $\xi = e_1$ almost surely. Then[1],

$$\begin{aligned} & \mathbb{E}_u\left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \xi < \tau_a \wedge T_{u+\varepsilon}^+ \wedge T_0^-\}}\right] \\ & = \mathbb{E}_u^1\left[e^{-qe_1} 1_{\{X_{e_1} \in \mathrm{d}y, e_1 < \tau_a \wedge T_{u+\varepsilon}^+ \wedge T_0^-\}}\right] \\ & = \mathbb{E}_u^1\left[e^{-qe_1} 1_{\{X_{e_1} \in \mathrm{d}y, e_1 < T_{u+\varepsilon}^+ \wedge T_0^-\}}\right] - \mathbb{E}_u^1\left[e^{-qe_1} 1_{\{X_{e_1} \in \mathrm{d}y, \tau_a < e_1 < T_{u+\varepsilon}^+ \wedge T_0^-\}}\right] \\ & = \mathbb{E}_u^1\left[e^{-qe_1} 1_{\{X_{e_1} \in \mathrm{d}y, e_1 < T_{u+\varepsilon}^+ \wedge T_0^-\}}\right] - \int_u^{u+\varepsilon} \mathbb{E}_u^1\left[e^{-q_1\tau_a} 1_{\{M_{\tau_a} \in \mathrm{d}z\}}\right] \mathbb{E}_{z-a}^1\left[e^{-qe_1} 1_{\{X_{e_1} \in \mathrm{d}y, e_1 < T_{u+\varepsilon}^+ \wedge T_0^-\}}\right]. \end{aligned} \tag{3.17}$$

[1]Recall that $\mathbb{E}_u^k[\cdot]$ $(k=1,2)$ implies that all processes and stopping times under the (conditional) expectation are those related to the process X^k.

Note that (3.17) is possibly non-zero only for $y \in (u-a, u+\varepsilon)$. It follows from Equations (3.3) and (3.4) that, for $y \in (u-a, u)$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon \mathrm{d}y} \mathbb{E}_u \left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \xi < \tau_a \wedge T^+_{u+\varepsilon} \wedge T^-_0\}} \right] &= \lambda_1 \left(W_1^{(q_1)\prime}(u-y) - \frac{W_1^{(q_1)\prime}(u)}{W_1^{(q_1)}(u)} W_1^{(q_1)}(u-y) \right) \\ &\quad - \frac{\sigma_1^2}{2} \left(\frac{\left(W_1^{(q_1)\prime}(a)\right)^2}{W_1^{(q_1)}(a)} - W_1^{(q_1)\prime\prime}(a) \right) \lambda_1 \frac{W_1^{(q_1)}(u-a) W_1^{(q_1)}(u-y)}{W_1^{(q_1)}(u)} \\ &= \lambda_1 \left(W_1^{(q_1)\prime}(u-y) - \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} W_1^{(q_1)}(u-y) \right),\end{aligned}$$

where the last step is due to the identity

$$\frac{\sigma_1^2}{2} \left(\frac{\left(W_1^{(q_1)\prime}(a)\right)^2}{W_1^{(q_1)}(a)} - W_1^{(q_1)\prime\prime}(a) \right) = \frac{W_1^{(q_1)}(u)}{W_1^{(q_1)}(u-a)} \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} - \frac{W_1^{(q_1)\prime}(u)}{W_1^{(q_1)}(u-a)},$$

which can be found in Equation (3.12) of [9]. Therefore,

$$\mathbb{E}_u \left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \xi < \tau_a \wedge T^+_{u+\varepsilon} \wedge T^-_0\}} \right] / \mathrm{d}y = \varepsilon \lambda_1 \left(W_1^{(q_1)\prime}(u-y) - \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} W_1^{(q_1)}(u-y) \right) 1_{\{u-a<y<u\}} + o(\varepsilon). \tag{3.18}$$

As for the second term in (3.16), we first condition on τ_a, at which moment the dynamic of X changes to X^2, and later condition on whether the dynamic of X will be changed back to X^1 before ξ or not. One obtains that, for $y \in (0, u+\varepsilon)$,

$$\begin{aligned}&\mathbb{E}_u \left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \tau_a < \xi < T^+_{u+\varepsilon} \wedge T^-_0\}} \right] \\ &= \int_u^{u+\varepsilon} \mathbb{E}_u^1 \left[e^{-q_1 \tau_a} 1_{\{M_{\tau_a} \in \mathrm{d}z\}} \right] \mathbb{E}^2_{z-a} \left[e^{-q e_2} 1_{\{X_{e_2} \in \mathrm{d}y, e_2 < T^+_z \wedge T^-_0\}} \right] \\ &+ \int_u^{u+\varepsilon} \mathbb{E}_u^1 \left[e^{-q_1 \tau_a} 1_{\{M_{\tau_a} \in \mathrm{d}z\}} \right] \mathbb{E}^2_{z-a} \left[e^{-q T^+_z} 1_{\{T^+_z < e_2 \wedge T^-_0\}} \right] \mathbb{E}_z \left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \xi < T^+_{u+\varepsilon} \wedge T^-_0\}} \right] \\ &= \int_u^{u+\varepsilon} \mathbb{E}_u^1 \left[e^{-q_1 \tau_a} 1_{\{M_{\tau_a} \in \mathrm{d}z\}} \right] \mathbb{E}^2_{z-a} \left[e^{-q e_2} 1_{\{X_{e_2} \in \mathrm{d}y, e_2 < T^+_z \wedge T^-_0\}} \right] + o(\varepsilon).\end{aligned} \tag{3.19}$$

Note that the last step in (3.19) is due to

$$\begin{aligned}&\int_u^{u+\varepsilon} \mathbb{E}_u^1 \left[e^{-q_1 \tau_a} 1_{\{M_{\tau_a} \in \mathrm{d}z\}} \right] \mathbb{E}^2_{z-a} \left[e^{-q T^+_z} 1_{\{T^+_z < e_2 \wedge T^-_0\}} \right] \mathbb{E}_z \left[e^{-q\xi} 1_{\{X_\xi \in \mathrm{d}y, \xi < T^+_{u+\varepsilon} \wedge T^-_0\}} \right] \\ &\leq \mathbb{P}^1_u (M_{\tau_a} < u+\varepsilon) \sup_{z \in (u, u+\varepsilon)} \mathbb{P}_z (\xi < T^+_{u+\varepsilon} \wedge T^-_0) \\ &= o(\varepsilon).\end{aligned}$$

By Equations (3.3) and (3.4), we have

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon \mathrm{d}y} \int_u^{u+\varepsilon} \mathbb{E}_u^1 \left[e^{-q_1 \tau_a} 1_{\{M_{\tau_a} \in \mathrm{d}z\}} \right] \mathbb{E}^2_{z-a} \left[e^{-q e_2} 1_{\{X_{e_2} \in \mathrm{d}y, e_2 < T^+_z \wedge T^-_0\}} \right] \\ &= \frac{\sigma_1^2}{2} \left(\frac{\left(W_1^{(q)\prime}(a)\right)^2}{W_1^{(q)}(a)} - W_1^{(q)\prime\prime}(a) \right) \lambda_2 \left(\frac{W_2^{(q_2)}(u-a) W_2^{(q_2)}(u-y)}{W_2^{(q_2)}(u)} - W_2^{(q_2)}(u-a-y) \right) \\ &= \lambda_2 W_2^{(q_2)}(u-y) \left(C_{q_1,q_2}(u-y) - C_{q_1,q_2}(u) \right) \mathrm{d}y.\end{aligned}$$

It follows from that (3.19) that

$$\mathbb{E}_u\left[e^{-q\xi}1_{\{X_\xi\in\mathrm{d}y,\tau_a<\xi<T^+_{u+\varepsilon}\wedge T^-_0\}}\right]/\mathrm{d}y=\varepsilon\lambda_2 W_2^{(q_2)}(u-y)\left(C_{q_1,q_2}(u-y)-C_{q_1,q_2}(u)\right)+o(\varepsilon). \tag{3.20}$$

Substituting (3.15), (3.18) and (3.20) into (3.13) yields

$$f^{(q)}(u,y)=e^{-\int_u^{u+\varepsilon}C_{q_1,q_2}(w)\mathrm{d}w}f^{(q)}(u+\varepsilon,y)+\varepsilon g_{q_1,q_2}(u,y)+o\left(\varepsilon\right),$$

where the function g_{q_1,q_2} is as defined in (3.12). Thus, one obtains the ODE

$$\frac{\mathrm{d}f^{(q)}(u,y)}{\mathrm{d}u}=C_{q_1,q_2}(u)f^{(q)}(u,y)-g_{q_1,q_2}(u,y),\quad u<b. \tag{3.21}$$

Together with the boundary condition $f^{(q)}(b,y)=0$, it is easy to show that (3.11) solves the ODE (3.21). ■

Remark 3.1 *Note that the proof of Proposition 3.2 uses an infinitesimal argument to simplify the derivation of the ODE (3.21). The primary advantage of such an approach resides in the convenience that no explicit expression needs to be given to the $o(\varepsilon)$ terms. Alternatively, one may use a renewal argument as in Section 4 for the more general Markov setting to prove this result.*

With the help of Proposition 3.2, we now provide a complete representation for the ETU V_a.

Theorem 3.1 *For $u\in[a,b]$ and $q\geq 0$, the ETU V_a is given by*

$$\begin{aligned}V_a(u)&=U(b)e^{-\int_u^b C_{q_1,q_2}(w)\mathrm{d}w}+U(0)\int_u^b e^{-\int_u^z C_{q_1,q_2}(w)\mathrm{d}w}D_{q_1,q_2}(z)\mathrm{d}z\\&\quad+\int_0^b U(y)\int_u^b e^{-\int_u^z C_{q_1,q_2}(w)\mathrm{d}w}g_{q_1,q_2}(z,y)\mathrm{d}z\mathrm{d}y,\end{aligned}$$

where C_{q_1,q_2}, D_{q_1,q_2} and g_{q_1,q_2} are given in (3.7), (3.8), and (3.12), respectively.

Proof. From (3.9) and Proposition 3.2, it remains to show that

$$\mathbb{E}_u\left[e^{-qT_b^+}1_{\{T_b^+<\xi\wedge T_0^-\}}\right]=e^{-\int_u^b C_{q_1,q_2}(w)\mathrm{d}w}, \tag{3.22}$$

and

$$\mathbb{E}_u\left[e^{-qT_0^-}1_{\{T_0^-<T_b^+\wedge\xi\}}\right]=\int_u^b e^{-\int_u^z C_{q_1,q_2}(w)\mathrm{d}w}D_{q_1,q_2}(z)\mathrm{d}z. \tag{3.23}$$

Note that (3.22) is immediate from (3.15) by replacing $u+\varepsilon$ by b. Moreover, (3.23) can be proved in the same manner. Indeed, by (2.2) and (3.6), we have

$$\begin{aligned}\mathbb{E}_u\left[e^{-qT_0^-}1_{\{T_0^-<T_b^+\wedge\xi\}}\right]&=\mathbb{E}_u\left[\mathbb{E}\left[e^{-qT_0^-}1_{\{T_0^-<T_b^+\wedge\xi\}}\,\middle|\mathcal{F}_{T_0^-}\right]\right]\\&=\mathbb{E}_u\left[\mathbb{E}\left[e^{-qT_0^-}e^{-\lambda_1\theta^1(T_0^-)-\lambda_2\theta^2(T_0^-)}1_{\{T_0^-<T_b^+\}}\,\middle|\mathcal{F}_{T_0^-}\right]\right]\\&=\mathbb{E}_u\left[e^{-q_1\theta^1(T_0^-)-q_2\theta^2(T_0^-)}1_{\{T_0^-<T_b^+\}}\right]\\&=\int_u^b e^{-\int_u^z C_{q_1,q_2}(w)\mathrm{d}w}D_{q_1,q_2}(z)\mathrm{d}z.\end{aligned} \tag{3.24}$$

The proof is therefore complete. ■

3.2 Risk management implication

To assess the performance of the DBRS model with state-dependent termination, we propose to compare the ETU for the DBRS risk process X with two other ETUs, namely those associated with the single regime processes X^1 and X^2. For that purpose, we exclusively consider the utility function $U(x) = 1_{\{x \geq b\}}$ for $x \in \mathbb{R}$ in this subsection, which leads an ETU of the form

$$V_a(u) = \mathbb{E}_u \left[e^{-qT_b^+} 1_{\{T_b^+ < T_0^- \wedge \xi\}} \right]. \tag{3.25}$$

This corresponds to an entity's desire to reach the target level b (in a timely manner) before either its surplus becomes negative or the state-dependent termination time ξ is triggered.

For the sake of comparison with the single regime models X^1 and X^2, we first extend the ETU $V_a(u)$ to $a \in (0, b]$. In fact, it is easy to show that $V_a(u) = \frac{W_1^{(q_1)}(u)}{W_1^{(q_1)}(a)} V_a(a)$ for $a \in (u, b]$, from which we deduce from Theorem 3.1 that

$$V_a(u) = \begin{cases} e^{-\int_u^b C_{q_1,q_2}(w)\mathrm{d}w}, & a \in (0, u], \\ \frac{W_1^{(q_1)}(u)}{W_1^{(q_1)}(a)} e^{-\int_a^b C_{q_1,q_2}(w)\mathrm{d}w}, & a \in (u, b]. \end{cases} \tag{3.26}$$

Note that from Proposition 6.3 of [9] and Lemma 3.1, we have

$$V_0(u) := \lim_{a \downarrow 0} V_a(u) = V^2(u) = \frac{W_2^{(q_2)}(u)}{W_2^{(q_2)}(b)}.$$

Also, it is easy to see from (3.26) and Lemma 3.1 that

$$V_b(u) = V^1(u) = \frac{W_1^{(q_1)}(u)}{W_1^{(q_1)}(b)}.$$

Hence, $V_b(u)$ and $V_0(u)$ can be regarded as the ETU (3.25) for the single regime processes X^1 and X^2, respectively.

As a basis of fair comparison, we assume that the processes X^1 and X^2 are defined such that $V_0(u) = V_b(u)$, or equivalently,

$$\frac{W_1^{(q_1)}(u)}{W_1^{(q_1)}(b)} = \frac{W_2^{(q_2)}(u)}{W_2^{(q_2)}(b)}. \tag{3.27}$$

The following proposition provides a sufficient (but not necessary) condition for the outperformance of the DBRS model.

Proposition 3.3 *For $u \in [0, b]$ and $q_1, q_2 > 0$, consider two underlying Brownian motions X^1 and X^2 satisfying Condition (3.27). If*

$$\left(\frac{c_2}{\sigma_2^2} - \frac{c_1}{\sigma_1^2} \right) \ln \frac{W_2^{(q_2)}(u)}{W_2^{(q_2)}(b)} - \left(\frac{q_1}{\sigma_1^2} - \frac{q_2}{\sigma_2^2} \right) (b - u) > 0, \tag{3.28}$$

then the DBRS model outperforms, in the sense that, there exists some $a \in (0, b)$ such that

$$V_a(u) > V_0(u) = V_b(u). \tag{3.29}$$

Proof. From Theorem 3.1, we have

$$V_a(u) = \mathbb{E}_u \left[e^{-qT_b^+} 1_{\{T_b^+ < \xi \wedge T_0^-\}} \right] = e^{-\int_u^b C_{q_1,q_2}(z)\mathrm{d}z}.$$

A sufficient (but not necessary) condition for (3.29) to hold is that

$$\lim_{a\downarrow 0}\frac{\partial}{\partial a}V_a(u)=\lim_{a\downarrow 0}\frac{\partial}{\partial a}e^{-\int_u^b C_{q_1,q_2}(z)\mathrm{d}z}>0,$$

or equivalently,

$$\lim_{a\downarrow 0}\frac{\partial}{\partial a}\int_u^b C_{q_1,q_2}(z)\mathrm{d}z=\lim_{a\downarrow 0}\int_u^b \frac{\partial}{\partial a}C_{q_1,q_2}(z)\mathrm{d}z<0.$$

Therefore, differentiating (3.7) at $s=q_1$ and $q=q_2$ with respect to a yields

$$\begin{aligned}\frac{\partial}{\partial a}C_{q_1,q_2}(z)=&\left(\frac{W_1^{(q_1)\prime\prime}(a)}{W_1^{(q_1)}(a)}-\frac{\left(W_1^{(q_1)\prime}(a)\right)^2}{\left(W_1^{(q_1)}(a)\right)^2}\right)\left\{1-\frac{\sigma_1^2}{2}\frac{W_2^{(q_2)}(z-a)}{W_2^{(q_2)}(z)}\left(W_1^{(q_1)\prime}(a)-W_1^{(q_1)\prime\prime}(a)\frac{W_1^{(q_1)}(a)}{W_1^{(q_1)\prime}(a)}\right)\right\}\\&+\frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}\frac{\sigma_1^2}{2}\frac{W_2^{(q_2)\prime}(z-a)}{W_2^{(q_2)}(z)}\left(W_1^{(q_1)\prime}(a)-W_1^{(q_1)\prime\prime}(a)\frac{W_1^{(q_1)}(a)}{W_1^{(q_1)\prime}(a)}\right)\\&-\frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}\frac{\sigma_1^2}{2}\frac{W_2^{(q_2)}(z-a)}{W_2^{(q_2)}(z)}\left(\left(W_1^{(q_1)\prime\prime}(a)\right)^2\frac{W_1^{(q_1)}(a)}{\left(W_1^{(q_1)\prime}(a)\right)^2}-W_1^{(q_1)\prime\prime\prime}(a)\frac{W_1^{(q_1)}(a)}{W_1^{(q_1)\prime}(a)}\right).\end{aligned}$$

From the definition of C_{q_1,q_2}, it follows that

$$\begin{aligned}\frac{\partial}{\partial a}C_{q_1,q_2}(z)=&\left(\frac{W_1^{(q_1)\prime\prime}(a)}{W_1^{(q_1)\prime}(a)}-\frac{W_2^{(q_2)\prime}(z-a)}{W_2^{(q_2)}(z-a)}\right)C_{q_1,q_2}(z)+W_1^{(q_1)\prime}(a)\frac{\frac{W_2^{(q_2)\prime}(z-a)}{W_2^{(q_2)}(z-a)}-C_{q_1,q_2}(z)}{W_1^{(q_1)}(a)}\\&+\frac{\sigma_1^2}{2}\frac{W_2^{(q_2)}(z-a)}{W_2^{(q_2)}(z)}\left(W_1^{(q_1)\prime\prime\prime}(a)-\frac{\left(W_1^{(q_1)\prime\prime}(a)\right)^2}{W_1^{(q_1)\prime}(a)}\right).\end{aligned}\tag{3.30}$$

When $a\to 0$, using a similar argument as in Proposition 6.3 in [9], one can show that

$$C_{q_1,q_2}(z)\to\frac{W_2^{(q_2)\prime}(z)}{W_2^{(q_2)}(z)},$$

and thus,

$$\lim_{a\downarrow 0}\frac{\frac{W_2^{(q_2)\prime}(z-a)}{W_2^{(q_2)}(z-a)}-C_{q_1,q_2}(z)}{W_1^{(q_1)}(a)}=\frac{1}{W_1^{(q_1)\prime}(a)}\left\{\frac{\left(W_2^{(q_2)\prime}(z)\right)^2-W_2^{(q_2)\prime\prime}(z)W_2^{(q_2)}(z)}{\left(W_2^{(q_2)}(z)\right)^2}-\lim_{a\downarrow 0}\frac{\partial}{\partial a}C_{q_1,q_2}(z)\right\}.\tag{3.31}$$

Substituting (3.31) into (3.30) yields

$$2\lim_{a\downarrow 0}\frac{\partial}{\partial a}C_{q_1,q_2}(z)=\frac{W_1^{(q_1)\prime\prime}(0)}{W_1^{(q_1)\prime}(0)}\frac{W_2^{(q_2)\prime}(z)}{W_2^{(q_2)}(z)}-\frac{W_2^{(q_2)\prime\prime}(z)}{W_2^{(q_2)}(z)}+\frac{\sigma_1^2}{2}\left(W_1^{(q_1)\prime\prime\prime}(0)-\frac{\left(W_1^{(q_1)\prime\prime}(0)\right)^2}{W_1^{(q_1)\prime}(0)}\right).$$

In the Brownian motion case, with the explicit scale function given in (3.1), it is easy to check that

$$\frac{W_1^{(q_1)\prime\prime}(0)}{W_1^{(q_1)\prime}(0)}=-2\frac{c_1}{\sigma_1^2},\qquad W_1^{(q_1)\prime\prime\prime}(0)-\frac{\left(W_1^{(q_1)\prime\prime}(0)\right)^2}{W_1^{(q_1)\prime}(0)}=\frac{4q_1}{\sigma_1^4},$$

and

$$\frac{W_2^{(q_2)\prime\prime}(x)}{W_2^{(q_2)}(x)}=2\left(\frac{q_2}{\sigma_2^2}-\frac{c_2}{\sigma_2^2}\frac{W_2^{(q_2)\prime}(x)}{W_2^{(q_2)}(x)}\right).$$

Therefore,

$$\lim_{a\downarrow 0}\frac{\partial}{\partial a}C_{q_1,q_2}(z)=\left(\frac{c_2}{\sigma_2^2}-\frac{c_1}{\sigma_1^2}\right)\frac{W_2^{(q_2)\prime}(z)}{W_2^{(q_2)}(z)}+\left(\frac{q_1}{\sigma_1^2}-\frac{q_2}{\sigma_2^2}\right),$$

and

$$-\lim_{a\downarrow 0}\left\{\frac{\partial}{\partial a}\int_u^b C_{q_1,q_2}(z)\mathrm{d}z\right\}=\left(\frac{c_2}{\sigma_2^2}-\frac{c_1}{\sigma_1^2}\right)\ln\frac{W_2^{(q_2)}(u)}{W_2^{(q_2)}(b)}-\left(\frac{q_1}{\sigma_1^2}-\frac{q_2}{\sigma_2^2}\right)(b-u)>0,$$

by condition (3.28). Therefore, we have $\lim_{a\downarrow 0}\frac{\partial}{\partial a}V_a(u)>0$ which implies that there exists some value of $a>0$ such that

$$V_a(u)>\lim_{a\downarrow 0}V_a(u)=V_0(u)=V_b(u),$$

where the last equality is due to assumption (3.27). ■

As an illustration, we consider two numerical examples for the ETU V_a in (3.25). The first example is chosen such that Condition (3.28) is satisfied, while the second one demonstrates a situation where the ETU $V_a(u)$ for all $0<a<b$ cannot do better than either of the single regime models X^1 and X^2. The parameter settings for these two examples are:

Example 1:

$$u=4, b=12, c_1=0.05, \sigma_1=0.5, c_2=0.07, \sigma_2=0.4579, \lambda_1=0.02, \lambda_2=0.024, q=0.$$

Example 2:

$$u=4, b=12, c_1=0.05, \sigma_1=0.5, c_2=0.055, \sigma_2=0.5893, \lambda_1=0.02, \lambda_2=0.024, q=0.$$

Note that the parameters for both examples are chosen so that Condition (3.27) is satisfied. As such, the left and right end points of both curves in Figure 2 coincide. Moreover, for both examples, considering a common practice of premium increase following significant insurance losses, the drift in regime 2 (c_2) is set to be larger than the drift in regime 1 (c_1). As a trade-off, the termination rate in regime 2 is set to be larger than the one in regime 1, i.e., $\lambda_2>\lambda_1$.

For Example 1 (left panel of Figure 2), we observe that the DBRS strategy for any $a\in(0,b)$ outperforms its counterparts in either single regime model (i.e., the two end points of the curve). In addition, we observe that there exists an optimal level a^* that maximizes the ETU V_a. Numerically, this value is found to be $a^*=2.35$ (with 2 decimal places of accuracy).

However, the DBRS model does not always outperform its single regime counterparts. As we can see for Example 2 (right panel of Figure 2), none of the DBRS processes do better than its counterparts in either single regime strategy. Intuitively, this can be explained by the fact that the drift-volatility trade-off for process X^2 is generally speaking less attractive under Example 2 than under Example 1 for the same given state-dependent killing rate λ_2.

4 Analysis under time-homogeneous Markov processes

In this section, we generalize the underlying processes X^1 and X^2 to two time-homogeneous Markov processes with possibly upward and downward jumps. More specifically, we assume X^1 and X^2 satisfy the strong Markov property (see Section III.8,9 of [15]), and exclude Markov processes with monotone paths.

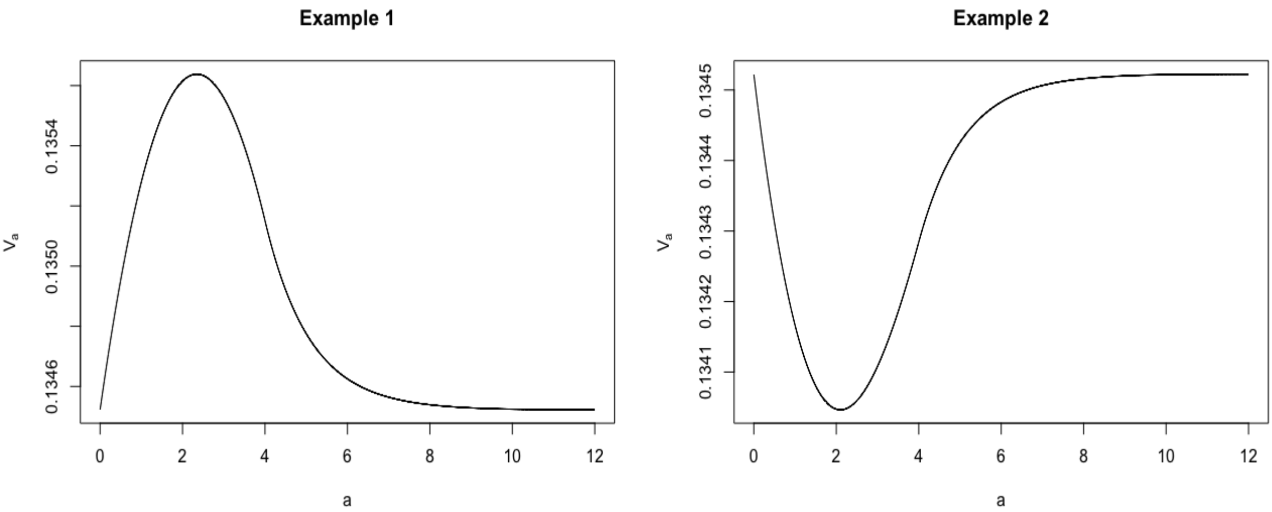

Figure 2: Change of V_a values with respect to a

To analyze the ETU V_a in this model setup, we first define the following three measures:

$$\begin{aligned}
F_{+}^{(q_1,q_2)}(u,\mathrm{d}y) &= \mathbb{E}_u\left[e^{-q_1\theta^1(T_b^+)-q_2\theta^2(T_b^+)}1_{\{T_b^+<T_0^-,X_{T_b^+}\in\mathrm{d}y\}}\right], \quad y\geq b,\\
F_{-}^{(q_1,q_2)}(u,\mathrm{d}y) &= \mathbb{E}_u\left[e^{-q_1\theta^1(T_0^-)-q_2\theta^2(T_0^-)}1_{\{T_0^-<T_b^+,-X_{T_0^-}\in\mathrm{d}y\}}\right], \quad y\geq 0,\\
F_{0}^{(q)}(u,\mathrm{d}y) &= \mathbb{E}_u\left[e^{-q\xi}1_{\{X_\xi\in\mathrm{d}y\}}1_{\{\xi<T_b^+\wedge T_0^-\}}\right], \quad y\in(0,b).
\end{aligned}$$

By (2.2) and using the same conditional argument as in (3.14) and (3.24), we have

$$\mathbb{E}_u\left[e^{-qT_b^+}1_{\{T_b^+<T_0^-\wedge\xi,X_{T_b^+}\in\mathrm{d}y\}}\right]=\mathbb{E}_u\left[e^{-q_1\theta^1(T_b^+)-q_2\theta^2(T_b^+)}1_{\{T_b^+<T_0^-,X_{T_b^+}\in\mathrm{d}y\}}\right]=F_{+}^{(q_1,q_2)}(u,\mathrm{d}y),$$

and

$$\mathbb{E}_u\left[e^{-qT_0^-}1_{\{T_0^-<T_b^+\wedge\xi,-X_{T_0^-}\in\mathrm{d}y\}}\right]=\mathbb{E}_u\left[e^{-q_1\theta^1(T_0^-)-q_2\theta^2(T_0^-)}1_{\{T_0^-<T_b^+,-X_{T_0^-}\in\mathrm{d}y\}}\right]=F_{-}^{(q_1,q_2)}(u,\mathrm{d}y).$$

Similarly as in (3.9), it follows that

$$\begin{aligned}
V_a(u) &= \mathbb{E}_u\left[e^{-qT_b^+}U\left(X_{T_b^+}\right)1_{\{T_b^+<T_0^-\wedge\xi\}}\right]+\mathbb{E}_u\left[e^{-qT_0^-}U\left(X_{T_0^-}\right)1_{\{T_0^-<T_b^+\wedge\xi\}}\right]+\mathbb{E}_u\left[e^{-q\xi}U\left(X_\xi\right)1_{\{\xi<T_b^+\wedge T_0^-\}}\right]\\
&= \int_{[b,\infty)}U\left(y\right)F_{+}^{(q_1,q_2)}(u,\mathrm{d}y)+\int_{[0,\infty)}U\left(-y\right)F_{-}^{(q_1,q_2)}(u,\mathrm{d}y)+\int_0^b U\left(y\right)F_0^{(q)}(u,\mathrm{d}y). \qquad (4.1)
\end{aligned}$$

For ease of notation, we will suppress the superscripts q_1,q_2,q of $F_{+/-/0}$ in what follows. From (4.1), it is clear that the analysis of V_a reduces to the characterization of the three measures $F_{+/-/0}$. More specifically, the main objective of this section is to derive and show the uniqueness of the solution to the associated integral equations for $F_{+/-/0}(u,A)$ in terms of u, where A is an arbitrary Borel subset of $[b,\infty)$, $[0,\infty)$ and $(0,b)$ for F_+, F_- and F_0, respectively.

By conditioning on τ_a and whether a recovery to the previous maximum will occur or not, one obtains

$$\begin{aligned}F_+(u,A) &= \mathbb{E}_u\left[e^{-q_1\theta^1(T_b^+)-q_2\theta^2(T_b^+)}1_{\{\tau_a<T_b^+<T_0^-,X_{T_b^+}\in A\}}\right]+\mathbb{E}_u\left[e^{-q_1\theta^1(T_b^+)-q_2\theta^2(T_b^+)}1_{\{T_b^+<T_0^-\wedge\tau_a,X_{T_b^+}\in A\}}\right]\\ &= \int_u^b\int_{[a,x)}\mathbb{E}_u^1\left[e^{-q_1\tau_a}1_{\{Y_{\tau_a}\in\mathrm{d}l,M_{\tau_a}\in\mathrm{d}x\}}\right]\int_{[x,b)}\mathbb{E}_{x-l}^2\left[e^{-q_2T_x^+}1_{\{T_x^+<T_0^-,X_{T_x^+}\in\mathrm{d}z\}}\right]F_+(z,A)+h_+(u,A;a,b),\end{aligned}\tag{4.2}$$

where

$$h_+(u,A;a,b)=\int_u^b\int_{[a,x)}\mathbb{E}_u^1\left[e^{-q_1\tau_a}1_{\{Y_{\tau_a}\in\mathrm{d}l,M_{\tau_a}\in\mathrm{d}x\}}\right]\mathbb{E}_{x-l}^2\left[e^{-q_2T_x^+}1_{\{T_x^+<T_0^-,X_{T_x^+}\in A\}}\right]+\mathbb{E}_u^1\left[e^{-q_1T_b^+}1_{\{T_b^+<\tau_a,X_{T_b^+}\in A\}}\right].$$

Using similar arguments, we also find that

$$\begin{aligned}F_-(u,A) &= \mathbb{E}_u\left[e^{-q_1\theta^1(T_0^-)-q_2\theta^2(T_0^-)}1_{\{\tau_a<T_0^-<T_b^+,-X_{T_0^-}\in A\}}\right]+\mathbb{E}_u\left[e^{-q_1\theta^1(T_0^-)-q_2\theta^2(T_0^-)}1_{\{\tau_a=T_0^-<T_b^+,-X_{T_0^-}\in A\}}\right]\\ &= \int_u^b\int_{[a,x)}\mathbb{E}_u^1\left[e^{-q_1\tau_a}1_{\{Y_{\tau_a}\in\mathrm{d}l,M_{\tau_a}\in\mathrm{d}x\}}\right]\int_{[x,b)}\mathbb{E}_{x-l}^2\left[e^{-q_2T_x^+}1_{\{T_x^+<T_0^-,X_{T_x^+}\in\mathrm{d}z\}}\right]F_-(z,A)+h_-(u,A;a,b),\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}F_0(u,A) =&\mathbb{E}_u\left[e^{-q\xi}1_{\{X_\xi\in A\}}1_{\{\tau_a<\xi<T_b^+\wedge T_0^-\}}\right]+\mathbb{E}_u\left[e^{-q\xi}1_{\{X_\xi\in A\}}1_{\{\xi<\tau_a\wedge T_b^+\}}\right]\\ =&\int_u^b\int_{[a,x)}\mathbb{E}_u^1\left[e^{-q_1\tau_a}1_{\{Y_{\tau_a}\in\mathrm{d}l,M_{\tau_a}\in\mathrm{d}x\}}\right]\int_{[x,b)}\mathbb{E}_{x-l}^2\left[e^{-q_2T_x^+}1_{\{T_x^+<T_0^-,X_{T_x^+}\in\mathrm{d}z\}}\right]F_0(x+z,A)+h_0\left(u,A;a,b\right),\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}h_-(u,A;a,b)=&\int_u^b\int_{[a,x)}\mathbb{E}_u^1\left[e^{-q_1\tau_a}1_{\{Y_{\tau_a}\in\mathrm{d}l,M_{\tau_a}\in\mathrm{d}x\}}\right]\mathbb{E}_{x-l}^2\left[e^{-q_2T_0^-}1_{\{T_0^-<T_x^+,-X_{T_0^-}\in A\}}\right]\\ &+\int_u^b\mathbb{E}_u^1\left[e^{-q_1\tau_a}1_{\{Y_{\tau_a}-x\in A,M_{\tau_a}\in\mathrm{d}x\}}\right].\end{aligned}$$

and

$$\begin{aligned}h_0\left(u,A;a,b\right)=&\int_u^b\int_{[a,x)}\mathbb{E}_u^1\left[e^{-q_1\tau_a}1_{\{Y_{\tau_a}\in\mathrm{d}l,M_{\tau_a}\in\mathrm{d}x\}}\right]\mathbb{E}_{x-l}^2\left[e^{-qe_2}1_{\{e_2<T_x^+\wedge T_0^-,X_{e_2}\in A\}}\right]\\ &+\mathbb{E}_u^1\left[e^{-qe_1}1_{\{e_1<\tau_a\wedge T_b^+,X_{e_1}\in A\}}\right].\end{aligned}$$

For ease of notation, we further define the following fundamental measures/functions[2] of the underlying

[2]We say $f, g_{+/-/0}, m_{+/-/0}$ are fundamental quantities because they only involve a single dynamics X^1 or X^2. Under some mild conditions, it is possible to further decompose those drawdown related quantities into only exit quantities; see, e.g., [12].

processes X^1 or X^2: for $x \geq u$, $b \geq u \geq 0$ and $l \geq a$,

$$\begin{aligned}
f(u, \mathrm{d}l, \mathrm{d}x) &= \mathbb{E}_u^1 \left[e^{-q_1 \tau_a} 1_{\{Y_{\tau_a} \in \mathrm{d}l, M_{\tau_a} \in \mathrm{d}x\}} \right], \\
g_+(u, A; 0, b) &= \mathbb{E}_u^2 \left[e^{-q_2 T_b^+} 1_{\{T_b^+ < T_0^-, X_{T_b^+} \in A\}} \right], \\
g_-(u, A; 0, b) &= \mathbb{E}_u^2 \left[e^{-q_2 T_0^-} 1_{\{T_0^- < T_b^+, -X_{T_0^-} \in A\}} \right], \\
g_0(u, A; 0, b) &= \mathbb{E}_u^2 \left[e^{-q e_2} 1_{\{e_2 < T_b^+ \wedge T_0^-, X_{e_2} \in A\}} \right], \\
m_+(u, A; a, b) &= \mathbb{E}_u^1 \left[e^{-q_1 T_b^+} 1_{\{T_b^+ < \tau_a, X_{T_b^+} \in A\}} \right], \\
m_-(u, A; a, b) &= \int_u^b \mathbb{E}_u^1 \left[e^{-q_1 \tau_a} 1_{\{Y_{\tau_a} - x \in A, M_{\tau_a} \in \mathrm{d}x\}} \right], \\
m_0(u, A; a, b) &= \mathbb{E}_u^1 \left[e^{-q e_1} 1_{\{e_1 < \tau_a \wedge T_b^+, X_{e_1} \in A\}} \right].
\end{aligned}$$

This allows us to rewrite (4.2), (4.3) and (4.4) into a unified integral equation form for $F_{+/-/0}$, namely

$$F_{+/-/0}(u, A) = \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) \int_{[x,b)} g_+(x - l, \mathrm{d}z; 0, x) F_{+/-/0}(z, A) + h_{+/-/0}(u, A; a, b), \tag{4.5}$$

where

$$h_{+/-/0}(u, A; a, b) = \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) g_{+/-/0}(x - l, A; 0, x) + m_{+/-/0}(u, A; a, b). \tag{4.6}$$

To show the existence and uniqueness of the solution to the integral equation (4.5), one can consider a mapping $\mathcal{L}_\psi$ on $\mathbb{M} = \{f(\cdot) : [a, b] \to [0, 1] \text{ is measurable}\}$: for any $\phi \in \mathbb{M}$,

$$\mathcal{L}_\psi \phi(u) = \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) \int_{[x,b)} \phi(z) g_+(x - l, \mathrm{d}z; 0, x) + \psi(u),$$

where $\psi \in \mathbb{M}$ is a given function such that

$$\int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) \int_{[x,b)} g_+(x - l, \mathrm{d}z; 0, x) + \psi(u) \leq 1, \quad \forall u \in [a, b]. \tag{4.7}$$

Lemma 4.1 *Given $\psi \in \mathbb{M}$ satisfying (4.7), suppose that*

$$\sup_{u \in [a,b]} \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) < 1. \tag{4.8}$$

Then there exists a unique solution $\phi \in \mathbb{M}$ satisfying the integral equation

$$\phi(u) = \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) \int_{[x,b)} \phi(z) g_+(x - l, \mathrm{d}z; 0, x) + \psi(u). \tag{4.9}$$

Proof. On $\mathbb{M}$, we define a metric: for $f, g \in \mathbb{M}$,

$$\boldsymbol{d}(f, g) = \sup_{u \in [a,b]} |f(u) - g(u)|.$$

Under condition (4.7), it is easy to see that $\mathcal{L}_\psi \phi \in \mathbb{M}$, and thus $\mathcal{L}_\psi$ is a self mapping on $\mathbb{M}$. Moreover, for any $\phi, \tilde{\phi} \in \mathbb{M}$,

$$
\begin{aligned}
\boldsymbol{d}(\mathcal{L}_\psi \phi, \mathcal{L}_\psi \tilde{\phi}) &= \sup_{u\in[a,b]} \left| \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) \int_{[b,x)} g_+(x-l, \mathrm{d}z; 0, x) \left[\phi(z) - \tilde{\phi}(z)\right] \right| \\
&\leq \boldsymbol{d}(\phi, \tilde{\phi}) \sup_{u\in[a,b]} \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) \int_{[b,x)} g_+(x-l, \mathrm{d}z; 0, x) \\
&\leq \boldsymbol{d}(\phi, \tilde{\phi}) \sup_{u\in[a,b]} \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x).
\end{aligned}
$$

It follows from (4.8) that $\mathcal{L}_\psi$ is a contraction mapping on $\mathbb{M}$. Thus, the existence and uniqueness of the solution to the integral equation (4.9) follows immediately from the Banach fixed point theorem. ■

It is easy to see that the integral equation (4.5) for $F_{+/-/0}$ all satisfy condition (4.7) of Lemma 4.1. Moreover, we further propose a very mild assumption on the underlying process, that is,

$$
\sup_{u\in[a,b]} \mathbb{E}_u^1 \left[e^{-q_1 \tau_a}\right] < 1, \tag{4.10}
$$

which is sufficient to guarantee condition (4.8) holds because

$$
\sup_{u\in[a,b]} \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) \leq \sup_{u\in[a,b]} \mathbb{E}_u^1 \left[e^{-q_1 \tau_a}\right] < 1.
$$

Theorem 4.1 *Suppose that (4.10) holds. For $u \in [a, b]$, the measures F_+, F_-, and F_0 are the unique solution to their corresponding integral equation in (4.5).*

In general, it is difficult to find explicit solution to the integral equation (4.5) for $F_{+/-/0}$ (or the fundamental quantities f, $g_{+/-/0}$ and $m_{+/-/0}$). However, when X^1 and X^2 are time-homogeneous Markov processes with no positive jumps, under some mild regularity conditions, it can be shown that $F_{+/-/0}$ satisfy some ODEs, which can be solved explicitly in terms of the fundamental quantities $f, g_{+/-/0}, m_{+/-/0}$. For illustration purposes, we consider the case where X^1 and X^2 are SNLPs in the next section. Other possible models with explicit expressions include linear diffusions, refracted SNLP, and some jump diffusions. Interested readers are referred to Section 3 of [12] for more details.

5 Example: spectrally negative Lévy models

In this section, we fully characterize the measures $F_{+/-/0}$ when X^1 and X^2 are two SNLPs. More precisely, we assume that X^k $(k = 1, 2)$ is a SNLP such that $|X^k|$ is not a subordinator and hence 0 is regular for $(0, \infty)$ (see Definition 6.4 and Theorem 6.5 of [7] for the definition and equivalent characterizations of the regularity).

The Laplace exponent of X^k is assumed to have the Lévy-Khintchine representation

$$
\psi_k(s) := \log \mathbb{E}\left[e^{sX_1^k}\right] = c_k s + \frac{1}{2}\sigma_k^2 s^2 + \int_{-\infty}^0 \left(e^{sx} - 1 - sx 1_{\{x>-1\}}\right) \Pi_k(\mathrm{d}x),
$$

for $s \geq 0$ where $c_k \in \mathbb{R}$, $\sigma_k \geq 0$ and the Lévy measure $\Pi_k(\cdot)$ is supported on $(-\infty, 0)$ such that

$$
\int_{-\infty}^0 (1 \wedge x^2) \Pi_k(\mathrm{d}x) < \infty.
$$

For any $q \geq 0$, let $\Phi_k(q) = \sup\{x \geq 0 : \psi_k(x) = q\}$, and define the q-scale function $W^{(q)} : \mathbb{R} \mapsto [0, \infty)$ as the unique function supported on $(0, \infty)$ with Laplace transform

$$\int_0^\infty e^{-sy} W_k^{(q)}(y)\mathrm{d}y = \frac{1}{\psi_k(s) - q}, \quad s > \Phi_k(q).$$

It is known that $W_k^{(q)}$ is continuous and strictly increasing on $(0, \infty)$. Henceforth, we assume that the jump measure Π_k has no atom which implies that $W_k^{(q)} \in C^1(0, \infty)$ (e.g., Lemma 2.4 of [6]).

We recall the following results for the SNLP X^k which will be useful in what follows. We refer the reader to e.g., [7], Chapter 8, for more details.

(a) For $q \geq 0$ and $0 \leq u \leq b$, the upward exit probability from $[0, b]$ is

$$\mathbb{E}_u^k \left[e^{-qT_b^+} 1_{\{T_b^+ < T_0^-\}} \right] = \frac{W_k^{(q)}(u)}{W_k^{(q)}(b)}. \tag{5.1}$$

(b) For $q \geq 0$, $0 \leq u \leq b$, and $y \in (0, b)$, the q-potential measure killed before exiting $[0, b]$ is

$$\int_0^\infty e^{-qt} \mathbb{P}_u^k \left(X_t \in \mathrm{d}y, t < T_b^+ \wedge T_0^- \right)$$

with density

$$\vartheta_k^{(q)}(u, y; 0, b) = \frac{W_k^{(q)}(u) W_k^{(q)}(b - y)}{W_k^{(q)}(b)} - W_k^{(q)}(u - y). \tag{5.2}$$

(c) For $q \geq 0$, $0 \leq u \leq b$, and $y \geq 0$, the downward exit density from $[0, b]$ is

$$\begin{aligned} \mathbb{E}_u^k \left[e^{-qT_0^-} 1_{\{T_0^- < T_b^+, -X_{T_0^-} \in \mathrm{d}y\}} \right] &= \int_0^\infty \vartheta_k^{(q)}(u, z; 0, b) \Pi_k(-\mathrm{d}y - z)\mathrm{d}z \\ &+ \frac{\sigma_k^2}{2} \left(W_k^{(q)\prime}(u) - \frac{W_k^{(q)}(u) W_k^{(q)\prime}(b)}{W_k^{(q)}(b)} \right) \delta_0 (\mathrm{d}y), \end{aligned} \tag{5.3}$$

where $\delta_0(\cdot)$ is the Dirac measure centered at 0.

Given that X^2 is a skip-free upward process and using (5.1), we can simplify (4.5) to

$$F_{+/-/0}(u, A) = \int_u^b \int_{[a,x)} f(u, \mathrm{d}l, \mathrm{d}x) \frac{W_2^{(q_2)}(x - l)}{W_2^{(q_2)}(x)} F_{+/-/0}(x, A) + h_{+/-/0}(u, A; a, b), \tag{5.4}$$

for $u \in [a, b]$, where from Equation (2.4) of [9],

$$f(u, \mathrm{d}l, \mathrm{d}x) = \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} e^{-\frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}(x-u)} \mathrm{d}x F_{Y_{\tau_a}}^{(q_1)}(\mathrm{d}l), \tag{5.5}$$

with

$$\begin{aligned} F_{Y_{\tau_a}}^{(q_1)}(\mathrm{d}l) &= \int_0^a \left(\frac{W_1^{(q_1)}(a)}{W_1^{(q_1)\prime}(a)} W_1^{(q_1)\prime}(z) - W_1^{(q_1)}(z) \right) \Pi_1(z - \mathrm{d}l)\mathrm{d}z \\ &+ \frac{W_1^{(q_1)}(a)}{W_1^{(q_1)\prime}(a)} W_1^{(q_1)}(0+)\Pi_1(-\mathrm{d}l) + \frac{\sigma_1^2}{2} \left(W_1^{(q_1)\prime}(a) - \frac{W_1^{(q_1)}(a)}{W_1^{(q_1)\prime}(a)} W_1^{(q_1)\prime\prime}(a) \right) \delta_a(\mathrm{d}l). \end{aligned}$$

Differentiating (5.4) in u, it follows that, for $u \in [a,b]$,

$$\begin{aligned} F'_{+/-/0}(u,A) &= -\int_{[a,u)} \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} F_{Y_{\tau_a}}^{(q_1)}(\mathrm{d}l) \frac{W_2^{(q_2)}(u-l)}{W_2^{(q_2)}(u)} F_{+/-/0}(u,A) \\ &\quad + h'_{+/-/0}(u,A;a,b) + \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} \left(F_{+/-/0}(u,A) - h_{+/-/0}(u,A;a,b)\right) \\ &= C_{q_1,q_2}(u)\, F_{+/-/0}(u,A) - p_{+/-/0}(u,A;a,b), \end{aligned} \tag{5.6}$$

where

$$C_{q_1,q_2}(u) = \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} \left(1 - \int_{[a,u)} \frac{W_2^{(q_2)}(u-l)}{W_2^{(q_2)}(u)} F_{Y_{\tau_a}}^{(q_1)}(\mathrm{d}l)\right), \tag{5.7}$$

and

$$p_{+/-/0}(u,A;a,b) = \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} h_{+/-/0}(u,A;a,b) - h'_{+/-/0}(u,A;a,b). \tag{5.8}$$

Note that (5.7) reduces to (3.7) with $s = q_1$ and $q = q_2$ when X^1 and X^2 are two Brownian motions.

For F_+, Theorem 3.1 of [9] showed that the solution to (5.6) with boundary condition $F_+(b,A) = 1_{\{b\in A\}}$ (as b is regular for (b,∞)) is given by

$$F_+(u,\mathrm{d}y) = e^{-\int_u^b C_{q_1,q_2(w)}\mathrm{d}w} \delta_b(\mathrm{d}y), \quad u \in [a,b],$$

where $\delta_b(\cdot)$ is the Dirac measure centered at b. For F_- and F_0, both measures satisfy their corresponding ODEs given in (5.6) with boundary condition $F_{-/0}(b,A) = 0$. Hence, their solution can be expressed as

$$F_{-/0}(u,\mathrm{d}y) = \int_u^b e^{-\int_u^z C_{q_1,q_2(w)}\mathrm{d}w} p_{-/0}(z,\mathrm{d}y;a,b)\mathrm{d}z, \quad u \in [a,b].$$

It remains to characterize $p_{-/0}(z,\mathrm{d}y;a,b)$. Substituting (4.6) and (5.5) into (5.8), one finds that

$$\begin{aligned} p_{-/0}(u,\mathrm{d}y;a,b) &= \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} m_{-/0}(u,\mathrm{d}y;a,b) - \frac{\mathrm{d}}{\mathrm{d}u} m_{-/0}(u,\mathrm{d}y;a,b) \\ &\quad + \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)} \int_{[a,u)} F_{Y_{\tau_a}}^{(q_1)}(\mathrm{d}l) g_{-/0}(u-l,\mathrm{d}y;0,u). \end{aligned} \tag{5.9}$$

We first tackle p_0. Noting that

$$g_0(u-l,\mathrm{d}y;0,u) = \lambda_2 \int_0^\infty e^{-q_2 t} \mathbb{P}^2_{u-l}\{t < T_u^+ \wedge T_0^-, X_t \in \mathrm{d}y\} = \lambda_2 \vartheta_2^{(q_2)}(u-l,y;0,u)\mathrm{d}y, \tag{5.10}$$

where $\vartheta_2^{(q_2)}$ is defined in (5.2), and

$$\begin{aligned} m_0(u,\mathrm{d}y;a,b) &= \mathbb{E}_u^1\left[e^{-qe_1} 1_{\{e_1<\tau_a\wedge T_0^-\wedge T_b^+, X_{e_1}\in\mathrm{d}y\}}\right] \\ &= \mathbb{E}_u^1\left[e^{-qe_1} 1_{\{e_1<T_0^-\wedge T_b^+, X_{e_1}\in\mathrm{d}y\}}\right] - \mathbb{E}_u^1\left[e^{-qe_1} 1_{\{\tau_a<e_1<T_0^-\wedge T_b^+, X_{e_1}\in\mathrm{d}y\}}\right] \\ &= \lambda_1 \vartheta_1^{(q_1)}(u,y;0,b) - \int_u^b \int_{[a,x)} f(u,\mathrm{d}l,\mathrm{d}x) \mathbb{E}_{x-l}^1\left[e^{-qe_1} 1_{\{e_1<T_0^-\wedge T_b^+, X_{e_1}\in\mathrm{d}y\}}\right] \\ &= \lambda_1 \left\{\vartheta_1^{(q_1)}(u,y;0,b) - \int_u^b \int_{[a,x)} f(u,\mathrm{d}l,\mathrm{d}x) \vartheta_1^{(q_1)}(x-l,y;0,b)\right\} \mathrm{d}y. \end{aligned} \tag{5.11}$$

Substituting (5.10) and (5.11) into (5.9) followed by some simple algebraic manipulations, we obtain

$$\begin{aligned}
p_0(u,\mathrm{d}y;a,b) &= \lambda_1\left(\frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}\vartheta_1^{(q_1)}(u,y;0,b)-\frac{\mathrm{d}}{\mathrm{d}u}\vartheta_1^{(q_1)}(u,y;0,b)\right)\mathrm{d}y\\
&\quad+\frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}\int_{[a,u)}F_{Y_{\tau_a}}^{(q_1)}(\mathrm{d}l)\left[\lambda_2\vartheta_2^{(q_2)}(u-l,y;0,u)-\lambda_1\vartheta_1^{(q_1)}(u-l,y;0,b)\right]\mathrm{d}y\\
&=\lambda_1\left(\frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}\vartheta_1^{(q_1)}(u,y;0,x)-\frac{W_1^{(q_1)\prime}(u)W_1^{(q_1)}(b-y)}{W_1^{(q_1)}(b)}+W_1^{(q_1)\prime}(u-y)\right)\mathrm{d}y\\
&\quad+\frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}\int_{[a,u)}F_{Y_{\tau_a}}^{(q_1)}(\mathrm{d}l)\left[\lambda_2\vartheta_2^{(q_2)}(u-l,y;0,u)-\lambda_1\vartheta_1^{(q_1)}(u-l,y;0,b)\right]\mathrm{d}y. \qquad (5.12)
\end{aligned}$$

Similarly for p_-, it follows from (5.3) that

$$\begin{aligned}
g_-(u-l,\mathrm{d}y;0,u) &= \int_0^\infty \vartheta_2^{(q_2)}(u-l,z;0,u)\Pi_2(-\mathrm{d}y-z)\mathrm{d}z\\
&\quad+\frac{\sigma_2^2}{2}\left(W_2^{(q_2)\prime}(u-l)-\frac{W_2^{(q_2)}(u-l)W_2^{(q_2)\prime}(u)}{W_2^{(q_2)}(u)}\right)\delta_0(\mathrm{d}y). \qquad (5.13)
\end{aligned}$$

Also,

$$m_-(u,\mathrm{d}y;a,b)=\int_u^b \mathbb{E}_u^1\left[e^{-q_1\tau_a}1_{\{Y_{\tau_a}\in x+\mathrm{d}y,M_{\tau_a}\in\mathrm{d}x\}}\right]=\int_u^b f\left(u,x+\mathrm{d}y,\mathrm{d}x\right). \qquad (5.14)$$

Substituting (5.13) and (5.14) into (5.9) leads to

$$\begin{aligned}
p_-(u,\mathrm{d}y;a,b) &= \frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}F_{Y_{\tau_a}}^{(q_1)}(u+\mathrm{d}y)\\
&\quad+\frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}\int_{[a,u)}F_{Y_{\tau_a}}^{(q_1)}(\mathrm{d}l)\int_0^\infty\vartheta_2^{(q_2)}(u-l,z;0,u)\Pi_2(-\mathrm{d}y-z)\mathrm{d}z\\
&\quad+\frac{W_1^{(q_1)\prime}(a)}{W_1^{(q_1)}(a)}\int_{[a,u)}F_{Y_{\tau_a}}^{(q_1)}(\mathrm{d}l)\frac{\sigma_2^2}{2}\left(W_2^{(q_2)\prime}(u-l)-\frac{W_2^{(q_2)}(u-l)W_2^{(q_2)\prime}(u)}{W_2^{(q_2)}(u)}\right)\delta_0(\mathrm{d}y). \qquad (5.15)
\end{aligned}$$

Finally, we shall verify that condition (4.10) is satisfied for the SNLP. We proceed by contradiction. For any $q_1>0$, $a>0$ and $0<\varepsilon<a$, if $\mathbb{E}^1[e^{-q_1\tau_a}]=1$, one must have $\mathbb{P}_x(T^-_{x+\varepsilon-a}<T^+_{x+\varepsilon})=\frac{W(a-\varepsilon)}{W(a)}=1$, which is in contradiction with the strictly increasing property of the scale function.

Now we are ready to give the full characterization of the ETU V_a in the DBRS model with spectrally negative Lévy dynamics.

Corollary 5.1 *Suppose that X^k $(k=1,2)$ are SNLPs. For $u\in[a,b]$ and $q\geq 0$, the ETU V_a is given by*

$$\begin{aligned}
V_a(u) &= U(b)\,e^{-\int_u^b C_{q_1,q_2}(w)\mathrm{d}w}+\int_{[0,\infty)}U(-y)\int_u^b e^{-\int_u^z C_{q_1,q_2}(w)\mathrm{d}w}p_-(z,\mathrm{d}y;a,b)\mathrm{d}z\\
&\quad+\int_0^b U(y)\int_u^b e^{-\int_u^z C_{q_1,q_2}(w)\mathrm{d}w}p_0(z,\mathrm{d}y;a,b)\mathrm{d}z.
\end{aligned}$$

where C_{q_1,q_2}, p_- and p_0 are as defined in (5.7), (5.15) and (5.12), respectively.

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References

[1] Asmussen, S.; Albrecher, H. Ruin Probabilities. Advanced Series in Statistical Science and Applied Probability 14, World Scientific, New Jersey, 2010, Second Edition.

[2] Avram, F.; Palmowski, Z.; Pistorius, M.R. (2007). On the optimal dividend problem for a spectrally negative Lévy process. The Annals of Applied Probability, 17(1), 156-180.

[3] Bernard, C.; Hardy, M.; MacKay, A. (2014). State-dependent fees for variable annuity guarantees. Astin Bulletin, 44(03), 559-585.

[4] Bertoin, J. Lévy processes. Cambridge University Press, Cambridge, 1996.

[5] Biffis, E.; Kyprianou, A.E. (2010). A note on scale functions and the time value of ruin for Lévy insurance risk processes. Insurance: Mathematics and Economics, 46(1), 85-91.

[6] Kuznetsov, A.; Kyprianou, A.E.; Rivero, V. The theory of scale functions for spectrally negative Lévy processes. Lévy matters II, 97–186, Lecture Notes in Math., 2061, Springer, Heidelberg, 2012.

[7] Kyprianou, A. E. Introductory lectures on fluctuations of Lévy processes with applications. Second edition. Universitext. Springer-Verlag, Heidelberg, 2014.

[8] Kyprianou, A. E.; Loeffen, R. L. (2010). Refracted Lévy processes. Ann. Inst. H. Poincaré, 46(1), 24-44.

[9] Landriault, D.; Li, B.; Li, S. (2015). Analysis of a drawdown-based regime-switching Lévy insurance model. Insurance: Mathematics and Economics, 60, 98-107.

[10] Landriault, D.; Li, B.; Zhang, H. (2015). On the frequency of drawdowns for brownian motion processes. Journal of Applied Probability, 52(01), 191-208.

[11] Landriault, D.; Li, B.; Zhang, H. (2017a). On magnitude, asymptotics and duration of drawdowns for Lévy models. Bernoulli, 23(1), 432-458.

[12] Landriault, D.; Li, B.; Zhang, H. (2017b). A unified approach for drawdown (drawup) of time-homogeneous Markov processes. Journal of Applied Probability, 54(02), 603-626.

[13] Lehoczky, J.P. (1977). Formulas for stopped diffusion processes with stopping times based on the maximum. The Annals of Probability, 5(4), 601-607.

[14] Rieder, U.; Wittlinger, M. (2014). On optimal terminal wealth problems with random trading times and drawdown constraints. Advances in Applied Probability, 46(1), 121-138.

[15] Rogers, L.C.G.; Williams, D. Diffusions, Markov processes, and Martingales: Volume 1, foundations. Second edition. Cambridge University Press, Cambridge, 2000.

[16] Schuhmacher, F.; Eling, M. (2011). Sufficient conditions for expected utility to imply drawdown-based performance rankings. Journal of Banking & Finance, 35(9), 2311-2318.

[17] Sekine, J. (2013). Long-term optimal investment with a generalized drawdown constraint. SIAM Journal on Financial Mathematics, 4(1), 452-473.

[18] Zhang, H.; Leung, T.; Hadjiliadis, O. (2013). Stochastic modeling and fair valuation of drawdown insurance. Insurance: Mathematics and Economics, 53(3), 840-850.