Classification of Finitely Generated Operator Systems

by

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Abstract

For the past few decades operator systems and their C^* −envelopes have provided an invaluable tool for studying the theory of C^* −algebras and positive maps. They provide the natural context in which to study the theory of completely positive maps (see [**?**], [15] and [29]). Furthermore, many of the important open problems in quantum information theory have found equivalent formulations in terms of operator systems (See [6] and [26]).

The question of the classification of operator systems and computing their C^* −envelopes have been the center of much interest (for example, see [2] and [1]).

Borrowing from the theory of representations of commutative C^* −algebras by affine maps, we construct a new tool for classifying certain types of finitely generated operator systems (see chapter 2). Using this tool, we show that all the information regarding such operator systems is usually encoded in the joint spectra of their generating operators.

Using this tool we completely classify operator systems generated by finitely many normal operators (see theorems 2.2.2 and 2.3.3 and remarks 2.2.8, 2.2.9 and 2.4.12). We also provide a different proof for the classification theorem of operator systems generated by a unitary with spectrum size different that 4 (see theorem 2.4.1). Furthermore, we settle the classification problem for operator systems generated by a single unitary with four points in its spectrum (see theorem 2.4.9). In addition, we compute the C^* −envelopes of such operator systems (see theorems 2.2.4, 2.3.4).

Furthermore, we apply this tool to the classification problem of those operator systems generated by a unilateral shift with arbitrary multiplicity or by an isometry and we compute their C^* −envelopes (see theorems 3.1.5, 3.2.5 and 3.3.3 and proposition 3.3.2).

Acknowledgements

Seven years ago, my knowledge in mathematics did not exceed a first-year calculus course. Since then, I have come a long way in both my knowledge and awareness of my ignorance of the depths of maths. For this, I'm indebted to my teachers throughout these years. First, I would like to thank my advisor Matthew Kennedy who masterfully and patiently guided my research and focus into problems that matter. Without him this present thesis would not have been possible. His support and mentorship have been inspiring throughout this journey. I am thankful to Benoit Collins form Kyoto University for his continuous support and many mathematical conversations. He was the first to initiate my journey into research and never seized to amaze me with the wealth of his mathematical creativity. I am indebted to Wojciech Jaworski form Carleton University who would spend hours on end teaching me during the masters program. His devotion to his students is incredible and he is responsible for teaching me most of the mathematics I'd learned before joining the doctorate program. Finally, The department of Pure Mathematics at Waterloo university has provided me with an exceptional environment in which to pursue a PhD. I have tremendously learned from the quality courses and seminars taught. I am also grateful for the positive environment it provided and for the friendships I formed. Above all, I thank my family for their continued support, sacrifice and encouragement.

Dedication

For Manal and Helena (of course).

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Chapter 1

Introduction

1.1 Main Results

Let H be a Hilbert space and denote by $\mathcal{B}(H)$ the space of bounded operators over H. A subspace of $\mathcal{B}(H)$ containing the identity and closed under the $*$-operation is called a concrete operator system or simply an operator system. Operator systems are characterized by a matrix order and an Archimedean matrix order unit. In other words, given an operator system $\mathcal{S}$, for each positive integer n, the subspace $\mathrm{M}_n(\mathcal{S})$ of $\mathcal{B}(H^{(n)})$ inherits the order of $\mathcal{B}(H^{(n)})$ and contains its Archimedean order unit, namely, the diagonal $n \times n$-matrix containing the identity along the diagonal. Furthermore, conjugation of a positive element in $\mathrm{M}_n(\mathcal{S})$ by a matrix in $\mathrm{M}_{m\times n}(\mathbb{C})$ the natural way, yields a positive matrix in $\mathrm{M}_m(\mathcal{S})$. This conjugation property, the matrix cone generated for each positive integer n and the Archimedean matrix order unit characterize an operator system (see [8]). Thus, two operator systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are unitally completely order isomorphic if and only if there exists a a unital bijective linear map ϕ from $\mathcal{S}_1$ onto $\mathcal{S}_2$; such that, an arbitrary $n \times n$ matrix $(s_{i,j})$ is positive in $\mathrm{M}_n(\mathcal{S}_1)$ if and only if $(\phi(s_{i,j}))$ is positive in $\mathrm{M}_n(\mathcal{S}_2)$; for each positive integer n.

One aspect that is not preserved under the unital complete order isomorphism is the C*-algebra generated by an operator system. Indeed, two unitally completely order isomorphic operator systems do not necessarily generate $*$-isomorphic C*-algebras (see theorem 3.1.5 for example). The smallest C*-algebras generated by an operator system $\mathcal{S}$ is called the C*-envelope of $\mathcal{S}$ and is denoted by $C_e^*(\mathcal{S})$. It was introduced by Arveson in [3]. We know that it exists and is unique within a $*$-isomorphism due to Hamana's work (see [19]).

The classification problem of operator systems is the general theme of this thesis. We deal with the classification of operator systems generated by finitely many commuting normal operators. We show that such operator systems are uniquely determined by the geometry of their joint spectra (see definition 1.3.1). Particularly, in the case of single normal operator, we show the following.

Theorem 1.1.1. *Let $N, M \in \mathcal{B}(H)$ be two normal operators. The operator systems $\mathcal{S}_N$ and $\mathcal{S}_M$, generated respectively by N and M, are unitally completely order isomorphic if and only if $\overline{\text{co}}(\sigma(N))$ and $\overline{\text{co}}(\sigma(M))$ are affinely homeomorphic.*

Using this result and some geometric tools we make various conclusions relating to the specific cases of normal operators with 4 or less points in their spectra (see remarks 2.2.9 and 2.4.12). Furthermore, we show that the C*-envelope of an operator system generated by a normal operator N coincides with $C(\partial_e\overline{\text{co}}(\sigma(N)))$. Generally, when we are dealing with multiple commuting normal operators, it turns out that the joint spectrum is the natural substitute for the concept of a spectrum of an operator. Denoting the joint spectrum of a set of finitely commuting normal operators N by $\sigma_J(N)$, we show the following theorem.

Theorem 1.1.2. *Let $N := \{N_1, \dots, N_n\}$ and $M := \{M_1, \dots, M_n\}$ be two sets of commuting normal operators. Let $\mathcal{S}_\mathcal{N}$ and $\mathcal{S}_\mathcal{M}$ denote the operator systems generated respectively by N and M. $\mathcal{S}_\mathcal{N}$ and $\mathcal{S}_\mathcal{M}$ are unitally completely order isomorphic if and only if $\overline{\text{co}}(\sigma_J(N))$ and $\overline{\text{co}}(\sigma_J(M))$ are affinely homeomorphic.*

Furthermore, we show that the C*-envelope of $\mathcal{S}_N$ is $C(\overline{\partial_e(\overline{\text{co}}(\sigma_J(N)))})$. In the first three sections of chapter 2 we deal with these operator systems and develop the geometric tools necessary for that.

In section 2.4 we use these geometric tools and provide an alternative proof for the classification theorem for operator systems generated by single unitary operators with three points or less and five points or more in their spectra (see theorem 2.4.1). Furthermore, we solve the classification problem relating to unitary operators with four points in their spectra. We use the concept of intersection coefficients (see lemma 1.2.4) to accomplish this classification.

Theorem 1.1.3. *Let $U, V \in \mathrm{B}(H)$ be two unitary operators. Assume the spectrum*

$$\sigma(U) := \{z_1, z_2, z_3, z_4\}$$

is listed in increasing order of arguments. Fix the unique $\alpha, \lambda \in (0,1)$ satisfying

$$\alpha z_1 + (1-\alpha) z_3 = \lambda z_2 + (1-\lambda) z_4$$

The following conditions are equivalent:

1. *$|\sigma(V)| = 4$ and $\{\alpha, \lambda\}$ are coefficients of intersection for the segments of lines $[w_1, w_3]$ and $[w_2, w_4]$; where w_1, w_2, w_3 and w_4 denote the points of the spectrum of V listed in increasing order of arguments (see the discussion preceding lemma 1.2.4).*

2. *$\overline{\mathrm{co}}(\sigma(U))$ is affinely homeomorphic to $\overline{\mathrm{co}}(\sigma(V))$.*

3. *$\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$.*

4. *There exists a $*$-isomorphism $\pi : C^*(U) \longrightarrow C^*(V)$ satisfying $\pi(U) = \alpha V + \beta V^* + \gamma\, 1_{\mathrm{C}^*(V)}$; where α, β and $\gamma \in \mathbb{C}$.*

Finally, in chapter 3, we address the classification problem of operator systems generated by single isometries. We show that this classification heavily depends on whether the unit circle is contained in the spectrum of the generating isometry. Using the Von Neuman - Wold decomposition allows us to breakdown this problem into simpler ones. Namely, the classification problem related to a unilateral shift with arbitrary multiplicity and the one related to the unitary. Furthermore, taking the quotient by a properly chosen ideal and doing some addition work we reduce the classification problem to only the unitary case and show the following.

Theorem 1.1.4. *Let V_1 and V_2 be two isometries. Denote by $\mathcal{S}_1$ and $\mathcal{S}_2$ the operator systems generated respectively by V_1 and V_2. One and only one of the following two cases is true:*

Case 1: $\mathbb{T} \subseteq \sigma(V_1)$. The following are equivalent:

- *$\mathcal{S}_1$ is completely order isomorphic to $\mathcal{S}_2$ via a unital complete order isometry assigning V_2 to V_1.*
- *$\mathbb{T} \subseteq \sigma(V_2)$.*

Case 2: $\sigma(V_1) \subsetneq \mathbb{T}$. Then, V_1 is a unitary and the following conditions are equivalent:

- *$\mathcal{S}_1$ is unitally completely order isomorphic to $\mathcal{S}_2$.*
- *V_2 is unitary and satisfies, together with V_1, the classification conditions for operator systems generated by single unitaries presented in section 2.4.*

Furthermore, we show that the C^*-envelope of an operator system generated by some isometry V coincides with $C(\partial_e(\overline{\text{co}}(\sigma(V))))$

1.2 Convexity and Affine Geometry

1.2.1 Algebraic and Finite Dimensional Affine Geometry

In this section we cover some elementary classic and new concepts from affine geometry. The material in this section presented starting at remark 1.2.2 and onward is new. Our focus will only be on those facts that we'll be using later on, particularly in Section 2.4.

All vector spaces are considered to be complex unless otherwise stated. At times we would need to view a complex vector space as a real vector space. Whenever this is the case we would state without confusion whether a vector space is over the complex or real scalar field. Let $\mathcal{V}$ be such a space. A subset $\mathcal{A}$ of $\mathcal{V}$ is said to be an affine space if and only if it is the translation of some real subspace of $\mathcal{V}$. Although the aforementioned definition of an affine space is not the most general one (see [35]), it is sufficient for our purposes. Clearly, if $\mathcal{W}$ is an affine space in $\mathcal{V}$ then for any vector $w \in \mathcal{W}$, the translation $\mathcal{V}_1 := -w + \mathcal{W}$ of $\mathcal{W}$ is a real vector subspace of $\mathcal{V}$. Conversely, if $-v + \mathcal{W}$ is a real vector subspace of $\mathcal{V}$ then v must be in $\mathcal{W}$. Furthermore, all translations by a vector sending $\mathcal{W}$ into a real vector subspace must send it to the same subspace. In other words, if $-v_1 + \mathcal{W}$ and $-v_2 + \mathcal{W}$ are real subspaces of $\mathcal{V}$ for some v_1 and v_2 in $\mathcal{V}$ then they must coincide. Indeed, suppose v_1 and v_2 are given as earlier. First, we note that since $0 \in (-v_1 + \mathcal{W}) \cap (-v_2 + \mathcal{W})$, we must have $v_1, v_2, v_1 - v_2$ and $v_2 - v_1$ in $(-v_1 + \mathcal{W}) \cap (-v_2 + \mathcal{W})$. Next, fix an arbitrary element $-v_1 + w$ in $-v_1 + \mathcal{W}$. Then,

$$-v_1 + w + (v_1 - v_2) \in -v_1 + \mathcal{W}.$$

On the other hand,

$$-v_1 + w + (v_1 - v_2) = w - v_2 \in -v_2 + \mathcal{W}.$$

However, $v_2 - v_1$ belongs to $-v_2 + \mathcal{W}$. Thus, putting the last two equalities together, we find that $-v_1 + w$ belongs to $-v_2 + \mathcal{W}$ as well. Then,

$$-v + 1 + \mathcal{W} \subseteq -v_2 + \mathcal{W}.$$

A symmetric argument shows $-v_2 + \mathcal{W} \subseteq -v_1 + \mathcal{W}$.

A linear combination, $\Sigma_{i=1}^{n} \alpha_i w_i$, of finitely many vectors $w_1, \ldots, w_n$ in $\mathcal{V}$ is said to be an affine combination whenever the coefficients $\alpha_1, \ldots, \alpha_n$ are all real and add up to 1. An affine combination whose scalar coefficients are all non-negative real numbers is called a convex combination.

Note that a non-empty subset $\mathcal{W}$ of the vector space $\mathcal{V}$ is an affine space if and only if it is closed under arbitrary affine combinations. Indeed, consider the affine combination $u := \Sigma_{i=1}^{n} \alpha_i w_i$ of vectors in the affine subspace $\mathcal{W}$. Then, for some vector v in $\mathcal{W}$, the translation $\mathcal{X} := -v + \mathcal{W}$ is a vector subspace. Thus, in order to show that $u \in \mathcal{W}$, it is enough to establish that $-v+u$ is in the subspace $\mathcal{X}$.

$$\begin{aligned} -v+u &= -1v + \Sigma_{i=1}^n \alpha_i w_i \\ &= (\Sigma_{i=1}^n \alpha_i)(-v) + \Sigma_{i=1}^n \alpha_i w_i \\ &= \Sigma_{i=1}^n \alpha_i(-v+w_i) \\ &\in \mathcal{X} \end{aligned}$$

Conversely, suppose $\emptyset \neq \mathcal{W} \subseteq \mathcal{V}$ is closed under finite affine combinations. It would be enough to show that $\mathcal{W}$ is the translation of some real vector subspace. Fix an arbitrary vector v in $\mathcal{W}$ and set $\mathcal{X} := -v + \mathcal{W}$. Then, the zero vector belongs to $\mathcal{X}$. We show that $\mathcal{X}$ is a real vector subspace of $\mathcal{V}$. For arbitrary coefficient α and vector $w \in \mathcal{W}$ we have

$$\begin{aligned} \alpha(-v+w) &= -v + ((1-\alpha)v + \alpha w) \\ &\in -v + \mathcal{W} = \mathcal{X}. \end{aligned}$$

It remains to show $\mathcal{X}$ is closed under addition. For arbitrary $w_1, w_2 \in \mathcal{W}$, we have

$$\begin{aligned} (-v+w_1) + (-v+w_2) &= 2(-v + (\frac{1}{2}w_1 + \frac{1}{2}w_2)) \\ &\in 2(-v+\mathcal{W}) = 2\mathcal{X} = \mathcal{X}. \end{aligned}$$

Consider the affine space $\mathcal{W}$ in $\mathcal{V}$ and some subset $\mathcal{W}_1$ of $\mathcal{W}$. We call $\mathcal{W}_1$ an affine subspace of $\mathcal{W}$ if and only it is the translation of some real vector subspace $\mathcal{V}_1$ of $\mathcal{V}$. Equivalently, if it is non-empty and closed under finite affine combinations. Note that if $\mathcal{W}_1$ is an affine subspace of $\mathcal{W}$ and $\mathcal{W}$ is the translation of some vector space $\mathcal{V}'$. Then, if $\mathcal{W}_1$ is the translation of some subspace $\mathcal{V}_1$ then $\mathcal{V}_1$ must be vector subspace of $\mathcal{V}'$.

The dimension of an affine subspace is defined to be that of its translation vector space.

A non-empty subset X of an affine space $\mathcal{W} \subseteq \mathcal{V}$ is said to be affinely dependent if and only if some vector of X can be expressed as an affine combination of finitely many other vectors in X. Otherwise, X is said to be affinely independent. Fix a vector v in X and consider the real vector subspace $-v + \mathcal{W}$. Then, the set X is affinely independent if and only if $-v + X$ is linearly independent in the real subspace $-v + \mathcal{W}$.

Indeed, let X be affinely dependent. Then, some vector v_0 in X can be expressed as the affine combination $\Sigma_{i=1}^{n}\alpha_i v_i$ of some n other vectors $v_1, \ldots, v_n$ in X. Without loss of generality assume that none of the α_i's are zero. Next, recall that translating $\mathcal{W}$ by the negation of any of its vectors sends $\mathcal{W}$ to a unique subspace $-v + \mathcal{W}$. In particular, for each of the i's, the vector $v_i - v_0$ belongs to $-v + \mathcal{W}$. Then, we obtain the following real linear dependency in $-v + \mathcal{W}$:

$$\begin{aligned}\Sigma_{i=1}^{n}\alpha_i(v_i - v_0) &= (\Sigma_{i=1}^{n}\alpha_i v_i) - (\Sigma_{i=1}^{n}\alpha_i)v_0 \\ &= (\Sigma_{i=1}^{n}\alpha_i v_i) - v_0 \\ &= v_0 - v_0 = 0\end{aligned}$$

Conversely, suppose that for some v in X we have $-v + X$ is real-linearly dependent. Then, for some vectors $v_1, \ldots, v_n$ in X and real scalars $\alpha_1, \ldots, \alpha_n$ that are not all zero we have $\Sigma_{i=1}^{n}\alpha_i(v_i - v) = 0$. If $\Sigma_{i=1}^{n}\alpha_i \neq 0$ then we obtain the affine dependence $v = \Sigma_{i=1}^{n}\frac{\alpha_i}{\Sigma_{i=1}^{n}\alpha_i}v_i$. On the other hand, if $\Sigma_{i=1}^{n}\alpha_i = 0$ then for some $\alpha_k \neq 0$ we have

$$\alpha_k = \Sigma_{i\in\{1,\ldots,n\}\setminus\{k\}} - \alpha_i$$

Thus, we obtain the affine dependence

$$v_k = \Sigma_{i\in\{1,\ldots,n\}\setminus\{k\}}\frac{-\alpha_i}{\alpha_k}v_i.$$

Let X and $\mathcal{V}$ be as earlier. Then, X is said to be convex if and only if it is closed under arbitrary finite convex combinations or, equivalently, if and only if the convex combination of any two vectors in X is still in X. An element of convex set X is extreme if and only if it can not be expressed as the convex combination of two distinct elements of X. A convex subset F of a convex set X is said to be a face of X if and only if whenever the convex combination of two vectors x_1 and x_2 of X belongs to F then both x_1 and x_2 must be in F.

The convex hull of a subset X of some affine space $\mathcal{W}$, denoted $\text{co}(X)$ is the smallest convex subset of $\mathcal{V}$ containing X, i.e., the intersection of all convex subsets of $\mathcal{V}$ containing X. Equivalently, $\text{co}(X)$ is the set composed of all convex combinations of vectors in X.

If the vector space $\mathcal{V}$ is Euclidean (i.e. isomorphic to $\mathbb{R}^n$ for some integer n) with origin O then we denote a vector $\overrightarrow{OP}$ in $\mathcal{V}$ by capital letter P and call it point P. Then, the convex hull of a set containing two points $\{P, Q\}$ is called the line segment with extreme points P and Q and is denoted by $[PQ]$. The length of $[PQ]$, the Euclidean distance from P to Q, is denoted by $|PQ|$.

The affine space generated by X is the smallest affine space containing X. Equivalently, it is the set of finite arbitrary affine combinations of vectors in X.

If $\mathcal{W}$ and $\mathcal{U}$ are affine spaces and $E \subseteq \mathcal{W}$ is convex. Then, a map $\phi : E \longrightarrow \mathcal{U}$ is said to be an affine map if and only if it preserves convex combinations . In other words, whenever $\lambda_1, \lambda_2 \in [0, 1]$, $v_1, v_2 \in E$ and $\lambda_1 + \lambda_2 = 1$ we have:

$$\phi(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \phi(v_1) + \lambda_2 \phi(v_2).$$

Note that the range of an affine map is always convex. Furthermore, the image of an extreme point of E is extreme in $\phi(E)$. Conversely, the pre-image of an extreme point of $\phi(E)$ is a face in E. Given such a map ϕ, denote by $\mathcal{V}_E$ the affine space generated by E. We can always extend ϕ to a unique map $\widetilde{\phi}$ over $\mathcal{V}_E$ that respects affine combinations. In order to accomplish such an extension, let v be some arbitrary vector in $\mathcal{V}_E$. Express v as the affine combination of some finitely many vectors in E, $v = \Sigma_{i=1}^n \alpha_i v_i$. Define $\widetilde{\phi}(v) := \Sigma_{i=1}^n \alpha_i \phi(v_i)$.

We show that $\widetilde{\phi}$ is well-defined. Suppose there is another affine combination of vectors in E, namely $\Sigma_{i=n+1}^m \alpha_i v_i$, that equals v. It is sufficient to prove that

$$\Sigma_{i=1}^n \alpha_i \phi(v_i) = \Sigma_{i=n+1}^m \alpha_i \phi(v_i). \tag{1.1}$$

Consider the equality

$$\Sigma_{i=1}^n \alpha_i v_i = \Sigma_{i=n+1}^m \alpha_i v_i \tag{1.2}$$

and note that the sums

$$\Sigma_{i=1}^n \alpha_i = \Sigma_{i=n+1}^m \alpha_i = 1 \tag{1.3}$$

Rearrange the terms of equation 1.3 so that only non-negative α_i's appear on each of its sides. Denote by m the sum of the numbers on each side. Next, rearranging the terms of equation 1.2 in the same manner yields an equation where only non-negative real scalars appear. Furthermore, by our preceding observation, dividing by m both sides of this new equation yields two equal convex combinations of vectors in E. Apply the affine map ϕ to both sides of the equality of convex combinations. Factoring ϕ through the convex combinations on each side yields a new equality of convex combinations of vectors in $\phi(E)$. Finally, simplifying m and rearranging the terms of the new equality back to the original arrangement that was presented in equation 1.2 yields equality 1.1.

We denote by $\mathrm{A}(\mathcal{V}, \mathcal{W})$ the set of all affine maps from the vector space $\mathcal{V}$ to $\mathcal{W}$.

If $\phi \in \mathrm{A}(\mathcal{V}, \mathcal{W})$ then there exists some vector $w \in \mathcal{W}$ and a real-linear map $T \in \mathcal{L}_{\mathbb{R}}(\mathcal{V}, \mathcal{W})$, where $\phi(v) = T(v) + w$ for any v in $\mathcal{V}$. Let w denote the vector $\phi(0)$. Define the map T assigning to each vector in v in $\mathcal{V}$ the vector $\phi(v) - w$. The function T is clearly well-defined and assigns the zero vector of $\mathcal{V}$ to that of $\mathcal{W}$. Furthermore, for an arbitrary vector v in $\mathcal{V}$ and non-zero real number r we have:

$$\begin{aligned} T(rv) &= \phi(rv) - w = \phi(rv + (1-r)0) - w = r\phi(v) + (1-r)\phi(0) - w \\ &= r\phi(v) + w - rw - w = r(\phi(v) - w) \\ &= rT(v). \end{aligned}$$

Finally, for any two vectors v and z in $\mathcal{W}$, we see that

$$\begin{aligned} T(v+z) &= 2\left(T\left(\frac{v+z}{2}\right)\right) = 2\left(\phi\left(\frac{v+z}{2}\right) - w\right) = 2\left(\frac{1}{2}\phi(v) + \frac{1}{2}\phi(z) - w\right) \\ &= \phi(v) + \phi(z) - 2w \\ &= T(v) + T(z). \end{aligned}$$

Conversely, If $T \in \mathcal{L}_{\mathbb{R}}(\mathcal{V}, \mathcal{W})$ and w is some vector in $\mathcal{W}$ then the map ϕ assigning to each vector in $\mathcal{V}$ the vector $T(v) + w$ is clearly and affine map.

Next, we present an assortment of classical theorems that we would use later on. The first of these theorems is usually referred to as Radon Theorem on Convex Sets and essentially states that any subset of $n+2$ distinct points of $\mathbb{R}^n$ can be partitioned into two disjoint sets whose convex hulls intersect at at least one point (see [12]). When $n = 2$, Radon Theorem asserts that any four distinct points in the Euclidean plane can be connected by line segments in such a way that yields two intersecting segments of line. The second is a necessary condition for four distinct concyclic (belonging to the same circle) points to be mapped onto four distinct concyclic points by an affine homeomorphism.

Theorem 1.2.1. *Let $E \subset \mathbb{R}^n$ be a finite set consisting of at least $n+2$ points. Then, there exist two disjoint subsets $F, G \subset E$ satisfying:*

1. $E = F \cup G$
2. $\text{co}(F) \cap \text{co}(G) \neq \emptyset$.

Proof. Let $E := \{v_1, \ldots, v_m\}$ and assume without loss of generality that $m = n+2$. Fix real numbers $x_1, \ldots, x_{n+2}$ satisfying $\sum_{i=1}^{m} x_i v_i = 0$ and $\sum_{i=1}^{m} x_i = 0$; such that not all the x_i's are zero. Indeed, this is possible to do since it is equivalent to solving a homogeneous system of $n+1$ equations in $n+2$ unknowns. Namely,

$$x_1 v_1 + \cdots + x_m v_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let M be the index subset of $1, \ldots, n+2$ corresponding to the x_i's that are non-negative and set $N := \{1, \ldots, n+2\} \setminus M$. Thus, we obtain the following equality which possesses only non-negative scalars:

$$\sum_{j \in M} x_i v_i = \sum_{i \in N} (-x_i) v_i$$

Setting $k = \sum_{i \in M} x_i$ we obtain a point $P := \sum_{i \in M} \frac{x_i}{k} v_i = \sum_{i \in N} \frac{-x_i}{k} v_i$ belonging to the intersection of the convex hulls of $F = \{v_i, i \in M\}$ and $G = \{v_i, i \in N\}$. $\square$

Remark 1.2.2. *Referring to the preceding theorem, we can show that the partition is unique in the case of a set of four concyclic points in $\mathbb{R}^2$. Indeed, suppose the coordinates of the points are $X_i := (x_i, y_i); i = 1, 2, 3, 4$. Consider again the homogeneous system of* *linear equations we had in the proof of 1.2.1. In our particular case, Its corresponding coefficient matrix is:*

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Clearly, any three column vectors in the above matrix are linearly independent. Thus, its rank is 3 *and its row reduction has the form:*

$$\begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & p \\ 0 & 0 & 1 & q \end{bmatrix},$$

where l, p, q *are some real numbers. Then the system reduces to:*

$$(-l\alpha_4)X_1 + (-p\alpha_4)X_2 + (-q\alpha_4)X_3 + \alpha_4 X_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Knowing that α_4 *is not null, we simplify it from the preceding vector equation. Rearranging the terms of this equation as in the proof of theorem 1.2.1, we obtain the unique partition we are after.*

Note that we do not need the four distinct points in question to be concyclic. The remark would still hold if we assume that no three points in the given set are collinear.

Theorem 1.2.3. *Let* A, B, C *and* D *be four distinct points in the Euclidean plane. By Radon Theorem of convex sets we may assume without loss of generality that* $[AC]$ *and* $[BD]$ *intersect through some point* I*. The points* A, B, C *and* D *are concyclic if and only if* $|A||IC| = |IB||ID|$*. Furthermore, if* T *is an affine map mapping* A, B, C *and* D *respectively onto* A_1, B_1, C_1 *and* D_1 *then* A_1, B_1, C_1 *and* D_1 *are concyclic if and only*

$$\frac{|AC|}{|BD|} = \frac{|A_1C_1|}{|B_1D_1|}$$

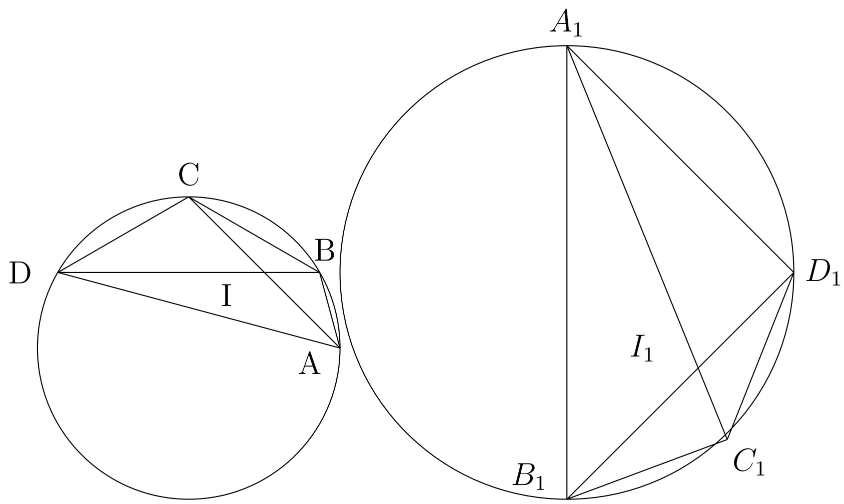

Figure 1.1: Mapping concyclic points onto concyclic points.

Proof. Refer to figure 1.1 and suppose that points A, B, C and D are concyclic.

Note that angles $\angle ADB$ and $\angle ACB$ intercept the same circle arc; thus, they have equal measures. Likewise, the measures of angles $\angle DAC$ and $\angle DBC$ are equal. Therefore, the triangles IAD and IBC are similar. The similarity ratio of proportion yields the result. Conversely, suppose that $|IA||IC| = |ID||IB|$. Re-arranging the terms in the previous equality yields $\dfrac{|IA|}{|ID|} = \dfrac{|IB|}{|IC|}$. Furthermore, observe that being opposite angles, the angles $\angle AID$ and $\angle BIC$ have equal measures. Therefore, by the (SAS) characterization of similar triangle we conclude that triangle IAD is similar to triangle IBD. Thus, angles $\angle ADB$ and $\angle ACB$ have equal measures. Furthermore, since $[CA] \cap [DB] \neq \phi$, point B must belong to the circle circumscribed about ADC.

Next, we move to proving the second assertion of the theorem. Denote by I_1 the image of I under T. Suppose

$$\begin{aligned}
\overrightarrow{OI} &:= \lambda \overrightarrow{OA} + (1-\lambda)\overrightarrow{OC}.\\
\overrightarrow{OI} &:= \alpha \overrightarrow{OB} + (1-\alpha)\overrightarrow{OD};\\
&\lambda, \alpha \in (0,1).
\end{aligned}$$

Since A, B, C and D are concyclic, using the previous part of this theorem we obtain:

$$\begin{aligned} |IA||IC| &= (1-\lambda)\lambda|AC|^2 \\ &= |IB||ID| \\ &= (1-\alpha)\alpha|BD|^2, \end{aligned}$$

which yields

$$\frac{|AC|^2}{|BD|^2} = \frac{(1-\alpha)\alpha}{(1-\lambda)\lambda}.$$

Since T is affine then I_1 is the intersection of $[A_1C_1]$ and $[B_1D_1]$. Furthermore, we have

$$\overrightarrow{OI_1} := \lambda\overrightarrow{OA_1} + (1-\lambda)\overrightarrow{OC_1}.$$
$$\overrightarrow{OI_1} := \alpha\overrightarrow{OB_1} + (1-\alpha)\overrightarrow{OD_1}.$$

Suppose A_1, B_1, C_1 and D_1 are concyclic. Then, similar reasoning to the preceding one shows that

$$\frac{|A_1C_1|^2}{|B_1D_1|^2} = \frac{(1-\alpha)\alpha}{(1-\lambda)\lambda},$$

which yields our result.

Conversely, suppose that

$$\frac{|AC|}{|BD|} = \frac{|A_1C_1|}{|B_1D_1|}.$$

In order to show that A_1, B_1, C_1 and D_1 are concyclic it is enough to show that $[I_1A_1][I_1C_1] = [I_1B_1][I_1D_1]$. Note that $[I_1A_1][I_1C_1] = (1-\lambda)\lambda[A_1C_1]^2$ and $[I_1B_1][I_1D_1] = (1-\alpha)\alpha[B_1D_1]^2$. Thus,

$$\begin{aligned} \frac{|I_1A_1||I_1C_1|}{|I_1B_1||I_1D_1|} &= \frac{(1-\lambda)\lambda[A_1C_1]^2}{(1-\alpha)\alpha[B_1D_1]^2} \\ &= \frac{(1-\lambda)\lambda}{(1-\alpha)\alpha}\frac{|AC|^2}{|BD|^2} \\ &= \frac{(1-\lambda)\lambda}{(1-\alpha)\alpha}\frac{(1-\alpha)\alpha}{(1-\lambda)\lambda} = 1. \end{aligned}$$

□

The context of the preceding theorem is that the four distinct concyclic points A, B, C and D are mapped via an affine homeomorphism respectively onto the four distinct concyclic A_1, B_1, C_1 and D_1. However, the theorem does not tell us when such a map exists. For the remainder of this section, we will provide necessary and sufficient conditions for the existence of such an affine homeomorphism. The context we are interested in is points on the unit circle. To this end, let $\{A, B, C, D\}$ and $\{A_1, B_1, C_1, D_1\}$ be two sets, each containing four distinct points of the unit circle. By Radon theorem we may assume without loss of generality that

$$[AC] \cap [BD] = \{I\}$$
$$[A_1C_1] \cap [B_1D_1] = \{I_1\}.$$

Consider the unique coefficients $\alpha, \lambda \in (0, 1)$ satisfying

$$\alpha A + (1-\alpha)C = \lambda B + (1-\lambda)D.$$

We shall call two positive numbers s and t coefficients of intersection for $[AC]$ and $[BD]$ if and only if

$$s \in \{\alpha, 1-\alpha, \lambda, 1-\lambda\}$$
$$\text{and}$$
$$t \in \{\alpha, 1-\alpha, \lambda, 1-\lambda\} \setminus \{s, 1-s\}.$$

Lemma 1.2.4. *Let $\{A, B, C, D\}$, $\{A_1, B_1, C_1, D_1\}$, I, I_1, λ and α be given as in the preceding discussion. There exists an affine homeomorphism T mapping the set $\{A, B, C, D\}$ onto $\{A_1, B_1, C_1, D_1\}$ if and only if $\{\alpha, \lambda\}$ are intersection coefficients of $[A_1C_1]$ and $[B_1D_1]$.*

Proof. Assume $T(\{A, B, C, D\}) = \{A_1, B_1, C_1, D_1\}$. Note that T has to map I onto I_1. Indeed,

$$\begin{aligned} T(I) &= T(\alpha A + (1-\alpha)C) \\ &= \alpha T(A) + (1-\alpha)T(C) \end{aligned}$$

and

$$T(I) = T(\lambda B + (1-\lambda)D)$$
$$= \lambda T(B) + (1-\lambda)T(D)$$

where $\alpha, \lambda \in (0,1)$. Then, $T(I) = [T(A)T(C)] \cap [T(B)T(D)]$. In addition, By remark 1.2.2, the partition of the the set $\{T(A), T(B), T(C), T(D)\} = \{A_1, B_1, C_1, D_1\}$ leading to two intersecting segments of lines is unique. Thus, the points $T(I)$ and I_1 coincide. Furthermore, we obtain

$$T(\{A, C\}) = \{A_1, C_1\}$$
$$T(\{B, D\}) = \{B_1, D_1\}$$

or

$$T(\{A, C\}) = \{B_1, D_1\}$$
$$T(\{B, D\}) = \{A_1, C_1\}.$$

In either of these cases we have our result.

Conversely, assume that $\{\alpha, \lambda\}$ are coefficients of intersection for $[A_1C_1]$ and $[B_1D_1]$. By our definition of coefficients of intersection, this hypothesis could mean that α and $1-\alpha$ are the coefficients expressing I_1 as a convex combination of A_1 and C_1 (taken in arbitrary order) and λ and $1-\lambda$ are the coefficients expressing I_1 as a convex combination of B_1 and D_1 (taken in arbitrary order). It could also mean that λ and $1-\lambda$ are the coefficients expressing I_1 as a convex combination of A_1 and C_1 (taken in arbitrary order) and α and $1-\alpha$ are the coefficients expressing I_1 as a convex combination of B_1 and D_1 (taken in arbitrary order). However, we restrict ourselves to the case when

$$\alpha A_1 + (1-\alpha)C_1 = \lambda B_1 + (1-\lambda)D_1.$$

The other cases are treated in an analogous fashion. Let T be the unique affine homeomorphism mapping A, B and D respectively onto A_1, B_1 and D_1. Finding such a T is possible. Indeed, since both sets of points are distinct and concyclic we have that each of the sets $\{\overrightarrow{AB}, \overrightarrow{AD}\}$ and $\{\overrightarrow{A_1B_1}, \overrightarrow{A_1D_1}\}$ is linearly independent. Thus there exists a unique invertible matrix M satisfying

$$M(\overrightarrow{AB}) = \overrightarrow{A_1B_1}$$
$$M(\overrightarrow{AD}) = \overrightarrow{A_1D_1}$$

Then, defining T as

$$T(X) = M(\overrightarrow{AX}) + A_1$$

for each X in $\mathbb{R}^2$ would do. Next, note that

$$\begin{aligned} T(I) &= T(\lambda B + (1-\lambda)D) \\ &= \lambda T(B) + (1-\lambda)T(D) \\ &= \lambda B_1 + (1-\lambda)D_1 \\ &= I_1. \end{aligned}$$

However,

$$I_1 = \alpha A_1 + (1-\alpha)C_1.$$

On the other hand, we have $T(A) = A_1$. Thus,

$$I_1 = \alpha T(A) + (1-\alpha)C_1.$$

Alternatively,

$$\begin{aligned} I_1 = T(I) &= T(\alpha A + (1-\alpha)C) \\ &= \alpha T(A) + (1-\alpha)T(C). \end{aligned}$$

Thus, we obtain

$$\alpha T(A) + (1-\alpha)T(C) = \alpha T(A) + (1-\alpha)C_1$$

which yields $T(C) = C_1$ □

Remark 1.2.5. *Note that we did not use the facts that A, B, C, D and A_1, B_1, C_1, D_1 are respectively concyclic. Indeed, the lemma would still apply if we simply assume that no three points of $\{A, B, C, D\}$ and no three points of $\{A_1, B_1, C_1, D_1\}$ are concyclic.*

1.2.2 Representation by Affine Maps

We specialize our discussion in this section to affine subspaces of a certain type of topological vector spaces; namely, locally convex Hausdorff spaces. Recall that a topological vector space is a vector space that is endowed with a topology which makes the binary operations of vector addition and scalar multiplication continuous. A locally convex space is a topological space whose topology is Hausdorff and induced by a family of semi-norms. In this context, we would be interested only in continuous affine maps. Thus, if $\mathcal{X}$ is a locally convex space and $K, M \subset \mathcal{X}$ are convex sets then $\mathrm{A}(K, M)$ denotes the continuous affine maps from K to M. Furthermore, when M is equal to $\mathbb{C}$ then we shall simply write $\mathrm{A}(K)$ to denote the set of continuous complex-valued affine maps; sometimes called affine functionals. On the other hand, when $M = \mathbb{R}$ we write $\mathrm{A}_{\mathbb{R}}(K)$. We will freely use many classical and known results about topological and locally convex spaces and their geometry without providing proofs. Among others, such results include the separation theorems and the Krein-Milman Theorem. We refer the reader to two classic books on the subject; namely, [10] and [32]. However, we will be providing the proofs of some results when we specialize the conversation to the geometry of measure spaces (see [17]).

Let K be a compact convex subset of a locally convex Hausdorff space $\mathcal{X}$. Note that $\mathrm{C}_{\mathbb{R}}(K)$ and $\mathrm{C}(K)$ are respectively a Banach algebra and a C^*-algebra under the $\|.\|_\infty$-norm and the pointwise-conjugation involution. It is easy to see that $\mathrm{A}(K)$ and $\mathrm{A}_{\mathbb{R}}(K)$ are respectively an ordered Banach space with an Archimedean order unit and a closed operator system (see Section 1.4.7).

Our aim is to present some essential important results on representations by affine functionals. The following results are classic and could be found in [17], [9] and [16]. To this end, let us begin by setting the stage. Let X be a compact Hausdorff space. Endow X with the Borel $\sigma-$algebra. Denote by $\mathrm{M}(X)$ the Banach space of complex Borel measures on X endowed with the total variation norm and denote by $\mathrm{C}(X)$ the C^*-algebra of continuous complex valued functions on X. Note that all such measures are regular. The Riesz Representation Theorem tells us that $\mathrm{M}(X)$ is isometrically isomorphic to the dual $\mathrm{C}(X)^*$, via the map ρ

$$\mathrm{M}(X) \ni \mu \longrightarrow \phi_\mu \in \mathrm{C}(X)^*.$$

Where ϕ_μ is the bounded functional assigning to each f in $\mathrm{C}(X)$ the number $\displaystyle\int_X f \,\mathrm{d}\,\mu$. In addition to ρ being an isometric isomorphism from $\mathrm{M}(X)$ onto $\mathrm{C}(X)^*$ it is also an isometric isomorphism from the Banach space of real-valued finite measures onto $\mathrm{C}_{\mathbb{R}}(X)^*$. On the other hand, $\mathrm{C}(X)^*$ is also endowed with the weak* topology making its unit ball $\mathrm{B}_1(\mathrm{C}(X)^*) = \{\phi \in \mathrm{C}(X)^*; \|\phi\| \leq 1\}$ and its set of states $\mathrm{S}(\mathrm{C}(X)) = \{\phi \in \mathrm{C}(X)^*; \|\phi\| = 1, \phi(1_{\mathrm{C}(X)}) = 1, \phi \geq 0\}$ compact. Note that in this context, having $\phi \geq 0$ is equivalent to ϕ assigning to each non-negative valued function over X a non-negative number. We will show in subsequent sections that the definition of a state we provided is equivalent to bounded functional being unital and having norm 1. Putting all these elementary results together, we see that endowing $\mathrm{M}(X)$ with the weak$-*$ topology inherited from $\mathrm{C}(X)^*$ makes the unit ball $\mathrm{M}_1(X)$ of $\mathrm{M}(X)$ and the set of probability measures $M_1^+(X)$ compact and respectively affinely homeomorphic to $\mathrm{B}_1(\mathrm{C}(X)^*)$ and $\mathrm{S}(\mathrm{C}(X))$. Thus, the extreme points of $\mathrm{M}_1^+(X)$ correspond exactly to the extreme states in $\mathrm{S}(\mathrm{C}(X))$. Such extreme states are called pure states.

Furthermore, it is noteworthy that $\mathrm{S}(\mathrm{C}(X))$ and $\mathrm{S}(\mathrm{C}_{\mathbb{R}}(X))$ coincide in the following sense. Each state ψ on $\mathrm{C}_{\mathbb{R}}(X)$ extends uniquely to a state $\widetilde{\psi}$ on $\mathrm{C}(X)$ and the following is a homeomorphism:

$$\iota : \mathrm{S}(\mathrm{C}(X)) \ni \psi \longrightarrow \psi|_{\mathrm{C}_{\mathbb{R}}(X)} \in \mathrm{S}(\mathrm{C}_{\mathbb{R}}(X)).$$

The fact that ι is a well-defined $\mathrm{wk}^* \,/\, \mathrm{wk}^*$ continuous map is clear. If we show that ι is injective and surjective then the wk^* compactness of $\mathrm{S}(\mathrm{C}(X))$ would imply the fact that ι is homeomorphic. That ι is injective and surjective is clear from the following two observations.

Let ψ_1 and ψ_2 be two states that agree on $\mathrm{C}_{\mathbb{R}}(X)$ and f arbitrary in $\mathrm{C}(X)$. Writing f in terms of its real and imaginary parts we obtain

$$\begin{aligned} \psi_1(f) &= \psi_1(f_r) + \mathrm{i}\,\psi_1(f_i) \\ &= \psi_2(f_r) + \mathrm{i}\,\psi_2(f_i) \\ &= \psi_2(f). \end{aligned}$$

Conversely, let ψ be a state on $\mathrm{C}_{\mathbb{R}}(X)$. Given an arbitrary f in $\mathrm{C}(X)$, express f uniquely in terms of its real and imaginary parts

$$f = f_r + \mathrm{i}\, f_c; f_r, f_c \in \mathrm{C}_{\mathbb{R}}(X).$$

Set

$$\widetilde{\psi}(f) = \phi(f_r) + \mathrm{i}\, \phi(f_c), \forall f \in \mathrm{C}(X).$$

It is clear that $\widetilde{\psi}$ is well-defined, unital positive and bounded. That $\widetilde{\psi}$ is unique is clear from the earlier comment on injectivity.

Consider $\mathrm{M}(X)$ with the weak$-*$ topology inherited from $\mathrm{C}(X)^*$ and consider the map assigning to each x in X the Dirac measure ϵ_x in $\mathrm{M}(X)$. Note that ϵ_x is a positive measure of total variation 1; thus, by the Riesz Representation Theorem ϵ_x is identified with a unique state in $\mathrm{C}(X)^*$, which we also denote by ϵ_x, assigning to each f in $\mathrm{C}(X)$ the number $\int_X f\, \mathrm{d}\, \epsilon_x = f(x)$. Now let's see how these objects relate to one another. First, note that the ϵ_x's are the extreme points of $\mathrm{M}^+(X)$ and the map

$$X \ni x \longrightarrow \epsilon_x \in \partial_e \, \mathrm{M}_1^+(X)$$

is a homeomorphism, where $\mathrm{M}(X)$ is endowed with the weak$-*$ topology inherited from $\mathrm{C}(X)^*$. In order to see that the map is continuous consider an arbitrary net x_λ in X converging to some x. Then, for each f in $\mathrm{C}(X)$ the net $f(x_\lambda) = \epsilon_{x_\lambda}(f)$ converges to $f(x) = \epsilon_x(f)$. Thus, ϵ_{x_λ} converges in the weak$-*$ topology to ϵ_x. The injectivity of the map is clear since $\mathrm{C}(X)$ separates the points of X. Thus, by the compactness of X we conclude that the map is homeomorphic onto its range.

It remains to show that this range is precisely the extreme points of the set of probability measures $\mathrm{M}_1^+(X)$.

Consider the Dirac measures ϵ_x at some x in X and suppose it could be expressed as the convex combination of two Borel probability measures μ_1 and μ_2 over X. Then, for every Borel subset E of the Borel set $X \setminus \{x\}$ and some non-negative α we have

$$0 = \alpha\mu_1(E) + (1 - \alpha)\mu_2(E).$$

However, both μ_1 and μ_2 are non-negative; thus, $\mu_1(E) = \mu_2(E) = 0$. Since both μ_1 and μ_2 are probability measures we conclude that

$$\mu_1(\{x\}) = \mu_2(\{x\}) = 1.$$

As a conclusion, μ_1 and μ_2 coincide with ϵ_x. Therefore, ϵ_x is extreme in $M_1^+(X)$.

Finally, we show that any extreme point of $\mathrm{M}_1^+(X)$ must be a Dirac measure. Let μ be an extreme probability measure. Then, μ may assume only one of the values 1 or 0 over any Borel subset of X. Indeed, if this were not true then there exists a proper Borel subset E of X satisfying $\mu(E) \in (0,1)$. Now consider the two probability measures $\mu_1 := \frac{1}{\mu(E)}\mu|_E$ and $\mu_2 := \frac{1}{1-\mu(E)}\mu|_{X\setminus E}$. Then, μ is obtained as the convex combination

$$\mu = \mu(E)\mu_1 + (1-\mu(E))\mu_2,$$

a contradiction. Next, Let E and F be two Borel subsets of X that have a non-zero measure under μ. Then, since μ is a probability measure and assumes only one of the two values 0 or 1, we have $\mu(E \cup F) = 1$. Thus, $\mu(E \cap F) = 1$. Then, the family of Borel sets with non-zero measure under μ is closed under finite intersection. In particular, The family of compact subsets of X that have a non-zero measure is closed under finite intersection. Thus, the intersection of all such compact sets, which we denote by K, is non-empty and we have $\mu(K) = 1$. Since, μ is a probability measure, it must be supported by K. Finally, assume that K is not a singleton and let $k \in K$. Since $\{k\}$ is compact then the minimality of K implies $\mu(\{k\}) = 0$ and $\mu(K \setminus \{k\}) = 1$. Since μ is regular, there exists a compact subset K' of $K \setminus \{k\}$ with a non-zero measure, a contradiction.

We shall make use of the following classical result which is due to Kadison (see [24]).

Theorem 1.2.6. *Let $\mathcal{X}$ be a locally convex space and $K \subseteq \mathcal{X}$ a compact convex subset. Define the map $\tau : K \longrightarrow \mathrm{S}(\mathrm{A}(K))$ assigning to each $x \in K$ the state ϵ_x; where, $\epsilon_x(f) := f(x)$ for each $f \in \mathrm{A}(K)$. Then, τ is an affine homeomorphism when its image is given the* wk*-*topology.*

Proof. Recall from our discussion in the beginning of this section that the map

$$\rho : K \ni k \longrightarrow \epsilon_k \in \partial_e \mathrm{M}_1^+(K) = \partial_e \mathrm{S}(\mathrm{C}(K))$$

is a K/wk^* −homeomorphism. Consider the restriction map R assigning to each ψ in $\mathrm{S}(\mathrm{C}(K))$ its restriction to $\mathrm{A}(K)$. Clearly, R is continuous and so is $R \circ \rho$. Note that $R \circ \rho$ is the map described in the statement of the theorem. We show that $R \circ \rho$ is bijective. Thus, by the compactness of K we conclude that $R \circ \rho$ is a homeomorphism. That $R \circ \rho$ is affine is clear from the convexity of both K and $\mathrm{S}(\mathrm{A}(K))$ and by computing the action of $R \circ \rho$ on convex combinations of elements of K. Thus, it remains to show that $R \circ \rho$ is bijective. Let $k_1 \neq k_2$ be two vectors in K and denote by ϵ_{k_1} and ϵ_{k_2} the extreme states over $\mathrm{C}(X)$ respectively corresponding to k_1 and k_2. Note that the continuous affine functionals separate k. Thus, there exists $f \in \mathrm{A}(K)$ satisfying

$$f(k_1) = \epsilon_{k_1}(f) \neq f(k_2) = \epsilon_{k_2}(f)$$

making $R \circ \rho$ injective. In order to see why $\mathrm{A}(K)$ separate K note that an application of the Hahn-Banach theorem yields a bounded linear functional ϕ over $\mathcal{X}$ satisfying $\phi(k_1) \neq \phi(k_2)$. The restriction of ϕ to K yields the desired separating affine map in $\mathrm{A}(K)$.

Finally, we show that $R \circ \rho$ is surjective. Let ψ be a pure state over $\mathrm{A}(K)$. Note that R is surjective (refer to the discussion on extending positive functionals in subsection 1.4.9). Since R is an affine continuous map, the pre-image $R^{-1}(\psi)$ is a closed face of the compact convex set $\mathrm{S}(\mathrm{C}(K))$. Thus, $R^{-1}(\psi)$ contains an extreme point of $\mathrm{S}(\mathrm{C}(K))$ which we denote by ψ'. However, ρ is a homeomorphism onto $\partial_e \mathrm{S}(\mathrm{C}(K))$. Thus, for some $k \in K$, we have $\rho(k) = \psi'$. Thus, $R \circ \rho$ maps k to ψ. Therefore, its range contains the extreme points of $\mathrm{S}(\mathrm{A}(K))$. Furthermore, since $R \circ \rho$ is continuous affine and its domain is compact convex, it must be a homeomorphism and its range must also be compact convex and by the Krein-Milman Theorem it must also be equal to $\mathrm{S}(\mathrm{A}(K))$. □

1.3 Joint Spectrum of Commuting Operators

Our aim in this section is to review the basic properties of the joint spectrum and functional calculus related to a set of commuting normal elements of a C*-algebra. We begin with a quick reminder of some properties of Banach and C*-algebras and their ideal spaces. For more details on this topic the reader is referred to [5], [27] and [25]. Let $\mathcal{B}$ be a Banach algebra. Recall that a left (right) ideal of a Banach algebra is said to be modular if and only if it has a right(left) modular unit. An element e is said to be a right (left) modular unit for left (right) ideal $\mathcal{I}$ of Banach algebra $\mathcal{B}$ if $be - b$ belongs to $\mathcal{I}$ whenever b is arbitrary in $\mathcal{B}$. Furthermore, every left (right) proper modular ideal is contained in a maximal proper left (right) modular ideal. Every such maximal ideal must be closed. We omit the use of left and right when the ideal and modular units at hand are two-sided. On the other hand, a non-zero representation ϕ of $\mathcal{B}$ over the complex numbers is necessarily continuous. Indeed, the quotient $\mathcal{B}/\ker(\phi)$ is homomorphic to the one-dimensional field $\mathbb{C}$; thus, making that quotient closed. Furthermore, since $\mathcal{B}$ is also closed, $\ker(\phi)$ must be closed as well. Thus, ϕ is continuous. Thus, every kernel of a complex non-trivial representation ϕ is a modular ideal of $\mathcal{B}$, with the modular unit being any pre-image of 1. Moreover, when the Banach algebra is commutative, the set of non-trivial complex representations over $\mathcal{B}$, denoted by $\Phi_{\mathcal{B}}$, is in one-to-one correspondence with the set of maximal modular ideals of $\mathcal{B}$. On the other hand, we know that for a Banach algebra $\mathcal{B}$, $\Phi_{\mathcal{B}}$ is a locally compact Hausdorff space under the wk* −topology; furthermore, it is compact when $\mathcal{B}$ is unital. In the case when $\mathcal{B}$ is commutative and unital the Gelfand Transform

$$\begin{aligned} \Gamma :&\mathcal{B} \longrightarrow \mathrm{C}(\Phi_{\mathcal{B}}) \\ &b \longrightarrow \Gamma_b ; \Gamma_b(\phi) = \phi(b) \textit{ for each } \phi \in \Phi_{\mathcal{B}} \end{aligned}$$

is a unital contractive homomorphism.

Let b be a fixed arbitrary element of the unital Banach algebra $\mathcal{B}$. Recall that the spectrum of b, denoted by $\sigma(b)$, is the set of all complex numbers λ making $b - \lambda 1_{\mathcal{B}}$ non-invertible. It is clear that

$$\sigma(b) \supseteq \{\phi(b); \phi \in \Phi_{\mathcal{B}}\}.$$

When, $\mathcal{B}$ is commutative and $\lambda \in \sigma(b)$, there exists a ϕ in $\Phi_{\mathcal{B}}$ whose kernel is the maximal ideal containing the ideal generated by $b - \lambda 1_{\mathcal{B}}$ and satisfying $\phi(b) = \lambda$.

Next, we shift the conversation to C*-algebras. Let $\mathcal{C}$ be a commutative unital C*-algebra and $c \in \mathcal{C}$. The elements of $\Phi_{\mathcal{C}}$ are in one-to-one correspondence with the set of maximal proper ideals of $\mathcal{C}$. Recall that all closed ideals in C*-algebras are necessarily

$*$-ideals; hence, our omission of any reference to the $*$-conjugation operation is justified. Furthermore, the Gelfand Transform

$$\Gamma : \mathcal{C} \longrightarrow \mathrm{C}(\Phi_{\mathcal{C}})$$
$$c \longrightarrow \Gamma_c; \Gamma_c(\phi) = \phi(c) \textit{ for each } \phi \in \Phi_{\mathcal{C}}$$

is a $*$-isomorphism. On the other hand, as in the case of Banach algebras, the spectrum of c, $\sigma(c)$ coincides with the set $\{\phi(c), \phi \in \Phi_{\mathcal{C}}\}$. Furthermore, when $\mathcal{C}$ coincides with the unital C*-algebra generated by c then $\sigma(c)$ is homeomorphic to $\Phi_{\mathcal{C}}$. This homeomorphism induces a $*$-isomorphism in the natural way of $\mathrm{C}(\sigma(c))$ onto $\mathrm{C}(\Phi_{\mathcal{C}})$. Thus, we obtain the functional calculus which states that $\mathcal{C}$ is $*$-isomorphic to $\mathrm{C}(\sigma(c))$ via a unital C^*-algebra isomorphism assigning c to the function $f(x) = x$ and $1_{\mathcal{C}}$ to the constant function 1.

It is noteworthy to mention that when $\mathcal{B}$ is not necessarily commutative, $\mathcal{A}$ is a C*-subalgebra of $\mathcal{B}$ and b is in $\mathcal{A}$ then $\sigma_{\mathcal{A}}(b)$ and $\sigma_{\mathcal{B}}(b)$ coincide.

Finally, recall that when c is unitary then $\sigma(c) \subseteq \mathbb{T}$ and when it is self-adjoint then $\sigma(c) \subseteq \mathbb{R}$. Furthermore, by the GNS construction every C*-algebra is $*$-isomorphic to a C*-subalgebra of $\mathcal{B}(H)$ where H is a Hilbert space.

Now, consider the case when $\mathcal{C}$ is the C*-algebra generated by a finite set of commuting normal elements $N := \{N_1, \dots, N_n\} \subset \mathcal{B}$. We will use the concept of joint spectrum to obtain a representation of $\mathcal{C}$ similar to the functional calculus that we obtain when $n = 1$. We begin by defining the concept of joint-spectrum of $N_1, \dots, N_n$, denoted by $\sigma_J(N)$. Then, we explore the properties of $\sigma_J(N)$ and its relationship to the individual spectra of each N_i. Finally, we establish the promised representation. For more information on this topic see [36] and [20].

Definition 1.3.1. *Let $N := \{N_1, \dots, N_n\} \in \mathcal{B}$ be a set of commuting normal operators in some commutative unital* C**-algebra $\mathcal{C}$. The joint spectrum of $N_1, \dots, N_n$, denoted by $\sigma_J(N)$ or $\sigma_J(N_1, \dots, N_n)$, is the subset of all $n-$tuples $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that the closed $*$-ideal $\langle N_1 - \lambda_1 1_{\mathcal{B}}, \dots, N_n - \lambda_n 1_{\mathcal{B}} \rangle$ is proper in $\mathcal{C}$.*

Lemma 1.3.2. *Let $N := \{N_1, \dots, N_n\} \in \mathcal{B}$ be as in definition 1.3.1 and denote by $\mathcal{C}$ the unital* C**-algebra generated by N. Then,*

$$\sigma_J(N) := \{(\phi(N_1), \dots, \phi(N_n)); \phi \in \Phi_{\mathcal{C}}\}.$$

Proof. Let $\underline{\lambda} := (\lambda_1, \ldots, \lambda_n)$ be arbitrary in $\sigma_J(N)$. Then, the closed $*$-ideal $\mathcal{I}$ generated by $\{N_1 - \lambda_1, \ldots, N_n - \lambda_n\}$ is proper in $\mathcal{C}$. Thus, it is contained in a maximal proper ideal $\overline{\mathcal{I}}$ of $\mathcal{C}$ and therefore it is the kernel of some ϕ in $\Phi_{\mathcal{C}}$. Hence, $\phi(N_i - \lambda_i 1) = 0$ for each integer i between 1 and n. Thus, the one way inclusion $\subseteq$ is established.

Conversely, suppose that $\underline{\lambda} := (\phi(N_1), \ldots, \phi(N_n)); \phi \in \Phi_{\mathcal{C}}$ for some ϕ in $\Phi_{\mathcal{C}}$. Then, for each integer i between 1 and n, the element $N_i - \phi(N_i)\, 1_{\mathcal{C}}$ belongs to the kernel of ϕ. But the kernel of ϕ is a proper closed maximal $*$-ideal in $\mathcal{C}$. Thus, the ideal generated by $\{N_1 - \lambda_1, \ldots, N_n - \lambda_n\}$ is proper and we have the reverse inclusion. □

Lemma 1.3.3. *Let N be as in definition 1.3.1 and for each integer k, $1 \leq k \leq n$, denote by $\pi_k : \mathbb{C}^n :\longrightarrow \mathbb{C}$ the projection assigning to each $n-$tuple its k_{th} component. Then, we have $\pi_k(\sigma_J(N)) = \sigma(N_k)$. Furthermore, $\sigma_J(N)$ is compact.*

Proof. Fix an arbitrary integer k between 1 and n and let $\mathcal{C}$ denote the unital C*-algebra generated by N.

Let $\underline{\alpha} := (\alpha_1, \ldots, \alpha_n)$ be arbitrary in $\sigma_J(N)$. Then, by lemma 1.3.2, there exists ϕ in $\Phi_{\mathcal{C}}$ satisfying $\underline{\alpha} := (\phi(N_1), \ldots, \phi(N_n))$. However, recall that $\sigma(N_k) = \{\psi(N_k); \psi \in \Phi_{\mathcal{C}}\}$. Then,

$$\pi_k(\underline{\alpha}) = \alpha_k = \psi(N_k) \in \sigma(N_k).$$

Conversely, if α is in $\sigma(N_k)$ then it corresponds to $\phi(N_k)$ for some ϕ in $\Phi_{\mathcal{C}}$. Hence, by Lemma 1.3.2, we have that $\underline{\alpha} = (\phi(N_1), \ldots, \phi(N_k))$ belongs to $\sigma_J(N)$ and α coincides with $\pi_k(\underline{\alpha})$.

That σ_J is compact is clear from the fact that whenever the sequence

$$\underline{\alpha}^{(k)} := \{(\alpha_1^{(k)}, \ldots, \alpha_n^{(k)})\}_{k\in\mathbb{N}}$$

is in $\sigma_J(N)$, then it must have a converging subsequence. Indeed, by the previous proven claim we know that $\{\pi_1(\underline{\alpha}^{(k)})\}_{k\in\mathbb{N}} = \{\alpha_1^{(k)}\}_{k\in\mathbb{N}}$ is in $\sigma(N_1)$ and so must have a converging subsequence. Passing to this converging subsequence we may assume without loss of generality that $\pi_1(\underline{\alpha})$ converges. Applying the same previous reasoning consecutively to $\pi_2(\underline{\alpha}^{(k)})$ through $\pi_n(\underline{\alpha}^{(k)})$ we see that $\{\underline{\alpha}^{(k)}\}_k$ has a converging subsequence. □

Lemma 1.3.4. *Let N be as in definition 1.3.1. Denote by $\mathcal{A}$ the commutative unital C^*-subalgebra generated by N. The ideal space $\Phi_\mathcal{A}$ of $\mathcal{A}$ is homeomorphic to $\sigma_J(N)$ via the homeomorphism:*

$$\tau : \Phi_\mathcal{A} \ni \phi \longrightarrow \tau(\phi) \in \sigma_J(N) \subset \mathbb{C}^n$$
$$\tau(\phi) := (\phi(N_1), \ldots, \phi(N_n))$$

Proof. That τ is well-defined surjective is clear from lemma 1.3.2. In order to see why it is injective, consider $\phi, \psi \in \Phi_\mathcal{A}$ satisfying $\tau(\phi) = \tau(\psi)$. By hypothesis, ϕ and ψ coincide over $\{N_1, \ldots, N_n\}$; thus, by the multiplicativity and additivity of ϕ and ψ, they must coincide over the algebra generated by N. Being multiplicative functions over a C^*-algebra, ϕ and ψ are self-adjoint and thus coincide over the $*$-algebra generated by N. Finally, by continuity, ϕ and ψ coincide over $\mathcal{A}$. Thus, τ is injective.

Finally, let $\{\phi_k\}_{k\in\mathbb{N}}$ be a net in $\Phi_\mathcal{A}$ converging in the wk^*-topology. Then, clearly $\{\tau(\phi_k)\}_{k\in\mathbb{N}} = \{(\phi_k(N_1), \ldots, \phi(N_n))\}_{k\in\mathbb{N}}$ converges in $\mathbb{C}^n$. Thus, τ is continuous and since its range is compact it must be homeomorphic. □

Theorem 1.3.5. *Let N, $\mathcal{A}$ and τ be as in Lemma 1.3.4. Denote by Γ the $*$-isomorphic Gelfand Transform of $\mathcal{A}$ onto $\mathrm{C}(\Phi_\mathcal{A})$. Denote by π_τ the $*$-isomorphism from $\mathrm{C}(\sigma_J(N))$ onto $\mathrm{C}(\Phi_\mathcal{A})$ induced by τ (see 1.3.4). Then, $\mathcal{A}$ is $*$-isomorphic to $\mathrm{C}(\sigma_J(N))$ via the unital $*$-isomorphism $\Gamma^{-1} \circ \pi_\tau$. Furthermore, if we denote by p_k the projection of σ_J onto the $k-th$ coordinate then the image of p_k under $\Gamma^{-1} \circ \pi_\tau$ is N_k for each integer k between 1 and n.*

Proof. Recall that the $*$-isomorphism π_τ induced by τ is simply the map

$$\pi_\tau : \mathrm{C}(\sigma_J) \longrightarrow \mathrm{C}(\Phi_\mathcal{A}),$$

where for each f in $\mathrm{C}(\sigma_J)$ the function $\pi_\tau(f)$ assigns to each ϕ in $\Phi_\mathcal{A}$ the value

$$\pi_\tau(f)(\phi) := f(\tau(\phi))$$

That π_τ is a unital bijective $*$-homomorphism is straightforward from the fact that τ is homeomorphic; thus, it is a $*-$isomorphism.

Given that the Gelfand Transform is a $*$-isomorphism settles the second claim of the Lemma. It remains to show that every p_k is mapped to N_k. First, note that $\pi_\tau(p_k)$ assigns to each ϕ in $\Phi_{\mathcal{A}}$ the value $\phi(N_k)$. Indeed,

$$\pi_\tau(p_k)(\phi) = p_k(\phi(N_1), \ldots, \phi(N_n)) = \phi(N_k).$$

However, $\phi(N_k)$ is nothing but $\Gamma_{N_k}(\phi)$; thus we are done. □

1.4 Operator Systems and Positive Maps

In this section we cover the necessary background related to operator systems and positive maps. The results presented in this section are fundamental and classic in the theory of completely and $k-$positive maps and we shall outline the results that are pertinent to this thesis. Operator systems provide a suitable framework for the theory of completely positive maps. The early study of this theory can be found in a landmark paper by Arveson (see [3]). Later the theory was developed through the work of Choi and Effros who provided the concept of an operator system with an abstract characterization (see [8]). This was done through the concepts of matrix cones and amplification of maps. The theory was developed further later through the works of Paulsen, Pisier, Blecher and others. For a more general treatise on this subject the reader is referred to books [29], [15] and [34]. Paulsen's book ([29]) is particularly accessible.

The first subsection covers the concept of an ordered vector space. In the second, we extend the order concept to complex vector spaces thus introducing the concept of an ordered $*$-vector space. The third subsection introduces positive maps and some of their properties. Next, we explore the relationship between the norm and positivity of maps over a special type of ordered $*$-vector spaces (i.e. the concrete operator systems). The following subsection covers the concepts of matrix cone, map amplification and $k-$positive maps. Then, we discuss some essential properties of completely positive and completely bounded maps. The last two subsections are devoted to two classical concepts. The first is the abstract operator system characterization of operator systems provided by Choi and Effros and the second is the concept of the C*-envelope of an operator system. We see that operator systems are characterized by their matrix cones and not merely the cone induced by their containing C*-algebras.

1.4.1 Ordered Vector Spaces

Definition 1.4.1. *Let $\mathcal{V}$ be a real vector space. $\mathcal{V}$ is said to be ordered if it is endowed with a partial anti-symmetric order $\leq$ such that, for any $v, w, z \in \mathcal{V}$, the following propositions hold true:*

1. *$rv \leq rw$ whenever $v \leq w$ and $r \geq 0$.*
2. *$v + z \leq w + z$ whenever $v \leq w$.*

Note that the anti-symmetric property in the aforementioned definition is not necessary. Indeed, many of the results we list below would still apply without this property. However, the specific spaces we will be dealing with do not allow for non-zero elements that are positive and negative at the same time. Thus, it becomes necessary to include the antisymmetric property. We denote by $(\mathcal{V}, \leq)$ the ordered vector space. Furthermore, suppose $\mathcal{V}$ contains an element, that we denote by $1_{\mathcal{V}}$, such that for any $v \in \mathcal{V}$ there exists $r \in \mathbb{R}$ satisfying $v \leq r\, 1_{\mathcal{V}}$. Then, the element $1_{\mathcal{V}} \in \mathcal{V}$ is called an order unit for $\mathcal{V}$ and if no confusion arises then we denote it by simply 1. The order unit 1 is said to be Archimedean if for every $v \in \mathcal{V}$, whenever the statement $v + r\,1 \geq 0$ for all $r > 0$ is true then we must have $v \geq 0$.

Note that it follows directly from definition 1.4.1 that $v \leq w \iff -w \leq -v$ for all $v, w \in \mathcal{V}$. Furthermore, an element 1 is an order unit for $(\mathcal{V}, \leq)$ if and only if for any $v \in \mathcal{V}$ there exists $r > 0$ satisfying $-r\,1 \leq v \leq r\,1$. Finally, an Archimedean order unit 1 is not unique. In fact, any multiple of 1 would be an Archimedean order unit.

The set of all non-negative vectors in $\mathcal{V}$ is referred to as the cone of positive elements or simply the cone of $\mathcal{V}$. The cone of positive elements is clearly closed under addition and non-negative scalar multiplication.

Conversely, given a real vector space $\mathcal{V}$ and a subset $\mathcal{C} \subset \mathcal{V}$. Then, $\mathcal{C}$ is said to be a cone if and only if

1. $\mathcal{C} \cap -\mathcal{C} = \{0\}$.

2. $\mathcal{C}$ is closed under addition.

3. $\mathcal{C}$ is closed under multiplication by non-negative real number.

A cone $\mathcal{C}$ in a real vector space $\mathcal{V}$ induces an order making $\mathcal{V}$ and ordered vector space. Indeed, for arbitrary $v, w \in \mathcal{V}$, define $v \leq w$ to mean $w - v \in \mathcal{C}$. It is easy to see that $\leq$ is an anti-symmetric partial order.

Example 1.4.2. *1. Let $\mathbb{R}^n$ be the real Euclidean vector space. Consider the relation $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ whenever, $x_i \leq y_i$ for all $1 \leq i \leq n$. Then, $(\mathbb{R}^n, \leq)$ is an ordered vector space via the coordinate-wise partial order. Furthermore, $(1, \ldots, 1)$ is an Archimedean order unit and the set $\{x \in \mathbb{R}^n; v(i) \geq 0, \forall i, 1 \leq i \leq n\}$ is its cone of positive elements.*

2. The space of all Hermitian $n \times n$ matrices forms an ordered real vector space. The cone of $\mathrm{M}_n(\mathbb{C})_h$ is the set of all positive semidefinite matrices. Recall, $A \in \mathrm{M}_n(\mathbb{C})$ is said to be positive semidefinite if and only if $\langle Av, v\rangle \geq 0$ for all $v \in \mathbb{C}^n$).

3. Let X be a locally compact Hausdorff space and let $\mathrm{C}_{\mathbb{R}}(X)$ the space of all real-valued continuous functions over X. Let $\leq$ denote the pointwise partial order on $C_{\mathbb{R}}(X)$. Then, $(C_{\mathbb{R}}(X), \leq)$ is an ordered vector space with Archimedean order unit being the function assigning to every element $x \in X$ the real number 1.

4. Let K be a compact convex subset of a locally convex Hausdorff space. Denote by $\mathrm{A}_{\mathbb{R}}(K)$ the real subspace of $C_{\mathbb{R}}(K)$ containing all the real affine maps. Note that $\mathrm{A}_{\mathbb{R}}(K)$ is $\|.\|_\infty-$closed ordered vector space with Archimedean order unit and order inherited from $(C_{\mathbb{R}}(K), \leq)$.

1.4.2 Ordered $*$-Vector Space

Next, we extend the concept of order to complex vector spaces. To this end, consider a complex vector space $\mathcal{V}$. Suppose that we have an additive map assigning to each vector v

in $\mathcal{V}$ some vector that we denote by v^* and satisfying $(\alpha v)^* = \overline{\alpha} v^*$ for each α in $\mathbb{C}$ and v in $\mathcal{V}$. Such a map is referred to as an involution map over $\mathcal{V}$ and a vector space equipped with an involution map is called a $*$-vector space.

An element $v \in \mathcal{V}$ is called Hermitian or self-adjoint whenever $v^* = v$. Denote by $\mathcal{V}_h$ the subset of $\mathcal{V}$ containing all Hermitian elements. Viewing $\mathcal{V}$ as a real vector space then one clearly sees that $\mathcal{V}_h$ is a real vector subspace of $\mathcal{V}$. Furthermore, when the real space $\mathcal{V}_h$ is ordered, $\mathcal{V}$ is said to be an ordered $*$-vector space.

Note that $\mathcal{V}_h$ coincides with the set $\{v + v^*, v \in \mathcal{V}\}$. Furthermore, every element $v \in \mathcal{V}$ can be expressed as linear combination of elements in $\mathcal{V}_h$. Indeed, let $v \in \mathcal{V}$ and write $v = \dfrac{v + v^*}{2} + i\dfrac{v - v^*}{2i}$.

Furthermore, assume that the real vector space $\mathcal{V}_h$ is partially ordered with order unit 1 and let $\mathcal{V}_h^+$ be the cone of positive elements corresponding to this order. Then, every element $v_h \in \mathcal{V}_h$ can be written as the difference of two elements in the positive cone. i.e. $v_h = \dfrac{v_h + r\,1}{2} - \dfrac{r\,1 - v_h}{2}$ for some $r > 0$. Thus, $\mathcal{V}_h^+$ spans $\mathcal{V}$.

Example 1.4.3. *1. Let X be a compact Hausdorff space and consider the space of continuous complex functions over X, $\mathrm{C}_{\mathbb{C}}(X)$. Defining the adjoint function as $f \longrightarrow f^* := \overline{f}$ for each $f \in \mathrm{C}_{\mathbb{C}}(X)$, we see that $\mathrm{C}_{\mathbb{C}}(X)$ is a $*$-vector space and its subspace $\mathrm{C}_{\mathbb{C}}(X)_h$ of Hermitian elements coincides with $\mathrm{C}_{\mathbb{R}}(X)$.*

2. *Let K be a convex compact set. Denote by $\mathrm{A}_{\mathbb{C}}(K)$, or simply $\mathrm{A}(K)$ when there is no confusion, the space of complex valued affine maps over K. Then, $\mathrm{A}(K)$ is a $*$-vector space. The ordered vector space described in part 4 of example 1.4.2 is the real subspace of $\mathrm{A}_{\mathbb{C}}(K)$ consisting of the Hermitian elements.*

3. *The real ordered vector subspaces described in parts 1 and 2 of example 1.4.2 are the real subspaces consisting of the Hermitian elements of respectively the ordered $*$-vector space $\mathbb{C}^n$ and $\mathrm{M}_n(\mathbb{C})$.*

4. *Let $\mathcal{A}$ be a unital C^*-algebra. Recall that an element $a \in \mathcal{A}$ is said to be positive if and only if $a = b^*b$ for some $b \in \mathcal{A}$ or alternatively a is self-adjoint, i.e. $a = a^*$, and the spectrum of a, $\sigma(a)$, is a subset of $\mathbb{R}_{\geq 0}$. Furthermore, using the GNS construction*

we may assume that $\mathcal{A}$ is embedded in $\mathrm{B}(H)$ for some Hilbert space H. Then, an element a in $\mathcal{A}$ is positive if and only if $\langle a\eta, \eta\rangle$ is non-negative for every η in H (see [10, Chapter 8, Section 3]). The set of all such positive elements in $\mathcal{A}$ is a cone that induces a partial order over the self-adjoint elements of $\mathcal{A}$.

Furthermore, we assert that the identity of $\mathcal{A}$, $1_{\mathcal{A}}$, is an Archimedean order unit for this order.

5. *Let $\mathcal{S}$ be a subspace of a unital C^*-algebra $\mathcal{A}$ that contains the unit and is closed under involution. $\mathcal{S}_h := \mathcal{S} \cap \mathcal{S}^*$ is a real subspace of $\mathcal{A}_h$ and inherits the partial order of $\mathcal{A}_h$. Thus, $\mathcal{S}$ is a partially ordered $*$-vector space with Archimedean order unit $1_{\mathcal{A}}$.*

Subspaces of unital C*-algebras which contains the unit and are closed under the involution operation (see the last part of the previous example) are referred to as concrete operator systems. Later, we will define the notion of abstract operator system and shall see that all abstract operator systems arise as concrete operator systems in some C*-algebra.

1.4.3 Positive Maps

Next, we move to the study of positive maps and how positivity and norm of a certain map relate to one another.

Definition 1.4.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be two partially ordered $*$-vector spaces with order units and $\phi : \mathcal{V} \longrightarrow \mathcal{W}$ a linear map. ϕ is said to be positive if it maps every positive element of $\mathcal{V}$ to a positive element of $\mathcal{W}$.*

Let ϕ be a positive map such as the one in 1.4.4. Note that a positive map must be self adjoint, i.e. $\phi(s^*) = \phi(s)^*$ for each s in $\mathcal{V}$. Then, ϕ maps Hermitian elements of $\mathcal{V}$ to Hermitian elements in $\mathcal{W}$. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are concrete operator systems. Then, both spaces inherit the C* −algebra norms. In such a context, a positive map ϕ must be continuous. Indeed $\|\phi\| \leq 2\|\phi(1_{\mathcal{V}})\|$. In order to see this, consider such a positive map ϕ.

For an arbitrary Hermitian operator h, we have $1\|h\| - h \geq 0$. Thus, $\phi(h) \leq \|h\|\phi(1)$. Thus, since $\phi(h)$ is Hermitian and $\phi(1)$ is positive, we we obtain

$$\|\phi(h)\| \leq \|h\|\|\phi(1)\|.$$

On the other hand, If $v \in \mathcal{V}$ is arbitrary in $\mathcal{V}$, then we express it as a linear combination of Hermitian operators as follows:

$$v := \frac{v + v^*}{2} - \mathrm{i}\, \frac{-v + v^*}{2\,\mathrm{i}}$$

Except for certain specific cases, the upper bound of $2\|\phi(1)\|$ given for the operator norm of an arbitrary positive map ϕ acting on a concrete operator system is the best we can do. Indeed, The following example which is due to Arveson (See [29, Chapter 2]) asserts this claim.

Let $\mathcal{S}$ denote the operator system generated by the polynomials $z, \overline{z}, 1$ in $\mathrm{C}(\mathbb{T})$. Let ϕ denote the map assigning the matrix $\begin{bmatrix} \gamma & 2\alpha \\ 2\beta & \gamma \end{bmatrix}$ in M_2 to the polynomial $\alpha z + \beta\overline{z} + \gamma$. Then, ϕ is positive. Indeed, let $q := \alpha z + \beta\overline{z} + \gamma$ be a polynomial in $\mathcal{S}$. Note that q is self-adjoint if and only if $\gamma \in \mathbb{R}$ and $\alpha = \overline{\beta}$. Furthermore, q is positive if and only if $\gamma \geq 2|\alpha|$. Then, when p is positive we obtain

$$\phi(p) = \begin{bmatrix} \gamma & 2\alpha \\ 2\alpha^* & \gamma \end{bmatrix}; \gamma \geq 2|\alpha|.$$

The above matrix is clearly self-adjoint and has a positive determinant; thus, it is positive.

On the other hand, we have

$$\|\phi(z)\| = 2 = 2\|\phi(1)\|$$

There are various factors impacting the norm of a positive map acting on a concrete operator system. The algebraic and analytical structures of the domain and range are some of these factors. Another factor that impacts this norm is whether the map is $2-$positive or completely positive. Conversely, given some map acting on concrete operator systems, the norm of such a map impacts its positivity. We shall summarize the results relating to these issues in the following sections but first let us look at some more important examples of positive maps. When we introduce the concepts of map amplification, $k-$positivity and $k-$boundedness we will revisit these positive maps and study their amplifications.

Example 1.4.5. *1. Consider the normalized trace map* $\frac{1}{n}\operatorname{Tr}$ *over* M_n. Tr *is a unital positive map. Indeed, let* $A \in \mathrm{M}_n^+$. *Let* $\lambda_1(A), \ldots, \lambda_n(A)$ *denote the non-decreasing list of the eigenvalues of* A *according to their algebraic multiplicities. Recall that that the trace of a matrix coincides with the sum of its eigenvalues. Thus,*

$$\frac{1}{n}\operatorname{Tr}(A) = \frac{\Sigma_{i=1}^n \lambda_i}{n} \geq 0.$$

2. *Let* H *be a Hilbert space and fix a vectors* $v \in H$. *Consider the complex valued map* ϕ_v *assigning to each bounded operator* $T \in \mathcal{B}(H)$ *the number* $\langle T(v), v\rangle$. *The map* ϕ_v *is positive and* $\phi(1_H) = \|v\|^2$. *Normalizing* v *turns* ϕ *into a unital map.*

3. *Let* X *be a compact Hausdorff space and* $Y \neq \phi$ *a compact subset of* Y. *The restriction map,* R, *assigning to each continuous function over* X *its restriction to* Y *is a unital positive map form* $\mathrm{C}(X)$ *to* $\mathrm{C}(Y)$. *In addition, by an application of Uryshon's Lemma,* R *is surjective. Furthermore, for fixed* $x \in X$ *the valuation map* ϕ_x *assigning to each function in* $C(X)$ *its value at* x, *is unital positive.*

4. *Consider the map* $\phi : \mathrm{M}_n \longrightarrow \mathrm{M}_n$, *due to Choi (see [7]), defined as follows:*

$$\forall M \in M_n; \phi(M) = (n-1)\operatorname{Tr}(M)\, 1_n - M, \tag{1.4}$$

where 1_n *is the unit of* M_n. *It is clear that* ϕ *is positive. In order to see why, note that when* M *is positive, its eigenvalues are all non-negative and its norm coincides with its largest eigenvalue. Furthermore,* $\operatorname{Tr}(M)$ *coincides with the sum of the eigenvalues of* M. *Then, we have* $M \leq 1_n\|M\| \leq 1_n \operatorname{Tr}(M)(n-1)$.

5. *Consider the transposition map* T *assigning to each matrix* $M \in \mathrm{M}_n$ *its transpose* M^t. T *is positive.*

1.4.4 Relating the Norm and Positivity of Maps

Consider the two concrete operator systems $\mathcal{V}$ and $\mathcal{W}$ and a map $\phi : \mathcal{V} \longrightarrow \mathcal{W}$. Let us look at the factors impacting the norm and positivity of ϕ. First, let us assume that ϕ is positive.

If ϕ is complex valued then its norm is equal to that of $\phi(\mathrm{id}_{\mathcal{V}})$. Indeed, in order to see this, recall that we have have already established in the discussion on the norm of a positive map in subsection 1.4.3 that for arbitrary hermitian element h in $\mathcal{V}$, the norm of $\phi(h)$ is dominated by $\|h\||\phi(\mathrm{id}_{\mathcal{V}})|$. Next, consider an arbitrary element v in $\mathcal{V}$ and let θ be the real number satisfying $\mathrm{e}^{\imath\theta}\,\phi(v) = |\phi(v)|$ and set $\lambda := \mathrm{e}^{\imath\theta}$ and let h denote the real part of λv, i.e., $\lambda v = h + \imath f$; where,

$$h = \frac{\lambda v + (\lambda v)^*}{2}$$
$$f = \frac{\lambda v - (\lambda v)^*}{2\imath}$$

Being positive, ϕ maps hermitian elements to real numbers. Then, we obtain the following computation:

$$\begin{aligned} |\phi(v)| &= \phi(\mathrm{e}^{\imath\theta}\,v) = |\phi(h)| \\ &\leq \|h\||\phi(\mathrm{id}_{\mathcal{V}})| \\ &\leq \|v\||\phi(\mathrm{id}_{\mathcal{V}})| \end{aligned}$$

Finally, since the norm of the unit element in $\mathcal{V}$ is 1 we obtain $\|\phi\| = |\phi(\mathrm{id}_{\mathcal{V}})|$.

More generally, if the range of ϕ is included in a commutative C*-algebra $\mathcal{A}$ then the norm of ϕ equals to that of $\phi(\mathrm{id}_{\mathcal{V}})$. Indeed, in order to see that, note that by the Gelfand transform $\mathcal{A}$ is $*-$isomorphic to $\mathrm{C}(X)$ for some compact Hausdorff set X. Then, fix an arbitrary x in X and consider the positive map ϕ_x assigning to each v in V the element $\phi(v)(x)$. By our previous result, we know that

$$|\phi_x(v)| \leq \|v\||\phi(\mathrm{id}_{\mathcal{V}})(x)|$$

Taking the supremum over all x in X, we obtain

$$\begin{aligned} \|\phi(v)\| &= \sup_{x\in X} |\phi_x(v)| \\ &\leq \sup_{x\in X} \|v\||\phi(\mathrm{id}_{\mathcal{V}})(x)| \\ &\leq \|v\|\|\phi(\mathrm{id}_{\mathcal{V}})\|. \end{aligned}$$

On the other hand, assume that the domain of ϕ is a unital commutative C^* −algebra then the norm of ϕ is also the same as that of $\phi(\mathrm{id}_{\mathcal{V}})$. One way to prove this is by using a standard partition of unity argument (see [29, Theorem 2.4]). Using this result we show that if the domain of ϕ is a unital $*$-subalgebra of some C^* −algebra then the norm of ϕ equals that of $\phi(\mathrm{id}_{\mathcal{V}})$. In order to see this, fix a contractive element v in the $*$−algebra $\mathcal{V}$ and consider the positive map ψ mapping every polynomial $p(z) + \overline{q(z)}$ in $\mathrm{C}(\mathbb{T})$ to the element $p(v) + \overline{q(v)}$ in $\mathcal{V}$. Then, the map $\phi_v := \phi \circ \psi$ in positive and by the Weierstrass approximation theorem it extends to a positive map $\widetilde{\phi_v}$ of $\mathrm{C}(\mathbb{T})$ into $\mathrm{C}^*(\mathcal{V})$. By our previous result the norm of ϕ_v equals to $\phi_v(1)$. Thus we obtain the computation:

$$
\begin{aligned}
\|\phi(v)\| &= \|\psi(\phi(z))\| \leq \|\phi_v(1)\|\|z\| \\
&\leq \|\phi_v(1)\| \\
&= \|\phi(\mathrm{id}_{\mathcal{V}})\|.
\end{aligned}
$$

Choosing v to be $\mathrm{id}_{\mathcal{V}}$ establishes our claim. Furthermore, as a corollary we obtain the following result, which is due to Russo and Dye (see [29]):

If the domain of ϕ is a unital C^*-algebra then the norm of ϕ equals that of $\phi(\mathrm{id}_{\mathcal{V}})$.

Next, assume that the map ϕ is unital, contractive and complex valued. Then, ϕ is necessarily positive.

In order to see this it would be enough to show that for arbitrary positive elements p in $\mathcal{V}$ of norm 1, we have $\phi(p) \geq 0$. By the positivity of p we have that the spectral radius of p coincides with its norm; furthermore, the spectrum of p is a compact subset of $[0, \infty)$. For contradiction's sake suppose that $\phi(p) \notin [0, \infty)$. Then, in particular, $\phi(p)$ is not in $\overline{\mathrm{co}}(\sigma(p))$. Being the convex hull of a compact subset of $\mathbb{R}^2$, the set $\overline{\mathrm{co}}(\sigma(p))$ is compact and it is equal to the intersection of all closed disks in $\mathbb{R}^2$ containing $\sigma(p)$. Therefore, there exists some closed disk $\mathrm{B}(z, r)$ centered at z with radius r including $\overline{\mathrm{co}}(\sigma(p))$ but not containing $\phi(p)$ (i.e. $|\phi(p) - z| > r$). Then the spectrum of $p - z\,\mathrm{id}_{\mathcal{V}}$ is included in the disk $\mathrm{B}(0, r)$ while $\phi(p) \notin \mathrm{B}(0, r)$. Putting these facts together with fact that $p - z\,\mathrm{id}_{\mathcal{V}}$ is hermitian, we obtain:

$$\begin{aligned}
&\|p - z\,\mathrm{id}_{\mathcal{V}}\| \\
&\qquad = \rho(p - z\,\mathrm{id}_{\mathcal{V}}) \leq r \\
&\qquad < |\phi(p) - z| \\
&\qquad = |\phi(p - z\,\mathrm{id}_{\mathcal{V}})|.
\end{aligned}$$

Thus, reaching the conclusion that ϕ is not contractive, a contradiction.

Using this last result, one can easily conclude that, generally, if ϕ is unital contractive then it is positive. Indeed, let ϕ be such a map and, without loss of generality, assume that the concrete operator system $\mathcal{W}$ is a subspace of $\mathrm{B}(H)$ for some Hilbert space H. Fix an arbitrary vector η of norm 1 in H. Let ϕ_h be the complex valued unital contractive map assigning to each element v in $\mathcal{V}$ the complex number $\langle \phi(v)\eta, \eta\rangle$. Then, by our previous result we conclude that ϕ_h is positive. Thus, for any positive element v in $\mathcal{V}$ we have

$$\phi_h(v) = \langle \phi(v)\eta, \eta\rangle \geq 0.$$

However, η was chosen arbitrarily. Thus, $\phi(v)$ is positive.

We shall have more to say about this relationship after we introduce the concept of complete positivity and complete boundedness.

1.4.5 Map Amplification, $k-$Positivity and $k-$Boundedness

We have already looked into some properties of concrete operator systems and uncovered some of the relationships between the norm and positivity of maps over such systems. As it turns out, this relationship and other properties are characteristic also to the positive matrix cone inherited from the ambient C*-algebra. In order to see this, we first need to define the concept of map amplification and matrices of operators.

We begin by introducing the concept of matrices of operators and their positive matrix cones.

Let $\mathcal{B}(H)$ be the C*-algebra of bounded operators over some Hilbert space H. Let n be an arbitrary fixed positive integer and consider the Hilbert space $H^{(n)}$. Recall that $H^{(n)}$

consists of all the $n-$tuples of elements of H with the addition and scalar multiplication operations defined entry-wise and the inner product being $\langle(\eta_1,\dots,\eta_n)^t,(\xi_1,\dots,\xi_n)^t\rangle_n = \sum_{i=1}^{n}\langle\eta_i,\xi_i\rangle$. Next, let $\mathrm{M}_n(\mathrm{B}(H))$ denote the set of all $n\times n-$matrices of operators over H. It is clear that $\mathrm{M}_n(\mathrm{B}(H))$ is a complex vector space when endowed with the usual entry-wise matrix addition and multiplication by a scalar. Furthermore, endowing it with the adjoint operator assigning to each matrix $(T_{i,j})_{1\le i,j\le n}$ the matrix $(T^*_{j,i})_{1\le i,j\le n}$ turns $\mathrm{M}_n(\mathrm{B}(H))$ into a $*$-vector space. Finally, taken with the usual matrix multiplication (where multiplication of matrix entries is operator composition) we see that $\mathrm{M}_n(\mathrm{B}(H))$ is a $*$-algebra.

Consider the C^*-algebra $\mathrm{B}(H)$ and define the $*$-homomorphism $\pi : \mathrm{M}_n(\mathrm{B}(H)) \longrightarrow \mathrm{B}(H^{(n)})$ assigning to each element $(T_{i,j})_{1\le i,j\le n}$ of $\mathrm{M}_n(\mathrm{B}(H))$ the operator $\phi((T_{i,j})_{1\le i,j\le n})$ which action on the vectors $\eta := (\eta_1,\dots,\eta_n)^t$ of $H^{(n)}$ is defined as:

$$\pi((T_{i,j})_{1\le i,j\le n})(\eta) = \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & & \vdots \\ T_{n,1} & \dots & T_{n,n} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} T_{1,i}(\eta_i) \\ \vdots \\ \sum_{i=1}^{n} T_{n,i}(\eta_i) \end{pmatrix}.$$

It is easily shown that π is a bijective $*$-homomorphism. Thus, endowing $\mathrm{M}_n(\mathrm{B}(H))$ with operator norm inherited via π from $\mathrm{B}(H^{(n)})$ turns it into a C*-algebra with unit $\mathrm{Diag}_n(1_{\mathrm{B}(H)},\dots,1_{\mathrm{B}(H)})$.

In a similar fashion, given a unital C*-algebra $\mathcal{A}$, one considers the C*-algebra $\mathrm{M}_n(\mathcal{A})$ consisting of the $n\times n$ matrices of elements of $\mathcal{A}$. Indeed, embedding $\mathcal{A}$ $*$-isomorphically in some $\mathrm{B}(H)$, we may consider the $*$-subalgebra $\mathrm{M}_n(A)$ of the C*-algebra $\mathrm{M}_n(\mathrm{B}(H))$. It is easily shown that $\mathrm{M}_n(A)$ is closed and thus is a C*-subalgebra with unit $\mathrm{Diag}_n(1_{\mathcal{A}},\dots,1_{\mathcal{A}})$.

Note that every positive matrix in $\mathrm{M}_n(\mathcal{A})$ can be expressed as a finite sum of matrices of the form $(a_i^* a_j)_{1\le i,j\le n}$ where $a_1,\dots,a_n$ are elements of $\mathcal{A}$.

Next, we continue to describe the matrix cone of a concrete operator system. Given any concrete operator system $\mathcal{S}$, with an ambient C*-algebra $\mathcal{A}$, the space of all $n\times n$-matrices of elements in $\mathcal{S}$ is a vector subspace of the C*-algebra $\mathrm{M}_n(\mathcal{A})$ which is closed under the

$*$-operation of matrices of operators and which contains the unit of $\mathrm{M}_n(\mathcal{A})$. Thus, $\mathrm{M}_n(\mathcal{S})$ inherits the order of $\mathcal{B}(H^{(n)})$ making it into a concrete operator system. Thus, every concrete operator system generates a sequence of positive cones $\{\mathrm{M}_n(\mathcal{S})^+\}_{n=1}^{\infty}$ with an Archimedean unit corresponding to each cone.

It is helpful to elaborate a little on the matrix cone of $\mathrm{M}_n(\mathcal{A})$ when $\mathcal{A}$ is a commutative unital C*-algebra. In such a case $\mathcal{A}$ is $*$-isomorphic to $\mathrm{C}(X)$, where X is a compact Hausdorff space. The C*-algebra $\mathrm{M}_n(\mathrm{C}(X))$ may be viewed as the C*-algebra of $n \times n$ matrix valued continuous functions over X. Then, the positive elements of $\mathrm{M}_n(\mathcal{A})$ are the continuous functions assigning to each $x \in X$ a positive $n \times n$-matrix.

Before we continue we digress and make a note regarding the algebraic structure of the space of matrices of operators. Consider the aforementioned C*-algebra $\mathrm{M}_n(\mathcal{A})$. Let $E_{i,j}; 1 \leq i, j \leq n$, denote a complete set of matrix units for M_n. It is easy to show that the space $\mathrm{M}_n(\mathcal{A})$ satisfies the universal property of the tensor product $\mathrm{M}_n \otimes \mathcal{A}$ whereby each elementary tensor $E_{i,j} \otimes a$ is identified with the matrix $(\delta_{i,j} a)$ in $\mathrm{M}_n(\mathcal{A})$ (the operator matrix containing a in the $(i,j)-$entry and the zero operator everywhere else). Recall that the multiplication and adjoint operators defined over the tensor $*$-algebra $\mathrm{M}_n \otimes \mathcal{A}$ are defined for arbitrary a, b, c, d in $\mathcal{A}$ as:

$$(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$$
$$(a \otimes b)^* = a^* \otimes b^*.$$

It is easy to see that the matrix multiplication and the matrix adjoint defined over $\mathrm{M}_n(\mathcal{A})$ make it $*$-isomorphic to the $*$-algebra $\mathrm{M}_n \otimes \mathcal{A}$. Therefore, $\mathrm{M}_n \otimes \mathcal{A}$ inherits the norm endowed on $\mathrm{M}_n(\mathcal{A})$ and becomes a C*-algebra. Note that under this C*-algebra norm, it is easily shown that elementary tensors of the form $M \otimes a$ in $\mathrm{M}_n \otimes \mathcal{A}$ have a norm equal to $\|M\|\|a\|$. Henceforth, depending on the contextual need, we will interchangeably use $\mathrm{M}_n \otimes \mathcal{A}$ and $\mathrm{M}_n(\mathcal{A})$ to denote the same C*-algebra.

Next, we continue to define the concept of map amplification. Let ϕ be a positive map from some concrete operator system $\mathcal{S}$ into another operator system $\mathcal{W}$. Since each operator system generates a sequence of positive matrix cones, It is natural to ask the question whether the positivity of ϕ and its bound *somehow carries* to the concrete operator systems $\mathrm{M}_n(\mathcal{S})$ and $\mathrm{M}_n(\mathcal{W})$ for a given natural number n. A natural way of formalizing such a question is to observe the entry-wise action of ϕ on elements of cone $\mathrm{M}_n(\mathcal{S})$. To this end, fix a natural number n and let $\phi^{(n)} : \mathrm{M}_n(\mathcal{S}) \longrightarrow \mathrm{M}_n(\mathcal{W})$ be the map assigning to each matrix of operators $(s_{i,j})$ in $\mathrm{M}_n(\mathcal{S})$ the matrix of operators $(\phi(s_{i,j}))_{i,j}$ in $\mathrm{M}_n(\mathcal{W})$. We say that ϕ is $n-$positive if and only of $\phi^{(n)}$ is positive. Furthermore, we denote the bound of $\phi^{(n)}$ by $\|\phi\|_n$. If $\phi^{(n)}$ is positive for any natural number n then ϕ is said to be completely positive. If all the norms $\|\phi\|_n$ are bounded above by the same positive real number r then ϕ is said to be completely bounded. In such a scenario, the supremum of all the $\|\phi\|_n'$s is denoted by $\|\phi\|_{cb}$.

In the context of tensor product, identifying $\mathrm{M}_n(\mathcal{S})$ and $\mathrm{M}_n(\mathcal{W})$ respectively with $\mathrm{M}_n \otimes \mathcal{S}$ and $\mathrm{M}_n \otimes \mathcal{S}$, we see that $\phi^{(n)}$ is simply the map $1_n \otimes \phi$ which action on the elementary tensors assigns $M \otimes \phi(s)$ to each $M \otimes s$ in $\mathrm{M}_n \otimes \mathcal{S}$.

Next, note that if the map $\phi : \mathcal{S} \longrightarrow \mathcal{W}$ is $n-$positive then it is also $m-$positive for each positive integer $m \leq n$. Furthermore, for arbitrary natural numbers n and m with $m \leq n$ then $\|\phi\|_m \leq \|\phi\|_n < \infty$ and $\|\phi\|_n \leq \|\phi\| n^2$. In order to see why the last inequality holds, consider an arbitrary matrix $(s_{i,j})$ in $\mathrm{M}_n \otimes \mathcal{S}$, its image $(\phi(s_{i,j})$ in $\mathrm{M}_n \otimes \mathcal{W}$ and suppose $\mathcal{W}$ sits in $\mathcal{B}(K)$ for some Hilbert space K. Then, we have

$$\begin{aligned}
\|(\phi(s_{i,j})\| &= \sup_{x,y \in \mathrm{B}_1(K^{(n)})} |\langle (\phi(s_{i,j}))x, y \rangle| \\
&\leq \sum_{1 \leq i,j \leq n} |\langle \phi(s_{i,j}) x_j, y_i \rangle| \\
&\leq \sum_{1 \leq i,j \leq n} \|\phi(s_{i,j})\| \|x_j\| \|y_i\| \\
&\leq \sum_{1 \leq i,j \leq n} \|\phi\| \|s_{i,j}\| \|x_j\| \|y_i\| \\
&\leq \|\phi\| \|s\| \sum_{1 \leq i,j \leq n} 1 = n^2 \|\phi\| \|s\|.
\end{aligned}$$

However, it is not necessarily true that $\|\phi\|_{cb} < \infty$. We present a map satisfying such criteria in the example below.

Next, let $\phi : \mathcal{S} \longrightarrow \mathcal{W}$ and suppose that ϕ and ϕ^{-1} are a unital completely positive maps. Note that ${\phi^{(n)}}^{-1} = {\phi^{-1}}^{(n)}$ is also bijective for each positive integer n. Then, an element $A_n \in M_n(\mathcal{S})$ is positive if and only if its image $\phi^{(n)}(A_n)$ is positive. Furthermore, ϕ maps the Archimedean matrix order unit of $\mathcal{S}$ onto the matrix order unit of $\mathcal{W}$. Hence, the concrete operator system $\mathcal{S}$ together with its matrix order and unit can be identified respectively with the operator system $\mathcal{W}$, its matrix order and its unit. In such a case, $\mathcal{S}$ is said to be unitally completely order isomorphic to $\mathcal{W}$ and ϕ is said to be a unital complete order isomorphism.

On the other hand, suppose $\phi^{(n)}$ is isometric for all $n \in \mathbb{N}$. Then, ϕ is said to be a complete isometry. On the other hand, if $\phi^{(n)}$ is contractive (i.e. $\|\phi^{(n)}\| \leq 1$) for all $n \in \mathbb{N}$ then, ϕ is said to be completely contractive.

Recall that in Section 1.4.4 we showed that a unital contractive map is positive. We shall see in Subsection 1.4.5 that a completely positive map has a completely bounded norm equal to that of the image of the identity. Thus, a complete order isomorphism and its inverse map are necessarily completely isometric. Conversely, if an invertible map and its inverse are both unital complete isometries then the map is a complete order isomorphism.

In the following section we shall uncover further relationships between completely bounded and completely positive maps. But first, let us see some examples of $k-$positive, completely positive, $k-$bounded and completely bounded maps.

Example 1.4.6. *1. The transposition map, T, described in part 5 of example 1.4.5 is not $2-$positive. Let $\{E^n_{i,j}; 1 \leq i, j \leq n\}$ be a complete set of matrix units for M_n and consider the positive matrix $P := (E^n_{i,j})_{1\leq i,j\leq 2} \in M_2(M_n)$.*

$$T^{(2)}(P) = (E^{n\ t}_{i,j})_{1\leq i,j\leq 2} = (E^n_{j,i})_{1\leq i,j\leq 2}.$$

Note that $\mathrm{Det}((E^n_{j,i})_{1\leq i,j\leq 2}) < 0$; thus, $T^{(2)}(P)$ is not positive. When H is infinite-dimensional and separable, the transposition map is bounded but $\|T\|_{cb} = \infty$.

2. Consider the map ϕ described in part 4 of example 1.4.5. M.D. Choi showed that ϕ is $(n-1)-$positive but not completely positive (see [7]). Indeed, letting $E^n_{i,j}$ denote then unit matrices in M_n, Choi showed that $\phi^{(n)}((E_{i,j})_{1\leq i,j\leq n})$ is not positive while $(E_{i,j})_{1\leq i,j\leq n}$ is positive. Thus, for each natural number $n > 1$, there exists an $(n-1)-$positive map over M_n that is not $n-$positive.

Clearly, we can generalize this result to finite dimensional C^**-algebras that are not* $*$*-isomorphic to* $\mathbb{C}^n$*. One only has to notice that every finite-dimensional* C^**-algebra* $\mathcal{A}$ *is* $*$*-isomorphic to* $\oplus_{k=1}^m M_{n_k}$ *(See [13, Chapter 3]).*

3. *It is clear that every* $*$*-homomorphism is positive and contractive. Thus, since every amplification of a* $*$*-homomorphism is also a* $*$*-homomorphism, we see that it is also completely positive and completely contractive.*

4. *Simple calculation shows that every positive functional is completely positive and every bounded functional is completely bounded by its norm bound. Specifically, the trace functional acting on* M_n *is completely positive and its* $\|.\|_{cb}$*-norm is* n *while the normalized trace map has a complete norm of* 1.

5. *Let* H *and* K *be two Hilbert spaces and consider a bounded linear function* $T \in B(H, K)$*. Denote by* $\mathrm{Ad}(T)$ *the map assigning to each operator* E *in* $B(K)$ *the operator* T^*ET *in* $B(H)$*. Since* $\mathrm{Ad}(T)$ *acts by conjugation it must be positive. Furthermore, it is easily seen that the map* T *is completely positive and completely bounded with* $\|\mathrm{Ad}(T)\|_{cb} = \|T^*T\|$.

6. *Our last example of completely positive maps involves the Schur Product of matrices. The Schur Product of two complex matrices* $(a_{i,j})$ *and* $(b_{i,j})$ *denoted by* $(a_{i,j}) * (b_{i,j})$ *is simply the matrix* $(a_{i,j}b_{i,j})$*. Fix a positive matrix* $A := (a_{i,j})$ *in* M_n *and denote by* S_A *the map assigning to each matrix* B *in* M_n *the Schur Product* $A * B$*. Then,* S_A *is completely positive and* $\| S_A \|_{cb} \leq \|A\|$.

1.4.6 Relating the $k-$Positivity and $k-$Boundedness

Our purpose in this section is to study how the results of subsection 1.4.4 can be applied to the amplifications of maps. We look at certain sufficient conditions that make a matrix of operators positive. These conditions will prove essential to relating complete positivity and complete boundedness.

For the remainder of this section, unless otherwise mentioned, we fix a map ϕ with domain some concrete operating system $\mathcal{V} \subseteq B(H)$ and range residing in a concrete operator system $\mathcal{W} \subseteq \mathrm{B}(K)$.

Let $p, q \in \mathcal{V}^+$ and a arbitrary in $\mathcal{V}$. Then, the matrix $A := \begin{bmatrix} p & a \\ a^* & q \end{bmatrix}$ is positive in $\mathrm{M}_2(\mathcal{V})$ if and only if

$$|\langle a(\eta), \xi \rangle|^2 \leq \langle q(\xi), \xi \rangle \langle p(\eta), \eta \rangle.$$

for arbitrary vectors η and ξ in H. Indeed, the positivity of A is equivalent to having

$$\langle A(v), v \rangle \geq 0$$

for every $v = (\eta, \xi)^T \in H^{(2)}$. Expanding the left-hand side of the inequality, we see that it is equivalent to having the matrix

$$\begin{bmatrix} \langle p(\eta), \eta \rangle & \langle a(\xi), \eta \rangle \\ \langle a^*(\eta), \xi \rangle & \langle q(\xi), \xi \rangle \end{bmatrix}$$

positive. This last matrix is self-adjoint, has a positive trace and thus is positive if and only if its determinant is non-negative. However, the positivity of the determinant is equivalent to having

$$|\langle a(\eta), \xi \rangle|^2 \leq \langle q(\xi), \xi \rangle \langle p(\eta), \eta \rangle.$$

When $p = 1_{\mathcal{V}}$ this last condition is equivalent to

$$a^* a \leq q.$$

Furthermore, if $q = 1_{\mathcal{V}}$ then the condition is equivalent to a being contractive.

This earlier condition of positivity is worth emphasizing and we will use it later; thus, we summarize it below:

Let p, q and a be three operators over some Hilbert space with p and q being positive. Then,

$$\begin{pmatrix} p & a \\ a^* & q \end{pmatrix} \iff |\langle av, w \rangle|^2 \leq \langle pw, w \rangle \langle qv, v \rangle. \tag{1.5}$$

for any $v, w \in H$.

Recall that in Subsection 1.4.4 we showed that the norm of $2\phi(1_{\mathcal{V}})$ is the best upper bound we can achieve for positive maps. The first corollary we conclude from the above conditions yields a lower bound on the norm of certain positive map. Specifically, if ϕ is $2-$positive then the norm of ϕ is $\|\phi(1_{\mathcal{V}})\|$. Indeed, if a is contractive then by the previous condition for positive matrices, the matrix $\begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}$ is positive. Thus, the $2-$positivity of ϕ yields $\begin{bmatrix} \phi(1) & \phi(a) \\ \phi(a)^* & \phi(1) \end{bmatrix} \geq 0$. Using the first of the above conditions yields $|\langle \phi(a)(\eta), \xi \rangle|^2 \leq \langle \phi(1)(\xi), \xi \rangle \langle \phi(1)(\eta), \eta \rangle$. Using a standard ϵ argument and the fact that the norm of any operator T over H coincides with the supremum of $|\langle \phi(\eta), \xi \rangle|$ over all the normalized vectors η and ξ yields $\|\phi(a)\| \leq \|\phi(1_{\mathcal{V}})\|$.

A similar reasoning yields the Schwartz Inequality, which states that

$$\phi(a)^* \phi(a) \leq \phi(a^* a), \forall a \in \mathcal{A}; \tag{1.6}$$

whenever ϕ is a unital $2-$positive map from some C*-algebra $\mathcal{A}$ into some C*-algebra $\mathcal{B}$.

Two powerful theorems that are straightforward corollaries from the earlier inequalities reveal the main goal of this section. In particular, the first states that if ϕ is completely positive then $\|\phi\|_{cb} = \|\phi\| = \|\phi(1_{\mathcal{V}})\|$. The second result states that if ϕ is unital and completely contractive then it is completely positive.

1.4.7 Abstract Operator Systems

After the short exposition on ordered $*$-vector spaces and positive maps over them we digressed and discussed the behavior of positive and bounded maps over concrete operator systems. Furthermore, we saw that the full properties of positive cones and positive maps on such systems is captured when considering amplification of maps and positive cones of matrices of operators. This in turn yielded a relationship between complete positivity and complete boundedness. Our aim in this section is to provide an abstract characterization of concrete operator systems which captures the concept of matrix cones. The results in this section are due to M.D. Choi and E. Effros (see [8]).

Let $\mathcal{S}$ be a $*$-vector space with Archimedean order identity $1_{\mathcal{V}} \in \mathcal{V}_h$. Let $\mathrm{M}_n(\mathcal{S})$ be the set of all $n \times n$ matrices with entries in $\mathcal{S}$. $M_n(\mathcal{S})$ is a complex vector space under the entry-wise scalar multiplication and addition. For each positive integer n define the adjoint map over $\mathrm{M}_n(\mathcal{S})$ by $(s_{i,j})^*_{1\leq i,j\leq n} = (s^*_{j,i})_{1\leq i,j\leq n}$. Thus, $\mathrm{M}_n(\mathcal{S})$ is a complex $*$-vector space. Furthermore, Note that $\mathrm{M}_n(\mathcal{S})$ is isomorphic to $\mathrm{M}_n \otimes \mathcal{S}$ and the adjoint map defined on $\mathrm{M}_n(\mathcal{S})$ induces an adjoint map over $\mathrm{M}_n \otimes \mathcal{S}$ characterized by $(A \otimes s)^* = A^* \otimes s^*; \forall A \in \mathrm{M}_n, \forall s \in \mathcal{S}$. Finally, we note that for each integer n, $M_n(\mathcal{S})_h$, the set of the of Hermitian matrices in $M_n(\mathcal{S})$, is a real subspace of $M_n(\mathcal{S})$.

Next, we extend the concept of order over $*$-vector spaces to that of matrices of elements of $*$-vector spaces. Recall, that the set of positive elements in $\mathcal{S}$ has a cone structure. (i.e. It is closed under addition and multiplication by non-negative real numbers and satisfying $\mathcal{S}^+ \cap -\mathcal{S}^+ = \{0\}$.) Similarly, $\{\mathcal{C}_n\}_{n=1}^{\infty}$ is said to be a matrix cone for $\mathcal{S}$ if for each integer n, $\mathcal{C}_n$ is a cone in $\mathrm{M}_n(\mathcal{S})_h$ satisfying the conjugation by complex matrices criteria. This criteria means that for every complex matrix $A \in \mathrm{M}_{m\times n}$ and $T \in \mathcal{C}_n$ the following holds true:

$$ATA^* \in \mathcal{C}_m.$$

As pointed earlier, $\mathcal{C}_n$ induces a vector space order on $\mathrm{M}_n(\mathcal{S})_h$. In such a case, the matrix cone is said to induce a matrix order over $\mathcal{S}$.

Furthermore, $1_{\mathcal{S}}$ is said to be an Archimedean matrix-order unit for the matrix order $\mathcal{C}_n$ if for each integer n, the diagonal $n \times n-$matrix with entries $1_{\mathcal{S}}$ along the diagonal, denoted by $\mathrm{Diag}_n(1_{\mathcal{S}})$, is an Archimedean order unit for the ordered $*$-vector space $M_n(\mathcal{S})_h$.

In summary, a $*-$vector space $\mathcal{S}$ with Archimedean order unit $1_{\mathcal{S}}$ is said to be an abstract operator system if it is endowed with a matrix cone $\{\mathcal{C}_n\}_{n=1}^{\infty}$ having a matrix Archimedean unit $1_{\mathcal{S}}$ and satisfying the above conjugation property.

Similar to the case of concrete operator systems we define and use the same notation for map amplification and the concept of $n-$positivity.

Next, note that the matrix cone induces a norm $\|.\|_n$ over $\mathrm{M}_n(\mathcal{S})$. Indeed, this norm is at the heart of the result by Choi and Effros identifying each abstract operator system with a concrete operator system. In particular, for each matrix in $A \in \mathrm{M}_n(\mathcal{S})$ we define $\|A\|_n$ to be the infimum of all positive real numbers r making the matrix $\left[\begin{array}{c|c} r\,\mathrm{Diag}_n(1_{\mathcal{S}}) & A^* \\ \hline A & r\,\mathrm{Diag}_n(1_{\mathcal{S}}) \end{array}\right]$ positive. The Archimedean property of $\mathrm{Diag}_{2n}(1_{\mathcal{S}})$ guarantees the existence of such an r. It is shown that this indeed is a norm. Furthermore, one may also show and that there exists a concrete operator system $\mathcal{N}$ in some C^*-algebra that is unitally completely order isomorphic to $\mathcal{S}$.

Theorem 1.4.7. *Let $\mathcal{S}$ be an abstract operator system with and Archimedean matrix order unit $1_{\mathcal{S}}$. There exists a Hilbert space H, a concrete operator system $\mathcal{S}_1 \subseteq \mathrm{B}(H)$ and a complete order isomorphism of $\mathcal{S}$ onto $\mathcal{S}_1$. Conversely, any concrete operator system $\mathcal{S}_1 \subseteq \mathrm{B}(H)$ is an abstract operator system.*

Henceforth, we assume without loss of generality that every operator system is a concrete one.

1.4.8 Conditions for CP and CB Maps

A natural question to ask at this point is why some positive (bounded) maps are completely positive (completely bounded). Furthermore, can one always find positive maps (bounded) that are not completely positive (completely bounded) over any operator system? Recall that certain positive maps, such as the conjugation map $\mathrm{Ad}(V)$, are completely positive and completely bounded due to the nature of their action. On the other hand, positive functionals are always completely positive and completely bounded. We identify three factors that affect the question stated earlier; namely, the structure of the domain, the structure of the range and the structure of the map itself.

W.F. Stinespring showed that if a positive map has a commutative C^*-algebra as its domain then it must be completely positive (See [33]). This, yields a sufficient condition on the domain of a positive map in order for it to be completely positive. On the other hand, one can show that every positive map mapping an operator system into a commutative C^*-algebra must be completely positive. This yields a sufficient condition on the range of a positive map in order for it to be completely positive.

In the previous two cases complete positivity and positivity of maps are equivalent. It turns out a certain type of matrix cones yields this equivalence. Let $\mathcal{V}$ and $\mathcal{W}$ be two operator systems. Note that if A is a positive $n \times n$ matrix and $s \in \mathcal{V}^+$ then $A \otimes s$ is positive. Consider the set

$$Q := \{M \otimes s; M \in \mathrm{M}_n^+, s \in \mathcal{V}^+\}$$

and denote by $\mathrm{M}_n^+ \otimes \mathcal{V}^+$ the set of linear combinations formed using elements of Q and only non-negative scalars. Clearly, $\mathrm{M}_n^+ \otimes \mathcal{V}^+$ has a cone structure and by our earlier remark it is a sub-cone of $\mathrm{M}_n(\mathcal{V})^+$. As a conclusion the closure of $\mathrm{M}_n^+ \otimes \mathcal{V}^+$ under the usual operator norm is included of $\mathrm{M}_n(\mathcal{V})^+$. We have no guarantee that reverse inclusion is true. However, one can show the following important proposition (see theorem 6.6 in [29]):

Proposition 1.4.8. $\mathrm{M}_n(\mathcal{V})^+ = \overline{\mathrm{M}_n^+ \otimes \mathcal{V}^+}$ *if and only if every positive map from* $\mathcal{V}$ *into* M_n *is necessarily completely positive.*

On the other hand, M.D. Choi showed that a map ϕ with domain M_n is completely positive if and only if it is $n-$positive. Furthermore, he showed that ϕ is completely positive if the operator matrix $(\phi(E_{i,j}))$ is positive (see [7]). A similar result holds when the range of ϕ is M_n and its domain is any operator system. Specifically, ϕ is $n-$ positive if and only if it is completely positive.

Finally, we move to describe the third factor making a positive map completely positive. Recall that every $*$-homomorphism π from some C*-algebra A into $\mathcal{B}(K)$ must be completely positive. Furthermore, if V is a bounded operator from some Hilbert space H into K, the map

$$\phi : \mathcal{A} \longrightarrow \mathcal{B}(H)$$

assigning to each a in $\mathcal{A}$ the operator $V^*\pi(a)V$ in $\mathcal{B}(H)$ is a completely positive map. Stinespring showed the converse of this example (see [33]). Namely, let ϕ be a linear map from some C*-algebra $\mathcal{A}$ into some $\mathcal{B}(H)$. Then, ϕ is completely positive if and only if there exist a Hilbert space K, a bounded map V in $\mathcal{B}(H, K)$ and a C*-algebra homomorphism π from $\mathcal{A}$ into $\mathcal{B}(K)$ satisfying

$$\phi(a) = V^*\pi(a)V$$

for each a in $\mathcal{A}$. Furthermore, K may be chosen to be minimal in the sense that

$$K = \overline{\text{span}}\{\pi(a)V(\eta); a \in \mathcal{A} \text{ and } \eta \in H\}.$$

In such a case, K would be unique within an isometric isomorphism.

Note that when ϕ is unital then V must be an isometry. Thus, identifying H with its isometric image in K, the above decomposition may be written as

$$\phi(a) = P_K \pi(a)|_K;$$

where P_K is the orthonormal projection over K.

A final comment about this theorem is that it is a generalization of the GNS construction (see [27] for a review of the inner workings of the GNS construction and the discussion following theorem 4.1 in [29]).

This theorem is referred to as the Stinespring Dilation Theorem. The other Stinespring Theorem stated in the beginning of Section 1.4.8 is simply referred to as Stinespring Theorem.

1.4.9 Extending Positive Maps

In this Section we consider the problem of extending completely positive maps. Given an operator system $\mathcal{S}$, let $\mathcal{M}$ and $\mathcal{N}$ be arbitrary operator systems such that $\mathcal{M}$ is a subsystem of $\mathcal{N}$. If ϕ is completely positive map from $\mathcal{M}$ into $\mathcal{S}$, is it possible to extend ϕ to a completely positive map $\overline{\phi}$ form $\mathcal{N}$ into $\mathcal{S}$ in such a way where the diagram below commutes (see figure 1.2)? Operator systems $\mathcal{S}$ where such extensions are always possible are called injective operator systems. This is an important concept and the idea of injectivity was used by Hamana to prove the existence of the C*-envelope (see [19]).

Note that a version of the Hahn-Banach Theorem still holds for positive functionals over real ordered vector spaces. To be specific, let $(\mathcal{V}, \leq)$ be a real ordered vector space with order unit $1_{\mathcal{V}}$ and $\mathcal{W}$ a real subspace of $\mathcal{V}$ containing $1_{\mathcal{V}}$. Clearly, $\mathcal{W}$ inherits the partial order of $\mathcal{V}$ and since it contains the order unit it is an order space admitting an

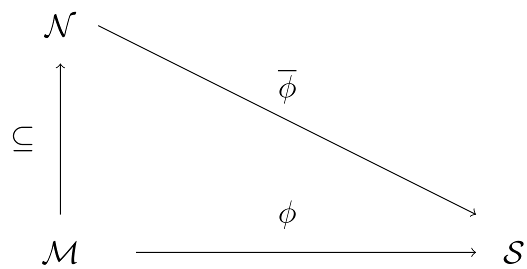

Figure 1.2: Extending maps.

order unit. Let ψ be a positive functional in $\mathcal{W}^*$. There exists a positive functional $\widetilde{\psi}$ in the dual of $\mathcal{V}$ extending ψ. Furthermore, if $\mathcal{V}$ is a normed space and ψ is bounded then $\widetilde{\psi}$ can be constructed in such a way to admit an operator norm equal to that of ψ (see [10, Secs. 3.6, 3.9]). The proof of this result parallels that of the Hahn-Banach Theorem. Furthermore, such a result can be easily extended to the case of operator systems instead of real ordered vector spaces.

In general not every completely positive map is extendible to a completely positive one. Furthermore, not every operator system is injective. Indeed, T. Huruya showed that there exists a non-injective C*-subalgebra of co-dimension 1 of an injective C*-algebra (see [21]). However, Arveson showed that when H is a Hilbert space, $\mathcal{B}(H)$ is injective (see [4]). Furthermore, Wittsctock showed a similar result for completely bounded maps (see [37]). We combine both results below.

Theorem 1.4.9. *Let $\mathcal{S}$ be an operator subsystem of an operator system $\mathcal{A}$ and ϕ a completely positive map from $\mathcal{S}$ to $\mathcal{B}(H)$. Then, ϕ extends to a completely positive map $\widetilde{\phi}$ from $\mathcal{A}$ into $\mathcal{B}(H)$. In other words, $\mathcal{B}(H)$ is injective. On the other hand, if $\mathcal{S}$ is a subspace of a space of operators over a Hilbert space $\mathcal{A}$ containing the unit and ϕ is completely bounded then ϕ extends to a completely bounded map from $\mathcal{A}$ to $\mathcal{B}(H)$ of the same cb-norm.*

1.4.10 C*-Envelopes

The goal of this section is to introduce the idea of the C*-envelope of an operator system. Given an abstract operator system $\mathcal{S}$, Section 1.4.7 tells us that it is unitally completely order isomorphic to some concrete operator subsystem of some $\mathcal{B}(H)$. Thus, it generates a C*-subalgebra in $\mathcal{B}(H)$. Furthermore, although this concrete realization is unique within a complete order isomorphism, there is no guarantee that $\mathcal{B}(H)$ or the C*-algebra generated by the copy of the operator system in $\mathcal{B}(H)$ are unique within a $*$-isomorphism. For example, in Section 2.4 we show that the C*-algebra generated by the operator system generated by the unilateral shift over some Hilbert space is not $*$-isomorphic to the one generated by the unitally completely order isomorphic image of this operator system in $\mathrm{C}(\mathbb{T})$. Thus, the C*-algebra generated by $\mathcal{S}$ or any of its unital completely order isomorphic copies are dependent on the ambient operator C*-algebra. In a sense, the C*-envelope of $\mathcal{S}$, which we denote by $\mathrm{C}^*_e(\mathcal{S})$, is the *smallest* C*-algebra that could be generated by a unitally completely order isomorphic copy of $\mathcal{S}$. Our exposition of the topic of the C*-envelope will be historical in nature. An alternative and at times more practical approach to the topic which was given by Hamana (see [19] and [18]).

In his famous paper (see [3]) William Arveson addressed a problem which is the extent to which a C*-algebra is determined by a generating subalgebra. In other words, if $\mathcal{A}$ is a subalgebra of some C*-algebra then what properties of $\mathrm{C}^*(\mathcal{A})$ are invariant when $\mathcal{A}$ is replaced by some unital completely isometric image. Arveson, showed that one of these invariant properties are a special kind of irreducible $*$-representations called the boundary representations. Furthermore, drawing from a motivation by the analogue of this question in the commutative case, Arveson defined and associated the concepts of boundary representations, boundary ideals and Silov boundaries in the context of generally non-commutative C*-algebras. For a comprehensive clear exposition of these topics in the commutative case see the book [30]. However, it is worth noting that this analogy carries only to a limited extent.

Next, we provide a short summary of these relevant results while stopping to provide proofs only for those theorems used later on. We begin by presenting various definitions and concepts then we move to tying all these together. Let $\mathcal{O}$ be a unital subspace of a C^*-algebra $\mathcal{A}$ containing the unit. Such unital subspaces are called concrete operator spaces or simply operator spaces when the ambient C^*-algebra is clear.

Suppose that $\mathcal{A} = C^*(\mathcal{O})$. An ideal $\mathcal{J}$ of $\mathcal{A}$ is said to be a boundary ideal of $\mathcal{O}$ with respect to $\mathcal{A}$, or simply boundary ideal of $\mathcal{O}$ when the context is clear, if the quotient map $q_{\mathcal{J}}$ is a unital complete isometry when restricted to $\mathcal{O}$. The Silov boundary is the largest such ideal. The C^*-envelope of $\mathcal{O}$ is $\mathcal{A}/\mathcal{J}$ when $\mathcal{J}$ is the Silov boundary. We denote this C^*-envelope by $C^*_e(\mathcal{O})$. Note that the definitions of the Silov boundary and the C^*-envelope do not guarantee their existence. Historically, Hamana was the first to show the existence of both the Silov boundary and the C^*-envelope using the concept of injective envelopes (see [?]). Below we will provide an answer to this existence problem using a different approach which draws upon the concept of boundary representations introduced by Arveson (see [3]) and the work of Davidson and Kennedy (see [14]).

Let π be an irreducible representation of $\mathcal{A}$ over some Hilbert space H. Let ϕ denote the restriction of π to $\mathcal{O}$. Then, π is said to be a boundary representation of $\mathcal{O}$ with respect to $\mathcal{A}$, or simply the boundary representation of $\mathcal{O}$ when the context is clear, if and only if π is the only completely positive map extending ϕ to $\mathcal{A}$.

Using the extension of completely contractive maps and Stinespring's Dilation Theorem, Arveson showed that boundary representations of $\mathcal{O}$ are invariant under unital completely isometric images of $\mathcal{O}$. Accurately speaking, let $\mathcal{A}$, $\mathcal{O}$ and π be defined as earlier. If $\mathcal{O}_1$ is an operator space in C^*-algebra $\mathcal{A}_1$, $\mathcal{A}_1 = C^*(\mathcal{O}_1)$ and

$$\phi\colon \mathcal{O} \longrightarrow \mathcal{O}_1$$

is a unital complete isometry of $\mathcal{O}$ onto $\mathcal{O}_1$ then there exists a boundary representation π_1 of $\mathcal{A}_1$ with respect to $\mathcal{O}_1$ that is unique within a unitary equivalence such that

$$\pi_1|_{\mathcal{O}_1} = \pi \circ \phi^{-1}$$

and

$$\pi_1(\mathcal{A}_1) = \pi(\mathcal{A}).$$

We denote by $\mathrm{Bd}_{\mathcal{A}}(\mathcal{O})$ the set containing a representative from each boundary representation equivalence class. When the context is clear, we simply denote it by $\mathrm{Bd}(\mathcal{O})$.

Furthermore, by the invariance of boundary representations with respect to unital completely isometric images, the elements of $\mathrm{Bd}_{\mathcal{A}_1}(\mathcal{O}_1)$ can be chosen in such a way that for each π_1 in $\mathrm{Bd}_{\mathcal{A}_1}(\mathcal{O}_1)$ there exists a unique π in $\mathrm{Bd}_{\mathcal{A}}(\mathcal{O})$ satisfying

$$\pi_1|_{\mathcal{O}_1} = \pi \circ \phi^{-1}$$

and

$$\pi_1(\mathcal{A}_1) = \pi(\mathcal{A}).$$

It is clear that if $\mathcal{J}$ is a boundary ideal for $\mathcal{O}$ with respect to $\mathcal{A}$ and π is a boundary representation for $\mathcal{A}$ over some Hilbert space H then

$$\mathcal{J} \subseteq \ker(\pi).$$

Indeed, note that $q_{\mathcal{J}}|_{\mathcal{O}}$ is a unital complete isometry. Then, by the preceding discussion there exists a boundary representation π' of $q_{\mathcal{J}}(A)$ over the Hilbert space H satisfying:

$$\pi|_{\mathcal{O}} = \pi' \circ q_{\mathcal{J}}|_{\mathcal{O}}$$

and

$$\pi(A) = p'(q_{\mathcal{J}}(A)).$$

However, $q_{\mathcal{J}}$ is a $*$-homomorphism and $q_{\mathcal{J}}(\mathcal{O})$ generates the C*-algebra $q_{\mathcal{J}}(\mathcal{A})$. Thus,

$$\pi = \pi' \circ q_{\mathcal{J}}. \tag{1.7}$$

Thus, $\mathcal{J} \subseteq \ker(\pi)$.

Denote by Ψ the direct sum of all boundary representations π in $\mathrm{Bd}_{\mathcal{A}}(\mathcal{O})$. Recall that, by definition, boundary representations are irreducible. Then, Ψ is an irreducible representation and $\ker(\Psi)$ coincides with the intersection of the kernels of all boundary representations of $\mathcal{A}$ with respect to $\mathcal{O}$. Furthermore, $\|\Psi(a)\|$ equals the supremum of the norms $\|\pi(a)\|$ as π ranges over $\mathrm{Bd}_{\mathcal{A}}(\mathcal{O})$. Thus, we have

$$\mathcal{J} \subseteq \ker(\Psi)$$

and thus the Silov boundary is included in the kernel of Ψ. Furthermore, since Ψ is a $*$-homomorphism it is completely contractive and so for each integer n we have

$$\|(s_{i,j})_{i,j}\|_n \geq \|\Psi(s_{i,j})_{i,j}\|_n = \sup_{\pi} \|\pi(s_{i,j})_{i,j}\|_n.$$

Arveson showed that under certain conditions enough boundary representations exist so that the above inequality becomes an equality; thus, making the kernel of Ψ the Silov boundary. However, the question of the existence of the Silov boundary remained open. In 1967 M. Hamana showed the existence of the Silov boundary using a different approach through the concepts of injective envelopes and C*-envelopes (see [19] and [18]). 5 However, the question of the existence of enough boundary representations to achieve the above equality remained open until 2015 when it was solved by K Davidson and M. Kennedy (see [14]). Thus, from now onward we take for granted the fact that the Silov boundary of $\mathcal{O}$ always exists and coincides with $\cap_{\pi \in \mathrm{Bd}_{\mathcal{A}}(\mathcal{O})} \ker(\pi)$. Furthermore, the quotient map corresponding to this ideal is the boundary representation $\oplus_{\pi \in \mathrm{Bd}_{\mathcal{A}}(\mathcal{O})} \pi$ and the image of $\mathcal{A}$ under this direct sum of boundary representations coincides with $\mathrm{C}^*_e(\mathcal{O})$.

Combining these results together we obtain the following proposition:

Proposition 1.4.10. *Let $\mathcal{O}$ and $\mathcal{O}_1$ be two unital operator spaces respectively contained in* C*-*algebras $\mathcal{A}$ and $\mathcal{A}_1$ and satisfying*

$$\mathcal{A} = \mathrm{C}^*(\mathcal{O}),$$
$$\mathcal{A}_1 = \mathrm{C}^*(\mathcal{O}_1).$$

Let ϕ be a unital complete isometry of $\mathcal{O}$ onto $\mathcal{O}_1$ and denote by $\mathcal{J}$ the Silov boundary of $\mathcal{O}$ with respect to $\mathcal{A}$. Then, there exists a boundary representation π from $\mathcal{A}_1$ onto $\mathrm{C}^*_e(\mathcal{O})$ such that $\pi \circ \phi(a)$ coincides with $q_{\mathcal{J}}(a)$ for each a in $\mathcal{O}$. Furthermore, the kernel of π is the Silov boundary of $\mathcal{O}_1$ with respect to $\mathcal{A}_1$ and $\mathrm{C}^*_e(\mathcal{O}_1) = \mathcal{A}/\mathcal{J} = \mathrm{C}^*_e(\mathcal{O})$. As a conclusion, the C*-*envelope is unique within $*$-isomorphism.*

Proposition 1.4.11. *Let $\mathcal{O}$ and $\mathcal{O}_1$ be two unital operator spaces respectively contained* *in* C*-*algebras* $\mathcal{A}$ *and* $\mathcal{A}_1$ *and satisfying*

$$\mathcal{A} = \mathrm{C}^*(\mathcal{O}),$$
$$\mathcal{A}_1 = \mathrm{C}^*(\mathcal{O}_1).$$

Furthermore, suppose that the Silov boundaries of $\mathcal{O}$ *and* $\mathcal{O}_1$ *with respect to* $\mathcal{A}$ *and* $\mathcal{A}_1$ *are both null. If* ϕ *is a unital complete isometry of* $\mathcal{O}$ *onto* $\mathcal{O}_1$ *then it extends to a unital* $*$*-isomorphism of* $\mathcal{A}$ *onto* $\mathcal{A}_1$.

Proof. Since the Silov boundary of $\mathcal{O}$ with respect to $\mathcal{A}$ is null and $\mathcal{O}$ is a unital completely isometric image of $\mathcal{O}_1$, by Proposition 1.4.10 there exists surjective $*$-homomorphism π of $\mathcal{A}_1$ onto $\mathcal{A}$ such that $\pi \circ \phi|_{\mathcal{O}}$ coincides with the identity map (see figure 1.3). Furthermore, the kernel of π coincides with the Silov boundary of $\mathcal{O}_1$. But the Silov boundary of $\mathcal{O}_1$ is null. Thus, π is a $*$-isomorphism extending ϕ^{-1}.

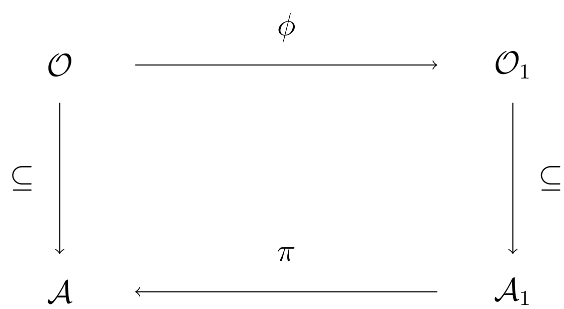

Figure 1.3: Extending a unital complete isometry.

□

Note that every operator system is a concrete operator space and recall that every order isomorphism is a complete isometry. Thus, the preceding result can be restated as:

Lemma 1.4.12. *Let* $q : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ *be a unital complete order isomorphism between two operator systems. Then,* q *extends to a* $*$*-isomoprhism between* $\mathrm{C}_e^*(\mathcal{S}_1)$ *and* $C_e^*(\mathcal{S}_2)$.

Another corollary of Proposition 1.4.10 is the following:

Corollary 1.4.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital* C^**-algebras and $q : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital complete order isomorphism. The map p must be a $*$-isomorphism.*

Finally, we note the following:

Lemma 1.4.14. *Suppose $V_1, \ldots, V_n \in \mathcal{B}(H)$. Define $\mathcal{X}$ to be the operator system generated by $\{V_1, \ldots, V_n, V_1V_1^*, \ldots, V_nV_n^*, V_1^*V_1, \ldots, V_n^*V_n\}$. Then, $\mathrm{C}_e^*(\mathcal{X})$ coincides with $\mathrm{C}^*(\mathcal{X})$.*

Proof. We will need the following lemma first:

Lemma 1.4.15. *Let $\mathcal{A}$ be a* C^**-algebra and H a Hilbert space. Suppose, $\phi : \mathcal{A} \longrightarrow \mathcal{B}(H)$ is a unital completely positive map. Let $a \in \mathcal{A}$ satisfy $\phi(aa^*) = \phi(a)\phi(a^*)$ and $\phi(a^*a) = \phi(a^*)\phi(a)$. Then, $\phi(ax) = \phi(a)\phi(x)$ and $\phi(xa) = \phi(x)\phi(a)$ for any x in $\mathcal{A}$.*

Proof. It is enough to show that $\phi(a^*a) = \phi(a^*)\phi(a)$ implies $\phi(xa) = \phi(x)\phi(a)$ for any x in $\mathcal{A}$. To this end, let a satisfy the antecedent of the preceding implication. Consider the matrix $B = \begin{pmatrix} a & x^* \\ 0 & 0 \end{pmatrix}$ in $\mathrm{M}_2(\mathcal{A})$. Then, applying the Schwartz Inequality 1.6 to the amplification $\phi^{(2)}$ of map ϕ to B we obtain

$$\phi^{(2)}(B^*)\phi^{(2)}(B) \leq \phi^{(2)}(B^*B).$$

Computing both sides of the inequality and noting that the positive map ϕ must also be self-adjoint preserving, we obtain:

$$\begin{pmatrix} \phi(a)^*\phi(a) & \phi(a)^*\phi(x)^* \\ \phi(x)\phi(a) & \phi(x)\phi(x)^* \end{pmatrix} \leq \begin{pmatrix} \phi(a^*a) & \phi(xa)^* \\ \phi(xa) & \phi(xx^*) \end{pmatrix}.$$

But $\phi(a^*a) = \phi(a)^*\phi(a)$. Thus we have,

$$\begin{pmatrix} 0 & \phi(xa)^* - \phi(a)^*\phi(x)^* \\ \phi(xa) - \phi(x)\phi(a) & \phi(xx^*) - \phi(x)\phi(x)^* \end{pmatrix} \geq 0$$

Then, by the property of positive matrices of operators stated in 1.5, we must have $\phi(xa) - \phi(x)\phi(a) = 0$ making $\phi(xa) = \phi(x)\phi(a)$. □

Next, we move to the proof of Lemma 1.4.14. Let $\mathcal{J}$ be the Silov boundary of $\mathcal{X}$ with respect to $\mathrm{C}^*(\mathcal{X})$. Then, the quotient map by the ideal $\mathcal{J}$

$$q : \mathrm{C}^*(\mathcal{X}) \longrightarrow \mathrm{C}^*_e(\mathcal{X})$$

is a unital complete order isomorphism when restricted to $\mathcal{X}$. Suppose that $\mathrm{C}^*(\mathcal{X})$ is a C^*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. Let ϕ denote the inverse of $q|_{\mathcal{X}}$. By Theorem 1.4.9, we extend the unital completely positive map ϕ to $\mathrm{C}^*_e(\mathcal{X})$ and denote it by $\widetilde{\phi}$. Let $\psi : \mathrm{C}^*(\mathcal{X}) \longrightarrow \mathcal{B}(H)$ be defined by $\psi := \widetilde{\phi} \circ \pi$. Note that ψ is a ucp that fixes $\mathcal{X}$. Thus,

$$\psi(V_iV_i^*) = \psi(V_i)\psi(V_i^*) = V_iV_i^* \textit{ and}$$
$$\psi(V_i^*V_i) = \psi(V_i^*)\psi(V_i) = V_i^*V_i$$

for each integer in $\{1, \ldots, n\}$. Thus, by Lemma 1.4.15 $\psi(TV_i) = \psi(T)\psi(V_i)$ and $\psi(V_iT) = \psi(V_i)\psi(T)$ for any T in $\mathrm{C}^*(\mathcal{X})$. Thus ψ is multiplicative over the $*$-algebra generated by $\mathcal{X}$ and being continuous and fixing $\mathcal{X}$ it coincides with the identity map over $\mathrm{C}^*(\mathcal{X})$. Thus, $\overline{\phi}$ is the left inverse of π, which makes the surjective $*$-homomorphism π a $*$-isomorphism. □

Chapter 2

Operator Systems Generated by Commuting Normal Operators

Operator subsystems of commutative C^* −algebras are often referred to as function systems. However, we won't be using this terminology. Instead, we will just refer to them as operator systems. Section 2.2 is a direct consequence of section 2.3. However, we present it first simply for pedagogical reasons. Indeed, the case of operator systems generated by a single normal operator leads to the study of affinely homeomorphic convex compact regions in the plane. In such a scenario, we have the Euclidean geometry to guide our intuition. On the other hand, the case of n commuting normal operators reduces to the study of affinely homeomorphic compact convex sets in $\mathbb{C}^n$.

2.1 Kadison's Representation Theorem

Building upon the results presented in Section 1.2.2 we work toward representing operator systems by affine maps. Let K be a compact convex subset of a locally convex Hausdorff space and consider the space of complex-valued continuous affine functionals $\mathrm{A}(K)$. Recall from Section 1.2.2 that $\mathrm{A}(K)$ is a closed subspace of the C^*-algebra $\mathrm{C}(K)$. Furthermore, it is clear that $\mathrm{A}(K)$ contains the unit of $\mathrm{C}(K)$ and is closed under the involution of $\mathrm{C}(K)$.

Thus, $\mathrm{A}(K)$ is a closed operator sub-system of $\mathrm{C}(K)$ inheriting the order of $\mathrm{C}(K)$ and containing its Archimedean order unit which we denote by $1_{\mathrm{A}(K)}$.

Theorem 2.1.1. *Let $K_1 \subseteq \mathcal{X}_1$ and $K_2 \subseteq \mathcal{X}_2$ be convex compact subsets of a complex locally convex spaces $\mathcal{X}_1$ and $\mathcal{X}_2$. The following statements are equivalent:*

1. *K_1 is affinely homeomorphic to K_2*
2. *$\mathrm{A}(K_1)$ is unitally completely order isomorphic to $\mathrm{A}(K_2)$.*

Proof. (1) $\implies$ (2):

Assume $\tau : K_1 \longrightarrow K_2$ is an affine homeomorphism. Define the map $T : \mathrm{A}(K_1) \longrightarrow \mathrm{A}(K_2)$ assigning to each $f \in \mathrm{A}(K_1)$ the affine functional $T(f) \in \mathrm{A}(K_2)$; where $T(f)(x_2) := f(\tau^{-1}(x_2)), \forall x_2 \in \mathcal{X}_2$. The fact that τ is a bijection that preserves convex combinations makes $T(f)$ a well-defined affine functional over K_2. Being the composition of two continuous maps $T(f) = f \circ \tau^{-1}$ is continuous.

That T is unital follows from the fact that $\forall x \in K_2$

$$T(1_{\mathrm{A}(K_1)})(x) = 1_{\mathrm{A}(K_1)}(\tau^{-1}(x)) = 1.$$

Next, we show that T is surjective. Let g be arbitrary in $\mathrm{A}(K_2)$. Define the affine functional $f \in \mathrm{A}(K_1)$ assigning to each $x \in K_1$ the value $f(x) := g(\tau(x))$. Being the composition of a continuous affine map and an affine functional, f is indeed in $\mathrm{A}(K_1)$. Clearly $T(f) = g$. Thus, T is surjective.

The fact that τ is a homeomorphism justifies the following computation: $\forall f \in \mathrm{A}(K_1)$,

$$\begin{aligned} \|T(f)\|_\infty &= \sup_{x \in K_2} |T(f)(x)| = \sup_{x \in K_2} |f(\tau^{-1}(x))| \\ &= \sup_{y \in K_1} |f(y)| = \|f\|_\infty. \end{aligned}$$

Thus T is an isometry.

Being a unital surjective isometry, T is invertible. Thus T and T^{-1} are unital isometries. However, since $\mathrm{A}(K_1)$ and $\mathrm{A}(K_2)$ are both operator systems respectively in the commutative C*-algebras $C(K_1)$ and $C(K_2)$ (see Section 1.4.8 or see [29, Theorem 3.9]), T is a unital complete isometry. Thus $\mathrm{A}(K_1)$ and $\mathrm{A}(K_2)$ are unitally completely order isomorphic (see [29, Theorem 3.5]).

(2) $\implies$ (1):

Let $T : \mathrm{A}(K_1) \longrightarrow \mathrm{A}(K_2)$ be a unital complete order isomorphism. Since T is an isometric isomorphism, so is $T^* : \mathrm{A}(K_2)^* \longrightarrow \mathrm{A}(K_1)^*$. Indeed, let ϕ be in $\mathrm{A}(K_2)^*$. The following simple computation shows that T^* is an isometry:

$$\begin{aligned}\|T^*(\phi)\| &= \sup_{f\in \mathrm{B}_1(\mathrm{A}(K_1))} |\langle T^*(\phi), f\rangle| \\ &= \sup_{f\in \mathrm{B}_1(\mathrm{A}(K_1))} |\langle \phi, T(f)\rangle| \\ &= \sup_{g\in \mathrm{B}_1(\mathrm{A}(K_2))} |\langle \phi, g\rangle| \\ &= \|\phi\|.\end{aligned}$$

Applying the preceding computation to the isometric isomorphism T^{-1} yields the fact that $(T^{-1})^*$ is an isometry. Furthermore, recall that, whenever T is invertible, the inverse $(T^*)^{-1}$ of T^* exists and coincides with $(T^{-1})^*$. Then, T^* is an isometric isomorphism.

Since T^* is an isometric isomorphism, we conclude two important facts. First, T^* maps the unit ball of $\mathrm{A}(K_2)^*$ onto that of $\mathrm{A}(K_1)^*$. Second, T^* is a homeomorphism from $\mathrm{A}(K_2)^*$ onto $\mathrm{A}(K_1)^*$ when both spaces are endowed with the wk*-topologies. As a conclusion, $T^*|_{\mathrm{B}_1(\mathrm{A}(K_2)^*)}$ is an affine homeomorphism of the unit ball of $\mathrm{A}(K_2)^*$ onto that of $\mathrm{A}(K_1)^*$.

Next, note that it is clear that since T is a complete order isomorphism, T^* maps the unital positive functionals in $\mathrm{A}(K_2)^*$ onto the unital positive functionals of $\mathrm{A}(K_2)^*$. Indeed, let ϕ be positive and unital in $\mathrm{A}(K_2)^*$ and f in $\mathrm{A}(K_1)^+$. Then, the fact that T is positive yields:

$$\langle T^*(\phi), f\rangle = \langle \phi, T(f)\rangle \geq 0.$$

The fact that T and ϕ are unital yields

$$\langle T^*(\phi), 1_{\mathrm{A}(K_1)}\rangle = \langle \phi, T(1_{\mathrm{A}(K_1)})\rangle = \phi(1_{\mathrm{A}(K_2)}) = 1.$$

Thus, T^* maps unital positive functionals in $\mathrm{A}(K_2)^*$ to unital positive functionals in $\mathrm{A}(K_1)^*$. Analogous reasoning shows that $(T^*)^{-1}$ maps unital positive functionals in $\mathrm{A}(K_1)^*$ to unital positive functionals in $\mathrm{A}(K_2)^*$.

Putting all these results together we see that $T^*|_{\mathrm{S}(\mathrm{A}(K_2))}$ is an affine homeomorphism from $\mathrm{S}(\mathrm{A}(K_2))$ onto $\mathrm{S}(\mathrm{A}(K_2))$, where both state spaces as endowed with the wk* −topologies. Thus, $\mathrm{S}(\mathrm{A}(K_2))$ is affinely homeomorphic to $\mathrm{S}(\mathrm{A}(K_1))$. However, by Theorem 1.2.6 we know that when $\mathrm{S}(\mathrm{A}(K_2))$ and $\mathrm{S}(\mathrm{A}(K_1))$ are endowed with the wk*-topologies, they are respectively affinely homeomorphic to K_2 and K_1. Thus, K_1 is affinely homeomorphic to K_2. □

Next, we represent operator systems on the space of the affine functionals over the state space. Lemma 2.1.4 is in a way a generalization of Theorem 1.2.6. Lemma 2.1.2 is well-known while the real-scalar version of Corollary 2.1.3 was shown by Jelett, see [22].

Lemma 2.1.2. *Let K be a convex compact subset of some locally convex Hausdorff space $\mathcal{X}$ and $\phi \in \mathrm{A}(K)^*$. There exist $r_1, r_2, r_3, r_4 \in \mathbb{R}_{\geq 0}$ and $x_1, x_2, x_3, x_4 \in K$; such that $\phi = (r_1\epsilon_{x_1} - r_2\epsilon_{x_2}) + \mathrm{i}(r_3\epsilon_{x_3} - r_4\epsilon_{x_4})$, where ϵ_x is the valuation map (see Theorem 1.2.6).*

Proof. Let K, $\mathcal{X}$ and ϕ be as in the statement of the Lemma. By Hahn-Banach Theorem, ϕ extends to a bounded functional ψ in $\mathrm{C}(K)^*$ of the same norm. By the Riesz Representation Theorem, there exists a complex measure $\mu \in \mathrm{M}_{\mathbb{C}}(K)$; satisfying $\psi(f) = \int_K f d\mu$ for each $f \in \mathrm{C}(K)$. Using the Jordan Decomposition Theorem and normalizing the measures we see that there exist $r_1, r_2, r_3, r_4 \in \mathbb{R}_{\geq 0}$ and probability measures μ_1, μ_2, μ_3 and μ_4 on K such that $\mu = r_1\mu_1 - r_2\mu_2 + \mathrm{i}(r_3\mu_3 - r_4\mu_4)$. Thus, using the Riesz representation theoreom again we see that $\psi = r_1\phi_1 - r_2\phi_2 + \mathrm{i}(r_3\phi_3 - r_4\phi_4)$, where $\phi_1, \ldots, \phi_4$ are states on $C(K)$. Restricting those states to $\mathrm{A}(K)$ and applying Theorem 1.2.6, we see that there exist $x_1, \ldots, x_4 \in K$; such that, $\epsilon_{x_1} = \phi_1|_{\mathrm{A}(K)}, \ldots, \epsilon_{x_4} = \phi_4|_{\mathrm{A}(K)}$. Thus, $\phi = \psi|_{\mathrm{A}(K)} = (r_1\epsilon_{x_1} - r_2\epsilon_{x_2}) + \mathrm{i}(r_3\epsilon_{x_3} - r_4\epsilon_{x_4})$. □

Corollary 2.1.3. *Let $\mathcal{X}$ be a locally convex Hausdorff space and $K \subseteq \mathcal{X}$ a compact convex subset. Let $\mathcal{S}$ be a subspace of $\mathrm{A}(K)$ that contains a constant function and separates the points of K. Thus, $\mathcal{S}$ is dense in $\mathrm{A}(K)$.*

Proof. Assume for the sake of contradiction that $\mathcal{S}_1 := \overline{\mathcal{S}}$ is a proper subset of $\mathrm{A}(K)$. Then, there exists a non-zero bounded functional ϕ on $\mathrm{A}(K)$ that vanishes on $\mathcal{S}_1$. By Lemma 2.1.2, there exist $x_1, \ldots, x_4 \in K$ and $r_1, \ldots, r_4 \in \mathbb{R}_{\geq 0}$ satisfying $\phi = (r_1\epsilon_{x_1} - r_2\epsilon_{x_2}) + \mathrm{i}(r_3\epsilon_{x_3} - r_4\epsilon_{x_4})$. Since $\mathcal{S}_1$ contains a constant function it must contain the unit. Then, we get:

$$\begin{aligned} 0 = \phi(1_{\mathrm{A}(K)}) &= ((r_1\epsilon_{x_1} - r_2\epsilon_{x_2}) + \mathrm{i}(r_3\epsilon_{x_3} - r_4\epsilon_{x_4}))(1_{\mathrm{A}(K)}) \\ &= r_1\, 1_{\mathrm{A}(K)}(x) - r_2\, 1_{\mathrm{A}(K)}(x) + \mathrm{i}(r_3 - r_4)\, 1_{\mathrm{A}(K)}(x) \\ &= r_1 - r_2 + \mathrm{i}(r_3 - r_4). \end{aligned}$$

Thus, $\phi = r_1(\epsilon_{x_1} - \epsilon_{x_2}) + ir_3(\epsilon_{x_3} - \epsilon_{x_4})$. On the other hand, since ϕ vanishes over $\mathcal{S}_1$, for every $f \in \mathcal{S}$, we have $\phi(f) = 0$. Hence, $r_1(f(x_1) - f(x_2)) + ir_3(f(x_3) - f(x_4)) = 0$ and, therefore, for every self-adjoint affine map g in $\mathcal{S}_{sa}$, we have

$$g(x_1) = g(x_2) \quad \text{and} \quad g(x_3) = g(x_4)$$

However, every $f \in \mathcal{S}$ can be expressed as a linear combination $g_1 + \mathrm{i}\, g_2$, where g_1 and g_2 are self adjoint in $\mathcal{S}$. Thus,

$$\forall f \in \mathcal{S}, f(x_1) = f(x_2) \quad \text{and} \quad f(x_3) = f(x_4)$$

Since $\mathcal{S}$ separates the points of K, we must have $x_1 = x_2$ and $x_3 = x_4$, making ϕ zero, a contradiction. □

Lemma 2.1.4 and Theorem 2.1.5 are due to Kadison and are usually referred to as Kadison's Representation Theorem (see [23]).

Lemma 2.1.4. *Let K be a compact Hausdorff space and $\mathcal{S}$ an operator system in $\mathrm{C}(K)$. The map assigning to each $s \in \mathcal{S}$, the affine valuation functional ϵ_s acting on $\mathrm{S}(\mathcal{S})$ is a unital complete order isomorphism onto its range.*

Proof. Let T denote the aforementioned map. That T is a unital linear map and positive is clear.

We show that T is injective. Let $s_1 \neq s_2 \in \mathcal{S}$. It is enough to show that there exists a state $\phi \in \mathrm{S}(\mathcal{S})$ satisfying $\phi(s_1) \neq \phi(s_2)$.

First, assume that s_1 and s_2 are self-adjoint. Consider the real vector space $\mathcal{V}$ generated by s_1, s_2 and 1. We can construct a positive unital real-linear functional ϕ' over $\mathcal{V}$ separating s_1 and s_2. Since $\mathcal{V}$ contains the order unit then it extends to a unital positive real-linear map ψ over $\mathcal{S}_{sa}$ (see Th 9.8, Ch 3 of [10]). Finally, consider the state $\phi(s) := \psi(s') + \mathrm{i}\,\psi(s'')$, where s' and s'' are respectively the real and complex parts of the function s, defined over $\mathcal{S}$. Then, we have

$$\phi(s_1) := \psi(s_1) \neq \psi(s_2) = \phi(s_2).$$

Next, assume s_1 and s_2 are arbitrary in $\mathcal{S}$ and express these functions in terms of their real and complex parts. Then, we have:

$$\begin{aligned} s_1 &= s_1' + \mathrm{i}\, s_1'' \\ s_2 &= s_2' + \mathrm{i}\, s_2''. \end{aligned}$$

Then, there exists a state ψ over $\mathcal{S}_{sa}$ such that $\psi(s_1') \neq \psi(s_2')$. Then, the state $\phi := \psi + \mathrm{i}\,\psi$ satisfies $\phi(s_1) \neq \phi(s_2)$.

Thus, T is injective.

Similar reasoning shows that for each $s \in \mathcal{S}$ if $\widehat{s}(\phi) = \phi(s) \geq 0, \forall \phi \in \mathrm{S}(\mathcal{S})$, then $s \geq 0$. Thus, ϕ^{-1} is unital positive.

Finally, since both C*-algebras $\mathrm{C}(K)$ and $\mathrm{C}(\mathrm{S}(\mathcal{S}))$ are commutative, we see that T and T^{-1} are unital completely contractive, making T a complete order isomorphism (see [29, Proposition 3.6 and Theorem 3.9] or Section 1.4.8). ☐

Theorem 2.1.5. *Let K be a compact Hausdorff space and $\mathcal{S} \subseteq \mathrm{C}(K)$ be an operator system. $\mathcal{S}$ is unitally completely order isomorphic to a dense operator sub-system of* $\mathrm{A}(\mathrm{S}(\mathcal{S}))$. *Hence, when $\mathcal{S}$ is closed, it is unitally completely order isomorphic to* $\mathrm{A}(\mathrm{S}(\mathcal{S}))$.

Proof. We may assume without loss of generality that $\mathcal{S}$ is closed. Let T denote the complete order isomorphism $s \longrightarrow \epsilon_s$ defined in Lemma 2.1.4. The image $T(\mathcal{S})$ of $\mathcal{S}$ is clearly a closed operator system. Furthermore, for each $s \in \mathcal{S}$, ϵ_x is the restriction of the linear map $\hat{s}$ to the compact convex set $\mathrm{S}(\mathcal{S})$; therefore, it is affine. Thus, $T(\mathcal{S})$ is a closed operator sub-system of $\mathrm{A}(\mathrm{S}(\mathcal{S}))$.

On the other hand, being an operator system containing the identity, $T(\mathcal{S})$ must contain all the constant functions on $\mathrm{S}(\mathcal{S})$. Furthermore, it separates the points of $\mathrm{S}(\mathcal{S})$. Thus, $T(\mathcal{S})$ coincides with $A(\mathrm{S}(\mathcal{S}))$ by Corollary 2.1.3. □

Corollary 2.1.6. *Let K_1, K_2 be compact Hausdorff spaces and $\mathcal{S}_1 \subseteq C(K_1)$, $\mathcal{S}_2 \subseteq C(K_2)$ be closed operator systems. Then, $\mathcal{S}_1$ is unitally completely order isomorphic to $\mathcal{S}_2$ if and only if $\mathrm{S}(\mathcal{S}_1)$ is affinely homeomorphic to $\mathrm{S}(\mathcal{S}_2)$.*

Proof. Recall that $\mathcal{S}_1$ and $\mathcal{S}_2$ are unitally completely order isomorphic respectively to $A(\mathrm{S}(\mathcal{S}_1))$ and $A(\mathrm{S}(\mathcal{S}_2))$ by Theorem 2.1.5. Thus, the fact that $A(\mathrm{S}(\mathcal{S}_1))$ and $A(\mathrm{S}(\mathcal{S}_2))$ are unitally completely order isomorphic is equivalent to $\mathcal{S}_1$ and $\mathcal{S}_2$ being unitally completely order isomorphic. Recall that $\mathrm{S}(\mathcal{S}_1)$ and $\mathrm{S}(\mathcal{S}_2)$ are convex and compact in the wk* topology. Hence, by Theorem 2.1.5, the claim that $A(\mathrm{S}(\mathcal{S}_1))$ and $A(\mathrm{S}(\mathcal{S}_2))$ are unitally completely order isomorphic is equivalent to $\mathrm{S}(\mathcal{S}_1)$ and $\mathrm{S}(\mathcal{S}_2)$ being affinely homeomorphic.

□

Definition 2.1.7. *Let K be a compact subset of a locally convex space and $f \in \mathrm{C}(K)$. f is said to be affinely extendible if it is the restriction of some affine map in $\mathrm{A}(\overline{\mathrm{co}}(K))$.*

Remark 2.1.8. *Let N be a normal operator and $\mathcal{S}_N$ the operator system generated by N. The unitally completely order isomorphic image of $\mathcal{S}_N$ in $\mathrm{C}(\sigma(N))$ under the functional calculus consists of functions that are affinely extendible to $\overline{\mathrm{co}}(\sigma(N))$. Indeed, this is due to the fact that this image is the operator system generated by z, $\overline{z}$ and 1 in $\mathrm{C}(\sigma(N))$ which are clearly affinely extendible.*

Theorem 2.1.9. *Let K be a compact subset of a locally convex Hausdorff space. Let $\mathcal{S}$ be a closed operator system in $\mathrm{C}(K)$ composed of functions that are affinely extendible to $\overline{\mathrm{co}}(K)$. The map assigning to each $s \in \mathcal{S}$ its affine extension to $\overline{\mathrm{co}}(K)$ is a well-defined unital complete order isomorphism onto its range.*

Proof. Denote by E the extension map assigning to each $s \in \mathcal{S}$ its affine extension $\overline{s} \in \mathrm{A}(\overline{\mathrm{co}}(K)$.

The function E is well-defined. Indeed, let $f, g \in \mathrm{A}(\overline{\mathrm{co}}(K))$ be two affine functions whose restrictions to K coincide. Let $M := \overline{\mathrm{co}}(K)$. By the Krein-Milman Theorem we know that $\partial_e(M) \subseteq K$; thus, f and g coincide over $\partial_e(M)$ and, by implication, over $\mathrm{co}(K)$. Therefore, by continuity of f and g, they must coincide over M. Hence E is well defined.

The map E is injective and the fact that it is unital is clear since the restriction of unit function of $\mathrm{A}(M)$ to K assigns to each $k \in K$ the number 1.

Furthermore, E is positive. Indeed, let $f \geq 0$ be in $\mathcal{S}$. By the Krein-Milman Theorem and the positivity of f, the value assigned by affine map $E(f)$ to each element in $\partial_e(M)$ is non-negative. Taking convex combinations of elements in $\partial_e(\overline{\mathrm{co}}(K))$, we see that $E(f)$ is non-negative over $\mathrm{co}(K)$. By continuity it is also non-negative over M. Thus, $E(f)$ is positive.

Next, consider E^{-1}, the restriction map that assigns to each affine map in the range of E its restriction to K. It is clear that E^{-1}, is unital and positive. Therefore, E and E^{-1} are unital positive maps. Thus, E is a complete order isomorphism (see [29, Proposition 3.6 and Theorem 3.9] or 1.4.8). □

2.2 A Single Normal Operator

In this and the following sections we use the results of the preceding section in order to classify some operator sub-systems of $\mathrm{C}(K)$ and compute their corresponding C*-envelopes. Among these operator systems are those generated by a single normal or unitary operator and multiple commuting normal operators. Furthermore, we will establish a link between the aforementioned results and Euclidean geometry over the spectra of certain operators. We shall exploit that link to support our classification results.

Proposition 2.2.1. *Let N be a normal operator. $\overline{\mathrm{co}}(\sigma(N))$ is affinely homeomorphic to $\mathrm{S}(\mathcal{S}_N)$ and $\mathrm{A}(\overline{\mathrm{co}}(\sigma(N)))$ is unitally completely order isomorphic to $\mathcal{S}_N$.*

Proof. Using the functional calculus we represent $\mathcal{S}_N$, via a unital complete order isometry, as the operator system $\mathcal{S}_z$ generated by z in $\mathrm{C}(\sigma(N))$. Note that z, $\overline{z}$ and 1 are respectively the restrictions of the polynomials $p(z) = z$, $\overline{p}(z) = \overline{z}$ and $1_{\mathrm{A}(\overline{\mathrm{co}}(\sigma(N)))}(z) = 1$ in $\mathrm{A}(\overline{\mathrm{co}}(\sigma(N)))$. Thus, they are affinely extendible. Then, by Theorem 2.1.9, we see that the operator system $\mathcal{S}_N$ is unitally completely order isomorphic to the operator system $\mathcal{S}'$ generated by z in $\mathrm{A}(\overline{\mathrm{co}}(\sigma(N)))$.

We claim that $\mathcal{S}'$ coincides with $\mathrm{A}(\overline{\mathrm{co}}(\sigma(N)))$. Indeed, S' separates the points of $\overline{\mathrm{co}}(\sigma(N))$ through the function z and contains all the constant functions. Furthermore, being finitely generated, $\mathcal{S}'$ is closed. Thus, by Corollary 2.1.3, $\mathcal{S}'$ coincides with $\mathrm{A}(\overline{\mathrm{co}}(\sigma(N)))$.

On the other hand, Theorem 2.1.5 implies that $\mathcal{S}_N$ is unitally completely order isomorphic to $\mathrm{A}(\mathrm{S}(\mathcal{S}_N))$. Therefore, applying Theorem 2.1.1, we conclude that $\mathrm{S}(\mathcal{S}_N)$ is homeomorphic to $\overline{\mathrm{co}}((\sigma(N))$. □

Theorem 2.2.2. *Let $N, M \in \mathcal{B}(H)$ be two normal operators. The operator systems $\mathcal{S}_N$ and $\mathcal{S}_M$, generated respectively by N and M, are unitally completely order isomorphic if and only if $\overline{\mathrm{co}}(\sigma(N))$ and $\overline{\mathrm{co}}(\sigma(M))$ are affinely homeomorphic.*

Proof. According to Proposition 2.2.1 we have that $\mathcal{S}_M$ and $\mathcal{S}_N$ are respectively affinely homeomorphic to $\mathrm{A}(\overline{\mathrm{co}}(\sigma(M)))$ and $A(\overline{\mathrm{co}}(\sigma(N)))$. Thus, by theoreom 2.1.1, $\mathcal{S}_N$ and $\mathcal{S}_M$ are completely order isomorphic if and only if $\overline{\mathrm{co}}(\sigma(N))$ and $\overline{\mathrm{co}}(\sigma(M))$ are affinely homeomorphic. □

Remark 2.2.3. *Let K be a non-empty compact convex subset of $\mathbb{R}^2$ and $E := \partial_e(K)$. Let $z \in \overline{E}$. Then, there exists a sequence of extreme points of K which we denote by $\{z_n\}$;*

$$\lim_{n\to\infty} z_n = z.$$

Then, $z \in K$.

Suppose z is not an extreme point of K. Then, there exists x and y in K satisfying:

$$z = \lambda x + (1 - \lambda)y.$$

Thus, since z_n converges to z in the Euclidean plane, there exists some integer m; for some $p > m$, the point z_p belongs to the close convex hull of $\{x, y, z_m\}$, a contradiction.

Theorem 2.2.4. *Let N be a normal operator over some Hilbert space and $K := \sigma(N)$ its spectrum. Let $\mathcal{S}$ denote the operator system generated by N. Then, the* C^**-envelope of* $\mathcal{S}_N$, $\mathrm{C}_e^*(\mathcal{S})$, *is* $\mathrm{C}(\partial_e(\overline{\mathrm{co}}(K)))$.

Proof. It is straightforward form Theorem 2.2.2 that $\mathcal{S}_N$ is unitally completely order isomorphic to $\mathcal{S}_z$ in $\mathrm{C}(E)$, where $E := \partial_e(\overline{\mathrm{co}}(\sigma(U)))$. Thus, by the uniqueness of the C^*-envelope within a complete order isomorphism, it is enough to show that

$$\mathrm{C}_e^*(\mathcal{S}_z) = \mathrm{C}(E).$$

To this end, we show that every boundary ideal for $\mathcal{S}_z$ in $\mathrm{C}(E)$ is trivial. Let $\mathcal{J}$ be a non-trivial ideal in $\mathrm{C}(E)$. Let q denote the restriction of its corresponding quotient map $q_{\mathcal{J}}$ to $\mathcal{S}_z$. It is sufficient to prove that q is not a complete order isomorphism, for then $\mathcal{J}$ would not be a boundary ideal. We will prove this by exhibiting a positive function in $\mathrm{C}(E)/\mathcal{J}$ whose pre-image under q in $\mathcal{S}_z$ is not positive.

First, note that by Remark 2.2.3, E is compact. Then, by the structure of ideals of C^*-algebras of C^*-algebras of continuous functions over Hausdorff compact spaces, there exists a closed non-empty proper subset of K; such that, the ideal $\mathcal{J}$ is precisely the set of all functions of $\mathcal{C}(E)$ that vanish over K. On the other hand, note that $E \setminus K$ is open in E and fix $a \in E \setminus K$. Thus, there exists an open set U containing a and has a compact closure $\overline{U}$ that is disjoint from K. Then, K and $\overline{U}$ can be completely separated. In other words, there exists a real-valued affine map over $\mathbb{C}$ whose restriction to E, denoted by ϕ, satisfies:

$$\sup_{x \in \overline{U}} \phi(x) < 0 < \inf_{x \in K} \phi(x).$$

It is clear that ϕ is in $\mathcal{S}_z$. Furthermore, since ϕ is continuous over E, there exists an open set V containing K; such that, $\phi|_V \geq 0$. Let

$$m := \sup_{x \in E \setminus V} |\phi(x)|$$

By Urysohn's Lemma, there exists a function $f \in \mathrm{C}(E)$ satisfying:

$$f|_{\overline{U}} = |m|$$
$$f|_K = 0$$

Thus, we have $f + \phi \geq 0$ and $\phi \notin \mathcal{S}_z^+$ but

$$q(\phi) = q_{\mathcal{J}}(\phi + f) \geq 0.$$

$\square$

Example 2.2.5. *Consider the normal matrices*

$$M_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2\,\mathrm{i} & 0 \\ 0 & 0 & 1-\mathrm{i} \end{bmatrix}, M_2 := \begin{bmatrix} 3+\mathrm{i} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5\,\mathrm{i}-2 \end{bmatrix}, M_3 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\,\mathrm{i} & 0 & 0 \\ 0 & 0 & 1-\mathrm{i} & 0 \\ 0 & 0 & 0 & 1/2+\mathrm{i} \end{bmatrix},$$

$$M_4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, M_5 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, M_6 := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Note that $\sigma(M_6) = \{\frac{\sqrt{3}\,\mathrm{i}-1}{2}, \frac{\sqrt{3}\,\mathrm{i}+1}{2}, 2\}$. *The regions* $\overline{\mathrm{co}}(\sigma(M_1))$, $\overline{\mathrm{co}}(\sigma(M_2))$ *and* $\overline{\mathrm{co}}(\sigma(M_3))$ *are all triangular in* $\mathbb{R}^2$. *Thus, they are all affinely homeomorphic. Therefore,* $\mathcal{S}_{M_1}$, $\mathcal{S}_{M_2}$ *and* $\mathcal{S}_{M_3}$ *are unitally completely order isomorphic by Theorem 2.2.2.*

Furthermore, by Theorem 2.2.4, the C^*-*envelopes of* $\mathcal{S}_{M_1}$, $\mathcal{S}_{M_2}$ *and* $\mathcal{S}_{M_3}$ *are* $*$*-isomorphic to* $(\mathbb{C}^3, \|.\|_\infty)$.

On the other hand, $\overline{\mathrm{co}}(\sigma(M_4)) = \overline{\mathrm{co}}\{1, i, -1, -i\}$ *is a square region; thus, it is not affinely homeomorphic to any triangular region since affine homeomorphisms preserve extreme points. Therefore,* $\mathcal{S}_{M_4}$ *is not unitally completely order isomorphic to* $\mathcal{S}_{M_1}$. *Finally, consider the matrices* M_4 *and* M_5. *The region* $\overline{\mathrm{co}}(\sigma(M_4))$ *is a square but* $\overline{\mathrm{co}}(\sigma(M_5))$ *is not even a parallelogram; thus,* $\overline{\mathrm{co}}(\sigma(M_5))$ *and* $\overline{\mathrm{co}}(\sigma(M_4))$ *are not affinely homeomorphic since affine homeomorphisms preserve parallel lines. Then,* $\mathcal{S}_{M_4}$ *and* $\mathcal{S}_{M_5}$ *are not unitally completely order isomorphic.*

However, we do have that $\mathrm{C}_e^*(M_4)$ *and* $\mathrm{C}_e^*(M_5)$ *coincide and are* $*$*-isomorphic to* $(\mathbb{C}^4, \|.\|_\infty)$.

Example 2.2.6. *Consider the measure space $[0,1]$ equipped with the inherited Borel $\sigma-$algebra and the Lebesgue measure and the continuous functions $f(x)=x^2$ and $g(x)=x^3+1$ over $[0,1]$. Denote by M_f and M_g the corresponding multiplication operators acting on the Hilbert space $L_2([0,1])$. Since f and g are continuous, we have $\sigma(f)=\{f(x);x\in[0,1]\}=[0,1]$ and $\sigma(g)=\{g(x);x\in[0,1]\}=[1,2]$. Being two segments of lines, clearly $\sigma(f)$ and $\sigma(g)$ are affinely homeomorphic making $\mathcal{S}_{M_f}$ and $\mathcal{G}_{M_g}$ unitally completely order isomorphic. Furthermore,*

$$\begin{aligned}\mathrm{C}_e^*(\mathcal{S}_f) \cong \mathrm{C}_e^*(\mathcal{S}_g) &= \mathrm{C}(\partial_e[0,1]) \\ &= (\mathbb{C}^2, \|.\|_\infty).\end{aligned}$$

Note that the multiplication operators corresponding to any two continuous real-valued functions over $[0,1]$ generate unitally completely order isomorphic operator systems. Indeed, let f and g be such functions. Then, the images of the compact connected $[0,1]$ under f and g must also be connected and compact; thus, intervals. Furthermore, those intervals are precisely the spectra of f and g and they are affinely homeomorphic.

On the other hand, when f or g are complex then the above conclusion need not follow. Consider for instance the maps $f(x)=x+x^2\,\mathrm{i}$ and $g(x)=x+x^3\,\mathrm{i}$. Consider the convex hulls of the spectra of f and g. Note, that

$$\begin{aligned}E &= \overline{\mathrm{co}}(\sigma(f)) = \overline{\mathrm{co}}\{x+x^2\,\mathrm{i}, x\in[0,1]\} \\ F &= \overline{\mathrm{co}}(\sigma(g)) = \overline{\mathrm{co}}\{x+x^3\,\mathrm{i}, x\in[0,1]\}.\end{aligned}$$

Clearly, E and F are not homeomorphic. Indeed, for contradiction's sake assume that there exists a homeomorphic affine map ϕ from E onto F. Being an affine map, ϕ maps $\partial_e E$ onto $\partial_e F$; furthermore, ϕ is the restriction of the real-linear map composed with a translation by a vector (see 1.2). Thus, there exists a $2\times 2-$matrix M and a vector $v\in\mathbb{R}^2$ satisfying

$$M(\partial_e E)+v=\partial_e F. \tag{2.1}$$

Set $M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $v=\begin{pmatrix} r \\ t\end{pmatrix}$ and substitute M and v by their set values in equation 2.1. We obtain

$$\begin{pmatrix} x \\ x^3\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x \\ x^2\end{pmatrix} + \begin{pmatrix} r \\ t\end{pmatrix}; \forall x\in[0,1]. \tag{2.2}$$

However, equation 2.2 and the fundamental theorem of algebra lead to a contradiction.

Example 2.2.7. *Let H be a separable infinite-dimensional Hilbert space with orthonormal basis $\{\eta_i\}_{i\in\mathbb{N}}$ and denote by P_i the projection over the span of η_i for each i in $\mathbb{N}$. Consider the unitary operators*

$$U := \sum_{i=1}^{\infty} e^{\mathrm{i}\,\pi/n} P_i$$
$$V := \sum_{i=1}^{\infty} e^{\mathrm{i}(\pi+\pi/n)} P_i.$$

Clearly, the spectrum of V is obtained from the spectrum of U by a rotation R of angle π around the origin of the complex plane. Furthermore, the points of $\sigma(U)$ and $\sigma(V)$ are respectively the extreme points respectively of $\overline{\mathrm{co}}(\sigma(U))$ and $\overline{\mathrm{co}}(\sigma(V))$. On the other hand, R is an affine homeomorphism. Putting all these facts together, we see that $\overline{\mathrm{co}}(\sigma(U))$ and $\overline{\mathrm{co}}(\sigma(V))$ are affinely homeomorphic. Thus, $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$ and

$$\mathrm{C}^*_e(U) = \mathrm{C}^*_e(V) = C_{\|.\|}(\overline{\sigma(U)}).$$

We shall see that the homeomorphism R is generic in the sense that it is the only possible homeomorphism between spectra of certain unitaries that generate unitally completely order isomorphic operator systems.

Remark 2.2.8. *Any two normal operators with each of their spectra composed of three non-collinear points generate unitally completely order isomorphic operator systems. Indeed, let N and M be two such normal operators. Thus, there exists an affine homeomorphism T mapping $\sigma(U)$ onto $\sigma(V)$ (see the proof of theorem 2.4.2). Thus, $T(\overline{\mathrm{co}}(\sigma(N))) = \overline{\mathrm{co}}(\sigma(M))$. Then, by theorem 2.2.2, $\mathcal{S}_N$ is unitally completely order isomorphic to $\mathcal{S}_M$.*

Generally speaking, any two normal operators with three extreme points in the convex hull of each of their spectra generate unitally completely order isomorphic operator systems.

Remark 2.2.9. *Let M and N be two normal operators. Recall that an affine homeomorphism ϕ map the extreme points of some convex set K onto the extreme points of $\phi(K)$. Then, by theorem 2.2.2, if $\mathcal{S}_N$ is unitally completely order isomorphic to $\mathcal{S}_M$ then $|\partial_e(\overline{\mathrm{co}}(\sigma(N)))| = |\partial_e(\overline{\mathrm{co}}(\sigma(N)))|$.*

2.3 Commuting Normal Operators

Our aim in this section is to use the tools we have built so far in order to characterize operator systems generated by finitely many commuting normal operators. The idea of affine extendibility introduced in Section 2.2 (see Theorem 2.1.9) and the technique of multi-variable functional calculus (see Section 1.3) will prove useful.

Throughout the following section we let $N := \{N_1, \dots, N_n\}$ be a set of commuting normal operators and denote by $\mathcal{S}_N$ the operator system generated by N. Recall that by the multi-variable functional calculus we know that $\mathrm{C}^*(N)$ is $*$-isomorphic to $\mathrm{C}(\sigma_J(N))$. Furthermore, under this $*$-isomorphism each N_k is mapped onto the $n-$variable polynomial p_k over $\sigma_J(N)$;

$$p_k(z_1, \dots, z_n) = z_k$$

for $k = 1, \dots, n$. We denote by $\mathcal{S}'_N$ the operator system generated by $p_1, \dots, p_n$ in $\mathrm{C}(\sigma_J(N))$ (i.e. the unitally completely order isomorphic image of $\mathcal{S}_N$ in $\mathrm{C}(\sigma_J(N))$. (see Section 1.3)

The preceding discussion and Theorem 2.1.9 yield the following Lemma:

Lemma 2.3.1. *The operator system $\mathcal{S}'_N$ is affinely extendible to a unitally completely order isomorphic image over $\overline{\mathrm{co}}(\sigma_J(N))$, which we denote by $\mathcal{S}$. Thus, $\mathcal{S}$ is unitally completely order isomorphic to $\mathcal{S}_N$. The ambient complete order isomorphism assigns N_k in $\mathrm{C}^*(N)$ the polynomial $\widetilde{p_k}(z_1, \dots, z_n) = z_k$ over $\overline{\mathrm{co}}(\sigma_J(N))$, for $k = 1, \dots, n$.*

Theorem 2.3.2. *The state space $\mathrm{S}(\mathcal{S}_N)$ is affinely homeomorphic to $\overline{\mathrm{co}}(\sigma_J(N))$.*

Proof. Let $K := \overline{\mathrm{co}}(\sigma_J(N))$. Refer to Lemma 2.3.1 and recall that the unital complete order isomorphism assigning to each N_k its corresponding image $\widetilde{p_k}$ in $\mathrm{C}(K)$ sends $\mathcal{S}_N$ onto $\mathcal{S}$. It is clear that $\mathcal{S} \subset \mathrm{A}(K)$, separates the points of K, contains all the constant affine functions and is closed. Hence, by Corollary 2.1.3, $\mathcal{S}$ coincides with $\mathrm{A}(K)$.

On the other hand, $\mathcal{S}_N$ is unitally completely order isomorphic to $\mathrm{A}(\mathrm{S}(\mathcal{S}_N))$ by Theorem 2.1.5. Thus, $\mathrm{A}(K)$ and $\mathrm{A}(\mathrm{S}(\mathcal{S}_N))$ are affinely homeomorphic. Therefore, by Theorem 2.1.1, $\mathrm{S}(\mathcal{S}_N)$ is affinely homeomorphic to $\overline{\mathrm{co}}(\sigma_J(N))$. ∎

Theorem 2.3.3. *Let $N := \{N_1, \dots, N_n\}$ and $M := \{M_1, \dots, M_n\}$ be two sets of commuting normal operators. Let $\mathcal{S}_\mathcal{N}$ and $\mathcal{S}_\mathcal{M}$ denote the operator systems generated respectively by N and M. $\mathcal{S}_\mathcal{N}$ and $\mathcal{S}_\mathcal{M}$ are unitally completely order isomorphic if and only if $\overline{\mathrm{co}}(\sigma_J(N))$ and $\overline{\mathrm{co}}(\sigma_J(M))$ are affinely homeomorphic.*

Proof. The fact that $\mathcal{S}_N$ is unitally completely order isomorphic to $\mathcal{S}_M$ is equivalent to $\mathrm{A}(\mathrm{S}(\mathcal{S}_N))$ being unitally completely order isomorphic to $\mathrm{A}(\mathrm{S}(\mathcal{S}_M))$ by Theorem 2.1.5. This, in turn, is equivalent to $\mathrm{S}(\mathcal{S}_N)$ being affinely homeomorphic to $\mathrm{S}(\mathcal{S}_M)$, according to Theorem 2.1.1. By Theorem 2.3.2, this is equivalent to $\overline{\mathrm{co}}(\sigma_J(N))$ being affinely homeomorphic to $\overline{\mathrm{co}}(\sigma_J(M))$. □

Theorem 2.3.4. *Let $K := \overline{\mathrm{co}}(\sigma_J(N))$. Then, $\mathrm{C}^*_e(N) = \mathrm{C}(\overline{\partial_e K})$.*

Proof. Let $K := \overline{\mathrm{co}}(\sigma_J(N))$ and $X := \overline{\partial_e K}$. We have already established that the operator system $\mathcal{S}_N$ is unitally completely order isomorphic to $\mathrm{A}(K)$. Hence, by the uniqueness of the C*-envelope (see Section 1.4.10), it is enough to show that $\mathrm{C}^*_e(\mathrm{A}(K)) = \mathrm{C}(X)$.

Let R denote the restriction function assigning to each element in $\mathrm{A}(K)$ its restriction to X.

The Krein-Milman Theorem tells us that R is injective and thus bijective onto its range. Furthermore, by Theorem 2.1.9 we know that R^{-1} is a unital complete order isomorphism. Thus, R is a unital complete order isomorphism.

On the other hand, for each $k = 1, \dots, n$, the image of each $\widetilde{p}_k$ under R is the $n-$variable polynomial over X, defined by $q_k(z_1, \dots, z_n) = z_k$. Denote by $\mathcal{S}'$ the range of R and note that it is the operator system generated by polynomials q_k; $k = 1, \dots n$. We show that the elements of the operator system $\mathcal{S}'$ separate the points of X and do not vanish at any point. Thus, by the Stone-Weierstrass Theorem, we conclude

$$\mathrm{C}^*(\mathcal{S}') = \mathrm{C}(X).$$

That the element of $\mathcal{S}'$ do not vanish over some point $\underline{x} \in X$ is clear form the fact that

$$1_{\mathcal{S}'}(x) = 1.$$

Let $\underline{x}$ and $\underline{y}$ be distinct tuples in X. Being distinct, they must disagree at some i^{th}-coordinate. Thus, we have

$$q_i(\underline{x}) = \underline{x}(i) \neq \underline{y}(i) = q_i(\underline{y}).$$

Therefore, by the uniqueness of the C^*-envelope, it is enough to prove that $\mathrm{C}^*_e(\mathcal{S}') = \mathrm{C}(X)$. Let $\mathcal{J}$ be an arbitrary proper non-trivial ideal in $\mathrm{C}^*(\mathcal{S}')$. Showing that $\mathcal{J}$ is not a boundary ideal will imply that the Silov boundary of $\mathcal{S}'$ in $\mathrm{C}(X)$ is trivial; thus, yielding our result. As in the proof of Theorem 2.2.4, let $q_{\mathcal{J}}$ denote the quotient map by $\mathcal{J}$ and q its restriction to $\mathcal{S}'$. Proceeding analogously shows that there exists a $\phi \in \mathcal{S}' \setminus \mathcal{S}'^{+}$ satisfying $q(\phi) \geq 0$. Thus, q is not a complete order isomorphism and we obtain our result.

□

2.4 A Single Unitary

The classification of operator systems generated by single unitary operators was investigated by Argerami et all, see [2]. They provided a classification for operator systems generated by single unitaries whose spectra contain at least 5 or at most 3 elements. They showed that two unitaries with at most three points in each of their spectra generate unitally completely order isomorphic operator systems if and only if their spectra have equal cardinalities. In the case when the cardinality of these spectra was at least five, they showed that the generated operator systems are unitally completely order isomorphic if and only if the spectra of the generating unitaries can be obtained from one another by a rotation (multiplication by a complex number in $\mathbb{T}$), a reflection across the x-axis (complex conjugation) or a combination of both. The main claim of this classification in the case of 5 points or more in the spectrum was reduced to a system of parametric equations which was relatively easy to solve. The system associated with the case when the spectrum contained exactly 4 points, though admits a solution in theory, was too complex to be translated into useful conditions. Thus, the case was left unresolved. Furthermore, the authors provided an example of two unitaries each of which has four points in its spectrum but the two unitaries generate operator systems that are not unitally completely order isomorphic.

Thus, showing that the case of four points in the spectrum is radically different from that of the three points. On the other hand, we will provide an example of two unitaries U and V such that $|\sigma(U)| = |\sigma(V)| = 4$ and $\sigma(U)$ can not be obtained from $\sigma(V)$ via any combination of rotations or reflections across the x-axis. However, these unitaries generate unitally completely order isomorphic operator systems. This provides us with evidence that the case of 4 points in the spectrum is radically different from that of at least 5 points in the spectrum.

In what follows, we will look deeper into this problem and explore the underlying reason behind this complexity arising in case 4. We will do this by reducing this classification problem to a problem involving the nature of affine homeomorphisms of two regions defined by concyclic quadrilaterals in the Euclidean plane. We perform this reduction using Proposition 2.2.1 and Theorem 2.2.2. Furthermore, using the same tools, we will provide an alternative proof for the cases of at most 3 and at least 5.

In Subsection 2.4.1 we state the Classification Theorem as it appeared in [2]. Then, we provide an alternative proof for it in hope of shedding some light on the 4−point case. We close this subsection by showing that case 4 is radically different from the other two cases. In Subsection 2.4.2 we provide a Classification Theorem for case 4. Then, we close this section by providing a sufficient condition that is very useful in practice (see Remark 2.4.10).

2.4.1 Cases $n \geq 5$ or $n \leq 3$

Theorem 2.4.1. *Let $U, V \in \mathrm{B}(H)$ be two unitary operators.*

If $|\sigma(U)| > 4$ then the following conditions are equivalent:

1. *$\sigma(U) = \lambda\sigma(V)$ or $\sigma(U) = \lambda\overline{\sigma(V)}$ for $\lambda \in \mathbb{T}$.*
2. *$\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$.*
3. *There exists a $*$-isomorphism $\pi : C^*(U) \longrightarrow C^*(V)$ satisfying $\pi(U) = \lambda V$ or $\pi(U) = \lambda V^*$, for some $\lambda \in \mathbb{T}$.*

On the other hand, if $\sigma(U) \leq 3$ then the following conditions are equivalent:

1. $|\sigma(U)| = |\sigma(V)|$.
2. $\mathcal{S}_U$ *is unitally completely order isomorphic to* $\mathcal{S}_V$.
3. $C^*(U)$ *is* $*$-*isomorphic to* $C^*(V)$ *via some* $*$-*isomorphism* π.

Before providing the new proof we need the following theorem:

Theorem 2.4.2. *Let T be an affine homeomorphism over $\mathbb{R}^2$. Let K be the number of distinct points mapped from $\mathbb{T}$ to $\mathbb{T}$ by T. If the number $K \geq 5$ then there exists $\lambda \in \mathbb{T}$ satisfying $T(\alpha) = \lambda\alpha$ or $T(\alpha) = \lambda\overline{\alpha}$ for each α in $\mathbb{T}$. On the other hand, any three distinct points of the unit circle can be mapped into three distinct points on the same circle via some affine homeomorphism.*

Proof. We begin by settling the second claim of the theorem. Fix a positive integer K. Denote by P_i, and Q_i some points of the unit circle for $1 \leq i \leq K$. Suppose further that the $P_i's$ are distinct from one another and the $Q_i's$ are distinct from one another.

Suppose $K = 3$. Consider the vectors $\overrightarrow{P_1P_2} \neq \overrightarrow{P_1P_3}$, and $\overrightarrow{Q_1Q_2} \neq \overrightarrow{Q_1Q_3}$. Since every point of $\mathbb{T}$ is extreme in the closed unit disk, each of the sets $B_1 := \{\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}\}$ and $B_2 := \{\overrightarrow{Q_1Q_2}, \overrightarrow{Q_1Q_3}\}$ form a basis for $\mathbb{R}^2$. Then, there exists an invertible real 2×2 matrix M mapping the set B_1 onto B_2. Consider the homeomorphism T assigning to each point P in $\mathbb{R}^2$ the point $M(\overrightarrow{P_1P}) + Q_1$. Clearly, $T(P_i) = Q_i$ for $i = 1, 2, 3$.

Suppose $K = 2$. Then, using the above reasoning it is clear to see that there exists a 2×2 invertible real matrix M mapping the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ respectively onto $\overrightarrow{OQ_1}$ and $\overrightarrow{OQ_2}$. Then, the affine homeomorphism assigning to each point P in $\mathbb{R}^2$ the point $M(\overrightarrow{OP}) + O$ is the sought affine homeomorphism. When $k = 1$, the translation vector $\overrightarrow{P_1Q_1}$ will do.

Next, suppose $K \geq 5$ and let T be as in the hypothesis satisfying $T(P_i) = Q_i$ for $1 \leq i \leq K$. Being an affine homeomorphism, T is the composition of multiplication by some real square matrix in $\mathrm{M}_2(\mathbb{R})$ and a translation by a fixed vector in $\mathbb{R}^2$. Expressing this last statement in terms of complex numbers, we see that there exist a fixed translation complex number γ and two complex numbers α and β satisfying $T(a) = \alpha a + \beta\overline{a} + \gamma 1$ for each $a \in \mathbb{C}$. Restricting T to the unit circle and expressing α, β and γ in terms of their respective moduli and arguments we obtain

$$T(e^{i\theta}) = |\alpha|e^{\mathrm{i}(\theta+\theta_\alpha)} + |\beta|e^{\mathrm{i}(-\theta+\theta_\beta)} + |\gamma|e^{i\theta_\gamma} \tag{2.3}$$

for each $\theta \in [0, 2\pi]$. Note that

$$T(e^{i\theta}) \in \mathbb{T} \iff T(e^{i\theta})e^{-\mathrm{i}(\frac{\theta_\alpha+\theta_\beta}{2})} \in \mathbb{T}$$

Multiplying equation (2.3) by $e^{-\mathrm{i}(\frac{\theta_\alpha+\theta_\beta}{2})}$ we obtain

$$T(e^{i\theta})e^{-\mathrm{i}(\frac{\theta_\alpha+\theta_\beta}{2})} = |\alpha|e^{\mathrm{i}(\theta+\frac{\theta_\alpha-\theta_\beta}{2})} + |\beta|e^{\mathrm{i}(-\theta-\frac{\theta_\alpha-\theta_\beta}{2})} + |\gamma|e^{i\theta_\gamma-\frac{\theta_\alpha+\theta_\beta}{2}} \tag{2.4}$$

However, as θ varies from 0 to 2π the curve charted by equation (2.4) is an ellipsoid centered at $|\gamma|e^{i\theta_\gamma-\frac{\theta_\alpha+\theta_\beta}{2}}$ with major symmetry axis parallel to the x-axis and minor symmetry axis parallel to the y-axis.

Indeed, in order to see why, consider the coordinates $x(\theta)$ and $y(\theta)$ of the point in the plane obtained from the right-hand side of (2.4). Writing the complex numbers in the right-hand side of (2.4) in terms of their real and complex parts we obtain

$$x(\theta) = (|\alpha| + |\beta|)\cos(\theta + \frac{\theta_\alpha - \theta_\beta}{2}) + |\gamma|\cos(\theta_\gamma - \frac{\theta_\alpha + \theta_\beta}{2}).$$
$$y(\theta) = (|\alpha| - |\beta|)\sin(\theta + \frac{\theta_\alpha - \theta_\beta}{2}) + |\gamma|\sin(\theta_\gamma - \frac{\theta_\alpha + \theta_\beta}{2}).$$

Thus, setting $l := |\gamma|\cos(\theta_\gamma - \dfrac{\theta_\alpha + \theta_\beta}{2})$ and $m = |\gamma|\sin(\theta_\gamma - \dfrac{\theta_\alpha + \theta_\beta}{2})$ we obtain

$$\frac{(x(\theta) - l)^2}{(|\alpha| + |\beta|)^2} + \frac{(y(\theta) - m)^2}{(|\alpha| - |\beta|)^2} = 1$$

Thus, having T map five or more distinct points of the unit circle onto the unit circle is equivalent to having the above ellipsoid intersect the unit circle at five or more distinct points. However, an ellipsoid is completely determined by five distinct points. Thus, the ellipsoid generated by T coincides with $\mathbb{T}$. Thus, $l = m = 0$ and $||\alpha| + |\beta|| = ||\alpha| - |\beta|| = 1$. Thus, we have

$$\begin{aligned} \gamma &= 0 \\ |\alpha| &= 1 \\ \beta &= 0 \end{aligned}$$

or

$$\begin{aligned} \gamma &= 0 \\ \alpha &= 0 \\ |\beta| &= 1. \end{aligned}$$

Thus, we are done. □

Next we provide the proof of Theorem 2.4.1.

Proof. Let K_1 and K_2 denote respectively $\overline{\text{co}}(\sigma(U))$ and $\overline{\text{co}}(\sigma(V))$ and identify the complex plane with the real vector space $\mathbb{R}^2$.

Since U and V are unitary operators, we have $\sigma(U), \sigma(V) \subseteq \mathbb{T}$. Thus $K_1, K_2 \subseteq \overline{\mathbb{D}}$; furthermore, every element of $\sigma(U)$ and $\sigma(V)$ is extreme in $\overline{\mathbb{D}}$. Thus, the points of $\sigma(U)$ and $\sigma(V)$ are extreme respectively in K_1 and K_2. On the other hand, by the Krein-Milman Theorem, the extreme points of K_1 and K_2 are contained respectively in the compacts $\sigma(U)$ and $\sigma(V)$. Thus,

$$\sigma(U) = \partial_e(K_1) \text{ and } \sigma(V) = \partial_e(K_2). \tag{2.5}$$

Case 1: Suppose $\sigma(U) \geq 5$.

$1 \iff 2:$

Assume $\sigma(U) = \lambda\sigma(V)$ for $\lambda \in \mathbb{T}$. Multiplying a complex number $\alpha := r_1 + \mathrm{i}\, r_2, r_1, r_2 \in \mathbb{R}$, by an element $\lambda \in \mathbb{T}$ corresponds to rotating the vector $(r_1, r_2)^t$ about the origin by an angle $\theta := \text{Arg}(\lambda)$. Let T denote this rotation. The matrix of T with respect to the standard basis in $\mathbb{R}^2$ is

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Then, $\sigma(U) \subset \mathbb{R}^2$ is obtained from $\sigma(V) \subset \mathbb{R}^2$ through T. Furthermore, being a rotation, T is an affine homeomorphism. Thus, it maps K_2 onto a compact convex set and it maps the extreme points of K_2 onto the extreme points of $T(K_2)$. Combining (2.5) with the fact that $T(\sigma(V)) = \sigma(U)$, we conclude that $T(K_2) = K_1$. Therefore, by Theorem 2.2.2, we conclude that $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$.

On the other hand, assume $\sigma(U) = \lambda\overline{\sigma(V)}$ for $\lambda \in \mathbb{T}$. Multiplying the conjugate of a complex number $\alpha := r_1 + \mathrm{i}\, r_2, r_1, r_2 \in \mathbb{R}$, by an element $\lambda \in \mathbb{T}$ corresponds to reflecting the vector $(r_1, r_2)^t$ across the x-axis then rotating it around the origin by an angle $\text{Arg}(\lambda)$. Thus $\sigma(U) \subset \mathbb{R}^2$ is obtained from $\sigma(V) \subset \mathbb{R}^2$ through a transformation $T \in \mathcal{M}(\mathbb{R}^2)$, which is the composition of a reflection and a rotation. More explicitly, the matrix of T with respect to the standard basis in $\mathbb{R}^2$ is

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}.$$

Furthermore, being the composition of a reflection and a rotation, T is an affine homeomorphism. Continuing in the same line of reasoning we followed in the case when $\sigma(U) = \lambda\sigma(V)$ we conclude that $T(K_2) = K_1$ and so $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$.

Conversely, suppose that $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$. By Theorem 2.2.2, K_1 is affinely homeomorphic to K_2. Thus, there exists an affine homeomorphism $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ mapping K_1 onto K_2. Since T is an affine homeomorphism, the extreme points of K_2 coincide with $T(\partial_e(K_1))$. Combining this fact with (2.5), we conclude:

$$T(\sigma(U)) = \sigma(V) \tag{2.6}$$

However, by our assumption we have $|\sigma(U)| > 4$; thus T maps five or more distinct points of $\mathbb{T}$ onto an equal number of points of $\mathbb{T}$. By Theorem 2.4.2, T must be a rotation around the origin or a composition of a reflection across the $x-$axis and a rotation around the origin. Thus, we have $\sigma(U) = \lambda\sigma(V)$ or $\sigma(U) = \lambda\overline{\sigma(V)}$, for some $\lambda \in \mathbb{T}$.

$2 \iff 3:$

Assume $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$. By the preceding proposition that we've just proven, namely $1 \iff 2$, there exist $\lambda \in \mathbb{T}$ and an affine homeomorphism T from $\sigma(V)$ onto $\sigma(U)$ satisfying:

$$T(\alpha) = \lambda\alpha \quad \text{or} \quad T(\alpha) = \lambda\overline{\alpha}; \forall \alpha \in \mathbb{C}.$$

Let π be the map from $\mathrm{C}(\sigma(U))$ to $\mathrm{C}(\sigma(V))$ assigning to each f the function $\pi(f)$, where

$$\pi(f)(\alpha) = f(T(\alpha)); \forall \alpha \in \sigma(V).$$

Note that π is a $*$-homomorphism and the fact that T is homeomorphic makes π into a $*$-isomorphism. Furthermore, by the definition of T, we have either $\pi(z) = \lambda z$ or $\pi(z) = \lambda\overline{z}$. Finally, using the functional calculus over $\mathrm{C}^*(U)$ and $\mathrm{C}^*(V)$ we extend π to a $*$-isomorphism $\widetilde{\pi}$ from $\mathrm{C}^*(U)$ onto $\mathrm{C}^*(V)$ satisfying $\widetilde{\pi}(U) = \lambda V$ or $\widetilde{\pi}(U) = \lambda V^*$.

Conversely, assume there exists a $*$-isomorphism $\pi : \mathrm{C}^*(U) \longrightarrow \mathrm{C}^*(V)$ satisfying $\pi(U) = \lambda V$ or $\pi(U) = \lambda V^*$, for some λ in $\mathbb{T}$. Then, set $\phi := \pi|_{\mathcal{S}_U}$. Being the restriction of a $*$-isomorphism, ϕ is a complete order isometry onto its range. That the range of ϕ is precisely $\mathcal{S}_V$ is clear from the facts that ϕ is unital and

$$\pi(U) = \lambda V$$
$$\pi(U^*) = \overline{\lambda}V^*$$

or

$$\pi(U) = \lambda V^*$$
$$\pi(U^*) = \overline{\lambda}V$$

Case 2: Suppose $\sigma(U) \leq 3$.

$1 \iff 2$: Suppose $|\sigma(U)| = |\sigma(V)|$. According to Theorem 2.4.2, there exists an affine homeomorphism T mapping the points of $\sigma(U)$ onto the points of $\sigma(V)$. Being an affine homeomorphism, T must map K_1 onto K_2. Thus, by Theorem 2.2.2, $\mathcal{S}_U$ and $\mathcal{S}_V$ are unitally completely order isomorphic.

Conversely, if $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$ then using Theorem 2.2.2 again, we conclude that K_1 and K_2 are affinely homeomorphic via some affine homeomorphism T. Then, T maps the extreme points K_1 one-to-one onto all the extreme points of K_2. Thus, T maps $\sigma(U)$ onto $\sigma(V)$.

$2 \iff 3$: Note that if $n := |\sigma(U)| \leq 3$ then we have

$$\mathrm{C}^*(U) = \mathbb{C}^n = \mathrm{span}\{U, U^*, 1\} = \mathcal{S}_U. \tag{2.7}$$

Suppose $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$ via a unital complete order isometry p. By our preceding proposition, namely that $1 \iff 2$, we conclude that $|\sigma(U)| = |\sigma(V)| \leq 3$. Thus, $\mathrm{C}^*(V) = \mathcal{S}_V$. Therefore, p is a u.c.i of $\mathrm{C}^*(U)$ onto $\mathrm{C}^*(V)$. However, by Lemma 1.4.13, we know that such complete order isomorphisms must be $*$-isomorphisms.

Conversely, assume that $\mathrm{C}^*(U)$ is $*$-isomorphic to $\mathrm{C}^*(V)$. Then, $|\sigma(U)| = |\sigma(V)| \leq 3$. Thus,

$$\mathcal{S}_V = \mathrm{C}^*(V),$$
$$\mathcal{S}_U = \mathrm{C}^*(U).$$

Since every $*$-isomorphism is a u.c.i., we obtain the complete order isomorphism between both operator systems.

□

Remark 2.4.3. *Note that one conclusion we can draw from the proof of the preceding theorem is that $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$ if and only if $\sigma(U)$ can be mapped onto to $\sigma(V)$ via an affine homeomorphism. Particularly, in the case when $|\sigma(U)| \leq 3$, this implies having the image $\pi(U)$ of U written as $\alpha V + \beta V^* + \gamma 1$. Furthermore, unlike the case of $|\sigma(U)| \geq 5$, neither α, β nor γ have to be null. This is due to the fact that any three distinct points of unit circle can be mapped by an affine homeomorphism onto any three distinct points of the same circle.*

Remark 2.4.4. *An alternative proof to Proposition* (1) $\implies$ (2) *in Case* 1 *above would be using Lemmas 1.4.12 and 1.4.14. Indeed, if $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$ via a u.c.i. ϕ then ϕ extends to a $*$-isomorphism from $\mathrm{C}^*_e(U)$ onto $\mathrm{C}^*_e(V)$. However, since U and V are unitaries, by 1.4.14 we have:*

$$\mathrm{C}^*(U) = \mathrm{C}^*_e(U),$$
$$\mathrm{C}^*(V) = \mathrm{C}^*_e(V).$$

Thus, ϕ extends to a $*$-isomorphism from $\mathrm{C}^*(U)$ onto $\mathrm{C}^*(V)$.

Example 2.4.5. *Note that when U is a unitary, Theorem 2.2.4 and Statement 2.5 in the preceding proof indicate that $\mathrm{C}^*_e(U) = \mathrm{C}(\sigma(U))$. Thus, in the case when $|\sigma(U)| = n$, we have $\mathrm{C}^*_e(U) = \mathbb{C}^n$. Thus, we can generate an example of operator systems that are not unitally completely order isomorphic but have the same C^*-envelope. Consider for instance the unitaries*

$$U := \mathrm{Diag}(1, \mathrm{e}^{\frac{\mathrm{i}\,\pi}{4}}, \mathrm{i}, -1, -\mathrm{i})$$
$$V := \mathrm{Diag}(1, \mathrm{e}^{\frac{\mathrm{i}\,\pi}{3}}, \mathrm{i}, -1, -\mathrm{i}).$$

Clearly, the spectrum of U is not obtained from the spectrum of V via a rotation about the origin, reflection nor a combination of both. Thus, by Theorem 2.4.2, $\overline{\mathrm{co}(\sigma(U))}$ is not affinely homeomorphic to $\overline{\mathrm{co}(\sigma(V))}$. Therefore, by Theorem 2.4, $\mathcal{S}_U$ is not unitally completely order isomorphic to $\mathcal{S}_V$. On the other hand, we have $\mathrm{C}^*_e(U) = \mathrm{C}^*_e(V) = \mathbb{C}^5$.

Remark 2.4.6. *Consider the proof of Theorem 2.4.1, specifically, the case when $|\sigma(U)| \geq 4$. We showed that a necessary and sufficient condition for $\mathcal{S}_U$ to be unitally completely order isomorphic to $\mathcal{S}_V$ is that $\sigma(V)$ is obtained from $\sigma(U)$ via a rotation and/or a reflection across the $x-$axis. Consider the following example, presented in [2, Section 4.2]:*

$$U := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{bmatrix} V := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1+\mathrm{i}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

We readily see that $\mathcal{S}_U$ is not unitally completely order isomorphic to $\mathcal{S}_V$. This is due to the facts that $\overline{\text{co}}(\sigma(U))$ is a square region while $\overline{\text{co}}(\sigma(V))$ is not even a parallelogram region. Hence, they can not be affinely homeomorphic since affine homeomorphisms preserve parallel lines. Thus, the case when the spectra contain four points each is different from the case of at most 3.

The following example provides a convincing argument that the case of four points in the spectrum is essentially different from the case of at least five points in the spectrum.

Example 2.4.7. *Consider the two unitary operators:*

$$U := \begin{pmatrix} e^{\mathrm{i}\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{\mathrm{i}\frac{3\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{\mathrm{i}\frac{5\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{\mathrm{i}\frac{7\pi}{4}} \end{pmatrix}$$

and

$$V := \begin{pmatrix} \frac{1}{2}\sqrt{\frac{7}{2}} + \frac{1}{2\sqrt{2}}\mathrm{i} & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\sqrt{\frac{7}{2}} + \frac{1}{2\sqrt{2}}\mathrm{i} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\sqrt{\frac{7}{2}} - \frac{1}{2\sqrt{2}}\mathrm{i} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{7}{2}} - \frac{1}{2\sqrt{2}}\mathrm{i} \end{pmatrix}.$$

The affine homomorphism:

$$T(z) = \frac{\sqrt{7}+1}{4} z + \frac{\sqrt{7}-1}{4}\overline{z}.$$

maps the spectrum of unitary U onto the spectrum of unitary V. Thus, $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$ by Theorem 2.2.2. However, it is clear that T is neither a rotation nor a composition of a rotation and a reflection across the $x-$axis. Indeed, T is not even angle preserving. Thus, the case when $n = 4$ is radically different from the case when $n \geq 5$.

2.4.2 Case when $n = 4$

The concluding part of the preceding section contained two examples that showed that case $n = 4$ is radically different from the cases $n < 4$ and $n > 4$. In this paragraph we present the characterisation for the case $n = 4$.

Remark 2.4.8. *Let z_1, z_2, z_3 and z_4 be four distinct complex numbers listed in increasing order of arguments. let p be the complex number associated to the point of intersection of segments of lines $[z_1, z_3]$ and $[z_2, z_4]$. Then, there exists $\alpha, \lambda \in (0, 1)$ satisfying*

$$p = \alpha z_1 + (1-\alpha) z_3 = \lambda z_2 + (1-\lambda) z_4. \tag{2.8}$$

Furthermore,

$$\alpha = \frac{\begin{vmatrix} \frac{z_4 - z_3 + \overline{z_4 - z_3}}{2} & \frac{z_4 - z_2 + \overline{z_4 - z_2}}{2} \\ \frac{z_4 - z_3 - \overline{z_4 - z_3}}{2} & \frac{z_4 - z_2 - \overline{z_4 - z_2}}{2} \end{vmatrix}}{\begin{vmatrix} \frac{z_1 - z_3 + \overline{z_1 - z_3}}{2} & \frac{z_4 - z_2 + \overline{z_4 - z_2}}{2} \\ \frac{z_1 - z_3 - \overline{z_1 - z_3}}{2} & \frac{z_4 - z_2 - \overline{z_4 - z_2}}{2} \end{vmatrix}} \quad \lambda = \frac{\begin{vmatrix} \frac{z_1 - z_3 + \overline{z_1 - z_3}}{2} & \frac{z_4 - z_3 + \overline{z_4 - z_3}}{2} \\ \frac{z_1 - z_3 - \overline{z_1 - z_3}}{2} & \frac{z_4 - z_3 - \overline{z_4 - z_3}}{2} \end{vmatrix}}{\begin{vmatrix} \frac{z_1 - z_3 + \overline{z_1 - z_3}}{2} & \frac{z_4 - z_2 + \overline{z_4 - z_2}}{2} \\ \frac{z_1 - z_3 - \overline{z_1 - z_3}}{2} & \frac{z_4 - z_2 - \overline{z_4 - z_2}}{2} \end{vmatrix}}.$$

In order to see why the last part is true, re-write equation (2.8) in terms of the real and imaginary parts of its components:

$$\begin{array}{l} \alpha \operatorname{Re}(z_1 - z_3) - \lambda \operatorname{Re}(z_2 - z_4) = \operatorname{Re}(-z_3 + z_4) \\ \alpha \operatorname{Im}(z_1 - z_3) - \lambda \operatorname{Im}(z_2 - z_4) = \operatorname{Im}(-z_3 + z_4) \end{array}.$$

Using Cramer's rule, we get the values of α and λ. Note that the solutions for α and λ do exist and their values fall in the interval $(0, 1)$.

Theorem 2.4.9. *Let $U, V \in \mathrm{B}(H)$ be two unitary operators. Assume the spectrum*

$$\sigma(U) := \{z_1, z_2, z_3, z_4\}$$

is listed in increasing order of arguments. Fix the unique $\alpha, \lambda \in (0, 1)$ satisfying

$$\alpha z_1 + (1-\alpha) z_3 = \lambda z_2 + (1-\lambda) z_4$$

(see remark 2.4.8). The following conditions are equivalent:

1. *$|\sigma(V)| = 4$ and $\{\alpha, \lambda\}$ are coefficients of intersection for the segments of lines $[w_1, w_3]$ and $[w_2, w_4]$; where w_1, w_2, w_3 and w_4 denote the points of the spectrum of V listed in increasing order of arguments (see the discussion preceding lemma 1.2.4).*

2. *$\overline{\text{co}}(\sigma(U))$ is affinely homeomorphic to $\overline{\text{co}}(\sigma(V))$.*

3. *$\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$.*

4. *There exists a $*$-isomorphism $\pi : C^*(U) \longrightarrow C^*(V)$ satisfying $\pi(U) = \alpha V + \beta V^* + \gamma\, 1_{C^*(V)}$; where α, β and $\gamma \in \mathbb{C}$.*

Before providing the proof let us consider the following. Let U and V be unitaries and $|\sigma(U)| = 4$. Furthermore, suppose $\mathcal{S}_V$ is unitally completely order isomorphic to $\mathcal{S}_U$. Then, by Theorem 2.2.2 we know that $\overline{\text{co}}(\sigma(U))$ is affinely homeomorphic $\overline{\text{co}}(\sigma(V))$ via some affine homeomorphism T. Since the spectra of both U and V are in $\mathbb{T}$, they are extreme points in respectively $\overline{\text{co}}(\sigma(U))$ and $\overline{\text{co}}(\sigma(V))$. By the Krein-Milman Theorem, they constitute all those extreme points. Then, $\sigma(U)$ is mapped onto $\sigma(V)$ by T. Thus, $|\sigma(V)| = 4$ and we see that the affine homeomorphism T maps a quadrilateral region inscribed in the unit circle onto another such region. Conversely, if $|\sigma(U)| = 4$ and $\overline{\text{co}}(\sigma(U))$ is affinely homeomorphic to $\overline{\text{co}}(\sigma(V))$ via T then, by the previous reasoning, we have that T maps a quadrilateral region inscribed in the unit circle onto another such region. Furthermore, we have that $\mathcal{S}_U$ is unitally completely order isomorphic to $\mathcal{S}_V$ by Theorem 2.2.2.

Thus, the problem at hand reduces to the question of finding a characterization of when two quadrilateral regions inscribed in the unit circle are affinely homeomorphic. Compositions of rotations and reflections across the $x-$axis affinely map such regions homeomorphically to one another but are there other types of affine homeomorphisms that do that?

Proof. $2 \iff 3$: This equivalence is a straightforward application of Theorem 2.2.2.

$1 \iff 2$: As in the proof of Theorem 2.4.1, let K_1 and K_2 respectively denote $\overline{\text{co}}(\sigma(U))$ and $\overline{\text{co}}(\sigma(V))$. Recall that we showed:

$$\sigma(U) = \partial_e(K_1) \text{ and } \sigma(V) = \partial_e(K_2). \tag{2.9}$$

Assume (2) is satisfied. Consider the affine homeomorphism T assigning to each z of the complex plane the number $\alpha z + \beta\overline{z} + \gamma$ and mapping K_1 onto K_2. Then, T maps the extreme points of K_1 onto the extreme points of K_2. Thus, T maps $\sigma(U)$ onto $\sigma(V)$. Thus, by lemma 1.2.4 we obtain our result.

On the other hand, assume (1) is true. Using lemma 1.2.4 again we see that there exists an affine homeomorphism T mapping $\sigma(U)$ onto $\sigma(V)$. Thus, T maps $\overline{\text{co}}(\sigma(U))$ onto $\overline{\text{co}}(\sigma(V))$.

That proposition (3) $\implies$ (4) is clear from the Lemmas 1.4.12 and 1.4.14. That (4) $\implies$ (3) is clear form the fact that the u.c.i. π maps the three-dimensional operator system $\mathcal{S}_U$ onto a three-dimensional operator system in the four-dimensional C^* −algebra $\text{C}^*(V)$. Then, since $\{V, V^*, 1\}$ is linearly independent, either V or V^* is in $\pi(\mathcal{S}_U)$. As a conclusion, $\pi(\mathcal{S}_U) = \mathcal{S}_V$.

□

Remark 2.4.10. *Theorem 1.2.3 provides an easy to use sufficient condition for four distinct concyclic points to be mapped onto four distinct concyclic points via and affine homeomorphism. Indeed, let A, B, C, D and A_1, B_1, C_1, D_1 be two sets of distinct concyclic points. By Radon theorem (see theorem 1.2.1), we may assume without loss of generality that*

$$[AC] \cap [BD] \neq \phi,$$

and

$$[A_1C_1] \cap [B_1D_1] \neq \phi.$$

If an affine homeomorphism T, which maps $\{A, B, C, D\}$ onto $\{A_1, B_1, C_1, D_1\}$, exists then, since T preserves intersection of segments of lines, we must have:

$$T(\{A, C\}) = \{A_1, C_1\}$$

and

$$T(\{B, D\}) = \{B_1, D_1\}$$

or

$$T(\{A,C\}) = \{B_1, D_1\}$$
$$\textit{and}$$
$$T(\{B,D\}) = \{A_1, C_1\}.$$

According to theorem 1.2.3, the first and second cases respectively yield

$$\frac{|AC|}{|BD|} = \frac{|A_1C_1|}{|B_1D_1|}$$

and

$$\frac{|AC|}{|BD|} = \frac{|B_1D_1|}{|A_1C_1|}.$$

Example 2.4.11. *Consider the following unitary matrices:*

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} + \mathrm{i}\,\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} - \mathrm{i}\,\frac{1}{\sqrt{2}} \end{bmatrix}, B := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{bmatrix},$$

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} - \mathrm{i}\,\frac{1}{2} \end{bmatrix}.$$

It is readily seen, from Remark 2.4.10, that the operator systems $\mathcal{S}_A$ and $\mathcal{S}_B$ are not unitally completely order isomorphic. Indeed,

$$\frac{1+1}{|\mathrm{i}+\mathrm{i}|} = 1 \neq \sqrt{2} = \frac{1+1}{\left|2\,\mathrm{i}\,\frac{1}{\sqrt{2}}\right|}$$

and

$$1 \neq \frac{1}{\sqrt{2}}.$$

On the other hand, the spectra of A and C satisfy the sufficient condition in remark 2.4.10. Indeed,

$$\frac{|1-(-1)|}{|\frac{1}{2}+\mathrm{i}\,\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}+\mathrm{i}\,\frac{1}{2}|}=\sqrt{2}=\frac{1+1}{\left|2\,\mathrm{i}\,\frac{1}{\sqrt{2}}\right|}.$$

However, the operator systems $\mathcal{S}_A$ and $\mathcal{S}_C$ are not unitally completely order isomorphic. In order to see why, we begin by computing the coefficients of intersection of the their spectra. The intersection of the segments of lines $[-1,1]$ and $[-\mathrm{i},\mathrm{i}]$ can be written as the convex combinations:

$$\frac{1+\sqrt{2}}{4}(1)+\frac{3-\sqrt{2}}{4}(-1)=\frac{1}{2}\left(\frac{1}{\sqrt{2}}+\mathrm{i}\,\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{1}{\sqrt{2}}-\mathrm{i}\,\frac{1}{\sqrt{2}}\right),$$

which yields intersection coefficients

$$C_1:=\left\{\frac{1}{2},\frac{1+\sqrt{2}}{4}\right\}$$

On the other hand, the intersection of segments of lines $[-1,1]$ and $[\frac{1}{2}+\mathrm{i}\,\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}-\mathrm{i}\,\frac{1}{2}]$ can be written as the convex combinations:

$$\frac{1+\sqrt{3}}{4}(1)+\frac{3-\sqrt{3}}{4}(-1)=$$
$$\frac{\sqrt{3}-1}{2}\left(\frac{1}{2}+\mathrm{i}\,\frac{\sqrt{3}}{2}\right)+\frac{3+\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}-\mathrm{i}\,\frac{1}{2}\right),$$

which yields intersection coefficients

$$C_2:=\left\{\frac{1+\sqrt{3}}{4},\frac{\sqrt{3}-1}{2}\right\}.$$

Clearly, neither of the elements of C_1 equals an element in C_2 nor could it be obtained as 1 minus an element of C_2.

Remark 2.4.12. *Let M and N be two normal operators, set $E_1:=\partial_e(\mathrm{co}(\sigma(M)))$ and $E_2:=\partial_e(\mathrm{co}(\sigma(N)))$ and suppose that $|E_1|=4$. By remark 2.2.9, a necessary condition for*

$\mathcal{S}_N$ *and* $\mathcal{S}_M$ *to be unitally completely order isomorphic is that* $|E_2| = 4$. *Refer to theorem 1.2.1 and remark 1.2.2 and note that each of the sets* E_1 *and* E_2 *can be partitioned uniquely in such a way that it produces a pair of intersecting segments of lines. Let* α *and* λ *be the intersection coefficients for* E_1. *Thus, by lemma 1.2.4 and remark 1.2.5,* $\mathcal{S}_N$ *and* $\mathcal{S}_M$ *are unitally completely order isomorphic if and only if* α *and* λ *are intersection coefficients for* E_2.

Remark 2.4.13. *Recall from the proof of theorem 2.4.1 that having* 5 *distinct points of the unit circle mapped onto* 5 *distinct points of the same circle via an affine homeomorphism* T *is equivalent to having an ellipsoid intersect the unit circle at* 5 *points. Specifically, this ellipsoid was defined to be the curve charted by* $T(e^{i\theta})$ *as* θ *varies from* 0 *to* 2π.

Similar reasoning shows that in the case when we have only four points, the problem at hand reduces to having the above ellipsoid intersect the unit circle at only four distinct points. Pursuing this line of reasoning, we can express these conditions as solutions to the zeros of a pencil of conics involving the equation of the ellipsoid and the equation of the unit circle. However, finding these zeros involves parametric third and second degree equations, the thing which is not useful when it comes to finding practical conditions.

Chapter 3

Unilateral Shifts

3.1 A Shift with Multiplicity 1

Let H denote an infinite dimensional separable Hilbert space and $S \in \mathcal{B}(H)$. The operator S is said to be a unilateral shift of multiplicity 1 if and only if there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ such that $Se_i = e_{i+1}$ for $i \in \mathbb{N}^+$. For each positive integer m, let P_m denote the orthogonal projection onto $\vee\{e_1, \ldots, e_m\}; m \in \mathbb{N}^+$. Let $\mathcal{K}(H) \subset \mathcal{B}(H)$ denote the norm-closed $*$-ideal of compact operators on H. For $v \in H$ and $\epsilon \in \mathbb{R}^+$, let $\mathrm{B}_\epsilon(v, H)$ denote the closed ball in H centered at v with radius ϵ. Denote by H^* the dual space of H. For $\phi \in H^*$ and $v, v_1 \in H$, we let $v \otimes \phi$ and $v \otimes v_1$ denote the rank-1 operators respectively defined as follows:

$$v \otimes \phi(h) := \phi(h)v$$
$$v \otimes v_1(h) := \langle h, v_1 \rangle v; \forall h \in H$$

Let $\mathcal{T}$, $\mathcal{S}_S$ and $q_{\mathcal{K}(H)}$ respectively denote the C*-algebra generated by S, the operator system generated by S and the quotient $*$-homomorphism of $\mathcal{B}(H)$ by the ideal $\mathcal{K}(H)$. Recall that S is an isometry, $\sigma(S) = \overline{\mathbb{D}}$ and the point spectrum $\sigma_p(S)$ of S is empty. For more on the properties of this operator see [11]. Our next aim is to show that the restriction of this quotient map to the operator system generated by S is a unital complete isometry. However, first we will need the following two Lemmas.

Lemma 3.1.1. *Let $S \in \mathcal{B}(H)$ be a unilateral shift of multiplicity* 1*, The* C*-algebra $\mathcal{T}$ *generated by S is irreducible. Furthermore, $\mathcal{K}(H)$ is a minimal ideal in $\mathcal{T}$.*

Proof. We begin by showing that every finite rank operator over H belongs to $\mathcal{T}$. $\forall m, n \in \mathbb{N}^+$ we have $e_n \otimes e_m \in \mathcal{T}$. Indeed, a routine computation shows that

$$e_n \otimes e_m(v) = S^{n-1}(1_{B(H)} - SS^*)(S^*)^{m-1}.$$

Thus, since $\mathcal{T}$ is closed, we have $v \otimes w \in \mathcal{C}_S$ for arbitrary v and w in H. Therefore, $\mathcal{T}$ contains all the rank-1 operators and, by extension, all the finite rank operators. Finally, since $\mathcal{K}(H)$ is the closure of the ideal of finite rank operators, we obtain

$$\mathcal{K}(H) \subseteq \mathcal{T}.$$

Next we show that $\mathcal{K}(H)$ is minimal in $\mathcal{T}$. Let $\mathcal{J}$ be a non-trivial closed $*$-ideal in $\mathcal{T}$. In order to prove that $\mathcal{K}(H) \subseteq \mathcal{J}$, it is enough to show that every rank-1 operator $v \otimes w$, such that $v, w \in H$, belongs to $\mathcal{J}$. Let $A \in \mathcal{J} \setminus \{0\}$ and $h \in H$, such that $Ah \neq 0$ and $\langle v, h \rangle \neq 0$. Set

$$T := A \circ (h \otimes h)$$
$$z := Tv \neq 0.$$

Note that $T \in \mathcal{J} \setminus \{0\}$. We have

$$v \otimes w = (\frac{1}{\|z\|^2} v \otimes z) \circ T \circ (v \otimes w) \in \mathcal{J}.$$

Finally, that $\mathcal{T}$ is irreducible is immediate from the facts that $\mathcal{K}(H) \subset \mathcal{T}$ and $\mathcal{K}(H)$ is weakly dense in $\mathcal{B}(H)$. □

The following lemma is straightforward. We will make use of it later on.

Lemma 3.1.2. *Consider the* C*-algebra $M_n(\mathcal{B}(H)) = \mathcal{B}(H^{(n)})$. *The following are true:*

1. $\mathcal{K}(H^{(n)}) = M_n[\mathcal{K}(H)]$.

2. $M_n[\mathcal{B}(H)]/\mathcal{K}(H^{(n)}) = M_n[\mathcal{B}(H)]/M_n[\mathcal{K}(H)]$.

3. $\mathcal{B}(H^{(n)})/M_n[\mathcal{K}(H)]$ *is* $*$*-isomorphic to* $M_n[(\mathcal{B}(H)/\mathcal{K}(H))]$.

Lemma 3.1.3. *The map* $q_{\mathcal{K}(H)}|_{\mathcal{S}_S}$ *is a unital complete isometry.*

Proof. Let $q := q_{\mathcal{K}(H)}|_{\mathcal{S}_S}$. We need to show that $q_n := 1_{\mathrm{M}_n} \otimes q$ is an isometry $\forall n \in \mathbb{N}^+$. Note that, $q_{\mathcal{K}(H)}$ is a $*$-homomorphism. Hence, it is completely contractive and, thus, so is q. Therefore, it is enough to show that for arbitrary $T \in M_n[\mathcal{S}_S]$, $\|q_n(T)\| \geq \|T\|$. By part (3) of Lemma 3.1.2 and the definition of the quotient norm, this is equivalent to showing that, $\inf_{K\in\mathcal{K}(H^{(n)})} \|T + K\| \geq \|T\|$.

To this end let $T \in \mathrm{M}_n(\mathcal{S}_S)$ and $K := (K_{i,j})_{1\leq i,j\leq n} \in \mathcal{K}(H^{(n)})$ be chosen arbitrarily. Writing each entry of T as a linear combination of S, S^* and the identity of $\mathcal{B}(H)$ we see that

$$T = A \otimes S + B \otimes S^* + C \otimes 1_{\mathcal{B}(H)} \tag{3.1}$$

for some $n \times n$ complex matrices A, B and C. Part (1) of Lemma 3.1.2 implies that each entry of K is compact.

Consider the orthonormal basis $\{f_{\underline{i}}\}_{\underline{i}\in\mathbb{N}^n}$ of $H^{(n)}$, where $f_{\underline{i}} = \frac{1}{\sqrt{n}}(e_{i_1}, \ldots, e_{i_n})^T$ for each $\underline{i} = (i_1, \ldots, i_n)$. Denote by P_m the projection onto the Hilbert subspace generated by $\vee\{e_{i_1,\ldots,i_n}; i_1, \ldots, i_n \leq m\}$.

We show that for each unit vector v in $H^{(n)}$, one can find another unit vector v_1 in the same Hilbert space satisfying,

$$\|Tv\| \leq \|(T + P_m K)v_1\|. \tag{3.2}$$

If this were true, then we would have $\|T\| \leq \|T + P_m K\|$. However, since K is compact, taking the limit of both sides of inequality 3.2 yields $\|T\| \leq \|T + K\|$. This clearly yields $\| 1_{M_n} \otimes q_{\mathcal{K}(H)}(T)\| = \|T\|$. Since n is arbitrary, we see that $q_{\mathcal{K}(H)}$ is a complete isometry.

Fix v and let $v_1 := 1_{\mathrm{M}_n} \otimes S^{m+1}v$. Note that by equation 3.1 $T(v_1)$ is orthogonal to $P_m K(v_1)$. Then, we have

$$\begin{aligned}
\|(T + P_m K)v_1\|^2 &\geq \|A \otimes S^{m+2} + B \otimes S^m + C \otimes S^{m+1}v\|^2 \\
&= \| 1_{\mathrm{M}_n} \otimes S^m(A \otimes S^2 + B \otimes 1_{\mathcal{B}(H)} + C \otimes S)v\|^2 \\
&= \|(A \otimes S^2 + B \otimes 1_{\mathcal{B}(H)} + C \otimes S)v\|^2 \\
&\geq \|(1_{\mathrm{M}_n} \otimes S^*)(A \otimes S^2 + B \otimes 1_{\mathcal{B}(H)} + C \otimes S)v\|^2 \\
&= \|Tv\|^2
\end{aligned}$$

□

Lemma 3.1.4. *The* C^**-algebra* $q_{\mathcal{K}(H)}(\mathcal{T})$ *is* $*$*-isomorphic to* $\mathrm{C}(\mathbb{T})$ *via a* $*$*-isomorphism assigning* $S + \mathcal{K}(H)$ *to* $z \in \mathrm{C}(\mathbb{T})$*.*

Proof. First, we note that $q_{\mathcal{K}(H)}(S)$ is a unitary. Indeed,

$$q_{\mathcal{K}(H)}(SS^* - S^*S) = q_{\mathcal{K}(H)}(SS^* - 1_{\mathcal{B}(H)}) = q_{\mathcal{K}(H)}(-e_1 \otimes e_1) = 0.$$

Therefore,

$$q_{\mathcal{K}(H)}(S)q_{\mathcal{K}(H)}(S)^* = q_{\mathcal{K}(H)}(S)^* q_{\mathcal{K}(H)}(S).$$

Combining this last equality with the fact that $q_{\mathcal{K}(H)}(S)^* q_{\mathcal{K}(H)}(S) = q_{\mathcal{K}(H)}(S^*S) = 1_{\mathcal{Q}}$ yields the result that $q_{\mathcal{K}(H)}(S)$ a unitary.

Next, note that the spectrum of $q_{\mathcal{K}(H)}(S)$ is the unit circle. The one-way inclusion is clear from the fact that $q_{\mathcal{K}(H)}(S)$ is a unitary. In order to prove the reverse inclusion we argue by contradiction as follows.

Let $S' = q_{\mathcal{K}(H)}(S)$ and suppose that $S' - \alpha 1$ is invertible for some α in $\mathbb{T}$. Then, there exists an operator A in $\mathcal{T}$ and a compact operator K such that,

$$A(S - \alpha\, 1_{\mathcal{B}(H)}) = 1_{\mathcal{B}(H)} + K. \tag{3.3}$$

On the other hand, since the point spectrum of S is empty and $\sigma(S) = \overline{\mathbb{D}}$, the approximate point spectrum of S must coincide with $\mathbb{T}$. Then, α is in the approximate point spectrum of S and there exists a sequence of unit vectors in $\{v_i\}_{i\in\mathbb{N}}$ in H, such that:

$$(S - \alpha\, 1_{\mathcal{B}(H)})v \neq 0, \forall v \in H \text{ and } \lim_{i\to\infty}(S - \alpha\, 1_{\mathcal{B}(H)})v_i = 0.$$

Furthermore, since K is compact, we may assume without loss of generality that $\{Kv_i\}$ converges to some w. Then, we have

$$\begin{aligned}\lim_{i\to\infty} v_i &= \lim_{i\to\infty}(1_{\mathcal{B}(H)} + K)v_i - \lim_{i\to\infty} Kv_i \\ &= \lim_{i\to\infty} A(S - \alpha\, 1_{\mathcal{B}(H)})v_i - w \\ &= -w\end{aligned}$$

Putting these results together, we obtain the contradiction:

$$(S - \alpha\, 1_{\mathcal{B}(H)})(-w) = (S - \alpha\, 1_{\mathcal{B}(H)}) \lim_{i\to\infty} v_i = \lim_{i\to\infty}(S - \alpha\, 1_{\mathcal{B}(H)})v_i = 0.$$

Finally, an application of the functional calculus shows the remainder of the lemma's claim. □

Theorem 3.1.5. *The map $q : \mathcal{S}_S \longrightarrow \mathrm{C}(\mathbb{T})$ assigning S to z and S^* to $\overline{z}$ is a unital complete isometry. Furthermore, $\mathrm{C}^*_e(\mathcal{T}) = \mathrm{C}(\mathbb{T})$.*

Proof. The first part of the theorem follows directly from Lemmas 3.1.3 and 3.1.4. The second part of the theorem follows directly from Theorem 2.2.4 □

3.2 Unilateral Shifts with Arbitrary Multiplicities

Next we characterize the operator system and the C*- envelope generated by a unilateral shift with arbitrary multiplicity μ. Let us first recall the definition of a unilateral shift with arbitrary multiplicity.

Definition 3.2.1. *$T \in \mathcal{B}(K)$ is said to be a unilateral shift with multiplicity μ if and only if there exists a sequence of Hilbert subspaces $\{K_i\}_{i\in\mathbb{N}}$ of K, such that:*

- $K_i \cong K_j, \forall i, j \in \mathbb{N}$.
- K_1 *has an orthonormal basis of cardinality* μ.
- $K_i \perp K_j, \forall i \neq j \in \mathbb{N}$.
- $K := \oplus_{i \in \mathbb{N}} K_i$.
- $T|_{K_i}$ *is a unitary from* K_i *onto* $K_{i+1}, \forall i \in \mathbb{N}$.

If $\mu = 1$ then definition 3.2.1 coincides with the definition of the unilateral shift of multiplicity 1 in the introduction of this chapter.

Lemma 3.2.2. *Let* T *be a unilateral shift of multiplicity* μ*. Then, it is the direct sum of* μ *unilateral shifts, each of multiplicity* 1*. Conversely, if* S *is a unilateral shift of multiplicity* 1 *over some separable Hilbert space* H *then* $\oplus_\mu S \in \mathcal{B}(H^{(\mu)})$ *is a unilateral shift of multiplicity* μ*.*

Proof. Let μ, T, K and $K_i, \forall i \in \mathbb{N}$, be as in definition 3.2.1. Let $\{k_\lambda\}_{\lambda \in \Lambda}$ be an orthonormal basis of K_1. Since T is a unilateral shift, the set

$$B_\lambda := \{T^i k_\lambda, i \in \mathbb{N}\}$$

is orthonormal for each $\lambda \in \Lambda$. Let $H_\lambda := \vee B_\lambda$. Note that H_λ is invariant for T, for each $\lambda \in \Lambda$. Furthermore, H_λ is reducing for T for each $\lambda \in \Lambda$. Indeed, Let P_λ be the orthogonal projection onto H_λ and note that $\{T^i k_\alpha\}_{i \in \mathbb{N}, \alpha \in \Lambda}$ forms an orthonormal basis of K. Then, for any vector k in basis B_λ of H_λ we have

$$TP_\lambda(T^i k) = P_\lambda T(T^i k), \forall i \in \mathbb{N}.$$

Thus,

$$TP_\lambda = P_\lambda T.$$

Note that $\cup_{\lambda \in \Lambda} B_\lambda$ forms a basis for K and $K = \oplus_{\lambda \in \Lambda} H_\lambda$. Set $S_\lambda := T|_{H_\lambda}$. Then, since each H_λ is reducing for T, we obtain $T = \oplus_{\lambda \in \Lambda} S_\lambda$. It is clear that each S_λ is a unilateral shift of multiplicity 1 acting over H_λ.

Conversely, Assume $S \in \mathcal{B}(H)$, is a unilateral shift with multiplicity 1 and $\{e_i\}_{i\in\mathbb{N}}$ an orthonormal basis of H satisfying $Se_i = e_{i+1}, \forall i \in \mathbb{N}$. Consider the Hilbert space $H^{(\mu)}$ and let Λ be a directed set of cardinality μ. Let $E := \{e_i, i \in \mathbb{N}\}$. Note that $\oplus_{\lambda\in\Lambda}E$ forms and orthonormal basis for H^{μ}. Denote the elements of this basis by e_i^{λ}; where

$$e_i^{\lambda}(\alpha) = \begin{cases} 0 & \alpha \neq \mu \\ e_i & \alpha = \mu \end{cases}.$$

Let

$$W_i := \vee\{e_i^{\lambda}, \lambda \in \Lambda\}$$

for each $i \in \mathbb{N}$. Then, it is clear that

$$\begin{aligned} &H^{(\mu)} = \oplus_{i\in\mathbb{N}} W_i, \\ &W_i \perp W_j, i \neq j, \\ &W_i \cong W_j, \forall i, j \in \mathbb{N}, \\ &\dim(W_i) = |\{e_i^{\lambda}, \lambda \in \Lambda\}| = |\Lambda| = \mu. \end{aligned}$$

Finally, fixing $i \in \mathbb{N}$ and setting $T := \oplus_{\mu} S$ we note that $T(e_i^{\lambda}) = e_{i+1}^{\lambda}$ and $T^*(e_i^{\lambda}) = e_{i-1}^{\lambda}$, $\forall \lambda \in \Lambda$, where $e_{-1}^{\lambda} = 0$. Thus, $T|_{W_i}$ is a unitary onto W_{i+1}. Thus $\oplus_{\mu} S$ is a unilateral shift with multiplicity μ. ☐

Lemma 3.2.3. *Let $T \in \mathcal{B}(K)$ and $V \in \mathcal{B}(L)$ be two unilateral shifts of respective multiplicities μ and ν. T is unitarily equivalent to V if and only if $\mu = \nu$.*

Proof. Assume $\exists U \in \mathcal{B}(L, K)$ unitary operator satisfying $U^*TU = V$. Note that $\mu = \dim(K \ominus T(K))$ and $\nu = \dim(V \ominus V(L))$. However,

$$\begin{aligned} L \ominus V(L) &= L \ominus U^*TU(L) = U^*(U(L) \ominus TU(L)) \\ &= U^*(K \ominus T(K)) \end{aligned}$$

and U^* is a unitary. Hence, we have $\mu = \nu$.

Conversely, assume that $\mu = \nu$. Set

$$K_i := T^{i-1}(K) \ominus T^i(K)$$
$$V_i := V^{i-1}(L) \ominus V^i(L).$$

Note that $K = \oplus_{i\in\mathbb{N}} K_i$ and $L = \oplus_{i\in\mathbb{N}} L_i$ and, since $\mu = \nu$, we have $K_i \cong L_j$ for $i, j \in \mathbb{N}$. Let $\{\eta_\lambda\}_{\lambda\in\Lambda}$ and $\{\xi_\lambda\}_{\lambda\in\Lambda}$ be orthonormal basis respective for K_0 and L_0. Then, we have that $\{T^i(\eta_\lambda)\}_{\lambda\in\Lambda}$ and $\{V_i(\xi_\lambda)\}_{\lambda\in\Lambda}$ are orthonormal basis respectively for K_i and L_i, $i \in \mathbb{N}$. For each natural number i there exists a unitary $U_i : K_i \longrightarrow L_i$ satisfying $U_i(T^i(\eta_\lambda)) = V^i(\xi_\lambda)$ for each λ in Λ. Then, for arbitrary $k_i \in K_i$, we have

$$\begin{aligned} U_i^* V U_i(k_i) &= U_i^* V \sum_\lambda \langle k_i, T^i \eta_\lambda \rangle U(T^i \eta_\lambda) \\ &= U_i^* V \sum_\lambda \langle k_i, T^i \eta_\lambda \rangle V^i \xi_\lambda \\ &= U_i^* \sum_\lambda \langle k_i, T^i \eta_\lambda \rangle V^{i+1} \xi_\lambda \\ &= \sum_\lambda \langle k_i, T^i \eta_\lambda \rangle T^{i+1} \xi_\lambda \\ &= T(k_i). \end{aligned}$$

As a conclusion, for arbitrary $k = \oplus_{i\in\mathbb{N}} k_i$ in K, we have

$$\begin{aligned} (\oplus_{i\in\mathbb{N}} U_i^* V \oplus_{i\in\mathbb{N}} U_i)(\oplus_i k_i) &= \oplus_{i\in\mathbb{N}} (U_i^* V U_i)(k_i) \\ &= \oplus_{i\in\mathbb{N}} T(k_i) \\ &= T(\oplus k_i). \end{aligned}$$

□

Corollary 3.2.4. *Let $S \in \mathcal{B}(H)$ be a unilateral shift of multiplicity 1 and T is a unilateral shift of multiplicity $\mu \in \mathcal{B}(K)$. Let $\phi : \mathcal{S}_S \longrightarrow \mathcal{S}_T$ be the linear unital map satisfying $\phi(S) = T$ and $\phi(S^*) = T^*$. Then, ϕ is a well defined complete order isomorphism.*

Proof. We adopt the notation of definition 3.2.1. When there's no confusion we set $1 := 1_{\mathcal{S}_S}$ and $1 := 1_{\mathcal{S}_T}$. Let $v \in K_1$ and recall that $T(v) \perp v$.

Note that $\{1, T, T^*\}$ is linearly independent. Indeed, assume

$$\alpha T + \beta T^* + \gamma 1 = 0$$

for some α, β and γ in $\mathbb{C}$. Then, $\alpha T(v) + \beta T^*(v) + \gamma v = \alpha T(v) + \gamma v = 0$. Thus, we must have $\alpha = \gamma = 0$. Therefore, α must be zero as well.

Since, T and μ are arbitrary, we must have $\{1, S, S^*\}$ linearly independent as well. Thus, ϕ is a well-defined unital bijective map.

We show that ϕ and ϕ^{-1} are both completely positive maps. By theorem 3.2.2, T is the direct sum of unilateral shifts each of which has multiplicity 1. Furthermore, each one of those shifts is unitarily equivalent to S via some unitary U_λ by theorem 3.2.3. Then, $A = \alpha T + \beta T^* + \gamma 1 \geq 0$ if and only if

$$\begin{aligned}
&\alpha \oplus_\lambda (U_\lambda^* S U_\lambda) + \beta \oplus_\lambda (U_\lambda^* S^* U_\lambda) + \gamma \oplus_\lambda 1 \\
&= \oplus_\lambda (\alpha U_\lambda^* S U_\lambda + \beta U_\lambda^* S^* U_\lambda + \gamma 1) \\
&= \oplus_\lambda U_\lambda^* (\alpha S + \beta S^* + \gamma 1) U_\lambda \geq 0.
\end{aligned}$$

The last operator in the equality is positive if and only if $\alpha S + \beta S^* + \gamma 1 \geq 0$.

Analogous reasoning shows that $M \otimes S + N \otimes S^* + P \otimes 1$ is positive if and only if

$$\begin{aligned}
&M \otimes \oplus_\lambda U_\lambda^* S U_\lambda + N \otimes \oplus_\lambda U_\lambda^* S^* U_\lambda + P \otimes \otimes_\lambda 1 \\
&= M \otimes T + N \otimes T^* + P \otimes 1 \geq 0
\end{aligned}$$

whenever, M, N and P are matrices in M_n for arbitrary non-negative integers n. Thus, ϕ and ϕ^{-1} are both unital and completely positive. As a conclusion, ϕ is a complete order isomorphism.

□

Theorem 3.2.5. *Let $T \in \mathcal{B}(K)$ and $V \in \mathcal{B}(W)$ be two unilateral shifts of respective multiplicities μ and ν. Then, $\mathcal{S}_T$ is unitally completely order isomorphic to $\mathcal{S}_V$ via a unital complete isometry assigning T to V and T^* to V^*. Furthermore, $\mathrm{C}_e^*(T)$ is $*$-isomorphic to $\mathrm{C}(\mathbb{T})$.*

Proof. This is immediate from the fact that both $\mathcal{S}_T$ and $\mathcal{S}_V$ are unitally completely order isomorphic to $\mathcal{S}_S$, the operator system generated by the unilateral shift of multiplicity 1. $\square$

Example 3.2.6. *A direct consequence of the preceding proposition is that two isometries need not be unitarily equivalent in order for them to have the same* C^**-envelope. Let* S_1 *and* S_2 *be unilateral shifts over some Hilbert space* H *with respective multiplicities* 1 *and* 2*. Then, by the preceding theorem we have*

$$\mathrm{C}_e^*(\mathcal{S}_{S_1}) = \mathrm{C}_e^*(\mathcal{S}_{S_2}) = C(\mathbb{T}).$$

3.3 A Single Isometry

Two of the operator systems we have characterized so far are those generated by a unilateral shift and those generated by a unitary operator. Our purpose in this present section is to use these two results in order to study operator systems generated by an isometry. The link allowing us to apply these previous results to our present endeavour is the Von Neumann-Wold decomposition. This decomposition asserts that every isometry can be written as the direct sum of a unilateral shift and a unitary operator with possibly either of the operators being trivial. Using the Von Neumann-Wold decomposition we will first compute the C^*-envelope of an operator system generated by an isometry, see Proposition 3.3.2. Next, we will characterize operator systems generated by an isometry, see Theorem 3.3.3. Below, we provide by way of reminder the statement of the decomposition and its proof. For more information see [28] or [11, Sec 23.7].

Theorem 3.3.1. *(Von Neumann-Wold Decomposition) Let* K *be an arbitrary Hilbert space and* $V \in \mathcal{B}(K)$ *an isometry. Let* H *denote the Hilbert-subspace* $\cap_{i=1}^{\infty} V^i K$ *of* K*. Then,* H *is reducing for* V*; furthermore,* $U := V|_H$ *is a unitary and* $S := V|_{H^\perp}$ *is a unilateral shift of multiplicity equal to the cardinality of the basis of* $H^\perp \ominus V H^\perp$*.*

Proof. Since V an isometry, $V^n(K)$ is closed for arbitrary integers n. Thus, H, the intersections of all such $V^n(K)'$s must be closed. On the other hand, it is clear that H is invariant for V. Indeed, for arbitrary η in H we have

$$\eta \in V^n(K)$$

for each integer n. Thus, $V(h)$ belongs to $V^{n+1}(K)$ for each integer n and so η must be in H. In order to see that H is reducing for H consider arbitrary vectors ξ in $H^\perp$ and η in H. It is enough to show that

$$\langle V^*(\eta), \xi \rangle = 0.$$

By definition of η we know that it belongs to $V^{n+1}(K)$ for each integer n. Thus, $V^*(\eta)$ belongs to $V^n(K)$ for each integer n and so it is in H. Setting

$$S = V|_{H^\perp} \textit{ and } T = V|_H,$$

we obtain

$$K = H^\perp \oplus H \textit{ and } V := \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}.$$

That T is a unitary is clear. Indeed, being the restriction of an isometry, T must be an isometry as well. We show that the range of T is $H^\perp$. To this end, let ξ be arbitrary in $H^\perp$. Since $H^\perp$ is reducing for V^* we have $V^*(\xi) = T^*(\xi)$ in $H^\perp$. Then, ξ is the image of $T^*(\xi) \in H^\perp$ under T and we're done.

Next, note that if $H^\perp$ is trivial then the proof is finished and S would be trivial. Assume that this is not the case. Then, in order to see that S is a unilateral shift, set $K_0 := H^\perp$ and for each positive integer n set

$$K_n := S^{n-1}(K_0) \ominus S^n(K_0).$$

It is enough to show that $S|_{K_n}$ is a unitary onto K_{n+1}, the Hilbert subspaces K_n's are pairwise orthogonal and $H^\perp = \oplus_{n=1}^\infty K_n$. In order to show that the K_n's are mutually orthogonal it suffices to show that none of the K_n's are trivial for then the orthogonality would be a consequence of the definition of the K_n's. Assume for the sake of contradiction that K_m is trivial for some fixed integer m. Then, $S^{m-1}(K_0)$ coincides with $S^m(K_0)$. However, since S is an isometry, applying $(S^*)^m$ to both sides of the preceding equation yields:

$$K_0 = S(K_0).$$

Then, V restricted to $H^\perp$ is a unitary. Thus we have

$$H^\perp = \cap_{n=0}^{\infty} V^n(H^\perp) \subseteq \cap_{n=0}^{\infty} V^n(K) = H.$$

By our assumption that $H^\perp$ is non-trivial we obtain a contradiction.

Finally, it is clear that since S is an isometry, it must map K_n onto K_{n+1}, making it a unitary onto its range. □

Note that by the proof of the preceding theorem we clearly see that the spectrum of an isometry V is either $\overline{\mathbb{D}}$, in which case the shift part of V is non-trivial, or a subset of $\mathbb{T}$ in which case the shift part of V is trivial.

Proposition 3.3.2. *Let V be an isometry over some Hilbert space K. Let $X := \partial_e(\overline{\text{co}}(\sigma(V)))$ be the set of extreme points of the closed convex hull of $\sigma(V)$. Denote by $p : \mathcal{S}_V \longrightarrow C(X)$ the map assigning z, $\overline{z}$ and 1 respectively to V, V^* and 1. Then, the following is true:*

- *p is a well-defined unital complete order isomorphism.*
- $\mathrm{C}_e^*(\mathcal{S}_V) = \mathrm{C}(X)$.
- *If the shift part of V is not trivial then $X = \mathbb{T}$.*
- *If the shift part of V is trivial then $X = \sigma(V)$.*

Proof. Let H, U and S be as in Theorem 3.3.1. If V is a unitary, in other words $H^\perp = \{0\}$, then we have our result by Theorem 2.2.4. If V is a unilateral shift, i.e. $H = \{0\}$, then the result follows from Theorem 3.2.5.

Therefore, we assume without loss of generality that neither H nor $H^\perp$ are trivial.

Let $\mathcal{S}_1 := \mathcal{S}_U \oplus \mathcal{S}_S$ and note that $\mathcal{S}_V \subseteq \mathcal{S}_1$. Consider the quotient map, $q_{\mathcal{K}(H^\perp)}$, by the C*-ideal of compact operators over $H^\perp$. According to Lemma 3.1.3 the map $q := q_{\mathcal{K}(H^\perp)}|_{\mathcal{S}_S}$

is a unital complete order isometry onto its range and the image of $\mathcal{S}_S$ under q is unitary with spectrum $\mathbb{T}$.

Next, consider the unital completely isometric map $1_{\mathcal{B}(H)} \oplus q : \mathcal{S}_1 \longrightarrow \mathcal{S}_U \oplus \mathcal{S}_{q_{\mathcal{K}(H)}(S)}$ and denote its restriction to $\mathcal{S}_V$ by q_1. Being the restriction of a unital complete order isometry, q_1 is a unital complete isometry onto its range which is the operator system generated by $q_1(V) = U \oplus q_{\mathcal{K}(H)}(S)$. Since both U and $q_{\mathcal{K}(H)}(S)$ are unitaries with $\sigma(q_{\mathcal{K}(H)}(S)) = \mathbb{T}$ and $\sigma(U) \subseteq \mathbb{T}$, we have, $q_1(V)$ is a unitary with spectrum equal to $\sigma(U) \cup \mathbb{T} = \mathbb{T}$.

Next, applying the functional calculus and using theorem 2.2.4 and the uniqueness of the C*-envelope, we obtain the rest of our claims. □

Theorem 3.3.3. *Let V_1 and V_2 be two isometries. Denote by $\mathcal{S}_1$ and $\mathcal{S}_2$ the operator systems generated respectively by V_1 and V_2. One and only one of the following two cases is true:*
Case 1: $\mathbb{T} \subseteq \sigma(V_1)$. The following are equivalent:

- *$\mathcal{S}_1$ is completely order isomorphic to $\mathcal{S}_2$ via a unital complete order isometry assigning V_2 to V_1.*
- *$\mathbb{T} \subseteq \sigma(V_2)$.*

Case 2: $\sigma(V_1) \subsetneq \mathbb{T}$. Then, V_1 is a unitary and the following conditions are equivalent:

- *$\mathcal{S}_1$ is unitally completely order isomorphic to $\mathcal{S}_2$.*
- *V_2 is unitary and satisfies, together with V_1, the classification conditions for operator systems generated by single unitaries presented in section 2.4.*

Proof. Let $X_1 := \partial_e(\overline{\text{co}(V_1)})$ and $X_2 := \partial_e(\overline{\text{co}(V_2)})$. Refer to proposition 3.3.2 and consider the unital completely isometric maps q_1 and q_2 mapping $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively into $\mathrm{C}(X_1)$ and $\mathrm{C}(X_2)$; such that, $q_1(V_1) = z_1 \in \mathrm{C}(X_1)$ and $q_2(V_2) = z_2 \in \mathrm{C}(X_2)$.

Case 1: $\mathbb{T} \subseteq \sigma(V_1)$. According to the Von-Neumann-Wold decomposition, we have either V_1 is a unitary with spectrum $\mathbb{T}$ or the shift part of V_1 is not trivial. In either of these cases we obtain $X_1 = \mathbb{T}$.

Assume $\mathcal{S}_1$ is completely order isomorphic to $\mathcal{S}_2$ via a unital complete order isometry assigning V_2 to V_1. Then, by the discussion in the beginning of this proof, we obtain that $\mathcal{S}_{z_1}$ and $\mathcal{S}_{z_2}$ are unitally completely order isomorphic. Thus, by theorem 2.4.1, we obtain $X_2 = \mathbb{T}$. Thus, by proposition 3.3.2, we have $\mathbb{T} \subseteq \sigma(V_2)$.

On the other hand, assume that $\mathbb{T} \subseteq \sigma(V_2)$. Thus, by proposition 3.3.2 and the hypothesis of case 1, we obtain $X_1 = X_2 = \mathbb{T}$. Thus, $\mathcal{S}_{z_1}$ is unitally completely order isomorphic to $\mathcal{S}_{z_2}$ via a unital complete order isomorphism assigning z_1 to z_2. Combining this result with the discussion in the beginning of this proof, we obtain our result.

Case 2: $\sigma(V_1) \subsetneq \mathbb{T}$. Then, by proposition 3.3.2, the shift part of V_1 is trivial; thus, V_1 is an isometry.

Assume that $\mathcal{S}_1$ is unitally completely order isomorphic to $\mathcal{S}_2$. Then, by the discussion beginning of this proof, we have $\mathcal{S}_{z_1}$ is unitally completely order isomorphic to $\mathcal{S}_{z_2}$. Thus, by theorems 2.4 and 2.4.9, we obtain that V_2 is a unitary its spectrum satisfies the conditions of the classification of operator systems generated by single unitaries.

Conversely, assume that V_2 is a unitary. Furthermore, assume that the spectra of V_1 and V_2 satisfy one of the conditions set forth in theorems 2.4.9 or 2.4.1, then $\mathcal{S}_1$ is unitally completely order isomorphic to $\mathcal{S}_2$. □

Example 3.3.4. *Let H be an infinite dimensional separable Hilbert space. Let S denote the unilateral shift over H and U a unitary over H with $\sigma(U) = \mathbb{T}$. Set $S_2 := S^2$ and let*

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{bmatrix}, B := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\mathrm{i}\frac{2\pi}{3}} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & e^{-\mathrm{i}\frac{\pi}{3}} & 0 \\ 0 & 0 & 0 & 0 & e^{i\frac{\pi}{6}} \end{bmatrix}.$$

and

$$V_1 := \begin{pmatrix} S & 0 \\ 0 & A \end{pmatrix}, V_2 := \begin{pmatrix} S_2 & 0 \\ 0 & B \end{pmatrix}, V_3 := \begin{pmatrix} S & 0 \\ 0 & B \end{pmatrix}.$$

By Theorem 3.3.3, $\mathcal{S}_S$ and $\mathcal{S}_U$ are unitally completely order isomorphic via uci assigning λU or $\overline{\lambda} U$ to S. However, it is evident as predicted by the same theorem that $\mathrm{C}^*(U)$ is not *unitally completely order isomorphic to* $\mathrm{C}^*(S)$*. Indeed, one only has to notice that while* $\mathrm{C}^*(U)$ *is commutative,* $\mathrm{C}^*(S)$ *is not. Similar reasoning shows that* $\mathcal{S}_S$, $\mathcal{S}_{S_2}$, $\mathcal{S}_U$, $\mathcal{S}_{V_1}$, $\mathcal{S}_{V_2}$ *and* $\mathcal{S}_{V_3}$ *are unitally completely order isomorphic to one another.*

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