This is a final draft of a work, prior to publisher editing and production, that appears in Siam J. Control Optim. Vol. 56, No 1, pp 120-147.
http://dx.doi.org/10.1137/140993417

LINEARIZED STABILITY OF PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATION TO STABILIZATION OF THE KURAMOTO-SIVASHINSKY EQUATION*

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Abstract. Linearization is a useful tool for analyzing the stability of nonlinear differential equations. Unfortunately, the proof of the validity of this approach for ordinary differential equations does not generalize to all nonlinear partial differential equations. General results giving conditions for when stability (or instability) of the linearized equation implies the same for the nonlinear equation are given here. These results are applied to stability and stabilization of the Kuramoto-Sivashinsky equation, a nonlinear partial differential equation that models reaction-diffusion systems. The stability of the equilibrium solutions depends on the value of a positive parameter ν. It is shown that if $\nu > 1$, then the set of constant equilibrium solutions is globally asymptotically stable. If $\nu < 1$ then the equilibria are unstable. It is also shown that stabilizing the linearized equation implies local exponential stability of the equation. Stabilization of the Kuramoto-Sivashinsky equation using a single distributed control is considered and it is described how to use a finite-dimensional approximation to construct a stabilizing controller. The results are illustrated with simulations.

Keywords: Stability; control; stabilization; Kuramoto-Sivashinsky; partial differential equations; linearized stability

1. Introduction. The Kuramoto-Sivashinsky (KS) equation was introduced by Kuramoto [34] in one space dimension for the theoretical study of a turbulent state in a distributed chemical reaction system. The KS equation is a mathematical model of reaction-diffusion systems and is related to various pattern formation phenomena where turbulence or chaos appear [4, 20, 24, 32, 37, 39, 47, 48].

Many researchers have studied the stability of the dynamics of the KS equation numerically; see for instance, [5, 10, 14, 17, 21, 22, 25, 30]. Lyapunov's indirect method was used to analyze the stability of the KS equation in [4, 12, 23, 40, 45]. Analytical results using the linearization as well as numerical studies of the dynamics of the KS equation have indicated that the KS equation is unstable for small values of a parameter ν. In [56], it was shown that the zero equilibrium solution of the KS equation, with periodic boundary conditions and odd initial condition, is globally exponentially stable for certain values of the instability parameter ν. A more general result will be obtained in this paper.

A number of papers on stabilization of the KS equation have been published. Boundary control of the KS equation has been widely explored [20, 32, 38, 39, 47]. The basic idea is to choose the boundary conditions so that the energy of the nonlinear system decays to zero exponentially. Distributed control of the KS equation has been approached by stabilizing the corresponding linearized system [1, 4, 33, 37, 40, 49].

Use of Lyapunov's indirect method for infinite-dimensional systems requires justification that the stability of the linearized systems reflects the stability of the nonlinear system. The proof for finite-dimensional systems does not generalize, and in fact stability of a linearized PDE does not always imply the same for the original PDE. In [16] it is shown that a nonlinear wave equation can fail to be exponentially stable

*The numerical results in section 6 of this paper were presented at the 2015 American Control Conference.

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This research was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

even when the linearization is exponentially stable. Another counter-example, due to Zwart [57], is given in this paper where the linearization is asymptotically stable but the nonlinear system is not. A further example is given in [58, sect. 18.3].

In this paper general conditions for when Lyapunov's indirect method can be used for infinite-dimensional systems are provided. It is proven that the KS equations satisfies these conditions. That is, it is proven that stabilizing the linearized KS equation will stabilize the nonlinear infinite-dimensional KS equation.

There are some results justifying the use of a linearization to analyze the stability of a nonlinear infinite-dimensional system. Most results assume that the nonlinear part of the generator is continuous, and satisfies additional assumptions. In [50] the following class of quasi-linear systems on a Banach space X is considered:

$$\begin{aligned} \dot{z}(t) &= Az(t) + f(z(t)), \\ z(0) &= z_0, \end{aligned}$$

where $z(t) \in X$ is the state and z_0 is the initial condition. The operator $A : \mathcal{D}(A) \subset X \to X$ is a linear operator that generates a C_0-semigroup in X and the nonlinear operator $f : X \to X$ is Fréchet differentiable. These assumptions imply that the nonlinear C_0-semigroup corresponding to the nonlinear system is continuously Fréchet differentiable [50, Theorem 11.18]. This was used to prove that if the system linearized at an equilibrium solution is exponentially stable, then the nonlinear system is locally exponentially stable [50, Theorem 11.22]. The conditions on f were relaxed in [28, Cor. 2.2]. This is a special case of the general result [28, Thm. 2.1] for which the conditions are difficult to check. In [51, Sect. VI.8] conditions for linearized stability where the nonlinear part may be discontinuous and the linear part of the generator is self-adjoint and non-negative are provided. However, the KS equation has a discontinuous nonlinearity that does not satisfy the assumptions in [51].

In the next section, general results on linearized stability analysis for dynamical systems on Banach spaces are presented. The key point is that the nonlinear semigroup must be Fréchet differentiable with derivative corresponding to the semigroup of the linearization. Also, the linearized system must be exponentially stable in order for a prediction of local exponential stability of the nonlinear system to be obtained. Earlier results justifying linearized stability analysis can be regarded as special cases of this result. The KS equation is then described along with some properties of the system. It is shown that if $\nu > 1$, then the set of constant equilibrium solutions is globally asymptotically stable. Fréchet differentiability of the C_0-semigroup corresponding to the controlled KS equation is proven in section 4. This is used to show that if $\nu < 1$ then the constant equilibria are unstable. In section 5, an approach to design of stabilizing feedback controllers for the KS equation using finite-dimensional approximations of the linearization is described. These results are illustrated in section 6 by a numerical example showing control of an unstable equation between different states.

2. Linearized stability of partial differential equations. Since it is often difficult to find a Lyapunov function, it is natural to use Lyapunov's indirect method to analyze the stability of nonlinear infinite-dimensional dynamical systems. However, the proof for ordinary differential equations that stability of the linearized system implies local stability of the original system relies on the finite-dimensionality of the underlying state-space; see for instance [31]. For infinite-dimensional systems, the asymptotic stability of the linearized system does not always imply the asymptotic stability of the original nonlinear infinite-dimensional system; see for instance [16] where it is shown that a nonlinear wave equation can fail to be exponentially stable even when the linearization is exponentially stable. This point is further illustrated here by the following counter-example found by Hans Zwart [57] .

EXAMPLE 2.1. *[57] Consider the nonlinear system defined on the Hilbert space* $l^2(\mathbb{C})$.

$$\begin{aligned} \dot{z}(t) &= F(z(t)), \ t \geq 0 \\ z(0) &= z_0. \end{aligned} \tag{2.1}$$

where $z = (z_1, z_2, \cdots, z_n, \cdots)$, $z_0 = (z_{01}, z_{02}, \cdots, z_{0n}, \cdots)$ *for* $, n = 1, \cdots, \infty$ *and*

$$F(z(t)) = -\begin{pmatrix} 1 & 0 & \dots & \\ 0 & \ddots & & \\ & & \frac{1}{n} & \\ & & & \ddots \end{pmatrix} z(t) + \begin{pmatrix} z_1^2(t) \\ \vdots \\ z_n^2(t) \\ \vdots \end{pmatrix}. \tag{2.2}$$

This system has infinitely many equilibrium solutions since $F(z) = 0$ *if and only if* $-\frac{1}{n}z_n + z_n^2 = 0$ *for* $n = 1, \cdots, \infty$. *This implies that* $z_n = 0$ *or* $z_n = \frac{1}{n}$. *The set of equilibria is therefore*

$$E = \left\{ z \in l^2 |\ z_n \in \left\{0, \frac{1}{n}\right\}, n = 1, \cdots, \infty \right\}.$$

Linearize the system (2.1) around the zero element $z = \{0, 0, \ldots\}$ *to obtain*

$$\begin{aligned} \dot{z}(t) &= Az(t), \ t \geq 0 \\ z(0) &= z_0, \end{aligned} \tag{2.3}$$

where

$$A = -\begin{pmatrix} 1 & 0 & \dots & \\ 0 & \ddots & & \\ & & \frac{1}{n} & \\ & & & \ddots \end{pmatrix}. \tag{2.4}$$

The C_0*-semigroup generated by the operator* A *is asymptotically stable.*

Choose an equilibrium $z_e \in E$ *with for some positive integer* n, $z_{e,n} = n$ *and all other components of* z_e *equal to zero; that is* $z_{e,m} = 0$, $m \neq n$. *If* $z(0) = z_e$ *then for all* t

$$\|z(t) - z_e\| = \frac{1}{n} \neq 0.$$

Hence, the zero equilibrium is not asymptotically stable.□

As mentioned in the introduction, further examples can be found in [16, 58]. Thus, the Lyapunov Indirect method cannot always be used for infinite-dimensional systems. Conditions for when the stability (or instability) of the linearized infinite-dimensional system implies the same stability as for the nonlinear system are needed.

Definition 2.2. *A family of operators $S(t)$, $t \geq 0$ on a Banach space X is said to be a nonlinear C_0-semigroup if*

- *For all $z \in X$, $t, \tau \geq 0$,*

$$S(t+\tau)z = S(t)S(\tau)z$$

- *S is a continuous operator from $X \times R^+$ into X.*

Consider the general nonlinear infinite-dimensional system defined on a Banach space X,

$$\begin{aligned} \dot{z}(t) &= F(z(t)), \\ z(0) &= z_0, \end{aligned} \tag{2.5}$$

where the nonlinear operator $F : \mathcal{D}(F) \subset X \to X$ generates a nonlinear C_0-semigroup $S(t)$. Let z_e be an equilibrium solution to the system. There are two basic definitions of derivatives.

Definition 2.3. *An operator $F : X \to X$ defined on a normed linear space X is* Fréchet differentiable *at z_0 if there exists a bounded linear operator $DF(z_0) : X \to X$ such that for all h*

$$\lim_{h \to 0} \frac{\|F(z_0 + h) - F(z_0) - DF(z_0)h\|}{\|h\|} = 0. \tag{2.6}$$

That is, there exists a function $f(z)$, $\lim_{z \to 0} f(z) = 0$ such that

$$F(z_0 + h) - F(z_0) = DF(z_0)h + f(h)\|h\|.$$

The operator F is said to be Fréchet differentiable *if it is Fréchet differentiable at every $z_0 \in X$.*

Definition 2.4. *Let $F : \mathcal{D}(F) \subset X \to X$ be an operator defined on a linear space $\mathcal{D}(F)$ contained in a Banach space X. The operator F is* Gâteaux differentiable *at $z_0 \in \mathcal{D}(F)$ if there exists a linear operator $F' : X \to X$ such that for $z_0, h \in \mathcal{D}(F)$,*

$$\lim_{\varepsilon \to 0} \frac{F(z_0 + \varepsilon h) - F(z_0)}{\varepsilon} = F'h.$$

The Fréchet derivative is a very strong definition. The unbounded nature of the generator in partial differential equations means that these generators are not generally Fréchet differentiable. Hence, the Gâteaux derivative is used to linearize the differential equation. However, even though the generator is not Fréchet differentiable, the semigroup is generally Fréchet differentiable. The derivative of the semigroup can be used to deduce local stability/instability. A similar approach was used in [19, Proposition 2.1], [28, Thm. 2.1], [50, Theorem 11.22] but the assumptions are weakened here.

Theorem 2.5. *Let z_e be an equilibrium point of the nonlinear system (2.5). Assume that for all $t \geq 0$, $S(t)$ is Fréchet differentiable at z_e with Fréchet derivative $T_{z_e}(t)$. If T_{z_e} is an exponentially stable semigroup, then z_e is a locally exponentially stable equilibrium of (2.5).*

Proof. To simplify the proof, set $z_e = 0$ without loss of generality. Several steps are required.

Step 1. Since $z_e = 0$ is an exponentially stable equilibrium solution of the linearized system, then there exists $M \geq 1$ and $\gamma > 0$ such that for all $z_0 \in X$

$$\|T_{z_e}(t) z_0\| \leq M e^{-\gamma t} \|z_0\|, \; t \geq 0. \tag{2.7}$$

Using the definition of Fréchet derivative (Definition 2.3) there is an operator $f(t, z_0)$ on X with $f(t, 0) = 0$ such that

$$S(t) z_0 = T_{z_e}(t) z_0 + f(t, z_0) \|z_0\|$$

with $\lim_{\|z_0\| \to 0} \|f(t, z_0)\| = 0$. Choose $\bar{t} = \frac{\ln(4M)}{\gamma} > 0$ and any $\delta > 0$. Since the C_0-semigroups $S(t)$ and $T_{z_e}(t)$ are continuous in t and z, so is f and there is $C_1 > 0$ such that $\|f(t, z)\| \leq C_1$ for $t \in [0, \bar{t}]$, $\|z_0\| \leq \delta$. It follows that for $\tau \in [0, \bar{t}]$,

$$\begin{aligned} \|S(\tau) z_0\| &\leq \|T_{z_e}(\tau) z_0\| + C_1 \|z_0\| \\ &\leq M e^{-\gamma \tau} \|z_0\| + C_1 \|z_0\| \\ &= \left(M e^{-\gamma \tau} + C_1\right) \|z_0\| \\ &= \underbrace{(M + C_1)}_{\mathbf{C}} \|z_0\|. \end{aligned} \tag{2.8}$$

Step 2. Using (2.7),

$$\begin{aligned} \|T_{z_e}(\bar{t}) z_0\| &\leq M e^{-\gamma \bar{t}} \|z_0\|, \\ &\leq \frac{1}{4} \|z_0\|. \end{aligned} \tag{2.9}$$

Using the definition of a Fréchet derivative (Definition 2.3), and the fact that $z_e = 0$ is an equilibrium point

$$\lim_{\|z\| \to 0} \frac{\|S(\bar{t}) z_0 - T_{z_e}(\bar{t}) z_0\|}{\|z_0\|} = 0.$$

Thus, there exists $\delta > 0$ such that if $\|z_0\| < \delta$, then

$$\|S(\bar{t}) z_0 - T_{z_e}(\bar{t}) z_0\| \leq \frac{1}{4} \|z_0\|. \tag{2.10}$$

Using (2.9) and (2.10),

$$\begin{aligned} \|S(\bar{t}) z_0\| &= \|S(\bar{t}) z_0 - T_{z_e}(\bar{t}) z_0 + T_{z_e}(\bar{t}) z_0\| \\ &\leq \|S(\bar{t}) z_0 - T_{z_e}(\bar{t}) z_0\| + \|T_{z_e}(\bar{t}) z_0\| \\ &\leq \frac{1}{2} \|z_0\| \\ &= e^{-\ln 2} \|z_0\|. \end{aligned} \tag{2.11}$$

For any positive integer $k > 0$ the semigroup property and (2.11) imply that

$$\begin{aligned} \|S(k\bar{t}) z_0\| &= \|S^k(\bar{t}) z_0\| \\ &\leq e^{-(\ln 2) k} \|z_0\|. \end{aligned} \tag{2.12}$$

It follows that the ball $\|z_0\| \leq \delta$ is invariant under $S(\bar{t})$.

Step 3. For any $t > 0$ define $k = \lfloor \frac{t}{\bar{t}} \rfloor$ and $\tau = t - k\bar{t}$. Then $\tau \in [0, \bar{t}]$ and using the semigroup property, (2.8) and (2.12),

$$\begin{aligned} \|S(t)\, z_0\| &= \|S(k\bar{t} + \tau)\, z_0\| \\ &\leq C e^{-(\ln 2)k} \|z_0\|. \end{aligned} \tag{2.13}$$

Now,

$$\begin{aligned} C \exp(-(\ln 2)\, k) &= C \exp(-\frac{\ln 2}{\bar{t}}(k\bar{t} + \tau)) \exp(\frac{\ln 2}{\bar{t}}\tau) \\ &\leq C \exp(-\left(\frac{\ln 2}{\bar{t}}\right) t) \exp(\ln 2). \end{aligned}$$

Defining $C_2 = 2C$, $\beta = \frac{\ln 2}{\bar{t}}$, (2.13) implies that there is $\delta > 0$ such that if $\|z_0\| < \delta$,

$$\|S(t)\, z_0\| \leq C_2 e^{-\beta t}.$$

Thus, the equilibrium solution z_e to the nonlinear system is locally exponentially stable. $\square$

It can similarly be shown that if the system linearized at an equilibrium point is unstable, then the nonlinear system is unstable at that equilibrium point. The following result is [19, Prop. 2.2] except that the assumption of continuous Fréchet differentiability is not needed, and the conclusion is slightly different. For completeness, the full proof is provided.

LEMMA 2.6. *Let V be a nonlinear operator on X that is Fréchet differentiable at a fixed point z_e, with Fréchet derivative U. If X can be decomposed as $X = X^+ \oplus X^-$ where X^+ and X^- are each $U-$invariant and there exist real numbers $1 < \theta < \eta$ such that for all $x_+ \in X^+$, $x_- \in X^-$,*

$$\|Ux_+\| \geq \eta \|x_+\|, \quad \|Ux_-\| \leq \theta \|x_-\|$$

then there is $\epsilon_0 > 0$ and a sequence of $z_n \in X$, $z_n \to z_e$ such that for each n there is n_k, $\|V^{n_k} z_n - z_e\| \geq \epsilon_0$.

Proof. To simplify the proof, set $z_e = 0$ without loss of generality. Let P^+ denote the projection of X onto X^+, and let P^- similarly denote the projection onto X^-. Also, the norm

$$\|z\| = \|P^+ z\| + \|P^- z\|$$

is equivalent to the original norm. From the definition of the Fréchet derivative, there is $\epsilon_0 > 0$ such that if $\|z\| \leq \epsilon_0$,

$$\|Uz - Vz\| \leq \frac{\eta - \theta}{4} \|z\|.$$

Define the set

$$S = \{z \in X,\ \|P^- z\| \leq \|P^+ z\|\}.$$

For any $z \in S$, with $\|z\| = \epsilon$ where $0 < \epsilon < \epsilon_0$,

$$\begin{aligned}\|P^+Vz\| &\geq \|P^+Uz\| - \|P^+(Vz - Uz)\| \\ &\geq \|UP^+z\| - \|Vz - Uz\| \\ &\geq \eta\|P^+z\| - \frac{\eta-\theta}{4}\|z\| \\ &\geq \eta\|P^+z\| - \frac{\eta-\theta}{2}\|P^+z\| \\ &= \frac{\eta+\theta}{2}\|P^+z\|.\end{aligned}$$

Also,

$$\begin{aligned}\|P^-Vz\| &\leq \|P^-Uz\| + \|P^-(Vz - Uz)\| \\ &\leq \|UP^-z\| + \|(Vz - Uz)\| \\ &\leq \theta\|P^-z\| + \frac{\eta-\theta}{4}\|z\| \\ &\leq \theta\|P^+z\| + \frac{\eta-\theta}{2}\|P^+z\| \\ &= \frac{\eta+\theta}{2}\|P^+z\|.\end{aligned}$$

Thus, $Vz \in S$.

Now, assume that for all positive integers k, and $z \in S$ with $\|z\| = \epsilon < \epsilon_0$, V^kz satisfies $\|V^kz\| < \epsilon_0$. Then by induction,

$$\|P^+V^kz\| \geq \left(\frac{\eta+\theta}{2}\right)^k \|P^+z\| \geq \left(\frac{\eta+\theta}{2}\right)^k \frac{1}{2}\|z\|.$$

Since $1 < \theta < \eta$, this approaches infinity . Thus, for at least one k, $\|V^kz\| \geq \epsilon_0$. Choose a sequence of positive $\epsilon_n \to 0$, $z_n \in S$ with $\|z_n\| < \epsilon_n$. Then for each z_n there is n_k so $\|V^{n_k}z_n\| \geq \epsilon_0$. The result follows. □

THEOREM 2.7. *Let z_e be an equilibrium solution of the nonlinear system (2.5) defined on a Hilbert space X. Assume that for $t \geq 0$, $S(t)$ is Fréchet differentiable at z_e with Fréchet derivative $T(t)$ where T is a C_0-semigroup with generator A. If X can be split into $X = X^+ \oplus X^-$ where X^+ is finite-dimensional, $T(t)$ is invariant with respect to each subspace, and*

1. *$\sigma(A|_{X^+})$ contains only eigenvalues, with $\sup \sigma(A|_{X^+}) > 0$,*
2. *the growth ω of $T|_{X^-}$ satisfies $\omega < \inf_{\lambda_i \in \sigma(A|_{X^+})} \mathrm{Re}\lambda_i$,*

then z_e is an unstable equilibrium of (2.5).

Proof. Let T^- , T^+ indicate T restricted to X^- and X^+ respectively, and indicate similarly A^-, A^+, σ^-, σ^+. Define $\alpha = \inf_{\lambda_i \in \sigma^+} \mathrm{Re}\lambda_i$. Since the generator of T^+ is bounded, with spectrum that consists only of eigenvalues, $\|T^+(t)z\| \geq e^{\alpha t}\|z\|$ for all $z \in X^+$. Also, $\|T^-(t)z\| \leq Me^{\omega t}\|z\|$ for all $z \in X^-$. Choose $t_0 > 0$ such that $Me^{\omega t_0} < e^{\alpha t_0}$. Define $\eta = e^{\alpha t_0}$, $V = S(t_0)$, $U = T(t_0)$ and choose $\theta > 1$ so that $Me^{\omega t_0} \leq \theta < \eta$. It follows that for all $z_+ \in X^+$, $z_- \in X^-$,

$$\|Uz_+\| \geq \eta\|z_+\|, \quad \|Uz_-\| \leq \theta\|z_-\|.$$

From Lemma 2.6, there is $\epsilon_0 > 0$, and a sequence $z_n \to z_e$, and integers n_k such that $\|V^{n_k}z_n - z_e\| \geq \epsilon_0$. In other words, there is an ϵ_0 such that for any $\epsilon > 0$ there is $z_n \in X$, $\|z_n - z_e\| < \epsilon$ such that for some time $t > t_0$, $\|S(t)z_n - z_e\| \geq \epsilon_0$. Thus, the equilibrium point is unstable. □

DEFINITION 2.8. *A satisfies the* spectrum decomposition assumption *at α if $\sigma(A)$ is the union of two parts σ^+ and σ^- such that a a rectifiable, simple, closed curve can be drawn so as to enclose an open set containing σ^+ in its interior, σ^- is in its exterior and also*

$$\sup_{\lambda \in \sigma^-} \sigma(A) < \alpha \leq \inf_{\lambda \in \sigma^+} \sigma(A).$$

COROLLARY 2.9. *Let z_e be an equilibrium solution of the nonlinear system (2.5) defined on a Hilbert space X. Assume that for $t \geq 0$, $S(t)$ is Fréchet differentiable at z_e with Fréchet derivative $T(t)$ where T is a C_0-semigroup with generator A. If A is a Riesz-spectral operator that satisfies the spectrum determined growth assumption for some $\alpha > 0$ then z_e is an unstable equilibrium point of (2.5).*

Proof. Since A satisfies the spectrum decomposition assumption for some $\alpha > 0$, X can be split into $X = X^+ \oplus X^-$ where X^+ is finite-dimensional, T is invariant on each subspace and generator A^+ of $T^+ = T|_{X^+}$ has spectrum σ^+ (see, for instance, [18, Lem.2.5.7]). The restriction of A to X^-, A^-, has spectrum $\sigma(A^-) = \sigma^-$. Since A is a Riesz-spectral operator, it and A^- satisfy the spectrum determined growth assumption and A^- generates a semigroup T^- satisfying $\|T^-\|(t) \leq Me^{\omega t})$ for some $M \geq 1$, $\omega < \alpha$. The conclusion then follows from Theorem 2.7. □

In summary, the Fréchet derivative of the nonlinear C_0-semigroup corresponding to the nonlinear system plays an important role in analyzing stability using Lyapunov's indirect method. If the equilibrium solution of the linearized system around the equilibrium solution is exponentially stable, then the equilibrium solution to the nonlinear system is locally exponentially stable. Furthermore, if the equilibrium solution to the linearized system is unstable, then the nonlinear system is also unstable. If the linearized system is only asymptotically stable at the equilibrium point then no conclusion about stability of the nonlinear system can be made.

Existing results for the linearized stability of quasilinear systems on a Banach space X can be obtained as special cases of the above theorems. Consider quasilinear systems on a Banach space X

$$\begin{aligned} \dot{z}(t) &= Az(t) + f(z(t)) \\ z(0) &= z_0, \end{aligned} \tag{2.14}$$

where $z(t) \in X$ is the state and z_0 is the initial condition. The operator $A : \mathcal{D}(A) \subset X \to X$ is a linear operator that generates a C_0-semigroup on X and the nonlinear operator $f : \mathcal{D}(f) \subset X \to X$ is Fréchet differentiable with $Df(z)$ the Fréchet derivative of f at z. It is straightforward to show that $A + Df(z)$ is the Gâteaux derivative of $Az + f(z)$ at z. The linearized system corresponding to (2.14) at the equilibrium point $z_e \in Z$ is

$$\frac{d\psi}{dt} = A\psi + Df(z_e)\psi. \tag{2.15}$$

Suppose that for some $r > 0$ the Fréchet derivative of f in (2.14) satisfies

$$||Df(z_1) - Df(z_2)|| \leq c(r)||z_1 - z_2||, \tag{2.16}$$

for all $||z_1|| \leq r, ||z_2|| \leq r$, where $c : [0, \infty) \to [0, \infty)$ is a continuous increasing function. Let z_e be an equilibrium point of (2.14). Section 3 of [28] can be used to show that these assumptions (in fact [28] has more general, but difficult to check, conditions) imply that the nonlinear semigroup is Fréchet differentiable at any equilibrium z_e, with generator $A + dF(z_e)$. In [28, Cor. 2.2] it is then shown that exponential stability of the linear semigroup implies local exponential stability of the original system, or Theorem 2.5 can be used.

The assumptions on f in the following theorem are slightly different to those above.

THEOREM 2.10. *[27] Let Z be a Hilbert space with norm $||\cdot||_Z$ and inner product $\langle\cdot,\cdot\rangle_Z$. Consider the quasilinear equation in (2.14) and suppose it generates a semigroup, $S(t)$. For any $p \in Z$ define*

$$N_{p,r} = \{z \in Z : ||p - z||_Z \leq r\}.$$

Assume $\text{Re}\langle Az, z\rangle_Z \leq 0$ *for all $z \in N_{p,r}$. and suppose f is Fréchet differentiable on $N_{z,r}$ and its derivative, Df, is locally Lipschitz continuous on $N_{p,r}$. Also, for some positive constant $K_{p,r}$ that depends on p and r, assume that*

$$\sup_{\eta \in N_{p,r}} ||Df(\eta)||_{op} = K_{p,r} < \infty$$

where $||\cdot||_{op}$ is the operator norm. Then (2.15) generates the semigroup $T_z(t)$ and for some $t_f > 0$,

$$T_z(t) = DS(z_0)(t), \quad 0 \leq t \leq t_f$$

where $DS(z_0)(t)$ is the Fréchet derivative of $S(t)$ at $z(0) = z_0$, $z_0 \in N_{p,r}$.

The approach in this section will now be used to analyze the local stability of the Kuramoto-Sivashinsky equation, and also to obtain locally exponentially stabilizing controllers.

3. The Kuramoto-Sivashinsky (KS) equation. Consider the controlled KS equation with a single state-feedback control and periodic boundary conditions

$$\begin{array}{l} \frac{\partial z}{\partial t} + \nu \frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial x^2} + z\frac{\partial z}{\partial x} = b\left(x\right)u\left(t\right) \\ \frac{\partial^n z}{\partial x^n}\left(-\pi, t\right) = \frac{\partial^n z}{\partial x^n}\left(\pi, t\right), \quad n = 0, 1, 2, 3 \\ z\left(x, 0\right) = z_0\left(x\right) \end{array} \tag{3.1}$$

where $\nu > 0$ is the instability parameter, $z \in L^2(-\pi, \pi)$ is the state of the system, the influence of the actuator is given by $b(x) \in L^2(-\pi, \pi)$ and $u \in \mathbb{C}$ is the controlled input to the KS equation. State-feedback control

$$u\left(t\right) = Kz\left(t\right), \tag{3.2}$$

where $K : L^2(-\pi, \pi) \to \mathbb{C}$ is defined by $Kz = \langle k, z\rangle$ with $k \in L^2(-\pi, \pi)$ will be considered.

For some $b \in L^2(-\pi, \pi)$ define the bounded linear operator $B : \mathbb{C} \to L^2(-\pi, \pi)$

$$Bu = b\left(x\right)u. \tag{3.3}$$

Also define the linear operators on $L^2(-\pi, \pi)$

$$Rz = \frac{\partial^2 z}{\partial x^2}, \qquad \mathcal{D}\left(R\right) = H^2_{per}(-\pi, \pi) \subset H^2[-\pi, \pi] \tag{3.4}$$

$$\hat{A}z = \nu\frac{\partial^4 z}{\partial x^4}, \qquad \mathcal{D}\left(\hat{A}\right) = H^4_{per}(-\pi, \pi) \subset H^4(-\pi, \pi) \tag{3.5}$$

$$Az = -\left(\hat{A} + R\right)z, \quad \mathcal{D}\left(A\right) = \mathcal{D}\left(\hat{A}\right) \tag{3.6}$$

and the nonlinear operator

$$F(z) = -z\frac{\partial z}{\partial x}, \quad \mathcal{D}(F) = H^1_{per}(-\pi,\pi) \subset H^1(-\pi,\pi). \tag{3.7}$$

The feedback controlled KS equation (3.1) can be written in the abstract form

$$\begin{aligned} \dot{z} &= Az + F(z) + BKz \\ z(0) &= z_0. \end{aligned} \tag{3.8}$$

The controlled KS equation (3.8) has a unique strong solution. This result is a special case of [42, Theorem 1.1] where the Galerkin method is used. For the proof see [2, Theorem 5.3.9].

THEOREM 3.1. *[2, Theorem 5.3.9]*
The feedback controlled KS equation with periodic boundary conditions has a unique strong solution $z(t) = S_B(t)z_0$, where $S_B(t)$ is a nonlinear C_0-semigroup. For any $T > 0$,

$$z \in C\left([0,T]; L^2(-\pi,\pi)\right) \cap L^2\left([0,T]; H^2_{per}(-\pi,\pi)\right).$$

The uncontrolled KS equation ((3.1) with $b = 0$) has an infinite number of equilibria. In particular, any constant function is an equilibrium solution to the KS equation. Define the closed set

$$Z_e = \{z_e : \ z_e \text{ is a constant function}\} \tag{3.9}$$

to be the set of constant equilibria. It is straightforward to verify the conservation of the space integral [13]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} z_0 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} z(x,t)dx.$$

Thus, the particular equilibrium is determined by the initial condition z_0: for initial condition z_0, the constant equilibrium $z_e = \frac{1}{2\pi}\int_{-\pi}^{\pi} z_0 dx$.

A set of equilibrium points can also be characterized as stable.

DEFINITION 3.2. *[53, Definition 2.6] (Stable Equilibrium Set)*
Let Z_e be the set of all equilibria to (2.5). The set Z_e is said to be stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $dist_X(z_0, Z_e) < \delta$, then

$$dist_X(z(t), Z_e) < \varepsilon, \quad t \geq 0$$

where $dist_X(z, Z_e) = inf\{\|z - y\| : y \in Z_e\}$.

THEOREM 3.3. *Consider the uncontrolled KS equation (3.1) with $b(x) = 0$. If the instability parameter $\nu > 1$, then the set of constant equilibrium solutions Z_e defined in (3.9) is globally asymptotically stable.*

Proof. Define

$$V(z) = \frac{1}{2}\|z\|^2. \tag{3.10}$$

For smooth functions satisfying the periodic boundary conditions, the Lyapunov derivative is

$$\begin{aligned}
\dot{V}(z) &= Re\langle z, \dot{z}\rangle, \\
&= -\nu\langle \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x^2}\rangle + \langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial x}\rangle - 0 \\
&= -\nu\|\frac{\partial^2 z}{\partial x^2}\|^2 + \|\frac{\partial z}{\partial x}\|^2. \qquad (3.11)
\end{aligned}$$

Using Poincaré's inequality [46, Lemma 1.8]) then yields

$$\begin{aligned}
\dot{V}(z) &\leq -(\nu - 1)\|\frac{\partial z}{\partial x}\|^2. \\
&\leq 0, \qquad (3.12)
\end{aligned}$$

since $\nu > 1$. If

$$-(\nu - 1)\|\frac{\partial z}{\partial x}\|^2 = 0,$$

then

$$\frac{\partial z}{\partial x} = 0.$$

This implies that z equals some constant function C; that is $z \in Z_e$. Since the C_0-semigroup generated by the uncontrolled KS equation is compact [48, Theorem 54.3], then the orbit $\gamma(z)$ is pre-compact for every $z \in \mathcal{D}(A)$. Therefore, by LaSalle's Invariance Principle, the solution of the KS equation converges to the invariant set Z_e. $\square$

If the instability parameter $\nu = 1$, then the Lyapunov derivative $\dot{V}(z)$ defined in (3.12) vanishes and therefore the equilibrium solution to the nonlinear KS equation is stable [52, Theorem 3.6 & 3.7]. In [48, Theorem 5.4.3] it was shown that the zero equilibrium is a global attractor.

Stability or instability of equilibria for the KS equation when the instability parameter $\nu < 1$ needs to be determined. This will be done by linearization of the KS equation around an equilibrium z_e and then analyzing the stability of the linearization. This approach can also be used to locally stabilize the system about an equilibrium, or in fact at any point.

4. Linearization of the Kuramoto-Sivashinsky equation. The feedback controlled KS equation (3.8) will be linearized at $z_0 \in \mathcal{D}(A)$, where the operator A and its domain is defined in (3.6). This is done by using the Gâteaux derivative [36, Definition 3.1.2].

We find the Gâteaux derivative of the nonlinear operator $F(z)$ defined in (3.7) at $z_0 \in \mathcal{D}(F)$, $F' : H^1(-\pi, \pi) \subset L^2(-\pi, \pi) \to L^2(-\pi, \pi)$

$$F'z = \lim_{\varepsilon \to 0} \frac{F(z_0 + \varepsilon z) - F(z_0)}{\varepsilon} = \frac{\partial}{\partial x}(z_0 z). \qquad (4.1)$$

Hence, the linearized controlled system of the KS equation around z_0 is

$$\begin{aligned}\dot{z}(t) &= Az(t) - \frac{\partial}{\partial x}\left(z_0 z(t)\right) + Bu\left(t\right) \\ &= A'z(t) + BKz(t)\end{aligned} \tag{4.2}$$

where

$$A'z = Az - \frac{\partial}{\partial x}\left(z_0 z\right) \tag{4.3}$$

with A defined in (3.6).

THEOREM 4.1. *[2, Theorem 5.2.1]*
The operator A' defined in (4.3), where z_0 is a constant function that does not depend on x, is a Riesz-spectral operator that has eigenvalues $\lambda_n = -\nu n^4 + n^2 - \imath n z_0$, $n \in \mathbb{Z}$ and the corresponding eigenvectors $\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$.

It will now be shown that C_0-semigroup $S_B(t)$ of the controlled nonlinear KS equation (3.8) is Fréchet differentiable at any $z_0 \in L^2(-\pi, \pi)$ and the Fréchet derivative equals the C_0-semigroup corresponding to the linearized KS equation at z_0. The main result is Theorem 4.3. A series of lemmas, placed in Appendix A, are used.

In [51, Section VI.8], it is shown that if $\langle z \frac{\partial y}{\partial x}, z \rangle = 0$, for every solution to the KS equation z and $y \in H^2(-\pi, \pi)$, then the nonlinear C_0-semigroup generated by the uncontrolled KS equation ((3.1) with $b(x) = 0$) is Fréchet differentiable. This assumption is not used here.

Note that the linear operator $BK : L^2(-\pi, \pi) \to L^2(-\pi, \pi)$ is bounded. That is, there exists $M > 0$ such that for all $z \in L^2(-\pi, \pi)$,

$$\|BKz\| \le M\|z\|.$$

Theorem 4.3 below is the key result of this section. It shows that the nonlinear C_0-semigroup corresponding to the open-loop controlled nonlinear KS equation (3.1) is Fréchet differentiable at every $z_0 \in L^2(-\pi, \pi)$ and the derivative is the linear C_0-semigroup corresponding to the linearized KS equation around z_0. The following lemma is needed.

LEMMA 4.2. *[2, Lem. 5.3.4] Consider the uncontrolled KS equation (3.1) with $u(t) = 0$. Let $S(t)$ be the C_0-semigroup generated by the nonlinear uncontrolled KS equation. Then,*

$$\|S(t)\, z_0\| \le e^{\frac{1}{\sqrt{2\nu}} t} \|z_0\|, \quad z_0 \in L^2(-\pi, \pi).$$

Proof. The uncontrolled KS equation is well-posed [48, Theorem 54.3] and the solution can be written

$$z(t) = S(t)\, z_0,$$

where $S(t)$ is a nonlinear C_0-semigroup in $L^2(-\pi, \pi)$ and z_0 is the initial condition.

It was shown above that (3.11)

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + \nu\|\frac{\partial^2 z}{\partial x^2}\|^2 = \|\frac{\partial z}{\partial x}\|^2. \tag{4.4}$$

Using the Cauchy-Schwarz inequality and Young's inequality,

$$
\begin{aligned}
\|\frac{\partial z}{\partial x}\|^2 &= \langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial x}\rangle \\
&= -\langle z, \frac{\partial^2 z}{\partial x^2}\rangle \text{ (Integration by parts)} \\
&\leq |\langle z, \frac{\partial^2 z}{\partial x^2}\rangle| \\
&\leq \|z\| \cdot \|\frac{\partial^2 z}{\partial x^2}\| \\
&\leq \frac{1}{4\nu}\|z\|^2 + \nu\|\frac{\partial^2 z}{\partial x^2}\|^2. \qquad (4.5)
\end{aligned}
$$

Using this inequality in equation (4.4) leads to

$$\frac{d}{dt}\|z\|^2 \leq \frac{1}{2\nu}\|z\|^2. \qquad (4.6)$$

Using Gronwall's lemma [55, Theorem 1.4.1] then implies that

$$\|z\|^2 \leq e^{\frac{1}{2\nu}t}\|z_0\|^2, \quad t \geq 0,$$

and so since $z(t) = S(t)z_0$, the result follows. □

Define the nonlinear operator $G : H^2_{per}(-\pi,\pi) \to L^2(-\pi,\pi)$ by

$$G(z) = Rz + F(z) - BKz, \qquad (4.7)$$

where the operators R, F are defined in (3.5), (3.7), respectively.

Theorem 4.3. *Consider the controlled KS equation (3.8). The nonlinear C_0-semigroup $S_B(t)$ is Fréchet differentiable at every $z_0 \in L^2(-\pi,\pi)$ and the derivative is the linear C_0-semigroup generated by the linearized KS equation at z_0.*

Proof. Consider the nonlinear controlled KS equation given by (3.8) with initial condition $y_0 \in L^2(-\pi,\pi)$. Let $M = \|BK\|$. Lemma 4.2 implies that for any $T > 0$, for $0 \leq t \leq T$, the L^2-norm of the solution $\|y(t)\| \leq e^{\left(\frac{1}{2\nu}+M\right)T}\|y_0\|$. For any $\varepsilon > 0$, define $r = (\|y_0\| + \varepsilon)e^{\left(\frac{1}{2\nu}+M\right)T}$. Then, for any z_0, $\|z_0 - y_0\| < \varepsilon$,

$$\sup_{t\in[0,T]} \|z(t)\| \leq r.$$

Subtracting the KS equation with initial condition y_0 from the equation with initial condition z_0, and letting $w(t) = z(t) - y(t)$ yields

$$
\begin{aligned}
\dot{w}(t) + \hat{A}w(t) &= -\left(G(z(t)) - G(y(t))\right) \\
w(0) &= z_0 - y_0 := w_0.
\end{aligned}
\qquad (4.8)
$$

Use the Gâteaux derivative to linearize the KS equation (4.8) around $y = S_B(t)y_0$

$$
\begin{aligned}
&\dot{\overline{w}}(t) = -\hat{A}\overline{w}(t) - R\overline{w}(t) - \tfrac{\partial}{\partial x}\left(y(t)\overline{w}(t)\right) + BK\overline{w}(t) \\
&\overline{w}(0) = w_0.
\end{aligned}
\qquad (4.9)
$$

Using [51, Theorem II.3.4] and [18, Lemma 3.1.5], the controlled linearized KS equation (4.9) has a unique strong solution

$$\overline{w}(t) \in L^2\left(0,T; H^2_{per}[-\pi,\pi]\right) \cap L^\infty\left(0,T; L^2[-\pi,\pi]\right), \quad \text{for } t \leq T < \infty.$$

That is, the solution can be written as

$$\overline{w}\,(t) = T_B\,(t)\,w_0, \tag{4.10}$$

where $T_B\,(t)$ is a C_0-semigroup on $L^2[-\pi,\pi]$.

The next step is to show that the nonlinear C_0-semigroup $S_B\,(t)$ is Fréchet differentiable at y_0 and $T_B\,(t)$ is its Fréchet derivative. Set $\phi = w - \overline{w}$ and use equations (4.8) and (4.9) and Lemma A.3 to obtain

$$\begin{aligned}
\dot{\phi}\,(t) &= \dot{w}\,(t) - \dot{\overline{w}}\,(t)\\
&= -\hat{A}\,(w\,(t) - \overline{w}\,(t)) - (G\,(z\,(t)) - G\,(y\,(t))) + R\overline{w}\,(t) + \frac{\partial}{\partial x}\,(y\,(t)\,\overline{w}\,(t))\\
&\qquad -BK\overline{w}\,(t)\\
&= -\hat{A}\phi\,(t) - R\phi\,(t) - \frac{\partial}{\partial x}\,(y\,(t)\,\phi\,(t)) + BK\phi\,(t) - F\,(w\,(t))\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\dot{\phi}\,(t) + \hat{A}\phi\,(t) = -R\phi\,(t) - \tfrac{\partial}{\partial x}\,(y\,(t)\,\phi\,(t)) + BK\phi\,(t) - F\,(w\,(t))\\
&\phi\,(0) = 0.
\end{aligned} \tag{4.11}$$

Take the inner product of the above system (4.11) with ϕ to obtain

$$\begin{aligned}
\langle\dot{\phi}\,(t)\,,\phi\,(t)\rangle + \langle\hat{A}\phi\,(t)\,,\phi\,(t)\rangle &= -\langle R\phi\,(t) + \frac{\partial}{\partial x}\,(y\,(t)\,\phi\,(t)) - BK\phi\,(t)\,,\phi\,(t)\rangle\\
&\quad -\langle F\,(w\,(t))\,,\phi\,(t)\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi\,(t)\,\|^2 + \nu\|\frac{\partial^2\phi}{\partial x^2}\,(t)\,\|^2 &= -\langle R\phi\,(t) + \frac{\partial}{\partial x}\,(y\,(t)\,\phi\,(t)) - BK\phi\,(t)\,,\phi\,(t)\rangle\\
&\quad -\langle F\,(w\,(t))\,,\phi\,(t)\rangle\\
&\le |\langle R\phi\,(t) + \frac{\partial}{\partial x}\,(y\,(t)\,\phi\,(t)) - BK\phi\,(t)\,,\phi\,(t)\rangle|\\
&\quad +|\langle F\,(w\,(t))\,,\phi\,(t)\rangle|.
\end{aligned} \tag{4.12}$$

Moreover, using the Cauchy-Schwarz inequality, Lemma (A.1), Lemma (A.2) and the Poincaré inequality leads to

$$\begin{aligned}
|\langle R\phi\,(t) + \frac{\partial}{\partial x}\,(y\,(t)\,\phi\,(t)) - BK\phi\,(t)\,,\phi\,(t)\rangle| &\le |\langle R\phi\,(t)\,,\phi\,(t)\rangle| + |\langle BK\phi\,(t)\,,\phi\,(t)\rangle|\\
&\quad +|\langle y\,(t)\,\frac{\partial\phi}{\partial x}\,(t)\,,\phi\,(t)\rangle| + |\langle\phi\,(t)\,\frac{\partial y}{\partial x}\,(t)\,,\phi\,(t)\rangle|.
\end{aligned}$$

That is,

$$|\langle R\phi(t) + \frac{\partial}{\partial x}(y(t)\phi(t)) - BK\phi(t), \phi(t)\rangle| \le M\|\phi(t)\|^2 + 3\|y(t)\|_\infty \|\phi(t)\| \cdot \|\frac{\partial \phi}{\partial x}(t)\| + \|\phi(t)\|\;\|\frac{\partial^2\phi}{\partial x^2}(t)\|$$

$$\le M\|\phi(t)\|^2 + (1+3\|y(t)\|_\infty)\|\phi(t)\|\,\|\frac{\partial^2\phi}{\partial x^2}(t)\|$$

$$\le M\|\phi\|^2 + (1+3r)\|\phi(t)\|\,\|\frac{\partial^2\phi}{\partial x^2}(t)\|$$

$$= M\|\phi(t)\|^2 + K_r\;\|\phi(t)\|\;\|\frac{\partial^2\phi}{\partial x^2}(t)\|,$$

where $K_r = 1+3r$.

With this result, the Cauchy-Schwarz inequality, Young's inequality and Lemma A.3, inequality (4.12) becomes

$$\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2 + \nu\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 \le K_r\;\|\phi(t)\|\;\|\frac{\partial^2\phi}{\partial x^2}(t)\| + M\|\phi(t)\|^2 + \|J(w(t))\|\cdot\|\phi(t)\|$$

$$\le \frac{\nu}{2}\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 + \left(\frac{K_r^2}{2\nu} + M\right)\|\phi(t)\|^2 + \frac{\nu}{2}\|J(w(t))\|^2 + \frac{1}{2\nu}\|\phi(t)\|^2$$

and so

$$\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2 + \nu\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 \le \frac{\nu}{2}\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 + \left(\frac{K_r^2+1}{2\nu} + M\right)\|\phi(t)\|^2 + \frac{\nu c^2}{8}\left(\|w(t)\|^2 + \|\frac{\partial^2 w}{\partial x^2}(t)\|^2\right)^2$$

$$\le \frac{\nu}{2}\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 + \left(\frac{K_r^2+1}{2\nu} + M\right)\|\phi(t)\|^2 + \frac{\nu c^2}{4}\left(\|w(t)\|^4 + \|\frac{\partial^2 w}{\partial x^2}(t)\|^4\right). \tag{4.13}$$

This implies that

$$\frac{d}{dt}\|\phi(t)\|^2 + \nu\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 \le \left(\frac{K_r^2+1}{\nu} + 2M\right)\|\phi(t)\|^2 + \frac{\nu c^2}{2}\left(\|w(t)\|^4 \;\; + \|\frac{\partial^2 w}{\partial x^2}(t)\|^4\right),$$

and so

$$\frac{d}{dt}\|\phi(t)\|^2 \le \left(\frac{K_r^2+1}{\nu} + 2M\right)\|\phi(t)\|^2 + \frac{\nu c^2}{2}\left(\|w(t)\|^4 + \|\frac{\partial^2 w}{\partial x^2}(t)\|^4\right). \tag{4.14}$$

Integrating with respect to t and using $\phi(0)=0$, Lemma A.4, leads to

$$\|\phi(t)\|^2 \le \left(\frac{K_r^2+1}{\nu} + 2M\right)\int_0^t \|\phi(s)\|^2 ds + \tilde{M}\|w_0\|^4 e^{4C_r t} \tag{4.15}$$

where $\tilde{M} = \frac{\nu c^2}{2}\left(\frac{8C_r}{\nu^2} + \frac{1}{4C_r}\right)$.

Using Gronwall's lemma and $\phi(0)=0$,

$$\|\phi(t)\|^2 \le \bar{C}^2\|w_0\|^4, \tag{4.16}$$

where

$$\bar{C}^2 = \tilde{M}e^{4C_rT} + \frac{\tilde{M}\nu}{4C_r\nu - K_r^2 + 1 - 2M\nu} e^{\frac{4C_r\nu - K_r^2 - 1 - 2M\nu}{\nu}T},$$

which implies that

$$\|\phi\,(t)\,\| \le \bar{C}\|w_0\|^2, \quad t \in [0, T].$$

Using the definitions of ϕ, w

$$\|\phi\,(t)\,\| = \|w\,(t) - \overline{w}\,(t)\,\| = \|z\,(t) - y\,(t) - \overline{w}\,(t)\,\| \le \bar{C}\|w_0\|^2 = \bar{C}\|z_0 - y_0\|^2, \quad z_0 \neq y_0.$$

That is,

$$\frac{\|z\,(t) - y\,(t) - \overline{w}\,(t)\,\|}{\|z_0 - y_0\|} \le \bar{C}\|z_0 - y_0\|, \tag{4.17}$$

or,

$$\frac{\|S_B\,(t)\,z_0 - S_B\,(t)\,y_0 - T_B\,(t)\,w_0\|}{\|w_0\|} \le \bar{C}\|w_0\|. \tag{4.18}$$

where $z_0 = y_0 + w_0$. Inequality (4.18) holds for every $z_0 \in L^2(-\pi, \pi)$ with $\|z_0 - y_0\| \le \varepsilon$ with $\varepsilon > 0$. Take the limit as $\|w_0\| \to 0$ to obtain

$$\lim_{\|w_0\| \to 0} \frac{\|S_B\,(t)\,(y_0 + w_0) - S_B\,(t)\,y_0 - T_B\,(t)\,w_0\|}{\|w_0\|} = \lim_{\|w_0\| \to 0} \bar{C}\|w_0\| = 0. \tag{4.19}$$

Thus, the nonlinear C_0-semigroup $S_B\,(t)$ generated by the controlled KS equation is Fréchet differentiable. Moreover, the Fréchet derivative of $S_B(t)$ is the C_0-semigroup generated by the linearized KS equation, $T_B\,(t)$. □

THEOREM 4.4. *Consider the uncontrolled KS equation (3.1) with $b\,(x) = 0$ at some constant equilibrium point z_e. If the instability parameter $\nu < 1$, then the equilibrium is unstable.*

Proof. Consider the KS equation linearized at a constant equilibrium point $z_i \in Z_e$. The generator A' (4.3) of the linearized semigroup is a Riesz-spectral operator with distinct eigenvalues $\lambda_n = n^2(1 - \nu n^2) - \imath z_e n$, $n \in \mathbb{Z}$ (Theorem 4.1). If $\nu < 1$, then there are eigenvalues with positive real part and the linearized system is unstable. The assumptions of Corollary 2.9 are satisfied. It follows that the uncontrolled nonlinear KS equation is unstable. □

The number of unstable eigenvalues depends on the value of the instability parameter ν which is a finite number. For a given $0 < \nu < 1$, let N be the smallest integer such that

$$N > \sqrt{\frac{1}{\nu}}. \tag{4.20}$$

The number of unstable eigenfunctions for the uncontrolled linearized KS equation at the equilibrium solution z_e is equal to N.

Stability of the equilibrium solutions to the uncontrolled KS equation ((3.1) with $b\,(x) = 0$) depends on the value of the instability parameter ν. If $\nu > 1$, it was shown in Theorem 3.3 that the set of all constant equilibrium solutions is globally asymptotically stable. If $\nu < 1$ then the constant equilibria are not stable.

In the next section linearization will be used to construct a locally stablizing feedback controller.

5. Stabilization of the Kuramoto-Sivashinsky Equation. It was shown in the previous sections that if $\nu < 1$ any constant equilibrium solution is unstable. As noted at the end of section 3, if $\nu = 1$, then the zero equilibrium solution is Lyapunov stable and not asymptotically stable. It is desired to design a feedback control to drive the solution of the KS equation to a desired state and more generally from one state to another.

If K is such that the controlled KS equation (3.1) is locally exponentially stable at a given equilibrium point, then K is said to locally *exponentially stabilize* the nonlinear KS equation. If such a K exists, the KS equation is said to be locally *exponentially stabilizable* at that equilibrium point.

Since the nonlinear C_0-semigroup corresponding to the controlled KS equation is Fréchet differentiable (shown in Theorem 4.3), then using Theorem 2.5, if the linearized controlled KS equation around a desired equilibrium solution generates an exponentially stable C_0-semigroup, then the same input-feedback control can be used to locally exponentially stabilize the nonlinear KS equation and hence can steer the solution of the KS equation to the desired state.

Note that this result is general and can be used to control the KS equation to any state, not necessarily a constant state. However, only constant equilibrium solutions are considered in this paper as it is easier to analyze the linearized KS equation around a constant equilibrium solution.

There are many ways to design a state-feedback controller that stabilizes linear infinite-dimensional partial differential equations; see for instance, [7, 9, 18, 35, 41, 44]. One approach is to design a linear quadratic controller [18, 54]. Another approach is H_∞-controller synthesis where the effect of the disturbance on the cost is considered instead of the initial condition [8, 29]. However, most controller design approaches, including these, cannot be implemented using the full partial differential equation. An approximation needs to be used in controller design and in simulations.

Approximations of controller design for infinite-dimensional systems do not always lead to reliable results; see for instance [11, 43, 44]. However, there are conditions for linear systems that guarantee that approximations yield stabilizing controllers and correctly predict closed-loop behaviour. Combined with Theorems 4.3 and 2.5, they lead to a method to design stabilizing controllers for the KS equation.

Write $Z = L^2(-\pi, \pi)$ and define a sequence of finite-dimensional subspaces $Z_n \subset H^1(-\pi, \pi)$ and the orthogonal projection $P_n : Z \to Z_n$. It is assumed that for all $z \in Z$, $\lim_{n\to\infty} \|P_n z - z\| = 0$. This assumption is satisfied by typical approximation methods, such as linear splines and also Fourier series expansions. The space Z_n is equipped with the norm inherited from Z. Define $B_n = P_n B$, and define the approximating generator $A_n : Z_n \to Z_n$ using some method. This leads to a sequence of finite-dimensional approximations

$$\frac{dz}{dt} = A_n z(t) + B_n u(t), \qquad z(0) = P_n z_0, \tag{5.1}$$

THEOREM 5.1. *Assume that the sequence of approximations (A_n, B_n) is stabilizable. Let K_n be a convergent sequence of controllers for the approximating systems such that the limit K exponentially stabilizes (A', B) defined in (4.2). Then for sufficiently high order n, the controllers K_n stabilize the KS equation.*

Proof.

$$A' - BK_n = A' - BK + B(K - K_n).$$

For any $\epsilon > 0$ there is N so $\|B(K - K_n)\| < \epsilon$ for all $n > N$. Thus, since $A' - BK$ generates an exponentially stable semigroup, there is N so $A' - BK_n$ generates an exponentially stable semigroup for all $n > N$. Theorem 2.5 then implies that the controlled KS equation is locally exponentially stable. $\square$

The key point in using the above theorem is to find a convergent sequence of stabilizing controllers for the finite-dimensional linearizations (A_n, B_n). However, there are a number of ways to do this. One possibility is $K_n = B_n^*$, that is $K_n = \langle P_n b, \cdot\rangle$. Also, the approach in [6] was extended in [43] to show that linear quadratic controller design yields such a sequence. An H_∞-controller design approach also yields a suitable sequence [26]. For a summary of this approach to controller design, see [44].

6. Example. Consider the nonlinear KS equation (3.1) with instability parameter $\nu = \frac{1}{2}$ and

$$b\,(x) = \frac{1}{0.3} \cdot \chi_{[r-0.15, r+0.15]}, \tag{6.1}$$

where $r \in (-\Pi + 0.15, \Pi - 0.15)$ and $\chi_{[a,b]}$ indicates the characteristic function with support on $[a, b]$.

Since $\nu < 1$, the uncontrolled system is unstable at any constant equilibrium point.

Since the eigenfunctions of the generator of the linearized system form an orthonormal basis for $L^2(-\pi, \pi)$ (Theorem 4.1), a truncation of the Fourier series can be used to approximate the solution. Let $\{\phi_n, \psi_n\}$, where $\phi_0 = \frac{1}{\sqrt{2\pi}}$, $\phi_n\,(\cdot) = \frac{1}{\sqrt{\pi}} \cos\,(n\cdot)$ and $\psi_n\,(\cdot) = \frac{1}{\sqrt{\pi}} \sin\,(n\cdot)$ for $n = 1, \cdots, \infty$ and define

$$\begin{aligned} b1_n &= \langle b, \phi_n\rangle, \text{ for } n = 0, 1, \cdots, \infty. \\ b2_n &= \langle b, \psi_n\rangle, \text{ for } n = 1, 2, \cdots, \infty. \end{aligned} \tag{6.2}$$

Defining Z_M to be the span of the first M functions: $Z_M = \text{span}\{\phi_0, \phi_i, \psi_i\}_{i=1}^M$, and the orthogonal projection $P_M : Z \to Z_M$, $\lim_{M\to\infty} \|P_M z - z\| = 0$ as discussed in the previous section. A Galerkin method with the eigenfunctions of the linearization as a basis for the approximating subspace will be used to approximate the solution to the uncontrolled nonlinear KS equation. For any $M > 0$ define

$$z_M\,(x, t) = a_0(t)\phi_0(x) + \sum_{i=1}^{M} a_i(t)\phi_i(x) + \sum_{i=1}^{M} c_i(t)\psi(x)$$

where $a_i\,(t)\,, c_i\,(t)$ yields the solution of the ODE system resulting from the Galerkin projection method. Note the approximation is of order $2M + 1$. Write the vector of coefficients

$$[z](t) = [a_0(t), a_1(t), \ldots a_M(t), c_1(t), \ldots c_M(t)].$$

Figure 6.1 is a 3-D landscape of the approximated solution with $M = 10$ and initial condition

$$z_0\,(x) = -\frac{1}{2}x^2 + 5x - 4. \tag{6.3}$$

The first 5 eigenfunctions (that is $M = 2$) in the KS equation with $\nu = \frac{1}{2}$ correspond to eigenvalues with positive real parts; see [2] for the calculations. The

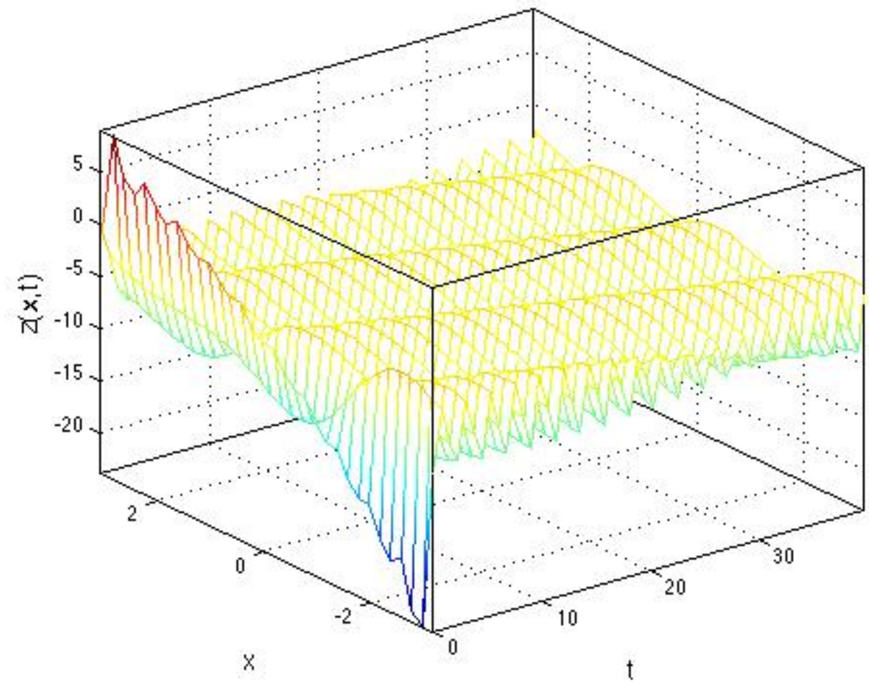

FIG. 6.1. *The response of the nonlinear KS equation with no control. The parameter* $\nu = \frac{1}{2}$ *and the initial condition* $z_0 = -\frac{1}{2}x^2 + 5x - 4$ *(6.3). The system is unstable with this value of* ν*.*

controllers will be designed using only the corresponding eigenfunctions, that is the approximation has $M = 2$. Stabilization to the equilibrium solution $z_e = 0$ will be considered first. Choose $r = 0.05$, then the actuator will have support at [-0.1,0.2]. Moreover, since $b1_n = \langle b, \phi_n \rangle \neq 0$ for $n = 0, 1, 2$ and $b2_n = \langle b, \psi_n \rangle \neq 0$ for $n = 1, 2$, the system is stabilizable. Linear-quadratic control is used here; that is the control u minimizes the quadratic cost function

$$J(u) = \int_0^\infty \left([z]^T (t) [z] (t) + u^T (t) u (t) \right) dt$$

subject to (5.1). This yields the control $u(t) = -K_0[z](t) = -\langle k_0, [z](t) \rangle$ where

$$k_0 = [-1.0000 \quad 3.2134 \quad 0.0264 \quad 3.2134 \quad 0.0264] \ .$$

The system simulated with a 21^{st}-order approximation ($M = 10$) and the controller designed with only 5 eigenfunctions as shown in Figure 6.2. The number of eigenfunctions used in the simulations is larger than the number of eigenfunctions used to design the controller, yet the feedback controller achieved the stabilization and there was no spillover.

Now, consider another equilibrium solution $z_e = 1$ and the control centred at $r = 1$. Again using LQ control with the same weights, the feedback control $K_1 = \langle k_1, \cdot \rangle$ where

$$k_1 = [1.0000 \quad 5.2073 \quad -0.0121 \quad 1.5929 \quad 0.0397].$$

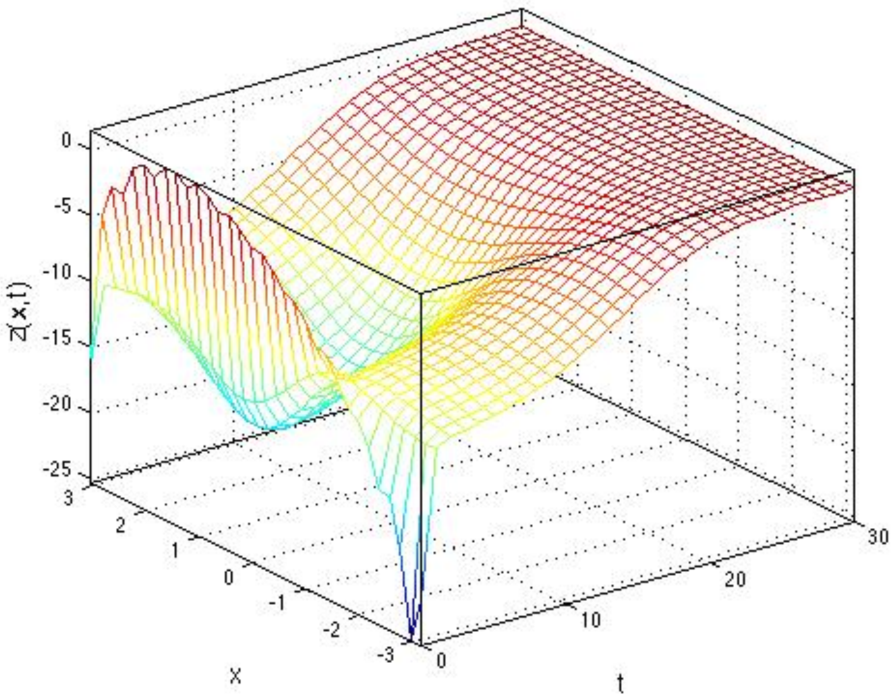

Fig. 6.2. *The controlled KS equation when $\nu = \frac{1}{2}$ with the same initial condition (6.3) as in Figure 6.1. The system is linearized at $z_e = 0$ to design the controller. Controller design is done using only the first 2 modes ($M = 2$) the simulations are with the first* 10 *modes for the equation ($M = 10$). There is no spillover and the solution converges to the zero equilibrium solution. The control causes the unstable equilibrium z_e to become a stable equilibrium.*

was calculated to stabilize the KS equation linearized around $z_e = 1$. Figure 6.3 is a 3-D landscape of the approximated controlled KS equation to the equilibrium solution $z_e = 1$. The figure shows that the approximated solution of the KS equation converging towards the desired equilibrium solution $z_e = 1$ illustrating that the controller stabilizes the nonlinear KS equation.

This same approach is now used to move the KS equation from one equilibrium state to another. Linearizing the KS equation at $z_e = 2$, a third controller $K_3 = \langle k_3, \cdot \rangle$ is calculated as above. Figure 6.4 is a 3-D landscape of the controlled nonlinear KS equation showing that applying the control K_1 followed by K_3 controls the state from the given initial condition to $z_e = 1$ and then from $z_e = 1$ to $z_e = 2$.

7. Summary. The Fréchet derivative of the semigroup corresponding to an infinite-dimensional dynamical system plays a key role in using the linearization to analyze stability and design controllers. If the semigroup is Fréchet differentiable, and the derivative is exponentially stable then the original system is locally exponentially stable. More particularly, if the spectrum-determined growth assumption holds, and the spectrum of the generator lies in the open left half plane, then the original system is locally exponentially stable. Similarly, if the derivative is unstable then the original system is unstable. if the linear generator has spectrum with positive real part, then

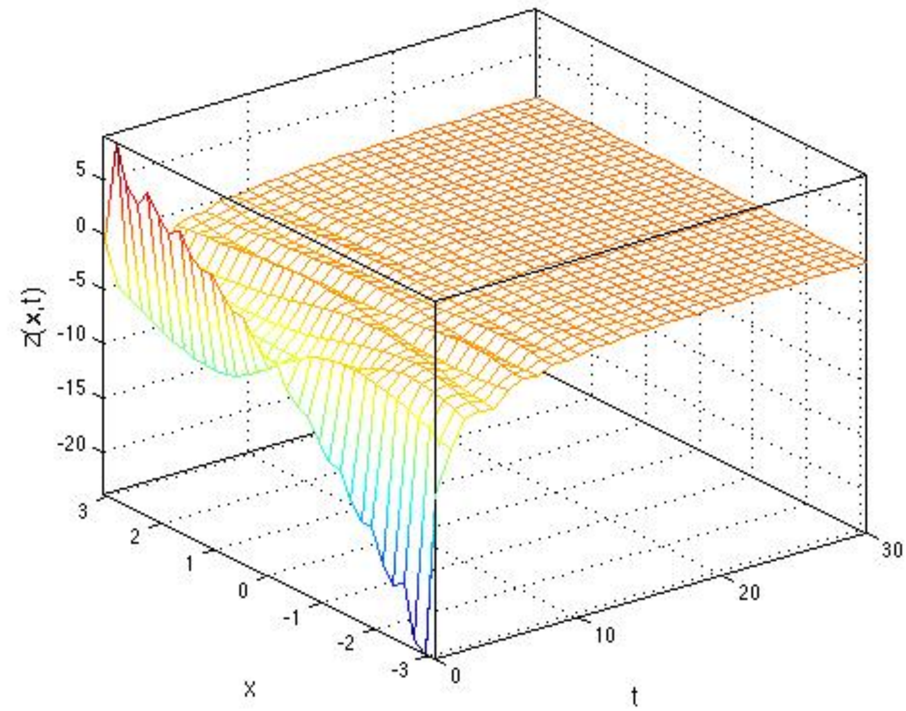

FIG. 6.3. *The controlled KS equation when $\nu = \frac{1}{2}$ with the same initial condition (6.3) as for Figures 6.1 and 6.2. Here the input-feedback control is designed using a linearization at the equilibrium $z_e = 1$ so that the solution converges to $z_e = 1$. As in Figure 6.2, 2 modes ($M = 2$) are used to design the controller, the simulations include 10 modes ($M = 10$). Again, the control causes the unstable equilibrium z_e to become a stable equilibrium.*

the original system is unstable. If the derivative is only asymptotically stable, no conclusion can be drawn.

Stability and stabilization of the KS equation with periodic boundary conditions is considered in detail in this paper. The set of all constant equilibria is shown to be asymptotically stable when the instability parameter $\nu > 1$. This is done using Lyapunov's theorem and LaSalle's invariance principle. It is shown that the semigroup corresponding to the KS equation is Frèchet differentiable. Lyapunov's indirect method can be used to analyze the stability. Constant equilibria for the KS equation are proven to be unstable when $\nu < 1$. The approach in [15] or reformulation of the state-space could be used to show that the set of constant equilibria is locally exponentially stable if $\nu > 1$.

Stabilization of the KS equation with a bounded control operator was then studied. The semigroup corresponding to the controlled KS equation is Frèchet differentiable. Furthermore, the generator of the linearized semigroup is the Gateaux derivative of the original generator. The linearization can therefore be used in controller design. It is proven further that finite-dimensional approximations of the linearized system can be used for controller synthesis. This means that the wide body of techniques available for linear finite-dimensional systems can be used. The effectiveness

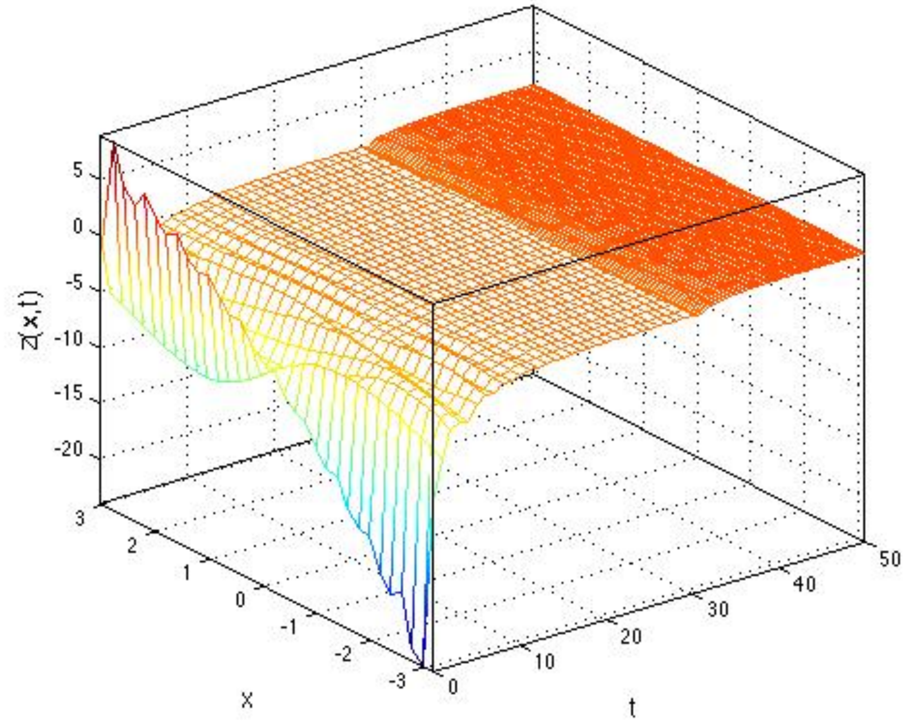

FIG. 6.4. *The controlled KS equation when $\nu = \frac{1}{2}$ with the same initial condition (6.3) as in Figures 6.1, 6.2, 6.3. The controller designed using the linearization at $z_e = 1$ is followed by the controller designed using the linearization at $z_e = 2$. The solution converges to the first equilibrium solution $z_e = 1$, then to $z_e = 2$.*

of the approach was illustrated by stabilization of an example where $\nu = \frac{1}{2}$. The state of the KS equation is driven to several different constant equilibrium solutions and from one equilibrium to another equilibrium.

Subsequent research has extended this approach to output feedback control of the KS equation [3].

Appendix A. Lemmas used to prove Theorem 4.3.

LEMMA A.1. *For every $z, w \in H^1_{per}(-\pi,\pi)$,*

$$|\langle z\frac{\partial w}{\partial x}, w\rangle| \leq \|z\|_\infty \ \|w\| \ \|\frac{\partial w}{\partial x}\|. \tag{A.1}$$

Proof. Let $z, w \in H^1_{per}(-\pi,\pi)$, then using Cauchy-Schwarz inequality [18, page

576] leads to

$$
\begin{aligned}
|\langle z\frac{\partial w}{\partial x},w\rangle| &= \left|\int_{-\pi}^{\pi}\frac{\partial w}{\partial x}z\overline{w}dx\right| \\
&\le \|z\|_\infty \int_{-\pi}^{\pi}\left|\overline{w}\frac{\partial w}{\partial x}\right|dx \\
&\le \|z\|_\infty\ \|w\|\ \|\frac{\partial w}{\partial x}\|. \qquad \square
\end{aligned}
$$

LEMMA A.2. *For every* $y,w\in H^1_{per}(-\pi,\pi)$,

$$
|\langle w\frac{\partial y}{\partial x},w\rangle| \le 2\ \|y\|_\infty\ \|w\|\ \|\frac{\partial w}{\partial x}\|. \tag{A.2}
$$

Proof. Let $y,w\in H^1_{per}(-\pi,\pi)$, then using integration by parts and the Cauchy-Schwarz inequality,

$$
\begin{aligned}
|\langle w\frac{\partial y}{\partial x},w\rangle| &= \left|\int_{-\pi}^{\pi}w\frac{\partial y}{\partial x}\overline{w}dx\right| \\
&= \int_{-\pi}^{\pi}\left|y\left(w\frac{\partial\overline{w}}{\partial x}\right)\right|+\left|y\left(\overline{w}\frac{\partial w}{\partial x}\right)\right|dx \\
&\le 2\ \|y\|_\infty\ \|w\|\ \|\frac{\partial w}{\partial x}\|. \qquad \square
\end{aligned}
$$

LEMMA A.3. *Define the nonlinear operator* $G: H^2_{per}(-\pi,\pi)\to L^2(-\pi,\pi)$

$$
G(z)=Rz+F(z)-BKz.
$$

For each $z,y\in H^2_{per}(-\pi,\pi)$,

$$
G(z)-G(y)=\frac{\partial}{\partial x}\left(y\left(z-y\right)\right)+\left(R-BK\right)\left(z-y\right)+F\left(z-y\right). \tag{A.3}
$$

Furthermore, for some $c>0$,

$$
\|F\left(z-y\right)\| \le \frac{c}{2}\ \|z-y\|_{H^1}. \tag{A.4}
$$

Proof. Let $z,y\in H^2_{per}(-\pi,\pi)$. Define $w=z-y$ and let $M=\|BK\|$. Using the definitions of G,F in (4.7) and (3.7) respectively leads to

$$
\begin{aligned}
G(z)-G(y) &= w\frac{\partial y}{\partial x}+Rw-BKw+z\frac{\partial w}{\partial x} \\
&= y\frac{\partial w}{\partial x}+w\frac{\partial y}{\partial x}+Rw-BKw+z\frac{\partial w}{\partial x}-y\frac{\partial w}{\partial x} \\
&= y\frac{\partial w}{\partial x}+w\frac{\partial y}{\partial x}+Rw-BKw+w\frac{\partial w}{\partial x}.
\end{aligned}
$$

Next, use Poincaré inequality [46, Lemma 1.8] and the multiplicative algebra property [51, Page 51] to obtain

$$
\begin{aligned}
\|F\left(z-y\right)\| &= \|\left(z-y\right)\left(\frac{\partial z}{\partial x}-\frac{\partial y}{\partial x}\right)\| \\
&= \frac{1}{2}\ \|\frac{\partial}{\partial x}\left(z-y\right)^2\| \\
&\le \frac{c}{2}\ \|z-y\|^2_{H^1},
\end{aligned}
$$

as was to be shown. $\square$

LEMMA A.4. *Consider the KS equation (3.8) with different initial conditions $z_0, y_0 \in L^2(-\pi,\pi)$. Let $w(t) = z(t) - y(t)$, where $z(t), y(t)$ are solutions to the KS equation corresponding to different initial conditions z_0, y_0, respectively. Then*

$$\int_0^t \|w(s)\|^4 ds \le \frac{1}{4C_r}\|w_0\|^4 e^{4C_r t} \tag{A.5}$$

$$\int_0^t \|\frac{\partial^2 w}{\partial x^2}(s)\|^4 ds \le \frac{8C_r}{\nu^2}\|w_0\|^4 e^{4C_r t}. \tag{A.6}$$

Moreover, $\|z\|_\infty \le r$ and $\|y\|_\infty \le r$, where $r = (\|y_0\| + \varepsilon)\max\{1, e^{\left(\frac{1}{2\nu}+M\right)T}\} > 0$.

Proof. Consider the nonlinear controlled KS equation given by (3.8) with different initial conditions $z_0, y_0 \in L^2(-\pi,\pi)$

$$\begin{aligned} \dot{z}(t) &= -\hat{A}z(t) - G(z(t)), \quad z(0) = z_0 \\ \dot{y}(t) &= -\hat{A}y(t) - G(y(t)), \quad y(0) = y_0, \end{aligned} \tag{A.7}$$

where the operators $\hat{A}$ and G are given in (3.5) and (4.7), respectively.

Since the operator BK is bounded, then there exists $M > 0$ such that $\|BKz\| \le M\|z\|$ for every $z \in L^2(-\pi,\pi)$. It can be shown that the L^2-norm of the solution $\|z(t)\| \le e^{\left(\frac{1}{2\nu}+M\right)T}\|z_0\|$, where $\nu > 0$ and $t \in [0,T]$. Suppose $\|z_0 - y_0\| \le \varepsilon$ for some $\varepsilon > 0$. Choose $r = (\|y_0\| + \varepsilon)\, e^{\left(\frac{1}{2\nu}+M\right)T}$. Then

$$\|y_0\| \le r - \varepsilon$$

$$\sup_{t\in[0,T]} \|y(t)\| \le r$$

$$\begin{aligned} \sup_{t\in[0,T]} \|z(t)\| &\le \sup_{t\in[0,T]} e^{\left(\frac{1}{2\nu}+M\right)t}(\|z_0 - y_0\| + \|y_0\|) \\ &\le r. \end{aligned}$$

Note that r does not depend on z_0. Subtracting the above two equations and letting $w(t) = z(t) - y(t)$,

$$\begin{aligned} &\dot{w}(t) + \hat{A}w(t) = -(G(z(t)) - G(y(t))) \\ &w(0) = z_0 - y_0 =: w_0. \end{aligned} \tag{A.8}$$

Moreover, it was shown in Lemma (A.3) that

$$G(z(t)) - G(y(t)) = Rw(t) + z(t)\frac{\partial w}{\partial x}(t) + w(t)\frac{\partial y}{\partial x}(t) - BKw(t). \tag{A.9}$$

Take the inner product of (A.8) with w to obtain

$$\langle \dot{w}(t), w(t)\rangle + \langle \hat{A}w(t), w(t)\rangle = -\langle (G(z(t)) - G(y(t))), w(t)\rangle.$$

That is,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \nu\|\frac{\partial^2 w}{\partial x^2}(t)\|^2 &= -\langle (G(z(t)) - G(y(t))), w(t)\rangle \\ &\le |\langle (G(z(t)) - G(y(t))), w(t)\rangle|. \end{aligned} \tag{A.10}$$

Using (A.9), Triangle inequality, Cauchy-Schwarz inequality, Lemma A.1, Lemma A.2 and the Poincaré inequality,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\left(t\right)\|^2+\nu\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^2 &\leq |\langle Rw\left(t\right),w\left(t\right)\rangle|+|\langle z\left(t\right)\frac{\partial w}{\partial x}\left(t\right),w\left(t\right)\rangle| \\ &\quad +|\langle w\left(t\right)\frac{\partial y}{\partial x}\left(t\right),w\left(t\right)\rangle|+|\langle BKw\left(t\right),w\left(t\right)\rangle| \\ &\leq \|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|\,\|w\left(t\right)\|+\|z\left(t\right)\|_\infty\|w\left(t\right)\|\,\|\frac{\partial w}{\partial x}\left(t\right)\| \\ &\quad +2\|y\left(t\right)\|_\infty\|w\left(t\right)\|\ \|\frac{\partial w}{\partial x}\left(t\right)\|+M\|w\left(t\right)\|^2 \\ &\leq (1+3r)\ \ \|w\left(t\right)\|\ \|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|+M\|w\left(t\right)\|^2 \\ &= K_r\ \|w\left(t\right)\|\ \|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|+M\|w\left(t\right)\|^2, \qquad \text{(A.11)}\end{aligned}$$

where $K_r = 1 + 3r$.

Using Young's inequality [46, Lemma 5.40],

$$\frac{1}{2}\frac{d}{dt}\|w\left(t\right)\|^2+\nu\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^2 \leq \frac{\nu}{2}\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^2+\left(\frac{K_r^2}{2\nu}+M\right)\|w\left(t\right)\|^2. \quad \text{(A.12)}$$

Multiply the above inequality by 2 and re-arrange the terms to obtain

$$\frac{d}{dt}\|w\left(t\right)\|^2+\nu\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^2 \leq 2C_r\|w\left(t\right)\|^2 \qquad \text{(A.13)}$$

where $C_r = \frac{K_r^2}{\nu} + M$ and so

$$\frac{d}{dt}\|w\left(t\right)\|^2 \leq 2C_r\|w\left(t\right)\|^2. \qquad \text{(A.14)}$$

This implies that

$$\|w\left(t\right)\|^2 \leq \|w_0\|^2 e^{2C_r t}, \quad t \geq 0 \qquad \text{(A.15)}$$

and so

$$\int_0^t \|w\left(s\right)\|^4 ds \leq \frac{1}{4C_r}\|w_0\|^4 e^{4C_r t}. \quad t \geq 0.$$

Combine inequalities (A.14) and (A.15) to obtain

$$\frac{d}{dt}\|w\left(t\right)\|^2 \leq 2C_r\|w_0\|^2 e^{2C_r t}.$$

Square the above inequality and integrate with respect to t to obtain

$$\begin{aligned}\int_0^t \left(\frac{d}{ds}\|w\left(s\right)\|^2\right)^2 ds &\leq 4C_r^2\|w_0\|^4\int_0^t e^{4C_r s}ds \\ &\leq C_r\|w_0\|^4 e^{4C_r t}. \qquad \text{(A.16)}\end{aligned}$$

Combine inequalities (A.13) and (A.15),

$$\frac{d}{dt}\|w\left(t\right)\|^2+\nu\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^2\leq 2C_r\|w_0\|^2 e^{2C_r t}. \tag{A.17}$$

Integrate (A.17) with respect to t to obtain

$$\|w\left(t\right)\|^2-\|w_0\|^2+\nu\int_0^t\|\frac{\partial^2 w}{\partial x^2}\left(s\right)\|^2 ds\leq\|w_0\|^2 e^{2C_r t}-\|w_0\|^2$$

which implies that

$$\int_0^t\|\frac{\partial^2 w}{\partial x^2}\left(s\right)\|^2 ds\leq\frac{1}{\nu}\|w_0\|^2 e^{2C_r t}. \tag{A.18}$$

Now, square inequality (A.17) and expand the perfect square on the left hand side to obtain

$$\left(\frac{d}{dt}\|w\left(t\right)\|^2\right)^2+2\nu\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^2\cdot\frac{d}{dt}\|w\left(t\right)\|^2+\nu^2\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^4\leq 4C_r^2\|w_0\|^4 e^{4C_r t}.$$

Re-arrange the terms and use Young's inequality ($|2a\cdot b|\leq 2a^2+\frac{1}{2}b^2$) to obtain

$$\nu^2\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^4\leq 4C_r^2\|w_0\|^4 e^{4C_r t}-\left(\frac{d}{dt}\|w\left(t\right)\|^2\right)^2-2\nu\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^2\cdot\frac{d}{dt}\|w\left(t\right)\|^2.$$

That is,

$$\begin{aligned}\nu^2\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^4 &\leq 4C_r^2\|w_0\|^4 e^{4C_r t}+\left(\frac{d}{dt}\|w\left(t\right)\|^2\right)^2+2\nu\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^2\cdot\left|\frac{d}{dt}\|w\left(t\right)\|^2\right|\\ &\leq 4C_r^2\|w_0\|^4 e^{4C_r t}+\left(\frac{d}{dt}\|w\left(t\right)\|^2\right)^2+2\left(\frac{d}{dt}\|w\left(t\right)\|^2\right)^2\\ &\quad+\frac{\nu^2}{2}\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^4.\end{aligned}$$

Re-arrange the terms to obtain

$$\frac{\nu^2}{2}\|\frac{\partial^2 w}{\partial x^2}\left(t\right)\|^4\leq 4C_r^2\|w_0\|^4 e^{4C_r t}+3\left(\frac{d}{dt}\|w\left(t\right)\|^2\right)^2.$$

Finally, integrate with respect to t and use inequality (A.16) to obtain

$$\begin{aligned}\frac{\nu^2}{2}\int_0^t\|\frac{\partial^2 w}{\partial x^2}\left(s\right)\|^4 ds &\leq C_r\|w_0\|^4 e^{4C_r t}+3C_r\|w_0\|^4 e^{4C_r t}\\ &=4C_r\|w_0\|^4 e^{4C_r t}.\end{aligned}$$

Hence,

$$\int_0^t\|\frac{\partial^2 w}{\partial x^2}\left(s\right)\|^4 ds\leq\frac{8C_r}{\nu^2}\|w_0\|^4 e^{4C_r t}. \quad\square$$

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