SIAM J. CONTROL OPTIM.
Vol. 42, No. 4, pp. 1244–1265

WELL-POSEDNESS OF BOUNDARY CONTROL SYSTEMS*

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Abstract. Continuity of the input/output map for boundary control systems is shown through the system transfer function. Our approach transforms the question of continuity of the input/output map of a boundary control system to uniform boundedness of the solution to a related elliptic problem. This is shown for a class of boundary control systems with Dirichlet, Neumann, or Robin boundary control.

Key words. infinite-dimensional systems, boundary control, input/output stability

AMS subject classifications. 35B30, 35B37, 93C20

DOI. 10.1137/S0363012902384916

1. Introduction.

Boundary control systems are an important class of infinite-dimensional control systems. Some important applications are control of annealing processes, control of structural vibrations, and active noise control.

Key questions are whether the mappings from input to state, input to output, initial state to input, and initial state to final state are well defined and bounded. When all four mappings are well defined and bounded, the system is said to be well-posed [19]. Salamon [20] showed that boundedness of the input/output map implies well-posedness of the control system with respect to some state space. (An alternative proof in [14] uses frequency domain analysis.) Since boundedness is equivalent to continuity for linear systems, ill-posedness of the input/output map indicates that the measured outputs are not continuously dependent on the inputs. This would lead to difficulties in the practical implementation of any such control system. Often, however, ill-posedness of the control system indicates modelling errors. An example illustrating this point is given in this paper. Thus, showing boundedness of the input/output map of a boundary control system is important. This problem is the focus of this paper.

Boundedness of the initial to final state map is equivalent to showing existence of a semigroup and is fairly well understood. A number of authors have obtained results on boundedness of the state/output map and input/state map. For more details see, e.g., [3, 8, 9, 10, 11, 12, 15, 16]

The literature on showing boundedness of the input/output map is less extensive. One technique for determining well-posedness is to use spectral expansion of the underlying semigroup. This technique is applicable to showing boundedness of the input/state and state/output maps as well as the input/output map. For example, in [7] it was shown that several examples of boundary control systems with one space dimension were well-posed. In [6], it was shown that the one-dimensional heat equation with Dirichlet boundary control and point observation is well-posed under a suitable choice of state space. In [18], well-posedness of an accelerometer control system was shown. The spectral expansion method requires the availability of the

*Received by the editors February 20, 2002; accepted for publication (in revised form) February 16, 2003; published electronically August 6, 2003. This research was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

http://www.siam.org/journals/sicon/42-4/38491.html

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eigenvalues and eigenvectors of the system (or at least estimates). Also, the eigenvectors must form a Riesz basis. For many multidimensional problems it is difficult to calculate the eigenfunctions and eigenvalues of the underlying semigroup. Hence there are difficulties in extending this method to more general problems.

Well-posedness of a structural acoustic control system has been considered in [1, 2]. The authors use a state-space formulation of the control system. Partial differential equation results lead to estimates of the regularity of the resolvant and hence of the transfer function in state-space form.

Another method to determine boundedness of the input/output map uses the system transfer function. The concept of the transfer function for finite-dimensional systems extends to general well-posed systems. This is discussed in detail in the next section. Curtain and Weiss [6] showed that the input/output map is bounded if and only if the transfer function is uniformly bounded in a right half-plane. In several papers [6, 18, e.g.] well-posedness is established by showing that the system transfer function is bounded in some right-half-plane. The difficulty with this approach is that the transfer function has been rigorously obtained for only a few systems.

In [3], boundedness of the input/output map was shown for a class of structural control systems with point measurement of acceleration by showing that the system transfer function is proper. However, unlike the examples given above, justification for the transfer function was not computed directly. Instead, it was shown that the fact that the infinitesimal generator generates an analytic semigroup implies properness of the system transfer function.

In the next section systems theory for boundary control systems is discussed. The nature of the input/output map and the transfer function for these systems is explained. We give a representation for the system transfer function purely in terms of the boundary control formulation.

In section 3 we present our approach. The question of boundedness of the input/output map of a boundary control system is transformed to uniform boundedness (in a sense defined later) of solutions to a related elliptic boundary value problem. We use this approach to obtain well-posedness of several large classes of boundary control systems. Section 4 contains some background on elliptic boundary value problems. In sections 5 and 6 we show boundedness of the input/output map for a several large classes of problems with Dirichlet, Neumann, or Robin boundary control.

Our approach has several advantages. It is not necessary to compute a state-space realization. Also, the analysis of an elliptic problem is simpler than that of the original problem, and the extensive literature available on boundary value problems may be used. Our method is particularly useful for multidimensional systems with variable coefficients where the state-space realization is tedious to obtain and the system transfer function is even more difficult to obtain from the realization.

2. Transfer functions for boundary control systems. We will use the following formal definition of a *boundary control system*:

$$\left.\begin{aligned} \tfrac{d}{dt}z(t) &= Lz, \qquad z(0) = z_0, \\ \Gamma z(t) &= u(t), \\ y(t) &= Kz(t). \end{aligned}\right\} \tag{2.1}$$

The operators $L \in \mathcal{L}(\mathcal{Z},\mathcal{H})$, $\Gamma \in \mathcal{L}(\mathcal{Z},\mathcal{U})$, and $K \in \mathcal{L}(\mathcal{Z},\mathcal{Y})$. The spaces $(\mathcal{Z},\|\cdot\|_{\mathcal{Z}}), (\mathcal{H},\|\cdot\|_{\mathcal{H}})$, $(\mathcal{U},\|\cdot\|_{\mathcal{U}}), (\mathcal{Y},\|\cdot\|_{\mathcal{Y}})$ are all Hilbert spaces, and $\mathcal{Z}$ is a dense subspace of $\mathcal{H}$ with continuous, injective embedding $\iota_{\mathcal{Z}}$. The triple (L,Γ,K) refers to a boundary control system with output operator K. We shall often refer to a boundary control system by the double (L, Γ). (The operator K is in this case understood to be the identity operator.) We will assume throughout this paper that a boundary control system (2.1) satisfies the following assumptions:

(A1) The operator Γ is onto, ker Γ is dense in $\mathcal{H}$, and there exists $\mu \in R$ such that $\ker(\mu I - L) \cap \ker\Gamma = 0$ and $\mu I - L$ is onto $\mathcal{H}$.

(A2) For any $z_0 \in \mathcal{Z}$ with $\Gamma z_0 = 0$, $u(\cdot) = 0$, there exists a unique solution of (Γ, L) in $C^1[0,T;\mathcal{H}] \cap C[0,T;\mathcal{Z}]$ depending continuously on z_0.

In this paper, we are solely interested in the boundedness of the input/output map from $u \in L^2(0,T;\mathcal{U})$ to $y \in L^2(0,T;\mathcal{Y})$.

DEFINITION 2.1. *The input/output map is bounded if for all times $T > 0$ and $u \in H^2(0,T;\mathcal{U})$, $z(0) = 0$, the output y is well defined and there is a constant c_T such that $\|y\|_{L^2(0,T;\mathcal{Y})} \leq c_T \|u\|_{L^2(0,T;\mathcal{U})}$.*

This implies that the input/output map $u \to y$ can be extended to a bounded map on all of $L^2(0,T;\mathcal{U})$. Alternatively, one can describe the relationship between the inputs and the outputs using the Laplace transform.

DEFINITION 2.2. *Let $\hat{y}(s)$ indicate the Laplace transform of the output of a system and indicate similarly the transform of the input by $\hat{u}(s)$. The system transfer function is the operator $G(s)$ such that*

$$\hat{y}(s) = G(s)\hat{u}(s)$$

for all s, Re $s > \sigma$ *for some real σ.*

Implicit in this definition is that the input/output map is well defined and that the output is Laplace transformable. Boundedness of the input/output map can be determined using the system transfer function.

THEOREM 2.3 (see [6]). *Let (L, Γ, K) define a boundary control system. The input/output map of the system is bounded if and only if there exists a real number σ such that the transfer function $G(s)$ associated with (L, Γ, K) satisfies*

$$\sup_{\mathrm{Re}\ s > \sigma} \| G(s) \|_{\mathcal{L}(\mathcal{U},\mathcal{Y})} < \infty.$$

The function $G(s)$ is said to be proper *if the above inequality holds.*

We now consider the definition of a transfer function for a boundary control system in detail. First, consider a control system in state-space form:

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0, \tag{2.2}$$

$$y(t) = Cz(t), \tag{2.3}$$

where A is an infinitesimal generator of a C_0-semigroup $T(t)$ on state space $\mathcal{H}$. Also, B and C are bounded operators: $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$. The input/output map is

$$y(t) = C \int_0^t T(t - \sigma) B u(\sigma)\, d\sigma. \tag{2.4}$$

Defining $g(t) = CT(t)B$, the output is simply the convolution of $g(t)$ and $u(t)$. Taking the Laplace transform on both sides of (2.4) gives

$$\hat{y}(s) = G(s)\hat{u}(s). \tag{2.5}$$

Here $G(s) = C(sI - A)^{-1}B$ is the system transfer function. Note that it is the Laplace transform of the function $g(t)$ that defines the input/output map. This is a direct generalization of the theory for finite-dimensional systems.

Any boundary control system can be written in state-space form (A, B, C) [19]. The operator A that generates the semigroup $T(t)$ in the state-space formulation is defined from the boundary control system as follows. Define

$$\mathcal{W} = \{\ z \in \mathcal{Z} \mid \Gamma z = 0\ \}, \tag{2.6}$$

and let ι denote the canonical injection from $\mathcal{W}$ to $\mathcal{Z}$. Then $A = L\iota$ and $\mathcal{W} = [D(A)]$, the completion of $D(A)$ in the graph norm of A. Assumptions (A1) and (A2) imply that A generates a C_0-semigroup on $\mathcal{H}$. Techniques to define B and C also exist [19], but the input and output operators are generally unbounded on the state space. The linear operator $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ is defined by $C = K\iota$ and $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$. The definition of B is more complicated and not needed here, but $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ where $\mathcal{V} = [D(A^*]'$, the dual space of $[D(A^*)]$. The operator A extends to an operator that generates a C_0-semigroup on $\mathcal{V}$ with domain $\mathcal{H}$. However, (2.3) is no longer well defined since $z(t)$ may not be in the domain of C.

In the following theorem we show that the output of a boundary control system is well defined, and that this output can be defined via the convolution of a Laplace-transformable distribution with the input. The following results will be required.

LEMMA 2.4 (see [19, Cor. 2.9]). *Let* (A1) *and* (A2) *be satisfied. Then for every* $z_o \in \mathcal{Z}$ *and every* $u \in H^2(0, T; \mathcal{U})$, *with* $\Gamma z_o = u(0)$, *there is a unique solution* $z(\cdot) \in C(0, T; \mathcal{Z}) \bigcap C^1(0, T; \mathcal{H})$ *of* (2.1).

THEOREM 2.5 (see [23, Theorem 6.5-1]). *Necessary and sufficient conditions for a function* $G(s) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ *to be the Laplace transform of a distribution whose support is bounded on the left at* $t = 0$ *are that* (1) *there exists some half-plane* $\operatorname{Re} s > \sigma$ *on which* $G(s)$ *is analytic and* (2) *that there is a polynomial* P *such that for* $\operatorname{Re} s > \sigma$

$$\|G(s)\|_{\mathcal{L}(\mathcal{U}, \mathcal{Y})} \leq P(|s|).$$

THEOREM 2.6. *The input/output map of any boundary control system* (2.1) *is well defined for all inputs* $u \in H^2(0, T; \mathcal{U})$, $u(0) = 0$. *This output can be written as*

$$y(t) = g(t) * u(t),$$

where $g(t)$ *is a distribution with Laplace transform* $G(s)$. *Let* $A = L\iota$ *with domain as in* (2.6). *The operator* $G(s) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ *for each* $s \in \rho(A)$ *and* $G(s)$ *is the system transfer function.*

Proof. First, as mentioned above, construct the state-space realization (A, B, C) using the procedure in [19]. Equation (2.2) is valid if we consider it as a differential equation on $\mathcal{V} = [D(A^*)]'$. Rewriting, we obtain, for any $\mu \in \rho(A)$,

$$\begin{aligned} z(t) &= (\mu I - A)^{-1}(\mu I - A)z(t) \\ &= (\mu I - A)^{-1}(\mu z(t) - \dot{z}(t)) + (\mu I - A)^{-1}Bu(t). \end{aligned} \tag{2.7}$$

For all initial conditions $z(0) = 0$ and smooth controls $u \in H^2(0, T; \mathcal{U})$ with $u(0) = 0$, the first term in (2.7) is in $\mathcal{W} \subset \mathcal{Z}$ for each time t (Lemma 2.4). Regarding A as a generator on $\mathcal{V}$ with domain $\mathcal{H}$, we obtain that $(\mu I - A)^{-1}B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$. Furthermore, for any $\mu \in \rho(A)$, $\operatorname{Range}(\mu I - A)^{-1}B \subset \mathcal{Z}$ and so $(\mu I - A)^{-1}B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ [19, Prop. 2.8]. Thus we may apply the operator K to the solution $z(t)$ to obtain the output $y(t)$:

$$y(t) = K(\mu I - A)^{-1}(\mu z(t) - \dot{z}(t)) + K(\mu I - A)^{-1}Bu(t). \tag{2.8}$$

Since $\mathcal{W} \subset \mathcal{Z}$, $K(\mu I - A)^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ and $K(\mu I - A)^{-1}B \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Since both u and z are Laplace transformable, we now take the Laplace transform of both sides of (2.8) to obtain

$$\hat{y}(s) = K(\mu I - A)^{-1}(\mu - s)(sI - A)^{-1}B\hat{u}(s) + K(\mu I - A)^{-1}B\hat{u}(s).$$

The system transfer function is thus

$$G(s) = K(\mu I - A)^{-1}(\mu - s)(sI - A)^{-1}B + K(\mu I - A)^{-1}B.$$

Setting $\mu = s$, we obtain that

$$G(s) = K(sI - A)^{-1}B \tag{2.9}$$

for any $s \in \rho(A)$. (This is formula (2.18) for the *generalized transfer function* in [19].)

For $s \in \rho(A)$, $G(s)$ is analytic and so condition (1) in Theorem 2.5 is satisfied. Since the norm on $\mathcal{H}$ is equivalent to the graph norm of A (as a generator on $\mathcal{V}$) on $\mathcal{V}$, $\|(s - A)^{-1}B\|_{\mathcal{L}(\mathcal{U},\mathcal{H})} \leq M$ for some constant M and all Re $s > \sigma$ for some σ. Thus, there is a polynomial $P(s)$ such that G satisfies condition (2) in Theorem 2.5. It follows from Theorem 2.5 that $G(s)$ is the Laplace transform of a distribution $g(t)$; hence the output $y(t)$ is the convolution of this distribution and the input. □

This representation of the input/output map is valid for any boundary control system and for $u \in H^2(0, T; \mathcal{U})$ with $u(0) = 0$. In order to extend the input/output map to all $u \in L^2(0, T; \mathcal{U})$ we need to show that the map is bounded or, equivalently, that the transfer function is proper.

We now obtain a representation of the transfer function of a boundary control system. This representation is based entirely on the boundary control description (2.1) and does not require construction of a state-space realization. The transfer function is defined in terms of an elliptic problem associated with the boundary control system.

DEFINITION 2.7. *The* abstract elliptic problem $(L, \Gamma)_e$ *corresponding to the boundary control system* (L, Γ), *as defined in* (2.1), *is*

$$\left.\begin{aligned} Lz &= sz, \quad s \in \mathbb{C}, \\ \Gamma z &= u. \end{aligned}\right\} \tag{2.10}$$

We denote the solution $z \in \mathcal{Z}$ by $z(s)$.

DEFINITION 2.8. *Let $\mathcal{T}(t)$ be a C_0-semigroup on $\mathcal{H}$. The constant α defined by*

$$\alpha = \inf_{t>0} \frac{1}{t} \log \|\mathcal{T}(t)\|$$

is called the growth bound *of the semigroup $\mathcal{T}(t)$.*

Let α indicate the growth bound of the semigroup associated with (L, Γ). The elliptic problem (2.13) has a unique solution $z(s)$ for all u and Re $s > \alpha$. The system transfer function may be described through the solutions to the abstract elliptic problem (2.13).

THEOREM 2.9. *Let (L, Γ, K) define a boundary control system. Define $\mathcal{W}, A$, and $D(A)$ be as above. Then there exists an $\alpha \in \Re$ such that the transfer function, $G(s)$, of the boundary control system (L, Γ, K) is given by*

$$G(s)u = Kz(s) \quad \textit{for all } s \in \mathbb{C}, \textit{ with } \mathrm{Re}\ s > \alpha, \tag{2.11}$$

where $z(s)$ is the solution to the abstract elliptic problem (2.10) *with input u.*

Proof. Let α denote the growth bound of the C_0-semigroup generated by A. Then for all $s \in \mathbb{C}$ with Re $s > \alpha$, $s \in \rho(A)$.

The transfer function $G(s)$ is given by (2.9). However, $(sI - A)^{-1}B$ is the solution operator of abstract elliptic problem (2.10) with input u [19, Prop. 2.8, eqn. 2.20].

Alternatively, for any given $u \in \mathcal{U}$, choose $z \in \mathcal{Z}$ so that $\Gamma z = u$. Then $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is defined by [19, Rem. 2.7]

$$G(s)\Gamma z = Kz - C(sI - A)^{-1}(sz - Lz). \tag{2.12}$$

For any $u \in \mathcal{U}$ and any $s \in \mathbb{C}$, with Re $s > \alpha$, let z solve the associated elliptic problem. From (2.12) we have

$$G(s)u = Kz(s).$$

This is precisely (2.11). □

Thus, the solution to (2.11) gives a representation of the transfer function of a boundary control system. The representation of $G(s)$ obtained above is not as surprising as the abstract elliptic problem (2.10) is the formal Laplace transform (with respect to t) of the boundary control system. Theorem 2.9 is a justification of such a process. Thus the abstract elliptic problem $(L, \Gamma)_e$ corresponding to the boundary control system (L, Γ) can be written as

$$\left.\begin{aligned} L\hat{z} &= s\hat{z}, \quad s \in \mathbb{C}, \\ \Gamma\hat{z} &= \hat{u}. \end{aligned}\right\} \tag{2.13}$$

As a simple example, we compute the transfer function for a heat transfer problem on a unit interval using (2.11).

Example 2.10 (one-dimensional heat equation with Neumann boundary control). One of the simplest examples of a well-posed boundary control system is the problem of temperature control in a one-dimensional rod of length 1 with a controlled heat flow at one end. The output is the temperature measured at x_1, $0 \le x_1 \le 1$. The system equations are

$$\left.\begin{array}{rcll} \frac{\partial z}{\partial t} &=& \frac{\partial^2 z}{\partial x^2}, & x \in [0,1], \\ z(x,0) &=& 0, & x \in [0,1], \\ \frac{\partial z}{\partial x}(0,t) &=& 0, & t > 0, \\ \frac{\partial z}{\partial x}(1,t) &=& u(t), & t > 0, \\ y(t) &=& z(x_1,t). & \end{array}\right\} \tag{2.14}$$

In this example,

$$\mathcal{Z} = \{z \in H^2(0,1); z'(0) = 0\},$$

with the norm inherited from $H^2(0,1)$, $\mathcal{U} = \mathcal{Y} = \Re$, and $\mathcal{H} = L^2(0,1)$. It is easy to verify that (A1) and (A2) are satisfied. The elliptic problem corresponding to (2.14) is

$$\left.\begin{array}{rcl} \frac{d^2\hat{z}}{dx^2} &=& s\hat{z}, \\ \hat{z}'(0) &=& 0, \\ \hat{z}'(1) &=& \hat{u}, \end{array}\right\} \tag{2.15}$$

with output equation

$$\hat{y} = \hat{z}(x_1).$$

The solution to the abstract elliptic problem is

$$\hat{z}(x,s) = \frac{\hat{u}\ \cosh(\sqrt{s}\,x)}{\sqrt{s}\ \sinh\sqrt{s}}.$$

For this problem, the growth bound $\alpha = 0$. By Theorem 2.9 we have for all $s \in \mathbb{C}$ with Re $s > 0$ that the transfer function of the system is given by

$$\begin{aligned} G(s)\hat{u} &= K\left(\frac{\hat{u}\ \cosh(\sqrt{s}\,x)}{\sqrt{s}\ \sinh\sqrt{s}}\right) \\ &= \frac{\hat{u}\ \cosh(\sqrt{s}\,x_1)}{\sqrt{s}\ \sinh\sqrt{s}}. \end{aligned}$$

This is exactly the transfer function one would obtain by formally taking the Laplace transform of (2.14). Moreover, the transfer function is proper; hence the input/output map is bounded.

The following example shows that if the boundary condition is not chosen correctly, it leads to an improper system transfer function. Hence examining the nature of the input/output map is useful in determining whether the mathematical model of the system is sensible. Some choices of sensing or control operations also lead to improper transfer functions.

Example 2.11 (Euler–Bernoulli beam with Kelvin–Voigt damping). Consider the Euler–Bernoulli beam with Kelvin–Voigt damping. The beam is assumed to be fixed at $x = 0$ and free at $x = 1$. Then the equation governing the motion of the transverse displacement is

$$(2.16)\qquad \left.\begin{array}{rcll} \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2} + c_d I\frac{\partial^3 w}{\partial x^2 t}\right] &=& 0, & x \in (0,1), \\ w(0,t) &=& 0, & t \geq 0, \\ \frac{\partial w}{\partial x}(0,t) &=& 0, & t \geq 0, \\ \frac{\partial^2 w}{\partial x^2}(1,t) &=& 0, & t \geq 0, \\ \frac{\partial^3 w}{\partial x^3}(1,t) &=& u(t), & t \geq 0, \\ y(t) &=& \frac{\partial w}{\partial t}(1,t), & \end{array}\right\}$$

where E, I, and c_d are positive constants. We shall compute the system transfer function via Theorem 2.9. First, we will rewrite the problem in the standard form (2.1). Define

$$z(x,t) = \begin{bmatrix} z_1(x,t) \\ z_2(x,t) \end{bmatrix} = \begin{bmatrix} w(x,t) \\ \frac{dw(x,t)}{dt} \end{bmatrix},$$

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -EI\frac{d^4}{dx^4} & -c_d I\frac{d^4}{dx^4} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \\ z_1'''(1,t) &= u(t), \\ y(t) &= z_2(1,t). \end{aligned}$$

For this problem,

$$\mathcal{Z} = \{(z_1,z_2) \in H^4(0,1) \times H^4(0,1); z_1(0) = z_1'(0) = z_1''(1) = 0\}.$$

The space $\mathcal{H} = \bar{H}^2(0,1) \times L^2(0,1)$, where

$$\bar{H}^2(0,1) = \{H^2(0,1); z(0) = z_1'(0) = 0\}$$

and $\mathcal{U} = \mathcal{Y} = R$. It can be verified that assumptions (A1) and (A2) are satisfied. The elliptic problem associated with (2.16) is, writing $w = z_1$ and noting that $\hat{z}_2 = s\hat{z}_1$,

$$\left.\begin{array}{rcl} (EI + sc_d I)\frac{\partial^4 \hat{w}}{\partial x^4} & = & -s^2\hat{w}, \\ \hat{w}'''(1) & = & \hat{u}. \end{array}\right\} \tag{2.17}$$

This is to be solved for $(\hat{w}, s\hat{w}) \in \mathcal{Z}$. The output equation $\hat{y} = s\hat{w}(1,s)$. The solution to the abstract elliptic problem is

$$\hat{w}(s,x) = A(s)\cosh(m(s)x) + B(s)\sinh(m(s)x) - A(s)\cos(m(s)x) - B(s)\sin(m(s)x),$$

where, letting $i = \sqrt{-1}$,

$$m(s) = \sqrt{i}\left(\frac{s^2}{EI + sc_d I}\right)^{\frac{1}{4}},$$
$$A(s) = \frac{-\hat{u}(\sinh(m(s)) + \sin(m(s)))}{2m^3(s)(1 + \cosh(m(s))\cos(m(s)))},$$
$$B(s) = \frac{-A(s)(\cosh(m(s)) + \cos(m(s)))}{\sinh(m(s)) + \sin(m(s))}.$$

Thus the system transfer function is

$$G(s) = \frac{s(\sinh(m(s))\cos(m(s)) - \cosh(m(s))\sin(m(s)))}{m^3(s)(1 + \cosh(m(s))\cos(m(s)))}.$$

One can show that for Re $s > 0$,

$$\lim_{|s|\to\infty} 4\exp\left(\sqrt{\frac{2}{i}}m(s)\right)(\sinh(m(s))\cos(m(s)) - \cosh(m(s))\sin(m(s))) = 1 - i,$$
$$\lim_{|s|\to\infty} 4\exp\left(\sqrt{\frac{2}{i}}m(s)\right)(1 + \cosh(m(s))\cos(m(s))) = 1.$$

Thus, for Re $s > 0$,

$$\lim_{|s|\to\infty} \frac{(\sinh(m(s))\cos(m(s)) - \cosh(m(s))\sin(m(s)))}{(1 + \cosh(m(s))\cos(m(s)))} = 1 - i.$$

Thus $G(s)$ is improper since $|\frac{s}{m^3(s)}|$ is unbounded as $|s| \to \infty$.

The appropriate boundary conditions should be on the bending moments and shear forces in the beam:

$$EI\frac{\partial^2 w}{\partial x^2} + c_d I\frac{\partial^3 w}{\partial x^2 t}(1,t) = 0, \qquad t \geq 0,$$
$$EI\frac{\partial^3 w}{\partial x^3} + c_d I\frac{\partial^4 w}{\partial x^3 t}(1,t) = u(t), \qquad t \geq 0.$$

The original set of boundary conditions is incorrect since the moment M is equal to $\frac{\partial w^2}{\partial x^2}$ only when there is no damping in the system. With these boundary conditions, the resulting transfer function is

$$G(s) = \frac{s\Big(\sinh(m(s))\cos(m(s)) - \cosh(m(s))\sin(m(s))\Big)}{m^3(s)\Big(EI + sc_d I\Big)\Big(1 + \cosh(m(s))\cos(m(s))\Big)}.$$

Now $G(s)$ is proper since

$$\lim_{|s|\to\infty} \frac{s}{m^3(s)(EI + sc_d I)} = 0.$$

3. Boundedness of the input/output map. Theorem 2.3 implies that the boundedness of the input/output map of a boundary control system can be determined from the properness of the system transfer function. For a given observation operator K, the properness of the transfer function depends entirely on the behavior of the solution to $(L, \Gamma)_e$ as the parameter s varies.

Since we will henceforth be working entirely with the Laplace transform, we shall drop the " ˆ "notation in the interest of clarity. The following theorem provides a sufficient condition for the properness of the transfer function of a boundary control system.

Definition 3.1. *Let $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be a normed linear space with $\mathcal{V} \subset \mathcal{H}$. We say that the solution, $\hat{z}(s)$, to the abstract elliptic problem* (2.13) *is* uniformly bounded with respect to the $\mathcal{V}$ norm *if there exist constants $\mu_1 \in \Re$ and $M \in \Re^+$ such that*

$$\|z(s)\|_{\mathcal{V}} \leq M\|u\|_{\mathcal{U}} \tag{3.1}$$

for all $u \in \mathcal{U}$ and for all $s \in \mathbb{C}$ with Re $s > \mu_1$.

The following sufficient condition for properness of the system transfer function is now immediate.

Theorem 3.2. *Let (L, Γ, K) define a boundary control system. Let $\mathcal{V}$ be a normed linear space satisfying $\mathcal{Z} \subset \mathcal{V} \subset \mathcal{H}$. If the solution to $(L, \Gamma)_e$ is uniformly bounded with respect to the $\mathcal{V}$ norm, then for all observation operators $K \in \mathcal{L}(\mathcal{V}, \mathcal{Y})$, the transfer function associated with the boundary control system (L, Γ, K) is proper.*

Proof. By assumption there exist constants μ_1 and M such that inequality (3.1) holds. Let A be as defined in Theorem 2.9 with growth bound ω_0. Choose $\mu = \max\{\mu_1, \omega_0\}$ and the result follows. □

Thus, continuity of the input/output map of a boundary control system can be established by determining uniform boundedness of the solution $z(s)$ to a family of elliptic problems. Continuity of the input/output map can be established without an explicit representation of the transfer function. Also, Theorem 3.2 states that uniform boundedness of the solution to the elliptic problem $(L, \Gamma)_e$ in the $\mathcal{V}$ norm implies boundedness of the input/output map for the class of boundary control systems $\{(L, \Gamma, K) \mid K \in \mathcal{L}(\mathcal{V}, \mathcal{Y})\}$. This is advantageous since there exist a large literature of results on solutions to elliptic partial differential equations, although not on uniform boundedness of the solution. A major advantage of this approach is that it is not required to compute the linear operators (A, B, C) of a state-space realization.

Example 3.3 (one-dimensional heat equation with Neumann boundary control continued). The solution to the corresponding elliptic problem is

$$z(x, s) = \frac{u\ \cosh(\sqrt{s}\ x)}{\sqrt{s}\ \sinh\sqrt{s}}.$$

Let $\mathcal{V} = H^1(0,1)$, $\mathcal{U} = \Re$, and $\mu_1 = 1$. Then for all $s \in \mathbb{C}$ with Re $s > 1$ we have

$$\|z\|^2_{L^2(0,1)} \leq \frac{|u|^2 \cosh 2}{16 \sinh 2} + \frac{|u|^2}{8 \sinh^2 2},$$
$$\left\|\frac{dz}{dx}\right\|^2_{L^2(0,1)} \leq \frac{|u|^2 \cosh 2}{2 \sinh 2} + \frac{|u|^2}{2 \sinh^2 2.}$$

Hence $\|z\|_{H^1(0,1)} \leq \sqrt{\frac{2 \cosh 2}{\sinh 2}}|u|$. Thus by Theorem 3.2, the input/output map is bounded for all $\mathcal{K} \in \mathcal{L}(\mathcal{H}^1(0,1), \Re)$. In particular, this holds for $Kz = z(x_1, t)$.

We now provide some conditions for uniform boundedness of the solution to $(L,\Gamma)_e$ with respect to $\mathcal{V}$ by rewriting $(L,\Gamma)_e$ as two subproblems.

Proposition 3.4. *Let (L,Γ) define a boundary control system as in* (2.1) *and let $\mathcal{V}$ be a normed linear space satisfying $\mathcal{Z} \subset \mathcal{V} \subset \mathcal{H}$. Let $\mu \in \Re^+$ and $\mu \notin \sigma(L)$ (spectrum of L), and define the problems $(L,\Gamma)_{e1}$ and $(L,\Gamma)_{e2}$ by*

$$(L,\Gamma)_{e1} := \begin{cases} Lf = \mu f, \\ \Gamma f = u. \end{cases} \tag{3.2}$$

$$(L,\Gamma)_{e2} := \begin{cases} Lw = sw + (s-\mu)f, \qquad s \in \mathbb{C}, \\ \Gamma w = 0. \end{cases} \tag{3.3}$$

The solution to $(L,\Gamma)_e$ is uniformly bounded with respect to the $\mathcal{V}$ norm if the following two conditions hold:

1. *There exists $f \in \mathcal{Z}$ such that f solves $(L,\Gamma)_{e1}$ and*

$$\|f\|_{\mathcal{V}} \leq C_1 \|u\|_{\mathcal{U}} \tag{3.4}$$

for some positive constant C_1.

2. *Let $f \in \mathcal{Z}$ denote the solution to $(L,\Gamma)_{e1}$. There exists $w \in \mathcal{Z}$ such that w solves $(L,\Gamma)_{e2}$ and*

$$\|w\|_{\mathcal{V}} \leq C_2 \|f\|_{\mathcal{V}} \tag{3.5}$$

for some positive constant C_2, independent of s.

Proof. The result is immediate by noting that $w + f$ solves the original elliptic problem $(L,\Gamma)_e$. ◻

4. Uniformly elliptic boundary value problems. In the remaining sections, we shall look at boundedness of solutions to uniformly elliptic boundary value problems. We concentrate on linear second order differential operators. Unfortunately, the traditional estimates on solutions to elliptic problems of the form (2.10) are dependent on the argument s. Our focus lies in obtaining estimates that are independent of s. We begin with some background theory and then show that under certain standard assumptions, solutions to uniformly elliptic boundary value problems of order 2 with either Dirichlet, Neumann, or Robin boundary control are uniformly bounded. The results generalize to higher order uniformly elliptic operators [5].

Let Ω be an open set in $\Re^n$. A linear second order differential operator in Ω is defined by

$$L(x,D) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) D_{ij} + \sum_{j=1}^{n} c_j(x) D_j + d(x). \tag{4.1}$$

We assume that the coefficients are sufficiently smooth and that the operator L is uniformly elliptic in Ω. More precisely,

[H1a] (smoothness condition 1) The coefficients $a_{ij}(x)$ are bounded and absolutely continuous in $\bar{\Omega}$, and the remaining coefficients are bounded and measurable in Ω.

[H1b] (uniform ellipticity) Define the principal part of L by

$$L^0(x, D) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) D_{ij} = D' A(x) D,$$

where $A(x)$ is an $n \times n$ positive definite matrix with components $a_{ij}(x)$. We assume that L is *uniformly elliptic in* Ω. That is, there exists a positive constant c_L such that for all $x \in \Omega$, $\xi \in \Re^n$,

$$L^0(x, \xi) \geq c_L |\xi|^2.$$

Our analysis is based on the boundary control system formulation. We shall no longer refer to the state-space realization. The boundary operator Γ is defined by

$$\Gamma(x, D) = b_0(x) + \sum_{i=1}^{n} b_{1i}(x) D_i = b_0(x) + B_1'(x) D, \tag{4.2}$$

where $B_1'(x) = \big(b_{11}(x), \ldots, b_{1n}(x)\big)$ and $D' = (D_1, \ldots, D_n)$. So $B_1'(x) = 0$ for Dirichlet boundary control and $b_0(x) = 0$ for Neumann boundary control. We impose the following condition on the operator Γ:

[H2] (smoothness condition 2) The coefficients of Γ are real. Also, $b_0(x) \in C^2(\partial\Omega)$ and $b_{1i}(x) \in C^1(\partial\Omega)$ for $i = 1, \ldots, n$.

Estimates of the solution to a uniformly elliptic boundary value problem depend on regularity of the region Ω.

DEFINITION 4.1 (see [4]). *Let Ω be an open set in $\Re^n$ with boundary $\partial\Omega$. Then Ω is said to be* uniformly regular of class C^m *if there exists a family of open sets $\{O_i\}$ of $\Re^n$ and of homeomorphisms $\{\Phi_i\}$ of O_i onto the unit ball $\{y : \|y\| < 1\}$ in $\Re^n$ and an integer N such that the following conditions are satisfied:*

[UR1] *For each i,*

$$\Phi_i (O_i \cap \Omega) = \{y \ : \ \|y\| < 1, y_1 > 0\},$$
$$\Phi_i (O_i \cap \partial\Omega) = \{y \ : \ \|y\| < 1, y_1 = 0\}.$$

[UR2] *Let $O_i' = \Phi_i^{-1} (\{y \in \Re^n : \|y\| < 1/2\})$. Then $\bigcup_{i=1}^{\infty} O_i'$ contains the $1/N$ neighborhood of $\partial\Omega$.*

[UR3] *Any $(N+1)$ distinct sets of $\{O_i\}$ have an empty intersection.*

[UR4] *Let $\Psi_i = \Phi_i^{-1}$. Then Ψ_i, Φ_i are mappings of class C^m. Let Φ_{ik}, Ψ_{ik} be the kth components of Φ_i, Ψ_i, respectively. Then*

$$|D^\alpha \Phi_{ik}(x)| \leq M, \quad |D^\alpha \Psi_{ik}(y)| \leq M, \quad |\Phi_{i1}(x)| \leq M \mathrm{dist}(x, \partial\Omega)$$

for $|\alpha| \leq m$, $x \in O_i$, $\|y\| < 1, k = 1, \ldots, n$, and $i = 1, 2, \ldots$.

In general, it is nontrivial to show that a region is uniformly regular of class C^m. For our work, we are concerned only with bounded sets Ω in $\Re^n$ and cylinders of the form $\Omega \times \Re$ in $\Re^{n+1}$. It was stated without details in [22, p. 237] that for bounded sets with sufficiently smooth boundary, there exist mappings $\{\Phi_i\}$ such that [UR2] holds. We give a more complete discussion of this point. If Ω is bounded, then there is a finite open cover for the boundary. If the boundary is sufficiently smooth, then it is possible to choose a covering such that [UR1] and [UR2] hold. Conditions [UR3] and [UR4] then hold trivially since the covering is finite. Thus we have the following result.

THEOREM 4.2. *If Ω is bounded with sufficiently smooth boundary, then $\Omega \times \Re$ is also uniformly regular.*

In addition to [H1a], [H1b], and [H2], we assume, unless stated otherwise, that Ω, L, and Γ also satisfy the following:

[H3] Ω is bounded and uniformly regular of class C^2.

[H4] (root condition) Let $L^0(x, D)$ denote the principal part of $L(x, D)$. For every pair of linearly independent real vectors ξ and η, the polynomial $L^0(x, \xi + \tau\eta)$ in τ has an equal number of roots with positive and negative imaginary parts.

[H5] (complementing condition) Let $B^0(x,D)$ denote the principal part of $\Gamma(x,D)$. Let x be an arbitrary point on $\partial\Omega$ and n be the outward normal unit vector to $\partial\Omega$ at x. For each tangential vector $\xi \neq 0$ to $\partial\Omega$ at x, let $\hat{\tau}$ be the root of the polynomial $L^0(x, \xi + \tau n)$ with positive imaginary part. Then $\hat{\tau}$ is not a root of $B^0(x, \xi + \tau n)$.

If $n \geq 3$, then the root condition is satisfied for all uniformly elliptic operators [21, p. 130]. If the coefficients of L are real, then the root condition is also satisfied when $n = 2$.

5. Uniformly elliptic operators with Dirichlet boundary control. It is well known that the one-dimensional heat equation on a unit interval with Dirichlet boundary control and point observations is not well-posed with respect to the usual choice of state space $L^2(0, 1)$ [6]. Thus, showing well-posedness of more general Dirichlet control problems with state-space methods is hampered by the difficulty of first obtaining an appropriate state space.

In this section we will show that a class of control problems with Dirichlet boundary control do have a bounded input/output map by showing that the associated elliptic problem is uniformly bounded and hence the transfer function is proper.

Let $\Omega \subset \Re^n$, $n = 1, 2, 3$, let L be a second order differential operator as defined in (4.1) with $d(x) \leq 0$, and define the boundary operator to be

$$\Gamma(x, D) = b_0(x), \quad b_0(x) \neq 0 \qquad \text{for all } x.$$

We shall show that if Ω, L, Γ satisfy hypotheses [H1]–[H5] and Ω satisfies an additional assumption, then the solution to the abstract elliptic problem

$$\left.\begin{array}{rclr} Lz & = & sz & \text{in } \Omega, \\ \Gamma z & = & u & \text{on } \partial\Omega \end{array}\right\} \tag{5.1}$$

is uniformly bounded with respect to the $\sup_{x\in\Omega} |\cdot|$ norm. This will imply boundedness of the input/output map for the corresponding boundary control system. The following definition is due to Browder [4].

DEFINITION 5.1. *Let Ω be an open set in $\Re^n$. If for any $a \in \partial\Omega$ the part of Ω, $\partial\Omega$ in some neighborhood of a is expressed as*

$$x_i > \psi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \qquad x_i = \psi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

respectively, for some $i = 1, \ldots, n$ and a C^{2m} function ψ, then Ω is called locally regular of class C^{2m}.

In addition to uniformly regularity of class C^2, we further assume the following:

[H6] Ω is locally regular of class C^4.

We will use Proposition 3.4 to show that the solution to the elliptic Dirichlet problem is uniformly bounded in the $C(\Omega)$-norm. The following result will be used to show that the solution to the subproblem $(L,\Gamma)_{e2}$ satisfies the second condition in Proposition 3.4.

THEOREM 5.2 (see [21, p. 216]). *Let $F \in C(\bar{\Omega})$ and consider*

$$\begin{aligned} Lw &= sw + F \qquad && \text{in } \Omega, \\ \Gamma w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The solution w exists, and $w \in C(\Omega)$. Furthermore, we have

$$\sup_{x\in\Omega} |w(x)| \le \frac{C}{|s|} \sup_{x\in\Omega} |F(x)|. \tag{5.2}$$

To prove boundedness of the input/output map we also require the maximum principle and existence of a solution to $Lf = 0$ with a Dirichlet boundary condition. This will be used to show that the first condition in Proposition 3.4 holds.

The following theorem is an immediate consequence of Theorems 8.6 and 8.12 in [13]. (The assumptions imposed on L and Ω in [13] are weaker than [H1]–[H6].)

THEOREM 5.3. *Let L and Ω satisfy assumptions* [H1]–[H6], *$\mu \in \Re^+$, $\mu \notin \sigma(L)$ be fixed, and $u \in H^2(\Omega)$; then there exists a unique $f \in H^2(\Omega)$ that solves*

$$\begin{aligned} Lf &= \mu f \qquad && \text{in } \Omega, \\ f &= u && \text{on } \partial\Omega. \end{aligned} \tag{5.3}$$

Proof. For $u \in H^1(\Omega)$, Theorem 8.6 in [13] guarantees that (5.3) has a unique (weak) solution $f \in H^1(\Omega)$. Since [H1]–[H3] hold and $u \in H^2(\Omega)$, by Theorem 8.12 in [13] the solution is in $H^2(\Omega)$. □

The norm $[\cdot]_{q-1/2,\partial\Omega}$ is defined by

$$[u]_{q-1/2,\partial\Omega} = \inf\{\|z\|_{H^q(\Omega)}; z \in H^q(\Omega), z = u \text{ on } \partial\Omega\}. \tag{5.4}$$

The space $H^{q-\frac{1}{2}}(\partial\Omega)$ is the space of functions defined on $\partial\Omega$ such that this norm is finite. For $u \in H^{q-\frac{1}{2}}(\partial\Omega)$, u may be extended to $\tilde{u} \in H^q(\Omega)$ such that $\tilde{u}|_{\partial\Omega} = u$ and $\|\tilde{u}\|_{H^q(\Omega)} = [u]_{q-1/2,\partial\Omega}$.

COROLLARY 5.4. *Let L and Ω satisfy assumptions* [H1]–[H6]. *For any $\mu \in \Re^+$, $\mu \notin \sigma(L)$, and $u \in H^{\frac{3}{2}}(\partial\Omega)$, there exists a unique $f \in H^2(\Omega)$ that solves*

$$\begin{aligned} Lf &= \mu f \qquad && \text{in } \Omega, \\ b_0(x)f &= u && \text{on } \partial\Omega. \end{aligned} \tag{5.5}$$

Proof. Since $b_0(x) \in C^2(\partial\Omega)$ and $b_0(x) \neq 0$ for all $x \in \partial\Omega$, we have $\tilde{u} = \frac{u}{b_0} \in H^{\frac{3}{2}}(\partial\Omega)$. Thus it can be extended to an element in $H^2(\Omega)$ which we shall denote by the same symbol. By Theorem 5.3 there exists a unique $f \in H^2(\Omega)$ that solves

$$\begin{aligned} Lf &= \mu f \qquad && \text{in } \Omega, \\ f &= \tilde{u} && \text{on } \partial\Omega. \qquad \square \end{aligned}$$

The following maximum principle is required. The stated assumptions are stronger than those given in [13].

THEOREM 5.5 (see, e.g., [13, Thm. 8.1]). *Let $f \in H^2(\Omega)$ satisfy $Lf - \mu f = 0$ in Ω. Then*

$$\sup_{x\in\Omega} f(x) \le \sup_{x\in\partial\Omega} \max\{f(x), 0\}.$$

We can now state our main theorem for this section. It implies in particular that Dirichlet boundary control with point observation is a well-posed control system.

THEOREM 5.6. *Consider the pair L, Γ with Dirichlet control $\Gamma = b_0(x)$. Assume that assumptions* [H1]–[H6] *are satisfied on the region Ω. The operators L, Γ define a boundary control system with $\mathcal{U} = H^{\frac{3}{2}}(\partial\Omega)$, $\mathcal{Z} = H^2(\Omega)$, and $\mathcal{H} = L^2(\Omega)$. The input/output map of the boundary control system* (5.1) *is bounded for all observation operators $K \in \mathcal{L}(C(\Omega), \mathcal{Y})$.*

Proof. Let $\mu \in \Re^+$ and $\mu \notin \sigma(L)$, and write (L, Γ) as $(L, \Gamma)_{e1}$ and $(L, \Gamma)_{e2}$ as in Proposition 3.4. We will use $\mathcal{V} = C(\Omega)$. Since $\Omega \subset R^n$, $n \le 3$, Sobolev's imbedding theorem, e.g., [21, Thm. 3.20], implies that $\mathcal{V} \subset \mathcal{Z}$. Define $C_1 = \sup_{x\in\partial\Omega} \frac{1}{|b_0(x)|}$. Using Theorem 5.5, we have

$$\begin{aligned}\sup_{x\in\Omega} |f(x)| &\le \sup_{x\in\partial\Omega} |f(x)| \\ &\le C_1 \sup_{x\in\partial\Omega} |u(x)| \\ &\le C_1 \|u\|_{H^{\frac{3}{2}}}.\end{aligned}$$

The latter inequality also follows from Sobolev's imbedding theorem. Thus, the solution to the subproblem $(L, \Gamma)_{e1}$ satisfies inequality (3.4). Inequality (3.5) then follows from inequality (5.2). Therefore, by Theorem 3.2 the system transfer function associated with (L, Γ, K) is proper for all observation operators $K \in \mathcal{L}(C(\Omega), \mathcal{Y})$. That is, the input/output map of the boundary control system (L, Γ, K) is bounded. □

6. Uniformly elliptic operators with Neumann or Robin boundary control. In this section we will show that a class of control problems with Neumann/Robin boundary control have a bounded input/output map. In the interests of clarity and brevity we will give only the proofs for second order elliptic operators. The generalization to higher order operators is straightforward. Details are in [5].

In special cases results have been obtained to these problems by transforming the boundary control system to state-space form and then using the analyticity of the underlying semigroup to show well-posedness of the input/output map. The transformation to state-space form is not necessary. As for Dirichlet problems, well-posedness for general Neumann problems is shown by direct analysis of the boundary control formulation.

Let L and Γ be defined as in (4.1) and (4.2). In this section we will assume $B_1'(x) \ne 0$. Hence Γ represents a Neumann boundary control when $b_0(x) = 0$ and a Robin boundary control otherwise. We shall show that if Ω, L, and Γ satisfy hypotheses [H1]–[H5], then the solution to the abstract elliptic problem is uniformly bounded with respect to the $H^1(\Omega)$ norm. This implies boundedness of the input/output map for the corresponding boundary control system.

It is not enough to use regularity of the solution to elliptic problems. We must show that the solution is uniformly bounded in the parameter s. We first state two theorems concerning estimates of solutions to elliptic problems. These theorems are key to showing uniform boundedness of solutions to Neumann/Robin boundary control problems.

THEOREM 6.1 (see [21, Thm. 4.10]). *Let Ω be uniformly regular of class C^2 and $L(x,D)$, $B(x,D)$ be defined as in* (4.1) *and* (4.2). Assume that $L(x,D)$ and $\Gamma(x,D)$ satisfy assumptions [H2]–[H5]. Then there exists a positive constant m_1 such that for all $z \in H^2(\Omega)$ the following inequality holds:

$$\|z\|_{H^2(\Omega)} \le m_1 \left[\|Lz\|_{L^2(\Omega)} + [\Gamma z]_{1/2,\partial\Omega} + \|z\|_{L^2(\Omega)} \right]. \tag{6.1}$$

THEOREM 6.2 (see [21, Lem. 5.7]). *Let L, Γ and Ω be as defined in* (4.1) *and* (4.2)*, and assume that they satisfy assumptions* [H1]–[H5]. *Let $\theta \in [-\pi, \pi)$ be fixed but arbitrary and t be a new real variable. Set*

$$Q = \Omega \times \Re,$$
$$\mathcal{L}_\theta(x,D) = \mathcal{L}_\theta(x,D_x,D_t) = L(x,D_x) + \exp(i\theta)D_t^2,$$

and define $\mathcal{B}(x,D_x)$ to be the extension of $\Gamma(x,D_x)$ to $\partial Q = \partial\Omega \times \Re$. If $\mathcal{L}_\theta, \mathcal{B}, Q$ also satisfy [H1]–[H5]*, then there exists a constant M_θ such that for any $z \in H^2(\Omega)$, $u \in H^{2-m_j}(\Omega)$*[1] *satisfying $\Gamma z = u$ on $\partial\Omega$ and any s satisfying* $\arg s = \theta$*, $|s| > M_\theta$, the following inequality holds:*

$$|s|^{1/2}\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \le M_\theta \Big[\|(L-s)z\|_{L^2(\Omega)} + |s|^{1-m_j/2}\,\|u\|_{L^2(\Omega)} + \|u\|_{H^{2-m_j}(\Omega)} \Big]. \tag{6.2}$$

The outline of the proof is as follows: For any $\theta \in [-\pi, \pi)$, define $Q, \mathcal{L}_\theta$, and $\mathcal{B}$ by

$$\left.\begin{aligned} Q \quad &:= \quad \Omega \times \Re, \\ \mathcal{L}_\theta(x,D) = \mathcal{L}_\theta(x,D_x,D_t) \quad &:= \quad L(x,D_x) + \exp(i\theta)D_t^2, \text{ and} \\ \mathcal{B}(x,D_x) \quad &:= \quad \text{the extension of } \Gamma(x,D_x) \text{ to } \partial Q = \partial\Omega \times \Re. \end{aligned}\right\} \tag{6.3}$$

From Theorem 6.2 we know that if $\{L, \Gamma, \Omega\}$ and $\{\mathcal{L}_\theta, \mathcal{B}, Q\}$ both satisfy [H1]–[H5], then there exists a constant M_θ such that the following a priori estimate holds for any $z \in H^2(\Omega)$, $u \in H^1(\Omega)$ satisfying $\Gamma z = u$ on $\partial\Omega$ and any s satisfying $\arg s = \theta$, $|s| > M_\theta$, $\theta \in [-\pi, \pi)$:

$$|s|^{1/2}\,\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \le M_\theta \left[\|(L-s)z\|_{L^2(\Omega)} + |s|^{1/2}\,\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right].$$

If z solves $Lz = sz$, then

$$\|z\|_{H^1(\Omega)} \le M_\theta \left(\|u\|_{L^2(\Omega)} + \frac{1}{|s|^{1/2}}\,\|u\|_{H^1(\Omega)} \right).$$

If in addition $|s| > 1$, then

$$\|z\|_{H^1(\Omega)} \le 2M_\theta\,\|u\|_{H^1(\Omega)}.$$

We will show that for $\theta \in [-\pi/2, \pi/2]$, M_θ can be chosen independently of θ. This will imply that the solution to the elliptic problem is uniformly bounded with respect to the H^1-norm and thus the input/output map is bounded for any observation operator $K \in \mathcal{L}(\mathcal{H}^1(\Omega), \mathcal{Y})$.

---

[1] m_j=0 if Γ is the Dirichlet boundary condition, and $m_j = 1$ if Γ is a Neumann or Robin boundary condition.

First we show that Q is uniformly regular of class C^2 and for each $\theta \in [-\pi/2, \pi/2]$, $\mathcal{L}_\theta, \mathcal{B}, Q$ satisfy assumptions [H1], [H2], [H4], and [H5]. This ensures the existence of M_θ.

LEMMA 6.3. *Let $L(x, D_x), \Gamma(x, D_x)$, and Ω satisfy assumptions* [H1]–[H5]. *For any $\theta \in [-\pi/2, \pi/2]$, define $\mathcal{L}_\theta, \mathcal{B}$, and Q be as in* (6.3). *Then Q is uniformly regular of class C^2 and $\{\mathcal{L}_\theta, \mathcal{B}\}$ satisfy assumptions* [H1], [H2], [H4], *and* [H5] *in Q.*

Proof. Since Ω satisfies [H3], Q is uniformly regular. Next we show that $\mathcal{L}_\theta$ is uniformly elliptic. That is, there exists a positive constant c_1 such that for all $(\xi, \eta) \in \Re^n \times \Re$ and $x \in \Omega$ the following inequality holds:

$$|\mathcal{L}_\theta^0(x, \xi, \eta)| \geq c_1 \left(|\xi|^2 + \eta^2\right).$$

By assumption, there exists a positive constant c_L such that for all $x \in \Omega$, $\xi \in \Re^n$

$$|L^0(x, \xi)| \geq c_L |\xi|^2.$$

Since the matrix A associated with L^0 is positive definite, this means $L^0(x, \xi) \geq 0$ for all $x \in \Omega$ and $\xi \in \Re^n$. Let $c = \min\{c_L^2, 1\}$. Then for any $(x, t) \in \Omega \times \Re$, $(\xi, \eta) \in \Re^n \times \Re$, and $\theta \in [-\pi/2, \pi/2]$, we have

$$\begin{aligned} \left|\mathcal{L}_\theta^0\big((x,t),(\xi,\eta)\big)\right|^2 &= |L^0(x,\xi) + \exp(i\theta)\eta^2|^2 \\ &= |L^0(x,\xi)|^2 + 2\cos(\theta) L^0(x,\xi)\eta^2 + \eta^4 \\ &\geq c_L^2 |\xi|^4 + \eta^4 \\ &\geq c\left(|\xi|^4 + \eta^4\right) \\ &\geq \frac{c}{2}\left(|\xi|^4 + 2|\xi|^2\eta^2 + \eta^4\right) \\ &= \frac{c}{2}\left(|\xi|^2 + \eta^2\right)^2. \end{aligned}$$

This implies the inequality

$$|\mathcal{L}_\theta^0(x, \xi, \eta)| \geq \sqrt{\frac{c}{2}} \left(|\xi|^2 + \eta^2\right),$$

which proves that $\mathcal{L}$ is uniformly elliptic in Q. Clearly [H2] holds. Also since $n \geq 2$, $n + 1 \geq 3$, the root condition holds. It remains to show that [H5] is satisfied. Let (x, t) be an arbitrary point on ∂Q, n_1 be the unit outward normal vector to $\partial\Omega$ at x, and ξ_1 be any nonzero tangential vector to $\partial\Omega$ at x. The outward normal unit vector to ∂Q at (x, t) is then $n = (n_1', 0)$ and any nonzero tangential vector has the form $\xi = (\xi_1', 0)$. Let $\hat{\tau}$ be a root of $\bar{B}^0(x, \xi + \tau n)$. Then $\hat{\tau}$ is a root of $B^0(x, \xi_1 + \tau n_1)$, which by assumption is not a root of $L^0(x, \xi_1 + \tau n_1)$. This implies that

$$\mathcal{L}(x, \xi + \hat{\tau} n) = L(x, \xi_1 + \hat{\tau} n_1) + \exp(i\theta)(\xi_2 + \hat{\tau} n_2)^2 = L(x, \xi_1 + \hat{\tau} n_1) \neq 0.$$

Hence $\hat{\tau}$ is not a root of $\mathcal{L}(x, \xi + \hat{\tau} n)$. So $\{\mathcal{L}, \mathcal{B}\}$ satisfies [H5]. □

For each $\theta \in [-\pi/2, \pi/2]$, $\mathcal{L}, \mathcal{B}, Q$ satisfy [H1], [H2], [H4], and [H5]; thus the hypotheses of Theorem 6.2 have been justified. It remains to show that M_θ may be chosen independent of θ in this range. The following lemma is needed to prove this claim.

LEMMA 6.4. *Let $\mathcal{L}_\theta(x, D)$ be defined as in* (4.1). *Then $\mathcal{L}_\theta$ is continuous in θ. That is, for any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|\theta_1 - \theta_2| < \delta$, $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$ we have*

$$\|\mathcal{L}_{\theta_1} v - \mathcal{L}_{\theta_2} v\|_{L^2(Q)} < \epsilon \|v\|_{H^2(Q)} \qquad \text{for all } v \in H^2(Q).$$

Proof. For any $0 < \epsilon < \sqrt{2}$, choose $\delta = \arccos(1 - \frac{\epsilon^2}{2})$, where arccos denotes the principal branch; then if $|\theta_1 - \theta_2| < \delta$ and $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$), we have

$$\begin{aligned}\|\mathcal{L}_{\theta_1} v - \mathcal{L}_{\theta_2} v\|_{L^2(Q)} &\le |\exp(i\theta_1) - \exp(i\theta_2)| \|v\|_{H^2(Q)} \\ &= \sqrt{(2 - 2\cos(\theta_1 - \theta_2))} \|v\|_{H^2(Q)} \\ &= \sqrt{(2 - 2\cos(|\theta_1 - \theta_2|))} \|v\|_{H^2(Q)}.\end{aligned}$$

Since $\epsilon < \sqrt{2}$, $\delta < \pi/2$; hence the function $f(x) = 2 - 2\cos(x)$ is nonnegative and monotone increasing on the interval $[0, \delta]$. Thus

$$\begin{aligned}\|\mathcal{L}_{\theta_1} v - \mathcal{L}_{\theta_2} v\|_{L^2(Q)} &< \sqrt{(2 - 2\cos(\delta))} \|v\|_{H^2(Q)} \\ &= \epsilon \|v\|_{H^2(Q)}.\end{aligned}$$

For any $\epsilon \ge \sqrt{2}$, choose $\delta = \pi/2$; then if $|\theta_1 - \theta_2| < \pi/2$ and $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$) we have

$$\begin{aligned}\|\mathcal{L}_{\theta_1} v - \mathcal{L}_{\theta_2} v\|_{L^2(Q)} &\le \sqrt{(2 - 2\cos(|\theta_1 - \theta_2|))} \|v\|_{H^2(Q)} \\ &< \sqrt{2} \|v\|_{H^2(Q)} \\ &< \epsilon \|v\|_{H^2(Q)}. \qquad \square\end{aligned}$$

Due to Theorem 6.1, for each $\theta \in [-\pi/2, \pi/2]$, there exists a constant m_θ such that for any $v \in H^2(Q)$,

$$\|v\|_{H^2(Q)} \le m_\theta \left(\|\mathcal{L}_\theta v\|_{L^2(Q)} + [\mathcal{B}v]_{0,\partial Q} + \|v\|_{L^2(Q)} \right). \tag{6.4}$$

For each θ, define $m(\theta) = \inf\{m_\theta : \text{inequality (6.4) holds}\}$. The infimum exists since clearly 1 is a lower bound for m_θ. The next theorem proves that $m(\theta)$ is bounded above.

THEOREM 6.5. *Let $m(\theta)$ be as defined above. Then $\{m(\theta); -\pi/2 \le \theta \le \pi/2\}$ is bounded above. Hence there exists a positive constant $\bar{m}$ such that the following inequality holds for all $\theta \in [-\pi/2, \pi/2]$:*

$$\|v\|_{H^2(Q)} \le \bar{m} \left(\|\mathcal{L}_\theta v\|_{L^2(Q)} + [\mathcal{B}v]_{1/2,\partial Q} + \|v\|_{L^2(Q)} \right). \tag{6.5}$$

Proof. Suppose not. Then for each n, there exists $\theta_n \in [-\pi/2, \pi/2]$ such that $m(\theta_n) > n$. The sequence $\{\theta_n\}$ is bounded; thus it contains a convergent subsequence $\{\theta_{k_n}\}$ which converges to $\bar{\theta} \in [-\pi/2, \pi/2]$. Theorem 6.1 ensures that $m(\bar{\theta})$ is positive and finite; thus there exists some n such that $m(\bar{\theta}) < n$. Let $\epsilon = \frac{1}{m(\bar{\theta})} - \frac{1}{n} > 0$. By Lemma 6.4, there exists $N > n$ such that for all $k_n > N$ (k_n are the indices of the convergent subsequence),

$$\|\mathcal{L}_{\bar{\theta}} v - \mathcal{L}_{\theta_{k_n}} v\|_{L^2(Q)} < \epsilon \|v\|_{H^2(Q)} \qquad \text{for all } v \in H^2(Q).$$

Pick a k_n such that $m(\theta_{k_n}) - 1 > n$. By definition, $m(\theta_{k_n})$ is the smallest constant such that for all $v \in H^2(Q)$, inequality (6.4) holds. Thus there exists some $v_0 \in H^2(Q)$ such that

$$\|v_0\|_{H^2(Q)} > (m_{\theta_{k_n}} - 1) \left(\|\mathcal{L}_{\theta_{k_n}} v_0\|_{L^2(Q)} + [\mathcal{B}v_0]_{1/2,\partial Q} + \|v_0\|_{L^2(Q)} \right).$$

But then

$$\begin{aligned}\epsilon\|v_0\|_{H^2(Q)} &= \left(\frac{1}{m(\bar\theta)} - \frac{1}{n}\right)\|v_0\|_{H^2(Q)}\\ &< \left(\frac{1}{m(\bar\theta)} - \frac{1}{m(\theta_{k_n})-1}\right)\|v_0\|_{H^2(Q)}\\ &< \left(\|\mathcal{L}_{\bar\theta}v_0\|_{L^2(Q)} + [\mathcal{B}v_0]_{1/2,\partial Q} + \|v_0\|_{L^2(Q)}\right)\\ &\quad - \left(\|\mathcal{L}_{\theta_{k_n}}v_0\|_{L^2(Q)} + [\mathcal{B}v_0]_{1/2,\partial Q} + \|v_0\|_{L^2(Q)}\right)\\ &\le \|\mathcal{L}_{\bar\theta}v_0 - \mathcal{L}_{\theta_{k_n}}v_0\|_{L^2(Q)}\\ &< \epsilon\|v_0\|_{H^2(Q)},\end{aligned}$$

a contradiction. Thus $m(\theta)$ is bounded above. Let $\bar m = \sup\{m(\theta), -\pi/2 \le \theta \le \pi/2\}$. Then for any $\theta \in [-\pi/2, \pi/2]$ and $v \in H^2(Q)$, inequality (6.5) holds. □

We now state a modification of Theorem 6.2.

THEOREM 6.6. *Let Ω, L, Γ, (4.1), (4.2) define a boundary control system with $\mathcal{H} = L^2(\Omega)$ and $\mathcal{U} = H^{\frac{1}{2}}(\partial\Omega)$. Assume that* [H1]–[H5] *are satisfied. Then there exists a positive constant R such that for any $z \in H^2(\Omega)$, $u \in \mathcal{U}$ satisfying $\Gamma z = u$ on $\partial\Omega$ and any complex number s on the open right half-plane $\mathbb{C}_{R^2} := \{s : \mathrm{Re}\ s > R^2\}$, the following inequality holds:*

$$|s|^{1/2}\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \le m\left[\|(L-s)z\|_{L^2(\Omega)} + |s|^{1/2}\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\right], \tag{6.6}$$

where m is a positive constant dependent only on L and Ω.

Proof. The proof is along the lines given in [21] except that we show that the constant is independent of θ. Let ζ be a function in $C^\infty(-\infty,\infty)$ such that $\zeta(t) = 0$ for $|t| > 1$, $\zeta(t) = 1$ for $|t| < 1/2$. Let m_1 be a constant chosen such that $\|\zeta\|_{H^2(\Re)} \le m_1$. Let $\bar m = \max\{m(\theta), -\pi/2 \le \theta \le \pi/2\}$ and $m_2 = \max\{\bar m, m_1\}$. Define

$$R := \text{largest root of the quadratic } r^2 - 6m_2^2 r - 6m_2^2.$$

We note that R is necessarily positive and real. In fact $R = \frac{6m_2^2 + m_2\sqrt{36m_2^2+24}}{2}$. Moreover, since $m(\theta)$ is bounded below by 1, $\bar m$ and hence m_2 is always greater than 1. Thus $R > 6$. For any $z \in H^2(\Omega)$ and any $s \in \mathbb{C}_{R^2}$, set $\theta = \arg s$, $r = |s|^{1/2}$, and $v(x,t) = \zeta(t)\exp(irt)z(x)$. Clearly $v \in H^2(Q)$; hence (6.5) implies

$$\begin{aligned}\|v\|_{H^2(Q)} &\le \bar m\left(\|\mathcal{L}_\theta v\|_{L^2(Q)} + [\mathcal{B}v]_{1/2,\partial Q} + \|v\|_{L^2(Q)}\right)\\ &\le m_2\left(\|\mathcal{L}_\theta v\|_{L^2(Q)} + [\mathcal{B}v]_{1/2,\partial Q} + \|v\|_{L^2(Q)}\right).\end{aligned} \tag{6.7}$$

Now a lower bound for $\|v\|_{H^2(Q)}$, an upper bound for $[\mathcal{B}v]_{1/2,\partial Q}$, and an upper bound for $\|\mathcal{L}_\theta v\|_{L^2(Q)}$ need to be computed. The final inequality is then obtained via simple algebra. First we compute a lower bound for $\|v\|_{H^2(Q)}$. By definition of $\|\cdot\|_{H^2(Q)}$ we have

$$\begin{aligned}\|v\|^2_{H^2(Q)} &= \sum_{|\alpha|+k\le 2}\int_{-\infty}^{\infty}\int_\Omega \left|D_x^\alpha D_t^k v(x,t)\right|^2 dxdt\\ &\ge \sum_{|\alpha|+k\le 2}\int_{-\frac12}^{\frac12}\int_\Omega \left|D_x^\alpha D_t^k \exp(irt)z(x)\right|^2 dxdt\end{aligned}$$

$$= \sum_{k=0}^{2} (r)^{2k} \sum_{|\alpha|+k\le 2} \int_\Omega \left| D_x^\alpha z(x) \right|^2 \, dx$$
$$= \sum_{k=0}^{2} (r)^{2k} \|z\|^2_{H^{2-k}(\Omega)}$$
$$\ge (r)^{2k} \|z\|^2_{H^{2-k}(\Omega)}$$

for any $k = 0, 1, 2$. Hence

$$\|v\|_{H^2(Q)} \ge (r)^k \|z\|_{H^{2-k}(\Omega)}$$

for any $k = 0, 1, 2$. Thus

$$3\|v\|_{H^2(Q)} \ge \sum_{k=0}^{2} (r)^k \|z\|_{H^{2-k}(\Omega)} . \tag{6.8}$$

Next we compute an upper bound for $[\mathcal{B}v]_{1/2,\partial Q}$. By definition of $[\cdot]_{1/2,\partial\Omega}$ we have for $\Gamma z \in H^2(\Omega)$ such that $z = u$ on $\partial\Omega$, and

$$\begin{aligned}
[\mathcal{B}v]^2_{1/2,\partial Q} &= [\zeta(t)\exp(irt)Bz(x)]^2_{1/2,\partial Q} \\
&= [\zeta(t)\exp(irt)u]^2_{1/2,\partial Q} \\
&\le \|\zeta(t)\exp(irt)u\|^2_{H^1(Q)} \\
&= \sum_{|\alpha|+k\le 1} \int_{-\infty}^{\infty} \int_\Omega \left| D_x^\alpha D_t^k \zeta(t)\exp(irt)u \right|^2 \, dxdt \\
&= \int_{-\infty}^{\infty} \int_\Omega |\zeta(t)\exp(irt)u|^2 \, dxdt + \int_{-\infty}^{\infty} \int_\Omega |\zeta(t)\exp(irt)Du|^2 \, dxdt \\
&\quad + \int_{-\infty}^{\infty} \int_\Omega |\zeta'(t)\exp(irt)u + ir\zeta(t)\exp(irt)u|^2 \, dxdt \\
&\le m_1^2 \|u\|^2_{L^2(\Omega)} + m_1^2 \|Du\|^2_{L^2(\Omega)} + m_1^2 \|u\|^2_{L^2(\Omega)} + 2rm_1^2 \|u\|^2_{L^2(\Omega)} \\
&\quad + r^2 m_1^2 \|u\|^2_{L^2(\Omega)} \\
&= 2m_1^2 \|u\|^2_{L^2(\Omega)} + m_1^2 \|Du\|^2_{L^2(\Omega)} + (2r + r^2) m_1^2 \|u\|^2_{L^2(\Omega)} .
\end{aligned}$$

Since $r = |s|^{1/2} > R > 6$, $2r < r^2$. Hence

$$\begin{aligned}
[\mathcal{B}v]^2_{1/2,\partial Q} &\le 2m_1^2 \left(\|u\|^2_{H^1(\Omega)} + r^2 \|u\|^2_{L^2(\Omega)} \right) \\
&\le 2m_1^2 \left(r\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right)^2 \\
&\le 2m_2^2 \left(r\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right)^2 .
\end{aligned}$$

Thus

$$[\mathcal{B}v]_{1/2,\partial Q} \le \sqrt{2} m_2 \left(r\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right) . \tag{6.9}$$

This is the upper bound on $[\mathcal{B}v]_{1/2,\partial Q}$. Now we calculate an upper bound on $\mathcal{L}_\theta v$. Substituting the expression for $v(x,t)$ into $\mathcal{L}_\theta v$, we find

$$\mathcal{L}_\theta v = \zeta(t)\exp(irt)(L - r^2\exp(i\theta))z + 2ir\exp(i\theta)\zeta'(t)\exp(irt)z + \exp(i\theta)\zeta''(t)\exp(irt)z.$$

Therefore

$$\begin{aligned}\|\mathcal{L}_\theta v\|_{L^2(Q)} &\le \left\|\zeta(t)\exp(irt)(L-r^2\exp(i\theta))z\right\|_{L^2(Q)} + 2\left\|r\exp(i\theta)\zeta'(t)\exp(irt)z\right\|_{L^2(Q)}\\ &\quad + \left\|\exp(i\theta)\zeta''(t)\exp(irt)z\right\|_{L^2(Q)}\\ &\le m_1\left(\left\|(L-r^2\exp(i\theta))z\right\|_{L^2(\Omega)} + 2r\|z\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)}\right)\\ &\le m_2\left(\left\|(L-r^2\exp(i\theta))z\right\|_{L^2(\Omega)} + 2r\|z\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)}\right). \qquad (6.10)\end{aligned}$$

Also,

$$\|v\|_{L^2(Q)} \le m_2\|z\|_{L^2(\Omega)}. \qquad (6.11)$$

Substituting inequality (6.8) into (6.7), we obtain

$$r^2\|z\|_{L^2(\Omega)} + r\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \le 3m_2\left(\|\mathcal{L}_\theta v\|_{L^2(Q)} + [\mathcal{B}v]_{1/2,\partial Q} + \|v\|_{L^2(Q)}\right). \qquad (6.12)$$

Next, substitute inequalities (6.9), (6.10), and (6.11) into inequality (6.12) to obtain

$$\begin{aligned}&r^2\|z\|_{L^2(\Omega)} + r\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)}\\ &\qquad\le 3m_2^2\left(\left\|(L-r^2\exp(i\theta))z\right\|_{L^2(\Omega)} + 2r\|z\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)}\right.\\ &\qquad\qquad\left. + \sqrt{2}r\|u\|_{L^2(\Omega)} + \sqrt{2}\|u\|_{H^1(\Omega)} + \|z\|_{L^2(\Omega)}\right). \qquad (6.13)\end{aligned}$$

After rearrangement we obtain

$$\begin{aligned}&(r^2 - 6m_2^2 r - 6m_2^2)\|z\|_{L^2(\Omega)} + r\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)}\\ &\qquad\le 3\sqrt{2}m_2^2\left(\left\|(L-r^2\exp(i\theta))z\right\|_{L^2(\Omega)} + r\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\right). \qquad (6.14)\end{aligned}$$

By definition of R we have $r^2 - 6m_2^2 r - 6m_2^2 \ge 0$. Hence (6.14) implies

$$r\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \le 3\sqrt{2}m_2^2\left(\left\|(L-r^2\exp(i\theta))z\right\|_{L^2(\Omega)} + r\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\right). \qquad (6.15)$$

Substituting back $s = r^2\exp(i\theta)$ above and defining $m = 3\sqrt{2}m_2^2$, we have the desired result. □

The boundedness of the input/output map for Neumann boundary control with observation now follows.

COROLLARY 6.7. *The input/output map of the boundary control system is bounded for all observation operators $K \in \mathcal{L}(H^1(\Omega), \mathcal{Y})$.*

Proof. By Theorem 6.6, the solution to the abstract elliptic problem (L, Γ) is uniformly bounded with respect to the $H^1(\Omega)$ norm. Hence by Theorem 3.2, the system transfer function associated with (L, Γ, K) is proper for all observation operators $K \in \mathcal{L}(H^1(\Omega), \mathcal{Y})$. Thus by Theorem 2.3 the input/output map is bounded to the boundary control system (L, Γ, K). □

Remark 6.8. The main result above is stated for a control space $\mathcal{U} = H^{\frac{1}{2}}(\partial\Omega)$. This space can be regarded as the traces of functions in $H^1(\Omega)$ (5.4). Consider the following characterization of these functions.

THEOREM 6.9 (see, e.g., [17, sect. 1.1.3]). *If a function u defined on Ω is absolutely continuous on almost all straight lines that are parallel to coordinate axes and the first classical derivatives of u belong to $L_2(\Omega)$, then $u \in H^1(\Omega)$.*

Thus, $H^{\frac{1}{2}}(\Omega)$ includes piecewise continuous functions, provided that Ω is such that we can extend u into the interior so that it satisfies the above theorem. The singularities on the boundary of Ω remain.

Remark 6.10. If Γ is Dirichlet boundary control, then $m_j = 0$ in Theorem 6.2. Using the same technique as Theorem 6.6 we can show that there exists a positive constant R such that for any $z \in H^2(\Omega)$, $u \in H^2(\Omega)$ satisfying $\Gamma z = u$ on $\partial\Omega$, and any complex number s on the open right half-plane $\mathbb{C}_{R^2} := \{s : \text{Re } s > R^2\}$, the following inequality holds:

$$|s|^{1/2} \|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \leq m \left[\|(L-s)z\|_{L^2(\Omega)} + |s| \, \|u\|_{L^2(\Omega)} + \|u\|_{H^2(\Omega)} \right],$$

where m is a positive constant dependent only on L and Ω. Unfortunately this implies the solution to $Lz = sz$ in Ω and $\Gamma z = u$ on $\partial\Omega$ satisfies only

$$\|z\|_{H^1(\Omega)} \leq m|s|^{1/2} \|u\|_{H^2(\Omega)}.$$

So we cannot conclude that the solution is uniformly bounded in the H^1-norm. In the case of Dirichlet boundary control on a one-dimensional rod, it can easily be shown that the solution to the elliptic problem is not uniformly bounded in the H^1-norm.

7. Conclusions. The input/output map and the transfer function are well defined for abstract boundary control systems. We showed that the question of continuity of the input/output map can be transformed to boundedness of solutions to a related elliptic problem. It is not necessary to construct a state-space realization.

This approach enabled us to show boundedness of the input/output map for general classes of boundary control systems involving uniformly elliptic operators with Dirichlet, Neumann, or Robin boundary control.

We are currently working on extending our approach to problems that are second order in time.

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