LARGE APERIODIC SEMIGROUPS

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We search for the largest syntactic semigroups of star-free languages having n left quotients; equivalently, we look for the largest transition semigroups of aperiodic finite automata with n states. We first introduce *unitary* semigroups generated by transformations that change only one state. In particular, we study *unitary-complete* semigroups which have a special structure, and show that each maximal unitary semigroup is unitary-complete. For $n \geqslant 4$ we exhibit a unitary-complete semigroup that is larger than any aperiodic semigroup known to date. We then present even larger aperiodic semigroups, generated by transformations that map a non-empty subset of states to a single state; we call such transformations and semigroups *semiconstant*. We examine semiconstant *tree* semigroups which have a structure based on full binary trees. The semiconstant tree semigroups are at present the best candidates for largest aperiodic semigroups.

Keywords: aperiodic, monotonic, semiconstant, transition semigroup, star-free language, syntactic complexity, unitary

1. Introduction

The *state complexity* of a regular language is the number of states in a complete minimal deterministic finite automaton (DFA) accepting the language [18]. An equivalent notion is that of *quotient complexity,* which is the number of left quotients of the language [1]; we prefer quotient complexity since it is a language-theoretic notion. The usual measure of complexity of an operation on regular languages [1, 18] is the maximal quotient complexity of the result of the operation as a function of the quotient complexities of the operands. This measure has some serious disadvantages, however. For example, as shown in [6], in the class of star-free languages all common operations have the same quotient complexity as they do in the class of arbitrary regular languages, with the exception of two cases discussed in [7]: reversal and a special case of product. Thus quotient complexity fails to differentiate between the very special class of star-free languages and the class of all regular languages.

It has been suggested that other measures of complexity may also be useful [2], in particular, the *syntactic complexity* of a regular language which is the cardinality of its syntactic semigroup [16]. This is the same as the cardinality of the *transition semigroup* of a minimal DFA accepting the language, and it is this latter representation that we use here. The transition semigroup is the set of all transformations induced by non-empty words on the set of states of the DFA. The *syntactic complexity of a class* of languages is the size of the largest syntactic semigroups of languages in that class as a function of the quotient complexities of the languages. Since the syntactic complexity of star-free languages is considerably smaller than that of regular languages, this measure succeeds in distinguishing the two classes.

The class of *star-free* languages [15] is the smallest class obtained from finite languages using only boolean operations and concatenation, but no star. By Schützenberger's theorem [17] we know that a language is star-free if and only if the transition semigroup of its minimal DFA is *aperiodic*, meaning that it contains no non-trivial subgroups. Equivalently, a transition semigroup is aperiodic if and only if no word over the alphabet of the DFA can induce a non-trivial permutation of any subset of two or more states.

Two aperiodic semigroups, monotonic and partially monotonic, were studied in [11]. That work was adapted to finite automata in [5], where nearly monotonic semigroups were also introduced; they are larger than the partially monotonic ones and were the largest aperiodic semigroups known to date for $n \leqslant 7$. For $n \geqslant 8$ the largest aperiodic semigroups known to date were those generated by DFAs accepting $\mathcal{R}$-trivial languages [4]. The syntactic complexity of $\mathcal{R}$-trivial languages is $n!$. As to aperiodic semigroups, tight upper bounds on their size were known only for $n \leqslant 3$.

The following are the main contributions of this paper:

(1) Using the method of [14], we enumerated all aperiodic semigroups for $n = 4$, and showed that maximal aperiodic semigroups have size 47, while the maximal nearly monotonic semigroup has size 41. This may seem like an insignificant result but it provided us with strong motivation to search for larger semigroups. The number of aperiodic transformations is $(n+1)^{n-1}$. For $n > 4$ the number of aperiodic semigroups is very large, and so it is difficult to check them all.
(2) We studied semigroups generated by transformations, which we call *unitary*, that change only one state. A transition semigroup of a DFA is *unitary-complete* if it is unitary and the addition of any new unitary transition results in a semigroup that is not aperiodic. We characterized unitary semigroups and computed their maximal sizes up to $n = 1,000$. For $n \geqslant 4$ the maximal unitary semigroups are larger than the maximal nearly monotonic ones and also larger than any previously known aperiodic semigroup.
(3) For each n we found a set of DFAs whose inputs induce *semiconstant tree* transformations that send a non-empty subset of the set of states to a single state, and which have a structure based on full binary trees. For $n \geqslant 4$, there is a semiconstant tree semigroup larger than the largest unitary-complete semigroup. We

computed the maximal size of these transition semigroups up to $n = 500$.

(4) We derived formulas for the maximal sizes of unitary-complete and semiconstant tree semigroups. We also provided recursive formulas characterizing the maximal unitary-complete and semiconstant tree semigroups; these formulas lead to efficient algorithms for computing the forms and sizes of such semigroups. These algorithms were used in the computations of (2) and (3) above.

Our results about aperiodic semigroups are summarized in Tables 1 and 2 for small values of n. Transformation $\mathbf{1}$ is the identity; it can be added to unitary and semiconstant transformations without affecting aperiodicity.

Additional information about the classes of semigroups in Tables 1 and 2 is given later. The classes are listed in the order of increasing size when n is large. The number in boldface shows the value of n for which the size of a given semigroup exceeds the sizes of all of the preceding ones. For example, the sizes of the largest semigroups of finite languages exceed the sizes of the preceding semigroups for $n \geqslant 12$.

There are two more classes of syntactic semigroups that have the same complexity as the semigroups of finite languages: those of cofinite and reverse definite languages. The tight upper bound $\lfloor e \cdot (n-1)! \rfloor$ for $\mathcal{J}$-trivial languages ([4]) is also a lower bound for definite languages ([5]). An upper bound of $n((n-1)! - (n-3)!)$ has been shown to hold [13] for definite and generalized definite languages [10], but it is not known whether this bound is tight.

Let $f_{pm}(n)$ be the size of the largest partially monotonic semigroups of transformations of n elements; then $f_{pm}(n)$ is asymptotically $A\frac{B^{2n-1}}{\sqrt{n}}$, where A and B are constants [5]. For nearly monotonic semigroups the size is $f_{pm}(n) + n - 1$.

Table 1. Large aperiodic semigroups.

n:	1	2	3	4	5	6	7	8
Monotonic $\binom{2n-1}{n}$	1	3	10	35	126	462	1,716	6,435
Part. mon. $f_{pm}(n)$	–	2	8	**38**	192	1,002	5,336	28,814
Near. mon. $f_{pm}(n)+n-1$	–	3	10	**41**	196	1,007	5,342	28,821
Finite $(n-1)!$	1	1	2	6	24	120	720	5,040
$\mathcal{J}$-trivial $\lfloor e \cdot (n-1)! \rfloor$	–	2	5	16	65	326	1,957	13,700
$\mathcal{R}$-trivial $n!$	1	2	6	24	120	720	5,040	**40,320**
Unitary-complete with $\mathbf{1}$	–	3	10	**45**	270	1,737	13,280	121,500
Semiconstant tree with $\mathbf{1}$	–	3	10	**47**	273	1,849	14,270	126,123
Aperiodic	1	3	10	47	?	?	?	?

The remainder of the paper is structured as follows. Section 2 presents our ter-

Table 2. Large aperiodic semigroups continued.

n:	9	10	11	12	13
Monotonic	24,310	92,378	352,716	1,352,078	5,200,300
Partially monotonic	157,184	864,146	4,780,008	26,572,086	148,321,344
Nearly monotonic	157,192	864,155	4,780,018	26,572,097	148,321,356
Finite	40,320	362,880	3,628,800	**39,916,800**	479,001,600
$\mathcal{J}$-trivial	109,601	**986,410**	9,864,101	108,505,112	1,302,061,345
$\mathcal{R}$-trivial	362,880	3,628,800	39,916,800	479,001,600	6,227,020,800
Unitary-complete with **1**	1,231,200	12,994,020	151,817,274	2,041,564,500	29,351,808,000
Semiconstant tree with **1**	1,269,115	14,001,629	169,410,932	2,224,759,333	31,405,982,419
Aperiodic	?	?	?	?	?

minology and notation. Our large aperiodic semigroups are defined in Section 3. The special case of unitary semigroups is then considered in Section 4, and semiconstant tree semigroups are the topic of Section 5. Section 6 concludes the paper.

A much abbreviated version of this work appeared in [8].

2. Terminology and Notation

Let Σ be a finite alphabet. The elements of Σ are *letters* and the elements of Σ^* are *words*, where Σ^* is the free monoid generated by Σ. The empty word is ε, and the set of all non-empty words is Σ^+. A *language* is any subset of Σ^*.

Suppose $n \geqslant 1$. Without loss of generality we assume that our basic set is $Q = \{0, 1, \dots, n-1\}$. A *deterministic finite automaton (DFA)* is a quintuple $\mathcal{D} = (Q, \Sigma, \delta, 0, F)$, where Q is a finite non-empty set of *states*, Σ is a finite non-empty *alphabet*, $\delta \colon Q \times \Sigma \to Q$ is the *transition function*, $0 \in Q$ is the *initial state*, and $F \subseteq Q$ is the set of *final states*. We extend δ to $Q \times \Sigma^*$ and to $2^Q \times \Sigma^*$ as usual.

A DFA $\mathcal{D}$ *accepts* a word $w \in \Sigma^*$ if $\delta(0, w) \in F$. The *language accepted* by $\mathcal{D}$ is $L(\mathcal{D}) = \{w \in \Sigma^* \mid \delta(0, w) \in F\}$. By the *language of a state* q of $\mathcal{D}$ we mean the language $L_q(\mathcal{D})$ accepted by the DFA $(Q, \Sigma, \delta, q, F)$. A state is *empty* (also called *dead* or a *sink*) if its language is empty. Two states p and q of $\mathcal{D}$ are *equivalent* if $L_p(\mathcal{D}) = L_q(\mathcal{D})$; otherwise, they are *distinguishable*. A state q is *reachable* if there exists a word $w \in \Sigma^*$ such that $\delta(0, w) = q$. A DFA is *minimal* if all its states are reachable and pairwise distinguishable.

A *transformation* of Q is a mapping of Q into itself. Let t be a transformation of Q; then qt is the *image* of $q \in Q$ under t. If P is a subset of Q, then $Pt = \{qt \mid q \in P\}$. An arbitrary transformation can be written in the form

$$t = \begin{pmatrix} 0 & 1 & \cdots & n-2 & n-1 \\ p_0 & p_1 & \cdots & p_{n-2} & p_{n-1} \end{pmatrix},$$

where $p_q = qt$ for $q \in Q$. We also use $t = [p_0, \dots, p_{n-1}]$ as a simplified notation. The *composition* of transformations t_1 and t_2 of Q is the transformation $t_1 \circ t_2$ such that $q(t_1 \circ t_2) = (qt_1)t_2$ for all $q \in Q$. We write $t_1 t_2$ for $t_1 \circ t_2$.

Let $\mathcal{T}_Q$ be the set of all n^n transformations of Q; then $\mathcal{T}_Q$ is a monoid under composition with $\mathbf{1}$ as the identity. A *permutation* of Q is a mapping of Q *onto* itself. For $k \geqslant 2$, a permutation t of a set $P = \{q_0, q_1, \dots, q_{k-1}\} \subseteq Q$ is a *k-cycle* if $q_0 t = q_1, q_1 t = q_2, \dots, q_{k-2} t = q_{k-1}, q_{k-1} t = q_0$; this k-cycle is denoted by $(q_0, q_1, \dots, q_{k-1})$. If a transformation t of Q acts like a k-cycle on some $P \subseteq Q$, we say that *t has a k-cycle*. A transformation of Q *has a cycle* if it has a k-cycle for some $k \geqslant 2$.

A transformation is *aperiodic* if it has no cycles.

In any DFA $\mathcal{D}$, each word $w \in \Sigma^*$ induces a transformation t_w of Q defined by $qt_w = \delta(q, w)$ for all $q \in Q$. The set of all transformations of Q induced in $\mathcal{D}$ by non-empty words is the *transition semigroup* of $\mathcal{D}$, which is a subsemigroup of $\mathcal{T}_Q$.

The *syntactic congruence* $\approx_L$ of any language L is defined as follows: $x \approx_L y$ if and only if $uxv \in L \Leftrightarrow uyv \in L$ for all $u, v \in \Sigma^*$. The set $\Sigma^+/{\approx_L}$ of equivalence classes of $\approx_L$ is a semigroup called the *syntactic semigroup* of L. A language is regular if and only if its syntactic semigroup is finite. The *syntactic complexity* of L is the cardinality of its syntactic semigroup.

If $\mathcal{D}$ is minimal, its transition semigroup is isomorphic to the syntactic semigroup of the language $L(\mathcal{D})$ [15, 16]. In this paper we deal only with transition semigroups.

If T is a set of transformations, then $\langle T \rangle$ is the semigroup generated by T. If $\mathcal{D} = (Q, \Sigma, \delta, 0, F)$ is a DFA, the transformations induced by letters of Σ are called *generators of the transition semigroup* of $\mathcal{D}$ or simply *generators* of $\mathcal{D}$.

3. Unitary and Semiconstant DFAs

We define a new class of aperiodic DFAs among which are found DFAs with the largest transition semigroups known. We also study several of its subclasses.

A *unitary* transformation t, denoted by $(p \to q)$, has $p \neq q$, $pt = q$ and $rt = r$ for all $r \neq p$. A DFA is *unitary* if each of its generators is unitary. A semigroup is *unitary* if it has a set of unitary generators.

A *constant* transformation t, denoted by $(Q \to q)$, has $pt = q$ for all $p \in Q$. A transformation t is *semiconstant* if it maps a non-empty subset P of Q to a single element q and leaves the remaining elements of Q unchanged. It is denoted by $(P \to q)$. A constant transformation is semiconstant with $P = Q$, and a unitary transformation $(p \to q)$ is semiconstant with $P = \{p\}$ (or $P = \{p, q\}$). A DFA is *semiconstant* if each of its generators is semiconstant. A semigroup is *semiconstant* if it has a set of semiconstant generators.

For each $n \geqslant 1$ we shall define several DFAs. Let m, $n_1, n_2, \dots, n_m$ be positive natural numbers. Also, let $n = n_1 + \cdots + n_m$, and for each i, $1 \leqslant i \leqslant m$, define r_i by $r_i = \sum_{j=1}^{i-1} n_j$. For $i = 1, \dots, m$, let $Q_i = \{r_i, r_i + 1, \dots, r_{i+1} - 1\}$; thus the cardinality of Q_i is n_i. Let $Q = Q_1 \cup \cdots \cup Q_m = \{0, \dots, n-1\}$; the cardinality of

Q is n. The sequence $(n_1, n_2, \ldots, n_m)$ is called the *distribution* of Q. The number $d(n)$ of different distributions of the n-element set Q is 2^{n-1}.

A binary tree is *full* if every vertex has either two children or no children. There are C_{m-1} full binary trees with m leaves, where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the Catalan number[a].

Let Δ_Q be a full binary tree with m leaves labeled $Q_1, \ldots, Q_m$ from left to right. To each node $v \in \Delta_Q$ we assign the union $\mathrm{Q}(v)$ of all the sets Q_i labeling the leaves in the subtree rooted at v. With each full binary tree we can associate different distributions. A full binary tree Δ_Q with a distribution $(n_1, n_2, \ldots, n_m)$ of Q is denoted by $\Delta_Q(n_1, n_2, \ldots, n_m)$ and is called the *structure* of Q, which will uniquely determine the transition function δ of the DFAs defined below. The number of possible structures of Q for a given n is the binomial transform of the Catalan number C_n[b]. We can denote the structure of Q as a binary expression. For example, the expression $((3,2),(4,1))$ denotes the full binary tree in which the leaves are labeled Q_1, Q_2, Q_3, and Q_4, where $|Q_1| = 3, |Q_2| = 2, |Q_3| = 4, |Q_4| = 1$, and the interior nodes are labeled by $Q_1 \cup Q_2$, $Q_3 \cup Q_4$ and $Q_1 \cup Q_2 \cup Q_3 \cup Q_4$. The expression $(((3,2),4),1)$ has interior nodes labeled $Q_1 \cup Q_2$, $Q_1 \cup Q_2 \cup Q_3$ and $Q_1 \cup Q_2 \cup Q_3 \cup Q_4$.

Definition 1 (Transformations) *Suppose $n > 1$, $(n_1, n_2, \ldots, n_m)$ is a distribution of Q, and $\Delta_Q(n_1, n_2, \ldots, n_m)$ is a structure of Q.*

Type 1: *For all $i = 1, \ldots, m$ and $q, q+1 \in Q_i$ the unitary transformations $(q \to q+1)$ and $(q+1 \to q)$ are* Type 1 *transformations.*

Type 2: *If $1 \leqslant i \leqslant m-1$ and $i < j \leqslant m$, for each $q \in Q_i$ and $p \in Q_j$, $(q \to p)$ is a* Type 2 *transformation.*

Type 3: *There are $m-1$ internal nodes. For each such node w, the semiconstant transformation $(\mathrm{Q}(w) \to \min(\mathrm{Q}(w)))$ is of* Type 3.

Type 4: *The identity transformation* $\mathbf{1}$ *on Q is of* Type 4.

For a fixed i there are $2n_i - 2$ transformations of Type 1 and $n_i(n_{i+1} + \cdots + n_m)$ transformations of Type 2; for $m = 1$ there are no transformations of Type 2. The number of all transformations of Type 3 is $m-1$. Note that the distribution $(n_1, n_2, \ldots, n_m)$ affects transformations of Types 1, 2, and 3, whereas the binary tree affects only transformations of Type 3.

In the following DFAs the transition function is defined by a set of transformations and the alphabet consists of letters inducing these transformation.

Definition 2 (DFAs) *Suppose $n > 1$.*

(1) Any DFA of the form $\mathcal{D}_u(n_1, \ldots, n_m) = (Q, \Sigma_u, \delta_u, 0, \{n-1\})$, where δ_u has all the transformations of Types 1 and 2, is a unitary-complete *DFA. Note that*

[a] `http://en.wikipedia.org/wiki/Catalan_number`
[b] `http://oeis.org/A007317`

the transition semigroup of a unitary-complete DFA is unitary-complete.

(2) $\mathcal{D}_{ui}(n_1,\dots,n_m) = (Q, \Sigma_{ui}, \delta_{ui}, 0, \{n-1\})$ *is* $\mathcal{D}_u(n_1,\dots,n_m)$ *with* **1** *added.*

(3) *Any DFA* $\mathcal{D}_{sct}(\Delta_Q(n_1,\dots,n_m)) = (Q, \Sigma_{sct}, \delta_{sct}, 0, \{n-1\})$*, where* δ_{sct} *has all the transformations of Types 1, 2 and 3, is a* semiconstant tree *DFA.*

(4) $\mathcal{D}_{scti}(\Delta_Q(n_1,\dots,n_m)) = (Q, \Sigma_{scti}, \delta_{scti}, 0, \{n-1\})$ *is* $\mathcal{D}_{sct}(\Delta_Q(n_1,\dots,n_m))$ *with* **1** *added.*

The directed graph $G(\mathcal{D}) = (V, E)$ of a unitary DFA $\mathcal{D}$ is defined as follows: $V = Q$, and for every unitary transformation $(p \to q)$ in $\mathcal{D}$, there is an edge (p, q) in E. A directed graph (V, E) is called a *bipath* (*bidirectional path*) [9] if $V = \{v_0, \dots, v_{k-1}\}$ for some $k \geqslant 1$, and for each $v_q, v_{q+1} \in V$ there are two edges (v_q, v_{q+1}) and (v_{q+1}, v_q), and there are no other edges. In the graph of a unitary-complete DFA the induced subgraph on Q_i is isomorphic to a bipath. Also, the graph of $\mathcal{D}_u(n_1,\dots,n_m)$ of Definition 2 can be viewed as a sequence $(Q_1,\dots,Q_m)$ of bipaths, where there are additional edges from every q in Q_i to every p in Q_j, if $i < j$.

Example 3. *Figure 1 shows three examples of unitary DFAs. In Fig. 1 (a) we have DFA* $\mathcal{D}_u(3)$*, where the letter* a_{pq} *induces the unitary transformation* $(p \to q)$*. In Fig. 1 (b) we show the graph* $G(\mathcal{D}_u(3))$*. Next, in Figs. 1 (c) and (d), we have* $G(\mathcal{D}_u(3,1))$ *and* $G(\mathcal{D}_u(2,2,2))$*, respectively. We shall return to these examples later.*

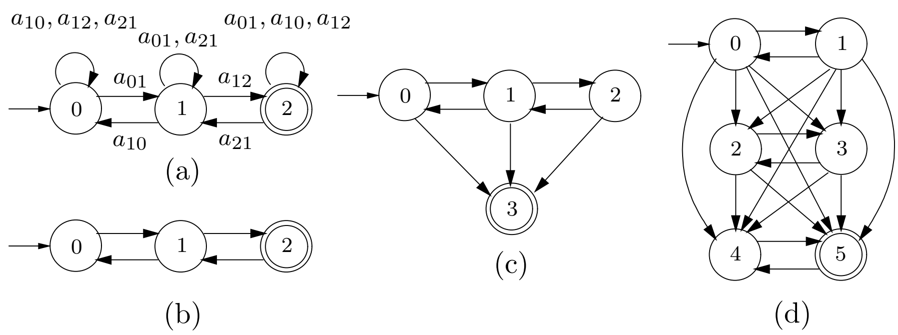

Fig. 1. Unitary DFAs: (a) $\mathcal{D}_u(3)$; (b) $G(\mathcal{D}_u(3))$; (c) $G(\mathcal{D}_u(3,1))$; (d) $G(\mathcal{D}_u(2,2,2))$.

Remark 4. *All four types of DFAs of Definition 2 are minimal as is easily verified. Hence the syntactic semigroup of the language of each DFA is isomorphic to the transition semigroup of the DFA.*

4. Unitary Semigroups

We study unitary semigroups because their generators are the simplest. We begin with three previously studied subsemigroups of unitary semigroups.

Monotonic semigroups: Monotonic semigroups were studied in [5, 11, 12]. Let $\preceq$ be a total order on Q. A transformation t of Q is *order-preserving* if for all $p, q \in Q$, $p \preceq q$ implies $pt \preceq qt$. Note that the identity transformation is order-preserving, and the composition of two order-preserving transformations is order preserving. A DFA is *monotonic* if each of its input transformations is order-preserving. A semigroup is *monotonic* if it has a set of order-preserving generators.

The following result of [11] is somewhat modified for our purposes:

Proposition 5 (Gomes and Howie) *The set M of all $\binom{2n-1}{n} - 1$ order-preserving transformations other than* $\mathbf{1}$ *is an aperiodic semigroup generated by*

$$G_M = \{(q \to q+1) \mid 0 \leqslant q \leqslant n-2\} \cup \{(q \to q-1) \mid 1 \leqslant q \leqslant n-1\},$$

and no smaller set of unitary transformations generates M.

Corollary 6. *The transition semigroup of $\mathcal{D}_{ui}(n)$ is the semigroup $M \cup \{\mathbf{1}\}$ of all order-preserving transformations.*

Note that $\mathcal{D}_u$ has transitions of Type 1 only, and D_{ui} has Type 1 and 4 only. Figure 1 (b) shows $\mathcal{D}_u(3)$ and $\mathcal{D}_{ui}(3)$, if $\mathbf{1}$ is added. The transition semigroup of $\mathcal{D}_{ui}(3)$ has ten elements and is the largest aperiodic semigroup for $n = 3$ [5].

Note also that there are monotonic semigroups that do not have unitary generating sets; each monotonic semigroup, however, is a subsemigroup of the transition semigroup of $\mathcal{D}_{ui}(n)$ consisting of all order-preserving transformations.

Partially monotonic semigroups: A *partial transformation* t of Q is a partial mapping of Q into itself. If t is defined for $q \in Q$, then qt is the image of q under t; otherwise, we write $qt = \square$. By convention, $\square t = \square$. The *domain* of t is the set $\mathrm{dom}(t) = \{q \in Q \mid qt \neq \square\}$. Given an order $\preceq$ on Q, a partial transformation is *order-preserving* if for all $p, q \in \mathrm{dom}(t)$, $p \preceq q$ implies $pt \preceq qt$.

Semigroups of order-preserving partial transformations were studied by Gomes and Howie [11] and adapted to automata in [5]. Suppose $Q = \{0, \dots, n-2\}$. We follow [5] by adding state $(n-1)$ for the undefined value $\square$ and defining $(n-1)t = n-1$ for all transformations. The semigroups of the obtained DFAs correspond naturally to semigroups of partial transformations. The transition semigroup of a DFA is *partially monotonic* if its corresponding semigroup of partial transformations has only order-preserving transformations. A DFAs with a partially monotonic semigroup is *partially monotonic*. The following is an adaptation of the results of [11]:

Proposition 7. *For $n \geqslant 2$, the DFA $\mathcal{D}_{ui}(n-1, 1) = (Q, \Sigma_{ui}, \delta_{ui}, 0, \{n-1\})$ has the following properties:*

(1) All partial transformations corresponding to the $3n-4$ generators of $\mathcal{D}_{ui}(n-$

1, 1) are order-preserving. Thus $\mathcal{D}_{ui}(n-1,1)$ is partially monotonic, and hence aperiodic.

(2) The transition semigroup of $\mathcal{D}_{ui}(n-1,1)$ corresponds to the semigroup PM_Q of all $f_{pm}(n)$ order-preserving partial transformations of Q, where

$$f_{pm}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{n+k-2}{k}. \tag{1}$$

(3) Each generator is idempotent, and $3n-4$ is the smallest number of idempotent generators of the transition semigroup of $\mathcal{D}_{ui}(n-1,1)$.

Example 8. *There are eight order-preserving partial transformations of the set $Q = \{0,1\}$, namely: $[\square,\square]$, $[0,\square]$, $[1,\square]$, $[\square,0]$, $[\square,1]$, $[0,0]$, $[0,1]$, $[1,1]$. When we replace $\square$ by 2, the partial transformations become total transformations $[2,2,2]$, $[0,2,2]$, $[1,2,2]$, $[2,0,2]$, $[2,1,2]$, $[0,0,2]$, $[0,1,2]$, $[1,1,2]$. The $9-4=5$ generators of $\mathcal{D}_{ui}(2,1)$ are: $(0 \to 1) = [1,1,2]$, $(0 \to 2) = [2,1,2]$, $(1 \to 0) = [0,0,2]$, $(1 \to 2) = [0,2,2]$ and $\mathbf{1}$. The DFA of Figure 1 (c) is an example of $\mathcal{D}_{ui}(3,1)$.*

For $n \geqslant 4$ the largest partially monotonic semigroup is larger than the semigroup of all order-preserving transformations.

There are partially monotonic semigroups that do not have unitary generating sets; each partially monotonic semigroup, however, is a subsemigroup of the transition semigroup of $\mathcal{D}_{ui}(n-1,1)$.

Other previously studied aperiodic semigroups: As we mentioned in the introduction, the syntactic complexity of five other language classes was studied previously. *Cofinite* languages are complements of finite languages, and therefore their minimal DFAs have the same transition semigroups as the DFAs of finite languages.

The reverse w^R of a word $w \in \Sigma^+$ is w spelled backwards and $\varepsilon^R = \varepsilon$. The reverse of a language L is $L^R = \{w^R \mid w \in L\}$. A language is *definite* if it has the form $E \cup \Sigma^* F$, where E and F are finite. It is *reverse definite* if its reverse is definite, that is, if it has the form $E \cup F\Sigma^*$, where E and F are finite. It was shown in [5] that the syntactic complexity of reverse definite languages is the same as that of finite languages. The syntactic complexity of definite languages remains open.

The well known Green relations define $\mathcal{R}$-trivial and $\mathcal{J}$-trivial monoids (semigroups with an identity). If M is a monoid, the relation $\mathcal{R}$ is defined by $s\mathcal{R}t \Leftrightarrow sM = tM$ for $s,t \in M$. A monoid is *$\mathcal{R}$-trivial* if $s\,\mathcal{R}\,t$ implies $s=t$. The relation $\mathcal{J}$ is defined by $s\,\mathcal{J}\,t \Leftrightarrow MsM = MtM$, and M is *$\mathcal{J}$-trivial* if $MsM = MtM$ implies $s=t$. Languages whose minimal DFAs have $\mathcal{R}$-trivial ($\mathcal{J}$-trivial) transition monoids are also called *$\mathcal{R}$-trivial* (*$\mathcal{J}$-trivial*).

Syntactic complexities of $\mathcal{R}$-trivial and $\mathcal{J}$-trivial languages were studied by Brzozowski and Li [4]. Consider the natural order $\leqslant$ on $Q = \{0,\dots,n-1\}$. We say that a transformation t is non-decreasing if $q \leqslant qt$ for all $q \in Q$. Let $\mathcal{F}_Q$ be the set of all non-decreasing transformations. The size of $\mathcal{F}_Q$ is $n!$.

It was shown in [3] that L is an $\mathcal{R}$-trivial language if and only if its minimal DFA is partially ordered, or equivalently, if its transition semigroup contains only non-decreasing transformations. Thus the largest semigroup generated by DFAs accepting $\mathcal{R}$-trivial languages is $\mathcal{F}_Q$.

Proposition 9. *The transition semigroup of* $\mathcal{D}_{ui}(1, 1, \ldots, 1)$ *is the semigroup* $\mathcal{F}_Q$ *of all non-decreasing transformations.*

Proof. DFA $\mathcal{D}_{ui}(1, 1, \ldots, 1)$ has only unitary transformations of Type 2. They generate only non-decreasing transformations, since each of them preserves the natural order. An arbitrary non-decreasing transformation has the form $t = \begin{pmatrix} 0 & 1 & \cdots & n-2 & n-1 \\ p_0 & p_1 & \cdots & p_{n-2} & n-1 \end{pmatrix}$,where $p_q \geqslant q$ for $q = 0, \ldots, n-2$. Since $\mathcal{D}_{ui}(1, 1, \ldots, 1)$ contains all unitary transformations of the form $(q \to p)$ for $q \leqslant p$, all transformations $t_q = (q \to p_q)$ are present. One verifies that applying $t_{n-2}t_{n-3} \cdots t_1 t_0$ results in t. □

There are semigroups with only non-decreasing transformations that do not have unitary generating sets, but each such semigroup is a subsemigroup of $\mathcal{F}_Q$. Since every $\mathcal{J}$-trivial language is also $\mathcal{R}$-trivial, the transition semigroups of all minimal DFAs accepting $\mathcal{J}$-trivial languages are also subsemigroups of $\mathcal{D}_{ui}(1, 1, \ldots, 1)$.

General unitary semigroups: A set $\{t_0, \ldots, t_{k-1}\}$ of unitary transformations is *k-cyclic* if it has the form $t_0 = (q_0 \to q_1)$, $t_1 = (q_1 \to q_2), \ldots, t_{k-2} = (q_{k-2} \to q_{k-1})$, $t_{k-1} = (q_{k-1} \to q_0)$, where the q_i are distinct.

Lemma 10. *Let T be a set of unitary transformations.*

(1) *If T has a k-cyclic subset $\{t_0, \ldots, t_{k-1}\}$ with $k \geqslant 3$, then $\langle T \rangle$ is not aperiodic.*
(2) *If T contains a subset $T_6 = \{t_{01}, t_{10}, t_{12}, t_{13}, t_{21}, t_{31}\}$ where $t_{i,j} = (q_i \to q_j)$ and $q_0, q_1, q_2, q_3 \in Q$, then $\langle T \rangle$ is not aperiodic.*

Proof. Without loss of generality, we can replace q_i by i in both claims.

(1) Suppose that T contains $t_0, \ldots, t_{k-1}$, where $k \leqslant n$, $t_q = (q, q+1)$ for $q = 0, \ldots, k-2$, and $t_{k-1} = (k-1 \to 0)$. Then $t_{k-2}t_{k-3} \ldots t_1 t_0 t_{k-1}$ maps 0 to $1, 1$ to $2, \ldots, k-3$ to $k-2, k-2$ to 0, and $k-1$ to 0, and does not affect any other states. Thus the set $\{0, 1, \ldots, k-2\}$ is cyclically permuted, which shows that $\langle T \rangle$ is not aperiodic.
(2) If $\{t_{01}, t_{12}, t_{13}, t_{10}, t_{21}, t_{31}\} \subseteq T$, then the transformation $t_{12}t_{01}t_{13}t_{21}t_{10}t_{31}$ transposes q_0 and q_1; hence $\langle T \rangle$ is not aperiodic. □

Theorem 11. *If $\mathcal{D} = (Q, \Sigma, \delta, 0, F)$ is a unitary DFA, the following are equivalent:*

(1) *$\mathcal{D}$ is aperiodic.*

(2) The set of generators of $\mathcal{D}$ does not contain any k-cyclic subsets with $k \geqslant 3$, and does not contain any sets of type T_6.

(3) Every strongly connected component of the graph of $\mathcal{D}$ is a bipath.

Proof. $1 \Rightarrow 2$: This follows from Lemma 10.

$2 \Rightarrow 3$: Consider a strongly connected component C of the graph of $\mathcal{D}$. If $|C| = 1$, the claim holds. Otherwise, suppose $p \in C$ and $(p \to q)$ is a transition. Then there must also be a directed path from q to p. If the last transition in that path is $(r \to p)$, where $r \neq q$, then the set of generators must contain a k-cyclic subset with $k \geqslant 3$, which is a contradiction. Hence the transition $(q \to p)$ must be present.

Next, suppose that there are transitions $(p \to q)$, $(p \to r)$, and $(p \to s)$. By the argument above there must also be transitions $(q \to p)$, $(r \to p)$, and $(r \to s)$. But then the set of generators contains a subset of type T_6, which is again a contradiction.

It follows that every strongly connected component is a bipath, and the graph of the transitions of $\mathcal{D}$ is a loop-free connection of such bipaths.

$3 \Rightarrow 1$: Suppose that there is a transformation t with a cycle $(q_0, \dots, q_{k-1})$. Every state q_i from the cycle lies in a single strongly connected component, which is a bipath. This bipath is monotonic with some order $\preceq$, which must be preserved by the transformation t restricted to the set of states of the cycle. If $q_0 \prec q_1$, then $q_1 = q_0 t \prec q_1 t = q_2$, etc. Thus we reach $q_{k-1} \prec q_0 \prec q_{k-1}$, which is a contradiction. Hence $\mathcal{D}$ is aperiodic. □

Theorem 12. *A maximal aperiodic unitary semigroup is isomorphic to the transition semigroup of a unitary-complete DFA $\mathcal{D}_u(n_1, \dots, n_m)$, where $(n_1, \dots, n_m)$ is some distribution of Q.*

Proof. We know that an aperiodic unitary DFA $\mathcal{D}$ is a loop-free connection of bipaths. Let $Q_1, \dots, Q_m$ be the bipaths of $\mathcal{D}$. There exists a linear ordering $\prec$ of them, such that there is no transformation $(p \to q)$ for $q \in Q_i, p \in Q_j, i \prec j$. If all possible transformations $(q \to p)$ for $q \in Q_i, p \in Q_j, i \prec j$ are present, then $\mathcal{D}$ is isomorphic to $\mathcal{D}_u(n_1, \dots, n_m)$. Otherwise we can add more unitary transformations of Type 2 and obtain a larger semigroup. □

For each distribution $(n_1, \dots, n_m)$, we calculate the size of the transition semigroup of $\mathcal{D}_{ui}(n_1, \dots, n_m)$.

Theorem 13. *The cardinality of the transition semigroup of $\mathcal{D}_{ui}(n_1, \dots, n_m)$ is*

$$\prod_{i=1}^{m} \left(\binom{2n_i - 1}{n_i} + \sum_{h=0}^{n_i - 1} \left(\sum_{j=i+1}^{m} n_j \right)^{n_i - h} \binom{n_i}{h} \binom{n_i + h - 1}{h} \right). \tag{2}$$

Proof. As above $\mathcal{D}_{ui}(n_1, \dots, n_m)$ is a loop-free connection of bipaths, and its generators are the transformations within each bipath, all transformations of the form $(p \to q)$ where $p \in Q_i$, $q \in Q_j$, $i < j$, and $\mathbf{1}$.

In the transition semigroup of $\mathcal{D}_{ui}(n_1, \dots, n_m)$, consider i such that $1 \leqslant i \leqslant m$, and a transformation t_i that

(1) acts as the identity on any state in Q_j for $j \neq i$,
(2) maps some number h of states of Q_i to Q_i,
(3) maps the remaining $n_i - h$ states of Q_i to some states in $Q_{i+1} \cup \dots \cup Q_m$.

It is convenient to temporarily consider the partial transformation t'_i defined on Q_i which for all $q \in Q_i$ has the property $qt'_i = qt_i$, if $qt_i \in Q_i$ and $qt'_i = \Box$, otherwise. Thus the images of the $n_i - h$ states mapped to the outside of Q_i are all lumped together into the undefined value $\Box$. The number of such partial transformations generated by the transitions in the bipath is $\binom{n_i}{h}\binom{n_i+h-1}{h}$ [11]; these are all the order-preserving partial transformations of Q_i that map exactly h states of Q_i to Q_i.

Returning to t_i, consider first the case $h = n_i$; then t'_i is the total transformation equal to t_i, and there are $\binom{2n_i-1}{n_i}$ such transformations. Otherwise, t_i maps $n_i - h$ states of Q_i to arbitrary states in $Q_{i+1} \cup \dots \cup Q_m$. If $k = n_{i+1} + \dots + n_m$ is the number of states in the bipaths below Q_i, then for each t'_i there are k^{n_i-h} transformations t_i. Altogether, for a fixed bipath Q_i, the number of transformations t_i is

$$\binom{2n_i - 1}{n_i} + \sum_{h=0}^{n_i-1} k^{n_i-h}\binom{n_i}{h}\binom{n_i+h-1}{h}. \tag{3}$$

If t is any transformation of $\mathcal{D}_{ui}(n_1, \dots, n_m)$, then it can be represented by $t = t_m \circ t_{m-1} \circ \dots \circ t_1$, where t_i maps Q_i into $Q_i \cup \ldots \cup Q_m$. Since the domains of $t_1, \dots, t_m$ are disjoint, there is a bijection between transformations t and the sets $\{t_1, \dots, t_m\}$. Hence we can multiply the numbers of different transformations t_i for each $1 \leqslant i \leqslant m$, and the formula in the theorem results. □

Note that each factor of the product in Theorem 13 depends only on n_i and on the sum $k = n_{i+1} + \dots + n_m$. Hence if $\mathcal{D}_{ui}(n_1, \dots, n_m)$ is maximal, then $\mathcal{D}_{ui}(n_2, \dots, n_m)$ is also maximal and so on. Consequently, we have

Corollary 14. *Let $m_{ui}(n)$ be the cardinality of the largest transition semigroup of DFA $\mathcal{D}_{ui}(n_1, \dots, n_m)$ with n states. If we define $m_{ui}(0) = 1$, then for $n > 0$*

$$m_{ui}(n) = \max_{j=1,\dots,n}\left(m_{ui}(n-j)\left(\binom{2j-1}{j} + \sum_{h=0}^{j-1}(n-j)^{j-h}\binom{j}{h}\binom{j+h-1}{h}\right)\right). \tag{4}$$

This leads directly to a dynamic algorithm taking $O(n^3)$ time for computing $m_{ui}(n)$ and the distributions $(n_1, \dots, n_m)$ yielding the maximal unitary semigroups. This holds assuming constant time for computing the internal terms in the summation and summing them, where, however, the numbers can be very large ($O(n^n)$).

The precise complexity depends on the algorithms used for multiplication, exponentiation and calculation of binomial coefficients.

We were able to compute the maximal $\mathcal{D}_{ui}$ up to $n = 1,000$. Here is an example of the maximal one for $n = 100$:

$$\mathcal{D}_{ui}(12, 11, 10, 10, 9, 8, 8, 7, 6, 5, 5, 4, 3, 2);$$

its syntactic semigroup size exceeds 2.1×10^{160}. Compare this to the previously known largest semigroup of an $\mathcal{R}$-trivial language; its size is 100! which is approximately 9.3×10^{157}. On the other hand, the maximal possible syntactic semigroup of any regular language for $n = 100$ is 10^{200}.

We were not able to compute the tight asymptotic bound on the maximal size of unitary semigroups. However, we computed a lower bound which is larger than $n!$, the previously known lower bound for the size of aperiodic semigroups.

Theorem 15. *For n even the size of the maximal unitary semigroup is at least*

$$\frac{n!(n+1)!}{2^n((n/2)!)^2}.$$

Proof. Let n be even and consider $\mathcal{D}_{ui}(2, 2, \dots, 2)$ consisting of $m = n/2$ bipaths. From Theorem 13 we have:

$$\begin{aligned}
&\prod_{i=1}^{m}\left(\binom{4-1}{2} + \sum_{h=0}^{1}\left(\sum_{j=i+1}^{m} 2\right)^{2-h}\binom{2}{h}\binom{2+h-1}{h}\right)\\
&= \prod_{i=1}^{m}\left(4(m-i)^2 + 8(m-i) + 3\right) = \prod_{i=1}^{m}\left((2i-1)(2i+1)\right)\\
&= (2m-1)!!(2m+1)!! = (2m-1)!!(2(m+1)-1)!!
\end{aligned}$$

By using the equality $(2k-1)!! = \frac{(2k)!}{2^k k!}$ we obtain:

$$\begin{aligned}
&= \frac{(2m)!}{2^m m!}\frac{(2(m+1))!}{2^{m+1}(m+1)!} = \frac{n!(n+2)!}{2^{n+1}(n/2)!((n/2)+1)!}\\
&= \frac{n!(n+2)(n+1)!}{2^{n+1}(n/2)!(n/2+1)(n/2)!} = \frac{n!(n+1)!}{2^n((n/2)!)^2}.
\end{aligned}$$

□

For $n = 100$ the bound exceeds 7.5×10^{158}. Larger lower bounds can also be found using increasing values of j in $\mathcal{D}_{ui}(j, j, \dots, j)$, but the complexity of the calculations increases, and such bounds are not tight.

5. Semiconstant Semigroups

We now consider our largest aperiodic semigroups, the semiconstant ones.

Nearly monotonic semigroups: Let NM_Q be the set of all transformations corresponding to order-preserving partial transformations and all constant transformations. A semigroup on Q is *nearly monotonic* if it is a subsemigroup of NM_Q. A DFA is *nearly monotonic* if its transition semigroup is nearly monotonic.

The semigroup of $\mathcal{D}_{scti}((n-1,1))$ is the largest nearly monotonic semigroup NM_Q, and for $n \geqslant 4$ it is larger than the largest partially monotonic semigroup. There are nearly monotonic semigroups without semiconstant generating sets, but each nearly monotonic semigroup is a subsemigroup of the transition semigroup of $\mathcal{D}_{scti}((n-1,1))$.

Semiconstant tree semigroups: An example of a maximal semiconstant tree DFA for $n = 6$ is $\mathcal{D}_{scti}(((2,2),2))$; its transition semigroup has 1,849 elements. For $n \geqslant 4$, the maximal semiconstant tree semigroup is the largest aperiodic semigroup known.

First we define a new operation on DFAs.

Definition 16. *Let $\mathcal{A} = (Q_\mathcal{A}, \Sigma_\mathcal{A}, \delta_\mathcal{A}, q_\mathcal{A}, F_\mathcal{A})$ and $\mathcal{B} = (Q_\mathcal{B}, \Sigma_\mathcal{B}, \delta_\mathcal{B}, q_\mathcal{B}, F_\mathcal{B})$ be DFAs, where $Q_\mathcal{A} \cap Q_\mathcal{B} = \emptyset$. Let $Q_\mathcal{C} = Q_\mathcal{A} \cup Q_\mathcal{B}$. The* semiconstant sum *of $\mathcal{A}$ and $\mathcal{B}$ is the DFA $\mathcal{C} = (\mathcal{A}, \mathcal{B}) = (Q_\mathcal{C}, \Sigma_\mathcal{C}, \delta_\mathcal{C}, q_\mathcal{A}, F_\mathcal{B})$. For each transition t in δ_A, we have a transition t' in δ_C such that $qt' = qt$ for $q \in Q_\mathcal{A}$ and $qt' = q$ otherwise. Dually, we have transitions defined by t in δ_B. Moreover we have a unitary transformation $(p \to q)$ for each $p \in Q_\mathcal{A}, q \in Q_\mathcal{B}$, and a constant transformation $(Q_\mathcal{C} \to q_\mathcal{A})$.*

For $m > 1$, each $\mathcal{D}_{scti}(\Delta_Q(n_1, \dots, n_m))$ is a semiconstant sum of two smaller semiconstant tree DFAs: $\mathcal{D}_{scti}(\Delta_{Q_{left}}(n_1, \dots, n_r))$, defined by the left subtree of $\Delta_Q(n_1, \dots, n_m)$, and $\mathcal{D}_{scti}(\Delta_{Q_{right}}(n_{r+1}, \dots, n_m))$, defined by the right subtree.

Lemma 17. *The semiconstant sum $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ is minimal if and only if every state of $\mathcal{A}$ is reachable from $q_\mathcal{A}$ in $\mathcal{A}$, the states of $\mathcal{B}$ are pairwise distinguishable, and $F_\mathcal{B} \neq \emptyset$.*

Proof. If $\mathcal{C}$ is minimal, then every state of $\mathcal{C}$ is reachable from $q_\mathcal{A}$ in $\mathcal{C}$. If q is not reachable from $q_\mathcal{A}$ in $\mathcal{A}$, then it must be reachable in $\mathcal{C}$ by a word that includes the constant transformation $(Q_\mathcal{C} \to q_\mathcal{A})$. But then q must be reachable from $q_\mathcal{A}$ in $\mathcal{A}$. Now consider two distinct states $p, q \in Q_\mathcal{B}$. Since $\mathcal{C}$ is minimal, p and q are distinguishable by some word w, and no letter of w can induce the constant transformation $(Q_\mathcal{C} \to q_\mathcal{A})$. Hence every letter of w induces a transformation that acts on $Q_\mathcal{B}$ either as the identity or as some $t' \in \delta_\mathcal{B}$. If we omit the letters that act as the identity, we obtain a word w' that distinguishes p and q in $\mathcal{B}$.

Conversely, distinct states $p \in Q_\mathcal{A}$, $q \in Q_\mathcal{C} \setminus F_\mathcal{B}$ are distinguishable as follows. Apply a unitary transformation t that takes p to a state in $F_\mathcal{B}$. Since q is not changed by t, p and q are distinguishable. If $p \in Q_\mathcal{A}$ and $q \in F_\mathcal{B}$ then p and q are already distinguished (by the empty word). If $p \in Q_\mathcal{B}$ and $q \in Q_\mathcal{B}$ then they are distinguishable by assumption. Every state of $\mathcal{A}$ is reachable from $q_\mathcal{A}$ in $\mathcal{A}$ by assumption. Also, any state in $q \in Q_\mathcal{B}$ is reachable from $q_\mathcal{A}$ by a unitary transformation. Hence all the states of $\mathcal{C}$ are reachable, and $\mathcal{C}$ is minimal. □

Lemma 18. *If $\mathcal{A}$ and $\mathcal{B}$ are aperiodic, then their semiconstant sum $(\mathcal{A}, \mathcal{B})$ is also aperiodic.*

Proof. Suppose that $\langle(\mathcal{A}, \mathcal{B})\rangle$ contains a cycle t. This cycle cannot include both a state from $\mathcal{A}$ and a state from $\mathcal{B}$, since the only way to map a state from $\mathcal{B}$ to a state from $\mathcal{A}$ in $(\mathcal{A}, \mathcal{B})$ is by a constant transformation, and a constant transformation cannot be used as a generator of a cycle. Hence all the cyclic states must be either in $Q_\mathcal{A}$ or $Q_\mathcal{B}$, which contradicts the assumption that $\mathcal{A}$ and $\mathcal{B}$ are aperiodic. □

An DFA is *complete* if it is aperiodic and adding any transition to it destroys aperiodicity. Note that the transition semigroup of a complete DFA contains all constant transformations.

Lemma 19. *If $\mathcal{A}$ and $\mathcal{B}$ are complete, their semiconstant sum $(\mathcal{A}, \mathcal{B})$ is also complete.*

Proof. We know from Lemma 18 that $(\mathcal{A}, \mathcal{B})$ is aperiodic. Suppose that a new transformation t can be added to $(\mathcal{A}, \mathcal{B})$ in such away that the resulting DFA remains aperiodic. We consider the following cases depending on the image $Q_\mathcal{B}t$.

If $|Q_\mathcal{B}t \cap Q_\mathcal{A}| = 0$ then $t = t_\mathcal{B} \circ u \circ t_\mathcal{A}$, where $t_\mathcal{A}$ and $t_\mathcal{B}$ are transformations of the DFA $(\mathcal{A}, \mathcal{B})$ changing only the states of $Q_\mathcal{A}$ and $Q_\mathcal{B}$, respectively, and acting as the identity on all other states, and u only maps some of the states of $Q_\mathcal{A}$ to $Q_\mathcal{B}$ and acts as the identity elsewhere. If t is new, then one of $t_\mathcal{B}$, or $t_\mathcal{A}$ or u is new. But we know that no new transition can be added to $\mathcal{A}$ or $\mathcal{B}$, and we have all possible transitions of type u.

If $|Q_\mathcal{B}t \cap Q_\mathcal{A}| > 0$ and $|Q_\mathcal{C}t| = 1$, then t is a constant transformation that we have already, because we have $(Q_\mathcal{B} \to q_\mathcal{A}), q_\mathcal{A} \in Q_\mathcal{A}$ from the construction of semiconstant sum, and each constant transformation on $Q_\mathcal{A}$, since $\mathcal{A}$ is complete.

If $|Q_\mathcal{B}t \cap Q_\mathcal{A}| > 0$ and $|Q_\mathcal{C}t| > 1$, then let $q_1 \in Q_\mathcal{B}$ be some state such that $q_1t = p_1 \in Q_\mathcal{A}$, and let $q_2 \in Q_\mathcal{C}$ be some state such that $q_2t = p_2$, where $q_1 \neq q_2$ and $p_2 \neq p_1$.

If $p_2 \in Q_\mathcal{B}$, let $c = (Q_\mathcal{B} \to q_1)$; since we cannot add any transformation to $\mathcal{B}$ and c is constant in $\mathcal{B}$, it must be present. Otherwise, let $c = (p_2 \to q_1)$. Note that c does not affect p_1. Similarly, if $q_2 \in Q_\mathcal{A}$, let $d = (Q_\mathcal{A} \to q_2)$; otherwise, let $d = (p_1 \to q_2)$. Note that d does not affect q_1. Then the transformation $t' = t \circ c \circ d$ is such that $q_1t' = q_2$ and $q_2t' = q_1$ and the DFA cannot be aperiodic. □

Corollary 20. *All semiconstant tree DFAs of the form $\mathcal{D}_{scti}(\Delta_Q(n_1, \dots, n_m))$ are complete.*

Proof. We use induction on m. For $m = 1$ we have $\mathcal{D}_{scti}(\Delta_Q(n_1)) = \mathcal{D}_{ui}(\Delta_Q(n_1))$, which is a bipath whose transformations preserve some order $\preceq$. Suppose that we add a new transformation t. Because the semigroup of $\mathcal{D}_{ui}(\Delta_Q(n_1))$ is maximal monotonic by Corollary 6, t must violate the order $\preceq$. Hence we have two states $p \prec q$ such that $pt \succ qt$. Thus there is an order-preserving transformation s that maps pt to q and qt to p. Now transformation ts contains the cycle (p,q). This contradicts the aperiodicity of the semigroup resulting from the addition of t.

Now suppose that $m > 1$; then $\mathcal{D}_{scti}(\Delta_Q(n_1,\dots,n_m))$ is the semiconstant sum of $\mathcal{D}_{scti}(\Delta_{Q_{\mathit{left}}}(n_1,\dots,n_r))$ and $\mathcal{D}_{scti}(\Delta_{Q_{\mathit{right}}}(n_{r+1},\dots,n_m))$. By the inductive assumption and Lemma 18 $\mathcal{D}_{scti}(\Delta_Q(n_1,\dots,n_m))$ is also aperiodic; by Lemma 19 we know that no transitions can be added. ☐

In order to count the size of the semigroup of a semiconstant sum, we extend the concept of partial transformations to k-partial transformations.

Definition 21. *A k-*partial transformation *of Q is a transformation of Q into $Q \cup \{\square_1,\square_2,\dots,\square_k\}$, where $\square_1,\square_2,\dots,\square_k$ are pairwise distinct, and distinct from all $q \in Q$.*

Each of the k undefined values corresponds to a different state outside Q. In a semiconstant sum $\mathcal{C} = (\mathcal{A},\mathcal{B})$, where $\mathcal{A} = (Q_\mathcal{A},\Sigma_\mathcal{A},\delta_\mathcal{A},q_\mathcal{A},F_\mathcal{A})$ and $\mathcal{B} = (Q_\mathcal{B},\Sigma_\mathcal{B},\delta_\mathcal{B},q_\mathcal{B},F_\mathcal{B})$, a $|Q_\mathcal{B}|$-partial transformation of $Q_\mathcal{A}$ corresponds to one of the transformations of $\mathcal{C}$ that is obtained from a transformation of $\mathcal{A}$. Thus, the number of the resulting transformations of $\mathcal{C}$ depends on the number of $|Q_\mathcal{B}|$-partial transformations induced by the transformations of $\mathcal{A}$.

We say that a k-partial transformation t is *consistent* for $\mathcal{A}$ if there exists t' in the transition semigroup of $\mathcal{A}$ such that if $qt \in Q$, then $qt = qt'$ for all $q \in Q$. The set of consistent k-partial transformations of a semigroup describes its potential for forming a large number of transformations when used in a semiconstant sum as $\mathcal{A}$. For a fixed $n \geqslant 6$, there exist aperiodic semigroups with smaller cardinalities than the maximal ones, but with larger numbers of consistent k-partial transformations for some k. So they result in a larger semiconstant sum than that composed from the maximal ones.

With the transition semigroup of $\mathcal{A}$ we associate a function $f_\mathcal{A} \colon \mathbb{N} \to \mathbb{N}$ counting all consistent k-partial transformations for a given k. This function is a polynomial in k of degree $|Q_\mathcal{A}|$. For example, for $k = 1$, $f_\mathcal{A}$ is the number of all consistent partial transformations for $\mathcal{A}$. For a DFA $\mathcal{A} = \mathcal{D}_{ui}(n_1,\dots,n_m)$, $f_\mathcal{A}(1)$ is the size of the transition semigroup of $\mathcal{D}_{ui}(n_1,\dots,n_m,1)$. From the proof of Theorem 13 we know that the number of consistent k-partial transformations for a bipath of size n having an identity transformation is

$$m_{bi}(n,k) = \binom{2n-1}{n} + \sum_{h=0}^{n-1} k^{n-h}\binom{n}{h}\binom{n+h-1}{h}.$$

Theorem 22. *Let $\mathcal{A} = (Q_\mathcal{A},\Sigma_\mathcal{A},\delta_\mathcal{A},q_\mathcal{A},F_\mathcal{A})$ and $\mathcal{B} = (Q_\mathcal{B},\Sigma_\mathcal{B},\delta_\mathcal{B},q_\mathcal{B},F_\mathcal{B})$ be strongly connected DFAs with n and m states, respectively. Let $f_\mathcal{A}(k)$ and $f_\mathcal{B}(k)$*

be the functions counting their consistent k-partial transformations. Then the function $f_{\mathcal{C}}$ counting the consistent k-partial transformations of the semiconstant sum $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ is

$$f_{\mathcal{C}}(k) = f_{\mathcal{A}}(m+k) f_{\mathcal{B}}(k) + n(k+1)^n((k+1)^m - k^m).$$

Proof. Suppose $|Q_{\mathcal{B}}t \cap Q_{\mathcal{A}}| = 0$. Let t be a k-partial transformation of $\mathcal{C}$. Then t can be uniquely represented as $t_{\mathcal{B}} \cup t_{\mathcal{A}}$, where is a k-partial transformation of $\mathcal{B}$, and $t_{\mathcal{A}}$ is a transformation from $Q_{\mathcal{A}}$ to $Q_{\mathcal{A}} \cup Q_{\mathcal{B}} \cup \{\square_1, \dots, \square_k\}$. We have $f_{\mathcal{B}}(k)$ possible $t_{\mathcal{B}}$ transformations, and $f_{\mathcal{A}}(m+k)$ possible $t_{\mathcal{A}}$ transformations, since each $t_{\mathcal{A}}$ corresponds to an $(m+k)$-partial transformation of $\mathcal{A}$. So we have $f_{\mathcal{A}}(m+k)f_{\mathcal{B}}(k)$ different pairs of transformations $t_{\mathcal{A}}$ and $t_{\mathcal{B}}$, which yield different k-partial transformations t in this case.

If $|Q_{\mathcal{B}}t \cap Q_{\mathcal{A}}| \geqslant 1$, then the constant generator $c = (Q_{\mathcal{B}} \to q_{\mathcal{A}}), q_{\mathcal{A}} \in Q_A$ must be used, since it is the only generator mapping a state from $Q_{\mathcal{B}}$ to a state from $Q_{\mathcal{A}}$. So the case $|Q_{\mathcal{B}}t \cap Q_{\mathcal{A}}| > 1$ is not possible. Therefore $|Q_{\mathcal{B}}t \cap Q_{\mathcal{A}}| = 1$, and we denote the single element of $Q_{\mathcal{B}}t \cap Q_{\mathcal{A}}$ by p. For each state $q \in Q_{\mathcal{A}}$ either qt is one of the k undefined values or $qt = p$. This yields $k+1$ possible values for a given p, and $(k+1)^n$ possibilities in total. Also, for each state $q \in Q_{\mathcal{B}}$ either qt is one of the k undefined values or $qt = p$. However, the latter case must occur for at least one $q \in Q_{\mathcal{B}}$. This yields $(k+1)^m - k^m$ possibilities in total. Because $\mathcal{A}$ is strongly connected, we have n possibilities for the selection of p. This yields $n(k+1)^n((k+1)^m - k^m)$ different k-partial transformations in this case.

Altogether, we have $f_{\mathcal{A}}(m+k)f_{\mathcal{B}}(k) + n(k+1)^n((k+1)^m - k^m)$. □

Corollary 23. *The number of k-partial transformations of $\mathcal{D}_{scti}(\Delta_Q(n_1, \dots, n_m))$ of size n is:*

$$f_{\mathcal{D}}(k) = \begin{cases} m_{bi}(n,k), & \text{if } m = 1; \\ f_{\mathcal{D}_{left}}(r+k) f_{\mathcal{D}_{right}}(k) + \ell(k+1)^{\ell}((k+1)^r - k^r), & \text{if } m > 1, \end{cases}$$

where $\mathcal{D}_{left}$ is the DFA defined by $\Delta_{Q_{left}}(n_1, \dots, n_i)$, the left subtree of the tree $\Delta_Q(n_1, \dots, n_m)$, $\mathcal{D}_{right}$ is defined by $\Delta_{Q_{right}}(n_{i+1}, \dots, n_m)$, the right subtree of $\Delta_Q(n_1, \dots, n_m)$, and ℓ, r are the numbers of states in $\mathcal{D}_{left}$ and $\mathcal{D}_{right}$, respectively.

Proof. This follows from Theorems 13 and 22. □

Corollary 24. *Let $m_{scti}(n,k)$ be the maximal number of k-partial transformations of a semiconstant DFA $\mathcal{D}_{scti}(\Delta_Q(n_1, \dots, n_m))$ with n states. Then*

$$m_{scti}(n,k) = \max \begin{cases} m_{bi}(n,k) \\ \max\limits_{s=1,\dots,n-1} \left\{ \begin{array}{l} m_{scti}(n-s, s+k) m_{scti}(s,k) \\ \quad + (n-s)(k+1)^{n-s}((k+1)^s - k^s) \end{array} \right\} \end{cases}. \quad (5)$$

Proof. A semiconstant tree DFA is either a bipath, or a semiconstant sum of two smaller semiconstant tree DFAs. Since its number of transformations depends only on the numbers of k-partial transformations of the smaller ones, we can use the maximal ones, and select the best split for the sum. □

The size of the semigroup of DFA $\mathcal{D}_{scti}(\Delta_Q(n_1,\dots,n_m))$ is $f_{\mathcal{D}}(0)$. The maximal size of semigroups of the DFAs $\mathcal{D}_{scti}$ with n states is $m_{scti}(n,0)$.

Instead of a bipath and the value $m_{bi}(n,k)$ we could use any strongly connected automaton with an aperiodic semigroup. If such a semigroup would have a larger number of k-partial transformations than our semiconstant tree DFAs for some k, then we could obtain even larger aperiodic semigroups.

The corollary results directly in a dynamic algorithm working in $O(n^3)$ time (assuming constant time for arithmetic operations and computing binomials) for computing $m_{scti}(n,0)$, and the distribution with the full binary tree yielding the maximal semiconstant tree semigroup.

We computed the maximal semiconstant tree semigroups up to $n = 500$. For $n = 100$ the syntactic semigroup of the DFA below exceeds 3.3×10^{160}.

$$\begin{aligned}\mathcal{D}_{scti} = (& (((((((2,2),(2,2)),((2,2),(2,2))),(((2,2),(2,2)),((2,2),3))),\\ & ((((2,2),3),(3,3)),((3,3),(3,3))))),((((3,2),(3,2)),((3,2),(2,2))),\\ & ((2,2),(2,2))))),(((3,3),(3,2)),((2,2),2))),\end{aligned}$$

6. Conclusions

We have found two new types of aperiodic semigroups. Maximal semiconstant semigroups of type $\mathcal{D}_{scti}(\Delta_Q(n_1,\dots,n_m))$ are currently the largest aperiodic semigroups known. A tight upper bound on the size of aperiodic semigroups remains unknown.

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References

[1] J. Brzozowski, Quotient complexity of regular languages, *J. Autom. Lang. Comb.* **15**(1/2) (2010) 71–89.
[2] J. Brzozowski, In search of the most complex regular languages, *Internat. J. Found. Comput. Sci.* **24**(6) (2013) 691–708.
[3] J. Brzozowski and F. E. Fich, Languages of $\mathcal{R}$-trivial monoids, *J. Comput. System Sci.* **20**(1) (1980) 32–49.
[4] J. Brzozowski and B. Li, Syntactic complexity of $\mathcal{R}$- and $\mathcal{J}$-trivial languages, *Internat. J. Found. Comput. Sci.* To appear.
[5] J. Brzozowski, B. Li and D. Liu, Syntactic complexities of six classes of star-free languages, *J. Autom. Lang. Comb.* **17**(2–4) (2012) 83–105.

[6] J. Brzozowski and B. Liu, Quotient complexity of star-free languages, *Internat. J. Found. Comput. Sci.* **23**(6) (2012) 1261–1276.

[7] J. Brzozowski and M. Szykuła, Large aperiodic semigroups, `http://arxiv.org/abs/1401.0157` (2014).

[8] J. Brzozowski and M. Szykuła, Large aperiodic semigroups, *CIAA 2014*, eds. M. Holzer and M. Kutrib *LNCS* **8587**, (Springer, 2014), pp. 124–135.

[9] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics,, Vol. 173, fourth edn. (Springer-Verlag, Heldelberg, 2010). http://diestel-graph-theory.com.

[10] A. Ginzburg, Abour some properties of definite, reverse definite and related automata, *IEEE Trans. Electronic Comput.* **EC–15** (1966) 806–810.

[11] G. Gomes and J. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum* **45** (1992) 272–282.

[12] J. M. Howie, Products of idempotents in certain semigroups of transformations, *Proc. Edinburgh Math. Soc.* **17**(2) (1971) 223–236.

[13] S. Iván and J. Nagy-György, On nonpermutational transformation semigroups with an application to syntactic complexity, `http://arxiv.org/abs/1402.7289` (2014).

[14] A. Kisielewicz and M. Szykuła, Generating small automata and the Černý conjecture, *CIAA 2013*, ed. S. Konstantinidis *LNCS* **7982**, (Springer, 2013), pp. 340–348.

[15] R. McNaughton and S. A. Papert, *Counter-Free Automata*, MIT Research Monographs,, Vol. 65 (The MIT Press, 1971).

[16] J.-E. Pin, Syntactic semigroups, *Handbook of Formal Languages, vol. 1: Word, Language, Grammar*, (Springer, New York, NY, USA, 1997), pp. 679–746.

[17] M. Schützenberger, On finite monoids having only trivial subgroups, *Inform. and Control* **8** (1965) 190–194.

[18] S. Yu, State complexity of regular languages, *J. Autom. Lang. Comb.* **6** (2001) 221–234.