On The Circuit Diameters of Some Combinatorial Polytopes

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contribuitions

The work presented here was done in collaboration with my advisor Laura Sanità and Kanstantsin Pashkovich. I was a major contributor to all the results contained in the chapters.

Abstract

The *combinatorial diameter* of a polytope P is the maximum value of a shortest path between two vertices of P, where the path uses the edges of P only. In contrast to the combinatorial diameter, the *circuit diameter* of P is defined as the maximum value of a shortest path between two vertices of P, where the path uses *potential* edge directions of P i.e., all edge directions that can arise by translating some of the facets of P.

In this thesis, we study the circuit diameter of polytopes corresponding to classical combinatorial optimization problems, such as the Matching polytope, the Traveling Salesman polytope and the Fractional Stable Set polytope. We also introduce the notion of the circuit diameter of a *formulation* of a polytope P. In this setting the circuits are determined from some external linear system describing P which may not be minimal with respect to its constraints. We use this notion to generalize other results of this thesis, as well as introduce new results about a formulation of the Spanning Tree polytope and a formulation of the Matroid polytope.

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Chapter 1

Introduction

For a polytope $P \subseteq \mathbb{R}^d$, the *1-skeleton* of P is the graph given by the set of vertices (0-dimensional faces) of P, and the set of edges (1-dimensional faces) of P. The *combinatorial diameter* of P is the maximum shortest path distance between two vertices in this graph. Giving bounds on the combinatorial diameter of polytopes is a central question in discrete mathematics and computational geometry. Combinatorial diameter is fundamental to the theory of linear programming due to the long standing open question about existence of a pivoting rule that yields a polynomial runtime for the Simplex algorithm. Indeed, existence of such a pivoting rule requires a general polynomial bound on the combinatorial diameter of a polytope.

The most famous conjecture in this context is the *Hirsch Conjecture*, proposed in 1957, which states that the combinatorial diameter of any d-dimensional polytope with f facets is at most $f - d$. While this conjecture has been disproved [20] [26], its *polynomial* version is still open i.e., it is not known whether there is some polynomial function of f and d which upperbounds the combinatorial diameter in general. Currently the best known upperbound on the diameter is exponential in d [28].

Recently researchers started investigating whether the bound $f - d$ is a valid upper-bound for some different (more powerful) notions of diameter for polytopes. The present work is concerned with one such notion of diameter: the *circuit diameter* of a polytope, formalized by Borgwardt et al. [5]. Given a polytope of the form $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}, B\boldsymbol{x} \leq \boldsymbol{d}\}$ for some rational matrices A and B and rational vectors $\boldsymbol{b}$ and $\boldsymbol{d}$, the *circuits* of P are the set of *potential* edge directions that can arise by varying $\boldsymbol{b}$ and $\boldsymbol{d}$ (see Chapter 2 for a formal definition). Starting from a point in P one is allowed to move along any circuit direction until the boundary of P is reached (see Chapter 2 for a formal definition). Since for every polytope the set of circuit directions contains all edge directions, the combinatorial diameter is always an upperbound on the circuit diameter. Thus even if the Hirsch Conjecture does not hold for the combinatorial diameter, its analogue may be true for the circuit diameter. In particular, Borgwardt et al. [5] conjectured that the circuit diameter is at most $f - d$ for every d-dimensional polytope with f facets. We refer the reader to [7] for recent progress on this conjecture.

Besides studies of upperbounds on combinatorial diameter for general polytopes, there is a long history of studies of such upperbounds for some special classes of polytopes. In particular, many researchers were working on the combinatorial diameter of polytopes corresponding to classical combinatorial optimization problems. Prominent examples of polytopes for which the combinatorial diameter has been widely studied are Transportation and Network Flow polytopes [2, 3, 4, 6, 8], Matching polytopes [3, 10], Traveling Salesman (TSP) polytopes [25, 18], and many others. In this context, there are some questions and conjectures regarding the tightness of the developed bounds which are open, and it is natural to investigate them using a more powerful notion of diameter, like the circuit diameter. In [5] upper bounds on the circuit diameter of Dual Transportation polytopes on bipartite graphs were given, while [4] gave upper bounds on the circuit diameter of Dual Network flow polytopes.

Our results. In this thesis, we study the circuit diameter of the Matching polytope, the Perfect Matching polytope, the TSP polytope, and the Fractional Stable Set polytope.

Our first result is an exact characterization of the circuit diameter of the Matching polytope (resp., Perfect Matching polytope), which is the convex hull of characteristic vectors of matchings (resp., perfect matchings) in a complete graph with n nodes. In particular, it is well-known that the combinatorial diameter of the Matching polytope equals $\lfloor \frac{n}{2} \rfloor$ [3, 10]. In Section 3.1, we show that the circuit diameter of the Matching polytope is upper bounded by a constant in contrast to the combinatorial diameter. In particular, we show that the circuit diameter of the Matching polytope equals 2 for all $n \geq 7$. To this aim, we show that for any two different matchings such that one is not contained in the other, the corresponding two vertices are either one circuit step away from each other or they have a common neighbor in the polytope. Therefore their circuit distance is always at most 2. In Section 3.2 we show that for the Perfect Matching polytope, for $n \neq 8$, the circuit diameter is 1, and if $n = 8$ the circuit diameter is 2. In contrast, the combinatorial diameter of the Perfect Matching polytope is known to be 2 for all $n \geq 8$ [23].

In Chapter 4, we give an exact characterization of the circuit diameter of the TSP polytope, which is the convex hull of all characteristic vectors of tours (i.e., Hamiltonian cycles) in a complete graph with n nodes. It is known that the combinatorial diameter of the TSP polytope is at most 4 [25]. In fact, Grötschel and Padberg conjectured in [18] that the combinatorial diameter of the TSP polytope is at most 2, and this conjecture is still open after more than 30 years. In Chapter 4, we show that this conjecture holds for the circuit diameter. In fact, the circuit diameter of the TSP polytope equals 1 whenever $n \neq 5$, while for $n = 5$ the circuit diameter is 2. This result is proven by showing that for every two tours in a complete graph, the corresponding vertices are one circuit step from each other whenever $n > 5$. Note that no linear description of the TSP polytope is known for general graphs. We achieve the above results for the TSP polytope by using only two famous classes of its facets: namely, *subtour* inequalities and (certain) *comb* inequalities [19].

In Chapter 5 we consider the Fractional Stable Set polytope. This is the polytope given by the standard LP relaxation of the stable set problem for a graph G with n nodes. The Fractional Stable Set polytope has been widely studied. In particular, it is known that this polytope is half-integral [1], and that the vertices of this polytope have a nice graph interpretation: namely, they can be mapped to subgraphs of G with all connected components being *trees* and 1-*trees*[1] [9, 11]. This graphical interpretation of vertices was used in [21] to prove that the combinatorial diameter of the Fractional Stable Set polytope is upper bounded by n. In Chapter 5, we provide a characterization for circuits of this polytope. Specifically, we show that every circuit corresponds to a connected (not necessarily induced) bipartite subgraph of G. Our characterization allows us to show that the circuit diameter of the Fractional Stable Set polytope can be essentially upper bounded by the *diameter* of the graph G, which is significantly smaller than n in many graphs.

Finally, in Chapter 6 we introduce the notion of circuits of a *formulation* of a polytope P. The combinatorial diameter is studied in part for its relevance to augmentation algorithms (like the Simplex method). Whereas the basis for the Simplex method is the edges of a polytope, there is potential for the circuits of a polytope to provide the basis for an alternative type of augmentation algorithm for solving optimization problems. We will discuss in Chapter 2 that to study the circuit diameter of a polytope *itself* requires a description of that polytope which is minimal with respect to its constraints. However, in a computational setting, polytopes corresponding to combinatorial optimization problems are usually modeled by well-known linear systems which are not necessarily minimal with respect to their constraints for general problem inputs. Thus, in the context of augmentation algorithms over polytopes modeling combinatorial optimization problems, it makes

[1] A 1-tree is a tree plus one edge between two nodes spanned by the tree.

sense to consider utilizing circuits derived from possibly redundant constraints if those constraints are present regardless.

In Section 6.1, we use this notion of formulations to extend our results of Chapter 3 to general graphs. In Section 6.2 we provide a bound on the circuit diameter of a classical formulation of the Matroid polytope, defined as the convex hull of all characteristic vectors of independent sets of a matroid. In particular, we show that the circuit diameter of the given formulation is at most 3. This is proven by showing that for the given formulation, all vectors with components in $\{1, 0, -1\}$ are circuits except for non-negative (resp. non-positive) vectors with more than one non-zero component. In Section 6.3 we provide a bound on the circuit diameter of a classical formulation of the Spanning Tree polytope, defined as the convex hull of all characteristic vectors of spanning trees of a graph. We show that for a graph G the circuit diameter of the given formulation is $\mathcal{O}(\Delta(G))$ where $\Delta(G)$ denotes the maximum degree of a vertex in G. To do this, we give a sufficient condition for the vertices corresponding to two spanning trees to be one circuit step away from each other.

Chapter 2

Preliminaries

Let P be a polytope of the form $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b},\, B\boldsymbol{x} \leq \boldsymbol{d}\}$ for rational matrices A and B and rational vectors $\boldsymbol{b}$ and $\boldsymbol{d}$.

Definition 1. *A non-zero vector $\boldsymbol{g} \in \mathbb{R}^n$ (whose components are normalized to be coprime integers) is a circuit of P if*

(i) $\boldsymbol{g} \in \mathrm{Ker}(A)$

(ii) $\mathrm{supp}(B\boldsymbol{g})$ *is not contained in any of the sets from the collection* $\{\mathrm{supp}(B\boldsymbol{y}) : \boldsymbol{y} \in \mathrm{Ker}(A), \boldsymbol{y} \neq \boldsymbol{0}\}$ *(i.e., $B\boldsymbol{g}$ is support-minimal).*

Here, $\mathrm{Ker}(A)$ denotes the kernel of A i.e., $\mathrm{Ker}(A) := \{\boldsymbol{y} \in \mathbb{R}^n : Ay = \boldsymbol{0}\}$. Furthermore, we denote by $\mathrm{supp}(\boldsymbol{x})$ the support of a vector $\boldsymbol{x}$.

In [5] and elsewhere, the definition of circuits requires that the components of a circuit be normalized to coprime integer components, and we have included that condition in the definition above. This is done to guarantee that for any given polytope, the set of circuits associated with that polytope is finite. Without this condition, for any circuit $\boldsymbol{g}$ of a polytope P, we have that $\alpha\boldsymbol{g}$ is a circuit of P for all real numbers $\alpha \neq 0$. With this condition, we have that for any circuit $\boldsymbol{g}$ of a polytope P, the only circuits parallel to $\boldsymbol{g}$ are $\boldsymbol{g}$ itself and $-\boldsymbol{g}$. That said, for convenience we may often informally refer to non-zero scalings of circuits as circuits themselves, even if the components are not coprime integers.

Given the notion of circuits, we can formally define *circuit steps*, *circuit walks*, and *circuit distance*.

Definition 2. *Given $\boldsymbol{x}' \in P$, we say that $\boldsymbol{x}'' \in P$ is one circuit step from $\boldsymbol{x}'$, if $\boldsymbol{x}'' = \boldsymbol{x}' + \alpha \boldsymbol{c}$ where $\boldsymbol{c}$ is a circuit of P and $\alpha > 0$ is chosen to be as large as possible so that $\boldsymbol{x}' + \alpha \boldsymbol{c} \in P$.*

Note that this definition does not specify that $\boldsymbol{x}'$ or $\boldsymbol{x}''$ are vertices of P.

Definition 3. *Given two points $\boldsymbol{x}'$ and $\boldsymbol{x}''$ in P, a circuit walk from $\boldsymbol{x}'$ to $\boldsymbol{x}''$ is a sequence of points in P, $\boldsymbol{x}' = \boldsymbol{z}^0, \boldsymbol{z}^1, \cdots, \boldsymbol{z}^{l-1}, \boldsymbol{z}^l = \boldsymbol{x}''$, where $\boldsymbol{z}^i$ is one circuit step from $\boldsymbol{z}^{i-1}$, for all $i = 1, \cdots, l$. We say such a circuit walk has length l.*

Definition 4. *Given two points $\boldsymbol{x}'$ and $\boldsymbol{x}''$ in P, the circuit distance from $\boldsymbol{x}'$ to $\boldsymbol{x}''$, called $\mathrm{cdist}(\boldsymbol{x}', \boldsymbol{x}'')$, is the length of a shortest circuit walk from $\boldsymbol{x}'$ to $\boldsymbol{x}''$.*

Note that from the latter two definitions, it follows that a circuit walk from $\boldsymbol{x}'$ to $\boldsymbol{x}''$ might not always be reversible. For example, let two points $\boldsymbol{x}'$ and $\boldsymbol{x}''$ be such that $\boldsymbol{x}''$ is one circuit step from $\boldsymbol{x}'$ i.e., we have that $\boldsymbol{x}'' = \boldsymbol{x}' + \alpha \boldsymbol{c}$ and $\alpha > 0$ is as large as possible so that $\boldsymbol{x}' + \alpha \boldsymbol{c} \in P$. However, it may be the case that $\boldsymbol{x}'' + \alpha'(-\boldsymbol{c}) \in P$ for some α' such that $\alpha' > \alpha$; and so $\boldsymbol{x}'$ is not one circuit step from $\boldsymbol{x}''$. Therefore, it may be the case that $\mathrm{cdist}(\boldsymbol{x}', \boldsymbol{x}'') \neq \mathrm{cdist}(\boldsymbol{x}'', \boldsymbol{x}')$. We refer to [16] for an extensive discussion about circuit distance.

Definition 5. *Given a polytope P, the circuit diameter of P, or $\mathcal{CD}(P)$, is the maximum circuit distance between any pair of vertices of P.*

When talking about the circuit diameter of a polytope P, unless specified we assume that the system of inequalities describing P is *minimal* with respect to its constraints i.e., each inequality of the above system defines a facet of P. Note that in contrast to the combinatorial diameter, the circuit diameter depends on the linear description of a polytope. In fact, redundant inequalities might become facet-defining after translating the corresponding hyperplanes.

Given a system of linear equations $\{A\boldsymbol{x} = \boldsymbol{0}\,, B\boldsymbol{x} = \boldsymbol{0}\}$, we say that a vector $\boldsymbol{c}$ is a unique (up to scaling) solution of the system, if every vector $\boldsymbol{y}$ satisfying $A\boldsymbol{y} = \boldsymbol{0}\,, B\boldsymbol{y} = \boldsymbol{0}$ is of the form $\boldsymbol{y} = \lambda \boldsymbol{c}$ for some $\lambda \in \mathbb{R}$. The following proposition gives an alternative definition of circuits, that will be useful later. It is an easy corollary of the results in [16], we report a proof here for completeness.

Proposition 1. *Given a polytope $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}\,, B\boldsymbol{x} \leq \boldsymbol{d}\}$, a non-zero vector $\boldsymbol{c} \in \mathbb{R}^n$ is a circuit if and only if $\boldsymbol{c}$ is a unique (up to scaling) non-zero solution of $\{A\boldsymbol{y} = \boldsymbol{0}\,, B'\boldsymbol{y} = \boldsymbol{0}\}$ where B' is a submatrix of B.*

Proof. Let us be given a non-zero vector $\boldsymbol{c}$ such that $A\boldsymbol{c} = 0$. Let B' be the maximal (with respect to the number of rows) submatrix of B such that $B'\boldsymbol{c} = \boldsymbol{0}$. Since P is a polytope the block matrix

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

has full column rank. Hence, there exists no non-zero vector $\boldsymbol{d}$ such that $A\boldsymbol{d} = \boldsymbol{0}$ and $\operatorname{supp}(B\boldsymbol{d}) \subset \operatorname{supp}(B\boldsymbol{c})$ only if there is a unique (up to scaling) non-zero solution of $\{A\boldsymbol{y} = \boldsymbol{0}\,, B'\boldsymbol{y} = \boldsymbol{0}\}$.

Now, let B' be a submatrix of B such that the system $A\boldsymbol{y} = \boldsymbol{0}\,, B'\boldsymbol{y} = \boldsymbol{0}$ has a unique (up to scaling) non-zero solution $\boldsymbol{c}$. Suppose for the sake of contradiction that $\boldsymbol{c}$ is not a circuit of P. Then there exists a non-zero vector $\boldsymbol{d}$ such that $A\boldsymbol{d} = \boldsymbol{0}$ and $\operatorname{supp}(B\boldsymbol{d}) \subset \operatorname{supp}(B\boldsymbol{c})$. In particular, this means that $A\boldsymbol{d} = \boldsymbol{0}\,, B'\boldsymbol{d} = \boldsymbol{0}$. Hence $\boldsymbol{d}$ is a scaling of $\boldsymbol{c}$; and thus $\boldsymbol{c}$ is a circuit as desired. □

The next lemma will be used in Chapter 3 to study the circuit diameter of polytopes with linear descriptions where the coefficients in each inequality are all non-negative or all non-positive.

Lemma 1. *Let $Q \subseteq \mathbb{R}^n$ be a polytope of the form $Q := \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b},\, B\boldsymbol{x} \leq \boldsymbol{d}\}$, where all entries of A are non-negative and all entries of B are non-positive. Then every circuit $\boldsymbol{c} \in \mathbb{R}^n$ of Q with $\boldsymbol{c} \geq \boldsymbol{0}$ or $\boldsymbol{c} \leq \boldsymbol{0}$ has exactly one non-zero coordinate.*

Proof. Suppose that $\boldsymbol{c}$ is a circuit of Q which has at least two non-zero coordinates. We may assume that $\boldsymbol{c} \geq \boldsymbol{0}$, as the case where $\boldsymbol{c} \leq \boldsymbol{0}$ is identical. Then by Proposition 1, $\boldsymbol{c}$ is the unique (up to scaling) non-zero solution of $A'\boldsymbol{y} = \boldsymbol{0}$, $B'\boldsymbol{y} = \boldsymbol{0}$ where A', B' are some submatrices of A, B respectively. Note that since all entries of A' and $\boldsymbol{c}$ are non-negative and $A'\boldsymbol{c} = \boldsymbol{0}$, we have that for every $i \in \operatorname{supp}(\boldsymbol{c})$ the i-th column of A' equals $\boldsymbol{0}$. Analogously, for every $i \in \operatorname{supp}(\boldsymbol{c})$ the i-th column of B' equals $\boldsymbol{0}$.

Let i be any index such that $\boldsymbol{c}_i > 0$. Define the vector $\boldsymbol{d}$ as

$$\boldsymbol{d}_j := \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

Then $\boldsymbol{d}$ is also a solution to $A'\boldsymbol{y} = \boldsymbol{0}$, $B'\boldsymbol{y} = \boldsymbol{0}$ and is not a scaling of $\boldsymbol{c}$, contradicting that $\boldsymbol{c}$ is a circuit. □

Chapter 3

Matching and Perfect Matching Polytopes

3.1 Matching Polytope

The Matching polytope is defined as the convex hull of all characteristic vectors of matchings in a complete graph i.e.,

$$P_{\text{MATCH}}(n) := \text{conv}\left\{\chi(M)\,:\, M \text{ is a matching in } K_n\right\}\,,$$

where $K_n = (V, E)$ denotes a complete graph with n nodes, and $\chi(M) \in \{0,1\}^E$ denotes the characteristic vector of a matching M.

The linear description of the Matching polytope is well-known and is due to Edmonds [12]. In particular, the following linear system constitutes a minimal linear description of $P_{\text{MATCH}}(n)$

$$\begin{aligned} &\boldsymbol{x}\left(E[S]\right) \leq (|S|-1)/2 && \text{for all } S \subset V,\ |S| \text{ is odd},\ |S| \geq 3 \\ &\boldsymbol{x}(\delta(v)) \leq 1 && \text{for all } v \in V \\ &\boldsymbol{x} \geq \mathbf{0}\,, \end{aligned} \tag{3.1}$$

where $E[S]$ denotes the set of edges with both endpoints in S, $\delta(v)$ denotes the set of edges with one endpoint being v, and $\boldsymbol{x}(F)$ denotes the sum $\sum_{e\in F} x_e$ for $F \subseteq E$. We note that the constraint matrix corresponding to the linear description 3.1 has all non-negative coefficients, and therefore Lemma 1 applies.

The combinatorial diameter of the Matching polytope $P_{\text{MATCH}}(n)$ equals $\lfloor n/2 \rfloor$ for all $n \geq 2$ [3, 10]. Our next Theorem provides the value of the circuit diameter of the Matching polytope $P_{\text{MATCH}}(n)$ for all possible n. In particular, it shows that the circuit diameter of the Matching polytope is substantially smaller than the combinatorial diameter.

Theorem 1. *For the Matching polytope we have:*

$$\mathcal{CD}(P_{\text{MATCH}}(n)) = \begin{cases} 1 & n = 2, 3 \\ 2 & n = 4, 5 \\ 3 & n = 6 \\ 2 & n \geq 7 \,. \end{cases}$$

The rest of the section is devoted to proving Theorem 1. We first recall the characterization of adjacency of vertices of the Matching polytope. In this thesis, we use the symbol Δ to represent the symmetric difference operator.

Lemma 2 ([3, 10])**.** *Consider matchings M_1, M_2 in K_n, $n \geq 2$. $\chi(M_1)$ and $\chi(M_2)$ are adjacent vertices of $P_{\text{MATCH}}(n)$ if and only if $(V, M_1 \triangle M_2)$ has a single non-trivial connected component*[1]*.*

The above lemma has a straightforward corollary.

Corollary 1. *Consider matchings M_1, M_2 in K_n, $n \geq 2$. If $(V, M_1 \triangle M_2)$ has a single non-trivial connected component, then $\boldsymbol{c} := \chi(M_1) - \chi(M_2)$ is a circuit of $P_{\text{MATCH}}(n)$.*

The next lemma shows that the set of circuits of the Matching polytope is much richer than the set of its edge directions. In particular, it shows that for two matchings to define a circuit their symmetric difference does not have to consist of one non-trivial component only. The circuit directions provided by this lemma will be extensively used to construct short circuit walks in the proof of Theorem 1.

Lemma 3. *Consider matchings M_1, M_2 in K_n, such that $M_1 \not\subseteq M_2$ and $M_2 \not\subseteq M_1$. If $(V, M_1 \Delta M_2)$ contains at three or more connected components, then $\boldsymbol{c} := \chi(M_1) - \chi(M_2)$ is a circuit of $P_{\text{MATCH}}(n)$.*

Before proving this lemma, we remark that the components of $M_1 \Delta M_2$ may be trivial i.e., consist of a single vertex.

[1]Trivial components are components consisting of a single node.

Proof. Suppose that $(V, M_1 \Delta M_2)$ contains at least three connected components. Let us assume for the sake of contradiction that $\boldsymbol{c} = \chi(M_1) - \chi(M_2)$ is not a circuit. Since $\boldsymbol{c}$ is not a circuit there exists a non-zero vector $\boldsymbol{y}$ such that $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, where D denotes the constraint matrix of the minimal linear description (3.1) for the Matching polytope.

Since the inequalities $\boldsymbol{x}_e \geq 0$, $e \in E$ are present in the minimal linear description (3.1) and $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, we have that $\boldsymbol{y}_e = 0$ for every edge e such that $\boldsymbol{c}_e = 0$. Let $e' = \{v_1, v_2\}$ be an edge such that $\boldsymbol{y}_{e'} \neq 0$. Without loss of generality, possibly using rescaling of the vector $\boldsymbol{y}$, we can assume $\boldsymbol{y}_{e'} = 1$. By exchanging the roles of M_1 with M_2 if necessary, we can assume that $\boldsymbol{c}_{e'} = 1$. Let C' be the connected component of $(V, M_1 \Delta M_2)$ containing the edge e'. Note that C' is either a path or a cycle. Moreover, for all nodes v with degree two in C' we have $\boldsymbol{c}(\delta(v)) = 0$. Since $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, we have that $\boldsymbol{c}(\delta(v)) = 0$ implies $\boldsymbol{y}(\delta(v)) = 0$, leading to $\boldsymbol{y}_e = \boldsymbol{c}_e$ for all $e \in C'$.

Now let $e'' = \{u_1, u_2\}$ be an edge such that $\boldsymbol{c}_{e''} = -1$. Note that such an edge e'' exists since $M_1 \not\subseteq M_2$ and $M_2 \not\subseteq M_1$. Let C'' be the connected component of $(V, M_1 \Delta M_2)$ containing the edge e''. Let us prove that $\boldsymbol{y}_e = \boldsymbol{c}_e$ for all $e \in C''$. If C' and C'' are the same connected component, then this readily follows from the previous paragraph. If not, let z be a node that belongs to a (possibly trivial) connected component $\tilde{C}$ of $(V, M_1 \Delta M_2)$ different from C' and C''. Let $S := \{z, u_1, u_2, v_1, v_2\}$ and note that $\boldsymbol{c}(E[S]) = 0$. Since $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, we get $\boldsymbol{y}(E[S]) = 0$, implying $\boldsymbol{y}_{e''} = \boldsymbol{c}_{e''} = -1$. As in the previous paragraph, C'' is either a path or a cycle, and for all $v \in V$ with degree two in C'' we have $\boldsymbol{c}(\delta(v)) = 0$. Since $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, necessarily $\boldsymbol{y}(\delta(v)) = 0$, implying $\boldsymbol{y}_e = \boldsymbol{c}_e$ for all $e \in C''$.

Now let $e''' = \{w_1, w_2\}$ be an edge not in C' and not in C'', but in the connected component C''' of $(V, M_1 \Delta M_2)$, such that $\boldsymbol{c}_{e'''} \neq 0$. If $\boldsymbol{c}_{e'''} = 1$ (resp. $\boldsymbol{c}_{e'''} = -1$), then we take the set $S := \{u_1, u_2, z, w_1, w_2\}$, where z is not in C'' and not in C''' (resp. $S := \{v_1, v_2, z, w_1, w_2\}$, where z is not in C' and not in C'''). Since $\boldsymbol{c}(E[S]) = 0$ and $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, we get that $\boldsymbol{y}(E[S]) = 0$. On the other side, $\boldsymbol{y}(E[S]) = 0$ implies $\boldsymbol{y}_{e'''} = \boldsymbol{c}_{e'''} = 1$ (resp. $\boldsymbol{y}_{e'''} = \boldsymbol{c}_{e'''} = -1$). Repeating this argument for all edges in the support of $\boldsymbol{c}$ we show that $\boldsymbol{y} = \boldsymbol{c}$, a contradiction. □

With the above lemma at hand, we are ready to prove Theorem 1.

Proof. (Proof of Theorem 1) The cases $n = 2$ and $n = 3$ are trivial. Indeed, $P_{\text{MATCH}}(2)$ and $P_{\text{MATCH}}(3)$ are simplices, and thus every two vertices of $P_{\text{MATCH}}(2)$ and $P_{\text{MATCH}}(3)$ form an edge.

For $n \geq 4$, we consider an empty matching M_1 and a matching M_2 consisting of two edges to establish

$$\mathcal{CD}(P_{\text{MATCH}}(n)) \geq 2\,.$$

Indeed, $\text{cdist}(\chi(M_1), \chi(M_2)) \geq 2$, because $\boldsymbol{c} := \chi(M_2) - \chi(M_1)$ satisfies $\boldsymbol{c} \geq 0$ and has two non-zero entries, and thus $\boldsymbol{c}$ is not a circuit by Lemma 1. Hence, the vertex $\chi(M_1)$ is not one circuit step away from the vertex $\chi(M_2)$, implying $\mathcal{CD}(P_{\text{MATCH}}(n)) \geq 2$.

For $n = 6$, the lower bound on the circuit diameter can be improved to

$$\mathcal{CD}(P_{\text{MATCH}}(6)) \geq 3\,.$$

Consider an empty matching M_1 and a perfect matching M_2. For a walk from $\chi(M_1)$ to $\chi(M_2)$ the first circuit step at the vertex $\chi(M_1) = \mathbf{0}$ corresponds to a circuit $\boldsymbol{c}$ with $\boldsymbol{c} \geq \mathbf{0}$. Thus, by Lemma 1 the first circuit step corresponds to $\boldsymbol{c}$ with exactly one non-zero coordinate. After the first circuit step we get a vertex $\chi(M')$, where M' is a matching consisting of a single edge e. Let us prove that $\boldsymbol{c}' := \chi(M_2) - \chi(M')$ is not a circuit and thus $\text{cdist}(\chi(M_1), \chi(M_2)) \geq 3$. If $e \in M_2$, the vector $\boldsymbol{c}'$ is not a circuit by Lemma 1. If $e \notin M_2$, let g be the edge in M_2 having no common vertex with the edge e. Then the vector $\boldsymbol{c}'$ is not a circuit, since the vector $D\chi(g)$ has a smaller support than $D\boldsymbol{c}'$, where D is the constraint matrix of the linear description (3.1) for $P_{\text{MATCH}}(6)$. Hence, we showed that any circuit step from $\chi(M_1)$ will always end in a vertex $\chi(M')$, which is at least two circuit steps from $\chi(M_2)$, implying $\mathcal{CD}(P_{\text{MATCH}}(6)) \geq 3$.

Now let us prove the corresponding upper bounds for $\mathcal{CD}(P_{\text{MATCH}}(n))$, $n \geq 4$. For $n = 4$, $n = 5$ and two matchings M_1 and M_2, $(V, M_1 \triangle M_2)$ has at most two non-trivial connected components. This fact together with Corollary 1 implies $\text{cdist}(M_1, M_2) \leq 2$. For $n = 6$ and two matchings M_1 and M_2, $(V, M_1 \triangle M_2)$ has at most three non-trivial connected components. Again, this fact together with Corollary 1 implies $\text{cdist}(M_1, M_2) \leq 3$.

For $n \geq 7$, consider the graph $(V, M_1 \Delta M_2)$ given by the symmetric difference of two matchings M_1 and M_2. If the symmetric difference contains one $e \in M_1$ and one $e' \in M_2$, then by Lemma 3 and Corollary 1, $\text{cdist}(M_1, M_2)$ is at most 2. Otherwise, the subset F of edges of $M_1 \Delta M_2$ satisfies either $F \subseteq M_1$ or $F \subseteq M_2$. If $|F| = 2$, the results again follows by Corollary 1. So assume $|F| \geq 3$. First, suppose $F \subseteq M_2$. Let e be any edge connecting two endpoints of two distinct edges in F, and let $\tilde{M} := M_1 \cup \{e\}$. Clearly, $\text{cdist}(M_1, \tilde{M}) = 1$. Now we claim that $\boldsymbol{c} := \chi(M_2) - \chi(\tilde{M})$ is a circuit. Indeed, $(V, \tilde{M} \Delta M_2)$ has at least three connected components: one path of length 3 and either at least two other edges, or one other edge plus at least one trivial connected component consisting of a single node (since $n \geq 7$). In both cases, Lemma 3 implies that $\boldsymbol{c} := \chi(M_2) - \chi(\tilde{M})$ is a circuit, leading to the result. Finally, suppose $F \subseteq M_1$. Similarly to the previous case, we set $\tilde{M} := M_2 \cup \{e\}$. Then, by Lemma 3 we get that $\chi(\tilde{M}) - \chi(M_1)$ is a circuit, and by Corollary 1 we get that $\chi(M_2) - \chi(\tilde{M})$ is a circuit, leading to the result. □

3.2 Perfect Matching Polytope

Let us define the Perfect Matching polytope

$$P_{\text{PERFECT MATCH}}(n) := \text{conv}\{\chi(M) \,:\, M \text{ is a perfect matching in } K_n\}\ ,$$

where $n \geq 4$ and n is even. In [12], Edmonds showed that the following linear system constitutes a minimal linear description of $P_{\text{PERFECT MATCH}}(n)$

$$\begin{aligned} &\boldsymbol{x}\,(\delta(S)) \geq 1 \quad \text{for all } S \subset V,\ |S| \text{ is odd}\,,\ |S| \geq 3 \\ &\boldsymbol{x}(\delta(v)) = 1 \quad \text{for all } v \in V \\ &\boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{3.2}$$

Theorem 2. *For the perfect matching polytope we have:*

$$\mathcal{CD}(P_{\text{PERFECT MATCH}}(n)) = \begin{cases} 1 & n = 4, 6 \\ 2 & n = 8 \\ 1 & n \geq 10\,. \end{cases}$$

The rest of this section is devoted to prove Theorem 2. First, let us recall the characterization of adjacency of the vertices of the Perfect Matching polytope.

Lemma 4 ([3, 10])**.** *Consider perfect matchings M_1, M_2 in K_n, $n \geq 2$. $\chi(M_1)$ and $\chi(M_2)$ are adjacent vertices of $P_{\text{PERFECT MATCH}}(n)$ if and only if $(V, M_1 \triangle M_2)$ has a single non-trivial connected component.*

The above lemma has a straightforward corollary.

Corollary 2. *Consider perfect matchings M_1, M_2 in K_n, $n \geq 2$. If $(V, M_1 \triangle M_2)$ has a single non-trivial connected component, then $\boldsymbol{c} := \chi(M_1) - \chi(M_2)$ is a circuit of $P_{\text{PERFECT MATCH}}(n)$.*

The next lemma shows that every pair of distinct matchings define a circuit whenever $n \geq 10$. The circuit directions provided by this lemma will be extensively used to construct short circuit walks in the proof of Theorem 2. The proof of Lemma 5 uses ideas similar to the ones in the proof of Lemma 3.

Lemma 5. *Consider two different perfect matchings M_1, M_2 in K_n, $n \geq 10$. Then $\boldsymbol{c} := \chi(M_1) - \chi(M_2)$ is a circuit of $P_{\text{PERFECT MATCH}}(n)$.*

Proof. Let us assume for the sake of contradiction that $\boldsymbol{c}$ is not a circuit. Then there exists a non-zero vector $\boldsymbol{y}$ such that $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, and $\boldsymbol{y} \in \text{Ker}\, F$, where D is the constraint matrix of the inequality constraints of (3.2), and F is the constraint matrix of the equality constraints of (3.2). Since the inequalities $\boldsymbol{x}_e \geq 0$, $e \in E$ are in the minimal linear description (3.2) and $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, we have $\boldsymbol{y}_e = 0$ for every edge e such that $\boldsymbol{c}_e = 0$.

Let $e' = \{v_1, v_2\}$ be such that $y_{e'} \neq 0$. Without loss of generality, possibly rescaling vector $\boldsymbol{y}$ we can assume $\boldsymbol{y}_{e'} = 1$. Let C' be the connected component of $(V, M_1 \Delta M_2)$ containing e'. By exchanging the roles of M_1 with M_2, we can assume $\boldsymbol{c}_{e'} = 1$. Moreover, for every node v since $\boldsymbol{y} \in \text{Ker}(A)$, we have $\boldsymbol{y}(\delta(v)) = 0$. Since C' is an even cycle and $\boldsymbol{y}(\delta(v)) = 0$, we have that $\boldsymbol{y}_e = \boldsymbol{c}_e$ for all edges $e \in C'$. In particular, for an edge $f = \{v_2, v_3\}$ different from e'such that $f \in (M_1 \Delta M_2)$, we have $\boldsymbol{y}_f = \boldsymbol{c}_f = -1$.

Now let C'' be a connected component of $(V, M_1 \Delta M_2)$ different from C'. Note that such C'' exists since otherwise $(V, M_1 \Delta M_2)$ contains only one non-trivial connected component, implying that $\boldsymbol{c}$ is a circuit by Lemma 4. Let $e'' = \{u_1, u_2\}$ be an edge in C'' such that $\boldsymbol{c}_{e''} = -1$. Again, since $\boldsymbol{y}(\delta(v)) = 0$ for every node v and since C'' is an even cycle, there exists γ such that $\boldsymbol{y}_e = \gamma \boldsymbol{c}_e$ for every edge e in C''.

Let z be a node that is not adjacent to any of the nodes u_1, u_2, v_1, v_2 in the graph $(V, M_1 \Delta M_2)$. Note that such a node exists, because each node in $(V, M_1 \Delta M_2)$ has degree exactly 2, and we have $n > 8$. Let us define $S := \{z, u_1, u_2, v_1, v_2\}$. It is straightforward to check that $\boldsymbol{c}(\delta(S)) = 0$. Indeed, since $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$ and the constraint $\boldsymbol{x}(\delta(S)) \geq 1$ is present in (3.2), we have that $\boldsymbol{y}(\delta(S)) = 0$. On the other side, $\boldsymbol{y}(\delta(S)) = -2 - 2\gamma = 0$, implying $\gamma = 1$ and therefore $\boldsymbol{y}_e = \boldsymbol{c}_e$ for all $e \in C''$. Repeating this argument for all non-trivial connected components of $(V, M_1 \Delta M_2)$, we get $\boldsymbol{y} = \boldsymbol{c}$, a contradiction. ☐

Now, with Lemma 5 at hand, we are ready to prove Theorem 2.

Proof. (Proof of Theorem 2) To show that the corresponding lower bounds for the circuit diameter hold, it is enough to show that

$$P_{\text{PERFECT MATCH}}(8) \geq 2\,.$$

To show this, let K_8 be the complete graph with the node set $\{v_1, \ldots, v_8\}$, and let us define the two perfect matchings

$$M_1 := \{v_1v_2, v_3v_4, v_5v_6, v_7v_8\} \quad \textit{and} \quad M_2 := \{v_1v_4, v_3v_2, v_5v_8, v_7v_6\}\,.$$

The vector $\boldsymbol{c} := \chi(M_1) - \chi(M_2)$ is not a circuit, since the vector $D\boldsymbol{c}$ has a larger support than $D\left(\chi(\{v_1v_2, v_3v_4\}) - \chi(\{v_1v_4, v_3v_2\})\right)$, where D is the linear constraint matrix of the linear description of $P_{\text{PERFECT MATCH}}(8)$. Hence, we have

$$\mathcal{CD}(P_{\text{PERFECT MATCH}}(8)) \geq 2\,.$$

Now let us prove the corresponding upper bounds for $\mathcal{CD}(P_{\text{MATCH}}(n))$, $n \geq 4$. For $n = 4$, $n = 6$ and two perfect matchings M_1 and M_2, $(V, M_1 \triangle M_2)$ has at most one non-trivial connected component. This fact together with Corollary 2 implies $\mathcal{CD}(P_{\text{MATCH}}(n)) \leq 1$ for $n = 4$, $n = 6$.

For $n = 8$ and two perfect matchings M_1 and M_2, $(V, M_1 \triangle M_2)$ has at most two non-trivial connected components. Again, this fact together with Corollary 2 implies $\mathcal{CD}(P_{\text{PERFECT MATCH}}(8)) \leq 2$. For $n \geq 10$, the upper bound follows from Lemma 5. □

Chapter 4

Traveling Salesman Polytope

The Traveling Salesman polytope is defined as the convex hull of all characteristic vectors of Hamiltonian cycles in a complete graph i.e.,

$$P_{\mathrm{TS}}(n) := \operatorname{conv}\{\chi(T) \,:\, T \text{ is a Hamiltonian cycle in } K_n\} \,.$$

Despite extensive studies of the Traveling Salesman polytope, no linear description of it is known for general n. In fact, any linear description of $P_{\mathrm{TS}}(n)$, which admits an efficient way to test whether a given linear constraint belongs to this description, would have consequences for the long-standing conjecture $\mathcal{NP} = co - \mathcal{NP}$ [24]. However, for some small values of n a linear description of the Traveling Salesman polytope is known. For example, $P_{\mathrm{TS}}(5)$ can be described by the constraints [17]

$$\begin{aligned} &\boldsymbol{x}\,(E[S])) \le |S| - 1 && \text{for all } S,\ S \subseteq V,\ 2 \le |S| \le |V| - 2 \\ &\boldsymbol{x}(\delta(v)) = 2 && \text{for all } v \in V \\ &\boldsymbol{x} \ge \mathbf{0}\,. \end{aligned} \tag{4.1}$$

Moreover, the linear inequalities from (4.1) define facets of the Traveling Salesman polytope $P_{\mathrm{TS}}(n)$ for all $n \ge 4$ [19]. For $n \ge 6$ the inequalities

$$\boldsymbol{x}_{uv} + \boldsymbol{x}_{vw} + \boldsymbol{x}_{wu} + \boldsymbol{x}_{uu'} + \boldsymbol{x}_{vv'} + \boldsymbol{x}_{ww'} \le 4 \quad \text{for distinct } u, v, w, u', v', w' \in V \tag{4.2}$$

also define facets of $P_{\mathrm{TS}}(n)$ [19]. The inequality (4.2) belongs to the well-known family of *comb inequalities*, which are valid for the Traveling Salesman polytope. Surprisingly, such scarce knowledge on linear description of the Traveling Salesman polytope is enough for us to prove the following theorem.

Theorem 3. *For the Traveling Salesman polytope we have:*

$$\mathcal{CD}(P_{\text{TS}}(n)) = \begin{cases} 1 & n = 3, 4 \\ 2 & n = 5 \\ 1 & n \geq 6\,. \end{cases}$$

The proof of Theorem 3 follows from a series of lemmas below.

Lemma 6. *For $n = 5$ we have $\mathcal{CD}(P_{\text{TS}}(n)) = 2$.*

Proof. Recall that the Traveling Salesman polytope $P_{\text{TS}}(5)$ admits the minimal linear description (4.1) [17].

For two Hamiltonian cycles T_1, T_2 in K_5 without a common edge (see Figure 4.1), the vector $\boldsymbol{c} := \chi(T_1) - \chi(T_2)$ is not a circuit of $P_{\text{TS}}(5)$. Indeed, $\operatorname{supp}(D\boldsymbol{y}) \subset \operatorname{supp}(D\boldsymbol{c})$ for the non-zero vector $\boldsymbol{y} := \chi(M_1) - \chi(M_2)$, where D is the constraint matrix of (4.1) and M_1, M_2 are two different matchings in K_5 on the same four nodes. Thus $\mathcal{CD}(P_{\text{TS}}(5)) \geq 2$.

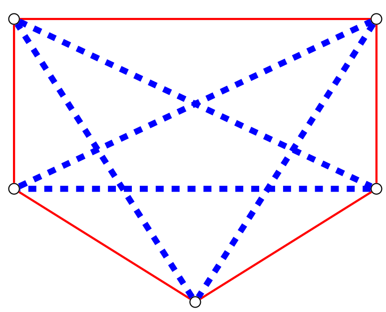


Figure 4.1: Hamiltonian cycles T_1 and T_2 in K_5 without a common edge. Here, the edges of T_2 are depicted as dashed edges.

The bound $\mathcal{CD}(P_{\text{TS}}(5)) \leq 2$ follows from the fact that for any two Hamiltonian cycles T_1, T_2 such that $T_1 \cap T_2 \neq \varnothing$, $\chi(T_1) - \chi(T_2)$ is a circuit of $P_{\text{TS}}(5)$. Indeed, up to symmetry we have two possible cases (see Figure 4.2) and in each of these cases $\chi(T_1) - \chi(T_2)$ is a circuit. □

Lemma 7. *For $n = 6$ we have $\mathcal{CD}(P_{\text{TS}}(n)) = 1$.*

Proof. Let us consider two different Hamiltonian cycles T_1 and T_2 in K_6, then up to symmetry and up to exchanging the roles of T_1 and T_2 we have one of the nine cases in Figure 4.3. In all these nine cases, $\chi(T_1) - \chi(T_2)$ is a circuit of $P_{\text{TS}}(6)$. □

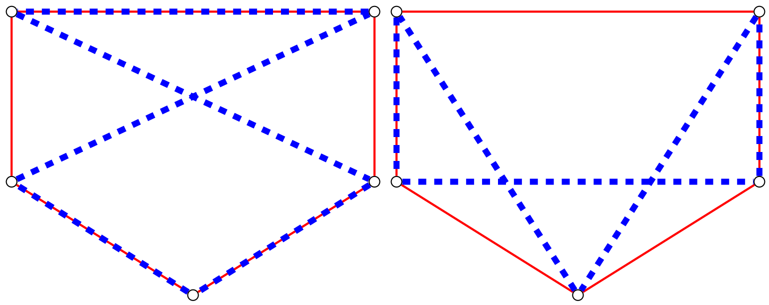


Figure 4.2: Hamiltonian cycles T_1 and T_2 in K_5 with a common edge. Here, the edges of T_2 are depicted as dashed edges.

Lemma 8. *For $n \geq 7$ we have $\mathcal{CD}(P_{\mathrm{TS}}(n)) = 1$.*

Proof. Consider two different Hamiltonian cycles T_1, T_2 in K_n, $n \geq 7$. For the sake of contradiction let us assume that $\boldsymbol{c} := \chi(T_1) - \chi(T_2)$ is not a circuit for the Traveling Salesman polytope $P_{\mathrm{TS}}(n)$. Thus there exists some $\neq \boldsymbol{c}$ satisfying $\mathrm{supp}(D\boldsymbol{y}) \subset \mathrm{supp}(D\boldsymbol{c})$, where D denotes the matrix of the linear constraints (4.1) and (4.2), since the linear inequalities in (4.1) and (4.2) define facets for $P_{\mathrm{TS}}(n)$, $n \geq 7$.

Case 1: T_1 and T_2 are not disjoint.

First, let us prove that $\boldsymbol{c}$ is a circuit when $T_1 \cap T_2 \neq \varnothing$. Then, there are two different nodes u and v such that $|\{e \in E \,:\, \boldsymbol{c}_e \neq 0,\, e \in \delta(u)\}| = |\{e \in E \,:\, \boldsymbol{c}_e \neq 0,\, e \in \delta(v)\}| = 2$ and $\boldsymbol{c}_{uv} = 0$.

Claim 1. *Consider w such that $|\{e \in E \,:\, \boldsymbol{c}_e \neq 0,\, e \in \delta(w)\}| = 4$ and two edges $e, g \in \delta(w)$ such that $\boldsymbol{c}_e = \boldsymbol{c}_g$. Then $\boldsymbol{y}_e = \boldsymbol{y}_g$ holds.*

Proof. For the values $\boldsymbol{c}_{uw}$ and $\boldsymbol{c}_{vw}$, we have (up to symmetry) four possibilities:

(i) $\boldsymbol{c}_{uw} = 1$ and $\boldsymbol{c}_{vw} = -1$

(ii) $\boldsymbol{c}_{uw} = 1$ and $\boldsymbol{c}_{vw} = 1$

(iii) $\boldsymbol{c}_{uw} = 0$ and $\boldsymbol{c}_{vw} = 1$

(iv) $\boldsymbol{c}_{uw} = 0$ and $\boldsymbol{c}_{vw} = 0$.

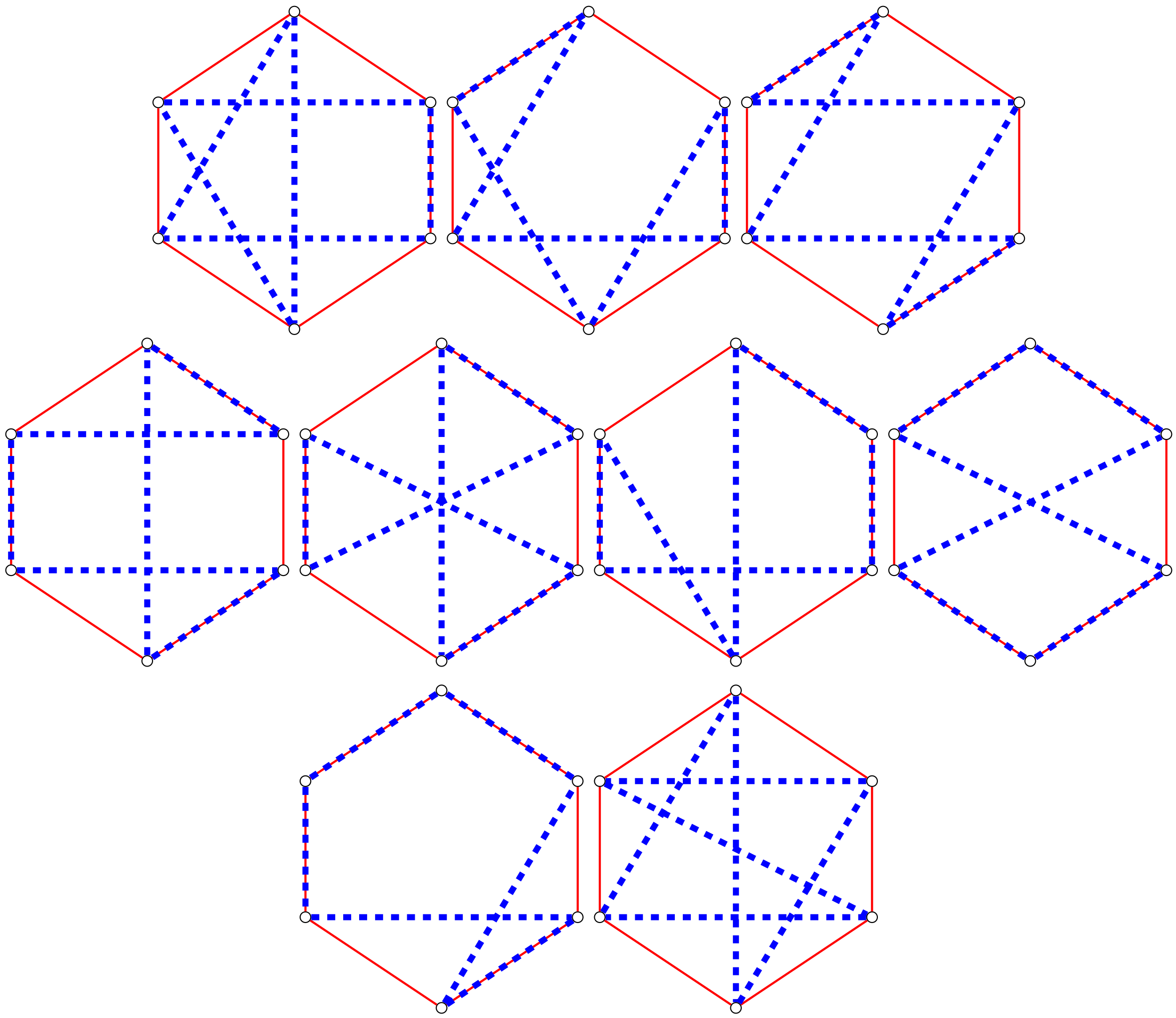
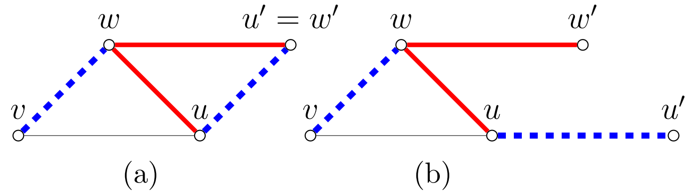


Figure 4.3: All possible cases (up to symmetry and up to exchanging the roles of T_1 and T_2) for two different Hamiltonian cycles T_1 and T_2 in K_6. Here, the edges of T_2 are depicted as dashed edges.

Case (i). Let u' be the node such that $\boldsymbol{c}_{uu'} = -1$; and w' be the node such that $\boldsymbol{c}_{ww'} = 1$ and $u \neq w'$. There are two possible cases: $u' = w'$ (see Figure 4.4a) and $u' \neq w'$ (see Figure 4.4b). In the first case (see Figure 4.4a), the statement of the Claim follows by considering $\boldsymbol{y}(\delta(u))$, $\boldsymbol{y}(\delta(w))$, $\boldsymbol{y}(E[\{u, v, w\}])$ and $\boldsymbol{y}_{ww'} + \boldsymbol{y}_{uw'} + \boldsymbol{y}_{wu} + \boldsymbol{y}_{wv} + \boldsymbol{y}_{ut} + \boldsymbol{y}_{w's}$, where $\boldsymbol{c}_{ut} = 0$, $\boldsymbol{c}_{w's} = 0$, $s \neq t$ and s, t are different from u, v, w, w'. (Note that such s, t

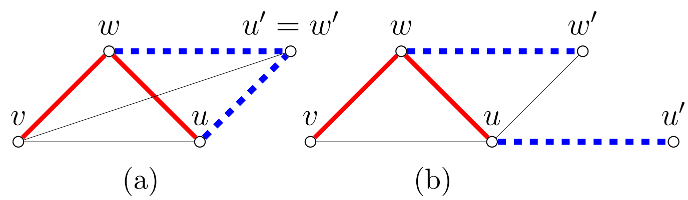


(a) (b)

Figure 4.4: Case 1 (i). The vector $\boldsymbol{c}$ has value -1 for blue dashed edges, 1 for red thick edges and 0 for thin edges. (The values of not depicted edges are not relevant for the proof.)

exist since there are at least 3 nodes in K_n different from u, v, w, w', because $n \geq 7$. For at most 2 nodes r of these 3 nodes, we have $\boldsymbol{c}_{w'r} \neq 0$. For every node r of these 3 nodes, we have $\boldsymbol{c}_{ur} = 0$.)

In the second case (see Figure 4.4b), the statement of the Claim follows by considering $\boldsymbol{y}(\delta(u))$, $\boldsymbol{y}(\delta(w))$, $\boldsymbol{y}(E[\{u, v, w\}])$ and $\boldsymbol{y}_{wu} + \boldsymbol{y}_{uv} + \boldsymbol{y}_{wv} + \boldsymbol{y}_{ww'} + \boldsymbol{y}_{uu'} + \boldsymbol{y}_{vs}$, where $\boldsymbol{c}_{vs} = 0$ and s is different from u, v, w, u', w'. (Note that such s exists since there are at least 2 nodes in K_n different from u, v, w, u', w', because $n \geq 7$. For at most 1 node r of these 2 nodes, we have $\boldsymbol{c}_{vr} \neq 0$.)



(a) (b)

Figure 4.5: Case 1 (ii). The vector $\boldsymbol{c}$ has value -1 for blue dashed edges, 1 for red thick edges and 0 for thin edges. (The values of not depicted edges are not relevant for the proof.)

Case (ii). Let u' be the node such that $\boldsymbol{c}_{uu'} = -1$; and w' be a node such that $\boldsymbol{c}_{ww'} = -1$. There are two possible cases: $u' = w'$ (see Figure 4.5a) and $u' \neq w'$ (see Figure 4.5b). In the first case (see Figure 4.5a), the statement of the Claim follows by considering $\boldsymbol{y}(\delta(u))$, $\boldsymbol{y}(\delta(v))$, $\boldsymbol{y}(\delta(w))$, $\boldsymbol{y}(E[\{v, w, w'\}])$ and $\boldsymbol{y}_{wu} + \boldsymbol{y}_{uv} + \boldsymbol{y}_{vw} + \boldsymbol{y}_{ww'} + \boldsymbol{y}_{ut} + \boldsymbol{y}_{vs}$, where $\boldsymbol{c}_{ut} = 0$, $\boldsymbol{c}_{vs} = -1$, $s \neq t$ and s, t are different from u, v, w, w'. (Note that such s, t trivially exist. The node s is uniquely defined, and for every node t different from u, v, w, w', s we have $\boldsymbol{c}_{ut} = 0$.)

In the second case (see Figure 4.5b), the statement of the Claim follows by considering $\boldsymbol{y}(\delta(u))$, $\boldsymbol{y}(\delta(w))$, $\boldsymbol{y}(E[\{u,w,w'\}])$ and $\boldsymbol{y}_{wu}+\boldsymbol{y}_{uv}+\boldsymbol{y}_{wv}+\boldsymbol{y}_{ww'}+\boldsymbol{y}_{uu'}+\boldsymbol{y}_{vs}$, where $\boldsymbol{c}_{vs}=0$ and s is different from u,v,w,u',w'. (Note that such s exists since there are at least 2 nodes in K_n different from u,v,w,u',w', because $n \geq 7$. For at most 1 node r of these 2 nodes, we have $\boldsymbol{c}_{vr} \neq 0$.)

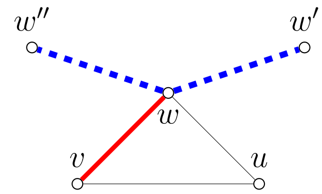


Figure 4.6: Case 1 (iii). The vector $\boldsymbol{c}$ has value -1 for blue dashed edges, 1 for red thick edges and 0 for thin edges. (The values of not depicted edges are not relevant for the proof.)

Case (iii) Let w', w'' be two different nodes such that $\boldsymbol{c}_{ww'}=-1$ and $\boldsymbol{c}_{ww''}=-1$(see Figure 4.6). The statement of the Claim follows by considering $\boldsymbol{y}(\delta(w))$ and $\boldsymbol{y}_{wu}+\boldsymbol{y}_{uv}+\boldsymbol{y}_{vw}+\boldsymbol{y}_{w\bar{w}}+\boldsymbol{y}_{ut}+\boldsymbol{y}_{vs}$ for each $\bar{w} \in \{w',w''\}$, where $\boldsymbol{c}_{ut}=0$, $\boldsymbol{c}_{vs}=0$, $s \neq t$ and s,t are different from $u,v,w,\bar{w}$. (Note that such s and t exist. Indeed, there are at least 3 nodes in K_n different from $u,v,w,\bar{w}$, because $n \geq 7$. For at most 2 nodes r of these 3 nodes, we have $\boldsymbol{c}_{ur} \neq 0$. For at most 1 node r of these 3 nodes, we have $\boldsymbol{c}_{vr} \neq 0$.)

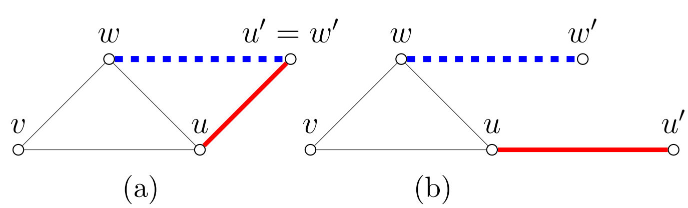


Figure 4.7: Case 1 (iv). The vector $\boldsymbol{c}$ has value -1 for blue dashed edges, 1 for red thick edges and 0 for thin edges. (The values of not depicted edges are not relevant for the proof.)

Case (iv). Consider a node w' and a node u' such that $\boldsymbol{c}_{ww'}=-\boldsymbol{c}_{uu'}$. To prove the Claim, it is enough to show that $\boldsymbol{y}_{ww'}=-\boldsymbol{y}_{uu'}$.

There are two possible cases: $u'=w'$ (see Figure 4.7a) and $u' \neq w'$ (see Figure 4.7b). In Figure 4.7, without loss of generality we assumed that $\boldsymbol{c}_{ww'}=-1$ and $\boldsymbol{c}_{uu'}=1$.) In the first case (see Figure 4.7a), we can consider $\boldsymbol{y}(E[\{w, u, u'\}])$ to establish $\boldsymbol{y}_{ww'} = -\boldsymbol{y}_{uu'}$.

In the second case (see Figure 4.7b), to establish $\boldsymbol{y}_{ww'} = -\boldsymbol{y}_{uu'}$ we can consider $\boldsymbol{y}_{wu} + \boldsymbol{y}_{uv} + \boldsymbol{y}_{vw} + \boldsymbol{y}_{ww'} + \boldsymbol{y}_{uu'} + \boldsymbol{y}_{vs}$ where $\boldsymbol{c}_{vs} = 0$ and s is different from u, v, w, u', w'. Such s exists unless $n = 7$ and we have the situations in Figure 4.8. (Note that otherwise such s exists. Indeed, there are at least 3 nodes in K_n different from u, v, w, u', w', if $n \geq 8$. For at most 2 nodes r of these 3 nodes we have $\boldsymbol{c}_{vr} \neq 0$.)

Now in the case in Figure 4.8 and $n = 7$, it is straightforward to establish that there are at least two nodes r such that $|\{e \in E : \boldsymbol{c}_e \neq 0, e \in \delta(r)\}| = 4$. Moreover, if $|\{e \in E : \boldsymbol{c}_e \neq 0, e \in \delta(w')\}| = 4$ then there are at least four nodes r such that $|\{e \in E : \boldsymbol{c}_e \neq 0, e \in \delta(r)\}| = 4$. Now it is straightforward to use already considered cases (i), (ii), (iii), to establish $\boldsymbol{y}_{ww'} = -\boldsymbol{y}_{uu'}$. □

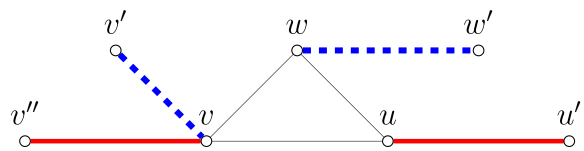


Figure 4.8: Case 1 (iv) (Special Case). The vector $\boldsymbol{c}$ has value -1 for blue dashed edges, 1 for red thick edges and 0 for thin edges. (The values of not depicted edges are not relevant for the proof.)

Using the above Claim for all nodes of degree 4 in a same connected component C of $T_1 \triangle T_2$, we establish that $\boldsymbol{y}_e = \boldsymbol{y}_g$ whenever $\boldsymbol{c}_e = \boldsymbol{c}_g$ and e, g are both in C. On the other side, we have $\boldsymbol{y}(\delta(v)) = 0$ for all nodes v. Hence, we also have $\boldsymbol{y}_e = -\boldsymbol{y}_g$ whenever $\boldsymbol{c}_e = -\boldsymbol{c}_g$ and e, g are both in C.

Moreover, $\boldsymbol{y}_e = -\boldsymbol{y}_g$ holds for all edges e, g such that $\boldsymbol{c}_e = -\boldsymbol{c}_g$. Indeed, let $e = vv'$ and $g = uu'$ be two edges from different connected components of $T_1 \triangle T_2$ such that $\boldsymbol{c}_e = -\boldsymbol{c}_g$. Consider the constraint $x(E[\{v, v', u, u'\}]) \leq 3$ from (4.1). Since $\boldsymbol{c}(E[\{v, v', u, u'\}]) = 0$, we have $\boldsymbol{y}(E[\{v, v', u, u'\}]) = \boldsymbol{y}_e + \boldsymbol{y}_g = 0$, implying $\boldsymbol{y}_e = -\boldsymbol{y}_g$.

Hence, for $n \geq 7$ we proved that $\chi(T_1) - \chi(T_2)$ is a circuit whenever $T_1 \cap T_2$ is not empty.

Case 2: T_1 and T_2 are disjoint. Let us prove that for $n \geq 7$, $\chi(T_1) - \chi(T_2)$ is a circuit whenever $T_1 \cap T_2 = \varnothing$.

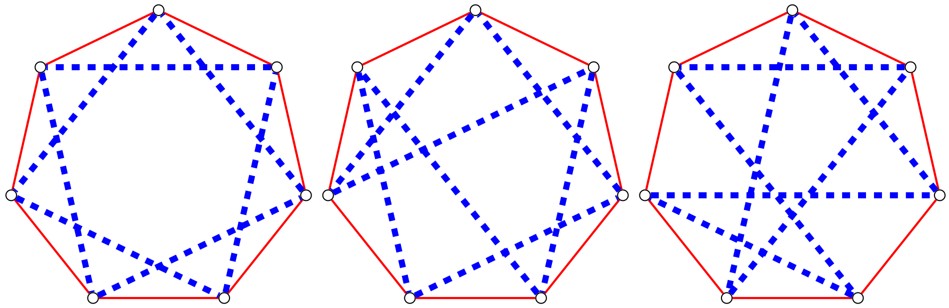


Figure 4.9: All possible cases (up to symmetry) for two different Hamiltonian cycles T_1 and T_2 in K_7 without a common edge. Here, the edges of T_2 are depicted as dashed edges.

For $n = 7$ we have (up to symmetry) three possibilities for two different Hamiltonian cycles T_1 and T_2 without a common edge (see Figure 4.9). In all these cases $\chi(T_1) - \chi(T_2)$ is a circuit.

For $n \geq 8$ let us show the following Claim.

Claim 2. *Let two edges $e, g \in \delta(w)$ be such that $\boldsymbol{c}_e = \boldsymbol{c}_g$. Then $\boldsymbol{y}_e = \boldsymbol{y}_g$ holds.*

Proof. Let e, g be wv, wu for some two nodes u, v. We may assume that u and v are different, since otherwise the statement of the Claim is trivial.

Without loss of generality, we may assume $\boldsymbol{c}_e = 1$ and $\boldsymbol{c}_g = 1$. There are two possible cases

(a) $\boldsymbol{c}_{uv} = -1$

(b) $\boldsymbol{c}_{uv} = 0$.

In the case (a), let w', w'' be two different nodes such that $\boldsymbol{c}_{ww'} = -1$ and $\boldsymbol{c}_{ww''} = -1$. For each $\bar{w} \in \{w', w''\}$, to establish $\boldsymbol{y}_{w\bar{w}} = \boldsymbol{y}_{uv}$ consider $\boldsymbol{y}_{wu} + \boldsymbol{y}_{uv} + \boldsymbol{y}_{wv} + \boldsymbol{y}_{w\bar{w}} + \boldsymbol{y}_{vs} + \boldsymbol{y}_{ut}$, where s, t, $s \neq t$ are two nodes different from u, v, w, $\bar{w}$ such that $\boldsymbol{c}_{vs} = 0$ and $\boldsymbol{c}_{ut} = 0$. (Note that such nodes s, t exist. Indeed, since $n \geq 8$ there are at least 4 nodes in K_n different from u, v, w, $\bar{w}$. There are at most 2 nodes r of these 4 nodes such that $\boldsymbol{c}_{vr} \neq 0$. Also there are at most 2 nodes r of these 4 nodes such that $\boldsymbol{c}_{ur} \neq 0$.) To establish $\boldsymbol{y}_e = \boldsymbol{y}_g$, now it is enough to consider $\boldsymbol{y}(E[\{s, w^\star, w\}])$ and $\boldsymbol{y}(\delta(w))$, where $s \in \{u, v\}$, $w^\star \in \{w', w''\}$ such that $\boldsymbol{c}_{sw^\star} = 0$. (Note that such nodes s, $w^\star$ exist, since otherwise T_2 has a subtour.).

In the case (b), Let u', u'' be two different nodes such that $\boldsymbol{c}_{uu'} = \boldsymbol{c}_{uu''} = -1$, and let v', v'' be two different nodes such that $\boldsymbol{c}_{vv'} = \boldsymbol{c}_{vv''} = -1$ (see Figure (4.10)). First note that $\{u', u''\} \neq \{v', v''\}$ as otherwise T_2 contains a subtour. Then we may assume that $v' \notin \{u', u''\}$ and $u' \notin \{v', v''\}$.

It follows that $\boldsymbol{y}_{uu'} = \boldsymbol{y}_{uu''}$ by considering $\boldsymbol{y}_{wu} + \boldsymbol{y}_{uv} + \boldsymbol{y}_{vw} + \boldsymbol{y}_{vv'} + \boldsymbol{y}_{u\bar{u}} + \boldsymbol{y}_{wz}$ for each $\bar{u} \in \{u', u''\}$, where $\boldsymbol{c}_{wz} = 0$ and z is different from u, v, w, v', and $\bar{u}$. (Note that such a z exists. Indeed there are at least 3 nodes in K_n different from u, v, w, v' and $\bar{u}$ if $n \geq 8$. For at most 2 nodes r of these 3 nodes, we have $\boldsymbol{c}_{wr} \neq 0$.). By symmetry, we also have that $\boldsymbol{y}_{vv'} = \boldsymbol{y}_{vv''}$.

There exists $\bar{u} \in \{u', u''\}$ such that $\boldsymbol{c}_{w\bar{u}} = 0$ as otherwise T_2 contains a subtour. Then it follows that $\boldsymbol{y}_{uw} = -\boldsymbol{y}_{u\bar{u}}$ by considering $\boldsymbol{y}(E[\{w, u, \bar{u}\}])$. Therefore, $\boldsymbol{y}_{uw} = -\boldsymbol{y}_{uu'}$ and $\boldsymbol{y}_{uw} = -\boldsymbol{y}_{uu''}$. Similarly, $\boldsymbol{y}_{vw} = -\boldsymbol{y}_{vv'}$ and $\boldsymbol{y}_{vw} = -\boldsymbol{y}_{vv''}$.

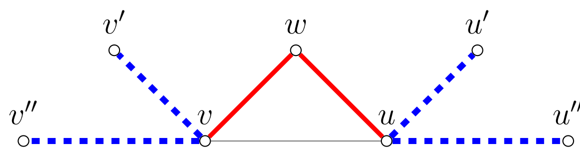


Figure 4.10: Case 2 (b) of Lemma 8. The vector $\boldsymbol{c}$ has value -1 for blue dashed edges, 1 for red thick edges and 0 for thin edges. (The values of not depicted edges are not relevant for the proof.)

Now, if $\boldsymbol{c}_{vu'} \neq 0$, then since $u' \notin \{v', v''\}$, we have that $\boldsymbol{c}_{vu'} = 1$. Then it follows that $\boldsymbol{y}_{vu'} = \boldsymbol{y}_{vw}$ by considering $\boldsymbol{y}(\delta(v))$. It follows that $\boldsymbol{y}_{vu'} = -\boldsymbol{y}_{uu'}$ by considering $\boldsymbol{y}(E[\{u, u', v\}])$. Then in this case we have that

$$\boldsymbol{y}_{uw} = -\boldsymbol{y}_{uu'} = \boldsymbol{y}_{vu'} = \boldsymbol{y}_{vw},$$

and therefore $\boldsymbol{y}_g = \boldsymbol{y}_e$, as desired.

Otherwise, $\boldsymbol{c}_{vu'} = 0$, and by symmetry we may assume that $\boldsymbol{c}_{uv'} = 0$ as well. There exists a node $v''' \neq u'$ such that $\boldsymbol{c}_{vv'''} = 1$. It follows that $\boldsymbol{y}_{vv'''} = \boldsymbol{y}_{vw}$ by considering $\boldsymbol{y}(\delta(v))$.

If $\boldsymbol{c}_{wu'} = 0$ (see Figure (4.11)), then it follows that $\boldsymbol{y}_{vv'''} = -\boldsymbol{y}_{uu'}$ by considering $\boldsymbol{y}_{uu'} + \boldsymbol{y}_{u'v} + \boldsymbol{y}_{vu} + \boldsymbol{y}_{vv'''} + \boldsymbol{y}_{u'w} + \boldsymbol{y}_{uv'}$. Then in this case we have that

$$\boldsymbol{y}_{uw} = -\boldsymbol{y}_{uu'} = \boldsymbol{y}_{vv'''} = \boldsymbol{y}_{vw},$$

and therefore $\boldsymbol{y}_g = \boldsymbol{y}_e$, as desired.

Otherwise, $\boldsymbol{c}_{wu'} = -1$. Then $\boldsymbol{c}_{wu''} = 0$, as otherwise T_2 contains a subtour.

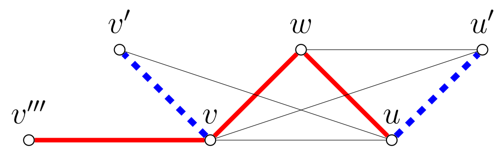


Figure 4.11: Case 2 (b) of Lemma 8: First sub-case. The vector $\boldsymbol{c}$ has value -1 for blue dashed edges, 1 for red thick edges and 0 for thin edges. (The values of not depicted edges are not relevant for the proof.)

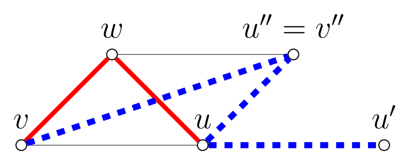


Figure 4.12: Case 2 (b) of Lemma 8: Second sub-case. The vector $\boldsymbol{c}$ has value -1 for blue dashed edges, 1 for red thick edges and 0 for thin edges. (The values of not depicted edges are not relevant for the proof.)

If $v'' = u''$ (see Figure (4.12)), it follows that $\boldsymbol{y}_{vw} = -\boldsymbol{y}_{uu'}$ by considering $\boldsymbol{y}_{uu''} + \boldsymbol{y}_{u''w} + \boldsymbol{y}_{wu} + \boldsymbol{y}_{uu'} + \boldsymbol{y}_{wv} + \boldsymbol{y}_{u''z}$, where $\boldsymbol{c}_{u''z} = 0$ and z is different from u, u'', w, u', and v. (Note that such a z exists. Indeed there are at least 3 nodes in K_n different from u, u'', w, u', and v if $n \geq 8$. For at most 2 nodes r of these 3 nodes, we have $\boldsymbol{c}_{u''r} \neq 0$.). Then in this case we have that

$$\boldsymbol{y}_{uw} = -\boldsymbol{y}_{uu'} = \boldsymbol{y}_{vw}$$

and therefore, $\boldsymbol{y}_g = \boldsymbol{y}_e$, as desired.

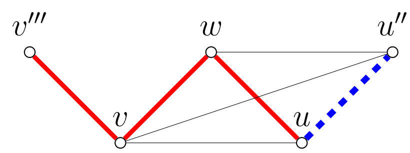


Figure 4.13: Case 2 (b) of Lemma 8: Third sub-case. The vector $\boldsymbol{c}$ has value -1 for blue dashed edges, 1 for red thick edges and 0 for thin edges. (The values of not depicted edges are not relevant for the proof.)

Otherwise, $v'' \neq u''$.

If $\boldsymbol{c}_{vu''} = 0$ (see Figure (4.13)), it follows that $\boldsymbol{y}_{vv'''} = -\boldsymbol{y}_{uu''}$ by considering $\boldsymbol{y}_{u''v} + \boldsymbol{y}_{vu} + \boldsymbol{y}_{uu''} + \boldsymbol{y}_{u''w} + \boldsymbol{y}_{vv'''} + \boldsymbol{y}_{uz}$, where $\boldsymbol{c}_{uz} = 0$ and z is different from u'', v, u, w, and v'''. (Note that such a z exists. Indeed there are at least 3 nodes in K_n different from u'', v, u, w, and v''' if $n \geq 8$. For at most 2 nodes r of these 3 nodes, we have $\boldsymbol{c}_{ur} \neq 0$.). Then in this case we have that

$$\boldsymbol{y}_{uw} = -\boldsymbol{y}_{uu''} = \boldsymbol{y}_{vv'''} = \boldsymbol{y}_{vw},$$

and therefore, $\boldsymbol{y}_g = \boldsymbol{y}_e$, as desired.

Finally, if instead $\boldsymbol{c}_{vu''} = 1$ (that is, $u'' = v'''$), then it follows that $\boldsymbol{y}_{vu''} = -\boldsymbol{y}_{uu''}$ by considering $\boldsymbol{y}(E[\{u, v, u''\}])$. Then in this case we have that

$$\boldsymbol{y}_{uw} = -\boldsymbol{y}_{uu''} = \boldsymbol{y}_{vu''} = \boldsymbol{y}_{vw},$$

and therefore $\boldsymbol{y}_g = \boldsymbol{y}_e$, as desired. □

Hence, we showed that for $n \geq 7$ and for any two different Hamiltonian cycles T_1, T_2, we have that $\boldsymbol{c} = \chi(T_1) - \chi(T_2)$ is a circuit for the Traveling Salesman polytope. □

Proof. (Proof of Theorem 3) The cases $n = 3$ and $n = 4$ are trivial. Indeed, $P_{\mathrm{TS}}(3)$ and $P_{\mathrm{TS}}(4)$ are simplices, and thus every two vertices of $P_{\mathrm{TS}}(3)$ and $P_{\mathrm{TS}}(4)$ form an edge. The cases, $n = 5$, $n = 6$ and $n \geq 7$ are covered by Lemma 6, Lemma 7 and Lemma 8, respectively. □

Chapter 5

Fractional Stable Set Polytope

Given a connected graph $G = (V, E)$, the Fractional Stable Set polytope is defined as follows

$$P_{\mathrm{FSTAB}}(G) := \{\boldsymbol{x} \in \mathbb{R}^V \,:\, \boldsymbol{x}_u + \boldsymbol{x}_v \leq 1 \text{ for all } uv \in E,\, \boldsymbol{x} \geq \mathbf{0}\}\,.$$

The Fractional Stable Set polytope is well studied. In particular, it is known that all vertices of it are half-integral [1] i.e., $\boldsymbol{x} \in \{0, 1/2, 1\}^V$ whenever $\boldsymbol{x}$ is a vertex of $P_{\mathrm{FSTAB}}(G)$. In [21], it is shown that the combinatorial diameter of $P_{\mathrm{FSTAB}}(G)$ is bounded from above by the number of nodes in G.

We first note that the linear description used in the definition of $P_{\mathrm{FSTAB}}(G)$ is minimal. That is, there is no constraint which can be removed and cause the set of feasible points to remain unchanged. If any constraint of the form $x_u \geq 0$ is removed, then the point defined by

$$\boldsymbol{x}_w := \begin{cases} -1 & \text{if } w = u \\ 0 & \text{otherwise} \end{cases}$$

becomes feasible where it previously was not. If any constraint of the form $x_u + x_v \leq 1$, $uv \in E$ is removed, then the point defined by

$$\boldsymbol{x}_w := \begin{cases} 1 & \text{if } w \in \{u, v\} \\ 0 & \text{otherwise} \end{cases}$$

becomes feasible where it previously was not. Hence, as stated in in [21], Hirsch conjecture holds for $P_{\mathrm{FSTAB}}(G)$.

Before we study the circuit diameter of the Fractional Stable Set polytope let us study the circuits of this polytope. Its circuits admit a nice characterization captured by the lemma below.

Lemma 9. *For a graph $G = (V, E)$, a vector $\boldsymbol{c} \neq \boldsymbol{0}$ is a circuit (or the non-zero scaling of a circuit) of $P_{\text{FSTAB}}(G)$ if and only if the graph G' with the node set $V' := \{v \in V \,:\, \boldsymbol{c}_v \neq 0\}$ and the edge set $E' := \{e \in E \,:\, e = uv,\ u, v \in V'$ and $\boldsymbol{c}_u + \boldsymbol{c}_v = 0\}$ is connected.*

Proof. If G' is not connected i.e., G' has a connected component C with a node set U, let us define the vector $\boldsymbol{c}' \in \mathbb{R}^V$ as

$$\boldsymbol{c}'_v := \begin{cases} \boldsymbol{c}_v & \text{if } v \in U \\ 0 & \text{otherwise} \end{cases}.$$

The vector $\boldsymbol{c}$ is not a circuit of $P_{\text{FSTAB}}(G)$ since the vector $D\boldsymbol{c}'$ has a smaller support than $D\boldsymbol{c}$, where D is the linear constraint matrix in the minimal description of $P_{\text{FSTAB}}(G)$.

On the other hand, it is straightforward to check that if G' is connected, then $\boldsymbol{c}$ is a unique (up to scaling) non-zero solution of

$$\begin{aligned} &\boldsymbol{y}_v = 0 && \text{for all } v \in V \text{ such that } \boldsymbol{c}_v = 0 \\ &\boldsymbol{y}_v + \boldsymbol{y}_u = 0 && \text{for all } uv \in E \text{ such that } \boldsymbol{c}_v + \boldsymbol{c}_u = 0\,, \end{aligned}$$

showing that $\boldsymbol{c}$ is a circuit of $P_{\text{FSTAB}}(G)$ by Proposition 1. □

To study the circuit diameter of the Fractional Stable Set polytope we need the following notation. For a node v, let $B(v, 0)$ be defined as $\{v\}$. For integer positive k, we define $B(v, k)$ to be the set of nodes which are at distance at most k from v. The set of nodes which are at distance exactly k from v is denoted by $N(v, k)$ i.e., $N(v, k) := B(v, k) \setminus B(v, k-1)$. The eccentricity $\varepsilon(v)$ of a node $v \in V$ is minimum k such that $V = B(v, k)$.

Lemma 10. *Let v be any node in a graph $G = (V, E)$. Then $\mathcal{CD}(P_{\text{FSTAB}}(G))$ is $\mathcal{O}(\varepsilon(v))$.*

Proof. Let $\boldsymbol{x}'$ and $\boldsymbol{x}''$ be two vertices of $P_{\text{FSTAB}}(G)$. Let us show that $\text{cdist}(\boldsymbol{x}', \boldsymbol{x}'')$ is at most $4\varepsilon(v) + 6$. To do this we construct a circuit walk from $\boldsymbol{x}'$ to $\boldsymbol{x}''$. The walk will be constructed in two different phases. In Phase I we construct a circuit walk from $\boldsymbol{x}'$ to some "well structured" point $\boldsymbol{y}'$, and in Phase II we move from $\boldsymbol{y}'$ to $\boldsymbol{x}''$ by a circuit walk.

To simplify the exposition, in the proof we assume that G is a non-bipartite graph. It will be clear from the analysis of the length of the circuit walk that the bound in the statement of the lemma is also satisfied in the bipartite case.

Phase I: Let us assume that b is the smallest k such that the subgraph of G induced by $B(v,k)$ is non-bipartite.

Start of Phase I: If b is odd, we first take a circuit walk from $\boldsymbol{x}'$ to a point $\boldsymbol{z}$ with

$$\boldsymbol{z}_u = \begin{cases} 0 & \text{if } u = v \\ \phi & \text{if } u \in N(v,1) \end{cases}$$

where $\phi := 1/2$ if $b = 1$ and $\phi := 1$ otherwise.

If b is even, we start by a circuit walk from $\boldsymbol{x}'$ to a point $\boldsymbol{z}$ with

$$\boldsymbol{z}_u = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \in N(v,1) \\ \phi & \text{if } u \in N(v,2) \end{cases}$$

where $\phi := 1/2$ if $b = 2$ and $\phi := 1$ otherwise.

We initialize $t := 1$ if b is odd and $t := 2$ if b is even.

Claim 3. *If $t = 1$ at the beginning of Phase I then 4 circuit steps are enough to walk from $\boldsymbol{x}'$ to $\boldsymbol{z}$.*

Proof. First suppose that $b = 1$. There are three possible cases:

1. $\boldsymbol{x}'_v = 1$
2. $\boldsymbol{x}'_v = 1/2$
3. $\boldsymbol{x}'_v = 0$.

In the case 1 we have that for all $w \in N(v,1)$, $\boldsymbol{x}'_v + \boldsymbol{x}'_w \leq 1$, and therefore $\boldsymbol{x}'_w = 0$. Then let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1) \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{x}'_u = 1 \\ 0 & \text{else} \end{cases} .$$

By Lemma 9, $\boldsymbol{c}$ is a circuit. Let $\boldsymbol{y} = \boldsymbol{x}' + \boldsymbol{c}$. Then clearly, $\boldsymbol{y}$ is feasible for $P_{\text{FSTAB}}(G)$. In particular, for any edge u_1u_2 with both endpoints in $N(v,1)$, we have that $\boldsymbol{y}_{u_1} + \boldsymbol{y}_{u_2} = 1$. Similarly, for any edge u_1u_2 with one endpoint in $N(v,1)$ and one endpoint in $N(v,2)$, we have that $\boldsymbol{y}_{u_1} + \boldsymbol{y}_{u_2} \leq 1$ (note that if $u_1 \in N(v,2)$, then by definition there exists $u_2 \in N(v,1)$ such that $u_1u_2 \in E$). Furthermore, the step from $\boldsymbol{x}'$ to $\boldsymbol{y}$ is maximal because $b = 1$ implies that there exists an edge u_1u_2 between two vertices of $N(v,1)$. Thus, we cannot increase $\boldsymbol{y}_{u_1}$ and $\boldsymbol{y}_{u_2}$ any further. Now, let $\boldsymbol{c}'$ be defined as

$$\boldsymbol{c}'_u = \begin{cases} -1/2 & \text{if } u = v \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}'$ is a circuit, and the desired point $\boldsymbol{z}$ is equal to $\boldsymbol{y} + \boldsymbol{c}'$. Then in this case, 2 circuit steps are enough to walk from $\boldsymbol{x}'$ to $\boldsymbol{z}$.

In the case 2 we have that for all $w \in N(v,1)$, $\boldsymbol{x}'_w \leq 1/2$. Then let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1),\ \boldsymbol{x}'_u = 0 \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{x}'_u = 1, \text{ and there exists } w \in N(v,1) \text{ with } uw \in E,\ \boldsymbol{x}'_w = 0 \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}$ is a circuit. Let $\boldsymbol{z} = \boldsymbol{x}' + \boldsymbol{c}$. Then clearly, $\boldsymbol{z}$ is feasible for $P_{\text{FSTAB}}(G)$. In particular, for any edge u_1u_2 with both endpoints in $N(v,1)$, we have that $\boldsymbol{z}_{u_1} + \boldsymbol{z}_{u_2} = 1$. Similarly, for any edge u_1u_2 with one endpoint in $N(v,1)$ and one endpoint in $N(v,2)$, we have that $\boldsymbol{z}_{u_1} + \boldsymbol{z}_{u_2} \leq 1$. Furthermore, the step from $\boldsymbol{x}'$ to $\boldsymbol{z}$ is maximal because $b = 1$ implies that there exists an edge u_1u_2 between two vertices of $N(v,1)$. Thus, we cannot increase $\boldsymbol{z}_{u_1}$ and $\boldsymbol{z}_{u_2}$ any further. Then $\boldsymbol{z}$ is the desired point, and in this case 1 circuit step is enough to walk from $\boldsymbol{x}'$ to $\boldsymbol{z}$.

In the case 3 we have that for all $w \in N(v,1)$, $\boldsymbol{x}'_w \leq 1$. Then let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1),\ \boldsymbol{x}'_u > 0 \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}$ is a circuit. Clearly, $\boldsymbol{x}' + \boldsymbol{c}$ is feasible for $P_{\text{FSTAB}}(G)$. Note however that in this case it may not be true that the step going from $\boldsymbol{x}'$ to $\boldsymbol{x}' + \boldsymbol{c}$ is maximal. In particular, it will not be maximal if for all $w \in N(v,1)$, $\boldsymbol{x}'_w \neq 1/2$.

First, suppose that the step from $\boldsymbol{x}'$ to $\boldsymbol{x}' + \boldsymbol{c}$ is maximal, and let $\boldsymbol{y} = \boldsymbol{x}' + \boldsymbol{c}$. Then let $\boldsymbol{c}'$ be defined as

$$\boldsymbol{c}'_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1),\ \boldsymbol{y}_u = 0 \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{y}_u = 1,\ \text{and there exists } w \in N(v,1) \text{ with } uw \in E,\ \boldsymbol{y}_w = 0 \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}'$ is a circuit, $\boldsymbol{z} = \boldsymbol{y} + \boldsymbol{c}'$ is feasible, and the step from $\boldsymbol{y}$ to $\boldsymbol{z}$ is maximal as $\boldsymbol{z}_v$=0. Then $\boldsymbol{z}$ is the desired point, and in this case 2 circuit steps are enough to walk from $\boldsymbol{x}'$ to $\boldsymbol{z}$.

Now, suppose that the step from $\boldsymbol{x}'$ to $\boldsymbol{x}'+\boldsymbol{c}$ is not maximal. In this case, let $\boldsymbol{y}' = \boldsymbol{x}'+2\boldsymbol{c}$. Then $\boldsymbol{y}'$ is feasible, and since $\boldsymbol{y}'_v = 1$, this is a maximal step. Note in this case that for all $w \in N(v,1)$, $\boldsymbol{y}_w = 0$. Let $\boldsymbol{c}''$ be defined as

$$\boldsymbol{c}''_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1) \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{y}'_u = 1 \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}''$ is a circuit. Let $\boldsymbol{y}'' = \boldsymbol{y}' + \boldsymbol{c}''$. Clearly, $\boldsymbol{y}''$ is feasible for $P_{\text{FSTAB}}(G)$. Furthermore, the step from $\boldsymbol{y}'$ to $\boldsymbol{y}''$ is maximal because $b = 1$. Finally, let $\boldsymbol{c}'''$ be defined as

$$\boldsymbol{c}'''_u = \begin{cases} -1/2 & \text{if } u = v \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}'''$ is a circuit. Let $\boldsymbol{z} = \boldsymbol{y}'' + \boldsymbol{c}$. Clearly, $\boldsymbol{z}$ is feasible for $P_{\mathrm{FSTAB}}(G)$, and the step from $\boldsymbol{y}''$ to $\boldsymbol{z}$ is maximal because $\boldsymbol{z}_v = 0$. Then $\boldsymbol{z}$ is the desired point, and in this case 3 circuit steps are enough.

Now, suppose $b > 1$. We have the same three cases as when $b = 1$, and we will refer to them identically.

In the case 1, we have that for all $w \in N(v,1), \boldsymbol{x}'_w = 0$. Then let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1) \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{x}'_u > 0 \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}$ is a circuit. Again, the step from $\boldsymbol{x}'$ to $\boldsymbol{x}' + \boldsymbol{c}$ may not be maximal. In particular, it will not be maximal if for all $w \in N(v,2)$ we have that $\boldsymbol{x}'_w \neq 1/2$.

First, suppose that the step from $\boldsymbol{x}'$ to $\boldsymbol{x}' + \boldsymbol{c}$ is maximal, and let $\boldsymbol{y} = \boldsymbol{x}' + \boldsymbol{c}$. Then let $\boldsymbol{c}'$ be the following vector:

$$\boldsymbol{c}'_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1) \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{y}_u > 0 \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}'$ is a circuit. Let $\boldsymbol{z} = \boldsymbol{y} + \boldsymbol{c}'$. Then $\boldsymbol{z}$ is feasible, the step from $\boldsymbol{y}$ to $\boldsymbol{z}$ is maximal, and $\boldsymbol{z}$ is the desired point. In this case, 2 circuit steps are enough.

Now, suppose that the step from $\boldsymbol{x}'$ to $\boldsymbol{x}' + \boldsymbol{c}$ is not maximal. Let $\boldsymbol{z} = \boldsymbol{x}' + 2\boldsymbol{c}$. Then $\boldsymbol{z}$ is feasible, the step from $\boldsymbol{x}'$ to $\boldsymbol{z}$ is maximal, and $\boldsymbol{z}$ is the desired point. In this case, 1 circuit step is enough.

In the case 2 we have that for all $w \in N(v,1)$, $\boldsymbol{x}'_w \leq 1/2$. Then let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1),\ \boldsymbol{x}'_u = 1 \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}$ is a circuit. Let $\boldsymbol{y} = \boldsymbol{x'} + \boldsymbol{c'}$. Then $\boldsymbol{y}$ is feasible, and the step from $\boldsymbol{x'}$ to $\boldsymbol{y}$ is maximal. Note that $\boldsymbol{y}$ satisfies the conditions of case 1. Then we can arrive at the point $\boldsymbol{z}$ in at most 2 more circuit steps, for a total of 3.

In the case 3 we have that for all $w \in N(v,1)$, $\boldsymbol{x'} \leq 1$. Then let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1),\ \boldsymbol{x}'_u > 0 \\ 0 & \text{else .} \end{cases}$$

By Lemma 9, $\boldsymbol{c}$ is a circuit. Then let $\boldsymbol{y^1} = \boldsymbol{x'} + \boldsymbol{c}$ and let $\boldsymbol{y^2} = \boldsymbol{x'} + 2\boldsymbol{c}$. As with earlier cases, there is exactly one $i \in \{1,2\}$ such that $\boldsymbol{y^i}$ is feasible and the step from $\boldsymbol{x'}$ to $\boldsymbol{y^i}$ is maximal. If $i = 1$, then $\boldsymbol{y^1}$ satisfies the conditions of case 2. Then we can arrive at the point $\boldsymbol{z}$ in at most 3 more circuit steps, for a total of 4. If $i = 2$, then $\boldsymbol{y^2}$ satisfies the conditions of case 1. Then we can arrive at the point $\boldsymbol{z}$ in at most 2 more circuit steps, for a total of 3.

Therefore, in all cases, we need at most 4 circuit steps to move from $\boldsymbol{x'}$ to $\boldsymbol{z}$, as desired. □

Claim 4. *If $t = 2$ at the beginning of Phase I, then 6 circuit steps are enough to walk from $\boldsymbol{x'}$ to $\boldsymbol{z}$.*

Proof. First suppose that $b = 2$. We have the same three cases as in Claim 3, and we will refer to them identically.

In the case 1, we have that for all $w \in N(v,1)$, $\boldsymbol{x}'_w = 0$. Then let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1) \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{x}'_u > 0 \\ 0 & \text{else .} \end{cases}$$

Then exactly one of $\boldsymbol{x'} + \boldsymbol{c}$ and $\boldsymbol{x'} + 2\boldsymbol{c}$ is feasible and constitutes a maximal circuit step from $\boldsymbol{x'}$. If the former is feasible and maximal, then let $\boldsymbol{y} = \boldsymbol{x'} + \boldsymbol{c}$, and let $\boldsymbol{c'}$ be defined as

$$\boldsymbol{c}'_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1) \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{y}_u > 0 \\ 0 & \text{else .} \end{cases}$$

Then $\boldsymbol{y}+\boldsymbol{c}'$ is feasible and constitutes a maximal circuit step from $\boldsymbol{y}$. Let $\boldsymbol{y^1} = \boldsymbol{y}+\boldsymbol{c}'$. In this case we have that $\boldsymbol{y}^{\mathbf{1}}_v = 0$, $\boldsymbol{y^1}(N(v,1)) = 1$, and $\boldsymbol{y^1}(N(v,2)) = 0$.

If instead, $\boldsymbol{x}'+2\boldsymbol{c}$ is feasible and maximal, then let $\boldsymbol{y^2} = \boldsymbol{x}'+2c$. In this case we also have that $\boldsymbol{y}^{\mathbf{2}}_v = 0$, $\boldsymbol{y^2}(N(v,1)) = 1$, and $\boldsymbol{y^2}(N(v,2)) = 0$. Then in either case we arrive at a point $\boldsymbol{y}'$ with those properties. Now, let $\boldsymbol{c}''$ be defined as

$$\boldsymbol{c}''_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1) \\ 1/2 & \text{if } u \in N(v,2) \\ -1/2 & \text{if } u \in N(v,3),\ \boldsymbol{y}'_u > 0 \\ 0 & \text{else .} \end{cases}$$

Let $\boldsymbol{y}'' = \boldsymbol{y}'+\boldsymbol{c}''$. Then $\boldsymbol{y}''$ is feasible and the step from $\boldsymbol{y}'$ to $\boldsymbol{y}''$ is maximal since $b = 2$. Now, let $\boldsymbol{c}'''$ be defined as

$$\boldsymbol{c}'''_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1) \\ 0 & \text{else .} \end{cases}$$

Then $\boldsymbol{z} = \boldsymbol{y}''+\boldsymbol{c}'''$ is the desired point. In this case, we can walk from $\boldsymbol{x}'$ to $\boldsymbol{z}$ in at most 4 circuit steps.

In the case 2 we have that for all $w \in N(v,1)$, $\boldsymbol{x}'_w \leq 1/2$. Let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1) \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{x}'_u > 0 \\ 0 & \text{else .} \end{cases}$$

Let $\boldsymbol{y} = \boldsymbol{x}'+c$. Let $\boldsymbol{c}'$ be defined as

$$\boldsymbol{c}'_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1) \\ 1/2 & \text{if } u \in N(v,2),\ \boldsymbol{y}_u = 0 \\ -1/2 & \text{if } u \in N(v,3),\ \boldsymbol{y}_u = 1,\ \text{and there exists } w \in N(v,2) \text{ with } uw \in E,\ \boldsymbol{y}_w = 0 \\ 0 & \text{else .} \end{cases}$$

Let $\boldsymbol{y}' = \boldsymbol{y} + \boldsymbol{c}'$. Let $\boldsymbol{c}''$ be defined as

$$\boldsymbol{c}''_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1),\ \boldsymbol{y}'_u > 0 \\ 0 & \text{else .} \end{cases}$$

Then $\boldsymbol{z} = \boldsymbol{y}' + \boldsymbol{c}''$ is the desired point. In this case, we can walk from $\boldsymbol{x}'$ to $\boldsymbol{z}$ in at most 3 circuit steps.

In the case 3, let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1),\ \boldsymbol{x}'_u = 1 \\ 0 & \text{else .} \end{cases}$$

Let $\boldsymbol{y^1} = \boldsymbol{x}' + \boldsymbol{c}$ and $\boldsymbol{y^2} = \boldsymbol{x}' + 2\boldsymbol{c}$. For exactly one $i \in \{1, 2\}$, $\boldsymbol{y^i}$ is feasible and the step from $\boldsymbol{x}'$ to $\boldsymbol{y^i}$ is maximal. If $i = 1$ then we are in case 2. If $i = 2$ then we are in case 1. In either case, we can arrive at the point $\boldsymbol{z}$ in at most 4 more circuit steps, for a total of 5.

Now suppose $b > 2$. We have the same three cases as before, and we will refer to them identically.

In the case 1, we can use the same circuit steps as in the case when $b = 2$ to get to a point $\boldsymbol{y}$ such that $\boldsymbol{y}_v = 0$, $\boldsymbol{y}(N(v,1)) = 1$, and $\boldsymbol{y}(N(v,2)) = 0$. This requires at most 3 circuit steps. Let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1) \\ 1/2 & \text{if } u \in N(v,2) \\ -1/2 & \text{if } u \in N(v,3),\ \boldsymbol{y}_u > 00 \\ & \text{else .} \end{cases}$$

Let $\boldsymbol{y}' = \boldsymbol{y} + \boldsymbol{c}$ and $\boldsymbol{z} = \boldsymbol{y} + 2\boldsymbol{c}$. If $\boldsymbol{z}$ is feasible and the step from $\boldsymbol{y}$ to $\boldsymbol{z}$ is maximal, then $\boldsymbol{z}$ is the desired point. If $\boldsymbol{y}'$ is feasible and the step from $\boldsymbol{y}$ to $\boldsymbol{y}'$ is maximal, then let

$\boldsymbol{c}'$ be defined as

$$\boldsymbol{c}_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1) \\ 1/2 & \text{if } u \in N(v,2) \\ -1/2 & \text{if } u \in N(v,3),\ \boldsymbol{y}'_u > 00 \\ \text{else .} & \end{cases}$$

Let $\boldsymbol{z}' = \boldsymbol{y}' + \boldsymbol{c}'$. Then $\boldsymbol{z}'$ is the desired point. In any case, at most 5 circuit steps is enough.

In the case 2 we have that for all $w \in N(v,1)$, $\boldsymbol{x}'_w \leq 1/2$. Let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} -1/2 & \text{if } u = v \\ 1/2 & \text{if } u \in N(v,1) \\ -1/2 & \text{if } u \in N(v,2),\ \boldsymbol{x}'_u > 0 \\ 0 & \text{else .} \end{cases}$$

Let $\boldsymbol{y} = \boldsymbol{x}' + c$. Let $\boldsymbol{c}'$ be defined as

$$\boldsymbol{c}'_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1) \\ 1/2 & \text{if } u \in N(v,2) \\ -1/2 & \text{if } u \in N(v,3),\ \boldsymbol{y}_u > 0 \\ 0 & \text{else .} \end{cases}$$

If $\boldsymbol{y} + \boldsymbol{c}'$ does not give a maximal step, then $\boldsymbol{z} = \boldsymbol{y} + 2\boldsymbol{c}'$ is the desired point. Otherwise Let $\boldsymbol{y}' = \boldsymbol{y} + \boldsymbol{c}'$. Let $\boldsymbol{c}''$ be defined as

$$\boldsymbol{c}''_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1) \\ 1/2 & \text{if } u \in N(v,2) \\ -1/2 & \text{if } u \in N(v,3),\ \boldsymbol{y}'_u > 0 \\ 0 & \text{else .} \end{cases}$$

Then $\boldsymbol{z} = \boldsymbol{y}' + \boldsymbol{c}''$ is the desired point. In this case, we can walk from $\boldsymbol{x}'$ to $\boldsymbol{z}$ in at most 3 circuit steps.

In the case 3, let $\boldsymbol{c}$ be defined as

$$\boldsymbol{c}_u = \begin{cases} 1/2 & \text{if } u = v \\ -1/2 & \text{if } u \in N(v,1),\ \boldsymbol{x}'_u = 1 \\ 0 & \text{else .} \end{cases}$$

Let $\boldsymbol{y^1} = \boldsymbol{x}' + \boldsymbol{c}$ and $\boldsymbol{y^2} = \boldsymbol{x}' + 2\boldsymbol{c}$. For exactly one $i \in \{1,2\}$, $\boldsymbol{y^i}$ is feasible and the step from $\boldsymbol{x'}$ to $\boldsymbol{y^i}$ is maximal. If $i = 1$ then we are in case 2. If $i = 2$ then we are in case 1. In either case, we can arrive at the point $\boldsymbol{z}$ in at most 5 more circuit steps, for a total of 6. □

Invariants for $\boldsymbol{z}$ and t in Phase I: During Phase I, we update $\boldsymbol{z}$ and t such that at each moment of time the following holds for all $u \in N(v,k)$, for all $k \leq t$:

$$\boldsymbol{z}_u = \begin{cases} 0 & \text{if } k \equiv b+1 \mod 2 \\ 1 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 1/2 & \text{if } k \geq b \text{ and } k \equiv b \mod 2. \end{cases} \tag{$\star$}$$

By construction, $\boldsymbol{z}$ and t defined at the beginning of Phase I satisfy condition ($\star$) for all $u \in B(v,t)$. At each step (except possibly the last one) of Phase I, t is increased by 2 and the point $\boldsymbol{z}$ is updated to satisfy ($\star$) for all $u \in B(v,t)$. In the end of Phase I, t equals $\varepsilon(v)$, and hence ($\star$) holds for all $u \in V$.

Step of Phase I: At each step we change coordinates of point $\boldsymbol{z}$ corresponding to the nodes in $N(v,t+1)$ and $N(v,t+2)$.

If $t < b-2$, we walk from $\boldsymbol{z}$ to the point $\boldsymbol{z}'$, such that for all $u \in N(v,k)$, for all $k \leq t+1$

$$\boldsymbol{z}'_u = \begin{cases} 1 & \text{if } k \equiv b+1 \mod 2 \\ 0 & \text{if } k \equiv b \mod 2. \end{cases}$$

Such a point $\boldsymbol{z}'$ can be reached from $\boldsymbol{z}$ in at most two circuit steps. From $\boldsymbol{z}'$ we walk to the point $\boldsymbol{z}''$ such that for all $u \in N(v,k)$, for all $k \leq t+2$

$$\boldsymbol{z}''_u = \begin{cases} 0 & \text{if } k \equiv b+1 \mod 2 \\ 1 & \text{if } k \equiv b \mod 2. \end{cases}$$

A point $\boldsymbol{z}''$ with above properties can be reached from $\boldsymbol{z}'$ in one circuit step. Thus, in this case we are able to define $\boldsymbol{z}''$ to be the new point $\boldsymbol{z}$ and increase t by 2 using at most three circuit steps.

If $t = b - 2$, we walk from $\boldsymbol{z}$ to the point $\boldsymbol{z}'$, such that for all $u \in N(v,k)$, for all $k \leq t+1$

$$\boldsymbol{z}'_u = \begin{cases} 1 & \text{if } k \equiv b+1 \mod 2 \\ 0 & \text{if } k \equiv b \mod 2. \end{cases}$$

Such a point $\boldsymbol{z}'$ can be reached from $\boldsymbol{z}$ in at most two circuit steps. From $\boldsymbol{z}'$ we walk to the point $\boldsymbol{z}''$ such that for all $u \in N(v,k)$, for all $k \leq t+2$

$$\boldsymbol{z}''_u = \begin{cases} 1/2 & \text{if } k \equiv b+1 \mod 2 \\ 1/2 & \text{if } k \equiv b \mod 2, \end{cases}$$

where k is such that $u \in N(v,k)$. A point $\boldsymbol{z}''$ with above properties can be reached from $\boldsymbol{z}'$ in one circuit step. From $\boldsymbol{z}''$ we walk to the point $\boldsymbol{z}'''$ such that for all $u \in N(v,k)$, for all $k \leq t+2$

$$\boldsymbol{z}'''_u = \begin{cases} 0 & \text{if } k \equiv b+1 \mod 2 \\ 1 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 1/2 & \text{if } k \geq b \text{ and } k = b \mod 2. \end{cases}$$

A point $\boldsymbol{z}'''$ with above properties can be reached from $\boldsymbol{z}''$ in one circuit step. Thus, in this case we are able to define $\boldsymbol{z}'''$ to be the new point $\boldsymbol{z}$ and increase t by 2 using at most four circuit steps.

If $t \geq b$, we walk from $\boldsymbol{z}$ to the point $\boldsymbol{z}'$, such that for all $u \in N(v,k)$, for all $k \leq t+1$

$$\boldsymbol{z}'_u = \begin{cases} 1/2 & \text{if } k \equiv b+1 \mod 2 \\ 1/2 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 0 & \text{if } k \geq b \text{ and } k \equiv b \mod 2. \end{cases}$$

Such a point $\boldsymbol{z}'$ can be reached from $\boldsymbol{z}$ in one circuit step. From $\boldsymbol{z}'$ we walk to the point $\boldsymbol{z}''$ such that for all $u \in N(v,k)$, for all $k \leq t+2$

$$\boldsymbol{z}''_u = \begin{cases} 0 & \text{if } k = b+1 \mod 2 \\ 1 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 1/2 & \text{if } k \geq b \text{ and } k \equiv b \mod 2. \end{cases}$$

A point $\boldsymbol{z}''$ with above properties can be reached from $\boldsymbol{z}'$ in one circuit step. Thus, in this case we are able to define $\boldsymbol{z}''$ to be the new point $\boldsymbol{z}$ and increase t by 2 using only two circuit steps.

Note, that if at the beginning of a Phase step we have $\varepsilon(v) = t + 1$, we are in the case $t \geq b$. In this case, we need only two circuits steps to update $\boldsymbol{z}$ and increase t by 1.

Phase II: We are now at the "well structured" point $\boldsymbol{y'} = \boldsymbol{z}$. In this Phase, we construct a circuit walk from the current point $\boldsymbol{z}$ to the vertex $\boldsymbol{x}''$. Recall that at the end of Phase I, $\boldsymbol{z}$ satisfies ($\star$) for all $u \in V$ and $t = \varepsilon(v)$.

Start of Phase II: If for $w \in N(v, t)$ we have $\boldsymbol{z}_w = 0$, then we first take two circuit steps from the current $\boldsymbol{z}$ to the point $\boldsymbol{z'}$, such that for all $u \in N(v, k)$, for all $k \leq t - 1$

$$\boldsymbol{z}'_u = \begin{cases} 0 & \text{if } k \equiv b + 1 \mod 2 \\ 1 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 1/2 & \text{if } k \geq b \text{ and } k \equiv b \mod 2, \end{cases}$$

and for $u \in N(v, t)$

$$\boldsymbol{z}'_u = \begin{cases} 1/2 & \text{if } \boldsymbol{x}''_u \in \{1/2, 1\} \\ 0 & \text{if } \boldsymbol{x}''_u = 0. \end{cases}$$

Now we take two circuit steps from $\boldsymbol{z'}$ to $\boldsymbol{z}''$, such that for all $u \in N(v, k)$, for all $k \leq t - 2$

$$\boldsymbol{z}''_u = \begin{cases} 0 & \text{if } k \equiv b + 1 \mod 2 \\ 1 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 1/2 & \text{if } k \geq b \text{ and } k \equiv b \mod 2, \end{cases}$$

for $u \in N(v, t - 1)$ we have

$$\boldsymbol{z}''_u = \begin{cases} 0 & \text{if } uw \in E \text{ for some } w \in N(v, t),\ \boldsymbol{x}''_w = 1 \\ 1/2 & \text{otherwise }, \end{cases}$$

and for $u \in N(v, t)$ we have $\boldsymbol{z}''_u = \boldsymbol{x}''_u$. Thus, we define $\boldsymbol{z}''$ to be the new point $\boldsymbol{z}$ and decrease t by 1 using at most four circuit steps.

Invariants for $\boldsymbol{z}$ and t in Phase II: During Phase II, we update $\boldsymbol{z}$ and t such that at each moment of time the following holds for all $u \in N(v, k)$, for all $k \leq t - 1$

$$\boldsymbol{z}_u = \begin{cases} 0 & \text{if } k \equiv b + 1 \mod 2 \\ 1 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 1/2 & \text{if } k \geq b \text{ and } k \equiv b \mod 2, \end{cases} \qquad (\star\star)$$

and for $u \in N(v,t)$, we have

$$z_u = \begin{cases} 0 & \text{if } \max\{x''_w : w \in N(v,t+1),\, uw \in E\} = 1 \\ 1/2 & \text{if } \max\{x''_w : w \in N(v,t+1),\, uw \in E\} = 1/2 \\ \phi & \text{otherwise }, \end{cases} \qquad (\star\star\star)$$

where $\phi := 1/2$ if $b \geq t$ and $\phi := 1$ if $b < t$. Moreover, for all $u \in N(v,k)$, $k > t$, we have $z_u = x''_u$. Again by construction, z and t defined at the beginning of Phase II satisfy condition ($\star\star$) for all $u \in B(v,t-1)$ and condition ($\star\star\star$) for all $u \in N(v,t)$. At each step (except possibly the last one) of Phase II, t is decreased by 2 and the point z is updated to satisfy condition ($\star\star$) for all $u \in B(v,t-1)$ and condition ($\star\star\star$) for all $u \in N(v,t)$. At every moment of Phase II, we have $t = b \mod 2$.

Step of Phase II:

For all points on the circuit walk in a step of Phase II, we have $z_u = x''_u$ for every $u \in N(v,k)$, for all $k > t$.

If at the beginning of a step of Phase II we have $t \geq b+2$, we take a circuit step from z to a point z', such that for all $u \in N(v,k)$, for all $k \leq t-1$

$$z'_u = \begin{cases} 1/2 & \text{if } k \equiv b+1 \mod 2 \\ 1/2 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 0 & \text{if } k \geq b \text{ and } k \equiv b \mod 2, \end{cases}$$

and for $u \in N(v,t)$, we have

$$z'_u = \begin{cases} 1/2 & \text{if } x''_u \in \{1/2, 1\} \\ 0 & \text{if } x''_u = 0. \end{cases}$$

From z' we take a circuit step to a point z'', such that for all $u \in N(v,k)$, for all $k \leq t-2$

$$z''_u = \begin{cases} 0 & \text{if } k \equiv b+1 \mod 2 \\ 1 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 1/2 & \text{if } k \geq b \text{ and } k \equiv b \mod 2, \end{cases}$$

for $u \in N(v,t-1)$, we have

$$z''_u = \begin{cases} 1/2 & \text{if } x''_u \in \{1/2, 1\} \\ 0 & \text{if } x''_u = 0, \end{cases}$$

and for $u \in N(v,t)$ we have $\boldsymbol{z}''_u = \boldsymbol{x}''_u$. From $\boldsymbol{z}''$ we take a circuit step to $\boldsymbol{z}'''$ such that for all $u \in N(v,k)$, for $k \le t-2$

$$\boldsymbol{z}'''_u = \begin{cases} 1/2 & \text{if } k \equiv b+1 \mod 2 \\ 1/2 & \text{if } k < b \text{ and } k \equiv b \mod 2 \\ 0 & \text{if } k \ge b \text{ and } k \equiv b \mod 2. \end{cases}$$

Moreover, for $u \in N(v,t-1) \cup N(v,t)$ we have $\boldsymbol{z}'''_u = \boldsymbol{x}''_u$. It is not hard to see, that from $\boldsymbol{z}'''$ it takes at most one more additional circuit step to a point satisfying condition ($\star\star$) for all $u \in B(v,t-3)$ and condition ($\star\star\star$) for all $u \in N(v,t-2)$. Thus, for $t \ge b+2$ it takes at most four circuit steps to update $\boldsymbol{z}$ and decrease t by 2.

In the case when $t < b+2$, in the same way t can be decreased by 2 and the point $\boldsymbol{z}$ can be updated in at most four circuit steps. Note that for $u \in V$ we have $\boldsymbol{x}''_u = 1/2$ only if $k \ge b$ or $\boldsymbol{x}''_w = 1/2$ for some $w \in N(v,k+1)$, $uw \in E$, where k is such that $u \in N(v,k)$. Furthermore, for the very last Phase II step we need only three circuit steps if $t=1$ and only one circuit step if $t=0$.

Number of Circuit Steps in the Constructed Walk: The total number of circuit steps needed in both Phases is at most $4\varepsilon(v)+c$ for some constant c.

Indeed, to start Phase I we need at most a constant number of circuit steps. With each step of Phase I, t increases by 2 and we use at most 4 circuit step. We stop when $t = \varepsilon(v)$.

But if $t = \varepsilon(v)-1$ in the beginning of a Phase I step, we still need at most 4 circuit steps to finish this Phase I step and increase t by 1. Moreover, in this case we also need at most 4 circuit steps to start Phase II by updating t to be equal to $\varepsilon(v)-1$.

With each step of Phase II, t decreases by 2 and we use at most 4 circuit steps; finally we stop when $t=0$. This gives the upper bound of $2\varepsilon(v)+2\varepsilon(v)+c$ for some constant c on the total number of circuit steps in the constructed circuit walk from $\boldsymbol{x}'$ to $\boldsymbol{x}''$. $\square$

Lemma 10 immediately implies an upper bound on the diameter of the Fractional Stable Set polytope in terms of the *diameter* of a graph, defined to be $\operatorname{diam}(G) := \max_{v\in V}\{\varepsilon(v)\}$.

Corollary 3. *For every graph G, $\mathcal{CD}(P_{\text{FSTAB}}(G))$ is $\mathcal{O}(\operatorname{diam}(G))$.*

Chapter 6

Formulations of Polytopes and Their Circuit Diameters

Upon introducing and defining circuits, we assumed that for a polytope P, the system of inequalities describing P was minimal – that each inequality was facet-defining. Indeed in Chapter 5 we show that all constraints are facet-defining, and in Chapters 3 and 4 we restrict ourselves to the case where our input graph is complete precisely to guarantee that each constraint is facet-defining. We will now expand upon this assumption, consider its significance, and ultimately formalize the notion of the circuit diameter of a *formulation* of a polytope.

As mentioned in Chapter 2, the assumption of having a minimal formulation of a polytope is not without loss of generality, and marks a significant departure from the combinatorial case. In particular, adding to a system constraints which are *not* facet-defining cannot reduce the combinatorial diameter of the polytope defined by that system. However, adding such constraints to a system can indeed add to the set of circuits, reducing the diameter. Thus, if we want to be able to say anything about the circuit diameter of a polytope that depends only on its *internal* description, it becomes necessary to forbid constraints which are not facet-defining. Otherwise, we may artificially "pad" a system with constraints which do nothing to define the set of points feasible for that system, but which reduce the diameter.

However, we should perhaps not discount the possibility that the importance of having facet-defining constraints is contextual. For example, one may reasonably feel that there is a not-insignificant difference between *adding* constraints to a system for the express purpose of reducing the circuit diameter, and merely *allowing* constraints which are not facet-defining, but which appear for some independently justifiable reason.

One such example is precisely the case we avoided in Chapters 3 and 4: the inclusion of constraints from a standard problem formulation which may or may not be facet-defining depending on the input graph. While it may not seem unreasonable to allow the inclusion of such constraints, we should ask whether there is any mathematical context which justifies considering the possibility. In particular, we may ask if there is any context in which the formulation we use for a problem is more important than the precise nature of the polytope itself.

To answer this question, we may return to one of the original motivations behind studying polytope diameters in the first place: augmentation algorithms to solve combinatorial optimization problems modeled by polyhedra. When solving a combinatorial optimization problem via linear programming, one often uses the full suite of constraints from the standard formulation of the problem with no effort made to cull constraints which are not facet-defining for the particular problem input.

Indeed, in practical settings we often accept that an LP solver like the Simplex method will spend computation time performing degenerate basis exchanges due to constraint redundancy. It seems reasonable enough, then, that if these constraints are being included indiscriminately – and indeed produce a small *hindrance* to augmentation algorithms based on basis exchanges – that we may *leverage* their information in an augmentation algorithm based on circuit walks. We do not discard this information when it slows us down. Why discard it when we might find it helpful?

To that end, one can define the notion of the circuit diameter of a *formulation* of a polytope. Let P be a polytope of the form $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b},\, B\boldsymbol{x} \leq \boldsymbol{d}\}$ for rational matrices A and B and rational vectors $\boldsymbol{b}$ and $\boldsymbol{d}$ where we assume the system is minimal with respect to its constraints, as before.

Definition 6. *The tuple $\mathcal{F} = (M, \boldsymbol{p}, N, \boldsymbol{q})$ is a formulation of P if for all $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \in P$ iff $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^n : M\boldsymbol{x} = \boldsymbol{p},\, N\boldsymbol{x} \leq \boldsymbol{q}\}$*

Note that given a system of linear constraints associated with a formulation $\mathcal{F}$, the polytope P for which $\mathcal{F}$ is a formulation is implicit (provided the system is feasible). The matrices defining this polytope can be obtained from $\mathcal{F}$ by removing redundant inequality constraints. We can now define the circuits and circuit diameter of the formulation $\mathcal{F}$ identically to the way they are defined for a polytope. For simplicity, we may informally refer directly to the system $\{\boldsymbol{x} \in \mathbb{R}^n : M\boldsymbol{x} = \boldsymbol{p},\, N\boldsymbol{x} \leq \boldsymbol{q}\}$ as the formulation $\mathcal{F}$ instead of explicitly referring to the tuple $(M, \boldsymbol{p}, N, \boldsymbol{q})$.

Definition 7. *A non-zero vector* $\boldsymbol{g} \in \mathbb{R}^n$ *(whose components are normalized to be coprime integers) is a circuit of* $\mathcal{F}$ *if*

(i) $\boldsymbol{g} \in \mathrm{Ker}(A)$

(ii) $\mathrm{supp}(M\boldsymbol{g})$ *is not contained in any of the sets from the collection* $\{\mathrm{supp}(M\boldsymbol{y}) : \boldsymbol{y} \in \mathrm{Ker}(A), \boldsymbol{y} \neq \mathbf{0}\}$.

Note that the set of circuits of $\mathcal{F}$ is a superset of the set of circuits of P. The definitions of circuit step, circuit walk, circuit distance, and circuit diameter for the formulation $\mathcal{F}$ are precisely the same as for the polytope P, using instead the circuits of $\mathcal{F}$. Note also that the statements and proofs of Proposition 1 and Lemma 1 do not depend on the fact that the constraints defining a polytope are constraint-minimal. Thus, they hold for formulations as well as for polytopes. We also note the following corollary of the above definitions:

Corollary 4. *If* $\mathcal{F}$ *is a formulation of a polytope* P*, then* $\mathcal{CD}(\mathcal{F}) \leq \mathcal{CD}(P)$.

6.1 Extending Previous Results

We can now extend our results from Chapter 3 to general (not necessarily complete) graphs using a formulation of the Matching polytope and Perfect Matching polytope. We will be using precisely the same linear systems as before to describe our formulations; the only difference being that for general graphs, some of these constraints may no longer be facet-defining. However, the proofs remain the same.

Let $G = (V, E)$ be a simple connected graph and let $n = |V|$. The Matching polytope of G is defined as the convex hull of all characteristic vectors of matchings in G i.e.,

$$P_{\text{MATCH}}(G) = \text{conv}\,\{\chi(M) \,|\, M \text{ is a matching in } G\}\,.$$

Corollary 5. *Consider the formulation* $\mathcal{F}_{\text{MATCH}}(G)$ *of* $P_{\text{MATCH}}(G)$ *defined by the following linear system:*

$$\begin{aligned}
&\boldsymbol{x}\,(E[S]) \leq (|S|-1)/2 && \textit{for all } S \subset V,\ |S| \textit{ is odd}\\
&\boldsymbol{x}(\delta(v)) \leq 1 && \textit{for all } v \in V\\
&\boldsymbol{x} \geq \mathbf{0}\,.
\end{aligned}$$

Then we have the following cases:

$$\mathcal{CD}(\mathcal{F}_{\text{MATCH}}(G)) \leq \begin{cases} 1 & n = 2, 3 \\ 2 & n = 4, 5 \\ 3 & n = 6 \\ 2 & n \geq 7\,. \end{cases}$$

Now, assume G admits a perfect matching. The Perfect Matching polytope of G is defined as the convex hull of all characteristic vectors of perfect matchings in G i.e.,

$$P_{\text{PERFECT MATCH}}(G) := \text{conv}\left\{\chi(M) \,|\, M \text{ is a perfect matching in } G\right\}\,.$$

Corollary 6. *Consider the formulation $\mathcal{F}_{\text{PERFECT MATCH}}(G)$ of $P_{\text{PERFECT MATCH}}(G)$ defined by the following linear system:*

$$\begin{aligned} &\boldsymbol{x}\,(\delta(S)) \geq 1 && \textit{for all } S \subset V,\ |S| \textit{ is odd} \\ &\boldsymbol{x}(\delta(v)) = 1 && \textit{for all } v \in V \\ &\boldsymbol{x} \geq \mathbf{0}. \end{aligned}$$

Then we have the following cases:

$$\mathcal{CD}(\mathcal{F}_{\text{PERFECT MATCH}}(G)) \leq \begin{cases} 1 & n = 4, 6 \\ 2 & n = 8 \\ 1 & n \geq 10\,. \end{cases}$$

6.2 Matroid Polytope

Now, let $M = (E, \mathcal{I})$ be a matroid, and let $rk(S)$ denote the rank of $S \subseteq E$. We assume that every singleton subset of E is independent. The Matroid polytope of M is defined as the convex hull of all characteristic vectors of independent sets in M i.e.,

$$P_{MAT}(M) = \text{conv}\{\chi(I)\,:\, I \text{ is an independent set of } M\}.$$

Let $\mathcal{F}_{MAT}(M)$ be the following well-known formulation of $P_{MAT}(M)$ [14]:

$$\begin{aligned} &\boldsymbol{x}\,(S) \leq rk(S) \quad \text{for all } S \subseteq E \\ &\boldsymbol{x} \geq \mathbf{0}\,. \end{aligned} \tag{6.1}$$

Lemma 11. *Let $\boldsymbol{c}$ be a vector with entries in $\{1, 0, -1\}$ such that if $\boldsymbol{c}$ has more than one non-zero entry, then $\boldsymbol{c}$ has at least one positive entry and at least one negative entry. Then $\boldsymbol{c}$ is a circuit of $\mathcal{F}_{MAT}$.*

Proof. Note that if $\boldsymbol{c}$ has exactly one non-zero entry, then $\boldsymbol{c}$ is a circuit of $\mathcal{F}_{MAT}$. Now, suppose $\boldsymbol{c}$ has more than one non-zero entry. Let us assume for the sake of contradiction that $\boldsymbol{c}$ is not a circuit. Then there exists a non-zero vector $\boldsymbol{y}$ such that $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, where D is the constraint matrix of (6.1). Since the inequalities $\boldsymbol{x}_e \geq 0$, $e \in E$ are in the linear description (6.1) and $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, we have that $\boldsymbol{y}_e = 0$ for every edge e such that $\boldsymbol{c}_e = 0$.

Let e be some element of E for which $\boldsymbol{y}_e \neq 0$. Without loss of generality, and by possibly rescaling vector $\boldsymbol{y}$, we can assume $\boldsymbol{y}_e = 1$. Furthermore, by possibly negating $\boldsymbol{c}$, we can assume $\boldsymbol{c}_e = 1$ as well.

For any $f \in E$ such that $\boldsymbol{c}_f = -1$ (of which there is at least one), we have that $\boldsymbol{c}(\{e, f\}) = 0$. Since $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, we have that $\boldsymbol{y}(\{e, f\}) = 0$, and thus $\boldsymbol{y}_f = -1$ as well. Given that $\boldsymbol{y}_f = -1$ for some f, if there are any other elements $f' \in E$ such that $\boldsymbol{c}_{f'} = 1$, then since $\boldsymbol{c}(\{f, f'\}) = 0$, we get that $\boldsymbol{y}_{f'} = 1$ as well. Thus, $\boldsymbol{y} = \boldsymbol{c}$, a contradiction. ☐

Theorem 4. $\mathcal{CD}(\mathcal{F}_{MAT}(M)) \leq 3$

Proof. Let $\boldsymbol{x}_1 = \chi(J_1)$ and $\boldsymbol{x}_2 = \chi(J_2)$ be two vertices of $F_{MAT}(M)$ where J_1 and J_2 are two unequal independent sets of M. If $\boldsymbol{x}_2 - \boldsymbol{x}_1$ has at least one positive and one negative entry, then $\boldsymbol{x}_2 - \boldsymbol{x}_1$ is a circuit of $F_{MAT}(M)$ by Lemma 11.

Otherwise, we first assume that $J_1 \subsetneq J_2$, $J_1 \neq \emptyset$. Let $e \in J_1$. Then $\boldsymbol{z} = \chi(J_2 \setminus \{e\})$ is one circuit step away from $\boldsymbol{x}_1$, and since $J_2 \setminus e$ is independent, this is a feasible point for $\mathcal{F}_{MAT}(M)$. Finally, $\boldsymbol{x}_2$ is one circuit step away from $\boldsymbol{z}$.

If $J_2 \subsetneq J_1$, $J_2 \neq \emptyset$, then the proof is similar. Let $e \in J_2$. Then $\boldsymbol{z} = \chi(J_1 \setminus \{e\})$ is one circuit step away from $\boldsymbol{x}_1$, and again, this is a feasible point for $\mathcal{F}_{MAT}(M)$. Finally, $\boldsymbol{x}_2$ is one circuit step away from $\boldsymbol{z}$.

If $J_1 = \emptyset$ and $J_1 \subsetneq J_2$, then for any $e \in J_2$, $\chi(\{e\})$ is one circuit step away from $\chi(J_1)$. Then as above, $\chi(\{e\})$ is two circuit steps away from $\chi(J_2)$.

If $J_2 = \emptyset$ and $J_2 \subsetneq J_1$, then first suppose $J_1 \neq E$. Let $e \notin J_2$. Then $\chi(\{e\})$ is one circuit step away from $\chi(J_1)$ by Lemma 11, and $\chi(\{e\})$ is one circuit step away from $\chi(\emptyset) = \chi(J_2)$. Now suppose that $J_1 = E$. Then for any $e \in J_1$, $\chi(J_1 \setminus \{e\})$ is one circuit step away from $\chi(J_1)$, $\chi(\{e\})$ is one circuit step away from $\chi(J_1 \setminus e)$ by Lemma 11, and finally $\chi(J_2) = \chi(\emptyset)$ is one circuit step away from $\chi(\{e\})$. ☐

We remark that it was shown in [15] that for connected matroids, an inequality of $\mathcal{F}_{MAT}(M)$ corresponding to a set F is facet-defining if and only if F is a flat such that the restriction of M to F and the contraction of M by F are both connected matroids (we refer to [22] for definitions). As the above proof does not restrict itself to such inequalities, for general matroids M it can only be said to be a result about the formulation $\mathcal{F}_{MAT}(M)$.

It is an interesting question whether any nice bound can be shown for the circuit diameter of $P_{MAT}(M)$ for general matroids M.

6.3 Spanning Tree Polytope

Let $G = (V, E)$ be a simple connected graph which is not a tree. The Spanning Tree polytope is defined as the convex hull of all characteristic vectors of spanning trees of G i.e.,

$$P_{\text{TREE}}(G) = \text{conv}\left\{\chi(T) \,:\, T \text{ is a spanning tree of } G\right\}.$$

Let $\mathcal{F}_{\text{TREE}}(G)$ be the following well-known formulation of $P_{\text{TREE}}(G)$ [13]:

$$\begin{aligned} &\boldsymbol{x}\left(E[S]\right) \leq |S| - 1 \quad \text{for all } S \subsetneq V \\ &\boldsymbol{x}\left(E\right) = |V| - 1 \\ &\boldsymbol{x} \geq \boldsymbol{0}\,, \end{aligned} \tag{6.2}$$

where $E[S]$ denotes the edges induced by S. Let $\Delta(G)$ denote the max degree of a node in G.

Theorem 5. *$\mathcal{CD}(\mathcal{F}_{\text{TREE}}(G))$ is $\mathcal{O}(\Delta(G))$.*

Before the proof of this theorem, we will give two lemmas. Let e_1 and e_2 be two consecutive edges of some cycle C of G. We define N to be the set of edges of G which share a vertex with either of e_1 or e_2, and N' to be $N - \{e_1, e_2\}$. We say that two subgraphs H and J of G agree on some set of edges $F \subseteq E$ if for every edge f in F, f is either in both H and J, or neither H nor J.

Lemma 12. *There exist two spanning trees of G, R_1 and R_2, such that R_1 contains e_1 and not e_2, R_2 contains e_2 and not e_1, and R_1 and R_2 agree on N'.*

Proof. First we construct R_1 by starting with the edges $C - e_2$ and then extending this to a spanning tree arbitrarily. Now we construct R_2 by adding e_2 and removing e_1. Since these two trees agree on every edge except e_1 and e_2, they agree on N'. $\square$

Lemma 13. *Let R_1 and R_2 be any two spanning trees of G which satisfy the hypotheses of Lemma 12. Then $\boldsymbol{c} = \chi(R_2) - \chi(R_1)$ is a circuit of $P_{\text{TREE}}(G)$.*

Proof. Note that $\boldsymbol{c}_{e_1} = -1$, $\boldsymbol{c}_{e_2} = 1$, and for any edge $f \in N'$, $\boldsymbol{c}_f = 0$. Let $e_1 = v_1 u$ and $e_2 = uv_2$. Let us assume for the sake of contradiction that $\boldsymbol{c}$ is not a circuit. Then there exists a non-zero vector $\boldsymbol{y}$ such that $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, where D is the constraint matrix of (6.2). Since the inequalities $\boldsymbol{x}_e \geq 0$, $e \in E$ are in the linear description (6.2) and $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, we have $\boldsymbol{y}_e = 0$ for every edge e such that $\boldsymbol{c}_e = 0$.

Since R_1 and R_2 agree on N', for any nodes w_1 and w_2 in G, the only edges in $E[\{v_1, u, w_1, w_2\}]$ or $E[\{u, v_2, w_1, w_2\}]$ for which $\boldsymbol{c}$ can have non-zero value are e_1 or e_2, respectively, and w_1w_2. Thus, since $\text{supp}(Dy) \subset \text{supp}(Dc)$, the same is true for $\boldsymbol{y}$. Let e be some edge for which $\boldsymbol{y}_e \neq 0$. Without loss of generality, and by possibly rescaling vector $\boldsymbol{y}$, we can assume $\boldsymbol{y}_e = 1$. Furthermore, by exchanging the roles of R_1 with R_2, we can assume $\boldsymbol{c}_e = 1$ as well. Note that if we exchange the roles of R_1 and R_2, we also exchange the roles of e_1 and e_2. Thus, it will always be the case that $\boldsymbol{c}_{e_1} = -1$.

Let $e = t_1t_2$. Then $\boldsymbol{c}(E[\{t_1, t_2, v_1, u\}]) = 0$. Since $\text{supp}(D\boldsymbol{y}) \subset \text{supp}(D\boldsymbol{c})$, we have $0 = \boldsymbol{y}(E[\{t_1, t_2, v_1, u\}]) = \boldsymbol{y}_e + \boldsymbol{y}_{e_1}$, and thus $\boldsymbol{y}_{e_1} = -1 = \boldsymbol{c}_{e_1}$.

Note that we do not require that $\{w_1, w_2\} \cap \{u, v_i\} = \emptyset$ in the constraints of 6.2, so the above argument is still valid even if $e = e_2$. Note that if $e = e_2$, we have that $\boldsymbol{y}_{e_2} = \boldsymbol{c}_{e_2}$. If $e \neq e_2$, then since $\boldsymbol{c}(E[\{v_1, u, v_2\}]) = 0$, we have that $\boldsymbol{y}_{e_2} = 1$. In either case, we have that $\boldsymbol{y}_{e_i} = \boldsymbol{c}_{e_i}$ for $i \in \{1, 2\}$.

Now, for any other edge $s_1s_2 \neq e_2$ such that $\boldsymbol{c}_{s_1s_2} = 1$, we get that $\boldsymbol{c}(E[\{s_1, s_2, v_1, u\}]) = 0$ and therefore $\boldsymbol{y}_{s_1s_2} = 1$. Similarly, we get that for any other edge $s_1s_2 \neq e_1$ such that $\boldsymbol{c}_{s_1s_2} = -1$, we get that $\boldsymbol{c}(E[\{s_1, s_2, u, v_2\}]) = 0$ and therefore $y_{s_1s_2} = -1$. Then $\boldsymbol{y} = \boldsymbol{c}$, a contradiction. □

With these two lemmas, we can now prove Theorem 5.

Proof. Let T_1 and T_2 be arbitrary spanning trees of G. As before, let C be an arbitrary cycle of G, and let e_1 and e_2 be two edges of C which share a vertex. Let R_1 and R_2 be *any* two spanning trees of G such that R_1 contains e_1 and not e_2, R_2 contains e_2 and not e_1, and R_1 and R_2 agree on N' (as defined above). We will first construct a path of combinatorial steps (steps which correspond to edges of the polytope) from $\chi(T_1)$ to another point $\chi(T_1')$, where T_1' is a spanning tree which agrees with R_1 on N. The same process gives a path of combinatorial steps from $\chi(T_2)$ to another point $\chi(T_2')$, where T_2' is a spanning tree which agrees with R_2 on N. By Lemma 13, $\chi(T_1')$ and $\chi(T_2')$ are one circuit step away from each other, and since the path from $\chi(T_2)$ to $\chi(T_2')$ only uses combinatorial steps, we can guarantee that the path is reversible. Thus, we can concatenate the paths from $\chi(T_1)$ to $\chi(T_1')$, from $\chi(T_1')$ to $\chi(T_2')$, and from $\chi(T_2')$ to $\chi(T_2)$, giving a path from $\chi(T_1)$ to $\chi(T_2)$.

We now demonstrate how to get from $\chi(T_1)$ to $\chi(T_1')$. Combinatorial steps correspond to adding an edge e to our tree and removing a different edge f in the unique cycle created by adding e [27]. For simplicity, we will speak about trees, and not the incidence vectors of trees. Similarly, we will speak about operations on trees which will correspond naturally to combinatorial steps in our polytope.

Let N and N' be defined as above. Let $M = T_1 \cap N$, and let $M' = R_1 \cap N$. Let f be any edge in $M' \setminus M$. We would like to add f to T_1 and remove some other edge f^- from T_1 such that f^- does not itself appear in M'. Let C be the unique cycle created by adding f to T_1, and suppose for the sake of contradiction that for any other edge f^- in C, f^- is also in M'. Then since M' contains f by assumption, M' contains a cycle, and thus so does R_1, a contradiction.

Then for any edge in M' which we wish to add to T_1, we can always find an edge to remove which doesn't interfere with our objective of making T_1 agree with R_1 on N. Note that since T_1 was arbitrary, and R_1 was chosen independently of our choice of T_1, we can repeatedly apply this process until we arrive at a spanning tree T_1^* such that $M' \subseteq M^* = N \cap T_1^*$.

Now to get from T_1^* to a tree T_1' which agrees with R_1 on N, we must remove any edges in M^* which are not in M'. Let f be any edge in $M^* \setminus M'$. We would like to remove f from T_1^* and add some other edge f^+ to T_1^* such that f^+ is not in N. We specify that f^+ is not in N because every edge in $M' \subseteq N$ is already in T_1^*. Thus any further edges we add must come from $E \setminus N$.

Let D_1 and D_2 be the two connected components of $T_1^* \setminus \{f\}$, and let F be the set of edges in E with one endpoint in D_1 and one endpoint in D_2. Suppose for the sake of contradiction that for any edge f^+ in $F \setminus \{f\}$, f^+ is also in N. Note that any such edge is not in M': Suppose $f^+ \neq f$ has one endpoint in D_1 and one endpoint in D_2 and f^+ is in M'. Then since $M' \subseteq T_1^*$, f^+ is also in T_1^*, and together f and f^+ create a cycle in T_1^*, a contradiction.

Then M' does not contain any edges from F since it also does not contain f. However, the edges of F define a cut of G, and if every edge of F is in N, but none are in M', then none are in R_1. Then R_1 is not connected, a contradiction.

Then for any edge in M^* which we wish to remove from T_1^*, we can always find an edge to add which doesn't interfere with our objective of making T_1^* agree with R_1 on N.

Again, we can repeatedly apply this process until we arrive at a spanning tree T_1' such that $T_1' \cap N = M'$, and therefore, T_1' agrees with R_1 on N. Note that this also means that T_1' contains e_1 and not e_2, as is the case with R_1. We can use the same process to determine our path from T_2 to T_2^*.

As per the explanation at the beginning of the proof, it follows that we get a path from T_1 to T_2. It remains to determine the length of this path. When moving from T_1 to T_1', we applied at most one step per edge in N. For each vertex in $\{v_1, u, v_2\}$, there are at most $\Delta(G)$ edges in N. Thus, there are at most $3\Delta(G)$ edges in N. However, this over-counts the edges e_1 and e_2 once each. Moving from T_2' to T_2 has the same bound, and moving from T_1' to T_2' takes one step. Thus, the total distance is at most $2(3\Delta(G)-2)+1 = 6\Delta(G)-3$, as desired. □

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