The Relation between Polygonal Gravity and 3D Loop Quantum Gravity

by

Paul Tiede

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Applied Mathematics

Waterloo, Ontario, Canada, 2017

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis, we explore the relation between 't Hooft polygonal gravity and loop quantum gravity (LQG) - two models of discrete gravity in 2+1 dimensions. While the relation between the two theories has been studied in the past, the relation between LQG and polygonal gravity remains unclear. Indeed we argue that each approach does not implement the same type of constraint at the kinematical level. Using a dual formulation of LQG, we show that polygonal gravity is then recovered by a gauge fixing in this framework. However, whether these gauge choices are possible in general is unanswered in this work. Therefore, we analyze a specific example given by the torus universe in each approach, using one and two polygon decompositions. By using the map from dual LQG to polygonal gravity, we express the physical variables of discrete gravity, or observables, in terms of polygonal gravity quantities. Once the constraints in polygonal gravity are implemented we find that physical observables are no longer independent, meaning that polygonal gravity cannot describe the torus universe using one and two polygon decompositions: the gauge fixing is actually over-constraining the theory. Faced with these results, we develop a dual version of 't Hooft gravity. The resulting theory is then proven to be equal to the kinematical phase space of LQG; therefore, dual 't Hooft gravity is free of the issues plaguing polygonal gravity.

Acknowledgements

I would like to thank numerous people for helping me through the last few years. First and foremost I would like to thank my supervisor, for guiding me though my master's degree and proposing this interesting line of research.

I would also like to thank Anthony Caterini who forced me to have a life outside of school and introducing me to the pump. I am indebted to my office mates and close neighbors Abdulmajid Osumanu, Yangang Chen and Jason Pye for the wonderful conversations about physics and other worldly events.

I also thank the Canadian Government for partially funding this research through the National Sciences and Engineering Research Council (NSERC) Canadian Graduate Scholarship (CGS-M).

Finally, but not least, I would like to thank Alex White for putting up with me these last few months, when research often overtook my life. Your loving support kept me going.

Dedication

This is dedicated to my parents for giving me the courage to explore the wonderful world of physics.

Table of Contents

List of Tables viii

List of Figures ix

1 Introduction 1

2 Introduction to 2+1 Canonical General Relativity 5

2.1 First order gravity . . . 5

2.1.1 Lagrangian formulation of first order gravity . . . 7

2.2 Canonical analysis . . . 9

2.2.1 First order gravity: Hamiltonian formulation . . . 9

3 Discrete Loop Gravity in Two Different Polarizations 13

3.1 Discretizating phase space . . . 13

3.2 LQG polarization . . . 19

3.3 Dual LQG polarization . . . 24

3.4 Conclusion . . . 28

4 't Hooft Polygonal Gravity 29

4.1 Kinematical structure and discretization . . . 29

4.2 Brackets and Constraints . . . 36

4.3 Conclusion . . . 39

5 Exploring the Relationship Between LQG and Polygonal Gravity 40

5.1 Comparing discretizations of polygonal gravity and LQG . . . 41

5.2 Gauge fixing dual LQG to polygonal gravity . . . 42

5.2.1 Gauge fixing to pure polygons . . . 43

5.3 The torus universe in LQG and polygonal gravity . . . 51

5.3.1 One polygon torus universe . . . 52

5.3.2 Two polygon torus universe 57
5.4 Conclusion 61

6 Duality in ’t Hooft Gravity **62**
6.1 Dual ’t Hooft gravity 63
6.2 Relation to LQG and Regge calculus 66
6.3 Conclusion 69

7 Conclusions and Future Work **71**

Bibliography **74**

A The Relation Between SO(2,1) and SU(1,1) **78**

B Symplectic Geometry and Constrained systems **80**
B.1 Symplectic manifolds 80
B.1.1 Momentum Maps 81
B.2 Dirac Bergman constraint analysis 83
B.2.1 An example of gauge fixing: the relativistic particle 86

List of Tables

3.1 Summary of the components of the triangulation Γ^* and dual graph Γ^* 14
3.2 Comparison between LQG, LQG* 28

5.1 Comparison between LQG, LQG* and 't Hooft gravity. 41

6.1 Comparison between 't Hooft polygonal gravity and dual 't Hooft gravity. . . 65

List of Figures

3.1 Components of the graphs Γ, Γ^* and the holonomy $g_c(x)$. Curvature and torsion sit on the vertices of Γ^*, i.e. v_i. The face c^* and $\bar{c}*$ share the edge $\ell^* = [\bar{v}v]$, the dual of the oriented edge $\ell = [\bar{c}c]$. 14
3.2 The constant holonomy $h_{\bar{c}c}$, relates the frames associated to the "center" of the two faces c and $\bar{c}$. 18
3.3 Gauss constraint associated to a vertex at v_0 26

4.1 The gauge fixing condition $t_i = t_j$ defines the boundaries between the polygons. The Θ and Θ' are the dihedral angles between the polygons. 30
4.2 The nodes and links of the graph Γ form the vertices and edges of the polygon respectively. The angles between the edges of the polygon are specified by the dihedral angle 2η, which is a boost. 31
4.3 Decomposition of holonomy between polygons. The $\phi_{v_2 v_1}$ orientates the frame $\mathcal{X}_{v_1}$ so that the y-axis is parallel with the polygon edge $[c_2 c_1]$. The parameter $2\eta_{v_2 v_1}$ is the boost around the edge $[c_2 c_1]$ and is the dihedral angle between the polygons v_1^*, v_2^*. Finally the angle $\bar{\phi}_{v_2 v_1}$ rotates the frame so that aligns with the frame $\mathcal{X}_{v_2}$. 32
4.4 The angles $\phi_{v_1 v_0}$, $\bar{\phi}_{v_0 v_2}$, specify the orientation of the edges $[c_2 c]$ and $[c c_1]$ respectively. They are related the angles $\alpha^c_{v_0}$ between the edges of the polygon, through the equation $\alpha^c_{v_0} = \phi_{v_1 v_0} + \bar{\phi}_{v_0 v_2}$. 33
4.5 Evolution of two links from at the node c. The growth rate of the edge i from the node c is given by $w_{c,i}$. 35

5.1 Using the decomposition (5.7), we can exactly recover the angles between the fluxes from the holonomy matrices. 47
5.2 By concatenating flatness constraints we can form F_λ, which enforces vanishing 3-curvature. Each polygon is extrinsically flat (grey fill), so we can derive polygonal gravity's constraint $C_{\alpha,f}$ constraint. 49

5.3 Left: Triangulation of torus and dual graph Γ. Opposite edges of triangulation are identified. Right: Dual polygon, found from cutting along the edges of the graph Γ. 52
5.4 Two polygon torus universe. 57

6.1 In dual 't Hooft gravity, observers sit at the nodes c_i of Γ, and describe the lengths of the edges of the triangulation Γ^* 64
6.2 Kinematical observables for the LQG polarization. These provide enough information to completely describe the kinematical phase space. 68

Chapter 1

Introduction

Physics in the 20$^{\text{th}}$ and 21$^{\text{st}}$ century has been about discovery and unification. However, most of modern physics is based off two separate ideas first proposed in the early 20$^{\text{th}}$ century: *quantum mechanics* and *general relativity*. Quantum mechanics was developed to explain some peculiar properties of matter, such as the photo-electric effect, black body radiation and the spectroscopy of atoms. The development of quantum mechanics led to an upheaval of the most basic laws of physics. No longer could precise statements be made about a particles trajectory; quantum mechanics has permanently made physics "fuzzy" or uncertain in nature. The ultimate culmination of quantum mechanics is the standard model. It is currently the most accurate physical theory and describes three of the four fundamental interactions in the Universe and all visible matter[1]. However, the fourth interaction, gravity, differs greatly.

The other major theory proposed in the early 1900s was *general relativity* (GR). General relativity is the most successful theory of gravitation ever devised. It has passed numerous tests, the most recent being the detection of gravitational waves by the LIGO collaboration [1]. Like quantum mechanics, GR has also changed the way we view physics. GR led to the realization that gravitation very different than the other interactions known in nature. Unlike the other forces that assume the existence of a background, usually Minkowski space, GR actually defines the background we live on. This is due to the special symmetry that GR obeys *general covariance* or *diffeomorphism symmetry*. The essence of this symmetry is that most quantities in nature are only described relative to each other. Despite all of these successes, it is thought that GR must break down at distances around the Planck length[2]. This occurs because general relativity is inconsistent with the principles

[1]The nature of dark matter, or if it even exists, is currently unclear at the time of this writing. We will not deal with this interesting problem in this thesis.

[2]The scale of the Planck length is 10^{-32} millimeters

of quantum mechanics. Therefore, it has long been suspected that we will need a theory of quantum gravity. The reasons for this are purely theoretical in nature, as there is no clear experimental evidence for such a theory.

However, quantum gravity has proven to be one of the most elusive theories in modern physics. In fact, the search for a quantum theory of gravity began immediately after the discovery of quantum mechanics and general relativity. Part of Dirac's work in [10] was dedicated to making Einstein's theory of general relativity more amenable to quantization. However, after almost a century of research, there is still no convincing solution. One reason for this elusiveness is a lack of understanding of the features that quantum gravity should have. Conceptually, the symmetries of general relativity (i.e. diffeomorphism invariance) make constructing observables extremely challenging [11, 12]. This has led to the famous "problem of time" in the quantum gravity community. Furthermore, technically constructing a model is arduous due to the nonlinear nature of general relativity.

When faced with such a problem, the first step is to simplify. This had led to the field of 2+1 quantum gravity. Conceptually, the theory is similar to 4D gravity; therefore, many lessons about the nature quantum gravity can be extracted from its study. Furthermore, the quantization of 3D gravity is technically simpler. This is due to the fact that the vacuum Einstein equations imply that there are no local degrees of freedom. The simplification of the resulting theory is so successful that there exists a myriad of approaches to the quantization of gravity in 2+1 dimensions (see [9] for a wonderful overview). As a result, it appears that a quantum theory of gravity may have a minimum scale. For instance, in loop quantum gravity it was discovered that there is a minimum area in 3+1 dimensions [44]. This has led to the exploration of discrete models of gravity. In 2+1 dimensions, discrete models can be exact, since there are no local degrees of freedom; however, there are many different ways to discretize 2+1 gravity.

The goal of this thesis is to study the relationship between two theories of discrete gravity. The first theory studied was developed by Gerard 't Hooft, and is called 't Hooft polygonal gravity, or just polygonal gravity. It was developed in a series of papers [46, 45, 47, 48], that analyzed the causal nature of 2+1 gravity with particles. The resulting theory of gravity, was described by an evolving surface composed of flat polygons. The theory was derived from purely geometric arguments, and provides a wonderful visualization of gravity. Unfortunately, as a result, its exact relation to 2+1 gravity is obscure. Moreover, the constraint algebra of the theory is immensely complicated [20], and the symmetries of the theory are poorly understood. Furthermore, there are disagreements in the literature when applying the theory to the torus universe [23, 54]. Finally, the quantization of this theory is currently unknown [45]. These issues have made it necessary to compare said theory to other models of discrete gravity, such as loop "quantum" gravity (LQG).

Loop quantum gravity (LQG) [43, 50] is one of the more successful proposals to quantize gravity in 4 dimensions. It has led to numerous results in quantum gravity, such as a calculation of black hole entropy [7, 41] and early universe cosmology [5]. Additionally, in 2+1 dimensions the theory is well-defined [38, 24]. The classical version of LQG in 2+1 dimensions is well-known and can also be viewed as a discrete model of gravity. Unlike polygonal gravity, the constraint and symmetry structure is well-known, and the application of the theory to the torus universe is well understood [53, 30]. Therefore, the **first goal** of the thesis is to explore the relation between LQG and polygonal gravity. Two previous attempts have been made [52, 28], to elucidate this relationship. However, we found both papers inadequately described the link between the two theories.

As mentioned above, the torus universe in polygonal gravity is not well understood [23, 54]. Therefore, a **secondary goal** of the thesis is to study whether polygonal gravity contains the torus universe. Part of the reason for the difficulty in describing the torus universe in polygonal gravity is the lack of known complete observables. Once these observables were found, we discovered that polygonal gravity may not be able to explain the torus universe.

Faced with these difficulties, we wondered whether a different formulation of polygonal gravity could be developed. Often times in physics, there can be two equivalent ways of looking at the same problem, also known as a duality. One of the most famous examples of this is AdS/CFT [31], which describes bulk gravitational physics in terms of a conformal field theory defined on the boundary. In discrete gravity there is another duality that was first explored in [16]. Therefore, the **final goal** of the thesis is to apply this duality to polygonal gravity. The resulting theory is called dual 't Hooft gravity.

In order to analyze these goals, the thesis begins with a quick review of classical 2+1 general relativity, in the first order formalism. The need for the formalism is then explained, and the Hamiltonian decomposition is discussed. In doing so, the choice of polarization will be highlighted. Polarization is the choice of how we split up phase space, i.e. what we choose to call configuration and momentum variables. This choice, while not changing the physical content of the theory, can greatly modify the appearance. This ambiguity leads to the duality mentioned above and will be important when defining dual 't Hooft gravity.

After this we will provide an explicit introduction to LQG, starting from the first order action and a triangulation of spacetime. The presentation of this chapter will follow [16], but the calculations have been made more explicit. Furthermore, we will develop the dual polarization of LQG. This will be needed when we attempt to answer the first goal of the thesis, i.e. the relation between LQG and polygonal gravity. The relation between LQG and its dual formulation will also be discussed. In continuum gravity, there are two sets of constraints: one enforces flatness and the other no torsion. Notably, the order in which the constraints are implemented will alter the geometric results of the theory. Furthermore, when quantizing, the representation of the quantum theory will change. For instance, in the LQG polarization, the torsion constraint is implemented first. This leads to SU(2) spin-networks. However, in the dual polarization, the flatness constraint is implemented first. The resulting quantum theory will be described by translation (non-commutative $\mathbb{R}^3$) spin-networks. Therefore, this notion of duality can be very important for any theory of quantum gravity.

Chapter 4 reviews 't Hooft's polygonal gravity; the presentation is adapted from [29, 45, 20]. The first part focuses on how to construct 't Hooft's foliation of spacetime in terms of flat polygons. The construction of the foliation necessarily requires a number of gauge choices. This is where the magic of 't Hooft's approach comes from, and is the main cause of the difficulty when relating it to LQG. The partial gauge fixing 't Hooft employs is also further discussed. In this regard, we will explore the two components of 't Hooft's gauge fixing. One amounts to a definition of time, and therefore a Hamiltonian. The other fixes the geometry of the Cauchy surface. What it very interesting about 't Hooft's theory is that once the Cauchy surface and gauge fixing is complete, we already have a notion of dynamics. This is due to 't Hooft's clever choice of time. Finally, in order to compare 't Hooft's theory with LQG, a symplectic structure is required. Therefore, the second part of chapter 4 will explain 't Hooft's choice of symplectic form. In doing so, we will also review the phase space constraints of polygonal gravity. The exact symmetries that these constraints encode, will also be discussed.

Chapter 5 analyzes the first goals of the thesis. The first part of the chapter describes the relation between 't Hooft gravity and LQG, or more specifically the dual form of LQG. This is part of the reason why previous works [52, 28] struggled to relate LQG to polygonal gravity. That is, they attempted to relate polygonal gravity to LQG, not its dual formulation. Dual LQG is the superior choice, because both it and polygonal gravity implement a notion of the flatness constraint first. The second part of this chapter will use the relation developed in the first part to analyze the torus universe in polygonal gravity. First, we will review the previous attempts of analyzing the torus universe [23, 54], then we will use the tools developed in this thesis to help elucidate the torus universe in polygonal gravity. In doing so, we will review the observables in LQG that relate to the torus universe.

The last chapter, 6, is dedicated to exploring a new theory of discrete gravity: dual 't Hooft. Similar to how there exist dual polarizations in LQG, we find the same duality exists in polygonal gravity. The first part of the chapter is dedicated to developing dual 't Hooft from purely geometrical arguments, analogous to polygonal gravity. We first solve a set of constraints in dual 't Hooft that are equivalent to the torsion constraint in LQG. This suggests that dual 't Hooft is related to LQG. Finally, we explore the relation between dual 't Hooft gravity and LQG.

Chapter 2

Introduction to 2+1 Canonical General Relativity

In this chapter we will review 2+1 gravity with no cosmological constant. The first part of the chapter will review the first order formulation of gravity. The choice for using this is twofold. First, in this language, 2+1 gravity can be viewed as a gauge theory, which makes it much easier to quantize. Secondly, the first order formulation can handle spinning particles, such as fermions.

In the final part, we will derive the Hamiltonian decomposition of 2+1 gravity in the first order formulation. In particular, we will emphasize the ambiguity that exists in defining the Liouville form or symplectic potential. This ambiguity leads to what is called "polarizations" in phase space. Namely, how one decides what are momentum and configuration variables. This choice of splicing leads to two different versions of LQG, as was first pointed out in [16].

2.1 First order gravity

General relativity is a theory about the geometry of spacetime. In Einstein's original formulation, the basic variable of the theory is the metric $g_{\mu\nu}$. We will take Greek indices μ as spacetime indices with values $0, 1, 2$. The Einstein-Hilbert action then encodes the dynamics of the theory, and is given by

$$S[g] = \int \mathrm{d}^3x \sqrt{-g}(R - 2\Lambda), \tag{2.1}$$

where g is the determinant of the metric and Λ is the cosmological constant. Here we have neglected any boundary terms that might be needed to make the integral well defined for spacetimes that are asymptotically flat or have a boundary (see [50] for a discussion). For the rest of the paper, we will set $\Lambda = 0$, and assume spacetime has no boundary. Furthermore, we will fix the signature of spacetime to $(-++)$. Computing the variation of the action, leads to the vacuum Einstein equations given by

$$R_{\mu\nu} = 0. \tag{2.2}$$

This equation is valid in all spacetime dimensions, however in 2+1 there is a simplification. In 2+1, $R_{\mu\nu} = 0$ implies that the Riemann curvature tensor vanishes. Therefore, from basic differential geometry (see [21] for a nice review), we know that, locally, the metric can be taken to be $\eta_{\mu\nu}$ i.e. the flat metric. This implies that spacetime is locally Minkowski space. In fact, this simplification leads to the notion of geometric structures. In more detail, this means that spacetime can be described as a series of Minkowski patches glued together by the isometry group e.g. $\mathrm{ISO}(2,1)$. This point will be key when formulating polygonal gravity. Unfortunately, even with these simplifications, the metric formulation is difficult to quantize. Furthermore, the metric formalism has difficulty incorporating fermions. For these reasons, we will use a different formulation of gravity - the *first order formulation*.

The first order formulation is given by two sets of variables. One is the triad e^I_μ, where I is an internal Minkowski indices that runs from 0 to 2. The triad e^I_μ, cam be interpreted as assigning a local inertial observer to every point of spacetime. Mathematically, $e^I = e^I_\mu \mathrm{d}x^\mu$ can be viewed as a Minkowski valued one form, also called the soldering form[1]. Furthermore, now $\mathrm{SO}(2,1)$ is a local symmetry of spacetime, because inertial frames are only defined up to a Lorentz transform. Finally, the triad is related to the metric of spacetime by

$$g_{\mu\nu} = e^I_\mu \eta_{IJ} e^J_\nu. \tag{2.3}$$

Note, that under a local gauge transformation, $e^I \to g^I{}_J e^J$, where $g \in \mathrm{SO}(2,1)$ that $g_{\mu\nu}$ is unaffected.

The other ingredient needed is given by a $\mathfrak{so}(2,1)$ connection $\omega^I_{\mu J}$. This connection will allow the parallel transport of local orthonormal frames around the manifold. Unlike the metric formulation, we will not assume that ω and e are related. This implies that spacetime may have torsion. Torsion T^I, is defined as the gauge-covariant derivative d_ω of the triad,

$$T^I = \mathrm{d}_\omega e^I = \mathrm{d}e^I + \omega^I{}_J \wedge e^J. \tag{2.4}$$

In terms of spacetime coordinates this is given by

$$T^I_{\mu\nu} = \partial_\mu e^I_\nu - \partial_\nu e^I_\mu + \omega^I_{\mu J} e^J_\nu - \omega^I_{\nu J} e^J_\mu. \tag{2.5}$$

If spacetime is torsion free then we recover the Levi-Civita condition relating e to ω,

$$\mathrm{d}e = -\omega \wedge e. \tag{2.6}$$

[1]The reason for this name is simple. If we view gravity as a local gauge theory, then e^I solders the local Minkowski space to the manifold so that it can be viewed as the tangent space.

As ω is a connection, it does not transform covariantly, instead the transformation is given by

$$\omega \to Ad(g)\omega + g\mathrm{d}g^{-1}. \tag{2.7}$$

The curvature form is given by,

$$F^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J. \tag{2.8}$$

However, in 2+1 dimensions we can further simplify this. The adjoint representation of $\mathfrak{so}(2,1)$ is equivalent to the fundamental one. Therefore we can use the following isomorphism to write the spin connection and curvature using one index

$$\omega^I = \frac{1}{2}\epsilon^{IJK}\omega_{JK} \qquad \omega^{IJ} = \epsilon^{IJK}\omega_K. \tag{2.9}$$

In fact, this is nothing but a vector space isomorphism from $\mathfrak{so}(2,1) \to \mathbb{R}^{2,1}$. In fact, it is more than that. It is also an isometry from the Lie algebra to Minkowski space, where the Lie algebra has the killing metric defined on it. We will take the killing metric to be $2\,\mathrm{tr}(\Sigma_I\Sigma_J) = \eta_{IJ}$ where Σ_I is a basis of $\mathfrak{so}(2,1)$.

2.1.1 Lagrangian formulation of first order gravity

Before we describe the dynamics of first order gravity, we will make one further transformation. This transformation will be moving from a local $\mathrm{SO}(2,1)$ gauge theory to a $\mathrm{SU}(1,1)$ gauge theory. This is possible because the standard action of $\mathrm{SO}(2,1)$ on $\mathbb{R}^{2,1}$ is isomorphic to the adjoint of $\mathrm{SU}(1,1)$. Moreover, $\mathrm{SU}(1,1)$ is the spin group, or universal cover of $\mathrm{SO}(2,1)$. The use of $\mathrm{SU}(1,1)$ is needed to include fermions, and simplifies calculations (for more on the relation see appendix A). To convert to a $\mathrm{SU}(1,1)$ gauge theory, we first rewrite our variables, $(e^I, \omega^I{}_J)$ in terms of the group $\mathrm{SU}(1,1)$. First, e^I is an $\mathbb{R}^{2,1}$ valued one form so we can use the isomorphism to $\mathfrak{su}(1,1)$ defined by

$$e^I \to e = e^I\tau_I,$$

where again the τ_I are the generators of $\mathfrak{g}$. Our conventions used (defined in appendix A), are such that the algebra is represented by $[\tau_I, \tau_J] = \epsilon^K_{IJ}\tau_K$. Furthermore, the killing form, or internal metric, is given by

$$\langle A, B\rangle = 2\,\mathrm{tr}(AB) \equiv \langle AB\rangle\,. \tag{2.10}$$

Next we take the $\mathfrak{su}(1,1)$ connection to be given by

$$\omega = \omega^I\tau_I = \frac{1}{2}\epsilon^{IJK}\omega_{IJ}.$$

Here we have used the isomorphism $\mathfrak{so}(2,1) \to \mathbb{R}^{2,1} \to \mathfrak{su}(1,1)$. Therefore, the basic variables for the theory are now (e,ω). Under a gauge transformation we then have that (ω, e) behave like

$$\omega \to \mathrm{Ad}(g)\omega + g\mathrm{d}g^{-1} \qquad e \to geg^{-1}. \tag{2.11}$$

At the infinitesimal level, these equations become

$$\omega \to \omega + \mathrm{d}\zeta + [\omega, \zeta] = \omega + \mathrm{d}_\omega \zeta, \qquad e \to e + [e, \zeta] \qquad \text{with } \zeta \text{ a } \mathfrak{g} \text{ valued scalar.} \tag{2.12}$$

Just like in the usual representation of gravity we will need to construct the curvature for the gauge theory. In this case the curvature is just given by the standard equation in gauge theory,

$$F = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]. \tag{2.13}$$

In terms of the basis τ^I, this is given by

$$F^I = d\omega^I + \frac{1}{2}\epsilon^I{}_{JK}\omega^J \wedge \omega^K, \tag{2.14}$$

The curvature in this formalism is equivalent to the usual 2+1 curvature $F^I{}_J$, and the relation is just given by

$$F^I = \frac{1}{2}\epsilon^{IJK}F_{JK}$$

This proof of this is,

$$\begin{aligned}
\frac{1}{2}\epsilon^I{}_{JK}F^{JK} &= d\Big(\frac{1}{2}\epsilon^I{}_{JK}\omega^{JK}\Big) + \frac{1}{2}\epsilon^I{}_{JK}\omega^J{}_L \wedge \omega^L{}_K \\
&= d\omega^I + \frac{1}{2}\epsilon^I{}_{JK}\epsilon_M{}^J{}_L\epsilon_N{}^L{}_L\omega^M \wedge \omega^N \qquad\qquad (\omega_{LM} = \epsilon_{ILM}\omega^I) \\
&= d\omega^I + \frac{1}{2}(\delta^I{}_M\delta_{KL} - \delta^I{}_L\delta_{KM})\epsilon_N{}^L{}_K\omega^M \wedge \omega^N \\
&= d\omega^I - \frac{1}{2}\epsilon_N{}^I{}_M\omega^M \wedge \omega^N \\
&= d\omega^I + \frac{1}{2}\epsilon^I{}_{NM}\omega^N \wedge \omega^M = F^I.
\end{aligned}$$

The dynamics of the 2+1 gravity in are specified by the action

$$S_{grav}(e, \omega) = -\int \langle e, F[\omega]\rangle = -\int_M e^I \wedge F_I[\omega]. \tag{2.15}$$

To see that this is equivalent to the original Einstein-Hilbert action without a cosmological constant, we will find the equations of motion[2]. First, varying the triad gives

$$F^I_{\mu\nu}[\omega] = 0,$$

i.e. spacetime is locally flat. These are almost the Einstein equations in 3D, except that we need to check if the connection is Levi-Civita. Varying the action with respect to ω,

[2] Note, that if one includes matter, the Einstein-Hilbert action and the first order gravity action can give different answers. For instance, with fermions first order gravity minimally coupled gives a spin-spin interaction and results in non-zero torsion

we get precisely the torsion free condition,

$$0 = \delta_\omega S_{grav} = -\int_M e^I \wedge d\delta\omega_I + \epsilon_{INM} e^i \wedge \delta\omega^N \wedge \omega^M$$
$$= -\int_M (de_N + \epsilon_{IMN} e^I \wedge \omega^M) \wedge \delta\omega^N,$$

where we used Stoke's theorem and the fact that there is no boundary component of spacetime.

The equations of motion in the first-order formulation are therefore given by,

$$F = 0, \qquad T \ \ = 0, \tag{2.16}$$

which just says that spacetime is locally flat and torsionless. This is precisely the solutions space for the usual metric formulation of gravity. Therefore, on shell, the two theories will agree classically. The symmetries of the theory, though, will be different. Both theories are invariant under diffeomorphisms. However, the first order action contains an additional symmetry; namely, it is invariant under local gauge transformations (2.12). In 3+1 gravity these are the symmetries in the first order formalism. However, in 2+1 gravity there is an additional symmetry. This symmetry generates local translations in spacetime,

$$\omega \to \omega, \qquad e \to e + \mathrm{d}_\omega N, \qquad \text{with } N \text{ a } \mathfrak{g} \text{ valued scalar.} \tag{2.17}$$

To see this, we replace e by its translated version, the action (2.15), becomes

$$\begin{aligned} S(\omega, e + \mathrm{d}_\omega\phi) &= -\int_M \langle (e + \mathrm{d}_\omega\phi) \wedge F\rangle \\ &= S(\omega, e) - \frac{1}{2}\int_M \langle \mathrm{d}_\omega\phi \wedge F\rangle \\ &= S(\omega, e) + \frac{1}{2}\int_M \langle \phi \mathrm{d}_\omega F\rangle + \text{boundary terms} \\ &= S(\omega, e). \end{aligned}$$

In the third step we used integration by parts, and in the last the Bianchi identity $\mathrm{d}_\omega F = 0$. The reason for the existence of this extra symmetry is precisely due to the fact the 2+1 gravity has no local degrees of freedom. That is, if we translate an observer, the theory will not change. The existence of this extra symmetry is what simplifies 2+1 gravity. One can show [8], that diffeomorphisms are just a combination of translations and gauge transformations in 2+1 dimensions. Implementing flatness and torsion, in a discrete theory of gravity is much simpler than trying to implement diffeomorphism constraints in a discrete theory.

2.2 Canonical analysis

2.2.1 First order gravity: Hamiltonian formulation

The first step in moving to a Hamiltonian decomposition, is to foliate spacetime into a family hypersurfaces Σ_t, characterized by time function t. For this to happen, we will assume that spacetime is globally hyperbolic. Furthermore, this implies that spacetime is of the form $\mathbb{R} \times \Sigma$, where Σ represents space. We will assume that Σ is 2D manifold without boundary. We will use the time function, denoting the hypersurface Σ_t as our time coordinate, meaning we write the coordinates as $t = x^0, x^1, x^2$. The next step is to separate the first order action into time and space parts. In components the first order action becomes,

$$\begin{aligned} S &= -\int_M e^I \wedge F_I = \frac{1}{2}\int_M e^I_\mu F_{I\rho\sigma} \mathrm{d}x^\mu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \\ &= -\frac{1}{2}\int_M e^I_\mu F_{I\rho\sigma} \tilde{\epsilon}^{\mu\rho\sigma} \mathrm{d}t\mathrm{d}^2x. \end{aligned} \tag{2.18}$$

At this point we must make a decision about the polarization we are going to choose. That is, we have to decide what variables to take as configuration variables and what will be the conjugate momenta. The action (2.15), depends on both the connection ω and the triad e. Note that, on shell we know that these two variables will be related through the torsion free condition $\mathrm{d}_\omega e = 0$. Therefore, it we should can pick one as our starting point. At the end of the day, both choices are acceptable since they will give the same theory. However, what does change is the conjugate momenta, and thus the symplectic potential or Liouville form (see appendix B for a definition of symplectic potential).

As an example, consider a non-relativistic particle with a Hamiltonian H. The action is given by

$$S = \int p\mathrm{d}q - H\mathrm{d}t, \tag{2.19}$$

implying the Liouville form Θ is $\Theta = p\mathrm{d}q$. By adding a total derivative to the action, i.e. $-\mathrm{d}(pq)$ we can change the Liouville form to $-q\mathrm{d}p$. Doing this, changes p to the configuration variable and q conjugate momentum. In fact, we can get a 1-parameter family of such Liouville forms modifying the action to be,

$$S = \int (p\mathrm{d}q - H\mathrm{d}t) - \alpha \int \mathrm{d}(pq)\,. \tag{2.20}$$

Taking $\alpha = 1$, we get the Liouville form $-\mathrm{d}qp$. If we take $\alpha = 1/2$ we get a "symmetric" potential $1/2(p\mathrm{d}q - q\mathrm{d}p)$. All of these forms give the same symplectic form (up to boundary terms in a field theory as we shall see). Furthermore, the equations of motion are unchanged since the total derivative gives a boundary term that vanishes. As a result, physically nothing will change from adding this boundary term.

Returning to the action (2.15), there seem to exist two choices of conjugate momenta,

$$\begin{aligned} \frac{\delta S_{\text{grav}}}{\delta \dot{\omega}_{aJ}} &\equiv \tilde{\mathbf{e}}^{aJ} = \tilde{\epsilon}^{0ab} e^J{}_b \\ \frac{\delta S_{\text{grav}}}{\delta \dot{e}_{aJ}} &\equiv \tilde{\boldsymbol{\omega}}^{aJ} = \tilde{\epsilon}^{0ba} \omega^J{}_b. \end{aligned} \tag{2.21}$$

Before making a choice of momenta, we will make the Hamiltonian decomposition easier by splitting the action (2.18) into space and time parts,

$$S = -\frac{1}{2}\int_M \left(e^I_0 F_{Iab}\tilde{\epsilon}^{0ab} + 2e^I_a F_{I0b}\epsilon^{a0b}\right)\mathrm{d}t\mathrm{d}x^2. \tag{2.22}$$

The second term, has a 2 because of the antisymmetry of F. We now set $\tilde{\epsilon}^{ab} = \tilde{\epsilon}^{0ab} = -\tilde{\epsilon}^{a0b}$, to simplify the expressions. The first term, enforces that curvature must vanish. Moreover, it contains no time derivatives suggesting it is a constraint. Focussing on the second term in (2.22), we get

$$e^I_a F_{I0b} = e^I_a \Big(\partial_0 \omega_{Ib} - \partial_b \omega_{I0} + \epsilon_I{}^{JK} \omega_{J0} \omega_{Kb}\Big). \tag{2.23}$$

Plugging this into (2.22) gives

$$\begin{aligned} S &= \int_M \left(\tilde{\epsilon}^{ab} e^I_a \partial_0 \omega_{Ib} - \tilde{\epsilon}^{ab} e^I_a \partial_b \omega_{I0} + \tilde{\epsilon}^{ab} e^I_a \epsilon_I{}^{JK} \omega_{J0}\omega_{Kb} - \frac{1}{2} e^I_0 F_{Iab} \tilde{\epsilon}^{ab}\right) \mathrm{d}t\mathrm{d}^2 x \\ &= \int_M \left(\tilde{\epsilon}^{ab} e^I_a \partial_0 \omega_{Ib} + \tilde{\epsilon}^{ab} \omega_{I0} \partial_b e^I_a + \tilde{\epsilon}^{ab} e^I_a \epsilon_I{}^{JK} \omega_{J0}\omega_{Kb} - \frac{1}{2} e^I_0 F_{Iab} \tilde{\epsilon}^{ab}\right) \mathrm{d}t\mathrm{d}^2 x \\ &= \int_M \left[\tilde{\epsilon}^{ab} e^I_a \partial_0 \omega_{Ib} + \omega_{J0}\Big(\tilde{\epsilon}^{ab} \partial_b e^J_a + \tilde{\epsilon}^{ab} \epsilon^J{}_{KI} e^I_a \omega^K_b\Big) - \frac{1}{2} e^I_0 F_{Iab} \tilde{\epsilon}^{ab}\right] \mathrm{d}t\mathrm{d}^2 x, \\ &= \int_M \left[\tilde{\epsilon}^{ab} e^I_a \partial_0 \omega_{Ib} - \frac{1}{2}\omega_{J0} \tilde{\epsilon}^{ab} T^I_{ab} - \frac{1}{2} e^I_0 \tilde{\epsilon}^{ab} F_{Iab}\right] \mathrm{d}t\mathrm{d}^2 x \end{aligned} \tag{2.24}$$

where for the second line we used integration by parts, and last $T^I = \mathrm{d}_\omega e^I$.

At this point, before we can continue, we need to pick a symplectic form. For this, we first introduce some notation. We will take δ to denote the variational differential [14] acting on the fields ω, e. This differential, will behave exactly like d.That is, it will square to zero $\delta^2 = 0$ and the product $\delta A \delta B$ will be antisymmetric. Again we stress that d acts on spacetime while δ acts on the fields (e, ω).

The action, as expressed in (2.24), suggests that the favored choice of polarization is to take ω_{Ia} as the configuration variable. In this case we get the Liouville form

$$\Theta_{\text{grav}} = \langle \tilde{\mathbf{e}}^a \delta\omega_a \rangle = -\langle e \wedge \delta\omega \rangle . \tag{2.25}$$

The symplectic current is then given by $\Omega_{\text{grav}} = \langle \delta\omega \wedge \delta e \rangle$.

However, we can add a boundary term to our action like in the particle example. Consider modifying the action by

$$S \to S - \alpha \int_M \mathrm{d}\Big(e^I \wedge \omega_I\Big) , \tag{2.26}$$

where $\alpha \in \mathbb{R}$. This is precisely the analogous to adding the term $\mathrm{d}(pq)$ to the action for a particle In the case of 2+1 gravity we get the following family of symplectic potentials,

$$\Theta_\alpha = (1-\alpha) \langle \tilde{\mathbf{e}}^a \delta\omega_a \rangle - \alpha \langle \tilde{\boldsymbol{\omega}}^a \delta e_a \rangle . \tag{2.27}$$

The symplectic current $\Omega = -\delta\Theta$, for our 1-parameter family is then given by

$$\begin{aligned} \Omega_\alpha &= -(1-\alpha)\delta \langle \tilde{\mathbf{e}}^a \delta\omega_a \rangle + \alpha\delta \langle \tilde{\boldsymbol{\omega}}^a \delta e_a \rangle \\ &= -(1-\alpha) \Big\langle \tilde{\epsilon}^{ba} \delta e_b \delta\omega_a \Big\rangle + \alpha \Big\langle \tilde{\epsilon}^{ba} \delta\omega_b \delta e_a \Big\rangle \\ &= (1-\alpha) \Big\langle \tilde{\epsilon}^{ab} \delta\omega_a \delta e_b \Big\rangle + \alpha \Big\langle \tilde{\epsilon}^{ab} \delta\omega_a \delta e_b \Big\rangle \\ &= \Omega_{\text{grav}}. \end{aligned}$$

As expected, the symplectic form independent of our choice of Θ_α. Inverting Ω_{grav}, we get the Poisson brackets for first order gravity,

$$\{\omega_{Ia}(x), \tilde{\epsilon}^{bc} e^J{}_c(y)\} = \{e^J{}_a(x), \tilde{\epsilon}^{cb}\omega_{Ic}(y)\} = \delta^J{}_I \delta^b{}_a \delta(x-y). \tag{2.28}$$

In this thesis, we will stick with two choices of polarization $\alpha = 0$, $\alpha = 1$. The first case, $\alpha = 0$ corresponds to the usual LQG choice. On the other hand, $\alpha = 1$, gives dual LQG, as was first discovered in [16][3].

Now that we have a symplectic structure, we need to define the dynamics of the theory. By varying e^I_0, ω_{I0}, in the action (2.24) you get 6 constraints,

$$\mathcal{F}^I = \tilde{\epsilon}^{ab} F^I_{ab} = 0, \qquad \mathcal{T}^J = \partial_a \tilde{\mathbf{e}}^{Ja} + \epsilon^J{}_{IK}\omega^I_a \tilde{\mathbf{e}}^{Ka} = 0. \tag{2.29}$$

These constraints force the spatial part of the curvature and torsion to vanish and encode the dynamics in 2+1 gravity. That is, the full 3-curvature and torsion must vanish on Σ. A standard calculation then shows that these constraints form a first class system. In fact, F_I and T^I are the momentum maps (see appendix B for a definition). To find the symmetries the constraints generate we smear the torsion and curvature $\mathcal{T} = \int \xi_K T^K$, $\mathcal{F} = \int N^K F_K$ constraints with appropriate functions ξ^K and N_K. A standard calculation shows that in the LQG polarization, i.e. $(\omega, \tilde{e})$ we get

$$\{\tilde{\mathbf{e}}, \mathcal{T}[\xi]\} = -[\tilde{\mathbf{e}}, \xi], \qquad \{\omega, \mathcal{T}[\xi]\} = -\mathrm{d}_\omega \xi \tag{2.30}$$

$$\{\tilde{\mathbf{e}}, \mathcal{F}[N]\} = \widetilde{\mathrm{d}_\omega N}, \qquad \{\omega, \mathcal{F}[N]\} = 0. \tag{2.31}$$

Referring to (2.12) and (2.17), we see that $\mathcal{T}$ and $\mathcal{F}$ generate gauge transformations and translations respectively.

[3] The symmetric case $\alpha = 1/2$ is related to the Chern-Simmons formulation of gravity [16]

Chapter 3

Discrete Loop Gravity in Two Different Polarizations

In the last chapter we reviewed continuum GR using the first order formalism. In this chapter, we will describe a discretization of first order gravity - loop "quantum" gravity LQG. Quantum is in quotes because this form of phase space was motivated by LQG, and the theory we will be describing will be purely classical. LQG is a well studied formulation of discrete gravity in 2+1, and is one of the best candidates for quantum gravity in 3+1.

We will begin by reviewing the discretization which will be used for LQG. In doing so, we will face the same ambiguity that was mentioned in the previous chapter; namely, the choice of polarization. This fact was first pointed out in [16], and much of the material of this chapter was taken from there. The polarization ambiguity will lead to two different theories: LQG and dual loop quantum gravity, or simply LQG*.

3.1 Discretizating phase space

We start with a triangulation Γ^*, which is dual to a 3-valent graph Γ. Inside each triangle, spacetime will be flat and torsionless. In fact, any curvature and/or torsion will be placed on the vertices of the triangulation. The vertices of the triangulation, will be denoted by $v_1, v_2, \ldots$ and the edges of Γ^* by $\tilde{\ell} = [v_2 v_1]$. The order of the edge is to be read from right to left. That is, for an edge starting at v_1 and ending at v_2 we would have $[v_2 v_1]$. While this ordering might seem reversed, it is the more natural choice when using the left action. Dual to the triangulation will be an oriented graph Γ, specified as follows: Inside each triangle $[v_3 v_2 v_1]$ of Γ^* a "center" point or node c is given. The duality is expressed by $c^* = [v_3 v_2 v_1]$. These center points will further denote the nodes of the graph Γ dual to the

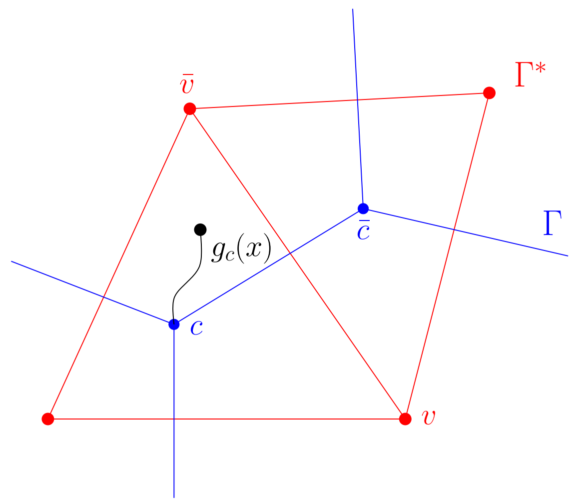

Figure 3.1: Components of the graphs Γ, Γ^* and the holonomy $g_c(x)$. Curvature and torsion sit on the vertices of Γ^*, i.e. v_i. The face c^* and $\bar{c}*$ share the edge $\ell^* = [\bar{v}v]$, the dual of the oriented edge $\ell = [\bar{c}c]$.

Triangulation Γ^*	Dual graph Γ
Triangle c^*	Node c
Edge ℓ^*	Link ℓ
Vertex v	Face f

Table 3.1: Summary of the components of the triangulation Γ^* and dual graph Γ^*

triangulation. The links of the graph are then denoted by $\ell = [c_2 c_1]$. The duality between the links of the graph and the edges of the triangulation are specified by

$$\ell^* = [c'c]^* = [v'v] \qquad \textbf{if} \qquad [v'v] = c^* \cap c'^*. \tag{3.1}$$

Pictorially, this expresses the fact that the dual link will be transverse to the edge of two adjacent triangles (see figure 3.1). This duality is actually between the oriented links and edges. The orientation on the edges of Γ^* are specified from a counterclockwise rotation from the respective dual links of Γ. This discretization scheme will form the basis for all discrete theories of gravity that will be described in this thesis.

Now that we have the triangulation and dual graph specified, we can discretize phase space. As curvature and torsion is concentrated on the vertices, we know that in each triangle, c^*, ω can be written as

$$\omega(x) \equiv (g_c \mathrm{d} g_c^{-1})(x) = -(\mathrm{d} g_c g_c^{-1})(x). \tag{3.2}$$

The group valued function $g_c(x)$, can be viewed as the holonomy from c to $x \in c^*$ with normalization $g_c(c) = 1$. Note, that since spacetime is flat and torsionless inside each triangle, the choice of path taken does not matter. For example, if we take any two paths from c to $x \in c^*$ and compose them to form a loop, then the holonomy around that loop must be the identity since the curvature vanishes.

With this parameterization, the torsion free condition implies that $g_c^{-1} e g_c$ is closed on c^*,

$$\begin{aligned} d(g_c^{-1} e_x g_c) &= (\mathrm{d}g_c^{-1} \wedge e) g_c + g_c^{-1} \mathrm{d}e g_c + g_c^{-1} + g_c^{-1}(e \wedge \mathrm{d}g_c) \\ &= -(g_c^{-1} \mathrm{d}g_c g_c^{-1}) \wedge e g_c + g_c^{-1} \mathrm{d}e g_c + g_c^{-1} e \wedge \mathrm{d}g_c \\ &= g_c^{-1}\Big(\mathrm{d}e - (\mathrm{d}g_c g_c^{-1}) \wedge e + e \wedge (\mathrm{d}g_c g_c^{-1})\Big) g_c \\ &= g_c^{-1}(\mathrm{d}e + \omega \wedge e - e \wedge \omega) g_c \\ &= g_c^{-1} T g_c \\ &= 0 \end{aligned}$$

In this calculation we used the following fact, that will be used time and time again in this thesis:

$$\mathrm{d}g^{-1} = -g^{-1} \mathrm{d}g g^{-1}. \tag{3.3}$$

From this calculation we see that $g_c^{-1} e g_c$ is closed on c^*, which implies the form is exact in the triangle. Therefore, we can introduce the Lie algebra valued function y_c on c^*, which satisfies

$$e(x) = \Big(g_c \mathrm{d}y_c g_c^{-1}\Big)(x) \tag{3.4}$$

The potential y_c can be viewed as an embedding of the triangle into flat Minkowski space.

The next step is to discretize the symplectic structure. Our starting point will be the symplectic form $\Omega_{\text{grav}} = \langle \delta e \wedge \delta\omega \rangle$, since this does not depend on our choice of polarization. Another reason we start with this, and not the Liouville form Θ_α (2.27), is that natural discretizations of density weight one variables, such as $\tilde{\omega}$, is not obvious. However, the discretization of the connection itself is immediate. Therefore, we will start with the discretization of ω and along the way try to identify the different polarizations mentioned in the previous chapter. As we will see, LQG i.e. $\alpha = 0$ and dual LQG $\alpha = 1$ will arise naturally.

In order to calculate Ω_{grav} in terms of our variables g_c the holonomy and $\mathrm{d}y_c$, we need to find the variations of ω and e in terms of $(g_c, \mathrm{d}y_c)$. First calculating $\delta\omega$, we get

$$\begin{aligned} \delta(g_c \mathrm{d}g_c^{-1}) &= \delta g_c \mathrm{d}g_c^{-1} + g_c \mathrm{d}\delta g_c^{-1} \\ &= \delta g_c \mathrm{d}g_c^{-1} - g_c \, \mathrm{d}\Big(g_c^{-1} \delta g_c g_c^{-1}\Big) \\ &= -g_c\Big(\mathrm{d}g_c^{-1} \delta g_c + g_c^{-1} \mathrm{d}\delta g_c\Big) g_c^{-1}. \end{aligned}$$

This implies another identity that will be used throughout this thesis.

$$\delta(g \mathrm{d}g^{-1}) = -g \, \mathrm{d}\Big(g^{-1} \delta g\Big) \, g^{-1}. \tag{3.5}$$

This only constitutes half of the variations considered. The others are the variations of the dyad in terms of the potential y_c and the holonomy g_c. In this case the variation is

$$\begin{aligned}\delta e &= \delta\Big(g_c \mathrm{d}y g_c^{-1}\Big) \\ &= \delta g_c \mathrm{d}y_c g_c^{-1} + g_c \mathrm{d}\delta y_c g_c^{-1} + g_c \mathrm{d}y_c \delta g_c^{-1} \\ &= \delta g_c \mathrm{d}y_c g_c^{-1} + g_c \mathrm{d}\delta y_c g_c^{-1} - g_c \mathrm{d}y_c g_c^{-1}\delta g_c g_c^{-1} \\ &= g_c\Big(\mathrm{d}\delta y + g_c^{-1}\delta g_c \mathrm{d}y_c - \mathrm{d}y_c g_c^{-1}\mathrm{d}g_c\Big)g_c^{-1} \\ &= g_c\Big(\mathrm{d}\delta y_c + [g_c^{-1}\delta g_c, \mathrm{d}y_c]\Big)g_c^{-1}. \end{aligned} \tag{3.6}$$

Discretizing Ω_{grav}, by smearing it over each triangle c^*, we get

$$\Omega_c = -\int_{c^*} \delta \left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}y_c \right\rangle. \tag{3.7}$$

To see this, we plug (3.5) and (3.6) into the symplectic density getting,

$$\begin{aligned}\Omega_{\text{grav}} &= \langle \delta\omega \wedge \delta e\rangle \\ &= \Big\langle - \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge \Big(\delta \mathrm{d}y_c + [g_c^{-1}\delta g_c, \mathrm{d}y_c]\Big)\Big\rangle \\ &= -\delta \left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}y_c \right\rangle. \end{aligned} \tag{3.8}$$

The last line can be verified by expanding the left hand side and remembering that $\delta A \delta B = -\delta B \delta A$ (if A and B are functions not forms),

$$\begin{aligned}-\delta \left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}y_c \right\rangle &= -\left\langle \mathrm{d}\delta\Big(g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}y_c \right\rangle - \left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}\delta y_c \right\rangle \\ &= -\left\langle \mathrm{d}\Big(\delta g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}y_c \right\rangle + \dots \\ &= \left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}y_c \right\rangle + \dots \\ &= \left\langle g_c^{-1}\delta g_c \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}y_c \right\rangle + \left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) g_c^{-1}\delta g_c \wedge \mathrm{d}y_c \right\rangle + \dots \\ &= -\left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}y_c g_c^{-1}\delta g_c \right\rangle + \left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge g_c^{-1}\delta g_c \mathrm{d}y_c \right\rangle \\ &\qquad - \left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge \mathrm{d}\delta y_c \right\rangle \\ &= -\left\langle \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) \wedge \Big(\mathrm{d}\delta y_c + [g_c^{-1}\delta g_c, \mathrm{d}y_c]\Big)\right\rangle. \end{aligned}$$

In the second last line, we used that the trace is cyclic and δ is antisymmetric.

While this calculation is tedious, (3.8) allows us to see that Ω_{grav} is an exact two form. This means we can pull a d out. However, now there is an ambiguity, namely how do we perform the integration? Each form will differ by a boundary term, however, when summing over all the triangles the boundary component will vanish. This is exactly the polarization ambiguity we described above, will lead to LQG and dual LQG.

The two integration choices we will make are

$$\Omega_c = -\int_{\partial c^*} \delta \left\langle g_c^{-1}\delta g_c \mathrm{d}y_c \right\rangle \tag{3.9}$$

$$= \int_{\partial c^*} \delta \left\langle y_c \mathrm{d}\Big(g_c^{-1}\delta g_c\Big)\right\rangle. \tag{3.10}$$

This gives us two different symplectic potentials and thus different polarizations,

$$\Theta_c = \int_{c^*} \left\langle g_c^{-1}\delta g_c \mathrm{d}y_c \right\rangle \tag{3.11}$$

$$\Theta_c^* = -\int_{c^*} \left\langle y_c \mathrm{d}\left(g_c^{-1}\delta g_c\right)\right\rangle . \tag{3.12}$$

The first form is the **LQG polarization** while the second the **dual LQG polarization**. If we continue with the discretization of the LQG polarization, we would end up with the standard LQG picture, as we will show below.

So far we only have the symplectic form for a single triangle, which can be seen as a sum of integrals over the edges ℓ^* of c^*. To form the complete phase space we need to to glue these triangles together. First, we will apply matching conditions to the triangles that enforce ω and e are continuous across triangles. Demanding that the connection ω is continuous across the edge implies that

$$g_c \mathrm{d}g_c^{-1}(x) = \omega(x) = g_{\bar{c}}\mathrm{d}g_{\bar{c}}^{-1}(x), \qquad \forall x \in \ell^*. \tag{3.13}$$

This equation requires that there exists a **constant** group element $h_{\bar{c}c}$ that relates the frames at c and $\bar{c}$,

$$g_c(x) = g_{\bar{c}}(x)h_{\bar{c}c}, \qquad \forall x \in \ell^*. \tag{3.14}$$

This equations says that the transformation from c to x, is the same as going from c to $\bar{c}$ and then to x (see figure 3.2), which is that case since curvature is contained at the vertices. The element $h_{c\bar{c}}$, represents the **holonomy** of the connection, ω, when moving from c to $\bar{c}$ along the link $\ell \in \Gamma$. This holonomy forms the first standard LQG variable. By flipping c and $\bar{c}$ in (3.13), one can easily see that $h_{\bar{c}c} = h_{c\bar{c}}^{-1}$.

The other matching condition, enforces that the dyad $e(x)$ is continuous across an edge,

$$g_c \mathrm{d}y_c g_c^{-1}(x) = e(x) = g_{\bar{c}}\mathrm{d}y_{\bar{c}}g_{\bar{c}}^{-1}(x), \qquad \forall x \in \ell^*. \tag{3.15}$$

Using (3.14) this means the potentials $y_c, y_{\bar{c}}$ must obey

$$\mathrm{d}y_{\bar{c}} = h_{\bar{c}c}\mathrm{d}y_c h_{\bar{c}c}^{-1}. \tag{3.16}$$

Integrating this equation we end up with,

$$y_{\bar{c}} = h_{\bar{c}c}(y_c + x_{\bar{c}c})h_{\bar{c}c}^{-1}. \tag{3.17}$$

The constant term $x_{\bar{c}c} \in \mathfrak{g}$ represents the **translational holonomy**, and is the edge vector connecting the two nodes c, and $\bar{c}$. This transformation simply reflects that the potentials y_c, transformation is given by the Poincaré group, with $(h_{\bar{c}c}, x_{\bar{c}c})$. In other words, when moving from c to $\bar{c}$ the frame is both rotated by $h_{\bar{c}c}$ and translated by $x_{\bar{c}c}$.

Using these relations, we can start gluing triangles together to get the complete discrete phase space of gravity. However, instead of dealing with triangles we will deal with the edges. The reason for this is that our symplectic form, after integration, is localized on

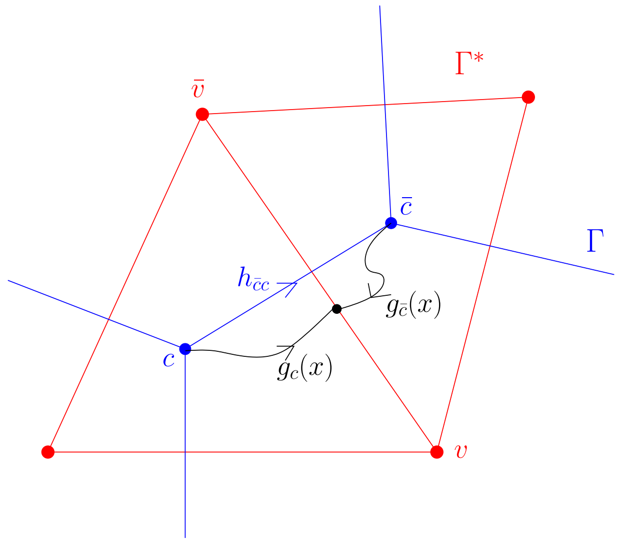

Figure 3.2: The constant holonomy $h_{\bar{c}c}$, relates the frames associated to the "center" of the two faces c and $\bar{c}$.

the edges (see (3.9)). Therefore, since each link of Γ is dual to an edge, we can write the total symplectic structure as the sum of contributions from each link $\ell = [\bar{c}c]$ of Γ. To do this, we first refer to figure 3.2, and the edge $\ell^* = [\bar{v}v]$, with dual link $[\bar{c}c]$. In this case, the symplectic form for $[\bar{v}v]$ will get contributions from c and $\bar{c}$. If we start with the LQG polarization, the symplectic potential for the edge ℓ will be given by

$$\Theta_\ell = \int_{\ell^*} \Big[g_c^{-1} \delta g_c \mathrm{d}y_c - g_{\bar{c}}^{-1} \delta g_{\bar{c}} \mathrm{d}y_c \Big]. \tag{3.18}$$

There are two terms, since each edge of the triangulation is shared by two triangles. Furthermore, the negative sign is due to the edge having the opposite orientation when viewed from the triangle $\bar{c}$. Similarly, the symplectic potential for the dual polarization will be given by

$$\Theta^*_\ell = -\int_{\ell^*} \Big[y_c \, \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) - y_{\bar{c}} \, \mathrm{d}\Big(g_{\bar{c}}^{-1}\delta g_{\bar{c}}\Big) \Big]. \tag{3.19}$$

The total symplectic potential is then given by summing all the links of the graph, i.e.

$$\Theta^{\text{total}} = \sum_{\ell \in \Gamma} \Theta_\ell. \tag{3.20}$$

These two potentials will form the basis for the next two sections. Before we move on, note that both of these symplectic potentials are related by a total derivative, namely

$$\Theta^*_\ell = \Theta^* - \int_{\ell^*} \mathrm{d}\Big(\Big\langle g_c^{-1}\delta g_c \Big\rangle \, y_c - \Big\langle g_{\bar{c}}^{-1} \delta g_{\bar{c}} y_{\bar{c}} \Big\rangle \Big). \tag{3.21}$$

After summing over all the links of the graph we therefore, get that this boundary term vanishes, confirming both polarizations will give the same symplectic form in the end.

3.2 LQG polarization

The starting point of the LQG or standard polarization is the Liouville form,

$$\Theta_\ell = \int_{\ell^*} \Big[g_c^{-1} \delta g_c \mathrm{d}y_c - g_{\bar{c}}^{-1} \delta g_{\bar{c}} \mathrm{d}y_c \Big]. \tag{3.22}$$

Since $g_{\bar{c}}(x)$ is a function of $h_{\bar{c}c}$ and $g_c(x)$ and $\mathrm{d}y_{\bar{c}}$ satisfies (3.16), we can rewrite the second term in (3.18) as

$$\begin{aligned} \Big\langle g_{\bar{c}}^{-1} \delta g_{\bar{c}} \mathrm{d}y_{\bar{c}} \Big\rangle &= \Big\langle h_{\bar{c}c} g_c^{-1}(x) \delta\big(g_c(x) h_{\bar{c}c}^{-1}\big) \Big(h_{\bar{c}c} \mathrm{d}y_c h_{\bar{c}c}^{-1} \Big) \Big\rangle \\ &= \Big\langle \Big(g_c^{-1}(x) \delta g_c(x) - h_{\bar{c}c}^{-1} \delta h_{\bar{c}c} \Big) \mathrm{d}y_c \Big\rangle . \end{aligned}$$

Therefore, the symplectic potential becomes

$$\Theta_\ell = \Big\langle h_{\bar{c}c}^{-1} \delta h_{\bar{c}c} \int_{\ell^*} \mathrm{d}y_c \Big\rangle , \tag{3.23}$$

where we were able to move the integral in since $h_{c\bar{c}}$ is constant over ℓ^*. The potential Θ_ℓ gives then the natural choice of variable dual to $h_{\bar{c}c}$. The dual variable is the standard LQG *flux* vector $\widetilde{E}^c_{\ell^*}$ defined as

$$\widetilde{E}^c_{\ell^*} = \int_{\ell^*} \mathrm{d}y_c = \int_{\ell^*} g_c e g_c^{-1} \equiv \int_{\ell} g_c \tilde{\mathbf{e}} g_c^{-1}. \tag{3.24}$$

Therefore, our phase space variables are $\widetilde{E}_{\ell^*}, \tilde{h}_\ell$. This provides a natural discretization of the $\Theta_{\text{grav}} = \langle \tilde{\mathbf{e}}^a \delta\omega_a \rangle$ polarization of gravity. The effect of the density being placed on e is that it encodes information of the triangulation. In the following section we will see a similar outcome for the $- \langle \tilde{\omega}^a \delta e_a \rangle$, where the holonomies live on the edges of the triangulation.

Before we continue, we should see how the flux vector transforms under a gauge transform. Consider the gauge transformation $\xi(x) \in \mathrm{SO}(2,1)$. Since we have chosen left multiplication for our group action, we know that the holonomy $g_c(x)$ will transform as $\xi(x) g_c(x) \xi^{-1}(c)$, and $e \to \xi(x) e(x) \xi(x)^{-1}$. Plugging these two equations into (3.24), we get that the flux transforms as

$$\widetilde{E}^c_\ell = \xi(c) \widetilde{E}_\ell \xi(c)^{-1}. \tag{3.25}$$

Using this transformation law, we can see that the flux vectors sit at the node c and in a sense represents the edge vector of ℓ^* as seen from c. One other point of interest is the orientation of ℓ^*. That is, choice of orientation is of arbitrary and therefore, we should find out how the flux changes if we reverse the orientation. In this regard, we replace $\ell = [\bar{c}c]$ with $-\ell = [c\bar{c}]$, which induces a change in orientation for ℓ^*, i.e. $\ell^* \to -\ell^* = [v\bar{v}]$. The holonomy variables then becomes $h_{c\bar{c}}$, and the corresponding flux becomes $\widetilde{E}^{\bar{c}}_{-\ell^*}$, and satisfies

$$\widetilde{E}^{\bar{c}}_{-\ell^*} = -h_{\ell^*} \widetilde{E}^c_{\ell^*} h_{\ell^*}^{-1}. \tag{3.26}$$

The discretized phase space variables are thus,

$$(h^v_{\bar{c}c} = g^{-1}_{v\bar{c}} g_{vc}, \widetilde{E}^c_{\ell^*}), \tag{3.27}$$

where $g_{vc} = g_c(v)$. Notice that in the above definition, we have set $h_{\bar{c}c} = g^{-1}_{\bar{c}}(v) g_c(v)$. This was done since h is a constant, and taking $x = v$ allows for a better comparison with dual LQG.

Using these phase space variables, the symplectic form is given by,

$$\begin{aligned} \Omega^{\text{LQG}}_\ell &= -\delta \left\langle h^{-1}_\ell \delta h_\ell \widetilde{E}_{\ell^*} \right\rangle \\ &= -\left\langle \delta(h^{-1}_\ell \delta h_\ell)\widetilde{E}_\ell - (h^{-1}_\ell \delta h_\ell)\delta \widetilde{E}_\ell \right\rangle \\ &= -\left\langle -(h^{-1}_\ell \delta h_\ell)(h^{-1}_\ell \delta h_\ell)\widetilde{E}_\ell - (h^{-1}_\ell \delta h_\ell)\delta \widetilde{E}_\ell \right\rangle . \end{aligned}$$

An equivalent expression is,

$$\Omega^{\text{LQG}}_\ell = \left\langle (h^{-1}_\ell \delta h_\ell)(h^{-1}_\ell \delta h_\ell)\widetilde{E}_\ell + (h^{-1}_\ell \delta h_\ell)\delta \widetilde{E}_\ell \right\rangle , \tag{3.28}$$

which is the standard symplectic form associated to $T^*\text{SU}(1,1)$ [3, 4]. This gives the symplectic structure for one edge of the graph Γ. Expanding this construction to all the links of our graph, we can get the entire phase space. The fluxes will sit at the nodes of Γ as (3.25) shows, but will depend on the edges of Γ^*. On the other hand, the holonomies sit on the links of Γ directly.

Before moving on, we remind the reader of some Lie group theory. The Lie algebra $\mathfrak{g} = T_e G$ of a Lie group G, is isomorphic to the set of left-invariant vector fields on G[1]. Left invariant vector fields $\boldsymbol{\nabla}^R_X$ are generated by right translation; namely, if $X \in \mathfrak{g}$, then the left invariant vector field associated to X is given by

$$\boldsymbol{\nabla}^R_E f(g) = f\left(g e^{tE}\right)\Big|_{t=0}, \tag{3.29}$$

where f is a function on G.

Every Lie group G also carries a natural one-form, called the Maurer-Cartan form. This one-form maps vector fields on TG to elements of the Lie algebra, and is given by $\omega_g(v) = \mathrm{d}L_{g^{-1}}(v_g)$. Often in literature, this form will be written as $g^{-1}\mathrm{d}g$, which is strictly true if G is a matrix Lie group. The Maurer-Cartan form is Left-invariant as is easily seen from

$$\begin{aligned} L^*_g \omega_{gh}(v) &= \omega_{gh}(L_g v) && (3.30)\\ &= \mathrm{d}L_{(gh)^{-1}} \mathrm{d}L_g v && (3.31)\\ &= \mathrm{d}L^{-1}_h v && (3.32)\\ &= \omega_h(v). && (3.33) \end{aligned}$$

[1]In this paper we have been using the left action of G so this definition is more natural for us. If we used the right action we would have used the right-invariant vector fields, but the choice is arbitrary.

In fact, using a similar calculation, one can show that the set of left invariant one forms is isomorphic to $\mathfrak{g}^*$.

In our case, $G = \mathrm{SU}(1,1)$ and we also have the killing metric $\langle X, Y\rangle = 2\,\mathrm{tr}(XY)$. This provides us with an isomorphism between $T^*\mathrm{SU}(1,1)$ and $T\mathrm{SU}(1,1)$. Therefore, we can define the dual of $Y \in \mathfrak{g}$, by $Y^* \in \mathfrak{g}^*$ where $Y^*(X) = 2\,\mathrm{tr}\{XY\}$. Thanks to this representation, we can then parameterize $\mathfrak{su}(1,1)^*$ with elements of $\mathfrak{g}$. Therefore, we can trivialize $T^*SU(1,1)$ using

$$\mathrm{SU}(1,1) \times \mathfrak{su}(1,1)^* \to T^*\mathrm{SU}(1,1) \tag{3.34}$$

$$(g, X) \to \Theta_g = \omega_g(X^*) = \left\langle X g^{-1}\mathrm{d}g \right\rangle. \tag{3.35}$$

Notice that this is exactly the symplectic potential of LQG (3.23). Using this trivialization, one is led to the following theorem for the Poisson brackets of $T^*\mathrm{SU}(1,1)$,

Theorem 3.1. *The symplectic two form Ω^{LQG} gives Poisson brackets*

$$\{H_Y, H_Z\} = H_{[Y,Z]}, \qquad \{H_Y, f(g)\} = \boldsymbol{\nabla}^R_X f(g), \qquad \{f_1(g), f_2(g)\} = 0, \tag{3.36}$$

where $H_Y : T^*G \to \mathbb{R}$, is the linear function $H_Y(Z^*) = \langle Y, Z\rangle$, and $f(g)$ is a smooth function on G.

For the proof of this theorem see [25].

The nice part of the Poisson brackets in theorem 3.1 is that they are independent of the basis we pick. However, when it comes to computations, this abstract form will not be of much use. Therefore, we will express the Poisson brackets in term of the orthonormal basis τ_I we described above. Using this basis we can identify $\mathfrak{su}(1,1)$ with $\mathbb{R}^{2,1}$. In fact, we have that $H_{\tau^I}(E) = \langle E, \tau^I\rangle = E^I$, is just the coordinate function. Therefore, the Poisson brackets for the fluxes become

$$\begin{aligned}\{H_{\tau^A}, H_{\tau^B}\}(\widetilde{E}_{\ell^*}) &= H_{[\tau^A,\tau^B]}(\widetilde{E}_{\ell^*})\\ &= H_{\epsilon^{ij}{}_k\tau^k}(\widetilde{E}_{\ell^*})\\ &= \epsilon^{AB}{}_C \widetilde{E}^C_{\ell^*}\end{aligned}$$

Similarly, if we consider the coordinate functions g_{ab} for $g \in \mathrm{SU}(1,1)$ we have that

$$\begin{aligned}\boldsymbol{\nabla}^R_{\tau^A} g_{mn} &= \frac{\partial g_{mn}}{\partial g_{ab}} \left.\frac{\mathrm{d}(g e^{t\tau^A})_{ab}}{\mathrm{d}t}\right|_{t=0}\\ &= (g\tau^A)_{mn}.\end{aligned}$$

Therefore, the second equation in (3.36), in terms of the coordinate functions, becomes

$$\{H_{\tau^A}(E), g_{mn}\} = (g\tau^A)_{mn}$$

Therefore, in the τ^A basis, the Poisson brackets for LQG becomes

$$\{\widetilde{E}^A_{\ell^*}, \widetilde{E}^B_{\ell^*}\} = \epsilon^{AB}{}_C \widetilde{E}^C_{\ell^*}, \qquad \{\widetilde{E}^A_{\ell^*}, h_\ell\} = h_\ell \tau^A, \qquad \{h_\ell, h_\ell\} = 0, \tag{3.37}$$

where h_ℓ needs to be interpreted as the coordinate function. Note, that the Poisson brackets between the holonomy and the flux with its orientation flipped, $\widetilde{E}_{-\ell^*}$, is given by

$$\{\widetilde{E}_{-\ell^*}, h_\ell\} = -\tau^i h_\ell^{-1}. \tag{3.38}$$

At this point we have can start to implement the dynamics or constraints of the theory. The first constraint arises naturally from the kinematical structure of our graph, and for this reason is known as the *kinematical constraint*. In the LQG formalism it is given by

$$\mathcal{J}_c \equiv \widetilde{E}_{[v_1 v_2]} + \widetilde{E}_{[v_2 v_3]} + \widetilde{E}_{[v_3 v_1]} = \int_{\partial c^*} \mathrm{d}y_c = \int_{c^*} \mathrm{d}\mathrm{d}y_c = 0. \tag{3.39}$$

The meaning of this constraint is apparent from its definition. It just reflects the fact that we assumed the interior of each triangle was torsion free. Therefore $\mathcal{J}_c$, the Gauss constraint, is a discrete version of the torsion constraint. In the dual LQG formulation, the kinematical constraint just reinforces the other assumption we made about the triangles, namely that they are locally flat.

Mathematically, what this constraint implies that our phase space variables $(h_\ell, \widetilde{E}_{\ell^*})$ are no longer free. Namely, different (h, E) could be related by a gauge transformation. Furthermore, by analogy with the continuum case, the expectation is that the constraint $\mathcal{J}_c$ generates local $\mathrm{SU}(1,1)$ gauge transformations. The constraint $\mathcal{J}_c$ is a abelian momentum map (see appendix B for a definition), therefore flow of the constraint is given by,

$$\delta^c_\alpha \widetilde{E}_{\ell^*} = \{\widetilde{E}_{\ell^*}, \mathcal{T}_c(\alpha)\}, \qquad \delta^c_\alpha h_\ell = \{h_\ell, \mathcal{T}_c(\alpha)\}, \tag{3.40}$$

where $\mathcal{J}_c(\alpha) = \xi^c_A \mathcal{J}^A_c$. Using the Poisson brackets in (3.37) one finds (an explicit derivation of a similar result will be provided in the next section)

$$\delta^c_\xi \widetilde{E}_{\ell^*} = [\xi_c, \widetilde{E}_{\ell^*}] \qquad \delta^c_\xi h_\ell = -h_\ell \xi_c. \tag{3.41}$$

These are precisely the infinitesimal $\mathrm{SU}(1,1)$ gauge transformations (3.25).

At this point we can recover the classical version of spin networks, aka the kinematical phase space of LQG $\mathcal{P}^{\mathrm{kin}}_{\mathrm{LQG}}$, using Marsden-Weinstein reduction [33],

$$\mathcal{P}^{\mathrm{kin}}_{\mathrm{LQG}} = (\times_{\ell\in\Gamma} T^*_\ell G) /\!/ (\times_{c\in\Gamma} \mathcal{J}_c^{-1}(0)).$$

Physically this means we have to do two things to find the kinematical phase space. First, we have to find a set of observables that are invariant under the flow of $\mathcal{J}_c$, i.e. $\mathrm{SU}(1,1)$ invariants. Second, the variables have to be a solution of $\mathcal{J}_c = 0$. Later in chapter 6, we will find a give a well-known description of a set of gauge invariant observables that describe the kinematical phase space.

The last constraint we will need to define is the constraint that defines the dynamics of the theory. In [51], it was shown that the Hamiltonian and diffeomorphism constraint in 2+1 are just a combination of the flatness and Gauss constraint. Therefore, by introducing a discrete form of the flatness constraint, will be able to recover the dynamics of 2+1 gravity. We will need to ensure that the constraint is first class in order for the theory to be related to gravity however. Luckily due to the holonomy, there is a canonical choice of constraint that enforces flatness that will be first class. Namely, that the holonomies around the faces of f in Γ are give the identity,

$$\mathcal{G}_f = \prod_{\ell \in f} h_\ell = 1. \tag{3.42}$$

This constraint, together with the kinematical constraints $\mathcal{J}$, form the complete system of constraints. Moreover, one can show, as will be done in the next section, that these constraints are first class, as required. The fact that $\{\mathcal{G}_f\}_{f\in\Gamma}$ is a first class constraint, immediately suggests that it generates some symmetries of the theory. However, here we run into a slight problem. The kinematical constraint was a straightforward abelian momentum map, which necessarily means that it is Lie algebra valued. In this case, the flatness constraint is Lie group valued. This constraint is an example of a **non-abelian momentum map** [2], see appendix B for additional information. Therefore, to see the symmetries generated by the momentum map we use the following formula [6],

$$\delta^v_\beta \widetilde{E}^A_{\ell^*} \equiv \langle \mathcal{G}_\lambda^{-1}\{\widetilde{E}^A_{\ell^*}, \mathcal{G}_\lambda\}, \beta^B_v \tau_B\rangle, \qquad \delta^v_\beta h_\ell \equiv \left\langle \mathcal{G}_\lambda^{-1}\{h_\ell, \mathcal{G}_\lambda\}, \beta^B_v \tau_B\right\rangle. \tag{3.43}$$

While this seems to depend on the choice of ordering for the flatness constraint, one can show that different orderings are just related by a redefinition of the gauge parameter β_v. In LQG the simplest loops are given by the faces dual to some vertex in Γ^*. For instance, if we consider the loop $\lambda = [c_1, c_n, \ldots, c_2, c_1]$ and define $\ell_i = [c_{i+1}, c_i]$, then the flow is

$$\delta^v_\beta \widetilde{E}_{\ell^*_i} = H_i \beta H_i^{-1}, \tag{3.44}$$

where $H_i = h_{\ell_i} h_{\ell_{i-1}} \cdots h_{\ell_1}$. This transformation is a discrete version of a translation, which matches the symmetry generated by the flatness constraint in continuum gravity.

The dynamical or reduced phase space, is then given by the Marsden-Weinstein theorem extended to Lie group valued momentum maps [2],

$$\mathcal{P}^{\text{dyn}}_{\text{LQG}} = \mathcal{P}^{\text{kin}}_{\text{LQG}} /\!\!/ (\times_{f\in\Gamma} \mathcal{G}_\lambda^{-1}(1)). \tag{3.45}$$

This phase space represents the true degrees of freedom of 3D gravity, i.e. its elements are the observables of gravity.

We can now do a sanity check to further ensure that our system is really gravity. The reduced phase space of gravity for genus g, $g > 1$, has dimension $12g - 12$ as previous studies have shown (see [9] for a review). Starting from the LQG phase space, lets assume that we have E links. Each link carries 3 degrees of freedom from the holonomies, and 3 degrees of freedom from the flux. Therefore, we have $6E$ degrees of freedom. Associated to every node of the graph, we have 3 constraints from the Gauss constraint $\mathcal{J}_c$. Furthermore, every face has three constrains, due to the flatness constraint. Therefore, since we also have to remove the degrees of freedom due to the flow of the constraint, the dimension of the reduced phase space of LQG is

$$6E - 2(3N - 3F) = -6(N - E + F) = -6\chi = 12g - 12, \tag{3.46}$$

where N, F are the number of nodes and faces respectively and χ is the Euler number. This matches the dimension of the reduced phase space of gravity.

3.3 Dual LQG polarization

In the last section we derived LQG in the standard polarization. This meant we started with (3.18). In this section we will flip the script and use the dual polarization Liouville form,

$$\Theta^*_\ell = -\int_{\ell^*} \left\langle y_c \, \mathrm{d}\Big(g_c^{-1}\delta g_c\Big) - y_{\bar{c}} \, \mathrm{d}\Big(g_{\bar{c}}^{-1}\delta g_{\bar{c}}\Big)\right\rangle . \tag{3.47}$$

Our goal will be to proceed in a similar direction as the previous section. The difference will be that now y_c will be our configuration variable instead of the holonomy. First, we simplify the second term in the symplectic potential by using (3.17), (3.14)

$$\begin{aligned}\left\langle y_{\bar{c}} \, \mathrm{d}\Big(g_{\bar{c}}^{-1}\delta g_{\bar{c}}\Big)\right\rangle &= \left\langle h_{\bar{c}c}(y_c + x_{\bar{c}c})h_{\bar{c}c}^{-1}\mathrm{d}\Big[h_{\bar{c}c}g_c^{-1}(x)\delta\Big(g_c(x)h_{\bar{c}c}^{-1}\Big)\Big]\right\rangle \\ &= \left\langle h_{\bar{c}c}(y_c + x_{\bar{c}c})\mathrm{d}\Big(g_c^{-1}(x)(\delta g_c(x)h_{\bar{c}}^{-1} - g_c(x)h_{\bar{c}c}^{-1}\delta h_{\bar{c}c}h_{\bar{c}c}^{-1})\Big)\right\rangle \\ &= \left\langle h_{\bar{c}c}(y_c + x_{\bar{c}c})\mathrm{d}\Big(g_c^{-1}(x)\delta g_c(x)\Big)h_{\bar{c}c}^{-1}\right\rangle ,\end{aligned}$$

where we used that $h_{\bar{c}c}$ is a constant function. Therefore, the symplectic potential becomes

$$\Theta^*_\ell = \int_{\ell^*} \left\langle x_{\bar{c}c}\mathrm{d}\Big(g_c^{-1}(x)\delta g_c(x)\Big)\right\rangle . \tag{3.48}$$

Recalling the translational holonomy $x_{\bar{c}c}$ is constant, we can perform this integration explicitly,

$$\Theta^*_\ell = \left\langle x_{\bar{c}c}\Big(g_{\bar{v}c}^{-1}\delta g_{\bar{v}c} - g_{vc}^{-1}\delta g_{vc}\Big)\right\rangle , \tag{3.49}$$

where $g_{vc} = g_c(v)$.

Defining the holonomy from v to $\bar{v}$ viewed from c, $\tilde{h}^c_{\bar{v}v} \equiv g_{\bar{v}c}g_{vc}^{-1}$, we can further simplify Θ^*,

$$\Theta^*_\ell = \left\langle (g_{\bar{v}c}x_{\bar{c}c}g_{vc})(\tilde{h}^c_{\bar{v}v})^{-1}\delta\tilde{h}^c_{\bar{v}v}\right\rangle . \tag{3.50}$$

From this we define the configuration variable,

$$E^v_\ell \equiv g_{vc}x_{\bar{c}c}g_{vc}^{-1}, \tag{3.51}$$

implying the symplectic potential is given by

$$\Theta^{\mathrm{LQG}^*}_\ell = \left\langle E^v_\ell(\tilde{h}^c_{\ell^*})^{-1}\delta\tilde{h}^c_{\ell^*}\right\rangle . \tag{3.52}$$

The holonomy $\tilde{h}$ provides the natural discretization of $\tilde{\boldsymbol{\omega}}$, (2.21), as now E^v_ℓ lives on the graph, and $\tilde{h}_{\ell^*}$ lives on the triangulation. Furthermore, since $x_{\bar{c}c}$ is the translational holonomy from c to $\bar{c}$ we see that E^v_ℓ can be interpreted at the vector connecting c to $\bar{c}$ as viewed from the vertex v.

To summarize, the variables of dual LQG are

$$\Big(E^v_\ell \equiv g_{vc}x_{\bar{c}c}g_{vc}^{-1}, \;\; \tilde{h}^c_{v\bar{v}} \equiv g_{\bar{v}c}g_{vc}^{-1}\Big). \tag{3.53}$$

Under a reverse of orientation, we need to understand how the fluxes change. Changing $\ell = [c_2c_1]$ to $[c_1c_2]$ again induces a change in orientation for the dual edge. Therefore, we actually have that the flux sits at a different vertex under a change of orientation,

$$E^{\bar{v}}_{-\ell} = -\tilde{h}^c_{\bar{v}v} E^v_\ell (\tilde{h}^c_{\bar{v}v})^{-1}. \tag{3.54}$$

In order to derive this, recall that $y_{\bar{c}} = h_{\bar{c}c}(y_c + x_{\bar{c}c})h^{-1}_{\bar{c}c}$, which gives

$$y_c = h_{c\bar{c}} y_{\bar{c}} h^{-1}_{c\bar{c}} - x_{c\bar{c}}. \tag{3.55}$$

As a result, we have $x_{c\bar{c}} = -h_{\bar{c}c} x_{\bar{c}c} h^{-1}_{\bar{c}c}$. The flux then becomes,

$$\begin{aligned}
E^{\bar{v}}_{-\ell} &= g_{\bar{v}\bar{c}} x_{c\bar{c}} g^{-1}_{\bar{v}\bar{c}} \\
&= -g_{\bar{v}\bar{c}} \Big(h^{\bar{v}}_{\bar{c}c} x_{c\bar{c}} (h^{\bar{v}}_{\bar{c}c})^{-1} \Big) g^{-1}_{\bar{v}\bar{c}} \\
&= -g_{\bar{v}\bar{c}} g_{\bar{c}v} g_{vc} x_{c\bar{c}} g_{cv} g_{v\bar{c}} g_{\bar{c}\bar{v}} \\
&= -\tilde{h}^{\bar{c}}_{\bar{v}v} E^v_\ell (\tilde{h}^{\bar{c}}_{\bar{v}v})^{-1}.
\end{aligned}$$

Note that these variables sit at the vertices of Γ^*, which is dual to the usual LQG variables. Therefore, we can think of have observers sitting at the vertices v locally describing the geometry they see around them.

Just like LQG, the symplectic structure is given by $T^*\mathrm{SU}(1,1)$. Inverting the symplectic form $\Omega^{\mathrm{LQG}^*}_\ell$ will be identical to LQG except with $\widetilde{E}_{\ell^*} \to E_\ell$ and $h_\ell \to \tilde{h}_{\ell^*}$, that is,

$$\{E^A_\ell, E^B_\ell\} = \epsilon^{AB}{}_C E^C_\ell \qquad \{E^A_\ell, \tilde{h}^c_{\ell^*}\} = \tilde{h}^c_{\ell^*}\tau^A \qquad \{\tilde{h}^c_{\ell^*}, \tilde{h}^c_{\ell^*}\} = 0. \tag{3.56}$$

This completes the symplectic structure for the dual LQG polarization. The next step will be to describe the constraints of the theory. In opposition to LQG, here the kinematical constraint enforces that the triangles of Γ^* are flat. Namely, for the triangle c^*, with vertices v_1, v_2, v_3 we have that

$$\mathcal{G}_c = \tilde{h}^c_{v_1v_3} \tilde{h}^c_{v_3v_2} \tilde{h}^c_{v_2v_1} = \mathbb{1}. \tag{3.57}$$

The flow for the constraint $\mathcal{G}_c$ is given by

$$\tilde{\delta}^c_\beta E^A_\ell = \langle \mathcal{G}^{-1}_c \{E^A_\ell, \mathcal{G}_c\}, \beta^B_c \tau_B \rangle, \qquad \tilde{\delta}^c_\beta \tilde{h}_{\ell^*} = \Big\langle \mathcal{G}^{-1}_c \{h_{\ell^*}, \mathcal{G}_c\}, \beta^B_c \tau_B \Big\rangle, \tag{3.58}$$

and gives the infinitesimal form of translation of the nodes of Γ. Just like the flatness constraint in LQG, this is an example of a non-abelian momentum map. If we consider a triangle c^* with vertices $[v_1v_2v_3]$ and let $\mathcal{G}_c = \tilde{h}^c_{\ell_3^*} \tilde{h}^c_{\ell_2^*} \tilde{h}^c_{\ell_1^*}$, where $\ell_i^* = [v_{i+1}v_i]$ then the flow is explicitly given by

$$\tilde{\delta}^c_\beta E^{v_3}_{\ell_3} = \tilde{h}^c_{\ell_3^*} \tilde{h}^c_{\ell_2^*} \beta_c (\tilde{h}^c_{\ell_2^*} \tilde{h}^c_{\ell_2^*})^{-1}, \qquad \tilde{\delta}^c_\beta E^{v_2}_{\ell_2} = \tilde{h}^c_{\ell_1^*} \beta_c (\tilde{h}^c_{\ell_1^*})^{-1}, \qquad \tilde{\delta}^c_\beta E^{v_1}_{\ell_1} = \beta_c. \tag{3.59}$$

To see how these were calculated, we consider the E_{ℓ_3} case. In this case we get that

$$\begin{aligned}\tilde{\delta}^c_\beta E^A_{\ell_3} &= \langle \mathcal{G}_c^{-1}\{E^A_{\ell_3}, \tilde{h}_{\ell_3^*}\tilde{h}_{\ell_2^*}\tilde{h}_{\ell_1^*}\}, \beta\rangle \\ &= \langle \tilde{h}^{-1}_{\ell_1^*}\tilde{h}^{-1}_{\ell_2^*}\tilde{h}^{-1}_{\ell_3^*}(\tilde{h}_{\ell_3^*}\tau^A)\tilde{h}_{\ell_2^*}\tilde{h}_{\ell_1^*}, \beta\rangle \\ &= \langle \mathrm{Ad}^*(\tilde{h}_{\ell_2^*}\tilde{h}_{\ell_1^*}), \beta\rangle \\ &= \langle \tau^A, \mathrm{Ad}(\tilde{h}_{\ell_2^*}\tilde{h}_{\ell_1^*})\beta\rangle \\ &= \langle \tau^A, \tau_C R^C{}_B(\tilde{h}_{\ell_2^*}\tilde{h}_{\ell_1^*})\beta^B\rangle \\ &= R^C{}_B(\tilde{h}_{\ell_2^*}\tilde{h}_{\ell_1^*})\beta^B\delta^A{}_C \\ &= [\mathrm{Ad}(\tilde{h}_{\ell_2^*}\tilde{h}_{\ell_1^*})\beta]^A.\end{aligned}$$

In the fourth and last line we use the isomorphism between the adjoint representation of SU(1, 1) and the standard representation of SO(2, 1).

The kinematical phase space of dual LQG is given by the symplectic reduction by the constraint $\mathcal{G}_c$,

$$\mathcal{P}^{\mathrm{kin}}_{\mathrm{LQG}^*} = (\times_{\ell^*\in\Gamma^*} T^*_{\ell^*}G) /\!\!/ \Big(\times_{c\in\Gamma}\mathcal{G}^{-1}(1)\Big). \tag{3.60}$$

If we implemented this constraint at the quantum level we would expect that we would get translational (non-commutative $\mathbb{R}^3$) spin networks. Namely, spin networks where the discrete translations have been factored out. As we will argue below, completely factoring out the gauge orbits is too strict when comparing LQG to polygonal gravity. Furthermore, unlike LQG where the reduction is mostly simple to implement, here it is quite difficult.

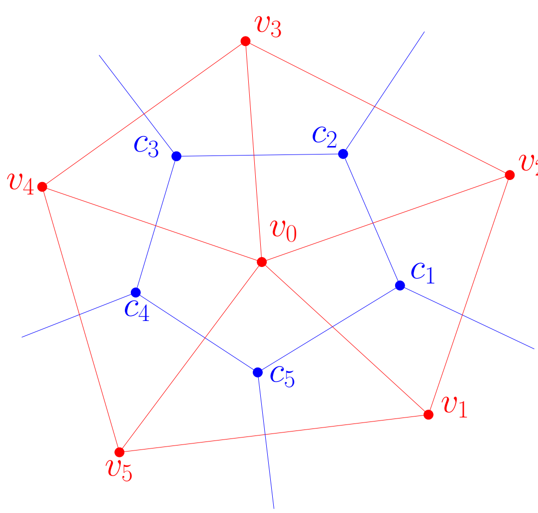

Figure 3.3: Gauss constraint associated to a vertex at v_0

Moving onto the dynamical constraint, we expect this to be related to the Gauss constraint. Consider a vertex v_0 surrounded by n vertices $(v_1, \ldots, v_n)$ ordered clockwise, and n nodes, or centers $(c_1, \ldots, c_n)$ (see figure 3.3). The labeling is chosen so that $c_i^* = [v_{i+1}v_iv_0]$, with $v_{n+1} = v_1$. Furthermore, to simplify notation, we denote $X^{v_0}_{[v_iv_0]^*} \equiv X_i$. Now we will construct an observable that will commute with $\mathcal{G}$. This means our observable at v_0 has to commute with the n flatness constraints $\mathcal{G}_{c_i}$. First, consider the quantity $E_1 + E_2 = E^{v_0}_{[v_1 v_0]^*} + E^{v_0}_{[v_2 v_0]^*}$. The first term in this quantity transforms as,

$$\tilde{\delta}^{c_1}_{\beta} E_1 = \beta, \tag{3.61}$$

if we choose $\mathcal{G}_{c_1} = \tilde{h}_{v_0 v_2} \tilde{h}_{v_2 v_1} \tilde{h}_{v_1 v_0}$. The second term is similar but there is a slight difference due to a change in orientation. For this recall that,[2]

$$\{E^A_\ell, \tilde{h}_{-\ell}\} = -\tau^A \tilde{h}_{-\ell^*}. \tag{3.62}$$

With this in hand, the second term is given by

$$\begin{aligned}
\tilde{\delta}^{c_1}_{\beta} E^A_2 &= \langle \mathcal{G}^{-1}_{c_1} \{E^A_2, \tilde{h}^{-1}_{v_2 v_0}\} \tilde{h}_{v_2 v_1} \tilde{h}_{v_1 v_0}, \beta \rangle \\
&= -\langle \mathcal{G}^{-1}_{c_1} \tau^A \mathcal{G}_{c_1}, \beta \rangle = -\langle \tau^A, \beta \rangle.
\end{aligned}$$

Therefore, the sum $E_1 + E_2$ commutes strongly with $\mathcal{G}_{c_1}$. Continuing this and summing all the fluxes E_i, we see that it will also strongly commute with every $\mathcal{G}_{c_i}$. Therefore, we will take the Gauss, or dynamical constraint, to be

$$\mathcal{J}_{v_0} = \sum_{i=1}^{n} E_a = 0, \tag{3.63}$$

which computes the integral of the torsion around the face f dual to v_0. To make sure this is related to the torsion constraint, we also need to compute the transformations it generates. Setting $\xi = \xi^A \tau_A$, and $\mathcal{J}_{v_0}(\xi_v) \equiv \xi_A \sum_j E^A_j$, we first get

$$\begin{aligned}
\delta^{v_0}_{\alpha} E^A_i &= \{E^A_i, \alpha_B \sum_j E^B_j\} \\
&= \alpha_B \{E^A_i, E^B_i\} \\
&= \alpha_B \epsilon^{AB}{}_C E^C_i \\
&= [\alpha, E_i]^A.
\end{aligned}$$

Similarly for the holonomy, where $\tilde{h}_i = \tilde{h}_{v_i v_0}$ we get

$$\delta^{v_0}_{\alpha} \tilde{h}_i = \{\tilde{h}_i, \xi_A \sum_j E^A_j\} = \xi_A \{\tilde{h}_i, E^A_j\} = -\tilde{h}_i \xi.$$

Therefore, putting these together we get that the transformations induced by the Gauss constraint $\mathcal{J}_v$ are given by

$$\delta^v_\xi E^v_\ell = [\xi_v, E^v_\ell], \qquad \delta^v_\xi \tilde{h}_{\ell^*} = -\tilde{h}_{\ell^*} \xi_v. \tag{3.64}$$

These are precisely the $\mathfrak{su}(1,1)$ gauge transformation, confirming that $\mathcal{J}_v$ is the discretized Gauss constraint. Therefore, the dynamics of LQG in the dual polarization will be given by the constraint $\mathcal{J}_v = 0$, and the physical phase space will be given by

$$\mathcal{P}^{\text{phys}}_{\text{LQG}^*} = \mathcal{P}^{\text{kin}}_{\text{LQG}^*} /\!\!/ (\times_{v \in \Gamma^*} \mathcal{J}^{-1}_v(0)). \tag{3.65}$$

[2]The derivation is $0 = \{E^A_\ell, \tilde{h}_{\ell^*} \tilde{h}_{-\ell^*}\} = \tilde{h}_{\ell^*} \tau^A \tilde{h}_{-\ell^*} + \tilde{h}_{\ell^*} \{E^A_\ell, \tilde{h}_{-\ell^*}\}$, which implies the result.

3.4 Conclusion

	LQG	(Dual) LQG
(Γ, Γ^*) variables	$(h_\ell \to \Gamma, \widetilde{E}^c_{\ell^*} \to \Gamma^*)$	$(E^v_\ell \to \Gamma, \tilde{h}_{\ell^*} \to \Gamma^*)$
kin constr.	$\mathcal{J}_c = 0$	$\mathcal{G}_c = 1$
	(triangle closure)	*(Flat triangles)*
dyn. constr.	$\mathcal{G}_\lambda = 1$	$\mathcal{J}_v = 0$
	(flat)	*(loop closure)*

Table 3.2: Comparison between LQG, LQG*

In this chapter, we described two versions of LQG: the standard and the dual polarizations. We can see from table 3.2 that the choice of polarization essentially flips the role of Γ and Γ^*. Namely, in the LQG polarization the holonomies live on the links of Γ, and the fluxes live on the dual edges. In the dual LQG polarization, we have the exact opposite. However, in both cases, the kinematical constraint lives on the nodes of the graph, and the dynamical relates to the faces of Γ. The meaning of the constraints is swapped between the two pictures. That is, the kinematical constraint in LQG is the Gauss constraint. On the other hand, the kinematical phase space of LQG* is given by enforcing a flatness constraint. This makes finding kinematical observables extremely difficult, as is demonstrated in chapter 5.

Chapter 4

't Hooft Polygonal Gravity

The first step in this thesis was to review 2+1 gravity. We showed that in 2+1 dimensions the equations of motion implied that spacetime is locally flat. This greatly simplifies the theory and makes it much easier to study, which is why so many different approaches to 2+1 quantum gravity exist. This local flatness will be crucial to this chapter.

In this chapter we will present 't Hooft's polygonal gravity, or simply polygonal gravity. Furthermore, this chapter will form the base for the latter parts of the thesis, where a dual version of polygonal gravity will be introduced. We will begin by using the locally flat nature of gravity to construct a special foliation of spacetime. This is the key feature that differentiates polygonal gravity from other approaches. In order to build the foliation, a partial gauge fixing must be made. In the literature, the consequences of this gauge fixing tend to be glossed over. Therefore, we will detail the consequences of this choice. This will be crucial when we try to relate the theory to LQG. Next, we will describe the constraint structure of the theory, and the symmetries they encode.

4.1 Kinematical structure and discretization

In this section, we will provide a review of 't Hooft's polygonal gravity [46, 48, 45]. Polygonal gravity, like classical LQG phase space, is built upon a discretization of spacetime. Unlike LQG however, the discretization requires a gauge fixation that greatly simplifies the theory, and uniquely specifies the dynamics. The idea behind polygonal gravity is to take advantage of the fact that, for 3D pure gravity, the Einstein equations imply that spacetime is flat and torsion free. This implies that M can be covered by a countable number of local Minkowski charts $\mathcal{X}$. Furthermore, the transition functions between intersecting charts will necessarily be the Poincaré group.

The next step is to construct a Cauchy surface, by taking advantage of these local Minkowski charts. To this end, consider two adjacent inertial charts $\mathcal{X}_1, \mathcal{X}_2 : M \to \mathbb{R}^3$ which overlap. Setting $\mathcal{X}_1 = (t_1, x_1, y_1)$ we can construct a constant time surface by setting $t_1 = \tau$, where $\tau \in \mathbb{R}$. Repeating the procedure for $\mathcal{X}_2 = (t_2, x_2, y_2)$, we end up with another constant time surface. However, there may be a time jump between these surfaces. In order to fix this, we set $t_1 = t_2$, which defines the straight boundary between the two charts, as is shown in figure 4.1. If we now intersect this surface with the constant time surface,

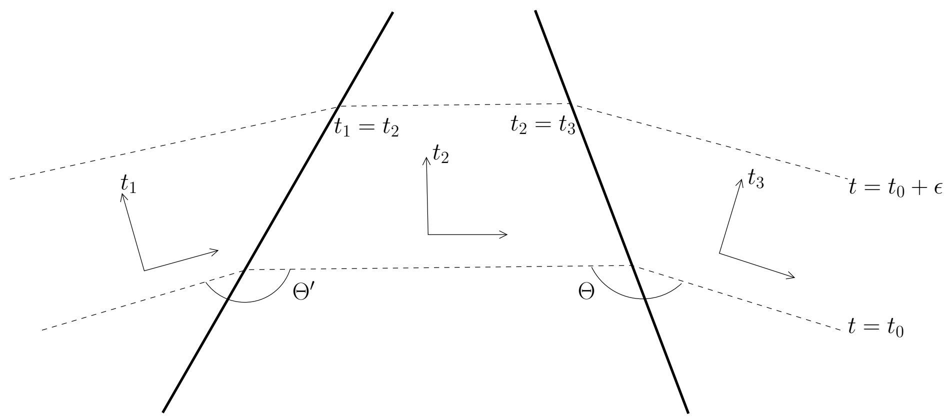

Figure 4.1: The gauge fixing condition $t_i = t_j$ defines the boundaries between the polygons. The Θ and Θ' are the dihedral angles between the polygons.

we can form an edge of a polygon without any time jumps. That is, the boundary of the polygons are defined by $t_1 = t_2$. Continuing this for all regions, i.e. setting

$$t_1 = t_2 = \cdots = \tau, \tag{4.1}$$

we end up with a piecewise flat Cauchy surface made of polygons. This construction, partially fixes the gauge in each polygon and defines a preferred Lorentz frame. In other words we have forced that time run equally fast on all polygons, by setting $t_i = t_j$, and forced the observers in each polygon to be a rest. This choice of time is itself an arbitrary distinction. While the matching conditions were used to construct a consistent Cauchy surface, different matching conditions could be chosen. For instance, if we have chosen $t_1 = at_2$, where a is a real number we would have a consistent Cauchy surface as well.

The relation between polygons are given by the transitions between the respective charts. Namely, since $\mathcal{X}_1, \mathcal{X}_2$ define the preferred inertial charts for each polygon, we have $\mathcal{X}_2 = P_{21}\mathcal{X}_1$, where $P_{21} = \mathcal{X}_2 \circ \mathcal{X}_1^{-1} \in \mathrm{ISO}(2,1)$. This is because the transition map sends inertial frames to inertial frames. This polygonal decomposition will be denoted by Γ. Following 't Hooft, we require, that Γ is dual to a triangulation Γ^*. This means that the vertices of the polygons will be three valent. This assumption is key for defining the theory as we will see. This discrete structure matches the last chapter, so we will use the same notation. Namely, the sides of the polygon will be called links $\ell = [c_2c_1]$, and the c_i, which

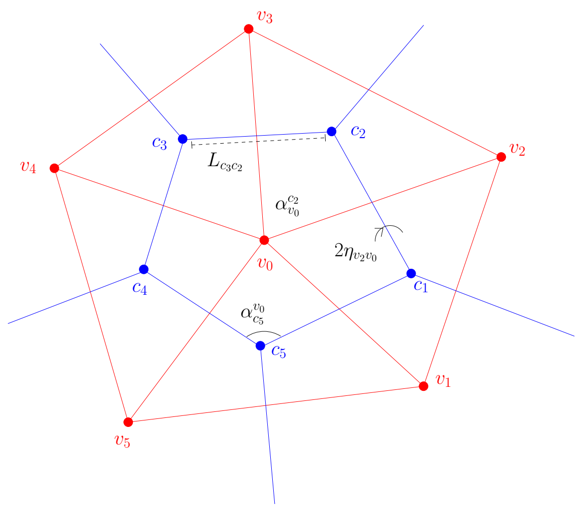

Figure 4.2: The nodes and links of the graph Γ form the vertices and edges of the polygon respectively. The angles between the edges of the polygon are specified by the dihedral angle 2η, which is a boost.

are the vertices of the polygons, will be called nodes. Therefore, a polygon will be given by v^* i.e. dual to a vertex v of a triangulation.

To see the consequences of the gauge fixing and 3-valent nodes, we consider a node c and dual triangle $[v_2v_1v_0]$. The edges of the triangle connect three the polygons with frames $\mathcal{X}_{v_0}, \mathcal{X}_{v_1}, \mathcal{X}_{v_2}$, and is shown in figure 4.4. The matching conditions between the polygons will be given by

$$\mathcal{X}_{v_2} = P_{v_2v_1} \triangleright \mathcal{X}_{v_1}, \qquad \mathcal{X}_{v_3} = P_{v_3v_2} \triangleright \mathcal{X}_{v_2}, \qquad \mathcal{X}_{v_1} = P_{v_1v_3} \triangleright \mathcal{X}_{v_3}. \tag{4.2}$$

Since the two polygons are related by a Poincaré transformation, we will parameterize elements of ISO(2,1) by $P = (\Lambda, a)$. In terms of this parameterization, the action will be given by $P \triangleright \mathcal{X} = \Lambda\mathcal{X} + a$, and $\Lambda \in \mathrm{SO}(2,1)$ and $a \in \mathbb{R}^{2,1}$. Furthermore, we will decompose Λ into

$$\Lambda_{v_jv_i} = R(\bar{\phi}_{v_jv_i})B(2\eta_{v_jv_i})R(\phi_{v_jv_i}) \tag{4.3}$$

where

$$R(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{pmatrix}, \qquad B(\Theta) = \begin{pmatrix} \cosh(\Theta) & \sinh(\Theta) & 0 \\ \sinh(\Theta) & \cosh(\Theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

This decomposition will be called the *'t Hooft polygonal decomposition* in the rest of the paper.

The quantities ϕ, $\bar{\phi}$, and 2η all have geometric interpretations. The quantity ϕ gives the orientation of the edge of the polygon from its observer. Namely, if we consider the holonomy $\Lambda_{v_2v_1}$, then $\phi_{v_2v_1}$ gives the orientation of the edge $[v_2v_1]^* = [c_2c_1]$ (see figure 4.3).

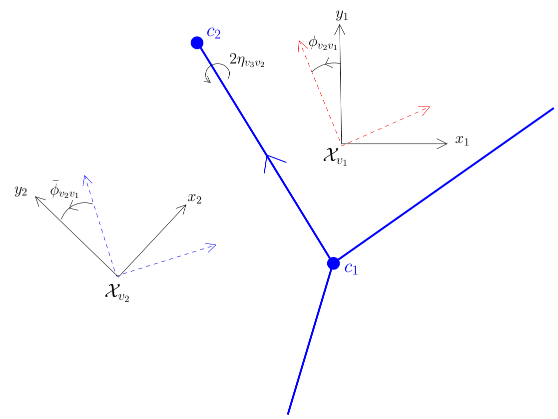

Figure 4.3: Decomposition of holonomy between polygons. The $\phi_{v_2v_1}$ orientates the frame $\mathcal{X}_{v_1}$ so that the y-axis is parallel with the polygon edge $[c_2c_1]$. The parameter $2\eta_{v_2v_1}$ is the boost around the edge $[c_2c_1]$ and is the dihedral angle between the polygons v_1^*, v_2^*. Finally the angle $\bar{\phi}_{v_2v_1}$ rotates the frame so that aligns with the frame $\mathcal{X}_{v_2}$.

In other words, if we view the edge as a vector $E^v_{c_2c_1}$ we can write this as [28]

$$E^v_{c_2c_1} = L_{c_2c_1} R(-\phi_{v_2v_1}) e_y, \tag{4.5}$$

where $e_y = (0,0,1)$. Therefore, $\phi_{v_2v_1}$ rotates the frame $\mathcal{X}_{v_1}$ so that the y-axis is parallel to the edge of the polygon $[c_2c_1]$. The boost parameter $2\eta_{v_2v_1}$, the dihedral angle between the faces of the polygon. Since assumed that the edge lies along e_y, we have that the boost is around this axis. Finally, the angle $\bar{\phi}_{v_2v_1}$ is the angle we must rotate the frame, after the boost, so that it is aligned with the frame $\mathcal{X}_{v_2}$. This decomposition/interpretation, has an implication when we reverse the orientation of the edge. First, note that we require that $\Lambda_{\ell^*}\Lambda_{-\ell^*} = 1$. However, when we replace $\ell \to -\ell$ there is a change of orientation of the edge vector. This means ϕ_{ℓ^*} and $\bar{\phi}_{\ell^*}$ will be anti-parallel to the $-\ell$ edge. Therefore, they need to be replaced by $\pi - \phi_\ell$. Therefore, in order to be consistent, we require that the holonomy for $\Lambda_{-\ell}$ be parameterized by

$$\Lambda_{-\ell^*} = R(\pi - \phi_{\ell^*}) B(2\eta_{-\ell^*}) R(\pi - \bar{\phi}_{\ell^*}). \tag{4.6}$$

The consistency condition $\Lambda_{-\ell^*}\Lambda_{\ell^*} = 1$ then requires that $\eta_{\ell^*} = \eta_{-\ell^*}$. Furthermore, this requires that

$$\phi_{\ell^*} + \bar{\phi}_{-\ell^*} = \pi, \tag{4.7}$$

again since the orientation of the edge will be reversed.

Assuming, for simplicity, that the transition between polygons $\mathcal{X}_1, \mathcal{X}_2$, P_{21} satisfies $\bar{\phi}_{v_2v_1} = \phi_{v_2v_1} = 0$,[1] then the matching conditions (4.2) are given by

$$\begin{aligned} t_2 &= \cosh(2\eta_{21})t_1 + \sinh(2\eta_{21})x_1 + a_0, \\ x_2 &= -\sinh(2\eta_{21})t_1 + \cosh(2\eta_{21})x_1 + a_1, \\ y_2 &= y_1 + a_3, \end{aligned} \tag{4.8}$$

[1] This is is always possible by a rotation around the observers respective time axes

where we dropped the v in the label for readability. By setting $\phi = \bar{\phi} = 0$, we are saying that the edge is parallel to the y-axis of both polygonal frames.

Using the gauge fix $t_1 = t_2$, that specifies the time evolution of the edge, we get

$$x_1(t_1) = \tanh{(\eta_{21})} t_1 + C_1, \tag{4.9}$$

where C_1 is a constant.

This provides us with a further interpretation of what η represents. The boost parameter $2\eta = \Theta$ defines the dihedral angle between the polygons, but η is also the rapidity of the edge of the polygon. Furthermore, the velocity of the edge is perpendicular to the direction of the edge. Note that because $\eta_{12} = \eta_{21}$, we have that the edge of the polygon appears to grow (or shrink) from either polygon frames. Furthermore, by construction, η does not evolve in time. Finally, the components of the translation will not be included in the description of the polygonal surface, since they are in fact related to the triangulation Δ, which will be used later in chapter 6.

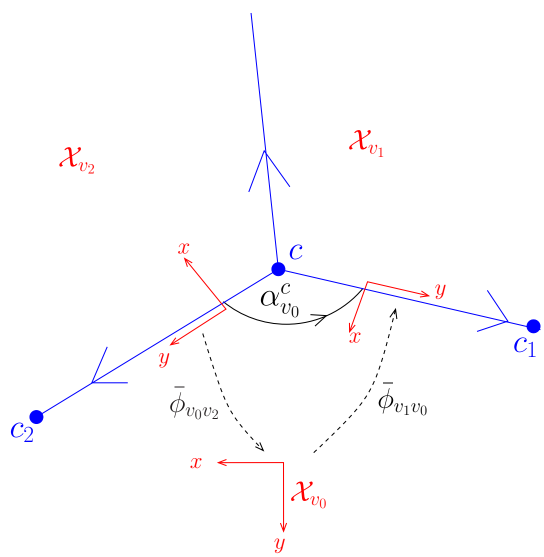

Figure 4.4: The angles $\phi_{v_1 v_0}$, $\bar{\phi}_{v_0 v_2}$, specify the orientation of the edges $[c_2 c]$ and $[c c_1]$ respectively. They are related the angles $\alpha^c_{v_0}$ between the edges of the polygon, through the equation $\alpha^c_{v_0} = \phi_{v_1 v_0} + \bar{\phi}_{v_0 v_2}$.

To see the consequence of the three valent nodes, recall the relation between the frames (4.2). Applying these in succession around the node, we get

$$P_{13} P_{32} P_{21} = \mathbb{1}. \tag{4.10}$$

Expanding the transition function as $P = (\Lambda, a)$ we get

$$\Lambda_{13} \Lambda_{32} \Lambda_{21} = \mathbb{1}, \tag{4.11}$$

$$\Lambda_{13} \Lambda_{32} a_{13} + \Lambda_{13} a_{32} + a_{13} = 0, \tag{4.12}$$

resembling the flatness constraint for the Chern-Simons connection. That is, the first constraint says that the three dimensional curvature at the node has to vanish, or that the triangles of Γ^* have to be flat. The second constraint is reminiscent of the abelian holonomy components from the Chern-Simons connection, and can be interpreted as a Gauss constraint for the triangle. However, the second constraint does not imply any relation between our variables ϕ and η, so we will ignore it as is usually done in the literature [29, 28]. The first vertex relation (4.11), gives *'t Hooft's vertex condition* for a node c,

$$V_c = B(2\eta^c_{v_1v_3})R(\alpha^c_{v_3})B(2\eta^c_{v_3v_2})R(\alpha^c_{v_2})B(2\eta^c_{v_2v_1})R(\alpha^c_{v_1}) = \mathbb{1}, \tag{4.13}$$

where we set $\Theta_{v_iv_j} = 2\eta_{v_iv_j}, \alpha^c_{v_1} = \phi^c_{v_2v_1} + \bar{\phi}^c_{v_1v_3}$, etc. Considering the node in figure 4.4, we see that $\alpha_{v_i} \in [0, 2\pi)$ are the angles between edges of the polygon incident to the node. Furthermore, both α and η are invariant under rotations about the time axis. The flatness constraint (4.13), gives a relation between the dihedral angles $\Theta_{v_iv_j} = 2\eta_{v_iv_j}$ at a node, and the angles α_{v_i}. Supposing that the triangle is given by $[v_3v_2v_1]$, with edges $\ell^*_i = [v_{i+1}v_i]$, where $v_{3+1} = v_1$, then we get

$$\frac{s_1}{\sigma_1} = \frac{s_2}{\sigma_2} = \frac{s_3}{\sigma_3} \tag{4.14}$$

$$\gamma_2 s_3 + s_1 c_2 + c_1 s_2 \gamma_3 = 0 \tag{4.15}$$

$$c_1 = c_2 c_3 - \gamma_1 s_2 s_3 \tag{4.16}$$

$$\gamma_1 = \gamma_2\gamma_3 + \sigma_2\sigma_3 c_1 \tag{4.17}$$

$$\left|\eta_{\ell^*_1}\right| + \left|\eta_{\ell^*_2}\right| \geq \left|\eta_{\ell^*_3}\right| \tag{4.18}$$

and all cyclic permutations, where we set

$$\sin\alpha_{v_i} = s_{i+1}, \qquad \cos\alpha_{v_i} = c_{i+1}, \qquad \sinh 2\eta_{\ell^*_i} = \sigma_i, \qquad \cosh 2\eta_{\ell^*_i} = \gamma_i, \tag{4.19}$$

to match the literature. Note, that this set is redundant, as shown in [27]. Furthermore, the equations do not uniquely specify α in terms of η. However, if we require that only one of the three angles α can be greater than π there we can specify α in terms of η. In fact, this requirement is necessary in order to have a Cauchy surface [46]. Therefore, one set of variables in polygonal gravity is taken to be the η. The angles, α are then functions of η. This set of equations, (4.13), also explains why 't Hooft chose to restrict the theory to 3-valent nodes. If the vertices where higher-valent, the equations for the rapidities η and the angles α would be overdetermined. Therefore, it would be difficult to define the theory in terms of just the rapidities.

The missing variables, will be given by the lengths, L, of the edges of the polygons. Like dual LQG the lengths will be assigned to the links ℓ of Γ. However, due to the gauge fixing we know that the edges will be purely spatial and lie in a spacelike hyperplane. Therefore, the lengths will just be the difference between the positions of the nodes of a link ℓ as seen from the polygon v^*. We will denote the length by L_ℓ. Furthermore, since the metric is assumed to be continuous, the lengths of the edges will be equal when viewed from either polygon [46]. As with η, these variables are invariant under rotations around the time axis of their respective polygons. Unfortunately, the lengths and rapidities will change highly non-trivially under boosts, [20], due to the partial gauge fixing.

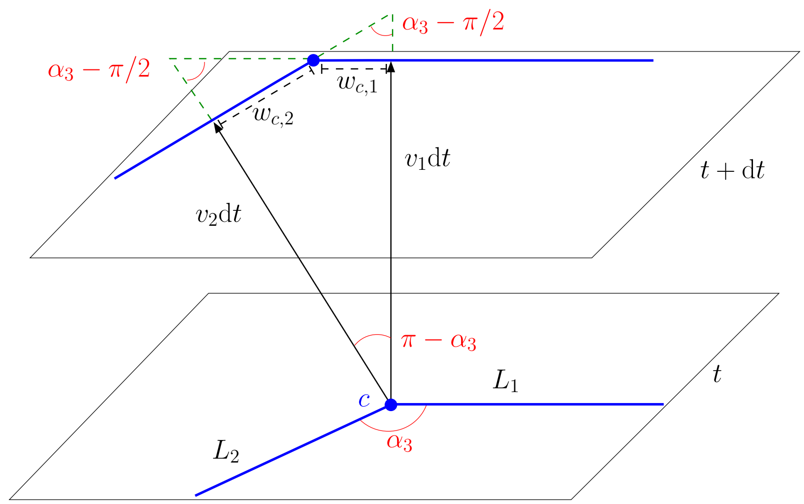

Figure 4.5: Evolution of two links from at the node c. The growth rate of the edge i from the node c is given by $w_{c,i}$.

While we do not yet have a symplectic structure for our space, we have specified a time. In constructing our Cauchy surface we chose a time that provided us with dynamics of the edges of the polygons. That is, the gauge fix $t_i = t_j$, allowed us to observe that the edges of the polygons must move perpendicularly to themselves with rapidity η. Using this gauge fixing, we can derive the dynamics of our phase space variables L and η. As mentioned above, we know that η will be constant in time, so those dynamics are trivial. For the lengths, we can derive their dynamics using that we known $\dot{x} = v = \tanh(\eta)$ for a polygon with frame (t, x, y). Following figure 4.5, we denote the angle the two edges makes by α_3 and the rapidities of the edges by η_1, η_2. The growth rate from the node c for the two edges will be denoted by $w_{c,i}$. Therefore, since the velocity of an edge is perpendicular to its orientation, we know that $\tan(\alpha_3 - \pi/2) = w_{c,2}/h_2 = w_{c,1}/h_1 + v_1\mathrm{d}t$. Here h_1, h_2 are the extra length we need to form a right angle triangle as shown in figure 4.9 (see the green dotted lines). We also know that $\cos(\pi - \alpha_3) = v_1\mathrm{d}t/(h_2 + v_2\mathrm{d}t) = v_2\mathrm{d}t/(h_1 + v_1\mathrm{d}t)$. After simple algebra we find

$$w_{c,1}\mathrm{d}t = \frac{v_2 + v_1\cos(\alpha_3)}{\sin(\alpha_3)}\mathrm{d}t, \qquad w_{c,2}\mathrm{d}t = \frac{v_1 + v_2\cos(\alpha_3)}{\sin(\alpha_3)}\mathrm{d}t. \tag{4.20}$$

Then since every edge will get two contributions, from the two nodes it connects, we get the evolution of the length L of the edge connecting the nodes A and B,

$$\frac{\mathrm{d}L_{AB}}{\mathrm{d}t} = w_{A,1} + w_{B,1}. \tag{4.21}$$

In order for this evolution to be well-defined, we need to do one more check. In our derivation, we assumed we had two edges of interest; however, all of our nodes will be

3-valent. Therefore, we need to ensure that

$$\frac{v_2 + v_1\cos(\alpha_3)}{\sin(\alpha_3)} = \frac{v_3 + v_1\cos(\alpha_2)}{\sin(\alpha_2)}. \tag{4.22}$$

This can be verified explicitly by using (4.16). This also means that we only have consistent dynamics when our η and α obey 't Hooft's vertex condition (4.13).

To summarize, we now have a way to describe a piece-wise flat Cauchy surface using lengths and rapidities/dihedral angles. These will parameterize the phase space of polygonal gravity. However, this is still an overcomplete description of the surface. This can be verified by counting the degrees of freedom of this model and comparing it to the reduced phase space for gravity. Therefore, there must exist constraints. However, before we discuss the constraints it will be pertinent to first discuss the symplectic structure.

4.2 Brackets and Constraints

In the last section, we found that the dynamics of the theory are specified by 't Hooft's partial gauge fixing. Namely, (4.9) shows that the edges grow linearly in time, and that η is constant in time[2]. However, this is not quite enough to ensure that this is a theory of gravity. In order to compare polygonal gravity with LQG we need a symplectic structure. In [45], 't Hooft asked the reduced Hamiltonian for node c to be

$$H_c = 2\pi - \sum_{i\,\mathrm{at}\,c} \alpha_i(\eta_j), \tag{4.23}$$

i.e. the deficit angle at the node. In order for this to generate the correct time evolution the symplectic structure is required to be

$$\{L_{\ell_i}, L_{\ell_j}\} = 0, \qquad \{L_{\ell_i}, 2\eta_{\ell_j^*}\} = -\delta_{ij}, \qquad \{\eta_{\ell_i^*}, \eta_{\ell_j^*}\} = 0. \tag{4.24}$$

This was verified in [45] by checking that $\{L_i, H_c\} = w_{c,i}$.

So far, thanks to solving the matching condition at the vertices for every edge of our polygon, we have two variables L, η. This means that if our polygonal decomposition has E edges that there are $2E$ degrees of freedom. This is much larger than the known reduced phase space of $2+1$ gravity, which for a surface of genus g, $g > 1$ has dimension $12g - 12$. Thus, as mentioned above, there must exist constraints. Furthermore, we just have two sets of variables to describe our polygons, (L, η) and the angles α, as determined from (4.13). Therefore, we need to ensure we can reconstruct our polygons from just this set. We will find, that in order to build polygons, some constraints will need to be imposed on on L and η.

[2]Note that this evolution is only for small time. For large enough time evolution, this can break down due to the gauge fixing. For instance, an edge's length can shrink to zero. When this happens, 't Hooft's transitions, described in [48], must be applied in order to maintain a Cauchy surface.

Suppose we have a polygon like in figure 4.2. Then the first step in reconstructing the polygon will be to find the edge vectors E_{ℓ_i} from $(L_{\ell_i}, \eta_{\ell_i^*}, \alpha^v_{c_i})$, where we set $\ell_i = [c_{i+1}c_i]$. To do this we will take advantage of the fact that the partial gauge fixing implies that the edge vectors E must lie in spacelike hyperplane. Furthermore, recall that L and η (and thus $\alpha = \alpha(\eta)$), are all invariant under rotation around the time axis of the polygon. Therefore, we assume that $E^{v_0}_{\ell_1}$ is parallel to the y-axis of the polygonal frame $\mathcal{X}_{v_0}$. Then to ensure that the polygon $f = v_0^*$ closes we require that

$$C_{G,f}(L_1) = L_1 e_y + \sum_{i=1}^{n} L_i R(\theta_i^{L_1}) e_y = 0, \qquad \theta_i^{L_s} = \sum_{j=s}^{i-1} (\pi - \alpha_j), \tag{4.25}$$

where $L_i = L_{\ell_i}$. In the literature, the constraint is often written in complex notation. Namely, the rotation matrices are replaced with complex exponentials. In this case the closure constraint is given by

$$C_{G,f}(L_1) = L_1 + \sum_{i=1}^{n} L_i \exp\left(\theta_i^{L_1}\right) = 0. \tag{4.26}$$

However, this is not quite enough to ensure that we have a true polygon. To have consistency, we need to make sure that we can choose any L_i to lie along the y-axis. Let us try L_2 instead of L_1. In this case the closure constraint is

$$C_{G,f}(L_2) = L_2 e_y + \sum_{i=3}^{n} L_i R(\theta_i^{L_2}) e_y + L_1 R\left(\sum_{j=2}^{n-1} \pi - \alpha_j\right) e_y. \tag{4.27}$$

Multiplying by $R(\pi - \alpha_1)$ we get that

$$R(\pi - \alpha_1) C_{G,f}(L_2) = L_2 R(\pi - \alpha_1) e_y + \sum_{i=3}^{n} L_i R(\theta_i^{L_1}) + L_1 R\left(\sum_{j=1}^{n} \pi - \alpha_j\right) e_y. \tag{4.28}$$

It is then easy to see that $C_{G,f} = 0$ will only be consistent if we also require that $C_{\alpha,f} = \sum_{j=1}^{n} (\pi - \alpha_j) - 2\pi = 0$. This has a nice geometric interpretation. By requiring that our polygons are co-planar, we have necessarily enforced that every polygon has zero extrinsic curvature; however, there can still be intrinsic curvature. Therefore, if we are indeed to have a solution of Einstein's equation we necessarily must require that the intrinsic curvature of the polygon vanishes which is precisely $C_{\alpha,f}$. In summary, in order to have true polygons we require the 't Hooft constraints:

$$C_{\alpha,f} = \sum_{j=1}^{N} (\pi - \alpha_j) - 2\pi \approx 0 \tag{4.29}$$

$$C_{G,f} = \sum_{j=1}^{N} L_j e^{i\theta_j} \approx 0, \qquad \theta_j = \sum_{k=1}^{j-1} (\pi - \alpha_k). \tag{4.30}$$

The constraint $C_{G,f}$, is reminiscent of the Gauss constraint in dual loop quantum gravity, except there are only 2 constraints since our gauge choice forced the edges to be purely spatial. In fact, in the next chapter we will prove they are the same. These constraints turn out to be first class [54, 20], however the constraint algebra does not form a Lie algebra since the structure "constants" depend on the phase space parameters. We will provide a proof that the constraint algebra is first class in the next chapter.

As these constraints are first class, we expect them to generate gauge transformations. The first constraint is argued to generate time translation. To see this, first recall the definition of the Euler characteristic $\chi = V - E + F = 2 - 2g$, where V, E, F are the number of vertices, edges and faces of Γ, and g is the genus of the surface. Then the first constraint is related to the total Hamiltonian $H = \sum_c H_c$ by

$$H = 2\pi(2 - 2g) - \sum_{f \in \Gamma} C_{\alpha,f}, \tag{4.31}$$

where we have summed over all the polygons. Note that this also means that the total Hamiltonian is constant when $C_{\alpha,f} = 0$ is satisfied. One reason to see why this is the case, is from the Gauss-Bonnet theorem. Namely, from Regge calculus for a 2D surface we know that

$$\int_\Sigma {}^{(2)}R\sqrt{{}^{(2)}g}\,\mathrm{d}^2x = \sum_c H_c = H, \tag{4.32}$$

However, the Gauss-Bonnet theorem gives

$$\int_\Sigma {}^{(2)}R\sqrt{{}^{(2)}g}\,\mathrm{d}^2x = 2\pi\chi. \tag{4.33}$$

Note, that computing the flow of the $C_{\alpha,f}$ constraint is simple. If we consider a polygon f, then a quick calculation show that the lengths of the polygon are invariant, i.e.

$$\delta_{C_{\alpha,f}} L = \{L, C_{\alpha,f}\} \approx 0. \tag{4.34}$$

However, it does change the lengths of the polygons adjacent to the polygon f, and after summing the constraints $C_{\alpha,f}$ which is why the sum of the constraints defines time evolution in 't Hooft time gauge.

The symmetries generated from the second constraint (4.30) are not so easily identified. In [45, 29] it is argued that it generates the two Lorentz boosts for the local frame in each polygon. This would seem to match the situation in LQG where the closure condition does generate local $\mathrm{SO}(2,1)$ transformations. However, we do not know of an explicit proof that the constraint (4.30) does in fact generate boosts. However, below we will show how one can recover this constraint from dual loop quantum gravity, proving 't Hooft's proposition.

As a conclusion to this chapter, we should stress an important point. Nowhere in our derivation did we ever use the Einstein-Hilbert action or the Einstein equations explicitly. That is, we used purely geometrical arguments to derive the symplectic structure and constraints. Therefore, we should do a quick counting argument to ensure that we get the correct number of degrees freedom for the reduced phase space. First if our graph has E

edges we then have $2E$ degrees of freedom. Every polygon has 3 constraints, two from $C_{G,f}$ and one from $C_{\alpha,f}$. However, the action of the symmetry group generated from $C_{\alpha,f}$ and $C_{G,f}$ is not free. Namely, if we apply the same Lorentz boost to every polygon frame the lengths and rapidities would not change. Therefore, we have that the number of degrees of freedom is seemingly given by

$$2E-3F-(3F-1) = 2E-6F+1 = 6E-6V-6F+1 = -6(V-E+F)+1 = 12g-11, \quad (4.35)$$

where we used the fact that the graph Γ is trivalent so $2E = 3V$. Therefore, it seems the dimension of phase space is an odd number. In [45], is is argued that this is due to the choice of time we picked as a gauge. This enforces a relation among the lengths and rapidities and decreases the number of degrees of freedom to the required $12g - 12$.

4.3 Conclusion

In this chapter polygonal gravity was introduced. Polygonal gravity is based on a special discretization of the Cauchy surface, where the faces or polygons of the graph Γ, are forced to be co-planar. As a result of this, the extrinsic curvature inside each polygon vanishes. In the next chapter, this will be the biggest hurdle when relating dual LQG to polygonal gravity. Another assumption necessary to define polygonal gravity is that the vertices of the graph are three valent. Namely, that the Cauchy surface has been triangulated. This is necessary, so that the rapidities of the edges, and the angles between them, have a well-defined relation. In other words, if the vertices have a higher-valence, solving the vertex condition (4.13) becomes more difficult.

The symplectic structure for the theory was then taken as a postulate, and was verified by ensuring that it generated the correct equations of motion. The constraints for the theory enforced that the 't Hooft polygons are "true" polygons. Firstly, the edges viewed as vectors have to sum to zero. This ensures that the polygon closes. Secondly, the sum of the angles of the polygon has to give the classical result, i.e. equation (4.30). In the next chapter, we will explicitly derive this structure from dual LQG.

Chapter 5

Exploring the Relationship Between LQG and Polygonal Gravity

In the previous two chapters, three models of discrete gravity were introduced. The first, LQG, is a discretization of first order gravity, where we have set $\alpha = 0$ in equation (2.27). The second was a dual version of LQG, and is the natural discretization of (2.27) with $\alpha = 1$. Therefore, both theories start from the same family of symplectic potentials. The third theory discussed was polygonal gravity. Unlike the previous two theories, polygonal gravity starts from a purely geometric standpoint, not from an action or equation of motion. The question then becomes, how the two classes of theories are related. This will answer the first goal of the thesis. While previous work has been done on this question [28, 52], we will explain why their solutions were incomplete, and further analysis was necessary.

The major obstacle in comparing the two theories will be the structure of the graph Γ, and how it relates to 't Hooft's polygons. Therefore, in the first section of this chapter we will compare the discretizations of LQG, LQG* and polygonal gravity. Here, we will mention why the approach taken in [52] fails to reduce LQG to polygonal gravity. Furthermore, this will fully motivate why using LQG* is the superior choice. The reason is that the flatness constraint is solved first in LQG* and polygonal gravity. In the following section, we will explore precisely how the dual loopy framework reduces to polygonal gravity. This will be related to work previously done in [28]. However, there are several differences between this work and this thesis. First, we do not restrict the theory to one polygon, making our presentation more general. Second, we correct a mistake presented in [28], where they stated that the constraints (4.29) and (4.30) are related to the Gauss

	LQG	(Dual) LQG	't Hooft gravity
(Γ, Γ^*) variables	$(h_\ell \to \Gamma, \widetilde{E}^c_{\ell^*} \to \Gamma^*)$	$(E^v_\ell \to \Gamma, \tilde{h}_{\ell^*} \to \Gamma^*)$	$L_\ell \to \Gamma, \eta_{\ell^*} \to \Gamma^*, \alpha_{\ell_1,\ell_2}$
kin constr.	$\mathcal{J}_c = 0$	$\mathcal{G}_c = 1$	$V_c = 1$
	(triangle closure)	*(Flat triangles)*	*(Flat triangles)*
gauge fixing.	–	–	$t_{v_1} = t_{v_2} = \cdots = \tau$
			(planar region & time choice)
dyn. constr.	$\mathcal{G}_\lambda = 1$	$\mathcal{J}_v = 0$	$C_G = 0, C_\alpha = 0$
	(flat)	*(closure constraint)*	*(polygon condition)*

Table 5.1: Comparison between LQG, LQG* and 't Hooft gravity.

constraint of LQG* (3.63).

Another issue that is overlooked in [28], is whether the gauge fixing required in polygonal gravity is possible. In order to answer this question, we apply polygonal gravity to the torus universe. In the literature there are conflicting statements regarding the nature of the torus universe in polygonal gravity. In [23], they claim to recover the reduced phase space for the torus universe (i.e. the cotangent bundle of the moduli space of the torus), by identifying the Teichmüller parameters. However, the authors of said paper never analyzed whether their proposed parameters where gauge invariant. This was first pointed out in [54], where "spacetime" arguments were used to suggest that said parameters were not gauge invariant. Therefore, in the final section of the chapter, we review both arguments and then use the map from dual LQG to attempt to answer whether the torus universe can be described in polygonal gravity.

5.1 Comparing discretizations of polygonal gravity and LQG

All the theories we have described in this thesis can effectively be described by the discretization scheme presented in section 3.1. Namely, we start by triangulating a spatial hypersurface Σ. The main difference is how the theories deal with the dual graph Γ. In regular LQG, the holonomies live on the graph Γ. These holonomies relate observers at the nodes of the graph. The fluxes $\widetilde{E}$, describe the edges of the triangulation by virtue of the Gauss constraint (3.39). Namely, any three vectors that sum to zero can be seen as describing the edges of a triangle.

In [52], the Gauss reduced phase space of LQG, $\mathcal{P}^{\text{LQG}}_{\text{kin}}$, is argued to be related to polygonal gravity. Their key insight was that the dihedral angle $\Theta_{c_2c_1}$ between two triangles $c_1^*c_2^*$ defined by (6.8) is canonically conjugate to the Gauss invariant length $L_{v_2v_1}$, (6.6) of the dual edge. While this does superficially seem to have the same symplectic structure as polygonal gravity there are some key differences. One difference is that, as mentioned above, the definitions of L, (6.6) and (6.8) are both Gauss invariant observables. Therefore, they will be unchanged under boosts. This is in stark contrast to L and η in polygonal gravity, which are clearly not invariant under boosts. Furthermore, LQG has exactly the opposite discretization as polygonal gravity. Namely, the edge vectors describe the edges of the dual triangles of Γ^* not the links of Γ. The authors of [52] realized this and attempted to deal with this by a complicated process, that involved joining together triangles together if the extrinsic curvature, i.e. Θ was zero, creating general polygons. However, by doing so they are unable to recover 't Hooft's vertex relations [52]. In fact, this is the cause for all the differences highlighted above. Polygonal gravity starts by solving the flatness constraint, not the Gauss constraint.

In light if these tribulations, it seems that the usual LQG formalism is not convenient to recover the polygonal formulation. Luckily, many of the difficulties we face is exactly due to the fact that the discretization of polygonal gravity is dual to the usual LQG choice, just like dual loop quantum gravity. Therefore, dual LQG is the correct theory to relate to polygonal gravity, since they both start by solving the flatness constraint. However, how this is done in polygonal gravity is not simple, as we will now show.

5.2 Gauge fixing dual LQG to polygonal gravity

Referring to table 5.1, we can see that LQG* and polygonal gravity are much more similar, at least at the level of the discretization, than LQG. In both LQG* and polygonal gravity, the holonomies live on the edges of the triangulation, and the kinematical constraint is the flatness constraint. In fact we can recover the polygonal variables α and η exactly in dual LQG. Consider a link $\ell = [c_2 c_1]$ with dual edge $\ell^* = [v_2 v_1]$. The holonomy $\tilde{h}^{c_1}_{v_2 v_1}$, then represents the change of frame from the faces of Γ as you travel through c_1^*. This suggests that we can use 't Hooft's decomposition of elements, (4.13). Note that since LQG is usually presented in using the spin group $\mathrm{SU}(1,1)$ of $\mathrm{SO}(2,1)$, the expressions for R and B will change,

$$R_s(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \qquad B_s(\Theta) = \begin{pmatrix} \cosh(\Theta) & \sinh(\Theta) \\ \sinh(\Theta) & \cosh(\Theta) \end{pmatrix}. \tag{5.1}$$

However, nothing else about the decomposition changes, i.e.

$$\tilde{h}^{c_1}_{v_2 v_1} = R_s(\bar{\phi}_{v_2 v_1}) B_s(2\eta_{v_2 v_1}) R_s(\phi_{v_2 v_1}). \tag{5.2}$$

At this point there seems to be an ambiguity in the choice of node c_1, c_2 for the holonomy $\tilde{h}_{v_2 v_1}$; namely, we can choose $\tilde{h}^{c_1}_{v_2 v_1} = g_{v_2 c_1} g_{c_1 v_1}$ or $\tilde{h}^{c_2}_{v_2 v_1} = g_{v_2 c_2} g_{c_2 v_1}$. However, recall that we assumed that curvature is focussed on the vertices of the triangulation. Furthermore, by composing $\tilde{h}^{c_1}_{v_1 v_2}$ and $\tilde{h}^{c_2}_{v_2 v_1}$ we end up with a closed curve, where the region inside the curve is flat. Therefore, both holonomies will be identical. Mathematically this can be expressed though the matching condition (3.14), which implies there exists a constant $h_{c_1 c_2}$ relating the two holonomies,

$$\begin{aligned}\tilde{h}^{c_1}_{v_2v_1} &= g_{v_2c_1}g_{c_1v_1}\\ &= g_{v_2c_1}h_{c_1c_2}h_{c_2c_1}g_{c_1v_1}\\ &= (g_{v_2c_1}h^{v_1}_{c_1c_2})(h^{v_2}_{c_2c_1}g_{c_1v_1})\\ &= g_{v_2c_2}g_{c_2v_1}\\ &= \tilde{h}^{c_2}_{v_2v_1},\end{aligned}$$

where in the third equality we used the fact that the holonomy is constant along the edge $[v_2v_1]$. Plugging the decomposition (5.2) into the flatness constraint $\mathcal{G}_c$, we then easily see that 't Hooft's vertex condition vanishes exactly when the flatness condition is satisfied. Therefore, η and α in dual LQG, obey the same conditions. In fact, η and α, are observables for the kinematical phase space of dual LQG, since components of the holonomies commute. This hints that polygonal gravity's phase space is related to the kinematical phase space of LQG*. However, this is where we start to run into issues. Namely, we do not yet have length variables. In fact, the construction of these spatial lengths, will be the main issue we have to deal with.

The expectation is that 't Hooft's lengths will be related to the fluxes of the dual loop picture. After all, both are assigned to the links of the graph Γ. Moreover, both are given by the difference between the two endpoints of the link as seen from the center of the polygon. In polygonal gravity, this is assumed. In dual LQG, this is true by definition of the flux

$$E^v_{[c_2c_1]} = g_{vc_1}x_{c_2c_1}g^{-1}_{vc_1}. \tag{5.3}$$

With this observation, it seems obvious choice of polygonal length in LQG* is $L_{c_1c_2} = \left|E^v_{c_1c_2}\right|$. Unfortunately, this choice does not seem to work since it is not a kinematical observable. That is, by using (3.59) we can see that L will change as

$$\tilde{\delta}^{c_1}_{\beta}L^2_{c_1c_2} = 2\langle E^v_{c_1c_2}, \tilde{\delta}^{c_1}_{\beta}E^v_{c_2c_1}\rangle = 2\langle E^v_{c_1c_2}, (H^{c_1})^{-1}\beta H^c\rangle, \tag{5.4}$$

where H^{c_1} is a holonomy that depends on the specific ordering of the flatness constraint. The geometrical reason this change for this is quite obvious. The flatness constraint at the node c_1 acts by translating the location of node inside the triangle c_1. Obviously, this will change the length of the edge unless we also move the node of c_2 by a corresponding amount. Faced with these difficulties we will try to define polygonal gravity's lengths after gauge fixing. The question becomes what is the correct choice of gauge.

5.2.1 Gauge fixing to pure polygons

Gauge fixing, as detailed in appendix B, means that we pick a unique representative of the gauge orbits of $\mathcal{G}_c$. The issue now is to find the correct gauge fixing that reduces LQG* to polygonal gravity. The answer is found in table 5.1. When 't Hooft constructed the theory, he imposed a gauge fixing. One of the implications of the gauge fixing, was that the faces of the graph Γ became planar. That is, all of the flux vectors had to live in some spacelike hyperplane. However, this is not generically true in either LQG or the dual formulation. Therefore, the correct gauge fixing is one that makes the polygons co-planar, as was first pointed out in [28]. In fact, there is an other reason for this gauge fixing. When 't Hooft defines his theory, as we mentioned above, he first imposes the applies the constraint $V_c = 1$. However, usually this also removing the gauge orbits of V_c from phase space. Unfortunately, it is clear whether or how this is done in polygonal gravity. Luckily, making a loop or face planar involves a gauge fixing related to the flatness constraint. To make the loop planar, the nodes must be translated into a spacelike hyperplane, which are the transformations generated by the flatness constraint. However, as we will argue below this does not completely remove all orbits of the flatness constraint.

We now need to find a gauge fixing condition obeying a few conditions. Mainly, we want to pick a gauge that applies to the flatness constraint only. Therefore, we need to ensure that the constraint commutes with the Gauss constraint. In this regard, considering the figure 3.3, one possible set of gauge fixing conditions is given by

$$O^v_{\ell_i\ell_j\ell_k} = E^v_{\ell_i} \cdot \left(E^v_{\ell_j} \wedge E^v_{\ell_k}\right) \approx 0, \tag{5.5}$$

for all permutations of the edge vectors. This set of constraints is naturally invariant under rotations since the triple product is invariant under $\mathrm{SO}(2,1)$ transformations. Unfortunately, this set of constraints is overcomplete. This implies certain Poisson brackets between the constraints will vanish, making it untenable for computing Dirac brackets. One solution is to find a subset of these constraints that, in general, enforce the all faces to be planar. While this is possible, the next problem is that computing the Dirac brackets is not straightforward. Furthermore, there is no guarantee that the theory at the end of the day is polygonal gravity. Furthermore, geometrically, the $O^v_{\ell_i\ell_j\ell_k}$ will not pick a single representative of the gauge orbits of $\mathcal{G}_c$. To see this, assume that we have satisfied the co-planar constraints. Computing the flow of the flatness constraint at every node we can translate every node in the time direction will preserve the co-planar condition. Of course this is not the only translation that preserves this gauge, but it is the simplest. In fact, it appears to be related to the time evolution in polygonal gravity. Another thing to note is that polygonal gravity assumes that observers are at rest in their polygon. This constitutes a second constraint. If we assume that the co-planar condition has been satisfied, then it means that there exists a unique normal to the polygon. Therefore, saying the observer is at rest in the polygon means that the normal is given by $(1,0,0)$. However, what if we do not assume the polygon is planar? Can we combine the two constraints? One possible constraint to do this would be given by forcing $E^0_\ell = 0$ for every $\ell \in \Gamma$. This constraint would require both translations and boosts. First, since polygons in general have more than 3 edges we again will need translations. Then we will need to rotate the frame in every polygon. This seems to combine both gauge fixings we mentioned above. Unfortunately, finding the Dirac brackets from applying this constraint will again be extremely complicated. Therefore, we did not attempt it in this thesis. However, we can talk about what the implications of this gauge fixing would be. If this gauge fixing is satisfied, we have a natural choice for the length

$$L_\ell^2 = E_\ell^i \delta_{ij} E_\ell^j. \tag{5.6}$$

That is, we just look at the spatial components of the fluxes. Furthermore, this gauge fixing also would still allow boosts to act on the surface. We cannot determine how they would act on the lengths without computing the Dirac brackets, but we would expect them to be highly complicated, like they are in polygonal gravity. Furthermore, as was first pointed out in [28] we can actually find the symplectic structure after assuming the gauge fixings have been satisfied.

Recall that polygonal gravity orients the edges of the polygons using components of the holonomy matrices. However, this is only possible after the two assumptions we just described are used. Furthermore, in order for this prescription to be consistent, we need ensure that the edges orientation satisfies a few consistency relations. First, by using the ϕ parameters, we were able to reconstruct the edge vector i.e. the flux using (4.5). In terms of $\mathrm{SU}(1,1)$ this is given by

$$E_\ell^{v_1} = L_\ell \mathrm{Ad}(R_s(-\phi_{\ell^*}))\tau_y \tag{5.7}$$

where $\phi, \bar{\phi}$ comes from 't Hooft decomposition of the holonomy (5.2). Furthermore, note that the lengths specified in the decomposition matches the definition we gave above for L_ℓ.

If we change the orientation of the edge, then we know the flux must be given by,

$$E_{-\ell} = -\mathrm{Ad}(\tilde{h}_{\ell^*})E_\ell. \tag{5.8}$$

Plugging in the decomposition (6.2), we get

$$E_{-\ell}^{v_2} = -\tilde{h}_{v_2 v_1} E_\ell^{v_1} \tilde{h}_{v_2 v_1}^{-1} = -L_\ell \mathrm{Ad}(R_s(\bar{\phi}_{v_0 v_2}))\tau_y, \tag{5.9}$$

where we used the fact that $B_s(\eta)$ leaves τ_y invariant, since it defines boosts around the τ_y axis.

In LQG, the angle between two fluxes is given by

$$\cos\alpha_{c_1}^{v_0}(E) = -\frac{\langle E_{c_2 c_1} E_{c_1 c}\rangle}{L_{c_1 c_2} L_{c_1 c}}. \tag{5.10}$$

In order for the 't Hooft definition of the angle to be consistent we should ensure that these two definitions of the angles match. A priori this does not have to be the case. In fact, the only reason they will be true is because of the gauge choices mentioned above.

Consider the situation depicted in figure 5.1. Using the decomposition (5.7), we get that the fluxes are given by

$$E_{c_2 c}^{v_0} = -L_{c_2 c}\mathrm{Ad}(R_s(\bar{\phi}_{v_0 v_2}))\tau_y, \qquad E_{c_1 c}^{v_0} = L_{c_1 c}\mathrm{Ad}(R_s(-\phi_{v_1 v_0}))\tau_y. \tag{5.11}$$

Following 't Hooft's vertex condition, the angle is given by $\alpha_v^{c_1}(h) = \phi_{v_1v_0} + \bar{\phi}_{v_0v_2}$. Substituting these decompositions into (5.10) gives

$$\begin{aligned}
\cos\alpha_{c_1}^{v_0}(E) &= -\frac{E_{c_2c_1}^{v_0}\cdot E_{cc_1}^{v_0}}{\left|E_{c_2c_1}^{v}\right|\left|E_{c_1c}^{v}\right|} \\
&= -\frac{-\left\langle L_{c_2c_1}\mathrm{Ad}(R_s(\bar{\phi}_{v_0v_2}))\tau_y, L_{c_1c}\mathrm{Ad}(R_s(-\phi_{v_1v_0}))\tau_y\right\rangle}{L_{c_2c_1}L_{c_1c}} \\
&= \left\langle \mathrm{Ad}(R_s(\bar{\phi}_{v_0v_2}))\tau_y, \mathrm{Ad}(R_s(-\phi_{v_1v_0}))\tau_y\right\rangle \\
&= \left\langle \tau_y, \mathrm{Ad}\Big(R_s(-\bar{\phi}_{v_0v_2}-\phi_{v_1v_0})\Big)\tau_y\right\rangle \\
&= \cos(\alpha_v^{c_1}(h)).
\end{aligned}$$

Therefore, it seems that the decomposition reproduces the correct result. However, we could have picked a different decomposition of $E_{c_1c_2}^{v_0}$ and have written it as

$$E_{c_1c_2}^{v_0} = L_{c_2c_1}\mathrm{Ad}(R_s(-\phi_{v_2v_0}))\tau_y. \tag{5.12}$$

In this case we have

$$\begin{aligned}
\cos\alpha_{c_1}^{v_0} &= -\left\langle \mathrm{Ad}(R_s(-\phi_{v_2v_0}))\tau_y, \mathrm{Ad}(R_s(\phi_{v_1v_0}))\tau_y\right\rangle \\
&= -\left\langle \tau_y, \mathrm{Ad}(R_s(\phi_{v_2v_0}-\phi_{v_1v_0}))\tau_y\right\rangle \\
&= -\cos(\phi_{v_2v_0}-\phi_{v_1v_0}) \\
&= -\cos\Big(\pi-\bar{\phi}_{v_0v_2}-\phi_{v_1v_0}\Big).
\end{aligned}$$

In the second last line we used (4.7). Therefore, we have that both decompositions of the edge give the same result, verifying that ϕ_{ℓ^*} can be used to orient the edge length in dual LQG with gauge fixing.

Note, that the explicit value of α depends on the order of the flatness constraint. Namely, whether the path around the triangle is in a clockwise or counter-clockwise fashion. To see this, consider the flatness constraint around c_1 in figure 5.1 given in the counter-clockwise orientation $\mathcal{G}_{c_1} = \tilde{h}_{v_0v_2}\tilde{h}_{v_2v_1}\tilde{h}_{v_1v_0}$. In this case we get the angles are given by

$$\alpha_{v_0}^{c_1} = \phi_{v_1v_0} + \bar{\phi}_{v_0v_2} \qquad \alpha_{v_1}^{c_1} = \phi_{v_2v_1} + \bar{\phi}_{v_1v_0} \qquad \alpha_{v_2}^{c_1} = \phi_{v_0v_2} + \bar{\phi}_{v_2v_1}. \tag{5.13}$$

If we replace the constraint with its inverse, i.e. $\tilde{h}_{v_0v_1}\tilde{h}_{v_1v_2}\tilde{h}_{v_2v_0}$, we get

$$\begin{aligned}
\alpha_{v_0}^{c_1} &= \phi_{v_2v_0} + \bar{\phi}_{v_0v_1} = \pi - \bar{\phi}_{v_0v_2} + \pi - \phi_{v_1v_0} \\
\alpha_{v_1}^{c_1} &= \phi_{v_0v_1} + \bar{\phi}_{v_1v_2} = \pi - \bar{\phi}_{v_1v_0} + \pi - \phi_{v_2v_1} \\
\alpha_{v_2}^{c_1} &= \phi_{v_1v_2} + \bar{\phi}_{v_2v_0} = \pi - \bar{\phi}_{v_2v_1} + \pi - \phi_{v_0v_2}.
\end{aligned}$$

Comparing this with α above, we see that the definitions differ by a factor of 2π. Of course the cosine of the angle will not change, but the interpretation will. In fact, the

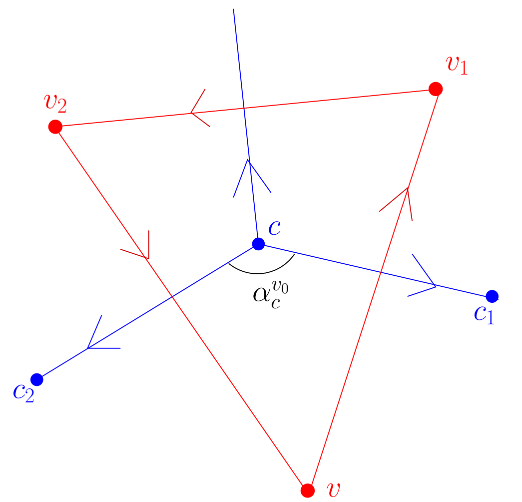

Figure 5.1: Using the decomposition (5.7), we can exactly recover the angles between the fluxes from the holonomy matrices.

sign ambiguity can be seen to arise from the fact that the rotation in the second case is performed clockwise instead of the usual counter-clockwise. In order to fixing this ambiguity we will always assume that the flatness constraint is taken in counter-clockwise orientation.

Now we can compute the symplectic potential of the theory. Following [28], we insert (5.2) and (5.7) into Θ^{LQG^*}. With this in mind, first note that $R^{-1}(\phi)\delta R(\phi) = \tau_0 \delta\alpha$ and $B^{-1}(2\eta)\delta B(2\eta) = 2\delta\eta\tau_y$. Note that τ_0 is the $\mathfrak{su}(1,1)$ basis element in the time direction (see appendix A for the definition). Then we have

$$\begin{aligned}
\Theta^{\mathrm{LQG}^*} &= \left\langle E_\ell \tilde{h}_{\ell^*}^{-1} \delta \tilde{h}_{\ell^*} \right\rangle \\
&= \left\langle L_\ell \mathrm{Ad}(R(-\phi_\ell))\tau_y \Big(\delta\phi\tau_0 + \delta\bar{\phi}\tau_0 + \mathrm{Ad}(R(-\phi_\ell))\tau_y 2\delta\eta_{\ell^*} \Big) \right\rangle \\
&= L_\ell 2\delta\eta_{\ell^*} \left\langle \mathrm{Ad}(R(-\phi_{\ell^*}))\tau_y, \mathrm{Ad}(R(-\phi_{\ell^*}))\tau_y \right\rangle \\
&= L_\ell 2\delta\eta_{\ell^*}.
\end{aligned}$$

This is exactly the symplectic structure for polygonal gravity. Therefore, it may seem we have reduced LQG* to polygonal gravity. In fact this is what the author [28] suggests. However, this is deceiving for a couple of reasons. One, is that we still do not quite understand what the gauge fixing entails. This means we still do not know what, translational degree of freedoms, i.e. the symmetries generated by $\mathcal{G}_c$, still remain after making the polygon planar. From the argument above, we expect that there are still some translational degree of freedoms left. Furthermore, this gauge fixing may not even be possible. Another issue is that we have not recovered the constraints of polygonal gravity, i.e. C_G and C_α. The issue is especially defined for C_α, which a first glance, does not seem to have a counterpart in dual loop quantum gravity.

First, let us see the form of the Gauss constraint $\mathcal{J}_v$ using the decomposition (5.7). Assuming we have the same orientation of edges as shown in figure 3.3, then every edge vector can be written as

$$E^{v_0}_{c_i c_{i-1}} = L_{c_i c_{i-1}} \mathrm{Ad}(R_s(-\phi_{v_i v_0}))\tau_y. \tag{5.14}$$

Therefore, the Gauss constraint becomes

$$\mathcal{J}_{v_0} = \sum_i L_i \mathrm{Ad}(R_s(-\phi_{v_i v_0}))\tau_y, \tag{5.15}$$

where we have set $L_i = L_{c_i c_{i-1}}$. At first glance, this looks nothing like polygonal gravity's closure constraint C_G. However, it is not, yet, expressed in terms of polygonal gravity's reduced variables L, η and $\alpha(\eta)$. In order to remedy this, we multiply the constraint by $\mathrm{Ad}(R_s(\bar{\phi}_{v_i v_0}))$,

$$\mathcal{J}_{v_0} = L_1 + \sum_{i=2} L_i \mathrm{Ad}(R_s(\phi_{v_1 v_0} - \phi_{v_i v_0})). \tag{5.16}$$

Considering the j^{th} term in the sum, we insert $\bar{\phi}_{v_0 v_k} - \bar{\phi}_{v_0 v_k}$, changing the rotation matrix to

$$\begin{aligned} R(\phi_{v_1 v_0} - \phi_{v_j v_0}) &= R\left(\phi_{v_1 v_0} + \sum_{n=2}^{j-1}(\bar{\phi}_{v_0 v_j} - \bar{\phi}_{v_0 v_j}) - \phi_{v_k v_0}\right) \\ &= R\Big((\phi_{v_1 v_0} + \bar{\phi}_{v_0 v_2}) + (\bar{\phi}_{v_0 v_3} - \bar{\phi}_{v_0 v_2}) + \cdots - \bar{\phi}_{v_0 v_{j-1}} - \phi_{v_j v_0}\Big) \\ &= R\left(\sum_{i=1}^{j-1}(\phi_{v_{i-1} v_0} + \bar{\phi}_{v_0 v_i} - \pi)\right) \\ &= R\left(\sum_{i=1}^{j-1}(\alpha^{c_i}_{v_0} - \pi)\right) \\ &= R(-\theta_j), \end{aligned} \tag{5.17}$$

where $\theta_j = \sum_{i=1}^{j-1}(\pi - \alpha^{c_i}_{v_0})$ is from (4.30). The reason we ended up with $-(\pi - \alpha)$, is because the constraint written in polygonal gravity's fashion assumes that the order of the edges is in a counter-clockwise fashion. However, for the Gauss constraint in LQG* the edges are summed in a clockwise fashion. Of course this choice is arbitrary so we indeed have that $C_{G,v^*} = \mathcal{J}_v$.

In polygonal gravity, it is usually stated, without proof, that $C_{G,f}$ generates local Lorentz boosts of each polygon f. However, we know of no exact proof of this statement. Furthermore, analyzing how this constraints acts on the lengths is not easy [20]. This showcases the power of deriving $C_{G,f}$ from dual LQG. In dual LQG it is easily shown that the spatial components of the Gauss constraint generate boosts, and so this will still be the case in polygonal gravity.

The remaining question, becomes the origin of the other polygonal gravity constraint $C_{\alpha,f}$? In [28], they try to argue that it is related to the time component of the Gauss constraint, but by forcing the polygons to be planar this component of the constraint vanishes. Furthermore, the gauge transformation generated by the time component of $\mathcal{J}_v$ in LQG* has already been factored out since our variables L, η are invariant with respect to these rotations. However, we know that it must be related to some constraint in LQG* if polygonal gravity does describe gravity. The meaning of the constraint is that the polygon must have a vanishing deficit angle. Geometrically this means its intrinsic curvature must vanish. Furthermore, since we assumed that the polygon is extrinsically flat by making it co-planar, this equivalently says that the polygon must have no 3-curvature. Therefore, it must be related to the flatness constraints. However, since we have already solved the local flatness constraints, it must be partially global in nature. Another way to see this is that we only get the correct value for the Gauss-Bonnet theorem if the constraints $C_{\alpha,f}$ for each face f is zero (4.31).

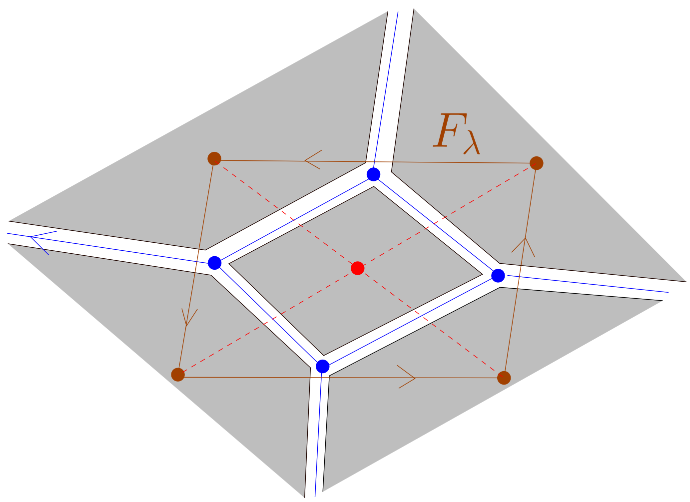

Figure 5.2: By concatenating flatness constraints we can form F_λ, which enforces vanishing 3-curvature. Each polygon is extrinsically flat (grey fill), so we can derive polygonal gravity's constraint $C_{\alpha,f}$ constraint.

Now consider a polygon f dual to a vertex v_0 (i.e. $f = v_0^*$), like in figure 3.3. For each node c_i, we have a local flatness constraint for the triangle dual to the node. However, we can concatenate these constraints together; in fact, the polygon v_i^* can be surrounded by a larger polygon formed from the vertices $\lambda = [v_1 v_5 v_4 v_3 v_2 v_2 v_1]$. The product of the holonomies of this polygon, $F_\lambda = \tilde{h}_{v_1 v_5}\tilde{h}_{v_5 v_4}\tilde{h}_{v_4 v_3}\tilde{h}_{v_3 v_2}\tilde{h}_{v_2 v_1}$, then must be trivial when $\mathcal{G}_{c_i} = 1$. To see this, consider the following ordering for the constraints $\mathcal{G}_{c_i} = \tilde{h}_{v_0 v_{i+1}}\tilde{h}_{v_{i+1} v_i}\tilde{h}_{v_i v_0}$. Taking the product of the constraints $\prod_i \mathcal{G}_{c_i}$, then we get

$$\prod_i \mathcal{G}_{c_i} = \prod_i \tilde{h}_{v_0 v_{i+1}}\tilde{h}_{v_{i+1} v_i}\tilde{h}_{v_i v_0} = F_\lambda. \tag{5.18}$$

Therefore, the area inside this region must be fully flat, as is expected in 2+1 gravity. Furthermore, the curve λ is homotopic to the edges of the polygon, so we require that the

3-curvature inside the polygon vanishes. However, we already know the region inside the polygon is extrinsically flat as in figure 5.2, therefore we require that the polygon must be intrinsically flat, which is precisely the $C_{\alpha,f}$ constraint.

One way to see this is to consider a loop in dual LQG like the one we constructed. We assume that the edges of the polygon within this loop have zero extrinsic curvature between them. The holonomies are therefore a rotation around the shared normal of the polygons, and F_λ becomes

$$F_\lambda = \exp\Big(\alpha_\lambda n^A \tau_A\Big), \tag{5.19}$$

where α_λ is the sum of the angles of the holonomy. Since spacetime is flat, we require that $\alpha_\lambda = 2\pi$. A calculation similar to (5.17) then shows that $\alpha_\lambda = \sum_{c_i \in \lambda}(\pi - \alpha_{c_i})$.

The form of the constraint also means that the flow it generates are time translations of the polygons. This what $C_{\alpha,f}$ is argued to generate in [45]. To see this, note that any flux vectors entirely contained in the loop λ Poisson commute with F_λ. This is precisely what was found for $C_{\alpha,f}$ in chapter 4. However, it will affect the polygons next to λ (e.g. next to v_0 in figure 5.2). In that case, the loop λ will intersect some of the edges of the polygon. This, again, is exactly what we found in the previous chapter. Namely, all variables associated to the polygon f commute with $C_{\alpha,f}$. Only the polygons or loops adjacent to f that are influenced by the $C_{\alpha,f}$ constraint. Again, we stress that this means that, unlike what was said in [28], the $C_{\alpha,f}$ constraint is not related to the Gauss constraint. Instead it is a global version of the flatness constraint. This also implies that polygonal gravity is not equal to the kinematical phase space of LQG*, as was argued in [28].

An issue we have not yet dealt with is the polygonal gravity Hamiltonian. Nowhere in our theory, did we require the introduction of a reduced Hamiltonian. This is because, we started with dual LQG as our starting point, which automatically provided us with the correct symplectic structure. In polygonal gravity's original formulation, a Hamiltonian is introduced to get a symplectic structure that gave the required equations of motion. As a consequence of this, an additional partial gauge fixing was required. This gauge choice can be seen as choosing a global time. In our case, we instead have a Hamiltonian constraint which, following Dirac's procedure, is given by

$$H = \sum_{f\in\Gamma} N_f C_{\alpha,f} + \sum_{f\in\Gamma} N_f^G C_{G,f}, \tag{5.20}$$

where f are the faces of Γ dual to the vertices v_0, and N_f, N_f^G are the Lagrange multipliers. Analyzing this constraint, 't Hooft's gauge fixing (4.2), is given by the choice of $N_f = 1, N_f^G = 0$ for all the faces of the polygon.

Summary

In this section, we attempted to derive 't Hooft's polygonal gravity starting from the dual LQG framework. This was mostly accomplished. Starting from dual LQG's symplectic form, we were able to recover polygonal gravity's symplectic structure, as was first done in [28]. This was done by forcing the faces dual to the vertices of the triangulation to be planar. The Gauss constraint, from dual LQG, was then shown to be equivalent to $C_{G,f}$ by an explicit calculation for the first time. The other constraint $C_{\alpha,f}$, was then recovered by concatenating a bunch of flatness constrains together. As a result of deriving these constraints from dual LQG, we immediately know the transformations they generate, proving it generates boosts. Another result is that $C_{\alpha,f}$ and $C_{G,f}$ are first class.

One issue that has not been answered is whether the gauge choice of making the faces co-planar is always possible. In fact, it is expected that this will not be the case. It is already known that time evolution in polygonal gravity leads to discontinuities, [46], leading to a series of transitions. Therefore, we postulate that the transitions occur precisely from the gauge fixing we described above. One approach to answering this question would be to compute the Dirac Brackets, and analyzing the resulting algebra. Furthermore, since the reduction, was not explicitly carried out, the full reduced phase space of gravity may not be contained in polygonal gravity. Therefore, to check whether the gauge fixing described above do not over constrain the system, we will analyze the torus universe.

5.3 The torus universe in LQG and polygonal gravity

The torus universe is the simplest spacetime one can consider in discrete gravity. In fact, the reduced phase space for the torus universe has been explicitly known for decades [37, 53]. The reason for this simplicity is that the flatness and torsion constraints are degenerate. In the LQG framework, this means it becomes much easier to identify independent degrees of freedom and write down explicitly the degrees of freedom and their Poisson brackets. However, this degeneracy will turn out to be problematic in polygonal gravity, where the nature of the degeneracy is not clear. This degeneracy led to two papers, [23] and [54], making different conclusions. In the paper [23], the authors claim to find the Teichmüller parameters of the theory, seemingly solving the torus universe. However, they fail to show that their proposed Teichmüller parameters are gauge invariant. In the other paper [54], the author analyzes this problem again and argues that the parameters in [23] are not invariant under boosts. However, the author also fails to identify the true degrees of freedom of the model and, instead relying on incomplete or “spacetime” arguments. Part of the reason for this is both that papers were unable to construct observables for the reduced phase space in terms of ’t Hooft’s variables. In fact, gauge invariant observables in polygonal gravity were unknown. In LQG however, the observables are well known [30, 35]. Therefore, using the map developed above, we will use the known observables of LQG and write them down in terms of polygonal gravity’s variables. Then the question becomes whether the observables in LQG are independent in ’t Hooft gravity. If they are, the torus universe will be contained in polygonal gravity.

5.3.1 One polygon torus universe

The first step in answering whether polygonal gravity can describe the torus universe is to choose a discretization. The simplest discretization that obeys 't Hooft's three-valent vertex condition is given in figure 5.3. As the figure shows, this discretization requires two triangles and a single dual face or polygon. Thus, we will refer to this discretization as the one polygon torus.

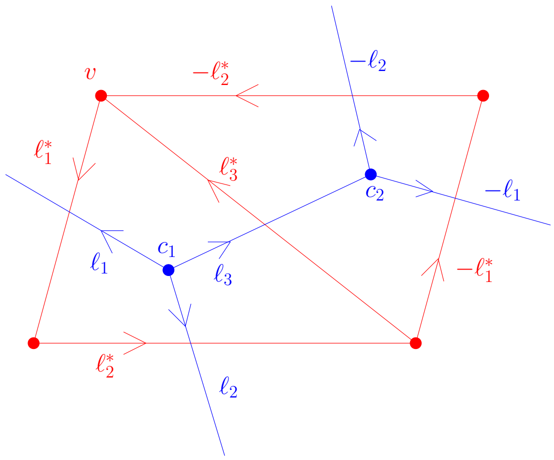

Figure 5.3: Left: Triangulation of torus and dual graph Γ. Opposite edges of triangulation are identified. Right: Dual polygon, found from cutting along the edges of the graph Γ.

The torus universe in the LQG formalism has been explicitly solved [37, 53]. The difference however, is that they they use a simpler discretization of the torus, and not the dual LQG polarization. Therefore, we will have identify the observables in the dual framework with our more complicated discretization. First, from figure 5.3, the constraints in LQG* will be given by

$$\mathcal{G}_{c_1} = \tilde{h}^{c_1}_{\ell_3^*}\tilde{h}^{c_1}_{\ell_2^*}\tilde{h}^{c_1}_{\ell_1^*}, \qquad \mathcal{G}_{c_2} = \tilde{h}^{c_2}_{-\ell_2^*}\tilde{h}^{c_2}_{-\ell_1^*}\tilde{h}^{c_2}_{-\ell_3^*}, \tag{5.21}$$

$$\mathcal{J}_v = E^v_{\ell_1} + E^v_{\ell_2} + E^v_{\ell_3} + E^v_{-\ell_1} + E^v_{-\ell_2} + E^v_{-\ell_3}. \tag{5.22}$$

Note that $\mathcal{G}_{c_2}$ is not an independent constraint. The triangles c_2^*, c_1^* share the same holonomies, implying $\mathcal{G}_{c_2} = 1$ exactly $\mathcal{G}_{c_1} = 1$. Furthermore, the constraints $\mathcal{J}_v$ and $\mathcal{G}_{c_1}$ are independent. To see this, note that the Gauss constraint can be written as

$$\mathcal{J}_v = (1 - \mathrm{Ad}(\tilde{h}_{\ell_1^*}))E^v_{\ell_1} + (1 - \mathrm{Ad}(\tilde{h}_{\ell_2^*}))E^v_{\ell_2} + (1 - \mathrm{Ad}(\tilde{h}_{\ell_3^*}))E^v_{\ell_3}. \tag{5.23}$$

Now consider the loop $[-\ell_2^* - \ell_1^*\ell_2^*\ell_1^*]$. From the two flatness constraints, the holonomy of this loop is trivial

$$\mathcal{G}_{c_2}\mathcal{G}_{c_1} = \tilde{h}_{-\ell_2^*}\tilde{h}_{-\ell_1^*}\tilde{h}_{\ell_2^*}\tilde{h}_{\ell_1^*} = 1.$$

This implies that $\tilde{h}_{\ell_1^*}$ and $\tilde{h}_{\ell_2^*}$ commute and therefore have the same rotation axis. If we denote this axis of rotation by $\tilde{n}_v$ then we have that

$$\langle n_v, \mathcal{J}_v \rangle = 0, \tag{5.24}$$

since the holonomies will leave the axis invariant. This means that the Gauss constraint enforces two independent constraints. Note, it still generates 3 symmetry transformations. To check whether we have found all the relations between the constraints, we will count the remaining degrees of freedom. Given there are 3 independent edges, this means we have $3 \times 3 \times \times 2 = 18$ variables from the holonomies and fluxes. The constraint $\mathcal{G}_{c_1}, \mathcal{G}_{c_2}, \mathcal{J}_v$ all generate independent transformations, which means $3 + 3 + 3 = 9$ degrees of freedom are just due to the orbits of the constraint. We also have $3 + 2$ constraints left from $\mathcal{G}_{c_1}$ and the two components of $\mathcal{J}_v$. Therefore, there are only $18 - 9 - 5 = 4$ degrees of freedom left, which is precisely the dimension of the reduced phase space of the torus.

The next step is to find the observables of the torus. The observables for this system have been discussed in the Chern-Simons formalism [36, 35, 34] and LQG in [30, 37, 53]. The first set is given by the traces of the holonomies of the loops ℓ_i^* of the torus. These are sometimes called the "mass" m_{ℓ_i}, and are defined by

$$\operatorname{tr} \tilde{h}_{\ell_i^*} = 2 \cosh\Big(m_{\ell_i^*}/2\Big). \tag{5.25}$$

Note that $m_{\ell_i^*}$, are not the 't Hooft rapidities η necessarily. In general, the trace of a holonomy in polygonal gravity is given by

$$\operatorname{tr}\Big(\tilde{h}_{\ell^*}\Big) = 2 \cos\left(\frac{\bar{\phi}_{\ell^*} + \phi_{\ell^*}}{2}\right) \cosh(\eta_{\ell^*}), \tag{5.26}$$

The other observables are sometimes called the **"spin"** of the holonomy. For a non-contractible loop λ they are given by

$$s_\lambda = \operatorname{tr}\left(\tilde{h}_\lambda E_\lambda\right), \tag{5.27}$$

where E_λ is the flux, or translational holonomy. Unlike the trace of the holonomy, these observables are harder to define for our discretization, since no single flux forms a loop. However, the loops defining the fundamental group of the torus, are given by ℓ_1^*, ℓ_2^*. We then need to construct the translational part of the holonomy using the links ℓ of Γ. From figure 5.3, we can see the two loops generating the fundamental group for the translational holonomy can be taken to be $[-\ell_3\ell_1]$ and $[-\ell_3\ell_2]$. The $\mathrm{SU}(1,1)$ holonomies corresponding to these loops are $\tilde{h}_{-\ell_2^*}$ and $\tilde{h}_{\ell_1^*}$ respectively. The two spin observables are then given by

$$s_1 = \operatorname{tr}\left[\tilde{h}_{\ell_1^*}\Big(E^v_{-\ell_3} + \operatorname{Ad}(\tilde{h}_{-\ell_1^*})E^v_{\ell_2}\Big)\right], \qquad s_2 = \operatorname{tr}\left[\tilde{h}_{-\ell_2^*}\Big(E^v_{-\ell_3} + E^v_{\ell_1}\Big)\right]. \tag{5.28}$$

To see that these Poisson commute with $\mathcal{G}_{c_1}, \mathcal{G}_{c_2}$, note that the flatness constraint flows are given by

$$\tilde{\delta}^{c_1}_\beta E_{\ell_1} = \beta, \qquad \tilde{\delta}^{c_2}_\beta E_{\ell_1} = -\tilde{h}_{\ell_2^*}\beta\tilde{h}^{-1}_{\ell_2^*}, \tag{5.29}$$

$$\tilde{\delta}^{c_1}_\beta E_{\ell_2} = \tilde{h}_{\ell_1^*}\beta\tilde{h}^{-1}_{\ell_1^*}, \qquad \tilde{\delta}^{c_2}_\beta E_{\ell_2} = -\beta, \tag{5.30}$$

$$\tilde{\delta}^{c_1}_\beta E_{\ell_3} = \tilde{h}^{-1}_{\ell_3^*}\beta\tilde{h}_{\ell_3^*}, \qquad \tilde{\delta}^{c_2}_\beta E_{\ell_3} = -\tilde{h}^{-1}_{\ell_3^*}\beta\tilde{h}_{\ell_3^*}. \tag{5.31}$$

The transformations for $E_{-\ell_i}$ are easily found by using $E_{-\ell_i} = -\tilde{h}_{\ell_1^*} E_{\ell_i} \tilde{h}_{\ell_i}$. It is then a simple exercise of plugging these transformation laws into the expression for s_1, s_2, and using the Ad-invariance of the trace to show that the s_i are observables.

One way to see that these are related to observables s_λ is to delete the ℓ_3 edge by a gauge transformation. In this case $h_{\ell_3} = 1$ and $E_{\ell_3} = 0$. The spin observables then become

$$s_1 = \mathrm{tr}\Big(E_{\ell_2} \tilde{h}_{\ell_1^*}\Big), \tag{5.32}$$

which is precisely s_λ.

To see what geometric information these observables encode, it is helpful to use a parameterization of the holonomy matrices. We will use a standard SU(1, 1) parameterization (which is reviewed in appendix A) given in terms of the exponential matrices $\exp\Big(\theta n^b \tau_b\Big)$. Depending on the nature of n, we get the following result

$$\exp\Big(\theta n^I \tau_I\Big) = \begin{cases} \cosh(\theta/2)1 + 2\sinh(\theta/2) n^b \tau_b & n^2 > 0 \text{ (spacelike)} \\ \cos(\theta/2)1 + 2\sin(\theta/2) n^b \tau_b & n^2 < 0 \text{ (timelike)} \\ 1 + n^b \tau_b & n^2 = 0 \text{(lightlike)}. \end{cases} \tag{5.33}$$

In this case we can see that the "mass" is just the angle or rapidity parameter for the transformation around the axis n. The nature of the holonomies associated the ell_i^* are hyperbolic in nature, since the torus is spacelike by assumption [9]. This implies that the rotation axis n is spacelike for ℓ_1^* and ℓ_2^*. Therefore, using exponential map parameterization, the spin variables become

$$\begin{aligned} s_1 &= 2\sinh\Big(m_{\ell_1^*}/2\Big)(E_{-\ell_3} + E_{\ell_2}) \cdot n, \\ s_2 &= 2\sinh\Big(m_{\ell_2^*}/2\Big)(E_{-\ell_3} + E_{\ell_1}) \cdot n. \end{aligned} \tag{5.34}$$

Notice that since $m_{\ell_i^*}$ is an observable, we can separate it from s_i to get the independent observables,

$$P_i = (E_{-\ell_3} + E_{\ell_i}) \cdot n, \qquad \text{i=1,2} \tag{5.35}$$

In the case of the torus universe, the Poisson brackets between the observables are easily calculated. First, note that

$$\{2\cosh(m_{\ell_i}/2), P_j\} = \sinh(m_{\ell_i}/2)\delta_{ij}. \tag{5.36}$$

This then implies that m_{ℓ_i} and the P_j are canonically conjugate. That is,

$$\{m_{\ell_i}, P_j\} = \delta_{ij}, \tag{5.37}$$

recovering the result from [37][1]. From the constraints of the system, it is also easily shown that the four observables are independent. Therefore, all torus universes can be described

[1]For higher genus, the Poisson brackets between the mass and spin observables becomes much more complicated.

by the values of these parameters. Now we can move to answering whether the torus universe with a one polygon description is contained in polygonal gravity.

The torus universe in the polygonal gravity was analyzed in [23, 54]. However, as mentioned above, the papers contradict each other. The reason is that both paper struggle is that the cannot find the observables in terms of polygonal gravity variables. The answer to whether we can describe the torus universe, will then be given by whether the 4 observables are independent on-shell. In order to check this, we need to analyze the nature of the constraints in the 1 polygon torus universe. Like LQG*, polygonal gravity's constraints are degenerate. Due to the identification of the links of the polygon in figure 5.3, the constraint (4.30) becomes

$$C_\alpha = 2(\alpha_1 + \alpha_2 + \alpha_3) - 4\pi. \tag{5.38}$$

Likewise, the Gauss constraint (4.29) becomes

$$\begin{aligned} C_G = L_{\ell_1} + L_{\ell_2} \exp(i(\pi - \alpha_3)) + L_{\ell_3} \exp(i(2\pi - \alpha_3 - \alpha_2)) + L_{\ell_1} \exp(i(3\pi - \alpha_3 - \alpha_2 - \alpha_1)) \\ + L_{\ell_2} \exp(i(4\pi - 2\alpha_3 - \alpha_2 - \alpha_1)) + L_{\ell_3} \exp(i(5\pi - 2\alpha_3 - 2\alpha_2 - \alpha_1)). \end{aligned} \tag{5.39}$$

Therefore, we see that $C_\alpha = 0$, implies $C_G = 0$. Therefore, we only have to ensure the $\alpha(\eta)$ satisfy $C_\alpha = 0$. This also means that all lengths of the polygon are unconstrained.

In [23], they give two possible solutions to C_α along with 't Hooft's vertex conditions. Since C_α implies that the 2D intrinsic curvature must vanish at the nodes, one solution is forcing all the rapidities to be zero. This gives the trivial solution and corresponds to the static torus [9]. This set of solutions is degenerate however, [9], so we will no longer consider it.

The second solution to the vertex conditions (4.13) and C_α, is when only one of the edges rapidities vanishes. In the literature, these are often called *quasi-static nodes*. Applying the constraint and 't Hooft's vertex conditions, we have that

$$\eta_{\ell_1^*} = \eta, \qquad \eta_{\ell_2^*} = \eta, \qquad \eta_{\ell_3^*} = 0, \tag{5.40}$$

$$\alpha_1 = \pi - \alpha, \qquad \alpha_2 = \alpha, \qquad \alpha_3 = \pi, \tag{5.41}$$

where α and η are free parameters. This solution is the point of departure for the two papers [23] and [54]. In [23] they take α as being a legitimate degree of freedom. In contrast, [54] view α as purely gauge and unphysical. Taking α as unphysical, gives the following degrees of freedom L_1, L_2, L_3, η. The author then argues that there are only 3 degrees of freedom left after gauge fixing. The problem with both papers is that even if 't Hooft's variables are not gauge invariant. Therefore, no conclusion can be made based on the dimension of the phase space, from the amount of non-gauge invariant observables. Therefore, we instead need to identify the true observables in the system to find the dimension of the reduced phase space.

To find these observables, we re-express the mass and spin observables in terms of 't Hooft's variables. Applying the two decompositions (5.2) and (5.7), we find that the mass observables are given by,

$$2\cosh\Big(m_{\ell_1^*}/2\Big) = -2\cos\left(\frac{\alpha_1+\alpha_2+\alpha_3}{2}\right)\cosh\Big(\eta_{\ell_1^*}\Big) \tag{5.42}$$
$$2\cosh\Big(m_{\ell_2^*}/2\Big) = -2\cos\left(\frac{\alpha_1+\alpha_2+\alpha_3}{2}\right)\cosh\Big(\eta_{\ell_2^*}\Big). \tag{5.43}$$

These where found by using (5.26) and the fact that the angles are related to the $\phi, \bar{\phi}$ by

$$\begin{aligned} \alpha_1 &= \phi_{\ell_3^*} + \bar{\phi}_{\ell_2^*} = \phi_{-\ell_3^*} + \bar{\phi}_{-\ell_2^*} \\ \alpha_2 &= \phi_{\ell_1^*} + \bar{\phi}_{\ell_3^*} = \phi_{-\ell_1^*} + \bar{\phi}_{-\ell_3^*} \\ \alpha_3 &= \phi_{\ell_2^*} + \bar{\phi}_{\ell_1^*} = \phi_{-\ell_2^*} + \bar{\phi}_{-\ell_1^*}. \end{aligned} \tag{5.44}$$

On-shell i.e. when C_α and 't Hooft's vertex condition are satisfied, we get that the observables reduce to

$$2\cosh\Big(m_{\ell_1^*}/2\Big) \approx 2\cosh(\eta) \tag{5.45}$$
$$2\cosh\Big(m_{\ell_2^*}/2\Big) \approx 2\cosh(\eta). \tag{5.46}$$

Therefore, in the 't Hooft gravity 1 polygon torus universe, we have that $2\eta = m_{\ell_1^*} = m_{\ell_2^*}$.

Moving to the spin variables (5.28), and subbing in the planar decomposition (5.7), we get that

$$\begin{aligned} s_1 &= \left[L_{\ell_2}\cos\left(\frac{\bar{\phi}_{\ell_1^*} - \phi_{\ell_1^*} + 2\phi_{\ell_2^*}}{2}\right) - L_{\ell_3}\cos\left(\frac{\bar{\phi}_{\ell_1^*} - 2\bar{\phi}_{\ell_3^*} - \phi_{\ell_1^*}}{2}\right)\right]\sinh\Big(\eta_{\ell_1^*}\Big) \\ s_2 &= \left[L_{\ell_1}\cos\left(\frac{\bar{\phi}_{\ell_2^*} - \phi_{\ell_2^*} + 2\phi_{\ell_1^*}}{2}\right) - L_{\ell_3}\cos\left(\frac{\bar{\phi}_{\ell_2^*} - 2\bar{\phi}_{\ell_3^*} - \phi_{\ell_2^*}}{2}\right)\right]\sinh(\eta_{\ell_2}). \end{aligned}$$

Rewriting this in terms of the polygon angles α, we get

$$\begin{aligned} s_1 = \left[L_{\ell_2}\cos(\alpha_3) - L_{\ell_3}\cos\left(\frac{\alpha_3-\alpha_2+\alpha_1}{2}\right)\right]\sinh(\eta_{\ell_1}) &\approx -(L_{\ell_2} + L_{\ell_3}\cos\alpha)\sinh\eta \\ s_2 = -[L_{\ell_1} + L_{\ell_2}\cos(\alpha_1)]\sinh(\eta_{\ell_1}) &\approx -(L_{\ell_1} - L_{\ell_2}\cos\alpha)\sinh\eta. \end{aligned} \tag{5.47}$$

Therefore, the observables for the torus, given in terms of 't Hooft's variables are given by

$$\begin{aligned} m_{\ell_1} &= 2\eta \qquad P_{\ell_1} = L_{\ell_2} + L_{\ell_3}\cos\alpha \\ m_{\ell_2} &= 2\eta \qquad P_{\ell_2} = L_{\ell_1} - L_{\ell_3}\cos\alpha. \end{aligned} \tag{5.48}$$

This shows that 't Hooft's one polygon torus universe only has three independent variables. Therefore, we cannot cover the reduced phase space for the torus universe with one 't Hooft polygon.

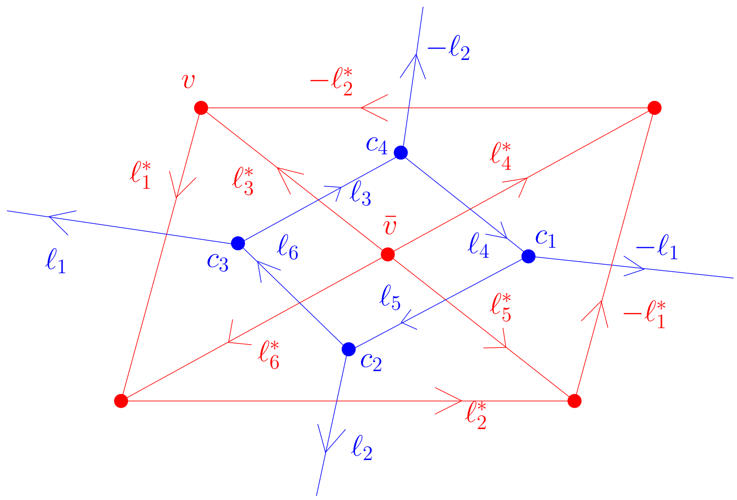

Figure 5.4: Two polygon torus universe.

5.3.2 Two polygon torus universe

The next simplest discretization in the torus universe is given by two polygons. The discretization we have chosen is given in figure 5.4. Note while other discretizations are possible, they are all just related to this one by a translation in the universal covering of the torus universe. With the two polygon discretization, the constraints for the system in LQG* are given by

$$\begin{aligned} \mathcal{G}_{c_1} &= \tilde{h}_{-\ell_4^*}\tilde{h}_{-\ell_1^*}\tilde{h}_{\ell_5^*} \qquad & \mathcal{G}_{c_2} &= \tilde{h}_{-\ell_5^*}\tilde{h}_{\ell_2^*}\tilde{h}_{\ell_6^*} \\ \mathcal{G}_{c_3} &= \tilde{h}_{-\ell_6^*}\tilde{h}_{\ell_1^*}\tilde{h}_{\ell_3^*} \qquad & \mathcal{G}_{c_4} &= \tilde{h}_{-\ell_3^*}\tilde{h}_{-\ell_2^*}\tilde{h}_{\ell_4^*} \end{aligned} \tag{5.49}$$

$$\begin{gathered} \mathcal{J}_v = E^v_{-\ell_3} + E^v_{\ell_1} + E^v_{-\ell_4} + E^v_{-\ell_2} + E^v_{-\ell_5} + E^v_{-\ell_1} + E^v_{-\ell_6} + E^v_{\ell_2} \\ \mathcal{J}_{\bar{v}} = E^{\bar{v}}_{\ell_3} + E^{\bar{v}}_{\ell_4} + E^{\bar{v}}_{\ell_5} + E^{\bar{v}}_{\ell_6}. \end{gathered} \tag{5.50}$$

If we concatenate the four flatness constraints together, we end up with a global constraint, which implies that $\tilde{h}_{\ell_1^*}$ and $\tilde{h}_{\ell_2^*}$ commute.

Like the one polygon torus, the constraints in the torus universe are not independent. To see this, note that we have 6 independent edges. This corresponds to 36 variables, 18 from the 6 holonomies and another 18 from the 6 fluxes. We also have 4 flatness constraints and two Gauss constraints. Since each constraint is three dimensional it seems that there are 18 constraint. Another 18 degrees of freedom are contained gauge orbits since the constraints are first class. This seemingly gives zero degrees of freedom. However, the constraints are not independent. Assuming that the flatness constraints for the nodes c_1, c_2, c_3 have been solved, we have $\mathcal{G}_{c_4} = 1$, as

$$\mathcal{G}_{c_4}^{-1} = \mathcal{G}_{c_1}\mathcal{G}_{c_2}\mathcal{G}_{c_3} = 1. \tag{5.51}$$

Therefore, one of the four flatness constraints is degenerate with the others.

From the one polygon universe we found that that the torsion constraint and the flatness constraints are not independent. In the same vein, we have that $\tilde{h}_{\ell_1^*}$ and $\tilde{h}_{\ell_2^*}$ commute, and thus define a common axis of rotation n_v. Taking the inner product between the normal and the Gauss constraint $\mathcal{J}_v$ gives

$$\langle n_v, \mathcal{J}_v \rangle = \left\langle E^v_{-\ell_3} + E^v_{-\ell_4} + E^v_{-\ell_5} + E^v_{-\ell_6}, n_v \right\rangle . \tag{5.52}$$

This is degenerate with $\mathcal{J}_{\bar{v}}$ so we see that one component of the two Gauss constraints is degenerate. Therefore 4 of the 18 constraints are not independent, giving the correct degrees of freedom.

The observables for the 2 polygon universe will be basically the same as the ones for the one polygon universe. The "mass" observables will again be given by

$$\operatorname{tr} \tilde{h}_{\ell_1^*} = 2\cosh(m_1/2), \qquad \operatorname{tr} \tilde{h}_{\ell_2^*} = 2\cosh(m_2/2). \tag{5.53}$$

The "spin" observables can also be identified in analogy to the one polygon universe. In this case, three fluxes are needed to form a loop, and a moment of refection will show that the spin observables are given by

$$s_1 = \operatorname{tr}\left[\tilde{h}_{\ell_1^*}\Big(E^v_{-\ell_3} + \operatorname{Ad}(\tilde{h}_{-\ell_1^*})(E^v_{-\ell_6} + E^v_{\ell_2})\Big)\right], \tag{5.54}$$

$$s_2 = \operatorname{tr}\left[\tilde{h}_{-\ell_2^*}\Big(E^v_{-\ell_3} + E^v_{\ell_1} + \operatorname{Ad}(\tilde{h}_{\ell_2})E^v_{-\ell_4}\Big)\right]. \tag{5.55}$$

Therefore, by analogy with the one polygon universe, we again have described the torus universe in the dual LQG framework. We can now proceed to the two polygon universe in 't Hooft's formalism.

First, we relate the angles of the 2 polygon universe to the orientations $\phi, \bar{\phi}$,

$$\begin{aligned}
\alpha^{\bar{v}}_{c_1} &= \phi_{\ell_5^*} + \bar{\phi}_{-\ell_4^*}, & \alpha^{\bar{v}}_{c_2} &= \phi_{\ell_6^*} + \bar{\phi}_{-\ell_5^*}, \\
\alpha^{\bar{v}}_{c_3} &= \phi_{\ell_3^*} + \bar{\phi}_{-\ell_6^*}, & \alpha^{\bar{v}}_{c_4} &= \phi_{\ell_4^*} + \bar{\phi}_{-\ell_3^*}, \\
\alpha^{v}_{c_4,32} &= \phi_{-\ell_3^*} + \bar{\phi}_{-\ell_2^*}, & \alpha^{v}_{c_3,13} &= \phi_{\ell_1^*} + \bar{\phi}_{\ell_3^*}, \\
\alpha^{v}_{c_3,61} &= \phi_{-\ell_6^*} + \bar{\phi}_{\ell_1^*}, & \alpha^{v}_{c_2,26} &= \phi_{\ell_2^*} + \bar{\phi}_{\ell_6^*}, \\
\alpha^{v}_{c_2,52} &= \phi_{-\ell_5^*} + \bar{\phi}_{\ell_2^*}, & \alpha^{v}_{c_1,15} &= \phi_{-\ell_1^*} + \bar{\phi}_{\ell_5^*}, \\
\alpha^{v}_{c_1,41} &= \phi_{-\ell_4^*} + \bar{\phi}_{-\ell_1^*}, & \alpha^{v}_{c_4,24} &= \phi_{-\ell_2^*} + \bar{\phi}_{\ell_4^*}.
\end{aligned} \tag{5.56}$$

The meaning of the labels is elucidated in figure 5.4. For example, the angle $\alpha^v_{c_1,32}$ is the angle at the node c_1 from the edges ℓ_3, ℓ_2, as seen from the polygon centered at v. Using these angles, the constraints, first for the polygon centered at $\bar{v}$, are given by

$$\begin{aligned}
C_{G,\bar{v}} = L_{\ell_6} + L_{\ell_5} \exp\Big(i(\pi - \alpha^{\bar{v}}_{c_2})\Big) &+ L_{\ell_4} \exp\Big(i(2\pi - \alpha^{\bar{v}}_{c_2} - \alpha^{\bar{v}}_{c_1})\Big) \\
&+ L_{\ell_3} \exp\Big(i(3\pi - \alpha^{\bar{v}}_{c_2} - \alpha^{\bar{v}}_{c_1} - \alpha^{\bar{v}}_{c_4})\Big)
\end{aligned} \tag{5.57}$$

$$C_{\alpha,\bar{v}} = 2\pi - \alpha^{\bar{v}}_{c_1} - \alpha^{\bar{v}}_{c_2} - \alpha^{\bar{v}}_{c_3} - \alpha^{\bar{v}}_{c_4}. \tag{5.58}$$

The deficit angle constraint, C_α^v, for the polygon centered at v is given by

$$C_{\alpha,v} = 6\pi - \alpha^v_{c_4,13} - \alpha^v_{c_3,32} - \alpha^v_{c_3,61} - \alpha^v_{c_2,26} - \alpha^v_{c_2,52} - \alpha^v_{c_1,15} - \alpha^v_{c_1,41} - \alpha^v_{c_4,24}. \tag{5.59}$$

The Gauss constraint for v isn't written down explicitly due its length. Using these expressions, the two mass observables can be written in terms of 't Hooft's variables as

$$\mathrm{tr}\Big(\tilde{h}_{\ell_1}\Big) = 2\cos\Big(\frac{1}{2}(\alpha^v_{c_1,15} + \alpha^v_{c_2,52} + \alpha^v_{c_4,24} + \alpha^v_{c_1,41})\Big)\cosh\Big(\eta_{\ell_1^*}\Big), \tag{5.60}$$

$$\mathrm{tr}\Big(\tilde{h}_{\ell_2}\Big) = 2\cos\Big(\frac{1}{2}(\alpha^v_{c_1,15} + \alpha^v_{c_2,52} + \alpha^v_{c_3,61} + \alpha^v_{c_2,26})\Big)\cosh\Big(\eta_{\ell_2^*}\Big). \tag{5.61}$$

In the one polygon case, the first step in analyzing the problem was solving the flatness constraints. In the two polygon case, explicitly solving the flatness constraint for each node is not as simple. One way forward is gauge fixing, the system to make it simpler. Our gauge choice is to set the rapidity of one of the edges to zero. This is only possible for edges that fail to form loops. Namely, the rapidities for $\eta_{\ell_1^*}, \eta_{\ell_2^*} \neq 0$. This is because a gauge transformation ϕ acting on the holonomies attached to these nodes would act by

$$\tilde{h}_{\ell_1^*} \to \phi(v)\tilde{h}_{\ell_1^*}\phi^{-1}(v). \tag{5.62}$$

In order for $\eta_{\ell_1^*} = 0$, the holonomy would have to be conjugate to a pure rotation, which is not possible. Therefore, only one of the edges dual to the polygon centered at $\bar{v}$ can have possibly have a vanishing rapidity. For definiteness, we set $\eta_{\ell_3} = 0$. In order for this gauge choice to be possible, we need to ensure that a solution to the constraints exist, and the gauge fixing can be reached. The latter point should be true, since C_G generates boosts. Therefore, all we have to do is ensure that a solution to the constraints exist.

First, setting $\eta_{\ell_3^*} = 0$ has a number of consequences. From (4.17), we have that $\eta_{\ell_1^*} = \eta_{\ell_6^*}$. This means that the node c_3 is again quasi-static and thus is described by

$$\begin{aligned} \eta_{\ell_1^*} &= \eta_{c_3} & \alpha^{\bar{v}}_{c_3} &= \pi - \alpha_{c_3} \\ \eta_{\ell_6^*} &= \eta_{c_3} & \alpha^v_{c_3,13} &= \alpha_{c_3} \\ \eta_{\ell_3^*} &= 0 & \alpha^v_{c_3,61} &= \pi. \end{aligned} \tag{5.63}$$

Similar reasoning hold for the node c_4, giving

$$\begin{aligned} \eta_{\ell_2^*} &= \eta_{c_4} & \alpha^{\bar{v}}_{c_4} &= \pi - \alpha_{c_4} \\ \eta_{\ell_4^*} &= \eta_{c_4} & \alpha^v_{c_4,32} &= \alpha_{c_4} \\ \eta_{\ell_3^*} &= 0 & \alpha^v_{c_4,24} &= \pi. \end{aligned} \tag{5.64}$$

Now we notice the rapidities for the two remaining triangles will be equal, i.e. the quantities $\eta_{c_4}, \eta_{c_3}, \eta_{\ell_5^*}$ will decorate the edges. Therefore, following 't Hooft's vertex condition (4.13), the angles will be the same. Namely, we will have

$$\alpha^v_{c_1,41} = \alpha^v_{c_2,26} = \hat{\alpha}, \tag{5.65}$$

$$\alpha^{\bar{v}}_{c_1} = \alpha^v_{c_2,52} = \alpha', \tag{5.66}$$

$$\alpha^{\bar{v}}_{c_2} = \alpha^v_{c_1,15} = \tilde{\alpha}. \tag{5.67}$$

Plugging these relations into the C_α constraints we get that

$$C_\alpha^{\bar{v}} = \alpha_{c_3} + \alpha_{c_4} - \alpha' - \tilde{\alpha}, \tag{5.68}$$
$$C_\alpha^{v} = 4\pi - \alpha_{c_3} - \alpha_{c_4} - 2\hat{\alpha} - \alpha' - \tilde{\alpha}. \tag{5.69}$$

Together these imply that,

$$2\pi = \hat{\alpha} + \alpha' + \bar{\alpha}, \tag{5.70}$$
$$2\pi = \alpha_{c_3} + \alpha_{c_4} + \hat{\alpha}. \tag{5.71}$$

These relations then simplify the Gauss constraints for both polygons. For $\bar{v}$ we get that

$$C_G^{\bar{v}} \approx L_{\ell_6} + L_{\ell_3}\exp(i\alpha_{c_3}) + L_{\ell_4}\exp(i(2\pi - \hat{\alpha})) + L_{\ell_5}\exp(i(\pi + \tilde{\alpha})), \tag{5.72}$$

and similarly for v,

$$\begin{aligned} C_G^{v} \approx\ & L_{\ell_6} + L_{\ell_1} + L_{\ell_5}\exp(i(\pi - \tilde{\alpha})) + L_{\ell_2}\exp(i(2\pi - \tilde{\alpha} - \alpha')) \\ & + L_{\ell_4}\exp(i(2\pi - \tilde{\alpha} - \alpha')) + L_{\ell_1}\exp(i\pi) + L_{\ell_3}\exp(i(2\pi - \alpha_{c_3})) \\ & + L_{\ell_2}\exp(i(3\pi - \alpha_{c_3} - \alpha_{c_4})) \\ \approx\ & L_{\ell_6} + L_{\ell_5}\exp(i(\pi - \tilde{\alpha})) + L_{\ell_4}\exp(i\hat{\alpha}) + L_{\ell_3}\exp(i(2\pi - \alpha_{c_3})). \end{aligned}$$

Therefore, we see that when the two C_α constraints are satisfied the Gauss constraints for v and $\bar{v}$ are equivalent.

However, there is an additional reduction. The expression (5.70), implies that the nodes c_2 and c_1 are quasi-static. This indicates that one of the rapidities must vanish, which necessarily gives that $\eta_{\ell_5^*} = 0$. From this, we get that $\hat{\alpha} = \pi$, and then that $\tilde{\alpha} = \pi - \alpha'$. Furthermore, since the vertex is quasi static, we get

$$\eta_{\ell_4^*} = \eta_{c_4} = \eta_{c_2} = \eta_{\ell_1^*}.$$

The result of this back reacts through the polygon. Setting $\eta_{c_3} = \eta$ and $\alpha_{c_3} = \alpha$ we get that,

$$\begin{array}{lll} \eta_{\ell_1^*} = \eta & \alpha_{c_3}^{\bar{v}} = \pi - \alpha & \alpha_{c_1}^{\bar{v}} = \alpha' \\ \eta_{\ell_2^*} = \eta & \alpha_{c_3,13}^{v} = \alpha & \alpha_{c_1,41}^{v} = \pi \\ \eta_{\ell_3^*} = 0 & \alpha_{c_3,61}^{v} = \pi & \alpha_{c_1,15}^{v} = \pi - \alpha' \\ \eta_{\ell_4^*} = \eta & \alpha_{c_4}^{\bar{v}} = \alpha & \alpha_{c_2}^{\bar{v}} = \pi - \alpha' \\ \eta_{\ell_5^*} = 0 & \alpha_{c_4,32}^{v} = \pi - \alpha & \alpha_{c_2,26}^{v} = \pi \\ \eta_{\ell_6^*} = \eta & \alpha_{c_4,24}^{v} = \pi & \alpha_{c_2,52}^{v} = \alpha' \end{array} \tag{5.73}$$

Therefore, α and α' are free parameters in the theory. Furthermore, in this case the Gauss constraint further simplifies to

$$C_G^{\bar{v}} = L_{\ell_6} + L_{\ell_3}\exp(i\alpha) - L_{\ell_4} + L_{\ell_5}\exp(-i\alpha'). \tag{5.74}$$

The solution to this constraint are then easily identified. A solution is given by setting $L_{\ell_6} = L_{\ell_4}$, $L_{\ell_3} = L_{\ell_5} = 0$. As a result, we have reduced the two polygon universe to the one polygon universe, by gauge fixing. Therefore, the observables for this system will be equal to the observables for the one polygon torus. Consequentially, the two polygon torus universe cannot contain reduced phase space of gravity.

The next step forward would be to ask whether N polygons could solve this issue. We suspect this will is not the case. It should always be possible to apply a gauge fixing and essentially delete an edge of a polygon. For instance with N polygons we could apply $N-1$ gauge fixes that force $N-1$ rapidities to vanish. In the end, we suspect, we will again get the one polygon universe. Therefore, we end with the following conjecture:

Conjecture 1. *Polygonal gravity cannot fully describe the torus universe.*

5.4 Conclusion

In this chapter we explored the relationship between LQG and polygonal gravity. First, we showed that due to the discrete structure of polygonal gravity, LQG* was the correct polarization to use. This is because polygonal gravity first solves the flatness constraint, which matches LQG*. This corrected a mistake in the literature [52]. The next step was reducing dual LQG to the polygonal gravity variables. This required gauge fixing the faces of Γ to be co-planar, so that they would form polygons. Unfortunately, how this was accomplished was not fully satisfactory. Namely, we assumed that such a gauge fixing was already done by a series of translations, but never attempted to see whether such a transformation was possible. The reason for this, is that Dirac brackets are very hard to compute in general; relatedly, observables for the kinematical phase space of LQG* are difficult to find. However, by assuming that the polygons were co-planar, we were able to reproduce the results in [28], but in the LQG* formalism. That is, we were able to recover the symplectic structure for polygonal gravity.

The $C_{G,f}$ constraint (4.30) was then shown to be equivalent to the Gauss law in LQG* (5.17). As a byproduct of this result, we corrected a mistake in [28] about the origin of the other constraint $C_{\alpha,f}$ (4.29). Therefore, we accomplished the first goal of this thesis.

In the following section, we used the relation between these two theories to attack the problem of whether the torus universe exists in polygonal gravity. This was done by expressing the physical variables of reduced discretized gravity, in terms of 't Hooft's variables. The result was that we did not have enough independent physical observables to cover the reduced phase space of the torus, when using one or two polygonal decompositions. We conjectured 1 that N polygons will also fail to capture the torus universe. The reasoning for this is that a N polygon decomposition of the torus can be reduced to the single polygon by gauge fixing. This suggests that polygonal gravity does not contain the torus universe, and gauge fixing from LQG* over constrains the theory.

Chapter 6

Duality in 't Hooft Gravity

In the previous chapter we explored two different approaches to discrete gravity and their relation. That is, we showed how 't Hooft's polygonal gravity is a gauge fixed version of dual LQG. The gauge fixing forced the faces of the graph Γ to be co-planar and, thus, on-shell form flat polygons. However, our explanation is not entirely satisfying. First, we were not able to entirely explain how the orbits of the flatness constraints (i.e. translations), are explicitly dealt with in dual LQG. Instead, we argued, somewhat heuristically, that by imposing the co-planar condition certain translations were factored out. Furthermore, we were not able to compute the Dirac brackets explicitly. Part of the issue is that solving the flatness constraint first is much more difficult than the Gauss constraint. This is partially the reason why the LQG polarization was developed before the dual version. In the LQG polarization, solving the Gauss constraint leads to Regge geometry [8, 13], a theory that was developed in the early 1960s [40]. This leads to the question of whether there exists a dual polarization of polygonal gravity that more closely mirrors Regge geometry and LQG.

The first part of this chapter will describe the dual 't Hooft formalism. Like polygonal gravity, it will be motivated from purely geometrical arguments. Therefore, we will not start with an action principle. Dual to polygonal gravity, the starting point will be the triangulation Γ^*. The theory will be defined in terms of the variables $(L, 2\eta, \alpha)$. However, the angles α will be given by the edge lengths L, not the dihedral angles 2η. Furthermore, the lengths will live on the edges of Γ^* not Γ, and vice versa for the dihedral angles. In the following section, we will describe the dynamical structure and will postulate that the symplectic structure for $L, 2\eta$ will be identical to polygonal gravity. This postulate is well motivated since our presentation suggests that dual 't Hooft is just a different polarization of polygonal gravity. In this regard, our dynamical constraint in dual 't Hooft is in opposition to polygonal gravity. Here, we find that the consistency condition for gluing triangles together is just a flatness constraint for the faces of the dual graph Γ.

In the second part of this chapter, we will show how dual 't Hooft gravity is related to the standard LQG polarization. We will argue the kinematical phase space of LQG is identical to dual 't Hooft's, when a specific gauge has been chosen. In this case, the reduction to the kinematical phase space is much simpler than LQG* to polygonal gravity. The reason for this is that finding observables for the kinematical phase space is much simpler.

6.1 Dual 't Hooft gravity

In polygonal gravity we started with assuming that observers sit a centers of polygons, which are related to the faces of Γ. The links of Γ then encoded information about the lengths of the polygons, and the edges of Γ^* described the transformation between the observers of adjacent polygons. Furthermore, in polygonal gravity we required that Γ^* was a triangulation. This was required to ensure that 't Hooft's vertex condition (4.13) would specify the angles of the polygon α in terms of rapidity η of the links. In dual 't Hooft, essentially the opposite occurs. First, in order to simplify the assumption we will assume that Γ, Γ^* are already embedded in a Cauchy surface. Therefore, we do not need to specify 't Hooft's matching condition (4.2). Dual to polygonal gravity, we will assume that observers sit in the center of the triangles of Γ^* not the faces of Γ. In other words, the nodes of Γ will represent observers, not the vertices of Γ^*. Note that since observers sit in triangles, we do not have to force them to be planar. In polygonal gravity, the polygons where described by lengths and angles, this will be opposite in dual 't Hooft. The lengths and angles in dual 't Hooft gravity are associated to the edges of the triangulation. Moreover, similar to polygonal gravity, we will assume that the observer in each triangle is at "rest". Therefore, the lengths L, will be purely spatial, and the normal to each triangle are given by $(1, 0, 0)$. Due to this structure, the angles of dual 't Hooft gravity are given by the lengths. Consider a triangle $c^* = [\ell_3^* \ell_2^* \ell_1^*]$ with side lengths given by $L^c_{\ell_i^*}$, then the angles are specified by

$$L^2_{\ell_i^*} = L^2_{\ell_j^*} + L^2_{\ell_k^*} - 2L_{\ell_j^*} L_{\ell_k^*} \cos\Big(\alpha^c_{v_i}\Big), \tag{6.1}$$

for every permutation of i, j, k. This relation is analogous to the vertex relation (4.13) for 't Hooft gravity. Additionally the lengths must obey the triangle identity, i.e. $L_{\ell_i^*} + L_{\ell_j^*} \geq L_{\ell_k^*}$, which is akin to 't Hooft's hyperbolic triangle identity $\Big|\eta_{\ell_i^*}\Big| + \Big|\eta_{\ell_j^*}\Big| \geq \Big|\eta_{\ell_k^*}\Big|$.

In terms of the construction of polygonal gravity, there is an interesting relation with dual 't Hooft gravity. Recall in 't Hooft gravity, consistency conditions are required when gluing the polygons together. This resulted 't Hooft's vertex condition (4.13). However, when defining this, we also had a translational part (4.2), that we ignored. This was because, the lengths/geometry of the dual triangles were not pertinent. In dual 't Hooft gravity this is reversed. In this case we are just concerned with the translational part of the triangles describing the lengths and angles. The non-translational parts are ignored contrasting polygonal gravity. Furthermore, the gluing condition for the translational part (4.12), gives the cosine law (6.1).

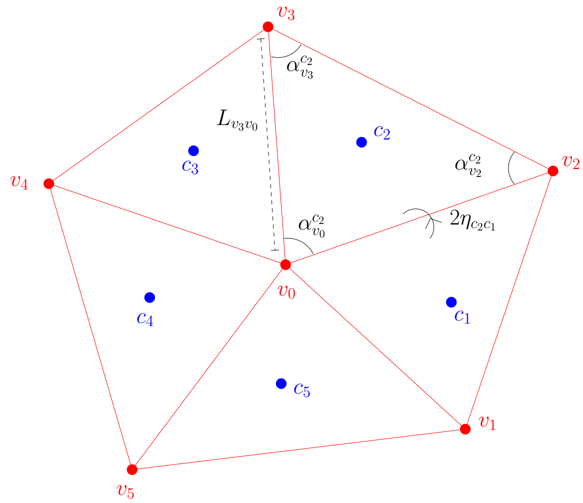

Figure 6.1: In dual 't Hooft gravity, observers sit at the nodes c_i of Γ, and describe the lengths of the edges of the triangulation Γ^*

So far we only have considered a single triangle in dual 't Hooft gravity. In order to describe the Cauchy surface we need to glue these triangles together. Each triangle is a local inertial observer, just like in 't Hooft gravity; therefore, we know that the observers will be related by a constant Poincaré transformation. However, we are only going to consider the SU(1, 1) part of the transformation. The translational part of the Poincaré transformation describes the lengths of the faces of Γ. However, analogous to polygonal gravity, our observers do not care about this information. Therefore, if we consider two triangles c_1^*, c_2^*, with the edge ℓ^* shared between them, the transition between them will be given by $h_{c_2c_1} \in \mathrm{SU}(1,1)$. Using the 't Hooft decomposition (4.3), we will write the holonomy as

$$h_{c_2c_1} = h_\ell = R_s(\bar{\phi}_\ell)B_s(2\eta_\ell)R_s(\phi_\ell). \tag{6.2}$$

Like in 't Hooft gravity, the rotation angles $\phi, \bar{\phi}$ will be interpreted as the orientation of the edge ℓ^* when viewed from c_1 or c_2 respectively. The boost parameter 2η, describes the dihedral angle between the two triangles. Finally just like polygonal gravity, we require the metric to be continuous between the triangles. This implies the length of the edge ℓ^* will be the same when viewed from c_1 or c_2.

The holonomies between triangles will not be independent in general. This can be seen from similar reasoning used to define 't Hooft's vertex condition. Consider an observer sitting in a triangle c_1 surrounded by a series of triangles c_i, as in figure 6.1. If this observer, $\mathcal{X}_{c_1}$, moves in a loop from triangle to triangle, then we require

$$\mathcal{X}_{c_1} = h_{c_1c_i}h_{c_ic_{i-1}}\cdots h_{c_2c_1}\mathcal{X}_{c_1}. \tag{6.3}$$

This implies that the holonomy if any loop $\lambda \in \Gamma$ is the identity, and is equivalent to

$$V_\lambda = B(2\eta_{c_1c_i})R(\pi - \alpha^{c_i}(L))B(2\eta_{c_ic_{i-1}})R(\pi - \alpha^{c_{i-1}}(L))\cdots B(2\eta_{c_2c_1})R(\pi - \alpha^{c_1}(L)) = 1. \tag{6.4}$$

	't Hooft polygonal gravity	Dual 't Hooft gravity
(Γ, Γ^*) variables	$L_\ell \to \Gamma, \eta_{\ell^*} \to \Gamma^*,\ \alpha(\eta)$	$L_\ell \to \Gamma,\ \alpha(L), \eta_{\ell^*} \to \Gamma^*$
kin constr.	$V_c = 1$ *(flat triangle)*	cosine law *(triangle closure)*
gauge fix	$t_{v_1} = t_{v_2} = \cdots = \tau$	$\hat{n} = (1, 0, 0)$
dyn. constr.	$C_G = 0, C_\alpha = 0$ *(polygon condition)*	$V_\lambda = 1$ *(flat)*

Table 6.1: Comparison between 't Hooft polygonal gravity and dual 't Hooft gravity.

Note, we took to the loop to be given by $\lambda = [c_1 c_i c_{i-1} \cdots c_1]$, and we left the functional dependence of α to reinforce they are functions of the length. This gives three constraints for dual 't Hooft gravity. The appearance of $\pi - \alpha$, comes from the fact that in LQG we are concerned with the angles between the edges of the triangulation, not the links of the dual graph. Finally, the constraint enforces the flatness constraint for the Cauchy surface. In our case, the loops will usually be given by the faces dual to the vertices of Γ^*.

The comparison between 't Hooft gravity and its dual is shown in table 6.1. At the kinematical level, the two theories mirror the duality of LQG and LQG*. That is, the location of the variables L, η are swapped between the two pictures. The gauge fixing in both pictures are similar as well. In polygonal gravity, the partial gauge fix implies three things. The first and second, enforce that the faces of Γ are co-planar and the observer is at rest in the polygon. The third, imposes that all polygonal clocks evolve at the same rate. For dual 't Hooft, we automatically have that the triangles are co-planar, and does not require any gauge fixing. We do require that observers in dual 't Hooft gravity are at rest. Finally, we have chosen not to apply 't Hooft's time gauge fix. The reason for using this gauge fixing is twofold. First, such choices of time may not be possible in principle. Secondly, the choice of time was made so that 't Hooft could recover the correct symplectic structure for gravity. However, we will take a different approach for dual 't Hooft gravity. We postulate that the symplectic structure is equal to one of polygonal gravity,

Proposition 6.1. *The symplectic structure needed for dual 't Hooft to be related to 2+1 gravity is given by,*

$$\{L_{\ell_i^*}, L_{\ell_j^*}\} = 0, \qquad \{L_{\ell_i^*}, 2\eta_{\ell_j}\} = -\delta_{ij}, \qquad \{2\eta_{\ell_i}, 2\eta_{\ell_j}\} = 0. \tag{6.5}$$

The argument for this proposition, is that the choice of polarization does not change the symplectic form.

This completes the discussion of dual 't Hooft gravity. In summary, we have a theory based on a set of lengths and angles which describe the geometry of triangles, and dihedral angles between them. This closely mirrors the situation in Regge calculus, or canonical simplicial gravity [13]. Furthermore, since polygonal gravity was related to LQG* we expect dual 't Hooft to be related to LQG. The exact nature of this relation is what we will tackle next.

6.2 Relation to LQG and Regge calculus

Dual 't Hooft gravity places its length variables on the edges of Γ^* which is similar to the fluxes of the LQG polarization. Therefore, superficially we expect the phase space of dual 't Hooft gravity to be related to the kinematical phase space of LQG $\mathcal{P}^{\text{kin}}_{\text{LQG}}$. Recall that this is given by solving the Gauss constraint (3.39). The exact relation between the two theories is what we will explore in this section. Recall, in polygonal gravity one of the issues was finding the Dirac brackets after partially gauge fixing the orbits of the flatness constraint. In dual 't Hooft gravity, however, we will be able to avoid this problem since finding a complete set of kinematical observables is much easier to do.

To start consider a triangle $c^* = [v_1 v_2 v_2]$. Following [8], the lengths and angles for the triangle are defined as

$$L_{\ell^*} = \left|\widetilde{E}^c_{\ell^*}\right|, \qquad \cos\Big(\alpha_{v_j}\Big) = -\frac{\widetilde{E}_{[v_i v_j]} \cdot \widetilde{E}_{[v_j v_k]}}{L_{[v_i v_j]} L_{[v_j v_k]}}. \tag{6.6}$$

For three-valent vertices, as we have assumed by necessity, the Gauss constraint, $\mathcal{J}_c$, implies that the three fluxes form the edges of a triangle. For higher valent vertices this is no longer true. This is why we had to, by hand, enforce the co-planar condition to hold in dual LQG. Moreover, the beautiful part of these observables is that they are well-defined on $\mathcal{P}^{\text{kin}}_{\text{LQG}}$, since both are gauge invariant. This starkly contrasts the situation in LQG* where the lengths do not form good observables for the kinematical phase space, prior to gauge fixing.

In order for these lengths and angles to be the same as the ones from dual 't Hooft, we need to ensure that the angles are related to the lengths by the cosine law. In fact, this is exactly what the Gauss constraint gives. To see, this consider a triangle $c^* = [v_1 v_2 v_3]$, with edges $\ell^*_i = [v_{i+1} v_i]$, where $v_{3+1} = v_1$. Taking the dot product of $\widetilde{E}_{\ell_3}$ with the Gauss constraint (3.39), we get that

$$\begin{aligned} 0 &\approx \widetilde{E}_{\ell_1} \cdot \Big(\widetilde{E}_{\ell_1} + \widetilde{E}_{\ell_2} + \widetilde{E}_{\ell_3}\Big) \\ &\approx \widetilde{E}_{\ell_1} \cdot \widetilde{E}_{\ell_1} - (\widetilde{E}_{\ell_2} + \widetilde{E}_{\ell_3}) \cdot \widetilde{E}_{\ell_2} - (\widetilde{E}_{\ell_2} + \widetilde{E}_{\ell_3}) \cdot \widetilde{E}_{\ell_2} \\ &= L^2_{\ell_1} - L^2_{\ell_2} - L^2_{\ell_3} + 2 L_{\ell_2} L_{\ell_3} \cos(\alpha_{v_1}), \end{aligned}$$

which is precisely the cosine law. Furthermore, the triangle inequality for the edge lengths can also be derived from the Gauss law. As such, we see that the dual 't Hooft's consistency conditions are just a form of the Gauss constraint. The lengths are also the same when viewed from other triangles since fluxes viewed from the adjacent triangle are related by (3.26). Therefore, it appears these lengths are precisely the ones in in dual 't Hooft gravity.

So far the L observables just describe the geometry of the local triangles. Therefore, we need some kinematical observables that describe how the triangles are glued together. This necessarily in require information from the LQG holonomies h_ℓ. The dihedral angles Θ between the triangles, will define the missing information. The first step in defining these kinematical observables will be to define the normal for each triangle. The normal to the triangle described above is given by

$$n^c = \frac{E_{\ell_1} \wedge E_{\ell_2}}{|E_{\ell_1} \wedge E_{\ell_2}|}. \tag{6.7}$$

Note that this definition of the normal is well-defined exactly when the Gauss constraint is satisfied. Namely, any two edges will give the same normal up to a term proportional to the Gauss constraint. The dihedral angle between two triangles is then given by

$$\cosh\left(\Theta^{\text{Min}}_{c_1 c_2}\right) = \left\langle n^{c_1}, \text{Ad}(h^{-1}_{c_2 c_1}) n^{c_2} \right\rangle. \tag{6.8}$$

Note that $\Theta_{c_1c_2} = \Theta_{c_2c_1}$ from the definition.

Since our gauge group is $\text{SU}(1,1)$ one may wonder if this definition if well-defined. That is, a possible concern may be that the normals may not be not be timelike. However, this is not the case because we embedded our graph in a spacelike Cauchy surface. To see this, note that ℓ is a spacelike curve. Writing the flux in terms of the continuum variables we have that

$$\widetilde{E}^A_\ell = \int R^A{}_B(g) e^B{}_a \dot{\ell}^a(s) \mathrm{d}s, \tag{6.9}$$

where s is the parameter for our curve and $R^A{}_B(g)$ is the gauge transformation g in the vectorial representation. Then we have

$$\begin{aligned}
\widetilde{E}^A \eta_{AB} \widetilde{E}^B &= R^A{}_C(g) \eta_{AB} R^B{}_D e^C{}_a e^B{}_b \dot{\ell}^a \dot{\ell}^b \\
&= \eta_{CD} e^C{}_a e^B{}_b \dot{\ell}^a \dot{\ell}^b \\
&= \dot{\ell}^a g_{ab} \dot{\ell}^b > 0,
\end{aligned}$$

where it is greater than zero since g_{ab} is the spatial metric. Therefore, our flux vectors will all be spacelike and the normals will be timelike.

The dihedral angles then form the other set of variables needed to describe the kinematical phase space of LQG. One way to check this is to count the degrees of freedom. First, there are two independent sets of kinematical observables for each link of Γ, L_{ℓ^*} and Θ_ℓ. Therefore, we have $2E$ variables, where E is the number of links of Γ. By solving the Gauss constraint we have $3F$ constraints, one flatness constraint for every face of Γ. Furthermore, we only have $3F$ degrees of freedom encoded the orbits of the flatness constraints. As a result, the number of degrees of freedom is

$$2E - 6F = 6E - 6V - 6F = 12g - 12, \tag{6.10}$$

where V is the number of vertices and we used that Γ^* is a triangulation. This again is the dimension of the reduced phase space of gravity for $g > 1$ surfaces, verifying we can describe the kinematical phase space using L and Θ.

Now the question is, why did we not use the same definition of Θ that was used in defining dual 't Hooft, i.e. (6.2). The dual 't Hooft definition, 2η, of the dihedral angle is inconvenient in our case since it is not $\text{SU}(1,1)$ gauge invariant. Therefore, in order for

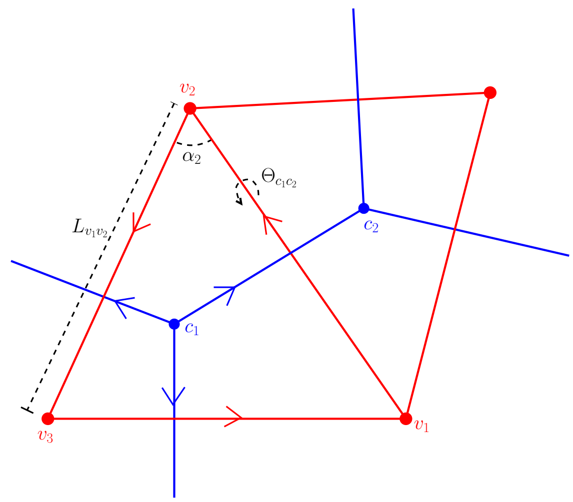

Figure 6.2: Kinematical observables for the LQG polarization. These provide enough information to completely describe the kinematical phase space.

it to describe dihedral angle between triangles we would need to apply a gauge fix, and then compute Dirac brackets to find the Poisson brackets for the kinematical phase space. This would lead to the same difficulties we found in 't Hooft gravity. Using the definition (6.8) instead makes it much easier to kind the Poisson brackets. The reason for this is that since L and Θ are kinematical observables their Poisson brackets can be calculated from the LQG Poisson brackets. In fact, we can explicitly calculate them, as was first done in [52]. To find the Poisson brackets, we first find brackets between the square of the lengths are given by

$$\{L^2_{\ell_i^*}, L^2_{\ell_j^*}\} = 4\delta_{ij}\widetilde{E}^A_{\ell_i^*}\widetilde{E}^B_{\ell_j^*}\epsilon_{ABC}\widetilde{E}^C_{\ell_i^*} = 0. \tag{6.11}$$

This implies that $\{L_{\ell_i^*}, L_{\ell_j^*}\} = 0$ for all $\ell_i^*, \ell_j^* \in \Gamma^*$. Similarly, it is easy to show that the dihedral angles Poisson commute, since holonomies Poisson commute in LQG.

For the remaining Poisson brackets, consider the two triangles shown in figure 6.2. In this case, it is immediate that the brackets between Θ_{ℓ_i} and $L_{\ell_j^*}$ will be zero if $i \neq j$. The only case of concern is when the curves intersect, e.g. ℓ_3 and ℓ_3^* in figure 6.2. To find this Poisson bracket, first we define the unnormalized normals for the triangles c_1^*, c_2^* by

$$\begin{aligned} N^{c_1} &= \widetilde{E}_{\ell_1^*} \wedge \widetilde{E}_{\ell_2} = [\widetilde{E}_{\ell_1^*}, \widetilde{E}_{\ell_2^*}] \\ N^{c_2} &= \widetilde{E}_{\ell_4^*} \wedge \widetilde{E}_{\ell_5} = [\widetilde{E}_{\ell_4^*}, \widetilde{E}_{\ell_5^*}]. \end{aligned} \tag{6.12}$$

It is then obvious that $\{\widetilde{E}_{\ell_3}, n^{c_i}\} = 0$, $i = 1, 2$. Next, we have that

$$\{L^2_{\ell_3}, |N^{c_1}||N^{c_2}| \cosh\Theta_{\ell_3}\} = 2L_{\ell_3}\sinh\Theta_{\ell_3}\{L_{\ell_3}, \Theta\}. \tag{6.13}$$

Conversely, plugging in the expressions for $L^2_{\ell_3}$ and $\cosh\Theta_{\ell_3}$ we get that

$$\begin{aligned}\{L^2_{\ell_3}, |N^{c_1}||N^{c_2}|\cosh\Theta_{\ell_3}\} &= \{\widetilde{E}^{c_1,A}_{\ell_3}\widetilde{E}^{c_1}_{\ell_3,A}, \langle \mathrm{Ad}(h_{\ell_3})N^{c_1}, N^{c_2}\rangle\} \\ &= 2\widetilde{E}^{c_1}_{\ell_3,A}\{E^{c_1,A}_{\ell_3}, \langle h_{\ell_3}N^{c_1}h^{-1}_{\ell_3}, N^{c_2}\rangle\} \\ &= 2\langle n^{c_2}, \mathrm{Ad}(h_{\ell_3})[\widetilde{E}^{c_1}_{\ell_3}, n^{c_1}]\rangle \\ &= 2(\mathrm{Ad}(h^{-1})N^{c_2})\cdot(\widetilde{E}_{\ell_3}\wedge N^{c_1}) \\ &= -2|N^{c_1}||N^{c_2}|\left|\widetilde{E}^{c_1}_{\ell_3}\right|\sinh\Theta_{\ell_3}\end{aligned}$$

from which the result $\{L_{\ell_3^*}, \Theta_{\ell_3}\} = -1$ follows. Therefore, the lengths and gauge invariant dihedral angles are canonically conjugate. However, this does not imply that $\Theta = 2\eta$ in general. However, precisely when dual 't Hooft's gauge fixing is satisfied, we will have this relation holds. The dual 't Hooft gauge fixing implied the normal of every was given by τ^0. This choice of gauge is easily possible in LQG, since it just is given by a local $\mathrm{SU}(1,1)$ transformation of each triangle frame. Furthermore, the Poisson bracket between L and Θ does not change after applying this gauge fix. This is precisely because L and Θ are kinematical observables. Therefore, we need to check whether the LQG L and Θ are equal to dual 't Hooft's L and 2η with this choice of gauge.

First the length variables trivially match, since after the gauge fix $n^I = (1,0,0)$, the fluxes become purely spatial. For the dihedral angles, we first decompose the LQG holonomy using (6.2), i.e. $h = R(\bar{\phi})B(2\eta)R(\phi)$. Then in this case since n^{c_1} and n^{c_2} are both just τ^0 we have that

$$\langle n^{c_1}, \mathrm{Ad}(h_{c_1c_2})n^{c_2}\rangle = \left\langle \tau^0, \mathrm{Ad}(B(2\eta_{c_1c_2}))\tau_0\right\rangle = \cosh(2\eta_{c_1c_2}), \tag{6.14}$$

implying that $\Theta_{c_1c_2} = 2\eta_{c_1c_2}$.

This almost completes the map from LQG to dual 't Hooft gravity. The only thing we have left to do is relate the LQG flatness constraint (3.42) to the dual 't Hooft constraint (6.4). In fact this is basically done, since the two constraints both enforce flatness for loops of Γ. The only point of contention may be the identification α in $\mathcal{G}_\lambda$ with the angle found from the cosine law from the Gauss constraint. However, in our gauge choice $n = \tau^0$ we already know that the α found from the holonomy and from the cosine law are equal, as a calculation similar to the one performed right after (5.7) shows. Therefore, the flatness constraints are equal. As a result, we have proved the postulate 6.1.

6.3 Conclusion

In this chapter, we introduced a new theory of discrete gravity - dual 't Hooft gravity. The motivation for this theory was to introduce a discrete theory of gravity from purely geometrical arguments, like polygonal gravity. However, unlike polygonal gravity, the hope was that the phase space structure and interpretation would be simplified. The idea was to look for a discrete version of gravity dual to polygonal gravity. This duality is expressed in table 6.1. In summary, the duality means swapping the location of the observers in 't

Hooft, from the center of polygons to the centers of the dual triangles. Furthermore, Gauss constraint is first solved in dual 't Hooft, like in LQG. The wonderful part this swap was, that far fewer gauge fixings are required to define the theory. This makes the comparison with other discrete models of gravity far easier. In fact, we were able to show that the phase space of dual 't Hooft is equal to the kinematical phase space of LQG. This implies that dual 't Hooft gravity is free of the problems that plagued polygonal gravity.

Chapter 7

Conclusions and Future Work

In this thesis we analyzed the relation between LQG and polygonal gravity. Using this relation, we then answered a long standing question about polygonal gravity: the nature of the torus universe in 't Hooft gravity. Furthermore, we analyzed whether polygonal gravity has a dual formulation similar to LQG.

In regards to the first question, we found that polygonal gravity was a gauge fixed version of LQG in its dual form (see table 5.1). Both polygonal gravity and dual LQG solve the flatness constraint around nodes first. These results challenged conclusions made in previous works. Namely, we refuted the claims made in [52] that state polygonal gravity is equal to the kinematical phase space of LQG. Furthermore, we found that the constraints of polygonal gravity arise from both the Gauss constraints and flatness constraints, refuting the claim made in [28]. That is, we did not find that polygonal gravity was equal to the kinematical phase space of dual LQG. Instead, there still exist remnants of the flatness constraint that are expressed in the polygonal constraint (4.29). One important issue to note is that we were not able to, after gauge fixing, find the Dirac brackets. This prevented us from finding whether polygonal gravity's gauge fix was always possible in principle. Moreover, we could not answer whether the reduced phase space of polygonal gravity was equal to the reduced phase space of LQG. In fact, as the solution to the second question (i.e. the torus universe) shows, this may not be the case.

In more detail, we found that polygonal gravity could not describe the torus universe using one or two polygons. This led to the conjecture 1 that the torus universe is not contained in polygonal gravity. The fact that the physical reduced gravity variables were not independent in the polygonal formalism led to this conclusion. Our results about the torus universe settled a question argued in previous works [23, 54] about the nature of polygonal gravity applied to the torus universe. Namely, we found that the author [54] was correct in suggesting that polygonal gravity could not describe the torus universe using one polygon. However, the reason for this differs from what the author suggested. Namely, while the angle α is not a full observable, this is not the reason polygonal gravity fails. Furthermore, we were able to extend these results to the two polygon model, finding that again polygonal gravity failed to describe the phase space of the torus universe. As our conjecture states, this raises an important question of whether polygonal gravity can describe the torus universe at all. If it cannot, this would signify that the various gauge fixes that were applied may not always be possible. A natural line of continued research would the standing of higher genus surfaces in polygonal gravity. This question has partially been answered [29]. However, they were unable to answer whether polygonal gravity covers the moduli space for genus g surfaces. Using the machinery developed in this thesis, namely the identification of the degrees of freedom in discrete gravity, an answer to this question may be possible.

A shortcoming about the research done in chapter 5 is that we were unable to analyze the origin of the transitions in polygonal gravity [46]. Namely, polygonal gravity contains discontinuities in its evolution, such as when an edge length shrinks to zero. In the literature [46, 22, 29], it is argued that these transitions are due to the gauge choices made in defining the theory. However, since we were unable to explicitly reduce the phase space in dual LQG to polygonal gravity using Dirac brackets, we could not answer these questions. Another issue left unexplored is whether adding particles to the above picture would change any of the results presented. Both issues provide interesting lines of future research.

In the last part of the thesis, we developed a dual formulation of polygonal or 't Hooft gravity, i.e. dual 't Hooft gravity. The point of this proposal was to develop a model of gravity similar to 't Hooft, but that escaped the issues that hindered polygonal gravity. The resulting theory expressed a similar notion of duality that is present in LQG. In fact, we explicitly showed that the phase space of dual 't Hooft gravity is equal to the kinematical phase space of LQG. This result implied that dual 't Hooft gravity is free of the problems that plagued the original formulation. Moreover, the kinematical phase space of LQG is known to be related to Regge geometry in 2+1 dimensions [8, 13]. Therefore, in this regard, dual 't Hooft gravity should be related to a Hamiltonian version of Regge geometry. This should make dual 't Hooft gravity more amenable to quantization. We hypothesize it should be related to the Ponzano-Regge model of quantum gravity [39, 24]. Another benefit with the dual 't Hooft framework is that one can recover a notion of a discrete diffeomorphism and Hamiltonian constraints, following the methods developed in [8]. In more detail [8], the Hamiltonian constraint would be given by

$$H^{\lambda,c} = 2\,\mathrm{tr}(n_c \mathcal{G}_{\lambda,c}), \tag{7.1}$$

where n_c is the normal to the triangle c^* and $\mathcal{G}_{\lambda,c}$ is a flatness constraint whose loop λ begins (and ends) at the node c. The diffeomorphism constraint is given by

$$D_\ell^{\lambda,c} = 2\,\mathrm{tr}\Big(\widetilde{E}_\ell^c \mathcal{G}_{\lambda,c}\Big). \tag{7.2}$$

Using this, one can recover a discrete version of Dirac's deformation algebra [49], which is universal in continuous gravity. As such, it appears that dual 't Hooft gravity can recover a notion of discrete ADM gravity.

An interesting next step in this formalism would be to add a cosmological constant. LQG, with a cosmological constant, has been explored extensively in literature [19, 15, 18, 17]. Therefore, the hope would be that by using the relation between LQG and 't Hooft developed in this thesis, a similar map could be developed.

Another potential line of research would be a symmetrization of polygonal gravity, or *Chern-Simons-'t Hooft gravity*. In [16], it is noted that Chern-Simons gravity can be viewed as the symmetric, or $\alpha = 1/2$, version of the Liouville form (2.27). This means, Chern-Simons gravity can be viewed as two copies of gravity; part in the LQG polarization and the other in the dual polarization. That is, the Chern-Simons symplectic form can be shown to be equivalent to

$$\Omega^{\mathrm{CS}} = \frac{1}{2}(\Omega^{\mathrm{LQG}} + \Omega^{\mathrm{LQG}^*}). \tag{7.3}$$

The question would be whether such a version of polygonal gravity exists. Returning to the dual graphs (Γ, Γ^*) the answer would appear to be yes. First, notice that in defining dual 't Hooft and polygonal gravity, we made two simplifications which are dual to each other. In polygonal gravity, we assumed that observers sat inside each face of Γ. The edges of Γ^* only contained $\mathrm{SU}(1,1)$ transition functions. The translational part was ignored, since we were not concerned with the geometry of the dual triangles. Instead we were only interested the $\mathrm{SU}(1,1)$, as it related the frames of the polygons. Furthermore, since we defined that the observer sat inside a polygon, we assumed the edges only contained translational degrees of freedom. In dual 't Hooft, the opposite was the case. Here, the observers sat in the triangles of Γ^*. Therefore, we were not concerned with the geometry of the dual faces (i.e. the translational degrees of freedom of Γ), but instead with the geometry of the triangles. The links of the polygons then contained $\mathrm{SU}(1,1)$ holonomies, since it encoded the relationship between the observers.

Chern-Simons-'t Hooft would then seemingly be constructed by combining these two theories. Namely, observers will be in both the faces and triangles. In this case, the edges and links of Γ^* and Γ, respectively, would both carry full $\mathrm{ISU}(1,1)$ holonomies. The expectation is that this formulation of 't Hooft gravity would be related to Chern-Simons gravity. The issue in formulating this theory, however, is the nature of the constraints. In [16] the independence of the constraint for their formulation of Chern-Simons gravity was not fully analyzed. Therefore, knowing the constraints needed in Chern-Simons-'t Hooft gravity may not be that simple. We leave this interesting line of research to future exploration.

Bibliography

[1] B. P. Abbott, R. Abbott, T. D. Abbott, et al. "Observation of Gravitational Waves from a Binary Black Hole Merger". In: *Phys. Rev. Lett.* 116.6 (Feb. 2016), p. 061102. DOI: 10.1103/PhysRevLett.116.061102.

[2] A. Y. Alekseev, A. Malkin, and E. Meinrenken. "Lie group valued moment maps". In: *J. Differ. Geom.* 48.3 (1998), pp. 445–495.

[3] D. Alekseevsky, J. Grabowski, G. Marmo, and P. W. Michor. "Poisson structures on the cotangent bundle of a Lie group or a principle bundle and their reductions". In: *J. Math. Phys.* 35.9 (Sept. 1994), pp. 4909–4927. DOI: 10.1063/1.530822.

[4] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. 2nd. Vol. 60. Graduate Texts in Mathematics. Springer-Verlag, 1989. DOI: 10.1007/978-1-4757-2063-1.

[5] A. Ashtekar, T. Pawlowski, and P. Singh. "Quantum nature of the big bang: Improved dynamics". In: *Phys. Rev. D* 74.8 (Oct. 2006), p. 084003. DOI: 10.1103/PhysRevD.74.084003.

[6] O. Babelon and D. Bernard. "Dressing symmetries". In: *Commun. Math. Phys.* 149.2 (Oct. 1992), pp. 279–306. DOI: 10.1007/BF02097626.

[7] E. Bianchi. "Entropy of Non-Extremal Black Holes from Loop Gravity". In: (Apr. 2012). arXiv: 1204.5122.

[8] V. Bonzom and B. Dittrich. "Dirac's discrete hypersurface deformation algebras". In: *Class. Quantum Gravity* 30.205013 (Apr. 2013). DOI: 10.1088/0264-9381/30/20/205013. arXiv: 1304.5983.

[9] S. Carlip. *Quantum Gravity in 2+1 dimensions*. Cambridge University Press, 2003.

[10] P. A. M. Dirac. *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, monograph series. Dover Publications, 2001.

[11] B. Dittrich. "Partial and complete observables for Hamiltonian constrained systems". In: *Gen. Relativ. Gravit.* 39.11 (Oct. 2007), pp. 1891–1927. DOI: 10.1007/s10714-007-0495-2.

[12] B. Dittrich. “How to construct diffeomorphism symmetry on the lattice”. In: *Proc. 3rd Quantum Gravity Quantum Geom. Sch. (QGQGS 2011).* 2011, p. 12. arXiv: 1201.3840 [gr-qc].

[13] B. Dittrich and P. A. Höhn. “Canonical simplicial gravity”. In: *Class. Quantum Gravity* 29.11 (June 2012), p. 115009. DOI: 10.1088/0264-9381/29/11/115009.

[14] W. Donnelly and L. Freidel. “Local subsystems in gauge theory and gravity”. In: *J. High Energy Phys.* 2016.9 (2016), p. 102. DOI: 10.1007/JHEP09(2016)102. arXiv: 1601.04744.

[15] Dupuis Maïté, F. Girelli, and E. R. Livine. “Deformed spinor networks for loop gravity: towards hyperbolic twisted geometries”. In: *Gen. Relativ. Gravit.* 46.11 (2014), pp. 1–22. DOI: 10.1007/s10714-014-1802-3. arXiv: 1403.7482.

[16] M. Dupuis, L. Freidel, and F. Girelli. “Discretization of 3d gravity in different polarizations”. In: (Jan. 2017). arXiv: 1701.02439.

[17] M. Dupuis and F. Girelli. “Observables in loop quantum gravity with a cosmological constant”. In: *Phys. Rev. D* 90.10 (Nov. 2014), p. 104037. DOI: 10.1103/PhysRevD.90.104037.

[18] M. Dupuis and F. Girelli. “Quantum hyperbolic geometry in loop quantum gravity with cosmological constant”. In: *Phys. Rev. D - Part. Fields, Gravit. Cosmol.* 87.12 (June 2013), p. 121502. DOI: 10.1103/PhysRevD.87.121502. arXiv: 1307.5461.

[19] M. Dupuis and F. Girelli. “Tensor Operators in Loop Quantum Gravity”. In: *Relativ. Gravit. 100 Years after Einstein Prague.* Ed. by J. Bičák and T. Ledvinka. Cham: Springer International Publishing, 2014, pp. 479–488. DOI: 10.1007/978-3-319-06761-2_68.

[20] J. Eldering. “The polygon model for 2+1D gravity: the constraint algebra and problems of quantization”. PhD thesis. June 2006. arXiv: 0606132 [gr-qc].

[21] T. Frankel. *The Geometry of Physics: An Introduction.* 3rd. Cambridge University Press, 2012.

[22] R. Franzosi and E. Guadagnini. “Particle decays and space-time kinematics in (2+1) gravity”. In: *Nucl. Phys. B* 450.1 (1995), pp. 327–354. DOI: 10.1016/0550-3213(95)00323-K.

[23] R. Franzosi and E. Guadagnini. “Topology and classical geometry in $(2 + 1)$ gravity”. In: *Class. Quantum Gravity* 13.3 (Mar. 1996), pp. 433–460. DOI: 10.1088/0264-9381/13/3/011.

[24] L. Freidel and D. Louapre. “Ponzano–Regge model revisited: I. Gauge fixing, observables and interacting spinning particles”. In: *Class. Quantum Gravity* 21.24 (Dec. 2004), pp. 5685–5726. DOI: 10.1088/0264-9381/21/24/002.

[25] L. Freidel and S. Speziale. “Twisted geometries: A geometric parametrization of SU(2) phase space”. In: *Phys. Rev. D* 82.8 (Oct. 2010), p. 084040. DOI: 10.1103/PhysRevD.82.084040.

[26] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*. Princeton paperbacks. Princeton University Press, 1992.

[27] H. R. Hollmann and R. M. Williams. "Hyperbolic geometry in 't Hooft's approach to (2 + 1)-dimensional gravity". In: *Class. Quantum Gravity* 16.5 (May 1999), pp. 1503–1518. DOI: 10.1088/0264-9381/16/5/304.

[28] Z. Kádár. "Polygon model from first-order gravity". In: *Class. Quantum Gravity* 22.5 (Mar. 2005), pp. 809–823. DOI: 10.1088/0264-9381/22/5/004.

[29] Z. Kádár and R. Loll. "(2+1) gravity for higher genus in the polygon model". In: *Class. Quantum Gravity* 21.9 (May 2004), pp. 2465–2491. DOI: 10.1088/0264-9381/21/9/020.

[30] R. Loll. "Independent loop invariants for 2 + 1 gravity". In: *Class. Quantum Gravity* 12.7 (July 1995), pp. 1655–1662. DOI: 10.1088/0264-9381/12/7/008.

[31] J. Maldacena. "The Large-N Limit of Superconformal Field Theories and Supergravity". In: *Int. J. Theor. Phys.* 38.4 (1999), pp. 1113–1133. DOI: 10.1023/A:1026654312961.

[32] J. E. Marsden and T. Raitui. *Introduction to Mechanics and Symmetry*. Second. Spinger-Verlag, 1999.

[33] J. Marsden and A. Weinstein. "Reduction of symplectic manifolds with symmetry". In: *Reports Math. Phys.* 5.1 (Feb. 1974), pp. 121–130. DOI: 10.1016/0034-4877(74)90021-4.

[34] C. Meusburger. "Cosmological measurements, time and observables in (2+1)-dimensional gravity". In: *Class. Quantum Gravity* 26.5 (Mar. 2009), p. 055006. DOI: 10.1088/0264-9381/26/5/055006.

[35] C. Meusburger. "Geometrical (2+1)-Gravity and the Chern-Simons Formulation: Grafting, Dehn Twists, Wilson Loop Observables and the Cosmological Constant". In: *Commun. Math. Phys.* 273.3 (June 2007), pp. 705–754. DOI: 10.1007/s00220-007-0255-x.

[36] C. Meusburger. "Grafting and Poisson Structure in (2+1)-Gravity with Vanishing Cosmological Constant". In: *Commun. Math. Phys.* 266.3 (Sept. 2006), pp. 735–775. DOI: 10.1007/s00220-006-0037-x.

[37] J. Nelson and T. Regge. "Homotopy groups and 2+1 dimensional quantum gravity". In: *Nucl. Phys. B* 328.1 (Dec. 1989), pp. 190–202. DOI: 10.1016/0550-3213(89)90099-0.

[38] K. Noui and A. Perez. "Three-dimensional loop quantum gravity: physical scalar product and spin-foam models". In: *Class. Quantum Gravity* 22.9 (May 2005), pp. 1739–1761. DOI: 10.1088/0264-9381/22/9/017.

[39] G. Ponzano and T. Regge. "Semiclassical Limit of Racah Coefficients." In: *Spectrosc. Gr. Theor. Methods Physics. Block, F. (ed.). New York, John Wiley Sons, Inc., 1968.* (Oct. 1969), pp. 1–58.

[40] T. Regge. “General relativity without coordinates”. In: *Nuovo Cim.* 19.3 (Feb. 1961), pp. 558–571. DOI: 10.1007/BF02733251.

[41] C. Rovelli. “Black Hole Entropy from Loop Quantum Gravity”. In: *Phys. Rev. Lett.* 77.16 (Oct. 1996), pp. 3288–3291. DOI: 10.1103/PhysRevLett.77.3288.

[42] C. Rovelli. “Partial observables”. In: *Phys. Rev. D* 65.12 (June 2002), p. 124013. DOI: 10.1103/PhysRevD.65.124013.

[43] C. Rovelli. *Quantum Gravity.* 1st. Cambridge University Press, 2008.

[44] C. Rovelli and L. Smolin. “Discreteness of area and volume in quantum gravity”. In: *Nucl. Phys. B* 442.3 (May 1995), pp. 593–619. DOI: 10.1016/0550-3213(95)00150-Q.

[45] G. ’t Hooft. “Canonical quantization of gravitating point particles in 2+1 dimensions”. In: *Class. Quantum Gravity* 10.8 (Aug. 1993), pp. 1653–1664. DOI: 10.1088/0264-9381/10/8/022.

[46] G. ’t Hooft. “Causality in (2+1)-dimensional gravity”. In: *Class. Quantum Gravity* 9.5 (May 1992), pp. 1335–1348. DOI: 10.1088/0264-9381/9/5/015.

[47] G. ’t Hooft. “Classical N-particle cosmology in 2+1 dimensions”. In: *Class. Quantum Gravity* 10.S (Dec. 1993), S79–S91. DOI: 10.1088/0264-9381/10/S/008.

[48] G. ’t Hooft. “The evolution of gravitating point particles in 2+1 dimensions”. In: *Class. Quantum Gravity* 10.5 (May 1993), pp. 1023–1038. DOI: 10.1088/0264-9381/10/5/019.

[49] C. Teitelboim. “How commutators of constraints reflect the spacetime structure”. In: *Ann. Phys. (N. Y).* 79.2 (Aug. 1973), pp. 542–557. DOI: 10.1016/0003-4916(73)90096-1.

[50] T. Thiemann. *Modern Canonical General Relativity.* Cambridge University Press, 2007.

[51] T. Thiemann. “Quantum spin dynamics (QSD): IV. Euclidean quantum gravity as a model to test Lorentzian quantum gravity”. In: *Class. Quantum Gravity* 15.5 (May 1998), pp. 1249–1280. DOI: 10.1088/0264-9381/15/5/011.

[52] H. Waelbroeck and J. A. Zapata. “(2 + 1) covariant lattice theory and ’t Hooft’s formulation”. In: *Class. Quantum Gravity* 13.7 (July 1996), pp. 1761–1768. DOI: 10.1088/0264-9381/13/7/009.

[53] H. Waelbroeck and F. Zertuche. “Homotopy invariants and time evolution in (2+1)-dimensional gravity”. In: *Phys. Rev. D* 50.8 (Oct. 1994), pp. 4966–4981. DOI: 10.1103/PhysRevD.50.4966.

[54] M. Welling. “The torus universe in the polygon approach to (2 + 1)-dimensional gravity”. In: *Class. Quantum Gravity* 14.4 (Apr. 1997), pp. 929–943. DOI: 10.1088/0264-9381/14/4/010.

Appendix A

The Relation Between SO(2,1) and SU(1,1)

In this thesis we mainly deal with the group, $\mathrm{SO}(2,1)$, which describes the local symmetry of Minkowski space minus the translational parts. Gravity can then be expressed as a local $\mathrm{SO}(2,1)$ gauge theory, where very point of spacetime has an inertial observer. In order to be able to add fermions to our theory however, we have to move from $\mathrm{SO}(2,1)$ to $\mathrm{spin}(\mathrm{SO}(2,1))$ which is $\mathrm{SU}(1,1)$. In this section we will review the relation and between the groups and some of the conventions we use.

The relation between $\mathrm{SO}(2,1)$ and $\mathrm{SU}(1,1)$ is that the standard representation of $\mathrm{SO}(2,1)$ equals the adjoint representation of $\mathrm{SU}(1,1)$, namely we have

$$gv^A\tau_A g^{-1} = \tau_A R^A{}_B(g)v^B. \tag{A.1}$$

Note that if we replace $g \to -g$ we get the same representation. The expresses the fact that $P\mathrm{SU}(1,1) \simeq \mathrm{SO}(2,1)$, or in other words, that $\mathrm{SU}(1,1)$ is the double cover of $\mathrm{SO}(2,1)$.

The elements τ_A are the (pseudo)-Pauli matrices or generators of the Lie algebra of $\mathrm{SU}(1,1)$ which we will assume satisfies the following algebra,

$$\tau_A\tau_B = \frac{1}{4}\eta_{AB} + \frac{1}{2}\epsilon_{AB}{}^C\tau_C, \tag{A.2}$$

An explicit representation of this algebra is given by

$$\tau^t = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \tau^x = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau^y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.3}$$

Using this it is easy to show that the Lie algebra $\mathfrak{su}(1,1)$ and $\mathfrak{so}(2,1)$ are isomorphic. Explicitly one can show that $2\tau^A \to (\Sigma_A)_{BC} = \epsilon_{ABC}$ is a Lie algebra isomorphism. Both Lie algebras contain a canonical metric, or the killing metric, which we will take to be given by $\langle A, B\rangle = 2\,\mathrm{tr}(AB)$.

These relations allow us to construct a further isomorphism $\mathfrak{su}(1,1) \simeq \mathbb{R}^{2,1(3)} \simeq \mathfrak{so}(2,1)$. For $\mathfrak{so}(2,1)$ the vector space isomorphism to $\mathbb{R}^{2,1}$ is given by $\xi^I = \frac{1}{2}\epsilon^{IJ}{}_K\xi^K{}_J$, or just the Levi-Civita symbol or an internal Hodge dual. Then the isomorphism to $\mathfrak{su}(1,1)$ is just given by $\xi = \xi^I\tau_I$. This isomorphism is in fact an isometry as well because the killing metric $2\,\mathrm{tr}(AB)$, $A, B \in \mathfrak{su}(1,1)$, since we have

$$2\,\mathrm{tr}(AB) = A^I B^J 2\,\mathrm{tr}(\tau_I\tau_J) = A^I B^J 2\,\mathrm{tr}(1/4\eta_{IJ}1) = A^I\eta_{IJ}B^J. \tag{A.4}$$

Using this representation one can integrate the Lie algebra using the exponential map $\exp : \mathfrak{su}(1,1) \to \mathrm{SU}(1,1)$ to get

$$\exp\Big(n^I\tau_I\Big) = \begin{cases} \cosh\frac{|n|}{2}1 + 2\sinh\frac{|n|}{2}\hat{n}^I\tau_I & \text{for } n^I n_I > 0 \\ \cos\frac{|n|}{2}1 + 2\sin\frac{|n|}{2}\hat{n}^I\tau_I & \text{for } n^I n_I < 0 \\ 1 + n^I\tau_I & \text{for } n^I n_I = 0 \end{cases} \tag{A.5}$$

For $\mathrm{SU}(1,1)$ note that the exponential map is not injective or even surjective. However, if we mod out by $\mathbb{Z}^2$ and form the projective group $P\mathrm{SU}(1,1) \simeq \mathrm{SO}(2,1)$ the exponential map is indeed surjective and so we can indeed use this as a parameterization of the group, since we are concerned with the adjoint representation of $\mathrm{SU}(1,1)$. To recover the matrix elements of $\mathrm{SO}(2,1)$ we simply use (A.1) to get

$$\langle \mathrm{Ad}(g)\tau_A, \tau_C\rangle^{\mathrm{Min}} = 2\,\mathrm{tr}\Big(g\tau_A g^{-1}\tau_C\Big) = \eta_{CB}R^B{}_A(g) \tag{A.6}$$

Applying these formulas to our parameterizations (A.5) we get that

$$R^A{}_B\Big(\exp\Big(n^I\tau_I\Big)\Big) = \begin{cases} \cosh|n|\delta^A{}_B + (1-\cosh|n|)\hat{n}^A\hat{n}_B - \sinh|n|\epsilon^A{}_{BC}\hat{n}^C & \text{for } n^I n_I > 0, \\ \cos|n|\delta^A{}_B + (1+\cos|n|)\hat{n}^A\hat{n}_B - \sin|n|\epsilon^A{}_{BC}\hat{n}^C & \text{for } n^I n_I < 0, \\ \delta^A{}_B - \epsilon^A{}_{BC}n^C & \text{for } n^I n_I = 0, \end{cases} \tag{A.7}$$

Therefore, physically this parameterization describes a rotation or boost by an amount $|n|$ around the axis $\hat{n}$. The nature of the transformation depends, critically, on the nature of n and is important when analyzing the torus universe.

Appendix B

Symplectic Geometry and Constrained systems

The mathematics of phase space is given by symplectic geometry. Therefore, we will provide a brief review of the symplectic geometry, needed in the thesis. Furthermore, since symmetries in field theories mean that constraints are present, we will also provide a brief introduction into Dirac's constraint analysis [10].

B.1 Symplectic manifolds

Phase space is given by a **symplectic manifold** (P, Ω) which is a manifold carrying a non-degenerate, closed two form, called the symplectic form Ω[1]. The wonderful aspect of symplectic geometry is that they provide a way to generate flows of observables H, through **Hamiltonian vector fields**, X_H, which are given by

$$\Omega(X_H, \cdot) = -\mathrm{d}H. \tag{B.1}$$

For example, if we take H to be the Hamiltonian system, then the corresponding Hamiltonian vector field generate time evolution. Symplectic manifolds also have a wonderful property- locally they are the same. That is, Darboux's theorem [4] states that there exist coordinates (q, p) (called Darboux or canonical) such that the symplectic form takes the form

$$\Omega = \mathrm{d}q \wedge \mathrm{d}p. \tag{B.2}$$

[1]If non-degeneracy is relaxed you end up with a pre-symplectic form, or in a different viewpoint a Poisson manifold.

This starkly contrasts the metric in GR, where they differ locally.

Symplectic forms also allow the construction of **Poisson brackets**, which are defined by

$$\{f, h\} = \Omega(X_f, X_h) = -\mathrm{d}f(X_h). \tag{B.3}$$

In canonical coordinates this is given by the familiar expression

$$\{f, h\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}. \tag{B.4}$$

This bracket is skew-symmetric and satisfies the Jacobi identity due to the fact that Ω is closed. In fact due to the fact that the symplectic form is non-degenerate, we can invert it and define a Poisson bivector $\Pi \in \wedge^2 TP$. In terms of Π, the Poisson bracket is given by

$$\{f, g\} = \Pi(\mathrm{d}f, \mathrm{d}h), \tag{B.5}$$

and Hamiltonian vector field can be expressed as

$$X_h = \Pi(\cdot, \mathrm{d}h). \tag{B.6}$$

In this thesis we only deal with a special symplectic manifold the cotangent bundle of configuration space. In physics we often start in a Lagrangian perspective. Namely, physics is based on the configuration manifold M. Then by performing a Legendre transform, we move to the Hamiltonian description. Mathematically the Legendre transform is a map from the tangent bundle TM of configuration space to the cotangent bundle T^*M. The benefit of doing this is that the cotangent bundle carries a canonical symplectic potential (often called Liouville form) Θ. In terms of canonical coordinates (q, p), Θ is given by $\Theta = p\mathrm{d}q$. The symplectic form is then just given by

$$\Omega = -\mathrm{d}\Theta. \tag{B.7}$$

Therefore, for phases spaces that or given by cotangent bundles, the symplectic form is exact. In this thesis, we will only deal phase space's that are this form.

B.1.1 Momentum Maps

Momentum maps encode symmetries of phase space. Namely, they are the phase space incarnation of Noether currents. Furthermore, the constraints of 2+1 gravity are themselves momentum maps, and generate $\mathrm{SO}(2, 1)$ transforms and translations. Therefore, we will give a brief introduction to the definition of a momentum map. For a more detailed presentation see [4, 32].

First we need to define what is a symmetry for a symplectic manifold. A symmetry for a symplectic manifold is a symplectomorphism, i.e. a map that preserves the symplectic form Ω. If we have a Hamiltonian, we also require that the symplectomorphism leave the Hamiltonian invariant. Symmetries of physical systems are usually captured by group actions. We will denote left group actions using the symbols $\triangleright : G \times P \to P$, and it will be written by $g \triangleright f$, where $g \in G$, the group and f is a function on the symplectic manifold P. A group action is then called **symplectic** if $g \triangleright \cdot : P \to P$ is a symplectomorphism for all $g \in G$.

The wonderful thing about Lie group actions, is that their local nature can be captured by their generators or Lie algebra $\mathfrak{g}$ of the Lie group G. In fact we can define the vector field $X \in \mathfrak{g}$ associated to the group action by

$$\delta_X(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp(tX) \triangleright f). \tag{B.8}$$

Abelian momentum maps of a group action G on P is then given by a Lie algebra valued function $J : P \to \mathfrak{g}$, such that

$$\mathrm{d}J(X) = -\omega(\delta_X, \cdot), \tag{B.9}$$

where $J(X) : P \to \mathbb{R}$ and is given by $J_p(X)$, where $X \in \mathfrak{g}$. In terms of Poisson brackets this is given by

$$\delta_X = \{\cdot, J(X)\}. \tag{B.10}$$

Therefore, momentum maps generate the symmetries of the system. Namely, their Hamiltonian vector field generate symmetries.

One restriction in this definition is that it assumes that the generator of the symmetry is abelian. More mathematically, we have assumed that the momentum map takes values in $\mathfrak{g}$ not the Lie group itself. In chapter 3, instead have that flatness constraint takes values in the group G. However, with analogy to the continuum case, we expect that the constraint should generate translations which are the symmetries of gravity. Therefore, we need a generalization of momentum maps to the non-abelian case.

For this consider $\mathfrak{g}$ and $\mathfrak{g}^*$, the Lie algebra of G and its dual the Lie algebra of the group G^*. In our case G^* will just be the group of translations which is abelian[2]. Let's denote e_A, e^B as the bases of $\mathfrak{g}, \mathfrak{g}^*$ respectively. In this case the **non-abelian momentum map** $\mathcal{M}$, will be given by [6],

$$\begin{aligned} &\mathcal{M} : P \to G^*, \\ &\qquad \xi \to g_*(\xi) = e^{-Q(\xi)}, \text{ such that } \delta_X f = \left\langle g_*^{-1}\{f, g_*\}, \xi \right\rangle, \end{aligned} \tag{B.11}$$

where $\langle \cdot, \cdot \rangle$ is the natural bilinear form between $\mathfrak{g}^*$ and $\mathfrak{g}$. $Q(\xi)$ is known as the charge generating the group action.

In this thesis the only phase space we deal with is $T * \mathrm{SU}(1,1) \simeq \mathrm{SU}(1,1) \times \mathfrak{su}(1,1)$. In this case $\mathrm{SU}(1,1)^* \simeq \mathfrak{su}(1,1)^* \simeq \mathbb{R}^3$. Furthermore, we used the killing form to identify

[2] In general this need not be the case. This is related to a mathematical framework called the *Heisenberg double* which generalizes the phase space T^*G.

$\mathfrak{g}$ and $\mathfrak{g}^*$. Therefore, the bases of $\mathfrak{g}$ are τ_A and $\mathfrak{g}^* = \tau^A$, which are defined in A. The two momentum maps of we are interested in are given by

$$\begin{aligned} \mathcal{J} : T^*G &\to \mathfrak{g}^* \simeq \mathbb{R}^3, & \mathcal{G} : T^*G &\to G \\ (G, X) &\to g_*(X) = e^{X^A \tau_A}, & (X, g) &\to g. \end{aligned} \tag{B.12}$$

The symmetry transformations they generate are given by,

$$\begin{aligned} \delta^A_{\mathcal{J}} X^B &= \left\langle g_*^{-1}\{X^B, g_*\}, \tau^A \right\rangle = \epsilon^{AB}{}_C X^C, & \delta^A_{\mathcal{J}} g &= \left\langle g_*^{-1}\{g, g_*\}, \tau^A \right\rangle = -g\tau^A \\ \delta^\beta_{\mathcal{G}} X^A &= \left\langle g^{-1}\{X^A, g\}, \beta \right\rangle = \mathrm{Ad}(g)\beta, & \delta^\beta_{\mathcal{G}} g &= \left\langle g^{-1}\{g, g\}, \beta \right\rangle = 0. \end{aligned} \tag{B.13}$$

Furthermore, since $\mathcal{J}$ is an abelian momentum map, we can see how the definition of a non-abelian momentum maps reduces to abelian momentum map definition.

B.2 Dirac Bergman constraint analysis

Constraints in Hamiltonian systems arise when the Legendre transform from the Lagrangian theory is singular. For example, lets assume our theory has a Lagrangian L; the conjugate momenta are given by

$$p_i = \frac{\partial L}{\partial \dot{q}}. \tag{B.14}$$

In order for this transformation to be well defined, one has to assume that this relation is invertible. However, e.g. in gravity, this is not may not be true in physical theories. In fact all gauge theories necessarily will be singular. The Hessian being singular means that there are relations among the phase space variables q, p. These relations are called **primary constraints** $\phi_m(q, p) = 0$.

At this point one may think that constructing a Hamiltonian may be impossible, however this is not the case and in fact we can still recover the canonical Hamiltonian given by

$$H_{\mathrm{C}} = p_m \dot{q}^m - L. \tag{B.15}$$

The reason this, is that H doesn't explicitly depend on $\dot{q}$ even in the singular case. This follows from

$$\begin{aligned} \delta H_C &= p_m \delta \dot{q}^m + \delta p_m \dot{q}^m - \frac{\partial L}{\partial \dot{q}^m} \delta \dot{q}^m - \frac{\partial L}{\partial q^m} \delta q^m \\ &= \delta p_m \dot{q}_m - \frac{\partial L}{\partial q^m} \delta q^m, \end{aligned}$$

which shows that H_C depends only on q and p. However, the Hamiltonian is not uniquely described in terms of p and q, since the δp_m are not all independent. Therefore, the canonical Hamiltonian is only defined on the constraint surface $\phi_m(q, p) = 0$ which we will denote by C. Recall that above we said that the Hamiltonian is not unique in phase space. This is because we have lots of different choices for how to extend the Hamiltonian off the constraint surface. One extension is given by the extended Hamiltonian H_E,

$$H_E = H_C + u^m \phi_m, \tag{B.16}$$

where the u^m are Lagrange multipliers. The u^m can be viewed as coordinates filling the part of the extended phase space not covered by (q, p). In fact, the Legendre transform from the space of $(q, \dot{q})$ to the surface of $\phi_m(q, p) = 0$ surface of the extended phase space (q, p, u) is invertible. Therefore, the natural Hamiltonian in this case is the extended one. Of course how to pick the u^m functions is ambiguous.

One thing we have not yet mentioned is whether the constraints ϕ_m are conserved in time? For our theory to be consistent we will require this to be true. Therefore, we need

$$\dot{\phi}_m = \{\phi_m, H_C\} - u^n\{\phi_n, \phi_m\} \approx 0. \tag{B.17}$$

Here we have used the notion of weak equivalence $\approx$ which means that the two terms are equivalent up to terms proportional to the constraints of the theory. This leads to two possible outcomes. Either we end up with an expression the involves the u^n, which at least partially determines what they must be, or we end up with a relation involving q and p. In the latter case we get **secondary constraints** that must be satisfied. This process of finding these extra constraints continues until no more constraints are found. In the end, the split between secondary and primary constraints isn't important so we will now just combine the primary and secondary constraints together.

More important than the distinction between primary and secondary constraints, are first and second class constraints. A phase space function F is said to be first class, if it Poisson commutes with all of the constraints, i.e. $\{F, \phi_m\} \approx 0$ for all m. If a function does not commute with all of the constraints then it is second class. It turns out that is almost all cases of physical interest [26] all first class constraints arise from gauge freedom in the theory. Moreover, first constraints generate symmetries, and are momentum maps. In fact, what we found in 2+1 first order gravity, where the flatness and torsion constraints generated translations and rotations. A potential problem with first class constraints are present is that time evolution is not unique longer unique. To see this note that the evolution of the constraints ϕ_m is given by,

$$\dot{\phi}_m = \{\phi_m, H_C\} + u^n\{\phi_m, \phi_n\} \approx 0. \tag{B.18}$$

As we have already added all of the primary and secondary constraints to ϕ_m, we know that we only have relations among the u^m left. Therefore, the natural question is whether we can find the Lagrange multipliers u^m. In the case of purely second class constraints the answer is yes. In this case the matrix $\Delta_{mn} = \{\phi_m, \phi_n\}$ is invertible and therefore we can find u^m. Time evolution of the system is then completely specified and we have a consistent Hamiltonian description. However, if even one constraint is first class, this is no longer true.

In this case the matrix Δ_{mn} is no longer invertible Physically however this is expected since they respond to gauge freedom. Therefore, different gauges will lead to seemingly different evolution in time, however physics or the observables will not change. The question then becomes how do we deal with the unknown functions u^m.

One way forward is to introduce the notion of an observable. A function $\mathcal{O}$ defines an **observable** if it, at least weakly, Poisson commutes will all first class observables. In this case the time evolution of an observable is defined unambiguously on the constraint surface since

$$\dot{\mathcal{O}} = \{\mathcal{O}, H_C\} + u^m\{\mathcal{O}, \phi_m\} \approx \{\mathcal{O}, H_C\} + u^n\{\mathcal{O}, \phi_n\}, \tag{B.19}$$

where ϕ_n are the second class constraints.

At this point we bring to the readers attention that in general covariant systems, like gravity, we often have that the Hamiltonian itself is a constraint, as an example below will show. In this thesis we always deal with generally covariant systems, we will have that the canonical Hamiltonian H_C is constrained to vanish. Note that something peculiar happens for observables in this case. As the canonical Hamiltonian is now a constraint we necessarily have that an observable must commute with it. This seems to suggest that observables in a generally covariant theory are "frozen" in time. This is the root of the problem of time in quantum gravity. For a expose of this issue and how to recover a notion dynamical evolution see [11, 42, 43]. A different problem with observables, is that they are extremely difficult to find. In fact, to find them one has to basically solve the theory. Luckily there is another way forward.

From above we know that if we have second class constraints we have well defined time evolution. Therefore, one way to deal with first class constraints is to introduce extra constraints, χ that make the first class constraints second class. This is known as **gauge fixing**. Geometrically this has a nice interpretation. The issue with first class constraints is are that the gauge orbits

$$\delta^\alpha_\phi F = \alpha\{F, \phi\}, \tag{B.20}$$

prevents time evolution from being unique. What gauge fixing does is pick a unique representative from the orbits. Therefore, we need our gauge fix χ to satisfy a couple of requirements. First, it must completely fix the gauge, i.e. $\{\chi, \phi\} \neq 0$ if ϕ is a first class constraint. This is a local condition. However, the gauge fix must also satisfy a global condition. The gauge fix must be possible. Namely, by applying gauge transformations we can satisfy the constraint. If either if these aren't possible, then that gauge choice fails. As a result, at least as many gauge fixes as first class constraints are required. Once we have done this, we have a representation of the reduced phase space. However, in this case the Poisson bracket must be modified. In order to preserve the constraint surface, **Dirac brackets** [10] need to be introduced. Dirac brackets are given by

$$\{f, g\}_D = \{f, g\} - \{f, \chi_A\}(\Delta^{AB})^{-1}\{\chi_B, g\}, \tag{B.21}$$

where χ^A are the second class constraints and $\Delta^{AB} = \{\chi^A, \chi^B\}$. This gives the Poisson bracket structure on the reduced phase space.

B.2.1 An example of gauge fixing: the relativistic particle

In order to see how Hamiltonian analysis of general covariant systems works, we will consider the example of a free relativistic particle. The Lagrangian of the system is given by

$$S = -m \int \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \mathrm{d}\tau, \tag{B.22}$$

where τ is some affine parameter. The Euler Lagrange equations are then

$$m\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{x}^\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\mu}}\right) = 0. \tag{B.23}$$

This action however, has contains a continuous symmetry. If we change our affine parameter from $\tau \to \bar{\tau}(\tau)$ we see that $\dot{x}^\mu$ changes as

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\bar{\tau}}\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}\tau},$$

and $\mathrm{d}\tau = \mathrm{d}\tau/\mathrm{d}\bar{\tau}\ \mathrm{d}\bar{\tau}$ which implies the action is invariant. From the discussion above this signifies that there will be a constraints in phase space.

Moving to the Hamiltonian picture, the conjugate momenta are given by

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}^\mu}{(-\dot{x}^\mu \dot{x}_\mu)^{1/2}}. \tag{B.24}$$

If we try to invert this relation, we would find that it isn't possible. One easy way to see this is that the momentum p^μ obeys $p^\mu p_\mu = -m^2$. This forms a primary constraint for the system which we will call $C = p^2 + m^2$. If we now attempt to find the canonical Hamiltonian for the system we get

$$H = p_\mu \dot{x}^\mu - \mathcal{L} = \frac{(-\dot{x}^\mu \dot{x}_\mu)^{1/2}}{m} p_\mu p^\mu + (-\dot{x}^\mu \dot{x}_\mu)^{1/2} = \frac{(-\dot{x}^\mu \dot{x}_\mu)^{1/2}}{m}\Big(p_\mu p^\mu + m^2\Big) = NC, \tag{B.25}$$

where $N = m^{-1}(-\dot{x}^\mu \dot{x}_\mu)^{1/2}$. Therefore, we see that the canonical Hamiltonian is proportional to the constraint C and so we have a totally constrained system as expected. Furthermore, the constraints are first class trivially.

The evolution of the system is then given by

$$\dot{x}^\mu = \{x^\mu, H\} \approx N\{x^\mu, C\} = 2Np^\mu \tag{B.26}$$

$$\dot{p}^\mu = \{p^\mu, H\} \approx 0, \tag{B.27}$$

which are the Euler Lagrange equations, but with an arbitrary function N, sometimes called the lapse function. Note, that these "evolution" equations are just the gauge orbits of the system.

Therefore, we don't yet have a picture of reduced phase space of the theory. To find the reduced phase space we will perform a gauge fixing. Here we will take our gauge choice to be $\Omega = x^0 - \tau$, which physically means choosing a time for the particle. To see if this was locally a good gauge choice, we compute the Poisson bracket of Ω with the constraint C. In this case we get

$$\{\Omega, C\} = \{x^0, p_\mu p^\mu\} = 2p^0 \neq 0. \tag{B.28}$$

Therefore, we have a good gauge choice, implying we can now specify the lapse function N, by requiring that Ω is preserved in time

$$0 = \frac{\mathrm{d}\Omega}{\mathrm{d}\tau} = \frac{\partial \Omega}{\partial \tau} + \{\Omega, H\} = -1 + 2Np^0. \tag{B.29}$$

which implies the lapse function is $N = 1/2p^0$. The equations of motion then become $\dot{x}^\mu = p^\mu/p_0$. Finally to find the Poisson structure on the reduced phase space after gauge fixing, we need to compute the Dirac brackets $\{\cdot,\cdot\}_D$. First, we find

$$\Delta^{AB} = \{\chi^A, \chi^B\} = 2p^0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{B.30}$$

where $\chi^A = (\Omega, C)$. Therefore, the Dirac brackets are

$$\{x^\mu, x^\nu\}_D = 0 \qquad \{p_\mu, p_\nu\}_D = 0 \qquad \{x^\mu, p_\nu\}_D = \delta^\mu_\nu - \frac{p^\mu}{p^0}\delta^0_\nu. \tag{B.31}$$

That last bracket is special. First, for purely spatial components we have that $\{x^i, p_j\}_D = \{x^i, p_j\}$, so the spatial Poisson brackets are unchanged. However, notice that

$$\{x^i, p_0\}_D = -\frac{p^i}{p^0} = \frac{p^i}{p_0} = \dot{x}^i, \tag{B.32}$$

i.e. our first component of the conjugate momenta has become the Hamiltonian. This was to be expected since the gauge choice forced x^0 as the time variable, and p_0 generates translations in x^0.

We can eliminate p^0 altogether by explicitly solving the constraint C. From C, we require $p_0 = \sqrt{p_i p^i + m^2}$. Therefore, in this instance the reduced Hamiltonian becomes

$$H_{\text{red}} = \sqrt{p_i p^i + m^2}, \tag{B.33}$$

which is just the energy of the particle. In fact if in the Lagrangian picture we had taken $\tau = x^0$, we would have arrived at this reduced Hamiltonian from the beginning.