Discriminators of Integer Sequences

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The discriminator of an integer sequence $\mathbf{s} = (s(n))_{n\geq 0}$, first introduced by Arnold, Benkoski and McCabe in 1985, is the function $D_s(n)$ that maps the integer $n \geq 1$ to the smallest positive integer m such that the first n terms of $\mathbf{s}$ are pairwise incongruent modulo m. In this thesis, we provide a basic overview of discriminators, examining the background literature on the topic and presenting some general properties of discriminators.

We also venture into various computational aspects relating to discriminators, such as providing algorithms to compute the discriminator, and establishing an upper bound on the discriminator growth rate. We provide a complete characterization of sequences whose discriminators are themselves, and also explore the problem of determining whether a given sequence is a discriminator of some other sequence with some partial results and algorithms.

We briefly discuss some k-regular sequences, characterizing the discriminators for the evil and odious numbers, and show that k-regular sequences do not necessarily have k-regular discriminators. We introduce the concept of shift-invariant discriminators, i.e. discriminators that remain the same even if the original sequence is shifted, and present a class of exponential sequences with this property. Finally, we provide a complete characterization of quadratic sequences with discriminator $p^{\lceil \log_p n \rceil}$ for primes $p \neq 3$, and provide some partial results for the case of $p = 3$.

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Dedication

This is dedicated to my parents and siblings, who have continuously encouraged me in my mathematical passions, and providing heavy emotional support in my pursuits.

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Chapter 1

Introduction

This thesis focuses primarily on the topic of discriminators of integer sequences. The discriminator for a sequence of integers is the function that maps each positive integer n to the smallest positive integer m such that the first n elements of the sequence are distinct modulo m. This topic was first introduced by Arnold, Benkoski, and McCabe [2] in 1985.

Their motivation was a problem in computer simulation about quickly determining the square roots of randomly generated positive perfect squares below a given upper bound. The idea was to pre-calculate all the squares within the range only once beforehand and then store them on an array where the entries indexed by perfect squares contain the corresponding square roots. This allows for quick lookup of the square roots (significantly faster than a square-root algorithm), but requires an array size of n^2 to store n square roots. A more space-efficient alternative is to store the square roots at indices corresponding to their square modulo a fixed constant k. This requires an array of size k while still performing fast lookups, so the only remaining question was how small k could be to ensure that each square root is stored in a distinct array entry. This translates directly into the problem of computing the discriminator for the sequence of positive perfect squares.

This idea could be applied to any arbitrary sequence, i.e., construct an array such that, given an arbitrary integer from the sequence, a lookup after a modulo function returns the index of that integer in the sequence. Regardless of the application, however, it is quite an interesting problem in number theory to compute the discriminator of a given sequence. Since its inception, several authors have published results relating to the discriminators of various different sequences. Most of the work so far was focused on sequences of powers and sequences of polynomials, along with a handful of results relating to exponential sequences.

This thesis provides a basic overview on the topic of discriminators while proving several new results using elementary number theory techniques. Chapter 1 contains the formal definitions and basic properties of discriminators, along with a review of previous work and a short guide on approaching the problem of computing the discriminator of some sequences. This is followed by a discussion on various different computational aspects of discriminators in Chapter 2.

The next three chapters are focused on the discriminators of different types of sequences. Chapter 3 is about the discriminators of k-regular sequences, such as the sequences of odious and evil numbers, respectively. Following that, Chapter 4 deals with exponential sequences, particularly a class of exponential sequences whose discriminators are shift-invariant, i.e., sequences whose discriminators are unchanged if the sequences are shifted by a constant. Finally, Chapter 5 considers the sequences of polynomials, most notably the quadratic sequences with discriminator $p^{\lceil \log_p n \rceil}$ for a prime number p.

This thesis is based, in part, on a published paper [10], a submitted paper [11], and an unpublished manuscript [9].

1.1 Discriminators

Let S be a set of integers. For any integer m such that the numbers in S are pairwise incongruent modulo m, we say that m *discriminates* S. In other words, computing each of the elements in S modulo m generates a set of the same cardinality as S.

Now let $\mathbf{s}$ be a sequence of distinct integers. The *discriminator* of the sequence $\mathbf{s}$ is the function that maps each integer $n \geq 1$ to the smallest positive integer m such that m discriminates the set of the first n elements of $\mathbf{s}$. The discriminator function is denoted by $D_{\mathbf{s}}(n)$ for sequence $\mathbf{s}$ and argument n, while the discriminators themselves can be represented as another sequence $(D_{\mathbf{s}}(n))_{n\geq 1}$. Note that while the definition for the discriminator is not dependent on how the sequence is indexed, the corresponding discriminator sequence generally begins with $D_{\mathbf{s}}(1)$, with 1 being the starting index unless $D_{\mathbf{s}}(0)$ is separately defined.

Let us consider the example of the sequence of positive squares, $(\mathrm{sq}(n))_{n\geq 1} = (n^2)_{n\geq 0} = 1, 4, 9, \ldots$ and compute the first few terms of its discriminator sequence:

- For $n = 1$, we consider the set with the first term only, $\{1\}$. As this is a singleton, it is discriminated by every positive integer, the smallest of which is 1. So $D_{\mathrm{sq}}(1) = 1$.
- For $n = 2$, the set with the first two terms is $\{1, 4\}$. Since $1 \equiv 4 \equiv 0 \pmod 1$, the number 1 does not discriminate $\{1, 4\}$. However, $1 \not\equiv 4 \pmod 2$, so the number 2 discriminates the set $\{1, 4\}$. There are other numbers which discriminate $\{1, 4\}$, such as the number 5, since $1 \not\equiv 4 \pmod 5$, but the number 2 is the smallest such positive integer amongst them, so $D_{\mathrm{sq}}(2) = 2$.
- For $n = 3$, the set with the first three terms is $\{1, 4, 9\}$. We observe that the numbers from 2 to 5 inclusive do not discriminate $\{1, 4, 9\}$ since

$$\begin{aligned} 9 &\equiv 1 \pmod 2, \\ 4 &\equiv 1 \pmod 3, \\ 9 &\equiv 1 \pmod 4, \\ 9 &\equiv 4 \pmod 5. \end{aligned}$$

But the number 6 discriminates $\{1, 4, 9\}$, and so, $D_{\text{sq}}(3) = 6$.

The sequence $(D_{\text{sq}}(n))_{n\geq 1} = 1, 2, 6, \ldots$ represents the discriminator sequence of the sequence of positive squares.

In this thesis, we use the following notation. For integers a and b, we denote $a|b$ to indicate that a divides b. Likewise, $a \nmid b$ denotes that a does not divide b. And for a set S, we let $|S|$ denote the cardinality of S, i.e., the number of elements in S.

1.2 Previous Work

The discriminator was first introduced by Arnold, Benkoski, and McCabe [2], who computed the discriminator for the sequence, $1, 4, 9, \ldots$ of positive squares. The motivation was based on a problem in computer simulation based on quickly determining the square roots of randomly generated perfect squares within a bounded range.

For $n > 4$, they proved that the discriminator is the smallest number greater or equal to $2n$ which is either a prime or twice or prime. The approach was focused on numbers in the range between $2n$ and $4n$ inclusive, showing that numbers less than $2n$ cannot be the discriminator, and then analyzing various possible cases of numbers in this range to prove that only those that are primes or twice a prime fail to divide the difference between any two of the first n squares. The existence of a prime in this range was ensured by Bertrand's Postulate, thus proving the proposed characterization of the discriminator.

Shortly after the concept of discriminators was introduced with the sequence of squares, it was followed by various results on discriminators of sequences of higher powers. Schumer and Steinig [19] defined the function $D(j, n)$ to represent the discriminator of the first n terms of the sequence of j-th powers, for $j \geq 1$ and $n \geq 1$. They then considered the case of $j = 2^h$ for $h \geq 2$, i.e., the discriminators of sequences of the form $1^{2^h}, 2^{2^h}, 3^{2^h}, \ldots$, such as the sequence of 4th powers or the sequence of 8th powers. They showed that for $n \notin \{1, 2, 4, 8\}$, the discriminator of the first n elements is the smallest number greater or equal to $2n$ of the form p or $2p$, where p is a prime number that is equivalent to 3 modulo 4. This characterization resembled the discriminator of the squares and was proved by a similar approach, except with slightly more complicated cases. The additional constraint of $p \equiv 3 \pmod 4$ arose from the fact that if $m = p$ or $m = 2p$ while $p \equiv 1 \pmod 4$, then $p = a^2 + b^2$ for some positive integers a and b, which implies that $a^4 \equiv b^4 \pmod m$, thus disqualifying m from being a possible discriminator for these sequences.

Aside from powers of 2, Schumer [18] characterized the discriminators for the sequence of cubes and the sequence of sixth powers, i.e., for $j = 3$ and $j = 6$. For the sequence of cubes, $D(3, n)$ is the smallest number m that is greater or equal to n such that m is squarefree and m has no prime divisors congruent to 1 modulo 3. The proof followed the same idea as the previous papers, but with the analyzed numbers ranging from n to $4n/3$ instead. Schumer noted that $b^3 - a^3 = (b - a)(a^2 + ab + b^2)$ for all integers a and b, and showed that the second factor, (a^2+ab+b^2), can only represent integers of the form s^2n_0, where n_0 is squarefree and not divisible by any prime $p \equiv 2 \pmod 3$. This is why the discriminator is required to be squarefree with no prime divisors congruent to 1 modulo 3. A result on primes in arithmetic progressions showed that for all $n \geq 32$, there exists a prime $p \equiv 2 \pmod 3$ satisfying $n \leq p < 4n/3$, thus ensuring that there is always at least one discriminator candidate in all such intervals for sufficiently large n.

Schumer also characterized the value of $D(6,n)$ as being the smallest number greater than $2n$ of the form p or $2p$ for prime $p \equiv 2 \pmod 3$. The approach was similar to the previous papers on sequences of squares or powers of 2^h, by analyzing numbers from the range $2n$ to $4n$. For any pair of integers a and b, we have $b^6 - a^6 = (b-a)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$, which is a multiple of b^3-a^3, so the results from the discriminator of cubes proved to be useful for eliminating many possible candidates.

These results on the discriminators of powers were generalized by Bremser, Schumer and Washington [6], who characterized $D(j,n)$ for all n sufficiently large and for all $j \geq 2$. For even j, the values of $D(j,n)$ follow a similar pattern as the case of $j=2$, namely that for n sufficiently large:

$$D(j,n) = \min\{k | k \geq 2n, k = p \text{ or } 2p, p \text{ prime}, \gcd(\varphi(k), j) = 2\}, \tag{1.1}$$

where the constraint of $\gcd(\varphi(k), j) = 2$ is a generalization that is consistent with the previous results on $j = 2^h$ and $j = 6$. This characterization is guaranteed to hold as long as there is a prime p with $\gcd(p-1, j) = 1$ in the range $2n < p < 3n$. Likewise, if j is odd, then the value of $D(j,n)$ resembles the case of $j = 3$, with

$$D(j,n) = \min\{k | k \geq n, k \text{ squarefree}, \gcd(\varphi(k), j) = 1\}, \tag{1.2}$$

for n sufficiently large, as long as there is a prime p with $\gcd(p-1, j) = 1$ and $n \leq p < 4n/3$.

All of the results on discriminators discussed so far involved characterizing $D(j,n)$ based on the value of n while j is fixed. A paper by Moree and Roskam [14] takes a different approach; instead of fixing j, they fixed the value of n. The case of $1 \leq n \leq 3$ is easily characterized, so their focus was on $n \geq 4$. The paper contained various interesting results involving functions $E(n)$ and $e(n)$, and a sequence $(a_k)_{k\geq 1}$, which were linked to the discriminator function $D(j,n)$. The function $E(n)$ maps n to the smallest number k such that $n|\varphi(k)$, while the function $e(n)$ maps n to the maximum of the exponents in the canonical prime factorization of n. Finally, $(a_k)_{k\geq 1}$ is the sequence such that a_k is the lowest common multiple of the first k totients, i.e., $a_k = \operatorname{lcm}(\varphi(1), \varphi(2), \ldots, \varphi(k))$.

One of the notable results in this paper was that $D(a_{e(p^\alpha)-1}, n) = E(p^\alpha)$ for prime p and integer α satisfying a specific set of conditions. Relaxing the conditions led to a weaker but simpler result that $D(a_{2p}, n) = 2p+1$ for all primes p such that $p \geq n \geq 4$ and $2p+1$. Such primes are called *Sophie Germain* primes. The paper then proceeded to show that for infinitely many values of n, there are primes p such that $D(j,n) = p^e$ for some integers j and e, where $p > n/2$ was a necessary condition for such a result. A stricter set of sufficient conditions on p, involving Fermat quotients, was also presented in the paper. Note that from Eqs. (1.1) and (1.2), the discriminator is squarefree for n sufficiently large, so the cases of $D(j,n) = p^e$ for $e \geq 2$ arise before n is sufficiently large enough for those equations to be applicable.

The first venture into the discriminators of sequences other than powers was by Moree and Mullen [13]. They observed a clear connection between the characterization of $D(j,n)$ and results about permutation polynomials over the ring $\mathbb{Z}/k\mathbb{Z}$ of integers modulo k. In particular, they noted that for $j > 1$, the power X^j induces a permutation on $\mathbb{Z}/k\mathbb{Z}$ if and only if k is squarefree and $\gcd(j, \varphi(k)) = 1$, which leads to an upper bound on $D(j,n)$ that matches with the characterization for odd j. A similar observation was made for even j as well. These observations led to the interest in considering the discriminators of general cyclic polynomials, which are given by Dickson (Chebyshev) polynomials:

$$g_j(X,a) = \sum_{i=0}^{\lfloor j/2 \rfloor} \frac{j}{j-i} \binom{j-i}{i} (-a)^i X^{j-2i},$$

where the sequence of X^j represents the case of $a = 0$. Before analyzing the discriminator of Dickson polynomials, the authors first improved the scope of Eq.(1.2), showing that it is applicable for n large enough such that there exists a prime p with $\gcd(p-1, j) = 1$ and

$$n \leq p \leq 2n - 5/2 + (-1)^n/2.$$

This can be more simply relaxed to $n \leq p < 2n-1$, which is still an improvement from the previous stricter condition of $n \leq p < 4n/3$. This result led to a characterization of the discriminator of Dickson polynomials for odd j such that $3 \nmid j$. Specifically, for $k = \prod_{p_i|k} p_i^{e_i}$, they defined $\psi_a(k) = \varphi(k) \prod_{p|k, p\nmid a}(p+1)$, and denoted the a-part of k as being the product $\prod_{p_i|a, p_i|k} p_i^{e_i}$, in order to construct the following function:

$$G_a(j,n) = \min\{k \geq n \,|\, \gcd(j, \psi_a(k)) = 1 \text{ and the } a\text{-part of } k \text{ is squarefree}\}.$$

They then showed that if $G_a(j,n) \leq 2n - 5/2 + (-1)^n/2$ for odd $j > 1$ and $3 \nmid j$, then the discriminator of the first n terms of the Dickson polynomial $(g_j(n,a))_{n \geq 1}$ is $G_a(j,n)$. In particular, this characterization holds if either $n \leq 3570$, n is sufficiently large, or $6 \nmid a$. They further derived a characterization for the discriminator in the case of j even, but only if $j \equiv 2 \pmod{12}$ or $j \equiv 10 \pmod{12}$. They remarked that the existence of the discriminator is not even ensured for other even values of j, since the sequence may contain duplicates. The paper concluded with some results on the growth of the discriminator of Dickson polynomials and some open problems like the discriminators for the cases of Dickson polynomials that were not considered, or for other types of polynomials with similar properties.

A generalization of polynomials was considered by Moree [12] soon afterwards. It was here that the discriminator function was generalized as $D_f(n)$ mapping $n \geq 1$ to the discriminator of the first n terms of the sequence defined by $f(n)$. The paper focused on polynomials $f \in \mathbb{Z}[x]$ such that the discriminator is given by

$$D_f(n) = \min\{k \geq n | f \text{ permutes } \mathbb{Z}/k\mathbb{Z}\} \tag{1.3}$$

for sufficiently large n. The powers x^j for odd j and Dickson polynomials $g_j(x, a)$ for odd j and $3 \nmid j$ that were discussed in previous literature were special cases that permute $\mathbb{Z}/p\mathbb{Z}$ for infinitely many primes p. For these functions, there are thus infinitely many primes in $K_f = \{k \geq 1 | f \text{ permutes } \mathbb{Z}/k\mathbb{Z}\}$. For a function f, any integer $k \geq n$ in K_f is an upper bound for $D_f(n)$, so only the lower bound remains to be proven, which can be quite difficult. One of the useful parameters that proved to be relevant in establishing the lower bound is denoted by $\gamma(f) = \limsup_{i\to\infty} k_{i+1}/k_i$, where $k_1, k_2, \ldots$ are the consecutive elements of K_f. This parameter indicates how "close" the asymptotically large values of K_f are to each other.

Through various lemmas, Moree showed that for sufficiently large n, Eq. (1.3) is satisfied by all functions f for which K_f either contains infinitely many primes, or contains all of the powers of at least two primes. For both of these cases, $\gamma(f) = 1$. For functions in which K_f contains all powers of exactly one prime, $\gamma(f) > 1$. If $\gamma(f) \leq 3/2$, then Eq. (1.3) is still satisfied, but otherwise there are known counterexamples. Eq. (1.3) is clearly not applicable to functions for which K_f is finite (where $\gamma(f) = \infty$), so characterizations of the discriminator of such functions was noted to be an open problem. Moree also presented some results on the densities of these different types of functions based on their degree.

Some of these results on functions satisfying Eq. (1.3) had references to manuscripts by Zieve, which were published later to expand on those results. Zieve [22] elaborated on various proofs involving functions for which $\gamma(f) = 1$ and improved the results for functions in which K_f contains all powers of exactly one prime. In particular, Zieve improved the sufficient condition of $1 \leq \gamma(f) \leq 3/2$ to $1 \leq \gamma(f) \leq 2$, while showing that there are counterexamples for any higher values of $\gamma(f)$, while also improving some other parameters involved in the characterization. Another general result is that if n is even, or a sufficiently large odd number, then $D_f(n) \in K_f$ if there is an element in K_f that is only slightly greater than n.

Zieve also proposed some generalizations to the concept of discriminators. One of them was to set the starting value of the sequence, by defining $D_f(a, n)$ as being the smallest positive integer k such that the terms $f(a + 1), f(a + 2), \ldots, f(a + n)$ are distinct modulo k. For functions f in which Eq. (1.3) is satisfied for sufficiently large n, it is also the case that $D_f(n) = D_f(a, n)$ for all integers a and sufficiently large n.

Another generalization involved the discriminator of rational functions of the form $f(X) = g(X)/h(X)$ where $g, h \in \mathbb{Z}[X]$ are coprime, where the discriminator $D_f(n)$ was defined to be the least positive integer k such that $f(1), f(2), \ldots, f(n)$ are distinct modulo k, with the additional constraint that $h(a)$ is coprime to k for all $a \in \mathbb{Z}$. Zieve showed that all of the results in the paper were applicable to the general case of rational functions.

Quite a while later, Sun [21] started looking into discriminators while seeking functions that take only prime values. Several results were based on a type of function that is closely related to the discriminator. These functions map the integer $n \geq 1$ to the smallest number $m > 1$ such that the first n terms of a given sequence are pairwise incongruent modulo m. These functions differ from the discriminator in that they enforce $m > 1$ instead of $m \geq 1$, which is only relevant for the case of $n = 1$, and are therefore referred to as $(m > 1)$-discriminators.

Sun conjectured that the $(m > 1)$-discriminator of two sequences, $\binom{2k}{k}_{k\geq 1}$ and $(k!)_{k\geq 1}$, is a prime number for all values of $n \geq 1$, with the sole exception of $n = 5$ for the second sequence $(k!)_{k\geq 1}$. Other sources verified the conjecture for $n \leq 5000$ for the first sequence, and $n \leq 10000$ for the second sequence. This conjecture also appears to be the first case in the literature in which discriminators were considered for sequences that grow exponentially.

Sun then proved various results relating to discriminators:

1. For $n \geq 1$, the $(m > 1)$-discriminator of the first n terms of $(2k(k-1))_{k\geq 1}$ is the least prime greater than $2n - 2$. In particular, the set of $(m > 1)$-discriminators is exactly the set of prime numbers.

2. For $n \geq 1$, the $(m > 1)$-discriminator of the first n terms of $(k(k-1))_{k\geq 1}$ is the least integer greater or equal to $2n - 1$ which is either a prime or a positive power of 2.

3. For $n \geq 1$, the discriminator $(m \geq 1)$ of the first n terms of $(k(k-1)/2)_{k\geq 1}$ (the triangular numbers) is given by $2^{\lceil \log_2 n \rceil}$.

4. For $d \in \{2, 3\}$ and $n \geq 1$, the discriminator of the first n terms of $(k(dk-1))_{k\geq 1}$ is given by $d^{\lceil \log_d n \rceil}$.

5. For $n \geq 4$, the discriminator of the first n terms of $(18k(2k-1))_{k\geq 1}$ is the least prime $p > 3n$ such that $p \equiv 1 \pmod 3$.

6. For $d \in \{4, 6, 12\}$ and $n \geq 3$, the $(m > 1)$-discriminator of the first n terms of $((2k-1)^d)_{k\geq 1}$ is the least prime $p \geq 2n - 1$ such that $p \equiv -1 \pmod d$.

7. For $n \geq 1$ and q an odd prime, the $(m > 1)$-discriminator of the first n terms of $(k^q(k-1)^q)_{k\geq 1}$ is the least prime $p \geq 2n - 1$ such that $p \not\equiv 1 \pmod q$.

8. Define $s_n = \sum_{k=1}^{n} (-1)^{n-k} p_k$ for all $n \geq 1$, where p_k denotes the k-th prime. Then, for $n \geq 1$, the first n terms of $(2s_k^2)_{k\geq 1}$ are pairwise incongruent modulo p_{n+1}.

The last result here gives an upper bound for the discriminator of $(2s_k^2)_{k\geq 1}$, which was also conjectured to be the lower bound as well, for $n \neq 1, 2, 4, 9$. This conjecture was verified for all $n \leq 10^5$. Sun also presented several other conjectures, relating to functions based on consecutive primes, and other specially defined functions, whose ($m > 1$)-discriminators might take only prime values.

More results on the discriminators of exponential sequences were presented by Moree and Zumalacárregui [15], who computed the discriminator of the Salajan sequence, which is given by $\left(\frac{|((-3)^n-5)|}{4}\right)_{n\geq 1}$ and named after Sabin Salajan, who conjectured the result. Moree and Zumalacárregui showed that the discriminator of the first n terms of the Salajan sequence is given by $\min(2^{\lceil \log_2(n) \rceil}, 5^{\lceil \log_5(5n/4) \rceil})$, through a rather long process. They started by showing that both $2^{\lceil \log_2(n) \rceil}$ and $5^{\lceil \log_5(5n/4) \rceil}$ discriminate the first n terms of the Salajan sequence, and also established an upper bound of $2n-1$ for the discriminator.

They then showed that all other numbers below $2n-1$ fail to be the discriminator. This involved several steps, the first of which involved considering the period of a sequence modulo d. A useful lemma showed that if the discriminator of sequence v is bounded from above by some non-decreasing function $g(n)$ for all $n \geq 1$, and that $g(\rho_v(d)) < d$ for some d, where $\rho_v(d)$ is the period of sequence v modulo d, then d cannot lie in the discriminator sequence of v. They applied this result to the Salajan sequence to show that the number 3 does not divide any element in the discriminator of the Salajan sequence. They then showed that the discriminator must be a prime power for all $n \geq 1$.

They then proceeded to define a function related to discriminators known as the incongruence index. Given a sequence v, the incongruence index is the function which maps the integer m to the integer k such that the first k terms of v are pairwise incongruent modulo m. In other words, it returns the maximum number of consecutive terms of the sequence that can be discriminated by the argument of the function. By analyzing the incongruence index of the Salajan sequence, along with other techniques, they showed that the discriminator cannot be the power of a prime number greater than 5. Since 3 was already excluded earlier, the only remaining discriminator candidates are powers of 2 and powers of 5, which leads to the desired result.

These results on the Salajan sequence were generalized by Ciolan and Moree [7], who computed the discriminator of Browkin-Salajan sequences, which is a generalization of the Salajan sequence introduced by Jerzy Browkin. For a prime $q \geq 5$ and setting $q^* = (-1)^{(q-1)/2} \cdot q$, the Browkin-Salajan sequences are of the form

$$(s_q(n))_{n\geq 1} = \left(\frac{3^n - q^*(-1)^n}{4}\right)_{n\geq 1},$$

where the original Salajan sequence represents the case $q = 5$. Browkin conjectured that the discriminator only takes values which are powers of 2 or powers of q. Ciolan and Moree showed that this conjecture is true for all cases except when $n = 5$ and $q \equiv \pm 1 \pmod{28}$, in which case the discriminator is 7. In particular, they characterized the discriminator of all Browkin-Salajan sequences based on the value of q, specifically on whether it is an

Artin, *Mirimanoff*, or *Fermat* prime. The discriminators were given as

$$D_{s_q}(n) = \begin{cases} \min\{2^e, q^f : 2^e \geq n, q^f \geq \frac{q}{q-1}n\} & \text{if } q \text{ is Artin, but not Mirimanoff;} \\ \min\{2^e, q : 2^e \geq n, q \geq n+1\} & \text{if } q \text{ is Artin, Mirimanoff, but not Fermat;} \\ \min\{2^e : 2^e \geq n\} & \text{if } q \text{ is Artin, Mirimanoff, and Fermat;} \\ \min\{2^e : 2^e \geq n\} & \text{if } q \text{ is not Artin,} \end{cases}$$

except for when $n = 5$ and $q \equiv \pm 1 \pmod{28}$ as noted earlier. They further showed that for the case in which q is Artin but not Mirimanoff, the value q^f occurs as a discriminator if and only if $\left\{ f\frac{\log q}{\log 2} \right\} > \frac{\log(q/(q-1))}{2}$, where $\{x\}$ denotes the fractional part of the real number x.

The process of proving these results was very similar to the proof for the discriminator of the Salajan sequence by Moree and Zumalacárregui [15], but with additional complications for the general case, such as how the period of the general Browkin-Salajan sequence modulo d can be odd for some d with $9 \nmid d$, whereas the proof for the Salajan sequence depended on the periods being even for all such d. There were also more cases to consider in some of the steps involved in proving the discriminator for the general Browkin-Salajan sequences.

More recently, another infinite family of sequences, known as the Lucas sequences, was studied by Faye, Luca, and Moree [8]. The Lucas sequences, denoted by $(u_k(n))_{n\geq 0}$ for integers $k \geq 1$, are defined by the recurrence $u_k(n+2) = (4k+2)u_k(n+1) - u_k(n)$ with initial values $u_k(0) = 0$ and $u_k(1) = 1$. For $k = 1$, they defined v_n as being the smallest power of 2 such that $v_n \geq n$, and w_n as the smallest integer of the form $2^a 5^b$ satisfying $w_n \geq 5n/3$ for positive integers a and b. They showed that the discriminator for $k = 1$ is given by $D_{u_1}(n) = \min\{v_n, w_n\}$.

The case of $k = 2$ was relatively simple, with the discriminator being the smallest integer greater or equal to n which is either a power of 2, or three times a power of 2. For $k > 2$, the discriminator appears to be very different from the case $k = 1$, and is based on the following sets:

$$\mathcal{A}_k = \begin{cases} \{m \text{ odd} : \text{ if } p|m, \text{then } p|k\} \text{ if } k \not\equiv 6 \pmod 9; \\ \{m \text{ odd}, 9 \nmid m : \text{ if } p|m, \text{then } p|k\} \text{ if } k \equiv 6 \pmod 9), \end{cases}$$

$$\mathcal{B}_k = \begin{cases} \{m \text{ even} : \text{ if } p|m, \text{then } p|k(k+1)\} \text{ if } k \not\equiv 2 \pmod 9; \\ \{m \text{ even}, 9 \nmid m : \text{ if } p|m, \text{then } p|k(k+1)\} \text{ if } k \equiv 2 \pmod 9). \end{cases}$$

Then the discriminator satisfies $D_{u_k}(n) \leq \min\{m \geq n : m \in \mathcal{A}_k \cup \mathcal{B}_k\}$, with equality established under certain conditions.

The proofs began with showing that for $n \geq 1$, all powers of 2 greater than n discriminate the first n terms of any Lucas sequence. For the powers of other primes, a useful parameter was introduced, denoted by $z(m)$, and referred to as the index of appearance of m in u_k. Given m, the value of $z(m)$ is the smallest positive integer such that

$u_k(z(m)) \equiv 0 \pmod{m}$. Through a long process of utilizing the properties of $z(m)$, and establishing various results based on the structure of the Lucas sequences, the discriminators of the Lucas sequences were established. The case $k = 1$ was quite different from the case $k > 1$, though the case $k = 2$ was especially simple and proved separately from the case $k > 2$.

The paper concluded with an analogy of $z(m)$ to polynomial sequences, noting how it led to the earlier established results by Moree [12] and Zieve [22].

1.3 Some Properties of Discriminators

In this section, we list various general properties of discriminators of sequences. Let $\mathbf{s} = (s(i))_{i\geq 0}$ be a sequence of distinct integers.

- **Fact 1:** $D_s(n)$ is the smallest positive integer m such that $m \nmid s(j) - s(i)$ for all $0 \leq i < j < n$.

This follows from the fact that

$$s(i) \equiv s(j) \pmod{m} \iff m|s(j) - s(i).$$

That is, m discriminates a set only when it does not divide the difference between any two integers in the set. This results in an alternative definition for discriminators.

- **Fact 2:** $D_s(1) = 1$.

A singleton is discriminated by every positive integer, the smallest of which is the number 1.

- **Fact 3:** The discriminator sequence is non-decreasing.

This can be proven by contradiction. Suppose $D_s(n) < D_s(n-1)$ for some $n \geq 2$. By definition, the number $D_s(n)$ discriminates the set

$$\{s(0), s(1), \dots s(n-1)\},$$

so the numbers in this set are pairwise incongruent modulo $D_s(n)$. But this implies that $D_s(n)$ also discriminates any subset of this set, such as the set of the first $n-1$ terms,

$$\{s(0), s(1), \dots, s(n-2)\}.$$

Since $D_s(n) < D_s(n-1)$, it follows that $D_s(n-1)$ is not the smallest number which discriminates this subset, thus contradicting the definition of the discriminator.

- **Fact 4:** $D_s(n) \geq n$ for all $n \geq 1$.

By definition of discriminator, the first n terms of $\mathbf{s}$ are pairwise incongruent modulo $D_s(n)$. This implies that these n terms lie in n different residue classes modulo $D_s(n)$. But if $D_s(n) < n$, then there are fewer than n possible residue classes modulo $D_s(n)$, which is a contradiction.

- **Fact 5:** If $\mathbf{s}$ is non-decreasing, then $D_s(n) \leq s(n-1) - s(0) + 1$ for all $n \geq 1$.

Since $\mathbf{s}$ is non-decreasing, it follows that from the set

$$\{s(0), s(1), \ldots, s(n-1)\},$$

the largest number is $s(n-1)$ and the smallest number is $s(0)$. The largest difference between two elements in the set is $s(n-1) - s(0)$. Any number that is greater than $s(n-1) - s(0)$ does not divide $s(n-1) - s(0)$ or any of the pairwise differences in the set, and so this number discriminates the sequence. Therefore, the discriminator can be at most $s(n-1) - s(0) + 1$.

- **Fact 6:** Let $\mathbf{s}'$ be a sequence such that $(s'(i))_{i\geq 0} = (s'(i) + a)_{i\geq 0}$ for a constant a. Then $D_{s'}(n) = D_s(n)$.

The difference between any two terms in $\mathbf{s}'$ is

$$s'(j) - s'(i) = s(j) + a - s(i) - a = s(j) - s(i).$$

Therefore, for $n \geq 1$, the smallest number which does not divide $s'(j) - s'(i)$ for all $0 \leq i < j < n$ is the same as the smallest number which does not divide $s(j) - s(i)$ in the same range. Thus, both the sequences $\mathbf{s}$ and $\mathbf{s}'$ have the same discriminator.

- **Fact 7:** Let $\mathbf{s}'$ be a sequence such that $((s'(i))_{i\geq 0} = (as'(i))_{i\geq 0}$ for a constant a. Then for any $n \geq 1$ such that $\gcd(a, D_s(n)) = 1$, we have $D_s(n) = D_{s'}(n)$.

This is proved in the following lemma:

Lemma 1. *Given a sequence $s(0), s(1), \ldots,$ and a non-zero integer a, let $s'(0), s'(1), \ldots,$ denote the sequence such that $s'(i) = as(i)$ for all $i \geq 0$. Then, for every n such that $\gcd(|a|, D_s(n)) = 1$, we have $D_{s'}(n) = D_s(n)$.*

Proof. From the definition of the discriminator, we know that for every $m < D_s(n)$, there exists a pair of integers i and j with $i < j < n$, such that $m | s(j) - s(i)$. Thus, for this same pair of i and j, we have

$$m | a(s(j) - s(i)) = as(j) - as(i) = s'(j) - s'(i).$$

Therefore, m cannot discriminate the set $\{s'(0), s'(1), \dots, s'(n-1)\}$ and so $D_{s'}(n) \geq D_s(n)$.

But for $m = D_s(n)$, we know that for all i and j with $i < j < n$, we have $m \nmid s(j) - s(i)$. Since $\gcd(m, |a|) = 1$, it follows that

$$m \nmid a(s(j) - s(i)) = as(j) - as(i) = s'(j) - s'(i)$$

for all i and j with $i < j < n$. Therefore, $m = D_s(n)$ discriminates the set

$$\{s'(0), s'(1), \dots, s'(n-1)\}$$

and so $D_{s'}(n) \leq D_s(n)$.

Putting these results together, we have $D_{s'}(n) = D_s(n)$. $\square$

1.4 Basic approaches in determining the discriminator of a sequence

In this section, we describe the basic approaches for computing the discriminator of a sequence.

1.4.1 Initial Approach

In general, it is a good idea to begin with numerically computing the discriminator for many terms in the sequence. This can be very useful to identify regular patterns while catching exceptional cases with unusual behavior. It is worth noting that in many cases, the discriminator sequence may not appear to follow any meaningful pattern until later on in the sequence. It is often useful to note down the prime numbers or prime powers that appear in the discriminator sequence, since they tend to play a significant role in governing the discriminator.

1.4.2 Characterizations of the discriminator

Based on the literature, it seems that are generally two common ways of characterizing discriminators, each with their own distinct but somewhat similar approaches. The first is to characterize the discriminator of n terms as being the smallest number within some range (at least a lower bound) that satisfies certain conditions, with the range depending on n. An example of this is the discriminator of the sequence of squares for $n \geq 4$, where $D_{\text{sq}}(n)$ is the smallest number greater or equal to $2n$ which is either a prime or twice a prime. The second type of characterization involves expressing the discriminator explicitly in a formula based on n. An example of this is the discriminator of the first n terms of the Salajan sequence, which is given by $\min(2^{\lceil \log_2(n) \rceil}, 5^{\lceil \log_5(5n/4) \rceil})$.

1. For approaches based on the former case, the default lower bound for the discriminator of n terms is n itself, which is applicable for all sequences. It is sometimes possible to prove a higher lower bound, depending on the sequence. Although an upper bound is not usually enforced in the characterization of the discriminator, it is often the case that an implicit upper bound is established.

   The upper bound tends to be closely related to the numbers that discriminate the first n terms of a sequences. In many cases, it can be shown that numbers satisfying certain conditions (which do not include an upper bound) discriminate the first n terms of a sequence. Then the upper bound manifests when proving that there exists at least one such number that lies below the upper bound. In some rare cases, the process is reversed, in that the upper bound is enforced as a condition on the discriminator, and is relied on to prove that numbers below the upper bound and fulfilling other conditions discriminate the first n terms of the sequence.

   Once the lower and upper bounds are established, and after finding sufficient conditions for a number to discriminate the first n terms, what remains to be shown is that the conditions are necessary. The upper and lower bounds are often important here, so that only numbers within these bounds can be considered. By showing that all numbers in this range that violate at least one of the given conditions fail to discriminate the first n terms of the sequence, the conditions are proven to be necessary, thus completing the characterization of the discriminator.

2. The second approach, which is based on an explicit formula to describe the discriminator, tends to have a more straightforward procedure. Each discriminator candidate generated by the formula represents both an upper bound and a lower bound of the discriminator, which are generally proven separately. The upper bound is established by showing that the candidate discriminates the first n terms of the sequence. Meanwhile, the lower bound is proven by showing that *all* numbers below the candidate fail to discriminate the same n terms of the sequence. This can sometimes be quite tricky, because there are often numbers less than the candidate which share similar properties to the candidate, so they need to be clearly distinguished.

Regardless of which type of characterization is pursued, it is often helpful to consider a general formula for $s(j) - s(i)$ for arbitrary integers i and j. This is because for any sequence $\mathbf{s}$ and for all $n \geq 1$, we have $D_s(n) \nmid s(j) - s(i)$ for all i and j in the range $0 \leq i < j < n$. Furthermore, for all $m < D_s(n)$, we have $m | s(j) - s(i)$ for at least one pair of i and j in the same range $0 \leq i < j < n$. It is also worth noting that the discriminator is a nondecreasing sequence, so it is sufficient to consider only the values of m in the range $D_s(n-1) \leq m < D_s(n)$ to prove a lower bound for $D_s(n)$.

The first type of characterization seems to occur more often in the literature, but the results in this thesis tend to follow the second type.

Chapter 2

Computational Aspects of Discriminators

In this chapter, we consider discriminators from a computational perspective. We present algorithms to compute the discriminator of the first n terms of a given sequence, and analyze the complexity of the algorithms. We also consider the computability of solving other problems relating to discriminators, such as determining whether there exists a sequence whose discriminator is a given sequence.

In this chapter, we use asymptotic order notations. The notation $f(n) \in O(g(n))$ indicates that there exists some $c > 0$ for which $|f(n)| \leq c|g(n)|$ for all n sufficiently large, which means the growth rate of $f(n)$ does not exceed $g(n)$. Likewise, $f(n) \in \Omega(g(n))$ indicates that there exists $c > 0$ for which $|f(n)| \geq c|g(n)|$ for all n sufficiently large. The notation $f(n) \in \Theta(g(n))$ means that $f(n)$ and $g(n)$ have the same growth rate, i.e., $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$.

For stricter bounds, the notation $f(n) \in o(g(n))$ indicates that $|f(n)| < c|g(n)|$ for all $c > 0$ and all n sufficiently large, which means $f(n)$ grows strictly slower than $g(n)$. Likewise, $f(n) \in \omega(g(n))$ indicates that $|f(n)| > c|g(n)|$ for all $c > 0$ and all n sufficiently large.

2.1 Discriminator Growth Rate

In the previous chapter, it was noted that the discriminator $D_s(n)$ of the first n terms of a sequence $(s(i))_{i\geq 0}$ satisfies the inequality $n \leq D_s(n) \leq s(n-1)+1$. Depending on how quickly the sequence $(s(i))_{i\geq 0}$ grows, this can be a very broad range. As noted in Chapter 1, the discriminator was first introduced for the purpose of improving space complexity, by reducing the size of an array from n^2 to d, where d is the discriminator of the first n positive squares, as studied by Arnold, Benkoski, and McCabe [2]. In that case, the discriminator of the squares grew linearly with n. Here we present a general result for the growth rate of the discriminator of all integer sequences.

Theorem 2. *Let $(s(i))_{i\geq 0}$ be a non-negative integer sequence of distinct terms such that $s(n) \in O(f(n))$ for a function $f(n)$. Then*

$$D_s(n) \in O(n^2 \log f(n)),$$

where $D_s(n)$ is the discriminator function of the sequence $(s(i))_{i\geq 0}$.

Proof. For $n \geq 1$, we define the functions $B_s(n)$ and $P_s(n)$ as follows:

$$B_s(n) = \max_{0\leq i<n} s(i),$$
$$P_s(n) = \prod_{0\leq i<j<n} s(j) - s(i).$$

Now, there exists a constant c such that for every positive integer N, there exists a number t that does not divide N such that $t \leq c \log_2 N$. This is shown by Pomerance, Robson, and Shallit [17] who established $c \leq 4.4$. Setting $N = P_s(n)$ implies that there exists $t \leq c \log_2 P_s(n)$ such that $t \nmid s(j) - s(i)$ for all integers i and j such that $0 \leq i < j < n$. It follows that t discriminates the set $\{s(0), s(1), \ldots, s(n-1)\}$, and thus, $D_s(n) \leq t \leq c \log P_s(n)$. Now

$$\begin{aligned} D_s(n) \leq c \log P_s(n) &= c \log\left(\prod_{0\leq i<j<n} s(j) - s(i)\right) \\ &\leq c \log\left(\prod_{0\leq i<j<n} B_s(n)\right) = c \sum_{0\leq i<j<n} \log B_s(n) = c \log B_s(n) \sum_{0\leq i<j<n} \qquad (1) \\ &= c\frac{n(n-1)}{2} \log B_s(n) \in O(n^2 \log B_s(n)). \end{aligned}$$

We have $B_s(n) \in O(f(n))$ since $s(n) \in O(f(n))$. Therefore, $D_s(n) \in O(n^2 \log f(n))$. $\square$

This result can be used to establish an upper bound on the growth rate of the discriminator of many sequences.

Corollary 3. *For sequences with growth rates in $O(n^k)$ for $k > 2$, the growth rates of the corresponding discriminator sequences are in $O(n^2 \log n)$.*

This result is also applicable for $k \leq 2$, but we already know that $D_s(n) \leq s(n) + 1$ for all $n \geq 1$ and so $D_s(n) \in O(n^k)$, which is a better upper bound for $k \leq 2$.

Corollary 4. *For exponential sequences with growth rate $O(a^n)$ for $a > 1$, the growth rates of their discriminators are in $O(n^3)$.*

Corollary 5. *Sequences with growth rate $O(n^n)$, such as $n!$, have discriminators with growth rates in $O(n^3 \log n)$.*

2.1.1 Discriminator growth rate of polynomials

As we have seen in Corollary 3, sequences with polynomial growth rate $O(n^k)$ have a discriminator growth rate in $O(n^2 \log n)$. However, it is not known whether there are any sequences with polynomial growth whose discriminator grows faster than linear. Furthermore, we can show that sequences of the form $(q(n))_{n\geq 0} = (\alpha n^2 + \beta n + \gamma)_{n\geq 0}$ have discriminators $D_q(n) \in O(n)$.

Theorem 6. *Let $(q(n))_{n\geq 0} = (\alpha n^2 + \beta n + \gamma)_{n\geq 0}$ be an integer-valued sequence. Then $D_q(n) \in O(n)$.*

Proof. By definition, the discriminator is the smallest number such that $D_q(n) \nmid q(j) - q(i)$ for all integers i and j such that $0 \leq i < j < n$. It suffices to find a prime number p such that $p \nmid q(j) - q(i)$ for all i and j in this range to show that $D_q(n) \leq p$. Now

$$\begin{aligned} q(j) - q(i) &= \alpha j^2 + \beta j - \alpha i^2 - \beta i = \alpha(j^2 - i^2) + \beta(j - i) \\ &= (j - i)(\alpha(i + j) + \beta). \end{aligned} \tag{2.1}$$

Note that $j - i \leq n$ and $\alpha(i+j)+\beta < 2\alpha n+\beta$. We define a constant $k = \max(1, 2\alpha+\beta, 2\alpha)$ so that $kn \geq j - i$ and $kn \geq \alpha(i + j) + \beta$. By Bertrand's Postulate, we know there exists a prime number p in the range $kn \leq p < 2kn$. Since $p > j - i$ and $p > \alpha(i + j) + \beta$, it follows that $p \nmid q(j) - q(i)$. Therefore, $D_q(n) \leq p < 2kn$ and thus, $D_q(n) \in O(n)$. □

Note that this proof applies only to sequences of the form $(\alpha n^2 + \beta n + \gamma)_{n\geq 0}$ and not to an arbitrary sequence with quadratic growth rate.

Conjecture 7. *All sequences with growth rates in $O(n^2)$ have discriminators in $\Theta(n)$.*

It is not known whether cubic sequences have discriminators with linear growth. The following conjecture is based on empirical results:

Conjecture 8. *The discriminators of the sequences $(n^3 + 11n^2 - 6n - 7)_{n\geq 0}$ and $(n^3 + 16n^2 - 39n + 22)_{n\geq 0}$ are in $\omega(n)$.*

2.2 Computing the discriminator of a given sequence

Given a sequence, we consider the problem of numerically computing the first n terms of the corresponding discriminator sequence. In general, there are two cases in how the input sequence is represented: either the first n terms are provided explicitly as input, or the sequence is described by a formula or recurrence relation. In this section, we provide algorithms for both scenarios.

2.2.1 Computing the discriminator given the terms of the sequence

We propose the following algorithm to compute the first n terms of the discriminator of a sequence for which the first n terms are given explicitly.

Algorithm 1: DiscFromSequence $(n, (s(t))_{0\le t<n})$

Input: Integer n, the values of the sequence $s(0), s(1), \ldots, s(n-1)$.
Output: Discriminator sequence $D_s(1), D_s(2), \ldots, D_s(n)$.

```
Initialize d ← 1, D_s(1) ← 1, and i ← 0;
Initialize R ← ∅;
for t ← 1 to n do
    while i < t do
        if s(i) mod d ∉ R then
            R ← R ∪ {s(i) mod d};
            i ← i + 1;
        else
            R ← ∅;
            d ← d + 1;
            i ← 0;
        end
    end
    Set D_s(t) ← d;
end
Return D_s(1), D_s(2), ..., D_s(n).
```

There are various data structures that can be used for the set R. One possibility is the binary trie, where each integer can be represented by traversing down a binary tree bit by bit.

Theorem 9 (DiscFromSequence Correctness)**.** *The DiscFromSequence algorithm returns the correct discriminator of the input sequence.*

Proof. We observe from the algorithm that the value of d is maintained for as long as the "else" condition is not reached. Within this duration, the set R is not cleared and the index i takes distinct values at each iteration of the while loop. This implies that the "else" condition is invoked if and only if there was an earlier value of i that added the current value of $s(i) \bmod d$ to the set R. The earlier addition must have been done for the same value of d, because R would have been cleared otherwise. This means that there are two values of i less than t for which $s(i) \bmod d$ are equivalent, and therefore, $D_s(t) \neq d$ for all t and d for which the "else" condition is reached.

This "else" condition resets i to 0 and increments d by 1 whenever $D_s(t) \neq d$. It follows that the "while" loop can only be exited by the smallest value of d for which the "if" condition is satisfied for all $0 \le i < t$, i.e., the numbers $s(i) \bmod d$ are distinct. In other words, $D_s(t) = d$ for all t and d in which the "while" loop terminates. $\square$

We now analyze the performance of the DiscFromSequence algorithm using the RAM computational model.

Theorem 10 (DiscFromSequence Time Complexity)**.** *Let $s(n) \in O(f(n))$. If the DiscFromSequence algorithm is implemented using a binary trie, then the runtime of the algorithm, $T(n, f(n))$, is in $O(n(\log f(n))D_s(n)(\log D_s(n)))$.*

Proof. For $n \ge 1$, this algorithm for computing the first n terms of the discriminator involves iterating the value of d from 1 to $D_s(n)$. For each value of d, there are at most n terms of the sequence being computed modulo d, with each result being looked up in the data structure and inserted if it's not present. Each mod operation is achieved in $O((\log f(n))(\log d))$ time in the RAM computational model, while each search/insert is achieved in $O(\log d)$ time (searching in a binary trie). So for each value of d, the runtime is in $O(n((\log f(n))(\log d) + (\log d))) \in O(n(\log f(n))(\log d))$.

Since d iterates from 1 to $D_s(n)$, the runtime of the entire algorithm is bounded by

$$\begin{aligned} T(n, f(n) &\in \sum_{d=1}^{D_s(n)} O(n(\log f(n))(\log d)) \in O\left(n(\log f(n)) \sum_{d=1}^{D_s(n)} (\log d)\right) \\ &\in O(n(\log f(n))D_s(n)(\log D_s(n))). \end{aligned}$$

$\square$

Corollary 11. *The runtime of DiscFromSequence, $T(n, f(n))$, using a binary trie implementation, is in $O(n^3(\log^2 f(n))((\log n) + (\log\log f(n))))$.*

Proof. Since $D_s(n) \in O(n^2 \log f(n))$, we have

$$\begin{aligned} T(n, f(n)) &\in O(n(\log f(n))D_s(n)(\log D_s(n))) \\ &\in O(n(\log f(n))n^2(\log f(n)) \log(n^2 \log f(n))) \\ &\in O(n^3(\log^2 f(n))((\log n) + (\log\log f(n)))). \end{aligned}$$

It may be worth noting that the runtime is in $O(n^{3+\epsilon_1}(\log^{2+\epsilon_2} f(n)))$ for all $\epsilon_1, \epsilon_2 > 0$. $\square$

The runtime is a polynomial function of the size of the input, where n is the number of terms in the input sequence, and $\log f(n)$ is an upper bound on the number of bits needed to represent each of the first n terms of the sequence. Therefore, the problem of computing the discriminator of the first n terms of a given input sequence lies in the class $\mathbf{P}$.

Theorem 12 (DiscFromSequence Space Complexity)**.** *Let $s(n) \in O(f(n))$. Then the space complexity of DiscFromSequence, using a binary trie implementation, and excluding the input and the output, is in $O(D_s(n))$.*

Proof. Aside from the input and output, the algorithm only stores a single value of d at any time, which takes $O(\log D_s(n))$ space, and a binary trie of residues modulo d. The space complexity of the binary trie depends on how many nodes it carries. At any time, the maximum height of the binary trie is equal to the number of bits of the largest number represented in it. The largest number is $d-1 \leq D_s(n)$, so the height is $\log_2 D_s(n)$ and thus, the maximum number of nodes in the trie is in $O(2^{\log_2 D_s(n)}) \in O(D_s(n))$, which dominates the complexity of storing a single value of d. $\square$

Corollary 13. *The space complexity of DiscFromSequence, using a binary trie implementation, and excluding the input and the output, is in $D(n^2 \log f(n))$.*

Proof. This follows immediately from the result that $D_s(n) \in O(n^2 \log f(n)$. $\square$

It may be worth noting, however, that the result $D_s(n) \in O(n^2 \log f(n))$ is not known to be tight, i.e., it is not known whether there are any sequences with discriminators in $\Theta(n^2 \log f(n))$. Therefore, the bounds that depend on $D_s(n)$ may be more useful.

In particular, note that the space complexity for storing the input is $O(n \log f(n))$, since there are n terms, each of which are in $O(f(n))$. In some cases, this may actually dominate the space complexity of the algorithm itself, which is in $O(D_s(n))$. For example, there are known exponential sequences whose discriminators grow linearly. In some of these cases, there may be a more space-efficient approach in computing the discriminators, as shown in the next subsection.

2.2.2 Generating the discriminator of sequences described by a formula

The second scenario is when the sequence is described by a formula or a set of formulas. This includes recurrence relations as well. An obvious approach to computing the discriminators of such sequences is to first compute the terms of the sequence and then call DiscFromSequence.

Algorithm 2: DiscFromFormula (n, Description of s)

Input: Integer n, description of the sequence $(s(n))_{n\geq 0}$.
Output: Discriminator sequence $D_s(1), D_s(2), \ldots, D_s(n)$.

1 Compute the first n terms of s, and store them as $s(0), s(1), \ldots, s(n-1)$;
2 Return DiscFromSequence (n, $(s(t))_{0\leq t<n}$).

Theorem 14 (DiscFromFormula Time Complexity)**.** *Let $s(n) \in O(f(n))$ and let $g(n)$ denote the time complexity of computing $s(n)$. Then the runtime of DiscFromFormula is in $O(n(\log f(n))D_s(n)(\log D_s(n)) + ng(n))$.*

Proof. The runtime for computing n terms of $s(n)$ is bounded by $O(ng(n))$. Including DiscFromSequence, the total runtime is in $O(n(\log f(n))D_s(n)(\log D_s(n)) + ng(n))$. □

Typically the runtime for computing all n terms of the sequence is dominated by the runtime for DiscFromSequence.

Theorem 15 (DiscFromFormula Space Complexity)**.** *Let $s(n) \in O(f(n))$. Then the space complexity of DiscFromFormula in in $O(n(\log f(n)) + D_s(n))$.*

Proof. Each term in the sequence can be represented by $O(\log f(n))$ bits, so the storage of n terms has a space complexity of $O(n \log f(n))$. Combined with the space complexity of DiscFromSequence, the total space complexity is in $O(n(\log f(n)) + D_s(n))$. □

As noted earlier, since discriminators tend to grow slower than the known upper bound of $O(n^2(\log f(n)))$, it is possible that the $n \log f(n)$ component of the space complexity (which stores the terms in the sequence) dominates $D_s(n)$. In some of these cases, however, it may be possible to describe the sequence using a linear recurrence, or a convenient formula composed of operations that are compatible with modular arithmetic. In these cases, we can generate the terms of the sequence modulo d instead of computing the exact terms.

Algorithm 3: DiscFromLinRec (n, Description of s)

Input: Integer n, the sequence $s(0), s(1), \dots, s(n-1)$ represented in binary.
Output: Discriminator sequence $D_s(1), D_s(2), \dots, D_s(n)$.

Initialize $d \leftarrow 1$, $D_s(1) \leftarrow 1$, and $i \leftarrow 0$;
Initialize R as an empty set;
for $t \leftarrow 1$ **to** n **do**
  while $i < t$ **do**
    Generate $s(i) \bmod d$;
    if $s(i) \bmod d \notin R$ **then**
      $R \leftarrow R \cup \{s(i) \bmod d\}$;
      $i \leftarrow i + 1$;
    else
      $R \leftarrow \varnothing$;
      $d \leftarrow d + 1$;
      $i \leftarrow 0$;
    end
  end
  Set $D_s(t) \leftarrow d$;
end
Return $D_s(1), D_s(2), \dots, D_s(n)$.

The algorithm is identical to DiscFromSequence, except $s(i) \bmod d$ is computed at each iteration of the while loop, since the values of $s(i)$ are not given as input.

In this case, the space complexity is no longer dependent on the growth rate of the given sequence. The time complexity now depends on the runtime of generating the terms modulo d, which may also be independent on the growth rate of the given sequence.

Theorem 16 (DiscFromLinRec Time Complexity)**.** *Let $h(n, d)$ denote the time complexity of computing $s(n) \bmod d$. Then the runtime for DiscFromLinRec satisfies*

$$T(n, h(t, d)) \in O\left(\sum_{d=1}^{D_s(n)} n(h(n, d) + \log d)\right)$$

Proof. For each value of d, there are n terms being generated modulo d, each having $h(n, d)$ runtime. Searching and inserting into the binary trie is in $O(\log d)$ time. Since d runs from 1 to $D_s(n)$, the desired result follows. □

If $s(n) \in O(f(n))$, then it is often the case that $h(n, d) \in O(\text{poly}(\log f(n)) \log d)$. This leads to a runtime that is comparable to the DiscFromSequence runtime.

Theorem 17 (DiscFromLinRec Space Complexity)**.** *The space complexity of DiscFromLinRec is in $O(D_s(n) + n(\log D_s(n)))$.*

Proof. The space complexity of the binary trie is the same as in DiscFromSequence, which is $O(D_s(n))$. In addition to that, each of the n terms in the sequence are stored modulo d for some value of $d \leq D_s(n)$ at any time. Therefore, the space complexity of storing the terms of the sequence modulo d is in $O(n(\log D_s(n)))$. Thus the total space complexity is in $O(D_s(n) + n(\log D_s(n)))$. □

We can use DiscFromLinRec to save space on many types of sequences. For example, exponential sequences of the form $(s(n))_{n\geq 0} = (a^n)_{n\geq 0}$ can grow very large very quickly, but they can be represented by the linear recurrence $s(n) = as(n-1)$. Thus we can apply DiscFromLinRec to achieve a space complexity of $O(D_s(n) + n(\log D_s(n)))$, which is better than the space complexity $O(D_s(n) + n \log f(n))$ from DiscFromSequence, especially if $D_s(n)$ grows much slower than $f(n)$.

2.3 Determining whether a given sequence is a discriminator of some other sequence

In this section, we are concerned with the following problem: given a sequence of positive integers $(d(n))_{n\geq 1}$, does there exist another integer sequence with distinct terms $(s(n))_{n\geq 1}$ such that $D_s(n) = d(n)$ for all $n \geq 1$? For this section, note that the sequences $(d(n))_{n\geq 1}$ and $(s(n))_{n\geq 1}$ both begin with the index 1, to maintain consistency.

In order to have $D_s(n) = (d(n))_{n\geq 1}$ for all $n \geq 1$, an obvious condition is that $d(1) = 1$, since $D_s(n) = 1$ for all integer sequences $(s(n))_{n\geq 1}$. The following lemmas indicate the sufficient and necessary conditions for $d(2)$ and $d(3)$.

Lemma 18. *Let $(d(n))_{n\geq 1}$ be a sequence of positive integers such that $d(1) = 1$. Then there exists an integer sequence with distinct terms $(s(n))_{n\geq 1}$ with $D_s(2) = d(2)$ if and only if $d(2) = p^k$ for prime p and integer $k \geq 1$.*

Proof. Suppose $d(2) = p^k$ for prime p and integer $k \geq 1$. Then let $(s(n))_{n\geq 1}$ be a sequence such that $s(2) = s(1) + \mathrm{lcm}(2, 3, \ldots, p^k - 1)$. Then for all $m < p^k$, we have $m | s(2) - s(1)$. Since $p^k \nmid s(2) - s(1)$, if follows that $D_s(2) = p^k = d(2)$.

For the other direction, let us suppose, to get a contradiction, that there exists a sequence of distinct integers $(s(n))_{n\geq 1}$ such that $D_s(2) = d(2)$ while $d(2)$ cannot be written in the form p^k for prime $p \geq 2$ and integer $k \geq 1$. Since $d(2) = D_s(2) \geq 2$, it follows that $d(2)$ can be written in the form $d(2) = qr$ where $q \geq 2$, $r \geq 2$ and $\gcd(q, r) = 1$. Then

$$d(2) = D_s(2) = qr \nmid s(2) - s(1) \implies q \nmid s(2) - s(1) \implies D_s(2) \leq q,$$

which is a contradiction, since $D_s(2) = d(2) = qr > q$. □

Lemma 19. *Let $(d(n))_{n\geq 1}$ be a sequence of positive integers such that $d(1) = 1$ and $d(2) = p^k$ for prime p and integer $k \geq 1$. Then there exists an integer sequence with distinct terms $(s(n))_{n\geq 1}$ with $D_s(n) = d(n)$ for $1 \leq n \leq 3$ if and only if $d(3) \geq p^k$, $d(3) \geq 3$, and one of the following conditions is true:*

1. *$d(3) = q^\ell$ for prime q and integer $\ell \geq 1$,*
2. *$d(3) = ap^k$ for integer $a \geq 1$ such that $\gcd(a, p) = 1$.*

Proof. The constraints that $d(3) \geq p^k$ and $d(3) \geq 3$ follow directly from the properties of the discriminator, namely that the discriminator is non-decreasing and that $D_s(n) \geq n$ for all $n \geq 1$. We now show that satisfying either of the two conditions as well is enough to ensure the existence of $(s(n))_{n\geq 1}$, while choosing any arbitrary starting value of $s(1)$.

1. $d(3) = q^\ell$ for prime q and integer $\ell \geq 1$. Let $u_1 = \mathrm{lcm}(1, 2, \ldots, p^k - 1)$ and choose $s(2) = s(1) + u_1$, so that $D_s(2) = d(2) = p^k$. If $d(3) = p^k$ as well, then choose $s(3)$ as any integer such that $s(1)$, $s(2)$, and $s(3)$ are in separate residue classes modulo $d(3)$, ensuring that $D_s(3) = D_s(2) = p^k = d(3)$.

   If $q^\ell \neq p^k$, let $u_2 = \mathrm{lcm}(p^k, p^k + 1, \ldots, q^\ell - 1)$. Clearly $q^\ell \nmid u_1$ and $q^\ell \nmid u_2$. If $q^\ell \nmid (u_1 + u_2)$, then choose $s(3) = s(2) + u_2$, so that q^ℓ is the smallest integer that does not divide $s(2) - s(1)$, $s(3) - s(2)$, or $s(3) - s(1)$, which are u_1, u_2, and $(u_1 + u_2)$ respectively, and so $D_s(3) = q^\ell = d(3)$. Note that $q = 2$ implies $q^\ell \nmid (u_1 + u_2)$, since the only multiple of $2^{\ell-1}$ in the range $[1, 2^\ell - 1]$ is $2^{\ell-1}$, which divides either u_1 or u_2, but not both.

If $q^\ell|(u_1+u_2)$, which also implies $q \neq 2$, then choose $s(3) = s(2) + 2u_2$, so that q^ℓ is the smallest integer that discriminates $\{s(1), s(2), s(3)\}$ and so, $D_s(3) = d(3)$.

2. $d(3) = ap^k$ for integer $a \geq 1$ such that $\gcd(a, p) = 1$. Let $u_1 = \mathrm{lcm}(1, 2, \ldots, p^k - 1, a, 2a, \ldots, a(p^k-1))$, i.e., the lcm of all numbers less than p^k as well as all multiples of a less than ap^k. Let u_2 be the lcm of all numbers between p^k and $ap^k - 1$ inclusive, but excluding every multiple of a.

   With this construction, every number less than ap^k divides either u_1 or u_2, with numbers less than p^k dividing u_1. Furthermore, we have $a|u_1$, $p^k \nmid u_1$, $a \nmid u_2$, and $p^k|u_2$, which implies that ap^k does not divide either u_1, u_2, or (u_1+u_2). Choose $s(2) = s(1) + u_1$ and $s(3) = s(2) + u_2$ so that $D_s(2) = p^k = d(2)$ and $D_s(3) = ap^k = d(3)$.

For the other direction, we show that there does not exist any sequence $(s(n))_{n\geq 1}$ with $D_s(3) = d(3)$ for $1 \leq n \leq 3$ if neither of the two conditions are satisfied. By contradiction, let us assume that such a sequence exists. For this sequence $(s(n))_{n\geq 1}$, let $v_1 = s(2) - s(1)$, $v_2 = s(3) - s(2)$, and $v_3 = s(3) - s(2) = v_1 + v_2$. Since $D_s(2) = d(2) = p^k$, it follows that $\mathrm{lcm}(1, 2, \ldots, p^k - 1)|v_1$ and $p^k \nmid v_1$. Note that any number that divides two of v_1, v_2, and v_3 divides the third as well.

If neither of the two conditions are satisfied, then there are several cases to consider:

Case 1: $d(3) = bp^t$ for $1 \leq t < k$ and $\gcd(b, p) = 1$. Since $p^t < p^k$, we have $p^t|v_1$. Since $D_s(3) > p^k$, we have either $p^k|v_2$ or $p^k|v_3$. Without loss of generality, assume $p^k|v_2$. Since $p^t|p^k$, it follows that $p^t|v_2$. Since $p^t|v_1$ as well, we have $p^t|v_3$. Finally, since $D_s(3) > b$, we have b dividing at least one of v_1, v_2, or v_3. But since p^t divides all three of those, it follows that bp^t divides at least one of those three as well, and so, $D_s(3) \neq bp^t = d(3)$, a contradiction.

Case 2: $d(3) = bp^t$ for $t > k$ and $\gcd(b, p) = 1$. Since $D_s(3) > p^k$, we have either $p^k|v_2$ or $p^k|v_3$, but not both, since $p^k \nmid v_1$. Without loss of generality, assume $p^k|v_2$ and $p^k \nmid v_3$. Since $p^k|bp^k$ and $p^k|p^t$, it follows that neither bp^k nor p^t divides either of v_1 or v_3. But since $D_s(3) > bp^k$ and $D_s(3) > p^t$, it follows that $bp^k|v_2$ and $p^t|v_2$, which implies $bp^t|v_2$ and so, $D_s(3) \neq bp^t$, a contradiction.

Case 3: $d(3) = qr$, where $q, r > 1$, $\gcd(q, r) = 1$, $p \nmid qr$, and $q < p^k$. Since $q < p^k$, we have $q|v_1$. If $r < p^k$, then we have $r|v_1$ and so, $qr|v_1$ and thus, $D_s(3) \neq qr$, a contradiction. Therefore, we have $r > p^k$. But then we have $d(3) > qp^k$, so it follows that qp^k divides either v_2 or v_3. Since $q|v_1$, this means q divides all three of v_1, v_2, and v_3. But $d(3) > r$ implies r divides one of v_1, v_2, and v_3, which means qr divides one of those three as well and thus, $D_s(3) \neq qr$, a contradiction.

Case 4: $d(3) = qr$ where $\gcd(q, r) = 1$, $p \nmid qr$, $q > p^k$, and $r > p^k$. Here, since $D_s(3) > p^k$, we have p^k dividing either v_2 or v_3, but not both. Without loss of generality, assume $p^k | v_2$. Since $D_s(3) > qp^k$, we have $qp^k | v_2$, and since $D_s(3) > rp^k$, we have $rp^k | v_2$. But this means $qr | v_2$, and so $D_s(3) \neq qr$, a contradiction.

Therefore, $D_s(3) \neq d(3)$ if neither of the two given conditions are satisfied. $\square$

This is a complete characterization for when the first three terms of a sequence can represent the first three values of the discriminator of some other sequence. However, generalizing the characterization to more than three terms proves to be quite a difficult task and remains as an open problem.

2.3.1 A class of sequences which are discriminators of other computable sequences

We consider the case when $(d(n))_{n\geq 1}$ that satisfies the following conditions:

1. $d(1) = 1$;
2. $d(i) | d(j)$ for all $j > i \geq 1$;
3. $d(n+1) \nmid \operatorname{lcm}(d(n), d(n)+1, \ldots d(n+1)-1)$ for all $n \geq 1$.

Examples of sequences that satisfy these conditions are the exponential sequences of the form $(p^{n-1})_{n\geq 1}$, where p is a prime number.

If these conditions are satisfied, we show that there exists a sequence $(s(n))_{n\geq 1}$ which can be computed sequentially such that $D_s(n) = d(n)$ for all $n \geq 1$.

Theorem 20. *Let $(d(n))_{n\geq 1}$ be an integer sequence that satisfies the above conditions, and let $(s(n))_{n\geq 1}$ be an integer sequence such that for $n \geq 1$, the following recurrence relation is satisfied:*

$$s(n+1) = s(n) + \operatorname{lcm}(d(n), d(n+1), \ldots, d(n+1)-1).$$

Then $D_s(n) = d(n)$ for all $n \geq 1$.

Proof. We prove this by induction on n.

Base Case For $n = 1$, we know $D_s(1) = 1 = d(1)$ from the first property of $d(n)$.

Inductive Step Suppose that for some $n \geq 1$, we have $D_s(n) = d(n)$. We show that $D_s(n+1) = d(n+1)$.

Since the discriminator is non-decreasing, we know $D_s(n+1) \geq D_s(n) = d(n)$. For all integers m in the range $d(n) \leq m < d(n+1)$, we know $m | \operatorname{lcm}(d(n), d(n+1), \ldots, d(n+1)-1) = s(n+1) - s(n)$. Therefore, m fails to discriminate the first $n+1$ terms of $(s(n))_{n\geq 1}$, and so, $D_s(n+1) \geq d(n+1)$.

In order to show that $D_s(n+1) \leq d(n+1)$, it suffices to show that $d(n+1) \nmid s(j)-s(i)$ for all $1 \leq i < j \leq n+1$. This can be done by considering each of the following cases:

Case 1: $j \leq n$ The inductive hypothesis suggests that for all integers i and j such that $1 \leq i < j \leq n$, we have $d(n) \nmid s(j) - s(i)$. Since $d(n)|d(n+1)$, it follows that $d(n+1) \nmid s(j) - s(i)$ for $1 \leq i < j \leq n$.

Case 2: $i = n, j = n+1$ Here, we have $s(j) - s(i) = s(n+1) - s(n)$. Note that

$$s(n+1) - s(n) = \operatorname{lcm}(d(n), d(n+1), \ldots, d(n+1) - 1).$$

The third condition on $(d(n))_{n\geq 1}$ implies that $d(n+1) \nmid s(n+1) - s(n)$ and thus $d(n+1) \nmid s(j) - s(i)$.

Case 3: $1 \leq i < n, j = n+1$ In this case,

$$\begin{aligned} s(j) - s(i) = s(n+1) - s(i) &= (s(n+1) - s(n)) + (s(n) - s(i)) \\ &= \operatorname{lcm}(d(n), d(n+1), \ldots, d(n+1) - 1) + (s(n) - s(i)). \end{aligned}$$

We know that $d(n)|\operatorname{lcm}(d(n), d(n+1), \ldots, d(n+1) - 1)$, while it follows from the inductive hypothesis that $d(n) \nmid s(n) - s(i)$. Therefore, it follows that $d(n) \nmid s(j) - s(i)$ and thus, $d(n+1) \nmid s(j) - s(i)$.

Putting these together, we have $D_s(n+1) = d(n+1)$.

Thus, by induction, we have $D_s(n) = d(n)$ for all $n \geq 1$. $\square$

This construction is applicable for any integer starting value of $s(1)$. For $n \geq 1$, the term $s(n+1)$ can be computed directly from the values of $s(n)$, $d(n)$, and $d(n+1)$, using the given recurrence relation. Note that this recurrence relation on $s(n)$ provides a sufficient condition for $D_s(n) = d(n)$ for all $n \geq 1$, but it may not be a necessary condition.

2.3.2 Other sequences as discriminators

For the class of sequences in the previous section, the condition that $d(i)|d(j)$ for all $j > i \geq 1$ ensured that all finite sequence of integers that are distinct modulo $d(i)$ are also distinct modulo $d(j)$. This allows $(s(n))_{n\geq 0}$ to be constructed in a manner that ensures that the choice of $s(k)$ for any $k \geq 1$ does not prevent a later element $d(k)$ from discriminating the the first k terms of $(s(n))_{n\geq 1}$.

However, in general, a sequence $(d(n))_{n\geq 1}$ may not have the property that later terms are multiples of previous terms. As a result, even if a finite sequence $s(1), s(2), \ldots, s(k)$ can be constructed such that $D_s(n) = d(n)$ for $1 \leq n \leq k$, it may be possible that a later term, $d(\ell)$ for $\ell > k$ might divide the difference $s(j) - s(i)$ for some $1 \leq i < j \leq k$, which means $d(\ell)$ cannot be the discriminator for any sequence with the prefix $s(1), s(2), \ldots, s(k)$. This makes it difficult to construct the terms of an infinite sequence $(s(i))_{n\geq 1}$ from a given general infinite sequence $(d(n))_{n\geq 1}$ such that $D_s(n) = d(n)$ for all $n \geq 1$.

There are many well-known sequences that lack the property of $d(i)|d(j)$ for all $j > i \geq 1$, such as the sequence of squares (or higher powers), the sequence of non-composite positive numbers (basically 1 and the prime numbers, since $d(1) = 1$ is required), and the sequence of Fibonacci numbers, and it is an open problem as to whether any of these sequences are discriminators of another sequence. More generally, it is also an open problem as to whether it is possible to characterize the infinite sequences which are also discriminators of other sequences.

2.4 Determining whether a finite sequence is the discriminator of a prefix of another sequence

The earlier section pointed out the difficulty of proving whether a general sequence is also the discriminator of another sequence. However, instead of considering an infinite sequence, it may be simpler to look at a finite prefix of a sequence. That is, given the first N terms of a sequence $(d(n))_{n\geq 1}$, find another sequence $(s(n))_{n\geq 1}$ such that $D_s(n) = d(n)$ for all $1 \leq n \leq N$.

2.4.1 Proposed Algorithms

A simple approach for solving this problem is to perform an exhaustive search of bounded integer sequences of length N and check if the discriminator for any sequence is $(d(n))_{1\leq n\leq N}$ for all N terms.

Algorithm 4: DiscToSeqBruteForce (N, $(d(n))_{1\leq n\leq N}$)

Input: Integer n, the values of the sequence $d(1), d(2), \ldots, d(N)$.
Output: Either a sequence $s(1), s(2), \ldots, s(N)$ such that $D_s(n) = d(n)$ for all $1 \leq n \leq N$, or "False" if none exist.

foreach *sequence* $(s(n))_{1\leq n\leq N}$ *of distinct integers such that* $s(1) = 0$ *and* $1 \leq s(n) \leq \mathrm{lcm}(1, 2, \ldots, d(N))$ *for all* $2 \leq n \leq N$ **do**
  if $(d(n))_{1\leq n\leq N}$ = *DiscFromSequence* $(n, (s(n))_{1\leq n\leq N})$ **then**
    Return $s(1), s(2), \ldots, s(N)$;
  end
end
Return "False";

This algorithm computes the discriminator of every sequence with values bounded between 1 and $\mathrm{lcm}(1, 2, \ldots, d(N))$, except for $s(1) = 0$, until it finds a sequence whose discriminator is $d(1), d(2), \ldots, d(N)$.

Theorem 21. *If there exists an integer sequence* $s(1), s(2), \ldots, s(N)$ *with discriminator* $d(1), d(2), \ldots, d(N)$*, then there exists an integer sequence* $s'(1), s'(2), \ldots s'(N)$ *with discriminator* $d(1), d(2), \ldots, d(N)$ *such that* $s'(1) = 0$ *and the values of* $s'(2), s'(3), \ldots, s(N)$ *are bounded between 1 and* $\operatorname{lcm}(1, 2, \ldots, d(N))$.

Proof. If $(s(n))_{1 \leq n \leq N}$ already satisfies the required conditions, then let $(s'(n))_{1 \leq n \leq N} = (s(n))_{1 \leq n \leq N}$. Otherwise, let $(s'(n))_{1 \leq n \leq N}$ be a sequence such that $s'(n) = (s(n) - s(1)) \bmod \operatorname{lcm}(1, 2, \ldots, d(N))$ for $1 \leq n \leq N$. Clearly $s'(1) = 0$. Note that the subtraction of $s(1)$ to all terms of $(s(n))_{1 \leq n \leq N}$ does not affect whether any two terms are congruent or incongruent modulo m for any positive integer m.

Furthermore, for any two terms $s(i)$ and $s(j)$ which are congruent or incongruent modulo some integer m, then $s(i)$ and $s(j)$ remain congruent or incongruent modulo m even if a multiple of m is subtracted from either or both of $s(i)$ and $s(j)$. If we have $1 \leq m \leq D_s(N)$, then $m | \operatorname{lcm}(1, 2, \ldots, d(N))$, and so, $s'(i)$ and $s'(j)$ are congruent modulo m if and only $s(i)$ and $s(j)$ are congruent modulo m, for all integers i and j. Therefore, $D_{s'}(n) = D_s(n) = d(n)$. Since the values of $(s'(n))_{1 \leq n \leq N}$ are bounded between 1 and $\operatorname{lcm}(d(1), d(2), \ldots, d(N))$, except for $s(1) = 0$, it follows that $(s'(n))_{1 \leq n \leq N}$ satisfies the required conditions. □

Corollary 22. *If there exists an integer sequence* $s(1), s(2), \ldots, s(N)$ *with discriminator* $d(1), d(2), \ldots, d(N)$*, then the DiscToBruteForce algorithm returns at least one of them.*

It is clear that this DiscToSeqBruteForce has very high time complexity, since the number of possible sequences being considered is $(\operatorname{lcm}(d(1), d(2), \ldots, d(N))^{N-1}$. The runtime could significantly be improved if the algorithm instead seeks to construct only a single sequence while ensuring that the discriminator matches the input. Such an algorithm is provided below, but it is not proven that the algorithm will always find such a sequence, even in cases where such a sequence exists.

Algorithm 5: DiscToSeqSeeker $(N,\ (d(n))_{1\le n\le N})$

Input: Integer n, the sequence $d(1), d(2), \ldots, d(N)$ represented in binary.
Output: Either a sequence $s(1), s(2), \ldots, s(N)$ such that $D_s(n) = d(n)$ for all $1 \le n \le N$, or "Failure".

```
Set s(1) ← 0;
for 2 ≤ n ≤ N do
    Initialize s(n) ← 1;
    while True do
        if s(n) > lcm(1, 2, ..., d(N)) then
            Return "Failure";
        end
        Initialize FirstCond ← True, SecondCond ← True;
        for 1 ≤ m < d(n) do
            if m discriminates the set {s(1), s(2), ..., s(n)} then
                FirstCond ← False;
                Break;
            end
        end
        if FirstCond is True then
            for n ≤ i ≤ N do
                if d(i) does not discriminate the set {s(1), s(2), ..., s(n)} then
                    SecondCond ← False;
                    Break;
                end
            end
        end
        if both FirstCond and SecondCond are True then
            Break;
        else
            s(n) ← s(n) + 1;
        end
    end
end
Return s(1), s(2), ..., s(N);
```

The basic idea of the DiscToSeqSeeker is that for each n from 2 to N, the value of $s(n)$ is initialized to 1 and then two conditions (recorded by FirstCond and SecondCond) are checked. The value of $s(n)$ is incremented until both conditions are satisfied, and the process is repeated for the next value of n. If $s(n)$ exceeds the upper bound of $\mathrm{lcm}(1, 2, \ldots, d(N))$, then the algorithm failed to find a sequence within the expected bounds and returns “Failure”.

Theorem 23. *If DiscToSeqSeeker returns a sequence* $(s(n))_{1\le n\le N}$*, then this sequence is the lexicographically least non-negative sequence with discriminator* $D_s(n) = d(n)$ *for all* $1 \le n \le N$*.*

Proof. The first value $s(1)$ is set to 0. For $2 \le n \le N$, the value of $s(n)$ is initialized to 1 and incremented until both of the given conditions are satisfied. If FirstCond is violated, then there is some $m < d(n)$ which discriminates $\{s(1), s(2), \ldots, s(n)\}$, and so $D_s(n) \le m$, which means $D_s(n) \neq d(n)$. If the second condition is violated, then there is some $n \le i \le N$ for which $d(i)$ does not discriminate $\{s(1), s(2), \ldots, s(n)\}$. Since $d(i)$ needs to discriminate the first i terms in order to be a candidate for $D_s(i)$, and we have $i \ge n$, it follows that $D_s(i) \neq d(i)$.

On the other hand, if both conditions are true, then FirstCond ensures that $D_s(n) \ge d(n)$, while SecondCond confirms that $D_s(n) \le d(n)$ from the case of $i = n$. Therefore, the sequence returned by DiscToSeqSeeker is the lexicographically least non-negative sequence $(s(n))_{1\le n\le N}$ whose discriminator is $D_s(n) = d(n)$ for all $1 \le n \le N$. □

Note that once the algorithm finds a prefix sequence $s(1), s(2), \ldots, s(i-1)$ for some $1 \le i < N$, for which each value satisfies FirstCond and SecondCond, these values are not modified again.

If there exists a sequence with discriminator $(d(n))_{1\le n\le N}$, then FirstCond and SecondCond are necessary conditions on $(s(n))_{1\le n\le N}$. However, even if there exists a sequence with discriminator $(d(n))_{1\le n\le N}$, it is not proven that enforcing these two conditions on $s(1), s(2), \ldots, s(i-1)$ for some $1 \le i < N$ is sufficient to find an integer $s(i)$ for which both FirstCond and SecondCond are true. As a result, it might be possible for DiscToSeqSeeker algorithm to return “Failure” when there actually does exist a sequence with the desired discriminator. However, there are no known examples for which DiscToSeqSeeker produces this false negative result.

Conjecture 24. *If DiscToSeqSeeker returns “Failure” for some input, then there does not exist any sequence of distinct integers whose discriminator sequence matches this input.*

Let us apply the DiscToSeqSeeker algorithm to find a finite sequence whose discriminator is $(n^2)_{1\le n\le 10}$. The lexicographically least such sequence is as follows:

$$
\begin{aligned}
s(1) &= 0,\\
s(2) &= 6,\\
s(3) &= 280,\\
s(4) &= 1710,\\
s(5) &= 552,\\
s(6) &= 526350,\\
s(7) &= 103230,\\
s(8) &= 1378,\\
s(9) &= 22014,\\
s(10) &= 47259.
\end{aligned}
$$

2.4.2 Sequences whose prefixes are discriminators of other sequences

The algorithms above can be used to determine whether a length-N prefix of an input sequence is a discriminator of the length-N prefix of another sequence. Rather than consider a specific value of N, it is an interesting problem to ask whether every prefix of a given input sequence is a discriminator of the prefix of some other sequence. Here we show that this property is maintained by the sequence of non-composite positive numbers, i.e., the sequence of 1 and the prime numbers.

Theorem 25. *Let $(d(n))_{n\geq 1} = 1, 2, 3, 5, 7, \ldots,$ be the sequence of non-composite positive integers, i.e., $d(1) = 1$ and $d(n)$ is the $(n-1)$-th prime number for $n \geq 2$. Then for each integer $N \geq 1$, there exists a finite sequence $(s_N(n))_{1\leq n\leq N}$ such that $D_{s_N}(n) = d(n)$ for all $1 \leq n \leq N$.*

Proof. We construct the first N terms of $s_N(n)$ by setting $s_N(n) = n$ for $1 \leq n \leq 3$ and enforcing the following conditions for $3 \leq n < N$:

1. $s_N(n+1) > s_N(n)$
2. $s_N(n+1) \equiv s_N(n) \pmod{\operatorname{lcm}(d(n), d(n)+1, \ldots, d(n+1)-1)}$
3. $s_N(n+1) \not\equiv s_N(a) \pmod{d(b)}$ for all integers a and b such that $1 \leq a \leq n < b \leq N$.

First, we show that there exists a sequence that can satisfy these three conditions. This can be seen by observing that $\operatorname{lcm}(d(n), d(n)+1, \ldots, d(n+1)-1)$, from the 2nd condition, is not divisible by any prime number that is greater or equal to $d(n+1)$, and that the modulus $d(b)$ in the third condition refers to a prime number greater or equal to $d(n+1)$.

It follows that the modulus in each of the constraints in the second and third conditions are all co-prime with each other, and thus, the Chinese Remainder Theorem can be applied to confirm the existence of a solution.

We induct on n to show that $D_{s_N}(n) = d(n)$ for $1 \leq n \leq N$.

Base Case For $1 \leq n \leq 3$, we have $D_{s_N}(n) = n = d(n)$.

Inductive Step Suppose that for all $n \leq k < N$, we have $D_{s_N}(n) = d(n)$. We show that $D_{s_N}(k+1) = d(k+1)$.

Since the discriminator is non-decreasing, we have $D_{s_N}(k+1) \geq D_{s_N}(k) = d(k)$. For all integers m in the range $d(k) \leq m < d(k+1)$, we know $m | \operatorname{lcm}(d(k), d(k)+1, \ldots, d(k+1)-1) | s_N(k+1) - s_N(k)$ from the second condition, thus disqualifying m from being the discriminator. So $D_{s_N}(k+1) \geq d(k+1)$.

In order to establish $D_{s_N}(k+1) \leq d(k+1)$, we show that $d(k+1) \nmid s_N(j) - s_N(i)$ for all integers i and j in the range $1 \leq i < j \leq k+1$. For all values in this range, we can apply the third condition for $n = j-1$, $a = i$, and $b = k+1$ to see that $s_N(j) \not\equiv s_N(i) \pmod{d(k+1)}$ and thus, $D_{s_N}(k+1) \leq d(k+1)$. Since $D_{s_N}(k+1) \geq d(k+1)$ as well, it follows that $D_{s_N}(k+1) = d(k+1)$.

Therefore, $D_{s_N}(n) = d(n)$ for all $1 \leq n \leq N$. $\square$

From this, we can deduce that every finite prefix of the sequence of non-composite integers is the discriminator of some finite sequence. An example of the construction of $(s_N(n))_{1 \leq n \leq N}$, for $N = 8$, is as follows:

$$\begin{aligned}
s_N(1) &= 1, \\
s_N(2) &= 2, \\
s_N(3) &= 3, \\
s_N(4) &= 39, \\
s_N(5) &= 279, \\
s_N(6) &= 10359, \\
s_N(7) &= 10755, \\
s_N(8) &= 98115.
\end{aligned}$$

However, it is not known whether there is an infinite sequence whose discriminator is the sequence of non-composite integers. If such a sequence $(s(n))_{n \geq 1}$ were to exist, then it would mean that for all $n \geq 2$, the set $\{s(1), s(2), \ldots, s(n)\}$ is discriminated by every prime number greater or equal to $D_s(n)$, which is the $(n-1)$-th prime number. In other words, the difference $s(j) - s(i)$ for all $1 \leq i < j$ does not contain any prime factors outside of the first $j-2$ primes. Constructing such an infinite sequence appears to be very difficult, and is conjectured to be impossible.

Conjecture 26. *There does not exist an infinite sequence whose discriminator is the sequence of non-composite integers.*

2.5 Self-Discriminators

An interesting problem to consider is to find sequences $(s(n))_{n\geq 1}$ such that $D_s(n) = s(n)$ for all $n \geq 1$. These are referred to as *self-discriminators*. In this section, we give a complete characterization of all self-discriminators.

Theorem 27. *Let $(s(n))_{n\geq 1}$ be an increasing sequence of integers. Then $(s(n))_{n\geq 1}$ is its own discriminator, i.e., $D_s(n) = s(n)$ for all $n \geq 1$, if and only if either $s(n) = n$ for all $n \geq 1$, or the following three conditions hold:*

1. *There exists an integer $t \geq 1$ such that $s(n) = n$ for $1 \leq n \leq t$ but $s(t+1) \neq t+1$,*
2. *$t+2 \leq s(t+1) \leq 2t+1$, and*
3. *$1 \leq s(n+1) - s(n) \leq t$ for all $n > t$.*

Proof. Note that $D_s(1) = 1$ for all sequences, so any self-discriminating sequence $(s(n))_{n\geq 1}$ should have $s(1) = 1$. If $s(n) = n$ for all $n \geq 1$, then it's clear that $D_s(n) = n$ for all $n \geq 1$, since we know $D_s(n) \geq n$ and the first n terms of $(s(n))_{n\geq 1}$ are all less than n, except the last one, which is 0 modulo n. Otherwise, the first condition is satisfied, where $t+1$ represents the index of the first element in which $s(n) \neq n$.

Clearly, $D_s(n) = n = s(n)$ for all $1 \leq n \leq t$. We show that, given the first condition, we have $D_s(n) = s(n)$ for all $n \geq 1$ if and only if the second and third conditions are also satisfied. There are two cases to consider here.

Case 1: $n = t+1$. If the second condition is satisfied, then for every m in the range $t+1 \leq m < s(t+1)$, we have $1 \leq s(t+1) - m \leq (2t+1) - (t+1) = t+1$. That is, $1 \leq s(t+1) - m \leq t$, and so, $m = s(t+1) - s(i)$ for some $1 \leq i \leq t$. Therefore, m fails to discriminate the first $t+1$ terms and so, $D_s(n) \geq s(t+1)$. However, $s(t+1)$ discriminates the first $t+1$ terms, since the first t terms are less than $s(t+1)$ and the last term is 0 modulo $s(t+1)$. Therefore, $D_s(t+1) = s(t+1)$.

If the second condition is not satisfied, then $s(t+1) \geq 2t+2$. In this case, choose $m = s(t+1) - (t+1)$. Since $s(t+1) \geq 2t+2$, we have $m \geq t+1$. So the first t terms are less than m while $s(t+1) \equiv t+1 \pmod{m}$, which means m discriminates the first $t+1$ terms. Therefore, $D_s(t+1) \leq m$ and so, $D_s(t+1) \neq s(t+1)$.

Case 2: $n > t+1$. If the third condition is satisfied, then for every m in the range $s(t+1) \leq m < s(n)$, there exists an index $i \leq n$ such that $1 \leq s(i) - m \leq t$ and so, m fails to discriminate the first n terms. The $s(n)$ discriminates the first n terms since they're all less than $s(n)$ except the last term, which is 0 modulo $s(t+1)$. Therefore, $D_s(n) = s(n)$.

Otherwise, if the third condition is not satisfied, then there is an integer $i > t$ such that $s(i+1) - s(i) > t$. Then choose $m = s(i+1) - (t+1) \geq s(i)$. Then m discriminates the first $i+1$ terms of the sequence, since $s(i+1) \equiv t+1 \pmod{m}$, which is not in the sequence, and all other terms of the sequence are less than m, unless $m = s(i)$, in which case $s(i) \equiv 0 \pmod{m}$, which is also not in the sequence and is distinct from $t+1 \pmod{m}$. Therefore, $D_s(i+1) \leq m$ and thus, $D_s(i+1) \neq s(i+1)$.

□

Corollary 28. *There are uncountably many increasing sequences of positive integers that are their own discriminators.*

Corollary 29. *For $1 \leq t \leq n$, the number of length-n finite self-discriminating sequences such that $s(i) = i$ for all $1 \leq i \leq t$ while $s(i) \neq i$ for $t < i \leq n$, is t^{n-t}. Hence the total number of finite sequences of length n that are self-discriminators is $\sum_{1 \leq t \leq n} t^{n-t}$.*

Proof. For these sequences, we have $s(i) = i$ for all $1 \leq i \leq t$. If $t = n$, there is exactly one such sequence. Otherwise, there are t possible values of $s(t+1)$, from $t+2$ to $2t+1$. For any subsequent terms (if there are any) of $s(i)$ for $i > t+1$, there are also t possibilities, from $s(i-1)+1$ to $s(i-1)+t$. After $s(t)$, there are $n-t$ remaining terms, which gives t^{n-t} possible extensions of length n. □

Note that the number of finite sequences of length n that are self-discriminators is given by sequence A026898 in Sloane's *On-Line Encyclopedia of Integer Sequences* [20].

2.6 Empirical results of common discriminators

We conclude this chapter with some empirical results that were obtained when trying to compute the discriminator prefixes of various sequences. These results are not particularly strong, with fewer than a hundred terms being considered for each sequence, but they can provide some insight on the complete discriminator sequences.

2.6.1 Discriminators of sequences in the OEIS

The first notable result arose by computing the discriminator prefix of the sequences in Sloane's *On-Line Encyclopedia of Integer Sequences* [20], and comparing them with other sequences in the OEIS. Out of the first 270,000 sequences in the OEIS, there were over 350 sequences with the discriminator sequence given by $(2^{\lceil \log_2 n \rceil})_{n \geq 0}$, which is A062383 in the OEIS.

Conjecture 30. *For all sequences of infinite length that are among the first 270,000 sequences in the OEIS, the most common discriminator sequence is* $(2^{\lceil \log_2 n \rceil})_{n \geq 0}$.

Of course, there are many different types of sequences in the OEIS, so it may be more useful to consider only certain classes of sequences. The later chapters in this thesis consider various sequences with discriminator $(2^{\lceil \log_2 n \rceil})_{n \geq 0}$.

2.6.2 Discriminators of general bounded sequences

For the more general case of considering all sequences, regardless of whether they are in the OEIS or not, it seems that the sequence $(2^{\lceil \log_2 n \rceil})_{n \geq 0}$, while still quite a popular discriminator sequence, does not seem to be the most common. In particular, we computed the discriminator of every possible finite increasing sequence of ten integers such that the first term is 1 and the remaining nine terms are bounded between 2 and 40 inclusive. There were 211,915,132 such sequences. Table 2.1 lists down the twenty most common discriminator sequences that were found.

Rank	Sequence	Frequency
1	1, 2, 4, 4, 8, 8, 8, 16, 16, 16	36,509
2	1, 2, 4, 7, 7, 7, 14, 14, 14, 14	36,360
3	1, 2, 4, 4, 8, 8, 16, 16, 16, 16	30,446
4	1, 2, 4, 4, 8, 8, 15, 15, 15, 15	24,516
5	1, 2, 4, 4, 8, 8, 14, 14, 14, 14	22,335
6	1, 2, 4, 7, 7, 7, 7, 14, 14, 14	22,244
7	1, 3, 5, 7, 7, 7, 14, 14, 14, 14	21,795
8	1, 2, 4, 4, 8, 8, 13, 13, 13, 13	20,418
9	1, 3, 3, 7, 7, 7, 14, 14, 14, 14	19,294
10	1, 2, 5, 7, 7, 7, 14, 14, 14, 14	19,121
11	1, 2, 3, 6, 6, 12, 12, 12, 12, 12	18,660
12	1, 2, 4, 4, 8, 8, 8, 15, 15, 15	18,586
13	1, 2, 3, 7, 7, 7, 14, 14, 14, 14	18,360
14	1, 2, 4, 7, 7, 14, 14, 14, 14, 14	18,312
15	1, 2, 5, 5, 7, 7, 14, 14, 14, 14	18,156
16	1, 2, 4, 8, 8, 8, 8, 16, 16, 16	17,721
17	1, 2, 4, 4, 7, 7, 14, 14, 14, 14	16,855
18	1, 3, 3, 6, 6, 12, 12, 12, 12, 12	16,556
19	1, 2, 4, 8, 8, 8, 15, 15, 15, 15	16,385
20	1, 2, 4, 4, 8, 8, 8, 8, 16, 16	16,369

Table 2.1: Most common discriminators of sequences bounded from 1 to 40

The most common discriminator sequence here is 1, 2, 4, 4, 8, 8, 8, 16, 16, 16. This was also the most common discriminator sequence when the bound was lowered to 30. This is the prefix of an infinite sequence, that we denote as $(d_1(n))_{n\geq 0} = 1, 2, 4, 4, \ldots$, which is the sequence with $d_1(0) = 1$ followed by the positive powers of 2 such that each power 2^i for $i > 0$ is repeated i times.

Conjecture 31. *Let N and B be positive integers. For all length-N increasing sequences with values bounded between 1 and B, the most common discriminator sequence is given by $(d_1(n))_{0\leq n<N}$.*

The sequence $(d_1(n))_{n\geq 0}$ does not seem to appear in the OEIS, but another sequence, A207872, is in the OEIS, which is similar to $(d_1(n))_{n\geq 0}$ except each power 2^i repeats $F(i)$ times instead of i times, where $(F(n))_{n\geq 0} = 1, 1, 2, 3, \ldots$, is the sequence of Fibonacci numbers.

Chapter 3

Discriminators of k-Regular Sequences

We now turn our attention to the discriminators of specific sequences or types of sequences. In this chapter, we consider the k-regular sequences. We characterize the discriminators for two 2-regular sequences, the so-called evil and odious numbers. In particular, we show that the odious numbers are the lexicographically least sequence with discriminator given by $2^{\lceil \log_2 n \rceil}$, which empirically seems to be the discriminator for many other sequences, as noted in the previous chapter. The discriminators for both the odious and evil numbers are also 2-regular, but we show that it is not necessarily the case that the discriminator of k-regular sequence is always k-regular, based on a counterexample. Finally, we conclude the chapter with a conjecture about the discriminator of Cantor numbers, which are also 2-regular sequences.

For this chapter, we use the following notation. Let Σ_k denote the k-letter alphabet $\{0, 1, \ldots, k-1\}$. If $x \in \Sigma_k^*$ is a string of digits, then $[x]_k$ denotes the value of x when considered as a base-k number. If n is an integer, then $(n)_k$ is the string giving the canonical base-k representation of n (with no leading zeroes). If x is a string of digits, then $|x|$ denotes the length of the string x, and $|x|_a$ denotes the number of occurrences of the letter a in x. Finally, $x^n = \overbrace{xx \cdots x}^{n}$ for $n \geq 0$.

By $S + i$, for S a set of integers and i an integer, we mean the set $\{x + i \ : \ x \in S\}$. For sets S and T, we write $S \sqcup T$ to denote the union of S and T, as well as the assertion that this union is actually disjoint.

3.1 k-regular sequences

Let $k \geq 2$ be an integer. The k-regular sequences, first introduced by Allouche and Shallit in 1992 [1], are an interesting class of sequences with notable closure properties. There are several equivalent ways of defining them, and here we provide three:

- They are the class of sequences $(s(n))_{n\geq 0}$ such that the set of subsequences of the form
$$\{(s(k^e n + i))_{n\geq 0} \;:\; e \geq 0 \text{ and } 0 \leq i < k^e\}$$
is a subset of a finitely-generated $\mathbb{Z}$-module.

- They are the class of sequences $(s(n))_{n\geq 0}$ for which there exist an integer $r \geq 1$, a $1 \times r$ row vector u, an $r \times 1$ column vector w, and an $r \times r$ matrix-valued morphism μ with domain Σ_k^* such that $s(n) = u\mu(v)w$ for all strings v with $[v]_k = n$.

- They are the class of sequences $(s(n))_{n\geq 0}$ such that there are a finite number of recurrence relations of the form
$$s(k^e n + i) = \sum_j a_j s(k^{e_j} n + i_j)$$
where $e \geq 0$, $e_j < e$, $0 \leq i < k^e$, and $0 \leq i_j < k^{e_j}$, that completely determine all but finitely many values of s.

The k-regular sequences also carry the following closure properties, as shown by Allouche and Shallit [1]:

Theorem 32. *Let* $\mathbf{r} = (r_i)_{i\geq 0}$ *and* $\mathbf{s} = (s_i)_{i\geq 0}$ *be two* k*-regular sequences of integers, and let* $m \geq 1$ *be an integer. Then so are*

(a) $\mathbf{r} + \mathbf{s} = (r_i + s_i)_{i\geq 0}$*;*

(b) $\mathbf{rs} = (r_i s_i)_{i\geq 0}$*;*

(c) $\mathbf{r} \bmod m = (r_i \bmod m)_{i\geq 0}$*.*

Note that the k-regular sequences are traditionally indexed as $n \geq 0$ whereas the discriminator is defined for $n \geq 1$. To maintain consistency for this chapter, we consider the discriminator sequence as being indexed as $n \geq 0$ as well, by setting $D_s(0) = 0$ for all integer sequences $(s(n))_{n\geq 0}$.

3.2 The evil and odious numbers

The so-called "evil" and "odious" numbers are two examples of 2-regular sequences; they are sequences A001969 and A000069 in Sloane's *On-Line Encyclopedia of Integer Sequences* [20], respectively. These numbers were named by Richard K. Guy c. 1976, and appear in the classic book *Winning Ways* [5, p. 431].

The evil numbers $(\mathrm{ev}(n))_{n\geq 0}$ are

$$0, 3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30, 33, 34, 36, 39, 40, 43, \ldots$$

and are those non-negative numbers having an even number of 1's in their base-2 expansion.

The odious numbers $(\mathrm{od}(n))_{n\geq 0}$ are

$$1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31, 32, 35, 37, 38, 41, \ldots$$

and are those non-negative numbers having an odd number of 1's in their base-2 expansion.

The names "evil" and "odious" are puns derived from "even" and "odd". Clearly the union of these two sequences is $\mathbb{N}$, the set of all non-negative integers.

To show that these two sequences are 2-regular, we note that both sequences satisfy the recurrence relations

$$\begin{aligned}
f(4n) &= -2f(n) + 3f(2n) \\
f(4n+1) &= -2f(n) + 2f(2n) + f(2n+1) \\
f(4n+2) &= \frac{2}{3}f(n) + \frac{5}{3}f(2n+1) \\
f(4n+3) &= 6f(n) - 3f(2n) + 2f(2n+1),
\end{aligned}$$

which can be proved by an induction using the characterization in [1, Example 12].

Let $\mathcal{O}_n = \{\mathrm{od}(i) : \mathrm{od}(i) < n\}$ (resp., $\mathcal{E}_n = \{\mathrm{ev}(i) : \mathrm{ev}(i) < n\}$) denote the set of all odious (resp., evil) numbers that are strictly less than n. The following lemma presents some properties of these sets.

Lemma 33. *(a) For $i \geq 1$ we have $|\mathcal{O}_{2^i}| = |\mathcal{E}_{2^i}| = 2^{i-1}$.*

(b) For $i \geq 1$ we have $\mathcal{O}_{2^{i+1}} = \mathcal{O}_{2^i} \sqcup (\mathcal{E}_{2^i} + 2^i)$.

(c) For $i \geq 1$ we have $\mathcal{E}_{2^{i+1}} = \mathcal{E}_{2^i} \sqcup (\mathcal{O}_{2^i} + 2^i)$.

Proof. (a) Let $0 \leq n < 2^i$. These n can be placed in 1–1 correspondence with the binary strings w of length i, using the correspondence $[w]_2 = n$. For each binary string x of length $i-1$, either $x0$ is odious and $x1$ is evil, or vice versa. Thus there are 2^{i-1} odious numbers less than 2^i, and 2^{i-1} evil numbers less than 2^i.

(b) Let $2^i \leq n < 2^{i+1}$. Consider $n - 2^i$. Since the base-2 expansion of $n - 2^i$ differs from that of n by omitting the first bit, clearly $n - 2^i$ is evil iff n is odious.

(c) Just like (b).

□

This gives the following corollary:

Corollary 34. *For integers $n \geq 0$ and $i \geq 1$ we have* $\mathrm{od}(n) \in \mathcal{O}_{2^i}$ *and* $\mathrm{ev}(n) \in \mathcal{E}_{2^i}$ *if and only if $n < 2^{i-1}$. Furthermore*

$$\mathrm{od}(2^{i-1}) = 2^i; \tag{3.1}$$
$$\mathrm{ev}(2^{i-1}) = 2^i + 1. \tag{3.2}$$

3.2.1 Discriminator of the odious numbers

We now turn our attention to the discriminators for the evil and odious numbers, starting with the odious numbers. First, we need to prove the following useful lemma.

Lemma 35. *Let $i \geq 1$ and $1 \leq m < 2^i$. Then there exist two odious numbers j, ℓ with $1 \leq j < \ell \leq 2^i$ such that $m = \ell - j$.*

Proof. Let $w = (m)_2$. There are three cases according to the form of w.

1. No 1 follows a 0 in w. Then $w = 1^a 0^b$, where $a \geq 1$, $b \geq 0$, and $a + b \leq i$. So $m = 2^b(2^a - 1)$. Take $\ell = 2^{a+b}$ and $j = 2^b$.
2. $w = x01y$, where $|xy|_1$ is odd. Take $j = 2^{|y|+1}$ and $\ell = m + 2^{|y|+1}$. Now $(\ell)_2 = x11y$, and clearly $|x11y|_1$ is odd, so ℓ is odious.
3. $w = x01y$, where $|xy|_1$ is even. Take $j = 2^{|y|}$ and $\ell = m + 2^{|y|}$. Now $(\ell)_2 = x10y$, and clearly $|x10y|_1$ is odd, so ℓ is odious.

□

With the help of this lemma, we can compute the discriminator for the sequence of odious numbers.

Theorem 36. *For the sequence of odious numbers, the discriminator $D_{\mathrm{od}}(n)$ satisfies the equation*

$$D_{\mathrm{od}}(n) = 2^{\lceil \log_2 n \rceil} \tag{3.3}$$

for $n \geq 1$.

Proof. The cases $n = 1, 2$ are left to the reader. Otherwise, let $i \geq 1$ be such that $2^i < n \leq 2^{i+1}$. We show $D_{\mathrm{od}}(n) = 2^{i+1}$. There are two cases:

Case 1: $n = 2^i + 1$. We compute the discriminator of $\mathrm{od}(0), \mathrm{od}(1), \ldots, \mathrm{od}(2^i) = 2^{i+1}$. By Lemma 35, for each $m < 2^{i+1}$, there exist two odious numbers j, ℓ with $1 \leq j \leq \ell \leq 2^{i+1}$ with $\ell - j = m$. So m does not discriminate $\{\mathrm{od}(0), \mathrm{od}(1), \ldots, \mathrm{od}(2^i)\}$ for $m < 2^{i+1}$. On the other hand, each of the numbers $\mathrm{od}(0), \mathrm{od}(1), \ldots, \mathrm{od}(2^i)$ are less than 2^{i+1} except $\mathrm{od}(2^i) = 2^{i+1} \equiv 0 \pmod{2^{i+1}}$, where 0 is not odious, thus implying that 2^{i+1} discriminates $\{\mathrm{od}(0), \mathrm{od}(1), \ldots, \mathrm{od}(2^i)\}$.

Case 2: $2^i + 1 < n \leq 2^{i+1}$. Since the discriminator is nondecreasing, we know $D_{\text{od}}(n) \geq 2^{i+1}$. It suffices to show that 2^{i+1} discriminates

$$\mathcal{O}_{2^{i+2}} = \{\text{od}(0), \text{od}(1), \ldots, \text{od}(2^{i+1} - 1)\}.$$

Now from Lemma 33(b), we have

$$\mathcal{O}_{2^{i+2}} = \mathcal{O}_{2^{i+1}} \sqcup (\mathcal{E}_{2^{i+1}} + 2^{i+1}).$$

If we now take both sides modulo 2^{i+1}, we see that the right-hand side is just $\mathcal{O}_{2^{i+1}} \sqcup \mathcal{E}_{2^{i+1}}$, which represents all integers in the range $[0, 2^{i+1})$.

□

As noted in the previous chapter, empirical results imply that there are many sequences of positive integers with discriminator $2^{\lceil \log_2 n \rceil}$. However, of all such sequences, the odious numbers play a special role: they are the lexicographically least.

Theorem 37. *The sequence of odious numbers is the lexicographically least increasing sequence of positive integers* $\mathbf{s}$ *such that* $D_s(n) = 2^{\lceil \log_2 n \rceil}$.

Proof. We prove this by contradiction. Suppose there exists a sequence of increasing positive integers, $s(0), s(1), \ldots$, that is lexicographically smaller than the sequence of odious numbers but shares the same discriminator, $D_s(n) = 2^{\lceil \log_2 n \rceil}$.

Let j denote the first index such that $s(j) \neq \text{od}(j)$, i.e., $s(j) < \text{od}(j)$, since s is a lexicographically smaller sequence than the odious numbers. We can see that $s(j)$ must be evil, because $\text{od}(j)$ is the next odious number after $\text{od}(j-1) = s(j-1)$. Note that since $\text{od}(0) = 1$ is the smallest positive integer, necessarily $j \geq 1$.

Now let $i \geq 0$ be such that $2^i \leq j < 2^{i+1}$. In that case, the discriminator of the sequence $s(0), s(1), \ldots, s(j)$ is $D_s(j+1) = 2^{\lceil \log_2 (j+1) \rceil} = 2^{i+1}$. However, $s(j)$ also discriminates this sequence, which implies that $s(j) \geq D_s(j+1) = 2^{i+1}$. Note that by the definition of j, this means that all odious numbers less than 2^{i+1} are present in the sequence $s(0), s(1), \ldots, s(j)$.

Furthermore, we have $s(j) < \text{od}(j) < \text{od}(2^{i+1}) = 2^{i+2}$. So $2^{i+1} \leq s(j) < 2^{i+2}$, which means that the largest power of 2 appearing in the binary representation of $s(j)$ is 2^{i+1}. Therefore $s(j) \bmod 2^{i+1} = s(j) - 2^{i+1}$ is odious. However, $s(j) \bmod 2^{i+1} < 2^{i+1}$. But the sequence $s(0), s(1), \ldots, s(j)$ contains all odious numbers less than 2^{i+1}, which therefore includes the result of $s(j) \bmod 2^{i+1}$. In other words, $s(j)$ is congruent to another number in this sequence modulo 2^{i+1}, i.e., $D_s(j+1) \neq 2^{i+1}$, which is a contradiction. □

3.2.2 Discriminator of the evil numbers

We now focus on the discriminator for the sequence of evil numbers. Here, we need to utilize a similar lemma as before.

Lemma 38. *Let $i \geq 3$ and $1 \leq m < 2^i - 3$. Then there exist two evil numbers j, ℓ with $0 \leq j < \ell \leq 2^i + 1$ such that $m = \ell - j$.*

Proof. Let $w = (m)_2$. There are several cases according to the form of w.

1. The number m is evil. Take $\ell = m$ and $j = 0$.
2. There are no 0's in w. Then $m = 2^a - 1$ where $0 < a < i$. Note that $a \neq i$. If $m = 1$, then take $\ell = 6$ and $j = 5$. Otherwise, take $\ell = 2^a + 2$ and $j = 3$.
3. No 1 follows a 0 in w and $|w|_0 > 0$. Then $w = 1^a 0^b$, where $a \geq 1$, $b \geq 1$, and $a+b \leq i$. So $m = 2^b(2^a - 1)$. Take $\ell = 2^{a+b} + 1$ and $j = 2^b + 1$.
4. There is exactly one 0 in w and w ends with 01. Then $w = 1^a 01$, where $1 \leq a \leq i-3$. So $m = 2^{a+2} - 3$. Take $\ell = 2^{a+2} + 2$ and $j = 5$.
5. There is exactly one 0 in w and w ends with 11. Then $w = 1^a 0 1^b$, where $a \geq 1$, $b \geq 2$, and $a + b \leq i - 1$. So $m = 2^{a+b+1} - 2^b - 1$. Take $\ell = 2^{a+b+1} + 1$ and $j = 2^b + 2$.
6. $w = x01y0z$, where $|xyz|_1$ is even. Take $j = 2^{|y|+|z|+1} + 2^{|z|}$ and $\ell = m + 2^{|y|+|z|+1} + 2^{|z|}$. So $(\ell)_2 = x10y1z$. We can see $|x10y1z|_1$ is even, so ℓ is evil.
7. $w = x0y01z$, where $|xyz|_1$ is even. Take $j = 2^{|y|+|z|+2} + 2^{|z|}$ and $\ell = m + 2^{|y|+|z|+2} + 2^{|z|}$. So $(\ell)_2 = x1y10z$. We can see $|x1y10z|_1$ is even, so ℓ is evil.

□

With the help of this lemma, we can compute the discriminator for the sequence of evil numbers.

Theorem 39. *For the sequence of evil numbers, the discriminator $D_{\mathrm{ev}}(n)$ satisfies the equation*

$$D_{\mathrm{ev}}(n) = \begin{cases} 2^{i+1} - 3, & \textit{if } n = 2^i + 1 \textit{ for odd } i \geq 2; \\ 2^{i+1} - 1, & \textit{if } n = 2^i + 1 \textit{ for even } i \geq 2; \\ 2^{\lceil \log_2 n \rceil}, & \textit{otherwise,} \end{cases} \tag{3.4}$$

for $n \geq 1$.

Proof. The cases $n = 1, 2, 3, 4$ are left to the reader. Otherwise, let $i \geq 2$ be such that $2^i < n \leq 2^{i+1}$. We show $D_{\mathrm{ev}}(n)$ satisfies the given equation. There are three cases presented in the equation:

Case 1: $n = 2^i + 1$ for odd $i \geq 2$. We compute the discriminator of $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i) = 2^{i+1} + 1$. By Lemma 38, for each $m < 2^{i+1} - 3$, there exist two evil numbers j, ℓ with $1 \leq j \leq \ell \leq 2^{i+1} + 1$ with $\ell - j = m$. So m does not discriminate $\{\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)\}$ for $m < 2^{i+1} - 3$.

Note that for odd $i \geq 2$, the only evil numbers in the range $[2^{i+1} - 3, 2^{i+1} + 1]$ are $2^{i+1} - 1$ and $2^{i+1} + 1$, easily observed from their binary representations. We can see that $2^{i+1} - 1 \equiv 2 \pmod{2^{i+1} - 3}$ and $2^{i+1} + 1 \equiv 4 \pmod{2^{i+1} - 3}$, where neither 2 nor 4 are evil. All the other numbers in the sequence $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)$ are less than $2^{i+1} - 3$, so it follows that $2^{i+1} - 3$ discriminates $\{\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)\}$.

Case 2: $n = 2^i + 1$ for even $i \geq 2$. We compute the discriminator of $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i) = 2^{i+1} + 1$. Just as in the previous case, Lemma 38 ensures that integers $m < 2^{i+1} - 3$ do not discriminate $\{\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)\}$.

For even $i \geq 2$, we see that both $2^{i+1} - 3$ and $2^{i+1} - 2$ are evil from their binary representations. Neither of them can discriminate the sequence since $m \bmod m = 0$ for either $m = 2^{i+1} - 3$ or $m = 2^{i+1} - 2$, while 0 is evil. Thus the discriminator must be at least $2^{i+1} - 1$. Since neither $2^{i+1} - 1$ nor 2^{i+1} are evil, we can see that each of the integers $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)$ are all less than $2^{i+1} - 1$ except $\mathrm{ev}(2^i) = 2^{i+1} + 1 \equiv 2 \pmod{2^{i+1} - 1}$, where 2 is not evil. Therefore, $2^{i+1} - 1$ discriminates $\{\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)\}$.

Case 3: $2^i + 1 < n \leq 2^{i+1}$. From the previous two cases, we know that $D_{\mathrm{ev}}(2^i + 1)$ is either $2^{i+1} - 3$ or $2^{i+1} - 1$. Since the discriminator is nondecreasing, we know $D_{\mathrm{ev}}(n) \geq 2^{i+1} - 3$. We see that the sequence $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(n-1)$ must include $\mathrm{ev}(2^i + 2) = 2^{i+1} + 2$, the next evil number after $2^{i+1} + 1$. We then observe that

$$
\begin{aligned}
2^{i+1} + 2 &\equiv 5 \pmod{2^{i+1} - 3}, \\
2^{i+1} + 1 &\equiv 3 \pmod{2^{i+1} - 2}, \\
2^{i+1} + 2 &\equiv 3 \pmod{2^{i+1} - 1},
\end{aligned}
$$

where the numbers 3 and 5 are evil. Therefore, the discriminator must be at least 2^{i+1}. It suffices to show that 2^{i+1} discriminates $\mathcal{E}_{2^{i+2}} = \{\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^{i+1}-1)\}$. Now from Lemma 33(c), we have

$$
\mathcal{E}_{2^{i+2}} = \mathcal{E}_{2^{i+1}} \sqcup (\mathcal{O}_{2^{i+1}} + 2^{i+1}).
$$

If we now take both sides modulo 2^{i+1}, we see that the right-hand side is just $\mathcal{E}_{2^{i+1}} \sqcup \mathcal{O}_{2^{i+1}}$, which represents all integers in the range $[0, 2^{i+1})$. Thus we have $D_{\mathrm{ev}}(n) = 2^{i+1} = 2^{\lceil \log_2 n \rceil}$ for $2^i + 1 < n \leq 2^{i+1}$.

$\square$

3.3 A k-regular sequence whose discriminator is not k-regular

The discriminators for the odious and evil numbers were 2-regular. This might raise the question of whether the discriminator of a k-regular sequence is always k-regular. In this section, we present a counterexample to show that this is not the case. In particular, the sequence of perfect squares, $(\mathrm{sq}(n))_{n\geq 0} = ((n+1)^2)_{n\geq 0} = 1, 4, 9, 16, \ldots$, is k-regular for all integers $k \geq 2$ [1, Example 5], but we show that its discriminator is not k-regular. Recall that the discriminator for the sequence of squares, given by Arnold, Benkoski, and McCabe [2], is given by

$$D_{\mathrm{sq}}(n) = \begin{cases} 1, & \text{if } n = 1; \\ 2, & \text{if } n = 2; \\ 6, & \text{if } n = 3; \\ 9, & \text{if } n = 4; \\ \min\{k \ : \ k \geq 2n \text{ and } (k = p \text{ or } k = 2p \text{ for some prime } p)\}, & \text{if } n > 4. \end{cases}$$

Theorem 40. *The discriminator sequence of the perfect squares is not k-regular for any k.*

Proof. We prove this by contradiction. Suppose $D_{\mathrm{sq}}(n)$ is k-regular. Then from Theorem 32 (c) we know that the sequence A given by $A(n) = D_{\mathrm{sq}}(n) \bmod 2$ is k-regular. From Theorem 32 (b) we know that the sequence $F(n) = A(n)D_{\mathrm{sq}}(n)$ is k-regular. From Theorem 32 (a) we know that the sequence $B(n) = 2 - 2A(n)$ is k-regular. From Theorem 32 (a) we know that the sequence $E(n) = F(n) + B(n)$ is k-regular. It is now easy to see that for $n > 4$ we have $E(n) = 2$ if $B(n)$ is even, while $E(n) = D_{\mathrm{sq}}(n)$ if $D_{\mathrm{sq}}(n)$ is odd. Thus $E(n)$ takes only prime values for $n > 4$.

We now argue that $(E(n))_{n\geq 0}$ is unbounded. To see this, it suffices to show that there are infinitely many indices n such that $D_{\mathrm{sq}}(n)$ is prime. By Dirichlet's theorem on primes in arithmetic progressions there are infinitely many primes p for which $p \equiv 1 \pmod 4$. For these primes consider $n = (p-1)/2$. Then $2n = p - 1$ is divisible by 4 and hence not twice a prime, but $2n + 1 = p$. Hence for these n we have $D_{\mathrm{sq}}(n) = p = 2n + 1$, and hence $E(n) = D_{\mathrm{sq}}(n)$. Thus $(E(n))_{n\geq 0}$ is unbounded.

Finally, we apply a theorem of Bell [4] to the sequence E. Bell's theorem states that any unbounded k-regular sequence must take infinitely many composite values. However, the sequence $(E(n))$ is unbounded and takes only prime values for $n > 4$. This contradiction shows that $D_{\mathrm{sq}}(n)$ cannot be k-regular. ∎

3.4 Discriminator of the Cantor numbers

In this section, we present a conjecture about the discriminator of another k-regular sequence, namely the Cantor numbers $(C(n))_{n\geq 0}$

$$0, 2, 6, 8, 18, 20, 24, 26, 54, 56, 60, 62, 72, 74, 78, 80, 162, 164, 168, 170, 180, \ldots$$

which are the numbers having only 0's and 2's in their base-3 expansion. This is sequence A005823 in Sloane's *On-Line Encyclopedia of Integer Sequences* [20]. It is 2-regular, as it satisfies the recurrence relations

$$\begin{aligned} C(2n) &= 3C(n) \\ C(2n+1) &= 3C(n) + 2. \end{aligned}$$

Based on some numerical computations, we have the following conjecture about the discriminator sequence $D_C(n)$ of the Cantor numbers:

$$\begin{aligned}
D_C(8n) &= \frac{13}{3} D_C(4n) - 2D_C(4n+1) + \frac{2}{3} D_C(4n+2) \\
D_C(8n+1) &= \frac{3}{2} D_C(2n) + \frac{7}{2} D_C(4n) - 2D_C(4n+1) + D_C(4n+2) \\
D_C(8n+2) &= \frac{10}{3} D_C(4n) - 2D_C(4n+1) + \frac{5}{3} D_C(4n+2) \\
D_C(8n+3) &= \frac{9}{2} D_C(2n) + \frac{11}{6} D_C(4n) - 3D_C(4n+1) + \frac{8}{3} D_C(4n+2) \\
D_C(8n+4) &= 6D_C(2n) - 2D_C(4n) + 2D_C(4n+1) + D_C(4n+2) \\
D_C(8n+5) &= 6D_C(2n) - 2D_C(4n) + D_C(4n+1) + 2D_C(4n+2) \\
D_C(8n+6) &= \frac{3}{2} D_C(2n) - \frac{1}{2} D_C(4n) - D_C(4n+1) + 4D_C(4n+2) \\
D_C(16n+7) &= -3D_C(2n) + D_C(4n) + 7D_C(4n+1) + 2D_C(4n+2) \\
D_C(16n+15) &= -9D_C(n) + \frac{27}{2} D_C(2n) - \frac{15}{2} D_C(4n) + 9D_C(4n+1) \\
&\quad - 6D_C(4n+2) + 10D_C(4n+3).
\end{aligned}$$

If true, this would mean that $D_C(n)$ is also 2-regular.

Chapter 4

Sequences with Shift-Invariant Discriminators

In the previous chapters, and in most of the previous work on discriminators, it is generally the first n terms of a sequence that is being considered, for $n \geq 2$. Therefore, the discriminator can depend crucially on the starting point of a given sequence. For example, although the discriminator for the first three positive squares, $\{1, 4, 9\}$, is 6, we can see that the number 6 does not discriminate the length-3 "window" into the shifted sequence, $\{4, 9, 16\}$, since $16 \equiv 4 \pmod 6$.

This chapter defines a special property in some sequences in that their discriminators are *shift-invariant*, i.e., independent of the starting point of the sequence. In other words, for all $n \geq 1$, the discriminator for the first n terms of a sequence with this property is also the discriminator for every n consecutive terms of this sequence. This idea was briefly introduced by Zieve [22], who considered sequences with discriminators that are shift-invariant for sufficiently large values of n.

This chapter presents a class of exponential sequences that are shift-invariant for all values of $n \geq 1$. There has been very little work on the discriminators of exponential sequences, with Sun [21] presenting some conjectures concerning certain exponential sequences, while Moree and Zumalacárrequi [15] computed the discriminator for the sequence $\left(\frac{|(-3)^j-5|}{4}\right)_{j\geq 0}$.

The discriminator for this class of exponential sequences is also of the form $2^{\lceil \log_2 n \rceil}$, which seems to be a popular discriminator sequence as seen from previous chapters. It may be worth noting that this discriminator sequence grows only linearly, despite discriminating a class of sequences that grow exponentially.

4.1 Shift-Invariant Discriminators

We say that the discriminator of a sequence is *shift-invariant* if the discriminator for the sequence is the same even if the sequence is shifted by any positive integer c, i.e., for all positive integers c the discriminator of the sequence $(s(n))_{n\geq 1}$ is the same as the discriminator of the sequence $(s(n+c))_{n\geq 0}$.

As a simple example, all sequences of the form $(\alpha n+\beta)_{n\geq 0}$ have shift-invariant discriminators. This is easily seen by observing that for any $c>0$, the sequence $(\alpha(n+c)+\beta)_{n\geq 0} = (\alpha n+\beta+\alpha c)_{n\geq 0}$ can be formed by adding αc to each term of the original sequence, $(\alpha n+\beta)_{n\geq 0}$, and so the discriminator is the same. The next chapter on quadratic sequences contains some more examples of sequences with shift-invariant discriminators.

The following lemma presents an interesting property of sequences with shift-invariant discriminators, though this property is not utilized anywhere in this thesis.

Lemma 41. *Let $n_0 \geq 1$ be an integer and let $(s(n))_{n\geq 0}$ be an integer sequence with a shift-invariant discriminator $D_s(n)$. Then the sequence of residues $(s(n) \bmod D_s(n_0))_{n\geq 0}$ is periodic for all $n_0 \geq 1$. Furthermore, the size of the period is the positive integer n_1 such that $D_s(n_0) = D_s(n_1)$ and $D_s(n_1) \neq D_s(n_1+1)$.*

Proof. From the property that the discriminator is shift-invariant, we know that for all $n \geq 0$, the integers $s(n+1), s(n+2), \ldots, s(n+n_1)$ are distinct modulo $D_s(n_1) = D_s(n_0)$. Therefore, $s(n+n_1) \not\equiv s(n+i) \pmod{D_s(n_1)}$ for all $1 \leq i \leq n_1 - 1$.

Now, we know that the integers $s(n), s(n+1), \ldots, s(n+n_1-1)$ are also distinct modulo $D_s(n_1) = D_s(n_0)$. If we suppose that $s(n) \not\equiv s(n+n_1) \pmod{D_s(n_1)}$, then it follows that $D_s(n_1)$ discriminates $\{s(n), s(n+1), \ldots, s(n+n_1)\}$. But since the discriminator is shift-invariant, this means that $D_s(n_1+1) = D_s(n_1)$, which is a contradiction. Therefore, we have $s(n) \equiv s(n+n_1) \pmod{D_s(n_1)}$ for all $n \geq 0$, and so, the sequence of $(s(n) \bmod D_s(n_1))_{n\geq 0} = (s(n) \bmod D_s(n_0))_{n\geq 0}$ is periodic with period n_1. □

4.2 A class of exponential sequences whose discriminators are shift-invariant

The main result of this chapter is to show that the discriminator of a certain class of exponential sequences is shift-invariant. We define this class as follows:

$$(\mathrm{ex}(n))_{n\geq 0} = \left(a\frac{(t^2)^n - 1}{2^b}\right)_{n\geq 0}$$

for odd positive integers a and t, where b is the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. A typical example is the sequence $\left(\frac{9^n-1}{8}\right)_{n\geq 0}$. We show that the discriminator for all sequences of this form is $D_{\mathrm{ex}}(n) = 2^{\lceil \log_2 n \rceil}$. Furthermore, we show that this discriminator is shift-invariant, i.e., it applies to every sequence $(\mathrm{ex}(n+c))_{n\geq 0}$ for $c \geq 0$.

Our approach involves proving the upper bound for the discriminator of the sequence $\left(\frac{(t^2)^n-1}{2^b}\right)_{n\geq 0}$ and all of shifts, and then separately proving some lemmas that are essential to establishing the lower bound of the discriminator, before combining the results to determine the discriminator of $(\mathrm{ex}(n))_{n\geq 0}$ and all of its shifts.

4.2.1 Upper bound

In this section, we derive an upper bound for the discriminator of the sequence $\left(\frac{(t^2)^n-1}{2^b}\right)_{n\geq 0}$ and all of its shifts. We start with some useful lemmas.

Lemma 42. *Let t be an odd integer, and let b be the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then $t^2 \equiv 2^b + 1 \pmod{2^{b+1}}$.*

Proof. Note that since every odd integer equals ± 1 modulo 4, we must have $b \geq 3$. From the definition of b, we have $t \equiv 2^{b-1} \pm 1 \pmod{2^b}$. Hence $t = 2^b c + 2^{b-1} \pm 1$ for some integer c. By squaring both sides of the equation, we get

$$\begin{aligned} t^2 &= 2^{2b}c^2 + 2^{2(b-1)} + 2^{2b}c \pm 2^{b+1}c \pm 2^b + 1 \\ &= 2^{b+1}\left(2^{b-1}c^2 + 2^{b-3} + 2^{b-1}c \pm c\right) \pm 2^b + 1, \\ \implies t^2 &\equiv \pm 2^b + 1 \pmod{2^{b+1}}, \\ \implies t^2 &\equiv 2^b + 1 \pmod{2^{b+1}}. \end{aligned}$$

□

Lemma 43. *Let t be an odd integer, and let b be the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then we have*

$$t^{2^k} \equiv 2^{k+b-1} + 1 \pmod{2^{k+b}} \tag{4.1}$$

for all integers $k \geq 1$.

Proof. By induction on k.

Base case: From Lemma 42, we have $t^2 \equiv 2^b + 1 \pmod{2^{b+1}}$.

Induction: Suppose Eq. (4.1) holds for some $k \geq 1$, i.e., $t^{2^k} \equiv 2^{k+b-1} + 1 \pmod{2^{k+b}}$. This means that $t^{2^k} = 2^{k+b}c + 2^{k+b-1} + 1$ for some integer c. Once again, by squaring both sides of the equation, we get

$$\begin{aligned} \left(t^{2^k}\right)^2 = t^{2^{k+1}} &= 2^{2k+2b}c^2 + 2^{2k+2b-2} + 1 + 2^{2k+2b}c + 2^{k+b+1}c + 2^{k+b} \\ &= 2^{k+b+1}\left(2^{k+b-1}c^2 + 2^{k+b-3} + 2^{k+b-1}c + c\right) + 2^{k+b} + 1, \\ \implies t^{2^{k+1}} &\equiv 2^{k+b} + 1 \pmod{2^{k+b+1}}. \end{aligned}$$

This shows that Eq. (4.1) holds for $k+1$ as well, thus completing the induction.

$\square$

This gives the following corollary.

Corollary 44. *Let t be an odd integer, and let b be the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then for $k \geq 1$, the powers of t^2 form a cyclic subgroup of order 2^k in $(\mathbb{Z}/2^{k+b})^*$.*

Proof. Let $\ell = k+1$. Since $\ell \geq 1$, we can apply Eq. (4.1) to get

$$\begin{aligned} (t^2)^{2^{\ell-1}} = t^{2^\ell} &\equiv 2^{\ell+b-1} + 1 \pmod{2^{\ell+b}}, \\ \implies (t^2)^{2^{\ell-1}} &\equiv 1 \pmod{2^{\ell+b-1}}, \\ \implies (t^2)^{2^k} &\equiv 1 \pmod{2^{k+b}}. \end{aligned}$$

Furthermore, by applying Eq. (4.1) directly, we get

$$\begin{aligned} (t^2)^{2^{k-1}} = t^{2^k} &\equiv 2^{k+b-1} + 1 \not\equiv 1 \pmod{2^{k+b}}, \\ \implies (t^2)^{2^{k-1}} &\not\equiv 1 \pmod{2^{k+b}}. \end{aligned}$$

Therefore, the order of the subgroup generated by t^2 in $(\mathbb{Z}/2^{k+b})^*$ is 2^k. $\square$

Lemma 45. *Let t be an odd integer, and let b be the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then for $k \geq 0$, the number 2^k discriminates every set of 2^k consecutive terms of the sequence $\left(\frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$.*

Proof. For every $i \geq 0$, it follows from Corollary 44 that the numbers

$$(t^2)^i, (t^2)^{i+1}, \ldots, (t^2)^{i+2^k-1}$$

are distinct modulo 2^{k+b}. By subtracting 1 from every element, we have that the numbers

$$(t^2)^i - 1, (t^2)^{i+1} - 1, \ldots, (t^2)^{i+2^k-1} - 1$$

are distinct modulo 2^{k+b}. Furthermore, these numbers are also congruent to 0 modulo 2^b because $t^2 \equiv 1 \pmod{2^b}$ from Lemma 42. It follows that the set of quotients

$$\left\{ \frac{(t^2)^i - 1}{2^b}, \frac{(t^2)^{i+1} - 1}{2^b}, \ldots, \frac{(t^2)^{i+2^k-1} - 1}{2^b} \right\}$$

consists of integers that are distinct modulo $\frac{2^{k+b}}{2^b} = 2^k$.

Such a set of quotients coincides with every set of 2^k consecutive terms of the sequence $\left(\frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$. Since the numbers in each set are distinct modulo 2^k, the desired result follows. $\square$

4.2.2 Lower bound

In this section, we establish some results useful for the lower bound on the discriminator of the sequence $\left(\frac{(t^2)^n-1}{2^b}\right)_{n\geq 0}$. We start with the following technical lemma:

Lemma 46. *Let m be a positive integer. Then $\log_3 m \leq \frac{m}{3}$.*

Proof. First, we prove that $m^3 \leq 3^m$ by induction for integers $m \geq 1$.

Base Case: By observation, this is true for $1 \leq m \leq 3$.

Induction: Suppose for some $m \geq 3$, it is true that $m^3 \leq 3^m$. We show that $(m+1)^3 \leq 3^{m+1}$ as well. Clearly, $(m+1)^3 = m^3 + 3m^2 + 3m + 1$. From the inductive hypothesis, we know $m^3 \leq 3^m$. Furthermore, since $m \geq 3$, we have

$$3m^2 \leq m^3 \leq 3^m,$$
$$3m + 1 \leq 3m + 6m = 9m = 3^2 m \leq m^3 \leq 3^m.$$

It follows that

$$(m+1)^3 = m^3 + 3m^2 + 3m + 1 \leq 3^m + 3^m + 3^m = 3(3^m) = 3^{m+1}.$$

Therefore, we have $m^3 \leq 3^m$. This implies that

$$m^3 \leq 3^m \implies m \leq 3^{m/3} \implies \log_3 m \leq \frac{m}{3}.$$

□

The main lemma for proving the lower bound is as follows:

Lemma 47. *Let t be an odd integer, and let b be the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then for all $k \geq 0$ and $1 \leq m \leq 2^{k+1}$, there exists a pair of integers, i and j, where $0 \leq i < j \leq 2^k$, such that $(t^2)^i \equiv (t^2)^j \pmod{2^b m}$.*

Proof. Let the prime factorization of m be

$$m = 2^x \prod_{1\leq \ell \leq u} p_\ell^{y_\ell} \prod_{1 \leq \ell \leq v} q_\ell^{z_\ell},$$

where $u, v, x, y_\ell, z_\ell \geq 0$, while $p_1, p_2, \ldots, p_u$ are the prime factors of m that also divide t, and $q_1, q_2, \ldots, q_v$ are the odd prime factors of m that do not divide t. For each $\ell \leq u$, let e_ℓ be the integer such that $p_\ell^{e_\ell} || t$, i.e., we have $p_\ell^{e_\ell} | t$ but $p_\ell^{e_\ell+1} \nmid t$.

We need to find a pair (i, j) such that $(t^2)^i \equiv (t^2)^j \pmod{2^b m}$. From the Chinese remainder theorem, we know it suffices to find a pair (i, j) such that

$$\begin{aligned}(t^2)^i &\equiv (t^2)^j \pmod{2^{x+b}},\\ (t^2)^i &\equiv (t^2)^j \pmod{p_\ell^{y_\ell}}, \text{ for all } 1 \le \ell \le u,\\ \text{and } (t^2)^i &\equiv (t^2)^j \pmod{q_\ell^{z_\ell}}, \text{ for all } 1 \le \ell \le v.\end{aligned}$$

For the first equation, we know from Corollary 44 that $(t^2)^i \equiv (t^2)^{i+2^x} \pmod{2^{x+b}}$. In other words, it suffices to have $2^x | (j-i)$ to satisfy $(t^2)^i \equiv (t^2)^j \pmod{2^{x+b}}$.

Next, we consider the u equations of the form $(t^2)^i \equiv (t^2)^j \pmod{p_\ell^{y_\ell}}$. Since $p_\ell^{e_\ell}$ is a factor of t, it follows that $(t^2)^{y_\ell/2e_\ell}$ is a multiple of $(p_\ell^{2e_\ell})^{y_\ell/2e_\ell} = p_\ell^{y_\ell}$. Therefore, $(t^2)^{y_\ell/2e_\ell} \equiv 0 \pmod{p_\ell^{y_\ell}}$. Any further multiplication by t^2 also yields 0 modulo $p_\ell^{y_\ell}$. Thus, it suffices to have $j > i \ge \frac{y_\ell}{2e_\ell}$ in order to ensure that $(t^2)^i \equiv (t^2)^j \pmod{p_\ell^{y_\ell}}$.

Finally, there are v equations of the form $(t^2)^i \equiv (t^2)^j \pmod{q_\ell^{z_\ell}}$. In each case, q_ℓ is co-prime to t, which means that $(t^2)^{\varphi(q_\ell^{z_\ell})/2} = t^{\varphi(q_\ell^{z_\ell})} \equiv 1 \pmod{q_\ell^{z_\ell}}$, where $\varphi(n)$ is Euler's totient function. Now $\frac{\varphi(q_\ell^{z_\ell})}{2} = \frac{q_\ell^{z_\ell - 1}(q_\ell - 1)}{2}$. Thus, it is sufficient to have $\frac{q_\ell^{z_\ell - 1}(q_\ell - 1)}{2} | (j-i)$ in order to ensure that $(t^2)^i \equiv (t^2)^j \pmod{q_\ell^{z_\ell}}$.

Merging these ideas together, we choose the following values for i and j:

$$\begin{aligned}i &= \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil,\\ j &= \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil + 2^x \prod_{1\le\ell\le v} \frac{q_\ell^{z_\ell-1}(q_\ell - 1)}{2},\end{aligned}$$

to ensure that $(t^2)^i \equiv (t^2)^j \pmod{2^b m}$. It is clear that $0 \le i < j$. In order to show that $j \le 2^k$, we first observe that

$$\begin{aligned}j &= \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil + 2^x \prod_{1\le\ell\le v} \frac{q_\ell^{z_\ell-1}(q_\ell - 1)}{2} = \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil + \frac{2^x}{2^v} \prod_{1\le\ell\le v} q_\ell^{z_\ell-1}(q_\ell - 1)\\ &\le \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{2^x}{2^v} \prod_{1\le\ell\le v} q_\ell^{z_\ell} = \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{m}{2^v \prod_{1\le\ell\le u} p_\ell^{y_\ell}}.\end{aligned}$$

We now consider the following two cases:

Case 1: $u = 0$**.** If $v = 0$ as well, then $j = 2^x = m < 2^{k+1}$, which means that $x \le k$ and thus $j \le 2^k$. Otherwise, if $v \ge 1$, then we have

$$j \le \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{m}{2^v \prod_{1\le\ell\le u} p_\ell^{y_\ell}} = \frac{m}{2^v} \le \frac{m}{2} < \frac{2^{k+1}}{2} = 2^k.$$

Case 2: $u \geq 1$. Let r be such that $y_r = \max_{1\leq \ell \leq u} y_\ell$, and thus, p_r is the corresponding prime number with exponent y_r. Since $p_r^{y_r} \geq p_r \geq 3$, we have

$$j \leq \max_{1\leq \ell\leq u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{m}{2^v \prod_{1\leq \ell\leq u} p_\ell^{y_\ell}} \leq \left\lceil \frac{y_r}{2} \right\rceil + \frac{m}{p_r^{y_r}} \leq \frac{y_r+1}{2} + \frac{m}{3} \leq \frac{y_r}{2} + \frac{1}{2} + \frac{m}{3}.$$

Note that $y_r \leq \log_{p_r} m \leq \log_3 m \leq \frac{m}{3}$ from Lemma 46, which means that

$$j \leq \frac{y_r}{2} + \frac{1}{2} + \frac{m}{3} \leq \frac{m}{6} + \frac{1}{2} + \frac{m}{3} = \frac{m}{2} + \frac{1}{2} = \frac{m+1}{2}.$$

Since both m and j are integers, this implies that

$$j \leq \left\lceil \frac{m}{2} \right\rceil \leq \left\lceil \frac{2^{k+1}}{2} \right\rceil \leq 2^k.$$

In both cases, we have $j \leq 2^k$, thus fulfilling the required conditions. □

4.2.3 Discriminator of $(\mathrm{ex}(n))_{n\geq 0}$ and its shifted counterparts

In this section, we combine the results of the previous sections to determine the discriminator for $(\mathrm{ex}(n))_{n\geq 1}$, as well as its shifted counterparts. First, we recall the following lemma from Chapter 1:

Lemma 1. *Given a sequence $s(0), s(1), \ldots,$ and a non-zero integer a, let $s'(0), s'(1), \ldots,$ denote the sequence such that $s'(i) = as(i)$ for all $i \geq 0$. Then, for every n such that $\gcd(|a|, D_s(n)) = 1$, we have $D_{s'}(n) = D_s(n)$.*

We now compute the discriminator for $(\mathrm{ex}(n))_{n\geq 0} = \left(a\frac{(t^2)^n-1}{2^b}\right)_{n\geq 0}$, and also for its shifted counterparts, which we denote by $(\mathrm{exs}(n,c))_{n\geq 0} = (\mathrm{ex}(n+c))_{n\geq 0}$ for some integer $c \geq 0$.

Theorem 48. *Let t, a, b, and c be integers such that a and t are odd, $c \geq 0$, and let b be the smallest integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then the discriminator for the sequence $(\mathrm{exs}(n,c))_{n\geq 0} = \left(a\frac{(t^2)^{n+c}-1}{2^b}\right)_{n\geq 0}$ is*

$$D_{\mathrm{exs}}(n) = 2^{\lceil \log_2 n \rceil}. \tag{4.2}$$

Proof. First we compute the discriminator for $a = 1$, where the sequence is of the form $(\mathrm{exs}(n))_{n\geq 0} = \left(\frac{(t^2)^{n+c}-1}{2^b}\right)_{n\geq 0}$.

The case for $n = 1$ is trivial. Otherwise, let $k \geq 0$ be such that $2^k < n \leq 2^{k+1}$. We show that $D_{\mathrm{exs}}(n) = 2^{k+1}$.

From Lemma 45, we know that 2^{k+1} discriminates the set,

$$\{\mathrm{ex}(c), \mathrm{ex}(c+1), \ldots, \mathrm{ex}(c+2^{k+1}-1)\},$$

as well as every smaller subset of these numbers. Therefore, 2^{k+1} discriminates

$$\{\mathrm{exs}(0,c), \mathrm{exs}(1,c), \ldots, \mathrm{exs}(n-1,c)\}.$$

In other words, $D_{\mathrm{exs}}(n) \le 2^{k+1}$.

Now let m be a positive integer such that $m < 2^{k+1}$. By Lemma 47, we know that there exists a pair of integers, i and j, such that

$$\begin{aligned}(t^2)^i \equiv (t^2)^j \pmod{2^b m} &\implies (t^2)^c (t^2)^i \equiv (t^2)^c (t^2)^j \pmod{2^b m},\\ &\implies (t^2)^{i+c} - 1 \equiv (t^2)^{j+c} - 1 \pmod{2^b m}.\end{aligned}$$

Note that since $(t^2) \equiv 1 \pmod{2^b}$ from Lemma 42, we have $(t^2)^{i+c} - 1 \equiv (t^2)^{j+c} - 1 \equiv 1 - 1 \equiv 0 \pmod{2^b}$. Therefore,

$$(t^2)^{i+c} - 1 \equiv (t^2)^{j+c} - 1 \pmod{2^b m} \implies \frac{(t^2)^{i+c} - 1}{2^b} \equiv \frac{(t^2)^{j+c} - 1}{2^b} \pmod{m}.$$

In other words, $\mathrm{exs}(i,c) \equiv \mathrm{exs}(j,c) \pmod{m}$ while both numbers are in the set

$$\{\mathrm{exs}(0,c), \mathrm{exs}(1,c), \ldots, \mathrm{exs}(n-1,c)\}$$

since $i < j \le 2^k < n$. Therefore, m fails to discriminate this set. Since this applies for all $m < 2^{k+1}$, we have $D_{\mathrm{exs}}(n) \ge 2^{k+1}$.

Since we have $2^{k+1} \le D_{\mathrm{exs}} \le 2^{k+1}$, this means that $D_{\mathrm{exs}}(n) = 2^{k+1}$ and thus $D_{\mathrm{exs}}(n) = 2^{\lceil \log_2 n \rceil}$, provided that $a = 1$.

Even for $a \ne 1$, we observe that the value of $2^{\lceil \log_2 n \rceil}$ is a power of 2 for all n, and so it is co-prime to all odd a. Therefore, we can apply Lemma 1 to prove that the discriminator remains unchanged for odd values of a, thus proving that the discriminator for the sequence, $(\mathrm{exs}(n,c))_{n\ge 0} = \left(a\frac{(t^2)^{n+c}-1}{2^b}\right)_{n\ge 0}$ is $D_{\mathrm{exs}}(n) = 2^{\lceil \log_2 n \rceil}$. $\square$

Chapter 5

Quadratic sequences with discriminator $p^{\lceil \log_p n \rceil}$ for prime p

So far, we encountered a few sequences whose discriminator is given by $2^{\lceil \log_2 n \rceil}$, most notably the odious numbers in Chapter 2 and the class of exponential sequences discussed in Chapter 3 such as $\left(\frac{9^n-1}{8}\right)_{n\geq 0}$. As noted in Chapter 2 from empirical results, there seem to be many sequences whose discriminators are given by $2^{\lceil \log_2 n \rceil}$.

In this chapter, we examine sequences given by a quadratic formula, and provide a complete characterization of all such sequences that have the discriminator $2^{\lceil \log_2 n \rceil}$. We then extend this idea to the general discriminator sequence of the form $p^{\lceil \log_p n \rceil}$ for any prime p. In particular, we show that there are no quadratic sequences with integer coefficients with such a discriminator for $p \geq 5$, and provide some necessary and sufficient conditions for the case of $p = 3$.

5.1 Approach

We denote quadratic sequences by $(q(n))_{n\geq 0} = (\alpha n^2 + \beta n + \gamma)_{n\geq 0}$, for rational numbers α, β, and γ. Our approach involves exploiting the property that for any $n \geq 1$, the discriminator $D_q(n)$ is the smallest integer that does not divide $q(j) - q(i)$ for all pairs of integers i and j such that $0 \leq i < j < n$. Here we can see that

$$\begin{aligned} q(j) - q(i) &= \alpha j^2 + \beta j - \alpha i^2 - \beta i = \alpha(j^2 - i^2) + \beta(j - i) \\ &= (j - i)(\alpha(i + j) + \beta). \end{aligned} \tag{5.1}$$

Eq. (5.1) is used to prove various results in later sections. Some of these results are used to show that some quadratic sequences have discriminator $D_q(n) = p^{\lceil \log_p n \rceil}$ for some prime p. This is accomplished by applying the following lemma:

Lemma 49. *Let $p \geq 2$ be a prime number and let $(s(n))_{n\geq 0}$ be a sequence of distinct integers that satisfies the following conditions:*

1. *For all pairs of integers k and m such that $k \geq 0$ and $0 \leq m < p^{k+1}$, there exists a pair of integers i and j such that $0 \leq i < j \leq p^k$ and $m|s(j) - s(i)$;*

2. *For all integers k, i, and j such that $k \geq 0$ and $0 \leq i < j < p^{k+1}$, we have $p^{k+1} \nmid s(j) - s(i)$.*

Then $D_s(n) = p^{\lceil \log_p n \rceil}$ for $n > 0$.

Proof. The case $n = 1$ follows from the fact that $D_s(1) = 1$ regardless of the given conditions. Otherwise, let $k \geq 0$ be such that $p^k < n \leq p^{k+1}$. From the first condition, we know that for all $0 \leq m < p^{k+1}$, there exists a pair of integers i and j such that $0 \leq i < j \leq p^k \leq n - 1$ and $m|s(j) - s(i)$. This means that m does not discriminate the set

$$\{s(0), s(1), \ldots, s(n-1)\},$$

and thus, $D_s(n) \geq p^{k+1}$ for all $p^k < n \leq p^{k+1}$.

Furthermore, from the second condition, we know that $p^{k+1} \nmid s(j) - s(i)$ as long as $0 \leq i < j < p^{k+1}$. So for $p^k < n \leq p^{k+1}$, we know p^{k+1} cannot divide $s(j) - s(i)$ for all i, j in the range $0 \leq i < j \leq n - 1$ since $n - 1 < p^{k+1}$. Therefore, $D_s(n) = p^{k+1} = p^{\lceil \log_2 n \rceil}$. □

Other results in later sections involve scenarios in which $D_s(n) \neq p^{\lceil \log_p n \rceil}$ for some prime p and integer $n \geq 1$. This is achieved with the help of the following lemma:

Lemma 50. *Let $p \geq 2$ be a prime number and let $(q(n))_{n\geq 0} = (\alpha n^2 + \beta n + \gamma)_{n\geq 0}$ be a quadratic sequence such that α, β, and γ are integers that satisfy any of the following conditions:*

1. *$p \nmid \alpha$;*

2. *$p|\beta$;*

3. *$\alpha = p^k c$ for some integer c such that $c \nmid \beta$.*

Then there exists a value of $n > 0$ such that $D_q(n) \neq p^{\lceil \log_p n \rceil}$.

Proof. We can assume $\gamma = 0$ since the discriminator does not depend on γ. We now consider the conditions one by one, while recalling from Eq. (5.1) that $q(j) - q(i) = (j - i)(\alpha(i + j) + \beta)$ for all pairs of integers i and j.

Case 1: $p \nmid \alpha$**.** For any integer $\ell \geq 2$, we show that there exists a pair of i and j such that $0 \leq i < j < p^\ell$ and $p^\ell | q(j) - q(i)$. Since $p \nmid \alpha$, this implies that α and p^ℓ are co-prime. We choose $i = 0$ and $j = -\beta(\alpha)^{-1} \bmod p^\ell$. Then

$$\begin{aligned} q(j) - q(i) = j(\alpha j + \beta) &\equiv j(\alpha(-\beta)(\alpha)^{-1} + \beta) \pmod{p^\ell} \\ &\equiv j(-\beta + \beta) \equiv 0 \pmod{p^\ell}. \end{aligned}$$

Since $p^\ell | q(j) - q(i)$ while $0 \leq i < j < p^\ell$, it follows that $D_q(p^\ell) \neq p^\ell$ and so, $D_q(n) \neq p^{\lceil \log_p n \rceil}$ for $n = p^\ell$.

Case 2: $p | \alpha$ **and** $p | \beta$**.** Again, for any integer $\ell \geq 2$, we show that there exists a pair of i and j such that $0 \leq i < j < p^\ell$ and $p^\ell | q(j) - q(i)$. Here, we choose $i = 0$ and $j = p^{\ell-1}$ to get

$$q(j) - q(i) = (j - i)(\alpha(i + j) + \beta) = p^{\ell-1}(p^{\ell-1}\alpha + \beta) = p^\ell(p^{\ell-2}\alpha + \frac{\beta}{p}),$$

noting that $p^{\ell-2}$ and $\frac{\beta}{p}$ are integers. Just as with Case 1, this implies that $D_q(p^\ell) \neq p^\ell$ and so $D_q(n) \neq p^{\lceil \log_p n \rceil}$ for $n = p^\ell$.

Case 3: $p | \alpha$**,** $p \nmid \beta$**, but** $c \nmid \beta$**, where** $\alpha = p^k c$ **for** $p \nmid c$**.** Let r be any prime number such that $r | c$ and $r \nmid \beta$. For any pair of integers i and j such that $0 \leq i < j < r$, we have $q(j) - q(i) = (j - i)(\alpha(i + j) + \beta)$. Since $j < r$, we have $r \nmid (j - i)$. We also have $r \nmid \alpha(i + j) + \beta$ since $r | \alpha$ but $r \nmid \beta$. Therefore, $r \nmid q(j) - q(i)$. It follows that $D_q(r) \leq r < p^{\lceil \log_p r \rceil}$, i.e., $D_q(n) \neq p^{\lceil \log_p n \rceil}$ for $n = r$.

In all cases, we have $D_q(n) \neq p^{\lceil \log_p n \rceil}$ for at least one value of $n \geq 1$. $\square$

Furthermore, to help generalize some results, we recall the following lemma from Chapter 1.

Lemma 1. *Given a sequence $s(0), s(1), \ldots$ and a non-zero integer c, let $s'(0), s'(1), \ldots$ denote the sequence such that $s'(i) = cs(i)$ for all $i \geq 0$. Then we have $D_{s'}(n) = D_s(n)$ for every n such that* $\gcd(|c|, D_s(n)) = 1$.

Finally, we also consider the cases in which the quadratic coefficients are not integers. Discriminators are only applicable to integer sequences, so we are interested in quadratic polynomials that are *integer-preserving* i.e. polynomials such that $q(n)$ is an integer if n is an integer. From a result of Pólya [16], we deduce that every integer-preserving quadratic polynomial can be written in the form

$$q(n) = c_1 \frac{n(n-1)}{2!} + c_2 n + c_3(1) = \frac{c_1}{2} n^2 + \frac{2c_2 - c_1}{2} n + c_3,$$

for integers c_1, c_2, and c_3. Note that if c_1 is even, then $2c_2 - c_1$ must also be even, and vice versa. Thus, we can express all integer-valued quadratic polynomials in the form

$$q(n) = \frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma,$$

for integers α', β', and γ, where α' and β' are either both even, or both odd. We denote this sequence as $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$. Here, for any integers i and j, we have

$$\mathrm{qr}(j) - \mathrm{qr}(i) = (j-i)(\alpha(i+j)+\beta) = \frac{(j-i)(\alpha'(i+j)+\beta')}{2} \tag{5.2}$$

We can extend Lemma 50 to apply to quadratic sequences with rational coefficients, except with odd primes $p \geq 3$ instead of $p \geq 2$.

Lemma 51. *Let $p \geq 3$ be a prime number and let $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$ for integers α', β', and γ be a quadratic sequence such that α' and β' are odd, and any of the following conditions are satisfied:*

1. *$p \nmid \alpha'$;*
2. *$p | \beta'$;*
3. *$\alpha' = p^k c$ for some integer c such that $c \nmid \beta'$.*

Then there exists a value of $n > 0$ such that $D_{\mathrm{qr}}(n) \neq p^{\lceil \log_p n \rceil}$.

Proof. The argument is identical to Lemma 50. For the first two cases, since p is odd, it follows that $p|(j-i)(\alpha'(i+j)+\beta')$ implies $p|\frac{(j-i)(\alpha'(i+j)+\beta')}{2} = \mathrm{qr}(j) - \mathrm{qr}(i)$. For the third case, it is clear that $r \nmid (j-i)(\alpha'(i+j)+\beta')$ implies $r \nmid \frac{(j-i)(\alpha'(i+j)+\beta')}{2} = \mathrm{qr}(j) - \mathrm{qr}(i)$. Thus the same arguments apply. $\square$

5.2 The case $p = 2$

In this section, we provide a complete characterization of all integer-valued quadratic sequences with discriminator $2^{\lceil \log_2 n \rceil}$. These quadratic sequences can be divided into two types, based on whether the quardatic coefficients are integers or not, i.e., whether α and β are integers for sequences of the form $(q(n))_{n\geq 0} = (\alpha n^2 + \beta n + \gamma)_{n\geq 0}$.

5.2.1 Integer quadratic coefficients

First we focus on the case in which α and β are integers. We begin by considering quadratic sequences of the form $(\mathrm{qd}(n))_{n\geq 0} = (2^t n^2 + bn)_{n\geq 0}$ for an integer $t > 0$ and odd integer b, and then extend the result later. For these sequences, we can apply Eq. (5.1) to get

$$\mathrm{qd}(j) - \mathrm{qd}(i) = (j-i)(2^t(i+j)+b). \tag{5.3}$$

We compute the discriminator for $(\mathrm{qd}(n))_{n\geq 0}$ using Lemma 49 for $p = 2$. The first condition for Lemma 49 is established by another lemma:

Lemma 52. *Let $k \geq 0$. For all positive integers $m < 2^{k+1}$, there exists at least one pair of integers, i and j, such that $0 \leq i < j \leq 2^k$ and $m|\,\mathrm{qd}(j) - \mathrm{qd}(i)$.*

Proof. We consider the different possible cases for the value of m.

1. m is a power of 2, i.e., $m = 2^\ell$ where $\ell \leq k$. Set $i = 0$ and $j = 2^i$ so that $j - i = 2^\ell$.

2. m is odd. Since m and 2^t are co-prime, this implies that 2^t has a multiplicative inverse modulo m. Let $x = -b(2^t)^{-1} \bmod m$. If $x \leq 2^k$, then we choose $i = 0$ and $j = x$. Otherwise, if $x > 2^k$, we choose $j = 2^k$ and $i = x - 2^k$. Since $x < m < 2^{k+1}$, it follows that $i < 2^{k+1} - 2^k = 2^k = j$. In both cases, we have $i + j = x$, and therefore,

$$\begin{aligned}\mathrm{qd}(j) - \mathrm{qd}(i) = (j-i)(2^t x + b) &\equiv (j-i)(2^t(-b)(2^t)^{-1} + b) \pmod{m}\\ &\equiv (j-i)(-b+b) \equiv 0 \pmod{m}.\end{aligned}$$

3. m is even, but not a power of 2. In this case, we can write $m = 2^\ell \cdot r < 2^{k+1}$ for $0 < \ell < k$, and odd $r > 2$. This implies $r < 2^{k+1-\ell}$. This time, let $x = -b(2^{t+1})^{-1} \bmod r$, and choose $i = (x - 2^{\ell-1}) \bmod r$ and $j = i + 2^\ell$, to get

$$\begin{aligned}\mathrm{qd}(j) - \mathrm{qd}(i) &= (i + 2^\ell - i)(2^t(i + i + 2^\ell) + b) = 2^\ell(2^t(2i + 2^\ell) + b)\\ &= 2^\ell(2^{t+1}(i + 2^{\ell-1}) + b),\end{aligned}$$

which is divisible by 2^ℓ. Also,

$$\begin{aligned}\mathrm{qd}(j) - \mathrm{qd}(i) &= 2^\ell(2^{t+1}(i + 2^{\ell-1}) + b) \equiv 2^\ell(2^{t+1}x + b) \pmod{r}\\ &\equiv 2^\ell(2^{t+1}(-b)(2^{t+1})^{-1} + b) \equiv 2^\ell(-b+b) \equiv 0 \pmod{r}.\end{aligned}$$

We now verify the conditions on i and j. It is clear that $0 \leq i < j$. Furthermore, we have $i < r < 2^{k+1-\ell}$ and $0 < \ell < k$. For $\ell = 1$, we have $j = i + 2 \leq (r-1) + 2 = r + 1 \leq (2^{k+1-1} - 1) + 1 = 2^k$. For $\ell > 1$, we have $r < 2^{k+1-2} = 2^{k-1}$, and so, $j = i + 2^\ell < r + 2^\ell < 2^{k-1} + 2^\ell \leq 2^{k-1} + 2^{k-1} = 2^k$. Therefore, the condition $0 \leq i < j \leq 2^k$ is fulfilled and $\mathrm{qd}(j) - \mathrm{qd}(i)$ is divisible by both 2^ℓ and r, and thus $m|\,\mathrm{qd}(j) - \mathrm{qd}(i)$.

In all cases, we have $m | \operatorname{qd}(j) - \operatorname{qd}(i)$ for some i and j in the required range. $\square$

The second condition of Lemma 49 is also satisfied, as shown by the next lemma:

Lemma 53. *Let $k \geq 0$. For all integers i and j satisfying $0 \leq i < j < 2^{k+1}$, we have $2^{k+1} \nmid \operatorname{qd}(j) - \operatorname{qd}(i)$.*

Proof. We know $\operatorname{qd}(j) - \operatorname{qd}(i) = (j - i)(2^t(i + j) + b)$, where $t > 0$. Here, the second factor is the sum of an even number and an odd number, and therefore must itself be odd and not divisible by 2. Therefore, any powers of 2 that divide $\operatorname{qd}(j) - \operatorname{qd}(i)$ must divide the first factor, $(j - i)$. But $j - i \leq j < 2^{k+1}$. Therefore, $2^{k+1} \nmid \operatorname{qd}(j) - \operatorname{qd}(i)$ for all i and j in the range $0 \leq i < j < 2^{k+1}$. $\square$

Therefore, the two conditions in Lemma 49 are satisfied for $(\operatorname{qd}(n))_{n\geq 0} = (2^t n^2 + bn)_{n\geq 0}$ for every integer $t > 0$ and odd integer b. It follows from Lemma 49 that $D_{\operatorname{qd}}(n) = 2^{\lceil \log_2 n \rceil}$ for $n > 0$. Along with Lemma 50 and Lemma 1, this is sufficient to characterize all quadratic sequences with integer coefficients that have discriminator $2^{\lceil \log_2 n \rceil}$, as shown in the following theorem:

Theorem 54. *For all quadratic sequences with integer coefficients, i.e., $(q(n))_{n\geq 0} = (\alpha n^2 + \beta n + \gamma)_{n\geq 0}$ for integers α, β, and γ, the discriminator $D_q(n)$ is equal to $2^{\lceil \log_2 n \rceil}$ for all $n \geq 0$ if and only if all of the following conditions are satisfied:*

1. *α is even, i.e., $\alpha = 2^t \cdot r$ for some $t \geq 1$ and odd r;*
2. *β is odd;*
3. *$r | \beta$.*

Proof. We assume $\gamma = 0$ since the discriminator does not depend on it. Now suppose conditions (1)-(3) hold. Then the resulting sequence, $(q(n))_{n\geq 0} = (2^t r n^2 + \beta n)_{n\geq 0}$ is equivalent to the sequence $(r \cdot \operatorname{qd}(n))_{n\geq 0} = (r(2^t n^2 + bn))_{n\geq 0}$ with $b = \frac{\beta}{r}$. For $r = 1$, we know $D_{\operatorname{qd}}(n) = 2^{\lceil \log_2 n \rceil}$ by an application of Lemma 49 for $p = 2$, with Lemmas 52 and 53 verifying that the conditions are fulfilled.

Since r is odd, it is co-prime to D_{qd} for all $n \geq 1$, and so we can apply Lemma 1 to show that $D_q(n) = D_{\operatorname{qd}}(n) = 2^{\lceil \log_2 n \rceil}$.

For the other direction, we observe that the violation of any one of these conditions implies the violation of a corresponding condition of Lemma 50 for $p = 2$, which showed that there exists a value of $n \geq 1$ for which $D_q(n) \neq 2^{\lceil \log_2 n \rceil}$. $\square$

Thus, we have provided a complete characterization of quadratic sequences with integer coefficients that have discriminator $2^{\lceil \log_2 n \rceil}$. We can further show that the discriminator for these sequences are shift-invariant. As defined in the previous chapter, the discriminator of $(q(n))_{n\geq 0}$ is said to be *shift-invariant* if it shares the same discriminator as $(q(n + c))_{n\geq 0}$ for all $c \geq 0$.

Theorem 55. *For quadratic sequences* $(q(n))_{n\geq 0} = (\alpha n^2 + \beta n + \gamma)_{n\geq 0}$ *for integers* α, β, *and* γ *with discriminator* $D_q(n) = 2^{\lceil \log_2 n \rceil}$, *the discriminator of the shifted sequence,* $(\mathrm{qs}(n,c))_{n\geq 0} = q(n+c)$ *for any integer* c *also satisfies* $D_{\mathrm{qs}}(n) = 2^{\lceil \log_2 n \rceil}$.

Proof. From Theorem 54, we know that $\alpha = 2^t \cdot r$ for some $t \geq 1$ and odd r, β is odd, and that $r|\beta$. Now, for any integer c, we have

$$\begin{aligned}\mathrm{qs}(n,c) = q(n+c) &= \alpha(n+c)^2 + \beta(n+c) + \gamma \\ &= \alpha n^2 + 2\alpha nc + \alpha c^2 + \beta n + \beta c + \gamma \\ &= \alpha n^2 + (2\alpha nc + \beta)n + (\alpha c^2 + \beta c + \gamma).\end{aligned}$$

The coefficient of n^2 is $\alpha = 2^t \cdot r$, which is even, while the coefficient of n is $2\alpha nc + \beta$, which is odd. Furthermore, since $r|\alpha$ and $r|\beta$, we have $r|2\alpha nc + \beta$. Therefore, the three conditions in Theorem 54 are fulfilled by $(\mathrm{qs}(n,c))_{n\geq 0}$ and so $D_{\mathrm{qs}}(n) = 2^{\lceil \log_2 n \rceil}$. ☐

5.2.2 Half-integer quadratic coefficients

We now consider quadratic sequences of the form $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$ for odd α' and β'. Recall from Eq. (5.2) that

$$\mathrm{qr}(j) - \mathrm{qr}(i) = (j-i)(\alpha(i+j)+\beta) = \frac{(j-i)(\alpha'(i+j)+\beta')}{2}. \qquad (5.2 \text{ revisited})$$

To characterize the discriminator of $(\mathrm{qr}(j))_{n\geq 0}$, we first consider the sequence of triangular numbers, $(\mathrm{tr}(n))_{n\geq 0} = \left(\frac{1}{2}n^2 + \frac{1}{2}n\right)_{n\geq 0}$ and extend the result to all $(\mathrm{qr}(j))_{n\geq 0}$. The discriminator for the sequence of triangular numbers was already shown by Sun [21] to be $2^{\lceil \log_2 n \rceil}$, but here we present an alternate proof that utilizes Lemma 49.

For the sequence of triangular numbers, Eq. (5.2) becomes

$$\mathrm{tr}(j) - \mathrm{tr}(i) = \frac{(j-i)(i+j+1)}{2}.$$

The first condition for Lemma 49 is established by the following lemma:

Lemma 56. *Let* $k \geq 0$. *For all positive integers* $m < 2^{k+1}$, *there exists at least one pair of integers,* i *and* j, *such that* $0 \leq i < j \leq 2^k$ *and* $m|\,\mathrm{tr}(j) - \mathrm{tr}(i)$.

Proof. We consider the different possible cases for the value of m.

1. m is a power of 2, i.e., $m = 2^\ell$ where $\ell \leq k$. Set $i = 2^\ell - 1$ and $j = 2^\ell$ to get

$$\mathrm{tr}(j) - \mathrm{tr}(i) = \frac{(2^\ell - 2^\ell + 1)(2^\ell - 1 + 2^\ell + 1)}{2} = \frac{2^{\ell+1}}{2} = 2^\ell = m.$$

2. m is odd. For $m = 1$, we set $i = 0$ and $j = 1$. Otherwise, we set $i = \lfloor \frac{m}{2} \rfloor - 1$ and $j = \lceil \frac{m}{2} \rceil$ to get

$$\mathrm{tr}(j) - \mathrm{tr}(i) = \frac{(\lceil \frac{m}{2} \rceil - \lfloor \frac{m}{2} \rfloor + 1)(\lfloor \frac{m}{2} \rfloor - 1 + \lceil \frac{m}{2} \rceil + 1)}{2} = \frac{2m}{2} = m.$$

3. m is even, but not a power of 2. In this case, we can write $m = 2^\ell(2r+1) < 2^{k+1}$ for $0 < \ell < k$ and $r > 0$. This implies $r < 2^{k-\ell} \le 2^{k-1}$. We have two further cases here. If $r \ge 2^\ell$, set $i = r - 2^\ell$ and $j = r + 2^\ell$ to get

$$\begin{aligned}\mathrm{tr}(j) - \mathrm{tr}(i) &= \frac{(r + 2^\ell - r + 2^\ell)(r - 2^\ell + r + 2^\ell + 1)}{2} \\ &= \frac{(2^{\ell+1})(2r+1)}{2} = 2^\ell(2r+1) = m.\end{aligned}$$

Otherwise, if $r < 2^\ell$, set $i = 2^\ell - r - 1$ and $j = r + 2^\ell$ to get

$$\begin{aligned}\mathrm{tr}(j) - \mathrm{tr}(i) &= \frac{(r + 2^\ell - 2^\ell + r + 1)(2^\ell - r - 1 + r + 2^\ell + 1)}{2} \\ &= \frac{(2r+1)(2^{\ell+1})}{2} = 2^\ell(2r+1) = m.\end{aligned}$$

In both cases, it is clear that $0 \le i < j$. Furthermore, since $r < 2^{k-\ell}$ and $0 < \ell < k$, we have $j = r + 2^\ell < 2^{k-1} + 2^\ell \le 2^{k-1} + 2^{k-1} = 2^k$, thus fulfilling the required constraints on i and j.

In all cases, we have $\mathrm{tr}(j) - \mathrm{tr}(i) = m$ for some i and j in the required range. □

The second condition of Lemma 49 is established by the following lemma.

Lemma 57. *Let $k \ge 0$. For all pairs of integers i and j satisfying $0 \le i < j < 2^{k+1}$, we have $2^{k+1} \nmid \mathrm{tr}(j) - \mathrm{tr}(i)$.*

Proof. If, for some i and j, we have $2^{k+1} \mid \mathrm{tr}(j) - \mathrm{tr}(i) = \frac{(j-i)(i+j+1)}{2}$, it follows that $2(2^{k+1}) = 2^{k+2} \mid (j-i)(i+j+1)$. Note that between the factors $(j-i)$ and $(i+j+1)$, one of them must be odd while the other is even. Therefore, at most only one of those factors can be a multiple of 2^{k+2} for any i and j. However, if $0 \le i < j < 2^{k+1}$, then $j - i < i + j + 1 \le 2j < 2(2^{k+1}) = 2^{k+2}$, and therefore, 2^{k+2} cannot divide either of those factors. In other words, $2^{k+1} \nmid \mathrm{tr}(j) - \mathrm{tr}(i)$ for all i and j such that $0 \le i < j < 2^{k+1}$. □

Thus, the two conditions of Lemma 49 are established for the sequence of triangular numbers through Lemma 56 and Lemma 57 respectively. Therefore, we can apply Lemma 49 to show that $D_{\mathrm{tr}}(n) = 2^{\lceil \log_2 n \rceil}$ for $n > 0$.

We now characterize all sequences of the form $(\mathrm{qr}(n))_{n \ge 0} = \left(\frac{\alpha'}{2} n^2 + \frac{\beta'}{2} n + \gamma\right)_{n \ge 0}$ for odd integers α' and β', and any integer γ, that have the discriminator $D_{\mathrm{tr}}(n) = 2^{\lceil \log_2 n \rceil}$.

Theorem 58. *For all quadratic sequences of the form* $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$ *for odd integers* α' *and* β'*, and any integer* γ*, the discriminator* $D_{\mathrm{qr}}(n)$ *is equal to* $2^{\lceil \log_2 n \rceil}$ *for all* $n \geq 0$ *if and only if* $\alpha' = \beta'$.

Proof. We know $D_{\mathrm{tr}}(n) = 2^{\lceil \log_2 n \rceil}$ from applying Lemma 49. Furthermore, since α is odd, we have $\gcd(|\alpha|, D_{\mathrm{tr}}(n)) = 1$ for all $n \geq 0$. Therefore, we can apply Lemma 1 to show that $D_{\mathrm{qr}}(n) = D_{\mathrm{tr}}(n) = 2^{\lceil \log_2 n \rceil}$.

On the other hand, if $\alpha' \neq \beta'$, we have two cases to consider:

Case 1: $|\alpha'| \neq |\beta'|$**.** Let k be any integer such that $2^k > |\alpha'|$ and $2^k > |\beta'|$. We show that there exists a pair of i and j such that $0 \leq i < j < 2^k$ and $2^k | \,\mathrm{qr}(j) - \mathrm{qr}(i)$. Let $x = -\beta'(\alpha')^{-1} \bmod 2^{k+1}$. If $x < 2^k$, we choose $i = 0$ and $j = x$. Otherwise, we choose $j = 2^k - 1$ and $i = x - j$. In both cases, we have $i + j = x$, and so,

$$\alpha'(i+j) + \beta' \equiv \alpha'(-\beta')(\alpha')^{-1} + \beta') \equiv -\beta' + \beta' \equiv 0 \pmod{2^{k+1}},$$

which implies that $2^k | \frac{\alpha'(i+j)+\beta'}{2}$ and therefore, $2^k | \,\mathrm{qr}(j) - \mathrm{qr}(i) = \frac{(j-i)(\alpha'(i+j)+\beta')}{2}$.

It is clear that $j < 2^k$ and $i \geq 0$ for both cases, and that $i < j$ if $x < 2^k$. We now verify that $i < j$ for $x \geq 2^k$. Since $|\alpha'| \neq |\beta'|$, and both $|\alpha'|$ and $|\beta'|$ are less than 2^k, it follows that $\alpha' \not\equiv \pm\beta' \pmod{2^{k+1}}$. Therefore, $x = -\beta'(\alpha')^{-1} \bmod 2^{k+1} \not\equiv \pm 1 \pmod{2^{k+1}}$, and so, $x < 2^{k+1} - 1$. Also, since both α' and β' are odd while 2^{k+1} is even, it follows that x is odd and thus, $x \leq 2^{k+1} - 3$. Therefore, $i = x - j \leq 2^{k+1} - 3 - 2^k + 1 = 2^k - 2 < j$.

Since $2^k | \,\mathrm{qr}(j) - \mathrm{qr}(i)$ for some i and j such that $0 \leq i < j < 2^k$, we have $D_{\mathrm{qr}}(2^k) \neq 2^k$, and therefore, $D_{\mathrm{qr}}(n) \neq 2^{\lceil \log_2 n \rceil}$ for $n = 2^k$.

Case 2: $\alpha' = -\beta'$**.** In this case, we have $\mathrm{qr}(0) = \mathrm{qr}(1) = 0$, and so, the sequence cannot even be discriminated.

Therefore, we have $D_{\mathrm{qr}}(n) = 2^{\lceil \log_2 n \rceil}$ if and only if $\alpha' = \beta'$. □

Unlike the case with integer coefficients, the discriminators for sequences of the form $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\alpha'}{2}n + \gamma\right)_{n\geq 0}$ are not shift-invariant. This is because for any integer c, we have

$$\begin{aligned}\mathrm{qr}(n+c) &= \frac{\alpha'}{2}(n+c)^2 + \frac{\alpha'}{2}(n+c) + \gamma = \frac{\alpha'}{2}(n^2 + 2nc + c^2) + \frac{\alpha'}{2}(n+c) + \gamma \\ &= \frac{\alpha'}{2}n^2 + \left(\frac{2\alpha' c}{2} + \frac{\alpha'}{2}\right)n + \left(\frac{\alpha'}{2}c^2 + \frac{\alpha'}{2}c + \gamma\right).\end{aligned}$$

where the coefficient of n^2 is $\frac{\alpha'}{2}$ while the coefficient of n is $\frac{2\alpha' c}{2} + \frac{\alpha'}{2} \neq \frac{\alpha}{2}$. Therefore, by Theorem 58, the discriminator for $(\mathrm{qr}(n+c)_{n\geq 0}$ cannot be $2^{\lceil \log_2 n \rceil}$, and thus, the discriminator is not shift-invariant.

5.3 The case $p \geq 5$

We leave the case of $p = 3$ to the next section, since the case of $p \geq 5$ is more straightforward and contains some results that are used for $p = 3$ as well.

In this section we show that for any prime $p \geq 5$, there are no sequences of the form $(q(n))_{n\geq 0} = (\alpha n^2 + \beta + \gamma)_{n\geq 0}$ with $\alpha \neq 0$, whose discriminator is $D_q(n) = p^{\lceil \log_p n \rceil}$ for all $n \geq 1$.

5.3.1 Integer quadratic coefficients

We begin by considering sequences of the form $(\mathrm{qp}(n))_{n\geq 0} = (c(p^k n^2 + bn))_{n\geq 0}$, for integers k, p, c, and b, where p is prime, $p^k \geq 5$, and $\gcd(b, p) = 1$. From Eq. (5.1), we see that for all i and j,

$$\mathrm{qp}(j) - \mathrm{qp}(i) = (j - i)(\alpha(i + j) + \beta) = c(j - i)(p^k(i + j) + b).$$

First, we present a lemma concerning the factor of $p^k(i + j) + b$ in the equation for $\mathrm{qp}(j) - \mathrm{qp}(i)$.

Lemma 59. *Let p, k, b, r be integers such that p and r are prime, $p^k \geq 5$, $\gcd(p, b) = 1$, $r > |b|$, and $r \equiv -b \pmod{p^k}$. Then $z = \frac{(p^k-1)r-b}{p^k}$ is the smallest non-negative integer such that $p^k z + b \equiv 0 \pmod{r}$.*

Proof. Since $r \equiv -b \pmod{p^k}$ and $\gcd(b, p) = 1$, it follows that $\gcd(r, p) = 1$. Now, for the equation $p^k z + b \equiv 0 \pmod{r}$, it is clear that $z \equiv (-b)(p^k)^{-1} \pmod{r}$, and thus, there is only one solution of z in the range $0 \leq z < r$. We now show that this single solution is $z = \frac{(p^k-1)r-b}{p^k}$.

First, we observe that for $z = \frac{(p^k-1)r-b}{p^k}$, the numerator is $p^k r-(r+b) \equiv 0 \pmod{p^k}$, since $r \equiv -b \pmod{p^k}$, thus ensuring that z is an integer. It is also clear that if $z = \frac{(p^k-1)r-b}{p^k}$, then $p^k z + b \equiv (p^k - 1)r \equiv 0 \pmod{r}$.

Now, since $p^k \geq 5$, it follows that $(p^k - 1)r \geq r$, which further implies $(p^k - 1)r - b > 0$ since $r > |b|$. Furthermore, it is clear that $r + b > 0$ and thus, $p^k r - (r + b) < p^k r$. In other words, we have $0 < z < \frac{p^k r}{p^k} = r$. Hence the result follows. $\square$

We now consider primes whose base-b representation, for some base b, have specified prefixes and suffixes. We let Σ_b denote the alphabet $\{0, 1, \ldots, b - 1\}$. The notation $[x]_b$ refers to the number that would be written as the string x in base-b.

Lemma 60. *Let $b \geq 2$ be an integer and let x and y be finite strings in Σ_b^* such that $\gcd(b, [y]_b) = 1$. Then there exist infinitely many strings $w \in \Sigma_b^*$ such that $[xwy]_b$ is prime.*

Proof. For an integer $n \geq 1$, let $\mathcal{P}_{x,y,b,n}$ denote the set of primes of the form $[xwy]_b$ for strings $w \in \Sigma_b^*$ such that $|wy| = n$. These numbers can be represented as $[x]_b \cdot b^n + [wy]_b$. Therefore, they are congruent to $[y]_b \bmod b$ and lie in the interval from $[x]_b \cdot b^n$ to $([x]_b + 1) \cdot b^n$ exclusive.

From Dirchlet's theorem on primes in arithmetic progressions, the number of primes less or equal to $[x]_b \cdot b^n$ and congruent to $[y]_b \bmod b$, denoted by $\pi([x]_b \cdot b^n, b, [y]_b)$, is approximated by

$$\pi([x]_b \cdot b^n, b, [y]_b) \approx \frac{1}{\varphi(b)} \operatorname{li}([x]_b \cdot b^n),$$

where $\varphi(b)$ is Euler's totient function, and $\operatorname{li}(m)$ is the logarithmic integral function where $\operatorname{li}(m) = \int_2^m dt/\log t \approx \frac{m}{\log m}$. Therefore,

$$\begin{aligned}
|\mathcal{P}_{x,y,b,n}| &= \pi(([x]_b + 1) \cdot b^n, b, [y]_b) - \pi([x]_b \cdot b^n, b, [y]_b) \\
&\approx \frac{1}{\varphi(b)} (\operatorname{li}(([x]_b + 1) \cdot b^n) - \operatorname{li}([x]_b \cdot b^n)) \\
&\approx \frac{1}{\varphi(b)} \left(\frac{([x]_b + 1) \cdot b^n}{\log([x]_b + 1) + \log b^n} - \frac{[x]_b \cdot b^n}{\log[x]_b + \log b^n} \right) \\
&\approx \frac{([x]_b + 1) \cdot b^n - [x]_b \cdot b^n}{\varphi(b)(\log[x]_b + n \log b)} \\
&= \frac{b^n}{\varphi(b)(\log[x]_b + n \log b)}.
\end{aligned}$$

As n grows large, this value approaches $\frac{b^n}{\varphi(b) n \log b}$. The error term for the approximation is known to be bounded by $O([x]_b \cdot b^n e^{-c\lambda([x]_b \cdot b^n)})$, where $\lambda(m) = (\log m)^{3/5} (\log \log m)^{-1/5}$.

$$\begin{aligned}
\lambda([x]_b \cdot b^n) &= (\log[x]_b \cdot b^n)^{3/5} (\log \log[x]_b \cdot b^n)^{-1/5} \\
&= (\log[x]_b + n \log b)^{3/5} (\log(\log[x]_b + n \log b))^{-1/5} \\
&= \left(\frac{(\log[x]_b + n \log b)^3}{\log(\log[x]_b + n \log b)} \right)^{1/5}.
\end{aligned}$$

As n grows large, we have

$$\begin{aligned}
\lambda([x]_b \cdot b^n) &= \left(\frac{(n \log b)^3}{\log(n \log b)} \right)^{1/5}, \\
\implies [x]_b \cdot b^n \cdot \exp(-c\lambda([x]_b \cdot b^n)) &= \frac{b^n}{\frac{1}{[x]_b} \exp(c \left(\frac{(n \log b)^3}{\log(n \log b)} \right)^{1/5})}.
\end{aligned}$$

Here, the denominator of the error term is an exponential function, with growth rate in $\Omega(e^{cn^{2/5}})$. This grows much faster than the denominator of $\frac{b^n}{\varphi(b) n \log b}$, i.e., $\varphi(b) n \log b$, which grows only linearly with n. In other words, as n grows large, the upper bound of the error term grows much slower than the approximation $|\mathcal{P}_{x,y,b,n}| \approx \frac{b^n}{\varphi(b) n \log b}$.

In other words, we have $|\mathcal{P}_{x,y,b,n}| > 0$ for n sufficiently large, i.e. $\mathcal{P}_{x,y,b,n}$ is non-empty. As there are infinitely many sets $\mathcal{P}_{x,y,b,n}$ with n large enough, it follows that there are infinitely many primes of the form $[xwy]_b$ for strings $w \in \Sigma_b^*$. $\square$

We use this lemma to prove the following lemma:

Lemma 61. *Let p, k, c, and b be integers such that p is prime, $p \geq 3$, $p^k \geq 5$, and $\gcd(p, b) = 1$. Then for all sequences of the form $(\mathrm{qp}(n))_{n\geq 0} = (c(p^k n^2 + bn))_{n \geq 0}$, there exists a pair of integers, r and ℓ such that $r < p^{\ell+1}$ and $r \nmid \mathrm{qp}(j) - \mathrm{qp}(i)$ for all i and j in the range $0 \leq i < j \leq p^\ell$.*

Proof. Here, we consider the prime numbers such that their first digit in base p^k is $p^k - 1$ and their last digit is equivalent to $-b$ modulo p^k. From Lemma 60, we know that there are infinitely many such primes that fulfil these conditions, so we choose r to be any of these primes such that $r > \max(|b|, |c|)$. Since the first digit of r in base p^k is $p^k - 1$, this means that there is an integer u such that $(p^k - 1)(p^k)^u \leq r < (p^k)^{u+1}$. Let $\ell = k(u+1) - 1$ so that $(p^k)^{u+1} = p^{\ell+1}$ and $(p^k)^u = p^{\ell-k+1}$. Therefore, we have $(p^k - 1)p^{\ell-k+1} \leq r < p^{\ell+1}$.

We now show that $r \nmid \mathrm{qp}(j) - \mathrm{qp}(i) = c(j - i)(p^k(i + j) + b)$ for all i and j in the range $0 \leq i < j \leq p^\ell$. It is clear that $r \nmid c$ since $r > |c|$, and that $r \nmid (j - i)$ since $j - i \leq j \leq p^\ell < r$. Thus, it suffices to show that $r \nmid p^k(i+j) + b$ to prove $r \nmid \mathrm{qp}(j) - \mathrm{qp}(i)$.

By contradiction, let us assume that there is some i and j in the range $0 \leq i < j \leq p^\ell$ such that $r | p^k(i + j) + b$. This implies that $p^k(i + j) + b \equiv 0 \pmod r$. Since $i + j$ must be non-negative, we can apply Lemma 59 to show that $i + j \geq \frac{(p^k-1)r-b}{p^k}$. Noting that $r > |b|$, $r \geq (p^k - 1)p^{\ell-k+1}$, and $p^k \geq 5$, we can deduce that

$$\begin{aligned} i + j &\geq \frac{(p^k - 1)r - b}{p^k} = \frac{p^k r - r - b}{p^k} \geq \frac{p^k r - r - r}{p^k} = \frac{(p^k - 2)r}{p^k}, \\ &\geq \frac{(p^k - 2)(p^k - 1)p^{\ell-k+1}}{p^k} = \frac{p^{\ell+1}(p^{2k} - 3p^k + 2)}{p^{2k}} \geq \frac{p^{\ell+1}(p^{2k} - 3p^k)}{p^{2k}} \\ &= p^{\ell+1}\left(1 - \frac{3}{p^k}\right). \end{aligned}$$

Now, if $p = 3$, then we have $p^k \geq 9$, and so

$$i + j \geq p^{\ell+1}\left(1 - \frac{3}{p^k}\right) \geq 3p^\ell\left(1 - \frac{3}{9}\right) = 2p^\ell.$$

Otherwise, if $p \geq 5$, then

$$i + j \geq p^{\ell+1}\left(1 - \frac{3}{p^k}\right) \geq 5p^\ell\left(1 - \frac{3}{5}\right) = 2p^\ell.$$

In both cases, we have $i + j \geq 2p^\ell$, which is a contradiction since $0 \leq i < j \leq p^\ell$. It follows that for all i and j in the range $0 \leq i < j \leq p^\ell$, we have $r \nmid \mathrm{qp}(j) - \mathrm{qp}(i)$. $\square$

Although this section is about the case of $p \geq 5$, the proof for Lemma 61 includes the case of $p = 3$, which is relevant to the next section.

We now show that the discriminator for $(\mathrm{qp}(n))_{n\geq 0}$ is not characterized by $p^{\lceil \log_p n \rceil}$.

Lemma 62. *Let p, k, c, and b be integers such that p is prime, $p \geq 3$, $p^k \geq 5$, and $\gcd(p, b) = 1$. Then, for every sequence of the form $(\mathrm{qp}(n))_{n\geq 0} = (c(p^k n^2 + bn))_{n\geq 0}$, there exists at least one value of $n \geq 1$ such that $D_{\mathrm{qp}}(n) \neq p^{\lceil \log_p n \rceil}$.*

Proof. From Lemma 61, we know there exists a pair of integers r and ℓ such that $r < p^{\ell+1}$ and $r \nmid \mathrm{qp}(j) - \mathrm{qp}(i)$ for all i and j in the range $0 \leq i < j \leq p^\ell$. This implies that r discriminates the first $p^\ell + 1$ terms of $(\mathrm{qp}(n))_{n\geq 0}$. Therefore, $D_{\mathrm{qp}}(p^\ell + 1) \leq r$. But since $r < p^{\ell+1}$, it follows that $D_{\mathrm{qp}}(p^\ell + 1) < p^{\ell+1} = p^{\lceil \log_p (p^\ell+1) \rceil}$. Thus, for $n = p^\ell + 1$, we have $D_{\mathrm{qp}}(n) \neq p^{\lceil \log_p n \rceil}$. $\square$

With Lemma 62, along with Lemma 50, we can show that there are no quadratic sequences with integer coefficients with discriminator $p^{\lceil \log_p n \rceil}$ for primes $p \geq 5$.

Theorem 63. *Let $p \geq 5$ be a prime number. Then for every quadratic sequence with integer coefficients, denoted by $(q(n))_{n\geq 0} = (\alpha n^2 + \beta + \gamma)_{n\geq 0}$ with $\alpha \neq 0$, there exists a value of $n \geq 1$ such that $D_q(n) \neq p^{\lceil \log_p n \rceil}$.*

Proof. We can apply Lemma 50 to show that there exists a value of $n \geq 1$ such that $D_q(n) \neq p^{\lceil \log_p n \rceil}$ if any of the following conditions are satisfied:

1. $p \nmid \alpha$;
2. $p | \beta$;
3. $\alpha = p^k c$ for some integer c such that $c \nmid \beta$.

If neither of these conditions are satisfied, then it follows that $(q(n))_{n\geq 0}$ is of the form $(\mathrm{qp}(n))_{n\geq 0} = (c(p^k n^2 + bn))_{n\geq 0}$ for $k \geq 1$, and $\gcd(p, b) = 1$. We can then apply Lemma 62 to show that $D_q(n) \neq p^{\lceil \log_p n \rceil}$ for some $n \geq 1$. $\square$

This means that for primes $p \geq 5$, there are no integer-valued quadratic sequences of the form $(q(n))_{n\geq 0} = (\alpha n^2 + \beta + \gamma)_{n\geq 0}$ such that $D_q(n) = p^{\lceil \log_p n \rceil}$ for all $n \geq 1$.

5.3.2 Half-integer quadratic coefficients

We further show that for the general case of $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$, there are still no such sequences with $D_{\mathrm{qr}}(n) = p^{\lceil \log_p n \rceil}$ for all $n \geq 1$ for any prime $p \geq 3$. Recall from Eq. (5.2) that for all integers i and j, we have

$$\mathrm{qr}(j) - \mathrm{qr}(i) = (j-i)(\alpha(i+j)+\beta) = \frac{(j-i)(\alpha'(i+j)+\beta')}{2}. \qquad (5.2 \text{ revisited})$$

Lemma 64. *Let $p \geq 3$ be a prime number and let $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$ for odd integers α', β', and γ be a quadratic sequence such that α' and β' are odd, and any of the following conditions are satisfied:*

1. *$\alpha' = \beta'$;*
2. *$p \nmid \alpha'$;*
3. *$p | \beta'$;*
4. *$\alpha' = p^k c$ for some integer c such that $c \nmid \beta'$;*
5. *$\alpha' = p^k c$ for some integer c such that $c|\beta'$ and $p^k \geq 5$.*

Then there exists a value of $n > 0$ such that $D_{\mathrm{qr}}(n) \neq p^{\lceil \log_p n \rceil}$.

Proof. If $\alpha' = \beta'$, then it was shown in Theorem 58 that the discriminator is $D_{\mathrm{qr}}(n) = 2^{\lceil \log_2 n \rceil}$. Thus the discriminator does not take values other than powers of 2. Otherwise, the middle three conditions are shown by Lemma 51.

For the final condition, we can express $(\mathrm{qr}(n))_{n\geq 0} = (\frac{1}{2}\,\mathrm{qp}(n))_{n\geq 0} = (\frac{1}{2}c(p^k n^2 + bn))_{n\geq 0}$, where $k \geq 1$ and b and c are integers with $\gcd(p, b) = 1$. From Lemma 61, we know that there exists a pair of integers r and ℓ such that $r < p^{\ell+1}$ and $r \nmid \mathrm{qp}(j) - \mathrm{qp}(i)$ for all i and j such that $0 \leq i < j \leq p^\ell$. It follows that $r \nmid \frac{1}{2}(\mathrm{qp}(j) - \mathrm{qp}(i))$ and so, $D_{\mathrm{qr}}(p^\ell + 1) \leq r < p^{\ell+1} = (p^\ell + 1)^{\lceil \log_p (p^\ell + 1) \rceil}$, and so, $D_{\mathrm{qr}}(n) \neq p^{\lceil \log_p n \rceil}$ for $n = p^\ell + 1$. $\square$

Corollary 65. *Let $p \geq 5$ be a prime number. Then for all quadratic sequences of the form $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$ for integers odd integers α' and β', and any integer γ, there exists a value of $n \geq 1$ such that $D_{\mathrm{qr}}(n) \neq p^{\lceil \log_p n \rceil}$.*

Proof. For $p \geq 5$, all possible cases are covered by Theorem 64. $\square$

5.4 The case $p = 3$

We finally turn to quadratic sequences with discriminator $3^{\lceil \log_3 n \rceil}$. In this section, we present a set of necessary conditions and a set of sufficient conditions for a quadratic sequence with integer coefficients to have discriminator $3^{\lceil \log_3 n \rceil}$.

5.4.1 Necessary conditions with integer quadratic coefficients

We begin with the set of necessary conditions, as described in the following theorem.

Theorem 66. *Let $(q(n))_{n\geq 0} = (\alpha n^2 + \beta + \gamma)_{n\geq 0}$ be a quadratic sequence with integer coefficients such that $D_q(n) = 3^{\lceil \log_3 n \rceil}$ for all $n \geq 1$. Then there exist integers b and c such that $\alpha = 3c$, $\beta = bc$, and $3 \nmid bc$. Furthermore, if b is even, then c is also even.*

Proof. From Lemma 50, we know that $D_q(n) \neq 3^{\lceil \log_3 n \rceil}$ for some $n \geq 1$ if certain conditions are satisfied. Violating the conditions of Lemma 50 implies that $p \nmid \beta$ and $\alpha = p^k c$ for integers k and c such that $k \geq 1$ and $c|\beta$. It follows that $D_q(n) = 3^{\lceil \log_3 n \rceil}$ implies that $(q(n))_{n\geq 0} = (p^k cn^2 + bcn)_{n\geq 0}$ where $3 \nmid bc$.

Now, from Lemma 62, we know that if $p^k \geq 5$, then $D_q(n) \neq p^{\lceil \log_p n \rceil}$ for some $n \geq 1$. Therefore, we must have $3^k = p^k < 5$ in order for $D_q(n) \neq 3^{\lceil \log_3 n \rceil}$. Since $k \geq 1$, it follows that $k = 1$ and thus, $\alpha = 3c$.

Finally, if $(q(n))_{n\geq 0} = (3cn^2 + bcn)_{n\geq 0}$, then $q(0) = 3c(0) + bc(0) = 0$ and $q(1) = 3c(1) + bc(1) = c(3 + b)$. If b is even, then $3 + b$ is odd. If c is also odd, then $q(1)$ is odd, which means that the number 2 discriminates $\{q(0), q(1)\}$, and so, $D_q(2) = 2 \neq 3$, which contradicts $D_q(2) = 3^{\lceil \log_3 (2) \rceil} = 3$. Therefore, if b is even, then c must also be even. $\square$

These conditions are not sufficient, however. For example, the discriminator of the first four terms of $(3n^2+7n)_{n\geq 0}$ is 7 instead of $3^{\lceil \log_3 4 \rceil} = 9$, even though the necessary conditions are fulfilled.

5.4.2 Sufficient conditions with integer quadratic coefficients

We now derive a set of sufficient conditions by considering the class of sequences of the form $(\mathrm{qt}(n))_{n\geq 0} = (3cn^2 + bcn)_{n\geq 0}$. Provided that b and c satisfy certain restrictions, we show that the discriminator sequence is $D_{\mathrm{qt}}(n) = 3^{\lceil \log_3 n \rceil}$. Some examples of such sequences are $(3n^2 + n)_{n\geq 0}$, $(6n^2 - 4n)_{n\geq 0}$, and $(21n^2 + 49n)_{n\geq 0}$.

Applying Lemma (5.1) to sequences of the form $(\mathrm{qt}(n))_{n\geq 0} = (3cn^2 + bcn)_{n\geq 0}$ yields

$$\mathrm{qt}(j) - \mathrm{qt}(i) = (j - i)(\alpha(i + j) + \beta) = (j - i)(3c(i + j) + bc) = c(j - i)(3(i + j) + b). \quad (5.4)$$

Before proving any results relating to the discriminator of such sequences, we first establish the following general lemmas:

Lemma 67. *For all positive integers ℓ and k such that $\ell < 2(3^k)$, there exists a pair of integers i and j such that $i + j = \ell$ and $0 \le i < j \le 3^k$.*

Proof. If $\ell \le 3^k$, choose $i = 0$ and $j = \ell$. Otherwise, if $\ell > 3^k$, choose $i = \ell - 3^k$ and $j = 3^k$. In this case, we have $i < j$ since $\ell < 2(3^k)$. □

Lemma 68. *Let u and v be integers such that $u \ge 3$, $v \ge 5$ and v is odd, and let $k \ge 2$ be such that $3^k \le uv < 3^{k+1}$. Then $u + v - 1 \le 3^k$.*

Proof. We leave the finitely many cases of $k = 2$ to the reader. Otherwise, if $k \ge 3$, we have $3^k \ge 27$. There are two cases here:

Case 1: $u = 3$. Since v is odd, we have $v \le 3^k - 2$. Therefore, $u + v - 1 \le 3 + 3^k - 2 - 1 = 3^k$.

Case 2: $u \ge 4$. Since $v \ge 5$, it follows that $(u - 4)(v - 4) \ge 0$. This implies that

$$u + v \le \frac{uv + 16}{4} < \frac{3^{k+1} + 16}{4} \le 3^k \cdot \frac{3}{4} + 4.$$

Since $3^k \ge 27$, we have $3^k \cdot \frac{3}{4} + 4 \le 3^k + 1$ and thus, $u + v - 1 \le 3^k$.

□

We now present the following lemma which enforces a set of constraints on the values of b and c in order to prove the lower bound of $D_{\mathrm{qt}}(n)$ for $n \ge 1$.

Lemma 69. *Let $k \ge 0$ and let $(\mathrm{qt}(n))_{n \ge 0} = (3cn^2 + bn)_{n \ge 0}$ be a quadratic sequence such that b and c are non-zero integers that satisfy all of the following conditions:*

1. *$b \ge -2$.*
2. *$3 \nmid bc$.*
3. *If b is even, then c is also even.*
4. *If b is odd and there exists an integer x such that $2(3^k) < 2^x < 3^{k+1}$, $2^x \le |b|$, and $(b \bmod 2^x) \equiv 0 \pmod 3$, then c is even.*
5. *For every prime number p such that $2(3^k) < p < 3^{k+1}$, $p \le |b|$, and $(b \bmod p) \equiv 0 \pmod 3$, we have $p | c$.*

Then for all positive integers $m < 3^{k+1}$, there exists integers i and j such that $0 \le i < j \le 3^k$ and $m | \mathrm{qt}(j) - \mathrm{qt}(i)$.

Proof. Let $m = 2^x 3^y r$ for $x \ge 0$, $y \ge 0$, $2 \nmid r$ and $3 \nmid r$.

There are several cases to consider for the value of m.

1. $m \leq 3^k$. Choose $i = 0$ and $j = m$ so that $m|(j-i)|\operatorname{qt}(j) - \operatorname{qt}(i)$.

2. m is a positive power of 2. We can split this further into three cases.

   (a) $m = 2^x < 2(3^k)$. Let $\ell = -b(3)^{-1} \bmod 2^x$, so that $m|(3\ell + b)$. If $\ell \neq 0$, then apply Lemma 67 so that $i + j = \ell$. Otherwise, if $\ell = 0$, then apply Lemma 67 so that $i + j = 2^x$. In both cases, we have $m|\operatorname{qt}(j) - \operatorname{qt}(i)$ and $0 \leq i < j \leq 3^k$.

   (b) $m = 2^x \geq 2(3^k)$, c is even. The case of $m = 2$ is trivial. Otherwise, for $m > 2$, let $\ell = -b(3)^{-1} \bmod 2^{x-1}$. Note that $m|2(3\ell+b)$. If $\ell \neq 0$, then apply Lemma 67 so that $i+j = \ell$. Otherwise, if $\ell = 0$, then apply Lemma 67 so that $i+j = 2^{x-1}$. In both cases, we have $m|\operatorname{qt}(j) - \operatorname{qt}(i)$ and $0 \leq i < j \leq 3^k$.

   (c) $m = 2^x \geq 2(3^k)$, c is odd. Note that from conditions 3 and 4 of the lemma statement, we have b odd and $(b \bmod 2^x) \not\equiv 0 \pmod 3$. For $m = 2$, choose $i = 0$ and $j = 1$ so that $3(i + j) + b$ is even. Otherwise, we have $m > 2$. If $b < 0$, let $z = b$. Otherwise, let $z = b \bmod 2^x$. Then it suffices to have $2^x|3(i + j) + z$ so that $m|\operatorname{qt}(j) - \operatorname{qt}(i)$. Note that z must be odd since b is odd. If $2^x \equiv z \pmod 3$, then let $\ell = \frac{2^x - z}{3}$. Otherwise, if $2^x \not\equiv z \pmod 3$, we have $2^{x+1} \equiv z \pmod 3$ since $2^x \not\equiv 0 \pmod 3$ and $z \not\equiv 0 \pmod 3$, so we let $\ell = \frac{2^{x+1} - z}{3}$. For both cases, note that $\ell > 0$ since z is odd, and also that $z \geq -1$. Therefore,

   $$\begin{aligned}\ell \leq \frac{2^{x+1} - z}{3} \leq \frac{2^{x+1} + 1}{3} = \frac{2(2^x) + 1}{3} \leq \frac{2(3^{k+1} - 1) + 1}{3} \\ = \frac{2(3^{k+1} - 1)}{3} < \frac{2(3^{k+1})}{3} = 2(3^k).\end{aligned}$$

   Thus, we can apply Lemma 67 so that $i+j = \ell$ and so, we have $m|\operatorname{qt}(j) - \operatorname{qt}(i)$ and $0 \leq i < j \leq 3^k$.

3. m is a prime ≥ 5 or twice such a prime. We also have three cases here.

   (a) m is prime and $5 \leq m < 2(3^k)$. Let $\ell = -b(3)^{-1} \bmod m$ so that $m|(3\ell + b)$. If $\ell \neq 0$, then apply Lemma 67 so that $i + j = \ell$. Otherwise, if $\ell = 0$, then apply Lemma 67 so that $i + j = m$. In both cases, we have $m|\operatorname{qt}(j) - \operatorname{qt}(i)$ and $0 \leq i < j \leq 3^k$.

   (b) m is prime and $m > 2(3^k)$. If $(b \bmod m) \equiv 0 \pmod 3$, we have $m|c$, and so, $m|\operatorname{qt}(j) - \operatorname{qt}(i)$ for any choice of i and j. Otherwise, we have $(b \bmod m) \not\equiv 0 \pmod 3$. If $b < 0$, let $z = b$. Otherwise, let $z = b \bmod m$. Then it suffices to have $m|(3(i + j) + z)$ so that $m|\operatorname{qt}(j) - \operatorname{qt}(i)$. If $m \equiv z \pmod 3$, then let $\ell = \frac{m-z}{3}$. Otherwise, if $m \not\equiv 0 \pmod 3$ and $z \not\equiv 0 \pmod 3$, then we have $2m \equiv z \pmod 3$, so we let $\ell = \frac{2m-z}{3}$. For either case, since $-2 \leq z < m$, we have $\ell \leq \frac{2m-z}{3} \leq \frac{2m+2}{3}$ and that $m < 3^{k+1} - 1$, since m is odd. Therefore,

   $$\ell \leq \frac{2m + 2}{3} = \frac{2(m + 1)}{3} < \frac{2(3^{k+1} - 1 + 1)}{3} = \frac{2(3^{k+1})}{3} = 2(3^k).$$

Thus, we can apply Lemma 67 so that $i+j=\ell$ and so, we have $m|\operatorname{qt}(j)-\operatorname{qt}(i)$ and $0\le i<j\le 3^k$.

(c) $m=2p$ for some prime $p\ge 5$. Let $\ell=-b(3)^{-1} \bmod p$ so that $p|3\ell+b$. We observe that

$$\ell<p=\frac{m}{2}<\frac{2m}{3}<\frac{2(3^{k+1})}{3}=2(3^k).$$

Now, if $\ell=0$, we apply Lemma 67 so that $i+j=p$. Otherwise, if $\ell\neq 0$, we apply Lemma 67 so that $i+j=\ell$. For either case, we have $p|3(i+j)+b$. If b is even, then c is also even. Otherwise, if b is odd, then either $(j-i)$ or $(3(i+j)+b)$ is even. Therefore, $m=2p|\operatorname{qt}(j)-\operatorname{qt}(i)$.

4. m does not have any prime factors except 2 and 3. We split this into four cases.

   (a) $m=3^y$ or $m=2(3^y)$. Clearly $y\le k$. Choose $i=0$ and $j=3^y$. If b is odd, then $3(i+j)+b$ is even. Otherwise, c is even. Therefore, $m|\operatorname{qt}(3^y)-\operatorname{qt}(0)$.

   (b) $m=4(3^y)$, b is even. Clearly $y<k$. Choose $i=0$ and $j=2(3^y)<3^{y+1}\le 3^k$. Since b is even, c must also be even, and so $m|c(j-i)$.

   (c) $m=2^x3^y$, $x\ge 3$, and b is even. Choose $i=0$ and $j=2^{x-2}3^y=\frac{m}{4}<\frac{3^{k+1}}{4}<3^k$. Then $j-i=2^{x-2}3^y$ while both c and $3(i+j)+b$ are even. Therefore, $m|\operatorname{qt}(j)-\operatorname{qt}(i)$.

   (d) $m=2^x3^y$, $x\ge 2$, and b is odd. Clearly $y<k$. Let $\ell=(-b(3)^{-1}-3^y) \bmod 2^x$ so that $2^x|3(\ell+3^y)+b$. Note that ℓ is even. If $\ell=0$, choose $i=2^{x-1}$. Otherwise, choose $i=\frac{\ell}{2}$. In both cases, choose $j=i+3^y$ so that $j-i=3^y$ while $i+j=2i+3^y\equiv\ell+3^y \pmod{2^x}$, and therefore, $m|(j-i)(3(i+j)+b)=\operatorname{qt}(j)-\operatorname{qt}(i)$. To verify that $j\le 3^k$, note that $i\le 2^{x-1}$ and so, $j\le 2^{x-1}+3^y$. Since $m=2^x3^y<3^{k+1}$, it follows that $2^{x-1}\le\frac{m}{6}<\frac{3^{k+1}}{6}=\frac{3^k}{2}$ and so, $j\le 2^{x-1}+3^y<\frac{3^k}{2}+3^{k-1}=\frac{5(3^k)}{6}<3^k$.

5. For all other possible cases of m, we can write $m=uv$ such that $u\ge 3$, $v\ge 5$, and $\gcd(v,6)=1$. In this case, let $\ell=(-b(3)^{-1}-u) \bmod v$ where $(3)^{-1}$ is the multiplicative inverse of 3 modulo v, so that $v|3(\ell+u)+b$. If ℓ is even, then choose $i=\frac{\ell}{2}$. Otherwise, if ℓ is odd, choose $i=\frac{\ell+v}{2}$. In both cases, choose $j=i+u$ so that $j-i=u$ and

$$3(i+j)+b=3(2i+j)+b\equiv 3(\ell+u)+b\equiv 0 \pmod v.$$

   Therefore, $m=uv|(j-i)(3(i+j)+b)|\operatorname{qt}(j)-\operatorname{qt}(i)$. To verify that $j\le 3^k$, note that $0\le i<v$ and so, $j\le u+v-1$. Thus, we can apply Lemma 68 to show that $j\le u+v-1\le 3^k$.

In all cases, we have $m|\operatorname{qt}(j)-\operatorname{qt}(i)$ for some i and j in the required range. $\square$

Before we move on to the upper bound, we provide some insight on the choices of conditions in Lemma 69.

- Condition 4 is due to the argument in Case 2c, where m is a power of 2 such that $2(3^k) < m < 3^{k+1}$, e.g., $m = 64$. Since b is odd, it is impossible for 2 to divide both $(j - i)$ and $(3(i + j) + b)$, so we need $m|c(3(i + j) + b)$. If $z \not\equiv 0 \pmod 3$ as described in Case 2c, then there is no problem. Otherwise, the smallest non-negative solution for $3\ell + b \equiv 0 \pmod m$ can have ℓ being anywhere from 0 to $m - 1$, which might be bigger than $2(3^k)$, thus making it impossible for $\ell = i + j$ in some cases. So we need c to be even to allow $m|c(3(i + j) + b)$ then.

- Likewise, Condition 5 is due to case 3b, through similar logic. An example that illustrates the need for both Conditions 4 and 5 is $(3n^2 + 75n)_{n\geq 0}$, where the discriminator of the first nineteen terms is 61 (a prime), while the discriminator of the first twenty terms is 64 (a power of 2).

- Conditions 4 and 5 also include $2^x \leq |b|$ and $p \leq |b|$ respectively. This is because if b is positive and greater than 2^x or p respectively, then we have $z = b$, which implies $z \not\equiv 0 \pmod 3$ due to Condition 2, making Cases 2c and 3b applicable. Furthermore, for any positive b, there are finitely many primes p such that $p \leq |b|$, so Condition 5 still ensures that c is bounded.

- However, if b is negative, there are some potential problems. This is because even if $3 \nmid b$, there are infinitely many values of m such that $b \bmod m \equiv 0 \pmod 3$ while $m = 2^x$ or p as described in Conditions 4 and 5, if b is negative. The constraints of $2^x \leq |b|$ and $p \leq |b|$ would no longer be sufficient to capture all such scenarios. Even if we try to expand these constraints, there might be infinitely many primes p for which the only solution of $3\ell + b \equiv 0 \pmod p$ involves $\ell > 2(3^k)$, making it impossible to bound c then.

- Despite this, the arguments in Cases 2c and 3b still work for $z \geq -2$. So if b is negative, we can let $z = b$, but we add Condition 1 to ensure that $z = b \geq -2$ in such cases.

Justification for Conditions 2 and 3 are covered by Theorem 66. We now move on to the upper bound on the discriminator of $(\mathrm{qt}(n))_{n\geq 0}$, which is handled by the following lemma.

Lemma 70. *Let $k \geq 0$. For all pairs of integers i and j satisfying $0 \leq i < j < 3^{k+1}$, we have $3^{k+1} \nmid \mathrm{qt}(j) - \mathrm{qt}(i)$ if $3 \nmid bc$.*

Proof. For $\mathrm{qt}(j) - \mathrm{qt}(i) = c(j - i)(3(i + j) + b)$, it is given that $3 \nmid c$. Since $3 \nmid b$, it follows that $3(i + j) + b \equiv b \not\equiv 0 \pmod 3$, and so, $3 \nmid 3(i + j) + b$. Therefore, any powers of 3 that divide $\mathrm{qt}(j) - \mathrm{qt}(i)$ must divide the $(j - i)$. But $j - i \leq j < 3^{k+1}$. Therefore, $3^{k+1} \nmid \mathrm{qt}(j) - \mathrm{qt}(i)$ for all i and j in the range $0 \leq i < j < 3^{k+1}$. □

We now compute the discriminator of $(\mathrm{qt}(n))_{n\geq 0} = (3cn^2 + bn)_{n\geq 0}$ with the same conditions as Lemma 69.

Theorem 71. *Let* $(\mathrm{qt}(n))_{n\geq 0} = (3cn^2 + bn)_{n\geq 0}$ *be a quadratic sequence such that* b *and* c *are non-zero integers that satisfy all of the following conditions:*

1. $b \geq -2$.
2. $3 \nmid bc$.
3. *If* b *is even, then* c *is also even.*
4. *If* b *is odd and there exists a pair of positive integers* x *and* k *such that* $2(3^k) < 2^x < 3^{k+1}$, $2^x \leq |b|$, *and* $(b \bmod 2^x) \equiv 0 \pmod 3$, *then* c *is even.*
5. *For every prime number* p *such that* $2(3^k) < p < 3^{k+1}$ *for a positive integer* k, $p \leq |b|$, *and* $(b \bmod p) \equiv 0 \pmod 3$, *we have* $p|c$.

Then the discriminator $D_{\mathrm{qt}}(n)$ *satisfies the equation*

$$D_{\mathrm{qt}}(n) = 3^{\lceil \log_3 n \rceil} \tag{5.5}$$

for $n \geq 1$.

Proof. This follows directly from an application of Lemma 49 on $(\mathrm{qt}(n))_{n\geq 0}$ for $p = 3$, where the two conditions of Lemma 49 are fulfilled by Lemmas 69 and 70 respectively. $\square$

We presented a set of conditions for which $D_{\mathrm{qt}}(n) = 3^{\lceil \log_3 n \rceil}$. These conditions, however, are not necessary. A simple example to illustrate this is $(3n^2 + 25n)_{n\geq 0}$, which satisfies all five conditions except Condition 5 for $p = 19$, where $2(3^2) < 19 < 3^3$, $19 \leq |25|$, and $25 \bmod 3 \equiv 0 \pmod 3$, but $19 \nmid 1$. The discriminator is still $3^{\lceil \log_3 n \rceil}$, however. This can be shown by observing that all cases in Lemma 69 are still applicable, except the case of $m = 19$ with $k = 2$. But even then, we can still set $i = 6$ and $j = 7$ to get $3(i+j)+25 = 38 \equiv 0 \pmod{19}$, and so, $m|\,\mathrm{qt}(7)-\mathrm{qt}(6)$, satisfying the result of Lemma 69.

It remains an open problem to close the gap between the necessary and sufficient conditions in order to provide a complete characterization of all quadratic sequences with discriminator $3^{\lceil \log_3 n \rceil}$.

5.4.3 Half-integer quadratic coefficients

We briefly discuss the case of $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$ for odd α' and β'. Recall Lemma 64,

Lemma 64. *Let* $p \geq 3$ *be a prime number and let* $(\mathrm{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$ *for odd integers* α', β', *and* γ *be a quadratic sequence such that* α' *and* β' *are odd, and any of the following conditions are satisfied:*

1. $\alpha' = \beta'$;

2. $p \nmid \alpha'$;

3. $p | \beta'$;

4. $\alpha' = p^k c$ *for some integer* c *such that* $c \nmid \beta'$;

5. $\alpha' = p^k c$ *for some integer* c *such that* $c|\beta'$ *and* $p^k \geq 5$.

Then there exists a value of $n > 0$ *such that* $D_{\text{qr}}(n) \neq p^{\lceil \log_p n \rceil}$.

This covers most cases of $(\text{qr}(n))_{n\geq 0}$ for odd α' and β'. The only remaining case is when $\alpha' = 3c$ for some integer c such that $c|\beta'$ and $3 \nmid c$. Unlike with integer coefficients, we were unable to find any examples for which $D_{\text{qr}}(n) = 3^{\lceil \log_3 n \rceil}$.

Conjecture 72. *Let* b *and* c *be non-zero integers such that* $3 \nmid bc$. *For all sequences of the form* $(\text{qr}(n))_{n\geq 0} = (\frac{3c}{2}n^2 + \frac{bc}{2}n)_{n\geq 0}$ *such that* b *and* c *are non-zero integers with* $3 \nmid bc$, *there exists at least one value of* $n \geq 1$ *such that* $D_{\text{qr}}(n) \neq 3^{\lceil \log_3 n \rceil}$.

Proving this conjecture would prove that $D_{\text{qr}} \neq 3^{\lceil \log_3 n \rceil}$ for all sequences of the form $(\text{qr}(n))_{n\geq 0} = \left(\frac{\alpha'}{2}n^2 + \frac{\beta'}{2}n + \gamma\right)_{n\geq 0}$ with odd α' and β'.

Chapter 6

Conclusion

In this chapter, we summarize the main results of the thesis and present some open problems and areas for future research.

6.1 Summary

Chapter 1 introduces the concept of discriminators, provides a background of past literature on the topic, and lists several basic properties of discriminators in general before concluding the chapter with a simple guide on the standard approaches to computing the discriminator of a sequence.

Chapter 2 discusses various different computational aspects of discriminators, such as providing bounds on the discriminator growth rate and providing several algorithms for computing the discriminator of a given sequence. The chapter also explores the problem of determining whether a given sequence, either an infinite sequence or a finite prefix, is the discriminator of any other sequence. Finally, the chapter concludes with a characterization of self-discriminators and some empirical results relating to discriminators.

Chapter 3 focuses specifically on k-regular sequences, which includes the characterization of the discriminators of odious and evil numbers respectively, and proving that a k-regular sequences do not necessarily have k-regular discriminators.

Chapter 4 introduces the concept of shift-invariant discriminators, and provides a specific class of exponential sequences whose discriminators are all shift=invariant.

Finally, Chapter 5 studies quadratic sequences with discriminator $p^{\lceil \log_p n \rceil}$ for prime p, where the case of $p = 2$ appears as a common discriminator in previous chapters. The chapter provides a complete characterization of the quadratic sequences with discriminator $p^{\lceil \log_p n \rceil}$ for $p = 2$, shows that there are no such quadratic sequences with this discriminator for $p \geq 5$, and provides partial results for the case of $p = 3$.

6.2 Open Problems

There are many open problems in the area of discriminators. In particular, every sequence of distinct integers has a discriminator sequence, and there are many notable sequences whose discriminators were not examined before. The literature so far contains detailed characterizations on the discriminators of sequences of fixed powers (squares, cubes, etc) and significant contributions on sequences of various polynomials, but very few results on exponential sequences and on other types of sequences.

Aside from the discriminator of specific sequences, it is also an open problem to seek and establish general properties of the discriminator. In particular, it would be interesting to find other conditions for which the discriminators of two different sequences would be the same or related in some other manner.

There are also several open areas in the computational side of discriminators, such as improving the upper bound on the growth rate for discriminators in general, or for specific types of sequences like cubic sequences. The problem of determining whether a given sequence is a discriminator of some other sequence is also open, with many finite or infinite sequences that could be explored, such as the conjecture that there are no infinite sequences whose discriminator is the sequence of non-composites. The current algorithms for checking if a finite sequence is a discriminator can be improved on either efficiency or proving correctness. The empirical results presented about common discriminator sequences are also open to improvement, or they can be solidified with theoretical justification.

Chapter 3 also considered the discriminators of k-regular sequences, raising several questions like on when the discriminator of a k-regular sequence is also k-regular, or on whether there are any properties of the discriminators of k-regular sequences. There are many k-regular sequences whose discriminators were not studied yet, with this chapter providing a conjecture on the discriminator of the sequence of Cantor numbers specifically.

The topic of shift-invariant discriminators is rarely touched on in the literature. There are many open questions in this area, such as on what other sequences have shift-invariant discriminators, and on what other properties are shared by such sequences.

Finally, Chapter 5 presented partial results on quadratic sequences with discriminator $3^{\lceil \log_3 n \rceil}$, so a complete characterization is still open. The results in Chapter 5 also lead to the natural problem of considering polynomials of higher degree, and characterizing those with discriminator $p^{\lceil \log_p n \rceil}$.

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