Generating Functions for Hyperbolic Plane Tessellations

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics & Optimization

Waterloo, Ontario, Canada, 2017

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

For a hyperbolic plane tessellation there is a generating function with respect to the distance. This generating function is the same as the growth function of a group of isometries of hyperbolic plane that acts regularly in the tessellation. For most of the tessellation the generating functions have a symmetric form. In this thesis we will show the computation of the generating function for the hyperbolic plane tessellation and find the tessellations that have a symmetric form in the generating functions.

Acknowledgements

I would like to thank my supervisor David Wagner for guiding me to the problem we discuss in this thesis, and his support and encouragement during my writing. And I'd like to thank my friends and family for there support during the study of the degree.

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Chapter 1

Introduction

For a finitely generated group G and a finite generating set Σ for G, we can define a norm in G called the word norm. A word is a finite string of elements $g_1 g_2 \cdots g_n$ where each g_i satisfies either $g_i \in \Sigma$ or $g_i^{-1} \in \Sigma$. For each element g in the group G, the word norm of g is

$$|g| = min\{n \mid g = g_1 \cdots g_n, g_i \in \Sigma \ or \ g_i^{-1} \in \Sigma\}.$$

Then we have a generating function for the word norm in the group

$$f(z) = \sum_{g \in G} z^{|g|}.$$

This generating function is called the growth function of the group G with respect to generator set Σ. We have another equivalent form of the growth function

$$f(z) = \sum_{n \geq 0} s_n z^n,$$

where s_n is the number of elements in G with word norm n.

The growth function of a finitely generating group is generally difficult to compute. It will help if we study the groups that reflect some geometry, and this has been studied by many authors. Let us mention the Coxeter groups [2, 3, 22], surface groups and Fuchsian groups [3, 27, 5, 12], and Weyl groups[18].

Our interest starts with the regular hyperbolic tessellations. Here a hyperbolic tessellation, sometimes called a hyperbolic tiling, is a covering of the hyperbolic plane $\mathbb{H}^2$, without gaps or overlappings, by polygons of the same shape and size. And we also require that there is a transitive group action on these polygons, that is to say, we have a group G of hyperbolic plane automorphisms, that can map a polygon in the tessellation to any other polygon. Regular tessellations are the tessellations such that the polygon is regular. It is well-known that regular hyperbolic tessellations exist. A good reference with introduction and lots of examples can be found in [7]. The basics about hyperbolic plane can be found in [20].

We can also define a norm in a tessellation. First we will fix one polygon C_0 as the origin and let it have norm zero, then we will let each polygon have norm one plus the minimum norm among its neighbors. So the norm $|C|$ of a polygon C will be the minimal d such that there exists a sequence of polygons $C_0, C_1, \ldots, C_d$, where $C_d = C$ and each C_i neighbors C_{i+1}. Then we will have a generating function on the tessellation as

$$f(z) = \sum_{C} z^{|C|}.$$

Since the group action is transitive, this generating function doesn't depend on the choice of C_0, and we will see that it coincides with the growth function of G if G acts regularly on the tessellation, with respect to the geometric generator set. Here we say a group acts regularly on the tessellation if it acts regularly on the face, which means for any two faces on the tessellation, there are exactly one group element that maps one face to the other. And we define the geometric generating set as $\Sigma = \{g \in G \mid gC_0 \cap C_0 \text{ is an edge of } C_0\}$.

Since in this thesis we will mainly focus on the combinatorial information of the tessellation, it will be convenient to use the terminology in graph theory to describe the tessellation. We can consider the tessellation as an infinite planar graph, so each polygon is a face, and each face has p edges and p vertices around it.

The growth function of regular hyperbolic plane tessellations was first studied in [27, 4, 3], then there are some more study by different approachs in [24, 23]. We will give a new computation in Chapter 2, and we will see that the generating functions are reciprocal and rational with denominator and numerator having the same degree, such that we have a symmetry form in the generating function in terms of $f(z) = f(z^{-1})$.

In [3, 4] Cannon shows that the growth functions of cocompact hyperbolic groups are rational. This leads to the more general questions about the computation of the growth function of general (usually not regular) hyperbolic tessellations and determining when will the function still have a similar symmetry form $f(z) = f(z^{-1})$. Following [12, 13], we will give the complete computation in Chapter 3, the proof of the symmetry in Chapter 4, and a brief discussion about some exceptional cases in Chapter 5.

Chapter 2

Regular Hyperbolic Plane Tessellations

2.1 Introduction

The simplest tessellations of the hyperbolic plane are the regular tessellations. The generating function of regular tessellations was first computed in [3] and in [27]. In this chapter we will define them, discuss the corresponding groups, and give a proof of the generating function.

We use the notation $\{p, q\}$, $p, q \geq 3$, from [8], for a regular tessellation of a sphere, an Euclidean plane or a hyperbolic plane with regular p-gons, q of them intersecting at each vertex. There are five types of regular tessellations on sphere and three on Euclidean plane. All the others with $1/p + 1/q < 1/2$ are tessellations on a hyperbolic plane, and we will only consider the hyperbolic plane case.

In this chapter we will first introduce some types of groups that act regularly on the regular tessellations, and the relation between growth functions of these groups and the generating functions in section 2.2. Then we will prove a recursion relation on angles in section 2.3 and calculate the generating function in section 2.4. The final result will be the following theorem.

Theorem 1. *For a regular hyperbolic plane tessellation $\{p, q\}$, when q is even, we let $m = q/2$. Then we have the generating function*

$$f(z) = \frac{z^m + 2z^{m-1} + \cdots + 2z + 1}{z^m - (p-2)z^{m-1} - \cdots - (p-2)z + 1}.$$

When q odd, we let $m = q-1$. Then we have the generating function $f(z)$ equal to

$$\frac{z^m + 2z^{m-1} + \cdots + 2z^{m/2-1} + 4z^{m/2} + 2z^{m/2+1} + \cdots + 2z + 1}{z^m - (p-2)z^{m-1} - \cdots - (p-2)z^{m/2-1} - (p-4)z^{m/2} - (p-2)z^{m/2+1} - \cdots - (p-2)z + 1}.$$

2.2 Group Action and Dual Graphs

There are several types of groups that act regularly on the regular tessellations. Here we give three examples of such groups and their geometric generating sets.

For the case when $p = q = 4g$ for $g \geq 2$, the tessellation can be seen as the universal covering space (see chapter 1 in [17]) of an orientable closed surface of genus g. The fundamental group of this surface is called the surface group:

$$G_g = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle.$$

This group can also be seen as the group of deck transformations of the universal covering space, or an automorphism group of the tessellation. There is an origin face on the tessellation such that any a_i, b_i will take the origin face to a neighboring face, so the set $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ is a geometric generating set. The growth function of this group was first computed in [3] and [27], and it's the first example of calculating generating functions of regular hyperbolic plane tessellations.

We can also consider Coxeter group generated by the reflections along edges. A Coxeter group (see [1]) is a group with the presentation

$$\langle x_1, \ldots, x_n \mid (x_i x_j)^{m_{ij}} = 1, 1 \leq i, j \leq n \rangle$$

where $m_{ii} = 1$ and $2 \leq m_{ij} \leq \infty$ for $i \neq j$. If $m_{ij} = \infty$, then it means there no relation of the form $(x_i x_j)^{m_{ij}}$ in the presentation. We will give further discussion about Coxeter group in section 5.1. Now let's fix one face, which will be called the origin face, and label the edges of this face from 1 to p clockwise. Here we let x_i be the reflection along the edge i. Then we have a Coxeter group

$$\langle x_1, \ldots, x_p \mid x_1^2, x_2^2, \ldots, x_p^2, (x_1 x_2)^q, (x_2 x_3)^q, \ldots, (x_p x_1)^q \rangle$$

for q odd and

$$\langle x_1, \ldots, x_p \mid x_1^2, x_2^2, \ldots, x_p^2, (x_1 x_2)^{q/2}, (x_2 x_3)^{q/2}, \ldots, (x_p x_1)^{q/2} \rangle$$

for q even. When q is even, this Coxeter group acts regularly on regular tessellation $\{p, q\}$ with geometric generating set $\{x_1, \dots, x_p\}$.

More generally, we have the Fuchsian groups. A Fuchsian group is a finitely generated discrete group of orientation-preserving noneuclidean automorphisms, which means a discrete group of Möbius transformations that maps the hyperbolic plane to itself.(see [20, II.7]) One example of Fuchsian groups that act on regular tessellations are groups generated by rotations around each vertex of a face. Again we fix an origin face and label the vertices clockwisely, and let a_i be the clockwise rotation around vertex i by $2\pi/q$. Then we have a Coxeter group

$$G = \langle a_1, \dots, a_p \mid a_1^q, \dots, a_p^q, (a_1a_2)^p, \dots, (a_pa_1)^p \rangle$$

for p odd, and

$$G = \langle a_1, \dots, a_p \mid a_1^q, \dots, a_p^q, (a_1a_2)^{p/2}, \dots, (a_pa_1)^{p/2} \rangle$$

for p even. Notice that these groups don't always act regularly on the tessellation. In [28] Robert shows an equivalent condition for the existence of a regular action by Fuchsian groups.

Theorem 2 (Theorem 1 in [28]). *The tessellation of type $\{p, q\}$ is a tessellation of the hyperbolic plane by fundamental domains of some Fuchsian group if and only if q has a prime divisor less than or equal to p.*

We will discuss more about fundamental domains in Chapter 3. For the more general groups of discrete, co-compact groups of isometries of $\mathbb{H}^2$, we still don't know what is the condition on $\{p, q\}$ such that a regular group action exists.

To describe the precise relation between the growth functions of the groups and the generating functions of the tessellations, we will introduce the dual graph Γ of the tessellation. By dual graph we mean a new graph Γ with the vertex set containing the faces of the tessellation. Two vertices in Γ are adjacent if and only if the two faces share a common edge in the tessellation. Any face in Γ will correspond to a vertex in the tessellation. If the origin face D corresponds to a vertex u_0 in Γ, then it's easy to see that a face with distance n to D in the tessellation will correspond to a vertex in Γ with distance n to u_0. So the generating function of tessellation is equal to $f(z) = \sum_{v \in V(\Gamma)} z^{d(u_0, v)}$, where $d(u_0, v)$ is the distance between u_0 and v in Γ.

When there is a group G acting regularly on the faces of a tessellation, we can see that G also acts regularly on the vertices of the dual graph Γ of the tessellation. In this case the graph Γ is exactly the undirected Cayley graph $X(G, \Sigma \cup \Sigma^{-1})$. An undirected Cayley graph $X(G, S)$ is a graph for G with respect to a subset S of G that is closed under taking inverse. The vertex set of this graph $X(G, S)$ is the set of elements of G, and two vertex u and v are adjacent if and only if $ug = v$ for some $g \in S$.

Proposition 3. *The Cayley graph $X(G, \Sigma \cup \Sigma^{-1})$ is isomorphic to Γ.*

Proof. Let's fix a vertex u_0 in Γ, then we want to show that the isomorphism ρ from $X(G, \Sigma \cup \Sigma^{-1})$ to Γ can be given by mapping the vertex g in $X(G, \Sigma \cup \Sigma^{-1})$ to $\rho(g) = gu_0$, the image of u_0 under g in Γ. It's easy to see that ρ is a proper bijection on vertex sets, so all we need to show is the correspondence between the edges. If $h \in G$ map u_0 to hu_0 in Γ, then since h is an isomorphism, h will also map the neighborhood of u_0 to the neighborhood of hu_0. Notice that all neighboring vertices of u_0 are in form of $g_i u_0$ or $g_i^{-1} u_0$ for $g_i \in \Sigma$, h will map $g_i u_0$ to $hg_i u_0$. This will correspond to the edge between h and hg_i in the Cayley graph. Similar for $g_i^{-1} u_0$. On the other hand for each edge between h and hg_j in the Cayley graph, there is a corresponding edge hu_0 and $hg_j u_0$ on Γ because h is an isomorphism. Thus ρ is an isomorphism. $\square$

Remark. *Notice that in the proof of Proposition 3, the order of multiplication in the definition of the Cayley graph is important. In Γ, for a $g_i \in \Sigma$, vertex hu_0 is always adjacent to $hg_i u_0$, but hu_0 is usually not adjacent to $g_i hu_0$.*

In a Cayley graph $X(G, \Sigma \cup \Sigma^{-1})$, for any element g, the distance between g and 1 is the same as the word norm of g with respect to Σ. Thus, following the discussion about the dual graph Γ and the Cayley graph $X(G, \Sigma \cup \Sigma^{-1})$, we know that the generating function of the a tessellation is the same as the growth function of a group that acts regularly on the tessellation with respect to a geometric generating set.

2.3 Angular Recursion

We will use recursion on angles to calculate the generating functions of the regular tessellations. This idea comes from Floyd and Plotnick in [13].

First we define $B(n)$ to be the part of regular tessellation consisting of faces with norm at most n. And we use $\partial B(n)$ to denote the boundary of $B(n)$, which means the edges separating faces with norm n and faces with norm $n + 1$. $B(0)$ is the origin face. All the angles on $\partial B(0)$ are equal, which we will call the primitive angle. On the $\partial B(n)$, every angle will be the sum of some copies of the primitive angle. When q is even, we use α_i to denote an angle equal to sum of $2i - 1$ primitive angles where $1 \le i \le q/2$. When q is odd, we use α_i to denote an angle equal to sum of $2i-1$ primitive angles where $1 \le i \le \lfloor q/2 \rfloor$ and $\alpha_{\lfloor q/2 \rfloor + i}$ to denote an angle equal to sum of $2i$ primitive angles where $1 \le i \le \lfloor q/2 \rfloor$.

We will let $m = q/2$ when q is even and $m = q-1$ when q is odd, so that we can always call the last angle α_m.

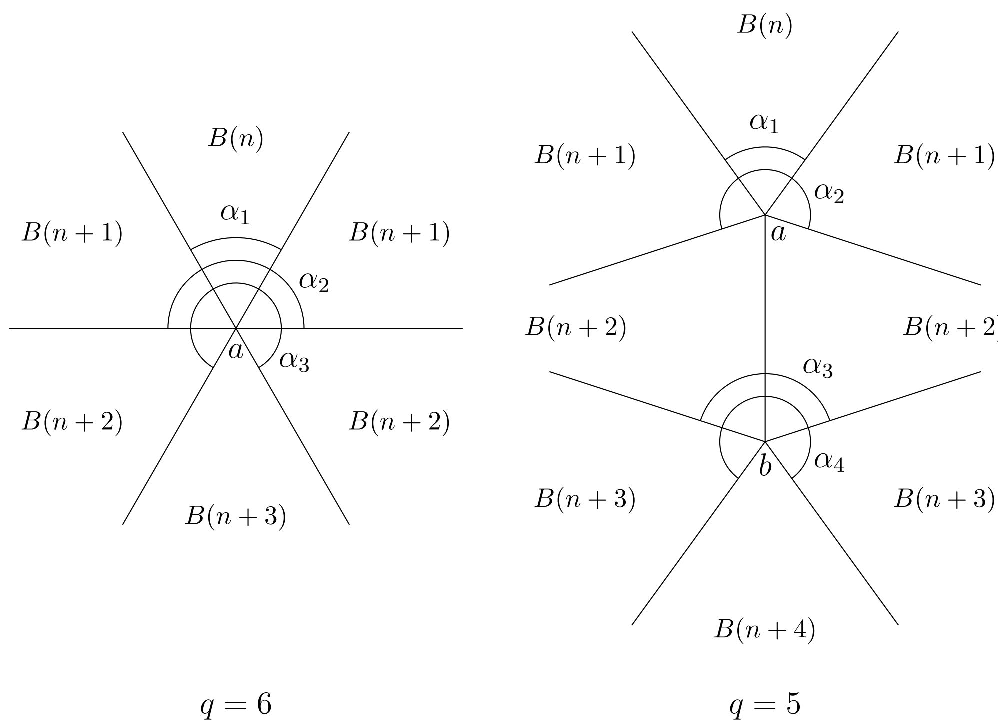

Figure 2.1: the local picture of two examples when $q = 6$ and $q = 5$.

In this thesis we will often use figures to show the local pictures of the tessellation or the dual graph like Figure 2.1, and we will always let the upper side of the figure be closer to the origin face and the lower side further from the origin face.

Every edge on $\partial B(n)$ will be an edge of a face with norm $n+1$. Thus $B(n+1)$ can be seen as $B(n)$ with new faces of norm $n+1$ glued onto the $\partial B(n)$. For any $n \ge 0$, take any angle α_i on $\partial B(n)$ with endpoint a, then a will change to a new angle on $\partial B(n+1)$. We will summarize these changes in the following proposition.

Proposition 4. *Let α_i be an angle on $\partial B(n)$ with endpoint a, for $n \ge 1$, .*

(a) When q is even, if $0 \le i < m$ then a will be the endpoint of an angle α_{i+1} on $\partial B(n)$. If $i = m$, then a will no longer be on $\partial B(n+1)$.

(b) When q is odd, if $0 \leq i < m$ and $i \neq m/2$ then a will be the endpoint of an angle α_{i+1} on $\partial B(n+1)$. If $i = m/2$ or m, then a will no longer be on $\partial B(n+1)$. But when $i = m/2$, a will be adjacent to the endpoint b of an angle $\alpha_{m/2+1}$ on $\partial B(n+1)$.

Proof. We will use induction on n to prove both the statements (a) and (b), and also the following two claims.

Claim 1. *When $p > 3$, for any $n \geq 0$ and any α_i on $\partial B(n+1)$ where $i > 1$, both of the two neighboring angles of α_i are α_1.*

Claim 2. *When $p = 3$ and q odd, for any $n \geq 0$ and for any α_m or $\alpha_{m/2}$ on $\partial B(n+1)$, both of the two neighboring angles are α_1.*

First we discuss the case when $p > 3$, with q either even or odd. We will prove Claim 1 and statement (a) and (b) for $p > 3$. For the base case $n = 0$, all angles on $\partial B(0)$ are primitive angles α_1.

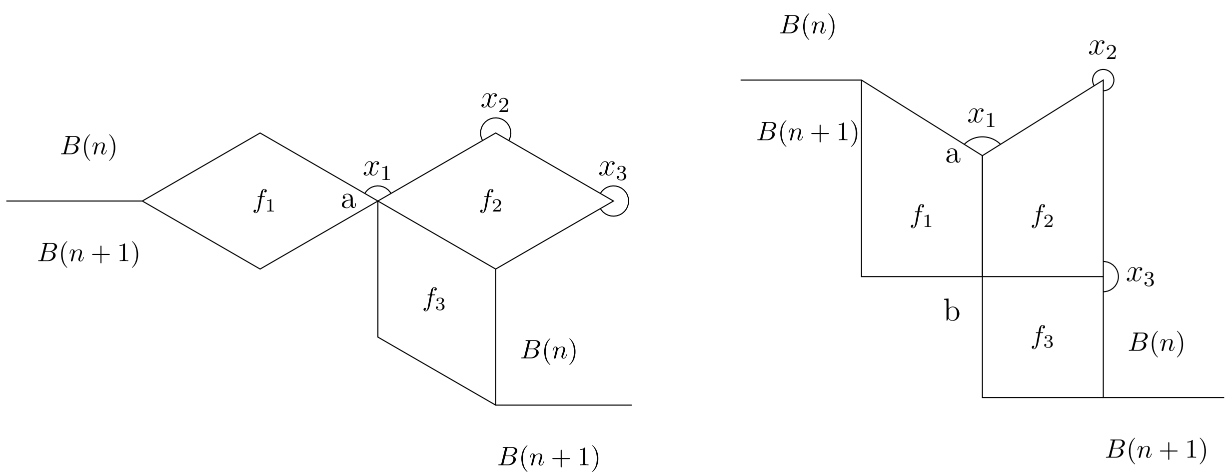

Figure 2.2: local picture for the induction around point a

Now assume that Claim 1 is true for some $n \geq 0$. We first check the statements (a) and (b) for α_1. Let's use x_1 to denote α_1. The statements will not hold only if there are new faces, other than the two new faces f_1 and f_2 with norm $n+1$ glued on the two sides of x_1, that intersect at a (or at b, a neighboring vertex of a when q odd and $x_1 = \alpha_{m/2}$)(see figure 2.2). If there exists such a third new face f_3, let's use x_2 to denote the angle neighboring x_1 on $\partial B(n)$ on the side closer to the third new face, and x_3 the angle neighboring x_2 on

$\partial B(n)$ such that $x_3 \neq x_1$. Then we can see that $x_2 = \alpha_m$, $x_3 = \alpha_m$ or $\alpha_{m/2}$, neither of them is α_1, which contradicts the Claim 1 on n. Notice that x_3 can equal to α_1 only if $m/2 = 1$, in which case $p = 4$, $q = 3$ and $x_1 = \alpha_1$, but $\{4, 3\}$ is not a hyperbolic plane tessellation.

Then we will check Claim 1 on $\partial B(n+1)$. So we assume that y_1 is an angle on $\partial B(n+1)$ that equal to α_{i+1} for $0 \leq i < m$, then from induction hypothesis we know that y_1 is generated from $x_1 = \alpha_i$ on $\partial B(n)$. Let y_2 be an angle neighboring y_1 on $\partial B(n+1)$, and again we use x_2 to denote the angle neighboring x_1 on $\partial B(n)$ on the side closer to y_2, and x_3 the angle neighboring x_2 on $\partial B(n)$ such that $x_3 \neq x_1$. All we need to show is y_2 is an angle α_1. If not, then we have $x_2 = \alpha_m$, $x_3 = \alpha_m$ or $\alpha_{m/2}$, neither of them is α_1, contradict to Claim 1 on n. Similarly, x_3 can equal to α_1 only if $p = 4$, $q = 3$ and $x_1 = \alpha_1$, which is not a hyperbolic plane tessellation. This conclude the induction on $p > 3$.

For the case $p = 3$ we are going to proof Claim 2 and statement (a), (b). Similarly if statement (a) or (b) doesn't hold near an angle x_1 with endpoint a on $\partial B(n)$, we have a third new face and x_2, x_3 on $\partial B(n)$ such that x_2 is equal to α_m and x_3 is not α_1. Since $q \geq 7$ we don't need to worry about the case when $m/2 = 1$.

At last we will check Claim 2. Let's take any two neighboring angles y_1 and y_2 on $\partial B(n+1)$ and assume that Claim 2 fails, y_1 is α_m or $\alpha_{m/2}$, but $y_2 = \alpha_i$ but $i \neq 1$. First we assume that i is also not $m/2 + 1$. We can see that the endpoint a of y_1 is the endpoint of an angle x_1 equal to α_{m-1}(when $y_1 = \alpha_m$) or $\alpha_{m/2-1}$(when $y_1 = \alpha_{m/2}$) on $\partial B(n)$, otherwise we can find two neighboring angles on $\partial B(n-1)$ that are not α_1 just like the proof of (a) and (b). Similarly the endpoint b of y_2 is the endpoint of an angle $x_2 = \alpha_{i-1}$ on $\partial B(n)$. Notice that the edge between a and b is the edge of a face f with norm $k+1$, and the other two edges of f are sides of α_{m-1} and α_{i-1} respectively. Thus the third vertex c of f(other than a and b) must be the endpoint of an angle $x_3 = \alpha_m$ on $\partial B(n)$. This x_3 is adjacent to x_1 on $\partial B(n)$. But x_1 is either α_{m-1} or $\alpha_{m/2-1}$, not α_1, a contradiction.

The proof is similar when y_2 is $\alpha_{m/2+1}$. The endpoint a of y_1 is again the endpoint of x_1 equal to α_{m-1} or $\alpha_{m/2-1}$ on $\partial B(n)$. Now the endpoint b of y_2 is adjacent to c, where c is the endpoint of an angle $x_2 = \alpha_{m/2}$ on $\partial B(n)$ and the third vertex of the face f with norm $k + 1$ containing both a and b. So we have $x_2 = \alpha_{m/2}$ adjacent to x_1, which is not α_1, a contradiction.

□

Corollary 5. *For any face f with norm $n + 1$, f will be either adjacent to one face in $B(n)$, or adjacent to two faces f_1, f_2 in $B(n)$, where $f \cap f_1 \cap f_2$ is the endpoint of an angle α_m on $\partial B(n)$.*

Proof. If f is a face with norm $n+1$, and f intersect with $B(n)$ on at least three consecutive edges, then any two neighboring edges in intersection will be the sides of an angle α_m on $\partial B(n)$. By Claim 1 and 2 in proof of Proposition 4, there are no neighboring α_m on $\partial B(n)$. Thus there are at most two edges in the intersection.

□

2.4 Generating Function

Finally we are going to prove Theorem 1. First we will calculate the number of angles of each type on $\partial B(n)$ for each n by showing that there are recursive relations among them. Let $N_i(n)$ be the number of angles of type α_i on the boundary of $B(n)$, where $1 \leq i \leq m$. The reason we try to compute $N_i(n)$ comes from the following proposition.

Proposition 6. *For every $n \geq 0$ we have $s_{n+1} = \sum_{i=1}^{m-1} N_i(n)$.*

Proof. By Corollary 5, every face with norm $n+1$ will intersect with $B(n)$ on either one edge, which corresponds to two non-α_m angles on the two end of the edge, or on the two side of an angle α_m, which correspond to an α_m neighboring two α_1 on $\partial B(n)$. On the other hand, from P position 4 we know that each α_i with $i < m$ on $\partial B(n)$ has two faces with norm $n+1$ glued on the two sides. In conclusion, each face with norm $n+1$ correspond to two non-α_m angles, and each α_i that is not α_m also correspond to two faces with norm $n+1$. So s_{n+1} is equal to the total numbers of α_i on $\partial B(n)$ for $i < m$, or $s_{n+1} = \sum_{i=1}^{m-1} N_i(n)$. □

We want to find, for each α_i on the boundary of $B(n)$, how many angles of each type will it contribute on the boundary of $B(n+1)$. From the previous analysis we know that there are two new faces corresponding to each angle α_i for $i < m$, and no new faces for α_m. Each new face has p α_1's before it is glued on $B(n)$. Thus each α_i with $i < m$ corresponds to $2p/2 = p$ α_1's since each face corresponds to two α_i with $i < m$.

Now we discuss what happens when we glue the new faces. In general, when $i \neq \frac{q-1}{2}$ or m, we have two new faces glued on the two sides of α_i, and two angles α_1 on the new faces glued at α_i, generating a new angle α_{i+1}(see Figure 2.3). Thus α_i on the boundary of $B(n)$ generates $(p-2)$ α_1's and one α_{i+1} on $\partial B(n+1)$. When q is odd and $i = \frac{q-1}{2}$, the two new faces have four angles glued into new angles. So in this case we have $(p-4)$ α_1's and one α_{i+1}. When $i = m$, only one angle is glued so α_m corresponds to (-1) α_1,

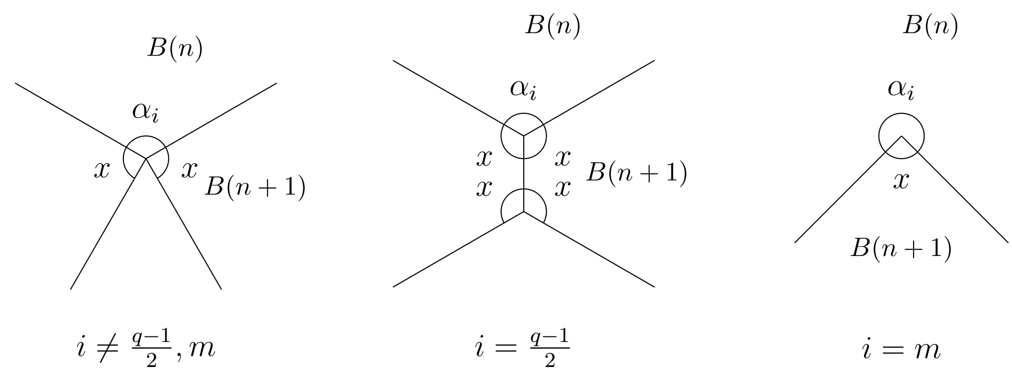

Figure 2.3: The angles that disappear after gluing(mark by x)

meaning we only lose one α_1.
Now, for $n > 0$, we have the following recursive relations:

$$N_i(n) = N_{i-1}(n-1) \tag{2.1}$$

where $i > 1$,

$$N_1(n) = (p-2)N_1(n-1) + \cdots + (p-2)N_{m-1}(n-1) - N_m(n-1) \tag{2.2}$$

when q is even, and

$$\begin{aligned} N_1(n) = (p-2)N_1(n-1) + \cdots + (p-2)N_{m/2-1}(n-1) + (p-4)N_{m/2}(n-1) \\ + (p-2)N_{m/2+1}(n-1) + \cdots + (p-2)N_{m-1}(n-1) - N_m(n-1) \end{aligned}$$

when q is odd.

For $n \geq m$ and q even, we can make it more uniform by using (2.1) and (2.2):

$$\begin{aligned} N_i(n) &= N_1(n-i+1) \\ &= (p-2)N_1(n-i) + \cdots + (p-2)N_{m-1}(n-i) - N_m(n-i) \\ &= (p-2)N_i(n-1) + \cdots + (p-2)N_i(n-m+1) - N_i(n-m) \end{aligned} \tag{2.3}$$

Since $s_{n+1} = \sum_{i=1}^{m-1} N_i(n)$, we have

$$s_n = (p-2)s_{n-1} + \cdots + (p-2)s_{n-m+1} - s_{n-m} \tag{2.4}$$

for q even and $n \geq m+1$. Similarly, for q odd and $n \geq m+1$, we have

$$s_n = (p-2)s_{n-1} + \cdots + (p-4)s_{m/2} + \cdots + (p-2)s_{n-m+1} - s_{n-m}.$$

For q even, we can get rid of the constraint on n by defining b_i as

$$b_n = s_n - (p-2)s_{n-1} - \cdots - (p-2)s_{n-m+1} + s_{n-m} \tag{2.5}$$

where we let $s_n = 0$ and $N_i(n) = 0$ for $n < 0$ and $1 \leq i \leq m$. Then $b_n = 0$ for $n \geq m+1$. Standard results in rational generating functions shows that (2.5) is equivalent to

$$\sum_{n\geq 0} s_n z^n = \frac{b_m z^m + \cdots + b_1 z + b_0}{z^m - (p-2)z^{m-1} - \cdots - (p-2)z + 1}$$

by clearing the denominator and comparing the coefficients.

Similarly, for q odd we have

$$f(z) = \sum_{n\geq 0} s_n z^n =$$

$$\frac{b_m z^m + \cdots + b_1 z + b_0}{z^m - (p-2)z^{m-1} - \cdots - (p-2)z^{m/2-1} - (p-4)z^{m/2} - (p-2)z^{m/2+1} - \cdots - (p-2)z + 1}$$

At last we need to solve b_i for $0 \leq i \leq m$. For the even case, first we observe that $s_0 = 1$ and $s_1 = p$. These are the only initial conditions that make (2.1) and (2.2) fail. We can see that $s_1 = N_1(0)$ comes from the p α_1's on the boundary of $B(0)$ which can not be deduced from $N_i(-1) = 0$, and s_0 is initial condition that can't be deduced from smaller s_i.

Thus when we try to use (2.4) for $n \leq m$, we implicitly use (2.3). But (2.3) doesn't hold for $n = i-1$ since $N_1(0)$ is not obtained from (2.2), so in the (2.3) the left-hand side $N_i(i-1) = p$ but the right-hand side is 0. This will cause, on the left-hand side of (2.4) for $n < m$, the s_n is larger than it should be by p. Notice that this will not affect $n = m$ since there is no term $N_i(i-1)$ in the sum $s_m = \sum_{i=1}^{m-1} N_i(m-1)$. For s_0, it should appear as 0 in (2.4) for $1 \leq n \leq m$ since it do not contribute to any angles on boundary of $B(n)$.

To correct these error, in (2.4) we replace s_0 with $s_0' = s_0 - 1 = 0$ and s_n with $s_n' = s_n - p$ for $1 \leq n < m$. Now we have a new version of (2.4) for $1 \leq n < m$:

$$s_n' = (p-2)s_{n-1} + \cdots + (p-2)s_1 + (p-2)s_0'$$

which implies

$$b_n = s_n - (p-2)s_{n-1} - \cdots - (p-2)s_1 - (p-2)s_0 = 2.$$

When $n = m$ we have

$$b_m = s_m - (p-2)s_{m-1} - \cdots - (p-2)s_1 + s_0 = 1.$$

At last we have $b_0 = s_0 = 1$. Thus we have conclude that when q is even

$$f(z) = \frac{z^m + 2z^{m-1} + \cdots + 2z + 1}{z^m - (p-2)z^{m-1} - \cdots - (p-2)z + 1}.$$

Similarly when q odd we have $f(z)$ equal to

$$\frac{z^m + 2z^{m-1} + \cdots + 2z^{m/2-1} + 4z^{m/2} + 2z^{m/2+1} + \cdots + 2z + 1}{z^m - (p-2)z^{m-1} - \cdots - (p-2)z^{m/2-1} - (p-4)z^{m/2} - (p-2)z^{m/2+1} - \cdots - (p-2)z + 1}.$$

Thus we prove the Theorem 1.

Chapter 3

General Hyperbolic Plane Tessellations

3.1 Introduction

In general, on tessellations that are not regular, which means the polygon are not regular, we can't expect describing the recursion in one single equation like (2.4).

We will define the tessellations of the hyperbolic plane by the action of discrete, cocompact groups G of isometries of $\mathbb{H}^2$. The detail of the following construction can be found in [16] and [20]. A group acting on $\mathbb{H}^2$ is discrete if whenever a subset $K \subseteq \mathbb{H}^2$ is compact, $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite. A group acting on $\mathbb{H}^2$ is cocompact if $\mathbb{H}^2/G$ is compact. For such a group G, we have a fundamental polygon D(also called the fundamental domain), which is a polygon in $\mathbb{H}^2$, that satisfies:

(a) $GD = \mathbb{H}^2$, the orbit of D covers $\mathbb{H}^2$;

(b) The interior of D doesn't intersect with any image of D under a non-identity element of G. We can see that (a) and (b) together require the group G acts regularly on the tessellation, as in chapter 1 and 2;

(c) For each edge s of D, there is a $g \in G$ such that $D \cap gD = s$, and

(d) A compact set $K \subseteq \mathbb{H}^2$ intersects only finitely many elements in the orbit of D.

One classic construction of a fundamental polygon on $\mathbb{H}^2$ is the Dirichlet region D with center x. Here we choose a point x in $\mathbb{H}^2$ that is not fixed by any non-identity elements of G, then we have

$$D = \{z \in \mathbb{H}^2 : d(z, x) \leq d(z, gx) \text{ for all } g \in G\},$$

where d is the standard distance in hyperbolic plane. From the definition of the fundamental polygon D we can see that the orbits of D under the action of G form a tessellation of $\mathbb{H}^2$. Since we only consider G finitely generated, the polygon D has finitely many edges. So we have a finite set $\Sigma = \{g \in G \mid gD \cap D \text{ is an edge of } D\}$. And Σ is a finite generating set of G closed under taking inverse. Again we call Σ the geometric generating set.

Here is an example of one type of tessellations and the corresponding groups. A Coxeter triangle group is a group $\Delta^*(p, q, r)$ with the presentation

$$\Delta^*(p, q, r) = \langle x, y, z \mid x^2, y^2, z^2, (xy)^p, (yz)^q, (zx)^r \rangle.$$

It can be seen as the group generated by reflections in the three edges of a triangle, with the angle between the edge reflected by x and the edge reflected by y is π/p, and similarly the other two angles equal to π/q and π/r. Figure 3.1 is an illustration of part of the tessellation generated by $\Delta^*(2, 3, 8)$. We let $\Delta(p, q, r)$ denotes the subgroup of $\Delta^*(p, q, r)$ consisting of all elements that are product of even number of reflections. Thus we can see that $\Delta(p, q, r)$ preserves the orientation of the $\mathbb{H}^2$, so $\Delta(p, q, r)$ is a Fuchsian group. We will call $\Delta(p, q, r)$ the triangle group.

Once we have a group G acting on the tessellation, we can determine a presentation of this group with respect to the geometric generating set, from the fundamental polygon D(see [21]). For any element g_i in the geometric generating set Σ, it will map D to g_iD with $D \cap g_iD$ an edge e of D. If we let $e' = g_i^{-1}e$ be the edge in D mapped by g_i to e, then we have paired e with e'. Each edge of D is paired by a unique element in Σ to an edge of D(can be the same edge), because we assume that the action of G is regular. Now start from any vertex a_1 in D, and an edge e_1 on D that is incident with a_1. Let $g_1 \in \Sigma$ be the element that pair (e_1, a_1) with $(g_1(e_1), g_1(a_1))$ where $g_1(e_1)$ is another edge on D. Let $a_2 = g_1(a_1)$, and let e_2 be the other edge on D that is incident with a_2 other than $g_1(e_1)$. Let $g_2 \in \Sigma$ maps (e_2, a_2) to $(g_2(e_2), g_2(a_2))$ for $g_2(e_2)$ an edge on D, $a_3 = g_2(a_2)$ and e_3 the other edge on D incident with a_2. Continuing in this fashion, we obtain a sequence of edges $e_1, e_2, \ldots$ and of vertices $a_1, a_2, \ldots$ on D. Let r be the smallest positive integer so that $e_{r+1} = e$, and $a_{r+1} = g_r g_{r-1} \cdots g_1 a_1 = a_1$. We will call the sequence $(a_1, a_2, \ldots, a_r)$ a vertex cycle through a. Notice that G acts transitively on the vertices on a vertex cycle, so every vertex in a vertex cycle has the same degree. Also notice that if g_i in the sequence is a reflection in an edge of D, then exactly two of $g_1, \ldots, g_r$ are reflections, and each vertex

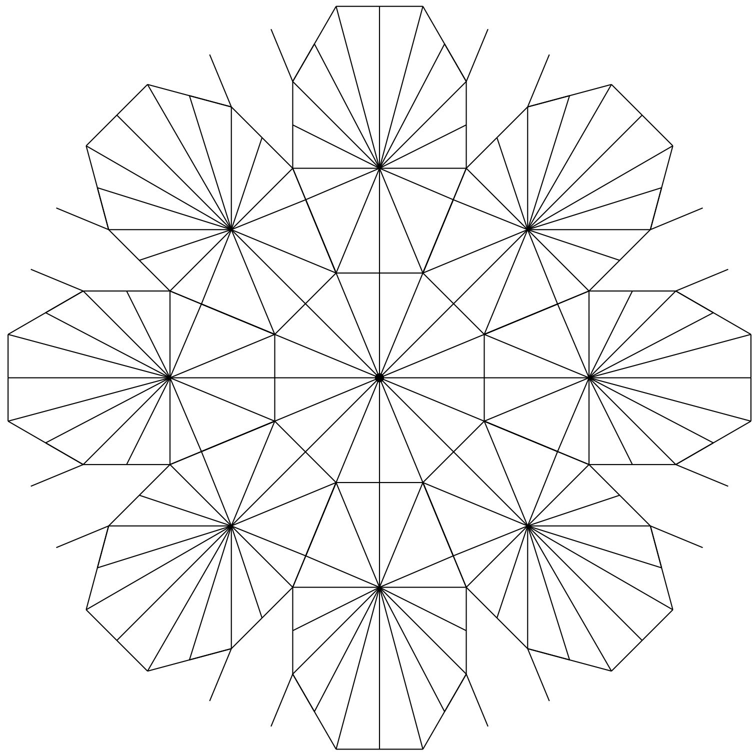

Figure 3.1: part of the tessellation generated by $\Delta^*(2, 3, 8)$

in the cycle appears exactly twice. If none of the g_i is a reflection, then each vertex appears only once.

If we consider the $\mathbb{H}^2$ as topological space and D correspond to the quotient space of $\mathbb{H}^2$ by the action of group G(with some vertices and edges of D identified), then the vertices on a vertex cycle will all correspond to a vertex of the quotient surface. Thus we can have a partition of the vertices of D by vertex cycles $[a_1], \ldots, [a_m]$. For a vertex cycle $[a_i]$, the sequence we construct above will determine a word $c_i = g_r \cdots g_1$, well-defined up to cyclic permutation or inversion. Notice that this c_i is an element in G that preserves the orientation of $\mathbb{H}^2$ and fixes vertex a_i, so it is a rotation about a_i of order n_i. Assume that $f_1, \ldots, f_s$ are all the reflection on edges in Σ, then we can conclude that G has the presentation

$$G = \langle \Sigma \mid (f_1)^2, \ldots, (f_s)^2, (c_1)^{n_1}, \ldots, (c_m)^{n_m} \rangle. \tag{3.1}$$

Here we give an example in figure 3.2, where e is a reflection and f, g, h are orientation preserving isometry. Then the group G will have a presentation:

$$G = \langle e, f, g, h \mid e^2, f^{-1}g^{-1}egfe, f^3, (hg)^2, h^2 \rangle$$

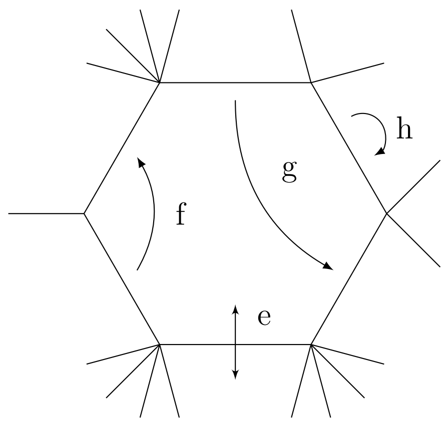

Figure 3.2: fundamental polygon and the generating set of G

If c_i contains two reflections, say e and f(we may have $e = f$), then we have $c_i = g^{-1}egf$ up to a cyclic permutation of the word c_i, for some word g. Also, since G is discrete, we can sum up the internal angles of word $(c_i)^{n_i}$ and get 2π. More specifically, for the word $c_i = g_r \cdots g_1$ and the sequence $(a_1, \dots, a_r)$, we can take the sum of the internal angles of D at all the $a_1, \dots, a_r$. Then we repeat the sum for n_i times, and the result should be 2π, since $(c_i)^{n_i} = 1$ means we complete the angle around the vertex. Conversely, suppose that we are given a polygon D with edge pairings Σ satisfying the cycle condition: the interior angles of D in a vertex cycle sum to $2\pi/n_i$ for some positive integer n_i. Then the Poincaré's Theorem for Fundamental Polygons in [21] shows that the edge pairings generate a discrete group with presentation as (3.1).

In this chapter we will give the computation of the generating functions of tessellations except for several exceptional cases, following [12] and [13]. We will first discuss the dual graph of the tessellation and propose a proposition that characterize the dual graph in section 3.2, then we will put the entire proof of this proposition in section 3.3. At the end we will give the angular recursion generalize the recursion in section 2.3 and summarize the result in a formula.

3.2 Building the Dual Graph

To calculate the generating function we first need to prove some properties of the tessellation. Describing these properties require using the dual graph Γ of the tessellation we define in Chapter 2. Recall that the vertex set of the dual graph of the tessellation is the set of faces of the tessellation, and each vertex in the tessellation correspond to a face in the dual graph. For our convenience we define a map φ from a tessellation to its dual graph, by mapping a face of the tessellation to the corresponding vertex, and a vertex to the corresponding face. To avoid confusion we will only use u, v, w to denote vertices in Γ, and Λ_a(or just Λ) to denote the face in Γ corresponding to a vertex a in the tessellation.

Again we can see that this graph Γ is isomorphic to the undirected Cayley graph $X(G, \Sigma \cup \Sigma^{-1})$, and the norm generating function of the tessellation is equal to the growth function of the group on word norm with respect to the geometric generating set.

Consider the vertex u_0 in Γ, which is $\varphi(D)$ for the fundamental polygon D in the tessellation. If D has p vertices and we denote them cyclically by $\{1, \ldots, p\}$, then there are p faces in Γ containing u_0, and we will call them $\Lambda_1, \ldots, \Lambda_p$ corresponding to the p vertices of D. We can consider them as p faces around the vertex u_0. If we look back at the presentation of G and the vertex cycles, we will see that each face around u_0 correspond to, possibly under a cyclic permutation or taking inversion, a word $(c_i)^{n_i}$ in the presentation. Here figure 3.3 is a example coming from the presentation in figure 3.2. In figure 3.3 the dashed line denote the fundamental polygon D in the tessellation, and the solid line denote the edges in the dual graph, corresponding to the elements marked in Figure 3.2.

Since the action of G is transitive on Γ, for every vertex v in Γ, which is the image of u_0 under the action of an element g of G, we always have p faces around v. And we can always cyclically denote these faces by $\Lambda'_1, \ldots, \Lambda'_p$ properly, so that we can have Λ'_i the image of Λ_i under action of g.

Now we will construct the graph Γ by a different approach and prove some condition on it. It's similar to the induction for $B(n)$ in chapter 2, but this time, instead of $B(n)$, we will use a series of subgraphs $\Gamma(n)$ that satisfy $\Gamma(0) \subseteq \Gamma(1) \subset \cdots \cup_{n=0}^{\infty} \Gamma(n) = \Gamma$. We will fix a vertex u_0 and let $\Gamma(0)$ be u_0. So u_0 is in every $\Gamma(n)$, and we will use u_0 as the origin vertex on every $\Gamma(n)$. This means for each $\Gamma(n)$ we will define a norm $|\cdot|_n$, and let $|v|_n$ be the distance between u_0 and v in $\Gamma(n)$. We will show later in Lemma 7 that for a vertex v and any n such that v is in $\Gamma(n)$, $|v|_n$ are actually the same as $|v|$ defined on Γ.

We already have $\Gamma(0)$ is a single vertex u_0. Then the $\Gamma(1)$ will be u_0 with the p faces $\Lambda_1, \ldots, \Lambda_p$ around it. Assume that we have $\Gamma(n-1)$. To obtain $\Gamma(n)$ we will look at all the vertices v on $\Gamma(n-1)$ that have norm $|v|_{n-1} = n-1$. For each v if the p faces around

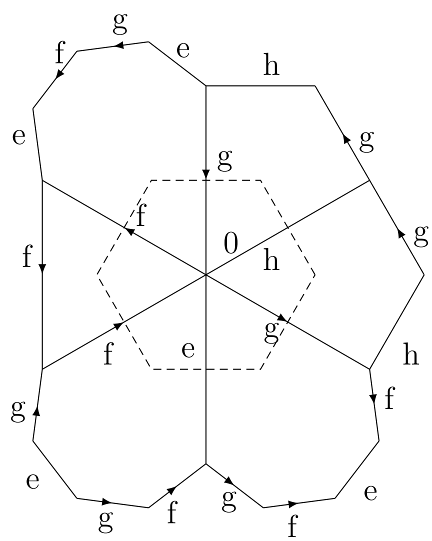

Figure 3.3: The p faces around u_0 in the dual graph

v are not all in $\Gamma(n-1)$, we will attach them to $\Gamma(n-1)$(we will give the detail later). Then $\Gamma(n)$ will be $\Gamma(n-1)$ with all the new faces attached.

Now we can unify the norm we use in all $\Gamma(n)$.

Lemma 7. *Each $\Gamma(n)$ contains all the vertices v with norm at most n. For a vertex u with norm $|u| = n$, if u is in $\Gamma(m)$ for some m, then $|u|_m = |u|$.*

With this lemma, we can always use the norm $|\cdot|$ in all $\Gamma(n)$, since it is equal to the norm $|\cdot|_n$ inside $\Gamma(n)$.

Proof. First we will use induction to prove the first part. We already know that $\Gamma(0)$ has only one vertex u_0 with $|u_0| = 0$. Observe that if $|v| > 0$ then v is adjacent to some w with $|w| = |v| - 1$, and we will call w a predecessor of v. If for $n > 0$ we have $\Gamma(n-1)$ already contains all vertices u with norm at most $n-1$, then for any v with norm n, v has predecessor w in $\Gamma(n-1)$, and the edge vw is in two faces around w. By the construction of $\Gamma(n)$, v is in $\Gamma(n)$.

For the second part, all we need to show is that, if $u \in \Gamma(m)$, then $|u|_m = |u|_{m+1}$. Because if so, then $|u|_m = |u|_n$ for any $n > m$. For large enough n such that $n \geq |u|_m$, we know that $\Gamma(n)$ contain all vertices v with $|v|$ at most n, then we have $|u|_m = |u|$, since

$\Gamma(n)$ must contain the shortest path from u_0 to u in Γ. In $\Gamma(m)$, one shortest path from u to u_0 will go through a face around u, which must contain a vertex v with least norm $|v|_m < m$. But if $|u|_{m+1} < |u|_m$, then the only possible change is that the new shortest path go through a new face attached to a vertex w with norm $|w|_m = m$ in the face, then there will be a vertex in the new face with norm greater than $|u|_m$ already, so the new path will be longer than the previous shortest path, a contradiction. □

We will inductively show that all $\Gamma(n)$ will satisfies five conditions. But these conditions will fail when the fundamental polygon D has three edges. So we will always assume that D has at least four edges in the following, and we will complete the analysis of triangular D in Chapter 5. Let $V(n)$ denote the vertices of $\Gamma(n)$. If u is a vertex with norm n, recall that we define the predecessors of u as any vertex v adjacent to u with norm $n-1$. We will call the vertices v with smallest norm in Λ the base vertices of Λ.

Proposition 8. *For any n, $\Gamma(n)$ satisfies the following five conditions.*

1. each Λ of $\Gamma(n)$ has base vertices in $V(n-1)$.

2. An edge in $\Gamma(n)$ joining a vertex of norm $\leq n$ to one of norm $< n$ belongs to exactly two of the Λ's in $\Gamma(n)$.

3. An edge of $\Gamma(n)$ joining two vertices of norm $\geq n$ belongs to exactly one Λ in $\Gamma(n)$.

4. Except for the case of $\Delta(2,3,q)$ with a five-sided D, a vertex u in $\Gamma(n)$ of norm m has at most two predecessors. If it has two predecessors, then all three vertices belong to the same Λ in $\Gamma(n)$, where u has the largest norm in Λ. For the $\Delta(2,3,q)$ some vertices of norm m have three predecessors.

5. Except for the case $\Delta(3,3,r)$ with a four-sided D, at most two vertices of a Λ can have equal norm. If Λ has two minimal(or maximal) norm vertices, then they are connected by an edge. For $\Delta(3,3,r)$ some Λ can have three minimal norm vertices, which are the base vertices.

We will give the proof of Proposition 8 in section 3.3. Here we illustrate the two exception cases and see how they fail some of the conditions. First we have a figure for the tessellation of the group $\Delta(3,3,8)$ in figure 3.4. The numbers on some faces denote the norm of those faces, with the face denoted 0 the origin face. Notice that there three faces with norm 4 around the vertex a. So in the dual graph, the face Λ_a corresponding to a will have three base vertices with norm 4, violating the condition 5 in Proposition 8. We can see that one of the three faces with norm 4 around a has no neighboring faces with norm greater than 4, and we will call it a buried domain. In general, gD is a buried domain for $|g| = n$, if gD is a subset of $int(B(n))$, the interior of $B(n)$. We will see that the buried domain is the reason that cause the generating function fail to be reciprocal.

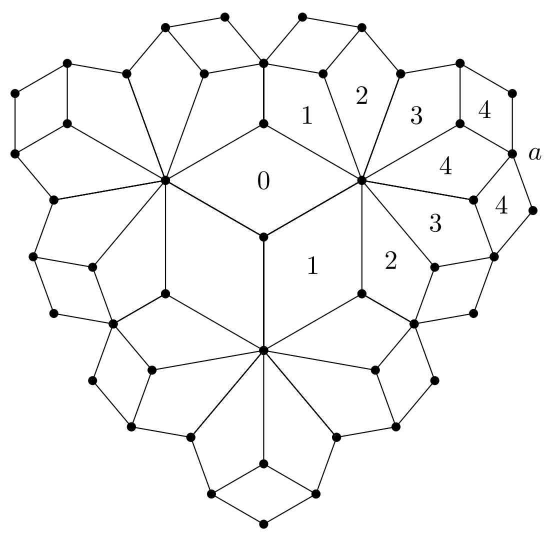

Figure 3.4: tessellation generated by $\Delta(3, 3, 8)$

For the second exceptional case $\Delta(2, 3, q)$, we give an example of $q = 8$ in figure 3.5. Here the dashed lines are the edges of tessellation of $\Delta^*(2, 3, 8)$ in figure 3.1, and the solid line are the edges of tessellation of $\Delta(2, 3, 8)$. Again the numbers denote the distance of the faces to the origin face marked by 0. Notice that the face marked by 5 has three predecessors, which violates the condition 4.

Now let's ignore the exception cases and complete the detail of construction of the dual graphs. For $n > 0$ assume that we have constructed the $\Gamma(n-1)$ satisfying the five conditions. Then a vertex $v \in \Gamma(n-1)$ of norm $n-1$, which is the image of u_0 under the action of an element $g \in G$, has either one or two predecessors.

Since we have exclude the case when D is a triangle, in Figure 3.6 the faces Λ, Λ', Λ'' are faces around v, but among the p faces around v we always miss some of them, unless we have triangle fundamental polygon D. We now add the remaining faces around v so that, under some proper cyclically labeling, we have $\Lambda'_1, \ldots, \Lambda'_p$. And each Λ'_i is the image of Λ_i around u_0 under the action of g. Notice that besides adding the faces we also identify the edges for pair of neighboring faces on their intersection on Γ, in particular the edges vw_1 and vw_2 are identified with two edges from the new faces.

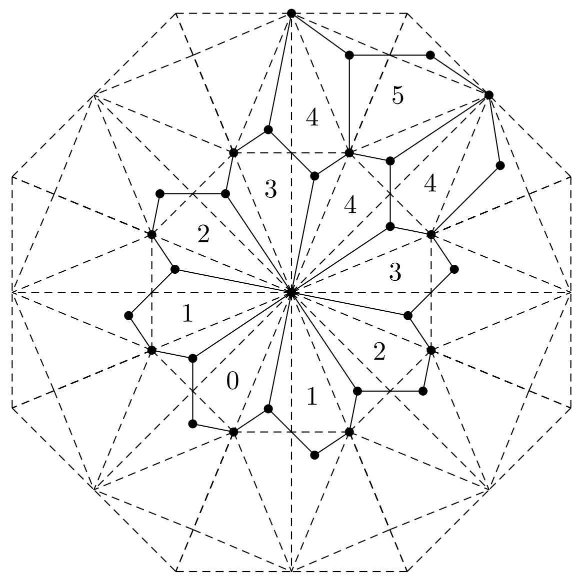

Figure 3.5: tessellation generated by $\Delta(2,3,8)$

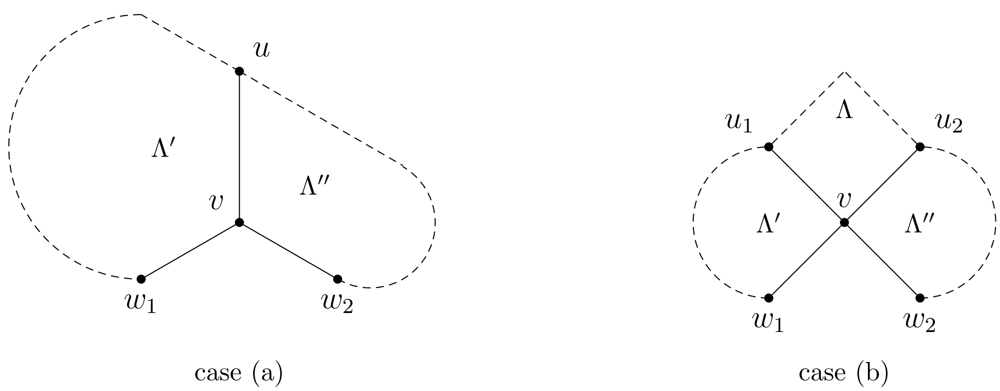

Figure 3.6: local picture around v

We will repeat this process for every $v \in \Gamma(n-1)$ of norm $n-1$, adding all the remaining faces around v, and call the resulting graph $\Gamma(n)$. If w_1(or w_2) has $|w_1| = n-1(|w_2| = n-1)$, then $w_1(w_2)$ will also add faces around it, including one neighboring face of $\Lambda'(\Lambda'')$. Since this face is also around v, we will identify this face with the one we add around v that is adjacent to $\Lambda'(\Lambda'')$. We can see that no further identification of edges or faces near v should be made.

3.3 Verification of the Five Conditions

Now we should prove Proposition 8 about the five conditions. All five conditions are satisfied when $n = 0$. So we let $n > 0$ and assume that $\Gamma(n-1)$ has been constructed satisfying the five conditions.

It's easy to see that condition 1 will hold in $\Gamma(n)$. For the other four conditions, we will consider the condition 5 first. Assume that there is a face Λ_1 which has at least three base vertices. If we consider the first time the face Λ_1 is added into $\Gamma(n)$ at v, then in figure 3.6, we can see that all three of v, w_1, w_2 has norm $n-1$ and are base vertices of Λ_1. In the case (a) of Figure 3.6, D will be forced to be three-sided, so we must have case (b), and D is four sided. By assumption v has minimal norm among all vertices producing Λ that has at least three base vertices, then both Λ' and Λ'' has length 3 or 4. We have the following lemma:

Lemma 9. *If the both vertices w_1 and w_2 in figure 3.6 has the same norm as v, then both Λ' and Λ'' are 3-cycles, unless G is $\Delta(2,4,q)$ with presentation $\langle a, b, c \mid a^2, c^4, b^q, abc\rangle$ and D is five sided.*

Proof. Without loss of generality we can assume that Λ'' is 4-cycles, and we can also assume that v has minimal norm among all vertices such that we have $|v| = |w_2|$ with Λ'' length 4 in the figure 3.6. Then Λ'' has two base vertices, and we have the figure 3.7.

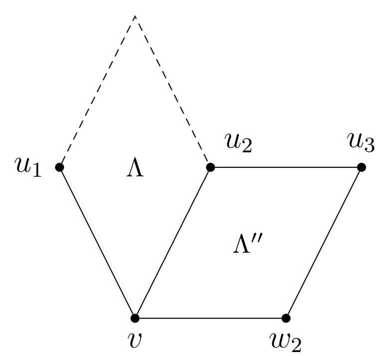

Figure 3.7: local picture around v

- Case 1: u_2 has one predecessor in Λ, which means Λ has length greater than 3. If u_2 has no other predecessor then we immediately have the fundamental polygon D

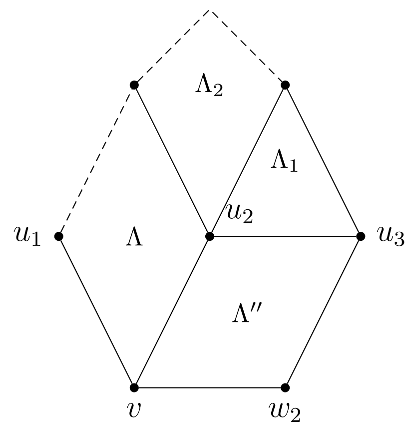

Figure 3.8: local picture around v

three sided. Hence u_2 has two predecessors and D has four sides. The local picture becomes figure 3.8

By minimality, Λ_1 is a 3-cycle. And Λ has length 4 or 5. Thus if we consider the the lengths of the cycles around u_2, we have two possibility $(3, 4, 4, *)$ and $(3, 4, 5, *)$ where $*$ is the length of the face Λ_2.

The four sided fundamental polygon is relative simple, there are only 6 possible groups of isometry of $\mathbb{H}^2$ that don't have elements of order 2 and 17 possible groups with order 2 elements. By examining these lists(see [13, Appendix] and [12, fig.9]), we can see that there is no such group fitting into the only possible configurations $(3, 4, 4, *)$ and $(3, 4, 5, *)$.

- Case 2: u_2 has no predecessor in Λ, which implies Λ is a 3-cycle. By condition 5 on induction hypothesis, there are only two subcases with x_2 has totally one or two predecessors.

- Case 2.1:u_2 has only one predecessor. So instead we have figure 3.9.

  Now one of Λ_1 and Λ_2 has length 3 or 4. Again checking the list, the only possibility is that Λ_2 has length 4, and we also know that Λ_1 has length at least 6, otherwise G will be finite. By transitivity of the G action on vertices, we can know the local picture around each vertices in Λ_1, which turns out to be the figure 3.10. And we have w contradicts the minimality of v.

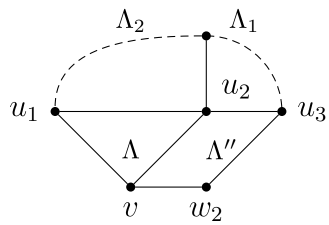

Figure 3.9: local picture around v

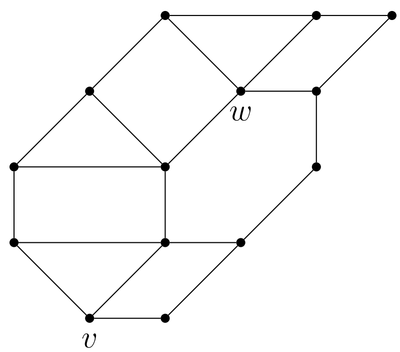

Figure 3.10: local picture around v

- Case 2.2: u_2 has totally 2 predecessor, so D is five sided. By the minimality we have figure 3.11.

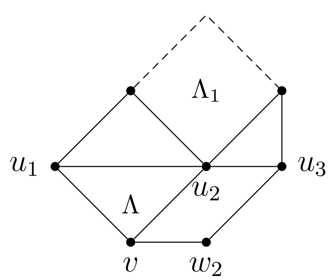

Figure 3.11: local picture around v

So this means that the only possible situation where we have Λ'' length 4 is D five sided which has vertices having degree $(3, 3, 4, 3, q)$ where q is the length of Λ_1 in figure 3.11. This forces the group to be $\Delta(2,4,q)$.

$\square$

Notice that the only tessellation having Λ'' length 4 is the $\Delta(2,4,q)$ with D five sided. But this will not violate the condition 5, since in $\Delta(2,4,q)$ where $|w_2| = |v| = n$ and Λ'' has length 4, v has degree 5. So the new face added around v has only two base vertices v and w_2.

Now we can assume that if v, w_1, w_2 having norm $n-1$ are the base vertices of a face, then in figure 3.6 both Λ' and Λ'' have length 3, and D is four-sided. Thus we have figure 3.12.

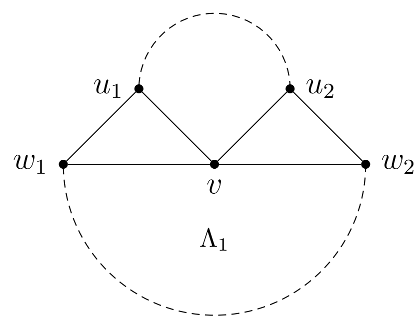

Figure 3.12: local picture around v

Again check the list of four sided D, we can find that the only possible group having such local picture is $\Delta(3,3,r)$, which is the only exception case in condition 5.

With condition 5, it will be easy to verify condition 2 and 3. And the next part we will check the condition 4, that any vertex w has at most two predecessor. Suppose that v is a predecessor of w, and $|w| = n$.

- Case 1: Every predecessor of w has a unique predecessor.

  If we look back at figure 3.6, and let v be the vertex v in the figure 3.6 with norm $n-1$, then either w is already in $\Gamma(n-1)$, in which case w is the w_1 or w_2 in figure 3.6, or w will be added to $\Gamma(n-1)$ along with the faces around v when we construct the $\Gamma(n)$.

- Case 1.1:w is in $\Gamma(n-1)$, and w is either w_1 or w_2 in figure 3.6.

  We can assume that w is the w_1 in figure 3.6. If w doesn't have another predecessor in Λ', then one of the other predecessor must be in a face sharing edge wv with Λ', which will force D to be three-sided. If w has anther predecessor v_1 in Λ', then we have figure 3.13.

  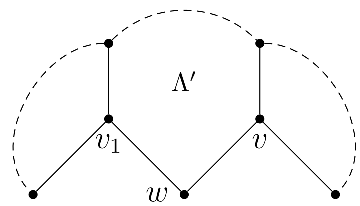

  Figure 3.13: local picture around v

  Notice that the local picture can only be figure 3.13 because we assume that both v and v_1 has only one predecessor. If w has a third predecessor, then it must be in a face sharing an edge, either wv or wv_1, with Λ'. Again, either case will lead to the local picture around v or v_1 forcing D to be three-sided, a contradiction.

- Case 1.2:w is not in $\Gamma(n-1)$, so w is not w_1 or w_2 in figure 3.6. Now consider whether the other two neighboring vertices w_1 and w_2 are the predecessors of w.

- Case 1.2.1:If both v_1 and v_2 are predecessor of w, we have figure 3.14.

  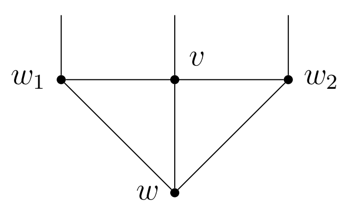

  Figure 3.14: local picture around v

  So we immediately have D 4-sided. If we go through the list of of groups with 4-sided D, we can see that the only possible match for having two length 3 faces around a vertex is the $\Delta(3,3,r)$, but in $\Delta(3,3,r)$ the two length 3 faces are not adjacent.

- Case 1.2.2 Assume that only one of w_1 and w_2 is the predecessor w, say it is w_2. Then we have a face Λ_1 with vertices v, w, w_2. Consider the third predecessor v_3 of w, the edge $v_3 w$ is in a face Λ_2 neighboring Λ_1, and Λ_2 will contain either w_1 or w_2. Thus we have the two possible local picture in figure 3.15.

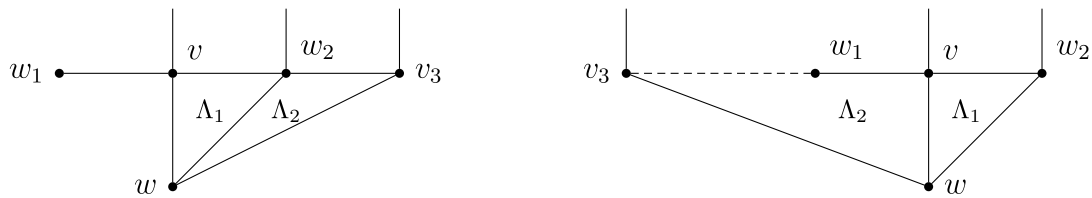

Figure 3.15: local picture around v

In the first case, wv_3 is in the same face Λ_2 with w_2, the local picture around w_2 is the same as case 1.2.1, which leads to contradiction. In the second case, wv_3 is in the same face Λ_2 with w_1. Recall that Λ_2 is added by base vertex v, so any vertex in Λ_2 will has norm no less than $|v| = n - 1$. But since v_3 is a predecessor of w, $|v_3| = n - 1$. So all the vertex in Λ_2 except w will have norm $n - 1$, and are base vertex. By condition 5, we will have λ_2 being length 3 unless the tessellation is in the exceptional case. And it's easy to check that the exceptional case doesn't match with this local picture.

- Case 1.2.3:Neither w_1 or w_2 are predecessors of w. Then we can assume that w has a predecessor v_2, and wv_2 is in the same face Λ_1 with edge vw. So Λ_1 must be through one of w_1 and w_2. Without loss of generality we assume that w_2 is in Λ_2, then we have figure 3.16.

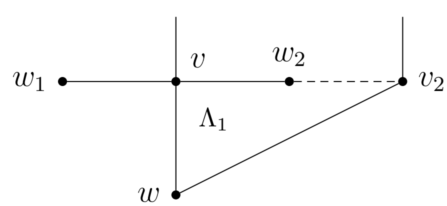

Figure 3.16: local picture around v

But similar to case 1.2.2, we must have Λ_1 length 4, which will force the tessellation being the exceptional case in condition 5. But this will leads to a contradiction with the assumption that w_1 is not a predecessor of w. And this will leads to the end of the case 1.

- Case 2:Some predecessor, v, of w has two predecessors. This means that v has degree 5, so D is five sided. Now let's consider the norms of w_1 and w_2 in figure 3.6, comparing to $|w| = n$.

- Case 2.1:Assume that $|w_1| = |w_2| = n - 1$, consider whether w_1 and w_2 are the predecessors of w.

- Case 2.1.1: If both w_1 and w_2 are predecessors of w, then D is five sided and by the argument in Lemma 9 we have both Λ' and Λ'' are length 3. The the local picture around v is figure 3.17.

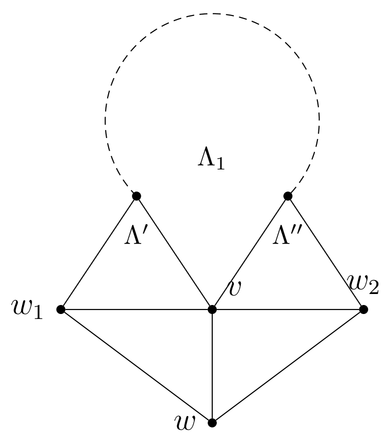

Figure 3.17: local picture around v

If the face Λ_1 has length q, then the five vertices of D have degrees $(3, 3, 3, 3, q)$, and the only possible tessellation with this structure is the $\Delta(2, 3, q)$. This is the only exceptional case is condition 5.

Before we move on the next case, here we make an observation that will be useful later.

Observation 10. *In Γ, if around a vertex v there are two adjacent length 3 faces, then the tessellation can only be the $\Delta(2, 3, q)$.*

Proof. This comes from summarizing case 1.2.1 and case 2.1.1. $\square$

- Case 2,1,2:If neither w_1 and w_2 are predecessors of w, then for another predecessor v_1 of w, we have wv_1 in the same face Λ_1 with vw, and this face contain either w_1 or w_2. Let's assume that w_2 is in Λ_1, then we have figure 3.18.

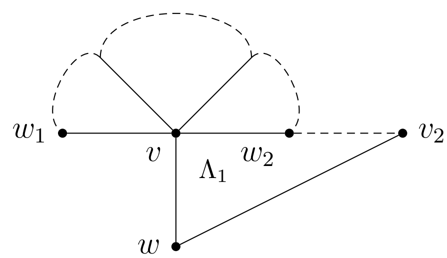

Figure 3.18: local picture around v

By condition 5 we know that Λ_1 can only be length 4, with three base vertices v, w_2, v_2. But that will forces D to be four-sided, contradict to our assumption.

- Case 2.1.3:If only one of w_1 and w_2 is the predecessor of w, say, w_1 is and w_2 is not. Then we have a three-sided face Λ_1 with vertices w, v, w_1. If w has a third predecessor v_3, zv_3 is in a face Λ_2 adjacent to Λ_1. Then one of w_1 and w_2 will be in Λ_2. If w_2 is in Λ_2 we will have a similar local picture as in figure 3.18, and a contradiction similar to case 2.1.2. If w_1 is in Λ_2 then we have figure 3.19.

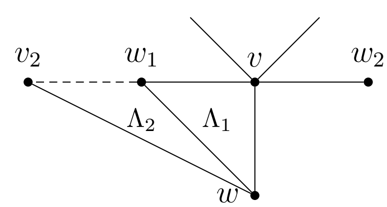

Figure 3.19: local picture around v

In figure 3.19, if there are other vertices between v_3 and w_1, we have a similar contradiction as in case 2.1.2. But if v_3 is adjacent to w_1, we can consider the number of predecessor of w_1. By induction assumption w_1 has only one or two predecessors. If w_1 has one predecessor we have the local picture around w_1 the same as figure 3.14, and we have a similar contradiction as case 1.2.1. If w_1 has two predecessors we have the local picture around w_1 the same as figure 3.17, and we have a similar contradiction as case 2.1.1.

- Case 2.2 Assume that $|w_1| = n$ and $w_1 \neq w$ in figure 3.6. If $|w_2| = n$, w can't have any other predecessors except v. So we can also assume that $|w_2| = n - 1$. Now we have two possibility:w_2 is a predecessor or not. But in both case, by the observation 10, we always have a face Λ_1 containing w that has three base vertices:If w_2 is a predecessor of w, then Λ_1 is the face containing the edge ww_2 but not v; If w_2 is not a predecessor of w, then Λ_1 is the face containing w, w_2 and v. Then this face w_2 will lead to a contradiction similar to either case 1.2.2 or 2.1.2, depending whether w_2 has two predecessors.

- Case 2.3 Assume that $|w_1| = n$ and $w_1 = w$ in figure 3.6. Now we have the local picture around v as figure 3.20.

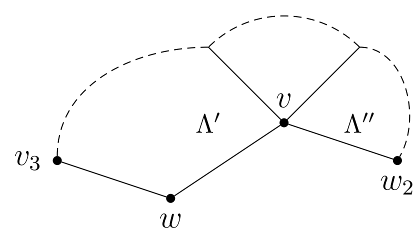

Figure 3.20: local picture around v

In figure 3.20, both v_3 and w_2 can be either of norm n or $n - 1$. If $|v_3| = n$, by condition 3 on induction hypothesis, we know that there is no other faces containing edge wv_3. So the other predecessors of w can not come from v_3. And if there are two other predecessors from w_2, we can have a similar contradiction as in case 2.2.

Now we can assume that $|v_3| = n-1$, and Λ' has length 4 or 5. If another predecessor is in a face containing v_3, then G must be $\Delta(3, 3, r)$, and this will lead to a contradiction with the faces around v and the fact that Λ' has length 4 or 5. If $|w_2| = n$, then we can't have three predecessors for w. So we can assume that $|w_2| = n - 1$. If there two predecessors come via v_2, we can have a similar contradiction as in case 2.2. Thus we conclude that the situation can only be figure 3.21.

Notice we have Λ_1 length 3, so if Λ'' has length 3, we will have observation 10. Therefore we can assume that Λ'' has length 4. But now means that we must have Λ' with length 4, and the group is $\Delta(2, 3, p)$, where p is the length of Λ_2. If we assume

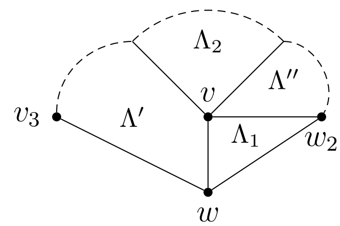

Figure 3.21: local picture around v

that w have minimal norm among all vertices with three predecessors, and since we know all faces around any vertices, we can read the local pictures of other vertices in Λ_2 with lower norm, then we have figure 3.22.

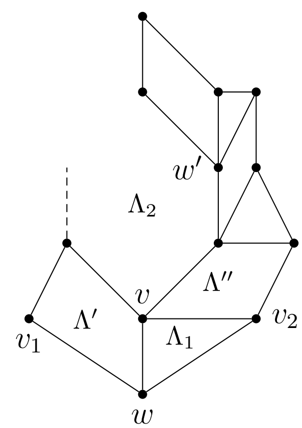

Figure 3.22: local picture around v

Notice that in figure 3.22, the face Λ_2 cannot have length p less than 6, otherwise the four faces around v will force the group to be a finite group, and the tessellation to be a tessellation of a ball. Thus in the situation we will have a vertex w' with norm $|w'| = |w| - 3$ having at least three predecessors, a contradiction with our assumption of minimality. And this will conclude the proof of the condition 4.

3.4 Generating Functions

Provided the five conditions in Proposition 8, now we can go back to $B(n)$ and prove some properties we need for calculating the generating functions. Recall that $B(n)$ is part of the tessellation consisting of faces with norm $\leq n$. First we show that $B(n)$ is a ball on the topological space of hyperbolic plane, in other words, we show that $B(n)$ has only one continuous closed boundary.

Lemma 11. *For every $n \geq 0$, $B(n)$ is a topological ball.*

Proof. First omit the two exceptional cases $\Delta(2,3,q)$ and $\Delta(3,3,r)$, so we can assume all five conditions in Proposition 8 hold. First $B(0)$ is D which is a topological ball. Since for each $n > 0$, $B(n)$ can be obtained from $B(n-1)$ by taking union with $\{b_1D, \ldots, b_kD\}$ where b_i are all the elements in G having norm n. It suffice to prove that, if $C_{i-1} = B(n-1) \cup b_1D \cup \cdots \cup b_{i-1}D$ is a ball, then so is $C_i = B(n-1) \cup b_1D \cup \cdots \cup b_iD$.

It's easy to see that all we need to prove is $b_i \cap C_{i-1}$ is connected. For large enough m, the condition 4 in $\Gamma(m)$ can be translated to the fact that b_iD has only one or two neighboring faces in $B(n-1)$. And if there are two faces neighboring D in $B(n-1)$, the image of all three under φ are in the same face Λ in Γ, so the two edges in $b_iD \cap B(n-1)$ intersect in one vertex. Similarly, the condition 5 implies that if $b_iD \cap b_jD \neq \emptyset$ for some $j \neq i$, then the image of b_iD and b_jD under φ are two vertices in one face Λ. So either b_iD and b_jD intersect at a vertex of an edge of $b_iD \cap B(n-1)$, or $b_iD \cap b_jD$ is an edge. If $b_iD \cap b_jD$ is an edge, then in dual graph Γ we have $\varphi(b_iD)$ and $\varphi(b_jD)$ are two adjacent vertices with maximal norm in a face Λ, so $b_iD \cap b_jD \cap B(n-1)$ is a vertex on the $\partial B(n-1)$.

We will prove the exceptional case $\Delta(3,3,r)$ in Chapter 5. Another exceptional case $\Delta(2,3,q)$ will fit in to the proof above, by noticing that in the dual graph Γ, if a vertex has three predecessors, they are in two neighboring faces. $\square$

Now we also need to construct a recursion on angles like Proposition 4. To obtain $B(n+1)$ from $B(n)$ we can simply attached the faces with norm $n+1$ on $\partial B(n)$, with each edge on $\partial B(n)$ corresponds to one face. The change of angles from $\partial B(n)$ to $\partial B(n+1)$ can be summarize in the following lemma. From now on we will exclude the exceptional case $\Delta(3,3,r)$.

Lemma 12. *Near any angle α with endpoint a on the boundary $\partial B(n)$ of $B(n)$, the change of α from $\partial B(n)$ to $\partial B(n+1)$ has one of the following three cases.*

1. *If the intersection of the two new faces corresponding to the two sides of α is exactly a. Then there is a new angle $\Phi(\alpha)$ on $\partial B(n+1)$ with endpoint a, and $\Phi(\alpha)$ is the sum of α with the two angles in the two new faces with endpoint a. In this case we will call α is of type* 1.

2. *If the intersection of the two new faces corresponding to the two sides of α is an edge between a and a vertex b on $\partial B(n+1)$. Then there is a new angle $\Phi(\alpha)$ on $\partial B(n+1)$ on b, which is the sum of the two angles of the new faces around b. In this case we will call α is of type* 2.

3. *If only one new face corresponds to the two sides of α, then no new angle generates near α, only one angle on the new face complete the angle α. In this case we will call α is of type* 3.

This lemma can be easily prove by condition 5 without the exceptional case. So now we have a recursion on angles like Proposition 4. But this time we have more than one primitive angle. More specifically, we will call the p angles of D the primitive angles $\alpha_1, \ldots, \alpha_p$(notice that we are using a different set of notation α_i compared to Chapter 2). Every angle on $\partial B(n)$ are sum of some of primitive angles. If we define $\mathcal{S}$ as the set of all possible angles on $\partial B(n)$, then $\mathcal{S}$ is a subset of the set combination of primitive angles $\{\alpha_{i_1} + \ldots + \alpha_{i_k} \mid 1 \leq i_1 \leq \cdots \leq i_k \leq p\}$. Actually most of the angles on each $\partial B(n)$ are equal to one primitive angle.

Notice that for any α on $\partial B(n)$, the next angle $\Phi(\alpha)$ is only determined by α, independent of n and the position on $\partial B(n)$. So Φ is a well-define map from $\mathcal{S}$ to $\mathcal{S}$. Another observation is about the exterior angles. For an angle α, we call $2\pi - \alpha$ the exterior angle of α. If $\beta_2 = \Phi(\beta_1)$ is the next angle of β_1, then the change from $2\pi - \beta_2$ to $2\pi - \beta_1$ follows the exact same rule as Lemma 12, but in the opposite direction, viewing the outside of $B(n)$ as inside of an area. In other words, we have $2\pi - \beta_1 = \Phi(2\pi - \beta_2)$. This means if $\beta_2 = \Phi(\beta_1)$ is the next angle of β_1, then not only β_2 is determined by β_1, but also β_1 is determined by β_2. Thus Φ is an injective map.(But not surjective, since primitive angle is not the next angle of any angle)

Here we give an example. Figure 3.23 shows part of a tessellation. Here we mark the origin face D with the two element in the geometric generating set. And four vertices of D have degree $8, 3, 8, 4$ respectively, so we will call the corresponding group $G_{(8,3,8,4)}$. Then we use α_1 to denote the angle of $\partial B(0)$ at the vertex with degree 3, so $\alpha_1 = 2\pi/3$. Similarly we denote $\alpha_2 = 2\pi/4$ and $\alpha_3 = 2\pi/8 = \alpha_4$. For both α_2 and α_3, the angles in the sequence they generated are all of type 1 or 3. So we will have

$$(\alpha_{3,1} = \alpha_3, \alpha_{3,2}, \alpha_{3,3}, \alpha_{3,4}) = (2\pi/8, 3 \cdot 2\pi/8, 5 \cdot 2\pi/8, 7 \cdot 2\pi/8).$$

Similar for the α_2. But we can see that $\alpha_{1,1} = \alpha_1$ is of type 2, so $\alpha_{1,2}$ is at a different vertex with the $\alpha_{1,1}$, and $\alpha_{1,2}$ will be equal to $2 \cdot 2\pi/8$.

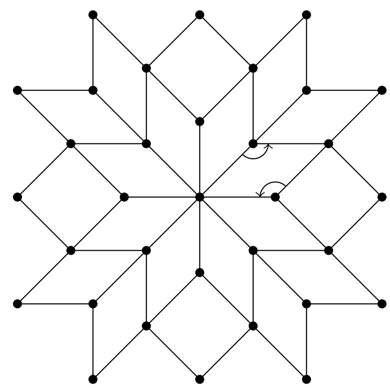

Figure 3.23: tessellation generated by $G_{(8,3,8,4)}$

Lemma 13. *In general, each primitive angle α_i on $\partial B(n)$ will generate a sequence of angles on $\partial B(n+1), \partial B(n+2), \ldots$, $\alpha_i = \alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,m_i}$. Here each $\alpha_{i,j+1}$ on $\partial B(n+j)$ equal to $\Phi(\alpha_{i,j})$ is the next angle of $\alpha_{i,j}$ on $\partial B(n+j-1)$ in the Lemma 12. The sequence will end when we have a type 3 angle α_{i,m_i}, where $\pi - \alpha_{i,m_i}$ is a primitive angle which complete the angle α_{i,m_i}.*

Proof. The construction of the sequence follows from Lemma 12. The sequence will always terminate because $\mathcal{S}$ is a finite set. If the sequence doesn't terminate, then there must be a cycle $(\beta_1, \beta_2 = \Phi(\beta_1), \ldots, \beta_k, \beta_1 = \Phi(\beta_k))$ in the sequence, with none of β_i equal to a primitive angle. But the first angle in the sequence is a primitive angle, which is not in the cycle. So if $\alpha_{i,k} = \beta_l$ is the first angle in the sequence that is in the cycle, then we have $\Phi(\alpha_{i,k-1}) = \alpha_{i,k} = \Phi(\beta_{l-1})$, with $\alpha_{i,k-1} \neq \beta_{l-1}$, a contradiction to our observation about the injectivity of Φ. $\square$

Now notice that the end of the sequence must be an angle α_{i,m_i} of case 3, which means $2\pi - \alpha_{i,m_i}$ is a primitive angle. Assume that this angle is α_j, then let's define a map $\sigma : \{1, \ldots, p\} \to \{1, \ldots, p\}$ by $\sigma(i) = j$. Soon we will show that σ is a involution.

A similar argument as in Lemma 13 can also show that, two sequences generated by two different primitive angles have all distinct angles. Thus, if the sequence generated by primitive angle α_i has length m_i, then we let $m = \sum_{i=1}^{p} m_i$, and any angle on any $\partial B(n)$ will be one of these m angles. Also notice that, if the sequence generated by α_i

ends on α_{i,m_i}, with the exterior angle $2\pi - \alpha_{i,m_i} = \alpha_j$, where j can be the same as i, then there is a correspondence between sequence generated by α_i and α_j. First we have $2\pi - \alpha_{i,m_i} = \alpha_j = \alpha_{j,1}$, then by the argument above, we have $\Phi(\alpha_{j,1}) = \alpha_{j,2} = 2\pi - \alpha_{i,m_i-1}$. Repeat this process we have $\alpha_{j,k} = 2\pi - \alpha_{i,m_i-k+1}$ for $1 \leq k \leq m_i$. So we also know that the sequence generated by α_j will end with a type 3 angle $\alpha_{j,m_i} = 2\pi - \alpha_{i,1} = 2\pi - \alpha_i$. Thus we know that σ is an involution, and we have $m_i = m_{\sigma(i)}$.

Now we can find the recursive relation on angles in $\mathcal{S}$. From Lemma 12, each angle on $\partial B(n)$ of type 1 or 2 will correspond to two new faces with norm $n+1$. And we can also show that each face with norm $n+1$ will correspond to two angles on $\partial B(n)$ of type 1 and 2 by condition 4 in Proposition 8. By condition 4, each face f with norm $n+1$ is adjacent to at most two faces in $B(n)$, three if it's the exceptional case $\Delta(2,3,q)$. If f is adjacent to only one face in $B(n)$ on an edge, then on $\partial B(n)$ the two angles on the two sides of this edge are both either type 1 or 2. If f is adjacent to two faces in $B(n)$, then also by condition 4, these three faces intersect at one vertex, and the angle at this vertex on $\partial B(n)$ is type 3. For $\Delta(2,3,q)$, we have two neighboring angles of type 3, and the other two neighbors of these two angles are of type 1. Thus we can conclude that, the number of angles of type 1 and 2 on $\partial B(n)$ is equal to the number of norm $n+1$ faces on the tessellation

$$s_{n+1} = \sum_{i=1}^{p} \sum_{j=1}^{m_i-1} s_{i,j}(n), \tag{3.2}$$

where s_{n+1} is the number of faces with norm $n+1$, and $s_{i,j}(n)$ is the number of angle $\alpha_{i,j}$ on $\partial B(n)$.

To find out the number of angles of each type on each $\partial B(n)$, we will consider how many angles of each type a angle on $\partial B(n)$ contribute to $\partial B(n+1)$. This is similar to the section 2.4. Since each angle of type 1 or 2 correspond to two new faces, it contribute two copies of each primitive angle. But this will count each primitive angle twice since a face correspond to two angles, so the actual number is one copy of each primitive angle. Then we need to subtract the angles disappear when we glue the new faces on $B(n)$, which are the angles mark as x on figure 3.24.

These information can be put into a matrix. We will use vectors $v \in \mathbb{Z}^m$ where $m = \sum_i^p m_i$ to record a collection of angles, where $v = (v_{1,1}, \ldots, v_{1,m_1}, v_{2,1}, \ldots, v_{p,m_p})$ means we have $v_{i,j}$ copies of angle $\alpha_{i,j}$. If the angles on $\partial B(n)$ can be represented by v, then the angles on $\partial B(n+1)$ are Av, where A is a $m \times m$ matrix with the following possible nonzero entries:

1. $A_{i,1;k,l}$ for $1 \leq i, k \leq p$ and $1 \leq l \leq m_k - 1$ denote how many copies of primitive angle

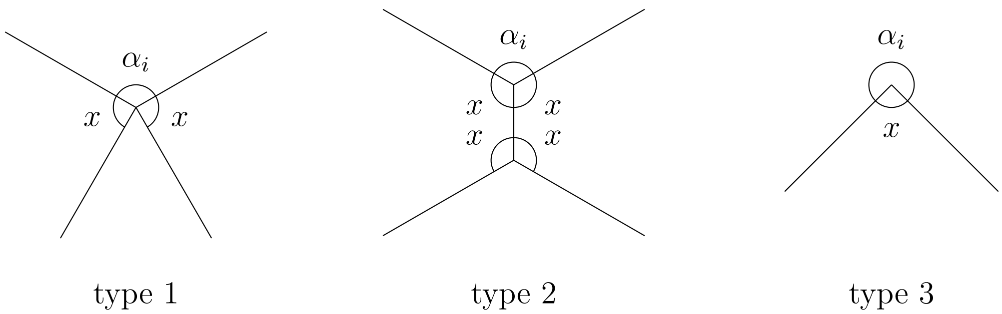

Figure 3.24: subtracted angles marked by x

α_i does $\alpha_{k,l}$(an angle of type 1 or 2) contribute, it is 1 minus number of α_i among the x angles.

2. $A_{i,j+1;i,j} = 1$ for $1 \leq i \leq p$ and $1 \leq j \leq m_i - 1$, because $\alpha_{i,j}$ will change into $\alpha_{i,j+1}$ by Lemma 12.

3. $A_{\sigma(i),1;i,m_i} = -1$ for $1 \leq i \leq p$ denote the case 3, where only one primitive angle is subtracted and no new angles are introduced.

So we can finally compute the generating function as

Theorem 14. *The generating function for the tessellation is*

$$f(z) = 1 + uz(1 - zA)^{-1}v.$$

Here v record the initial collection of angles. Since $B(0)$ has one copy of each primitive angles, we can denote this by $v = (v_{1,1}, \ldots, v_{p,m_p})^T$ where $v_{i,1} = 1$ and $v_{i,j} = 0$ for $j > 1$. So $\partial B(n)$ has angles $A^n v$.

By the correspondence between number of angles and number of faces in (3.2), we know that $s_{n+1} = uA^n v$ for $u = (u_{1,1}, \ldots, u_{p,m_p})$ with $u_{i,m_i} = 0$ and $u_{i,j} = 1$ for $1 \leq j < m_i$.

Then the Theorem 14 follows from

$$\begin{aligned} f(z) &= 1 + \sum_{n=0}^{\infty} uA^n v z^{n+1} \\ &= 1 + uz(\sum_{n=0}^{\infty}(Az)^n)v \\ &= 1 + uz(1 - zA)^{-1}v. \end{aligned}$$

Recall the example we gave in figure 3.23 corresponding to the group $G_{8,3,8,4}$, we can calculate its transition matrix as

$$A = \left[\begin{array}{ccccc|cc|cccc} -1 & 1 & 1 & -1 & -1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & -1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \tag{3.3}$$

Here we omit the α_4 because $\alpha_3 = \alpha_4$, and we can simply count α_3 twice in the initial data v, so we have $v_8 = 2$ in

$$v^T = \left[\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \end{array}\right], \tag{3.4}$$

then we have

$$f(z) = 1 + uz(1 - zA)^{-1}v$$

for

$$u = \left[\begin{array}{ccccccccccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{array}\right]. \tag{3.5}$$

Chapter 4

Symmetry of the Generating Functions

Aside from the exceptional case $\Delta(3,3,r)$, we always have symmetry $f(z) = f(z^{-1})$ in the generating functions, which means both the polynomials in the denominator and numerator are reciprocal and have the same degree. Our main result is:

Theorem 15 (Planar reciprocity theorem in [13])**.** *For the generating function for the hyperbolic plane tessellations, we always have symmetry $f(z) = f(z^{-1})$, except for the tessellation generated by $\Delta(3,3,r)$ for $r \geq 3$, and the tessellation with fundamental polygon a triangle with angles $2\pi/3, \pi/p, \pi/p$.*

We will discuss the $\Delta(3,3,r)$ and the triangle fundamental polygon in Chapter 5. Now we will prove that Theorem 15 always holds when D is not triangle, except for $\Delta(3,3,r)$. To prove this, we will find an equivalent statement of the Theorem. Since we know that

$$f(z) = 1 + \sum_{n=0}^{\infty} uA^n v z^{n+1} = 1 + u(1 - zA)^{-1} zv, \tag{4.1}$$

we have

$$\begin{aligned}
f(z^{-1}) - 1 &= u(1 - z^{-1}A)^{-1}z^{-1}v \\
&= u(z - A)^{-1}v \\
&= u(A^{-1}z - 1)^{-1}A^{-1}v \\
&= -u(1 - A^{-1}z)^{-1}A^{-1}v \\
&= -u(\sum_{n=0}^{\infty} A^{-(n+1)}z^n)v
\end{aligned} \tag{4.2}$$

Comparing (4.1) and (4.2), we can see that $f(z) = f(z^{-1})$ is equivalent to the following two conditions:

$$uA^{-1}v = 0 \tag{4.3}$$

$$uA^n v = -uA^{-(n+2)}v \text{ for all } n \geq 0. \tag{4.4}$$

First we look at the A^{-1}. Since A is quite sparse, we can easily compute A^{-1} explicitly, the only possible nonzero entries of A^{-1} are:

1. $A_{i,1;k,l} = (A^{-1})_{\sigma(i),m_i;k,l+1}$ for $1 \leq i, k \leq p$ and $1 \leq l \leq m_k - 1$.
2. $(A^{-1})_{i,j;i,j+1} = 1$ for $1 \leq i \leq p$ and $1 \leq j \leq m_i - 1$.
3. $(A^{-1})_{\sigma(i),1;i,m_i} = -1$ for $1 \leq i \leq p$.

Recall the example we give in section 3.4 about group $G_{(8,3,8,4)}$, and later we compute the matrix A in (4.5). Now we give the inverse of the transition matrix of this tessellation.

$$A = \left[\begin{array}{ccccc|cc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & -1 & -1 & 0 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0
\end{array}\right] \tag{4.5}$$

We can see that the only nonzero elements in $A^{-1}v$ are $(A^{-1}v)_{i,m_i} = -1$ for $1 \leq i \leq p$. So immediately we have $uA^{-1}v = 0$.

Next we will prove (4.4). But first we will describe the angle duality, which turns out to be the crucial reason for the symmetry of the generating function.

Proposition 16. *(Angle duality) $A_{i,1;k,l} = A_{i,1;\sigma(k),m_k-l}$, for $1 \leq l \leq m_k - 1$.*

Proof. Recall that when we build the sequence $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,m_i}$, the exterior angles of α_{i,m_i} generates a sequence $\alpha_{\sigma(i),1}, \alpha_{\sigma(i),2}, \ldots, \alpha_{\sigma(i),m_i}$, with $\alpha_{\sigma(i),j} = 2\pi - \alpha_{i,m_i-j+1}$. Notice that $A_{i,1;k,l}$ is determined only by the number of $\alpha_{i,1}$ among the x angle on the two new faces on the two sides of $\alpha_{k,l}$, and the fact that angle $\alpha_{\sigma(k),m_k-1}$ share the same two new faces and the same x angles. So $A_{i,1;k,l} = A_{i,1;\sigma(k),m_k-l}$. □

Observe that in A, any row with index (i,j) for $j > 1$ has only one nonzero entry $A_{i,j;i,j-1} = 1$, which comes from that fact that $\alpha_{i,j}$ can only be generated from $\alpha_{i,j-1}$. So we know that for any $w \in \mathbb{Z}^m$ and $j > 1$, A will shift the elements of w by $(Aw)_{i,j} = w_{i,j-1}$ for $1 < j \leq m_i$. Similarly we have $(A^{-1}w)_{i,j} = w_{i,j+1}$ for $1 \leq j < m_i$.

Since $u_{i,j} = 1$ for $1 \leq j \leq m_i - 1$ and $u_{i,m_i} = 0$, we have

$$uA^n v = \sum_{i=1}^{p} \sum_{j=1}^{m_i-1} A^n v_{i,j},$$

and

$$u(A^{-(n+2)})v = \sum_{i=1}^{p} \sum_{j=1}^{m_i-1} (A^{-(n+2)}v)_{i,j} = \sum_{i=1}^{p} \sum_{j=1}^{m_i-1} (A^{-(n+1)}v)_{i,j+1}.$$

To prove 4.4 it suffice to prove that $(A^n v)_{i,j} = -(A^{-(n+1)}v)_{\sigma(i),m_i-j+1}$ for $1 \leq i \leq p$ and $1 \leq j < m_i$. And we will prove this by induction on n.

For $n = 0$ we have already shown that the only nonzero elements in $A^{-1}v$ are $(A^{-1}v)_{i,m_i} = -1$ for $1 \leq i \leq p$, so $(A^0 v)_{i,j} = -(A^{-1}v)_{\sigma(i),m_i-j+1}$ holds. If this holds for n, then for $1 \leq i \leq p$ and $1 \leq j \leq m_i - 1$ we have

$$(A^{n+1}v)_{i,j+1} = (A^n v)_{i,j} = -(A^{-(n+1)}v)_{\sigma(i),m_i-j+1} = -(A^{-(n+2)}v)_{\sigma(i),m_i-j}$$

by shifting the elements. So all we need to prove is $(A^{n+1}v)_{i,1} = -(A^{-(n+2)}v)_{\sigma(i),m_i}$, which comes from the following lemma.

Lemma 17. *If $w_{i,j} = \bar{w}_{\sigma(i),m_i+j-1}$, then $(Aw)_{i,1} = (A^{-1}\bar{w})_{\sigma(i),m_i}$*

Proof. First we have

$$(Aw)_{i,j} = \sum_{k=1}^{p} \sum_{l=1}^{m_k-1} A_{i,1;k,l} w_{k,l} - w_{\sigma(i),m_i}$$

and

$$(A^{-1}\bar{w})_{\sigma(i),m_i} = \sum_{k=1}^{p} \sum_{l=1}^{m_k-1} (A^{-1})_{\sigma(i),m_i;k,l+1} \bar{w}_{k,l+1} - \bar{w}_{i,1}$$

But from the computation of A^{-1} we have $(A^{-1})_{\sigma(i),m_i;k,l+1} = A_{i,1;k,l}$, and by Proposition 16 $A_{i,1;k,l} = A_{i,1;\sigma(k),m_k-l}$. From hypothesis we have $\bar{w}_{k,l+1} = w_{\sigma k,m_k-l}$ and $w_{\sigma(i),m_i} = \bar{w}_{i,1}$. So

$$(A^{-1}\bar{w})_{\sigma(i),m_i} = \sum_{k=1}^{p} \sum_{l=1}^{m_k-1} A_{i,1;\sigma(k),m_k-l} w_{\sigma k,m_k-l} - w_{\sigma(i),m_i}$$

Since σ is a involution in $1, \ldots, p$, this sum is the same as in $(Aw)_{i,j}$. $\square$

Chapter 5

Exceptional Cases

Our proof is not complete without a discussion of the exceptional cases, when the fundamental polygon D is a triangle or when the group G is $\Delta(3,3,r)$. In this chapter we will discuss those exceptional cases, and we will focus on the determining which of them has reciprocal growth function. First we will calculate the growth function for Coxeter group and groups having a Coxeter-like presentation. Then we will use these two results to finish the computation of the exceptional cases.

5.1 Coxeter Groups

In this section we will give a brief introduction to the computation of the growth function for Coxeter group with a set of generators, following the chapter 1,2 and 7 of [1]. Here we call a Coxeter system (W,S) for a Coxeter group W with its Coxeter generators S. The Coxeter system can be defined by a Coxeter matrix m for S, which is a matrix $m : S \times S \to \mathbb{Z}^+ \cup \{\infty\}$ such that $m(s,s') = m(s',s)$ and $m(s,s') = 1$ if and only if $s = s'$. Let

$$S^2_{fin} = \{(s,s') \in S^2 \mid m(s,s') \neq \infty\}.$$

Then the matrix m determine a group W with the presentation

$$\langle S \mid (s,s')^{m(s,s')} = 1, \text{ for all } (s,s') \in S^2_{fin} \rangle$$

For the tessellation we discuss in Chapter 3, if it is generated by a Coxeter group, then the fundamental polygon D is a d-sided polygon on $\mathbb{H}^2$. If we cyclically denote the

d vertices by $a_1, \ldots, a_d$, and assume that the angle of D at a_i is π/λ_i, the reflection along the edge $a_i a_{i+1}$ is s_i, then we have the Coxeter system (W, S), with $S = \{s_1, \ldots, s_d\}$ and

$$W = \langle S \mid (g_i)^2, (g_1 g_2)^{\lambda_1}, \ldots, (g_{d-1} g_d)^{\lambda_{d-1}}, (g_d g_1)^{\lambda_d} \rangle.$$

For a Coxeter system (W, S), we can define the length $|w|$ for each element w just as the word norm we defined before, by letting $|w|$ be the smallest k such that w can be written as $w = s_1 \cdots s_k$ for each $s_i \in S$. Such a expression is called a reduced word or a reduced expression. For each w in the Coxeter group, we define it's right descent set to be $\{s \in S \mid |ws| < |w|\}$. The reason for this terminology can be partially answer by the following lemma.

Lemma 18. *For all $s \in S$ and $w \in W$, $s \in D_R(w)$ if and only if some reduced expression for w ends with s.*

Now we define some subsets of a Coxeter group that will be useful when finding the generating functions. The first is the parabolic subgroups. For $J \subseteq S$, let W_J be the subgroup of W generated by the set J. Notice that although we can define the length $|w|_J$ in W_J, but we always have $|w| = |w|_J$ for any $w \in W_J$. So we can just use the notation of length of W on W_J.

The next notation is the right descent classes $\mathcal{D}_I^J$ for $I \subseteq J \subseteq S$:

$$\mathcal{D}_I^J = \{w \in W \mid I \subseteq D_R(w) \subseteq J\}.$$

A special case of right descent class is

$$\mathcal{D}_I = \mathcal{D}_I^I,$$

and another is the quotients W^J for $J \subseteq S$.

$$\begin{aligned} W^J &= D_\emptyset^{S \setminus J} \\ &= \{w \in W \mid |ws| > |w| \text{ for all } s \in J\} \end{aligned}$$

We have a similar lemma as Lemma 18.

Lemma 19. *An element belongs to W^J if and only if no reduced expression for w ends with a letter from J.*

And now is the proposition that will be used for finding the generating functions.

Proposition 20. *Let $J \subseteq S$. Then the following hold:*

(i) Every $w \in W$ has a unique factorization $w = w^J \cdot w_J$ such that $w^J \in W^J$ and $w_J \in W_J$.

(ii) For this factorization, $|w| = |w^J| + |w_J|$

This proposition can be easily proved by Lemma 18 and Lemma 19.

We can also use the right descent set to characterize the finiteness of W.

Proposition 21. *(i) If W is finite, there exist an element $w_0 \in W$ such that $|w_0| \geq |w|$ for all $w \in W$.*

(ii) Conversely, suppose that W is a Coxeter group with generating set S, $x \in W$ and $D_R(x) = S$, then W is finite and $x = w_0$.

Now we can define the generating function for every subset A of W as

$$A(z) = \sum_{w \in A} z^{|w|}.$$

From the Proposition 20 we immediately have this lemma.

Lemma 22. *Let $J \subseteq S$, then*

$$W(z) = W^J(z) W_J(z).$$

This lemma reduces the problem of computing $W(z)$ to computing $W^J(z)$ and $W_J(z)$. However, although the parabolic subgroup W_J is a Coxeter group itself, so we can reduce the problem, possibly by induction, the quotient W^J is usually not a Coxeter group. Here is the proposition that solve this problem by Principle of Inclusion-Exclusion.

Proposition 23. *Let $I \subseteq J \subseteq S$, then*

$$\mathcal{D}_I^J(z) = \sum_{J \setminus I \subseteq K \subseteq I} (-1)^{|J \setminus K|} W^{S \setminus K}(z).$$

Proof. Since

$$\begin{aligned} W^{S\setminus K} &= \{w \in W | D_R(w) \subseteq K\} \\ &= \bigcup_{L\subseteq K} \{w \in W \mid D_R(w) = L\} \\ &= \bigcup_{L\subseteq K} \mathcal{D}_L, \end{aligned}$$

we have

$$W^{S\setminus K}(z) = \sum_{L\subseteq K} \mathcal{D}_L(z)$$

for all $K \subseteq S$. Hence,

$$\sum_{J\setminus I\subseteq K\subseteq J} (-1)^{|J\setminus K|} W^{S\setminus K}(z) = \sum_{L\subseteq J} \mathcal{D}_L(z) \sum_{(J\setminus I)\cup L\subseteq K\subseteq J} (-1)^{|J\setminus K|}. \tag{5.1}$$

However, by the Principle of Inclusion-Exclusion,

$$\sum_{(J\setminus I)\cup L\subseteq K\subseteq J} (-1)^{|J\setminus K|} = \begin{cases} 1, & \text{if } (J \setminus I) \cup L = J \\ 0, & \text{otherwise.} \end{cases}$$

Apply this to (5.1) we can see the the only term that left in the first sum are those L satisfying $I \subseteq L$, so

$$\sum_{J\setminus I\subseteq K\subseteq J} (-1)^{|J\setminus K|} W^{S\setminus K}(z) = \sum_{I\subseteq L\subseteq J} \mathcal{D}_L(z) = D_I^J(z).$$

□

Now we set $I = J = S$ in Proposition 23, and divide both sides by $W(z)$, we have:

$$\sum_{K\subset S} \frac{(-1)^{|K|}}{W_K(z)} = \frac{\mathcal{D}_S(z)}{W(z)}.$$

Now use the Proposition 21, we have the following formula for calculating generating function.

Corollary 24. *(i) If W is finite, we have a element w_0 such that $|w_0| \geq |w|$ for all $w \in W$, then*

$$\sum_{K \subset S} \frac{(-1)^{|K|}}{W_K(z)} = \frac{z^{|w_0|}}{W(z)}. \tag{5.2}$$

(ii) If W if infinite, we know there is no element in $\mathcal{D}_S$, so

$$\sum_{K \subset S} \frac{(-1)^{|K|}}{W_K(z)} = 0. \tag{5.3}$$

With (5.2) and (5.3), now we can calculate the growth functions.

Theorem 25. *For $G = \langle S \mid (g_i)^2, (g_1 g_2)^{\lambda_1}, \ldots, (g_{d-1} g_d)^{\lambda_{d-1}}, (g_d g_1)^{\lambda_d} \rangle$, we have generating function*

$$f(z) = \frac{[2][\lambda_1] \cdots [\lambda_d]}{(\sum_{i=1}^{d} [\lambda_i] \cdots \widehat{[\lambda_i]} \cdots [\lambda_d]) + [\lambda_1] \cdots [\lambda_d](z + 1 - d)}$$

where $[i] = 1 + z + z^2 + \cdots + z^{i-1}$. Further more, we have $f(z) = f(z^{-1})$.

Proof. Apply (5.2) and (5.3). Notice that the only finite parabolic subgroups are the trivial group, $G_{\{s_i\}}$, $G_{\{s_i, s_{i+1}\}}$ and $G_{\{s_d, s_1\}}$. And we have $G_{\{s_i\}}(z) = z + 1 = [2]$, $G_{\{s_i, s_{i+1}\}}(z) = [2][\lambda_i]$. ∎

5.2 Coxeter-type Presentations

If G is a Coxeter group, we can notice that each relation has even length. Thus if we have a tessellation generated by group G with fundamental polygon D and geometric generating set Σ, we call it a Coxeter-type presentation if the presentation has all even length relations. We study this type of tessellation because the following property:

Lemma 26. *Let (G, Σ) have a Coxeter-like presentation. Then in the construction of the dual graph in Chapter 3, each face Λ in Γ has a unique base vertex.*

Proof. Notice in condition 5 in Proposition 8, if two vertices u_1 and u_2 are the base vertices of a face Λ, then they are adjacent, and there have the maximal norm in another face Λ'. Also we can check that if a even-length face has only one base vertex, then it has only one maximal norm vertex. Thus if we consider the face Λ_1 with $\varphi^{-1}(\Lambda_1)$ having minimal norm among all faces Λ with two base vertices u_1 and u_2, then we know that there is another face Λ' containing u_1 and u_2, and Λ' has odd length.

A more general way to proof this was contributed by W.Parry. Since all relations have even length, the map $\varepsilon : \Sigma \to \{-1\} \subset \mathbb{Z}_2$ extends to a homomorphism $\varepsilon : G \to \mathbb{Z}_2$. This is a generalization of the map ε in Coxeter group in [1, p. 25]. Since u_1 and u_2 differ by an element in Σ, they must have different parity on norms. □

We again consider the sequences of angles generated by primitive angles in Lemma 13. Consider a primitive angle at a generating a sequence of angles. By the Lemma 26, we can see that each angle is of either type 1 or type 3 in Lemma 12. Thus all the angles in this sequence are at the same vertex a. This vertex correspond to a face $\Lambda = \varphi(a)$ with length λ. And since the face correspond to a relation in the presentation of G, λ is even, and each internal angle at a is $2\pi/\lambda$. Recall that when we calculate the matrix A, the outcome of $A_{i,1;k,l}$, the number of primitive angle α_i on $\partial B(n+1)$ that an angle $\alpha_{j,k}$ contributes, only depend on the subtracted angle x. But in this case, the two x angle are always equal to $2\pi/\lambda$. Thus, when calculating the linear relation among number of angles, we can identify primitive angles corresponding to faces having the same length λ, thus also identify the sequences of angles generated by these primitive angles. Each angle of ∂D occurs in a unique cycle of even length. These lengths give a collection of distinct even number $\lambda_1, \dots, \lambda_k$, each representing a set of angles we identified with multiplicity μ_i. We use α_i to denote the set of identified angles corresponding to cycle of length λ_i, so $\alpha_i = 2\pi/\lambda_i$. Now we can work with only k primitive angle(with the sequence of angles generated by them), each having multiplicity μ_i, with $\sum_{i=1}^{k} \mu_i = p$. Notice that the sequence generated ay α_i has length $\lambda_i/2$.

Theorem 27. *Let (G, Σ) have a Coxeter-type presentation, with primitive angles $\alpha_1, \dots, \alpha_k$ of multiplicity $\mu_1, \dots, \mu_k$. Then*

(i) The generating function $f(z)$ only depends on $(\alpha_i, \mu_i)_{1 \le i \le k}$, and not on the edge pairings.

(ii) The generating function is the same as the generating function of the Coxeter group generated by reflections in the sides of a d-sided polygon with μ_i angles equal to alpha$_i$.

Proof. To prove the (i) we can follow the same steps as we calculate the matrix A in Chapter 3. Notice that the involution σ are always identity, and the x angle are already known. We now have A a block matrix $(A_{i,k})_{1\leq i,k\leq d}$, with

$$A_{i,i} = \begin{bmatrix} \mu_i - 2 & \mu_i - 2 & \dots & \mu_i - 2 & -1 \\ 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ 0 & & & 1 & \end{bmatrix}$$

and

$$A_{i,j} = \begin{bmatrix} \mu_i & \mu_i & \dots & \mu_i & 0 \\ & & & & \\ & & 0 & & \\ & & & & \end{bmatrix}.$$

Since now we have primitive angles with multiplicities, we will also need to change u and v. We have v a vector

$$v = (v_{1,1}, \dots, v_{1,\lambda_1/2}, v_{2,1}, \dots, v_{k,1}, \dots, v_{k,\lambda_k/2})^T,$$

with $v_{i,1} = \mu_i$ because we have μ_i copies of α_i on ∂D and $v_{i,j} = 0$ for $j \neq 1$. And

$$u = (u_{1,1}, \dots, u_{1,\lambda_1/2}, u_{2,1}, \dots, u_{k,1}, \dots, u_{k,\lambda_k/2}),$$

with $u_{i,j} = 1$ for $j < \lambda_i/2$ and $u_{i,\lambda_i/2} = 0$. Thus we can have (i). And we can easily see (ii) follows from (ii). $\square$

5.3 Exceptional Cases

In this section we will finish the computation of the two cases we didn't compute in Chapter 3, the triangle group $\Delta(3, 3, r)$ and the tessellations with triangle fundamental polygon D. We will only give the sketch of the computation, and focus on determining whether the generating function is reciprocal or not. The complete proof can be found on [13, section 6].

Since in these two cases, the Proposition 8 may not hold, so we no longer have Lemma 12. In these two cases, when we attach the new faces gD with $|gD| = n+1$ on the boundary of $B(n)$, the result may not always a ball in the hyperbolic plane. More specifically, a new face can intersect $B(n)$ on disconnected subset on the boundary, then we will have more than one connected components of boundary.

To solve this problem, we will use the recursive structure introduces in [12]. Here we will not use angle types for recursion, instead we will simply denote the faces on different types, and find a similar linear recursive relation as in section 3.4. Then again we can use a matrix to calculate how many new faces of each types on each $\partial B(n)$. Here we will let the origin face D be a unique type, and denote all the other types by $\{t_1, \ldots, t_m\}$.

First for the $\Delta(3, 3, r)$, we have $2r + 1$ types. Since $\Delta(3, 3, r)$ still satisfies the condition 3 in Proposition 8, we know that each face gD with norm $n + 1$ will intersect $B(n)$ on one or two edges. In the following, besides defining the types of faces, we will also prove that each $B(n)$ is a topological ball by induction. Clearly $B(0)$ is a ball and the type is defined. If $gD \cap B(n)$ is one edge with vertices t_1 and t_2, we always have the angle of $\partial B(n)$ at t_1 or t_2 is $2\pi/3$. Let's assume that it is t_1. Since $B(n)$ is a ball, there is a unique vertex v neighboring t_1 and not equal to t_2 on $\partial B(n)$. Let's assume that on $\partial B(n)$, the angle is x at v and x_2 at v_2, then

$$\text{the face } gD \text{ is of type} = \begin{cases} t_i & \text{if } x < (2r-1)\pi/r \text{ and } x_2 = i\pi/r, \\ t_{2r-1} & \text{if } x = (2r-1)\pi/r \text{ then } x_2 = 2\pi/r. \end{cases}$$

If $gD \cap B(n)$ are two edges with a vertex a in common, then the angle on $\partial B(n)$ at a is either $4\pi/3$ or $(2r - 1)\pi/r$, and the two neighboring vertices of a on $\partial B(n)$ have the same angle x.

$$\text{The face } gD \text{ is of type} = \begin{cases} t_{2r} & \text{if } x = (2r-1)\pi/r, \\ t_{2r-1} & \text{if } x = 4\pi/3. \end{cases}$$

From the definition we can see that the four faces neighboring D are all of type t_1. Notice that the type t_{2r} is the buried domain that cause the condition 5 in Proposition 8 to fail, because it has two neighboring faces with same norm that share the same vertex a, and then the cycle Λ_x will have three base vertices. But if we add the type t_{2r} faces first when we construct $B(n + 1)$ from $B(n)$, we can easily prove that $B(n + 1)$ is also a topological ball.

So we can have the transition matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1/2 & 0 & 0 & 0 \end{bmatrix}$$

and

$$f(z) = 1 + u(1 - zA)^{-1} zv,$$

for $u = (1, 1, \dots, 1)$ and $v = (4, 0, 0, \dots, 0)^T$. Let

$$\alpha = 4(1 - z - \dots - z^{r-1} + z^r),$$

and

$$\beta = 4(1 + z + \cdots z^{r-2} - 2z^{r-1}).$$

Let

$$w = \begin{bmatrix} z\alpha - 4z^r + z^{r+1}\beta + 4z^{2r} \\ z^2\beta - 4z^r + 4z^{2r} \\ z^2\alpha \\ z^3\beta \\ z^3\alpha \\ \vdots \\ z^r\beta \\ z^r\alpha \\ z^r\alpha/2 \\ z^{r+1}\beta/2 \end{bmatrix}$$

Then we have $(I - zA)w = ((\alpha - 4z^{r-1} + z^r\beta + 4z^{2r-1})/4)zv$. Thus

$$\begin{aligned}
f(z) &= 1 + u(I - zA)^{-1}zv \\
&= 1 + 4uw/(\alpha - 4z^{r-1} + z^r\beta + 4z^{2r-1}) \\
&= \frac{(1-z)(1+4z+\cdots+(3r-5)z^{r-2}}{(1-z)(1-z^2-2z^3-\cdots-(r-3)z^{r-2}} \\
&\qquad \frac{+(3r-3)z^{r-1}+(3r-3)z^{3r}+\cdots+3z^{2r-1}-2z^{2r-1})}{-(r-1)z^{r-1}-(r-3)z^r-\cdots-z^{2r-4}+z^{2r-2}} \\
&= \frac{1+4z+\cdots+(3r-5)z^{r-2}}{(1+z+\cdots+z^{r-2})} \\
&\qquad \frac{+(3r-3)z^{r-1}+(3r-3)z^{3r}+\cdots+3z^{2r-1}-2z^{2r-1}}{\cdot(1-z-\cdots-z^{r-1}+z^r)}.
\end{aligned}$$

Now we will treat the case when D is a triangle. First we will consider the degree of the three vertices on D. If all the three vertices have even degree, then the group is a Coxeter group, and we can apply the result in Theorem 25. There are still three cases left, we summarize the result as follows.

1. All vertices of D have degree $3p$, for $p \geq 3$ odd. In this case, we have $3p$ types of faces. And the generating function is

$$f(z) = \frac{1+2z+\cdots+2z^{k-1}+4z^k+2z^{k+1}+\cdots+2z^{3p-2}+z^{3p-1}}{1-z-\cdots-z^{k-1}+z^k-z^{k+1}-\cdots-z^{3p-2}+z^{3p-1}},$$

   where $k = (3p-1)/2$, and it's reciprocal.

2. Two vertices of D have degree $2p$ for $p \geq 3$ and the third has degree $2q+1$ for $q \geq 2$. In this case, there are $4p + 2q - 2$ types. But in this case we can prove that the tessellation will satisfies the Lemma 12. So we the proof of the reciprocity in the Chapter 4 will apply, and the generating function is reciprocal.

3. Two vertices of D have degree $2p$ for $p \geq 3$ and the third has degree 3. In this case there are $4p + 1$ types, and it is not reciprocal. This tessellation is similar to the $\Delta(3, 3, r)$ we discuss above, and its generating function is not reciprocal because it has buried domains as in $\Delta(3, 3, r)$. This is the second counterexample in Theorem 15.

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