Costly Actions, External Incentives and Prediction Markets

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Consider a prediction market of multiple rounds with a security contingent on a certain event whose final outcome is decided by the agents who also trade in the market. One such prediction market is one in which two agents, Alice and Bob, are trading on the likelihood of a project both are working on complete. Prior research either only considers the expected rewards in the prediction market or if external incentives are present, then only a low number of rounds in the prediction market, to our knowledge at most 2. In addition, the existing literature assumes that when external incentives exist, there is no net difference between the cost of different actions agents may take outside of the prediction market. For example, it is the same cost for either Alice to work hard to complete the project as it is for her to "loaf" and not work hard. In this work we consider a 2-round round setting in which agents' cost of external actions differ. We show that when external action costs differ but are within a proper range, a prediction market is incentive compatible regardless of the initial market estimate, something that currently is not shown in the existing literature.

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Chapter 1

Background Introduction

A prediction market is created to aggregate information from individuals about uncertain events of interest. It is generally assumed that the agents who participate in the market by trading may have superior information about the relevant event, but no direct control over the outcome. However, prediction markets are often used in situations where this assumption is violated [7]. In this study we consider the impact of external incentives on the efficacy of prediction markets, especially when the external incentives require costly actions. Prediction markets that are deployed in corporate settings consist of a market maker, a center with whom all participants (agents) trade, that is present to facilitate trade and boost market liquidity. In our study, the market maker, is also the market participants' employer, does not want agents to take undesirable actions at work that impact the outcome of the traded event.

There is a mountain of evidence that prediction markets can help produce forecasts of event outcomes with a lower prediction error than conventional forecast methods [1]. However, it is also possible that agents might bluff and deceive other players by not revealing their true beliefs, hoping to correct the prediction probability and benefit from it later [8]. In addition to bluffing to maximize prediction market payoff, an agent may also change her behavior outside of the prediction market to maximize her reward within the market as well as outside of the market. For example, employees might have incentives to "slack off" when working on a project just because their prediction market position is favorable to the project being delayed and would somehow benefit more in the prediction market by working less hard in the workplace. Society is also worried that terrorists might have a higher incentive to actually conduct attacks if they trade in a relevant prediction market [16]. In reality, terrorist markets were shut down by congress in the United States of America over these concerns. There are additional canonical real-world examples showing that having agents trade in a prediction market distorts the same agents' incentives within and outside of the prediction market [6]. In fact, there is no ex post way to determine how the presence of a prediction market changes agents' probability estimates, without considering the equilibrium strategies of agents within prediction markets. In this study, we use equilibrium analysis to see if prediction market may indeed cause deviant agent behavior when the actions of agents external to the market are costly.

In conducting equilibrium analysis with the presence of external incentives in prediction markets, we find that when external actions have asymmetric costs, one costs more or less than another, then these asymmetries may actually lead agents to behave truthfully. Specifically, we determine the equilibrium strategies for two agents in a 2-round setting where the agents trade in a prediction market with a final value contingent on an event that the same two agents have a direct impact on the likelihood of the traded event. We prove that the final equilibrium strategy shows that participants will always take desirable actions/undesirable actions related to the project (work hard/loaf) and be truthful when reporting in the prediction market, as they would have had the prediction market not existed.

In the remainder of this document we introduce related work in Chapter 2 and define our general model in Chapter 3. We next show that when the cost of exerting high effort is positive, agents do not work hard, but are truthful in the prediction market in Section 4.2; similarly we show that when the cost of exerting high effort is negative, agents work hard and are sill truthful in the prediction market in Section 4.3. However, we show in Section 4.4 that when there is no cost for efforts, there is possibility that agents do not always work hard and they do not report truthfully in the prediction market. In Chapter 5 we conclude and discuss future research directions.

Chapter 2

Literature Review

There has been a lot of research on prediction market with scoring rules that do not consider outside incentives. [4] defines the quadratic scoring rule that assumes agents cannot influence the outcome of the predicted event. [3] compares three commonly used strictly proper scoring rules: quadratic, spherical and logarithmic scoring rules. [18] demonstrate that different strictly proper scoring rules yield considerably different rankings of forecasters based on their scoring rule scores. We, on the other hand, assume agents interact in a market scoring rule market proposed by [14], derived from the difference of sequential scoring rules. Hanson's market scoring rule (MSR) incentivizes risk-neutral, myopic agents to truthfully reveal their probabilistic estimates by ensuring that truthful reporting/betting maximizes their expected payoffs (is incentive compatible). [10] implies that market making can serve as an effective trading strategy for individual agents who do not possess superior information but are willing to learn from prices. Unlike the work of [10], we consider MSR prediction markets in which forward-looking agents may take costly external actions external of the market to influence the likelihood of the traded event. The agents in our setting, are informed and do not simply learn from the traded prices.

It is generally assumed that the agents who participate in the market by trading may have superior information about the relevant event, but no direct control over the outcome as noted by [7]. However, prediction markets are often used in situations where this assumption is violated to a greater or lesser degree, and we also assume agents may influence the outcome of the traded event and therefore have outside incentives similar to [6, 12, 15, 22]. [22] indicate a potential downside of prediction markets is that they may incentivize agents to take undesirable actions, and prove that there exist principle-aligned prediction mechanisms that do not incentivize undesirable actions with an 'overpayment result'. In particular, unlike our work, [22] does not use a market scoring rule mechanism, instead uses sequential scoring rules, and has linear subsidy (in the number of agents). [6] give a two-round example to understand when markets may be prone to manipulation due to different outside incentives and how much to trust the resulting prediction probabilities. However, [6] does not consider costly actions as our work do. [15] assumes profit-indifferent manipulators and proposes a modification to market scoring rules in the form of trade limits, in order to reduce the extent of manipulation of prediction markets due to external incentives. However, the limitation of trade amounts may also interfere with enabling agents' true beliefs, a limitation we do not have in our work as we do not bound agents' budgets. [9] employ a two-player market scoring rule setting where a manipulator with outside incentives trade first, followed by a truthful trader. We also have the two-player market setting but the two agents in our model are both strategic traders with outside incentives. Unlike the papers cited in this paragraph, we show that when non-myopic agents' expected payoffs not only consist of payoffs from the prediction market, but also include the costs of their related actions which decide the outcome of the market, the quadratic scoring rule used as the market reward mechanism incentivize agents to report truthfully in the prediction market and take actions as if the prediction market were not present.

Chapter 3

Model Description

In this section, we propose and describe our setting formally. In a company C, two employees are assigned to complete a time-limited project E together, and we call them Alice and Bob. We consider each week as a round from the beginning of this project and the scheduled time for conducting this project is T weeks. Because we are considering our model in a multi-round setting, the number of rounds should be equal or larger than 2. In every round, Alice and Bob will separately decide whether to give high, later denoted as 1, or low, later denoted as 0, effort to project E during this week. After T rounds, Alice and Bob's total efforts will determine the likelihood of the project's success (e.g., meet its scheduled delivery date). The project E has binary outcome as E occurs or E does not occur. If the project succeeds by the end of T rounds, we say E occurs; if the project fails by the end of T rounds, we say E does not occur. Also at the end of every round, every high effort will bring some payoff scores (negative scores are equivalent to net costs of high efforts and positive scores are equivalent to net rewards) that is a function of the effort in that round to the player who exerted this effort, but low efforts will not bring any payoff scores to the players. [1]

[1]Payoff scores need not be linear in effort and the constant α is used to convert the functional form of effort to payoff scores, in order to be compared with scores earned in the prediction market. When payoff score is negative, high efforts bring net costs to agents; when payoff score is positive, high efforts bring net rewards to agents. The total scores earned from exerting efforts and reporting in the prediction market could be converted to some financial costs or rewards that will be given to the agents.

At the same time, in a prediction market, Alice and Bob also trade in security F whose ultimate value is contingent on the outcome of E. We assume that the prediction market is a market scoring rule market. If there is no related prediction market in C, then employees will be inspired to devote high efforts to E in order to gain maximal expected payoffs. However, when a prediction market contingent on E exists, the employer may worry that employees will change their effort levels in order to benefit more through the rewards procured in the prediction market.

For every round i ($i = 1, ..., T$), Alice and Bob devote efforts $e_A^{(i)}$ and $e_B^{(i)}$ to project E and report prediction probabilities as $r_A^{(i)}$ and $r_B^{(i)}$ in the prediction market, respectively. When E happens, a report of $r_A^{(i)}$ earns Alice a net score $\rho_s^{(i)}(r_A^{(i)}) = s(E, r_A^{(i)}) - s(E, r_B^{(i-1)})$, where s is some scoring rule; when E does not happen, the report earns Alice a net score $\rho_f^{(i)}(\overline{r}_A^{(i)}) = s(E, \overline{r}_A^{(i)}) - s(E, \overline{r}_B^{(i-1)})$ instead ($\rho_s^{(i)}(r_B^{(i)})$ and $\rho_f^{(i)}(\overline{r}_B^{(i)})$ are similarly defined).

In the previous sentence, $\overline{r}_A^{(i)} = 1 - r_A^{(i)}$, in the remainder of this document we will analogously define $\overline{w} = 1 - w$ for any variable w. In addition, $s(\cdot)$ is said to be a proper scoring rule if for a risk-neutral agent with belief p and report r on an event, then:

$$\frac{d}{dr}[p \cdot s(r) + \overline{p} \cdot s(\overline{r})] = 0|_{r=p}, \tag{3.1}$$

and

$$\frac{d^2}{dr^2}[p \cdot s(r) + \overline{p} \cdot s(\overline{r})] \leq 0. \tag{3.2}$$

When an agent's score maximizing report is equal to her true belief, a proper scoring rule, and in turn a market scoring rule, is said to *incentive compatible.* Here we have the assumption that Alice (Bob) would assume the other player to be myopic when she (he) sees his (her) previous prediction in the market and we use ν_A ($0 < \nu_A < 1$) to describe Alice's impact on likelihood of success of project E. We denote $h_A^{(i)}$ ($h_B^{(i)}$) as the accumulated number of high efforts that have been taken by Alice (Bob) by the end of round i. $p_A^{(i)}$ ($p_B^{(i)}$) is the belief on the likelihood of E's final occurring held by Alice (Bob) in round i before she (he) takes any actions in this round. $\pi_A^{(i)}$ ($\pi_B^{(i)}$) is the payoff score earned from the current round i to the final round T by agent Alice (Bob) from exerting efforts to E and also from making reports in the prediction market. $I_A^{(i)}$ ($I_B^{(i)}$) is the system stage that Alice (Bob) has in round i after she (he) observes the most recent prediction probability and before she (he) takes any actions in that round. $\mathbb{E}[\pi_A^{(i)}(I_A^{(i)}, a_A^{(i)})]$ ($\mathbb{E}[\pi_B^{(i)}(I_B^{(i)}, a_B^{(i)})]$) is the expected payoff score given the current system state $I_A^{(i)}$ ($I_B^{(i)}$) and the action set $a_A^{(i)}$ ($a_B^{(i)}$) that agent would take in round i, while $\mathbb{E}[\pi_A^{(i)*}(I_A^{(i)}, a_A^{(i)})]$ ($\mathbb{E}[\pi_B^{(i)*}(I_B^{(i)}, a_B^{(i)})]$) is the corresponding maximal expected payoff score for a given stage $I_A^{(i)}$ ($I_B^{(i)}$). $a_A^{(i)*} = (e_A^{(i)*}, r_A^{(i)*})$ ($a_B^{(i)*} = (e_B^{(i)*}, r_B^{(i)*})$) is the optimal action set Alice (Bob) would take in round i in order to get $\mathbb{E}[\pi_A^{(i)*}(I_A^{(i)}, a_A^{(i)})]$ ($\mathbb{E}[\pi_B^{(i)*}(I_B^{(i)}, a_B^{(i)})]$). Figure 3.1 shows the dynamics of our model. We assume that in each round Alice first determines her effort level, $e_A^{(i)}$, then she makes a report in the prediction market, $r_A^{(i)}$; next, Bob determines the effort level he exerts in this round, $e_B^{(i)}$, and finally the round concludes by Bob reporting in the prediction market, $r_B^{(i)}$. However, for a certain round, one agent decides her (his) effort level and reported belief of this round at the same time, though the actions exerted to the project and exerted in the prediction market are conducted sequentially. Alice's and Bob's reports in the prediction market are always common knowledge to both agents in all cases. Effort levels, however, are not common knowledge.

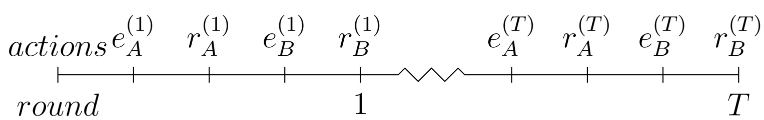

Figure 3.1: Model dynamics timeline

Given that in each round two risk-neutral forward-looking agents are maximizing their expected profits from the current round to the end of the project horizon, T, we can write down the Bellman equation to determine the payoff for each round for each agent. Before we write down the equations fo each round, we first define the payoffs each agent will receive in each round. In round i, Alice (Bob) will receive $\rho_e^{(i)}(e_A^{(i)})$ ($\rho_e^{(i)}(e_B^{(i)})$) from devoting effort $e_A^{(i)}$ ($e_B^{(i)}$) to the project, and receive $p_A^{(i)} \cdot \rho_s^{(i)}(r_A^{(i)}) + \overline{p}_A^{(i)} \cdot \rho_f^{(i)}(\overline{r}_A^{(i)})$ ($p_A^{(i)} \cdot \rho_s^{(i)}(r_B^{(i)}) + \overline{p}_B^{(i)} \cdot \rho_f^{(i)}(\overline{r}_B^{(i)})$) from reporting probability estimate $r_A^{(i)}$ ($r_B^{(i)}$) in the prediction market, with each probability estimate bounded between 0 and 1. As previously introduced, the value of payoff function of exerted efforts, $\rho_e^{(i)}(e_A^{(i)})$ ($\rho_e^{(i)}(e_B^{(i)})$), is negative when high efforts bring net costs and positive when high efforts bring net rewards. In our model we assume that the likelihood of E occurring is determined by the total number of high efforts of all rounds, weighed by agents' impacts on determining the likelihood of E:

$$\frac{\nu_A \cdot \sum_{n=1}^{T} e_A^{(n)} + (1-\nu_A) \cdot \sum_{n=1}^{T} e_B^{(n)}}{T} \tag{3.3}$$

For Alice in round i before she takes any actions in this round, the number of accumulated high efforts exerted by her and by the end of round $i-1$ ($h_A^{(i-1)}$) is known to herself, but the number of accumulated high efforts exerted by Bob and by the end of round $i-1$ ($h_B^{(i-1)}$) is not observed by her directly. However, $\sum_{n=i+1}^{T} \tilde{e}_A^{(n)}$ and $\sum_{n=i}^{T} \tilde{e}_B^{(n)}$ are all future efforts, and $e_A^{(i)}$ is the effort level decision she needs to make in this current round i. For agent Bob before he takes any actions in round i, the situation is slightly different. Because Alice has already taken her actions in this round, the number of accumulated high efforts exerted by her is $h_A^{(i)}$ instead. To be more specific, we can write down the definitions of $p_A^{(i)}$ and $p_B^{(i)}$ as:

$$p_A^{(i)} = \frac{\nu_A(h_A^{(i-1)} + e_A^{(i)} + \sum_{n=i+1}^{T} \tilde{e}_A^{(n)}) + (1-\nu_A) \cdot (h_B^{(i-1)} + \sum_{n=i}^{T} \tilde{e}_B^{(n)})}{T}, \tag{3.4a}$$

$$p_B^{(i)} = \frac{\nu_A(h_A^{(i)} + \sum_{n=i+1}^{T} \tilde{e}_A^{(n)}) + (1-\nu_A) \cdot (h_B^{(i-1)} + e_B^{(i)} + \sum_{n=i}^{T} \tilde{e}_B^{(n)})}{T}. \tag{3.4b}$$

In equation (3.4a) we see the formal definition of $p_A^{(i)}$ as the total efforts of Alice, past and future, $h_A^{(i-1)} + \sum_{n=i}^{T} \tilde{e}_A^{(n)}$, plus the total efforts of Bob, past and future, $h_B^{(i-1)} + \sum_{n=i}^{T} \tilde{e}_B^{(n)}$, all weighted by ν_A. From Alice's perspective, she knows her past effort levels, thus $h_A^{(i-1)}$ is known and is some natural number between 0 and $i-1$. Similarly, as we will see in our analysis, Alice may infer Bob's effort levels from his reports in the prediction market, and again $h_B^{(i-1)}$ is known to Alice. The $\tilde{e}_A^{(n)}$ and $\tilde{e}_B^{(n)}$ effort values, for $n \geq i+1$, are not necessarily binary, and are instead real numbers over $[0,1]$ to account for the fact that

Alice's and Bob's equilibrium effort strategies are mixed. As effort levels are not common knowledge, we have to define how each agent interprets the reported probability of the other agent. One way to interpret reported probabilities is to use Bayesian updating, given current prior beliefs. However, as the probability of E is dependent not only on market estimates but current and future effort levels, defining the Bayesian updating policy is quite convoluted. To simplify our analysis, we define $r_B(A)^{(i)}$ as Bob's estimate on the likelihood of E up to and including round i after observing Alice's last report. However, when Alice makes the report in round i, Bob has not yet taken actions in this round, the expectation of his effort value of round i perceived by Alice is denoted as $\mathbb{E}_A(\tilde{e}_B^{(i)})$. We similarly define $r_A(B)^{(i)}$ as Alice's estimate of the likelihood of E's occurring up to and including round i after observing Bob's last report. We formally define the two notations below as:

$$r_B(A)^{(i)} = \frac{\nu_A h_A^{(i)} + (1-\nu_A)(h_B^{(i-1)} + \mathbb{E}_A(\tilde{e}_B^{(i)}))}{i}, \tag{3.5a}$$

$$r_A(B)^{(i)} = \frac{\nu_A h_A^{(i)} + (1-\nu_A)h_B^{(i)}}{i}. \tag{3.5b}$$

Here we assume that at the beginning of round i, Alice has no information of predicting Bob's effort level of this round, which implies $\mathbb{E}_A(\tilde{e}_B^{(i)}) = 0.5$ for any round i. Then the notations of $r_B(A)^{(i)}$ and $r_A(B)^{(i)}$ can be further defined as:

$$r_B(A)^{(i)} = \frac{\nu_A h_A^{(i)} + (1-\nu_A)(h_B^{(i-1)} + 0.5)}{i}, \tag{3.6a}$$

$$r_A(B)^{(i)} = \frac{\nu_A h_A^{(i)} + (1-\nu_A)h_B^{(i)}}{i}. \tag{3.6b}$$

Note that $r_B(A)^{(i)} = r_A^{(i)}$ and $r_A(B)^{(i)} = r_B^{(i-1)}$, as we are simply presenting how each of the agents interprets the observed prediction market probabilities of the other agent. In (3.4a) Alice's number of previous high efforts, $h_A^{(i-1)}$, is known to herself; and the number of Bob's previous high efforts, $h_B^{(i-1)}$, could not be observed directly but could be inferred using his last prediction probability $r_B^{(i-1)}$ as $h_B^{(i-1)} = \frac{(i-1)\cdot r_B^{(i-1)} - \nu_A h_A^{(i-1)}}{1-\nu_A}$. For (3.4b) the unobservable number of previous high efforts exerted by Alice could also be inferred using $r_A^{(i)}$ as $h_A^{(i)} = \frac{i\cdot r_A^{(i)} - (1-\nu_A)h_B^{(i-1)} + 0.5}{\nu_A}$. By inserting $h_B^{(i-1)} = \frac{(i-1)\cdot r_B^{(i-1)} - \nu_A h_A^{(i-1)}}{1-\nu_A}$ and $h_A^{(i)} = \frac{i\cdot r_A^{(i)} - (1-\nu_A)(h_B^{(i-1)} + 0.5)}{\nu_A}$

into (3.4a) and (3.4b) separately, we can get a further expression of the agents' beliefs on the likelihood of E's final occurring as:

$$
p_A^{(i)} = \frac{(i-1)\cdot r_B^{(i-1)} + \nu_A(e_A^{(i)} + \sum_{n=i+1}^{T} \tilde{e}_A^{(n)}) + (1-\nu_A)\sum_{n=i}^{T} \tilde{e}_B^{(n)}}{T}, \tag{3.7a}
$$

$$
p_B^{(i)} = \frac{i\cdot r_A^{(i)} - 0.5(1-\nu_A) + \nu_A \sum_{n=i+1}^{T} \tilde{e}_A^{(n)} + (1-\nu_A)(e_B^{(i)} + \sum_{n=i+1}^{T} \tilde{e}_B^{(n)})}{T}. \tag{3.7b}
$$

With the payoff scores collected in each round, each agent maximizes the current round's payoff scores plus the discounted future rounds' payoff scores.

$$
\begin{aligned}
\mathbb{E}[\pi_A^{(i)*}(I_A^{(i)}, a_A^{(i)})] &= \max_{(e_A^{(i)}, r_A^{(i)})} \{ \underbrace{\delta \cdot \mathbb{E}[\pi_A^{(i+1)*}]}_{\text{discounted future payoff}} + \underbrace{\rho_e^{(i)}(e_A^{(i)})}_{\text{effort payoff}} + \underbrace{p_A^{(i)} \cdot \rho_s^{(i)}(r_A^{(i)}) + \overline{p}_A^{(i)} \cdot \rho_f^{(i)}(\overline{r}_A^{(i)})}_{\text{prediction market payoff}} \}, \\
\mathbb{E}[\pi_B^{(i)*}(I_B^{(i)}, a_B^{(i)})] &= \max_{(e_B^{(i)}, r_B^{(i)})} \{ \delta \cdot \mathbb{E}[\pi_B^{(i+1)*}] + \rho_e^{(i)}(e_B^{(i)}) + p_B^{(i)} \cdot \rho_s^{(i)}(r_B^{(i)}) + \overline{p}_B^{(i)} \cdot \rho_f^{(i)}(\overline{r}_B^{(i)}) \}.
\end{aligned} \tag{3.8}
$$

In the remainder of this document, we will determine the values of $\mathbb{E}[\pi_A^{(i)*}(I_A^{(1)}, a_A^{(1)})]$ and $\mathbb{E}[\pi_B^{(1)*}(I_B^{(1)}, a_B^{(1)})]$ and the equilibrium strategies of both players.

Chapter 4

Result and Analysis

This section first introduces one of the most commonly used scoring rules, the quadratic scoring rule, that will be used in all following cases. Then we give mathematical analysis of case 1, high efforts bring net cost to agents, and case 2, high efforts bring net reward instead and case 3, external incentives do not exist. We assume $T = 2$ for all cases.

In this work, we show that with external incentives (net payoff scores are given to exerted efforts), the quadratic scoring applied in the prediction market is incentive-compatible (agents will report truthfully in the prediction market). In fact, we prove that in a 2-round setting, agents will take desirable actions (always exert high efforts) when high efforts bring net rewards in case 1; and agents will take undesirable actions (always exert low efforts) when high efforts bring net costs in case 2. When external incentives exist, agents will behave as if the prediction market were not present, which indicates that the prediction market will not change agents' incentives outside of the market. However, when external incentive does not exist, we prove in case 3 that agents will have incentives to bluff (report untruthfully) in the prediction market when they are forward-looking. This result is aligned with the conclusion from previous work [8, 11].

4.1 Application of the quadratic scoring rule

In this section, we use one of the most popular scoring rules, the quadratic scoring rule, as the reward mechanism in the prediction market. As we consider extreme reports in our results, we cannot use another, perhaps more popular, scoring rule, the logarithmic scoring rule. Then $s(E, r)$ introduced in Chapter 3 is defined as:

$$\begin{aligned} s(E, r) &= 2r - r^2 - \overline{r}^2 = -2r^2 + 4r - 1 \\ s(E, \overline{r}) &= 2\overline{r} - r^2 - \overline{r}^2 = -2r^2 + 1 \end{aligned} \tag{4.1}$$

$s(E, r)$ is a proper scoring rule as (3.1) and (3.2) are satisfied. We have already defined the scores earned by reporting in the prediction market in Chapter 3 as:

$$\begin{aligned} \rho_s(r_A^{(i)}) &= s(E, r_A^{(i)}) - s(E, r_B^{(i-1)}) \\ \rho_f(\overline{r}_A^{(i)}) &= s(E, \overline{r}_A^{(i)}) - s(E, \overline{r}_B^{(i-1)}) \\ \rho_s(r_B^{(i)}) &= s(E, r_B^{(i)}) - s(E, r_A^{(i)}) \\ \rho_f(\overline{r}_B^{(i)}) &= s(E, \overline{r}_B^{(i)}) - s(E, \overline{r}_A^{(i)}) \end{aligned} \tag{4.2}$$

Using $s(E, r)$ defined in (4.1), we can further write the scores as:

$$\begin{aligned} \rho_s(r_A^{(i)}) &= 4r_A^{(i)} - 2(r_A^{(i)})^2 - 4r_B^{(i-1)} + 2(r_B^{(i-1)})^2 \\ \rho_f(\overline{r}_A^{(i)}) &= -2(r_A^{(i)})^2 + 2(r_B^{(i-1)})^2 \\ \rho_s(r_B^{(i)}) &= 4r_B^{(i)} - 2(r_B^{(i)})^2 - 4r_A^{(i)} + 2(r_A^{(i)})^2 \\ \rho_f(\overline{r}_B^{(i)}) &= -2(r_B^{(i)})^2 + 2(r_A^{(i)})^2 \end{aligned} \tag{4.3}$$

4.2 Case 1: High efforts bring net costs to agents

In case 1 we assume that two agents' efforts together decide the outcome of E and the ultimate value of F but high efforts will bring net costs to the agents who exert them. In this setting, we assume the payoff function of exerted efforts to be $\rho_e(e) = -\alpha \cdot e^2$ ($\alpha > 0$):

$$\rho_e(e_A^{(i)}) = -\alpha \cdot (e_A^{(i)})^2, \tag{4.4a}$$

$$\rho_e(e_B^{(i)}) = -\alpha \cdot (e_B^{(i)})^2. \tag{4.4b}$$

In this section we use the definition of Alice's (Bob's) belief on the likelihood E's final occurrence in each round when maximizing her (his) expected total payoff from Chapter 3, and apply the quadratic scoring rule introduced in section 4.1. From (4.4), (3.7) and (4.3), we know that $I_A^{(i)} = r_B^{(i-1)}$ and $I_B^{(i)} = r_A^{(i)}$ when deciding the values of $\mathbb{E}[\pi_A^{(i)*}(I_A^{(i)}, a_A^{(i)})]$ and $\mathbb{E}[\pi_B^{(i)*}(I_B^{(i)}, a_B^{(i)})]$, respectively. After inserting the payoff functions of exerted efforts (4.4), the belief on the likelihood of E's occurring (3.7), and the payoff functions in the prediction market (4.3) into the maximization equations (3.8) from the first round to the final round, we get the following maximization equations for Alice and Bob separately when $T = 2$:

For agent Alice:

$$\begin{aligned}
\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)}, a_A^{(1)})] = & \max_{a_A^{(1)}=(e_A^{(1)}, r_A^{(1)})} \{\delta \cdot \mathbb{E}[\pi_A^{(2)*}(\tilde{r}_B^{(1)}, a_A^{(2)})] - \alpha \cdot (e_A^{(1)})^2 \qquad (4.5a) \\
& + \frac{\nu_A(e_A^{(1)} + \tilde{e}_A^{(2)}) + (1-\nu_A)\sum_{n=1}^{2} \tilde{e}_B^{(n)}}{2} \cdot [4r_A^{(1)} - 2(r_A^{(1)})^2 - 4r_B^{(0)} + 2(r_B^{(0)})^2] \\
& + (1 - \frac{\nu_A(e_A^{(1)} + \tilde{e}_A^{(2)}) + (1-\nu_A)\sum_{n=1}^{2} \tilde{e}_B^{(n)}}{2}) \cdot [-2(r_A^{(1)})^2 + 2(r_B^{(0)})^2]\} \\
= & \max_{a_A^{(1)}=(e_A^{(1)}, r_A^{(1)})} \{\delta \cdot \mathbb{E}[\pi_A^{(2)*}(\tilde{r}_B^{(1)}, a_A^{(2)})] - \alpha \cdot (e_A^{(1)})^2 \qquad (4.5b) \\
& + \frac{\nu_A(e_A^{(1)} + \tilde{e}_A^{(2)}) + (1-\nu_A)\sum_{n=1}^{2} \tilde{e}_B^{(n)}}{2} \cdot [4r_A^{(1)} - 4r_B^{(0)}] \\
& + [-2(r_A^{(1)})^2 + 2(r_B^{(0)})^2]\}, \\
\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)}, a_A^{(2)})] = & \max_{a_A^{(2)}=(e_A^{(2)}, r_A^{(2)})} \{-\alpha \cdot (e_A^{(2)})^2 \qquad (4.5c) \\
& + \frac{1 \cdot r_B^{(1)} + \nu_A \cdot e_A^{(2)} + (1-\nu_A) \cdot \tilde{e}_B^{(2)}}{2} \cdot [4r_A^{(2)} - 4r_B^{(1)}] \\
& + [-2(r_A^{(2)})^2 + 2(r_B^{(1)})^2]\}.
\end{aligned}$$

For agent Bob:

$$\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)}, a_B^{(1)})] = \max_{a_B^{(1)}=(e_B^{(1)}, r_B^{(1)})} \{\delta \cdot \mathbb{E}[\pi_B^{(2)*}(\tilde{r}_A^{(2)}, a_B^{(2)})] - \alpha \cdot (e_B^{(1)})^2 \tag{4.6a}$$
$$+ \frac{r_A^{(1)} - 0.5(1-\nu_A) + \nu_A \cdot \tilde{e}_A^{(2)} + (1-\nu_A) \cdot (e_B^{(1)} + \tilde{e}_B^{(2)})}{2}$$
$$\cdot [4r_B^{(1)} - 4r_A^{(1)}] + [-2(r_B^{(1)})^2 + 2(r_A^{(1)})^2]\},$$

$$\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)}, a_B^{(2)})] = \max_{a_B^{(2)}=(e_B^{(2)}, r_B^{(2)})} \{-\alpha \cdot (e_B^{(2)})^2 \tag{4.6b}$$
$$+ \frac{2 \cdot r_A^{(2)} - (1-\nu_A) \cdot 0.5 + (1-\nu_A) \cdot e_B^{(2)}}{2} \cdot [4r_B^{(2)} - 4r_A^{(2)}]$$
$$+ [-2(r_B^{(2)})^2 + 2(r_A^{(2)})^2]\}.$$

From the timeline we know that Alice is the first agent to make action decisions in round 1. Although for Alice in round 1, $\tilde{r}_B^{(1)}, \tilde{e}_A^{(2)}, \tilde{e}_B^{(1)}$ and $\tilde{e}_B^{(2)}$ are values for future actions, which are not known to Alice now, but could be inferred by her, for agents are assumed to be rational, forward-looking and strategic in our model. To be more specific, Alice can play the whole game in her mind knowing that both agents want to maximize the expected scores earned from the current round to the final round in any round, and infer what future optimal actions will be after exerting effort $e_A^{(1)}$ and making report $r_A^{(1)}$ in round 1. Following this logic, we can use backwards induction to solve this problem, i.e., determine the values of (4.5) and (4.6).

Define $f_B^{(i)}$ ($f_A^{(i)}$) as the function of Bob's (Alice's) expected payoff scores earned from the current round i and $f_B^{(i)*}$ ($f_A^{(i)*}$) is the corresponding optimal function value. First consider agent Bob in round 2, there is no future score for Bob since this is the last round, so we have $f_B^{(2)}$ as:

$$f_B^{(2)} = \mathbb{E}[\pi_B^{(2)}(r_A^{(2)}, a_B^{(2)})] = -\alpha \cdot (e_B^{(2)})^2$$
$$+ \frac{2 \cdot r_A^{(2)} - (1-\nu_A) \cdot 0.5 + (1-\nu_A) \cdot e_B^{(2)}}{2} \cdot [4r_B^{(2)} - 2(r_B^{(2)})^2 - 4r_A^{(2)} + 2(r_A^{(2)})^2] \tag{4.7}$$
$$+ (1 - \frac{2 \cdot r_A^{(2)} - (1-\nu_A) \cdot 0.5 + (1-\nu_A) \cdot e_B^{(2)}}{2}) \cdot [-2(r_B^{(2)})^2 + 2(r_A^{(2)})^2]$$

Lemma 1. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ *and* $r_A^{(2)} \leqslant \frac{1}{4}(1-\nu_A)$*, then* $(e_B^{(2)*}, r_B^{(2)*}) = (0,0)$ *and* $\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)}, a_B^{(2)})] = -2(r_A^{(2)})^2 + (1-\nu_A)r_A^{(2)}$.

Proof. As we can see from the equation above, $f_B^{(2)}$ is a function of Alice's most recent report ($r_A^{(2)}$) and his actions of this round ($a_B^{(2)} = (e_B^{(2)}, r_B^{(2)})$). To determine the optimal action set of Bob in round 2 ($a_B^{(2)*} = (e_B^{(2)*}, r_B^{(2)*})$) and the optimal function value ($f_B^{(2)*}$) we need to check the convexity of the function $f_B^{(2)}$ by checking the Hessian of this function:

$$\nabla^2 f_B^{(2)}(e_B^{(2)}, r_B^{(2)}) = \begin{bmatrix} \dfrac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} & \dfrac{\partial^2 f_B^{(2)}}{\partial e_B^{(2)} \partial r_B^{(2)}} \\ \dfrac{\partial^2 f_B^{(2)}}{\partial r_B^{(2)} \partial e_B^{(2)}} & \dfrac{\partial^2 f_B^{(2)}}{\partial (r_B^{(2)})^2} \end{bmatrix} = \begin{bmatrix} -2\alpha & \dfrac{4(1-\nu_A)}{2} \\ \dfrac{4(1-\nu_A)}{2} & -4 \end{bmatrix}$$

The first principle minor of $\nabla^2 f_B^{(2)}(e_B^{(2)}, r_B^{(2)})$ is negative for $\det \begin{bmatrix} \frac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} \end{bmatrix} = -2\alpha < 0$. The second principle minor is positive when:

$$\det \begin{bmatrix} \dfrac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} & \dfrac{\partial^2 f_B^{(2)}}{\partial e_B^{(2)} \partial r_B^{(2)}} \\ \dfrac{\partial^2 f_B^{(2)}}{\partial r_B^{(2)} \partial e_B^{(2)}} & \dfrac{\partial^2 f_B^{(2)}}{\partial (r_B^{(2)})^2} \end{bmatrix} = 8\alpha - \frac{16(1-\nu_A)^2}{4} > 0 (\alpha > \frac{(1-\nu_A)^2}{2}).$$

In fact, when $\alpha > \frac{\nu_A^2}{2}$, we can prove that $f_A^{(2)}$ (which could be similarly defined as $f_B^{(2)}$) is also a concave function. it is reasonable to assume $\max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\} < \alpha \leqslant 2$ for $0 < \max\{\frac{1-\nu_A}{2}, \frac{\nu_A}{2}\} < \frac{1}{2}$. If we assume $\alpha \leqslant \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, then the range of scores earned from efforts in each round is between $[0, \frac{1}{2})$, while the range of scores earned from reports in each round is between $[0, 2]$. Under this assumption we find $f_B^{(2)}$ to be a concave function. We take the first derivative of $f_B^{(2)}$ regarding $e_B^{(2)}$ and $r_B^{(2)}$ respectively and let the two derivatives equal to 0 simultaneously. If the solution pair $a_B^{(2)} = (e_B^{(2)}, r_B^{(2)})$ is a feasible solution, then it is the optimal solution. However, we get $e_B^{(2)} = \frac{-(1-\nu_A)^2}{2(2\alpha-(1-\nu_A)^2)} < 0$

$e_B^{(2)}$	$r_B^{(2)}$	$f_B^{(2)}$
0	$r_A^{(2)} + \frac{1}{4}(\nu_A - 1)$	$\frac{1}{8}(\nu_A^{(1)} - 1)^2$
1	$r_A^{(2)} + \frac{1}{4}(1 - \nu_A)$	$-\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2$
0	0	$-2(r_A^{(2)})^2 + (1 - \nu_A)r_A^{(2)}$
0	1	$-2(r_A^{(2)})^2 + (5 - \nu_A)r_A^{(2)} + \nu_A - 3$
1	0	$-2(r_A^{(2)})^2 + (\nu_A - 1)r_A^{(2)} - \alpha$
1	1	$-2(r_A^{(2)})^2 + (\nu_A + 3)r_A^{(2)} - \alpha - \nu_A - 1$

Table 4.1: KKT points of $f_B^{(2)}$ in case 1

when $\alpha > \frac{(1-\nu_A)^2}{2}$, and it is not a feasible solution for the value of $e_B^{(2)}$ is between $[0, 1]$. In this situation, the way to find the maximal value $f_B^{(2)*}$ is to find its Karush–Kuhn–Tucker (KKT) points and compare the values of $f_B^{(2)}$ of these points, for $f_B^{(2)*}$ must be attained in one of these KKT points. We write down in table 4.1 the KKT points of $f_B^{(2)}$ and the corresponding function values. However, to simplify our analysis, unfeasible KKT points are not shown in this table. If $r_A^{(2)} \leqslant \frac{1}{4}(1 - \nu_A)$, by comparing the values of $f_B^{(2)}$ of feasible KKT points in table 4.1, we find that $f_B^{(2)*}$ is attained as $f_B^{(2)*} = -2(r_A^{(2)})^2 + (1 - \nu_A)r_A^{(2)}$ when $(e_B^{(2)*}, r_B^{(2)*}) = (0, 0)$. So lemma 1 is true. □

As there are conditions on when Lemma 1 is applicable we use $(e_{B,1}^{(2)*}, r_{B,1}^{(2)*})$ and $\mathbb{E}[\pi_{B,1}^{(2)*}(r_A^{(2)}, a_B^{(2)})]$ to denote the optimal values as determined in the lemma. Similarly for all subsequent lemmas, say lemma i, we use $(e_{j,i}^{(k)*}, r_{j,i}^{(k)*})$, $E[\pi_{j,i}^{(k)*}]$, and $f_{j,i}^{(k)*}$ to denote the equilibrium decisions, total expected payoff (from round k to the last round), and round payoff, of player j, $j \in \{A, B\}$ in round k according to the conditions of lemma i, respectively.

Lemma 2. *If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $r_A^{(2)} \geqslant \frac{1}{4}(1 - \nu_A)$, then $(e_{B,2}^{(2)*}, r_{B,2}^{(2)*}) = (0, r_A^{(2)} + \frac{1}{4}(\nu_A - 1))$ and $\mathbb{E}[\pi_{B,2}^{(2)*}(r_A^{(2)}, a_B^{(2)})] = \frac{1}{8}(\nu_A^{(1)} - 1)^2$.*

Proof. Similar as the proof for lemma 1, if $\frac{1}{4}(1 - \nu_A) \leqslant r_A^{(2)}$, by comparing the values of $f_B^{(2)}$ of feasible KKT points in table 4.1, we find that $f_B^{(2)*}$ is attained as $f_{B,2}^{(2)*} = \frac{1}{8}(\nu_A^{(1)} - 1)^2$ when $(e_{B,2}^{(2)*}, r_{B,2}^{(2)*}) = (0, r_A^{(2)} + \frac{1}{4}(\nu_A - 1))$. So lemma 2 is true. □

Given the two cases for Bob's equilibrium decisions in round 2, we may now determine Bob's round 2 decisions:

Theorem 1. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, *then*

$$(e_B^{(2)*}, r_B^{(2)*}) = \begin{cases} (0,0) & \text{if } 0 \leqslant r_A^{(2)} \leqslant \frac{1}{4}(1-\nu_A) \\ (0, r_A^{(2)} + \frac{1}{4}(\nu_A - 1)) & \text{if } \frac{1}{4}(1-\nu_A) \leqslant r_A^{(2)} \leqslant 1 \end{cases}$$

$$f_B^{(2)*} = \begin{cases} -2(r_A^{(2)})^2 + (1-\nu_A)r_A^{(2)} & \text{if } 0 \leqslant r_A^{(2)} \leqslant \frac{1}{4}(1-\nu_A) \\ \frac{1}{8}(\nu_A^{(1)} - 1)^2 & \text{if } \frac{1}{4}(1-\nu_A) \leqslant r_A^{(2)} \leqslant 1 \end{cases}$$

Proof. With lemma 1 and lemma 2, theorem 1 is true. □

Then consider Alice's expected payoff score in round $i = 2$. We know from theorem 1 that $e_B^{(2)*} = 0$ no matter what value $r_A^{(2)}$ is. After inserting $\tilde{e}_B^{(2)} = e_B^{(2)*} = 0$ into the function of $\mathbb{E}[\pi_A^{(2)}(r_B^{(1)}, a_A^{(2)})]$, we get $f_A^{(2)}$ as:

$$\begin{aligned} f_A^{(2)} = \mathbb{E}[\pi_A^{(2)}(r_B^{(1)}, a_A^{(2)})] &= -\alpha \cdot (e_A^{(2)})^2 \\ &+ \frac{1 \cdot r_B^{(1)} + \nu_A \cdot e_A^{(2)} + (1-\nu_A) \cdot 0}{2} \cdot [4r_A^{(2)} - 4r_B^{(1)}] \\ &+ [-2(r_A^{(2)})^2 + 2(r_B^{(1)})^2] \end{aligned}$$

Theorem 2. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, *then Alice's optimal action set in round 2 is* $(e_A^{(2)*}, r_A^{(2)*}) = (0, \frac{1}{2}r_B^{(1)})$, *and her optimal expected payoff score of round 2 is* $\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)}, a_A^{(2)})] = \frac{1}{2}(r_B^{(1)})^2$.

Proof. To find the optimal function value $f_A^{(2)*}$, we still need to check the Hessian of function $f_A^{(2)}$:

$$\nabla^2 f_A^{(2)}(e_A^{(2)}, r_A^{(2)}) = \begin{bmatrix} \frac{\partial^2 f_A^{(2)}}{\partial (e_A^{(2)})^2} & \frac{\partial^2 f_A^{(2)}}{\partial e_A^{(2)} \partial r_A^{(2)}} \\ \frac{\partial^2 f_A^{(2)}}{\partial r_A^{(2)} \partial e_A^{(2)}} & \frac{\partial^2 f_A^{(2)}}{\partial (r_A^{(2)})^2} \end{bmatrix} = \begin{bmatrix} -2\alpha & 2\nu_A \\ 2\nu_A & -4 \end{bmatrix}$$

$e_A^{(2)}$	$r_A^{(2)}$	$f_A^{(2)}$
0	$\frac{1}{2}r_B^{(1)}$	$\frac{1}{2}(r_B^{(1)})^2$
1	$\frac{1}{2}(r_B^{(1)}+\nu_A)$	$-\alpha+\frac{1}{2}(r_B^{(1)})^2$
$\frac{\nu_A(1-r_B^{(1)})}{\alpha}$	1	$2(r_B^{(1)}-1)+\frac{\nu_A^2(r_B^{(1)}-1)^2}{\alpha}$
0	0	0
0	1	$2(r_B^{(1)}-1)$
1	0	$-2\nu_A r_B^{(1)}-\alpha$
1	1	$2(1-\nu_A)(r_B^{(1)}-1)-\alpha$

Table 4.2: KKT points of $f_A^{(2)}$ in case 1

We find that $f_A^{(2)}$ is a concave function because the value of the first principle minor of $\nabla^2 f_A^{(2)}$ is negative as $-2\alpha < 0$, and the value of the second principle minor of $\nabla^2 f_A^{(2)}$ is positive as when $\alpha > \frac{\nu_A^2}{2}$. We take the first derivative of $f_A^{(2)}$ regarding $e_A^{(2)}$ and $r_A^{(2)}$ respectively and let the two derivative equations equal to 0 simultaneously, then we get $e_A^{(2)} = \frac{-\nu_A r_B^{(1)}}{2\alpha-\nu_A^2}$. However, it is not a feasible solution when $r_B^{(1)} \neq 0$, so we still need to discuss the KKT points of $f_A^{(2)}$ in order to get $a_A^{(2)*}$ and $f_A^{(2)*}$. We denote in table 4.2 the KKT points of $f_B^{(2)}$ and the corresponding function values. However, to simplify our analysis, unfeasible KKT points are not shown in this table.

By comparing the values of $f_A^{(2)}$ in table 4.2, we find that $f_A^{(2)}$ is maximized as $f_A^{(2)*} = \frac{1}{2}(r_B^{(1)})^2$ when $(e_A^{(2)*}, r_A^{(2)*}) = (0, \frac{1}{2}r_B^{(1)})$, regardless of the value of $r_B^{(1)}$. So theorem 2 is true. □

Consider Bob in round 1. From theorem 1 and theorem 2 we have $e_A^{(2)*} = e_B^{(2)*} = 0$.

Knowing $\tilde{e}_A^{(2)} = \tilde{e}_B^{(2)} = 0$, we get the expression of $f_B^{(1)}$ as:

$$\begin{aligned} f_B^{(1)} &= -\alpha \cdot (e_B^{(1)})^2 \\ &+ \frac{r_A^{(1)} - 0.5(1-\nu_A) + \nu_A \cdot 0 + (1-\nu_A)\cdot(e_B^{(1)} + 0)}{2} \cdot [4r_B^{(1)} - 4r_A^{(1)}] \\ &+ [-2(r_B^{(1)})^2 + 2(r_A^{(1)})^2] \end{aligned} \tag{4.8}$$

Lemma 3. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ *, then*

$$(e_B^{(2)*}, r_B^{(2)*}) = \begin{cases} (0,0) & \text{if } 0 \leqslant r_B^{(1)} \leqslant \frac{1}{2}(1-\nu_A) \\ (0, r_A^{(2)} + \frac{1}{4}(\nu_A - 1)) & \text{if } \frac{1}{2}(1-\nu_A) \leqslant r_B^{(1)} \leqslant 1 \end{cases}$$

$$f_B^{(2)*} = \begin{cases} -2(r_A^{(2)})^2 + (1-\nu_A) r_A^{(2)} & \text{if } 0 \leqslant r_B^{(1)} \leqslant \frac{1}{2}(1-\nu_A) \\ \frac{1}{8}(\nu_A^{(1)} - 1)^2 & \text{if } \frac{1}{2}(1-\nu_A) \leqslant r_B^{(1)} \leqslant 1 \end{cases}$$

Proof. According to theorem 2, we know that $r_A^{(2)*} = \frac{1}{2} r_B^{(1)}$. Together with the statement of theorem 1, we infer lemma 3. □

We now move to Bob's optimal actions in the first round. We find that Bob will exert low effort and report as 0 in round 1 no matter what report Alice makes in round 1.

Theorem 3. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ *and* $0 < \nu_A < \frac{1}{3}$*, then Bob's optimal action set in round 1 is* $(e_B^{(1)*}, r_B^{(1)*}) = (0,0)$*, and his expected total payoff score from this round to the final round is* $\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)}, a_B^{(1)})] = r_A^{(1)}(1-\nu_A)$.

Proof. According to the definitions of $\mathbb{E}[\pi_B^{(1)}(r_A^{(1)}, a_B^{(1)})]$, $f_B^{(1)}$ and $f_B^{(2)*}$, we know that $\mathbb{E}[\pi_B^{(1)}(r_A^{(1)}, a_B^{(1)})] = f_B^{(1)} + \delta \cdot f_B^{(2)*}$, and $f_B^{(2)*}$ is a fixed value for a given $r_A^{(2)*}$. From lemma 3 we know that if $r_B^{(1)} \leqslant \frac{1}{2}(1-\nu_A)$, then $(e_B^{(2)*}, r_B^{(2)*}) = (0,0)$ and $f_{B,1}^{(2)*} = -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}(1-\nu_A) r_B^{(1)}$; if $r_B^{(1)} \geqslant \frac{1}{2}(1-\nu_A)$, then $(e_B^{(2)*}, r_B^{(2)*}) = (0, r_A^{(2)} + \frac{1}{4}(\nu_A - 1))$ and $f_{B,2}^{(2)*} = \frac{1}{8}(\nu_A^{(1)} - 1)^2$.

We first discuss the situation where $f_{B,1}^{(2)*}$ is a function of $r_B^{(1)}$ as $f_{B,1}^{(2)*} = -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}(1-\nu_A) r_B^{(1)}$ when $r_B^{(1)} \leqslant \frac{1}{2}(1-\nu_A)$. Under this condition, $f_B^{(1)} + \delta \cdot f_{B,1}^{(2)*}$ is also a concave function if $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$. Compare the KKT points of $f_B^{(1)} + \delta \cdot f_{B,1}^{(2)*}$, we get $f_B^{(1)} + \delta \cdot f_B^{(2)*}$ will be maximized as $r_A^{(1)}(1-\nu_A)$ when $(e_B^{(1)}, r_B^{(1)}) = (0,0)$. We secondly discuss the situation

$e_B^{(1)}$	$r_B^{(1)}$	$f_B^{(1)} + \delta \cdot f_{B,2}^{(2)*}$
0	$\frac{1}{2}r_A^{(1)} + \frac{1}{4}(\nu_A - 1)$	$\frac{1}{2}r_A^{(1)}(r_A^{(1)} + 1 - \nu_A) + \frac{1}{8}(1+\delta)(\nu_A^{(1)} - 1)^2$
1	$\frac{1}{2}r_A^{(2)} + \frac{1}{4}(1 - \nu_A)$	$-\alpha + \frac{1}{2}r_A^{(1)}(r_A^{(1)} - 1 + \nu_A) + \frac{1}{8}(1+\delta)(\nu_A^{(1)} - 1)^2$
0	1	$(r_A^{(1)} - 1)(3 - \nu_A) + \frac{1}{8}\delta(\nu_A^{(1)} - 1)^2$
1	1	$(r_A^{(1)} - 1)(\nu_A + 1) - \alpha + \frac{1}{8}\delta(\nu_A^{(1)} - 1)^2$

Table 4.3: KKT points of $f_B^{(1)} + \delta \cdot f_{B,2}^{(2)*}$ in case 1 when $\frac{1}{2}(1 - \nu_A) \leqslant r_B^{(1)} \leqslant 1$

where $f_{B,2}^{(2)*}$ is a constant as $f_{B,2}^{(2)*} = \frac{1}{8}(\nu_A^{(1)} - 1)^2$ when $\frac{1}{2}(1 - \nu_A) \leqslant r_B^{(1)} \leqslant 1$. Under this condition, $f_B^{(1)} + \delta \cdot f_{B,2}^{(2)*}$ is also a concave function if $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$. Similarly we denote the KKT points of $f_B^{(1)} + \delta \cdot f_{B,2}^{(2)*}$ in table 4.3. We conclude from table 4.3 that when $r_A^{(1)} \geqslant \frac{3}{2}(1 - \nu_A)$, $f_B^{(1)} + \delta \cdot f_B^{(2)*}$ will be maximized as $\frac{1}{2}r_A^{(1)}(r_A^{(1)} + 1 - \nu_A) + \frac{1}{8}(1+\delta)(\nu_A^{(1)} - 1)^2$ when $(e_B^{(1)}, r_B^{(1)}) = (0, \frac{1}{2}r_A^{(1)} + \frac{1}{4}(\nu_A - 1))$, which is larger than $r_A^{(1)}(1 - \nu_A)$; however, when $0 < \nu_A < \frac{1}{3}$, this situation does not exist for $r_A^{(1)} \leqslant 1$. When $r_A^{(1)} < \frac{3}{2}(1 - \nu_A)$, the rest three points' values are all less than $r_A^{(1)}(1 - \nu_A)$.

In conclusion, the optimal value of $\mathbb{E}[\pi_B^{(1)}(r_A^{(1)}, a_B^{(1)})]$ is achieved as $\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)}, a_B^{(1)})] = r_A^{(1)}(1 - \nu_A)$ when $(e_B^{(1)*}, r_B^{(1)*}) = (0, 0)$. So theorem 3 is true. □

At last, consider *Alice* in round 1. We have already proved in theorem 1, theorem 2 and theorem 3 that $e_B^{(1)*} = e_A^{(2)*} = e_B^{(2)*} = r_B^{(1)*} = 0$ and $f_A^{(2)*} = \frac{1}{2}(r_B^{(1)})^2 = 0$ no matter what effort *Alice* exerts and no matter what report she makes in round 1. In this situation $f_A^{(2)*} = 0$ is a constant and we get the expression of $f_A^{(1)} + \delta f_A^{(2)*}$ as:

$$\begin{aligned} f_A^{(1)} + f_A^{(2)*} &= \delta \cdot 0 - \alpha \cdot (e_A^{(1)})^2 \\ &+ \frac{\nu_A(e_A^{(1)} + 0) + (1 - \nu_A) \cdot (0 + 0)}{2} \cdot [4r_A^{(1)} - 4r_B^{(0)}] \\ &+ [-2(r_A^{(1)})^2 + 2(r_B^{(0)})^2] \end{aligned} \tag{4.9}$$

We now move to *Alice*'s optimal actions in the first round. We find that *Alice* will exert low effort and report as 0 in round 1 no matter what initial market estimate is.

Theorem 4. *If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, then Alice's optimal action set in round 1 is $(e_A^{(1)*}, r_A^{(1)*}) = (0, 0)$, and her expected total payoff score from this round to the final round is $\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)}, a_A^{(1)})] = 2(r_B^{(0)})^2$.*

$e_A^{(1)}$	$r_A^{(1)}$	$f_A^{(1)} + \delta \cdot f_A^{(2)*}$
0	0	$2(r_B^{(0)})^2$
1	$\frac{1}{2}\nu_A$	$-\alpha + \frac{1}{2}\nu_A^2 - 2\nu_A r_B^{(0)} + 2(r_B^{(0)})^2$
$\frac{\nu_A(1-r_B^{(0)}}{\alpha}$	1	$-2 + \frac{\nu_A^2(r_B^{(0)}-1)^2}{\alpha} + 2(r_B^{(0)})^2$
1	0	$-\alpha - 2\nu_A r_B^{(0)} + 2(r_B^{(0)})^2$
1	1	$-\alpha - 2\nu_A r_B^{(0)} + 2(r_B^{(0)})^2 + 2(\nu_A - 1)$

Table 4.4: KKT points of $f_A^{(1)} + \delta \cdot f_A^{(2)*}$ in case 1

Proof. We write down the KKT points of the concave function $f_A^{(1)} + \delta \cdot f_A^{(2)*}$ in table 4.4. By comparing the function values of the KKT points in table 4.4, we immediately conclude that the optimal value of $f_A^{(1)} + \delta \cdot f_A^{(2)*}$ is achieved as $2(r_B^{(0)})^2$ when $(e_A^{(1)*}, r_A^{(1)*}) = (0, 0)$. So theorem 4 is true. □

From theorem 4, theorem 3, theorem 2, and theorem 1, we infer the set of equilibrium strategies for both agents of 2 rounds in table 4.5:

Round, i	Player, j	Optimal action, $a_j^{(i)*} = (e_j^{(i)*}, r_j^{(i)*})$
1	A	$a_A^{(1)*} = (0, 0)$
1	B	$a_B^{(1)*} = (0, 0)$
2	A	$a_A^{(2)*} = (0, 0)$
2	B	$a_B^{(2)*} = (0, 0)$

Table 4.5: Equilibrium strategies in case 1

And we have table 4.6 of equilibrium payoffs in case 1:

We can immediately conclude that if $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $0 < \nu_A < \frac{1}{3}$, then not only will all of the agents' reports be 0, but so will all of their effort values. This suggests that even when agents are strategic and external incentives exist, the proposed market scoring rule is incentive compatible regardless of the initial market estimate, so long as we

Round, i	Player, j	Optimal expected payoff, $E[\pi_j^{(i)*}()]$
1	A	$\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)}, a_A^{(1)}] = 2(r_B^{(0)})^2$
1	B	$\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)*} = 0, a_B^{(1)})] = 0$
2	A	$\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)*} = 0, a_A^{(2)})] = 0$
2	B	$\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)*} = 0, a_B^{(2)})] = 0$

Table 4.6: Equilibrium payoffs in case 1

set a proper range for α and ν_A. In the next section we derive the symmetric result when the cost of high effort is negative.

4.3 Case 2: High efforts bring net rewards to agents

In case 2 we assume that two agents' efforts together decide the outcome of E and the ultimate value of F but high efforts will bring negative net costs (equivalent to positive net rewards) to the agents who exert them. We further assume the payoff function of exerted effort to be $\rho_e(e) = \alpha \cdot e^2 (\alpha > 0)$:

$$\rho_e(e_A^{(i)}) = \alpha \cdot (e_A^{(i)})^2, \tag{4.10a}$$

$$\rho_e(e_B^{(i)}) = \alpha \cdot (e_B^{(i)})^2. \tag{4.10b}$$

In order to compare case 2 with case 1, we set the same range for α as $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$. In this section we set Alice's impact on deciding the likelihood of E as $\nu_A \in (0, 1)$ and have the same definition of Alice's (Bob's) belief on the likelihood of E's final occurring held in each round as that of case 1. Here we continue using the quadratic scoring rule in the prediction market. In case 2 we still have $I_A^{(i)} = r_B^{(i-1)}$ and $I_B^{(i)} = r_A^{(i)}$ as in case 1. After inserting the reward functions of exerted efforts (4.10), the belief on the likelihood of E's occurring (3.7) perceived by agents, and the reward functions in the prediction market (4.3) into the maximization equations (3.8) for Alice and Bob separately, we get the following maximization equations when $T = 2$:

For agent Alice:

$$\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)}, a_A^{(1)})] = \max_{a_A^{(1)}=(e_A^{(1)},r_A^{(1)})} \{\delta \cdot \mathbb{E}[\pi_A^{(2)*}(\tilde{r}_B^{(1)}, a_A^{(2)})] + \alpha \cdot (e_A^{(1)})^2 \tag{4.11a}$$
$$+ \frac{\nu_A(e_A^{(1)} + \tilde{e}_A^{(2)}) + (1-\nu_A)\sum_{n=1}^{2} \tilde{e}_B^{(n)}}{2} \cdot [4r_A^{(1)} - 4r_B^{(0)}]$$
$$+ [-2(r_A^{(1)})^2 + 2(r_B^{(0)})^2]\},$$

$$\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)}, a_A^{(2)})] = \max_{a_A^{(2)}=(e_A^{(2)},r_A^{(2)})} \{\alpha \cdot (e_A^{(2)})^2 \tag{4.11b}$$
$$+ \frac{1 \cdot r_B^{(1)} + \nu_A \cdot e_A^{(2)} + (1-\nu_A) \cdot \tilde{e}_B^{(2)}}{2} \cdot [4r_A^{(2)} - 4r_B^{(1)}]$$
$$+ [-2(r_A^{(2)})^2 + 2(r_B^{(1)})^2]\}.$$

For agent Bob:

$$\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)}, a_B^{(1)})] = \max_{a_B^{(1)}=(e_B^{(1)},r_B^{(1)})} \{\delta \cdot \mathbb{E}[\pi_B^{(2)*}(\tilde{r}_A^{(2)}, a_B^{(2)})] + \alpha \cdot (e_B^{(1)})^2 \tag{4.12a}$$
$$+ \frac{r_A^{(1)} - 0.5(1-\nu_A) + \nu_A \cdot \tilde{e}_A^{(2)} + (1-\nu_A) \cdot (e_B^{(1)} + \tilde{e}_B^{(2)})}{2}$$
$$\cdot [4r_B^{(1)} - 4r_A^{(1)}] + [-2(r_B^{(1)})^2 + 2(r_A^{(1)})^2]\},$$

$$\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)}, a_B^{(2)})] = \max_{a_B^{(2)}=(e_B^{(2)},r_B^{(2)})} \{\alpha \cdot (e_B^{(2)})^2 + \frac{2 \cdot r_A^{(2)} - (1-\nu_A) \cdot 0.5 + (1-\nu_A) \cdot e_B^{(2)}}{2} \tag{4.12b}$$
$$\cdot [4r_B^{(2)} - 4r_A^{(2)}] + [-2(r_B^{(2)})^2 + 2(r_A^{(2)})^2]\}.$$

When $i = 2$, there are no future rounds. Consider Bob's function of expected payoff scores in round 2 as $f_B^{(2)}$:

$$\begin{aligned} f_B^{(2)} &= \mathbb{E}[\pi_B^{(2)}(r_A^{(2)}, a_B^{(2)})] = \alpha \cdot (e_B^{(2)})^2 \\ &+ \frac{2 \cdot r_A^{(2)} - (1-\nu_A)\cdot 0.5 + (1-\nu_A)\cdot e_B^{(2)}}{2} \cdot [4r_B^{(2)} - 4r_A^{(2)}] \\ &+ [-2(r_B^{(2)})^2 + 2(r_A^{(2)})^2] \end{aligned} \tag{4.13}$$

Lemma 4. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ *and* $r_A^{(2)} \leqslant \frac{3+\nu_A}{4}$*, then* $(e_{B,4}^{(2)*}, r_{B,4}^{(2)*}) = (1, r_A^{(2)} + \frac{1}{4}(1-\nu_A))$ *and* $\mathbb{E}[\pi_{B,4}^{(2)*}(r_A^{(2)}, a_B^{(2)})] = \alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2$.

Proof. Because Bob is the last agent to take actions in round 2, there are no future action values that need to be inserted into the equation of $f_B^{(2)}$. To determine the optimal value of $f_B^{(2)}$ and the optimal action set $a_B^{(2)}$ for a given report $r_A^{(2)}$, we need to check the convexity of the function $f_B^{(2)}$ by checking the Hessian of this function:

$$\nabla^2 f_B^{(2)}(e_B^{(2)}, r_B^{(2)}) = \begin{bmatrix} \frac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} & \frac{\partial^2 f_B^{(2)}}{\partial e_B^{(2)} \partial r_B^{(2)}} \\ \frac{\partial^2 f_B^{(2)}}{\partial r_B^{(2)} \partial e_B^{(2)}} & \frac{\partial^2 f_B^{(2)}}{\partial (r_B^{(2)})^2} \end{bmatrix} = \begin{bmatrix} 2\alpha & 2(1-\nu_A) \\ 2(1-\nu_A) & -4 \end{bmatrix}$$

The first principle minor of $\nabla^2 f_B^{(2)}(e_B^{(2)}, r_B^{(2)})$ is positive for $\det \begin{bmatrix} \frac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} \end{bmatrix} = 2\alpha > 0$. the second principle minor is negative because:

$$\det \begin{bmatrix} \frac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} & \frac{\partial^2 f_B^{(2)}}{\partial e_B^{(2)} \partial r_B^{(2)}} \\ \frac{\partial^2 f_B^{(2)}}{\partial r_B^{(2)} \partial e_B^{(2)}} & \frac{\partial^2 f_B^{(2)}}{\partial (r_B^{(2)})^2} \end{bmatrix} = -8\alpha - 4(1-\nu_A)^2 < 0.$$

We find that the Hessian of $f_B^{(2)}$ is an indefinite matrix so we still need to find its KKT points and compare the corresponding values of $f_B^{(2)}$, for $f_B^{(2)*}$ must be attained in

$e_B^{(2)}$	$r_B^{(2)}$	$f_B^{(2)}$
0	$r_A^{(2)} + \frac{1}{4}(\nu_A - 1)$	$\frac{1}{8}(\nu_A^{(1)} - 1)^2$
1	$r_A^{(2)} + \frac{1}{4}(1 - \nu_A)$	$\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2$
$\frac{r_A^{(2)}(1-\nu_A)}{\alpha}$	0	$-2(r_A^{(2)})^2 + r_A^{(2)}(1-\nu_A) - \frac{(r_A^{(2)})^2(1-\nu_A)^2}{\alpha}$
0	0	$-2(r_A^{(2)})^2 + (1-\nu_A)r_A^{(2)}$
0	1	$-2(r_A^{(2)})^2 + (5-\nu_A)r_A^{(2)} + \nu_A - 3$
1	0	$-2(r_A^{(2)})^2 + (\nu_A - 1)r_A^{(2)} + \alpha$
1	1	$-2(r_A^{(2)})^2 + (\nu_A + 3)r_A^{(2)} + \alpha - \nu_A - 1$

Table 4.7: KKT points of $f_B^{(2)}$ in case 2

one of these KKT points. We write down in table 4.7 the KKT points of $f_B^{(2)}$ and the corresponding function values. However, to simplify our analysis, unfeasible KKT points are not shown in this table. We find that if $r_A^{(2)} \leqslant \frac{3+\nu_A}{4}$, $f_B^{(2)*}$ is attained as $\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2$ when $(e_B^{(2)}, r_B^{(2)}) = (1, r_A^{(2)} + \frac{1}{4}(1-\nu_A))$. So lemma 4 is true.

□

Lemma 5. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ *and* $r_A^{(2)} \geqslant \frac{3+\nu_A}{4}$, *then* $(e_{B,5}^{(2)*}, r_{B,5}^{(2)*}) = (1,1)$ *and* $\mathbb{E}[\pi_{B,5}^{(2)*}(r_A^{(2)}, a_B^{(2)})] = -2(r_A^{(2)})^2 + (\nu_A+3)r_A^{(2)} + \alpha - \nu_A - 1$.

Proof. Similar as the proof for lemma 4, if $r_A^{(2)} \geqslant \frac{3+\nu_A}{4}$, by comparing the values of $f_B^{(2)}$ of feasible KKT points in table 4.6, we find that $f_B^{(2)*}$ is attained as $f_{B,5}^{(2)*} = -2(r_A^{(2)})^2 + (\nu_A + 3)r_A^{(2)} + \alpha - \nu_A - 1$ when $(e_{B,5}^{(2)*}, r_{B,5}^{(2)*}) = (1,1)$. So lemma 5 is true. □

Given the two cases for Bob's equilibrium decisions in round 2, we may now determine Bob's round 2 decisions:

Theorem 5. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, *then*

$$(e_B^{(2)*}, r_B^{(2)*}) = \begin{cases} (1, r_A^{(2)} + \frac{1}{4}(1-\nu_A)) & \text{if } 0 \leqslant r_A^{(2)} \leqslant \frac{3+\nu_A}{4} \\ (1,1) & \text{if } \frac{3+\nu_A}{4} \leqslant r_A^{(2)} \leqslant 1 \end{cases}$$

$$f_B^{(2)*} = \begin{cases} \alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2 & \text{if } 0 \leqslant r_A^{(2)} \leqslant \frac{3+\nu_A}{4} \\ -2(r_A^{(2)})^2 + (\nu_A+3)r_A^{(2)} + \alpha - \nu_A - 1 & \text{if } \frac{3+\nu_A}{4} \leqslant r_A^{(2)} \leqslant 1 \end{cases}$$

Proof. With lemma 4 and lemma 5, theorem 5 is true. □

Then consider *Alice* in round $i = 2$ when the only future effort is $\tilde{e}_B^{(2)}$. We know from theorem 5 that $\tilde{e}_B^{(2)} = e_B^{(2)*} = 1$. After inserting $\tilde{e}_B^{(2)} = 1$ into $\mathbb{E}[\pi_A^{(2)}(r_B^{(1)}, a_A^{(2)})]$ we get:

$$\begin{aligned} f_A^{(2)} &= \mathbb{E}[\pi_A^{(2)}(r_B^{(1)}, a_A^{(2)})] = \alpha \cdot (e_A^{(2)})^2 \\ &+ \frac{(2-1) \cdot r_B^{(1)} + \nu_A \cdot e_A^{(2)} + (1-\nu_A) \cdot 1}{2} \cdot [4r_A^{(2)} - 4r_B^{(1)}] \\ &+ [-2(r_A^{(2)})^2 + 2(r_B^{(1)})^2] \end{aligned} \tag{4.14}$$

We now move to *Alice*'s optimal actions in round 2. We find that *Alice* will exert high effort and report as 1 in round 2 no matter what *Bob* reports in round 1.

Theorem 6. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$*, Alice's optimal action set in round* 2 *is* $(e_A^{(2)*}, r_A^{(2)*}) = (1, \frac{1}{2}(r_B^{(1)} + 1))$*, and the optimal expected payoff score of round* 2 *is* $\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)}, a_A^{(2)})] = \alpha + \frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2} - r_B^{(1)}$.

Proof. To find the value of $f_A^{(2)*}$, we need to check the Hessian of function $f_A^{(2)}$:

$$\nabla^2 f_A^{(2)}(e_A^{(2)}, r_A^{(2)}) = \begin{bmatrix} \dfrac{\partial^2 f_A^{(2)}}{\partial (e_A^{(2)})^2} & \dfrac{\partial^2 f_A^{(2)}}{\partial e_A^{(2)} \partial r_A^{(2)}} \\ \dfrac{\partial^2 f_A^{(2)}}{\partial r_A^{(2)} \partial e_A^{(2)}} & \dfrac{\partial^2 f_A^{(2)}}{\partial (r_A^{(2)})^2} \end{bmatrix} = \begin{bmatrix} 2\alpha & \dfrac{4\nu_A}{2} \\ \dfrac{4\nu_A}{2} & -4 \end{bmatrix}$$

We find that the Hessian of $f_A^{(2)}$ is also an indefinite matrix because the value of its first principle minor is positive as $2\alpha > 0$, and the value of its second principle minor is negative as $-8\alpha - 4(\nu_A)^2 < 0$. We write down in table 4.8 the KKT points of $f_A^{(2)}$ and the corresponding function values. We find that $f_A^{(2)}$ is maximized as $f_A^{(2)*} = \alpha + \frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2} - r_B^{(1)}$ when $(e_A^{(2)}, r_A^{(2)}) = (1, \frac{1}{2}(r_B^{(1)} + 1))$. So theorem 6 is true.

□

$e_A^{(2)}$	$r_A^{(2)}$	$f_A^{(2)}$
0	$\frac{1}{2}r_B^{(1)} + \frac{1}{2}(1-\nu_A)$	$\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}(\nu_A - 1)^2 + r_B^{(1)}(\nu_A - 1)$
1	$\frac{1}{2}(r_B^{(1)} + 1)$	$\alpha + \frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2} - r_B^{(1)}$
$\frac{\nu_A r_B^{(1)}}{\alpha}$	0	$2r_B^{(1)}(\nu_A - 1) - \frac{(\nu_A r_B^{(1)})^2}{\alpha}$
0	0	$2r_B^{(1)}(\nu_A - 1)$
0	1	$2\nu_A(r_B^{(1)} - 1)$
1	0	$-2r_B^{(1)} + \alpha$
1	1	α

Table 4.8: KKT points of $f_A^{(2)}$ in case 2

Consider Bob in round 1. From theorem 5 and theorem 6 we know that $e_A^{(2)*} = e_B^{(2)*} = 1$. Knowing $\tilde{e}_A^{(2)} = \tilde{e}_B^{(2)} = 1$, we get the expression of $f_B^{(1)}$ as:

$$
\begin{aligned}
& f_B^{(1)} = \alpha \cdot (e_B^{(1)})^2 \\
& + \frac{r_A^{(1)} - 0.5(1-\nu_A) + \nu_A \cdot 1 + (1-\nu_A) \cdot (e_B^{(1)} + 1)}{2} \cdot [4r_B^{(1)} - 4r_A^{(1)}] \\
& + [-2(r_B^{(1)})^2 + 2(r_A^{(1)})^2]
\end{aligned} \tag{4.15}
$$

Lemma 6. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, *then*

$$
(e_B^{(2)*}, r_B^{(2)*}) = \begin{cases} (1, r_A^{(2)} + \frac{1}{4}(1-\nu_A)) & \text{if } 0 \leqslant r_B^{(1)} \leqslant \frac{1}{2}(1+\nu_A) \\ (1,1) & \text{if } \frac{1}{2}(1+\nu_A) \leqslant r_B^{(1)} \leqslant 1 \end{cases}
$$

$$
f_B^{(2)*} = \begin{cases} \alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2 & \text{if } 0 \leqslant r_B^{(1)} \leqslant \frac{1}{2}(1+\nu_A) \\ -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}r_B^{(1)} + \frac{1}{2}r_B^{(1)}\nu_A - \frac{1}{2}\nu_A + \alpha & \text{if } \frac{1}{2}(1+\nu_A) \leqslant r_B^{(1)} \leqslant 1 \end{cases}
$$

Proof. According to theorem 6, we know that $r_A^{(2)*} = \frac{1}{2}(r_B^{(1)} + 1)$. Together with the statement of theorem 5, we infer lemma 6. □

Lemma 7. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ *and* $r_A^{(1)} \leqslant \frac{\nu_A+1}{2}$, *then* $(e_{B,7}^{(1)*}, r_{B,7}^{(1)*}) = (1, \frac{1}{2}r_A^{(1)} + \frac{1}{4}(3 - \nu_A))$ *and* $\mathbb{E}[\pi_{B,7}^{(1)*}(r_A^{(1)}, a_B^{(1)})] = \frac{1}{2}r_A^{(1)}(r_A^{(1)} - 3 + \nu_A) + \alpha + \frac{1}{8}\nu_A^{(2)} - \frac{3}{4}\nu_A + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2)$.

$e_B^{(1)}$	$r_B^{(1)}$	$f_B^{(1)} + \delta \cdot f_{B,4}^{(2)*}$
0	$\frac{1}{2}r_A^{(1)} + \frac{1}{4}(\nu_A + 1)$	$\frac{1}{2}r_A^{(1)}(r_A^{(1)} - 1 - \nu_A) + \frac{1}{8}(\nu_A^{(1)} + 1)^2 + \delta(\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2)$
1	$\frac{1}{2}r_1^{(2)} + \frac{1}{4}(3 - \nu_A)$	$\frac{1}{2}r_A^{(1)}(r_A^{(1)} - 3 + \nu_A) + \alpha + \frac{1}{8}\nu_A^{(2)} - \frac{3}{4}\nu_A + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2)$
$\frac{r_A^{(1)}(1-\nu_A)}{\alpha}$	0	$-r_A^{(1)}(1 + \nu_A) - \frac{(r_A^{(1)})^2(\nu_A - 1)^2}{\alpha} + \delta(\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2)$
0	0	$-r_A^{(1)}(\nu_A + 1) + \delta(\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2)$
1	0	$r_A^{(1)}(\nu_A - 3) + \alpha + \delta(\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2)$

Table 4.9: KKT points of $f_B^{(1)} + \delta \cdot f_{B,4}^{(2)*}$ in case 2 when $r_B^{(1)} \leqslant \frac{1}{2}(1 + \nu_A)$

Proof. From lemma 6 we know that $f_B^{(2)*}$ is attained as $f_{B,4}^{(2)*} = \alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2$ when $r_B^{(1)} \leqslant \frac{1}{2}(1 + \nu_A)$; and attained as $f_{B,5}^{(2)*} = -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}r_B^{(1)} + \frac{1}{2}r_B^{(1)}\nu_A - \frac{1}{2}\nu_A + \alpha$ when $r_B^{(1)} \geqslant \frac{1}{2}(1 + \nu_A)$.

We first discuss the situation where $f_B^{(2)*}$ is a constant as $f_{B,4}^{(2)*} = \alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2$ when $r_B^{(1)} \leqslant \frac{1}{2}(1 + \nu_A)$. Similarly we denote the KKT points of $f_B^{(1)} + \delta \cdot f_{B,4}^{(2)*}$ in table 4.9. We find that when $r_A^{(1)} \leqslant \frac{1+\nu_A}{2}$, $f_B^{(1)} + \delta \cdot f_{B,4}^{(2)*}$ is maximized as $\frac{1}{2}r_A^{(1)}(r_A^{(1)} - 3 + \nu_A) + \alpha + \frac{1}{8}\nu_A^{(2)} - \frac{3}{4}\nu_A + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2)$, which we denote as $\mathbb{E}[\pi_{B,7}^{(1)}(r_A^{(1)}, a_B^{(1)})]$, when $(e_B^{(1)}, r_B^{(1)}) = (1, \frac{1}{2}r_A^{(1)} + \frac{1}{4}(3 - \nu_A))$.

We secondly discuss the situation where $f_{B,5}^{(2)*}$ is a function of $r_B^{(1)}$ as $f_{B,5}^{(2)*} = -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}r_B^{(1)} + \frac{1}{2}r_B^{(1)}\nu_A - \frac{1}{2}\nu_A + \alpha$ when $r_B^{(1)} \geqslant \frac{1}{2}(1 + \nu_A)$. Using the same method, we find that $f_B^{(1)} + \delta \cdot f_{B,5}^{(2)*}$ is maximized as $(1 - \nu_A)(1 - r_A^{(1)}) + \alpha(1 + \delta)$, which we denote as $\mathbb{E}[\pi_{B,8}^{(1)}(r_A^{(1)}, a_B^{(1)})]$, when $(e_B^{(1)}, r_B^{(1)}) = (1, 1)$. Through calculation we find that if $r_A^{(1)} \leqslant \frac{\nu_A+1}{2}$, $\mathbb{E}[\pi_{B,7}^{(1)}(r_A^{(1)}, a_B^{(1)})]$ is larger than $\mathbb{E}[\pi_{B,8}^{(1)}(r_A^{(1)}, a_B^{(1)})]$, so $\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)}, a_B^{(1)})]$ is achieved as $\mathbb{E}[\pi_{B,7}^{(1)}(r_A^{(1)}, a_B^{(1)})] = \frac{1}{2}r_A^{(1)}(r_A^{(1)} - 3 + \nu_A) + \alpha + \frac{1}{8}\nu_A^{(2)} - \frac{3}{4}\nu_A + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2)$ when $(e_{B,7}^{(1)*}, r_{B,7}^{(1)*}) = (1, \frac{1}{2}r_A^{(1)} + \frac{1}{4}(3 - \nu_A))$. So lemma 7 is true. $\square$

Lemma 8. *If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $r_A^{(1)} \geqslant \frac{\nu_A+1}{2}$, then $(e_{B,8}^{(1)*}, r_{B,8}^{(1)*}) = (1, 1)$ and $\mathbb{E}[\pi_{B,8}^{(1)*}(r_A^{(2)}, a_B^{(2)})] = (1 - \nu_A)(1 - r_A^{(1)}) + \alpha(1 + \delta) + \delta\nu_A(\frac{1}{2} - \nu_A)$.*

Proof. Similar as the proof for lemma 7, we find that if $r_A^{(1)} \geqslant \frac{\nu_A+1}{2}$, then $\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)}, a_B^{(1)})]$ is achieved as $\mathbb{E}[\pi_{B,8}^{(1)}(r_A^{(2)}, a_B^{(2)})] = (1-\nu_A)(1-r_A^{(1)}) + \alpha(1+\delta) + \delta\nu_A(\frac{1}{2} - \nu_A)$ when $(e_{B,8}^{(1)*}, r_{B,8}^{(1)*}) = (1, 1)$. So lemma 8 is true. $\square$

Given the two cases for Bob's equilibrium decisions in round 1, we may now determine Bob's round 1 decisions:

Theorem 7. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$*, then*

$$(e_B^{(1)*}, r_B^{(1)*}) = \begin{cases} (1, \frac{1}{2}r_A^{(1)} + \frac{1}{4}(3-\nu_A)) & \textit{if } 0 \leqslant r_A^{(1)} \leqslant \frac{\nu_A+1}{2} \\ (1,1) & \textit{if } \frac{\nu_A+1}{2} \leqslant r_A^{(1)} \leqslant 1 \end{cases}$$

$$f_B^{(1)*} = \begin{cases} \frac{1}{2}r_A^{(1)}(r_A^{(1)} - 3 + \nu_A) + \alpha + \frac{1}{8}\nu_A^{(2)} - \frac{3}{4}\nu_A + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A^{(1)} - 1)^2) & \textit{if } 0 \leqslant r_A^{(1)} \leqslant \frac{\nu_A+1}{2} \\ (1-\nu_A)(1-r_A^{(1)}) + \alpha(1+\delta) + \delta\nu_A(\frac{1}{2} - \nu_A) & \textit{if } \frac{\nu_A+1}{2} \leqslant r_A^{(1)} \leqslant 1 \end{cases}$$

Proof. With lemma 7 and lemma 8, theorem 7 is true. □

Finally, consider Alice in round 1. We have already proved in theorem 5, theorem 6 and theorem 7 that $e_B^{(1)*} = e_A^{(2)*} = e_B^{(2)*} = 1$ and $\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)}, a_A^{(2)})] = \alpha + \frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2} - r_B^{(1)}$. After inserting $\tilde{e}_B^{(1)} = \tilde{e}_A^{(2)} = \tilde{e}_B^{(2)} = 1$ and $\mathbb{E}[\pi_A^{(2)*}(\tilde{r}_B^{(1)}, a_A^{(2)})]$ into $\mathbb{E}[\pi_A^{(1)}(r_B^{(0)}, a_A^{(1)})]$, we get the function of $f_A^{(1)} + \delta f_A^{(2)*}$ as:

$$\begin{aligned} & f_A^{(1)} + \delta f_A^{(2)*} = \mathbb{E}[\pi_A^{(1)}(r_B^{(1)}, a_A^{(2)})] = \delta(\alpha + \frac{1}{2}(1)^2 + \frac{1}{2} - 1) \\ & + \alpha \cdot (e_A^{(1)})^2 + \frac{\nu_A(e_A^{(1)} + 1) + (1-\nu_A)(1+1)}{2} \cdot [4r_A^{(1)} - 4r_B^{(0)}] \\ & + [-2(r_A^{(1)})^2 + 2(r_B^{(0)})^2] \end{aligned} \tag{4.16}$$

We now move to Alice's optimal actions in the first round. We find that Alice will exert high effort and report as 1 in round 1 no matter what initial market estimate is.

Theorem 8. *If* $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$*, then Alice's optimal action set in round 1 is* $(e_A^{(1)*}, r_A^{(1)*}) = (1,1)$*, and her expected total payoff score from this round to the final round is* $\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)}, a_A^{(1)})] = 2(r_B^{(0)} - 1)^2 + (1+\delta)\alpha$.

Proof. To find the optimal value of $\mathbb{E}[\pi_A^{(1)}(r_B^{(0)}, a_A^{(1)})]$, we check the Hessian of function $f_A^{(1)} + \delta f_A^{(2)*}$ and find it is also an indefinite matrix. Still we need to denote the KKT points of $f_A^{(1)} + \delta f_A^{(2)*}$ in table 4.10. We infer from the table that the optimal value of $f_A^{(1)} + \delta \cdot f_A^{(2)*}$ is achieved as $2(r_B^{(0)} - 1)^2 + (1+\delta)\alpha$ when $(e_A^{(1)*}, r_A^{(1)*}) = (1,1)$. So theorem 8 is true. □

$e_A^{(1)}$	$r_A^{(1)}$	$f_A^{(1)} + \delta f_A^{(2)*}$
0	$1 - \frac{1}{2}r_B^{(0)}$	$\frac{1}{2}(2r_B^{(0)} + \nu_A)^2 + 2(1-\nu_A) - 4r_B^{(0)} + \delta\alpha$
1	1	$2(r_B^{(0)} - 1)^2 + (1+\delta)\alpha$
$\frac{\nu_A r_B^{(0)}}{\alpha}$	0	$2r_B^{(0)}(r_B^{(0)} + \nu_A - 2) - (\nu_A r_B^{(0)})^2 + \delta\alpha$
0	0	$2r_B^{(0)}(r_B^{(0)} + \nu_A - 2) + \delta\alpha$
0	1	$2r_B^{(0)}(r_B^{(0)} + \nu_A - 2) + 2(1-\nu_A) + \delta\alpha$

Table 4.10: KKT points of $f_A^{(1)} + \delta f_A^{(2)*}$ in case 2

From theorem 8, theorem 7, theorem 6, and theorem 5, we infer the set of optimal equilibrium strategies for both agents of 2 rounds in table 4.11:

Round, i	Player, j	Optimal action, $a_j^{(i)*} = (e_j^{(i)*}, r_j^{(i)*})$
1	A	$a_A^{(1)*} = (1,1)$
1	B	$a_B^{(1)*} = (1,1)$
2	A	$a_A^{(2)*} = (1,1)$
2	B	$a_B^{(2)*} = (1,1)$

Table 4.11: Equilibrium strategies in case 2

And we also have the equilibrium payoffs in table 4.12:

Round, i	Player, j	Optimal expected payoff, $E[\pi_j^{(i)*}()]$
1	A	$\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)}, a_A^{(1)}] = 2(r_B^{(0)} - 1)^2 + (1+\delta)\alpha$
1	B	$\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)*}, a_B^{(1)})] = \alpha(1+\delta) + \delta\nu_A(\frac{1}{2} - \nu_A)$
2	A	$\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)*}, a_A^{(2)})] = \alpha$
2	B	$\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)*}, a_B^{(2)})] = \alpha$

Table 4.12: Equilibrium payoffs in case 2

We can immediately conclude that if $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, then not only will all of the agents' reports be 1, but so will all of their effort values.

The results in sections 4.2 and 4.3 generalize to the 4 round setting, but it is still unclear if they generalize for any value of T. The analysis required for the generalized 4 round setting is similar to the 2 round setting, and as such we omit it from this document.

4.4 Case 3: External incentives do not exist

In case 3 we assume that two agents' efforts together decide the outcome of E and the ultimate value of F but their efforts will bring no payoffs to the agents who exert them ($\rho_e(e) = 0$). Previous work has shown that when external incentives do not exist outside of the prediction market, non-myopic agents have incentives to lie in the prediction market. We show in case 3 that this result also applies for our model. Without losing generality, we assume $\nu_A = \frac{1}{2}$ (Alice and Bob have the same impact on deciding the occurring probability of E) and $r_B^{(0)} = \frac{3}{4}$ (the initial market estimate is set to be $\frac{3}{4}$). Under the above assumptions we then have Alice and Bob's bellman equations in the 2-round setting as:
For agent Alice:

$$\begin{aligned}\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)} = \frac{3}{4}, a_A^{(1)})] = \max_{a_A^{(1)}=(e_A^{(1)}, r_A^{(1)})} &\{\delta \cdot \mathbb{E}[\pi_A^{(2)*}(\tilde{r}_B^{(1)}, a_A^{(2)})] \\ &+ \frac{\nu_A(e_A^{(1)} + \tilde{e}_A^{(2)}) + (1-\nu_A)(\tilde{e}_B^{(1)} + \tilde{e}_B^{(2)})}{2} \\ &\cdot [4r_A^{(1)} - 4r_B^{(0)}] + [-2(r_A^{(1)})^2 + 2(r_B^{(0)})^2]\},\end{aligned} \tag{4.17a}$$

$$\begin{aligned}\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)}, a_A^{(2)})] = \max_{a_A^{(2)}=(e_A^{(2)}, r_A^{(2)})} &\{\frac{r_B^{(1)} + \nu_A \cdot \tilde{e}_A^{(2)} + (1-\nu_A) \cdot \tilde{e}_B^{(2)}}{2} \\ &\cdot [4r_A^{(2)} - 4r_B^{(1)}] + [-2(r_A^{(2)})^2 + 2(r_B^{(1)})^2]\}.\end{aligned} \tag{4.17b}$$

For agent Bob:

$$\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)}, a_B^{(1)})] = \max_{a_B^{(1)}=(e_B^{(1)}, r_B^{(1)})} \{\delta \cdot \mathbb{E}[\pi_B^{(2)*}(\tilde{r}_A^{(2)}, a_B^{(2)})] \tag{4.18a}$$

$$+ \frac{r_A^{(1)} - 0.5(1-\nu_A) + \nu_A \cdot \tilde{e}_A^{(2)} + (1-\nu_A) \cdot (e_B^{(1)} + \tilde{e}_B^{(2)})}{2} \tag{4.18b}$$

$$\cdot [4r_B^{(1)} - 4r_A^{(1)}] + [-2(r_B^{(1)})^2 + 2(r_A^{(1)})^2]\},$$

$$\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)}, a_B^{(2)})] = \max_{a_B^{(2)}=(e_B^{(2)}, r_B^{(2)})} \{\frac{2r_A^{(2)} - (1-\nu_A) \cdot 0.5 + (1-\nu_A) \cdot e_B^{(2)}}{2} \tag{4.18c}$$

$$\cdot [4r_B^{(2)} - 4r_A^{(2)}] + [-2(r_B^{(2)})^2 + 2(r_A^{(2)})^2]\}.$$

Using backwards induction as has been explained in detail for case 1 and case 2, we infer the set of equilibrium strategies for both agents of 2 rounds in table 4.13:

Round, i	Player, j	Optimal action, $a_j^{(i)*} = (e_j^{(i)*}, r_j^{(i)*})$
1	A	$a_A^{(1)*} = (0, \frac{1}{2})$
1	B	$a_B^{(1)*} = (1, \frac{7}{8})$
2	A	$a_A^{(2)*} = (0, \frac{11}{16})$
2	B	$a_B^{(2)*} = (1, \frac{13}{16})$

Table 4.13: Equilibrium strategies in case 3

And we denote the equilibrium payoffs in table 4.14:

So in case 3 where external incentives do not exist, we observe directly from the optimal actions that agents do not necessarily exert high efforts to E and they also do not report truthfully in the prediction market.

Round, i	Player, j	Optimal expected payoff, $E[\pi_j^{(i)*}()]$
1	A	$\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)} = \frac{3}{4}, a_A^{(1)})] = \frac{1}{8}$
1	B	$\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)*} = \frac{1}{2}, a_B^{(1)})] = \frac{1}{32}$
2	A	$\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)*} = \frac{7}{8}, a_A^{(2)})] = \frac{9}{128} + \frac{1}{8}\delta$
2	B	$\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)*} = \frac{11}{16}, a_B^{(2)})] = \frac{1}{32}(1 + \delta)$

Table 4.14: Equilibrium payoffs in case 3

Chapter 5

Discussion and Conclusion

In this work we show that with costly actions that determine the outcome of a prediction market traded event, the promised costs or rewards of external actions actually have a great influence on changing agents' behaviour than potential prediction market benefits. We find that in a 2-round setting, when agents are forward-looking and want to maximize their total expected payoffs gained from exerting efforts towards realizing the traded event as well as from trading in the prediction market, asymmetric action costs results in agents avoiding taking the costliest actions. However if a market maker rewards her preferred action the most, agents will take the desired action. We find that the value of net reward to each desirable action should be larger than a certain amount, which is determined by the value of ν_A in our 2-round setting. Perhaps unexpectedly, even though agents' actions are influenced by external costs, agents will always report truthfully during each prediction market round, so long as we set a proper range of scores given to the related actions. This observation, shows that our proper scoring rule is incentive-compatible even with external incentives and costly actions.

In the past, decision and policy makers have expressed concern that the existence of a prediction market will inspire undesirable actions for the agents who trade in the prediction market and also have a direct impact on deciding the likelihood of the predicted event. However, previous research does not take into account the potential payoffs (could be net costs or net rewards) to the agents who exert such actions. We base our research on the assumption that forward-looking agents wish to maximize their total expected payoffs not only from the prediction market but also from their actions related to the traded event. With our finding, a market maker who cares about the result of such predicted event can in fact inspire agents' desirable actions by rewarding preferred actions. A market maker can also gain true information about agents' actions from agents' reports in the prediction market for the market reward mechanism is incentive-compatible.

In this work we set a range for expected payoff scores of exerted efforts, whose absolute value should be larger than a certain value in each round, in order to inspire certain actions and truthful reports. We do not discuss whether the prediction market will still be incentive-compatible or not when this range is violated. However, we do show that when efforts are not costly actions (payoff scores are 0 for exerted efforts), agents will bluff in the prediction market. More importantly, in this work we set the number of rounds (T) to be 2, which guarantees our assumption that agents are forward-looking. However, we do have the need to extend our model to a finite round setting with T being a large number to see whether our conclusion still holds if agents take actions and trade in the long run.

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APPENDICES

α constant that represents amount of cost (reward) for efforts, we assume $0 < \alpha \leqslant 2$.

i number of round, $i = 1, \ldots, T$.

$r_A^{(i)}$ the reported prediction probability of E's occurring in round i if the report has already been made by Alice, $r_A^{(i)} \in [0,1], \overline{r}_A^{(i)} = 1 - r_A^{(i)}$.

$r_B^{(i)}$ the reported prediction probability of E's occurring in round i if the report has already been made by Bob, $r_B^{(i)} \in [0,1], \overline{r}_B^{(i)} = 1 - r_B^{(i)}$.

$\tilde{r}_A^{(i)}$ the reported prediction probability of E's occurring in round i if the report has not yet been made by Alice.

$\tilde{r}_B^{(i)}$ the reported prediction probability of E's occurring in round i if the report has not yet been made by Bob.

$r_B^{(0)}$ the beginning market estimate set in the prediction market by the market maker.

$e_A^{(i)}$ value of Alice's effort in round i if the effort action has already happened, $e_A^{(1)} = 1$ if Alice devotes high effort in round i, otherwise 0, $\overline{e}_A^{(i)} = 1 - e_A^{(i)}$.

$e_B^{(i)}$ value of Bob's effort in round i if the effort action has already happened, $e_B^{(1)} = 1$ if Bob devotes high effort in round i, otherwise 0, $\overline{e}_B^{(i)} = 1 - e_B^{(i)}$.

$\tilde{e}_A^{(i)}$ value of Alice's future effort in round i if the effort action has not yet taken place.

$\tilde{e}_B^{(i)}$ value of Bob's future effort in round i if the effort action has not yet taken place.

$\mathbb{E}_A(\tilde{e}_B^{(i)})$ the expectation of Bob's future effort value of round i perceived by Alice when she reports in round i.

$a_A^{(i)}$ the action set that Alice takes in round i, $a_A^{(i)} = (e_A^{(i)}, r_A^{(i)})$.

$a_A^{(i)*}$ the optimal action set that Alice takes in round i, $a_A^{(i)} = (e_A^{(i)*}, r_A^{(i)*})$.

a_A the policy that Alice takes for all rounds, $a_A := (a_A^{(1)}, ..., a_A^{(T)})$.

a_A^* the optimal policy for Alice, $a_A^* := (a_A^{(1)*}, ..., a_A^{(T)*})$.

$\mathcal{A}_A^{(i)}$ the action set that Alice has in round i, $a_A^{(i)} \in \mathcal{A}_A^{(i)}$.

$\mathcal{A}_A$ the whole action set that Alice has, $\mathcal{A}_A = \mathcal{A}_A^{(1)} \times ... \times \mathcal{A}_A^{(i)}$ and $a_A \in \mathcal{A}_A$.

$a_B^{(i)}$ the action set that Bob takes in round i, $a_B^{(i)} = (e_B^{(i)}, r_B^{(i)})$.

$a_B^{(i)*}$ the optimal action set that Bob takes in round i, $a_B^{(i)*} = (e_B^{(i)*}, r_B^{(i)*})$.

a_B the policy that Bob takes for all rounds, $a_B := (a_B^{(1)}, \ldots, a_B^{(T)})$.

a_B^* the optimal policy for Bob, $a_B^* := (a_B^{(1)*}, \ldots, a_B^{(T)*})$.

$\mathcal{A}_B^{(i)}$ the action set that Bob has in round i, $a_B^{(i)} \in \mathcal{A}_B^{(i)}$.

$\mathcal{A}_B$ the whole action set that Bob has, $\mathcal{A}_B = \mathcal{A}_B^{(1)} \times ... \times \mathcal{A}_B^{(i)}$ and $a_B \in \mathcal{A}_B$.

$h_A^{(i)}$ number of high efforts Alice has exerted in total from round 1 to round i, $h_A^{(0)} = 0$.

$h_B^{(i)}$ number of high efforts Bob has exerted in total from round 1 to round i, $h_B^{(0)} = 0$.

ν_A Alice's impact on deciding the likelihood of project E's occurring, $0 < \nu_A < 1$.

$p_A^{(i)}$ Alice's belief on the likelihood of E's occurring in round i after she observes Bob's most recent report and before she takes any actions in that round, taking into account actions in future rounds.

$p_B^{(i)}$ Bob's belief on the likelihood of E's occurring in round i after he observes Alice's most recent report and before he takes any actions in that round, taking into account actions in future rounds.

$\rho_e^{(i)}(\cdot)$ function of payoff scores of efforts devoted in round i.

$\rho_s^{(i)}(\cdot)$ function of payoff scores earned from moving the probability in prediction market in round i if E occurs (succeeds) after T rounds.

$\rho_f^{(i)}(\cdot)$ function of payoff scores earned from moving the probability in prediction market in round i if E does not occur (fails) after T rounds.

δ discounting factor on the future profits, $0 < \delta < 1$.

$\pi_A^{(i)}$ payoff scores earned by Alice from round i to round T, including scores earned in this round and discounting future scores.

$\pi_B^{(i)}$ payoff scores earned by Bob from round i to round T, including scores earned in this round and discounting future scores.

$I_A^{(i)}$ the system state that Alice has in round i, after she observes Bob's most recent report in the prediction market, and before she takes any actions in that round.

$\mathbb{E}[\pi_A^{(i)}(I_A^{(i)}, a_A^{(i)})$] the expected value of $\pi_A^{(i)}$ if the current state is $I_A^{(i)}$ and the current round's action set is $a_A^{(i)}$.

$\mathbb{E}[\pi_A^{(i)*}(I_A^{(i)}, a_A^{(i)})$] the optimal expected value of $\pi_A^{(i)}$ given the current state $I_B^{(i)}$.

$I_B^{(i)}$ the system state that Bob has in round i, after he observes Alice's most recent report in the prediction market, and before he takes any actions in that round.

$\mathbb{E}[\pi_B^{(i)}(I_B^{(i)}, a_B^{(i)})$] the expected value of $\pi_B^{(i)}$ given if the current state is $I_B^{(i)}$ and the current round's action set is $a_B^{(i)}$.

$\mathbb{E}[\pi_B^{(i)*}(I_B^{(i)}, a_B^{(i)})$] the optimal expected value of $\pi_B^{(i)}$ given the current state $I_B^{(i)}$.