Condition-based Inspection and Maintenance of Medical Devices

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Inspection and Maintenance of medical devices are essential for modern health services, but the low availability of devices or unnecessary maintenance can cause major problems. A proper maintenance program can significantly reduce operational costs and increase device availability. For any maintenance program, two questions arise: 1) What kinds of devices should be included? and 2) How and when should they be inspected and maintained? This thesis proposes methods to solve those two problems.

For the first question, numerous classification and prioritization models have been suggested to evaluate medical devices, but most are empirical scoring systems, which can not be widely used. To build a generalized scoring system, we propose a risk level classification model. More specifically, we select three important risk factors (Equipment function, Location of use and Frequency of use), then use provided data to find the relationship between risk factors and risk levels. Four different classification models (Linear regression, Logistic regression, Classification tree and Random forest) are used to analyze the problem, and all of them are effective.

For the second question, some inspection and maintenance models have been developed and widely used to assure the performance of medical devices. However, those models are restricted to a few specific kind of problems. In contrast, our model provide a more comprehensive response to current maintenance problems in the healthcare industry, by introducing a condition-based multi-component inspection and maintenance model. We first present a parameter estimation method to predict the deterioration rate of a system. We use provided data and expectation-maximization algorithm to estimate the transition matrix of system conditions. Then, we use Markov decision processes to solve the decision model, which consists of two decisions: the next inspection time and whether to repair the devices. The inspection interval is non-periodic in our model, and this flexibility of non-periodic inspection model can avoid unnecessary inspections. We use relative value iteration to find the optimal inspection and maintenance strategies and the long-run average cost. Changing the parameter of cost and the structure of the system clarified the influence of these parameters. Our model achieves lower minimal average costs for complex systems than previous periodic inspection models.

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Chapter 1

Introduction

Medical devices have been always an important part of health services, but their management has become increasingly complex. One issue hospitals and healthcare organizations face is to ensure the proper performance of their medical devices. To achieve this objective, a number of management strategies have been used, *i.e.* inspection and preventive maintenance. Those strategies must be improved continuously if they are to keep up with the increasing expectations and complexity of healthcare organizations.

1.1 Motivation of the Problem

Medical devices play an increasingly important role in modern medicine, especially with the development of technologically advanced devices and increasing medical research activities. Although management strategies have been widely used in hospitals and healthcare organizations, most organizations do not benefit from these strategies, which tend to include all medical devices and follow the manufacturers' recommendations for inspection and preventive maintenance. In fact, most of the inspection and maintenance is unnecessary [25].

The Winnipeg Regional Health Authority (WRHA) is one of the largest and most diverse health regions in Canada. It operates or funds over 200 health service facilities and programs, which manage more than 3000 beds and approximately 10,000 different types of medical devices. These devices are used in all aspects of health services [46]. A comprehensive study indicates that, on average, each hospital has acquired 15-20 pieces of medical devices for each staffed bed, and the annual medical-device maintenance and management cost is approximately 1% of the total hospital budget. Thus, for a 500-bed hospital, the total maintenance cost could be around $5 million per year [25]. Finding the optimal maintenance strategy for minimizing the cost is a major focus of this thesis.

1.2 Problems of Interest

Performance Assurance (PA) program is a medical device risk management system provided by WRHA [47, 14], which is a comprehensive framework for management of medical devices. In WRHA's framework, risk classification system is the foundation for its regular inspection activity, but the lack of a generalized classification system is a problem. Moreover, the inspection and maintenance activities are followed by the manufactures' recommendation, which may not be the best strategy for the operation. WRHA wants to develop an efficient program, which could be more suitable for the organization.

In general, an efficient medical device management program should be able to solve two problems:

1. Making decisions about whether a device should be included;

2. How should the devices be inspected and maintained.

With limited resources and time, we need to decide whether a device should be included into a maintenance program. Normally, healthcare organizations have a wide variety of medical devices: some of them provide primary health service to patients and cause serious damage if they failed (*e.g.* defibrillator); while some devices provide ancillary service to public (*e.g.* thermometer). Moreover, we need to know what kinds of factors should be considered to make the decision. Many attributes affect the risk levels of medical devices (*e.g.* function, cost, age *et al.*), while the situation of the operating environment should be considered as well. For example, the maintenance strategies could be different for defibrillators used in emergency departments and those used in general patient care areas or clinics.

The second problem is an optimization problem, which is based on the result of the first problem. The medical devices would be corrupted while using it. Proper inspection and maintenance can ensure the performance of the devices. Obviously, Frequent inspection and maintenance could ensure the safety of the devices, but would affect the operation of the health services and cost numerous money and manpower. Base on the motivation outlined in section 1.1. Unnecessary inspection and maintenance waste hospitals' budget. So our objective is to reduce the percentage of unnecessary inspection and maintenance, and to minimize the maintenance cost by using the optimal inspection and maintenance strategies.

1.3 Basic Ideas

To solve the problems raised in section 1.2, we need to identify the subject of the research problem first. In this section, it will be clarified that the subject is a group of medical devices that provide health service as a system rather than a single device. The operation of the system requires the combination and cooperation of different devices, so the optimal inspection and maintenance of a single device is not enough to ensure the optimal performance of a system.

Since the subject of our problem is a system with multiple devices, it could be viewed as a multi-component system. Cho and Parlar [9] define multi-component maintenance as follows: Multi-component maintenance models are concerned with optimal maintenance policies for a system consisting of several units of machines or many pieces of equipment, which may or may not depend on each other. The dependence between components affects the complexity of the systems. According to Thomas's paper [41], the dependence between components in the systems has three different types: economic, structural and probabilistic dependence. In our research, we focus on economic dependence, a typical system with economic dependence is *k-out-of-n* systems, which has wide applications in industry. A *k-out-of-n* system works if at least k components works. When we evaluate the risk level of the medical devices in a hospital, we need to consider the backup ratio of the devices, which could be viewed as a multi-component system. There are several machines for each type in hospital (*e.g.* Defibrillator, Infusion pump and Exerciser treadmill), they have to make sure that there are k minimal function machines available in each type, which implies the *k-out-of-n* systems. Meanwhile, the combining inspection and maintenance actions between the machines can also yield a lower total cost than inspecting or maintaining a single machine, which implies the economic dependence between components.

For the risk level classification problem. We will review existing literature and use provided data to select several essential factors. Then the relationship between the factors and the risk level will be found out by using four different data analysis methods (*e.g.* Linear regression, Logistic regression, Classification tree and Random forest). A generalized classification system is obtained after the analysis and comparison.

For the inspection and maintenance optimization problem. The maintenance strategy in our problem is condition-based maintenance (CBM), which has been widely accepted in recent decades. CBM is an efficient strategy, as it reduces unnecessary maintenance actions and eliminate the risks associated with preventive maintenance actions [1]. The research of Olde Keizer et al., [27] is the first paper to consider CBM for multi-unit systems with economic dependencies and redundancy (*k-out-of-n* systems). But they did not consider the impact of inspection frequency. Inspection plays an important role in CBM model, the inspection cost and frequency would be considered in our model, and the inspection frequency becomes a decision variable. We implement a suitable method with the provided data to estimate the deterioration rate. Then, we applied Markov decision processes model for CBM model considering the deterioration rate and estimated costs. The optimal inspection and maintenance strategy is obtained by solving the model.

1.4 Scope of Research

Based on the motivation in Section 1.1 and basic ideas in Section 1.3, this thesis focuses on management of medical devices. Two main problems will be addressed.

1. *The classification of medical devices.* It is important to evaluate the medical devices before looking for the optimal inspection and maintenance policy. Based on several key risk factors, by using linear regression, logistic regression, classification tree and random forest, we classify the medical devices into three different risk levels. Healthcare organizations and hospitals could make decisions on whether include a medical devices in their maintenance program based on the risk level.

2. *Inspection and maintenance of medical devices.* Once the evaluation of medical devices has been finished, the optimal inspection and maintenance policy would be discussed. The goal is to find out the minimal cost policy. Different decisions will generate different costs and affect the condition of the devices, so we applied Markov Decision Processes to solve this problem. The final result provides the optimal policy and the long-run minimal average cost of the policy.

1.5 Outline of the Thesis

This thesis consists of five chapters. Following this introduction, Chapter 2 is a literature review on Risk Level Classification Model and Inspection and Maintenance Model.

In Chapter 3, the classification models are introduced and built in this part. Four different models (Linear Regression, Logistic Regression, Classification Tree and Random Forest) are discussed and compared. We use all four models to evaluate 10 different kinds of devices.

Chapter 4 is the second part of this thesis, the inspection and maintenance model is constructed. Specifically, a parameter estimation method is introduced at the beginning of the inspection and maintenance model, followed by a decision model, which is a Markov Decision Processes model. The optimal solutions are obtained and analyzed. A number of numerical examples and a case study are presented at the end of this chapter to show some interesting patterns of the optimal decisions.

The last chapter consists of a summary of the main results of this thesis and a brief discussion on some problems for future research.

Chapter 2

Literature Review

The problems discussed in Chapter 1 include medical devices classification and optimal inspection and maintenance policy. Thus, in this chapter, we assess the status of research on these two problems. The review of the literature is divided into two main categories, which are classification of medical devices and device inspection and maintenance.

First, we review the medical devices classification topic. The focus of the review will be on risk-based classification, which has become an essential task for healthcare organizations. The purpose of this part is to know existing risk factors and scoring system. A brief review of statistical classification methods is presented at the end of this subsection. Inspired by existing literature and some successfully application, we developed our own scoring system in Section 3.

Second, the topics related to maintenance will be reviewed. Both the maintenance model and the decision-making model are discussed. The focus of the review will be on condition-based maintenance (CBM) model. For the decision-making model, a brief review on Markov decision processes will is presented.

2.1 Classification Models

2.1.1 Risk Factors

Joint Commission on Accreditation of Healthcare Organizations (JCAHO) in US has made a medical standard for hospitals to decide which medical device should be involved in Medical Equipment Management Program (MEMP) [24].They consider three factors, which are maintenance requirements of medical device, equipment function and physical risk. These three factors were originally introduced by Fennigkoh and Smith [12] in 1989 and then be widely accepted in a lot of research works [15, 39, 40]. They classified medical devices by scoring the devices equipment number (EN) as follows,

$$EN = Function + Risk + Required\ Maintenance$$

For each factor, they have a criteria. If the devices EM number is greater than 12, then it should be involved in MEMP. Although this method has been widely accepted and used, the method to evaluate a device is not appropriate, because it calculate an arithmetic average over three factor without considering the weights of them. JCAHO also explained that hospitals may use different strategies for different items as appropriate. Although the three factors have been widely accepted, an increasing number of new factors have been considered as indispensable factors to make medical equipment managerial decisions. Wang and Levenson [45] added another factor called equipment utilization rate to JCAHOs system, and their scoring equation is called Equipment Management Rating (EMR),

$$EMR = [Utilization\ rate \times (mission\ critical + 2 \times maintenance)] + 2 \times risk$$

The American Society for Healthcare Engineering(ASHE) [26] presented a classification method according to the five factors: equipment function (E),clinical application (A), preventive maintenance requirements (P),probability of equipment failure (F), and environmental use (U). The total score (T) is calculated by:

$$T = E + A + \left(\frac{P + F + U}{3}\right)$$

ASHE considered more factors and their related weights, especially, the environment use factor distinguish the risk level of same type of device with different locations. But, the scoring system is based on a sole or a few experts opinion, its hard to be used generally. Obviously, the classification of medical devices is a multi-criteria decision-making (MCDM) problem, different experts opinion should be considered rather than considering only one sole experts evaluation. Taghipour et al. [39] used a MCDM scoring system for medical devices classification based on six factors: risk, age, equipment function, mission criticality, recalls and hazard alerts and maintenance requirements. They used analytical hierarchy process to evaluate the devices. Furthermore, Jamshidi et al. [26] introduced a maintenance framework for medical devices. The main idea of this research is a novel fuzzy multi-criteria decision making approach to medical devices classification problem.

For more risk factors, we refer to following literature: location of equipment use [16], age of equipment [39, 26], meantime between repair [28, 26], utilization [39, 40], available of substitute [16, 40, 26]. Please see Appendix A for the summary of risk factors.

2.1.2 Classification Methods

According to the review above, the existing methods of medical devices evaluation are very simple and most of them are based on experts' empirical opinions. Meanwhile, many statistical classification methods from data science area have been applied in many practical problems (*e.g.* Pattern recognition, Credit scoring and Medical image analysis). The advantage of this kind of method is explicit expression and automation. By using historical data, we can have a explicit function to evaluate the devices and the evaluation procedure could be automatically finished by computer. In healthcare area, these methods have been applied to solve many issues. Kononenko [29] reviewed the application of data analysis in medical diagnosis problem, this paper emphasized on three classification methods, naive Bayesian, Neural networks and Decision trees. A comparison of the performance in classifying patients between logistic regression and decision tree is discussed in literature [31].

However, there are few paper to discuss the application of statistical classification methods in medical device evaluation problem. We briefly review several statistical classification and evaluation methods as follows.

1. Linear Regression. The regression analysis deals with finding the best linear relationship between Y and x, quantifying the strength of that relationship, and using methods that allow for prediction of the response values Y given values of the regressor x [44]. Linear regression has been rigorously studied and widely used [48]. This is because, comparing with non-linear model, linear model is easier to to fit and the parameters are easier to determine.

2. Logistic Regression. Logistic regression is an appropriate regression analysis to classify a categorical outcome using a linear function of independent variables. The idea was proposed by statistician David Cox [10]. If the number of outcomes is greater than two, the problem should be analyzed in multinomial logistic regression, and if the categories are ordered, ordered logistic regression should be applied [43].

3. Classification Tree. Classification tree is a method commonly used in data mining, classification tree model can predict the value of a target variable based on several input variables [34].

4. Random Forest. Random forest is an ensemble machine learning method for classification. Based on the idea of classification tree, it constructs a multitude of classification trees at training time and outputting the class that is the mode of the classes [18].

These four statistical classification methods have been widely applied in many fields. In our research, we are going to use all of them to solve our problem in Chapter 3.

2.2 Inspection and Maintenance Model

The literature on the inspection and maintenance optimization model is abundant. The focus of these papers are changing as the times change. Traditionally, maintenance decisions are based on elapsed time, which is also called time-based maintenance (TBM) model [36, 32, 17]. In recent years, condition-based maintenance (CBM) model has attracted more attention as the development of computer and monitoring technologies. Compare with TBM, which makes decisions based on historical data, CBM recommends maintenance decision based on the current condition of the devices. In the review, we will focus on the CBM model.

We will review the CBM model based on two major characters, 1) Non-periodic inspection model, 2) Multi-component maintenance model. Then we will briefly review the methodology, which is mainly about Markov decision process.

2.2.1 Non-periodic Inspection Models

Inspection is an essential part of CBM model, because inspection is the approach to acquire the condition of the system. Many researches focus on the influence of inspection quality and inspection frequency. For inspection quality, most of the literature assume perfect inspection, which means the exact condition of the system can be revealed by inspection. We can also find some literature about imperfect inspection. Zequeira and Berenguer [49] assume there are three types of inspection available and three different failure types, one of the failure types can only be detected by perfect inspection. The other two inspection types are imperfect inspection and partial inspection. He et al. [17] considered a system with periodic imperfect inspection. In this thesis, we assume the inspections are perfect.

For inspection frequency, most of the literature considered periodic inspection interval [32, 23, 38, 2]. However, non-periodic inspection is a more efficient than periodic inspection, especially if the inspection procedure is costly for the system. For CBM, the inspection interval could be made based on the condition of the system. More inspections should be conducted when the system is in a bad condition, while longer inspection interval should be determined when the condition of system is pretty new. Barker and Newby [3] studied a non-periodic inspection model with multivariate degradation character. Zhao et al. [50] compared the difference of periodic inspection and non-periodic inspection in their model. In they examples, they verified that periodic inspection is more expensive than non-periodic inspection. Castanier et al. [8] considered a two-unit condition-based maintenance model with non-periodic inspection, they used the semi-regenerative property to calculate the cost, but for multi-unit system, they have to consider Monte Carlo simulation method.

2.2.2 Muiti-component Maintenance Models

Most existing literature on CBM model concerned single component system. However, the system could be more complex with multiple components in reality. According to [41], there are three types of interactions between components:

1. **Economic dependence**: The cost of maintenance has interdependencies between components, which means the cost of joint maintenance of a group components is not equal to the total cost of individual maintenance. For example, due to economies of scale, the average cost of maintenance falls as components increase.

2. **Structural dependence**: The components structurally form a part, one has to maintain or at least dismantle some working components when repair a failure one.

3. **Probabilistic dependence** (a.k.a. Stochastic dependence): The failure of one components would affect other components' condition or failure rate.

In our problem, we focus on the economies dependence.

In the study of Castanier et al. [8], the economic dependence between two components was discussed. Olde Keizer et al. [27] considered a condition-based maintenance for multi-components system with redundancy and economic dependencies, they developed a Markov decision processes model to find the optimal maintenance strategy, but they did not consider the impact of inspection frequency and cost. Zhu et al. [51] studied a multi-component system with high maintenance setup cost. High maintenance setup cost would stronger the economic dependence, so joint maintenance is introduced in their model. They proposed a nested approach to find out the optimal policy.

Meanwhile, other studies study maintenance models with probability dependence. Li et al. [30] proposed a condition based maintenance model for multi-component system with both economic dependence and stochastic dependence. Hong et al. [20] investigated a condition-based model considering the dependency among the degradation of components and different risk attitudes of the decision maker, they used a joint probability distribution called copula function to characterize the dependency.

2.2.3 Markov Decision Processes

In the situation with uncertainty transition between states, the states can be controlled by taking sequential actions or policy, and Markov decision processes are versatile and powerful mathematical tool for solving probabilistic sequential decision problem. Howard [22] develop this tool by using both Markov chain theory and dynamic programming principles. Markov decision processes have been used in inspection and maintenance model in some studies [2, 27]. In our research, we refer to [33, 42, 4] for some important properties about Markov decision processes.

Chapter 3

Risk Level Classification

Risk classification is an essential process for the inspection and maintenance of medical devices. Different inspection and maintenance strategies are performed for devices with different risk level. Currently, in Winnipeg Regional Health Authority, they developed two mathematical model functions to transform risk factors score to final risk levels score, but further investigation on the models are necessary. In this chapter, we use statistical classification methods to find out the functions, the selection of risk factors and the development of risk scoring models are addressed.

First, we examine risk factors in existing literature. According to their definitions and the provided data we selected several factors, then according to correlation and the availability of the data, in the end, three risk factors (*e.g.* Equipment Function, Location of Use and Frequency of Use) are selected.

Second, we introduce four existing scoring systems to find out the relationship between the risk level and risk factors. Different methods explain the relationship form different prospective, we can have better understanding of the relationship with all those methods and identifies important factors and provides motivation to investigate these factors.

Last, we test the risk scoring systems by using data provided by Winnipeg Regional Health Authority.

3.1 Risk Factors

The first major task is the selection of critical risk factors. Some previous researches related to medical equipments risk factors are summarized here.

Three basic factors were introduced by Fennigkoh and Smith in 1989 [12]: equipment function, risk and required maintenance. Those three factors have been widely accepted in the follow-up research works [15, 39, 40].

In addition, an increasing number of new factors have been considered as indispensable factors to make medical equipment managerial decisions: location of equipment use [16], age of equipment [39, 26], meantime between repair [28, 26], utilization [39, 40, 26], available of substitute [16, 40, 26], etc.

After reviewing all the factors (see Appendix A), we summarize a few repeated mentioned factors in existing literature according to their definitions (see Table 3.1):

Table 3.1: Selected factors

Factors	Definition
Age	Actual age of a device and its predictable life span.
Utilization	The total hours a device is used on average in a hospital.
Backup safety ratio	Number of available identical devices.
Location	The area in which it is primarily used
Maintenance requirement	All aspects that affects the requirement for intervention
Function	The main purpose for which it is to be used.
Meantime between failures	The mean time interval between two consecutive failures
Meantime between repair	The mean time interval between two consecutive repairs

Among those selected factors, after considering the availability of the data (Table 3.2), we calculate the correlation between those factors. From the data, we can see that there are seven different risk factors variables (*e.g.* Equipment Function, Location of Use, Locational risk, Frequency of Use, Condition of use, PA Need and Physical Consequs).

Table 3.2: Sample data from WRHA

Term	Asset	PA Required	Risk Level	Equipment Function	Location of Use	Locational risk	Frequency of Use	Condition of use	PA Need	Physical Consequ
Thermometer, infrared, ear	606Q	Required	Low	2	4	1	1	3	2	2
Patient monitor, vital signs	KN016982	Required	Low	4	2	1	1	3	2	2
Patient monitor, vital signs	01557 Vital Signs Monitor	Required	Low	4	2	1	1	3	2	2
Suction unit, transportable	016987	Required	Medium	5	2	2	2	2	3	2
Suction unit, transportable	017676	Required	Medium	5	2	1	1	2	3	2
Suction unit, transportable	KN109875	Required	Medium	5	2	2	2	2	3	2
Suction unit, transportable	SC104898	Required	Medium	5	2	2	2	2	3	2
Infusion pump, general-purpose	KN040130	Required	Medium	5	4	1	1	3	3	3
Infusion pump, general-purpose	KN050182	Required	Medium	5	4	1	1	3	3	3
Infusion pump, general-purpose	KN002452	Required	Medium	5	4	1	1	3	3	3
Infusion pump, general-purpose	KN028885	Required	Medium	5	4	1	1	3	3	3

From Table 3.3, we can see that PA Need is highly related (greater than 0.85) to Equipment Function, while Physical Consequ is highly related (greater than 0.79) to Location of Use. According to the data notes provided by WRHA, Condition of use also depends on the location, so we eliminated this factor. Last, the correlation between Location risk and Frequency of Use is 0.97, which means they are not independent at all. Some of selected factors will be used in the inspection optimization. Eventually, we select three factors from them to do data analysis: Utilization for Frequency of Use, Location for Location of Use, and Function for Equipment Function.

Table 3.3: The correlation between provided factors

	Equipment Function	Location of Use	Locational risk	Frequency of Use	Condition of use	PA Need	Physical Consequ
Equipment Function	1.00						
Location of Use	0.46	1.00					
Locational risk	-0.06	-0.15	1.00				
Frequency of Use	-0.08	-0.20	0.97	1.00			
Condition of use	0.54	0.55	-0.22	-0.25	1.00		
PA Need	0.85	0.63	-0.05	-0.09	0.30	1.00	
Physical Consequ	0.64	0.79	-0.06	-0.10	0.37	0.75	1.00

3.2 Model Definition

We first define the basic variables in our model.

1. The risk level of device i, denoted by R_i.

$$R_i = \begin{cases} 1, \ risk\ level\ is\ low \\ 2, \ risk\ level\ is\ medium \\ 3, \ risk\ level\ is\ high \end{cases}$$

   In the provided data, there are three different levels of risk. Level three implies the highest risk level and one for the lowest level.

2. Equipment function of device i, denoted by X_{1i}.

   There are six different levels of equipment functions. Level six implies the function is critical while one means that the function is not important, i.e., $X_{1i} \in \{1, 2, 3, 4, 5, 6\}$. Specifically, 1. Miscellaneous – patient related; 2. Ancillary – diagnostic; 3. Ancillary – therapeutic; 4. Essential / critical – diagnostic; 5. Essential / critical – therapeutic; 6. Life Support.

3. Location of use of device i, denoted by X_{2i}.

   The location of use would also affect the risk level of the devices. We have five different level of locations. Location level 1 means this place is comparatively safer than other places and the consequence of failure has less influence, while location level five means that this place is more risky than other places. i.e., $X_{2i} \in \{1, 2, 3, 4, 5\}$. Specifically, 1. Non-patient areas; 2. General care areas; 3. Wet location/labs/exam areas; 4. Critical care areas; 5. Anesthetizing locations.

4. Frequency of use of device i, denoted by X_{3i}.

   In the data, frequency of use of device are divided into three levels. Level three means the usage of the device is more frequent, and level two means that the frequency is low and level one is the lowest frequency. i.e., $X_{3i} \in \{1, 2, 3\}$, Specifically, 1. Low; 2. Medium; 3. High.

Then we are going to find out the relationship between the risk levels and the three risk factors. The relation is denoted as: $R_i = f(X_{1i}, X_{2i}, X_{3i})$.

3.3 Methodologies

In order to find out the relationship, we are going to use four different methods to build models (*i.e.* Linear Regression, Logistic Regression, Classification Tree and Random Forest).

3.3.1 Data Processing

Before we build those four models, we have to process the data. Normally, we split data into training set and testing set. We use training data to build the models, and use testing data to assess the performance of our models. No matter which model we used, the details of the model is based on the training data. For classification tree, the testing data plays another essential role: preventing the overfitting model. Overfitting means the model we built based on the training data is too complex to predict new sample data. So we use testing data to implement the model and score the performance of the model. According to the performance of different size of models, we can select the right size with the best performance.

In our case, we randomly separate the data to 75% training data (4097 observations) and 25% testing data (1366 observations). Then we have a learning data set L which comprises n couples of observations (r_1, $\mathbf{x}_1$),...,(r_n, $\mathbf{x}_n$), where $\mathbf{x}_i = (x_{1i},\ x_{2i},\ x_{3i})$ is a set of independent variables and r_i is the dependent variable associated with $\mathbf{x}_i$.

3.3.2 Linear Regression

In the linear regression model, we have the vector $\mathbf{b} = (b_1, \dots , b_n)$ as coefficient and b_0 as intercept. In our case, $\mathbf{x}_i = (x_{1i},\ x_{2i},\ x_{3i})$ are the independent variables, r_i is response variable, then the regression model is:

$$\hat{r} = \ b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3, \tag{3.1}$$

where $\hat{r}$ is the estimated response value.

For completeness and convenience, we provide details on the Linear regression method.

We use ordinary least squares to fit the data, which means using the coefficients **b** to minimize the residual sum of squares (SSE) between the observed data and the responses predicted by the model.

$$SSE = \sum_{i=1}^{n} (r_i - \widehat{r}_i)^2 = \sum_{i=1}^{n} (r_i - b_0 - b_1 x_{1i} - b_2 x_{2i} - b_3 x_{3i})^2. \tag{3.2}$$

Then differentiate SSE with respect to b_0, b_1, b_2, b_3 and equal to zero 0, get 4 new equations. Solve the new equations then get a set of coefficients of the model.

$$\begin{cases} \sum_{i=1}^{n} r_i - nb_0 - b_1 \sum_{i=1}^{n} x_{1i} - b_2 \sum_{i=1}^{n} x_{2i} - b_3 \sum_{i=1}^{n} x_{3i} = 0 \\ \sum_{i=1}^{n} x_{1i} r_i - b_0 \sum_{i=1}^{n} x_{1i} - b_1 \sum_{i=1}^{n} x_{1i}^2 - b_2 \sum_{i=1}^{n} x_{1i} x_{2i} - b_3 \sum_{i=1}^{n} x_{1i} x_{3i} = 0 \\ \sum_{i=1}^{n} x_{2i} r_i - b_0 \sum_{i=1}^{n} x_{2i} - b_1 \sum_{i=1}^{n} x_{2i} x_{1i} - b_2 \sum_{i=1}^{n} x_{2i}^2 - b_3 \sum_{i=1}^{n} x_{2i} x_{3i} = 0 \\ \sum_{i=1}^{n} x_{3i} r_i - b_0 \sum_{i=1}^{n} x_{3i} - b_1 \sum_{i=1}^{n} x_{3i} x_{1i} - b_2 \sum_{i=1}^{n} x_{3i} x_{2i} - b_3 \sum_{i=1}^{n} x_{3i}^2 = 0 \end{cases} \tag{3.3}$$

We can get the coefficients by solving Equation (3.3). Usually, when the number of variables exceeds two, we rewrite the above equation in matrix form and express the solution in matrix form, which can facilitate the mathematical manipulation considerably. The matrix notations are presented below.

For equation (3.1), we denote $\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{21} & \cdots & x_{k1} \\ 1 & x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1n} & x_{2n} & \cdots & x_{kn} \end{bmatrix}$,

$\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$.

Then equation (3.2) can be expressed as:

$$SSE = (\mathbf{r} - \mathbf{X}\mathbf{b})'(\mathbf{r} - \mathbf{X}\mathbf{b}). \tag{3.4}$$

Differentiate the SSE, we have $\frac{\partial}{\partial \mathbf{b}}(SSE) = 0$. Equation (3.3) is simplified as:

$$\left(\mathbf{X}'\mathbf{X}\right)\mathbf{b} = \mathbf{X}'\mathbf{r}. \tag{3.5}$$

If the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular, then we have the solution:

$$\mathbf{b} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{r}. \tag{3.6}$$

In this report, we only discuss the steps of matrices form manipulation. For more detail of matrices manipulation, we refer to [44].

In our model, we get b_0 = -0.225, b_1 = 0.291, b_2 = 0.183, b_3 = 0.046.

Table 3.4: Linear regression results

Dep. Variable:	y		R-squared:	0.813	
Model:	OLS		Adj. R-squared:	0.813	
Method:	Least Squares		F-statistic:	5941	
No. Observations:	4097		Prob (F-statistic):	0	
Df Residuals:	4093		Log-Likelihood:	700.23	
Df Model:	3		AIC:	-1392	
Covariance Type	nonrobust		BIC:	-1367	
	coef	std err	t	$P > \|t\|$	[95.0% Conf. Int.]
const	-0.2252	0.018	-12.837	0.00	[-0.260, -0.191]
x0	0.2909	0.003	91.029	0.00	[0.285, 0.297]
x1	0.1832	0.004	47.048	0.00	[0.176, 0.191]
x2	0.0458	0.004	11.137	0.00	[0.038, 0.054]

Although this is a multiple linear regression, we selected the variables before our model building, and the result shows all three variables are important, which means the three independent variables are supposed to be chosen. We test the adequacy of this model, the coefficient of determination:

$$R^2 = \frac{SSR}{SST} = \frac{\sum_{i=1}^{n}\left(\widehat{r_i} - \overline{r}\right)^2}{\sum_{i=1}^{n}\left(r_i - \overline{r}\right)^2}, \tag{3.7}$$

where $\overline{r}$ is the mean of response variables, $\hat{r}$ is the estimated response value.

The result shows R^2= 0.818, which means this model fits the data quite well.

Then we test the importance of individual coefficient, the result shows all of them have significant influence on the model, the p-value less than 0.005. Here, we use t-distribution with 4093 degrees of freedom (df). We have

$$t = \frac{b_i}{s\sqrt{c_{ii}}}, \tag{3.8}$$

where $s^2 = \frac{SSE}{df}$,and $c_{ii}$is the i-th diagonal element of matrix $(X'X)^{-1}$ and

$$X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{31} \\ \vdots & & \ddots & \vdots \\ 1 & x_{1n} & \cdots & x_{3n} \end{bmatrix}, \tag{3.9}$$

We assume that the relationship between risk level and risk factors is linear in this model, but since the risk levels are categorical variables, we round the results to the nearest integer. Then we get the model as:

$$r^* = [-0.225 + \ 0.291x_1 + \ 0.183x_2 + \ 0.046x_3]. \tag{3.10}$$

We summarize the computation steps as follows:

Step 1: Randomly separate the data to training data and testing data.

Step 2: Use training data to build the model. Risk factors are independent variables $\mathbf{x}$ and risk levels are response variables r.

Step 3: Use matrix form function to compute coefficients $\mathbf{b}$ (see equation ((3.6))), and then get the linear model as equation (3.1).

Step 4: Test the model by the coefficient of determination using equation (3.7).

Step 5: Test the individual independent variable using t-test of equation (3.8).

Step 6: Round the result to get the model as equation (3.10)

Step 7: Test the accuracy of the model using testing data. The result is shown in result section. If the accuracy is good we can use the model to predict the risk level of other devices, otherwise, we need to consider changing the model or using more data to train the model.

Step 8: The risk factors could be input in the model (3.10) as x_1, x_2, x_3 respectively and the risk level is the result r*.

It is hard to say whether a specific accuracy rate is good, it depends on the problem. Comparing with the accuracy of the model in the project poster (67%) [14], we used almost same factors but different models, our result is very good (98%).

3.3.3 Logistic Regression

In our problem, the objective is to find the relationship between the risk level and the risk factors. Meanwhile, the risk level is a categorical outcome variable, which is quite suitable to be analyzed by logistic model.

In logistic model, we will get the probability that a data sample allocated to a class. Thus the model is written in terms of probability. For each class, we have a function as follows

$$p = \frac{1}{1 + e^{-(b_0 + b_1 x_1 + \ldots + b_k x_k)}}, \tag{3.11}$$

where k is the number of independent variables, x_i is independent variables, p is the probability that the sample belongs to this class.

However, there is another form of logistic regression model which is the odd ratio.

$$\frac{p}{1-p} = e^{-(b_0 + b_1 x_1 + \ldots + b_k x_k)}. \tag{3.12}$$

Then we take 'logit' of the odd ratio.

$$\log\left(\frac{p}{1-p}\right) = b_0 + b_1 x_1 + \ldots + b_k x_k. \tag{3.13}$$

The original problem is to find the relationship between y and $\{x_1, x_2, \ldots, x_k\}$. However, y takes integer values $\{0, 1\}$. We cannot apply the standard linear regression method directly. The idea of this method is to transform an integer variable y problem into a continuous variable problem (3.13) such that we can use the standard linear regression method to estimate parameters {b_0, b_1, ..., b_k}. Then use equation (3.11) to estimate p for {x_1, x_2, ..., x_k}. Final, we use p to find y (=0 or 1). If p is in certain range, y = 1; otherwise, y = 0. This logit model looks like the linear regression. The bigger the logit is the bigger probability p is.

In order to get the coefficients, we need a cost function, we denote function (3.11) in matrix form,

$$h_\theta(x) = \frac{1}{1 + e^{-\theta \mathbf{x}^T}}, \tag{3.14}$$

where θ is the coefficients, $\theta = [b_0, b_1, ..., b_k]$, $h_\theta(x)$ is the estimated value, y is the response and x is independent variables $\mathbf{x}= [1, x_1, x_2, ..., x_k]$.

Then we denote the cost as follows:

$$\mathrm{J}(\theta) = \frac{1}{m} \sum_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{cost}\left(h_\theta\left(x^{(i)}\right), y^{(i)}\right), \tag{3.15}$$

where

$$\mathrm{cost}(h_\theta(x), y) = \begin{cases} -log(h_\theta(x)) & if\ y = 1 \\ -log(1 - h_\theta(x)) & if\ y = 0 \end{cases},$$

and $y = 0$ or 1 always, m is the number of training samples, $\mathrm{x}^{(i)}$ is the independent variable of i-th observation, $\mathrm{y}^{(i)}$ is the response of the i-th observation.

There are only two possible cases, y = 1 or y = 0. We can simplify the cost function as follows:

$$\mathrm{J}(\theta) = -\frac{1}{m} \sum_{\mathrm{i=1}}^{\mathrm{m}} \left(y^{(i)} \log\left(h_\theta\left(x^{(i)}\right)\right) + \left(1 - y^{(i)}\right) \log\left(1 - h_\theta\left(x^{(i)}\right)\right) \right). \tag{3.16}$$

To minimize the above function over the coefficients θ (3.16), Sklearn package uses a coordinate descent algorithm based on Liblinear. For more details about the cost function of logistic regression model, please refer to [21].

In our model, since the risk levels are ordered categorical outcome, so we used ordinal logistic regression [21]. The idea is the same as logistic regression, since there are three risk levels, we need to define the possible events as follow:

1. Being in risk level 1;
2. Being in risk level 1 or 2;
3. Being in risk level 1, 2 or 3.

Then we could use general logistic model to predict the probability of each events. The risk factors are the independent variables **x** as defined in Section 2.

We have three functions to calculate the probability for each event.

$$P\,(R{=}\,1) = \frac{1}{1{+}e^{-(47.22-7.34x_1-6.84x_2-0.30x_3)}}, \tag{3.17}$$

$$P\,(R = 1 \;or\; 2) = \frac{1}{1 + e^{-(73.35-7.34x_1-6.84x_2-0.30x_3)}}, \tag{3.18}$$

$$P\,(R = 1,\; 2 \;or\; 3) = \;1. \tag{3.19}$$

Then function (3.17) is the probability of $(R{=}\,1)$, the difference between functions (3.18) and (3.17) is the probability of $(R{=}\,2)$, the difference between function (3.19) and (3.18) is the probability of $(R{=}\,3)$. Then the risk level R* of the device with {x1, x2, x3} is the value of R with the biggest probability.

The computation steps are as follows:

Step 1: Randomly separate the data to training data and testing data.

Step 2: Use training data to build the model. Risk levels are the categories and risk factors are the independent variables. We define three events according to the risk levels.

Step 3: For each event, we construct a binary variable, 1 for this event happens and 0 for not happen.

Step 4: Solve a minimization problem of (3.16) by using Newton's method to get coefficients for an event, get the model as equation (3.11).

Step 5: Repeat Step 2 and Step 3 for other two events, the probability of last event is definitely 1.

Step 6: The final model includes three equations (3.17)(3.18)(3.19).

Step 7: Take the difference of the equations and we can get the probability of each risk level, and choose the one with the biggest as the risk level of the device with {x_1, x_2, x_3}.

Step 8: Test the accuracy of the model with testing data. The results are present in the results section.

If the accuracy is good (e.g., great than 67%) [14], we can use the model to predict the risk level of other devices, otherwise, we need to consider changing the model or using more data to train the model. The result of this model is pretty good, so the risk factors could be input in the models (equation 3.16,3.17,3.18) as x_1, x_2, x_3 respectively and we can get the probabilities of the sample belonging to every risk levels.

3.3.4 Classification Tree

Classification tree builds a model in the form of a tree structure. The final result is a tree with decision nodes and terminal nodes. Normally, a decision node has two branches and terminal node represents a classification.

There are two processes when building the classification tree: tree growing and tree pruning. We randomly acquire a set of training data from the original data to build the classification tree.

In the tree growing process, learning data L is recursively divided into two subsets by binary split until the terminal nods are achieved. In order to decrease the impurity of the node, the split selection is chosen according to Gini Index. Normally, we select the point that most significant reducing impurity as the split point. The Gini index is shown as follows (3.20):

$$Impurity\,(t) = 1 - \sum_{k=1}^{n} p_k^2, \tag{3.20}$$

where p_k: the proportion of cases in node t that belong to class k. $Impurity\,(t) = 0$ when all cases belong to the same class in node t.

Then we need to calculate the information gain, which is based on the decrease in impurity after a decision node is split on a split point. We will find the split point that returns the highest information gain by a tree algorithm CART [5].

Let t_1 and t_2 be the two children nodes of node t after splitting, and $H(t)$ be the number of cases in node t. Then we have

$$\text{Information gain} = \text{impurity}(t) - \frac{\text{impurity}(t_1) \times H(t_1) + \text{impurity}(t_2) \times H(t_2)}{H(t)} \tag{3.21}$$

After tree growing, each node is full pure, the tree may be too complicated and over-fitting. Consequently, it should be pruned back, using cost complexity to choose the best tree, however setting the minimum number of samples required at a leaf node or setting the maximum depth of the tree are commonly used. In our case, there is no over-fitting problem, so we didn't prune the tree.

For evaluation of model performance, confusion matrix is a proper method, which could visualize the performance of the model, but we only use the accuracy to evaluate the performance of the model.

$$\text{Accuracy} = \frac{\text{The number of correct prediction}}{\text{The number of total samples}} \tag{3.22}$$

The computation steps are as follows:

Step 1: Randomly separate the data to training data and testing data.

Step 2: Recursively partition the learning data into two sub-sets by binary split according the Gini index (3.20).

2.1 For splitting, all features and all possible split points are evaluated. (e.g., x_1<= 4.5 or x_1>4.5).

2.2: Calculate the impurity of the node before and after the partition for each candidate split points.

2.3: Calculate the information gain by subtracting impurity after split from the impurity before the split and select the split point with the largest information gain as the new decision node. (3.21)

Step 3: Repeat Step 2 until the depth of the tree arrives the maximum depth, then we get the final model.

Step 4: Test the accuracy of the model with testing data (3.22). The results shows in the result section.

Step 5: The result of this model is pretty good, we can use the value of the variables to make decision according to the tree (Figure 3.1), and then we can get the risk level once we get a terminal node.

If the accuracy is good (e.g., great than 67%) [14], we can use the model to predict the risk level of other devices, otherwise, we need to consider changing some parameters of the model or using more data to training the model.

In this model, we can predict the risk level of a device based on its equipment function and location of use, which means we only need two risk factors to predict the risk level in the classification tree model. Figure 3.1 is the classification tree.

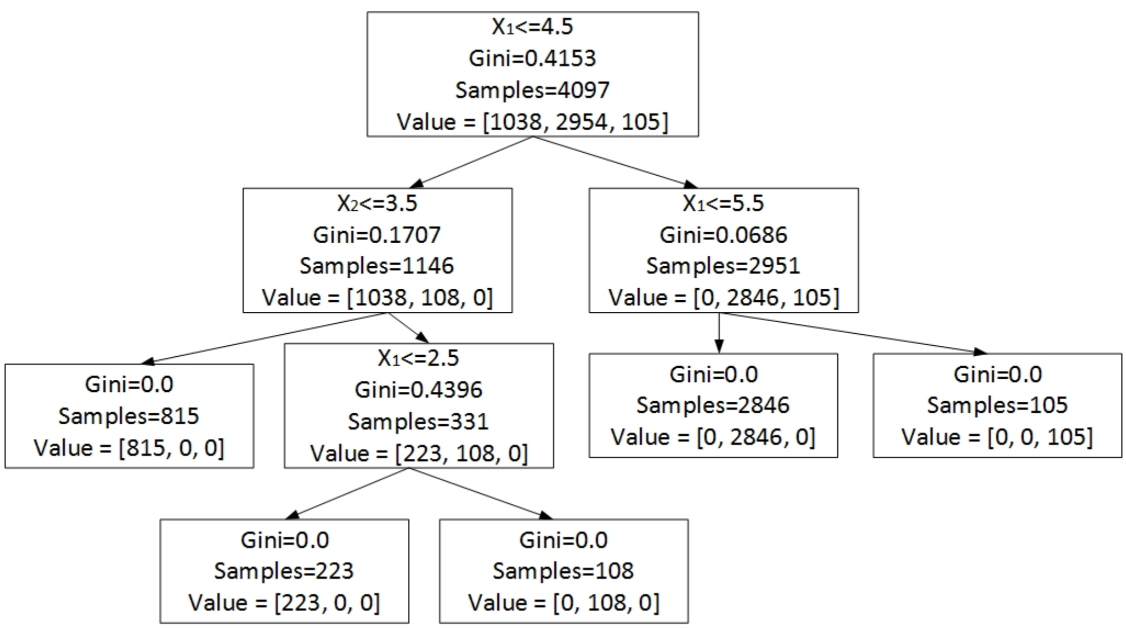


Figure 3.1: Classification tree

3.3.5 Random Forest

Random forest takes the idea of classification tree further and grows many trees. To classify a new object from an input vector, put the input vector down in each of the trees in the forest. Each tree gives a classification, and we say the tree 'votes' for that class. The forest chooses the classification having the most votes over all the trees in the forest [7].

But the trees in Random Forest are different from the tree in Classification Tree. In Random Forest, each tree is built from a random sample drawn with replacement (i.e., a bootstrap sample) from the training data set. Moreover, the split selection is chosen form the best split of a random subset of the features. Although the bias of one tree slightly increases, but the bias are normalized by dividing by the number of tree, hence normally yielding a better model.

For the trees building, there is a step called tree bagging, which means selecting a random sample with replacement of the training set and fits trees to the samples. This method was proposed by Breiman [6]. For completeness, we describe the idea as follow,

1. Tree bagging:

   For k = 1, ..., K, is a random sample with replacement of the training set L which comprises n couples of observations (r_1, $\mathbf{x}_1$),...,(r_n, $\mathbf{x}_n$), where $\mathbf{x}_i = (x_{1i},\ x_{2i},\ x_{3i})$ is a set of independent variables and r_i is the dependent variable associated with $\mathbf{x}_i$, call it L_k. Then we can train a tree on L_k, call f_k. Then prediction can be made by the majority vote of the trees.

2. Feature bagging:

   There is another bagging technique to improve the accuracy of random forest, which is feature bagging. A random subset of the features is selected for each tree f_k [19]. The reason to do this is reduce the correlation between different trees.

3. Voting:

   After we have a lot of trees, we need to make decisions based on these trees. A typical method is that the prediction can be made by the majority vote of these trees.

We summarize the computation steps as follows:

Step 1: Randomly separate the data to training data and testing data.

Step 2: Set the number of trees in the model (e.g., 100).

Step 3: For each tree, get a bootstrap sample from the training data to build the tree. The bootstrap sample size is always the same as the original training data size but the samples are drawn with replacement.

Step 4: Follow the step of Classification Tree to build each tree, but in the split selection step (Step 2 in Classification Tree), choose the best split from a random subset of all features. Then there are 100 different trees.

Step 5: For each $\{x_1, x_2, ..., x_k\}$, every tree can have a classification decision, the model choose the classification R having the most votes.

Step 6: Test the accuracy of the model with testing data, the results shows in the results section.

If the accuracy is good (e.g., great than 67%) [14], we can use the model to predict the risk level of other devices, otherwise, we need to consider changing the model or using more data to train the model. The result of this model is good. The model is not explicitly present in formula, so normally we use program to calculate the risk level of a given sample data.

3.4 Results and Comparison

In this section, we present a few sample prediction results of those four methods, then we compare the accuracy rate of those methods.

Table 3.5 shows ten sample results of four models. The results of linear regression model has some disagreement with provided data, while logistic regression, classification tree model and random forest model have 100% agreement with provided data.

Table 3.5: Sample data of the medical devices

Device i	Asset #	Provided R_i	Risk Factors			Results of Scoring models			
			X_{1i}	X_{2i}	X_{3i}	linear	logistic	tree	forest
1	00123	High (3)	6	5	3	3	3	3	3
2	00923	High (3)	6	5	1	**2**	3	3	3
3	02132	High (3)	6	5	3	3	3	3	3
4	KN039397	Medium (2)	5	4	3	2	2	2	2
5	KN039400	Medium (2)	5	4	3	2	2	2	2
6	KN060102	Medium (2)	5	4	2	2	2	2	2
7	KN060198	Medium (2)	5	4	2	2	2	2	2
8	B02792	Low (1)	2	1	1	1	1	1	1
9	B02793	Low (1)	2	1	1	1	1	1	1
10	B03452	Low (1)	2	1	1	1	1	1	1

The distribution of risk levels and the accuracy rate of each model are presented in Table 3.6 and Figure 3.2. All of the models have quite high accuracy rate, logistic regression, classification tree and random forest have 100% accuracy rate.

Table 3.6: Accuracy rate of the models

Models	Accuracy rate
Linear regression	0.98
Logistic regression	1.00
Classification tree	1.00
Random Forest	1.00

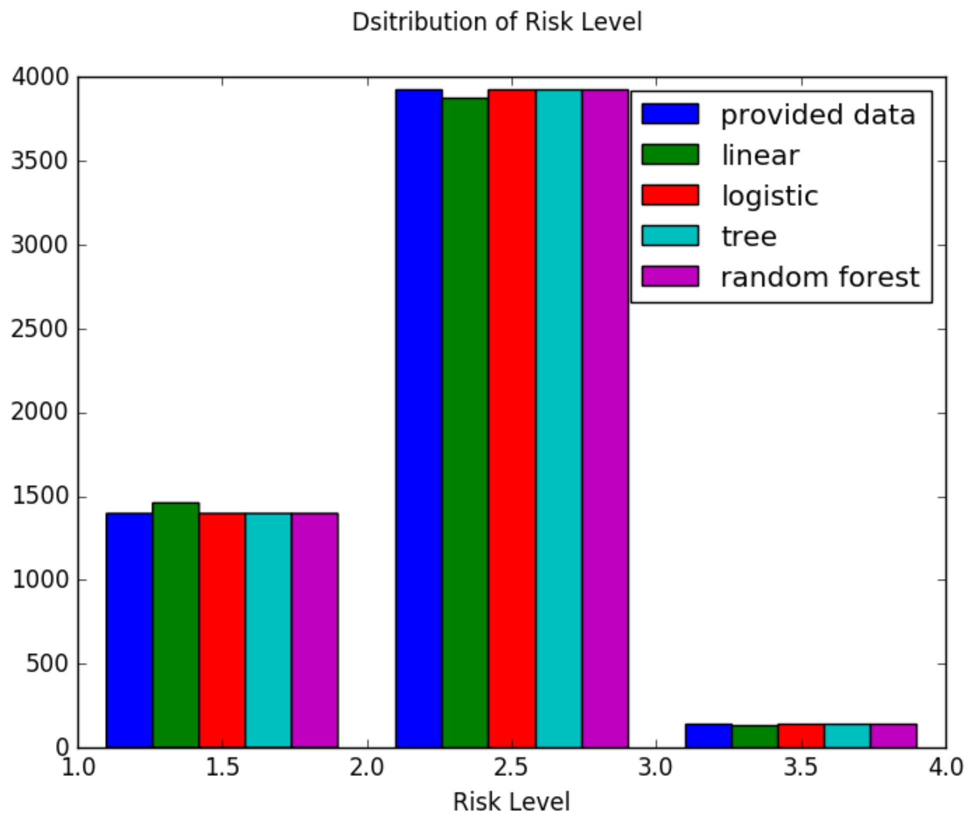


Figure 3.2: Distribution of the risk levels of the medical devices

3.5 Conclusion

Through different methods, we can have different insights into the relationship of risk factors and risk levels. According to the classification tree, we could explicitly find out that how Equipment Function and Location of use classify the devices to different risk levels. By Linear Regression and Logistic regression models, we can see that all those three risk factors have critical influence on the risk levels.

By comparison, we find out that logistic regression, classification tree and random forest have better performance on risk levels classification in our case. However, the results are obtained from the sample data, and the data is not very diverse (10 types of devices), therefore, models need to be tested further with more diverse data.

Chapter 4

Inspection and Maintenance Model

Based on the result of classification from Chapter 3, in this chapter, we are going to find out the best policy to manage important devices. Inspection and maintenance model is a data-driven optimization model, we use data to acquire the condition of the device and to predict the deterioration rate, then, we use a decision model to get the optimal decisions and the minimal average cost.

This chapter is organized as follows. Section 4.1 briefly introduce the main idea of the model and a few basic assumptions of the model. In Section 4.2, the estimation of transition matrix is discussed, a few practical examples from WRHA are presented. In Section 4.3, we discuss and solve the Markov decision processes model. In the numerical examples, we use the result of the example in Section 4.2 to find the optimal decisions and the minimal long-run average cost. Furthermore, we extend the models to situation with *k-out-of-n* systems in the numerical examples and compare our model with periodic inspection model. A case study with heterogeneous devices is presented in Section 4.4.

4.1 Model of Interest

In this section, a brief description of the inspection and maintenance model is presented at first, then we list a number of assumptions about the model.

4.1.1 Model Description

The inspection and maintenance model can be roughly divided into two parts: the deterioration rate estimation and a decision model [13]. The deterioration rate estimation is about the uncertain time to failure and the transition probabilities between different states, and the decision model uses the result of estimated deterioration rate to determine the optimal time and action of inspection and maintenance.

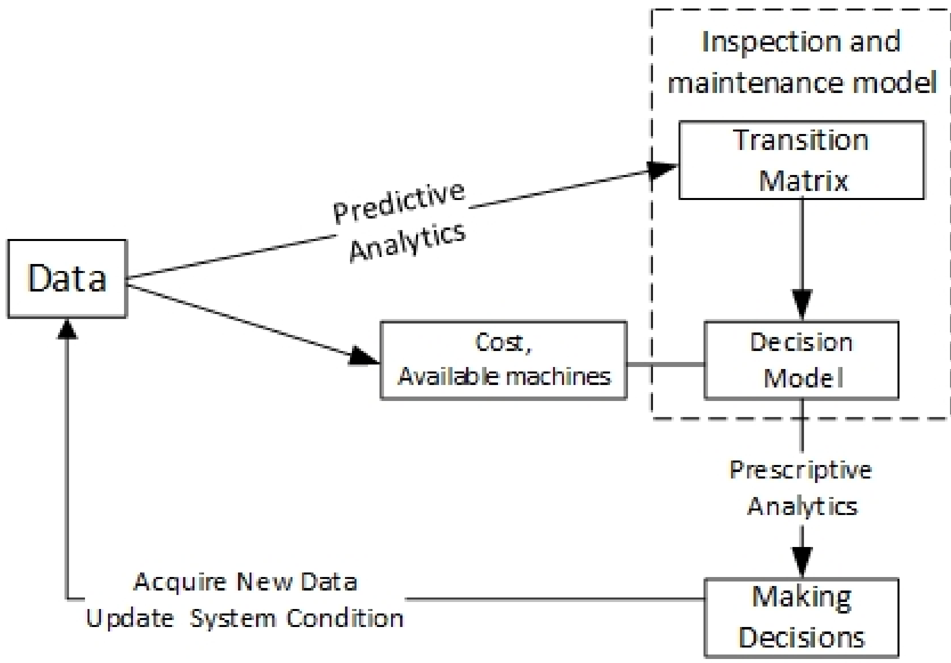


Figure 4.1: The inspection and maintenance model

In recent years, many research projects have been conducted on the prediction and estimation of deterioration rate or matrix. Normally, some partially observable historical data is available, but different type of historical data fits in with different models. In our decision model, the inspection frequency is non-periodic inspection, so the data comes from unequal observation intervals [11], some transitions would be unobservable. Without complete information of the system, the EM algorithm will be used to estimate the transition probabilities.

Both maintenance and inspection decisions would be made, and through the inspection process, we can collect the states of the system, which can be used in the deterioration model. The stochastic dynamic programming model or discrete-time Markov decision model that will be used for making decisions has been widely applied in maintenance, inventory and telecommunication among others.

The combination of deterioration rate estimation and decision model can continuously make decisions and generate required data to update the system condition. We discuss these two parts separately in Section 4.2 and Section 4.3.

4.1.2 Basic Assumptions

We list a few assumptions about the model as follows. Assumptions 1 to 3 are about a single device, which would be used in the estimation of transition matrix, the rest of the assumptions are about the system, which would be used in the decision model.

1. There are three possible deterioration states $T = \{0, 1, 2\}$: a healthy state 0, an unhealthy operational state 1, and a failure state 2.

2. States are unobservable without inspection.

3. The deterioration process of single device is a Markov chain $\{X_n, n = 0, 1, 2, ...\}$ with 3 states and transition matrix P.

4. The system is a discrete-state discrete-time stochastic deteriorating system.

5. The state of the system with n devices can be denoted as a vector $\mathbf{x} = [x_0, x_1, ..., x_n]$, x_0 is the time for the next inspection on the system, x_1 to x_n are the deterioration level of the machine 1 to machine n. The deterioration level ranges from 0 (initial new state) to 2 (failure state).

6. The system is not inspected each epoch. The possible inspection epochs range from 0 (inspect right now) to S (the largest inspection interval). The condition of the system can be perfectly observed through inspection.

7. The repair duration is negligible and the repair cost is supposed to be higher for corrective repair than for preventive repair.

8. The repair always leads to an 'as good as new' machine.

4.2 Estimation of the Transition Matrix

In this section, based on the assumptions 1 to 3 in previous section, we are going to find out the transition matrix P, which is a discrete-time homogeneous Markov chain.

The discrete-time homogeneous Markov chain is used to describe the state deterioration process, and it is a very popular model used for investigate the condition of machines, since it is a powerful model to describe the pattern of failure. According to some existing researches, more than 70% of the failure patterns are constant probability of failure at all ages or infant mortality, followed by a constant or very slowly increasing failure probability [37], which indicates the homogeneous property is a reasonable assumption for the failing system. Meanwhile, this model is simple and easy to track through matrix analysis.

An Estimation-Maximization (EM) algorithm will be introduced later and a few numerical examples will be presented.

4.2.1 The Estimation-Maximization Algorithm

Craig and Sendi [11] developed an EM algorithm to estimate the transition matrix when the observation intervals are unequal. We applied this method in our model.

Consider the system on the state-space $T = \{0, 1, 2\}$, where the state 0-1 are called transit states, and state 2 means failure state, which is also the absorbing state. Suppose the transition matrix P is

$$P = \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ 0 & p_{11} & p_{12} \\ 0 & 0 & 1 \end{pmatrix}$$

Since the state of the system can never improve without intervention, so $p_{ij} = 0$ for $i > j$.

First, we consider the situation with complete information of the transition. Suppose the inspection interval is one-time-unit, and we want to get a one-time-unit transit matrix. Based on the observation, we have an observed transition count matrix:

$$N = \begin{pmatrix} n_{00} & n_{01} & n_{02} \\ 0 & n_{11} & n_{12} \\ 0 & 0 & n_{22} \end{pmatrix}$$

where n_{ij} is the number of occurrences that transition from state i to state j after one-time-unit.

The maximum likelihood estimate of the transition matrix is the row proportions of N.

$$\hat{P} = \begin{pmatrix} \frac{n_{00}}{\sum_{j=0}^{2} n_{0j}} & \frac{n_{01}}{\sum_{j=0}^{2} n_{0j}} & \frac{n_{02}}{\sum_{j=0}^{2} n_{0j}} \\ 0 & \frac{n_{11}}{\sum_{j=0}^{2} n_{1j}} & \frac{n_{12}}{\sum_{j=0}^{2} n_{1j}} \\ 0 & 0 & 1 \end{pmatrix}$$

Then, we consider the situation without complete information of the transition, which means the inspection intervals may be unequal and there are S inspection intervals which are integer multiples (1, 2, 3,...,n) of the time-unit. The missing data are the unobservable states for the system at the non-inspection period. So the EM algorithm needs to estimate these states at the non-inspection period, and the expected number of transitions. Normally, an initial transition matrix is needed for the first iteration. However, convergence is not guaranteed, so several different initial transition matrix are recommended. There are two steps in EM algorithm, which are Expectation step and Maximization step.

For the E-step, we use the estimated matrix (or the initial matrix for the first iteration) to compute the probability of each path a sample subject could have. Here the Chapman-Kolmogorov equations are used to denote the probabilities. We have defined the one-time-unit transition probabilities p_{ij}. According to Chapman-Kolmogorov equations, the n-time-unit transition probability is p_{ij}^{n}, which means a sample subject in state i will be in state j after n-time-unit. Obviously, if the inspection interval is n-time-unit, then the transition matrix would be

$$P^{(n)} = \begin{pmatrix} p_{00}^{n} & p_{01}^{n} & p_{02}^{n} \\ 0 & p_{11}^{n} & p_{12}^{n} \\ 0 & 0 & 1 \end{pmatrix}$$

Denote the number of occurrences where the initial state i to state j after k-time-unit as n_{ij}^k. We need to estimate the possible path over k-time-unit. For example, the probability of a sample following the path $(0 \to 1 \to 1)$ over 2-time-unit is

$$p\left(0 \to 1 \to 1 | 0 \to ? \to 1\right) = \frac{p_{01}p_{11}}{p_{01}^2}$$

If there are n_{01}^2 samples transit from state 0 to state 1 after 2-time-unit, then the expected number of samples to follow the path$(0 \to 1 \to 1)$is

$$n_{01}^2 \frac{p_{01}p_{11}}{p_{01}^2}$$

Based on the expected number of paths, we could get the expected one-time-unit transitions. For example, in the path $(0 \to 1 \to 1)$, there are one $(0 \to 1)$transition and one $(1 \to 1)$ transition. Then a new one-time-unit transition count matrix is generated.

For the M-step, the maximum likelihood estimation method introduced with complete information is used again, then a new transition matrix is created, which would be used to estimate the probability of each path in the next iteration.

In general, there are three steps:

1. Initial step: Choose an initial transition matrix as current matrix,

2. E-step: Use the current matrix and the observed data to estimate the number of paths and one-time-unit transitions n_{ij}.

3. M-step: Use the estimated number of one-time unit transitions n_{ij} to generate a new transition matrix as the current transition matrix then back to E-step, stop the iteration if $||M_n - M_{n-1}||_1$ is sufficiently small or the maximum iteration limit was reached. (Note: $||A||_1 = \max_{1 \le j \le n}\{\sum_{i=1}^n |a_{i,j}|\}$ for matrix $A = (a_{i,j})$.)

4.2.2 Numerical Examples

Example 4.2.1 *In this example, we simulate a data set from a three state model with two different inspection intervals. A one year transition matrix is desired.*

Initial state	*Final state*		
	0	*1*	*2*
One year			
0	*227*	*22*	*21*
1	*0*	*70*	*23*
2	*0*	*0*	*0*
Two years			
0	*214*	*45*	*41*
1	*0*	*82*	*101*
2	*0*	*0*	*0*

The first iteration of the EM algorithm is shown below:

Initial transition matrix:

$$M_0 = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \end{pmatrix}$$

E-step:

$$n_{00} = 227 + 2 \times 214 + 45 \times \left(\frac{p_{00}p_{01}}{p_{01}^2} \right) + 41 \times \left(\frac{p_{00}p_{02}}{p_{02}^2} \right);$$

$$n_{01} = 22 + 45 + 41 \times \left(\frac{p_{01}p_{12}}{p_{02}^2} \right);$$

$$n_{02} = 21 + 41 \times \left(\frac{p_{02} + p_{00}p_{02}}{p_{02}^2} \right);$$

$$n_{11} = 70 + 2 \times 82 + 101 \times \left(\frac{p_{11}p_{12}}{p_{12}^2} \right);$$

$$n_{12} = 23 + 101 + 41 \times \left(\frac{p_{01}p_{12}}{p_{02}^2} \right).$$

M-step:

$$p_{00} = \frac{n_{00}}{n_{00} + n_{01} + n_{02}}; p_{01} = \frac{n_{01}}{n_{00} + n_{01} + n_{02}}; p_{02} = \frac{n_{02}}{n_{00} + n_{01} + n_{02}};$$

$$p_{11} = \frac{n_{11}}{n_{11}+n_{12}}; p_{12} = \frac{n_{12}}{n_{11}+n_{12}}.$$

Then we get:

$$M_1 = \begin{pmatrix} 0.8407 & 0.0916 & 0.0676 \\ 0 & 0.6986 & 0.3014 \\ 0 & 0 & 1 \end{pmatrix}$$

After 10 iterations:

$$M_{10} = \begin{pmatrix} 0.9682 & 0.0239 & 0.0079 \\ 0 & 0.6622 & 0.3378 \\ 0 & 0 & 1 \end{pmatrix}$$

After 100 iterations, the transition matrix is stable.

$$M_{100} = \begin{pmatrix} 0.9682 & 0.0239 & 0.0079 \\ 0 & 0.6582 & 0.3418 \\ 0 & 0 & 1 \end{pmatrix}$$

In this example, after 100 iteration, the transition matrix stabilizes. However, we only present one possible initial transition matrix, actually, the convergence is not guaranteed. With different initial transition matrices, the result would be slightly different.

Example 4.2.2 *In this example, we use the provided data from Winnipeg Health Reginal Authority. We collect the data set of defibrillator with three different inspection interval, the model of this device is 'Lifepak 20'. After cleaning the data, we get the raw sample data as follows*

Table 4.1: The sample data from WHRA

Model	Asset	Date	WO	Result	Interval	year	Condition
Lifepak 20 ˆ	KN031471	2010-10-04	A	Repair	307.9377	1	1
Lifepak 20 ˆ	KN031471	2011-10-25	B	Inspection	256.1514	1	0
Lifepak 20 ˆ	KN031471	2012-10-29	B	Repair	370.0084	1	1
Lifepak 20 ˆ	KN031471	2013-10-03	A	Repair	339.0581	1	1
Lifepak 20 ˆ	KN031471	2014-10-24	B	Inspection	385.8316	1	0
Lifepak 20 ˆ	KN031471	2015-11-20	B	Inspection	392.0418	1	0
Lifepak 20 ˆ	KN030356	2010-10-29	B	Repair	332.9872	1	1
Lifepak 20 ˆ	KN030356	2011-10-25	B	Inspection	361.0983	1	0
Lifepak 20 ˆ	KN030356	2012-10-29	B	Repair	370.0084	1	1
Lifepak 20 ˆ	KN030356	2013-10-25	B	Inspection	360.9365	1	0
Lifepak 20 ˆ	KN030356	2014-10-24	B	Inspection	363.9532	1	0
Lifepak 20 ˆ	KN030356	2016-01-04	B	Inspection	437.1581	1	0
Lifepak 20 ˆ	KN030359	2010-09-14	A	Failure	288.1373	1	2
Lifepak 20 ˆ	KN030359	2011-10-25	B	Repair	361.0983	1	1
Lifepak 20 ˆ	KN030359	2012-10-29	B	Repair	370.0084	1	1
Lifepak 20 ˆ	KN030359	2014-10-24	B	Inspection	724.8897	2	0
Lifepak 20 ˆ	KN030359	2015-11-02	B	Repair	374.1223	1	1

In the Table 4.1, 17 samples of the work-order after 2010 of three machines are shown. In the column WO, B means inspection and A means repair, we interpret it as that B means that after previous action, the machine is healthy, while A means that after previous action, the machine is not healthy. In the column Result, Inspection means the machine is healthy, Repair means the machine is unhealthy but operational, Failure means the machine is failure. So we can see three different results by A action and two different results by B action. Since our model in a discrete-time model, we rounded the time interval to year time unit. The Condition column is directly from the Result column.

After counting all transitions, we get the transition count matrix as follows. A one-time-unit transition matrix is desired.

Initial state	Final state		
	0	1	2
One year			
0	106	42	3
1	0	35	7
2	0	0	0
Two years			
0	2	1	0
1	0	0	0
2	0	0	0
Three years			
0	0	0	0
1	0	1	1
2	0	0	0

We set the initial transition matrix as:

$$M_0 = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}$$

By the algorithm introduced above, we can get the transition matrix of this type of machines.

$$M^{EM} = \begin{pmatrix} 0.7222 & 0.2596 & 0.0181 \\ 0 & 0.8306 & 0.1694 \\ 0 & 0 & 1 \end{pmatrix}$$

The transition matrix has been obtained, the result will be used in decision model in Section 4.3 for optimal inspection and maintenance decision making.

4.3 Decision Model

In this section, we will introduce the decision model. First, we introduce some basic function and variables, followed by an example with a single device. Then, an algorithm to calculate long-run average cost will be presented. Last, some numerical examples are presented and discussed.

4.3.1 Introduction

The inspection duration is negligible. After each inspection, $n+1$ decisions are made: a) the next inspection time; b) whether to repair the machines. Denote as $\mathbf{D} = [d_0, d_1, ..., d_n]$, where d_0 is the next inspection time, d_1 to d_n are binary variables to decide whether to repair the machines.

$$d_0 \in \{0, 1, ..., S\},\ d_n = \begin{cases} 1, & \text{if repairing the machine } n, n > 0 \\ 0, & \text{if not repairing the machine } n, n > 0. \end{cases}$$

During non-inspection epoch, no decision would be made. We set the decision $\mathbf{D} = [0, 0, ..., 0]$ for convenience, which means no change for next inspection time and no repair action occurred.

$$\mathbf{D} = \begin{cases} [0, 0, ..., 0], & \text{if } x_0 > 0 \\ [d_0, d_1, ..., d_n], d_0 > 0, & \text{if } x_0 = 0 \end{cases}$$

At the beginning of time t, after the decision and repair, the state of the system is:

$$\bar{\mathbf{x}}^t = [x_0^t + d_0^t, x_1^t(1 - d_1^t), x_2^t(1 - d_2^t), ..., x_n^t(1 - d_n^t)]$$

Between two consecutive time units, the state transition of one machine follows the transition matrix introduced in section 4.2:

$$P_1 = \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ 0 & p_{11} & p_{12} \\ 0 & 0 & 1 \end{pmatrix}$$

where p_{ij} is the transition probability from state i to state j.

So the transition probability of n machines is the Kronecker product of n transition matrices $P_1, P_2, ..., P_n$. Denote as P_N.

$$P_N = P_1 \otimes P_2 \otimes ... \otimes P_n$$

So there are 3^n states in the matrix P_N. Denote the set of the states as Ω.

We introduce $R(\cdot)$, which equals 1 if the machine i is functioning, and 0 if it has failed. Hence,

$$R_i(x_i) = \begin{cases} 1, & \text{if } x_i < 2, \\ 0, & \text{if } x_i \geq 2. \end{cases}$$

We introduce $Q(\cdot)$, which equals 1 if the system is working, and 0 if it is down.

$$Q(\mathbf{x}) = \begin{cases} 1, \text{ if } \sum_{i=1}^{n} R_i(x_i) \geq k, \\ 0, \text{ if } \sum_{i=1}^{n} R_i(x_i) < k, \end{cases}$$

where k is the minimum functioning components for the system, which means if there are less than k functioning components, the system has failure.

Again, we introduce $I(\cdot)$, which equals 1 if the device is in inspection, and 0 if it is not. Hence,

$$I(\mathbf{x}) = \begin{cases} 1, \text{if } x_0 = 0, \\ 0, \text{if } x_0 > 0. \end{cases}$$

The costs of the system are given by

c_n: Inspection cost, once an inspection happens, this cost is incurred. (per time)

c_c: Corrective cost, once a repair for a failure device happens, this cost is incurred. (per time, per device)

c_p: Preventive cost, once a repair for an unhealthy operational device happens, this cost is incurred. (per time, per device)

c_s: Setup cost, once a repair happens, this cost is incurred. (per time)

p_c: Penalty cost, once the system is down, this cost is incurred. (per time)

Let $C^{t*}(\mathbf{x}^t)$ denote the optimum cumulative cost from time t onward. Since the transition matrix is a discrete-state matrix, the cost function can be expressed as follows

$$\begin{aligned}
&C^{t*}\left(\mathbf{x}^t\right) \\
&\quad = \min_{\mathbf{D}} \Bigg\{ I\left(\mathbf{x}^t\right) c_n + p_c\left(1 - Q(\mathbf{x}^t)\right) + c_s\left(1 - \prod_{i=1}^{n}\left(1 - d_i^t\right)\right) \\
&\qquad + \sum_{i=1}^{n} d_i R_i\left(x_i^t\right) c_p + \sum_{i=1}^{n} d_i\left(1 - R_i\left(x_i^t\right)\right) c_c \\
&\qquad + \sum_{[x_1^{t+1}, x_2^{t+1}, \ldots, x_n^{t+1}] \in \Omega} C^{t+1*}\left([\bar{x}_0^t - 1, x_1^{t+1}, x_2^{t+1}, \ldots, x_n^{t+1}]\right) p_{[\bar{x}_1^t, \bar{x}_2^t, \ldots, \bar{x}_n^t],[x_1^{t+1}, x_2^{t+1}, \ldots, x_n^{t+1}]} \Bigg\}
\end{aligned} \tag{4.1}$$

where $[\bar{x}_0^t - 1, x_1^{t+1}, x_2^{t+1}, \ldots, x_n^{t+1}]$ is the state of $\mathbf{x}^{t+1}$, $\mathbf{x}^t$ is the condition at time t, d_i^t is the repair decision for device i at time t.

4.3.2 An Example of a Single Device

For convenience, we give an example about the Markov decision processes of this model with only one machine.

Suppose the initial state $\mathbf{x}^t = [0, 0]$, so $I(\mathbf{x}^t) = 1$, $R_1(x_1^t) = 1$, $Q(\mathbf{x}^t) = 1$.

Let $C^t(\mathbf{x}^t)$ denote the arbitrary cumulative cost from time t onward. If the decision is $\mathbf{D}^t = [1, 0]$, then $\bar{\mathbf{x}}^t = [1, 0]$.

$$\begin{aligned}
C^t\left(\mathbf{x}^t\right) &= 1 \times c_n + p_c(1 - 1) + c_s \times 0 + 0 \times 1 \times c_p + 0 \times (1 - 1) c_c + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^t x_1^{t+1}} C^{t+1}\left([0, x_1^{t+1}]\right) \\
&= c_n + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^t x_1^{t+1}} C^{t+1}\left([0, x_1^{t+1}]\right) \\
&= c_n + p_{00} C^{t+1}([0, 0]) + p_{01} C^{t+1}([0, 1]) + p_{02} C^{t+1}([0, 2]).
\end{aligned} \tag{4.2}$$

Suppose $x_1^{t+1} = 1$, so $I(\mathbf{x}^t) = 0,\ R_1\left(x_1^{t+1}\right) = 1,\ Q(\mathbf{x}^{t+1}) = 1$.

Let $C^{t+1}(\mathbf{x}^{t+1})$ denote the arbitrary cumulative cost from time $t+1$ onward.

If the decision is $\mathbf{D}^t = [1, 0]$, then $\bar{\mathbf{x}}^t = [1, 1]$.

$$\begin{aligned} C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 1 \times c_n + p_c(1-1) + c_s \times 0 + 0 \times 1 \times c_p + 0 \\ &\quad \times (1-1)\, c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left([0, x_1^{t+2}]\right) \\ &= c_n + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t} x_1^{t+1}} C^{t+1}\left([0, x_1^{t+1}]\right) \\ &= c_n + p_{11} C^{t+1}([0, 1]) + p_{12} C^{t+1}([0, 2]) \end{aligned} \tag{4.3}$$

If the decision is $\mathbf{D}^t = [1, 1]$, then $\bar{\mathbf{x}}^t = [1, 0]$.

$$\begin{aligned} C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 1 \times c_n + p_c(1-1) + c_s \times 1 + 1 \times 1 \times c_p + 1 \\ &\quad \times (1-1)\, c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left([0, x_1^{t+2}]\right) \\ &= c_n + c_s + c_p + p_{00} C^{t+2}([0, 0]) + p_{01} C^{t+2}([0, 1]) + p_{02} C^{t+2}([0, 2]) \end{aligned} \tag{4.4}$$

Suppose $x_1^{t+1} = 2$, so $I(\mathbf{x}^t) = 0,\ R_1\left(x_1^{t+1}\right) = 0,\ Q(\mathbf{x}^{t+1}) = 0$.

Let $C^{t+1}(\mathbf{x}^{t+1})$ denote the arbitrary cumulative cost from time $t+1$ onward.

If the decision is $\mathbf{D}^t = [1, 0]$, then $\bar{\mathbf{x}}^t = [1, 2]$.

$$\begin{aligned} C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 1 \times c_n + p_c(1-0) + c_s \times 0 + 0 \times 0 \times c_p + 0 \\ &\quad \times (1-0)\, c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left([0, x_1^{t+2}]\right) \\ &= c_n + p_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t} x_1^{t+1}} C^{t+1}\left([0, x_1^{t+1}]\right) \\ &= c_n + p_c + p_{22} C^{t+1}([0, 2]) \end{aligned} \tag{4.5}$$

If the decision is $\mathbf{D}^t = [1, 1]$, then $\bar{\mathbf{x}}^t = [1, 0]$.

$$\begin{aligned} C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 1 \times c_n + p_c\left(1-0\right) + c_s \times 1 + 1 \times 0 \times c_p + 1 \\ &\quad \times \left(1-0\right)c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1}x_1^{t+2}} C^{t+2}\left(\left[0, x_1^{t+2}\right]\right) \\ &= c_n + p_c + c_s + c_c + p_{00}C^{t+2}\left([0,0]\right) + p_{01}C^{t+2}\left([0,1]\right) + p_{02}C^{t+2}\left([0,2]\right) \end{aligned} \tag{4.6}$$

By Equations (4.2) to (4.6), we can get the costs and conditions for one time unit interval. We summarize the results in Table 4.2. Furthermore, we present explicitly the recursive functions for all cases when the inspection interval is greater than one in Appendix B. Here, we summarize all situations in Table 4.2.

Note that in Table 4.2, the first column is the condition of the system, the second column is the decision, the third and fourth column are corresponding costs and possible next conditions given current condition and decision. For example, if the condition of the system is [0,1], and the decision is [1,1], which means we inspect the system, and the condition of the device is unhealthy and we decide to repair it, therefore the costs consist of inspection cost (c_n), set-up cost (c_s) and preventive repair cost (c_p). After the repair, the condition of the system is [0,0], so the next possible conditions are {[0,0],[0,1],[0,2]}.

Table 4.2: Summary of the one-machine system

Condition	Decision	Costs	Next Condition
If the inspection interval is one.			
[0,0]	1,0	c_n	[0,0],[0,1],[0,2]
[0,1]	1,0	c_n	[0,1],[0,2]
[0,1]	1,1	$c_n + c_s + c_p$	[0,0],[0,1],[0,2]
[0,2]	1,0	$c_n + p_c$	[0,2]
[0,2]	1,1	$c_n + p_c + c_s + c_c$	[0,0],[0,1],[0,2]
If the inspection interval is two.			
[0,0]	2,0	c_n	[1,0],[1,1],[1,2]
[1,0]	0,0	0	[0,0],[0,1],[0,2]
[1,1]	0,0	0	[0,1],[0,2]
[1,2]	0,0	p_c	[0,2]
[0,1]	2,0	c_n	[1,1],[1,2]
[0,1]	2,1	$c_n + c_s + c_p$	[1,0],[1,1],[1,2]
[0,2]	2,0	$c_n + p_c$	[1,2]
[0,2]	2,1	$c_n + p_c + c_s + c_c$	[1,0],[1,1],[1,2]
If the inspection interval is n.			
[0,0]	n,0	c_n	[n-1,0],[n-1,1],[n-1,2]
[n-1,0]	0,0	0	[n-2,0],[n-2,1],[n-2,2]
[n-1,1]	0,0	0	[n-2,1],[n-2,2]
[n-1,2]	0,0	p_c	[n-2,2]
[0,1]	n,0	c_n	[n-1,1],[n-1,2]
[0,1]	n,1	$c_n + c_s + c_p$	[n-1,0],[n-1,1],[n-1,2]
[0,2]	n,0	$c_n + p_c$	[n-1,2]
[0,2]	n,1	$c_n + p_c + c_s + c_c$	[n-1,0],[n-1,1],[n-1,2]

4.3.3 Long-run Average Cost

In the previous section, we use an example to present the Markov Decision Processes, in this section, we are going to show an algorithm to solve this model.

First, we clarify that the object is minimizing the long-run average cost. Based on the assumptions in Section 4.1, we can see that 1) the costs and transition probabilities are stationary and time homogeneous; 2) for any states and actions, the cost is bounded; 3) the state space is finite. Under these three assumptions, all stationary policies would generate a single irreducible class Markov chain. Before we illustrate the algorithm, we show the long-run average cost for a stationary policy. In our problem, for each state, a decision related to inspection and maintenance would be made, which is denoted as **D** in the previous section. For a given set of decisions, we have a stationary policy, say R. We say this policy is a unichain policy, which means this is a finite-state Markov chain with single recurrent class. Therefore, the Markov chain will visit a recurrent state in finite transitions. We denote n-step transition probability under policy R as $P_{ij}^n(R)$.

$$P_{ij}^n(R) = P\{X_{n+k} = j | X_k = i\}, n \geq 0, i, j \in \Omega$$

where Ω is the set of states. By Chapmen-Kolmogorov equations, we have

$$P_{ij}^{n+m}(R) = \sum_{k=0}^{\infty} P_{ik}^n(R) P_{kj}^m(R) \text{ for all } n, m \geq 0, i, j \in \Omega$$

Denote by $V_n(i, R)$ the total expected costs during first n decision epochs when the starting state is i and policy is R, we have

$$V_n(i, R) = \sum_{t=0}^{n-1} \sum_{j \in \Omega} P_{ij}^t(R) c_j(R)$$

where $c_j(R)$ is the total cost in state j under policy R. Then we have the average cost function

$$g_i(R) = \lim_{n \to \infty} \frac{1}{n} V_n(i, R), i \in \Omega$$

By the theorem of stationary probability, we can prove that $g(R) = g_i(R)$, which means the average costs are independent of the initial state. Next, we will present an algorithm to calculate the minimal long-run average cost and optimal policy.

There are several method to calculate the optimal policy, *e.g.* Relative Value Iteration, Policy Iteration and Linear Programming [33].

The method we applied is the relative value iteration. This method is a very mature method. We refer to literature [33] for the functions.

Let $\mathbf{v}^n$ be the optimal n-stage costs, $r_{\mathbf{D}}$ be a vector for the cost under decision $\mathbf{D}$ in all states and $P_{\mathbf{D}}$ be the transition matrix under decision $\mathbf{D}$. We present the algorithm as follows, refer to literature [33, 42, 4] for more details.

Step 1. Choose arbitrary values for $\mathbf{v}^0$. For simplicity, set $\mathbf{v}^0 = 0$. Choose a fixed state s. Specify $\varepsilon > 0$, set $\mathbf{w}^0 = \mathbf{v}^0 - \mathbf{v}^0(s)\,\mathbf{e}$, where $\mathbf{e}$ is the vector of ones, and set $n = 0$, and maximum iteration number N.

Step 2. Set $\mathbf{v}^{n+1} = \min_{\mathbf{D}\in R}\{r_{\mathbf{D}} + P_{\mathbf{D}}\mathbf{w}^n\}$, and $\mathbf{w}^{n+1} = \mathbf{v}^{n+1} - \mathbf{v}^{n+1}(s)\,\mathbf{e}$.

Step 3. If $sp(\mathbf{v}^{n+1} - \mathbf{v}^n) < \varepsilon$ or $n > N$ go to Step 4, otherwise, increment n by 1 and return to Step 2.

Step 4. Choose $\mathbf{D}^* = \arg\min_{\mathbf{D}\in R}\{r_{\mathbf{D}} + P_{\mathbf{D}}\mathbf{v}^n\}$.

However, in our problem, for any policy with inspection interval greater than 1, the Markov chain is not aperiodic, or we say the Markov chain is not ergodic [35]. So we have to apply some transformation to make sure all the policies with aperiodic transition probability matrix as follows.

Choose $\gamma \in (0, 1)$, denote $\tilde{r}_{\mathbf{D}}$ be the transformed cost under decision $\mathbf{D}$ and $\tilde{P}_{\mathbf{D}}$ be the transformed transition matrix under decision $\mathbf{D}$. We have

$$\tilde{P}_{\mathbf{D}} = (1-\gamma)I + \gamma P_{\mathbf{D}}$$

$$\tilde{r}_{\mathbf{D}} = \gamma r_{\mathbf{D}}$$

Then for all stationary policies, the transition matrix have strictly positive diagonal entries and are aperiodic. The optimal decisions would be the same as before, but the average cost has changed since the cost has multiplied by γ, so the result should be devided by γ to get the desired average cost.

4.3.4 Numerical Examples

Before the numerical examples, we summarize the computation steps as follows:

Step 1: Using maintenance data, estimate the transition matrix by the EM algorithm. (See Section 4.2)

Step 2: Model the problem to a Markov decision processes. Calculate the costs and transition probability for given conditions and decisions. (See Section 4.3.1 and Section 4.3.2)

Step 3: Calculate the optimal policy and the minimal long-run average cost by relative value iteration algorithm. (See Section 4.3.3)

Example 4.3.1 (*1-out-of-2* system) *In a location called Resus Room, two 'Lifepak 20' defibrillator machines are available, suppose at least one machine should be operational for this location. So this is a 1-out-of-2 system. For the system, the components are identical. For each components, there are 3 states (0, 1, and 2), and state 2 is the failure state. By the Markov decision processes, the optimal maintenance strategy will be found.*

The transition matrix is:

$$M^{EM} = \begin{pmatrix} 0.7222 & 0.2596 & 0.0181 \\ 0 & 0.8306 & 0.1694 \\ 0 & 0 & 1 \end{pmatrix}$$

For the cost parameters, we suppose the inspection cost is 100 per time, the set up cost is 50 per time, the preventive repair cost is 150 per time per device, the corrective repair cost is 200 per time per device and the penalty cost is 1000 per time. Denoted as: $c_n : 100$*;* $c_c : 200$*;* $c_p : 150$*;* $c_s : 50$*;* $p_c : 1000$*.*

The optimal maintenance decisions are shown in Table 4.3, and the average cost is 116.0181. The percentage of downtime of the system is shown in Figure 4.2.

Table 4.3: The optimal decisions for *1-out-of-2* system

Condition [x_1,x_2]	Repair [d_1,d_2]	Inspection interval[d_0]
0,0	[0,0]	4
0,1	[0,0]	3
0,2	[0,1]	4
1,0	[0,0]	3
1,1	[1,1]	4
1,2	[1,1]	4
2,0	[1,0]	4
2,1	[1,1]	4
2,2	[1,1]	4

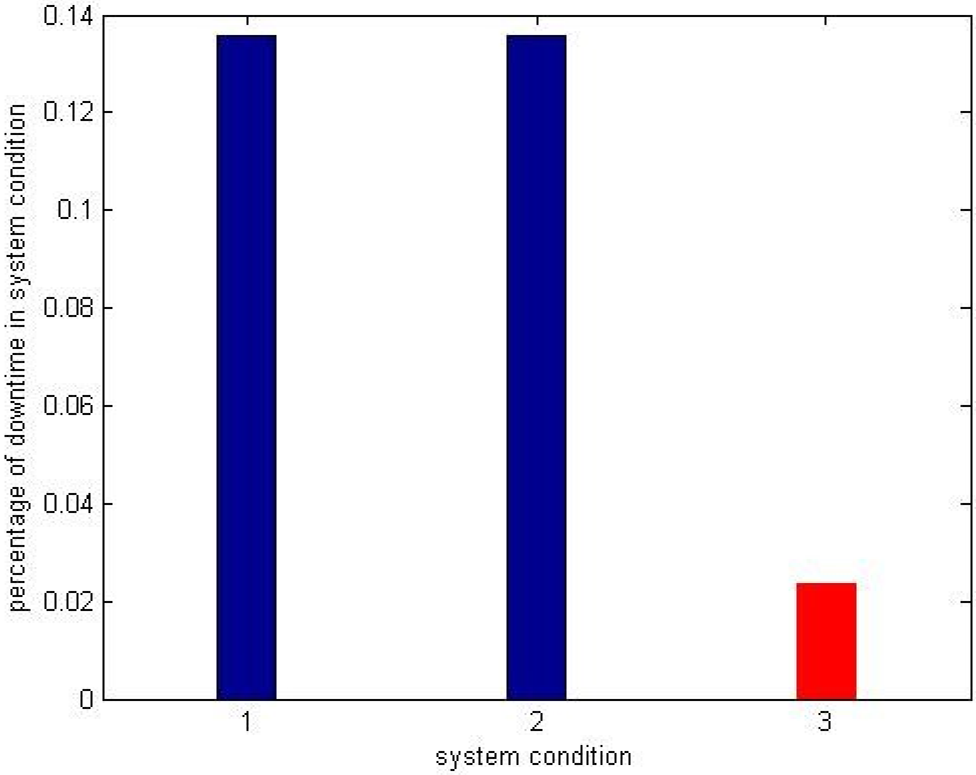


Figure 4.2: The percentage of downtime of the *1-out-of-2* system.
(The blue bars are for the machines, the red bar is for the system)

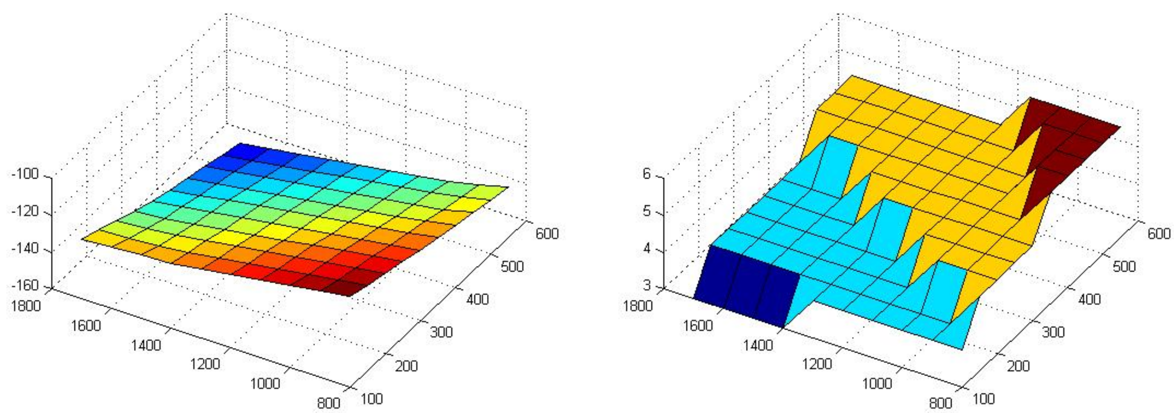
From Table 4.3, we can see that the inspection interval is actually a function of the condition after repair, we denote it as $d_0 = f([\bar{x}_1, \bar{x}_2, ..., \bar{x}_n])$. In this example, the relationship is:

$$d_0 = \begin{cases} 4, & \text{if } [\bar{x}_1, \bar{x}_2] = [0,0], \\ 3, & \text{if } [\bar{x}_1, \bar{x}_2] = [0,1] \text{or } [\bar{x}_1, \bar{x}_2] = [1,0]. \end{cases}$$

According to the optimal decisions, we can see that the inspection intervals are four years for most conditions, when only one machine is unhealthy, the inspection interval is three years. For the maintenance decisions, a machine would be repaired if the condition is failure or another machine is not healthy.

Example 4.3.2 (Inspection and Penalty Cost Analysis) *Based on Example 4.3.1, the parameters of the model will be changed to see the difference of the costs and the optimal decisions. First, we let the inspection cost to be a set of values {100, 150, 200, 250, 300, 350, 400, 450, 500, 550}, and the penalty cost to be a set of values {800, 900, 1000, 1100, 1200, 1300, 1400, 1500, 1600, 1700}. All the other parameters are the same as the Example 4.3.1.*

The minimal average costs and d_0 for new system are shown in Figure 4.3a and Figure 4.3b. With the increasing of the inspection cost and penalty cost, the minimal average cost increase.



(a) The minimal average costs

(b) Inspection interval for new system

Figure 4.3: The optimal results with different c_n and p_c.

There are five different strategies for different parameters. For convenience, they are named as S1 to S5 and presented in Table 4.4.

Table 4.4: Five different strategies for different c_n and p_c

Condition	S1			S2			S3			S4			S5		
[x1,x2]	d0	d1	d2	d0	d1	d2	d0	d1	d2	d0	d1	d2	d0	d1	d2
00	3	0	0	4	0	0	4	0	0	5	0	0	5	0	0
01	2	0	0	2	0	0	3	0	0	3	0	0	4	0	0
02	3	0	1	4	0	1	4	0	1	5	0	1	5	0	1
10	2	0	0	2	0	0	3	0	0	3	0	0	4	0	0
11	3	1	1	4	1	1	4	1	1	5	1	1	5	1	1
12	3	1	1	4	1	1	4	1	1	5	1	1	5	1	1
20	3	1	0	4	1	0	4	1	0	5	1	0	5	1	0
21	3	1	1	4	1	1	4	1	1	5	1	1	5	1	1
22	3	1	1	4	1	1	4	1	1	5	1	1	5	1	1

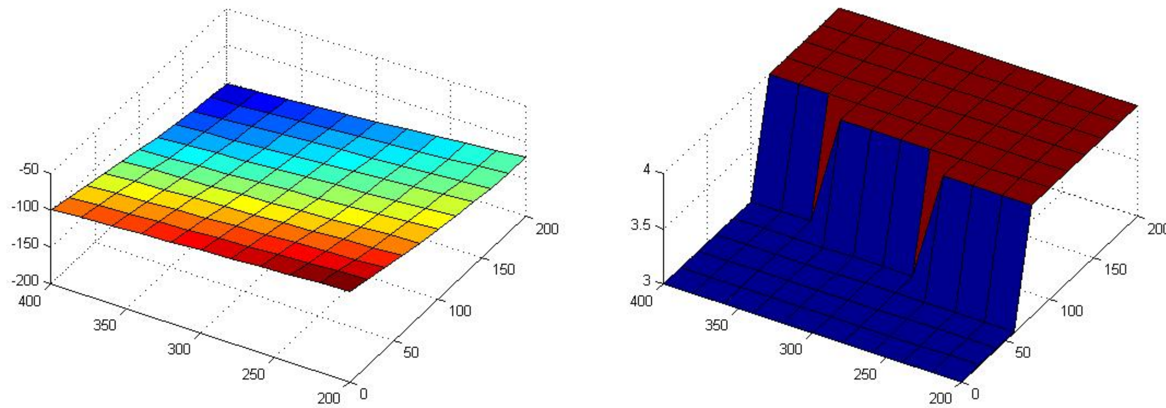
For different inspection cost and penalty cost, we obtain different optimal strategy (Table 4.5). Obviously, the higher the inspection cost is, the longer the inspection interval is. If the penalty cost is higher, the inspection interval would be shorter. Because more frequent inspection would reduce the percentage of the downtime of the system.

Table 4.5: The optimal strategies for different inspection cost and penalty cost

p_c \ c_n	100	150	200	250	300	350	400	450	500	550
800	S3	S3	S3	S3	S4	S4	S4	S5	S5	S5
900	S3	S3	S3	S3	S3	S3	S3	S4	S4	S4
1000	S3	S3	S3	S3	S3	S3	S3	S3	S3	S3
1100	S3	S3	S3	S3	S3	S3	S3	S3	S3	S3
1200	S3	S3	S3	S3	S3	S3	S3	S3	S3	S3
1300	S3	S3	S3	S3	S3	S3	S3	S3	S3	S3
1400	S1	S3	S3	S3	S3	S3	S3	S3	S3	S3
1500	S1	S1	S3	S3	S3	S3	S3	S3	S3	S3
1600	S1	S1	S1	S2	S3	S3	S3	S3	S3	S3
1700	S1	S1	S1	S1	S2	S3	S3	S3	S3	S3

Example 4.3.3 (Preventive Repair and Corrective Repair Cost Analysis) *We let the preventive repair cost to be a set of values {0, 20, 40, 60, 80, 100, 120, 140, 160, 180, 200}, and the corrective repair cost to be a set of values {200, 220, 240, 260, 280, 300, 320, 340, 360, 380, 400}. All the other parameters are the same as the Example 4.3.1.*

The minimal average costs and d_0 for new system are shown in Figure 4.4a and Figure 4.4b. Obviously, the minimal average cost would increase with the increasing of the repair costs.



(a) The minimal average costs (b) Inspection interval for new system

Figure 4.4: The optimal results with different c_p and c_c.

There are seven different strategies for different parameters. For convenience, they are named as S0 to S6 and presented in Table 4.6.

Table 4.6: Seven different strategies for different c_p and c_c

Condition	S0			S1			S2			S3			S4			S5			S6		
[x1,x2]	d0	d1	d2	d0	d1	d2	d0	d1	d2	d0	d1	d2	d0	d1	d2	d0	d1	d2	d0	d1	d2
00	3	0	0	4	0	0	4	0	0	4	0	0	4	0	0	4	0	0	4	0	0
01	3	0	1	4	0	1	3	0	0	3	0	0	3	0	0	4	0	1	3	0	0
02	3	0	1	4	0	1	4	0	1	4	0	1	4	0	1	2	0	0	2	0	0
10	3	1	0	4	1	0	3	0	0	3	0	0	3	0	0	4	1	0	3	0	0
11	3	1	1	4	1	1	4	1	1	3	0	1	2	0	0	4	1	1	4	1	1
12	3	1	1	4	1	1	4	1	1	3	1	0	3	0	1	4	1	1	4	1	1
20	3	1	0	4	1	0	4	1	0	4	1	0	4	1	0	2	0	0	2	0	0
21	3	1	1	4	1	1	4	1	1	3	1	0	3	1	0	4	1	1	4	1	1
22	3	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1

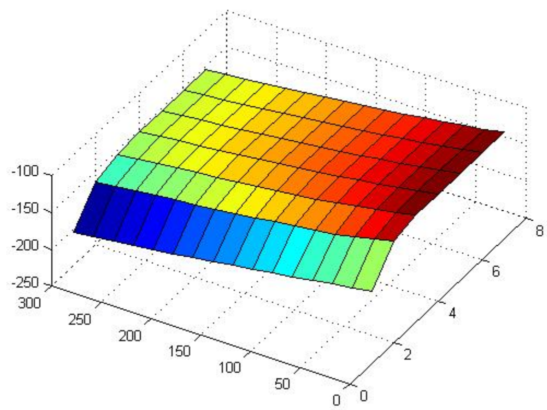
For different preventive repair cost and corrective repair cost, we have different best strategy (Table 4.7). We can see a trend in the table from the lower left corner (S0) to the upper right corner (S5). From S0 to S6, the optimal strategies are becoming more and more risky, which means the repair action is less frequently conducted and the inspection interval is longer.

Table 4.7: The optimal strategies for different preventive repair cost and corrective repair cost

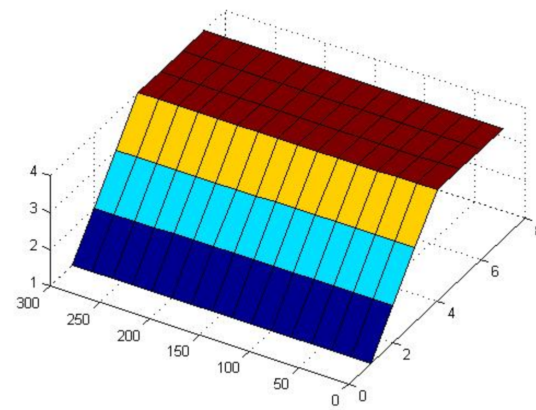
c_p / c_c	0	20	40	60	80	100	120	140	160	180	200
200	S0	S0	S0	S0	S1	S2	S2	S2	S3	S4	S4
220	S0	S0	S0	S0	S1	S2	S2	S2	S2	S3	S4
240	S0	S0	S0	S0	S1	S1	S2	S2	S2	S3	S4
260	S0	S0	S0	S0	S1	S1	S2	S2	S2	S3	S4
280	S0	S0	S0	S0	S0	S1	S2	S2	S2	S2	S3
300	S0	S0	S0	S0	S0	S1	S1	S2	S2	S2	S3
320	S0	S0	S0	S0	S0	S1	S1	S2	S2	S2	S3
340	S0	S0	S0	S0	S0	S1	S1	S2	S2	S2	S2
360	S0	S0	S0	S0	S0	S0	S1	S2	S2	S2	S2
380	S0	S0	S0	S0	S0	S0	S1	S1	S2	S2	S2
400	S0	S0	S0	S0	S0	S0	S1	S5	S6	S6	S6

Example 4.3.4 (Potential inspection interval and Setup cost) *We let the potential inspection interval to be a set of values {1, 2, 3, 4, 5, 6, 7}, and the setup cost to be a set of values {0, 20, 40, 60, 80, 100, 120, 140, 160, 180, 200, 220, 240, 260, 280, 300}. All the other parameters are the same as the Example 4.3.1.*

The minimal average costs and d_0 for new system are shown in Figure 4.5. We can see that sufficient potential inspection interval is important for the cost optimizing. While the setup cost does not have great influence on the problem when the cost is not high.



(a) The minimal average costs



(b) Inspection interval for new system

Figure 4.5: The optimal results with different inspection interval and c_s.

A *k-out-of-n* system is a typical system with economic dependence, a *k-out-of-n* system works if at least k components works. The *1-out-of-2* system has been discussed, but in practice, both k and n are important system parameters, and they could be any positive integer numbers as long as k less than or equal to n. In this section, we will use the same data as before, but change the value of these two parameters. Some interesting patterns are founded, which are presented by the following two examples.

Example 4.3.5 (Constant k with variable n) *we consider a system with a constant k but variable n. Following the Example 4.3.1, 1-out-of-1, 1-out-of-3 and 1-out-of-4 systems would be discussed.*

For *1-out-of-1* system, the optimal maintenance decisions are shown in Table 4.8, and the average cost is 146.4859. The percentage of downtime of the system is shown in Figure 4.6a. The percentage of downtime of the system is 0.0468.

Table 4.8: The optimal decision for *1-out-of-1* system

Condition	Repair	Inspection
0	0	2
1	1	2
2	1	2

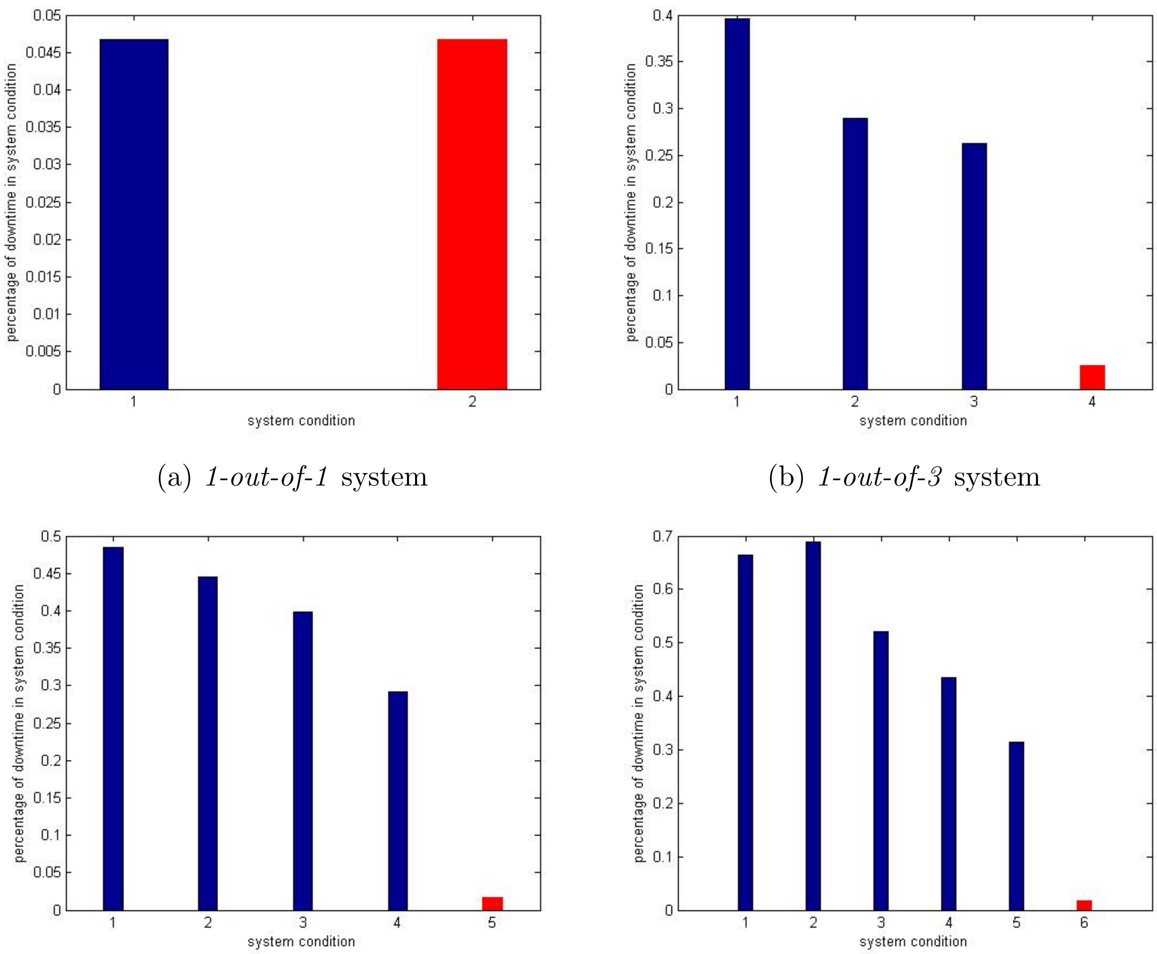
For *1-out-of-3* system, the optimal maintenance decisions are shown in Table 4.9, and the average cost is 108.8438. The percentage of downtime of the system is shown in Figure 4.6b. The percentage of downtime of the system is 0.0247.

Table 4.9: The optimal decision for *1-out-of-3* system

Condition	Repair	Inspection	Condition	Repair	Inspection	Condition	Repair	Inspection
0,0,0	[0,0,0]	5	1,0,0	[0,0,0]	5	2,0,0	[0,0,0]	4
0,0,1	[0,0,0]	5	1,0,1	[0,0,0]	4	2,0,1	[0,0,0]	3
0,0,2	[0,0,0]	4	1,0,2	[0,0,0]	3	2,0,2	[0,0,1]	4
0,1,0	[0,0,0]	5	1,1,0	[0,0,0]	4	2,1,0	[0,0,0]	3
0,1,1	[0,0,0]	4	1,1,1	[0,0,0]	3	2,1,1	[1,0,0]	4
0,1,2	[0,0,0]	3	1,1,2	[0,0,1]	4	2,1,2	[1,0,1]	5
0,2,0	[0,0,0]	4	1,2,0	[0,0,0]	3	2,2,0	[0,1,0]	4
0,2,1	[0,0,0]	3	1,2,1	[0,1,0]	4	2,2,1	[1,1,0]	5
0,2,2	[0,0,1]	4	1,2,2	[0,1,1]	5	2,2,2	[0,1,1]	4

For *1-out-of-4* system, the average cost is 108.2322. The percentage of downtime of the system is shown in Figure 4.6c. The percentage of downtime of the system is 0.0165.

For *1-out-of-5* system, the average cost is 108.2322. The percentage of downtime of the system is shown in Figure 4.6d. The percentage of downtime of the system is 0.0165.



(a) *1-out-of-1* system

(b) *1-out-of-3* system

(c) *1-out-of-4* system

(d) *1-out-of-5* system

Figure 4.6: The percentage of downtime of the *1-out-of-n* systems.

We summarize the results of the systems with different number of machines as follows:

Table 4.10: Summary the results of the systems with different number of machines

Systems	Average cost	Percentage of downtime
1-out-of-1	146.4859	0.0468
1-out-of-2	116.0181	0.0235
1-out-of-3	108.8438	0.0247
1-out-of-4	108.2322	0.0165
1-out-of-5	108.2322	0.0165

Example 4.3.6 (Constant n with variable k) *We consider a system with a constant n but variable k. The 1-out-of-3, 2-out-of-3 and 3-out-of-3 systems would be compared and discussed.*

For *2-out-of-3* system, the average cost is 174.7510. The optimal decisions and the percentage of downtime presented below:

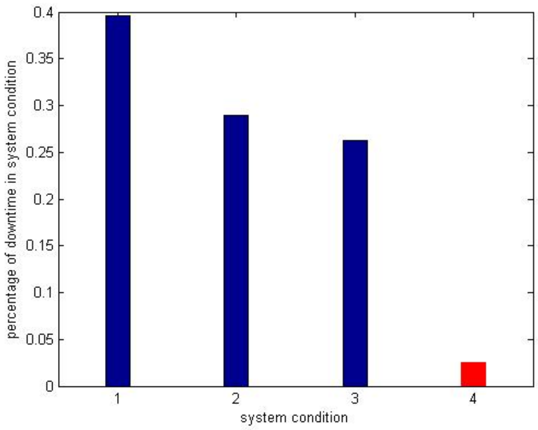
Table 4.11: The optimal decision for *2-out-of-3* system

Condition	Repair	inspection	Condition	Repair	inspection	Condition	Repair	inspection
0,0,0	[0,0,0]	3	1,0,0	[0,0,0]	2	2,0,0	[1,0,0]	3
0,0,1	[0,0,0]	2	1,0,1	[1,0,1]	3	2,0,1	[1,0,1]	3
0,0,2	[0,0,1]	3	1,0,2	[1,0,1]	3	2,0,2	[1,0,1]	3
0,1,0	[0,0,0]	2	1,1,0	[1,1,0]	3	2,1,0	[1,1,0]	3
0,1,1	[0,1,1]	3	1,1,1	[1,1,1]	3	2,1,1	[1,1,1]	3
0,1,2	[0,1,1]	3	1,1,2	[1,1,1]	3	2,1,2	[1,1,1]	3
0,2,0	[0,1,0]	3	1,2,0	[1,1,0]	3	2,2,0	[1,1,0]	3
0,2,1	[0,1,1]	3	1,2,1	[1,1,1]	3	2,2,1	[1,1,1]	3
0,2,2	[0,1,1]	3	1,2,2	[1,1,1]	3	2,2,2	[1,1,1]	3

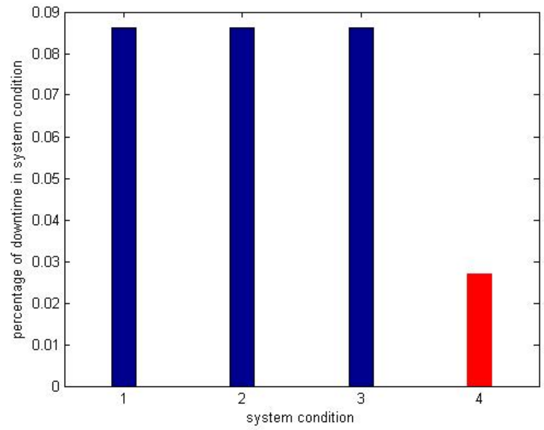
If $k = 3$, then we have a *3-out-of-3* system, the average cost is 312.5183. The optimal decisions and the percentage of downtime presented below:

Table 4.12: The optimal decision for *3-out-of-3* system

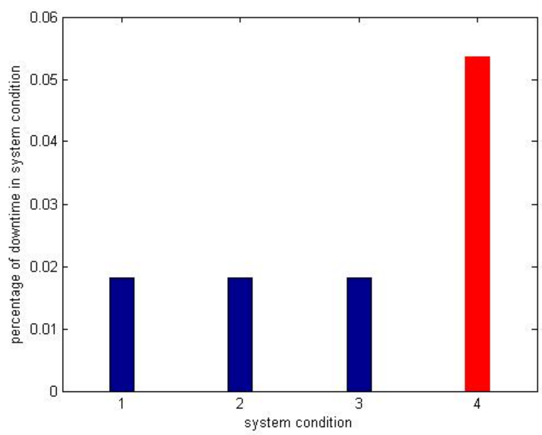
Condition	Repair	inspection	Condition	Repair	inspection	Condition	Repair	inspection
0,0,0	[0,0,0]	1	1,0,0	[1,0,0]	1	2,0,0	[1,0,0]	1
0,0,1	[0,0,1]	1	1,0,1	[1,0,1]	1	2,0,1	[1,0,1]	1
0,0,2	[0,0,1]	1	1,0,2	[1,0,1]	1	2,0,2	[1,0,1]	1
0,1,0	[0,1,0]	1	1,1,0	[1,1,0]	1	2,1,0	[1,1,0]	1
0,1,1	[0,1,1]	1	1,1,1	[1,1,1]	1	2,1,1	[1,1,1]	1
0,1,2	[0,1,1]	1	1,1,2	[1,1,1]	1	2,1,2	[1,1,1]	1
0,2,0	[0,1,0]	1	1,2,0	[1,1,0]	1	2,2,0	[1,1,0]	1
0,2,1	[0,1,1]	1	1,2,1	[1,1,1]	1	2,2,1	[1,1,1]	1
0,2,2	[0,1,1]	1	1,2,2	[1,1,1]	1	2,2,2	[1,1,1]	1



(a) *1-out-of-3* system



(b) *2-out-of-3* system



(c) *3-out-of-3* system

Figure 4.7: The percentage of downtime of the *k-out-of-3* systems.

We summarize the results of the systems with a constant n and variable k as follows:

Table 4.13: Summary the results of the systems with different number of k

Systems	Average cost	Percentage of downtime
1-out-of-3	108.8438	0.0274
2-out-of-3	174.7510	0.0269
3-out-of-3	312.5183	0.0536

Example 4.3.7 (Non-identical machines) *Before, we assume the machines are identical. In this part, we discussed the system with non-identical machines. If the machines have different deterioration rate and repair cost, the optimal decisions would not be similar to the system with identical machines.*

Suppose there are two machines in the system, one is the same as the Example 4.3.1, another one has a different deterioration rate. We have two deterioration matrix:

$$M_1 = \begin{pmatrix} 0.7222 & 0.2596 & 0.0181 \\ 0 & 0.8306 & 0.1694 \\ 0 & 0 & 1 \end{pmatrix}$$

,

$$M_2 = \begin{pmatrix} 0.8950 & 0.0668 & 0.0382 \\ 0 & 0.8393 & 0.1607 \\ 0 & 0 & 1 \end{pmatrix}$$

.

Apparently, the second machine has lower deterioration rate. The optimal maintenance decisions are shown in Table 4.14, and the average cost is 95.5293. The percentage of downtime of the system is shown in Figure4.8.

Table 4.14: The optimal decision for *1-out-of-2* system with different devices

Condition [x1,x2],	Repair[d1,d2],	Inspection interval[d0]
0,0	[0,0]	4
0,1	[0,1]	4
0,2	[0,1]	4
1,0	[0,0]	3
1,1	[1,1]	4
1,2	[1,1]	4
2,0	[1,0]	4
2,1	[1,1]	4
2,2	[1,1]	4

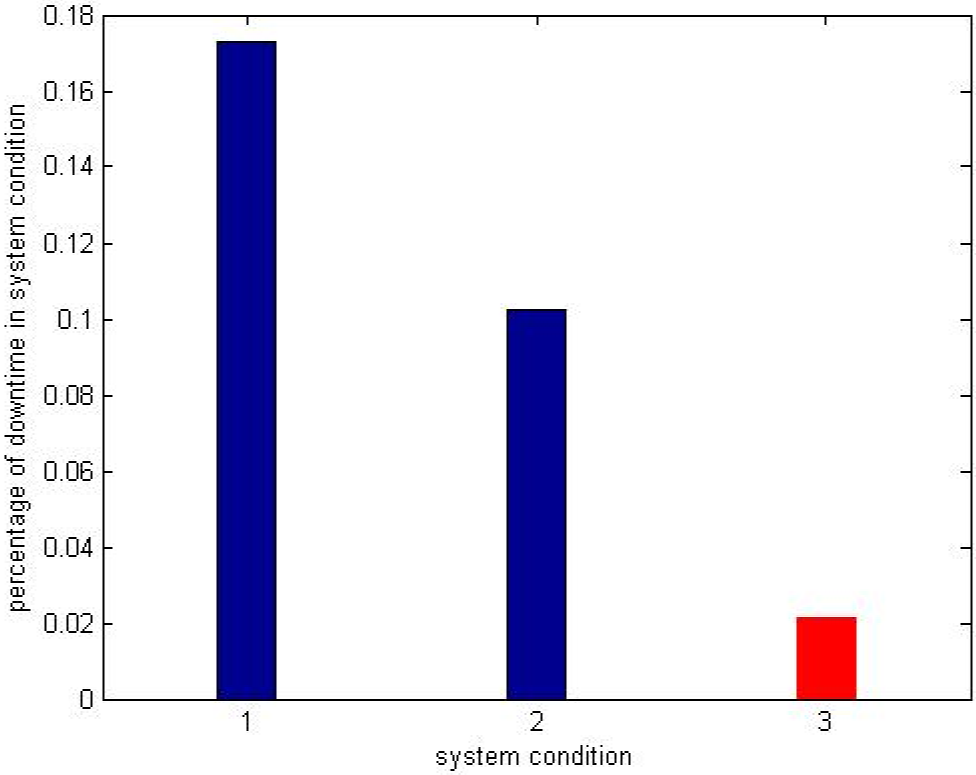


Figure 4.8: The percentage of downtime of the *1-out-of-2* system with different devices.

By the optimal decisions, we can see that the inspection intervals are four years for most conditions, only when the first one machine is unhealthy, the inspection interval is three years. For the maintenance decisions, the first machine would be repaired if the condition is failure or both machines are not healthy, the second machine would be repair if the machine is not healthy.

Example 4.3.8 (Comparison with periodical inspection) *The model we adapted is a non-periodical inspection model, which means the inspection interval could be different for different conditions. Some research studied periodical inspection model, which means the inspection interval would be the same for all different conditions. Apparently, non-periodical inspection model is a more efficient model, since the inspection interval is more flexible,* *and the strategies of non-periodical inspection model include the strategies of periodical inspection model. We will use this numerical example to show the result.All the parameters are the same as the Example 4.3.1.*

The minimal average cost for both models are show in Figure 4.9. In Table 4.3, we get the optimal maintenance strategy for non-periodical model, and the inspection interval could be 3 or 4 in that example. In the Figure 4.9, the blue line represents the value of non-periodical inspection model, and it arrives the minimal value when the max inspection interval greater than 3. The green line represents the value of periodical inspection model, the optimal value is 118.1126 when inspection interval is 3, but it is still higher than the minimal value of non-periodical inspection model, which is 118.0181.

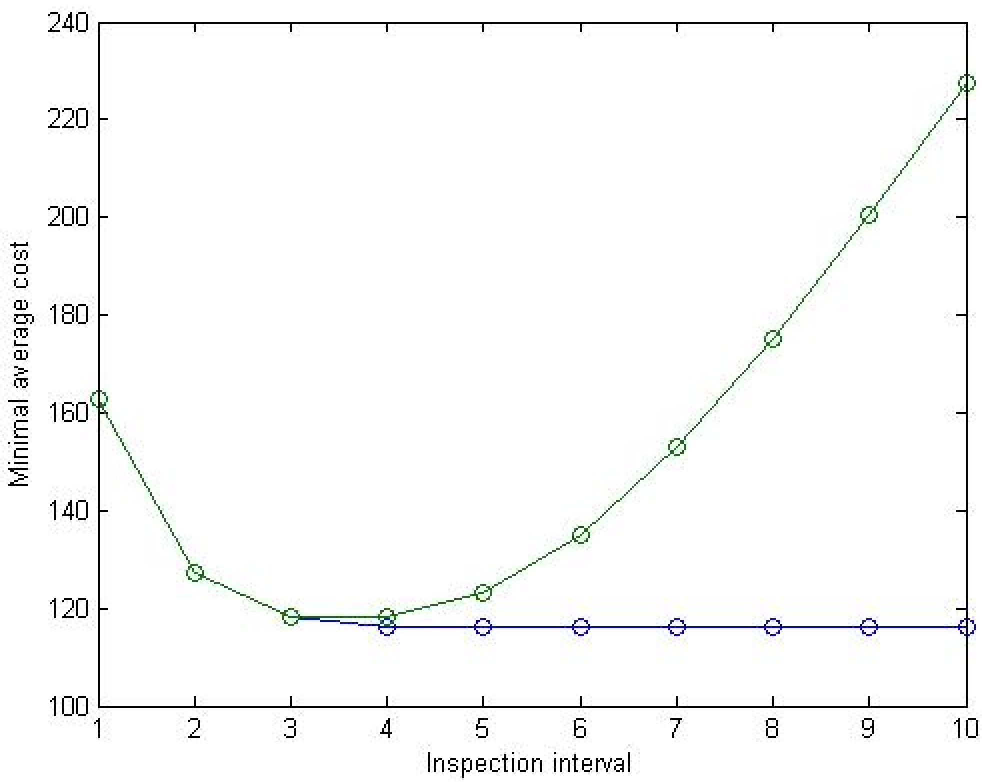
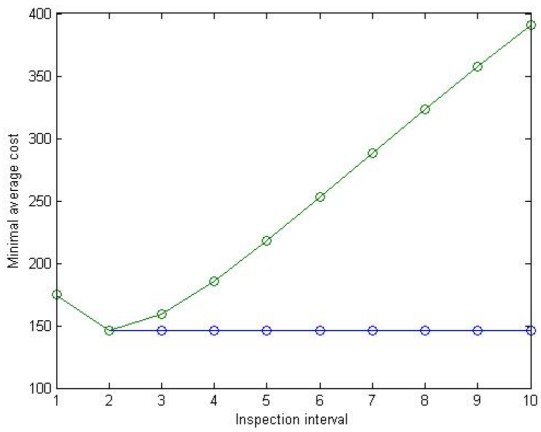
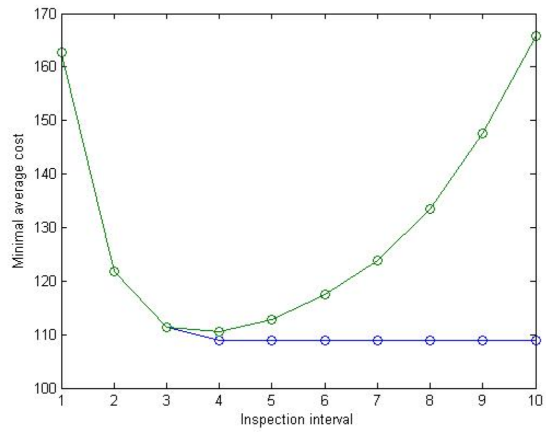


Figure 4.9: The comparison with periodical inspection for the *1-out-of-2* system.

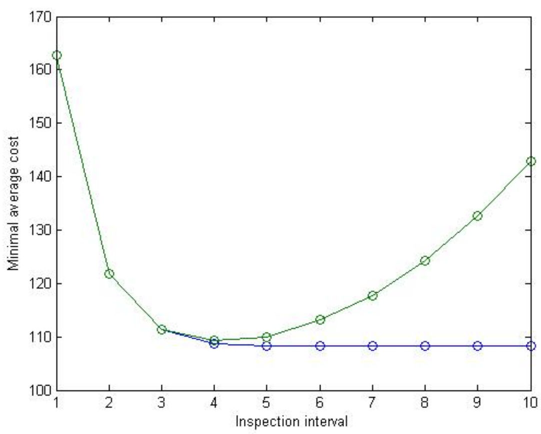
We can consider more examples with different n in Example 4.3.5 and compare the result with periodical inspection model.(See Figure 4.10).



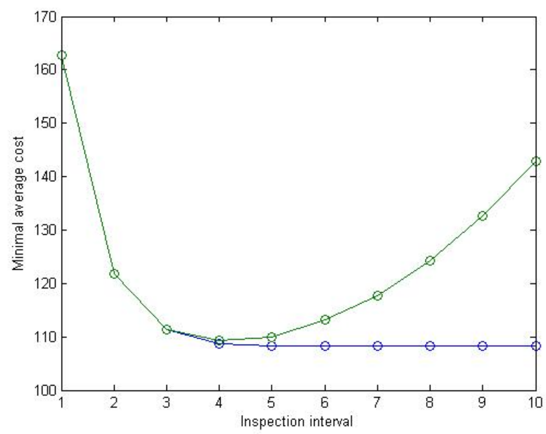
(a) *1-out-of-1* system



(b) *1-out-of-3* system



(c) *1-out-of-4* system



(d) *1-out-of-5* system

Figure 4.10: The comparison with periodical inspection for the *1-out-of-n* systems.

We summarize the results in Table 4.15.

Table 4.15: The summary of the comparison results

Number of machine	Periodical inspection	Non-periodical inspection	Improvement
1	146.4859	146.4859	0.00%
2	118.1126	118.0181	0.08%
3	110.4999	108.8438	1.50%
4	109.3225	108.2322	1.00%
5	109.3225	108.2322	1.00%

4.4 Case Study

In this section, we analyze a real case by our models.

There is an area called Emergency Resuscitation in some hospitals. Patients can get immediate care in cardiac arrest, airway, breathing and circulation compromise. All resuscitative equipment (monitors, defibrillators, airway, intubation and surgical equipment) available in the 'Resus' area.

In the data, we find there are two kinds of medical devices available, including three defibrillators and a suction unit. We need to find out the optimal management strategy for this 'Resus' area.

Table 4.16: The devices in 'Emergency Resus'

Term	Manufacturer	Model	Asset	Description	Location
Defibrillator/monitor	Philips Medical Systems	M3535A ˆHeartStart MRx	18641	Emergency	Emergency Resus
Defibrillator/monitor	Philips Medical Systems	M3535A ˆHeartStart MRx	18647	Emergency	Emergency Resus
Defibrillator/monitor	Philips Medical Systems	M3535A ˆHeartStart MRx	18648	Emergency	Emergency Resus
Suction unit	Devilbiss Healthcare	7305P-D	KN053264	Emergency Physician	Emergency Resus

Table 4.17: Risk factors of the medical devices

Term	Asset	Location	Equipment Function	Location of Use	Frequency of Use
Defibrillator/monitor	18641	Emergency Resus	6	5	3
Defibrillator/monitor	18647	Emergency Resus	6	5	3
Defibrillator/monitor	18648	Emergency Resus	6	5	3
Suction unit	KN053264	Emergency Resus	5	2	3

By the classification result (e.g. the classification tree 3.1 in Chapter 3), the defibrillator/monitors are classified into high risk level, and the suction unit is classified into medium risk level.

Then we assume both high and medium risk levels medical devices should be considered into the maintenance program, and at least one working device available for each type of devices to ensure the operating of the Emergency Resus.

Then we need to find out the transition matrices for both types of devices the method introduced in Section 4.2.

The model of the defibrillators is Heartstart mrx m3535a. After cleaning the data, we get the transition count matrix as follows. A one year transition matrix is desired.

Initial state	Final state		
	0	1	2
One year			
0	119	20	6
1	0	126	64
2	0	0	0
Two years			
0	15	1	2
1	0	6	5
2	0	0	0

Initial transition matrix:

$$M_0 = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}$$

After 1000 iterations, the estimated stable transition matrix is:

$$M_1^{EM} = \begin{pmatrix} 0.9098 & 0.0699 & 0.0203 \\ 0 & 0.6901 & 0.3099 \\ 0 & 0 & 1 \end{pmatrix}$$

The model of this suction unit device is 7305P-D. After cleaning the data, we get the transition count matrix as follows. A one year transition matrix is desired.

Initial state	Final state		
	0	1	2
One year			
0	602	132	9
1	0	55	71
2	0	0	0
Two years			
0	44	26	1
1	0	9	18
2	0	0	0
Three years			
0	1	3	0
1	0	0	0
2	0	0	0

Initial transition matrix:

$$M_0 = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}$$

After 1000 iterations, the estimated stable transition matrix is:

$$M_2^{EM} = \begin{pmatrix} 0.8551 & 0.1372 & 0.0077 \\ 0 & 0.4940 & 0.5060 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that this is no longer a *k-out-of-n* structure system, but a system with more general structure.

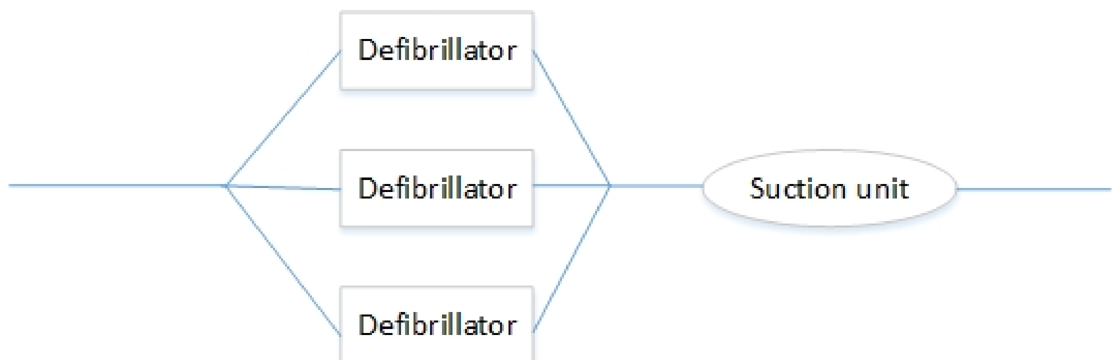


Figure 4.11: The structure of the system of 'Emergency Resus'

By the model introduced in Section 4.3, we can get the optimal maintenance strategy and the minimal average cost.

Let x_1 to x_3 denote the deterioration level of the three defibrillator devices and x_4 denote the deterioration level of the suction unit. Then we need to change the function $Q(\cdot)$, which equals 1 if the system is up, and 0 if it is down.

$$Q(\mathbf{x}) = \begin{cases} 1, \text{ if } \sum_{i=1}^{3} R_i(x_i) \geq 1 \textit{ and } R_4(x_4) = 1 \\ 0, \text{ if } \sum_{i=1}^{3} R_i(x_i) < 1 \textit{ or } R_4(x_4) = 0 \end{cases}$$

For the cost parameters, we suppose the inspection cost is 100 per time, the set up cost is 50 per time, the preventive repair cost is 150 per time per device for defibrillator and 100 per time per device for suction unit, the corrective repair cost is 200 per time per device for defibrillator and 150 for suction unit, and the penalty cost is 1000 per time. Denoted as: $c_n : 100$; $\mathbf{c}_c$: $[200, 200, 200, 150]$; $\mathbf{c}_p$: $[150, 150, 150, 100]$; $c_s : 50$; $p_c : 1000$.

Then Equation (4.1) would be expressed as follows

$$\begin{aligned} & C^{t*}\left(\mathbf{x}^t\right) \\ & = \min_{\mathbf{D}} \Bigg\{ I\left(\mathbf{x}^t\right) c_n + p_c\left(1 - Q(\mathbf{x}^t)\right) + c_s \left(1 - \prod_{i=1}^{n} (1 - d_i^t)\right) \\ & \quad + \sum_{i=1}^{n} d_i R_i\left(x_i^t\right) \mathbf{c}_{pi} + \sum_{i=1}^{n} d_i \left(1 - R_i\left(x_i^t\right)\right) \mathbf{c}_{ci} \\ & \quad + \sum_{[x_1^{t+1}, x_2^{t+1}, ..., x_n^{t+1}] \in \Omega} C^{t+1*}\left([\bar{x}_0^t - 1, x_1^{t+1}, x_2^{t+1}, ..., x_n^{t+1}]\right) p_{[\bar{x}_1^t, \bar{x}_2^t, ..., \bar{x}_n^t], [x_1^{t+1}, x_2^{t+1}, ..., x_n^{t+1}]} \Bigg\} \end{aligned} \tag{4.7}$$

where $[\bar{x}_0^t - 1, x_1^{t+1}, x_2^{t+1}, ..., x_n^{t+1}]$ is the state of $\mathbf{x}^{t+1}$, $\mathbf{c}_{pi}$ and $\mathbf{c}_{ci}$ is the preventive repair cost and corrective repair cost for device i.

The optimal maintenance decisions are shown in Table 4.18, and the average cost is 154.4509. The percentage of downtime of the system is shown in Figure 4.12.

Table 4.18: The optimal decisions for the system

Condition **x**	Maintenance **D**	Condition **x**	Maintenance **D**	Condition **x**	Maintenance **D**
0000	20000	1000	20000	2000	20000
0001	20001	1001	20001	2001	20001
0002	20001	1002	20001	2002	20001
0010	20000	1010	20000	2010	20010
0011	20001	1011	20011	2011	20011
0012	20001	1012	20011	2012	20011
0020	20000	1020	21000	2020	20010
0021	20001	1021	21001	2021	20011
0022	20001	1022	21001	2022	20011
0100	20000	1100	20000	2100	20100
0101	20001	1101	20101	2101	20101
0102	20001	1102	21001	2102	20101
0110	20000	1110	20110	2110	20110
0111	20011	1111	20111	2111	20111
0112	20011	1112	21011	2112	20111
0120	20100	1120	21100	2120	20110
0121	20101	1121	21101	2121	20111
0122	20101	1122	21101	2122	20111
0200	20000	1200	21000	2200	20100
0201	20001	1201	21001	2201	20101
0202	20001	1202	21001	2202	20101
0210	20010	1210	21010	2210	20110
0211	20011	1211	21011	2211	20111
0212	20011	1212	21011	2212	20111
0220	20010	1220	21010	2220	20110
0221	20011	1221	21011	2221	20111
0222	20011	1222	21011	2222	20111

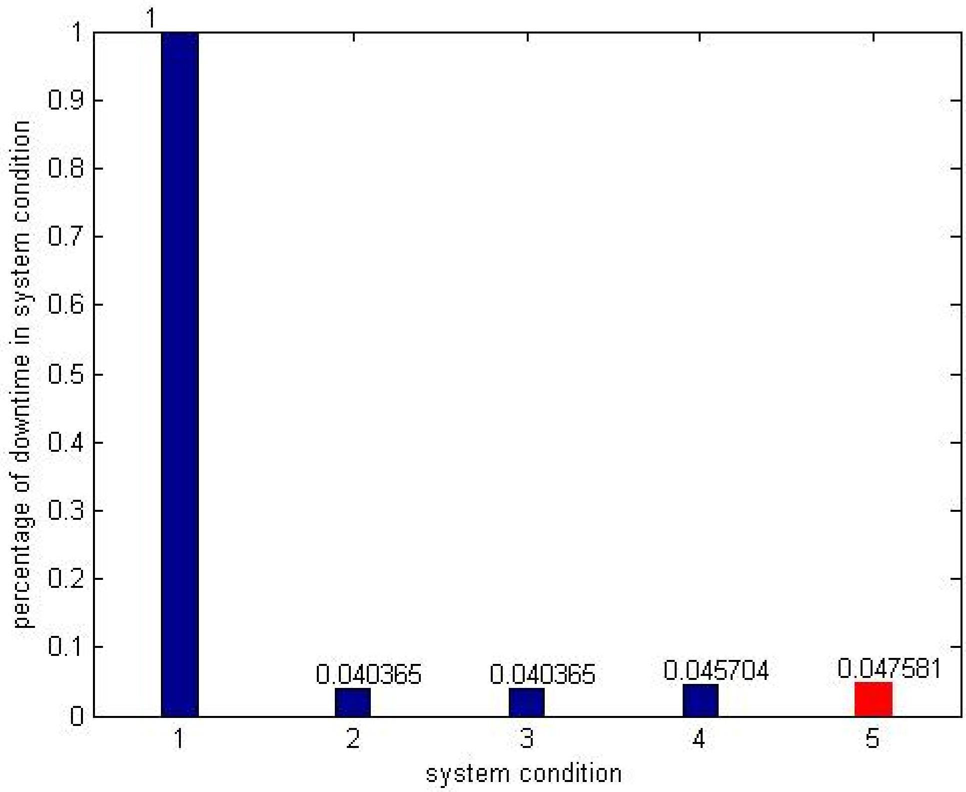


Figure 4.12: The percentage of downtime of the system at Emergency Resus. (The blue bars are for the machines, the red bar is for the system)

From the result, we find out that one of the defibrillators is redundant, and since there is only one suction unit, the inspection frequency is quite often. Obviously, more suction unit or increasing the backup ratio of suction unit could improve the performance of this system.

Chapter 5

Conclusion

5.1 Summary of the Thesis

Management of medical device has obtained many attentions in both research and practical field, but there is still a big gap between theoretical research and real life application. In this thesis, we started from a real problem proposed by WRHA, then integrated the real problem with some proper methods from operations research and data analysis. We looked into the challenge that the healthcare organizations are facing. We proposed our own models, and eventually obtained some interesting results, which include the classification of medical devices and making inspection and maintenance decisions.

(a) *Classification of medical devices.* In this part, we summarized the existing risk factors and scoring system and methods. We developed a generalized and useful evaluation model. Based on the given data, we divided the devices into three categories: High risk level, Medium risk level and Low risk level. We selected three important factors (Equipment function, Location of use and Frequency of use) in our model according to their definition and their correlations, which are calculated by using given data. Then, we built four different classification models (Linear regression, Logistic regression, Classification tree and Random forest) to analyze the relationship between the risk factors and the risk levels. Through cross-validation method, all four models are performed pretty well. The three risk factors have critical influence on the risk levels, and according to the classification tree, we could explicitly find out that how the devices are classified into different risk levels by Equipment Function and Location of use.

The result could be further used in maintenance work. By the classification, we can easily decide what kinds of devices are suppose to be included in the devices maintenance program.

(b) *An inspection and maintenance model.* The second part of this thesis is a condition-based multi-component inspection and maintenance model. In this part, we first studied the deterioration rate by using the provided data from WRHA and EM algorithm to estimate a transition matrix of the system's conditions. The transition matrix is a discrete time and discrete states Markov chain, which is suitable for the decision model. Then, we proposed a Markov decision processes model. Through this model, we can find out the optimal inspection and maintenance decisions and the minimal long-run average cost. The algorithm we used is relative value iteration. Another feature worth mentioning is that we consider the inspection interval as a dimension of the condition, so that we can obtain the optimal inspection interval and maintenance strategy at the same time.

The optimal inspection and maintenance decisions depend significantly on the cost parameters and structure of the system. In the numerical part, changing parameters can result in completely different optimal decisions. Moreover, we compared the minimal cost of our model with the result of periodic inspection model, our model can achieve lower minimal cost for complicated systems.

5.2 Future Research

The proposed models can be enriched by more realistic assumptions and considerations, *e.g.* imperfect inspection, imperfect repair and finite life time of devices. Some other techniques can also be used to improve our research.

(a) Connection between the two parts. In this thesis, the connection between the two parts is not close. A possible improvement is constructing a decision-making model based on risk level directly. For example, we can use dynamic risk factors and partial observable Markov decision processes model to estimate the system's condition and make maintenance decisions.

(b) *Classification of medical devices.* We selected our risk factors based on previous researches, more research and investigation are required to select risk factors. The diversity of the provided data is limited, the classification models need to be tested further with more data.

(c) *An inspection and maintenance model.*

We only consider the discrete-time situation, while in practice, the inspection and maintenance could happen in continuous-time. For the deterioration transition matrix, the convergence is not guaranteed (may converge to local maximum), with different initial transition matrix, we may have different final result. Further research is required for the EM algorithm in our problem.

For the inspection and maintenance model, we discussed multi-component system. But with the increasing of the number of devices, the number of states in our model would soar. By Kronecker product, the number of states increase exponentially with the increase of multiplier. Therefore, it is hard to solve large problem by this model. How to reduce the number of states is a problem to investigate in the future. Another possible solution is using heuristic or approximation methods. Although we have observed certain patterns and properties by the numerical examples, a formal proof of the relationship is required.

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APPENDICES

Appendix A

Review of the Risk Factors

Researcher(s)	Factors	Definition	Intuitively understanding
Fennigkoh and Smith (1989)	Equipment Function	The function of this medical device. Function has been subdivided into four main categories: therapeutic, diagnostic, analytic, and miscellaneous.	Defibrillator used to control heart fibrillation; x-ray machines are used to take pictures of dense tissues such as bones and teeth.

	Risk	Physical risks associated with a device's clinical application. A device malfunction could result in: Patient Death, Patient or Operator Injury, Inappropriate Therapy or Misdiagnosis, No Significant Risks.	What are the possible consequence to the patient and/or the operator in the event of equipment failure or malfunction?
	Required Maintenance	Maintenance requirements of device types. Three general levels: extensive, average, and minimal.	What kind of testing and the maintenance interval. (performance verification, safety testing, visual inspection, extensive maintenance)
Collins (2001)	Function Risk	Classify risks associated with medical devices as they affect patient care and physiology reflects actual risk.	Consider both function of the device and the care settings.
	Physical Risk		
	Maintenance Requirement	The total repair times per device were ranked from high to low and divided into six arbitrary levels.	
Capuano and Koritko (1996)	Function	The function category characterizes the device contribution to patient outcome.	

	Consequence	Consequence speculates what effect a malfunctioning device would have on a patient or staff member.	
	Maintenance	The maintenance category indicates all aspects that affect the need for technical intervention on a regular basis.	
	Protection	Protection category increases the risk level according to protection factors not provided by the devices.	
	Lethality	The lethality rating indicates the presence of dangerous outputs from the device.	
	Use	Use typifies how much a device is used, which influences its failure potential.	

Anderson (1992)	Risk	Risk code: the degree of risk associated with failure of various types of equipment and devices.	Will the failure of this item injure a patient or staff member? How will the failure affect the quality of the patient's medical care? What will be the severity of the injuries (if any) resulting from the item's failure?
Gullikson et al. (1996)	Static risk characteristics: equipment function, Physical risk	Function: equipment's purpose; Physical risk: an estimate of the worst-case effect if the equipment does not perform as expected. Their values usually remain the same over the life cycle of an item of medical equipment.	

	Dynamic risk-factor modules: Maintenance requirement, risk points	Maintenance requirement measure works on the assumption that an increased number of interventions by technicians both indicates and causes risk. The risk point measure combines various risk factors such as failures, reasons for failure, or poor performance with regard to such criteria as the meantime between failures (MTBFs) or the American Hospital Association's Useful Life standard.	
Khalaf et al. (2010)	Effectiveness	Effectiveness: outcome (uptime, failure rate, patient incidents, mean time between failures, turnaround time, mean time to repair and failure codes)	
	Efficiency	Efficiency: financial figures (total cost of ownership, total cost of maintenance per acquisition, clinical engineering costs for a group for which maintenance strategy was changed, cost of clinical engineering per occupied beds, cost of clinical engineering per patient discharged and service contracts as % of total maintenance and management costs)	

Taghipour et al. (2011)	Function	The function of a device is the main purpose for which it is to be used.	
	Mission Criticality	Mission criticality or operational impact describes the extent to which a device is crucial to the care delivery process of a hospital. Depends on utilization and availability of similar or alternative devices. Utilization: the total hours a device is used on average in a hospital. Availability of alternative devices: a functions of the number of similar or backup devices and their demand per unit time.	
	Age	Actual age of a device and its predictable life span.	
	Risk	The risk of all its failure modes. Frequency: the likelihood of a failure occurrence. Consequence: the total consequence of each failure mode. (operational (downtime), non-operational (cost), safety and environment) Detectability: the ability to detect a failure when it occurs.	

	Recalls and Hazard Alerts	The number and class of recalls and the number of hazard alerts that may occur for a device.	
	Maintenance Requirement		
Hamdi et al. (2012)	Equipment Function	the functional category of the device	
	Location of equipment use	the area in which it is primarily used	
	Hospital load	The product of hospital capacity (given by the number of beds within the hospital) and bed occupancy may be used as an indicator of the hospital load	
	Time	The time since the maintenance request was issued (T) is also considered in the system	
	Distance to nearest alternative	The geographical location of the hospital and the distance to the nearest hospital with an appropriate substitute for the failed device is accounted for within the system	
	Available of substitute	substitute or alternate of an equipment	
Corciova et al. (2013)	Clinical function	Function	

	Problem avoidance probability	Detectability	
	Incident history		
	Manufacturers requirements for specific schedules	Maintenance requirement	
Tawfik et al. (2013)	Mission Criticality	The criticality of a given device within the hospital global mission. Utilization rate: the equipment's operating hours relative to the departmental total working hours. Equipment importance: consequences of equipment unavailability Backup safety ratio: the number of available alternatives in case of a given piece of equipment unavailability.	x-ray machines are more critical for emergency hospitals than for primary care centers.
	Equipment function	Main purpose	MC factor reflects the importance of equipment usage in the hospital rather than the value of the equipment itself.

	Maintenance Requirement			
	Physical risk		Consequence attributed to a device because of its use, unavailability, and failure.	
Jamshidi et al. (2015)	First Step	Detectability	The probability of detection of a potential failure before it occurs.	
		Occurrence	The frequency of failures or probability of occurrence estimates the frequency of potential failures or risks for a given device.	
		Failure's consequences	The total consequences of each failure mode, all its potential impacts need to be assessed. (patient safety, operator safety, time cost, economic loss)	
	Second Step	Age	Reliability of a medical device is a function of the age of a component or system.	
		Usage-related hazards	Hazards associated with device usage are a common and serious problem.	misuse

		Utilization	Utilization is a compound measure based on the weighted sum of two indicators. The first indicator is the average daily utilization rate of the device, and the second indicator, is calculated as the proportion between the number of patients served per day and the maximum number of patients that the device may treat.	
		Number of available identical devices	Taghipour et al.	
		Recalls and hazard alerts		
		Function	Life support, Therapeutic, Diagnosis, Analysis, Others	
		Maintenance requirements		

Appendix B

Examples of Inspection Interval Greater Than One

Here, we consider the situation when the inspection interval is greater than one.

If the decision is $\mathbf{D}^t = [2, 0]$, then $\bar{\mathbf{x}}^t = [2, 0]$.

$$\begin{aligned} C^t\left(\mathbf{x}^t\right) &= 1 \times c_n + p_c\left(1-1\right) + c_s \times 0 + 0 \times 1 \times c_p + 0 \times \left(1-1\right) c_c + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^t x_1^{t+1}} C^{t+1}\left(\left[1, x_1^{t+1}\right]\right) \\ &= c_n + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^t x_1^{t+1}} C^{t+1}\left(\left[1, x_1^{t+1}\right]\right) \\ &= c_n + p_{00} C^{t+1}\left([1,0]\right) + p_{01} C^{t+1}\left([1,1]\right) + p_{02} C^{t+1}\left([1,2]\right) \end{aligned} \tag{B.1}$$

There is no decision to be made at time $t+1$, but from the equation, we can get the expected penalty cost, which is $p_{02}C^{t+1}\left([1,2]\right)$.

If $x_1^{t+1} = 0$, so $I\left(\mathbf{x}^{t+1}\right) = 0$, $R_1\left(x_1^{t+1}\right) = 1$,$Q\left(\mathbf{x}^{t+1}\right) = 1$.

$$\begin{aligned} C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 0 \times c_n + p_c\left(1-1\right) + c_s \times 0 + 0 \times 1 \times c_p + 0 \\ &\quad \times \left(1-1\right)c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[1, x_1^{t+2}\right]\right) \\ &= \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[1, x_1^{t+2}\right]\right) \\ &= p_{00} C^{t+2}\left(\left[1,0\right]\right) + p_{01} C^{t+2}\left(\left[1,1\right]\right) + p_{02} C^{t+2}\left(\left[1,2\right]\right) \end{aligned} \tag{B.2}$$

If $x_1^{t+1} = 1$, so $I\left(\mathbf{x}^{t+1}\right) = 0$, $R_1\left(x_1^{t+1}\right) = 1$, $Q\left(\mathbf{x}^{t+1}\right) = 1$.

$$\begin{aligned} C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 0 \times c_n + p_c\left(1-1\right) + c_s \times 0 + 0 \times 1 \times c_p + 0 \\ &\quad \times \left(1-1\right)c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[1, x_1^{t+2}\right]\right) \\ &= \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[1, x_1^{t+2}\right]\right) \\ &= p_{11} C^{t+2}\left(\left[1,1\right]\right) + p_{12} C^{t+2}\left(\left[1,2\right]\right) \end{aligned} \tag{B.3}$$

If $x_1^{t+1} = 2$, so $I\left(\mathbf{x}^{t+1}\right) = 0$, $R_1\left(x_1^{t+1}\right) = 0$, $Q\left(\mathbf{x}^{t+1}\right) = 0$.

$$\begin{aligned} C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 0 \times c_n + p_c\left(1-0\right) + c_s \times 0 + 0 \times 0 \times c_p + 0 \\ &\quad \times \left(1-0\right)c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[1, x_1^{t+2}\right]\right) \\ &= \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[1, x_1^{t+2}\right]\right) \\ &= p_c + p_{22} C^{t+2}\left(\left[1,2\right]\right) \end{aligned} \tag{B.4}$$

Combining the equations, we get

$$
\begin{aligned}
C^t\left(\mathbf{x}^t\right) &= 1\times c_n + p_c\left(1-1\right) + c_s\times 0 + 0\times 1\times c_p + 0\times\left(1-1\right)c_c + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^t x_1^{t+1}} C^{t+1}\left([1, x_1^{t+1}]\right) \\
&= c_n + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^t x_1^{t+1}} C^{t+1}\left([1, x_1^{t+1}]\right) \\
&= c_n + p_{00}(p_{00}C^{t+2}\left([0,0]\right) + p_{01}C^{t+2}\left([0,1]\right) + p_{02}C^{t+2}\left([0,2]\right)) \\
&\quad + p_{01}(p_{11}C^{t+2}\left([0,1]\right) + p_{12}C^{t+2}\left([0,2]\right)) + p_{02}(p_c + p_{22}C^{t+2}\left([0,2]\right)) \\
&= c_n + p_{00}^2C^{t+2}\left([0,0]\right) + p_{01}^2C^{t+2}\left([0,1]\right) + p_{02}^2C^{t+2}\left([0,2]\right) + p_{02}p_c
\end{aligned} \tag{B.5}
$$

Suppose $x_1^{t+2} = 1$, so $R_1\left(x_1^{t+2}\right) = 1$, $Q\left(\mathbf{x}^{t+2}\right) = 1$.

Let $C^{t+2}\left(\mathbf{x}^{t+2}\right)$ denote the arbitrary cumulative cost from time t+2 onward.

If the decision is $\mathbf{D}^{t+2} = [2, 0]$, then $\bar{\mathbf{x}}^{t+2} = [2, 1]$.

$$
\begin{aligned}
C^{t+2}\left(\mathbf{x}^{t+2}\right) &= 1\times c_n + p_c\left(1-1\right) + c_s\times 0 + 0\times 1\times c_p + 0 \\
&\quad \times\left(1-1\right)c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
&= c_n + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
&= c_n + p_{11}C^{t+3}\left([1,1]\right) + p_{12}C^{t+3}\left([1,2]\right)
\end{aligned} \tag{B.6}
$$

There is no decision to be made at time t+3, but from the equation, we can get the expected penalty cost, which is $p_{12}C^{t+3}\left([1,2]\right)$.

By the same procedure we can get

$$
\begin{aligned}
C^{t+2}\left(\mathbf{x}^{t+2}\right) &= 1 \times c_n + p_c\left(1-1\right) + c_s \times 0 + 0 \times 1 \times c_p + 0 \\
&\quad \times \left(1-1\right)c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
&= c_n + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
&= c_n + p_{11}(p_{11}C^{t+4}\left([0,1]\right) + p_{12}C^{t+4}\left([0,2]\right)) + p_{12}(p_c + p_{22}C^{t+4}\left([0,2]\right)) \\
&= c_n + p_{11}^2 C^{t+4}\left([0,1]\right) + p_{12}^2 C^{t+4}\left([0,2]\right) + p_{12}p_c
\end{aligned}
\tag{B.7}
$$

If the decision is $\mathbf{D}^t = [2,1]$, then $\bar{\mathbf{x}}^t = [2,0]$.

$$
\begin{aligned}
C^{t+2}\left(\mathbf{x}^{t+2}\right) &= 1 \times c_n + p_c\left(1-1\right) + c_s \times 1 + 1 \times 1 \times c_p + 1 \\
&\quad \times \left(1-1\right)c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
&= c_n + c_s + c_p + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
&= c_n + c_s + c_p + p_{00}C^{t+3}\left([1,0]\right) + p_{01}C^{t+3}\left([1,1]\right) + p_{02}C^{t+3}\left([1,2]\right)
\end{aligned}
\tag{B.8}
$$

There is no decision to be made at time $t{+}3$, but from the equation, we can get the expected penalty cost, which is $p_{02}C^{t+3}\left([1,2]\right)$.

By the same procedure we can get

$$
\begin{aligned}
C^{t+2}\left(\mathbf{x}^{t+2}\right) &= 1\times c_n + p_c\left(1-1\right) + c_s\times 1 + 1\times 1\times c_p + 1 \\
&\quad \times\left(1-1\right)c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2}x_1^{t+3}} C^{t+3}\left(\left[1, x_1^{t+3}\right]\right) \\
&= c_n + c_s + c_p + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2}x_1^{t+3}} C^{t+3}\left(\left[1, x_1^{t+3}\right]\right) \\
&= c_n + c_s + c_p + p_{00}(p_{00}C^{t+4}\left([0,0]\right) + p_{01}C^{t+4}\left([0,1]\right) + p_{02}C^{t+4}\left([0,2]\right)) \\
&\quad + p_{01}(p_{11}C^{t+4}\left([0,1]\right) + p_{12}C^{t+4}\left([0,2]\right)) + p_{02}(p_c{+}p_{22}C^{t+4}\left([0,2]\right)) \\
&= c_n + c_s + c_p + p_{00}^2C^{t+4}\left([0,0]\right) + p_{01}^2C^{t+4}\left([0,1]\right) + p_{02}^2C^{t+4}\left([0,2]\right) + p_{02}p_c
\end{aligned} \tag{B.9}
$$

Suppose $x_1^{t+2} = 2$, so $R_1\left(x_1^{t+2}\right) = 0$, $Q\left(\mathbf{x}^{t+2}\right) = 0$.

Let $C^{t+2}\left(\mathbf{x}^{t+2}\right)$ denote the arbitrary cumulative cost from time $t{+}2$ onward.

If the decision is $\mathbf{D}^{t+2} = [2, 0]$, then $\bar{\mathbf{x}}^{t+2} = [2, 2]$.

$$
\begin{aligned}
C^{t+2}\left(\mathbf{x}^{t+2}\right) &= 1\times c_n + p_c\left(1-0\right) + c_s\times 0 + 0\times 0\times c_p + 0 \\
&\quad \times\left(1-0\right)c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2}x_1^{t+3}} C^{t+3}\left(\left[1, x_1^{t+3}\right]\right) \\
&= c_n + p_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2}x_1^{t+3}} C^{t+3}\left(\left[1, x_1^{t+3}\right]\right) \\
&= c_n + p_c + p_{22}C^{t+3}\left([1,2]\right)
\end{aligned} \tag{B.10}
$$

There is no decision to be made at time $t{+}3$, but from the equation, we can get the expected penalty cost, which is $p_{22}C^{t+3}\left([1,2]\right)$.

By the same procedure we can get

$$
\begin{aligned}
C^{t+2}\left(\mathbf{x}^{t+2}\right) = & 1 \times c_n + p_c\,(1-0) + c_s \times 0 + 0 \times 0 \times c_p \\
& + 0\,(1-0)\,c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
= & c_n + p_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
= & c_n + p_c + p_{22}(p_c + p_{22} C^{t+4}\,([0,2])) \\
= & c_n + p_c + p_{22}^2 C^{t+4}\,([0,2]) + p_{22} p_c
\end{aligned}
\tag{B.11}
$$

If the decision is $\mathbf{D}^t = [2,1]$, then $\bar{\mathbf{x}}^t = [2,0]$.

$$
\begin{aligned}
C^{t+2}\left(\mathbf{x}^{t+2}\right) = & 1 \times c_n + p_c\,(1-0) + c_s \times 1 + 1 \times 0 \times c_p + 1 \\
& \times (1-0)\,c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
= & c_n + p_c + c_s + c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left([1, x_1^{t+3}]\right) \\
= & c_n + p_c + c_s + c_c + p_{00} C^{t+3}\,([1,0]) + p_{01} C^{t+3}\,([1,1]) + p_{02} C^{t+3}\,([1,2])
\end{aligned}
\tag{B.12}
$$

There is no decision to be made at time t+3, but from the equation, we can get the expected penalty cost, which is $p_{02} C^{t+3}\,([1,2])$.

By the same procedure we can get

$$
\begin{aligned}
C^{t+2}\left(\mathbf{x}^{t+2}\right) &= 1 \times c_n + p_c\left(1-0\right) + c_s \times 1 + 1 \times 0 \times c_p + 1 \\
&\quad \times \left(1-0\right) c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left(\left[1, x_1^{t+3}\right]\right) \\
&= c_n + p_c + c_s + c_c + \sum_{x_1^{t+3}=0}^{2} p_{\bar{x}_1^{t+2} x_1^{t+3}} C^{t+3}\left(\left[1, x_1^{t+3}\right]\right) \\
&= c_n + p_c + c_s + c_c + p_{00}(p_{00} C^{t+4}\left([0,0]\right) + p_{01} C^{t+4}\left([0,1]\right) + p_{02} C^{t+4}\left([0,2]\right)) \\
&\quad + p_{01}(p_{11} C^{t+4}\left([0,1]\right) + p_{12} C^{t+4}\left([0,2]\right)) + p_{02}(p_c + p_{22} C^{t+4}\left([0,2]\right)) \\
&= c_n + p_c + c_s + c_c + p_{00}^2 C^{t+4}\left([0,0]\right) + p_{01}^2 C^{t+4}\left([0,1]\right) + p_{02}^2 C^{t+4}\left([0,2]\right) + p_{02} p_c
\end{aligned} \tag{B.13}
$$

If the decision is $\mathbf{D}^t = [n, 0]$, then $\bar{\mathbf{x}}^t = [n, 0]$.

$$
\begin{aligned}
C^{t}\left(\mathbf{x}^{t}\right) &= 1 \times c_n + p_c\left(1-1\right) + c_s \times 0 + 0 \times 1 \times c_p + 0 \\
&\quad \times \left(1-1\right) c_c + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^{t} x_1^{t+1}} C^{t+1}\left(\left[n-1, x_1^{t+1}\right]\right) \\
&= c_n + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^{t} x_1^{t+1}} C^{t+1}\left(\left[n-1, x_1^{t+1}\right]\right) \\
&= c_n + p_{00} C^{t+1}\left([n-1,0]\right) + p_{01} C^{t+1}\left([n-1,1]\right) + p_{02} C^{t+1}\left([n-1,2]\right)
\end{aligned} \tag{B.14}
$$

There is no decision to be made from time $t+1$ to time $t+n-1$, but from the equations below, we can get the expected penalty cost.

If $x_1^{t+1} = 0$, so $R_1\left(x_1^{t+1}\right) = 1$, $Q\left(\mathbf{x}^{t+1}\right) = 1$.

$$\begin{aligned}
C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 0 \times c_n + p_c\left(1-1\right) + c_s \times 0 + 0 \times 1 \times c_p + 0 \\
&\quad \times \left(1-1\right) c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[n-2, x_1^{t+2}\right]\right) \\
&= \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[n-2, x_1^{t+2}\right]\right) \\
&= p_{00} C^{t+2}\left(\left[n-2, 0\right]\right) + p_{01} C^{t+2}\left(n-2, 1\right]) \\
&\quad + p_{02} C^{t+2}\left(\left[n-2, 2\right]\right)
\end{aligned} \tag{B.15}$$

If $x_1^{t+1} = 1$, so $R_1\left(x_1^{t+1}\right) = 1$, $Q\left(\mathbf{x}^{t+1}\right) = 1$.

$$\begin{aligned}
C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 0 \times c_n + p_c\left(1-1\right) + c_s \times 0 + 0 \times 1 \times c_p + 0 \\
&\quad \times \left(1-1\right) c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[n-2, x_1^{t+2}\right]\right) \\
&= \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[n-2, x_1^{t+2}\right]\right) \\
&= p_{11} C^{t+2}\left(\left[n-2, 1\right]\right) + p_{12} C^{t+2}\left(\left[n-2, 2\right]\right)
\end{aligned} \tag{B.16}$$

If $x_1^{t+1} = 2$, so $R_1\left(x_1^{t+1}\right) = 0$, $Q\left(\mathbf{x}^{t+1}\right) = 0$.

$$\begin{aligned}
C^{t+1}\left(\mathbf{x}^{t+1}\right) &= 0 \times c_n + p_c\left(1-0\right) + c_s \times 0 + 0 \times 0 \times c_p + 0 \\
&\quad \times \left(1-0\right) c_c + \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[n-2, x_1^{t+2}\right]\right) \\
&= \sum_{x_1^{t+2}=0}^{2} p_{\bar{x}_1^{t+1} x_1^{t+2}} C^{t+2}\left(\left[n-2, x_1^{t+2}\right]\right) \\
&= p_c + p_{22} C^{t+2}\left(\left[n-2, 2\right]\right)
\end{aligned} \tag{B.17}$$

Combining the equations, we get that

$$
\begin{aligned}
C^t\left(\mathbf{x}^t\right) &= 1\times c_n + p_c\left(1-1\right) + c_s\times 0 + 0\times 1\times c_p + 0 \\
&\quad \times\left(1-1\right)c_c + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^t x_1^{t+1}} C^{t+1}\left(\left[n-1, x_1^{t+1}\right]\right) \\
&= c_n + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^t x_1^{t+1}} C^{t+1}\left(\left[n-1, x_1^{t+1}\right]\right) \\
&= c_n + p_{00}(p_{00}C^{t+2}\left([n-2,0]\right) + p_{01}C^{t+2}\left([n-2,1]\right) + p_{02}C^{t+2}\left([n-2,2]\right)) \\
&\quad + p_{01}(p_{11}C^{t+2}\left([n-2,1]\right) + p_{12}C^{t+2}\left([n-2,2]\right)) + p_{02}(p_c + p_{22}C^{t+2}\left([n-2,2]\right)) \\
&= c_n + p_{00}^2 C^{t+2}\left([n-2,0]\right) + p_{01}^2 C^{t+2}\left([n-2,1]\right) + p_{02}^2 C^{t+2}\left([n-2,2]\right) + p_{02}p_c
\end{aligned}
\tag{B.18}
$$

Follow the process, we can finally get the equation

$$
\begin{aligned}
C^t\left(\mathbf{x}^t\right) &= 1\times c_n + p_c\left(1-1\right) + c_s\times 0 + 0\times 1\times c_p + 0\times\left(1-1\right)c_c \\
&\quad + \sum_{x_1^{t+1}=0}^{2} p_{\bar{x}_1^t x_1^{t+1}} C^{t+1}\left(\left[n-1, x_1^{t+1}\right]\right) \\
&\ldots \\
&= c_n + p_{00}^2 C^{t+2}\left([n-2,0]\right) + p_{01}^2 C^{t+2}\left([n-2,1]\right) + p_{02}^2 C^{t+2}\left([n-2,2]\right) + p_{02}p_c \\
&= c_n + p_{00}^2(p_{00}C^{t+3}\left([n-3,0]\right) + p_{01}C^{t+3}\left([n-3,1]\right) + p_{02}C^{t+3}\left([n-3,1]\right)) \\
&\quad + p_{01}^2(p_{11}C^{t+3}\left([n-3,1]\right) + p_{12}C^{t+3}\left([n-3,1]\right)) \\
&\quad + p_{02}^2(p_c + p_{22}C^{t+3}\left([n-3,2]\right)) + p_{02}p_c \\
&= c_n + p_{00}^3 C^{t+3}\left([n-3,0]\right) + p_{01}^3 C^{t+3}\left([n-3,1]\right) + p_{02}^3 C^{t+3}\left([n-3,2]\right) + p_{02}p_c + p_{02}^2 p_c \\
&\ldots \\
&= c_n + p_{00}^n C^{t+n}\left([0,0]\right) + p_{01}^n C^{t+n}\left([0,1]\right) + p_{02}^n C^{t+n}\left([0,2]\right) + \sum_{i=1}^{n-1} p_{02}^i p_c
\end{aligned}
\tag{B.19}
$$

Suppose $x_1^{t+n} = 1$, so $R_1\left(x_1^{t+n}\right) = 1$, $Q\left(\mathbf{x}^{t+n}\right) = 1$.

Let $C^{t+n}\left(\mathbf{x}^{t+n}\right)$ denote the arbitrary cumulative cost from time $t+n$ onward.

If the decision is $\mathbf{D}^{t+n} = [n, 0]$, then $\bar{\mathbf{x}}^{t+n} = [n, 1]$.

$$
\begin{aligned}
C^{t+n}\left(\mathbf{x}^{t+n}\right) &= 1 \times c_n + p_c\left(1-1\right) + c_s \times 0 + 0 \times 1 \times c_p + 0 \times \left(1-1\right) c_c \\
&\quad + \sum_{x_1^{t+n+1}=0}^{2} p_{\bar{x}_1^{t+n} x_1^{t+n+1}} C^{t+n+1}\left([n-1, x_1^{t+n+1}]\right) \\
&= c_n + \sum_{x_1^{t+n+1}=0}^{2} p_{\bar{x}_1^{t+n} x_1^{t+n+1}} C^{t+n+1}\left([n-1, x_1^{t+n+1}]\right) \\
&= c_n + p_{11}(p_{11}C^{t+n+2}\left([n-2,1]\right) + p_{12}C^{t+n+2}\left([n-2,2]\right)) \\
&\quad + p_{12}(p_c + p_{22}C^{t+n+2}\left([n-2,2]\right)) \\
&= c_n + p_{11}^2 C^{t+n+2}\left([n-2,1]\right) + p_{12}^2 C^{t+n+2}\left([n-2,2]\right) + p_{12}p_c \\
&\ldots \\
&= c_n + p_{11}^n C^{t+2n}\left([0,1]\right) + p_{12}^n C^{t+2n}\left([0,2]\right) + \sum_{i=1}^{n-1} p_{12}^i p_c
\end{aligned}
\tag{B.20}
$$

If the decision is $\mathbf{D}^{t+n} = [n, 1]$, then $\bar{\mathbf{x}}^{t+n} = [n, 0]$.

$$
\begin{aligned}
C^{t+n}\left(\mathbf{x}^{t+n}\right) &= 1 \times c_n + p_c\left(1-1\right) + c_s \times 1 + 1 \times 1 \times c_p + 1 \times \left(1-1\right) c_c \\
&\quad + \sum_{x_1^{t+n+1}=0}^{2} p_{\bar{x}_1^{t+n} x_1^{t+n+1}} C^{t+n+1}\left([n-1, x_1^{t+n+1}]\right) \\
&= c_n + c_s + c_p + \sum_{x_1^{t+n+1}=0}^{2} p_{\bar{x}_1^{t+n} x_1^{t+n+1}} C^{t+n+1}\left([n-1, x_1^{t+n+1}]\right) \\
&= c_n + c_s + c_p \\
&\quad + p_{00}(p_{00}C^{t+n+2}\left([n-2,0]\right) + p_{01}C^{t+n+2}\left([n-2,1]\right) + p_{02}C^{t+n+2}\left([n-2,2]\right)) \\
&+ p_{01}(p_{11}C^{t+n+2}\left([n-2,1]\right) + p_{12}C^{t+n+2}\left([n-2,2]\right)) \\
&\quad + p_{02}(p_c + p_{22}C^{t+n+2}\left([n-2,2]\right)) \\
&= c_n + c_s + c_p + p_{00}^2 C^{t+n+2}\left([n-2,0]\right) + p_{01}^2 C^{t+n+2}\left([n-2,1]\right) \\
&\quad + p_{02}^2 C^{t+n+2}\left([n-2,2]\right) + p_{02}p_c \\
&\ldots \\
&= c_n + c_s + c_p + p_{00}^n C^{t+2n}\left([0,0]\right) + p_{01}^n C^{t+2n}\left([0,1]\right) + p_{02}^n C^{t+2n}\left([0,2]\right) + \sum_{i=1}^{n-1} p_{02}^i p_c
\end{aligned}
\tag{B.21}
$$

Suppose $x_1^{t+n} = 2$, so $R_1\left(x_1^{t+n}\right) = 0$, $Q\left(\mathbf{x}^{t+n}\right) = 0$.

Let $C^{t+n}\left(\mathbf{x}^{t+n}\right)$ denote the arbitrary cumulative cost from time $t+n$ onward.

If the decision is $\mathbf{D}^{t+n} = [n, 0]$, then $\bar{\mathbf{x}}^{t+n} = [n, 2]$.

$$\begin{aligned}
C^{t+n}\left(\mathbf{x}^{t+n}\right) &= 1 \times c_n + p_c\left(1-0\right) + c_s \times 0 + 0 \times 1 \times c_p + 0 \times \left(1-1\right) c_c \\
&\quad + \sum_{x_1^{t+n+1}=0}^{2} p_{\bar{x}_1^{t+n} x_1^{t+n+1}} C^{t+n+1}\left(\left[n-1, x_1^{t+n+1}\right]\right) \\
&= c_n + p_c + \sum_{x_1^{t+n+1}=0}^{2} p_{\bar{x}_1^{t+n} x_1^{t+n+1}} C^{t+n+1}\left(\left[n-1, x_1^{t+n+1}\right]\right) \\
&= c_n + p_c + p_{22}(p_c + p_{22} C^{t+n+2}\left([n-2, 2]\right)) \\
&= c_n + p_c + p_{22}^2 C^{t+n+2}\left([n-2, 2]\right) + p_{12} p_c \\
&\ldots \\
&= c_n + p_c + p_{22}^n C^{t+2n}\left([0, 2]\right) + \sum_{i=1}^{n-1} p_{22}^i p_c
\end{aligned} \tag{B.22}$$

If the decision is $\mathbf{D}^{t+n} = [n, 1]$, then $\bar{\mathbf{x}}^{t+n} = [n, 0]$.

$$
\begin{aligned}
C^{t+n}\left(\mathbf{x}^{t+n}\right) &= 1\times c_n + p_c\left(1-0\right) + c_s\times 1 + 1\times 0\times c_p + 1\times\left(1-0\right)c_c \\
&\quad + \sum_{x_1^{t+n+1}=0}^{2} p_{\bar{x}_1^{t+n}x_1^{t+n+1}} C^{t+n+1}\left(\left[n-1, x_1^{t+n+1}\right]\right) \\
&= c_n + p_c + c_s + c_c + \sum_{x_1^{t+n+1}=0}^{2} p_{\bar{x}_1^{t+n}x_1^{t+n+1}} C^{t+n+1}\left(\left[n-1, x_1^{t+n+1}\right]\right) \\
&= c_n + p_c + c_s + c_c \\
&\quad + p_{00}(p_{00}C^{t+n+2}\left([n-2,0]\right) + p_{01}C^{t+n+2}\left([n-2,1]\right) + p_{02}C^{t+n+2}\left([n-2,2]\right)) \\
&\quad + p_{01}(p_{11}C^{t+n+2}\left([n-2,1]\right) + p_{12}C^{t+n+2}\left([n-2,2]\right)) \\
&\quad + p_{02}(p_c + p_{22}C^{t+n+2}\left([n-2,2]\right)) \\
&= c_n + p_c + c_s + c_c + p_{00}^2 C^{t+n+2}\left([n-2,0]\right) \\
&\quad + p_{01}^2 C^{t+n+2}\left([n-2,1]\right) + p_{02}^2 C^{t+n+2}\left([n-2,2]\right) + p_{02}p_c \\
&\dots \\
&= c_n + p_c + c_s + c_c + p_{00}^n C^{t+2n}\left([0,0]\right) + p_{01}^n C^{t+2n}\left([0,1]\right) \\
&\quad + p_{02}^n C^{t+2n}\left([0,2]\right) + \sum_{i=1}^{n-1} p_{02}^i p_c
\end{aligned}
\tag{B.23}
$$