Techniques in operator algebras: classification, dilation and non-commutative boundary theory

by

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

I am the sole author of Chapters 1, 2 and 3. Chapter 4 is joint work with Daniel Markiewicz, Chapter 5 is joint work with Guy Salomon and Chapter 6 is a variation on joint work with Kenneth Davidson, Orr Shalit and Baruch Solel.

Abstract

In this thesis we bring together several techniques in the theory of non-self-adjoint operator algebras and operator systems. We begin with classification of non-self-adjoint and self-adjoint operator algebras constructed from C*-correspondence and more specifically, from certain generalized Markov chains. We then transitions to the study of non-commutative boundaries in the sense of Arveson, and their use in the construction of dilations for families of operators arising from directed graphs. Finally, we discuss connections between operator systems and matrix convex sets and use dilation theory to obtain scaled inclusion results for matrix convex sets.

We begin with classification of non-self-adjoint operator algebras. In Chapter 3 we solve isomorphism problems for tensor algebras arising from weighted partial dynamical systems. We show that the isometric isomorphism and algebraic / bounded isomorphism problems are two distinct problems, that require separate criteria to be solved. Our methods yield an alternative solution to Arveson's conjugacy problem, first solved by Davidson and Katsoulis.

A natural bridge between operator algebras / systems and C*-algebras is the C*-envelope, which is a non-commutative generalization of the notion of *Shilov boundary* from the theory of function algebras. In Chapter 4 we investigate C*-envelopes arising from the operator algebras of stochastic matrices via subproduct systems. We identify and classify these non-commutative boundaries in terms of the matrices, and exhibit new examples of C*-envelopes of non-self-adjoint operator algebras arising from a subproduct system construction.

In Chapter 5 we apply Arveson's non-commutative boundary theory to dilate every Toeplitz-Cuntz-Krieger family of a directed graph G to a full Cuntz-Krieger family for G. We also obtain a generalization of our dilation result to the context of colored directed graphs, which relies on the complete injectivity of amalgamated free products of operator algebras.

The last part of this thesis is devoted to the interplay between matrix convex sets and operator systems, inspired by the work of Helton, Klep and McCullough. In Chapter 6 we establish a functorial duality between finite dimensional operator systems and matrix convex sets that recovers many interpolation results of completely positive maps in the literature. We proceed to investigate dual, minimal and maximal matrix convex sets, and relate them to dilation theory, scaled inclusion results and operator systems. By using dilation theory, we provide *rank-independent* optimal scaled inclusion results for matrix convex sets satisfying a certain symmetry condition, and prove the existence of an essentially unique *self-dual* matrix convex set.

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Dedication

This thesis is dedicated to the memory of my father, Alexander Dor-On, who passed away on November 7th 2015, during my time as a PhD student at University of Waterloo.

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Chapter 1

Introduction

The study of operator algebras, and particularly C^*-algebras, has been a very active area of research in recent years, spearheaded by Elliott's classification program for large classes of simple C^*-algebras. Structural results for operator algebras often establish connections to classical dynamical theories. One very good example for this phenomenon is the work of Elliott in [51] and [52], on classification of approximately finite dimensional and real-rank zero circle C^*-algebras in terms of K-theory. Using this work, Giordano, Putnam and Skau [56] were able to classify Cantor minimal $\mathbb{Z}$-systems in terms of orbit equivalence. However, operator algebras need not be simple nor self-adjoint in general, yet they still yield many interesting invariants for the underlying dynamics. The first part of this thesis is comprised of Chapters 3, 4 and 5, and deals with non-commutative boundaries for different classes of non-self-adjoint operator algebras along with the classification of these operator algebras and their associated boundaries.

In the realm of real algebraic geometry and convex optimization, many applications were found for matricial domains defined by a linear matrix inequality, especially in the work of Helton, Klep and McCullough. A good instance of this is the solution of Helton, Klep, McCullough and Schweighofer [64] to the matrix cube problem in optimization, which was considered by Ben-Tal and Nemirovski [16]. More precisely, using dilation, they find optimal scales $\theta(m)$ such that for any LMI domain $\mathcal{D}_B$, with B comprised of matrices whose ranks are at most m, and $[-1,1]^d \subseteq \mathcal{D}_B(1)$, we have that every contractive d-tuple $X = (X_1, ..., X_d)$ belongs to $\theta(m)\mathcal{D}_B$. On the other hand, in operator algebras, CP maps are the fabric for exactness, amenability and nuclearity-type properties. In [80, 81], Kavruk, Paulsen, Todorov and Tomforde systematically study nuclearity related properties of operator systems, and relate them to many important problems from quantum information theory and operator algebras. One such problem is the well-known Connes' embedding conjecture (See [96]), for which they obtain many new and interesting equivalent formulations and simplify the proofs for existing ones. The second part of this thesis is comprised of Chapter 6, dealing with the connections between matrix convex sets, dilation theory and operator systems.

We next give an introduction for this thesis, starting with the first part. Aside from this introduction, more specific details can be found in the introductions to any of the non-preliminary chapters.

The origin of the classification of non-self-adjoint operator algebras is in the work of Arveson [4] and Arveson and Josephson [13]. Peters [102] continued this investigation where he introduced his semi-crossed product algebra, and generalized the Arveson–Josephson classification. Hadwin and Hoover [60] improved Peters' classification by removing some of the restrictions on fixed points, and for decades it was unknown if a restrictions on fixed points was necessary. This problem came to be known as Arveson's conjugacy problem.

Tensor algebras of C^*-correspondences have been the subject of a deep study by Muhly and Solel [90, 91, 92] which has led to a far-reaching non-commutative generalization of function theory. Some of the first successful attempts to classify subclasses of tensor algebras outside the scope of [92] came from tensor algebras associated to countable directed graphs. At around the same time, Solel [116] and Katsoulis and Kribs [73] independently introduced methods of representations into upper triangular 2×2 matrices to solve isomorphism problems for graph tensor algebras. We will provide an alternative proof of their results when the graphs G and G' are finite and multiplicity-free (See Corollary 3.6.4).

Arveson's conjugacy problem was finally resolved by Davidson and Katsoulis [34], by adapting the methods of Hadwin and Hoover in [60], and the methods of representations into upper triangular 2×2 matrices in [73, 116]. They prove that for two continuous maps $\sigma : X \to X$ and $\tau : Y \to Y$ on locally compact spaces X and Y respectively, Peter's semi-crossed product algebras $C_0(X) \times_\sigma \mathbb{Z}_+$ and $C_0(Y) \times_\tau \mathbb{Z}_+$ are algebraically / bounded / isometrically isomorphic if and only if σ and τ are conjugate. We will give an alternative proof of this result in the case when X and Y are compact (See Corollary 3.6.5).

In Chapter 3, which is based on [41], we provide classification results for tensor algebras $\mathcal{T}_+(\sigma, w)$ arising from weighted partial systems (σ, w) (See Section 3.2) up to isometric / bounded isomorphism and in some cases up to algebraic isomorphism, in terms of branch-transition and weighted-orbit conjugacy. Our objective is to show that weighted partial systems (WPS for short) yield tensor algebras which are still completely classifiable up to bounded / isometric isomorphisms, while covering many examples of such classification results. The following is a shorthand version of Theorems 3.5.6 and 3.5.7. Suppose (σ, w) and (τ, u) are WPS over compact X and Y respectively.

1. $\mathcal{T}_+(\sigma, w)$ and $\mathcal{T}_+(\tau, u)$ are isometrically isomorphic if and only if if (σ, w) and (τ, u) are *branch-transition conjugate.*

2. $\mathcal{T}_+(\sigma, w)$ and $\mathcal{T}_+(\tau, u)$ are bounded isomorphic if and only if (σ, w) and (τ, u) are *weighted-path conjugate.*

Using this theorem, we show that the isometric isomorphism and algebraic / bounded isomorphism problems for tensor algebras are two distinct problems, that require separate criteria to be solved.

By an *operator algebra* in this thesis we mean a (not necessarily self-adjoint) closed subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$. Recall that given an operator algebra $\mathcal{A}$, a C^*-cover is a pair $(\mathcal{B}, \iota)$ where $\mathcal{B}$ is a C^*-algebra and $\iota : \mathcal{A} \to \mathcal{B}$ is a completely isometric homomorphism, such that $C^*(\iota(\mathcal{A})) = \mathcal{B}$. A C^*-cover is called the *C^*-envelope* for $\mathcal{A}$ if for any other C^*-cover $(\mathcal{B}', \iota')$, the map $\iota'(a) \mapsto \iota(a)$ extends uniquely to a surjective $*$-homomorphism $\mathcal{B}' \to \mathcal{B}$. In this precise sense, the C^*-envelope is the smallest C^*-algebra which contains a completely isometric copy of $\mathcal{A}$, and usually the algebra $\mathcal{B}$ is denoted $C^*_e(\mathcal{A})$ and the map ι is suppressed. The theory of C^*-envelopes and non-commutative boundaries was introduced and applied by Arveson in his seminal "Subalgebras of C^*-algebras" papers [5, 6] and [9].

The existence of the C^*-envelope of a unital operator algebra was first proven by Hamana [61], by way of proving the existence of an injective envelope for operator systems. An alternative proof via dilation theory was found by Dritschel and McCullough in [45]. The new dilation idea in this alternative proof allowed Arveson [11] to follow the original strategy he envisioned in [5, 6] to prove the existence of the C^*-envelope via boundary representations in the separable case. Davidson and Kennedy finally realized Arveson's vision in full in [36] by providing a simpler proof without the assumption of separability.

Given a C^*-correspondence E, the operator algebras associated to shift operators (also called creation operators) over the Fock correspondence $\mathcal{F}(E)$ have been the subject of considerable attention by many researchers. The non-self-adjoint operator algebra generated by these shifts is the aforementioned tensor algebra, denoted by $\mathcal{T}_+(E)$, and it provides a very successful prototype for the study of operator algebras. It is closely related to the Toeplitz algebra $\mathcal{T}(E)$, which is the C^*-algebra generated by the shifts, and its celebrated universal quotient, the Cuntz-Pimsner-Katsura algebra $\mathcal{O}(E)$ (See [78] and [104]). In fact, by a theorem of Katsoulis and Kribs [76] we know that $C^*_e(\mathcal{T}_+(E)) \cong \mathcal{O}(E)$.

Analogously, given a a subproduct system X in the sense of Shalit and Solel [113] of C^*-correspondences over a C^*-algebra $\mathcal{A}$ (See Subsection 2.2.2), one obtains the operator algebras associated to shifts on $\mathcal{F}(X)$: the tensor algebra $\mathcal{T}_+(X)$, the Toeplitz algebra

$\mathcal{T}(X)$ and the Cuntz-Pimsner algebra $\mathcal{O}(X)$, where the latter was defined in [119]. This new framework generalizes the previous one, in the sense that a C^*-correspondence E gives rise to a *product* system X of tensor iterates of E, whose Fock correspondence and associated operator algebras are precisely the ones discussed in the previous paragraph. However, when X is not a product system, as opposed to $\mathcal{O}(E)$ mentioned in the previous paragraph, it is no longer clear if $\mathcal{O}(X)$ has a universal property or a co-universal property in the form of a gauge-invariant uniqueness theorem.

There has been important work on the operator algebras arising from subproduct systems over $\mathbb{C}$, or equivalently, the special case of subproduct systems whose C^*-correspondence fibers are actually *Hilbert spaces*, see for example [113, 37, 72]. In [42], we turned to the simplest case for which the fibers of the subproduct system are not Hilbert spaces. Namely, we considered the case of subproduct systems of C^*-correspondences over $\ell^\infty(\Omega)$ when Ω is countable with more than one point. Such a subproduct system and its associated operator algebras are conveniently parametrized by a stochastic matrix P over the state space Ω, and in [42] we resolve isomorphism problems of the tensor algebras $\mathcal{T}_+(P)$ associated to stochastic matrices P, via these subproduct systems. We denote by $\mathcal{T}(P)$ and $\mathcal{O}(P)$ the Toeplitz and Cuntz algebras associated to P via these subproduct systems.

In Chapter 4, which is based on joint work with Daniel Markiewicz [43], we show that for an irreducible stochastic matrix P, the C^*-envelope $C^*_e(\mathcal{T}_+(P))$ fits in the following sequence of quotient maps

$$\mathcal{T}(P) \longrightarrow C^*_e(\mathcal{T}_+(P)) \xrightarrow{\pi_P} \mathcal{O}(P)$$

where for appropriate choices for P all three algebras $\mathcal{T}(P)$, $C^*_e(\mathcal{T}_+(P))$ and $\mathcal{O}(P)$ are pairwise non stably isomorphic. It is known that for general subproduct systems X, we may have that the algebras $\mathcal{T}(X)$, $C^*_e(\mathcal{T}_+(X))$ and $\mathcal{O}(X)$ are all distinct. However, the subproduct system arising from an irreducible stochastic matrix P is *minimal*. Furthermore, if one is to have a co-universal property as one has for the C^*-envelope, this perhaps suggest that a different definition for $\mathcal{O}(X)$ is needed when X is not a product system.

Our concrete description of the C^*-envelope leads to an unexpected richness of possibilities. We classify $C^*_e(\mathcal{T}_+(P))$ up to $*$-isomorphism and stable isomorphism and obtain the following results. For an irreducible stochastic matrix P, denote by n_P the number of columns of P whose entries are only in $\{0,1\}$. For a large class of irreducible stochastic matrices P and Q over state sets Ω^P and Ω^Q we obtain that

1. $C^*_e(\mathcal{T}_+(P))$ and $C^*_e(\mathcal{T}_+(Q))$ are stably isomorphic if and only if $n_P = n_Q$ (See Theorem 4.4.10).

2. $C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are $*$-isomorphic if and only if $|\Omega^P| = |\Omega^Q|$, $n_P = n_Q$ and up to a reordering of Ω^Q, the matrices P and Q have the same *column nullity* in every column (See Definition 4.4.8 and Theorem 4.4.11).

These results are proven by determining the K-theory of these C^*-algebras and by combining this information with results due to Paschke and Salinas [97] in extension theory.

In [28], Cuntz and Krieger introduce C^*-algebras associated to finite, sourceless, sinkless and multiplicity free graphs, and use them to study and produce invariants of subshifts of finite type. They show that the stabilized Cuntz-Krieger algebra is an invariant of flow equivalence of subshifts of finite type, and that the classical Bowen-Franks groups arise as the extension groups of the Cuntz-Krieger algebras.

It is natural to ask about the relationship between $C_e^*(\mathcal{T}_+(P))$ and the Cuntz-Krieger graph algebra $\mathcal{O}_{\mathrm{Gr}(P)}$, where $\mathrm{Gr}(P)$ is the unweighted directed graph obtained from a finite irreducible stochastic matrix P. We compare these two algebras arising from an irreducible stochastic matrix P, and show that $\mathcal{O}_{\mathrm{Gr}(P)}$ and $C_e^*(\mathcal{T}_+(P))$ generally yield distinct invariants for $\mathrm{Gr}(P)$.

A *directed graph* G is a quadruple (V, E, s, r) consisting of a set V of vertices, a set E of edges and two maps $s, r : E \to V$, called the source map and range map, respectively. If $v = s(e)$ and $w = r(e)$ we say that v emits e and w receives it.

For a directed graph $G = (V, E, s, r)$ a *Toeplitz-Cuntz-Krieger G-family* (P, S) is a set of mutually orthogonal projections $P := \{P_v : v \in V\}$ and a set of partial isometries $S := \{S_e : e \in E\}$ which satisfy the Toeplitz-Cuntz-Krieger relations:

(I) $S_e^* S_e = P_{s(e)}$ for every $e \in E$, and

(TCK) $\sum_{e \in F} S_e S_e^* \leq P_v$ for every finite subset $F \subseteq r^{-1}(v)$.

We say that (P, S) is a *Cuntz-Krieger G-family* if, in addition, we have

(CK) $\sum_{r(e)=v} S_e S_e^* = P_v$ for every $v \in V$ with $0 < |r^{-1}(v)| < \infty$.

The universal C^*-algebra $\mathcal{T}(G)$ generated by Toeplitz-Cuntz-Krieger G-families is called the *Toeplitz-Cuntz-Krieger algebra* of the graph G, and the universal C^*-algebra $\mathcal{O}(G)$ generated by Cuntz-Krieger families is called the *Cuntz-Krieger algebra* of the graph G. The *tensor algebra* $\mathcal{T}_+(G)$ is then just the norm-closed operator algebra generated by a universal Toeplitz-Cuntz-Krieger family, and is a subalgebra of $\mathcal{T}(G)$. These algebras provide a rich supply of examples of operator algebras. By results of Enomoto and Watatani [53] we have that $\mathcal{O}(G)$ coincides with the classical Cuntz-Krieger algebras $\mathcal{O}_G$ defined in [28] when G is finite, sourceless, sinkless and multiplicity free. We recommend [106] for an excellent account of the theory of graph C^*-algebras. In fact, in [106, Chapter 8] it is explained how $\mathcal{T}_+(G)$, $\mathcal{T}(G)$ and $\mathcal{O}(G)$ arise as the Tensor, Toeplitz and Cuntz-Pimsner-Katsura algebras of a canonical C^*-correspondence associated with G.

Chapter 5 is based on joint work with Guy Salomon [44], where we apply Arveson's non-commutative boundary theory to show that every TCK family dilates to a *full* CK family. More precisely, given a TCK G-family (P, S) on a Hilbert space $\mathcal{H}$, we show there exists a CK G-family (Q, T) on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that

(CKF) $\quad \text{SOT-}\sum_{r(e)=v} T_e T_e^* = Q_v$, for every $v \in V$ with $r^{-1}(v) \neq \emptyset$.

and for any polynomial $f \in \mathbb{C}\langle V, E\rangle$ in non-commuting variables, we have that $f(P, S) = P_{\mathcal{H}} f(Q, T)|_{\mathcal{H}}$ where $P_{\mathcal{H}}$ is the projection from $\mathcal{K}$ to $\mathcal{H}$. This yields an alternative proof to a classical result of Katsoulis and Kribs [75] that $C_e^*(\mathcal{T}_+(G))$ is $*$-isomorphic to $\mathcal{O}(G)$.

We push our dilation results further, and leverage them to the free setting. We prove a complete injectivity result for operator algebras amalgamated over any common C^*-algebra, generalizing a result of Armstrong, Dykema, Exel and Li in [3]. We then use complete injectivity together with a special joint completely positive extension result in [21] to obtain similar dilation results for TCK families associated to *colored* directed graphs. Universal non-self-adjoint operator algebras associated to colored directed graphs were investigated by [47] where their C^*-envelopes were said to coincide with free products of associated Cuntz-Krieger algebras. However, in [32] a gap in [46] (on which [47] relied) was found and corrected. Our complete injectivity result provides another way to fix this gap, and we recover these results for colored directed graphs as well.

We next discuss the second part of this thesis, on matrix convex sets and finite dimensional operator systems. In linear programming we are given n linear inequalities $\sum_{i=1}^d a_i^{(j)} x_i \leq b_j$ and an linear objective function $f(x) = \sum_{i=1}^d c_i x_i$. The goal is then to find a global maximum / minimum for the function f inside the convex polyhedral domain

$$\mathcal{D} := \{ \ x = (x_1, ..., x_d) \in \mathbb{R}^d \mid \sum_{i=1}^d a_i^{(j)} x_i \leq b_j, \ \forall 1 \leq j \leq d \ \}.$$

Another, equivalent way to express the above domain $\mathcal{D}$ is as follows: we define $A_i = \operatorname{diag}(a_i^{(1)}, ..., a_i^{(n)})$ and $B = \operatorname{diag}(b_1, ..., b_n)$ and we have $x = (x_1, ..., x_d) \in \mathcal{D}$ if and only if

$$\sum_{i=1}^d x_i A_i \leq B. \tag{1.1}$$

Now, if we wanted to arrive at the notion of a *semidefinite program*, we enlarge the type of convex domains that we may optimize a linear objective function on. We do this by allowing for not-necessarily diagonal matrices A_i in equation (1.1), to obtain what is known as a *linear matrix inequality* domain or *spectrahedron*. A main emerging branch of convex optimization in the last 20 years is that of semidefinite programming [94], and is based on linear matrix inequalities and *spectrahedra*, as opposed to just a convex polyhedral domains.

In order to simplify optimization problems related to spectrahedra, it is often useful to check if a complicated spectrahedron is included in a simpler spectrahedron (such as a polyhedral domain). However, Ben-Tal and Nemirovski [16] show that even for the cube $[-1,1]^d$, there are spectrahedra $\mathcal{D}$ such that determining whether or not containment $[-1,1]^d \subseteq \mathcal{D}$ holds is an NP-hard problem. Instead of this, Helton, Klep and McCullough [63] introduce a relaxation of the problem by considering *free* spectrahedra. Indeed, for a d-tuple of self-adjoint matrices $A = (A_1, ..., A_d)$ we define

$$\mathcal{D}_A^{sa}(n) = \{ \; X = (X_1, ..., X_d) \in M_n(\mathbb{C})_{sa}^d \mid \sum_{i=1}^d A_i \otimes X_i \leq I \}$$

where $\otimes$ is the usual Kronecker tensor. It is then shown, for another self-adjoint d-tuple of matrices B that the simultaneous containment $\mathcal{D}_A^{sa} \subseteq \mathcal{D}_B^{sa}$ (meaning that $\mathcal{D}_A^{sa}(n) \subseteq \mathcal{D}_B^{sa}(n)$ for all $n \in \mathbb{N}$) can be verified in "approximately polynomial time" (using a feasibility semidefinite program). Clearly $\mathcal{D}_A^{sa} \subseteq \mathcal{D}_B^{sa}$ implies $\mathcal{D}_A^{sa}(1) \subseteq \mathcal{D}_B^{sa}(1)$, so this is a relaxation of the containment problem of spectrahedra. Helton, Klep and McCullough also provide examples where $\mathcal{D}_A^{sa}(1) \subseteq \mathcal{D}_B^{sa}(1)$ while $\mathcal{D}_A^{sa} \not\subseteq \mathcal{D}_B^{sa}$.

One highly important feature of free spectrahedral containment is that it encodes the interpolation of unital completely positive maps between finite dimensional operator systems. More precisely, in [63, Theorem 3.5] it is shown, assuming boundedness of the spectrahedron $\mathcal{D}_A$, that we have containment $\mathcal{D}_A \subseteq \mathcal{D}_B$ if and only if there is a unital completely positive map $\phi : \mathcal{V}_A \to \mathcal{V}_B$ mapping A_i to B_i, where $\mathcal{V}_A$ and $\mathcal{V}_B$ are the operator system generated by the A_i's and B_i's respectively.

In [49], Effros and Winkler provide a Hahn–Banach separation type theorem for *matrix convex sets* introduced by Wittstock [121]. These quantize the respective notions and results from convex theory in Banach spaces. A prominent example of a matrix convex set is none other than the free spectrahedra $\mathcal{D}_A$. Hence, it comes as no surprise that the Effros–Winkler Hahn–Banach separation theorem has been useful for studying spectrahedra.

In Chapter 6, which is based on joint work with Kenneth Davidson, Orr Shalit and Baruch Solel [31], we unify, expand and simplify the theory for (finite dimensional) matrix convex sets, by establishing a categorical duality functor between them and finite dimensional operator systems. Our categorical duality is then used to relate maximal and minimal operator systems structures on Archimedean order unit spaces in the sense of [100] with maximal and minimal matrix convex sets with a given first level convex set, together with dual operator system structures in the sense of [25] with dual matrix convex sets.

We accomplish our results by replacing free spectrahedra $\mathcal{D}_A$ with their matrix dual matrix ranges $\mathcal{W}(A)$ of A, and appealing to the techniques of Arveson [6] and those of Effros and Winkler [49]. For a d-tuple of operators $A = (A_1, ..., A_d)$ on a Hilbert space $\mathcal{H}$ we define $\mathcal{W}(A) = (\mathcal{W}_n(A))$ via

$$\mathcal{W}_n(A) = \{ \ X = (\phi(A_1), ..., \phi(A_d)) \mid \phi \in UCP(\mathcal{V}_A, B(\mathcal{H})) \ \}.$$

We show that under certain minimality assumptions on A, the matrix convex set $\mathcal{W}(A)$ is a complete unitary / approximate unitary invariant for A. Using duality one can recover a generalization of a result of [63] in terms of $\mathcal{D}_A$, for when A is a matrix d-tuple.

In [64], Helton, Klep, McCullough and Schweighofer tried to turn LMI relaxation of the matrix cube inclusion around. They found numerical estimates on the optimal scale $\theta(m) \geq 1$, such that

$$[-1,1]^d \subseteq \mathcal{D}_B^{sa}(1) \Longrightarrow \mathfrak{C} \subseteq \theta(m)\mathcal{D}_B^{sa}$$

for all d-tuples B of matrices of ranks at most m, and $\mathfrak{C}$ the matrix convex set of d-tuples of contractions. They accomplish this feat by simultaneously dilating, up to some scaling, *all* contractive $m \times m$ matrices to a subset of commuting contractions on an infinite dimensional Hilbert space.

We were inspired by their results, and decided to try and obtain an inclusion scale that depends on the dimension d as opposed to the ranks of B. In the abstract setting, we relate minimal and maximal matrix convex sets and operator systems with the general scaled inclusion problem. We use dilation techniques to provide a rich class of "symmetric" convex sets such that for any two d-tuple of operators $A = (A_1, ..., A_d)$ and $B = (B_1, ..., B_d)$ on Hilbert space, with $\mathcal{D}_A(1)$ in the class, we have

$$\mathcal{D}_A^{sa}(1) \subseteq \mathcal{D}_B^{sa}(1) \Longrightarrow \mathcal{D}_A^{sa} \subseteq d \cdot \mathcal{D}_B^{sa}.$$

As mentioned earlier, the matrix convex set inclusion $\mathcal{D}_A \subseteq \mathcal{D}_B$ is helpful for verifying the spectrahedra inclusion $\mathcal{D}_A^{sa}(1) \subseteq \mathcal{D}_B^{sa}(1)$. As opposed to that, the inclusion $\mathcal{D}_A^{sa} \subseteq d \cdot \mathcal{D}_B^{sa}$ is then helpful to *rule out* the spectrahedra inclusion $\mathcal{D}_A^{sa}(1) \subseteq \mathcal{D}_B^{sa}(1)$. For spectrahedra $\mathcal{D}_A^{sa}$ with $\mathcal{D}_A(1)$ the ball, we show that the above scaling is in fact optimal.

After figuring out duality of matrix convex sets and its relation to that of finite dimensional operator systems, it is natural to ask for a *self-dual* matrix convex set. In the operator systems category, this was accomplished in [95], where a self-dual operator system structure SOH was put on Pisier's self-dual operator space OH. From the viewpoint of matrix convex sets, we show the existence of an essentially unique self-dual matrix convex set. As a final application, we use this uniqueness in the matrix convex set category to show, for any fixed dimension, that the operator system of matrix-affine maps on this self-dual matrix convex set is unitally completely order isomorphic to SOH.

The following is an overview of the chapters of this thesis. This thesis has six chapters, including this introduction.

In Chapter 2 we give the necessary preliminaries on Arveson's non-commutative boundary theory, subproduct systems and their operator algebras, topological and dynamical constructs to be used in this thesis and on matrix convex sets and operator systems axiomatics.

In Chapter 3 we introduce weighted partial systems, their C^*-correspondences and their associated tensor algebras. We characterize different isomorphism between the C^*-correspondences, and then use character space techniques to get our classification results for the tensor algebras. We finish this chapter with alternative proofs to two previously-obtained results in classification of operator algebras.

In Chapter 4 we compute the Cuntz-Pimsner algebra of an irreducible stochastic matrix via a subproduct systems along with its extension theory. We use this to compute the C^*-envelope of the non-self-adjoint tensor algebra of this subproduct system, and classify them up to stable and $*$-isomorphism. We conclude this chapter with examples and a comparison with the classical Cuntz-Krieger algebra associated to the graph of the stochastic matrix.

In Chapter 5 we apply Arveson's non-commutative boundary theory to obtain optimal dilations of Toeplitz-Cuntz-Krieger families to full Cuntz-Krieger families. This gives us an alternative proof to a C^*-envelope result of Katsoulis and Kribs for graph tensor algebras. We prove a complete injectivity result for amalgamated free products of operator algebras which allows us to generalize our results to the free setting, in the context of families of operator associated to *colored* directed graphs.

Finally, in Chapter 6 we establish a categorical duality between (finite dimensional) matrix convex sets and finite dimensional operator systems. We use this to describe and relate minimal, maximal and dual structures in these categories. This then comes into play for obtaining scaled inclusion results, especially for matrix convex sets with ground floor satisfying a certain symmetry condition. Finally, we get optimal inclusion results for matrix balls, along with a description of a *self-dual* matrix convex set that coincides with the self-dual operator system SOH up to categorical duality.

Chapter 2

Preliminaries

2.1 Non-commutative boundaries

2.1.1 C*-envelope, unique extension property and maximality

Operator algebras and operator systems can be given an axiomatic definition, as was shown in [20] and [25] respectively. This means that there is an intrinsic operator structure to these objects that is preserved by any completely isometric homomorphism / unital complete order embedding. We will survey the theory of non-commutative boundaries for unital operator algebras and operator systems, and we refer the reader to [5, 6, 10, 18] for a more in-depth treatment of the theory. We will denote operator algebras and C*-algebras by $\mathcal{A}$ and $\mathcal{B}$, operator systems by $\mathcal{V}$ and $\mathcal{W}$.

Let $\mathcal{M}$ be an operator algebra or an operator system. We say that the pair $(\iota, \mathcal{B})$ is a *C*-cover* for the operator algebra / operator system $\mathcal{M}$, if $\iota : \mathcal{M} \to \mathcal{B}$ is a completely isometric homomorphism / unital complete order embedding respectively, and $C^*(\iota(\mathcal{M})) = \mathcal{B}$.

There is always a unique, smallest C*-cover for an operator algebra / operator system $\mathcal{M}$. This C*-cover $(C^*_e(\mathcal{M}), \kappa)$ is called the *C*-envelope* of $\mathcal{M}$ and it satisfies the following universal property: given any other C*-cover $(\mathcal{B}, \iota)$ for $\mathcal{M}$, there exists a (necessarily unique and surjective) $*$-homomorphism $\pi : \mathcal{B} \to C^*_e(\mathcal{M})$, such that $\pi \circ \iota = \kappa$. We will sometimes identify $\mathcal{M}$ with its image $\iota(\mathcal{M})$ under a given C*-cover $(\iota, \mathcal{B})$ for $\mathcal{M}$.

For an operator algebra / operator system $\mathcal{M}$ generating a C*-algebra $\mathcal{B}$, an ideal $\mathcal{J}$ of $\mathcal{B}$ is called *a boundary ideal* for $\mathcal{M}$ if the quotient map $\mathcal{B} \to \mathcal{B}/\mathcal{J}$ is a unital complete isometry on $\mathcal{M}$. The largest boundary ideal $\mathcal{J}_{\mathcal{M}}$ of $\mathcal{M}$ in $\mathcal{B}$ is called *the Shilov ideal* of $\mathcal{M}$

in $\mathcal{B}$, and its importance in our context is that it gives a way of computing the C*-envelope. Namely, the C*-envelope of $\mathcal{M}$ is always isomorphic to $\mathcal{B}/\mathcal{J}_{\mathcal{M}}$.

Suppose $\mathcal{M}$ is a *unital* operator algebra / operator system generating a C*-algebra $\mathcal{B}$. We say that a unital complete contraction $\rho : \mathcal{M} \to B(\mathcal{H})$ has the *unique extension property* if the only unital completely positive extension to $\mathcal{B}$ is a $*$-representation.

When $\mathcal{V}$ is an operator system, by [99, Proposition 3.5] any unital complete contraction must be completely positive, and when $\mathcal{A}$ is an operator algebra, any unital completely contractive map ρ can be extended to a unital completely positive map on the operator system $\mathcal{A}^* + \mathcal{A}$. Hence, when $\mathcal{M}$ is a unital operator algebra / operator system, by Arveson's extension theorem any unital complete contraction $\rho : \mathcal{M} \to B(\mathcal{H})$ has *some* unital completely positive extension $\phi : \mathcal{B} \to B(\mathcal{H})$. When $\mathcal{M}$ is a unital operator algebra and ρ has the unique extension property, it is automatically multiplicative.

We will mostly be interested in unital complete contractions that arise from restrictions of $*$-homomorphisms on the C*-cover. When $\pi : \mathcal{B} \to B(\mathcal{H})$ is a $*$-representation such that $\pi|_{\mathcal{M}}$ has the unique extension property, then any boundary ideal of $\mathcal{M}$ in $\mathcal{B}$ is annihilated by π. We will call such π a boundary representation if it is also irreducible. The boundary theorem of Davidson and Kennedy [36] then describes the Shilov ideal as the intersection of all kernels of boundary representations, providing another way of computing the C*-envelope.

For a unital operator algebra / operator system $\mathcal{M}$ and a unital complete contraction $\varphi : \mathcal{M} \to B(\mathcal{H})$, a unital complete contraction $\psi : \mathcal{M} \to B(\mathcal{K})$ is said to *dilate* φ if there is an isometry $V : \mathcal{H} \to \mathcal{K}$ such that for all $a \in \mathcal{M}$ we have $\varphi(a) = V^*\psi(a)V$. Since V is an isometry, we can identify $\mathcal{H} \cong V(\mathcal{H})$ as a subspace of $\mathcal{K}$, so that ψ dilates φ if and only if there is a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that for all $m \in \mathcal{M}$ we have that $\varphi(m) = P_{\mathcal{H}}\psi(m)|_{\mathcal{H}}$ where $P_{\mathcal{H}}$ is the projection onto $\mathcal{H}$. We say that a unital complete contraction $\rho : \mathcal{M} \to B(K)$ is *maximal* if whenever π is a unital complete contraction dilating ρ, then in fact $\pi = \rho \oplus \psi$ for some unital complete contraction ψ.

Based on ideas of Muhly and Solel from [89], Dritschel and McCullough [45, Theorem 1.1] (See also [10]) showed that a unital complete contraction $\rho : \mathcal{M} \to B(\mathcal{K})$ is maximal if and only if it has the unique extension property. Dritschel and McCullough [45, Theorem 1.2] then used this to show that every unital complete contraction ρ on $\mathcal{M}$ can be dilated to a *maximal* unital complete contraction π on $\mathcal{M}$. This provided the first dilation-theoretic proof for the existence of the C*-envelope for unital operator algebras, operator systems, and for unital operator spaces (See [71]).

In [12] Arveson investigated a closely related notion for C*-covers called hyperrigidity. One of the equivalent formulations for hyperrigidity of a unital operator algebra / operator system $\mathcal{M}$ in a C*-cover $(\iota, \mathcal{B})$ is that for every $*$-representation $\pi : \mathcal{B} \to B(\mathcal{H})$, the restriction $\pi|_{\mathcal{M}}$ has the unique extension property. In particular, since the Shilov ideal $\mathcal{J}_{\mathcal{M}}$ is contained in the intersection of kernels of all $*$-representations, it must be trivial, so that the C*-envelope must be $B = C_e^*(\mathcal{M})$.

2.1.2 Boundary theory for non-unital algebras

We explain how to define the notions of maximality and the unique extension property for representations of non-unital operator algebras, in a way that yields essentially the same theory as in the unital case. For an operator algebra $\mathcal{A}$, we will say that a map $\varphi : \mathcal{A} \to B(\mathcal{H})$ is a *representation* of $\mathcal{A}$ if it is a completely contractive homomorphism.

If $\mathcal{A} \subseteq B(\mathcal{H})$ is a non-unital operator algebra generating a C*-algebra $\mathcal{B} = C^*(\mathcal{A})$, a theorem of Meyer [88, Section 3] (see also [18, Corollary 2.1.15]) states that every representation $\varphi : \mathcal{A} \to B(\mathcal{K})$ extends to a unital representation φ^1 on the *unitization* $\mathcal{A}^1 = \mathcal{A} \oplus \mathbb{C} I_{\mathcal{H}}$ of $\mathcal{A}$ by specifying $\varphi^1(a + \lambda I_{\mathcal{H}}) = \varphi(a) + \lambda I_{\mathcal{K}}$. This theorem allows one to show that every representation ϕ has a completely contractive and completely positive extension to $\mathcal{B}$ via Arveson's extension theorem. In fact, this is a version of Arveson's extension theorem for non-unital operator algebras. Meyer's theorem also shows that $\mathcal{A}$ has a *unique* (one-point) unitization, in the sense that if $(\iota, \mathcal{B})$ is a C*-cover for the operator algebra $\mathcal{A}$, and $\mathcal{B} \subseteq B(\mathcal{H})$ is some faithful representation of $\mathcal{B}$, then the operator-algebraic structure on $\mathcal{A}^1 \cong \iota(\mathcal{A}) + \mathbb{C}1_{\mathcal{H}}$ is independent of the C*-cover and the faithful representation of $\mathcal{B}$.

Next, we discuss how to extend the notions of maximality and the unique extension property to non-unital operator algebras.

Definition 2.1.1. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be an operator algebra generating a C*-algebra $\mathcal{B}$.*

1. *We say that a representation $\rho : \mathcal{A} \to B(\mathcal{K})$ has the* unique extension property *(UEP for short) if every completely contractive and completely positive map $\pi : \mathcal{B} \to B(\mathcal{K})$ extending ρ is a $*$-representation.*

2. *We say that a representation $\rho : \mathcal{A} \to B(\mathcal{K})$ is* maximal *if whenever π is a representation dilating ρ, then $\pi = \rho \oplus \psi$ for some representation ψ.*

Remark 2.1.2. When the maps in the definitions above are not assumed multiplicative, there are instances where the UEP is satisfied vacuously. We thank Raphaël Clouâtre for bringing these issues to our attention.

Indeed, Suppose $\mathcal{A}$ is a non-unital operator algebra containing a self-adjoint positive element P and let $\rho : \mathcal{A} \to \mathcal{B}$ be a completely contractive homomorphism. The map $-\rho$

is completely contractive, but cannot be extended to a completely contractive completely positive map on $\mathcal{B} = C^*(\mathcal{A})$, as ρ must send P to $-P$. Hence, $-\rho$ vacuously has the UEP. Furthermore, when ρ is *not* maximal, the map $-\rho$ is a completely contractive map that admits a non-trivial completely contractive dilation, coming from the one for ρ. Hence, $-\rho$ is also not maximal. Thus, we see that if we drop the multiplicativity assumptions on our definitions above, the UEP and maximality would not be equivalent.

By a similar proof to [10, Proposition 2.2], and by the Arveson extension theorem for non-unital operator algebras via Meyer's theorem, we get that maximality is equivalent to the UEP.

Consequentially, since maximality does not depend on the choice of C*-cover, the unique extension property for representations does not depend on the choice of C*-cover, even for non-unital operator algebras. We will often refer to this fact as the "*invariance of the UEP*".

For a representation ρ it is easy to see that ρ is maximal if and only if ρ^1 is maximal. Hence, as maximality is equivalent to the UEP, we see that a representation ρ on $\mathcal{A}$ has the UEP if and only if its unitization ρ^1 has the UEP.

Suppose $\mathcal{A}$ is an operator subalgebra of $B(\mathcal{H})$, and $\rho : \mathcal{A} \to B(\mathcal{K})$ is a representation. We can write $\rho := \rho_{nd} \oplus 0^{(\alpha)}$, where $0 : \mathcal{A} \to \mathbb{C}$ is the zero map and α is some multiplicity, such that ρ_{nd} is the non-degenerate part in the sense that $\rho_{nd}(a) = \rho(a)|_L$ with $L := C^*(\rho(\mathcal{A}))\mathcal{K}$.

When $\mathcal{A}$ is unital, we get that any completely contractive completely positive extension of $0 : \mathcal{A} \to \mathbb{C}$ to $\mathcal{B} = C^*(\mathcal{A})$ must be 0. As the direct sum of representations with the UEP still has the UEP, we see that ρ has the UEP if and only if the *unital* representation ρ_{nd} has the UEP.

In the case where $\mathcal{A}$ is separable, non-unital and contains a positive approximate identity, we let $0^1 : \mathcal{A}^1 \to \mathbb{C}$ be the unitization of the zero map, which is a unital representation. Since this map extends uniquely to a map on the operator system $\mathcal{S} = \mathcal{A}^1 + (\mathcal{A}^1)^*$, which we still denote by 0^1, and as $\mathcal{A} \cap \mathcal{A}^*$ contains a positive approximate identity, by [12, Theorem 6.1] we see that 0^1 has the UEP when restricted to $\mathcal{A}^1$. Hence, the restriction $0 = 0^1|_{\mathcal{A}}$ has the UEP.

Hence, if we assume that $\mathcal{A}$ is separable and has a positive approximate identity, we still have that ρ has UEP if and only if ρ_{nd} has UEP. These assumptions will be satisfied by all non-unital operator algebras discussed in this thesis.

The C*-envelope of a non-unital operator algebra can also be computed from the C*-envelope of its unitization. More precisely, as the pair $(C^*_e(\mathcal{A}), \iota)$ where $C^*_e(\mathcal{A})$ is the C*-

subalgebra generated by $\iota(\mathcal{A})$ inside the C*-envelope $(C_e^*(\mathcal{A}^1), \iota)$ of the (unique) unitization $\mathcal{A}^1$ of $\mathcal{A}$. By the proof of [18, Proposition 4.3.5] this C*-envelope of an operator algebra $\mathcal{A}$ has the desired universal property, that for any C*-cover $(\iota', \mathcal{B}')$ of $\mathcal{A}$, there exists a (necessarily unique and surjective) $*$-homomorphism $\pi : \mathcal{B}' \to C_e^*(\mathcal{A})$, such that $\pi \circ \iota' = \iota$.

As to representations with the UEP, when $\mathcal{A}$ is an operator algebra generating a C*-algebra $\mathcal{B}$, using these unitization tricks, the theorem of Dritschel and McCullough in the unital case shows that $C_e^*(\mathcal{A})$ is again the image of a $*$-representation $\rho : \mathcal{B} \to B(\mathcal{K})$ such that $\rho|_{\mathcal{A}}$ is completely isometric and has the unique extension property.

Let $\mathcal{A}$ be an operator algebra generating a C*-algebra $\mathcal{B}$. We say that $\mathcal{A}$ has the unique extension property in $\mathcal{B}$ if for any faithful $*$-representation $\pi : \mathcal{B} \to B(\mathcal{H})$ we have that $\pi|_{\mathcal{A}}$ has the unique extension property. By taking a direct sum of π with a given $*$-representation of $\mathcal{B}$, it is easy to show that the faithfulness assumption can be dropped, and in particular, we must have that $\mathcal{B} \cong C_e^*(\mathcal{A})$.

We will need the following result on the existence of a largest sub-representation with the UEP. Let $\phi : \mathcal{B} \to B(\mathcal{H})$ be a completely contractive completely positive map on a C*-algebra $\mathcal{B}$, and let $\mathcal{K} \subseteq \mathcal{H}$ be a reducing subspace for $\phi(\mathcal{A})$. Let $\phi_{\mathcal{K}} : \mathcal{B} \to B(\mathcal{K})$ denote the restriction $\phi_{\mathcal{K}}(b) = \phi(b)|_{\mathcal{K}}$.

Proposition 2.1.3. *Let $\mathcal{A}$ be an operator algebra generating a C*-algebra $\mathcal{B}$ and let $\pi : \mathcal{B} \to B(\mathcal{H})$ be a $*$-representation. Then there is a unique largest reducing subspace $\mathcal{K}$ for π such that $\pi_{\mathcal{K}}|_{\mathcal{A}}$ has the unique extension property.*

Proof. If there is no such non-trivial reducing subspace, we take $\mathcal{K} = \{0\}$. Otherwise, let $\mathfrak{C}$ be the (non-empty) collection of non-trivial reducing subspaces $\mathcal{L}$ for π such that $\pi_{\mathcal{L}} : \mathcal{B} \to B(\mathcal{L})$ has the UEP when restricted to $\mathcal{A}$. Set $\mathcal{K} := \bigvee_{\mathcal{L} \in \mathfrak{C}} \mathcal{L}$. Since every $\mathcal{L} \in \mathfrak{C}$ is reducing for π, we must have that $\mathcal{K}$ is reducing for π as well. It remains to show that $\pi_{\mathcal{K}}|_{\mathcal{A}}$ has the UEP. To this end, let $\phi : \mathcal{B} \to B(\mathcal{K})$ be a completely contractive completely positive extension of $\pi_{\mathcal{K}}|_{\mathcal{A}}$. Then for every $\mathcal{L} \in \mathfrak{C}$ the map $P_{\mathcal{L}}\phi(\cdot)|_{\mathcal{L}}$ is a completely contractive completely positive map from $\mathcal{B}$ to $B(\mathcal{L})$ and $P_{\mathcal{L}}\phi(a)|_{\mathcal{L}} = P_{\mathcal{L}}\pi_K(a)|_{\mathcal{L}} = \pi_{\mathcal{L}}(a)$ for every $a \in \mathcal{A}$. As $\pi_{\mathcal{L}}$ has the UEP we have that $P_{\mathcal{L}}\phi(b)|_{\mathcal{L}} = \pi_{\mathcal{L}}(b)$ for every $b \in \mathcal{B}$. In addition, by Schwarz inequality, for every $\mathcal{L} \in \mathfrak{C}$ and $b \in \mathcal{B}$,

$$\begin{aligned}
0 &\leq P_{\mathcal{L}}\phi(b)^*(I_{\mathcal{K}} - P_{\mathcal{L}})\phi(b)P_{\mathcal{L}} \\
&= P_{\mathcal{L}}\phi(b)^*\phi(b)P_{\mathcal{L}} - P_{\mathcal{L}}\phi(b)^*P_{\mathcal{L}}\phi(b)P_{\mathcal{L}} \\
&\leq P_{\mathcal{L}}\phi(b^*b)P_{\mathcal{L}} - P_{\mathcal{L}}\phi(b)^*P_{\mathcal{L}}\phi(b)P_{\mathcal{L}} \\
&= \pi_{\mathcal{L}}(b^*b) - \pi_{\mathcal{L}}(b)^*\pi_{\mathcal{L}}(b) = 0
\end{aligned}$$

so that $P_{\mathcal{L}}\phi(b)P_{\mathcal{L}} = \phi(b)P_{\mathcal{L}}$ for every $b \in \mathcal{B}$. Thus, for every $n \in \mathbb{N}$, $\mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathfrak{C}$, and $\xi_1 \in \mathcal{L}_1, \ldots, \xi_n \in \mathcal{L}_n$ we have

$$\begin{aligned}\pi_{\mathcal{K}}(b)\left(\sum_{i=1}^{n}\xi_i\right) &= \sum_{i=1}^{n}\pi_{\mathcal{K}}(b)\xi_i = \sum_{i=1}^{n}\pi_{\mathcal{L}_i}(b)\xi_i \\ &= \sum_{i=1}^{n}P_{\mathcal{L}_i}\phi(b)\xi_i = \sum_{i=1}^{n}\phi(b)\xi_i = \phi(b)\left(\sum_{i=1}^{n}\xi_i\right).\end{aligned}$$

As sums $\sum_{i=1}^{n}\xi_i$ are dense in $\mathcal{K}$, we have that $\pi_{\mathcal{K}}(b) = \phi(b)$ for every $b \in \mathcal{B}$. Hence, $\pi_{\mathcal{K}}|_{\mathcal{A}}$ has the unique extension property, and $\mathcal{K}$ is the unique largest subspace with this property.

□

2.2 Subproduct systems and their operator algebras

2.2.1 C*-correspondences

We assume that the reader is familiar with the basic theory of Hilbert C*-modules, which can be found in [84, 85, 97]. We only give a quick summary of basic notions and terminology as we proceed, so as to clarify our conventions.

Definition 2.2.1. *Let $\mathcal{A}$ be a C*-algebra, E is called an inner product module over $\mathcal{A}$ if it is a right $\mathcal{A}$-module, with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ on $E \times E$, such that the following conditions are satisfied for all $x, y, z \in E$, $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$.*

1. *$\mathcal{A}$-linearity in the second variable:*

$$\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle,$$

$$\langle x, ya\rangle = \langle x, y\rangle a;$$

2. *Hermitian symmetry : $\langle x, y\rangle = \langle y, x\rangle^*$;*
3. *Positivity: $\langle x, x\rangle \geq 0$;*
4. *Definiteness: $\langle x, x\rangle = 0$ implies $x = 0$.*

If E is an inner product module over $\mathcal{A}$, then a norm on E is given by $||x|| = ||\langle x, x\rangle||^{1/2}$, and if E is complete with respect to this norm, then E is called a Hilbert C*-module *over $\mathcal{A}$.*

Let E and F be Hilbert C*-modules over $\mathcal{A}$, and let $T : E \to F$ be a map. Then T is called *adjointable* if there is a map $T^* : F \to E$ such that for all $x \in E$ and $y \in F$, $\langle y, Tx\rangle = \langle T^*y, x\rangle$. Unlike in the Hilbert space case, not all bounded linear maps on a Hilbert C*-module are adjointable. The set of all adjointable maps from E to F is denoted by $\mathcal{L}(E, F)$, and we denote $\mathcal{L}(E) := \mathcal{L}(E, E)$ the adjointable operators on E. An adjointable map is automatically a bounded $\mathcal{A}$-module map by the Uniform Boundedness Principle.

Definition 2.2.2. *Let $\mathcal{A}$ be a C*-algebra and E a Hilbert C*-module over $\mathcal{A}$. If in addition, E has a left $\mathcal{A}$-module structure given by a $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{L}(E)$, we call E a* C*-correspondence over $\mathcal{A}$. *We will say that E is* faithful *if ϕ is faithful, and that E is* essential *if $\phi(\mathcal{A})E = E$.*

Let E be a C*-correspondence over $\mathcal{A}$ with left action $\varphi : \mathcal{A} \to \mathcal{L}(E)$, and let $\mathcal{J}$ be an ideal of $\mathcal{A}$. We may define a quotient C*-correspondence $E^{\mathcal{J}}$ over $\mathcal{A}/\mathcal{J}$ as follows. Let $q_{\mathcal{J}} : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ be the quotient map. We define

$$\langle \xi, \eta\rangle_{\mathcal{J}} = q_{\mathcal{J}}(\langle \xi, \eta\rangle)$$

and define the C*-module $E(\mathcal{J})$ to be the quotient of E by the closed right $\mathcal{A}$ submodule $E_{\mathcal{J}} := \phi(\mathcal{J})E + E\mathcal{J}$. We define the left action $\phi_{\mathcal{J}} : \mathcal{A}/\mathcal{J} \to \mathcal{L}(E(\mathcal{J}))$ given by $\phi_{\mathcal{J}}(a + \mathcal{J})(\xi + E_{\mathcal{J}}) = \phi(a)(\xi) + E_{\mathcal{J}}$ for $\xi \in E$ and $a \in \mathcal{A}$, which is a well-defined left action of $\mathcal{A}$ on $E(\mathcal{J})$. Together with this left action, $E(\mathcal{J})$ becomes a C*-correspondence over $\mathcal{A}/\mathcal{J}$. We then have a natural quotient $\mathcal{A}$ - $\mathcal{A}/\mathcal{J}$ bimodule map $q_{\mathcal{J}} : E \to E(\mathcal{J})$ given by $\psi_{\mathcal{J}}(\xi) = \xi + E_{\mathcal{J}}$.

A key notion of C*-correspondences is the internal tensor product. If E is a C*-correspondence over $\mathcal{A}$ with left action ϕ, and F is a C*-correspondence over $\mathcal{A}$ with left action ψ, then on the algebraic tensor product $E \otimes_{alg} F$ one defines an $\mathcal{A}$-valued pre-inner product satisfying $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle y_1, \psi(\langle x_1, x_2\rangle)y_2\rangle$ on simple tensors. The usual completion process with respect to the norm induced by this inner product, yields the internal Hilbert C*-module tensor product of E and F, denoted by $E \otimes F$, $E \otimes_{\psi} F$ or $E \otimes_{\mathcal{A}} F$, which is a C*-correspondence over $\mathcal{A}$ with left action $\phi \otimes Id_F$.

If E and F are C*-correspondences over $\mathcal{A}$ and $\mathcal{J}$ is a closed ideal of $\mathcal{A}$, it is easy to show that the map $\iota : E(\mathcal{J}) \otimes_{\mathcal{A}/\mathcal{J}} F(\mathcal{J}) \to (E \otimes_{\mathcal{A}} F)(\mathcal{J})$ given by $\iota((\xi + E_{\mathcal{J}}) \otimes (\eta + F_{\mathcal{J}})) =$

$\xi \otimes \eta + (E \otimes F)_{\mathcal{J}}$ is an isometric $\mathcal{A}/\mathcal{J}$-bimodule map. Hence, we may always think of $E(\mathcal{J}) \otimes_{\mathcal{A}/\mathcal{J}} F(\mathcal{J})$ as a closed sub-C*-correspondence of $(E \otimes_{\mathcal{A}} F)(\mathcal{J})$ via ι. Henceforth, unless otherwise specified we will suppress notation and write $a \cdot \xi := \phi(a)\xi$ for the left action of a given C*-correspondence over $\mathcal{A}$.

We now discuss certain types of morphisms between C*-correspondences that arise naturally in the context of tensor algebras.

Definition 2.2.3. *Let E and F be C*-correspondences over the C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, let $\rho : \mathcal{A} \to \mathcal{B}$ be a $*$-isomorphism. Then we define the following:*

1. *A ρ-bimodule map $V : E \to F$ is a map satisfying $V(a\xi b) = \rho(a)V(\xi)\rho(b)$.*
2. *A ρ-bimodule map $V : E \to F$ is called ρ-adjointable if there exists ρ^{-1}-bimodule adjoint $V^* : F \to E$. That is, for $\xi \in F$ and $\eta \in E$,*

$$\langle V^*(\xi), \eta \rangle = \rho^{-1}(\langle \xi, V(\eta) \rangle).$$

We note that a ρ-adjointable map $V : E \to F$ is again automatically bounded by the Uniform Boundedness Principle, where the ρ-adjoint $V^* : F \to E$ is a bounded ρ^{-1}-bimodule map.

Given a $*$-isomorphism $\rho : \mathcal{A} \to \mathcal{B}$ and a C*-correspondence F over $\mathcal{B}$, we may define a C*-correspondence structure F_ρ over $\mathcal{A}$ on the set F. For $a \in \mathcal{A}$ and $\xi \in F$, we define left and right actions given by

$$a \cdot \xi := \rho(a)\xi \quad \text{and} \quad \xi \cdot a := \xi\rho(a)$$

and $\mathcal{A}$-valued inner product, given for $\xi, \eta \in F$ by

$$\langle \xi, \eta \rangle_\rho = \rho^{-1}(\langle \xi, \eta \rangle).$$

This construction turns F into a C*-correspondence over $\mathcal{A}$, satisfies $(F_\rho)_{\rho^{-1}} = F$ as C*-correspondences over $\mathcal{B}$, and behaves well with respect to internal tensor products. That is, if F, F' are C*-correspondences over $\mathcal{B}$ and $\rho : \mathcal{A} \to \mathcal{B}$ is a $*$-isomorphism, then $(F \otimes_{\mathcal{B}} F')_\rho$ is unitarily isomorphic to $F_\rho \otimes_{\mathcal{A}} F'_\rho$.

Next we show that tensor products of bounded ρ-bimodule maps exist even when the maps are not necessarily adjointable.

Proposition 2.2.4. *Let E, E' be C^*-correspondences over $\mathcal{A}$ and F, F' be C^*-correspondences over $\mathcal{B}$. Suppose $V : E \to F$, $W : E' \to F'$ are bounded ρ-bimodule maps for some $*$-isomorphism $\rho : \mathcal{A} \to \mathcal{B}$. Then there exists a unique* bounded *ρ-bimodule map $V \otimes W : E \otimes E' \to F \otimes F'$ defined on simple tensors by $(V \otimes W)(\xi \otimes \eta) = V\xi \otimes W\eta$, and moreover $\|V \otimes W\| \leq \|V\| \cdot \|W\|$.*

Proof. By the preceding discussion, looking at F_ρ, F'_ρ and $(F \otimes_{\mathcal{B}} F')_\rho \cong F_\rho \otimes_{\mathcal{A}} F'_\rho$ instead, we may assume without loss of generality that $\mathcal{A} = \mathcal{B}$ and that $\rho = Id_{\mathcal{A}}$. Then, by item (1) of [18, Subsection 8.2.12] the desired result follows. □

Hence, if $V : E \to F$ is a bounded ρ-bimodule map, the maps $V^{\otimes n} : E^{\otimes n} \to F^{\otimes n}$ are bounded ρ-bimodule maps, with $\|V^{\otimes n}\| \leq \|V\|^n$. If in addition to that there exists $C > 0$ such that for all $n \in \mathbb{N}$ we have $\|V^{\otimes n}\| \leq C$, we say that V is *tensor-power bounded*.

Definition 2.2.5. *Let E and F be C^*-correspondences over the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, and let $\rho : \mathcal{A} \to \mathcal{B}$ be a $*$-isomorphism.*

1. *A bounded ρ-bimodule map $V : E \to F$ is called a ρ-isomorphism if V is bijective.*
2. *A ρ-bimodule map $V : E \to F$ is called a ρ-similarity if V is bijective and V and V^{-1} are tensor-power bounded.*
3. *A map $U : E \to F$ is called a ρ-unitary if U is a surjective isometric ρ-bimodule map.*

We will say that E and F are isomorphic / similar / unitarily isomorphic if there exist a $*$-isomorphism $\rho : \mathcal{A} \to \mathcal{B}$ and a ρ - isomorphism / similarity / unitary $V : E \to F$ respectively.

Remark 2.2.6. It turns out that $U : E \to F$ is ρ-unitary if and only if U is ρ-adjointable and $U^*U = Id_E$ and $UU^* = Id_F$. In this case, we also see that U is a ρ-similarity.

If V is a *ρ-adjointable* ρ-isomorphism, then $V^*V \in \mathcal{L}(E)$ is an Id-isomorphism and $V|V|^{-1}$ defines a ρ-unitary between E and F. Hence, we note that in general, we do not assume that bounded ρ-bimodule maps are ρ-adjointable. In fact, in Example 3.2.28 we will see a ρ-similarity which is not ρ-adjointable.

2.2.2 Subproduct systems

The following is a C*-algebraic version of [113, Definition 1.1] for the semigroup $\mathbb{N}$. It was also given in [118, Definition 1.4] for *essential* C*-correspondences.

Definition 2.2.7. *Let $\mathcal{A}$ be a C*-algebra, let $X = \{X_n\}_{n\in\mathbb{N}}$ be a family of C*-correspondences over $\mathcal{A}$ and let $U = \{U_{n,m} : X_n \otimes X_m \to X_{n+m}\}$ be a family of bounded bimodule maps. We will say that (X, U) is a* subproduct system *over $\mathcal{A}$ if the following conditions are met:*

1. $X_0 = \mathcal{A}$.
2. *The maps $U_{0,n}$ and $U_{n,0}$ are given by the left and right actions of $\mathcal{A}$ on X_n respectively.*
3. *$U_{n,m}$ is an* adjointable *coisometric map for every* non-zero $n, m \in \mathbb{N}$.
4. *For every $n, m \in \mathbb{N}$ we have the associativity identity*

$$U_{n+m,p}(U_{n,m} \otimes Id_{X_p}) = U_{n,m+p}(Id_{X_n} \otimes U_{m,p}).$$

In case the maps $U_{n,m}$ are unitaries for non-zero $n, m \in \mathbb{N}$, *we say that X is a* product system.

Example 2.2.8. *If E is a C*-correspondence over $\mathcal{A}$, define* $\mathrm{Prod}(E) = \{\mathrm{Prod}(E)_n\}$ *by* $\mathrm{Prod}(E)_n = E^{\otimes n}$ *and $U^E = \{U^E_{n,m}\}$ the natural associativity unitaries $U^E_{n,m} : E^{\otimes n} \otimes E^{\otimes m} \to E^{\otimes(n+m)}$ when n, m are non-zero. Then $(Prod(E), U^E)$ is a product system.*

Example 2.2.9. *Let $\mathcal{H}$ be a Hilbert space as a C*-correspondence over $\mathbb{C}$. Let p_n be the projection of $\mathcal{H}^{\otimes n}$ onto the symmetric subspace of $\mathcal{H}^{\otimes n}$ given by*

$$p_n(\xi_1 \otimes ... \otimes \xi_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma^{-1}(1)} \otimes ... \otimes \xi_{\sigma^{-1}(n)}.$$

We define $SSP(\mathcal{H}) = \{SSP(\mathcal{H})_n\}$ by $SSP(\mathcal{H})_n = p_n(H^{\otimes n})$, with subproduct maps $U_{n,m} : SSP(\mathcal{H})_n \otimes SSP(\mathcal{H})_m \to SSP(\mathcal{H})_{n+m}$ are given by

$$U_{n,m}(x \otimes y) = p_{n+m}(x \otimes y).$$

Then $(SSP(\mathcal{H}), U)$ is a subproduct system which is not a product system.

Definition 2.2.10. *Let* (X, U^X) *and* (Y, U^Y) *be two subproduct systems over* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively. A family* $V = \{V_n\}_{n\in\mathbb{N}}$ *of maps* $V_n : X_n \to Y_n$ *is called a* morphism *of subproduct systems from* (X, U^X) *to* (Y, U^Y) *if*

1. *The map* $\rho := V_0 : \mathcal{A} \to \mathcal{B}$ *is a *-isomorphism,*
2. *For all* $n \neq 0$ *the map* V_n *are* uniformly *bounded* ρ*-bimodule morphisms in the sense that* $\sup_{n\in\mathbb{N}} \|V_n\| < \infty$*,*
3. *For all* $n, m \in \mathbb{N}$ *the following identity hold:*

$$V_{n+m} \circ U^X_{n,m} = U^Y_{n,m} \circ (V_n \otimes V_m).$$

When the family V *is a family of*

1. ρ*-isomorphisms, such that* $V^{-1} := \{V_n^{-1}\}$ *is a morphism from* (Y, U^Y) *to* (X, U^X)*, we say that* X *and* Y *are* isomorphic *via* V *and write* $X \sim Y$*.*
2. ρ*-unitaries, we say that* X *and* Y *are* unitarily isomorphic *via* V *and write* $X \cong Y$*.*

We next show that that whenever (X, U) is a product system, it is in fact unitarily isomorphic to a product system of the form $(\mathrm{Prod}(E), U^E)$ as in Example 2.2.8, for the C*-correspondence $E = X_1$, and that any isomorphism $V = \{V_n\}$ between product systems is determined by V_1.

Proposition 2.2.11. *Let* (X, U) *be a product system over a C*-algebra* $\mathcal{A}$*. Then* (X, U) *is unitarily isomorphic to* $(\mathrm{Prod}(X_1), U^{X_1})$*. Furthermore, if* $(\mathrm{Prod}(E), U^E)$ *and* $(\mathrm{Prod}(F), U^F)$ *are product systems, and* $V = \{V_n\}$ *an isomorphism / unitary isomorphism between them. Then* $V_n = W^{\otimes n}$ *for a* ρ*-similarity /* ρ*-unitary* W *respectively, where* $W = V_1$ *and* $\rho = V_0$*.*

Proof. We construct a morphism of subproduct systems $W : (\mathrm{Prod}(X_1), U^{X_1}) \to (X, U)$ comprised of Id-unitaries $\{W_n : X_1^{\otimes n} \to X_n\}$ which, by associativity of $U = \{U_{n,m}\}$, are uniquely determined inductively by the equations $W_1 = Id_{X_1}$ and $W_{n+m} = U_{n,m} \circ (W_n \otimes W_m)$. Each W_n is an Id-unitary, and by their inductive definition they intertwine the associativity unitary U^{X_1} and U. Hence, $(\mathrm{Prod}(X_1), U^{X_1})$ and (X, U) are unitarily isomorphic.

Next, when $V = \{V_n\}$ is an isomorphism / unitary isomorphism between $(\mathrm{Prod}(E), U^E)$ and $(\mathrm{Prod}(F), U^F)$, then $V_1 : E \to F$ is a ρ-isomorphism / unitary respectively, and by the intertwining property of morphisms between subproduct systems, we see that $V_{n+m}U^E_{n,m} =$

$U^F_{n,m} \circ (V_n \otimes V_m)$. However, since $U^E_{n,m}$ and $U^F_{n,m}$ are associativity unitaries, this simply means that $V_{n+m} = V_n \otimes V_m$ for all $m, n \in \mathbb{N}$. Hence we get that $V_n = V_1^{\otimes n}$ by induction, and in the case where V is an isomorphism of subproduct systems, as $\sup_{n\in\mathbb{N}} \|V_n\| < \infty$ and $\sup_{n\in\mathbb{N}} \|V_n^{-1}\| < \infty$ we get that V_1 and V_1^{-1} are tensor-power bounded. □

Suppose (X, U) is a subproduct system over a C*-algebra $\mathcal{A}$, and that $\mathcal{J}$ is a closed ideal of $\mathcal{A}$. Then for each X_n we have a quotient map $q_{\mathcal{J}} : X_n \to X_n(\mathcal{J})$, and we may induce an adjointable coisometric $\mathcal{A}/\mathcal{J}$ -bimodule map $U^{\mathcal{J}}_{n+m} : (X_n \otimes X_m)(\mathcal{J}) \to X_{n+m}(\mathcal{J})$ by setting $U^{\mathcal{J}}_{n+m}(\xi \otimes \eta + (X_n \otimes X_m)_{\mathcal{J}}) = U_{n+m}(\xi \otimes \eta) + (X_{n+m})_{\mathcal{J}}$. When the adjointable isometric map $(U^{\mathcal{J}}_{n+m})^*$ has range in $X_n(\mathcal{J}) \otimes X_m(\mathcal{J})$ considered as a subspace of $(X_n \otimes X_m)_{\mathcal{J}}$, the pair $(X^{\mathcal{J}}, U^{\mathcal{J}})$ given by $X^{\mathcal{J}} = \{X^{\mathcal{J}}_n\}$ with $U^{\mathcal{J}} = \{U^{\mathcal{J}}_{n,m}\}$ becomes a subproduct system in its own right. Hence, we make the following definition, which is the C*-analogue of [42, Definition 6.19].

Definition 2.2.12. *Let (X, U) be a subproduct system over a C*-algebra $\mathcal{A}$. We say that an ideal $\mathcal{J}$ of $\mathcal{A}$ is* X-invariant *if $(U^{\mathcal{J}}_{n+m})^*$ has range in $X_n(\mathcal{J}) \otimes X_m(\mathcal{J})$. We say that (X, U) is* minimal *if $\mathcal{A}$ has no non-trivial X-invariant ideals.*

2.2.3 Operator algebras arising from subproduct systems

We next describe the construction of the tensor, Toeplitz and Cuntz-Pimsner algebras arising from subproduct systems (see [118, 119]).

Let (X, U) be a subproduct system over a C*-algebra $\mathcal{A}$. There is a canonical *product system* containing (X, U) as a subproduct subsystem as follows.

We define $E := X_1$, so that $(\mathrm{Prod}(E), U^E)$ constitutes a product systems as in Example 2.2.8 where $U^E_{n,m}$ are the usual associativity unitaries. One can then construct a morphism of subproduct systems $V : (\mathrm{Prod}(E), U^E) \to (X, U)$ comprised of *adjointable coisometries* $\{V_n : E^n \to X_n\}$ which, by associativity of $U = \{U_{n,m}\}$, are uniquely determined inductively by the equations $V_1 = Id_{X_1}$ and $V_{n+m} = U_{n,m} \circ (V_n \otimes V_m)$. The X-Fock correspondence is the C*-correspondence direct sum of the fibers of the subproduct system

$$\mathcal{F}_X := \bigoplus_{n\in\mathbb{N}} X_n \tag{2.1}$$

Denote by $Q_n \in \mathcal{L}(\mathcal{F}_X)$ the projection of $\mathcal{F}_X$ onto the n-th fiber X_n, and define $Q_{[0,n]} = Q_0 + ... + Q_n$, and $Q_{[n,\infty)} := Id_{\mathcal{F}_X} - Q_{[0,n-1]}$. We then obtain an adjointable coisometric map $V : \mathcal{F}_{\mathrm{Prod}(E)} \to \mathcal{F}_X$ given by $V = \oplus_{n=0}^{\infty} V_n$.

The X-shifts are the operators $S_\xi^{(n)} \in \mathcal{L}(\mathcal{F}_X)$ uniquely determined between fibers by $S_\xi^{(n)}(\eta) := U_{n,m}(\xi \otimes \eta)$ where $n, m \in \mathbb{N}$ and $\xi \in X_n$, $\eta \in X_m$.

We note that $S_\xi^{(n)} = VS_{V_n^*(\xi)}^{(n)}V^*$, so that $S_\xi^{(n)}$ is adjointable with adjoint given by $S_\xi^{(n)*} = VS_{V_n^*(\xi)}^{(n)*}V^*$, where $S_{V_n^*(\xi)}^{(n)}$ is a product system shift and is hence an adjointable operator in $\mathcal{L}(\mathcal{F}_{\mathrm{Prod}(X)})$.

Definition 2.2.13. *The tensor and Toeplitz algebras are the* norm-closed *non-self-adjoint and self-adjoint subalgebras of $\mathcal{L}(\mathcal{F}_X)$ generated by a copy of $\mathcal{A}$ and all X-shifts respectively,*

$$\mathcal{T}_+(X) := \overline{Alg}(\mathcal{A} \cup \{S_\xi^{(n)} | \xi \in X_n, n \in \mathbb{N}\})$$

$$\mathcal{T}(X) := C^*(\mathcal{A} \cup \{S_\xi^{(n)} | \xi \in X_n, n \in \mathbb{N}\}).$$

The algebra $\mathcal{L}(\mathcal{F}_X)$ admits a group homomorphism α of the unit circle $\mathbb{T}$ called the gauge action, defined by $\alpha_\lambda(T) = W_\lambda T W_\lambda^*$ for all $\lambda \in \mathbb{T}$ where $W_\lambda : \mathcal{F}_X \to \mathcal{F}_X$ is the unitary defined by

$$W_\lambda(\oplus_{n=0}^\infty \xi_n) = \oplus_{n=0}^\infty \lambda^n \xi_n.$$

Since $\alpha_\lambda(S_\xi^{(n)}) = S_{\lambda^n \xi}^{(n)} = \lambda^n S_\xi^{(n)}$ and $\alpha_\lambda(a) = a$ for $a \in \mathcal{A}$ and $\xi \in E$, it follows that both the Toeplitz algebra and tensor algebra are α-invariant closed subalgebras, so that α restricts to a completely isometric circle action on each of them, which we still denote by α. One then shows that for every $S \in \mathcal{T}(X)$, the function $f(\lambda) = \alpha_\lambda(S)$ is norm continuous, and this enables the definition of a conditional expectation Φ given by

$$\Phi(S) = \int_{\mathbb{T}} \alpha_\lambda(S) d\lambda,$$

where $d\lambda$ is the normalized Haar measure on $\mathbb{T}$.

Let $\{k_n\}_{n=1}^\infty$ denote Fejer's kernel function defined for $\lambda \in \mathbb{T}$ by

$$k_n(\lambda) = \sum_{j=-n}^{n} \big(1 - \frac{|j|}{n+1}\big)\lambda^j.$$

Note that for $S \in \mathcal{T}(X)$, the existence of the canonical conditional expectation Φ permits the definition of Fourier coefficients for an element $S \in \mathcal{T}(X)$ by

$$\Phi_n(S) = \int_{\mathbb{T}} \alpha_\lambda(S) \lambda^{-n} d\lambda.$$

Then define the Cesaro sums,

$$\sigma_n(S) := \sum_{j=-n}^{n} \big(1 - \frac{|j|}{n+1}\big)\Phi_j(S) = \int_{\mathbb{T}} \sum_{j=-n}^{n} \big(1 - \frac{|j|}{n+1}\big)\alpha_\lambda(S)\lambda^{-j} d\lambda = \int_{\mathbb{T}} \alpha_\lambda(S) k_n(\lambda) d\lambda.$$

Every tensor algebra is then graded by the spaces

$$\mathcal{T}_+(X)_n = \Phi_n(\mathcal{T}_+(X)) = \overline{\mathrm{Sp}}\{ \ S_\xi^{(n)} \ \ | \ \ \xi \in X_n \ \}.$$

We will denote $\mathcal{T}_+(E) := \mathcal{T}_+(\mathrm{Prod}(E))$ for the tensor algebra defined by a single correspondence, and note that whenever X is a product system, the isomorphism $X \cong \mathrm{Prod}(X_1)$ given in 2.2.11 promotes to an isomorphism $\mathcal{T}_+(X) \cong \mathcal{T}_+(X_1)$.

The following is a folklore result for tensor algebras that relates the above notions. We refer the reader to [42, Proposition 6.2] for a proof of this result in the case of subproduct systems over W*-algebras, which is easily adapted to our context.

Proposition 2.2.14. *Let (X, U) be a subproduct system over $\mathcal{A}$. For every $n \in \mathbb{N}$ we have that X_n is isometrically isomorphic as a Banach $\mathcal{A}$-bimodule to $\mathcal{T}_+(X)_n$ via the map determined uniquely by $\xi \mapsto S_\xi^{(n)}$.*

Therefore, every element $T \in \mathcal{T}_+(X)$ has a unique representation as an infinite series $T = \sum_{n=0}^\infty S_{\xi_n}^{(n)}$ where $\xi_n \in X_n$ satisfies $\Phi_n(T) = S_{\xi_n}^{(n)}$ (called its Fourier series representation for short), and the series converges Cesaro to T in norm: if $\sigma_N(T) = \sum_{n=0}^N \Big(1 - \frac{n}{N+1}\Big) S_{\xi_n}^{(n)}$, then we have that $\lim_{N\to\infty} \|\sigma_N(T) - T\| = 0$. Furthermore, if $T, T' \in \mathcal{T}_+(X)$ have Fourier series representations $T = \sum_{i=0}^\infty S_{\xi_i}^{(i)}$ and $T' = \sum_{i=0}^\infty S_{\eta_i}^{(i)}$, then

$$TT' = \sum_{n=0}^{\infty} S_\zeta^{(n)}, \quad \textit{where} \quad \zeta = \sum_{k=0}^{n} U_{k,n-k}(\xi_k \otimes \eta_{n-k}).$$

Another algebra associated to the subproduct system, introduced by Viselter in [118], arises as a special quotient of $\mathcal{T}(X)$, and was defined for subproduct systems (X, U) such that each X_n is a *faithful and essential* C*-correspondence. Let $\mathcal{C} \subseteq \mathcal{L}(\mathcal{F}_X)$ be given by

$$\mathcal{C} = \{ \ T \in \mathcal{L}(\mathcal{F}_X) \ | \ \lim_{n\to\infty} \|TQ_n\| = 0 \ \}.$$

Viselter showed in [119, Theorem 2.5] that $\mathcal{J}(X) := \mathcal{C} \cap \mathcal{T}(X)$ is a closed two-sided ideal inside $\mathcal{T}(X)$.

Definition 2.2.15. *Let* (X, U) *be a subproduct system. The* Cuntz-Pimsner *ideal of* $\mathcal{T}(X)$ *is* $\mathcal{J}(X) := \mathcal{C} \cap \mathcal{T}(X)$*, and the Cuntz-Pimsner algebra of* X *is then* $\mathcal{O}(X) := \mathcal{T}(X)/\mathcal{J}(X)$.

When $X = \text{Prod}(E)$ for a faithful and essential C*-correspondence E, By [119, Proposition 2.8] we have that $\mathcal{O}(X) \cong \mathcal{O}_E$, where $\mathcal{O}_E$ is the well known Cuntz-Pimsner algebra [104].

We note that the circle action on $\mathcal{T}(X)$ passes naturally to $\mathcal{O}(X)$ since $\mathcal{J}(X)$ is gauge invariant. We shall later need the following formula for the norm of an element in the quotient $M_s(\mathcal{O}(X))$, in terms of representatives in $M_s(\mathcal{T}(X))$. Denote by $q : \mathcal{T}(X) \to \mathcal{O}(X)$ the canonical quotient map. When $Q_n \in \mathcal{T}(X)$, it follows from item (1) of [119, Theorem 3.1] that $\{I_s \cdot Q_{[0,m]}\}$ is an approximate identity for $M_s(\mathcal{J}(X))$, and one may then invoke [7, Exercise 1.8.C] to obtain the following.

Corollary 2.2.16. *Let* (X, U) *be a subproduct system, and suppose that* $Q_n \in \mathcal{T}(X)$ *for all* $n \in \mathbb{N}$*. Then for any* $T = [T_{ij}] \in M_s(\mathcal{T}(X))$ *we have*

$$\|q^{(s)}(T)\| = \lim_{m\to\infty} \|[T_{ij}Q_{[m,\infty)}]\|.$$

2.3 Topological and Dynamical constructs

2.3.1 Topological graphs and quivers

Another type of object that turns up in our analysis is the topological quiver in the sense of [93]. See [93, Subsection 3.3] for some classes of examples generalized by topological quivers.

We slightly modify and generalize the definitions of topological quiver as in [93, Definition 3.1] and topological graph in the sense of Katsura [77] to fit our setting. Our choice of range and source maps is reversed to the one in [93] but is consistent with the work of Kwasniewski [83] on Exel crossed products, with the work of Kumjian and Pask in [82] on higher-rank graphs and with topological graphs in the sense of Katsura [77, 79].

Definition 2.3.1. *A* directed graph G *is a quadruple* (E^0, E^1, s, r) *consisting of a set* V *of vertices, a set* E *of edges and two maps* $s, r : E^1 \to E^0$*, called the source map and range map, respectively. If* $v = s(e)$ *and* $w = r(e)$ *we say that* v *emits* e *and* w *receives it. A directed graph is said to be* row-finite *if every vertex receives at most finitely many edges, and is* sourceless *if every vertex receives at least one edge.*

A topological graph *is a quadruple $\mathcal{Q} = (E^0, E^1, r, s)$ such that E^0 and E^1 are compact Hausdorff spaces of vertices and edges respectively, such that the source and range maps $r, s : E^1 \to E^0$ are continuous.*

We will call a collection of measures $\lambda = \{\lambda_v\}_{v \in E^0}$ an s-system if they are Radon measures on E^1 such that $\mathrm{supp}(\lambda_v) \subseteq s^{-1}(v)$ *and for every $\xi \in C(E^1)$ the map $v \mapsto \int_{E^1} \xi(e) d\lambda_v(e)$ is in $C(E^0)$. Together with an s-system, we call the quintuple $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ a* topological quiver*.*

Note that we do not assume that E^0 and E^1 are second countable as in [93, Definition 3.1], or that s is an open map as in [83, Definition 3.29], but we do assume that E^0 and E^1 are compact. An s-system is used to form a C*-correspondence from a topological graph. Note also that the only asymmetry in Definition 2.3.1 is in the definition of a topological quiver. In [93, Definition 3.1], second countability along with the assumption that s is open were used to ensure the existence of an s-system $\lambda = \{\lambda_v\}_{v \in E^0}$ such that $\mathrm{supp}(\lambda_v) = s^{-1}(v)$. In the definition of an s-system, both in [93, Definition 3.1] and [83, Definition 3.29] it is required that $\mathrm{supp}(\lambda_v) = s^{-1}(v)$. However, in order to create a C*-correspondence from a topological quiver, s need not be open, and the weaker condition $\mathrm{supp}(\lambda_v) \subseteq s^{-1}(v)$ is enough.

If we require that s is a local homeomorphism in Definition 2.3.1, by [93, Example 3.20] we get the notion of a topological graph in the sense of Katsura [77], and there is a natural s-system λ_v given by counting measure on $s^{-1}(v)$ for each $v \in E^0$. Note again that in our definition of a topological graph, we do not even assume that s is open.

Next we describe how to construct a C*-correspondence from a topological quiver $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$. We define a $C(E^0)$-valued semi-inner product and bimodule actions on $C(E^1)$ for $v \in E^0$, $\xi, \eta \in C(E^1)$ and $f, g \in C(E^0)$ by setting

$$\langle \xi, \eta \rangle(v) := \int_{s^{-1}(v)} \overline{\xi(e)} \eta(e) d\lambda_v(e) \quad \text{and} \quad (f \cdot \xi \cdot g)(e) := f(r(e))\xi(e)g(s(e)).$$

Taking the Hausdorff completion of $C(E^1)$ yields a C*-correspondence over $C(E^0)$, denoted by $\mathcal{X}_{\mathcal{Q}}$, and we call it the quiver C*-correspondence associated to $\mathcal{Q}$. When s is a local homeomorphism and $\{\lambda_v\}$ are counting measures on $s^{-1}(v)$, this C*-correspondence coincides with the standard one that is associated to a topological graph as defined and associated in [77]. When $\mathrm{supp}(\lambda_v) = s^{-1}(v)$ and s is open, we get the C*-correspondence associated to a topological quiver as in [93] with range and source reversed.

Definition 2.3.2. *Let $\mathcal{Q} = (E^0, E^1, r, s)$ be a topological graph. A* path *in $\mathcal{Q}$ is a finite sequence of edges $\mu = \mu_n ... \mu_1$ with $r(\mu_i) = s(\mu_{i+1})$ for $1 \leq i \leq n-1$. We say that such* *a path has length $|\mu| := n$. Let E^n denote the collection of paths of length n. We extend the maps r and s to E^n by setting $r(\mu) = r(\mu_n)$ and $s(\mu) = s(\mu_1)$. We endow E^n with the topology inherited from $E^1 \times ... \times E^1$. Since s and r are continuous on E^1, we see that this persists when s and r are considered as maps on E^n.*

We will often deal with multiplicity free topological graphs, so we make a definition.

Definition 2.3.3. *Let $\mathcal{Q} = (E^0, E^1, r, s)$ be a topological graph. We say that $\mathcal{Q}$ is multiplicity free if for any edges $e, e' \in E^1$, if $r(e) = r(e')$ and $s(e) = s(e')$ then $e = e'$.*

The advantage of multiplicity free topological graphs is that they can be identified as *closed* subsets of $E^0 \times E^0$ in a canonical way. If $\mathcal{Q} = (E^0, E^1, r, s)$ is multiplicity free, we define a map $r \times s : E^1 \to E^0 \times E^0$ given by $(r \times s)(e) = (r(e), s(e))$. As $r \times s$ is an injective continuous map on a compact space, we see that E^1 is homeomorphic to its image under $r \times s$, which is compact inside $E^0 \times E^0$, and is hence *closed*. Hence, $\mathcal{Q}$ is isomorphic to the topological graph $\mathcal{Q}' := (E^0, (r \times s)(E^1), \pi_r, \pi_s)$ where $\pi_r, \pi_s : E^0 \times E^0 \to E^0$ are given by $\pi_r(y, x) = y$ and $\pi_s(y, x) = x$. Additionally, if $\lambda = \{\lambda_v\}$ is an s-system for $\mathcal{Q}$, we denote $\lambda' = \{\lambda'_v\}$ the s-system on $\mathcal{Q}'$ given by $\lambda'_v(E) = \lambda_v((r \times s)^{-1}(E))$, so that $\mathcal{Q}$ and $\mathcal{Q}'$ are isomorphic as topological quivers with λ and λ' respectively. We will often identify multiplicity free topological quivers $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ with $E^1 = (r \times s)(E^1)$ as closed subspace of $E^0 \times E^0$, where $r = \pi_r$, $s = \pi_s$ and $\lambda = \lambda'$.

2.3.2 Markov-Feller operators

Let X be a compact space. We denote by $M_+(X)$ the collection of all finite positive Radon measures on X, identified with positive bounded linear functionals on $C(X)$ via the Riesz-Markov theorem. We also denote by $\mathrm{Pr}(X)$ the w*-compact convex subset of Radon probability measures on X.

Definition 2.3.4. *Let X and Y be compact spaces. A* positive measure valued *map (p.m.v. map for short) is a continuous map $P : X \to M_+(Y)$, where $M_+(Y)$ is imbued with the weak* topology. We set $P_x := P(x)$ for the finite measure P associates to $x \in X$. If in addition for all $x \in X$ we have $P_x \in \mathrm{Pr}(Y)$ is a probability measure, then we call $P : X \to \mathrm{Pr}(Y)$ a Markov-Feller map.*

We note that since a p.m.v. map $P : X \to M_+(Y)$ is always assumed continuous, $P_x(Y)$ is continuous in $x \in X$. Since X is compact, there is always an $M > 0$ such that $P_x(Y) \leq M$, and the measures $\{P_x\}_{x \in X}$ are uniformly finite measures.

Example 2.3.5 (Continuous map)**.** *Let $h : X \to Y$ be a continuous map, then h naturally induces a Feller map $P : X \to \Pr(Y)$ given by $P_x = \delta_{h(x)}$. So we see that Feller maps are a generalization of continuous functions from X to Y, where we switch Y with $\Pr(Y)$ in the range of h, and we think of Y up to homeomorphism as the closed subspace $\{\delta_y | y \in Y\}$ inside $\Pr(Y)$ with the w^*-topology.*

Example 2.3.6 (Stochastic matrix)**.** *Let X be a finite set. Then every Markov-Feller map $P : X \to M_+(X)$ is a stochastic matrix. Indeed, if $i \in X$, then $P_i(j) \geq 0$ and defining $P_{ij} = P_i(j)$ gives rise to an $|X| \times |X|$ non-negative matrix such that $\sum_{j \in X} P_{ij} = 1$.*

Example 2.3.7 (Continuously weighted systems)**.** *Let X be a compact space. For $\sigma_1, ..., \sigma_n : X \to X$ continuous maps and $f_1, ..., f_n : X \to \mathbb{R}_+$ continuous functions, the map $P(\sigma, f) : X \to M_+(X)$ given by*

$$P(\sigma, f)_x = \sum_{i=1}^{n} f_i(x) \cdot \delta_{\sigma_i(x)}$$

is a p.m.v. map on X. When $\sum_{i=1}^{n} f_i(x) = 1$ for all $x \in X$, we see that $P(\sigma, f)$ is a Feller map. We think of each point $x \in X$ as possessing a probability vector $(f_1(x), ..., f_n(x))$ that determines the probability of applying σ_i in the next iteration of the process.

Remark 2.3.8. The notion of Feller transition probability function on a compact metric space X, as defined in [122, Chapter 1], turns out to coincide with the notion of a Feller map $P : X \to \Pr(X)$.

P.m.v. maps can be composed. More precisely, if X, Y and Z are compact spaces and $P : X \to M_+(Y)$ and $Q : Y \to M_+(Z)$ are p.m.v. maps, composition is defined as

$$(QP)_x(A) = \int_Y Q_y(A) dP_x(y),$$

for all $x \in X$ and $A \in \mathcal{B}_Z$.

The following is a more general reformulation of [122, Theorem 1.1.5] and its preceding discussion (for compact X), given for Feller maps, that yields a generalization of the commutative Gelfand-Naimark duality. For a positive map $S : C(Y) \to C(X)$, denote by $S^* : M_+(X) \to M_+(Y)$ the dual map induced between the sets of positive finite Radon measures, when we identify linear functionals on $C(X)$ and $C(Y)$ with finite complex Radon measures on X and Y resp. via the Riesz-Markov representation theorem.

Theorem 2.3.9 (Commutative Gelfand-Naimark duality for positive maps)**.** *There is a contravariant equivalence between the category of compact spaces with p.m.v. maps and the category of commutative unital C^*-algebras with positive linear maps. More specifically, for compact spaces X and Y,*

1. *if* $P : X \to M_+(Y)$ *is a p.m.v. map then* $S_P : C(Y) \to C(X)$ *defined by*

$$S_P(f)(x) = \int_Y f(y) dP_x(y)$$

is a positive map, and;

2. *if* $S : C(Y) \to C(X)$ *is a positive map, we define* $P_S : X \to M_+(Y)$ *by setting*

$$(P_S)_x = S^*(\delta_x)$$

which yields a p.m.v. map.

Operations (1) and (2) are inverses of each other where unital maps are identified with Feller maps and unital $*$*-endomorphisms are associated with p.m.v. maps arising from composition by continuous functions as in Example 2.3.5.*

Due to the above theorem, we sometimes abuse notation and write $P : C(Y) \to C(X)$ to mean both the original Feller map P and the positive map $P(f)(x) := S_P(f)(x) = \int_Y f(y)dP_x(y)$ for all $x \in X$.

We next describe how to construct C*-correspondences from (completely) positive maps. One way is via the GNS (or KSGNS) construction associated to a completely positive map on a unital C*-algebra. This construction is done in detail in [84, Chapter 5]. We apply this construction only when the underlying C*-algebra is commutative.

Let X be a compact space, and let P be a (completely) positive map on $C(X)$. The GNS representation of P is a pair (F_P, ξ_P) consisting of a C*-correspondence F_P and a vector $\xi_P \in F_P$ such that $P(a) = \langle \xi_P, a\xi_P \rangle$.

F_P is defined as the C*-correspondence $C(X) \otimes_P C(X)$ which is the Hausdorff completion of the algebraic tensor product $C(X) \otimes C(X)$ with respect to the inner product and bimodule actions given respectively for $a, b, c, d \in C(X)$ by

$$\langle a \otimes b, c \otimes d \rangle = b^* P(a^* c) d \quad \text{and} \quad a \cdot (b \otimes c) \cdot d = ab \otimes cd.$$

The reconstructing vector of this correspondence is then given by $\xi_P = 1 \otimes 1$, and clearly satisfies $P(a) = \langle \xi_P, a\xi_P \rangle$.

For an up-to-date account on the GNS construction and its associated Toeplitz, Cuntz-Pimsner and other algebras, see [83, Section 3].

An alternative way to get a C*-correspondence from a p.m.v. map is as follows. We may define a topological quiver, as in Definition 2.3.1, associated to a p.m.v. map $P : X \to M_+(X)$. We define a quintuple $\mathcal{Q}_P = (X, \mathrm{Gr}(P), r, s, \lambda_x)$ where $\mathrm{Gr}(P)$ is the closure of $\{ \ (y, x) \mid y \in \operatorname{supp} P_x \ \}$ inside $X \times X$, the maps $r, s : \mathrm{Gr}(P) \to X$ are the restriction of the left and right coordinate maps on $X \times X$ to $\mathrm{Gr}(P)$ and $\lambda = \{\lambda_x\}$, such that $\operatorname{supp} \lambda_x \subseteq s^{-1}(x)$, is given by $\lambda_x(\{x\} \times U) = P_x(U)$. That this is a topological quiver according to Definition 2.3.1 is a consequence of [83, Lemma 3.30], and it is clearly multiplicity free. It turns out that the C*-correspondence associated to the topological quiver $\mathcal{Q}_P$ coincides with F_P as is shown in [83, Proposition 3.32].

Proposition 2.3.10. *Let $P : C(X) \to C(X)$ be a positive map, and $P : X \to M_+(X)$ its associated p.m.v. map. The map $a \otimes b \mapsto W(a \otimes b)$ given by $W(a \otimes b)(y, x) = a(y)b(x)$ is an isometric Id-bimodule map on the linear span of simple tensors inside F_P and extends to an Id-unitary from F_P to $\mathcal{X}_{\mathcal{Q}_P}$.*

It is important to mention at this point that by [83, Example 3.35], the topological quiver $\mathcal{Q}_P$ associated to a p.m.v. map P may fail to have either an open source map s or $\operatorname{supp} \lambda_x = s^{-1}(x)$. This prompted us to give Definition 2.3.1 without assuming that s is open and that $\operatorname{supp} \lambda_x = s^{-1}(x)$ as in [83, Definition 3.29].

2.3.3 Stochastic matrices

We next discuss some of the preliminaries on stochastic matrices, and the results in [42] for subproduct systems associated to stochastic matrices. For the basic theory of stochastic matrices and Markov chains on discrete spaces, we recommend [48, Chapter 6] and [112].

Definition 2.3.11. *Let Ω be a countable set. A stochastic matrix is a function $P : \Omega \times \Omega \to \mathbb{R}_+$ such that for all $i \in \Omega$ we have $\sum_{j \in \Omega} P_{ij} = 1$. Elements of Ω are called* states *of P.*

To every stochastic matrix, one can associate a set of edges $\mathrm{Gr}(P) := \{ \ (i, j) \mid P_{ij} > 0 \ \}$ and a $\{0, 1\}$ - adjacency matrix $\mathrm{Adj}(P)$ representing the directed graph of P as an incidence matrix by way of

$$\mathrm{Adj}(P)_{ij} = \begin{cases} 1 & : P_{ij} > 0 \\ 0 & : P_{ij} = 0 \end{cases}$$

Many dynamical properties of P can be put in terms of the directed graph $\mathcal{Q}_P := (\Omega, Gr(P), r, s)$ of P, where $s(i, j) = i$ and $r(i, j) = j$. We note immediately that in the context of stochastic matrices in this subsection, and in Chapter 4, we take *reversed* range and source convention to the one taken in Subsection 2.3.1 and the definition of the graph of a Markov-Feller operator as in Subsection 2.3.2.

Definition 2.3.12. *Let P be a stochastic matrix over Ω. We will say that a state i leads to a state j if there is a path γ in $\mathcal{Q}_P$ such that $s(\gamma) = i$ and $r(\gamma) = j$.*

Definition 2.3.13. *Let P be a stochastic matrix over Ω, and let $i \in \Omega$.*

1. *The* period *of i is $t(i) = gcd\{ \ n \mid P_{ii}^{(n)} > 0 \ \}$. If no finite such $t(i)$ exists, or if $t(i) = 1$ we say that i is aperiodic.*
2. *P is said to be irreducible if for any pair $i, j \in \Omega$, we have that i leads to j (and so j also leads to i).*

If P is an irreducible stochastic matrix over Ω, it turns out that every state $i \in \Omega$ is of the same periodicity t, so we define the periodicity of P to be t.

Let us recall the statement of the cyclic decomposition of irreducible stochastic matrices [112, Theorem 1.3] which justifies the notion of periodicity of an irreducible stochastic matrix P.

Theorem 2.3.14. *(Cyclic decomposition for periodic irreducible matrices)*
Let P be an irreducible stochastic matrix over a state set Ω with period t, and let $\omega \in \Omega$. For each $\ell = 0, \ldots t-1$, let $\Omega_\ell = \{j \in \Omega \mid P_{\omega j}^{(n)} > 0 \implies n \equiv \ell \mod t\}$. Then,

1. *The family $(\Omega_\ell)_{\ell=0}^{t-1}$ is a partition of Ω.*
2. *If $j \in \Omega_\ell$ then there exists $N(j)$ such that for all $n \geq N(j)$ we have $P_{\omega j}^{(nt+\ell)} > 0$.*
3. *Up to re-enumeration of Ω, there exist* rectangular *stochastic matrices $P_0, \ldots P_{t-1}$ such that P has the following cyclic block decomposition:*

$$\begin{bmatrix} 0 & P_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{t-2} \\ P_{t-1} & \cdots & 0 & 0 \end{bmatrix}$$

   where the rows (columns) of P_ℓ in this matrix decomposition are indexed by Ω_ℓ ($\Omega_{\ell+1}$ respectively) for all $\ell \in \mathbb{Z}_t$, where $\mathbb{Z}_t$ is the cyclic group of order t.

In this subsection and in Chapter 4 we shall restrict our attention to *finite irreducible* stochastic matrices, or equivalently, those matrices whose associated graph is *transitive* and finite. For this class of stochastic matrices, we have that the more generally stated [42, Theorem 2.10], combined with[112, Theorem 4.1] and [48, Theorem 6.7.2], yields the following clean formulation.

Theorem 2.3.15. *(Convergence theorem for finite irreducible matrices)*
Let P be a finite irreducible *stochastic matrix over Ω with period $t \geq 1$, and $\Omega_0, ..., \Omega_{t-1}$ a cyclic decomposition for it as in item (3) of Theorem 2.3.14. Then there exists a unique probability vector $\nu = (\nu_i)_{i\in\Omega}$ so that when we are given $i \in \Omega_{l_1}$ and $j \in \Omega_{l_2}$, for $0 \leq \ell < t$ such that $\ell \equiv (l_2 - l_1) \mod t$, we have that*

$$\lim_{m\to\infty} P_{ij}^{(mt+\ell)} = \nu_j t.$$

Let Ω be a finite set and $\ell^\infty(\Omega) = C(\Omega) = \mathbb{C}^\Omega$ the C*-algebra of finite sequences indexed by Ω. We denote by $\{p_j\}_{j\in\Omega}$ the collection of pairwise perpendicular projections on $C(\Omega)$ given by $p_j(i) = \delta_{ij}$.

Notation 2.3.16. We denote by $*$ the Schur (entrywise) multiplication of matrices $A = [a_{ij}]$ and $B = [b_{kl}]$ given by $A * B = [a_{ij}b_{ij}]$, and let Diag be the map on matrices given by $\text{Diag}([a_{ij}]) = (a_{ii})_{i\in\Omega} \in C(\Omega)$.

Next, for a non-negative matrix $P = [P_{ij}]$ indexed by Ω, we denote by $\sqrt{P}$ and $P^\flat$ the matrices with (i, j)-th entry given by

$$(\sqrt{P})_{ij} := \sqrt{P_{ij}}, \qquad \text{and} \qquad (P^\flat)_{ij} := \begin{cases} (P_{ij})^{-1}, & \text{if } P_{ij} > 0 \\ 0, & \text{else} \end{cases}$$

In [42, Theorem 3.4] the Arveson-Stinespring subproduct system associated to a stochastic matrix P on countable Ω was computed. When Ω is finite, we arrive at the following simpler version of the theorem.

Theorem 2.3.17. *Let P be a stochastic matrix over* finite Ω. *The following is a subproduct system* $\text{Arv}(P)$ *over $C(\Omega) = \ell^\infty(\Omega) = \mathbb{C}^{|\Omega|}$ and is the one given in [42, Theorem 3.4].*

1. *The n-th fiber is a C*-correspondence over $C(\Omega)$ given by*

$$\text{Arv}(P)_n := \{\ [a_{ij}] \mid a_{ij} = 0 \ \ \textit{if} \ \ (i, j) \notin \text{Gr}(P^n)\ \}$$

with left and right actions of $C(\Omega)$ on $\text{Arv}(P)_n$ *as a bimodule are given by diagonal left and right matrix multiplication and the $C(\Omega)$-valued inner product is given by*

$$\langle A, B\rangle = \text{Diag}\left[A^* B\right]$$

for $A, B \in$ $\text{Arv}(P)_n$.

2. The subproduct maps are given by

$$U_{n,m}(A \otimes B) = (\sqrt{P^{n+m}})^\flat * \left[(\sqrt{P^n} * A) \cdot (\sqrt{P^m} * B)\right]$$

for $n, m \neq 0$ *and* $A \in \mathrm{Arv}(P)_n$ *and* $B \in \mathrm{Arv}(P)_m$.

Remark 2.3.18. Since the subproduct systems we shall consider in this work will be with finite dimensional fibers and over finite dimensional C*-algebras, they will automatically be W^*-correspondences. Hence, the theories of subproduct systems over C*-algebras and their operator algebras discussed here and of subproduct systems over W^*-algebras and their operator algebras discussed in [42] will coincide.

Remark 2.3.19. Let P be a stochastic matrix over *finite* Ω. We next characterize the $\mathrm{Arv}(P)$-invariant ideals in $C(\Omega)$. In this case, the $\mathrm{Arv}(P)$-invariant ideals of $C(\Omega)$ correspond to $\mathrm{Arv}(P)$-reducing projections $p \in C(\Omega)$, according to [42, Definition 6.19]. By [42, Proposition 7.4], there is a 1-1 bijection between reducing projections p and subsets $C_p \subseteq \Omega$ such that whenever $\gamma = e_0 ... e_\ell$ is a path in $\mathcal{Q}_P$ with $e_k \in Gr(P)$ such that $s(\gamma), r(\gamma) \in C_p$, then for every $1 \leq k \leq \ell$ we have $s(e_k) \in C_p$.

Using this, it is easy to show that a finite stochastic matrix P over Ω is irreducible if and only if $(\mathrm{Arv}(P), U)$ is minimal according to Definition 2.2.12.

2.3.4 Extension theory

We recall some facts from the theory of primitive ideal spectra and extension theory for C*-algebras. More details on primitive ideal spectra of C*-algebras can be found in [40, Chapter 3] and [7, Section 1.5]. For an account on the Busby invariant and extension theory for C*-algebras see [8], [17, Section 15], [23, Section 1], [50, Section 2] and [98]. Finally, for K-theory of C*-algebras, we recommend [111].

Let $\mathcal{A}$ be a C*-algebra. We denote by $\hat{\mathcal{A}}$ the collection of unitary equivalence classes of irreducible representations of $\mathcal{A}$. On the other hand, we define $\mathrm{Prim}(\mathcal{A})$ to be the set of primitive ideals of $\mathcal{A}$, where a primitive ideal is the kernel of an irreducible representation of $\mathcal{A}$.

The set $\mathrm{Prim}(\mathcal{A})$ comes equipped with a lattice structure determined by set inclusion. Next, since any two unitarily equivalent *-representations have the same kernel, the map $\pi \mapsto \mathrm{Ker}\,\pi$ factors through to yield a surjective map $\kappa : \hat{\mathcal{A}} \to \mathrm{Prim}(\mathcal{A})$.

It turns out that a C*-algebra is type I if and only if the above map κ is a injective [57]. This means that up to unitary equivalence, an irreducible representation π is completely determined by its kernel $\mathrm{Ker}\,\pi$.

When we have a *-isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ between two C*-algebras, we denote by $\varphi_* : \mathrm{Prim}(\mathcal{B}) \to \mathrm{Prim}(\mathcal{A})$ the induced lattice isomorphism between the spectra.

Suppose we have the following exact sequence of C*-algebras

$$0 \to \mathcal{K} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} \mathcal{B} \to 0. \tag{2.2}$$

Then there is a $*$-homomorphism $\theta : \mathcal{A} \to M(\mathcal{K})$ into the multiplier algebra of $\mathcal{K}$, uniquely determined by $\theta(a)c = \iota^{-1}(a\iota(c))$ for $c \in \mathcal{K}$ and $a \in \mathcal{A}$. Denote by $q : M(\mathcal{K}) \to M(\mathcal{K})/\mathcal{K} =: \mathcal{Q}(\mathcal{K})$ the Calkin map. Hence, a $*$-homomorphism $\eta : \mathcal{B} \to \mathcal{Q}(\mathcal{K})$ will be induced from θ, and we call η the *Busby invariant* of the exact sequence in (2.2). We say that the exact sequence (2.2) is *essential* if $\mathcal{K}$ is an essential ideal in $\mathcal{A}$, that is, if the intersection of $\mathcal{K}$ with any non-trivial ideal in $\mathcal{A}$ is non-trivial.

The above association turns out to be a bijection between exact sequences of C*-algebras given as in equation (2.2) and $*$-homomorphisms $\eta : \mathcal{B} \to \mathcal{Q}(\mathcal{K})$. Indeed, the inverse map sends a $*$-homomorphism $\eta : \mathcal{B} \to \mathcal{Q}(\mathcal{K})$ to the exact sequence where the pre-image $\mathcal{A} := q^{-1}(\eta(\mathcal{B}))$ under the Calkin quotient q yield an exact sequence as in (2.2), and π replaced by the restriction of q to $\mathcal{A}$. Under this bijection, an exact sequence as in (2.2) is essential if and only if its associated Busby invariant is an *injective* $*$-homomorphism.

Definition 2.3.20. *Suppose $\mathcal{K}_i, \mathcal{A}_i, \mathcal{B}_i$ are C*-algebras for $i = 1, 2$, and that*

$$0 \to \mathcal{K}_1 \xrightarrow{\iota_1} \mathcal{A}_1 \xrightarrow{\pi_1} \mathcal{B}_1 \to 0 \quad \text{and} \quad 0 \to \mathcal{K}_2 \xrightarrow{\iota_2} \mathcal{A}_2 \xrightarrow{\pi_2} \mathcal{B}_2 \to 0 \tag{2.3}$$

are two short exact sequences. We say that these two short exact sequences are isomorphic *if there exists a $*$-isomorphism $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\alpha(\iota_1(\mathcal{K}_1)) = \iota_2(\mathcal{K}_2)$.*

Suppose η_1 and η_2 are Busby maps for exact sequences as in equation (2.3). [50, Theorem 2.2] then yields that these two short exact sequences are isomorphic if and only if there exist *-isomorphisms $\kappa : \mathcal{K}_1 \to \mathcal{K}_2$ and $\beta : \mathcal{B}_1 \to \mathcal{B}_2$ such that

$$\widetilde{\kappa}\eta_1 = \eta_2\beta$$

where $\widetilde{\kappa} : \mathcal{Q}(\mathcal{K}_1) \to \mathcal{Q}(\mathcal{K}_2)$ is the induced $*$-isomorphism between the Calkin algebras.

In the context of extensions by a single copy of compact operators on separable infinite dimensional Hilbert space, that is when $\mathcal{K} = \mathcal{K}(\mathcal{H})$, the Calkin quotient map $q : M(\mathcal{K}(\mathcal{H})) \to \mathcal{Q}(\mathcal{K}(\mathcal{H}))$ discussed above is just the regular quotient map into the Calkin algebra, since $M(\mathcal{K}(\mathcal{H})) = B(\mathcal{H})$, so that $M(\mathcal{K}(\mathcal{H}))/\mathcal{K}(\mathcal{H}) = \mathcal{Q}(\mathcal{H})$.

Let $\mathcal{B}$ be a C*-algebra. We write $E(\mathcal{B})$ for the collection of all injective $*$-homomorphisms of $\mathcal{B}$ into $\mathcal{Q}(\mathcal{H})$. We call elements in $E(\mathcal{B})$ extensions, as they are in bijection, under (the inverse of) the Busby map, with essential exact sequences of C*-algebras of the form

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{A} \to \mathcal{B} \to 0.$$

We then say that two extensions $\eta_1, \eta_2 \in E(\mathcal{B})$ are

1. Strongly (unitarily) equivalent if there is a unitary $U \in B(\mathcal{H})$ such that $\eta_1(b) = q(U)\eta_2(b)q(U^*)$ for all $b \in \mathcal{B}$.
2. Weakly (unitarily) equivalent if there is a unitary element $u \in \mathcal{Q}(\mathcal{H})$ such that $\eta_1(b) = u\eta_2(b)u^*$ for all $b \in \mathcal{B}$.

When $\mathcal{B}$ is unital we write $\mathrm{Ext}_s(\mathcal{B})$ and $\mathrm{Ext}_w(\mathcal{B})$ for the strong and weak equivalence classes of *unital* extensions in $E(\mathcal{B})$, respectively. When $\mathcal{B}$ is non-unital, we write $\mathrm{Ext}_s(\mathcal{B})$ and $\mathrm{Ext}_w(\mathcal{B})$ for the strong and weak equivalence classes of *all* extensions in $E(\mathcal{B})$ respectively. In the latter case however, $\mathrm{Ext}_s(\mathcal{B}) = \mathrm{Ext}_w(\mathcal{B})$ by [17, Proposition 15.6.4]. We denote by $[\eta]_s$ and $[\eta]_w$ the equivalence classes of an extension η in $\mathrm{Ext}_s(\mathcal{B})$ and $\mathrm{Ext}_w(\mathcal{B})$, respectively.

Given $\eta_1, \eta_2 \in E(\mathcal{B})$, we may define $\eta_1 \oplus \eta_2 \in E(\mathcal{B})$ (via some fixed identification $\mathcal{Q}(\mathcal{H}) \oplus \mathcal{Q}(\mathcal{H}) \subseteq \mathcal{Q}(\mathcal{H} \otimes \mathbb{C}^2) \cong \mathcal{Q}(\mathcal{H})$) by specifying $(\eta_1 \oplus \eta_2)(b) = \eta_1(b) \oplus \eta_2(b)$. This operation induces a well-defined addition $+$ on $\mathrm{Ext}_s(\mathcal{B})$ and $\mathrm{Ext}_w(\mathcal{B})$ given for two extensions η_1 and η_2 by $[\eta_1]_s + [\eta_2]_s := [\eta_1 \oplus \eta_2]_s$ and $[\eta_1]_w + [\eta_2]_w := [\eta_1 \oplus \eta_2]_w$, and makes them into abelian semigroups.

An extension τ is called *trivial* if it lifts to a *-homomorphism $\hat{\tau} : \mathcal{B} \to B(\mathcal{H})$. That is, if $q \circ \hat{\tau} = \tau$. Such a trivial extension τ is called *strongly unital* if the map $\hat{\tau}$ can be chosen to be unital (in particular this is relevant only when $\mathcal{B}$ itself is unital and τ is unital). Trivial extensions correspond to *split essential* exact sequences via (the inverse of) the Busby map. It is straightforward to construct injective *-homomorphisms of a C*-algebra $\mathcal{B}$ into $B(\mathcal{H})$ which do not intersect $\mathcal{K}(\mathcal{H})$, hence trivial extensions always exist. Moreover, the same argument yields strongly unital trivial extensions.

Voiculescu [120] showed that when $\mathcal{B}$ is separable, the semigroup $\mathrm{Ext}_s(\mathcal{B})$ has a zero element. When $\mathcal{B}$ is non-unital, the zero element consists precisely of the trivial extensions. When $\mathcal{B}$ is unital, it consists of the strongly unital trivial extensions. For more details, see [17, Section 15.12], and especially [17, Theorem 15.12.3].

Although $\mathrm{Ext}_s(\mathcal{B})$ and $\mathrm{Ext}_w(\mathcal{B})$ are not always groups, it follows from a theorem of Choi and Effros that when $\mathcal{B}$ is separable and nuclear, both semigroups are actually groups (see [17, Corollary 15.8.4]).

Suppose now that $\mathcal{B}$ is unital. There is an action ϵ of $\mathbb{Z}$ on $\mathrm{Ext}_s(\mathcal{B})$ given by $\epsilon(m)[\eta]_s = [\mathrm{Ad}_u \circ \eta]_s$ where $u \in \mathcal{Q}(\mathcal{H})$ is a unitary of Fredholm index $-m$, and $\mathrm{Ad}_u(a) = u^*au$ for $a \in \mathcal{Q}(\mathcal{H})$. By definition of addition, we have that $\epsilon(n+m)([\eta_1]_s+[\eta_2]_s) = [\mathrm{Ad}_{u\oplus v}(\eta_1 \oplus \eta_2)] = \epsilon(n)[\eta_1]_s+\epsilon(m)[\eta_2]_s$ where u and v are unitaries in $\mathcal{Q}(\mathcal{H})$ of indices $-n$ and $-m$ respectively. In particular, if τ is a strongly unital trivial extension then $\epsilon(m)[\eta]_s = \epsilon(0+m)([\eta]_s+[\tau]_s) = [\eta]_s + \epsilon(m)[\tau]_s$. Hence, when we denote by $\lambda_{\mathcal{B}} : \mathrm{Ext}_s(\mathcal{B}) \to \mathrm{Ext}_w(\mathcal{B})$ the canonical quotient map, we have that $\mathrm{Ker}\,\lambda_{\mathcal{B}} = \{\epsilon(m)[\tau]_s \mid m \in \mathbb{Z}\}$.

Let $\gamma_{\mathcal{B}} : \mathrm{Ext}_w(\mathcal{B}) \to \mathrm{Hom}(K_1(\mathcal{B}), \mathbb{Z})$ denote the so-called *index invariant* of $\mathcal{B}$, given by $\gamma_{\mathcal{B}}([\eta]_w) = \mathrm{ind} \circ \eta_*$, where $\eta_* : K_1(\mathcal{B}) \to K_1(\mathcal{Q}(\mathcal{H}))$ is the map induced between the K_1 groups and $\mathrm{ind} : K_1(\mathcal{Q}(\mathcal{H})) \to \mathbb{Z}$ is the Fredholm index. Hence, for a unital C*- algebra $\mathcal{B}$, we always have the following sequence of maps

$$\mathrm{Ext}_s(\mathcal{B}) \xrightarrow{\lambda_{\mathcal{B}}} \mathrm{Ext}_w(\mathcal{B}) \xrightarrow{\gamma_{\mathcal{B}}} \mathrm{Hom}(K_1(\mathcal{B}), \mathbb{Z}). \tag{2.4}$$

We next give the details of two particular examples, which will turn out to be useful to us in Chapter 4.

Example 2.3.21. *Take $\mathcal{B} = C(\mathbb{T})$. In this case $\mathcal{B}$ is nuclear and separable, so both the weak and strong extension semigroups are groups. We note that $\mathrm{Hom}(K_1(\mathcal{B}), \mathbb{Z}) \cong \mathbb{Z}$ as $K_1(\mathcal{B}) \cong \mathbb{Z}$, and every homomorphism is determined on the generator 1. We next show that in this case, the map $\gamma_{\mathcal{B}} \circ \lambda_{\mathcal{B}}$ is surjective. Indeed, for every $m \in \mathbb{Z}$ there is a unitary $u \in \mathcal{Q}(\mathcal{H})$ with $\sigma(u) = \mathbb{T}$, and Fredholm index m, so we may define a *-homomorphism $\eta_m : C(\mathbb{T}) \to \mathcal{Q}(\mathcal{H})$ given by $\eta_m(z \mapsto z) = u$ which implements a *-isomorphism $C(\mathbb{T}) \cong C^*(u)$. Thus we obtain an extension with index invariant $k \mapsto k \cdot m \in \mathrm{Hom}(K_1(\mathcal{B}), \mathbb{Z})$.*

Next, we show that $\gamma_{\mathcal{B}} \circ \lambda_{\mathcal{B}}$ is injective. Indeed, if $\gamma_{\mathcal{B}} \circ \lambda_{\mathcal{B}}[\eta]_s = 0$, then $\mathrm{ind}(\eta(z \mapsto z)) = 0$ and hence there is a unitary $U \in B(\mathcal{H})$ with $\sigma(U) = \mathbb{T}$ s.t $q(U) = \eta(z \mapsto z)$. Thus, η lifts to a unital $*$-homomorphism $\hat{\eta} : C(\mathbb{T}) \to B(\mathcal{H})$, so that η is a strongly unital trivial extension, and the map $\gamma_{\mathcal{B}} \circ \lambda_{\mathcal{B}}$ is injective.

We conclude that $\mathrm{Ext}_s(C(\mathbb{T})) \cong \mathrm{Ext}_w(C(\mathbb{T})) \cong \mathbb{Z}$, and that $\epsilon(n)$ acts trivially on $\mathrm{Ext}_s(C(\mathbb{T}))$ for each n.

Example 2.3.22. *Take $\mathcal{B} = M_d(\mathbb{C})$. Again in this case $\mathcal{B}$ is nuclear and separable so that both weak and strong extension semigroups are groups. We already know that $K_1(M_d(\mathbb{C})) \cong \{0\}$, so that the right most group in equation (2.4) vanishes. Let $\eta : M_d(\mathbb{C}) \to \mathcal{Q}(\mathcal{H})$*

be a unital extension. We reiterate the construction in [17, Example 15.4.1 (b)] lifting $\eta : M_d(\mathbb{C}) \to \mathcal{Q}(\mathcal{H})$ *to a *-homomorphism* $\hat{\eta} : M_d(\mathbb{C}) \to B(\mathcal{H})$*, and measuring how far* $\hat{\eta}$ *is from being unital. That is, how far is* η *from being a strongly unital trivial extension.*

Let $\{\overline{e_{ij}}\}$ *be a system of matrix units for* $\eta(M_d(\mathbb{C}))$*. By standard essential spectrum arguments, one can find projections* $p_{ii} \in B(\mathcal{H})$ *that lift each* $\overline{e_{ii}}$*. Next, by appealing to [98, Lemma 1.1], for all* $2 \leq i \leq d$ *we may find partial isometries* e_{1i} *lifting* $\overline{e_{1i}}$ *such that* $e_{1i}^* e_{1i} \leq p_{ii}$ *and* $e_{1i}e_{1i}^* \leq p_{11}$*. We set* $e_{ij} = e_{1i}^* e_{1j}$ *so that* $\{e_{ij}\}$ *is a lifted set of matrix units in* $pB(\mathcal{H})p$*, where* $p = \sum e_{ii}$*. We note that* p *is a projection of finite dimensional cokernel, say of dimension* ℓ*, so that by adding a homomorphism from* $M_d(\mathbb{C})$ *to* $(1-p)B(\mathcal{H})(1-p)$ *if necessary, we may arrange for* $0 \leq \ell < d$*.*

The defect of η *is then defined to be* $\ell \in \mathbb{Z}_d$*, and up to strong equivalence it is independent of the choice made in the process above. It is then easy to show that two unital extensions* $\eta_1, \eta_2 \in E(M_d(\mathbb{C}))$ *are strongly equivalent if and only if they have the same defect, and that they are always weakly equivalent. Hence, we conclude that* $\mathrm{Ext}_s(M_d(\mathbb{C})) \cong \mathbb{Z}_d$ *and* $\mathrm{Ext}_w(M_d(\mathbb{C})) \cong \{0\}$*.*

2.4 Matrix positivity and convexity

2.4.1 Matrix convex sets

We will often denote $n \times n$ matrix algebras over complex numbers by M_n, and by $B(\mathcal{H})$ / $\mathcal{K}(\mathcal{H})$ all bounded / compact operators on Hilbert space. For a C*-algebra $\mathcal{A}$ we denote by $\mathcal{A}^d$ the d-tuples of operators from $\mathcal{A}$ and $\mathcal{A}^d_{sa}$ the d-tuples of *self-adjoint* operators from $\mathcal{A}$. Whenever $X = (X_1, ..., X_d) \in B(\mathcal{H})^d$ and $V \in B(\mathcal{H})$ is some operator, we denote $VXV^* := (VX_1V^*, ..., VX_dV^*)$, and for another d-tuple $Y = (Y_1, ..., Y_d) \in B(\mathcal{K})^d$ we will denote $X \oplus Y = (X_1 \oplus Y_1, ..., X_d \oplus Y_d) \in (B(\mathcal{H}) \oplus B(\mathcal{K}))^d \subseteq B(\mathcal{H} \oplus \mathcal{K})^d$

Definition 2.4.1. *For* $d \in \mathbb{N}$*, a collection of subsets* $\mathcal{S} = (\mathcal{S}_n)$ *where* $\mathcal{S}_n \subseteq M_n^d$ *is called a* free set *in* d *dimensions. When each* $\mathcal{S}_n \subseteq (M_n)^d_{sa}$*, we will say that* $\mathcal{S}$ *is* self-adjoint*. We will say that a free set* $\mathcal{S} = (\mathcal{S}_n)$ *is an* nc set *if when* $A \in \mathcal{S}_n$ *and* $B \in \mathcal{S}_m$ *then* $A \oplus B \in \mathcal{S}_{n+m}$*. We will say that a free nc set in* d *dimensions* $\mathcal{S} = (\mathcal{S}_n)$ *is* matrix convex *in* d *dimensions if for any isometry* $V : \mathbb{C}^n \to \mathbb{C}^k$ *and* $X \in \mathcal{S}_k$ *we have that* $VXV^* \in \mathcal{S}_n$*.*

We note that a matrix convex set $\mathcal{S} = (\mathcal{S}_n)$ is always *unitarily invariant* in the sense that for any $X \in \mathcal{S}_n$ and unitary matrix $U \in M_n(\mathbb{C})$ we have $UXU^* \in \mathcal{S}_n$. Furthermore, from the Choi-Krauss theorem [99, Proposition 4.7] a free nc set $\mathcal{S} = (\mathcal{S}_n)$ is matrix convex set if and only if for any UCP map $\varphi \in UPC(M_n, M_k)$ and a d-tuple $X \in \mathcal{S}_k$ we have $\varphi(X) \in \mathcal{S}_n$. The above definition of matrix convex set coincides with the one given by Effros and Winkler in [49, Section 3], for a matrix convex set in $\mathbb{C}^d$ (or a matrix convex set in $\mathbb{R}^d$ when $\mathcal{S}$ is self adjoint). We will say that a free set $S = (\mathcal{S}_n)$ is *open / closed / bounded* if $\mathcal{S}_n$ is open / closed / bounded respectively for each $n \in \mathbb{N}$.

Proposition 2.4.2. *Let $\mathcal{S}$ be a matrix convex set. The following are equivalent:*

1. *$\mathcal{S}$ is uniformly bounded.*
2. *$\mathcal{S}$ is bounded.*
3. *$\mathcal{S}_1$ is bounded.*

Proof. If $\mathcal{S}$ is uniformly bounded, there exists $r > 0$ such that for every $n \in \mathbb{N}$ and $X \in \mathcal{S}_n$ we have $\|X_i\| \leq r$, it is clear that each $\mathcal{S}_n$ is bounded, and in particular $\mathcal{S}_1$ is bounded. Conversely, suppose towards contradiction that $\mathcal{S}$ is unbounded while $\mathcal{S}_1$ is. Then there is $X^{(m)} \in \mathcal{S}_{n_m}$ such that $\|X_i^{(m)}\|$ is arbitrarily large for some $1 \leq i \leq d$. Since the numerical radius is an equivalent norm to the operator norm, we see that there are norm-one column vectors v_m such that $|v_m^* X_i^{(m)} v_m|$ is also arbitrarily large. Since $v^* X^{(m)} v \in \mathcal{S}_1$, we see that $\mathcal{S}_1$ is also unbounded. □

Given a d-tuple of operators $A = (A_1, ..., A_d) \in B(\mathcal{H})^d$ we may define a free set $\mathcal{D}_A = (\mathcal{D}_A(n))$ by specifying

$$\mathcal{D}_A(n) = \{ \ X = (X_1, ..., X_d) \in M_n^d \ | \ \operatorname{Re}\big(\sum_{i=1}^{d} A_j \otimes X_j\big) \leq I \ \}$$

or alternatively, when $A \in B(\mathcal{H})^d_{sa}$, we define the *self-adjoint* free set

$$\mathcal{D}_A^{sa}(n) = \{ \ X = (X_1, ..., X_d) \in (M_n)^d_{sa} \ | \ \sum_{i=1}^{d} A_j \otimes X_j \leq I \ \}$$

where $\otimes$ is the usual Kronecker tensor product, which coincides with the standard tensor product inside the C*-algebra $B(\mathcal{H}) \otimes M_n$.

The free sets $\mathcal{D}_A$ and $\mathcal{D}_A^{sa}$ are called *free operator spectrahedra*. When the d-tuple A is of matrices from M_n, we will call $\mathcal{D}_A$ and $\mathcal{D}_A^{sa}$ free *matrix* spectrahedron. In the literature, the notion of free spectrahedron, or free LMI domain is reserved for free *matrix* *spectrahedra*, where A is comprised of matrices. We will allow for operator coefficients for our free spectrahedra.

In the literature, free spectrahedra are often given as the positivity domains of a non-commutative *linear matrix / operator inequality.* A (monic) linear pencil defined by $A = (A_1, ..., A_d) \in B(\mathcal{H})^d$ is a function on $\mathbb{C}^d$ of the form

$$L_A(x) = I - \sum_{i=1}^{d} A_j \cdot x_j$$

which allows for evaluation on non-commuting d-tuples $X = (X_1, ..., X_d) \in M_n^d$ by way of

$$L_A(X) = I \otimes I_n - \sum_{i=1}^{d} A_j \otimes X_j$$

so we see that $\mathcal{D}_A$ and $\mathcal{D}_A^{sa}$ are also given alternatively by

$$\mathcal{D}_A(n) = \{\ X = (X_1, ..., X_d) \in M_n^d \mid \ \operatorname{Re} L_A(X) \geq 0\ \}.$$

and

$$\mathcal{D}_A^{sa}(n) = \{\ X = (X_1, ..., X_d) \in (M_n)_{sa}^d \mid L_A(X) \geq 0\ \}.$$

Clearly $\mathcal{D}_A$ and $\mathcal{D}_A^{sa}$ are closed free nc sets, but they are also matrix convex. For instance if $X \in \mathcal{D}_A(n)$ and $\varphi \in UCP(M_n, M_k)$ we have that $I \otimes \varphi$ is UCP. So we get that

$$\operatorname{Re} L_A(\varphi(X)) = \operatorname{Re}(I \otimes \varphi)(L_A(X)) = (I \otimes \varphi) \operatorname{Re} L(X) \geq 0$$

since UCP maps respect real and imaginary parts. One then similarly shows that $\mathcal{D}_A^{sa}$ is matrix convex.

Example 2.4.3 (Matrix cube and polydisc)**.** *The* matrix cube $\mathfrak{C}^{(d)}$ *is the self-adjoint spectrahedron* $\mathcal{D}_C^{sa}$ *determined by the* $2d \times 2d$ *matrices* $C_j = \left(\begin{smallmatrix} E_{jj} & 0 \\ 0 & -E_{jj} \end{smallmatrix}\right)$*, where* E_{jj} *is the diagonal* $d \times d$ *matrix with* 1 *on the* j*-th diagonal entry, and* 0 *everywhere else. Then* $X \in \mathcal{D}_C^{sa}$ *if and only if*

$$0 \leq I - \sum_{j=1}^{d} C_j \otimes X_j = \sum_{j=1}^{d} \begin{pmatrix} E_{jj} \otimes I & 0 \\ 0 & E_{jj} \otimes I \end{pmatrix} - \begin{pmatrix} E_{jj} \otimes X_j & 0 \\ 0 & -E_{jj} \otimes X_j \end{pmatrix} = \begin{pmatrix} E_{jj} \otimes (I - X_j) & 0 \\ 0 & E_{jj} \otimes (I + X_j) \end{pmatrix},$$

which is equivalent to $\|X_j\| \leq 1$ *for* $1 \leq j \leq d$.

The matrix polydisc $\mathfrak{D}^{(d)}$ *is the the spectrahedron* $\mathcal{D}_D$ *determined by* $2d \times 2d$ *matrices* $D_j = E_{jj} \otimes \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$ *for* $1 \leq j \leq d$. *So that* $X \in \mathcal{D}_D$ *if and only if*

$$0 \leq \mathrm{Re}\Big(I - \sum_{j=1}^{d} E_{jj} \otimes \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) \otimes X_j\Big) = \sum_{j=1}^{d} E_{jj} \otimes \Big(\begin{smallmatrix} I & -X_j \\ -X_j^* & I \end{smallmatrix}\Big).$$

Which is again equivalent to $\|X_j\| \leq 1$ *by [99, Lemma 3.1].*

Example 2.4.4 (Real and complex matrix balls)**.** *The* real matrix ball *and the* complex matrix ball *are the free sets defined by*

$$\mathfrak{B}_{\mathbb{R}}^{(d)} = \{X \in \cup_n (M_n)_{sa}^d : \sum_j X_j^2 \leq I\},$$

and

$$\mathfrak{B}_{\mathbb{C}}^{(d)} = \{X \in \cup_n M_n^d : \sum_j X_j X_j^* \leq I\}.$$

These sets are the free spectrahedra $\mathcal{D}_{\mathrm{Re}(B)}^{sa}$ *and* $\mathcal{D}_B$ *respectively, determined by the matrices* $B_j = E_{1(j+1)}$ *for* $1 \leq j \leq d$, *where* E_{ik} *are canonical* $(d+1) \times (d+1)$ *matrix units for* $1 \leq i, k \leq d+1$. *The details are similarly verified using [99, Lemma 3.1], as in the previous example.*

The matrix range of a single operator, which was introduced by Arveson in [6, Section 2.4] easily generalizes to several variables. Let $A = (A_1, ..., A_d) \in B(\mathcal{H})^d$ be a d-tuple of operators. We denote by $\mathcal{V}_A := \mathrm{Sp}\{I, A_1, A_1^*, ..., A_d, A_d^*\}$ the finite dimensional operator system spanned by A. We define the *matrix range* of A as the free set $\mathcal{W}(A) = (\mathcal{W}_n(A))$ given by

$$\mathcal{W}_n(A) = \{\ (\varphi(A_1), ..., \varphi(A_d)) \mid \varphi \in UCP(\mathcal{V}_A, M_n)\ \}.$$

Since $\mathcal{W}_n(A) \subseteq (M_n)_{sa}^d$ if and only if $A \in B(H)_{sa}^d$, we will not require separate notation for matrix ranges in the self-adjoint setting. We will also denote $C^*(I, A) := C^*(I, A_1, ..., A_d)$ the *unital* C*-algebra generated by A.

Clearly $\mathcal{W}(A)$ is a free nc set that is closed under the application of UCP maps from M_n to M_k and is hence a matrix convex set. Furthermore, by [99, Theorem 7.4] we know that $UCP(\mathcal{V}_A, M_n)$ is compact in the BW topology. Hence, we see that each $\mathcal{W}_n(A)$ is compact in the weak* topology on each coordinate, which coincides with the norm topology on each coordinate as M_n is finite dimensional. Hence, we obtain that,

Proposition 2.4.5. *For any $A \in B(\mathcal{H})^d$, the free set $\mathcal{W}(A)$ is a closed and bounded matrix convex set.*

When $N = (N_1, ..., N_d)$ is a d-tuple of *normal commuting* operators, we will simply call it a *normal* d-tuple. In this case $C^*(I, N)$ is a unital commutative C*-algebra which by the Gelfand-Naimark theorem is *-isomorphic, via the Gelfand transform τ to $C(X)$ for some compact Hausdorff space X. If we denote $f_i := \tau(N_i) \in C(X)$, these functions define a homeomorphism $x \mapsto (f_1(x), ..., f_d(x))$ that identifies X homeomorphically as a subset $\sigma(N)$ of $\mathbb{C}^d$. This compact space $\sigma(N)$ is called the joint spectrum of the normal d-tuple N. It is readily seen by the above definitions that $\sigma(N) \subseteq \mathcal{W}_1(N)$. We will say that a representation $\rho : C(X) \to B(\mathcal{H})$ is *diagonal* if there is an orthonormal basis $\{\xi_i\}$ for $\mathcal{H}$ such that for each function $f \in C(X)$, each ξ_i is an eigenvector for $\rho(f)$.

Theorem 2.4.6. *Let $N = (N_1, ..., N_d)$ be a normal d-tuple. Then for every $n \in \mathbb{N}$ we have*

$$\mathcal{W}_n(N) = \left\{ \sum_{i=1}^{m} \lambda^{(i)} K_i \mid m \in \mathbb{N},\ K_i \in (M_n)_+,\ \sum K_i = I_n \right\} \tag{2.5}$$

where m in the above equality can be taken to be at most $2n^4d + n^2$. In particular, $\mathcal{W}_1(N) = \operatorname{conv}(\sigma(N))$*, and $\mathcal{W}(N)$ is the smallest matrix convex set containing $\sigma(N)$.*

Proof. For $\lambda^{(i)} = (\lambda_1^{(i)}, ..., \lambda_d^{(i)}) \in \sigma(N)$ and $K_i \in (M_n)_+$ that satisfy $\sum_{i=1}^m (K_i^{\frac{1}{2}})^* K_i^{\frac{1}{2}} = \sum_{i=1}^m K_i = I_n$, we have that

$$\sum_{i=1}^{m} \lambda^{(i)} K_i = \sum_{i=1}^{m} (K_i^{\frac{1}{2}})^* \lambda^{(i)} K_i^{\frac{1}{2}} \in \mathcal{W}_n(N).$$

So certainly the right-hand side is a subset of the left-hand side in equation (2.5).

We now show the reverse inclusion. Let $\varphi \in UCP(\mathcal{V}_N, M_n)$. By Arveson's extension theorem followed by Stinespring's theorem, we know that φ extends to a UCP map on $C^*(I, N) \cong C(\sigma(N))$ that is the compression of a $*$-representation $\pi : C(\sigma(N)) \to B(\mathcal{H})$ via an isometry $V : \mathbb{C}^n \to \mathcal{H}$. Up to taking the direct sum of π with itself, we may assume that $\mathcal{H}$ is separable and infinite dimensional. By the Weyl-von Neumann-Berg Theorem (see [30, Corollary II.4.5]), we know that π can be approximated in the point-norm topology by a diagonal representation $\rho : C(\sigma(N)) \to B(\mathcal{K})$, with diagonalizing orthonormal basis $\{\xi_i\}$. Hence, ρ has the form $\rho(f) = \sum_{i=1}^{\infty} f(\lambda^{(i)}) P_i$ where P_i is the projection onto $\operatorname{Sp}\{\xi_i\}$ and $\{\lambda^{(i)}\}$ is dense in $\sigma(N)$. A second approximation allows us to pick m such points

$\{\lambda^{(i)}\}_{i=1}^m$ such that $f \mapsto \sum_{i=1}^m f(\lambda^{(i)})P_i$ approximates ρ in the point-norm topology. Thus, φ can be arbitrarily approximated by a map of the form $\tau : f \mapsto \sum_{i=1}^m f(\lambda^{(i)})K_i$ where K_i is an $n \times n$ positive matrix compression of P_i such that $\sum_{i=1}^m K_i = I_n$.

Hence, for a general d-tuple $\varphi(N) := (\varphi(N_1), ..., \varphi(N_d)) \in \mathcal{W}_n(N)$, as each N_i is identified with the coordinate function $(z_1, ..., z_d) \mapsto z_i$, we see that the d-tuple $\varphi(N)$ is arbitrarily close to $(\tau(z_1), ..., \tau(z_d)) = \sum_{i=1}^m \lambda^{(i)} K_i$. Hence, the closure of the right-hand side of equation (2.5) is $\mathcal{W}_n(N)$.

Eliminating the closure is obtained by a more refined analysis as in [5, Theorem 1.4.10]. It is shown there that the extreme points of $UCP(C(\sigma(N)), M_n)$ are exactly maps of the form $\tau : f \mapsto \sum_{i=1}^m f(\lambda^{(i)})K_i$ where $\lambda^{(i)} \in \sigma(N)$ are m distinct points and $K_1, ..., K_m$ are positive operators such that $\sum_{i=1}^m K_i = I_n$ where the subspaces $K_i M_n K_i$ are linearly independent. In particular, we must have $m \leq n^2$, and that $\mathcal{W}_n(N)$ has at most n^2 extreme points. Since $\mathcal{W}_n(N) \subseteq M_n^d$ and M_n^d is a $2n^2d$-dimensional as a *real* vector space, Caratheodory's Theorem assures us that every point in $\mathcal{W}_n(N)$ is a convex combination of at most $2n^2d + 1$ extreme points. Putting these estimates together, we see that at most $2n^4d + n^2$ terms are required in the right hand side of equation (2.5) for each element in $\mathcal{W}_n(N)$.

Finally, if $\mathcal{S}$ is a matrix convex set such that $\sigma(N) \subseteq \mathcal{S}_1$, as $\sum_{i=1}^m (K_i^{\frac{1}{2}})^* \lambda^{(i)} K_i^{\frac{1}{2}} \in \mathcal{S}_n$ for m points $\lambda^{(i)} \in \sigma(N)$ and positive matrices $K_i \in (M_n)_+$ with $\sum_{i=1}^m K_i = I_n$, it is clear that $\mathcal{W}_n(N) \subseteq \mathcal{S}_n$, so $\mathcal{W}(N) \subseteq \mathcal{S}$.

□

2.4.2 Operator system axiomatics

We recall some definitions and results about operator system structures on Archimedean ordered unit spaces, as discussed in the work of Paulsen, Todorov and Tomforde [100].

A $*$-vector space is a complex vector space $\mathcal{V}$ together with a map $* : \mathcal{V} \to \mathcal{V}$ such that $(v^*)^* = v$ and $(\lambda v + w)^* = \overline{\lambda} v^* + w^*$ for all $v, w \in \mathcal{V}$ and $\lambda \in \mathbb{C}$. We will denote by $\mathcal{V}_{sa} := \{x \in V | x^* = x\}$ the Hermitian / self-adjoint elements in the $*$-vector space $\mathcal{V}$.

An *ordered* $*$-vector space is a pair $(\mathcal{V}, \mathcal{V}_+)$ such that $\mathcal{V}$ is a $*$-vector space and $\mathcal{V}_+$ is a cone in $\mathcal{V}_{sa}$ such that $\mathcal{V}_+ \cap -\mathcal{V}_+ = \{0\}$. This induces a partial order on $\mathcal{V}$ by specifying $a \leq b$ if and only if $b - a \in \mathcal{V}_+$. Such a cone $\mathcal{V}_+$ is called the cone of positive elements in $\mathcal{V}$.

For an ordered $*$-vector space $(\mathcal{V}, \mathcal{V}_+)$, we call an element $e \in \mathcal{V}$ an *order unit* if for all $v \in \mathcal{V}_{sa}$ there is $r > 0$ such that $re \geq v$. If additionally we have that $re + v \geq 0$ for all $r > 0$ implies $v \geq 0$ we say that e is *Archimedean*.

When $e \in \mathcal{V}$ is an Archimedean order unit for an ordered $*$-vector space $(\mathcal{V}, \mathcal{V}_+)$, we will call the triple $(\mathcal{V}, \mathcal{V}_+, e)$ an *Archimedean ordered *-vector space* or *AOU space* for short.

When $(\mathcal{V}, \mathcal{V}_+, e)$ is an AOU space, we may define the order norm on $\mathcal{V}_{sa}$ via

$$\|v\| = \inf\{ \ t \in \mathbb{R}_+ \mid -te \leq v \leq te \ \}.$$

It was shown in [101] that $\| \cdot \|$ can be extended to a norm on $\mathcal{V}$, but even though this extension is not unique, all such extensions yield equivalent norms. We call the topology induced by any extension of $\| \cdot \|$ to a norm the *order topology* induced from $\mathcal{V}_+$ on $\mathcal{V}$.

Let $(\mathcal{V}, \mathcal{V}_+, e)$ be an AOU space. We denote by $\mathcal{V}'$ the collection of *continuous* linear functionals $f : \mathcal{V} \to \mathbb{C}$ with the order topology on $\mathcal{V}$ induced by $\mathcal{V}_+$. We may then define a $*$-operation $f \mapsto f^* \in \mathcal{V}'$ given by $f^*(v) = \overline{f(v^*)}$. This turns $\mathcal{V}'$ into a $*$-vector space. The set $\mathcal{V}'_+ \subseteq \mathcal{V}'$ of all *positive* linear functionals contains the set $S(\mathcal{V})$ of *states* on $\mathcal{V}$comprised of those positive linear functionals f such that $f(e) = 1$.

When $\mathcal{V}$ is a $*$-vector space, the set of all $n \times n$ matrices $M_n(\mathcal{V})$ also becomes a $*$-vectors space with the $*$ operation $[v_{ij}]^* = [v_{ji}^*]$ for $v_{ij} \in \mathcal{V}$. We say that $\mathcal{P} := \{\mathcal{P}_n\}$ is a *matrix ordering* for $\mathcal{V}$ if $(M_n(V), \mathcal{P}_n)$ is an ordered $*$-vector space, and for every $n, m \in \mathbb{N}$ and $X \in M_{n,m}$ we have $X^* \mathcal{P}_n X \subseteq \mathcal{P}_m$, and we call $(\mathcal{V}, \mathcal{P})$ a matrix ordered space. Given a matrix ordering $\{\mathcal{P}_n\}$ on a $*$-vector space we will say that $e \in \mathcal{V}$ is a *matrix ordered unit* if $e_n = \text{diag}(e, e, ...e)$ is an order unit for $(M_n(\mathcal{V}), \mathcal{P}_n)$ for all $n \in \mathbb{N}$ and that it is *matrix Archimedean* order unit if each e_n is an Archimedean order unit for $(M_n(\mathcal{V}), \mathcal{P}_n)$ for all $n \in \mathbb{N}$.

When $(\mathcal{V}, \{\mathcal{P}_n\})$ and $(\mathcal{W}, \{\mathcal{R}_n\})$ are matrix ordered spaces, we say that a linear map $\phi : \mathcal{V} \to \mathcal{W}$ is *completely positive* if for each $[v_{ij}] \in \mathcal{P}_n$ we have that $[\phi(v_{ij})] \in \mathcal{R}_n$. We say that ϕ is a complete order isomorphism if ϕ is bijective with a completely positive inverse.

Definition 2.4.7. *A triple $(\mathcal{V}, \{\mathcal{P}_n\}, e)$ is called an (abstract)* operator system *if $(\mathcal{V}, \{\mathcal{P}_n\})$ is a matrix ordered space, and e is a matrix Archimedean order unit for it.*

The definition justifies itself by a theorem of Choi and Effros [25, Theorem 4.4]. It shows that if $(\mathcal{V}, \{\mathcal{P}_n\}, e)$ is an abstract operator system, then there exists a Hilbert space $\mathcal{H}$, a norm-closed unital $*$-subspace $\mathcal{W} \subseteq B(\mathcal{H})$ and a complete order isomorphism $\phi : \mathcal{V} \to \mathcal{W}$ such that $\varphi(e) = I_{\mathcal{H}}$. We will often refer to $\mathcal{V}$ as an operator system where the Archimedean matrix order unit and matrix order cones are understood and denoted by I and $M_n(\mathcal{V})_+$ respectively.

The concrete representation ϕ of an operator system $\mathcal{V}$ in $B(\mathcal{H})$ imbues $\mathcal{V}$ with a complete norm structure. This structure is then independent of the representation, and is given intrinsically by defining for $V \in M_n(\mathcal{V})$ the norm,

$$\|V\|_n = \inf\left\{ r > 0 \mid \begin{bmatrix} re_n & V \\ V^* & re_n \end{bmatrix} \in M_{2n}(\mathcal{V})_+ \right\}.$$

This complete norm structure on $\mathcal{V}$ is called the canonical operator space structure on $\mathcal{V}$, and will be understood as the complete norm structure of an operator system if not otherwise specified through a concrete representation.

We wish to investigate different operator system structures on an AOU space $(\mathcal{V}, \mathcal{V}_+, e)$. We say that a matrix ordering $\mathcal{P} = \{\mathcal{P}_n\}$ on $\mathcal{V}$ is an operator system structure on $(\mathcal{V}, \mathcal{V}_+, e)$ if $(\mathcal{V}, \{\mathcal{P}_n\}, e)$ is an operator system with $\mathcal{P}_1 = \mathcal{V}_+$. In [100], Paulsen, Todorov and Tomforde show that there are two extremal operator system structures on any AOU space, in the sense that every operator system structure on $(\mathcal{V}, \mathcal{V}_+, e)$ must land between them.

Definition 2.4.8. *Let (V, V_+, e) be an AOU space.*

1. *For each $n \in \mathbb{N}$ we set*

$$\mathcal{P}_n^{min}(\mathcal{V}) = \{[v_{ij}] \in M_n(\mathcal{V}) \mid \sum_{i,j=1}^{n} \overline{\lambda_i} v_{ij} \lambda_j \in \mathcal{V}_+ \text{ for } \lambda_i \in \mathbb{C} \}$$

or alternatively, $[v_{ij}] \in \mathcal{P}_n^{min}(\mathcal{V})$ if and only if $(s(v_{ij})) \in M_n(\mathbb{C})_+$ for any state $s \in S(\mathcal{V})$, and set $\mathcal{P}^{min}(\mathcal{V}) = \{\mathcal{P}_n^{min}\}$.

2. *For each $n \in \mathbb{N}$ we set*

$$\mathcal{D}_n(\mathcal{V}) = \{ X \operatorname{diag}(v_1, ..., v_n) X^* \mid X \in M_{n,m}(\mathbb{C}), v_i \in V_+ \}$$

and let $\mathcal{P}_n^{max}(\mathcal{V})$ be the closure of $\mathcal{D}_n(\mathcal{V})$ in the weak topology induced from states in $M_n(S(\mathcal{V}))$. Or alternatively,

$$\mathcal{P}_n^{max}(\mathcal{V}) = \{ V \in M_n(\mathcal{V}) \mid re_n + V \in \mathcal{D}_n(\mathcal{V}) \text{ for all } r > 0 \}$$

and set $\mathcal{P}^{max}(\mathcal{V}) = \{\mathcal{P}_n^{max}\}$.

Paulsen, Todorov and Tomforde [100] show that the above induce an operator system structure on (V, V_+, e) denoted by $\mathrm{OMIN}(\mathcal{V}) = (\mathcal{V}, \mathcal{P}^{min}(\mathcal{V}), e)$ and $\mathrm{OMAX}(\mathcal{V}) = (\mathcal{V}, \mathcal{P}^{max}(\mathcal{V}), e)$ with the following universal properties:

1. for any operator system $(\mathcal{W}, \{\mathcal{R}_n\}, f)$, any positive map $\phi : \mathcal{W} \to \mathrm{OMIN}(\mathcal{V})$ such that $\phi(f) = e$ is automatically *completely* positive.

2. for any operator system $(\mathcal{W}, \{\mathcal{R}_n\}, f)$, any positive map $\phi : \text{OMAX}(\mathcal{V}) \to \mathcal{W}$ such that $\phi(e) = f$ is automatically *completely* positive.

3. In particular, if $\{\mathcal{P}_n\}$ is some operator system structure on the Archimedean ordered unit space (V, V_+, e), then $\mathcal{P}_n^{max}(\mathcal{V}) \subseteq \mathcal{P}_n \subseteq \mathcal{P}_n^{min}(\mathcal{V})$ for all $n \in \mathbb{N}$.

Finally, we describe how to induce an operator system structure on $\mathcal{V}'$, when $\mathcal{V}$ is *finite dimensional*. For every $f \in M_n(\mathcal{V})'$ we let $f_{ij} : \mathcal{V} \to \mathbb{C}$ be given by $f_{ij}(v) = f(v \otimes E_{ij})$ where $\{E_{ij}\}$ are canonical matrix units for M_n. Then each $f_{ij} \in \mathcal{V}'$, so we may identify $f \in M_n(\mathcal{V})'$ with $[f_{ij}] \in M_n(\mathcal{V}')$, where the inverse operation is given for $[f_{ij}] \in M_n(\mathcal{V}')$ by $f([v_{ij}]) = \sum_{ij} f_{ij}(v_{ij})$, for $[v_{ij}] \in M_n(\mathcal{V})$. Hence, to specify a matrix ordering on $\mathcal{V}'$, we define cones in each $M_n(\mathcal{V})'$. Following [100], given an operator system $(\mathcal{V}, \{\mathcal{P}_n\}, e)$, we set

$$\mathcal{P}_n' = \{ \ f : M_n(\mathcal{V}) \to \mathbb{C} \mid f(\mathcal{P}_n) \subseteq \mathbb{R}_+ \ \}.$$

Then $(\mathcal{V}', \{\mathcal{P}_n'\})$ becomes a matrix ordered space on the $*$-vector space $\mathcal{V}'$ with $\mathcal{P}_1' = \mathcal{V}_+'$. The following proposition shows that unital complete order isomorphism of operator systems promotes to a complete order isomorphism between their dual matrix ordered spaces.

Proposition 2.4.9. *Let $\mathcal{V}$ and $\mathcal{W}$ be operator systems and $\phi : \mathcal{V} \to \mathcal{W}$ a linear map. If ϕ is completely positive then $\phi' : \mathcal{W}' \to \mathcal{V}'$ given by $\phi'(f) = f \circ \phi$ is completely positive.*

Proof. Suppose ϕ is completely positive. Let $[f_{ij}] \in M_n(\mathcal{W}')_+$. We need to show that $[\phi'(f_{ij})] = [f_{ij} \circ \phi]$, as a map sending $v \in \mathcal{V}$ to $[f_{ij}\phi(v)]$, is completely positive. Let $[v_{kl}] \in M_m(\mathcal{V})_+$. Since ϕ is completely positive, we have that $[\phi(v_{kl})]$ is positive in $M_m(\mathcal{W})$. As $[f_{ij}]$ is positive, we have that $[f_{ij}(\phi(v_{kl}))]$ is positive in $M_n(M_m(\mathbb{C}))$. Hence, $[\phi'(f_{ij})] = [f_{ij} \circ \phi]$ is completely positive, and we are done. ☐

In general, it is not clear what functional $f \in \mathcal{V}'$ plays the role of an ordered Archimedean matrix order unit for $(\mathcal{V}', \{\mathcal{P}_n'\})$. However, when $\mathcal{V}$ is *finite dimensional*, a theorem of Choi and Effros [25, Corollary 4.5] gives us a candidate. Recall that a linear functional $\tau : \mathcal{V} \to \mathbb{C}$ is said to be *strictly positive* if for any $s \in \mathcal{V}_+$ with $s \neq 0$ we have $\tau(s) > 0$.

Theorem 2.4.10. *Let $(\mathcal{V}, \{\mathcal{P}_n\}, e)$ be a* finite dimensional *operator system. Then, there exists a strictly positive functional τ which is an Archimedean matrix order unit for $(V', \{\mathcal{P}_n'\})$. Hence, $(V', \{\mathcal{P}_n'\}, \tau)$ is an abstract operator system.*

Proof. We note that any $f \in \mathcal{V}'_{sa}$ may be written as $f = p - q$ for positive functionals $p, q \in \mathcal{V}'$. To see this, embed $\mathcal{V} \subseteq B(\mathcal{H})$ as an operator system, and extend f to a self-adjoint weak*-continuous linear functional on $B(\mathcal{H})$. Considered as a trace class operator, f can then be decomposed as a difference of two positive trace class operators. Restrict back to $\mathcal{V}$ to obtain the desired positive functionals p and q such that $f = p - q$.

We saw that $\mathcal{V}'$ is spanned by positive elements. Let $p_1, ..., p_d$ be a basis of positive functionals in $\mathcal{V}'$. It is now clear that $\tau = \sum_i p_i$ is strictly positive on $\mathcal{V}$. Indeed, it is clearly positive, and if it isn't strictly positive, then there is $v \in \mathcal{V}_+ \setminus \{0\}$ such that $p_i(v) = 0$, so that $f(v) = 0$ for all $f \in \mathcal{V}'$. As $\mathcal{V}'$ separates points, this is impossible.

Now, suppose $\tau = \sum_i p_i$ for some basis $p_1, ..., p_d$ of positive functions in $\mathcal{V}'$. Let $[f_{ij}] \in M_n(\mathcal{V}')_{sa}$, and write

$$[f_{ij}] = A_1 \otimes p_1 + ... + A_d \otimes p_d$$

where each A_i is self-adjoint. Let $r = \max\{\rho(A_i)\}$ where $\rho(A_i)$ denotes the spectral radius of A_i. It is then clear that

$$[f_{ij}] \leq r\big[I \otimes p_1 + ...I \otimes p_d\big] = r \cdot \tau^{(n)}$$

so that $\tau^{(n)}$ is an order unit for $M_n(\mathcal{V}')$. Finally, since p_i is a basis, we may form its dual basis $\{v_i\}$ for $\mathcal{V}$. Hence, the positivity of $r \cdot \tau^{(n)} + [f_{ij}]$ for all $r \geq 0$ would imply the positivity of each A_i by evaluating on v_i. Hence, we see that $[f_{ij}]$ is positive, and $\tau^{(n)}$ is Archimedean. Hence, we see that τ is an Archimedean matrix order unit for $\mathcal{V}'$. $\square$

Example 2.4.11. *Let $\mathcal{V} \subseteq M_n(\mathbb{C})$ be a concrete finite dimensional operator system realized in finite dimensions. By the above theorem we know there is a strictly positive linear functional $\tau \in \mathcal{V}'$ that makes $(\mathcal{V}', \{M_n(\mathcal{V})'\}, \tau)$ into an abstract operator system. We will see in Corollary 6.3.13 that a positive functional τ is strictly positive if and only if it is an Archimedean matrix order unit for $\mathcal{V}'$. Since the normalized trace* $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ *restricted to $\mathcal{V}$ is easily seen to be a strictly positive, we get that $(\mathcal{V}', \{M_n(\mathcal{V})'_+\}, \mathrm{tr}\,|_{\mathcal{V}})$ is an abstract operator system.*

Chapter 3

Isomorphisms of tensor algebras arising from weighted partial systems

3.1 Introduction

Non-self-adjoint operator algebras associated to dynamical / topological / analytic objects, and their classification via these objects, have been the subject of study by many authors for almost 50 years, beginning with the work of Arveson [4] and Arveson and Josephson [13].

The main theme of this line of research, as is the main theme of this chapter, is to identify the extent to which the dynamical objects classify their associated non-self-adjoint operator algebras. We shall mainly focus on classification of non-self-adjoint tensor operator algebras arising from a single C*-correspondence over a commutative C*-algebra, although a profusion of results have been obtained in other contexts [33, 37, 38, 42, 58, 72, 73, 109] to mention only some.

In this chapter, which is based on [41], we provide classification results for tensor algebras arising from weighted partial systems (WPS for short). Our objective is to show that WPS yield tensor algebras that are still completely classifiable up to bounded / isometric isomorphisms, while covering many examples of such classification results. For instance, those for multiplicity free finite directed graphs [73, 116] and for Peters' semi-crossed product [34].

A weighted partial system on a compact space X is a pair (σ, w) of d-tuples $(\sigma_1, ..., \sigma_d)$ and $(w_1, ..., w_d)$ of partially defined continuous functions $\sigma_i : X_i \to X$ and $w_i : X_i \to (0, \infty)$

for X_i clopen. WPS generalize many classical constructions such as non-negative matrices, continuous function on a compact space, multivariable systems, distributed function systems, graph directed systems and more.

To each WPS (σ, w) we associate a multiplicity free topological quiver (in the sense of [93]) that encodes some information on it. This topological quiver gives rise to a C*-correspondence $C(\sigma, w)$, as constructed in [93]. We completely characterize these C*-correspondences up to unitary isomorphism and similarity, in terms of conjugacy relations between the WPS that we call *branch-transition conjugacy* and *weighted-orbit conjugacy* respectively.

We then associate a tensor algebra $\mathcal{T}_+(\sigma, w)$ to $C(\sigma, w)$ as one usually does for general C*-correspondences [90, 91, 92], which coincides with $\mathcal{T}_+(\mathrm{Prod}(C(\sigma, w)))$ as in subsection 2.2.3. Characterization of the C*-correspondences allows for classification of these tensor algebras up to isometric / bounded isomorphism and in some cases up to algebraic isomorphism, in terms of the WPS (σ, w).

The following are our main results (See Theorems 3.5.6 and 3.5.7). Suppose (σ, w) and (τ, u) are WPS over compact spaces X and Y respectively.

1. $\mathcal{T}_+(\sigma, w)$ and $\mathcal{T}_+(\tau, u)$ are isometrically isomorphic if and only if $C(\sigma, w)$ and $C(\tau, u)$ are unitarily isomorphic if and only if (σ, w) and (τ, u) are branch-transition conjugate.

2. $\mathcal{T}_+(\sigma, w)$ and $\mathcal{T}_+(\tau, u)$ are boundedly isomorphic if and only if $C(\sigma, w)$ and $C(\tau, u)$ are similar if and only if (σ, w) and (τ, u) are weighted-path conjugate. If in addition the clopen sets X_i (which are the domains of each σ_i) cover X, the above is equivalent to having an algebraic isomorphism between $\mathcal{T}_+(\sigma, w)$ and $\mathcal{T}_+(\tau, u)$.

The solution to these isomorphism problems requires an adaptation of a new method in the analysis of character spaces due to Davidson, Ramsey and Shalit in [37], used in the solution of isomorphism problems of universal operator algebras associated to tuples of operators subject to homogeneous polynomial constraints.

One of the main thrusts of the work in this chapter is the use of these classification results to show that, in general, the (completely) isometric isomorphism and algebraic / (completely) bounded isomorphism problems are distinct in the sense that they require separate criteria to be solved (See Example 3.5.8).

This chapter contains six sections, including this introductory section. In Section 3.2 we introduce the notion of a weighted partial system, and define three different notions of conjugacy between WPS called branch-transition conjugacy, weighted-orbit conjugacy and graph conjugacy. We then associate a C*-correspondence to every WPS in such a way that the three conjugacy relations above correspond to unitary isomorphism, similarity and isomorphism between the C*-correspondences. We give examples that show that these three conjugacy relations are distinct. In Section 3.3 we discuss the general theory of tensor algebras arising from C* correspondences, and develop the theory of semi-graded isomorphisms by building on ideas from [42, Section 6] and [92, Section 5]. In Section 3.4 we compute the character space of a tensor algebra associated to a WPS by adapting the methods of [60], and providing a useful characterization of semi-gradedness in terms of the character space. In Section 3.5 we reduce the general isomorphism problem to the problem on semi-graded isomorphisms, and using Sections 3.4, 3.3 and 3.2 in tandem with a new character space technique due to [37], we conclude our main classification results. Finally, in Section 3.6 we compare and apply our our theorems to other tensor algebra constructions arising from non-negative matrices, single variable dynamics and partial systems with disjoint graphs. We show how in some cases we can apply our results to recover some previously obtained results in the literature.

3.2 Weighted partial systems

We define the notion of weighted partial system, and examine it associate a completely positive map and a topological quiver.

Definition 3.2.1. *Let X be a compact space. A d-variable* weighted partial system *(WPS for short) is a pair (σ, w) where $\sigma = (\sigma_1, ..., \sigma_d)$ is comprised of continuous maps $\sigma_i : X_i \to X$ where each X_i is clopen in X, and $w = (w_1, ..., w_d)$ is comprised of continuous non-vanishing weights $w_i : X_i \to (0, \infty)$.*

When $w_i = 1$ for all $1 \leq i \leq d$, then the information on the weights is redundant, and in this case we replace $(\sigma, 1)$ by σ and call it a d-variable (clopen) *partial system.* Partial systems were used under the name of "quantised dynamical systems" by Kakariadis and Shalit to classify tensor algebras associated to monomial ideals in the ring of polynomials in non-commuting variables (See [72, Corollary 8.12]).

Weighted partial systems provide us with concrete examples of Markov-Feller maps and topological quivers.

Definition 3.2.2. *Let (σ, w) be a d-variable WPS over a compact X.*

1. *The operator associated to* (σ, w) *is a positive linear map* $P(\sigma, w) : C(X) \to C(X)$ *given by*

$$P(\sigma, w)(f)(x) = \sum_{i:x\in X_i} w_i(x) f(\sigma_i(x)).$$

2. *The quiver associated to* (σ, w) *is the quintuple* $\mathcal{Q}(\sigma, w) = (X, \mathrm{Gr}(\sigma), r, s, P(\sigma, w))$ *where* $\mathrm{Gr}(\sigma)$ *is the (union) cograph of* σ*, i.e. the union of the cographs of* σ_i *given by*

$$\mathrm{Gr}(\sigma) = \cup_{i=1}^{d} \{\ (\sigma_i(x), x) \mid x \in X_i\ \}.$$

The range and source maps are given by $r(\sigma_i(x), x) = \sigma_i(x)$*,* $s(\sigma_i(x), x) = x$ *and Radon measures*

$$P(\sigma, w)_x = \sum_{i:x\in X_i} w_i(x)\delta_{(\sigma_i(x),x)}.$$

Note first that the source map $s : \mathrm{Gr}(\sigma) \to X$ is an open map, and that the graph we have constructed is multiplicity-free, even though the original system (σ, w) need not be multiplicity-free. More specifically, if for some $x \in X$ we have an index $1 \le i \le d$ such that $\sigma_i(x) = \sigma_j(x)$ for some other index $j \ne i$, then σ is not multiplicity free, yet in $Gr(\sigma)$ we have $(\sigma_i(x), x) = (\sigma_j(x), x)$. Instead, part of the information on the multiplicity of $\sigma_i(x)$ is encoded in the measure $P(\sigma, w)_x$, depending on the weights $w_j(x)$ for these j that satisfy $\sigma_j(x) = \sigma_i(x)$. In fact, $\mathcal{Q}(\sigma, w)$ is just the topological quiver associated to $P(\sigma, w)$ as in Subsection 2.3.2.

We also abused notation above and decided to denote both the positive map and the collection of Radon measures of the topological quiver of (σ, w) in the same way, the reason being the 1-1 correspondence between positive maps $P : C(X) \to C(X)$ and p.m.v. maps P as in Theorem 2.3.9.

Remark 3.2.3. Since each w_i never vanishes, we see that for every $x \in X$ we have

$$\mathrm{supp}(P(\sigma, w)_x) = \{\ (\sigma_i(x), x) \mid x \in X_i\ \} = s^{-1}(x) \tag{3.1}$$

and so by [83, Lemma 3.30] we have that $\mathcal{Q}(\sigma, w)$ is a topological quiver with $s^{-1}(x) = \mathrm{supp}(P(\sigma, w)_x)$. If we were to alter Definition 3.2.1 to allow some w_i to vanishes on X_i, as σ_i is defined on X_i, we would arrive at a situation where the equation (3.1) fails. This is because it is then possible for $\mathrm{supp}(P(\sigma, w)_x)$ not to contain an edge $(\sigma_i(x), x)$ while $s^{-1}(x)$ always does. See [83, Example 3.35] for this phenomenon and [83, Section 3] for other complications that arise in for general p.m.v. maps and their topological quivers when s is not open, and when $\mathrm{supp}(P(\sigma, w)_x)$ is a proper subset of $s^{-1}(x)$.

Weighted partial systems encompass many different classical dynamical objects. When they have simpler forms, we describe their associated topological quiver and positive map.

Example 3.2.4 (Non-negative matrices). *If $A = [A_{ij}]$ is a non-negative matrix indexed by a finite set Ω, we associate a $|\Omega|$-variable WPS (σ^A, w^A) to it by specifying $\Omega_i^A := \{ j \in \Omega \mid A_{ij} > 0 \}$ and define $\sigma_i^A : \Omega_i^A \to \Omega$ by setting $\sigma_i^A(j) = i$, and $w_i^A(j) = A_{ij}$. Note that some σ_i^A may be the empty set function. This way the graph of the WPS is given by $Gr(\sigma) = Gr(A) := \{ (i,j) \mid A_{ij} > 0 \}$, the Radon measures by $P(\sigma^A, w^A)_j = \sum_{i\in\Omega} A_{ij}\delta_{(i,j)}$ and the positive map $P(\sigma^A, w^A)$ by $P(\sigma^A, w^A)(f)(j) = \sum_{i\in\Omega} A_{ij} f(i)$.*

Example 3.2.5 (Finite directed graphs). *Let $G = (V, E, r, s)$ be a directed graph with finitely many edges and vertices. We can regard every $v \in V$ as comprising a clopen subset $\{v\}$ of V, and each edge $e \in E$ as (the unique) map from $\{s(e)\}$ to $\{r(e)\}$. With $w_e = 1$, the collection $\sigma^E = \{e\}_{e\in E}$ becomes a (weighted) partial system. We then see that $Gr(\sigma^E)$, being the regular union in $V \times V$, yields the* multiplicity free *directed graph associated to (V, E, r, s), That is, $Gr(\sigma^E) := \{(r(e), s(e))|e \in E\}$. Denote by $m_{w,v} = |s^{-1}(v) \cap r^{-1}(w)|$ the multiplicity of edges starting at v and ending at w. Then the Radon measures are given by $P(\sigma^E)_v = \sum_{w:(w,v)\in Gr(\sigma^E)} m_{w,v}\delta_{(w,v)}$, and the positive map is given by*

$$P(\sigma^E)(f)(v) = \sum_{w:(w,v)\in Gr(\sigma^E)} m_{w,v} f(w)$$

We note that when G has finitely many edges and vertices, it can be encoded as a non-negative matrix $A_G = [m_{w,v}]$ indexed by V, and then the topological quiver, positive maps for G and for A_G as in Example 3.2.4 coincide.

Example 3.2.6 (Partially defined continuous maps). *For a compact space X, a clopen subset $X' \subseteq X$ and a continuous map $\sigma : X' \to X$, we have that (X, σ) is a partial system. The positive map $P(\sigma)(f) = f \circ \sigma$ is a *-homomorphism on $C(X)$, and in fact, all *-homomorphisms on $C(X)$ arise in this way via the commutative Gelfand-Naimark duality. The graph of the partial system σ is then just $Gr(\sigma) = \{ (\sigma(x), x) \mid x \in X' \}$, and the Radon measures are just Dirac measures $P(\sigma)_x = \delta_{(\sigma(x),x)}$.*

Example 3.2.7 (Multivariable systems). *When $\sigma = (\sigma_1, ..., \sigma_d)$ is a d-tuple of continuous maps defined on* all *of X, the graph of σ is just the union of the graphs of σ_i as in Definition 3.2.2, but the Radon measures yield the simpler form $P(\sigma)_x = \sum_{i=1}^d \delta_{(\sigma_i(x),x)}$ for all $x \in X$. The positive operator associated to σ is then given by $P(\sigma)(f)(x) = \sum_{i=1}^d f(\sigma_i(x))$.*

Example 3.2.8 (Distributed function systems). *If $\sigma = (\sigma_1, ..., \sigma_d)$ is a multivariable system on a compact metric space X, and $p = (p_1, ..., p_d)$ are continuous non-vanishing probabilities in the sense that for each $x \in X$ we have that $\sum_{i=1}^d p_i(x) = 1$ and $p_i(x) > 0$ for* *all $x \in X_i$, we call (σ, p) a distributed function system. The positive operator associated to (σ, p) given by $P(\sigma)(f)(x) = \sum_{i=1}^{d} p_i(x) f(\sigma_i(x))$ yields a unital positive map on $C(X)$, and this yields a Markov-Feller map via Theorem 2.3.9.*

When the p_i are constant and each σ_i is a 1-1 strict contraction, we call (σ, p) a distributed iterated function system. *These systems were used by Hutchinson in [67] and by Barnsley in [14, 15] to construct certain invariant measures on self-similar sets coming from σ.*

Example 3.2.9 (Graph directed systems)**.** *Let $G = (V, E, r, s)$ be a directed graph with finitely many vertices and edges, $\{X_v\}_{v \in V}$ a (finite) set of compact metric spaces and $\{\sigma_e\}_{e \in E}$ a (finite) set of 1-1 strict contractions $\sigma_e : X_{s(e)} \to X_{r(e)}$. Then we call the data $(G, \{X_v\}_{v \in V}, \{\sigma_e\}_{e \in E})$ a* graph directed system *or* Mauldin-Williams graph. *If we set $X = \sqcup_{v \in V} X_v$, then $(\sigma_e)_{e \in E}$ becomes a partial system over X.*

See [86], where Mauldin-Williams graphs are used to construct self-similar sets and iterated limit sets. Also see [87] where the Hausdorff dimension of such iterated limit sets is computed in some cases.

Our next goal is to define the main conjugacy relations between WPS in this chapter. One particular conjugacy relation that we call *branch-transition* conjugacy, will turn out to arise from isometric isomorphism of the associated operator algebra.

We say that (σ, w) and (τ, u) d-variable and d'-variable WPS over compact spaces X and Y respectively are conjugate if one is a homeomorphic image of the other up to some permutation. That is, $d = d'$ and there is a homeomorphism $\gamma : X \to Y$ and a permutation $\alpha \in S_d$ such that $\gamma^{-1}\tau_{\alpha(i)}\gamma = \sigma_i$ and $u_{\alpha(i)} \circ \gamma = w_i$ for all $1 \leq i \leq d$.

Conjugation of WPS is the most rigid equivalence relation between WPS. We define a weaker equivalence relation, that loses some information about multiplicities and weights of the WPS.

For an s-variable WPS (τ, u) on a compact space Y and a homeomorphism $\gamma : X \to Y$ denote $\tau^\gamma = \gamma^{-1}\tau\gamma := (\gamma^{-1}\tau_1\gamma, ..., \gamma^{-1}\tau_s\gamma)$ and $u^\gamma = u\gamma = (u_1\gamma, ..., u_s\gamma)$.

Definition 3.2.10. *Let σ and τ be partial systems on compact spaces X and Y respectively. We say that σ and τ are* graph conjugate *if there exists a homeomorphism $\gamma : X \to Y$ such that $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau^\gamma)$. Equivalently, there exists a homeomorphism $\gamma : X \to Y$ such that the map $\gamma \times \gamma : \mathrm{Gr}(\sigma) \to \mathrm{Gr}(\tau)$ is a homeomorphism.*

Some natural sets that arise while considering graphs of partial systems are the sets of points for which some of the maps in the system coincide and / or sets of points for which they "branch out".

Definition 3.2.11. *Let σ be a d-variable partial system.*

1. *A point $x \in X$ is a* branching point *for σ if there is some net $x_\lambda \to_\lambda x$ and two distinct indices $i, j \in \{1, ..., d\}$ such that $x_\lambda \in X_i \cap X_j$ and $\sigma_i(x_\lambda) \neq \sigma_j(x_\lambda)$ for all λ while $\sigma_i(x) = \sigma_j(x)$.*

2. *An edge $e \in \mathrm{Gr}(\sigma)$ is a* branching edge *for $\mathrm{Gr}(\sigma)$ if there are two nets $\{e_\lambda\}$ and $\{f_\lambda\}$ converging to e such that $s(e_\lambda) = s(f_\lambda)$ while $r(e_\lambda) \neq r(f_\lambda)$ for all λ.*

Remark 3.2.12. If e is a branching edge, then by taking subnets if necessary, we see that $s(e)$ is a branching point. However, if $s(e)$ is a branching point, e may not be a branching edge. Still, every branching point is the source of *some* branching edge. Moreover, we see that if two partial systems σ and τ are graph conjugate via γ, then σ and τ^γ have the same sets of branching points and branching edges.

Definition 3.2.13. *Let σ be a d-variable partial system and $I \subseteq \{1, ..., d\}$ a non-empty subset of indices.*

1. *The coinciding set of I is the set*

$$C(I) = \{\ x \in \cap_{i \in I} X_i \mid \sigma_i(x) = \sigma_j(x)\ \forall i, j \in I\ \}.$$

2. *$x \in X$ is a* coinciding point *for $\mathrm{Gr}(\sigma)$ if there is some $I \subseteq \{1, ..., d\}$ with $|I| \geq 2$ such that $x \in C(I)$.*

We also denote $B(I) := \partial C(I)$ the topological boundary of $C(I)$ inside X.

For $I \subseteq \{1, ..., d\}$, since the maps σ_i of a partial system are defined on clopen sets X_i, we see that $\cap_{i \in I} X_i$ is clopen, so that both $C(I)$ and $B(I)$ are in fact closed subsets of $\cap_{i \in I} X_i$.

We next characterize branching points and branching edges in terms of boundaries of coinciding sets.

Proposition 3.2.14. *Let σ be a d-variable (clopen) partial system.*

1. *$x \in X$ is a branching point if and only if for some $I \subseteq \{1, ..., d\}$ we have $x \in B(I)$.*

2. *$e \in \mathrm{Gr}(\sigma)$ is a branching edge for $\mathrm{Gr}(\sigma)$ if and only if $s(e) \in B(I)$ for some $I \subseteq \{1, ..., d\}$ so that $r(e) = \sigma_i(s(e))$ for some (and hence all) $i \in I$.*

Proof. We first prove (1). If $x \in B(I)$ for some $I \subseteq \{1, ..., d\}$, there exists a net $\{x_\lambda\}$ in $\cap_{i\in I} X_i$ converging to x such that for every $\lambda \in \Lambda$ there exist $i_\lambda \neq j_\lambda$ in I such that $\sigma_{i_\lambda}(x_\lambda) \neq \sigma_{j_\lambda}(x)$. By passing to a subnet, we may arrange that $i_{\lambda_1} = i_{\lambda_2}$ and $j_{\lambda_1} = j_{\lambda_2}$ for all $\lambda_1, \lambda_2 \in \Lambda$ and so x is a branching point.

For the converse, if $x \in X$ is a branching point, let $i, j \in \{1, ..., d\}$ be two distinct indices and $\{x_\lambda\}$ a net in $X_i \cap X_j$ converging to x such that $\sigma_i(x_\lambda) \neq \sigma_j(x_\lambda)$, while $\sigma_i(x) = \sigma_j(x)$. Then by taking $I = \{i, j\}$ we have that $x \in C(I)$, and the existence of the above net shows that $x \in B(I)$.

Next, we prove (2). Suppose $s(e) \in B(I)$ for some $I \subseteq \{1, ..., d\}$ so that $r(e) = \sigma_i(s(e))$ for some (and hence all) $i \in I$. Then by the above we have a net $\{x_\lambda\}$ in $\cap_{i\in I} X_i$ converging to $s(e)$, and two distinct indices i, j in I such that $\sigma_i(x_\lambda) \neq \sigma_j(x_\lambda)$ for all λ, while $\sigma_i(s(e)) = \sigma_j(s(e)) = r(e)$. Then the nets of edges $e_\lambda = (\sigma_i(x_\lambda), x_\lambda)$ and $f_\lambda = (\sigma_j(x_\lambda), x_\lambda)$ converge to e, and have the same sources, and different ranges for every λ.

Conversely, if we have two nets $\{e_\lambda\}$ and $\{f_\lambda\}$ converging to e such that $s(e_\lambda) = s(f_\lambda)$ while $r(e_\lambda) \neq r(f_\lambda)$ for all λ, by taking subnets as necessary, we may assume that $r(e_\lambda) = \sigma_i(s(e_\lambda))$, and $r(f_\lambda) = \sigma_j(s(f_\lambda))$ while $r(e) = \sigma_i(s(e)) = \sigma_j(s(e))$ for i, j distinct. By taking $I = \{i, j\}$ we see that $s(e) \in B(I)$, while $r(e) = \sigma_i(s(e))$, and we are done. □

For an edge $e \in \mathrm{Gr}(\sigma)$, we denote $I(e, \sigma) = \{\ i \mid \sigma_i(s(e)) = r(e),\ s(e) \in X_i\ \}$, which is the set of all indices of maps that send $s(e)$ to $r(e)$.

Definition 3.2.15. *Let (σ, w) be a WPS over X. The weight induced on the graph of σ is a function $w : \mathrm{Gr}(\sigma) \to (0, \infty)$ given for any edge $e = (y, x) \in \mathrm{Gr}(\sigma)$ by*

$$w(e) = \sum_{i\in I(e,\sigma)} w_i(s(e)) = \sum_{i:\sigma_i(x)=y,\ x\in X_i} w_i(x).$$

Proposition 3.2.16. *Let (σ, w) be a WPS over X. Then $w : \mathrm{Gr}(\sigma) \to (0, \infty)$ is discontinuous at $e \in \mathrm{Gr}(\sigma)$ if and only if e is a branching edge for $\mathrm{Gr}(\sigma)$. Moreover, $w : \mathrm{Gr}(\sigma) \to (0, \infty)$ is bounded from above and from below.*

Proof. $\Rightarrow$: If e is not a branching edge for σ, there exist a neighborhood U of e inside $\mathrm{Gr}(\sigma)$ such that for any $f \in U$ we have $I(f, \sigma) = I(e, \sigma)$. Hence, for $f \in U$ we have

$$w(f) = \sum_{i\in I(f,\sigma)} w_j(s(f)) = \sum_{i\in I(e,\sigma)} w_j(s(f))$$

so we see that w is continuous at e by continuity of w_j.

$\Leftarrow$: If $e \in \mathrm{Gr}(\sigma)$ is a branching edge, without loss of generality, and perhaps by taking a subnet, there is a net $e_\lambda \to_\lambda e$ indexed by Λ with $I := I(e_{\lambda_1}, \sigma) = I(e_{\lambda_2}, \sigma) \subsetneq I(e, \sigma)$ for all $\lambda_1, \lambda_2 \in \Lambda$. Hence we obtain that

$$w(e_\lambda) = \sum_{i \in I} w_i(s(e_\lambda)) \to \sum_{i \in I} w_i(s(e))$$

by continuity of w_i for all $1 \leq i \leq d$. Yet on the other hand,

$$w(e) = \sum_{i \in I(e,\sigma)} w_i(s(e)) > \sum_{i \in I} w_i(s(e)) = \lim_\lambda w(e_\lambda)$$

since I is a proper subset of $I(e, \sigma)$, and w_i are bounded from below on the clopen sets X_i.

Finally, since for every $1 \leq i \leq d$ we have that w_i, being continuous on X_i, is bounded above by M_i and below by C_i. Let $M = \max\{M_1, ..., M_d\}$ and $C = \min\{C_1, ..., C_d\}$. Thus, if $e \in \mathrm{Gr}(\sigma)$, there is some $i \in \{1, ..., d\}$ with $\sigma(s(e)) = r(e)$ so that $w(e) \geq w_i(s(e)) \geq C > 0$, and of course $w(e) \leq |I(e, \sigma)| \cdot M \leq d \cdot M$, and we see that $w : \mathrm{Gr}(\sigma) \to (0, \infty)$ is bounded below by C and above by $d \cdot M$. □

Definition 3.2.17. *Let (σ, w) and (τ, u) be WPS on compact spaces X and Y respectively. We say that (σ, w) and (τ, u) are* branch-transition conjugate *if σ and τ are graph conjugate via some homeomorphism $\gamma : X \to Y$ and we have that the weighted transition function $\frac{u^\gamma}{w} : \mathrm{Gr}(\sigma) \to (0, \infty)$ from w to u^γ given by*

$$\frac{u^\gamma}{w}(e) := \frac{u^\gamma(e)}{w(e)}$$

is continuous at e for any branching edge $e \in \mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau^\gamma)$.

We interpret the above to mean that the discontinuities for w and u^γ, which can only be at branching edges, are of the same proportions, so that the weighted transition function becomes continuous at every branched edge, and hence everywhere on $\mathrm{Gr}(\sigma)$.

Example 3.2.18. *Graph conjugacy does not imply branch-transition conjugacy, not even when the weights w and u are constant. If we take $X = [0, 1]$ and $\sigma_1(x) = x$, $\sigma_2(x) = 0$, and pick two pairs of constant weights $u = (\frac{1}{2}, \frac{1}{2})$ and $w = (\frac{1}{3}, \frac{2}{3})$, then (σ, w) and (σ, u) are not branch-transition conjugate. Indeed, if σ is graph conjugate to itself via γ, as γ sends branching points to themselves, and 0 is the only branching point, we then must have $\gamma(0) = 0$ which means that γ must be non-decreasing. This means that*

$$\frac{u^\gamma}{w}(y, x) = \begin{cases} \frac{3}{2} & \textit{if } x > 0 \ \& \ y = x \\ \frac{3}{4} & \textit{if } x > 0 \ \& \ y = 0 \\ 1 & \textit{if } x = 0 \end{cases}$$

so that $\frac{u^\gamma}{w}$ *is not continuous at the branching edge* $e = (0,0)$*, and so* (σ, w) *and* (σ, u) *are not branch-transition conjugate.*

Corollary 3.2.19. *Let* (σ, w) *and* (τ, u) *be d-variable and s-variable WPS on* X *and* Y *respectively. If* σ *and* τ *are graph conjugate, then there exists some* $K \geq 1$ *such that*

$$\frac{1}{K} \leq \frac{u^\gamma}{w} \leq K.$$

If in addition, (σ, w) *and* (τ, u) *are branch-transition conjugate, then* $\frac{u^\gamma}{w}$ *is continuous on* $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau^\gamma)$*.*

Proof. Without loss of generality we assume that $\gamma = Id_X$. Assuming $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau)$, since both w and u are bounded from above and below by the last part of Proposition 3.2.16, we see that there is a $K \geq 1$ such that $\frac{1}{K} \leq \frac{u}{w} \leq K$.

Lastly, by Proposition 3.2.16 again, both w and u are continuous at edges which are not branching points for $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau)$, and by branch-transition conjugacy we see that $\frac{u}{w}$ is continuous on all of $\mathrm{Gr}(\sigma)$. □

We now focus on the second conjugacy relation arising from our operator algebras, which we call *weighted-orbit conjugacy.* We give an example of two WPS which are not weighted-orbit conjugate, and an example of weight-orbit conjugate WPS which are not branch-transition conjugate. We also provide a simple criterion for when graph, weighted-orbit and branch-transition conjugacy coincide. It will turn out that weighted-orbit conjugacy arises from bounded isomorphism of the associated operator algebra.

Definition 3.2.20. *Let* (σ, w) *and* (τ, u) *be WPS on compact spaces* X *and* Y *respectively. We say that* (σ, w) *and* (τ, u) *are* weighted-orbit conjugate *with constant* $C \geq 1$ *if* σ *and* τ *are graph conjugate via some homeomorphism* $\gamma : X \to Y$ *and there exists* $H \in C(\mathrm{Gr}(\sigma))$ *such that for any* $n \in \mathbb{N}$ *and any path* $\mu = \mu_n ... \mu_1 \in \mathrm{Gr}(\sigma^n)$ *we have*

$$\frac{1}{C} \leq \Pi_{k=1}^n \Big[\frac{u^\gamma}{w}(\mu_k) H(\mu_k) \Big] \leq C.$$

Intuitively, this means that multiplying by some continuous function H on $\mathrm{Gr}(\sigma)$ makes the gaps introduced by $\frac{u^\gamma}{w}$ uniformly bounded on paths of *any* length. Note that when $C = 1$, the above implies the continuity of $\frac{u^\gamma}{w}$ so that in this case (σ, w) and (τ, u) are branch-transition conjugate.

Example 3.2.21. *It turns out that the weighted multivariable systems of Example 3.2.18 are not even weighted-orbit conjugate, despite being graph conjugate. Indeed, for every* $x \in [0,1]$ *one can construct a path of length* n *comprised of the same edge* $e = (x,x) \in \mathrm{Gr}(\sigma)$ *at every step. In this case, if* $H \in C(\mathrm{Gr}(\sigma))$ *and a (necessarily non-decreasing) homeomorphism* γ *realize weighted-orbit conjugacy with constant* C*, we have that for* $e = (x,x)$ *such that* $x > 0$ *and* $n \in \mathbb{N}$*,*

$$\frac{1}{C} \leq \Pi_{k=1}^{n}\Big[\frac{3}{2} \cdot H(e)\Big] \leq C$$

so this forces $H(e) = \frac{2}{3}$*. On the other hand, if* $x = 0$ *we have*

$$\frac{1}{C} \leq \Pi_{k=1}^{n} H(e) \leq C$$

which forces $H(e) = 1$*, and* H *cannot be continuous since* $H(x,x)$ *does not converge to* $H(0,0)$ *as* $(x,x) \to (0,0)$ *in* $\mathrm{Gr}(\sigma)$*.*

Example 3.2.22. *Weighted-orbit conjugacy does not imply branch-transition conjugacy, not even when the weights* w *and* u *are the constant* 1*. If we take* $X = [0,1]$ *and* $\sigma_1(x) = \chi_{[0,\frac{1}{2}]}(x) + 2(1-x)\chi_{(\frac{1}{2},1]}$*,* $\sigma_2(x) = 0$*, then the multivariable systems* $\sigma = (\sigma_1, \sigma_2, \sigma_2)$ *and* $\tau = (\sigma_1, \sigma_1, \sigma_2)$*, considered as WPS* (σ, w) *and* (τ, u) *with constant weights* $w = u = 1$*, are not branch conjugate. Indeed, suppose* σ *is graph conjugate to* τ *via* γ*. Then* $\gamma(1) = 1$ *as* 1 *is the only branching point for both* σ *and* σ'*, and so* γ *must be non-decreasing. Hence, we have that*

$$\frac{u^{\gamma}}{w}(y,x) = \begin{cases} 2 & \text{if } x < 1 \ \& \ y = \sigma_1(x) \\ \frac{1}{2} & \text{if } x < 1 \ \& \ y = \sigma_2(x) = 0 \\ 1 & \text{if } x = 1. \end{cases}$$

So we see that $\frac{u^{\gamma}}{w}$ *is not continuous at the branching edge* $e = (0,1)$*, so that* (σ, w) *and* (σ, u) *are not branch transition conjugate.*

However, we show that (σ, w) *and* (τ, u) *are weighted-orbit conjugate via* $\gamma = Id_{[0,1]}$*. Note first that* $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau)$ *so that* σ *and* τ *are graph conjugate via* Id_X*. Next, we define the following continuous* $H \in C(\mathrm{Gr}(\sigma))$ *by setting*

$$H(y,x) = \begin{cases} \frac{1}{2} & \text{if } x < \frac{1}{2} \ \& \ y = \sigma_1(x) \\ x & \text{if } \frac{1}{2} \leq x \leq 1 \ \& \ y = \sigma_1(x) \\ 2 & \text{if } x < \frac{1}{2} \ \& \ y = \sigma_2(x) = 0 \\ -2x + 3 & \text{if } \frac{1}{2} \leq x \leq 1 \ \& \ y = \sigma_2(x) = 0 \\ 1 & \text{if } x = 1 \end{cases}$$

and by our definition of H for $e \in \mathrm{Gr}(\sigma)$ with $s(e) \leq \frac{1}{2}$ we have $H(e)\frac{u}{w}(e) = 1$. The important thing to note here is that every path beginning at some $x \in [0,1]$ must be comprised, from the third edge on, by edges e with $r(e), s(e) \in \{0, \frac{1}{2}\}$. Indeed, if $\mu = \mu_n ... \mu_1$ is a path of length $|\mu| \geq 3$, suppose that $s(\mu_1) = x$, then $r(\mu_1) \in [0, \frac{1}{2}]$, and this forces $r(\mu_2) \in \{0, \frac{1}{2}\}$.

Hence, we see that for any $n \in \mathbb{N}$ and any path $\mu = \mu_n ... \mu_1$ we have

$$\Pi_{k=1}^{n} \frac{u(\mu_k) H(\mu_k)}{w(\mu_k)} = \Pi_{k=1}^{2} \frac{u(\mu_k) H(\mu_k)}{w(\mu_k)}$$

Since both H and $\frac{w}{u}$ have values only in the interval $[\frac{1}{2}, 2]$, we see that

$$\left(\frac{1}{2}\right)^4 \leq \Pi_{k=1}^{n} \frac{u(\mu_k) H(\mu_k)}{w(\mu_k)} \leq 2^4$$

and so (σ, w) and (σ, u) are weighted-orbit conjugate via $Id_{[0,1]}$.

It is easy to see that the conjugacy relations we have defined between two WPS have a natural hierarchy. By definition, if (σ, w) and (τ, u) are two WPS over X and Y respectively, then each condition below implies the one after it:

1. (σ, w) and (τ, u) are conjugate.
2. (σ, w) and (τ, u) are branch-transition conjugate.
3. (σ, w) and (τ, u) are weighted-orbit conjugate.
4. (σ, w) and (τ, u) are graph conjugate.

As we have seen, the different conjugacy relations are distinct, but in some subclasses, it is possible to identify some of them.

1. For partially defined continuous functions as in Example 3.2.6, graph conjugacy implies conjugacy.
2. For non-negative matrices as in Example 3.2.4, graph conjugacy implies branch-transition conjugacy.

In general we have the following in the case when there are no branching points, which tells us that information on the weights can only be detected if the WPS have branching points.

Corollary 3.2.23. *Let (σ, w) and (τ, u) be WPS over compact X and Y respectively. Suppose either σ or τ have no branching points. Then σ and τ are graph conjugate if and only if (σ, w) and (τ, u) are branch-transition conjugate.*

Proof. If σ and τ are graph conjugate and either σ or τ have no branching points, then both have no branching points by Remark 3.2.12. Hence by Proposition 3.2.16 we see that both w and u are continuous, and so for a homeomorphism $\gamma : X \to Y$ such that $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau^\gamma)$ we have that $\frac{u^\gamma}{w}$ is continuous, and (σ, w) is branch-transition conjugate to (τ, u). □

Our final goal for this section is to identify isomorphism classes of the C*-correspondence associated to a WPS (σ, w).

For the GNS correspondence $F_{\sigma,w} := F_{P(\sigma,w)}$, for any $f, g, h, k \in C(X)$ the inner product formula and bimodule actions for simple tensors are given respectively by

$$\langle f \otimes g, h \otimes k\rangle(x) = \sum_{i:x\in X_i} \overline{g(x)f(\sigma_i(x))} w_i(x)h(\sigma_i(x))k(x) \quad \text{and} \quad f \cdot (g \otimes h) \cdot k = fg \otimes hk.$$

Next, we denote by $C(\sigma, w) := \mathcal{X}_{\mathcal{Q}(\sigma,w)}$ the quiver correspondence of $\mathcal{Q}(\sigma, w)$. The notation for weights of edges gives a nice formula for the Radon measures $P(\sigma, w)$ by $P(\sigma, w)_x = \sum_{s(e)=x} w(e)\delta_e$, so that for any $\xi, \eta \in C(\mathrm{Gr}(\sigma))$ and $f, g \in C(X)$ we have left and right $C(X)$ actions given by

$$(f \cdot \xi \cdot g)(e) = f(r(e))\xi(e)g(s(e))$$

and inner product

$$\langle \xi, \eta\rangle_w(x) = \sum_{s(e)=x} \overline{\xi(e)} w(e)\eta(e) = \sum_{i:x\in X_i} \overline{\xi(\sigma_i(x), x)} w_i(x)\eta(\sigma_i(x), x).$$

We denote $f \odot g \in C(\mathrm{Gr}(\sigma))$ the function given by $(f \odot g)(e) = f(r(e))g(s(e))$, which "behaves like" the element $f \otimes g$ in $F_{\sigma,w}$, as the following proposition demonstrates.

Proposition 3.2.24. *Let (σ, w) be a d-variable WPS on a compact space X. Then the map $f \otimes g \mapsto f \odot g$ uniquely extends to a unitary isomorphism between $F_{\sigma,w}$ and $C(\sigma, w)$. Moreover, the supremum norm on $C(\mathrm{Gr}(\sigma))$ and the norm induced by the inner product on $C(\sigma, w)$ are equivalent, so that $C(\mathrm{Gr}(\sigma))$ is complete with respect to the norm induced by the inner product.*

Proof. The first assertion of the proposition follows easily from Proposition 2.3.10. For the second part, we show that the norm $\|\cdot\|_w := \|\langle\cdot,\cdot\rangle_w^{\frac{1}{2}}\|$ defined on $C(\mathrm{Gr}(\sigma))$ is equivalent to the supremum norm on it. Indeed, for $\xi \in C(\mathrm{Gr}(\sigma))$ we have

$$\sup_{i,\ x\in X_i} w_i(x)|\xi(\sigma_i(x),x)|^2 \leq \sup_x \sum_{i:x\in X_i} w_i(x)|\xi(\sigma_i(x),x)|^2 \leq d\cdot \sup_{i,\ x\in X_i} w_i(x)|\xi(\sigma_i(x),x)|^2.$$

Since for each $1 \leq i \leq d$ we have that w_i is positive and continuous on X_i, and X_i is compact, there exists $C > 0$ such that for all $1 \leq i \leq d$ and $x \in X_i$ we have $\frac{1}{C} \leq w_i(x) \leq C$, so that

$$\frac{1}{C}\cdot\|\xi\|^2_{\mathrm{Gr}(\sigma)} \leq \|\xi\|^2_w \leq dC\cdot\|\xi\|^2_{\mathrm{Gr}(\sigma)}.$$

□

As it turns out, the notation for paths in topological graphs is a good fit for computing the internal tensor iterates of $C(\sigma, w)$.

Recall that the collection of paths in $\mathcal{Q}(\sigma, w)$ of length n is given by

$$Gr(\sigma^n) := \{\ \mu = \mu_n...\mu_1 \mid r(\mu_k) = s(\mu_{k+1})\ \ \forall 1 \leq k < n\ \}.$$

These can alternatively be identified with the closed set of orbits of length $n+1$ given by $n+1$-tuples $(x_{n+1}, x_n, ..., x_1)$ in X^{n+1} such that for all $1 \leq m < n$ there is some $1 \leq i \leq d$ such that $\sigma_i(x_m) = x_{m+1}$ and $x_m \in X_i$.

Next, for functions $\xi, \eta \in C(\mathrm{Gr}(\sigma^n))$ and $f, g \in C(X)$, left and right actions of $C(X)$ on $C(\mathrm{Gr}(\sigma^n))$ are given by

$$(f\cdot\xi\cdot g)(\mu) = f(r(\mu))\xi(\mu)g(s(\mu))$$

and the inner product by

$$\langle\xi,\eta\rangle(x) = \sum_{s(\mu)=x} \overline{\xi(\mu)}w(\mu)\eta(\mu)$$

where $w(\mu) := w(\mu_n)\cdot ... \cdot w(\mu_1)$ is the extended definition of the weights of edges to weights of paths. Since $\mathcal{Q}(\sigma, w)$ has open source map and $\mathrm{supp}(P(\sigma, w)_x) = s^{-1}(x)$ for every $x \in X$, by the discussion at the beginning of Section 6 of [93], the above yields the C*-correspondence $\mathcal{X}_{\mathcal{Q}(\sigma,w)^n}$ associated to the topological quiver $\mathcal{Q}(\sigma, w)^n$ on the space of n-paths. By the discussion preceding [93, Remark 6.3] we then get that the C*-correspondences $\mathcal{X}^{\otimes n}_{\mathcal{Q}(\sigma,w)}$ and $\mathcal{X}_{\mathcal{Q}(\sigma,w)^n}$ are Id-unitarily isomorphic via a map sending simple tensors $\xi_n \otimes ... \otimes \xi_1$ to the function $\mu_n...\mu_1 \mapsto \xi_n(\mu_n)\cdot ...\xi_1(\mu_1)$.

Proposition 3.2.25. *Let (σ, w) be a d-variable WPS. The map sending simple tensors $\xi_n \otimes ... \otimes \xi_1 \in C(\sigma, w)^{\otimes n}$ to the function $\xi_n \odot ... \odot \xi_1 : \mu_n...\mu_1 \mapsto \xi_n(\mu_n) \cdot ... \cdot \xi_1(\mu_1)$ in $C(\mathrm{Gr}(\sigma^n))$ extends uniquely to an (Id-)unitary isomorphism between $C(\sigma, w)^{\otimes n}$ and $\mathcal{X}_{\mathcal{Q}(\sigma,w)^n}$. Moreover, the supremum norm on $C(\mathrm{Gr}(\sigma^n))$ and the norm induced by the inner product on $\mathcal{X}^{\otimes n}_{\mathcal{Q}(\sigma,w)}$ are equivalent. So in particular, $C(\mathrm{Gr}(\sigma^n))$ is complete with respect to the norm induced by the inner product.*

Proof. The first part follows from the preceding discussion. For the second part of the proposition, we note that for $x \in X$, the number of paths of length n emanating from x is at most d^n, and so, for every element $\xi \in C(\mathrm{Gr}(\sigma^n))$, and an arbitrary path $\mu = \mu_n...\mu_1$ of length n emanating from $s(\mu)$ we have

$$\xi(\mu)w(\mu) \leq \sum_{s(\nu)=s(\mu)} |\xi(\nu)|^2 w(\nu) \leq d^n \sup_{\nu \in \mathrm{Gr}(\sigma^n)} \xi(\nu)w(\nu).$$

By Proposition 3.2.16 there is some $K > 0$ such that $\frac{1}{K^n} \leq w(\nu) = w(\nu_n)...w(\nu_1) \leq K^n$ for any $\nu = \nu_n...\nu_1 \in \mathrm{Gr}(\sigma^n)$ so that

$$\frac{1}{K^n}\xi(\mu) \leq \sup_{x \in X} \sum_{s(\nu)=x} |\xi(\nu)|^2 w(\nu) \leq d^n K^n \cdot \sup_{\nu \in \mathrm{Gr}(\sigma^n)} \xi(\nu).$$

Since μ was an arbitrary path, we see that

$$\frac{1}{K^n} \sup_{\mu \in \mathrm{Gr}(\sigma^n)} \xi(\mu) \leq \sup_{x \in X} \sum_{s(\nu)=x} |\xi(\nu)|^2 w(\nu) \leq d^n K^n \cdot \sup_{\nu \in \mathrm{Gr}(\sigma^n)} \xi(\nu)$$

and so the norm induced by the inner product and the supremum norm on $\mathcal{X}^{\otimes n}_{\mathcal{Q}(\sigma,w)} = C(\mathrm{Gr}(\sigma^n))$ are equivalent. □

We now characterize branch-transition / weighted-orbit conjugacy in terms of unitary isomorphism / similarity of associated C*-correspondences respectively.

For γ implementing branch-transition / weighted-orbit conjugacy and ρ implementing unitary isomorphism / similarity, we may often assume without loss of generality that $\gamma = Id_X$ and / or that $\rho = Id_{C(X)}$.

Indeed, if $V : C(\sigma, w) \to C(\tau, u)$ is a ρ-bimodule map, with $\gamma : Y \to X$ the homeomorphism such that $\rho(f) = f \circ \gamma^{-1}$, we may define a ρ-unitary $\widetilde{\rho} : C(\tau^\gamma, u^\gamma) \to C(\tau, u)$ given by $\widetilde{\rho}(\xi)(y, v) = \xi(\gamma^{-1}(y), \gamma^{-1}(v))$. $\widetilde{\rho}$ then satisfies $\widetilde{\rho^{-1}} = \widetilde{\rho}^{-1}$ where $\widetilde{\rho^{-1}}$ is a ρ^{-1}-unitary.

Hence, by composing we get an Id-bimodule map $\widetilde{\rho}^{-1} \circ V : C(\sigma, w) \to C(\tau^\gamma, u^\gamma)$, and V is a ρ-similarity / ρ-unitary if and only if $\widetilde{\rho}^{-1} \circ V$ is an Id-similarity / Id-unitary respectively.

Further, on the conjugacy side, note that (σ, w) and (τ, u) are graph / weighted-orbit / branch-transition conjugate via γ if and only if (σ, w) and (τ^γ, u^γ) are graph / weighted-orbit / branch transition conjugate via Id_X respectively.

Proposition 3.2.26. *Let (σ, w) and (τ, u) be WPS on compact spaces X and Y respectively. Suppose that $\gamma : X \to Y$ is a homeomorphism and $\rho : C(X) \to C(Y)$ is the *-isomorphism given by $\rho(f) = f \circ \gamma^{-1}$.*

1. *If (σ, w) and (τ, u) are weighted-orbit conjugate with $C \geq 1$ via γ, then there exists a ρ-similarity $V : C(\sigma, w) \to C(\tau, u)$ with*
$$\sup_n \max\{\|V^{\otimes n}\|^2, \|(V^{-1})^{\otimes n}\|^2\} \leq C.$$
2. *If $V : C(\sigma, w) \to C(\tau, u)$ is a ρ-similarity, then (σ, w) and (τ, u) are weighted-orbit conjugate via γ and constant*
$$C = \sup_n \max\{\|V^{\otimes n}\|^2, \|(V^{-1})^{\otimes n}\|^2\}.$$

Proof. We first show (1). Assume without loss of generality that $\gamma = Id_X$, so that $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau)$. Let $H \in C(\mathrm{Gr}(\sigma))$ be such that for any path $\mu = \mu_n ... \mu_1$ we have

$$\frac{1}{C} \leq \Pi_{k=1}^n \frac{u}{w}(\mu_k) H(\mu_k) \leq C.$$

We define $V : C(\sigma, w) \to C(\tau, u)$ by setting $V(\xi)(e) = \xi(e)\sqrt{H(e)}$. It is easily seen that V is a $C(X)$-bimodule map, and we show that V is an Id-isomorphism. Indeed, for $\xi \in C(\mathrm{Gr}(\sigma))$ we have

$$\|V(\xi)\|^2 = \sup_{x \in X} \sum_{s(e)=x} |\xi(e)|^2 u(e) H(e) \leq C \cdot \sup_{x \in X} \sum_{s(e)=x} |\xi(e)|^2 w(e) = C \|\xi\|^2$$

and the symmetric argument shows that $\|\xi\| \leq C \|V(\xi)\|$. Hence $V : C(\sigma, w) \to C(\tau, u)$ is an Id-isomorphism. To show that it is an Id-similarity, we repeat the above for the tensor iterates which are identified with $C(\mathrm{Gr}(\sigma^n)) = \mathcal{X}_{\mathcal{Q}(\sigma,w)}$ for $n \in \mathbb{N}$ by Proposition 3.2.25.

Indeed, fix $n \in \mathbb{N}$, and $\xi \in C(\mathrm{Gr}(\sigma^n))$. By Proposition 3.2.25 and the definition of V, we must have that $V^{\otimes n}(\xi)(\mu_n ... \mu_1) = \xi(\mu_n ... \mu_1)\Pi_{k=1}^n \sqrt{H(\mu_k)}$. Thus, we compute,

$$\|V^{\otimes n}(\xi)\|^2 = \sup_{x \in X} \sum_{s(\mu_n ... \mu_1) = x} |\xi(\mu_n ... \mu_1)|^2 \Pi_{k=1}^n H(\mu_k) u(\mu_k) \leq$$

$$C \cdot \sup_{x \in X} \sum_{s(\mu_n ... \mu_1) = x} |\xi(\mu_n ... \mu_1)|^2 \Pi_{k=1}^n w(\mu_k) = C\|\xi\|^2.$$

So that V is tensor-power bounded by $\sqrt{C}$ and the symmetric argument shows that V^{-1} is also tensor power bounded by $\sqrt{C}$.

We now show (2). Without loss of generality we assume that $\rho = Id_{C(X)}$ (so that we need $\gamma = Id_X$). Denote by $\zeta = V(1 \odot 1) \in C(\tau, u)$. For any $f, g \in C(X)$ we have $f \cdot \zeta \cdot g = V(f \odot g)$ and then

$$\sup_{x \in X} \sum_{s(e) = x} |f(r(e))|^2 |\zeta(e)|^2 |g(s(e))|^2 u(e) = \|V(f \odot g)\| \leq$$

$$\|V\| \|f \odot g\| = \|V\| \sup_{x \in X} \sum_{s(e) = x} |f(r(e))|^2 |g(s(e))|^2 w(e)$$

so we see that for $(y, x) \in \mathrm{Gr}(\sigma)$, by taking infimum over $f, g : X \to [0, 1]$ with $f(y) = 1$ and $g(x) = 1$ which vanish outside arbitrarily small neighborhoods of y and x respectively, we have that $(y, x) \in \mathrm{Gr}(\tau)$, for otherwise the right hand side would vanish while the left hand side would not. The symmetric argument then shows that $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau)$, and σ and τ are graph conjugate via Id_X.

By Proposition 3.2.24, convergence in $C(\sigma, w)$ is equivalent to uniform convergence on $C(\mathrm{Gr}(\sigma))$, and since $\zeta(e) \cdot (f \odot g)(e) = (f \cdot \zeta \cdot g)(e) = V(f \odot g)(e)$ for every $e \in \mathrm{Gr}(\sigma)$, we then must have that $V(\xi)(e) = \zeta(e) \cdot \xi(e)$ for every $\xi \in C(\mathrm{Gr}(\sigma))$ and $e \in \mathrm{Gr}(\sigma)$.

Next, since for every $\xi_k \in C(\mathrm{Gr}(\sigma)) = C(\mathrm{Gr}(\tau))$ for $1 \leq k \leq n$ we have that

$$\|\xi_n \otimes ... \otimes \xi_1\|^2 \leq \|(V^{-1})^{\otimes n}\|^2 \|V^{\otimes n}(\xi_n \otimes ... \otimes \xi_1)\|^2$$

and

$$\|V^{\otimes n}(\xi_n \otimes ... \otimes \xi_1)\|^2 \leq \|V^{\otimes n}\|^2 \|\xi_n \otimes ... \otimes \xi_1\|^2.$$

We obtain that

$$\sup_{x \in X} \sum_{s(\mu_n ... \mu_1) = x} \Pi_{k=1}^n |\xi_k(\mu_k)|^2 w(\mu_k) \leq$$

$$\|(V^{-1})^{\otimes n}\|^2 \sup_{x\in X} \sum_{s(\mu_n...\mu_1)=x} \Pi_{k=1}^n |\xi_k(\mu_k)|^2 |\zeta(\mu_k)|^2 u(\mu_k)$$

and

$$\sup_{x\in X} \sum_{s(\mu_n...\mu_1)=x} \Pi_{k=1}^n |\xi_k(\mu_k)|^2 |\zeta(\mu_k)|^2 u(\mu_k) \leq$$

$$\|V^{\otimes n}\|^2 \sup_{x\in X} \sum_{s(\mu_n...\mu_1)=x} \Pi_{k=1}^n |\xi_k(\mu_k)|^2 w(\mu_k).$$

If we *fix* a path $\nu = \nu_n...\nu_1$, we can take infimum over functions ξ_k that vanish outside arbitrarily small neighborhoods of ν_k for each k and are equal to 1 at ν_k, to get

$$\Pi_{k=1}^n w(\nu_k) \leq \|(V^{-1})^{\otimes n}\|^2 \Pi_{k=1}^n |\zeta(\nu_k)|^2 u(\nu_k)$$

and

$$\Pi_{k=1}^n |\zeta(\nu_k)|^2 u(\nu_k) \leq \|V^{\otimes n}\|^2 \Pi_{k=1}^n w(\nu_k)$$

so that with

$$C = \max\{\sup_n \|V^{\otimes n}\|^2, \sup_n \|(V^{-1})^{\otimes n}\|^2\}$$

which by our assumptions is finite, we get

$$\frac{1}{C} \leq \frac{\Pi_{k=1}^n H(\nu_k) u(\nu_k)}{\Pi_{k=1}^n w(\nu_k)} = \Pi_{k=1}^n \frac{u}{w}(\nu_k) H(\nu_k) \leq C$$

where $H = |\zeta|^2 \in C(\mathrm{Gr}(\sigma))$, as required. □

As a corollary to Proposition 3.2.26, we obtain a characterization for branch-transition conjugacy.

Corollary 3.2.27. *Let (σ, w) and (τ, u) be WPS on compact spaces X and Y respectively. Suppose that $\gamma : X \to Y$ is a homeomorphism and $\rho : C(X) \to C(Y)$ is the *-isomorphism given by $\rho(f) = f \circ \gamma^{-1}$.*

1. *If (σ, w) and (τ, u) are branch-transition conjugate via γ, then there exists a ρ-unitary $U : C(\sigma, w) \to C(\tau, u)$.*
2. *If $U : C(\sigma, w) \to C(\tau, u)$ is a ρ-unitary, then (σ, w) and (τ, u) are branch-transition conjugate via γ.*

Proof. To show (1), we use Corollary 3.2.19 to see that $H = \frac{w}{u^\gamma}$ is continuous, and realizes weighted-orbit conjugacy with $C = 1$, so that the ρ-similarity U arising from Proposition 3.2.26 satisfies $\|U\|, \|U^{-1}\| \leq 1$ and is hence a ρ-unitary.

To show (2), without loss of generality we assume that $\rho = Id_{C(X)}$ (So that we need $\gamma = Id_X$). Denote by $\zeta = V(1 \odot 1) \in C(\tau, u)$. For any $f, g \in C(X)$ we have $f \cdot \zeta \cdot g = U(f \odot g)$ and then

$$\sup_{x \in X} \sum_{s(e)=x} |f(r(e))|^2 |\zeta(e)|^2 |g(s(e))|^2 u(e) = \|U(f \odot g)\| =$$

$$\|f \odot g\| = \sup_{x \in X} \sum_{s(e)=x} |f(r(e))|^2 |g(s(e))|^2 w(e)$$

so we see that for $e = (y, x) \in \mathrm{Gr}(\sigma)$, by taking infimum over $f, g : X \to [0, 1]$ with $f(y) = 1$ and $g(x) = 1$ which vanish outside arbitrarily small neighborhoods of y and x respectively, we obtain $|\zeta(e)|^2 u(e) = w(e)$ so that $e \in \mathrm{Gr}(\sigma)$ if and only if $e \in \mathrm{Gr}(\tau)$ and σ and τ are graph conjugate via Id. Moreover, since $\frac{u}{w} = \frac{1}{|\zeta|^2}$ is a continuous function on $C(\mathrm{Gr}(\sigma))$, it must be continuous on each branching edge in particular, and hence (σ, w) and (τ, u) are branch-transition conjugate. □

Example 3.2.28. *As a consequence of Proposition 3.2.26 and Corollary 3.2.27 we see from Example 3.2.22 that there are WPS which have similar C*-correspondences that can not be unitarily isomorphic. In particular, by Remark 2.2.6 we see that between the two correspondences arising from the weighted multivariable systems of Example 3.2.22, no ρ-isomorphism can be ρ-adjointable.*

Remark 3.2.29. Using the theory we have developed so far, and the first part of Corollary 3.2.19, one can show that for two WPS (σ, w) and (τ, u) over compact spaces X and Y respectively, we have that σ and τ are graph conjugate via γ if and only if $C(\sigma, w)$ and $C(\tau, u)$ are ρ-isomorphic. Hence, ρ-isomorphism does not detect any information regarding the weights of the WPS, and only detects the graphs of the systems.

3.3 Tensor algebras

In this section we relate isomorphisms of product systems to graded isomorphisms, and to *semigraded* isomorphism of associated tensor algebras.

Definition 3.3.1. *Let E and F be C*-correspondences over $\mathcal{A}$ and $\mathcal{B}$ respectively. An isomorphisms $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ that satisfies $\varphi(\mathcal{T}_+(E)_n) = \mathcal{T}_+(F)_n$ for all $n \in \mathbb{N}$ is called* graded.

For a C*-correspondence E, let $\Psi_E : E \to \mathcal{T}_+(E)_1$ be the isometric Banach bimodule isomorphism given by $\Psi_E(\xi) = S_\xi^{(1)}$. In the following theorem it is important that we do not require ρ-similarities to be adjointable in item (2) of Definition 2.2.5.

Theorem 3.3.2. *Let E and F be be C*-correspondences over commutative C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. Then,*

1. *If $V : E \to F$ is a ρ-similarity for some $*$-isomorphism ρ between $\mathcal{A}$ and $\mathcal{B}$, then there exists a graded completely bounded isomorphism $\operatorname{Ad}_V : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ such that $\operatorname{Ad}_V|_{\mathcal{A}} = \rho$ with*

$$\max\{\| \operatorname{Ad}_V \|_{cb}, \| \operatorname{Ad}_V^{-1} \|_{cb}\} \leq \sup_{n\in\mathbb{N}} \|V^{\otimes n}\| \cdot \sup_{n\in\mathbb{N}} \|(V^{-1})^{\otimes n}\|.$$

2. *If $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ is a bounded graded isomorphism, then $\rho_\varphi := \varphi|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ is a *-isomorphism and $V_\varphi : E \to F$ uniquely determined by $S_{V_\varphi(\xi)} = \varphi(S_\xi)$ for $\xi \in E$ yields a ρ_φ-similarity satisfying*

$$\sup_{n\in\mathbb{N}} \|(V_\varphi)^{\otimes n}\| \leq \|\varphi\| \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|(V_\varphi^{-1})^{\otimes n}\| \leq \|\varphi^{-1}\|.$$

Moreover, the operations (1) and (2) are inverses of each other in the sense that $\varphi = \operatorname{Ad}_{V_\varphi}$ and $V = V_{\operatorname{Ad}_V}$, and in particular every bounded graded isomorphism φ is completely bounded with $\|\varphi\|_{cb} \leq \|\varphi\| \cdot \|\varphi^{-1}\|$.

Proof. (1) Suppose $V : E \to F$ is a ρ-similarity. Define a bounded ρ-bimodule map W_V from $\mathcal{F}_E$ to $\mathcal{F}_F$ by $W_V = \oplus_{n=0}^{\infty} V^{\otimes n}$ which is well defined since $\sup_{n\in\mathbb{N}} \|V^{\otimes n}\| < \infty$. Furthermore, we have that W_V is invertible with $W_V^{-1} = W_{V^{-1}}$ (which is also a well-defined bounded ρ^{-1}-bimodule map since $\sup_{n\in\mathbb{N}} \|(V^{-1})^{\otimes n}\| < \infty$) and

$$\|W_V\| \cdot \|W_V^{-1}\| \leq \sup_{n\in\mathbb{N}} \|V^{\otimes n}\| \cdot \sup_{n\in\mathbb{N}} \|(V^{-1})^{\otimes n}\|$$

We define $\operatorname{Ad}_V : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ by setting $\operatorname{Ad}_V(T) = W_V T W_V^{-1}$ which then satisfies

$$\max\{\| \operatorname{Ad}_V \|_{cb}, \| \operatorname{Ad}_V^{-1} \|_{cb}\} \leq \sup_{n\in\mathbb{N}} \|V^{\otimes n}\| \cdot \sup_{n\in\mathbb{N}} \|(V^{-1})^{\otimes n}\|$$

(2) Now suppose that $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ is a bounded graded isomorphism. Note that ρ_φ is a *-isomorphism since $\mathcal{A}$ and $\mathcal{B}$ are assumed commutative. We define the map $V_\varphi : E \to F$ by $V_\varphi = \Psi_F^{-1} \varphi \Psi_E$ which is a ρ-correspondence map by virtue of gradedness

of φ and the fact that ρ_φ is a *-isomorphism. Then it is easily verified that for all $n \in \mathbb{N}$ we have $(V_\varphi)^{\otimes n} = (\Psi_F^{-1})^{\otimes n} \circ \varphi \circ (\Psi_E)^{\otimes n}$ so that $\|(V_\varphi)^{\otimes n}\| \leq \|\varphi\|$ and V_φ is tensor-power bounded with $\sup_n \|(V_\varphi)^{\otimes n}\| \leq \|\varphi\|$. One then shows that $V_\varphi^{-1} = V_{\varphi^{-1}} = \Psi_E^{-1}\varphi^{-1}\Psi_F$ is also a ρ^{-1}-correspondence map which is similarly tensor power-bounded with $\sup_n \|(V_\varphi^{-1})^{\otimes n}\| \leq \|\varphi^{-1}\|$, as required. □

We then get as an easy corollary, the corresponding theorem for the isometric case.

Theorem 3.3.3. *Let E and F be C*-correspondences over commutative C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively.*

1. *If $U : E \to F$ is a ρ-unitary for some $*$-isomorphism ρ between $\mathcal{A}$ and $\mathcal{B}$, then there exists a graded completely isometric isomorphism $\mathrm{Ad}_U : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ with $\mathrm{Ad}_U|_{\mathcal{A}} = \rho$.*

2. *If $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ is a graded isometric isomorphism, then ρ_φ is a $*$-isomorphism and there exists a ρ_φ-unitary $U_\varphi : E \to F$ with $\varphi|_{\mathcal{A}} = \rho_\varphi$.*

Moreover, the operations (1) and (2) are inverses of each other in the sense that $\varphi = \mathrm{Ad}_{V_\varphi}$ and $V = V_{\mathrm{Ad}_V}$, and in particular every isometric graded isomorphism φ must be completely isometric.

We consider two special classes of isomorphisms which will provide a convenient framework for addressing isomorphism problems. When our isomorphisms fit into these classes, we can often use this to extract information more readily from the isomorphism.

Notation 3.3.4. *If E and F are C*-correspondences over C*-algebra $\mathcal{A}$ and $\mathcal{B}$ respectively and $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ is an algebraic isomorphism, we denote by $\rho_\varphi := \Phi_0 \circ \varphi|_{\mathcal{A}}$ the homomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

Definition 3.3.5. *Let E and F be C*-correspondences over C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. We say that an algebraic isomorphism $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ is* base-detecting *if $\rho_\varphi : \mathcal{A} \to \mathcal{B}$ is a *-isomorphism and $\rho_\varphi^{-1} = \rho_{\varphi^{-1}}$.*

Base detection is usually the first thing we check for, since it usually implies that the base algebras can be detected from the isomorphism.

We note that for a graded isomorphism φ as considered in Theorem 3.3.2, ρ_φ is automatically an isomorphism, and since it is between commutative C*-algebras, ρ_φ has to be a $*$-isomorphisms. This means that graded isomorphisms are always base-detecting.

Isometric isomorphisms are also automatically base detecting. Indeed, let E and F be C*-correspondences over C*-algebras $\mathcal{A}$ and $\mathcal{B}$ and let $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ be an isometric isomorphism. Since $\mathcal{T}_+(F) \subseteq \mathcal{T}(F)$, we can regard φ as a map into the Toeplitz C*-algebra. Thus, $\varphi|_{\mathcal{A}} : \mathcal{A} \to \mathcal{T}(F)$ is an isometric homomorphism, and is hence necessarily positive and preserves the involution from $\mathcal{A}$ to $\mathcal{T}(F)$. Thus, $\varphi(\mathcal{A}) = \varphi(\mathcal{A})^* \subseteq \mathcal{T}_+(F)^* \subseteq \mathcal{T}(F)$, and we must have that $\varphi(\mathcal{A}) \subseteq \mathcal{T}_+(F) \cap \mathcal{T}_+(F)^* = \mathcal{B}$. Thus we have in fact that $\varphi(\mathcal{A}) \subseteq \mathcal{B}$, and the symmetric argument shows that $\varphi^{-1}(\mathcal{B}) \subseteq \mathcal{A}$, and so $\rho_{\varphi^{-1}}$ is the inverse of ρ_φ, and φ is base-detecting.

In the forthcoming definition, we relax the assumption of gradedness of an isomorphism while maintaining base-detection. The following concept of semi-gradedness appeared in the work of Muhly and Solel in section 5 of [92, Section 5] where they resolve the isometric isomorphism problem for tensor algebras arising from *aperiodic* C*-correspondences, and was also used in [42] to provide classification for tensor algebras arising from stochastic matrices, in terms of the matrices.

Definition 3.3.6. *Let E and F be C*-correspondences over C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, and suppose $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ is an algebraic isomorphism. We say that φ is* semi-graded *if $\varphi(\operatorname{Ker} \Phi_0) = \operatorname{Ker} \Phi_0$.*

The following has similar proof to the one in [42, Proposition 6.15], but we provide it for posterity.

Proposition 3.3.7. *Let E and F be C*-correspondences over commutative C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, and $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ is a semi-graded bounded isomorphism. Then φ is automatically base-detecting.*

Proof. Let Φ_0^E and Φ_0^F denote the conditional expectations on $\mathcal{T}_+(E)$ and $\mathcal{T}_+(F)$ respectively. As φ is semi-graded, for any $T \in \mathcal{T}_+(E)$ we have,

$$\Phi_0^F \varphi(T) = \Phi_0^F \varphi \Phi_0^E(T)$$

Hence, we must have that $\rho_\phi = \Phi_0^F \varphi|_{\mathcal{A}}$ is surjective. The same argument then works for φ^{-1}, and we have for every $a \in \mathcal{A}$ that

$$\rho_{\varphi^{-1}} \circ \rho(a) = \Phi_0^E \varphi^{-1} \Phi_0^F \varphi(a) = \Phi_0^E \varphi^{-1} \varphi(a) = a$$

Thus, we see that $\rho_{\varphi^{-1}} = (\rho_\varphi)^{-1}$. As $\mathcal{A}$ and $\mathcal{B}$ are commutative, ρ_φ and $\rho_{\varphi^{-1}}$ must both be contractive, and hence *-preserving, so that φ is base-detecting. □

Definition 3.3.8. *Let E be a C^*-correspondence over $\mathcal{A}$. The minimal degree of an element $0 \neq T \in \mathcal{T}_+(E)$, denoted* $\mathrm{md}(T)$*, is the smallest $n \in \mathbb{N}$ with $\Phi_n(T) \neq 0$.*

We will need the following criterion for semi-gradedness of *bounded* isomorphisms.

Proposition 3.3.9 (Criterion for semi-gradedness)**.** *Let E and F be C^*-correspondences over $\mathcal{A}$ and $\mathcal{B}$ respectively, and let $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ be a* bounded *base-detecting isomorphism. The following are equivalent:*

1. $\mathrm{md}(\varphi(T)) = \mathrm{md}(T)$ *for all $T \in \mathcal{T}_+(E)$.*
2. *φ is semi-graded.*
3. $\mathrm{md}(\varphi(S^{(1)}_\xi)) \geq 1$ *for every $\xi \in E$.*

Proof. It is clear that (1) implies (2) which implies (3).

We show that (3) implies (1). We first note that for $\eta \in E^{\otimes n}$ we have that $\mathrm{md}(\varphi(S^{(n)}_\eta)) \geq n$. Indeed, if we take $\eta = \xi_1 \otimes ... \otimes \xi_n$ with $\xi_i \in E$, since $S^{(n)}_\eta = S^{(1)}_{\xi_1} \cdot ... \cdot S^{(1)}_{\xi_n}$ we get $\mathrm{md}(S^{(n)}_\eta) \geq \mathrm{md}(S^{(1)}_{\xi_1}) + ... + \mathrm{md}(S^{(1)}_{\xi_n}) \geq n$. Next, since the collection of elements $\eta' := \sum_{i=1}^{\ell} \xi^{(i)}_1 \otimes ... \otimes \xi^{(i)}_n$ is dense in $E^{\otimes n}$, and as we saw, $\Phi_m(\varphi(S^{(n)}_{\eta'})) = 0$ for all $m < n$, by continuity of φ and Φ_m we get that $\Phi_m(\varphi(S^{(n)}_\eta)) = 0$ for *any* $\eta \in E^{\otimes n}$, so that $\mathrm{md}(\varphi(S^{(n)}_\eta)) \geq n$ for any $\eta \in E^{\otimes n}$.

We now show that $\mathrm{md}(\varphi(T)) \geq \mathrm{md}(T)$ for any $T \in \mathcal{T}_+(E)$. Indeed, let $T \in \mathcal{T}_+(E)$ be an operator with $\mathrm{md}(T) = n \geq 0$. Then we can write $T = \sum_{k=n}^{\infty} \Phi_k(T)$ as a norm converging Cesaro sum, and by boundedness of φ, we obtain that $\varphi(T) = \sum_{k=n}^{\infty} \varphi(\Phi_k(T))$ converging Cesaro. Since $\Phi_k(T) = S^{(k)}_{\xi_k}$ is of minimal degree at least $k \geq n$, so would be $\varphi(\Phi_k(T))$. Then, by continuity of Φ_k and φ, we have that $\varphi(T)$ is of minimal degree at least n, and we see that $\mathrm{md}(\varphi(T)) \geq \mathrm{md}(T)$.

To show that $\mathrm{md}(\varphi(T)) = \mathrm{md}(T)$, we will show that $\mathrm{md}(\varphi^{-1}(S^{(1)}_\xi)) \geq 1$ for $\xi \in F$ and repeat the above argument to get that $\mathrm{md}(\varphi^{-1}(T)) \geq \mathrm{md}(T)$. Together with the above we obtain $\mathrm{md}(\varphi(T)) = \mathrm{md}(T)$.

We show $\mathrm{md}(\varphi^{-1}(S^{(1)}_\xi)) \geq 1$ for $\xi \in F$. Indeed, let $\xi \in F$ and write $\varphi^{-1}(S^{(1)}_\xi) = a + T$ with $a \in \mathcal{A}$ and $\mathrm{md}(T) \geq 1$. We have already shown that $\mathrm{md}(\varphi(T)) \geq 1$, so that

$$\rho_\varphi(a) = \Phi_0(\varphi(a)) = \Phi_0(\varphi(a) + \varphi(T)) = \Phi_0(S^{(1)}_\xi) = 0$$

Since φ is base-detecting, ρ_φ is a *-isomorphism, and we have that $a = 0$. This means that $\mathrm{md}(\varphi^{-1}(S^{(1)}_\xi)) \geq 1$, and we are done. □

We next prove an analogue of [42, Proposition 6.17] in the discussion on semi-graded isomorphisms, that yields a reduction of our isomorphism problems.

Proposition 3.3.10. *Let E and F be C^*-correspondences over commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, and let $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ be a semi-graded* bounded *isomorphism. There is a unique bounded homomorphism $\widetilde{\varphi} : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ satisfying*

$$\widetilde{\varphi}(S_\xi^{(1)}) = \Phi_1(\varphi(S_\xi^{(1)})), \quad \xi \in E$$

and $\widetilde{\varphi}$ is a graded completely bounded isomorphism such that $\widetilde{\varphi}^{-1} = \widetilde{\varphi^{-1}}$, and $\|\widetilde{\varphi}\|_{cb} \leq \|\varphi\| \cdot \|\varphi^{-1}\|$.

Proof. First note that since φ is semi-graded, by Proposition 3.3.7 it must be base-detecting. Hence, by Proposition 3.3.9, for any $T \in \mathcal{T}_+(E)$ with $\mathrm{md}(T) = n$ we must have $\Phi_n^F \varphi(T) = \Phi_n^F \varphi \Phi_n^E(T)$. It follows that for all $n \in \mathbb{N}$ and any $S \in \mathcal{T}_+(E)_n$ we must have

$$S = \Phi_n^E(S) = \Phi_n^E \varphi^{-1} \varphi(S) = \Phi_n^E \varphi^{-1} \Phi_n^F \varphi(S) \tag{3.2}$$

Set $\rho = \rho_\varphi = \Phi_0^F \varphi|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$, which is a $*$-isomorphism, and define a ρ-bimodule map $V_n : E^{\otimes n} \to F^{\otimes}$ by setting $V_n(\xi) = (\Psi_n^F)^{-1} \Phi_n^F \varphi \Psi_n^E(\xi)$, where $\Psi_n(\xi) = S_\xi^{(n)}$. Note that V_n is clearly well-defined with $\|V_n\| \leq \|\varphi\|$ so that V_n is a bounded ρ-bimodule map. One similarly defines a bounded ρ^{-1}-bimodule map $V_n' : F^{\otimes} \to E^{\otimes n}$ satisfying $\|V_n'\| \leq \|\varphi^{-1}\|$ that satisfies $V_n^{-1} = V_n'$ by equation (3.2). We now wish to show that $V = V_1$ is a ρ-similarity, so we show that V is tensor power bounded by showing that $V_n = V^{\otimes n}$, and a similar argument would then work for V^{-1}. We show by induction that $V_n = V^{\otimes n}$. Indeed, suppose $V_k = V^{\otimes k}$ for all $k < n + m$ with $n, m \geq 1$. Let $\xi \in E^{\otimes n}$ and $\eta \in E^{\otimes m}$, then by semi-gradedness and the definition of V_n, V_m and V_{n+m}, we have the following chain of equalities

$$S_{V_{n+m}(\xi \otimes \eta)}^{(n+m)} = \Phi_{n+m}^F \varphi(S_{\xi \otimes \eta}^{(n+m)}) = (\Phi_n^F(\varphi(S_\xi^{(n)}))\Phi_m^F(\varphi(S_\eta^{(m)}))) =$$
$$S_{V_n(\xi)}^{(n)} S_{V_m(\eta)}^{(m)} = S_{V^{\otimes n}(\xi)}^{(n)} S_{V^{\otimes m}(\eta)}^{(m)} = S_{V^{\otimes(n+m)}(\xi \otimes \eta)}^{(n+m)}$$

so that by applying $(\Psi_{n+m}^F)^{-1}$ to both sides of this equation we obtain that $V_{n+m}(\xi \otimes \eta) = V^{\otimes(n+m)}(\xi \otimes \eta)$, so that $V_{n+m} = V^{\otimes(n+m)}$.

Thus, we have constructed a ρ-similarity $V : E \to F$ satisfying $S_{V(\xi)}^{(1)} = \Phi_1^F \varphi(S_\xi^{(1)})$ for all $\xi \in E$ with the tensor iterates of V and V^{-1} bounded in norm by the norms of φ and φ^{-1} respectively. By item (1) of Theorem 3.3.2 the ρ-similarity V promotes to a graded completely bounded isomorphism $\widetilde{\varphi} = \mathrm{Ad}_V : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ uniquely determined by satisfying $S_{V(\xi)}^{(1)} = \widetilde{\varphi}(S_\xi^{(1)})$ for all $\xi \in E$, with $\|\widetilde{\varphi}\| \leq \|\varphi\| \|\varphi^{-1}\|$. So we see that

$\widetilde{\varphi}(S_\xi^{(1)}) = \Phi_1^F \varphi(S_\xi^{(1)})$ for all $\xi \in E$ and that $\widetilde{\varphi}$ is uniquely determined by this property as required. ☐

Corollary 3.3.11. *Let E and F be C*-correspondences over commutative C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, and let $\varphi : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ be a semi-graded* isometric *isomorphism. There is a unique bounded homomorphism $\widetilde{\varphi} : \mathcal{T}_+(E) \to \mathcal{T}_+(F)$ satisfying*

$$\widetilde{\varphi}(S_\xi^{(1)}) = \Phi_1(\varphi(S_\xi^{(1)})), \quad \xi \in E$$

and $\widetilde{\varphi}$ is a graded completely isometric *isomorphism such that $\widetilde{\varphi}^{-1} = \widetilde{\varphi^{-1}}$.*

Remark 3.3.12. We developed the theory in this section for tensor algebras arising from a single C*-correspondence. We note, however, that everything proven in this section can be readily adapted to general subproduct systems where ρ-unitary / ρ-similarity of a single C*-correspondence is replaced with unitary isomorphism / isomorphism of subproduct systems respectively.

3.4 Character space

In this subsection, we adapt methods of Hadwin and Hoover [60], which were used in the solution of Arveson's conjugacy problem [34]. We compute the character space of $\mathcal{T}_+(\sigma, w) := \mathcal{T}_+(C(\sigma, w))$ for any WPS (σ, w), and then use this to show that every algebraic isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ is automatically base-detecting, so that the base spaces X and Y can be identified. Finally, we provide a criterion to detect semi-gradedness from the induced homeomorphism on the character spaces.

Let (σ, w) be a WPS on compact X. Denote by $\mathcal{M}(\sigma, w)$ the space of multiplicative linear functionals on $\mathcal{T}_+(\sigma, w)$ with its weak* topology. $\mathcal{M}(\sigma, w)$ is partitioned by X since for every $\theta \in \mathcal{M}(\sigma, w)$ there is a unique $x \in X$ such that $\theta|_{C(X)} = \delta_x$. We denote by $\mathcal{M}(\sigma, w)_x$ the weak* closed subset of $\theta \in \mathcal{M}(\sigma, w)$ satisfying $\theta|_{C(X)} = \delta_x$, and we let $\theta_{x,0}$ be the unique element in $\mathcal{M}(\sigma, w)_x$ such that $\theta_{x,0}(\operatorname{Ker} \Phi_0^P) = \{0\}$. Denote by $W_{(\sigma,w)} := S_1^{(1)} = S_{1\odot 1}^{(1)}$, the shift operator by the constant function $1 = 1 \odot 1 \in C(\sigma, w)$. Note that since $\operatorname{Ker} \Phi_0^P$ is the closed two sided ideal generated by $W_{(\sigma,w)}$, we have that $\theta_{x,0}$ is the unique element in $\mathcal{M}(\sigma, w)_x$ such that $\theta_{x,0}(W_{(\sigma,w)}) = 0$.

Definition 3.4.1. *Let σ be a d-variable partial system on compact X. We say that $x \in X$ is a* fixed point *for σ if $\sigma_i(x) = x \in X_i$ for some $1 \le i \le d$. We denote by* $\operatorname{Fix}(\sigma)$ *the closed set of fixed points of σ.*

Lemma 3.4.2. *Let (σ, w) be a WPS on compact X, $x \in X$, $\theta \in \mathcal{M}(\sigma, w)_x$ and $\xi \in C(\sigma, w)$. Then we have*

1. *If $x \in \mathrm{Fix}(\sigma)$ then $\theta(S_\xi^{(1)}) = \xi(x, x)\theta(W_{(\sigma,w)})$.*
2. *If $x \notin \mathrm{Fix}(\sigma)$ then $\theta(S_\xi^{(1)}) = 0$.*

In particular, when $x \notin \mathrm{Fix}(\sigma)$, we have $\mathcal{M}(\sigma, w)_x = \{\theta_{x,0}\}$.

Proof. First we show (1). Let $x \in X$ be a fixed point for σ. For every open neighborhood U of x, by Urysohn's Lemma, there is a continuous function $f_U : X \to [0, 1]$ with $f_U(x) = 1$ and $f_U(y) = 0$ for $y \notin U$. Thus, for $\theta \in \mathcal{M}(\sigma, w)_x$ and U, V open neighborhoods of x we have,

$$|\theta(S_\xi^{(1)} - \xi(x, x)W_{(\sigma,w)})|^2 = |\theta(f_U \cdot (S_\xi^{(1)} - \xi(x, x)W_{(\sigma,w)})) \cdot f_V)|^2 = |\theta(S_{f_U \cdot (\xi - \xi(x,x)1) \cdot f_V})|^2 \leq$$

$$\|f_U \cdot (\xi - \xi(x, x)1) \cdot f_V\|^2 = \sup_{y \in X} \sum_{i: y \in X_i} |f_U(\sigma_i(y))|^2 |\xi(\sigma_i(y), y) - \xi(x, x)|^2 |f_V(y)|^2 w_i(y) \leq$$

$$\sup_{y \in V} \sum_{i: y \in X_i} |f_U(\sigma_i(y))|^2 |\xi(\sigma_i(y), y) - \xi(x, x)|^2 w_i(y)$$

Taking infimum over all open neighborhoods V of x we get

$$|\theta(S_\xi^{(1)} - \xi(x, x)W_{(\sigma,w)})|^2 \leq \sum_{i: x \in X_i} |f_U(\sigma_i(x))|^2 |\xi(\sigma_i(x), x) - \xi(x, x)|^2 w_i(x) \leq$$

$$\sum_{i:\ x \in X_i,\ \sigma_i(x) \in U} |\xi(\sigma_i(x), x) - \xi(x, x)|^2 w_i(x)$$

Taking infimum over all U open neighborhoods of x, we obtain

$$|\theta(S_\xi^{(1)} - \xi(x, x)W_{(\sigma,w)})|^2 \leq \sum_{i: \sigma_i(x) = x \in X_i} |\xi(\sigma_i(x), x) - \xi(x, x)|^2 w_i(x) = 0$$

and we must have that $\theta(S_\xi^{(1)}) = \xi(x, x)W_{(\sigma,w)}$.

In order to show (2), note that if $x \notin \mathrm{Fix}(\sigma)$, a similar chain of inequalities, replacing $\xi(x, x)W_{(\sigma,w)}$ by 0 above, would yield that for all $\theta \in \mathcal{M}(\sigma, w)_x$ we have $\theta(S_\xi^{(1)}) = 0$.

Finally, if $x \notin \mathrm{Fix}(\sigma)$, we have that $\theta(W_{(\sigma,w)}) = 0$ for all $\theta \in \mathcal{M}(\sigma, w)_x$ so that $\theta(\mathrm{Ker}\,\Phi_0) = 0$ for all $\theta \in \mathcal{M}(\sigma, w)_x$. Now since $\theta_{x,0}$ is the only element in $\mathcal{M}(\sigma, w)_x$ with $\theta_{x,0}(\mathrm{Ker}\,\Phi_0) = 0$, we must then have that $\theta = \theta_{x,0}$ and $\mathcal{M}(\sigma, w)_x = \{\theta_{x,0}\}$. ∎

Now, in the case where $x \in X$ is a fixed point for σ, we are interested to know how $\theta \in \mathcal{M}(\sigma, w)_x$ acts on iterates $S_\xi^{(n)}$ for $\xi \in C(\sigma, w)^{\otimes n} \cong C(\mathrm{Gr}(\sigma^n))$. Recall the discussion preceding Proposition 3.2.25 where we identified $Gr(\sigma^n)$ with the collection of orbits of length $n+1$ inside X^{n+1}, that is the collection of sequences $(x_{n+1}, ..., x_1)$ such that for every $1 \leq m \leq n$ there is some $1 \leq i \leq d$ with $\sigma_i(x_m) = x_{m+1} \in X_i$.

Thus, take $\xi^{(1)}, ..., \xi^{(n)} \in C(\sigma, w)$ and note that by Lemma 3.4.2,

$$\theta(S^{(n)}_{\xi^{(1)} \otimes ... \otimes \xi^{(n)}}) = \theta(S^{(1)}_{\xi^{(1)}}) \cdot ... \cdot \theta(S^{(1)}_{\xi^{(n)}}) = \xi^{(n)}(x,x) \cdot ... \cdot \xi^{(1)}(x,x) \cdot \theta(W_{(\sigma,w)})^n.$$

By supremum norm approximation we obtain for every $\xi \in C(\mathrm{Gr}(\sigma^n))$ that

$$\theta(S_\xi^{(n)}) = \xi(x, ..., x) \cdot \theta(W_{(\sigma,w)})^n$$

due to density of the linear span of elements of the form $\xi^{(1)} \otimes ... \otimes \xi^{(n)}$ in $C(\sigma, w)^{\otimes n} \cong C(\mathrm{Gr}(\sigma^n))$, with the supremum norm, established by Proposition 3.2.25.

The next proposition is an adaptation of the methods of [34, Section 3], originally used by Hadwin and Hoover in [60]. For a WPS (σ, w), recall that we defined the weight of an edge $(y, x) \in \mathrm{Gr}(\sigma)$ to be $w(y, x) = \sum_{i:\sigma_i(x)=y,\ x \in X_i} w_i(x)$.

Proposition 3.4.3. *Let X be a compact space, (σ, w) a WPS on X and $x \in \mathrm{Fix}(\sigma)$. Then $\mathcal{M}(\sigma, w)_x \cong \overline{\mathbb{D}}_{r_x^w}$ via the map $\theta \mapsto \theta(W_{(\sigma,w)})$, where $\overline{\mathbb{D}}_{r_x^w}$ is the closed disc of radius $r_x^w = \sup_{\theta \in \mathcal{M}(\sigma,w)_x} |\theta(W_{(\sigma,w)})| = \sqrt{w(x,x)}$.*

Moreover if $\Theta_x : \overline{\mathbb{D}}_{r_x^w} \to \mathcal{M}(\sigma, w)_x$ is the homeomorphism above, then it is in fact pointwise analytic on $\mathbb{D}_{r_x^w}$, in the sense that for every $T \in \mathcal{T}_+(\sigma, w)$, the function $\Theta_x(\cdot)(T) : \mathbb{D}_{r_x^w} \to \mathbb{C}$ is analytic.

Proof. We first define a character $\theta_{x,z}$ for every $z \in \mathbb{C}$ with $|z| < \sqrt{w(x,x)}$. Let $T \in \mathcal{T}_+(\sigma, w)$. By Proposition 2.2.14 we get that T has a Fourier series representation as $T = \sum_{n=0}^{\infty} S^{(n)}_{\xi_n}$ converging Cesaro. We then define

$$\theta_{x,z}(T) = \sum_{n=0}^{\infty} \xi_n(x, ..., x) z^n.$$

Since $|z| < \sqrt{w(x,x)}$ and $|\xi_n(x, ..., x)| \leq \frac{\|\xi_n\|}{\sqrt{w(x,x)^n}} = \frac{\|\Phi_n(T)\|}{\sqrt{w(x,x)^n}}$ we get

$$|\theta_{x,z}(T)| \leq \sum_{n=0}^{\infty} |\xi_n(x, ..., x)||z|^n \leq \sum_{n=0}^{\infty} \|\Phi_n(T)\| \Big(\frac{|z|}{\sqrt{w(x,x)}} \Big)^n \leq \|T\| \sum_{n=0}^{\infty} \Big(\frac{|z|}{\sqrt{w(x,x)}} \Big)^n$$

so that the above is a well-defined multiplicative linear functional on $\mathcal{T}_+(\sigma, w)$. Indeed, $\theta_{x,z}$ is linear and multiplicative due to multiplication of Fourier series given in Proposition 2.2.14 and due to the identification of Proposition 3.2.25.

We show that for every $\theta \in \mathcal{M}(\sigma, w)_x$, one must have $|\theta(W_{(\sigma,w)})| \leq \sqrt{w(x,x)}$. Indeed, for every open neighborhood of x, by Urysohn's Lemma, there is a continuous function $f_U : X \to [0,1]$ with $f_U(x) = 1$ and $f(y) = 0$ for $y \notin U$. Thus, for U, V open neighborhoods of x we have,

$$|\theta(W_{(\sigma,w)})|^2 = |\theta(f_U \cdot W_{(\sigma,w)} \cdot f_V)|^2 = |\theta(S_{f_U \odot f_V})|^2 \leq \|S_{f_U \odot f_V}\|^2 =$$

$$\sup_{x\in X} \sum_{i:x\in X_i} |f_U(\sigma_i(x))|^2 |f_V(x)|^2 w_i(x) \leq \sup_{x\in V} \sum_{i:x\in X_i} |f_U(\sigma_i(x))|^2 w_i(x).$$

Taking infimum over all open neighborhoods V of x we get that

$$|\theta(W_{(\sigma,w)})|^2 \leq \sum_{i:x\in X_i} |f_U(\sigma_i(x))|^2 w_i(x) \leq \sum_{i:x\in X_i,\ \sigma_i(x)\in U} w_i(x).$$

Taking infimum over all U open neighborhoods of x, we obtain

$$|\theta(W_{(\sigma,w)})|^2 \leq \sum_{i:\sigma_i(x)=x\in X_i} w_i(x) = w(x,x).$$

Thus, we see that $|\theta(W_{(\sigma,w)})| \leq \sqrt{w(x,x)}$ and so the range of the map $\theta \mapsto \theta(W_{(\sigma,w)})$ contains the open disc $\mathbb{D}_{r_x^w}$ which is dense in $\overline{\mathbb{D}}_{r_x^w}$.

Hence, the function from $\mathcal{M}(\sigma, w)_x$ to $\overline{\mathbb{D}}_r$ given by $\theta \mapsto \theta(W_{(\sigma,w)})$ is a continuous injective map between compact spaces that has dense range, and thus must be a homeomorphism.

For the last part, we see that the inverse of the above homeomorphism restricted to the open disc $\Theta_x : \mathbb{D}_{r_x^w} \to \mathcal{M}(\sigma, w)_x$ is given by

$$\Theta_x(z)(T) = \theta_{x,z}(T) = \sum_{n=0}^{\infty} \xi_n(x, ..., x) z^n$$

for $T \in \mathcal{T}_+(\sigma, w)$ with Fourier series $T = \sum_{n=0}^{\infty} S_{\xi_n}^{(n)}$. So that $\Theta_x(\cdot)(T)$ is analytic on $\mathbb{D}_{r_x^w}$ for every fixed $T \in \mathcal{T}_+(\sigma, w)$.

□

Let us call a subset of $\mathcal{M}(\sigma, w)$ an *analytic disc* if it is the range of a pointwise analytic injective map $\Theta : \mathbb{D}_s \to \mathcal{M}(\sigma, w)$, for $s > 0$. For $f \in C(X)$ we must have that $\Theta(z)(\overline{f}) = \overline{\Theta(z)(f)}$, since for every $z \in \mathbb{D}_s$ there is some $x \in X$ such that $\Theta(z)|_{C(X)} = \delta_x$. Thus, due to analyticity, $\Theta(\cdot)(f) : \mathbb{D}_s \to \mathbb{C}$ must be constant $f(x)$, and so $\Theta(\mathbb{D}_s)$ is contained in $\mathcal{M}(\sigma, w)_x$ for some $x \in X$. Proposition 3.4.3 tells us that for every fixed point $x \in X$ of σ, the interior of $\mathcal{M}(\sigma, w)_x$ is an analytic disc, and is hence maximal in the collection of analytic discs, due to the above observation and the fact that every analytic disc contained in $\mathcal{M}(\sigma, w)_x$ must be open due to the Open Mapping Theorem, and is hence contained in $\Theta_x(\mathbb{D}_{r_x^w})$.

One can use maximal analytic discs together with the computation of the character space to obtain automatic base-detection for isomorphisms between tensor algebras associated to WPS. For any linear homomorphism θ between operator algebras $\mathcal{A}$ and $\mathcal{B}$, we denote by $\theta^* : \mathcal{M}_{\mathcal{B}} \to \mathcal{M}_{\mathcal{A}}$ the map induced between their character spaces.

Proposition 3.4.4. *Let (σ, w) and (τ, u) be WPS on compact X and Y respectively and let $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ be an algebraic isomorphism. Then φ is base-detecting and in fact, ρ_φ^* is a bijection that sends fixed points of τ to those of σ.*

Proof. Let φ be as in the statement of the proposition, so that φ induces a homeomorphism $\varphi^* : \mathcal{M}(\tau, u) \to \mathcal{M}(\sigma, w)$. It is easily verified that φ^* sends maximal analytic discs to maximal analytic discs, since it preserves the lattice of inclusion of analytic discs. Hence we obtain a bijection between the maximal analytic discs of $\mathcal{M}(\tau, u)$ and $\mathcal{M}(\sigma, w)$ which extends to a bijection between closures of such analytic discs. That is, to every $y \in Y$ there is a unique $\gamma_\varphi(y) \in X$ such that φ^* restricted to $\mathcal{M}(\tau, u)_y$ is a homeomorphism onto $\mathcal{M}(\sigma, w)_{\gamma_\varphi(y)}$, and furthermore, we must have that $\gamma_{\varphi^{-1}} = \gamma_\varphi^{-1}$ and that γ_φ is a bijection between fixed points of τ and fixed points of σ.

To show that φ is base detecting, let $\iota_X : C(X) \to \mathcal{T}_+(\sigma, w)$ be the canonical inclusion. By noting that $\iota^* : \mathcal{M}(\sigma, w) \to X$ is the canonical quotient map sending every element in $\mathcal{M}(\sigma, w)_x$ to $\theta_{x,0}$ (which is identified with $x \in X$), that $\Phi_0^* : Y \to \mathcal{M}(\tau, u)$ is the map $\Phi_0^*(y) = \theta_{y,0}$ and that

$$\gamma_\varphi = \iota^* \circ \varphi^* \circ \Phi_0^* = (\Phi_0 \circ \varphi \circ \iota)^* = \rho_\varphi^*$$

we see that ρ_φ is a *-isomorphism satisfying $\rho_\varphi^{-1} = \rho_{\varphi^{-1}}$ by using the commutative Gelfand-Naimark functorial duality, with $\rho_\varphi^* = \gamma_\varphi$ inducing a bijection between the fixed points of τ and those of σ. ☐

Proposition 3.4.4 enables an important reduction of our isomorphisms problems. Indeed, if (σ, w) and (τ, u) are WPS on X and Y respectively and $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ is a bounded / isometric isomorphism. Let $\gamma = (\rho_\varphi^{-1})^* : X \to Y$ be the induced map on the base spaces. Then obviously the WPS (τ^γ, u^γ) is conjugate to (τ, u) via γ^{-1}, and so one can see that (σ, w) is weighted-orbit / branch-transition conjugate to (τ, u) via γ if and only if (σ, w) is weighted-orbit / branch-transition conjugate to (τ^γ, u^γ) via id_X respectively. Moreover, the conjugation between (τ^γ, u^γ) and (τ, u) promotes to a completely isometric graded isomorphism $\widetilde{\gamma} : \mathcal{T}_+(\tau^\gamma, u^\gamma) \to \mathcal{T}_+(\tau, u)$ and so $\psi = \widetilde{\gamma}^{-1} \circ \varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau^\gamma, u^\gamma)$ is a bounded / isometric isomorphism (resp. to what φ is) where the WPS (σ, w) and (τ^γ, u^γ) are on the *same space* X with $\rho_\psi^* = Id_X$. Our goal is then reduced to establishing weighted-orbit / branch-transition conjugation of (σ, w) and (τ^γ, u^γ) via Id_X from a bounded / isometric isomorphism $\psi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau^\gamma, u^\gamma)$ respectively, with $\rho_\psi^* = Id_X$. This motivates the following definition

Definition 3.4.5. *Let (σ, w) and (τ, u) be partial systems on X. We say that an isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$* covers *$X$ if $\rho_\varphi^* = Id_X$ which is equivalent to having $\rho_\varphi = \Phi_0 \circ \varphi|_{C(X)} = Id_{C(X)}$.*

Next, we characterize semi-graded isomorphisms between tensor algebras arising from WPS, in terms of the induced homeomorphism on the character spaces. We show how this can be used to get automatic semi-gradedness of a general isomorphism between WPS comprised of strict contractions on compact perfect metric spaces.

If (σ, w) and (τ, u) are WPS on X, and $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ is an algebraic isomorphism covering X, by Proposition 3.4.4 we must have that $f_x^\varphi(\cdot) := \varphi^*(\theta_{x,\cdot})(W_{(\sigma,w)}) : \mathcal{M}(\tau, u)_x \to \mathcal{M}(\sigma, w)_x$. If $x \notin \mathrm{Fix}(\tau) = \mathrm{Fix}(\sigma)$ we must have that $f_x^\varphi(\theta_{x,0}) = \theta_{x,0}$ since $\mathcal{M}(\sigma, w)_x = \{\theta_{x,0}\}$.

Next, we note that if $x \in \mathrm{Fix}(\tau)$ is not an interior point of $\mathrm{Fix}(\tau)$ in X then $f_x^\varphi(0) = 0$ due to continuity of $\varphi^*(\theta_{x,0})(W_{(\sigma,w)})$ in $x \in X$, and the fact that for points $x' \notin \mathrm{Fix}(\tau)$ we have $\varphi^*(\theta_{x',0}) = \theta_{x',0}$. Hence, the only "problematic" points are those in the interior of $\mathrm{Fix}(\tau)$. Thus we obtain the following characterization of semi-gradedness.

Proposition 3.4.6. *Let (σ, w) and (τ, u) be WPS on compact X. A bounded isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ covering X is semi-graded if and only if $f_x^\varphi(0) = 0$ for all x in the interior of* $\mathrm{Fix}(\tau)$. *In particular, if either* $\mathrm{Fix}(\sigma)$ *or* $\mathrm{Fix}(\tau)$ *have empty interior, then every bounded isomorphism φ is semi-graded.*

Proof. If φ is semi-graded, then $\varphi(W_{(\sigma,w)}) \in \mathrm{Ker}\,\Phi_0$ and so

$$f_x^\varphi(0) = \varphi^*(\theta_{x,0})(W_{(\sigma,w)}) = \theta_{x,0}(\varphi(W_{(\sigma,w)})) = \theta_{x,0}(\Phi_0(\varphi(W_{(\sigma,w)}))) = 0$$

and so $f_x^\varphi(0) = 0$.

Conversely, if $f_x^\varphi(0) = 0$ for all $x \in X$ and φ covers X, we have that $\varphi^*(\theta_{x,0}) = \theta_{x,0}$ for all $x \in X$, and by Proposition 3.3.9 it suffices to show that for any $\xi \in C(\sigma, w)$ we have $\mathrm{md}(\varphi(S_\xi^{(1)})) \geq 1$. Indeed, write $\varphi(S_\xi^{(1)}) = h + T$ with $\mathrm{md}(T) \geq 1$ and $h \in C(X)$. Since for $x \in X$ we have that $h(x) = \theta_{x,0}(\varphi(S_\xi^{(1)})) = \theta_{x,0}(S_\xi^{(1)}) = 0$, we are done. □

As a corollary to the above, we show that every isomorphism is automatically semi-graded between tensor algebras arising from distributed iterated function systems and graph-directed systems as in Examples 3.2.8 and 3.2.9 respectively, when the spaces are with no isolated points.

Corollary 3.4.7. *Let (σ, w) and (τ, u) be d-variable and d'-variable WPS on a metric compact* perfect *spaces X and Y respectively, such that either σ or τ is comprised of strict contractions. If $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ is a bounded / isometric isomorphism, then it is automatically semi-graded.*

Proof. Let r be the metric on X. Without loss of generality, σ is comprised of contractions. Since for any two points $x, y \in X$ we must have $r(\sigma_i(x), \sigma_i(y)) < r(x, y)$ for all $i \in \{1, ..., d\}$, we see that σ_i can have at most one fixed point, and so $\mathrm{Fix}(\sigma)$ has at most d points. Since X is perfect, $\mathrm{Fix}(\sigma)$ must have empty interior, and so by Proposition 3.4.6 φ must be semi-graded. □

Remark 3.4.8. We record here that all methods used in this section work equally well when the WPS (σ, w) is replaced with any p.m.v. map P as in subsection 2.3.2.

3.5 Isomorphisms

In this section we adapt a new method in the analysis of character spaces due to Davidson, Ramsey and Shalit in [37], and use this to construct a bounded / isometric semi-graded isomorphism from any bounded / isometric isomorphism of our tensor algebras respectively. We then use this to provide two theorems that separately deal with classification up to bounded isomorphism and classification up to isometric isomorphism, which turn out to yield two distinct equivalences.

We first provide a criterion for automatic continuity, that will help answer the algebraic isomorphism problem for our tensor algebras, under the assumption that the union of X_i covers X, where X_i are the clopen domains of definition for σ_i's. We will follow the ideas of Davidson, Katsoulis and Kribs used in [34, 73]. For operator algebras $\mathcal{A}$ and $\mathcal{B}$ suppose we have a surjective homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$. Let

$$\mathcal{S}(\varphi) = \{ \ b \in \mathcal{B} \mid \text{there is a sequence } (a_n) \text{ in } \mathcal{A} \text{ with } a_n \to 0 \text{ and } \varphi(a_n) \to b \ \}$$

It is readily verified that the graph of φ is closed if and only if $\mathcal{S}(\varphi) = \{0\}$, hence, by the closed graph theorem φ is continuous if and only if $\mathcal{S}(\varphi) = \{0\}$. The following is an adaptation of a lemma by Sinclair, the origins of which can be traced back to [114].

Lemma 3.5.1 (Sinclair)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a surjective algebraic homomorphism. Let $(b_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{B}$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$\overline{b_1 b_2 ... b_n \mathcal{S}(\varphi)} = \overline{b_1 b_2 ... b_N \mathcal{S}(\varphi)} \quad \textit{and} \quad \overline{\mathcal{S}(\varphi) b_n ... b_2 b_1} = \overline{\mathcal{S}(\varphi) b_N ... b_2 b_1}.$$

For every WPS (σ, w) we can define the weight function of the system to be $w_\sigma(x) = P(\sigma, w)(1)(x) = \sum_{i:x\in X_i} w_i(x)$ which is positive continuous function that vanishes only on $X - \cup_{i=1}^d X_i$.

Definition 3.5.2. *Let σ be a partial system on X. We say σ is* well-supported *if $\{X_i\}$ covers X, where X_i are the clopen domain of definition for σ_i.*

When we have a well-supported (σ, w), we define the normalized WPS $(\sigma, \widetilde{w})$ by setting $\widetilde{w} = (\frac{w_1}{w_\sigma}, ..., \frac{w_n}{w_\sigma})$, and we say that (σ, w) is normalized if $w_\sigma = 1$. Note that when (σ, w) is a well-supported normalized system, we must have that $P(\sigma, w)$ is a unital map, or in other words a Markov-Feller map.

Theorem 3.5.3. *Let (σ, w) and (τ, u) be WPS operating on X and Y respectively such that either σ or τ are well-supported. Then every algebraic isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ is automatically a bounded isomorphism.*

Proof. Suppose without loss of generality that τ is well-supported. Since for every edge $e \in \mathrm{Gr}(\tau)$ we have that $\frac{\widetilde{u}}{u}(e) = u_\tau(s(e))^{-1}$, we see that $\frac{\widetilde{u}}{u}$ is continuous on $\mathrm{Gr}(\tau)$ so that (τ, u) and $(\tau, \widetilde{u})$ are branch-transition conjugate. By Corollary 3.2.27 and Theorem 3.3.3 used in tandem, $\mathcal{T}_+(\tau, u)$ is graded completely isometrically isomorphic to $\mathcal{T}_+(\tau, \widetilde{u})$. So we assume without loss of generality that (τ, u) is normalized. In this case, the constant function $1 = 1 \odot 1 \in C(\mathrm{Gr}(\tau))$ gives rise to an isometry $W_{(\tau,u)} := S_1^{(1)} = S_{1\odot 1}^{(1)} \in \mathrm{Ker}\,\Phi_0 \subseteq \mathcal{T}_+(\tau, u)$, since (τ, u) is well-supported and normalized.

Now suppose towards contradiction that there is $0 \neq T \in \mathcal{S}(\varphi)$. Since $W_{(\tau,u)}$ is an isometry, we have that $W^n_{(\tau,u)}T \neq 0$ for all $n \in \mathbb{N}$. By Sinclair's lemma there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$W^N_{(\tau,u)}\mathcal{S}(\varphi) = W^n_{(\tau,u)}\mathcal{S}(\varphi) \subseteq \bigcap_{k<n} \operatorname{Ker}\Phi_k.$$

So in fact we must have that $W^N_{(\tau,u)}\mathcal{S}(\varphi) = \cap_{k\in\mathbb{N}} \operatorname{Ker}\Phi_k = \{0\}$, in contradiction to having $W^N_{(\tau,u)}T \neq 0$ as shown above. □

Our next goal is to reduce the existence of a bounded / isometric isomorphism between WPS to the existence of a *graded* such isomorphism. Let (σ, w) be a WPS on X. Recall the gauge group action $\alpha : \mathbb{T} \to \operatorname{Aut}(\mathcal{T}_+(\sigma, w))$ uniquely determined on generators by $\alpha_\lambda(S^{(1)}_\xi) = \lambda S^{(1)}_\xi$ and $\alpha_\lambda(f) = f$ for $\xi \in C(\sigma, w)$ and $f \in C(X)$. Now, if (σ, w) and (τ, u) are WPS on X, and $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ is an algebraic isomorphism covering X, by Proposition 3.4.4 we have that $\varphi^*|_{\mathcal{M}(\tau,u)_x} : \mathcal{M}(\tau, u)_x \to \mathcal{M}(\sigma, w)_x$.

Next, if $x \in \operatorname{Fix}(\tau)$, by Proposition 3.4.3 we can identify $\varphi^*|_{\mathcal{M}(\tau,u)_x}$ with a bijective biholomorphism $f^\varphi_x := \Theta_x^{-1} \circ \varphi^* \circ \Theta_x : \mathbb{D}_{r^u_x} \to \mathbb{D}_{r^w_x}$ which then must be of the form given by $f^\varphi_x(z) = r^w_x \hat{f}^\varphi_x((r^u_x)^{-1}z)$ where $\hat{f}^\varphi_x$ is a biholomorphism of the *unit* disc $\mathbb{D}$ given by

$$\hat{f}^\varphi_x(z) = e^{i\theta_x}\frac{w_x - z}{1 - \overline{w_x}z}$$

for some $\theta_x \in [0, 2\pi]$ and $w_x \in \mathbb{D}$. Note also that since $f^\varphi_x(0) = \varphi^*(\theta_{x,0})(W_{(\sigma,w)}) = r^w_x e^{i\theta_x} w_x$, and since $\varphi^*(\theta_{x,0})(W_{(\sigma,w)})$ depends continuously on $x \in X$, we can extend f^φ_x continuously to be 0 for $x \notin \operatorname{Fix}(\tau)$. Further, if $\psi : \mathcal{T}_+(\tau, u) \to \mathcal{T}_+(\pi, v)$ is another algebraic isomorphism covering X we have that $\hat{f}^{\psi\circ\varphi}_x = \hat{f}^\varphi_x \circ \hat{f}^\psi_x$.

We now wish to examine an isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ covering X for which there exists $x \in X$ an interior point of $\operatorname{Fix}(\sigma) = \operatorname{Fix}(\tau)$, with $f^\varphi_x(0) \neq 0$.

Fix an element $x \in \operatorname{Fix}(\tau)$ with $f^\varphi_x(0) \neq 0$. One can then find $\lambda_x, \gamma_x \in \mathbb{T}$ such that the isomorphism $\psi = \varphi \circ \alpha_{\lambda_x} \circ \varphi^{-1} \circ \alpha_{\gamma_x} \circ \varphi$ satisfies $f^\psi_x(0) = 0$. Indeed, for $\lambda \in \mathbb{T}$, since $\hat{f}^{\varphi\circ\alpha_\lambda}_x(0) = \lambda \cdot \hat{f}^\varphi_x(0)$, we get that $C = \{\hat{f}^{\varphi\circ\alpha_\lambda}_x(0) | \lambda \in \mathbb{T}\}$ is a circle centered around 0. Since $\hat{f}^{\varphi^{-1}}_x$ is a Möbius map of the form described above, it must send C to a circle *through* the origin. That is, $C' = \hat{f}^{\varphi^{-1}}_x(C) = \{\hat{f}^{\varphi\circ\alpha_\lambda\circ\varphi^{-1}}_x(0) | \lambda \in \mathbb{T}\}$ is a circle through the origin, since for $\lambda = 0$ we get $0 = \hat{f}^{Id}_x(0) = \hat{f}^{\varphi\circ\alpha_\lambda\circ\varphi^{-1}}_x(0) \in C'$. If we again take an arbitrary $\gamma \in \mathbb{T}$ and do this, we can "fill the circle". That is, since $\hat{f}^{\varphi^{-1}\circ\alpha_\gamma}_x(C) = \gamma \cdot f^{\varphi^{-1}}_x(C) = \gamma \cdot C'$, the region bounded by C', which we denote by $\operatorname{ins}(C')$, is a subset of $\{\ \hat{f}^{\varphi\circ\alpha_\lambda\circ\varphi^{-1}\circ\alpha_\gamma}_x(0) \mid \lambda, \gamma \in \mathbb{T}\ \}$.

Once more, since $\hat{f}_x^{\varphi}$ is the inverse of $\hat{f}_x^{\varphi^{-1}}$, being a Möbius map, it must send C' back to C, and so it must send $\mathrm{ins}(C')$ to $\mathrm{ins}(C)$. Thus we obtain that the set

$$\{ \ \hat{f}_x^{\varphi\circ\alpha_\lambda\circ\varphi^{-1}\circ\alpha_\gamma\circ\varphi}(0) \mid \lambda, \gamma \in \mathbb{T} \ \}$$

contains the origin, and hence there is some choice of λ_x and γ_x with which

$$\hat{f}_x^{\varphi\circ\alpha_{\lambda_x}\circ\varphi^{-1}\circ\alpha_{\gamma_x}\circ\varphi}(0) = 0.$$

We now wish to show that a choice of *continuous* functions $x \mapsto \lambda_x$ and $x \mapsto \gamma_x$ from X to $\mathbb{T}$ can be found such that $f_x^{\varphi\circ\alpha_{\lambda_x}\circ\varphi^{-1}\circ\alpha_{\gamma_x}\circ\varphi}(0) = 0$ for all $x \in D \subseteq \mathrm{Fix}(\tau)$ where D is a closed set for which $x \mapsto |w_x|^2$ is continuous on D.

Proposition 3.5.4. *Let (σ, w) and (τ, u) be WPS on compact X, and let $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ be an algebraic isomorphism covering X. If $D \subseteq \mathrm{Fix}(\tau)$ is a closed set on which $x \mapsto |w_x|^2$ is continuous and* non-zero*, there exist continuous functions $\lambda, \gamma : X \to \mathbb{T}$ such that for all $x \in D$ we have*

$$(\hat{f}_x^{\varphi} \circ \hat{f}_x^{\alpha_{\gamma_x}} \circ \hat{f}_x^{\varphi^{-1}} \circ \hat{f}_x^{\alpha_{\lambda_x}} \circ \hat{f}_x^{\varphi})(0) = 0. \tag{3.3}$$

Proof. First note that since the map $x \mapsto |w_x|^2$ is continuous on D, and since $|w_x|^2 < 1$ for all $x \in D$, we may extend it to a continuous function $h : X \to [0, 1]$ so that $\|h\|_\infty < 1$ still. Next, we simplify equation (3.3) to the following equivalent form:

$$(\hat{f}_x^{\alpha_{\lambda_x}} \circ \hat{f}_x^{\varphi})(0) = (\hat{f}_x^{\varphi} \circ \hat{f}_x^{\alpha_{\overline{\gamma_x}}} \circ \hat{f}_x^{\varphi^{-1}})(0)$$

which is equivalent to having for all $x \in D$ that

$$\lambda_x e^{i\theta_x} w_x = \hat{f}_x^{\varphi}(\overline{\gamma_x} w_x) = e^{i\theta_x} \frac{w_x - \overline{\gamma_x} w_x}{1 - \overline{\gamma_x} |w_x|^2}. \tag{3.4}$$

It then suffices to find continuous functions $\gamma, \lambda : X \to \mathbb{T}$ such that for any $x \in D$,

$$\lambda_x = \frac{1 - \overline{\gamma_x}}{1 - \overline{\gamma_x} h(x)} = \frac{\gamma_x - 1}{\gamma_x - h(x)} \tag{3.5}$$

as multiplying both sides by $e^{i\theta_x} w_x$ yields equation (3.4) for all $x \in D$.

Since $h(x) < 1$ for all $x \in X$, we see that $\gamma_x - h(x) \neq 0$ for all $x \in X$, so we may define

$$\gamma_x = \Big(\frac{1 + h(x)}{2}, \sqrt{1 - (\frac{1 + h(x)}{2})^2}\Big) \quad \text{and} \quad \lambda_x = \frac{\gamma_x - 1}{\gamma_x - h(x)}.$$

As $|\gamma_x - 1| = |\gamma_x - h(x)|$ for all $x \in X$, we see that γ and λ are well-defined continuous functions from X into $\mathbb{T}$ satisfying equation (3.5), and we are done. □

Finally, we are at the point where we can reduce general isomorphism problems to corresponding semi-graded isomorphism problems. Recall that for a d'-variable WPS (τ, u) on X and an index set $I \subseteq \{1, ..., d'\}$ we defined the coinciding set of I to be

$$C(I) = \{\ x \in \cap_{i \in I} X_i \mid \tau_i(x) = \tau_j(x)\ \}$$

where for each $1 \leq i \leq d'$ we have $\tau_i : X_i \to X$, with X_i clopen.

Theorem 3.5.5. *Let (σ, w) and (τ, u) be d-variable and d'-variable WPS respectively, on the same compact space X, and let $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ be a bounded / isometric isomorphism covering X. Then there exists a semi-graded bounded / isometric isomorphism $\psi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ covering X respectively.*

Proof. Suppose $k \geq 0$ for which there exists ψ such that $f_x^\psi(0) = 0$ for all $x \in \text{Fix}(\tau) \cap \bigcup_{|I| \geq k+1} C(I)$ where we range over all subsets $I \subseteq \{1, ..., d'\}$ of size at least $k+1$. Our assumptions guarantee that such a k exists and that $k \leq d'$, since φ certainly satisfies $f_x^\varphi(0) = 0$ for all $x \in \text{Fix}(\tau) \cap \bigcup_{|I| \geq d'+1} C(I) = \emptyset$.

If there exists ψ for which we can take $k = 0$, then $f_x^\psi(0) = 0$ for all $x \in \text{Fix}(\tau) \cap \bigcup_{I \subseteq \{1,...,d'\}} C(I) = \text{Fix}(\tau)$, we will be done by Proposition 3.4.6.

Thus, suppose φ is a bounded / isometric isomorphism and $k > 0$ such that $f_x^\varphi(0) = 0$ for all $x \in \text{Fix}(\tau) \cap \bigcup_{|I| \geq k+1} C(I)$ but $f_x^\varphi(0) \neq 0$ for some $x \in \text{Fix}(\tau) \cap \bigcup_{|I| \geq k} C(I)$. We will construct ψ_k for which $f_x^{\psi_k}(0) = 0$ for all $x \in \text{Fix}(\tau) \cap \bigcup_{|I| \geq k} C(I)$, so that for ψ_k there is a smaller $k' < k$ for which $f_x^{\psi_k}(0) = 0$ for all $x \in \text{Fix}(\tau) \cap \bigcup_{|I| \geq k'} C(I)$. By successive iterations of this procedure we keep decreasing k, so that we would eventually get ψ for which we can take $k = 0$, and be done by the previous paragraph.

We claim that under our current assumptions on φ, on the closed set $D_k = \text{Fix}(\tau) \cap \bigcup_{|I|=k} C(I)$ we have that $x \mapsto |w_x|^2$ is continuous. We know that $\varphi^*(\theta_{x,0})(W_{(\sigma,w)}) = f_x^\varphi(0) = r_x^w e^{i\theta_x} w_x$ depends on $x \in X$ continuously, so we restrict it to D_k. By Proposition 3.4.3 we have that $r_x^w = \sqrt{w(x,x)} = \sqrt{\sum_{i:\sigma_i(x)=x \in X_i} w_i(x)}$ and as a function of x is bounded below on $\text{Fix}(\tau) = \text{Fix}(\sigma) \supset D_k$, and is hence non-zero on D_k. Moreover, the only discontinuities $x \mapsto r_x^w$ can have on D_k are those arising from branching points $B(J) \cap \text{Fix}(\tau)$ in D_k for subsets $J \supsetneq I$ and $|I| \geq k$, and so $x \mapsto |w_x|^2 = \frac{f_x^\varphi(0)^2}{(r_x^w)^2}$ is continuous at every $x \in D_k$ which is not a point in $B(J) \cap \text{Fix}(\tau)$ for some $J \supsetneq I$ and $|I| \geq k$.

Next, for a point $y \in B(J) \cap \text{Fix}(\tau)$ inside D_k for some $J \supsetneq I$ and $|I| \geq k$, our assumptions guarantee that $0 = f_x^\varphi(0) = r_x^w e^{i\theta_x} w_x$ for all $x \in C(J)$ since $|J| > |I| \geq k$, so that $|w_y|^2 = 0$, since $x \mapsto r_x^w$ is non-zero for all $x \in D_k$.

Now, since $x \mapsto r_x^w$ is bounded below on D_k, say by ϵ, we have that $|f_x^\varphi(0)|^2 \geq \epsilon^2 |w_x|^2$, and by continuity of $x \mapsto f_x^\varphi(0)$ at y, we see that $|w_x|^2 \to 0$ as $x \to y$. This means that $x \mapsto |w_x|^2$ is continuous at y inside D_k, so that $x \mapsto |w_x|^2$ is continuous on all of D_k.

Using Proposition 3.5.4 we have two continuous maps $L : x \mapsto \lambda_x$ and $G : x \mapsto \gamma_x$ from X to $\mathbb{T}$ that satisfy equation (3.3) for any $x \in D_k$. Define two unitaries U_L on $C(\sigma, w)$ and U_G on $C(\tau, u)$ given by $U_L(\xi) = L \cdot \xi$ and $U_G(\eta) = G \cdot \eta$ for $\xi \in C(\sigma, w)$ and $\eta \in C(\tau, u)$ using the left action by continuous functions. Next, use Theorem 3.3.3 to promote U_L and U_G to (completely) isometric graded automorphisms $\mathrm{Ad}_{U_L} : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\sigma, w)$ and $\mathrm{Ad}_{U_G} : \mathcal{T}_+(\tau, u) \to \mathcal{T}_+(\tau, u)$ such that for any point $x \in D_k$ we have

$$\hat{f}_x^{\mathrm{Ad}_{U_L}}(z) = \lambda_x z \quad \text{and} \quad \hat{f}_x^{\mathrm{Ad}_{U_G}}(z) = \gamma_x z.$$

Thus, we get for all $x \in D_k$ that $\hat{f}_x^{\mathrm{Ad}_{U_L}}(z) = \hat{f}_x^{\alpha_{\lambda_x}}(z)$ and $\hat{f}_x^{\mathrm{Ad}_{U_G}}(z) = \hat{f}_x^{\alpha_{\gamma_x}}(z)$. Next, we define $\psi_k = \varphi \circ \mathrm{Ad}_{U_L} \circ \varphi^{-1} \circ \mathrm{Ad}_{U_G} \circ \varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$, and since for every $x \in D_k$,

$$\hat{f}_x^{\psi_k}(0) = \hat{f}_x^{\varphi \circ \mathrm{Ad}_{U_L} \circ \varphi^{-1} \circ \mathrm{Ad}_{U_G} \circ \varphi}(0) = (\hat{f}_x^\varphi \circ \hat{f}_x^{\mathrm{Ad}_{U_G}} \circ \hat{f}_x^{\varphi^{-1}} \circ \hat{f}_x^{\mathrm{Ad}_{U_G}} \circ \hat{f}_x^\varphi)(0) =$$

$$(\hat{f}_x^\varphi \circ \hat{f}_x^{\alpha_{\gamma_x}} \circ \hat{f}_x^{\varphi^{-1}} \circ \hat{f}_x^{\alpha_{\lambda_x}} \circ \hat{f}_x^\varphi)(0) = 0$$

we obtain that ψ_k is a bounded / isometric isomorphism (respectively to what φ is) such that $f_x^{\psi_k}(0) = 0$ for all $x \in D_k = \mathrm{Fix}(\tau) \cap \bigcup_{|I| \geq k} C(I)$, and we have managed to find ψ_k for which we can take $k' < k$ such that $f_x^{\psi_k}(0) = 0$ for all $x \in \mathrm{Fix}(\tau) \cap \bigcup_{|I| \geq k'} C(I)$. □

The following two theorems resolve algebraic / bounded / isometric isomorphism problems for tensor algebras arising from WPS, and classify them up to bounded / isometric isomorphisms.

Theorem 3.5.6 (Algebraic / Bounded isomorphisms). *Let (σ, w) and (τ, u) be WPS over X and Y respectively. The following are equivalent*

1. *(σ, w) and (τ, u) are weighted-orbit conjugate.*
2. *$C(\sigma, w)$ and $C(\tau, u)$ are similar C^*-correspondences.*
3. *There exists a graded completely bounded isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$.*
4. *There exists a bounded isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$.*

Moreover, if either σ or τ are well-supported, any of the above conditions are equivalent to the existence of an algebraic isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$.

Proof. The equivalence between (1) and (2) and (3) ran on Proposition 3.2.26 and Theorem 3.3.2. (3) implies (4) trivially. To show that (4) implies (3), let $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ be a bounded isomorphism. By Theorem 3.5.5, there is a semi-graded bounded isomorphism $\psi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$. Then by Proposition 3.3.10 we obtain a completely bounded graded isomorphism $\widetilde{\psi} : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$.

For the last part, if either σ or τ are well-supported, and $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ is an algebraic isomorphism, then by Theorem 3.5.3, either φ or φ^{-1} is bounded, but then by the Open Mapping Theorem in Banach spaces, both are bounded. □

Theorem 3.5.7 (Isometric isomorphisms)**.** *Let (σ, w) and (τ, u) be WPS over X and Y respectively. The following are equivalent*

1. *(σ, w) and (τ, u) are branch-transition conjugate.*
2. *$C(\sigma, w)$ and $C(\tau, u)$ are unitarily isomorphic C*-correspondences.*
3. *There exists a graded completely isometric isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$.*
4. *There exists an isometric isomorphism $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$.*

Proof. The equivalence between (1) and (2) and (3) ran on Corollary 3.2.27 and Theorem 3.3.3. (3) implies (4) trivially. To show that (4) implies (3), let $\varphi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$ be an isometric isomorphism. By Theorem 3.5.5, there is a semi-graded isometric isomorphism $\psi : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$. Then by Corollary 3.3.11 we obtain a completely isometric graded isomorphism $\widetilde{\psi} : \mathcal{T}_+(\sigma, w) \to \mathcal{T}_+(\tau, u)$. □

Example 3.5.8. *By the above two theorems and Example 3.2.22 we see that there are two WPS for which there exists an algebraic / bounded isomorphism of their tensor algebras but there is no* isometric *isomorphism between their tensor algebras.*

This shows that the isometric isomorphism problem for general tensor algebras cannot be solved just by extracting information from algebraic / bounded isomorphism invariants of the tensor algebras, such as representations into upper triangular 2×2 matrices, which were used in [34, 35, 39, 73, 116].

3.6 Applications and comparisons

We start this section by proving a universal property for the Toeplitz algebra $\mathcal{T}(\sigma, w) := \mathcal{T}(C(\sigma, w))$ arising from a WPS (σ, w), as a C*-algebra generated by certain set elements satisfying certain relations. This enables us to think of the non-self-adjoint tensor algebra $\mathcal{T}_+(\sigma, w) := \mathcal{T}_+(C(\sigma, w))$ as a the norm closed operator subalgebra of the universal C*-algebra $\mathcal{T}(\sigma, w)$ generated by the same set of elements.

Recall that for a partial system (σ, w), the positive operator $P(\sigma, w) : C(X) \to C(X)$ used to construct the GNS C*-correspondence of (σ, w) was given by

$$P(\sigma, w)(f)(x) = \sum_{i:x \in X_i} w_i(x) f(\sigma_i(x)).$$

Definition 3.6.1. *Let (σ, w) be a WPS on compact X. A* representation *of (σ, w) is a pair (π, T) with $\pi : C(X) \to B(\mathcal{H})$ a unital $*$-representation and an operator $T \in B(\mathcal{H})$ such that*

$$T^* \pi(f) T = \pi\big(P(\sigma, w)(f)\big) \quad \text{for all} \quad f \in C(X).$$

Denote by $C^*(\pi, T)$ the C-algebra and $\overline{\mathrm{Alg}}(\pi, T)$ the norm-closed algebra generated by the image of π and T inside $B(\mathcal{H})$.*

The following universal description shows that we can think of $\mathcal{T}_+(\sigma, w)$ as a certain "semi-crossed product" by the positive map $P(\sigma, w)$.

Theorem 3.6.2 (Universal description)**.** *Let (σ, w) be a WPS on compact X. Then the Toeplitz algebra $\mathcal{T}(\sigma, w)$ and the tensor algebra $\mathcal{T}_+(\sigma, w)$ are the universal C*-algebra and operator algebra (respectively) generated by a universal representation (π_u, T_u) of (σ, w).*

Proof. Since representations of (σ, w) are exactly representations of $(C(X), P(\sigma, w))$ in the sense of [83, Definition 3.1], by [83, Proposition 3.10], these are in bijection with *isometric* (in the sense of [90, Definition 2.11]) representations $(\pi, \pi_{P(\sigma,w)})$ of the GNS C*-correspondence $F_{\sigma,w}$, that satisfy $\pi_{P(\sigma,w)}(f \otimes g) = \pi(f) T \pi(g)$. By [90, Theorem 2.12], these are in bijection with the representations $\tau_{(\pi, \pi_{P(\sigma,w)})}$ of $\mathcal{T}(\sigma, w)$ that send $\mathcal{T}_+(\sigma, w)$ to $\overline{\mathrm{Alg}}(\pi(C(X)) \cup \pi_{P(\sigma,w)}(F_{\sigma,w}))$. Hence, if (π, T) is a representation of (σ, w), it promotes to a representation $\tau_{(\pi,T)}$ of $\mathcal{T}(\sigma, w)$ that sends $\mathcal{T}_+(\sigma, w)$ to $\overline{\mathrm{Alg}}(\pi, T)$, and every such representations π of $\mathcal{T}(\sigma, w)$ arises in this way, and must send $\mathcal{T}_+(\sigma, w)$ to $\overline{\mathrm{Alg}}(\pi, T)$. □

We next apply our results to certain subclasses of WPS, by computing what the conjugation relations yield for these classes. For some classes of WPS, our tensor algebras coincide with previously-investigated operator algebras, and we recover some classification results on non-self-adjoint operator algebras.

When we have a non-negative matrix $A = [A_{ij}]$ indexed by a finite set Ω, we associated a d-variable WPS (σ^A, w^A) to it as in Example 3.2.4, to which we associate a topological quiver $\mathcal{Q}(A) := \mathcal{Q}(\sigma^A, w^A) = (\mathrm{Gr}(A), P(A))$ given by $\mathrm{Gr}(A) = \mathrm{Gr}(\sigma^A) = \{ \ (i,j) \mid A_{ij} > 0 \ \}$ with Radon measures $P(A)_j := P(\sigma^A, w^A)_j = \sum_{i \in \Omega} A_{ij} \delta_{(i,j)}$. This topological quiver encodes the information of the entries of A into the Radon measures, since the entries $A_{ij} = \int_{\mathrm{Gr}(A)} \chi_{\{(i,j)\}} dP(A)_j$ can be detected by integration against a characteristic function of a singleton. We note that $\mathrm{Gr}(A)$ has no sinks if and only if (σ^A, w^A) is well-supported.

Since the tensor algebra associated to the C*-correspondence arising from the topological quiver $\mathcal{Q}(A)$ is $\mathcal{T}_+(\sigma^A, w^A)$, we just write $\mathcal{T}_+(A) := \mathcal{T}_+(\sigma^A, w^A)$.

Thus, for two non-negative matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ indexed by Ω^A and Ω^B respectively, we see that the graphs $\mathrm{Gr}(A)$ and $\mathrm{Gr}(B)$ are isomorphic directed graphs if and only if (σ^A, w^A) and (σ^B, w^B) are graph conjugate, if and only if they are branch-transition conjugate, since the topology on $\mathrm{Gr}(\sigma^A) = \mathrm{Gr}(A) = \{ \ (i,j) \mid A_{ij} > 0 \ \}$ is discrete, and so the weight-transition functions will always be continuous. Hence, we obtain the following:

Corollary 3.6.3. *Let A and B be non-negative matrices indexed by a finite set Ω. Then* $\mathrm{Gr}(A)$ *and* $\mathrm{Gr}(B)$ *are isomorphic directed graphs if and only if $\mathcal{T}_+(A)$ and $\mathcal{T}_+(B)$ are (completely) bounded / (completely) isometrically isomorphic. Moreover, if either* $\mathrm{Gr}(A)$ *or* $\mathrm{Gr}(B)$ *have no sinks, the above is equivalent to $\mathcal{T}_+(A)$ and $\mathcal{T}_+(B)$ being algebraically isomorphic.*

When the non-negative matrix is given as the incidence matrix $A_G = [m_{w,v}]$ of some finite directed graph $G = (V, E, r, s)$ as in Example 3.2.5, such that $m_{w,v}$ is either 0 or 1, then G is multiplicity free. The topological quiver $\mathcal{Q}(A_G)$ associated to A_G is then just the topological quiver structure we associate to the original graph G, that is with edges $\mathrm{Gr}(A_E) := \{ \ (r(e), s(e)) = e \mid e \in E \ \}$ and with Radon measures given by counting measure $P(G)_v = P(A_E)_v = \sum_{w:(w,v) \in \mathrm{Gr}(A_G)} \delta_{(w,v)}$ on $s^{-1}(v)$ (See [93, Example 3.19], with reversed source and range maps). This means that the tensor algebra $\mathcal{T}_+(G)$ associated to G as in [73], coincides with $\mathcal{T}_+(C(\mathcal{Q}(A_G)))$, and we recover results of Katsoulis and Kribs in [73] and Solel in [116], for the case of finite multiplicity-free graphs.

Corollary 3.6.4. *Let G and G' be* finite multiplicity free *graphs. Then G and G' are isomorphic as directed graphs if and only if $\mathcal{T}_+(G)$ and $\mathcal{T}_+(G')$ are (completely) bounded / (completely) isometrically isomorphic. Moreover, if either G or G' have no sinks, the above is equivalent to $\mathcal{T}_+(G)$ and $\mathcal{T}_+(G')$ being algebraically isomorphic.*

For a continuous map σ on a compact space X, we can associate an operator algebra $C(X) \times_\sigma \mathbb{Z}_+$ called *Peters' semi-crossed product* to it as done originally by Peters in [102].

We do this here by giving a universal definition. We call a pair (ρ, T) a representation of (X, σ) if $\rho : C(X) \to B(\mathcal{H})$ is a *-representation and $S \in B(\mathcal{H})$ a contraction such that $\rho(f)S = S\rho(f \circ \sigma)$. We say that a (ρ, S) is *isometric* if in addition S is an isometry.

Peters' semi-crossed product $C(X) \times_\sigma \mathbb{Z}_+$ of the system (X, σ) is the norm closed algebra generated by the image of a universal *isometric* representation (ρ_u, S_u) for (X, σ). Note that for any isometric representation (ρ, S) we have that $S^* \rho(f) S = \rho(f \circ \sigma)$ and we obtain a representation of the WPS $(\sigma, 1)$ as in Definition 3.6.1.

Muhly and Solel show in [90] that every representation (ρ, S) of (X, σ) dilates to an isometric representation, so that Peters' semi-crossed product is also the norm closed algebra generated by the image of a (contractive) universal representation.

When we look at $\sigma = (\sigma, 1)$ as a WPS, by [83, Proposition 3.21] any representation (π, T) of the WPS $(\sigma, 1)$ satisfies $\pi(f)T = T\pi(f \circ \sigma)$, so in fact we have obtained a representation of the system (X, σ) as above. We conclude that $\mathcal{T}_+(\sigma, 1) \cong C(X) \times_\sigma \mathbb{Z}_+$.

On the other hand, for two continuous maps σ and τ on compact spaces X and Y respectively, we have that $(\sigma, 1)$ and $(\tau, 1)$ are graph-conjugate if and only if σ and τ are conjugate, and we obtain the following alternative proof, assuming our spaces are compact, of a theorem first proven by Davidson and Katsoulis (See [34, Corollary 4.7]).

Corollary 3.6.5. *Let σ and τ be continuous maps on compact spaces X and Y respectively. Then $C(X) \times_\sigma \mathbb{Z}_+$ and $C(X) \times_\sigma \mathbb{Z}_+$ are algebraically / (completely) bounded / (completely) isometrically isomorphic if and only there is some homeomorphism $\gamma : X \to Y$ such that $\gamma^{-1}\tau\gamma = \sigma$.*

When $\sigma = (\sigma_1, ..., \sigma_d)$ is a partially defined system so that $\sigma_i : X_i \to X$, we think of it as a WPS by specifying $w = (1, ..., 1)$. The weight induced on $\mathrm{Gr}(\sigma)$ then becomes $m_\sigma(e) := w(e) = \sum_{i \in I(e,\sigma)} 1 = |I(e, \sigma)|$ is just the multiplicity of $e \in \mathrm{Gr}(\sigma)$. Denote by $\mathcal{T}_+(\sigma) := \mathcal{T}_+(\sigma, 1)$ the tensor algebra of a partially defined system σ. In this case, we have the following characterization of isometric isomorphism between tensor algebras.

Theorem 3.6.6. *Let σ and τ be partially defined systems on compact X and Y respectively. Then $\mathcal{T}_+(\sigma)$ and $\mathcal{T}_+(\tau)$ are isometrically isomorphic if and only if there is a homeomorphism $\gamma : X \to Y$ such that $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau^\gamma)$ and the function $\frac{m_{\tau^\gamma}}{m_\sigma} : \mathrm{Gr}(\sigma) \to (0, \infty)$ is locally constant.*

Proof. By Theorem 3.5.7 we have that $\mathcal{T}_+(\sigma)$ and $\mathcal{T}_+(\tau)$ are isometrically isomorphic if and only if there is a homeomorphism $\gamma : X \to Y$ such that $\mathrm{Gr}(\sigma) = \mathrm{Gr}(\tau^\gamma)$ and the function $\frac{m_{\tau^\gamma}}{m_\sigma} : \mathrm{Gr}(\sigma) \to (0, \infty)$ is continuous. Since $\frac{m_{\tau^\gamma}}{m_\sigma}$ can only attain finitely many values, it is continuous if and only if it is locally constant. □

Chapter 4

C*-envelopes of tensor algebras arising from stochastic matrices

4.1 Introduction

In this chapter, which is based on joint work with Daniel Markiewicz [43], we focus on the C*-envelope of the tensor algebra associated to finite irreducible stochastic matrices. We remind the reader that in this Chapter we take a *reversed* range and source convention to the one taken in Subsection 2.3.1 and the definition of the graph of a Markov-Feller operator as in Subsection 2.3.2. We were motivated by the known results in the determination of the C*-envelope of the tensor algebra of a subproduct system:

1. Given a C*-correspondence E, we have that $C^*_e(\mathcal{T}_+(E)) = \mathcal{O}(E)$. This was first proven by Muhly and Solel [90, Corollary 6.6] when the left action on E is faithful, essential and acts by compacts, and in the general case (without extra assumptions) by Katsoulis and Kribs [76].

2. Let I be a homogeneous ideal in the ring of polynomials in finitely many commuting variables. The universal commuting row contraction subject to the polynomial relations in I gives rise to a subproduct system of Hilbert spaces X^I, and it was shown by Davidson, Ramsey and Shalit [37] that $C^*_e(\mathcal{T}_+(X^I)) = \mathcal{T}(X^I)$.

3. Let I be a monomial ideal in the ring of polynomials in non-commuting variables. Similarly to the commutative case, a subproduct system X^I can be defined. Kakariadis and Shalit [72] have shown that for many cases (depending on the monomial ideal) either $C^*_e(\mathcal{T}_+(X^I)) = \mathcal{T}(X^I)$ or $C^*_e(\mathcal{T}_+(X^I)) = \mathcal{O}(X^I)$.

In summary, for all these cases, when the subproduct system X is irreducible in the appropriate sense (i.e. no nontrivial X-invariant ideals, see Definition 2.2.12) , $C_e^*(\mathcal{T}_+(X))$ has been found to be isomorphic either to $\mathcal{T}(X)$ or $\mathcal{O}(X)$. In [119, Section 6], this phenomenon was observed, and it was asked if one can find a general description for the behavior of C*-envelopes of tensor algebras associated with subproduct systems. In this chapter we shed some light on this question: we show that the evidence for the dichotomy witnessed above is misleading, and that the situation is more mysterious than previously thought. We do this by providing an example of stochastic matrix P with an associated subproduct system $X := \mathrm{Arv}(P)$ such that the $C_e^*(\mathcal{T}_+(X))$ is not $*$-isomorphic to either $\mathcal{T}(X)$ or $\mathcal{O}(X)$ (See Example 4.3.18).

Our first main result in this chapter is as follows. Let P be a finite irreducible stochastic matrix. In this case we show that the C*-envelope lands between the Toeplitz and Cuntz-Pimsner algebras in the sense that it fits in the following sequence of quotient maps:

$$\mathcal{T}(P) \longrightarrow C_e^*(\mathcal{T}_+(P)) \xrightarrow{\pi_P} \mathcal{O}(P). \qquad (*)$$

Moreover, in the case when P has the multiple arrival property (see Definition 4.3.13), we identify the boundary representations of $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$, also known as the non-commutative Choquet boundary in the sense of Arveson [5]. This enables us to describe the C*-envelope $C_e^*(\mathcal{T}_+(P))$ in terms of boundary representations. For details see Corollary 4.3.14 and Theorem 4.3.15.

The fact that the Cuntz-Pimsner algebra $\mathcal{O}(X)$ as defined by Viselter [119] is not always isomorphic to the C*-envelope of the tensor algebra $\mathcal{T}_+(X)$ in the subproduct system case, and even a dichotomy as above fails to hold, suggests that perhaps an alternative definition of Cuntz-Pimsner algebra for subproduct systems is desirable.

The concrete description of the C*-envelope and lack of dichotomy lead to an unexpected richness of possibilities. Our second main result in this chapter concerns with the classification of C*-envelopes up to $*$-isomorphism and stable isomorphism, so as to clarify the situation. For a finite irreducible stochastic matrix P over Ω^P, the ideal $\mathrm{Ker}(\pi_P)$ in the sequence of equation $(*)$ is $*$-isomorphic to a direct sum of $n_P \leq |\Omega^P|$ copies of the algebra of compact operators. Given two irreducible stochastic matrices P and Q over finite state sets Ω^P and Ω^Q we have that

1. $\mathcal{T}_+(P)$ and $\mathcal{T}_+(Q)$ have stably isomorphic C*-envelopes if and only if $n_P = n_Q$. For more details see Theorem 4.4.10.

2. $\mathcal{T}_+(P)$ and $\mathcal{T}_+(Q)$ have $*$-isomorphic C*-envelopes if and only if $|\Omega^P| = |\Omega^Q|$, $n_P = n_Q$ and up to a reordering of Ω^Q, the matrices P and Q have the same column nullity in every column. For more details, see Definition 4.4.8 and Theorem 4.4.11.

Therefore, we see that instead of a dichotomy, we actually have a profusion of possibilities.

These results are obtained by leveraging the surprisingly simple form of the Cuntz-Pimsner algebra as obtained in [42, Corollary 5.16] to compute the K-theory of $C^*_e(\mathcal{T}_+(P))$. We compute the K-theory of $C^*_e(\mathcal{T}_+(P))$ and use extension theory (especially work by Paschke and Salinas [98]) to obtain our classification results. We should note that Dilian Yang pointed out to us that there was a gap in the proof of [42, Corollary 5.16], which we resolve in Section 4.2 of this chapter.

Finally, it is natural to ask about the relationship between $C^*_e(\mathcal{T}_+(P))$ and the Cuntz-Krieger algebra $\mathcal{O}_{\mathrm{Adj}(P)}$ associated to the $\{0,1\}$-adjacency matrix obtained from a finite irreducible stochastic matrix P. We apply our classification results for the C*-envelope and K-theory for Cuntz-Krieger algebras to show that these two objects are generally incomparable in the sense that we exhibit 3×3 irreducible stochastic matrices P, Q and R such that

$$\begin{aligned} C^*_e(\mathcal{T}_+(P)) \not\sim C^*_e(\mathcal{T}_+(Q)) \cong C^*_e(\mathcal{T}_+(R)) \\ \text{and} \qquad \mathcal{O}_{\mathrm{Adj}(P)} \cong \mathcal{O}_{\mathrm{Adj}(Q)} \not\sim \mathcal{O}_{\mathrm{Adj}(R)} \end{aligned}$$

where $\cong$ stands for $*$-isomorphism and $\sim$ stands for stable isomorphism.

Hence, the significance of this chapter is that it provides a new class of C*-algebras, via the C*-envelope, that are amenable to classification machinery.

This chapter has five sections including this introduction. In Section 4.2 we fill the gap pointed out to us by Dilian Yang in the proof of [42, Corollary 5.16] and compute the extension groups for the Cuntz-Pimsner algebra of a finite irreducible stochastic matrix. In Section 4.3 we determine the non-commutative Choquet boundary of $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$, which then allows us to compute C*-envelopes $C^*_e(\mathcal{T}_+(P))$ associated to finite irreducible stochastic matrices. In Section 4.4 we compute the K-theory of $C^*_e(\mathcal{T}_+(P))$ in terms of boundary representations and use extension theory to prove our main classification results mentioned above. Finally, we compare $C^*_e(\mathcal{T}_+(P))$ and $\mathcal{O}_{\mathrm{Adj}(P)}$ as invariants for the graph of P.

4.2 Cuntz-Pimsner algebra of a stochastic matrix

In this section, we close a gap in [42], that was kindly pointed out to us by Dilian Yang, in the proof of the characterization of the Cuntz-Pimsner algebra of a finite stochastic matrix. Recall Theorem 2.3.17 that characterizes the subproduct system associated to a finite stochastic matrix P over Ω. We denote by $\mathcal{O}(P) := \mathcal{O}(\mathrm{Arv}(P))$ the Cuntz-Pimsner algebra of the subproduct system $\mathrm{Arv}(P)$. The result at stake, which corresponds to [42, Corollary 5.16], is as follows

Theorem 4.2.1. *Let P be an irreducible stochastic matrix of size d. Then $\mathcal{O}(P) \cong M_d(\mathbb{C}) \otimes C(\mathbb{T})$.*

The main issue is that the cyclic decomposition of periodic irreducible stochastic matrices need not be realized in square blocks as claimed in [42, Remark 2.9]. Consider the following example kindly brought to our attention by Dilian Yang: let $\Omega = \{1,2,3\}$ and set

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & P_0 \\ P_1 & 0 \end{bmatrix}, \qquad \text{where} \quad P_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The matrix P has period 2, $\Omega_0 = \{1,2\}$ and $\Omega_1 = \{3\}$, and both P_0 and P_1 are not square.

Therefore, the results in [42, Section 5] after [42, Proposition 5.7], and in particular [42, Proposition 5.15], only apply in the case of square blocks. Therefore a gap remains in the proof of Theorem 4.2.1 in its stated form, and we now provide a different proof for it, which works in all cases, and resolves any remaining gaps with the rest of [42, Section 5]. The issue above does not affect the remainder of that paper, namely [42, Sections 6 and 7].

We recall the different algebras $\mathcal{T}(P)$ and $\mathcal{T}^{\infty}(P)$ that were involved in the computation of $\mathcal{O}(P)$. For $A \in \mathrm{Arv}(P)_n$ in [42] we defined the shift operator $S_A^{(n)}$ uniquely determined on fibers by $S_A^{(n)}(B) = U_{n,m}(A \otimes B)$ for $B \in \mathrm{Arv}(P)_m$. The tensor and Toeplitz algebras of $\mathrm{Arv}(P)$ are given respectively by

$$\mathcal{T}_+(P) = \overline{Alg}\Big(C(\Omega) \cup \{ \ S_A^{(n)} \mid n \in \mathbb{N}, \ A \in \mathrm{Arv}(P)_n \ \} \Big)$$

and

$$\mathcal{T}(P) = C^*\Big(C(\Omega) \cup \{ \ S_A^{(n)} \mid n \in \mathbb{N}, \ A \in \mathrm{Arv}(P)_n \ \} \Big).$$

We note that $\mathcal{T}_+(P)$ and $\mathcal{T}(P)$ are generated (as a norm-closed algebra and as a C*-algebra respectively) by $\{p_i\}_{i\in\Omega}$ and $\{S_{E_{ij}}\}_{(i,j)\in \mathrm{Gr}(P)}$, where $E_{ij} = [\delta_{ij}(k,l)]$ is the zero matrix, except for the (i,j) entry at which it is 1. Indeed, since P is a finite matrix, each $S_A^{(n)}$ can be written as a finite linear combination of $S_{E_{ij}}^{(n)}$ with $(i,j) \in \mathrm{Gr}(P^n)$. Then choose a path of length n, say $i = j_0 \to j_1 \to ... \to j_n = j$, and we have that $S_{E_{ij}}^{(n)} = c \cdot S_{E_{j_0 j_1}}^{(1)} \cdot ... \cdot S_{E_{j_{n-1} j_n}}^{(1)}$ for some $c > 0$.

Next, for a finite stochastic matrix P over Ω, and for every $n \in \mathbb{N}$ and $A \in \mathrm{Arv}(P)_n$ we defined operators in $\mathcal{L}(\mathcal{F}_{\mathrm{Arv}(P)})$ mapping each $\mathrm{Arv}(P)_m$ to $\mathrm{Arv}(P)_{n+m}$, one denoted by $T_A^{(n)}$ and given by $T_A^{(n)} = S^{(n)}_{(\sqrt{P^n})^\flat * A}$, and the other denoted by $W_A^{(n)}$ which is uniquely determined on fibers $\mathrm{Arv}(P)_m$ by $W_A^{(n)}(B) = A \cdot B$. For the purposes of computing the Cuntz-Pimsner algebra, we defined in [42, Section 5] the auxiliary C*-algebra

$$\mathcal{T}^\infty(P) := C^*\Big(C(\Omega) \cup \{\ W_A^{(n)} \mid n \in \mathbb{N}, \quad A \in \mathrm{Arv}(P)_n\ \}\Big)$$

and we noted that due to finiteness of P we have that

$$\mathcal{T}(P) = C^*\Big(C(\Omega) \cup \{\ T_A^{(n)} \mid n \in \mathbb{N}, \quad A \in \mathrm{Arv}(P)_n\ \}\Big).$$

We also defined the two-sided ideal

$$\mathcal{J}(\mathcal{T}^\infty(P)) := \{\ T \in \mathcal{T}^\infty(P) \mid \lim_{n\to\infty} \|TQ_n\| = 0\ \}$$

in $\mathcal{T}^\infty(P)$, and by [42, Proposition 5.6] we get that $\mathcal{O}(P)$ is $*$-isomorphic to the quotient $\mathcal{T}^\infty(P)/\mathcal{J}(\mathcal{T}^\infty(P))$. This reduced the computation of the Cuntz-Pimsner algebra to computing a quotient of an algebra generated by operators $W_A^{(n)}$ which depend only on the non-zero entries of P.

Recall that the adjoint $W_A^{(n)*}$ of $W_A^{(n)}$, which maps $\mathrm{Arv}(P)_{n+m}$ to $\mathrm{Arv}(P)_m$, is uniquely determined on fibers by

$$W_A^{(n)*}(B) = \mathrm{Adj}(P^m) * (A^*B), \qquad B \in Arv(P)_{m+n}$$

where the reason for Schur-multiplying $A^* \cdot B$ with $\mathrm{Adj}(P^m)$ is to make sure that the product lands in $\mathrm{Arv}(P)_m$ with its given entry constraints (See the discussion preceding [42, Proposition 5.7]).

We note that $C(\Omega)$ acts on $\mathrm{Arv}(P)_m$ as left multiplication by diagonal matrices. Therefore, $W^{(0)}_{E_{kk}} = p_k$ as the adjointable operator on $\mathcal{F}_{\mathrm{Arv}(P)}$.

The following proposition, which works in all cases, replaces [42, Remark 5.10] and the discussion preceding it.

Proposition 4.2.2. *Let P be an t-periodic irreducible stochastic matrix over Ω. Let $q \in \mathbb{N}$ and suppose that $A \in \mathrm{Arv}(P)_q$. Then there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ we have that*

$$W_A^{(q)*}(B) = A^*B, \qquad \forall B \in \mathrm{Arv}(P)_{q+m}.$$

That is, if $m \geq m_0$ and $B \in \mathrm{Arv}(P)_{q+m}$, then the matrix A^*B has support contained in the support of P^m.

Proof. Suppose this fails. Then there exists a sequence of matrices $B(n) \in \mathrm{Arv}(P)_{q+m_n}$, with $n \mapsto m_n$ increasing, such that the support of $A^*B(n)$ is not contained in the support of P^{m_n}. By finiteness of P, perhaps by replacing $B(n)$ by a subsequence, we may assume that there exist $i, j, k \in \Omega$ independent of n such that $B(n) \in \mathrm{Arv}(P)_{q+m_n}$ and both $p_i A^* p_k \neq 0$ and $p_k B(n) p_j \neq 0$ while $P_{ij}^{(m_n)} = 0$. Again by moving to a subsequence, we may assume that there exists $0 \leq \ell < t$ independent of n such that $m_n \equiv \ell \mod t$.

Let $\Omega_0, \ldots, \Omega_{t-1}$ be the cyclic decomposition of P with respect to k. Note that by item (2) of Theorem 2.3.14, we must have that $P_{ij}^{(m)} = 0$ for *all* m such that $m \equiv \ell \mod t$. Therefore there are no paths from i to j whose length is of residue $\ell \mod t$.

Let $\sigma(i), \sigma(j)$ be such that $i \in \Omega_{\sigma(i)}$ and $j \in \Omega_{\sigma(j)}$. Since $p_i A^* p_k \neq 0$, we have that $p_k A p_i \neq 0$ and hence $P_{ki}^{(q)} > 0$. It follows from the cyclic decomposition theorem that paths from k to i have length with residue $\sigma(i)$ (mod t, which we will suppress). Since paths from k to k must have lengths with zero residue by periodicity, we must have that paths from i to k will have length with residue $t - \sigma(i)$. Therefore paths from i to j have lengths with residue $t - \sigma(i) + \sigma(j) \equiv \sigma(j) - \sigma(i) \mod t$.

Next, since $A \in \mathrm{Arv}(P)_q$, and $p_k A p_i \neq 0$, we have by the definition of the cyclic decomposition that $\sigma(i) \equiv q \mod t$. Similarly, since $B(n) \in \mathrm{Arv}(P)_{q+m_n}$, and $p_k B(n) p_j \neq 0$, we have that $\sigma(j) \equiv q + m_n \equiv q + \ell \mod t$. Therefore, $\sigma(j) - \sigma(i) \equiv q + \ell - q \equiv \ell \mod t$ and we conclude that all paths from i to j must have residue ℓ mod t. But this is impossible since we have noted before that there are *no* paths from i to j whose length is of residue $\ell \mod t$. ☐

Definition 4.2.3. *Let P be a finite irreducible t-periodic stochastic matrix over Ω of size d. We will say that a cyclic decomposition $\Omega_0, ..., \Omega_{t-1}$ for P is* properly enumerated *if Ω is enumerated in such a way that for every $0 \leq m < k < t$, $i \in \Omega_m$ and $j \in \Omega_k$ we have that $i < j$. For $i \in \Omega$, denote by $\sigma(i)$ the unique index $0 \leq \sigma(i) < t$ such that $i \in \Omega_{\sigma(i)}$.*

Given a properly enumerated cyclic decomposition $\Omega_0, ..., \Omega_{t-1}$ for P, we define operators U and $(S_{ij})_{i,j\in\Omega}$ in $\mathcal{L}(\mathcal{F}_{\mathrm{Arv}(P)})$ as follows. The operator U has degree t with respect to the grading, i.e. for every $m \in \mathbb{N}$, $U(\mathrm{Arv}(P)_m) \subseteq \mathrm{Arv}(P)_{m+t}$, and it is uniquely determined by

$$U(B) = \mathrm{Adj}(P^{m+t}) * B, \qquad m \in \mathbb{N}, B \in \mathrm{Arv}(P)_m.$$

If $i \leq j$, then $\sigma(i) \leq \sigma(j)$, and denote by $\ell = \sigma(j) - \sigma(i)$. Then S_{ij} is an operator of degree ℓ, i.e. for all $m \geq 0$, $S_{ij}(\mathrm{Arv}(P)_m) \subseteq Arv(P)_{m+\ell}$ and it is given by

$$S_{ij}(B) = \mathrm{Adj}(P^{m+\ell}) * (E_{ij} \cdot B), \qquad m \in \mathbb{N}, B \in \mathrm{Arv}(P)_m.$$

If $i > j$ we define $S_{ij} = S_{ji}^*$. The family $(U, (S_{ij})_{i,j\in\Omega})$ is called the standard family *associated to the properly enumerated cyclic decomposition $\Omega_0, ..., \Omega_{t-1}$.*

We denote by $\overline{T} \in \mathcal{O}(P)$ the image of $T \in \mathcal{T}^\infty(P)$ under the associated canonical quotient map $q : \mathcal{T}^\infty(P) \to \mathcal{O}(P) \cong \mathcal{T}^\infty(P)/\mathcal{J}(\mathcal{T}^\infty(P))$.

Lemma 4.2.4. *Let P be an irreducible t-periodic stochastic matrix over Ω of size d with properly enumerated cyclic decomposition $\Omega_0, ..., \Omega_{t-1}$, and let $(U, (S_{ij})_{i,j\in\Omega})$ be its associated standard family.*

1. *Let $i, j \in \Omega$ be such that $i \leq j$ in the properly enumerated decomposition of Ω, so that $\ell := \sigma(j) - \sigma(i) \geq 0$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have*
$$S_{ij} = W_{I_d}^{(nt)*} W_{E_{ij}}^{(nt+\ell)} \quad \text{and} \quad U = W_{I_d}^{(nt)*} W_{I_d}^{(nt+t)}.$$
Hence, $U \in \mathcal{T}^\infty(P)$ and $S_{ij} \in \mathcal{T}^\infty(P)$ for all $i, j \in \Omega$.
2. *Let $i, j \in \Omega$. There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, and $B \in \mathrm{Arv}(P)_m$ we have that*
$$S_{ij}(B) = E_{ij}B, \qquad U(B) = B \qquad \text{and} \qquad U^*(B) = B.$$
3. *For all $i, j, s, k \in \Omega$ we have $S_{ij}S_{sk} - \delta_{js}S_{ik} \in \mathcal{J}(\mathcal{T}^\infty(P))$.*
4. *$U^*U - I, \;\; UU^* - I \in \mathcal{J}(\mathcal{T}^\infty(P))$.*
5. *For all $i, j \in \Omega$ we have $S_{ij}U - US_{ij} \in \mathcal{J}(\mathcal{T}^\infty(P))$.*
6. *The family $(\overline{U}, \{\overline{S_{ij}}\}_{i,j\in\Omega})$ generates $\mathcal{O}(P)$.*

Therefore, $\{\overline{S_{ij}}\}_{i,j\in\Omega}$ is a system of $d \times d$ matrix units in $\mathcal{O}(P)$ and $\overline{U}$ is a unitary in $\mathcal{O}(P)$ that commutes with them and together they generate $\mathcal{O}(P)$.

Proof.

1. Let $i, j \in \Omega$ be such that $i \leq j$ in the properly enumerated decomposition of Ω, so that $\ell := \sigma(j) - \sigma(i) \geq 0$. By item (2) of the cyclic decomposition Theorem 2.3.14, there exists $n_0 \in \mathbb{N}$ such that $E_{ij} \in \mathrm{Arv}(P)_{nt+\ell}$ and $I_d \in \mathrm{Arv}(P)_{nt}$ for all $n \geq n_0$. Then for all $n \geq n_0$, $m \in \mathbb{N}$ and $B \in \mathrm{Arv}(P)_m$ we have
$$W_{I_d}^{(nt)*} W_{E_{ij}}^{(nt+\ell)}(B) = \mathrm{Adj}(P^{m+\ell}) * (E_{ij}B) = S_{ij}(B)$$
$$W_{I_d}^{(nt)*} W_{I_d}^{(nt+t)}(B) = \mathrm{Adj}(P^{m+t}) * B = U(B)$$
so that
$$S_{ij} = W_{I_d}^{(nt)*} W_{E_{ij}}^{(nt+\ell)} \quad \text{and} \quad U = W_{I_d}^{(nt)*} W_{I_d}^{(nt+t)}.$$

2. Let $i \leq j \in \Omega$ be given. By the previous item and by Proposition 4.2.2 there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and $B \in \mathrm{Arv}(P)_m$ we have

$$S_{ij}(B) = W_{I_d}^{(nt)*} W_{E_{ij}}^{(nt+\ell)}(B) = E_{ij}B$$
$$U(B) = W_{I_d}^{(nt)*} W_{I_d}^{(nt+t)}(B) = B$$

Similarly, by taking adjoints, we have that

$$S_{ji}(B) = W_{E_{ji}}^{(nt+\ell)*} W_{I_d}^{(nt)}(B) = E_{ji}B$$
$$U^*(B) = W_{I_d}^{(nt+t)*} W_{I_d}^{(nt)}(B) = B$$

proving the statement in all cases.

3. Let $i, j, s, k \in \Omega$ be given. By item (2), there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and $B \in \mathrm{Arv}(P)_m$ we have that

$$S_{ij}S_{sk}(B) = E_{ij}E_{sk}B = \delta_{js}E_{ik}B = \delta_{js}S_{ik}(B).$$

Thus we have that $S_{ij}S_{sk} - \delta_{js}S_{ik} \in \mathcal{J}(\mathcal{T}^\infty(P))$.

4. By item (2), there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and $B \in \mathrm{Arv}(P)_m$ we have that

$$U^*U(B) = U^*(B) = B \in \mathrm{Arv}(P)_m \qquad \text{and} \qquad UU^*(B) = U^*(B) = B \in \mathrm{Arv}(P)_m.$$

Thus we have that $U^*U - I, UU^* - I \in \mathcal{J}(\mathcal{T}^\infty(P))$.

5. Let $i, j \in \Omega$ be given. By item (2), there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and $B \in \mathrm{Arv}(P)_m$ we have the following element in $\mathrm{Arv}(P)_{m+t+\ell}$ where $\ell = \sigma(j) - \sigma(i)$.

$$S_{ij}U(B) = S_{ij}(B) = E_{ij}B = U(E_{ij}B) = US_{ij}(B).$$

Thus we have that $S_{ij}U - US_{ij} \in \mathcal{J}(\mathcal{T}^\infty(P))$.

6. We first observe that since we are dealing with stochastic matrices over a *finite* state space Ω, it is in fact the case that $\mathcal{T}^\infty(P)$ is generated by $C(\Omega)$ and $\{W_{E_{ij}}^{(1)}\}_{(i,j)\in Gr(P)}$. Indeed, since every $\mathrm{Arv}(P)_n$ is finite dimensional, every $W_A^{(n)}$ can be written as a linear combination of elements of the form $W_{E_{ik}}^{(n)}$. Now, if $W_{E_{ik}}^{(n)}$ is non-zero, this means that $P_{ik}^{(n)} > 0$ and so there is a path of length n from i to k given by $i = j_0 \to$

$j_1 \to ... \to j_n = k$ and we would have that $W^{(n)}_{E_{ik}} = W^{(1)}_{E_{j_0 j_1}} \cdot ... \cdot W^{(1)}_{E_{j_{n-1} j_n}}$ so that every element $W^{(n)}_{E_{ik}}$ is in the algebra generated by $C(\Omega)$ and $\{W^{(1)}_{E_{ij}}\}_{(i,j)\in Gr(P)}$, and so

$$\mathcal{T}^\infty(P) = C^*\big(C(\Omega) \cup \{W^{(1)}_{E_{ij}}\}_{(i,j)\in Gr(P)}\big).$$

Therefore $\mathcal{O}(P)$ is generated as a C*-algebra by the images of $C(\Omega)$ and $\{W^{(1)}_{E_{ij}}\}_{(i,j)\in Gr(P)}$ under $q : \mathcal{T}^\infty(P) \to \mathcal{O}(P)$.

Let us denote by $\mathcal{A}$ the C*-subalgebra of $\mathcal{O}(P) \cong \mathcal{T}^\infty(P)/\mathcal{J}(\mathcal{T}^\infty(P))$ generated by $\overline{U}$ and $\overline{S_{ij}}$ for $i, j \in \Omega$. It follows from item (2) that $S_{ii} - p_i \in \mathcal{J}(\mathcal{T}^\infty(P))$, therefore, we have that $q(C(\Omega)) \subseteq \mathcal{A}$. In order to complete the proof that $\mathcal{A} = \mathcal{O}(P)$, it suffices to show that $\overline{W^{(1)}_{E_{ij}}} \in \mathcal{A}$ for all $(i, j) \in Gr(P)$.

Let $(i, j) \in Gr(P)$, and suppose that $t > 1$. If $i \le j$, then we must have by the cyclic decomposition theorem that $\sigma(j) - \sigma(i) = 1$ and S_{ij} is an operator of degree one and by item (2) we have that $S_{ij} - W^{(1)}_{E_{ij}} \in \mathcal{J}(\mathcal{T}^\infty(P))$. On the other hand, if $i > j$, then also by the cyclic decomposition theorem we must have $\sigma(i) - \sigma(j) = t - 1$ and in that case S_{ij} is an operator of degree $-(t - 1)$. Therefore US_{ij} has degree 1, and by item (2) we have that $US_{ij} - W^{(1)}_{E_{ij}} \in \mathcal{J}(\mathcal{T}^\infty(P))$. Therefore, in both cases we obtain that $\overline{W^{(1)}_{E_{ij}}} \in \mathcal{A}$.

Finally, if $(i, j) \in Gr(P)$, and $t = 1$, we have that S_{ij} is an operator of degree zero and by item (2) we have that $US_{ij} - W^{(1)}_{E_{ij}} \in \mathcal{J}(\mathcal{T}^\infty(P))$. Therefore, we also obtain that $\overline{W^{(1)}_{E_{ij}}} \in \mathcal{A}$. □

Recall from the discussion preceding [42, Proposition 5.5] that there is a natural gauge group action α on $\mathcal{T}^\infty(P)$ uniquely determined by $\alpha_\lambda(W^{(n)}_A) = \lambda^n W^{(n)}_A$. Since $\mathcal{J}(\mathcal{T}^\infty(P))$ is gauge invariant by [42, Theorem 5.6], this gauge action passes to the quotient $\mathcal{O}(P)$, and we denote by $\mathcal{O}(P)_0$ the fixed point algebra of this action.

Proof of Theorem 4.2.1. First note that $M_d(\mathbb{C}) \otimes C(\mathbb{T})$ is the universal C*-algebra generated by a system of $d \times d$ matrix units e_{ij} and a unitary u that commutes with them. Hence, by Lemma 4.2.4 we obtain a surjective *-homomorphism $\psi : M_d(\mathbb{C}) \otimes C(\mathbb{T}) \to \mathcal{O}(P)$ that sends e_{ij} to $\overline{S_{ij}}$ and u to $\overline{U}$. It remains to show that ψ is injective.

Let $\mathcal{A} = \bigoplus_{\ell=0}^{t-1} M_{|\Omega_\ell|}(\mathbb{C}) \otimes 1 \subseteq M_d(\mathbb{C}) \otimes C(\mathbb{T})$. First we note that ψ restricted to $\mathcal{A}$ is injective, since ψ is already injective when restricted to the larger simple subalgebra $M_d(\mathbb{C}) \otimes 1$.

We now show that $\psi(\mathcal{A}) = \mathcal{O}(P)_0$. First note that $\mathcal{O}(P)_0$ is generated by monomials of degree zero (according to the gauge action) in the matrix units $(\overline{S_{ij}})$ and the unitary $\overline{U}$, which commutes with the latter. Let $X \in \mathcal{O}(P)_0$ be such a monomial. Products of matrix units are also matrix units, therefore there exists $i, j \in \Omega$, $n \in \mathbb{Z}$ such that $X = \overline{S_{ij}}\,\overline{U}^n$. Hence, the only way that X has degree zero is if $n = 0$ and $\sigma(i) = \sigma(j)$. Moreover, $\mathcal{A}$ is precisely generated by all e_{ij}, with $i, j \in \Omega$ such that $\sigma(i) = \sigma(j)$. Hence $\psi(\mathcal{A}) = \mathcal{O}(P)_0$.

Next, we show ψ is injective on the entire algebra $M_d(\mathbb{C}) \otimes C(\mathbb{T})$. Given the identifications $M_d(\mathbb{C}) \otimes C(\mathbb{T}) \cong C(\mathbb{T}; M_d(\mathbb{C}))$ and $M_d(\mathbb{C}) \otimes 1 \cong M_d(\mathbb{C})$, let us consider the *faithful* conditional expectation $\Gamma_0 : M_d(\mathbb{C}) \otimes C(\mathbb{T}) \to M_d(\mathbb{C}) \otimes 1$ given by

$$\Gamma_0(T) = \int_{\mathbb{T}} T(z)\, dz$$

where dz represents normalized Haar measure on the circle. Note in particular that, for all $i, j \in \Omega$ and $n \in \mathbb{Z}$, we have $\Gamma_0(e_{ij}u^n) = \delta_{0,n}\, e_{ij}$.

We now take E_0 to be the *faithful* conditional expectation from $M_d(\mathbb{C}) \otimes 1$ to $\bigoplus_{\ell=0}^{t-1} M_{|\Omega_\ell|}(\mathbb{C}) \otimes 1$, and let $\Phi_0 : \mathcal{O}(P) \to \mathcal{O}(P)_0$ denote the canonical conditional expectation into the fixed point algebra associated with the gauge action. We then have that $\Phi_0\psi = \psi E_0 \Gamma_0$. Indeed, since for all $i, j \in \Omega, n \in \mathbb{N}$,

$$\begin{aligned}\Phi_0\psi(e_{ij}u^n) &= \Phi_0(\overline{S_{ij}}\,\overline{U}^n) = \delta_{0,n}\delta_{\sigma(i),\sigma(j)}\,\overline{S_{ij}} = \delta_{0,n}\delta_{\sigma(i),\sigma(j)}\,\psi(e_{ij}) = \delta_{0,n}\psi(E_0(e_{ij})) \\ &= \psi(E_0(\Gamma_0(e_{ij}u^n))),\end{aligned}$$

and since the linear span of monomials is dense, we have $\Phi_0 \circ \psi = \psi E_0 \Gamma_0$.

Finally, suppose towards contradiction that ψ is not injective. Then there exists a positive non-zero $T \in M_d(\mathbb{C}) \otimes C(\mathbb{T})$ such that $\psi(T) = 0$. In that case $\Phi_0(\psi(T)) = 0$. Hence $\psi(E_0(\Gamma_0(T))) = \Phi_0(\psi(T)) = 0$. By injectivity of ψ on the image of E_0, which is the algebra $\mathcal{A}$, we obtain $E_0(\Gamma_0(T)) = 0$. We reach a contradiction since E_0 and Γ_0 are faithful conditional expectations. □

Now that we have filled the gap in the computation of the Cuntz-Pimsner algebra of a finite irreducible stochastic matrix, we compute the extension groups for it, which will be useful to us later in Section 4.4.

Based on the work of [98], one has a description of $\mathrm{Ext}_s(\mathcal{B} \otimes M_d)$ for any unital C*-algebra $\mathcal{B}$, for which $\mathrm{Ext}_s(\mathcal{B})$ contains no elements of order d, as follows. For any unital extension $\eta \in E(\mathcal{B} \otimes M_d)$, we define a map $[\eta]_s \mapsto ([\iota_*\eta]_s, [j_*\eta]_s)$ into $\mathrm{Ext}_s(\mathcal{B}) \otimes \mathbb{Z}_d$ by setting $\iota_*\eta = \eta|_{\mathcal{B}\otimes I}$ and $j_*\eta = \eta|_{I\otimes M_d}$. Then [98, Proposition 2.2] shows that this map induces an isomorphism of semigroups

$$\mathrm{Ext}_s(\mathcal{B}\otimes M_d) \cong \{\ (d[\eta] + \epsilon(\ell)[\tau], \ell) \in \mathrm{Ext}_s(\mathcal{B}) \otimes \mathbb{Z}_d \mid \eta \in E(\mathcal{B}),\ \ell \in \mathbb{Z}\ \}$$

where τ is a trivial strongly unital extension. By Example 2.3.21 we have that $\epsilon(\ell)[\eta]_s = [\eta]_s$ for all $\eta \in \mathrm{Ext}_s(C(\mathbb{T}))$, so that

$$\mathrm{Ext}_s(C(\mathbb{T})\otimes M_d) \cong \{(ds, \ell) \in \mathbb{Z}\times\mathbb{Z}_d \mid s \in \mathbb{Z},\ \ell \in \mathbb{Z}\ \}$$

so that $\mathrm{Ext}_s(C(\mathbb{T})\otimes M_d) \cong d\mathbb{Z}\times\mathbb{Z}_d$ and $\mathrm{Ext}_w(C(\mathbb{T})\otimes M_d) \cong \mathbb{Z}$ as the projection (and division by d) onto the first coordinate of $\mathrm{Ext}_s(C(\mathbb{T})\otimes M_d)$. Since $\mathrm{Ext}_w(C(\mathbb{T})\otimes M_d) \cong \mathbb{Z}$ is the quotient of $\mathrm{Ext}_s(C(\mathbb{T})\otimes M_d) \cong d\mathbb{Z}\times\mathbb{Z}_d$ by the subgroup $\{\ \epsilon(n)[\tau]_s \mid n \in \mathbb{Z}\ \} \cong \mathbb{Z}_d$, we can identify the subgroup $\{\ \epsilon(n)[\tau]_s \mid n \in \mathbb{Z}\ \}$ of $\mathrm{Ext}_s(C(\mathbb{T})\otimes M_d)$ with the image $\{\ [j_*\eta]_s \mid [\eta]_s \in \mathrm{Ext}_s(C(\mathbb{T})\otimes M_d)\ \} \cong \mathbb{Z}_d$.

Note that any automorphism β of $C(\mathbb{T})\otimes M_d$ induces an automorphism β_s of $\mathrm{Ext}_s(C(\mathbb{T})\otimes M_d)$ by composition $[\eta]_s \mapsto [\eta\circ\beta]_s$. Furthermore, every unitary element $u \in \mathcal{U}(C(\mathbb{T})\otimes M_d)$ defines an automorphism Ad_u of $C(\mathbb{T})\otimes M_d$ by way of $\mathrm{Ad}_u(f)(z) = u^*(z)f(z)u(z)$ for $f \in C(\mathbb{T}; M_d)$ and $z \in \mathbb{T}$. Denote by $\mathrm{Aut}_{C(\mathbb{T})}(C(\mathbb{T})\otimes M_d)$ the collection of $C(\mathbb{T})$-bimodule *-automorphisms of $C(\mathbb{T})\otimes M_d$.

Proposition 4.2.5. *Let η be a unital extension and let $\beta \in \mathrm{Aut}(C(\mathbb{T})\otimes M_d)$ be an automorphism. Up to the identification $\mathrm{Ext}_s(C(\mathbb{T})\otimes M_d) \cong d\mathbb{Z}\otimes\mathbb{Z}_d$ given above, we have that either $\beta_s[\eta] = [\eta] = ([\iota_*\eta], [j_*\eta])$ or $\beta_s[\eta] = (-[\iota_*\eta], [j_*\eta])$.*

Proof. Let $\beta \in \mathrm{Aut}(C(\mathbb{T})\otimes M_d)$ be some *-automorphism. Then β induces an automorphism β_* on the primitive ideal spectrum $\mathbb{T}$, which then induces an automorphism $(\beta_*)^*$ back on $C(\mathbb{T})\otimes M_d$ given by $(\beta_*)^*(f)(z) = f(\beta_*^{-1}(z))$. It is easy to see that $[j_*\eta] = [j_*\eta\circ(\beta_*)^*]$ since $(\beta_*)^*(I\otimes M_d) = I\otimes M_d$. Since the induced map $((\beta_*)^*)_s$ on $\mathrm{Ext}_s(C(\mathbb{T})\otimes M_d) \cong d\mathbb{Z}\times\mathbb{Z}_d$ is the identity on the second coordinate $\mathbb{Z}_d$, we must have that $[\iota_*(\eta\circ(\beta_*)^*)]$ is either $[\iota_*\eta]$ or $-[\iota_*\eta]$. Hence, by composing with the inverse of $(\beta_*)^*$ if necessary, we may assume that $\beta_* = Id_{\mathbb{T}}$.

By [108, Corollary 5.46] we have that $\beta \in \mathrm{Aut}_{C(\mathbb{T})}(C(\mathbb{T})\otimes M_d)$, so that by [108, Lemma 4.28], there is a point-norm continuous map $\sigma : \mathbb{T} \to \mathrm{Aut}(M_d)$ such that $\beta(f)(z) = \sigma_z(f(z))$. Since the second cohomology group of the torus $H^2(\mathbb{T}; \mathbb{Z})$ vanishes, by [108, Theorem 5.42], there is a unitary element $u \in \mathcal{U}(C(\mathbb{T})\otimes M_d)$ such that $\beta = \mathrm{Ad}_u$. Then Ad_u induces a map on $\mathrm{Ext}_s(C(\mathbb{T})\otimes M_d)$, so that by the homomorphism property of the Fredholm index, we get that,

$$[\iota_*\eta\circ\mathrm{Ad}_u] = \mathrm{ind}(\eta(\mathrm{Ad}_u(z\otimes I))) = \mathrm{ind}(\eta(z\otimes I)) = [\iota_*\eta].$$

Next, since the image $\{ [j_*\eta]_s \mid [\eta]_s \in \mathrm{Ext}_s(C(\mathbb{T}) \otimes M_d) \} \cong \mathbb{Z}_d$ can be identified with the subgroup $\{ \epsilon(n)[\tau]_s \mid n \in \mathbb{Z} \}$, in order to show that $[j_*\eta \circ \beta] = [j_*\eta]$, it will suffice to show that $\beta_s(\epsilon(n)[\tau]_s) = \epsilon(n)[\tau]_s$. However, since β_s commutes with $\epsilon(n)$, it will suffice to show that $\beta_s([\tau]_s) = [\tau]_s$. But β_s is a group homomorphism, so it must send $[\tau]_s$ to itself. Hence, we obtain that $[j_*\eta \circ \beta] = [j_*\eta]$. □

4.3 C*-envelope and boundary for $\mathcal{T}_+(P)$

In this section we determine the irreducible representations of $\mathcal{T}(P)$ for a finite irreducible stochastic matrix P, and find which of those are boundary representations with respect to $\mathcal{T}(P)$. We show that any representation annihilating $\mathcal{J}(P) := \mathcal{J}(\mathrm{Arv}(P))$ has the unique extension property when restricted to $\mathcal{T}(P)$, and find conditions that guarantee when an irreducible representation supported on $\mathcal{J}(P)$ is boundary, or not. A good reference for the theory used in this section is [7].

As given in [42, Theorem 5.6], the C*-algebra $\mathcal{T}^c(P)$ is the one generated by both $\mathcal{T}^\infty(P)$ and $\mathcal{T}(P)$, and it too has a gauge action which is the restriction of the gauge action of $\mathcal{L}(\mathcal{F}_{\mathrm{Arv}(P)})$, which satisfies $\alpha_\lambda(S_A^{(n)}) = \lambda^n S_A^{(n)}$ and $\alpha_\lambda(W_A^{(n)}) = \lambda^n W_A^{(n)}$, so that $\mathcal{T}^c(P)$ is gauge invariant, and $\mathcal{J}(\mathcal{T}^c(P))$ is a closed gauge invariant two-sided ideal by [42, Theorem 5.6].

As discussed in [42, Section 4] for general subproduct systems, Fourier coefficients Φ_k on $\mathcal{T}^c(P)$ may be defined in such a way that every $T \in \mathcal{T}^c(P)$ can be written as $\sum_{k=-\infty}^{\infty} \Phi_k(T)$, where this sum converges Cesaro. That is, where $\sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \Phi_k(T)$ converges in norm to T.

Proposition 4.3.1. *Let P be an irreducible stochastic matrix on Ω of size d. Then $\mathcal{J}(\mathcal{T}^c(P))$ is the two sided ideal generated by $\{Q_n\}_{n\in\mathbb{N}}$ inside $\mathcal{T}^c(P)$.*

Proof. By [42, Proposition 5.2] we see that $Q_n \in \mathcal{T}(P) \subseteq \mathcal{T}^c(P)$, and since $\|Q_n Q_m\| \to 0$ as m goes to infinity, we see that $Q_n \in \mathcal{J}(\mathcal{T}^c(P))$.

For the reverse inclusion, let $T \in \mathcal{J}(\mathcal{T}^c(P))$, and write $T = \sum_{k=-\infty}^{\infty} \Phi_k(T)$ as a Cesaro convergent sum where $\Phi_k(T)$ maps $\mathrm{Arv}(P)_n$ to $\mathrm{Arv}(P)_{n+k}$ if $n + k \geq 0$ and $\{0\}$ otherwise. Further notice that $\Phi_k(T) \in \mathcal{J}(\mathcal{T}^c(P))$ for all $k \in \mathbb{Z}$, since by [42, Theorem 5.6] we have that $\mathcal{J}(\mathcal{T}^c(P))$ is gauge invariant. In this case, we have that $\|\Phi_k(T) Q_{[n+1,\infty)}\| = \sup_{m\geq n+1} \|\Phi_k(T) Q_m\| \to 0$. Hence, since $\Phi_k(T) Q_{[0,n]}$ is in the ideal generated by $\{Q_n\}_{n\in\mathbb{N}}$, we see that $\Phi_k(T)$ is in the closed ideal generated by $\{Q_n\}_{n\in\mathbb{N}}$ and so must be T by Cesaro approximation. □

For a finite irreducible stochastic matrix P with state set Ω of size d, we have that $C(\Omega)$ is faithfully represented in $B(\ell^2(\Omega))$ by diagonal matrix multiplication on columns. Hence by [108, Corollary 2.74], this faithful *-representation promotes to a faithful *-representation $\pi : \mathcal{L}(\mathcal{F}_{\mathrm{Arv}(P)}) \to B(\mathcal{F}_{\mathrm{Arv}(P)} \otimes_{id} \ell^2(\Omega))$ given by $\pi(T)(\xi \otimes h) = T\xi \otimes h$. Note that $\mathcal{F}_{\mathrm{Arv}(P)} \otimes_{id} \mathbb{C}e_k$ is a reducing subspace for $\pi(\mathcal{T}^c(P))$ for each $k \in \Omega$.

Notation 4.3.2. *For a state $k \in \Omega$ we will find it useful to denote $\mathrm{Arv}(P)_{n,k} := \mathrm{Arv}(P)_n \otimes \mathbb{C}e_k$, and $\mathcal{F}_{P,k} := \oplus_{n=0}^{\infty} \mathrm{Arv}(P)_{n,k} = \mathcal{F}_{\mathrm{Arv}(P)} \otimes_{id} \mathbb{C}e_k$, the reducing Hilbert space for $\pi(\mathcal{T}^c(P))$ mentioned above, so that $\mathcal{F}_{\mathrm{Arv}(P)} \otimes \ell^2(\Omega) = \oplus_{k\in\Omega}\mathcal{F}_{P,k}$. For fixed n we also denote for $i \in \Omega$ with $(i,k) \in Gr(P^n)$ the elements $e_{ik}^{(n)} := E_{ik} \otimes e_k \in \mathrm{Arv}(P)_{n,k}$ which comprise a finite orthonormal basis for each $\mathrm{Arv}(P)_{n,k}$, so that for varying $n \in \mathbb{N}$ and $i \in \Omega$ with $(i,k) \in Gr(P^n)$ the collection $\{e_{ik}^{(n)}\}$ is an orthonormal basis for $\mathcal{F}_{P,k}$.*

Proposition 4.3.3. *Let P be an irreducible stochastic matrix over Ω of size d. Then for each $\pi_k : \mathcal{T}^c(P) \to B(\mathcal{F}_{P,k})$ given by $\pi_k(T) = \pi(T)|_{\mathcal{F}_{P,k}}$ we have that $\pi_k(\mathcal{T}(P))$ is an irreducible subalgebra of $B(\mathcal{F}_{P,k})$.*

Proof. By [42, Proposition 5.2] we see that $Q_n \in \mathcal{T}(P)$ for every $n \in \mathbb{N}$. Let $0 \neq \mathcal{H}' \subseteq \mathcal{F}_{P,k}$ be some non-zero invariant subspace. Since $\{\pi_k(Q_{[0,n]})\}$ converges SOT to the identity on $\mathcal{F}_{P,k}$, there is some minimal $n_0 \in \mathbb{N}$ such that $\pi_k(Q_{n_0})\xi \neq 0$ for some $\xi \in \mathcal{H}'$. In this case, $0 \neq \pi_k(Q_{n_0})\xi = A \otimes e_k \in \mathcal{H}' \cap \mathrm{Arv}(P)_{n_0,k}$ for some $A \in \mathrm{Arv}(P)_{n_0}$, so that there exists $j \in \Omega$ and some non-zero scalar $c \in \mathbb{C}$ with $0 \neq e_{jk}^{(n_0)} = c \cdot \pi_k(p_j Q_{n_0})\xi \in \mathcal{H}'$ where $(j,k) \in Gr(P^{n_0})$. This means that $e_{kk}^{(0)} = c_1 \pi_k(S_{E_{jk}}^{(n_0)*})(e_{jk}^{(n_0)}) \in \mathcal{H}'$, where $c_1 > 0$ is some scalar.

Thus, for $m \geq 0$ if $e_{ik}^{(m)}$ is some vector in $\mathrm{Arv}(P)_{m,k}$, we see that $e_{ik}^{(m)} = c_2 \pi_k(S_{E_{ik}}^{(m)})(e_{kk}^{(0)}) \in \mathcal{H}'$ where $c_2 > 0$ is some scalar. This shows that the set of elements $e_{ik}^{(m)}$ for all $m \geq 0$ and $(i,k) \in Gr(P^m)$ is in $\mathcal{H}'$, and this set of elements is an orthonormal basis for $\mathcal{F}_{P,k}$, and so $\mathcal{H}' = \mathcal{F}_{P,k}$. ☐

Hence, we see that π decomposes into $d = |\Omega|$ irreducible representations π_k as above, so that $\pi = \oplus_{k\in\Omega}\pi_k : \mathcal{T}^c(P) \to \oplus_{k\in\Omega} B(\mathcal{F}_{P,k})$. We next show that each $\pi_k|_{\mathcal{T}(P)}$ is in a distinct unitary equivalence class of irreducible representations for $\mathcal{T}(P)$.

Proposition 4.3.4. *Let P be a finite irreducible stochastic matrix on Ω and $k, k' \in \Omega$ be* distinct *indices. Then $\pi_k|_{\mathcal{T}(P)}$ and $\pi_{k'}|_{\mathcal{T}(P)}$ are not unitarily equivalent.*

Proof. Suppose that $k, k' \in \Omega$ are such that $\pi_k|_{\mathcal{T}(P)}$ and $\pi_{k'}|_{\mathcal{T}(P)}$ are unitarily equivalent. Then there is a unitary $U : \mathcal{F}_{P,k} \to \mathcal{F}_{P,k'}$ such that $U\pi_k(T) = \pi_{k'}(T)U$ for all $T \in \mathcal{T}(P)$.

For $j \in \Omega$, we have that $p_j Q_0 \in \mathcal{T}(P)$, so that $U\pi_k(p_j Q_0) = \pi_{k'}(p_j Q_0)U$. Apply this operator to $e^{(0)}_{kk} \in \mathrm{Arv}(P)_{0,k} \subseteq \mathcal{F}_{P,k}$ and get

$$\pi_{k'}(p_j Q_0)U(e^{(0)}_{kk}) = U\pi_k(p_j Q_0)(e^{(0)}_{kk}) = U(\delta_{jk} e^{(0)}_{kk}).$$

On the other hand $\pi_{k'}(Q_0)U(e^{(0)}_{kk})$ must have image in $\mathrm{Arv}(P)_{0,k'}$ so that $\pi_{k'}(Q_0)U(e^{(0)}_{kk}) = c \cdot e^{(0)}_{k'k'}$ for some non-zero $c \in \mathbb{C}$. But after applying $\pi_{k'}(p_j)$ we would obtain that

$$\pi_{k'}(p_j Q_0)U(e^{(0)}_{kk}) = c \cdot \pi_{k'}(p_j)(e^{(0)}_{k'k'}) = c \cdot \delta_{jk'} e^{(0)}_{k'k'}.$$

Thus, we see that if $k \neq k'$ then by taking $j = k$ we would obtain that $0 = c \cdot \delta_{jk'} e^{(0)}_{k'k'} = U(\delta_{jk} e^{(0)}_{kk}) \neq 0$ in contradiction. Hence, $\pi_k|_{\mathcal{T}(P)}$ and $\pi_{k'}|_{\mathcal{T}(P)}$ are not unitarily equivalent. ☐

Proposition 4.3.5. *Let P be a finite irreducible stochastic matrix on Ω. Then $\mathcal{J}(\mathcal{T}(P)) = \mathcal{J}(\mathcal{T}^c(P))$ and is *-isomorphic to $\oplus_{k\in\Omega}\mathcal{K}(\mathcal{F}_{P,k})$. Thus, we have that $\mathcal{T}^\infty(P) \subseteq \mathcal{T}^c(P) = \mathcal{T}(P)$.*

Proof. By Proposition 4.3.1, we have that $\mathcal{J}(\mathcal{T}^c(P))$ is the ideal generated by $\{Q_n\}_{n\in\mathbb{N}}$ inside $\mathcal{T}^c(P)$, and since $\pi(Q_n)$ is a finite rank operator, we see by Proposition 4.3.3 that $\pi_k(\mathcal{J}(\mathcal{T}^c(P)))$ and $\pi_k(\mathcal{J}(\mathcal{T}(P)))$ are irreducible compact operator subalgebras of $B(\mathcal{F}_{P,k})$ and hence by [7, Theorem 1.3.4] they must both be equal to $\mathcal{K}(\mathcal{F}_{P,k})$. Write the identity representation $Id : \pi(\mathcal{J}(\mathcal{T}^c(P))) \to \oplus_{k\in\Omega} B(\mathcal{F}_{P,k})$ as a direct sum of irreducible representations with multiplicity $Id = \bigoplus n(\zeta)\cdot\zeta$, where each ζ is a representative in the equivalence class of irreducible representation given by restriction to some $\mathcal{F}_{P,k}$ for some k. Then by Proposition 4.3.4 we have that $n(\zeta) = 1$ for all ζ and that $Id|_{\pi(\mathcal{J}(\mathcal{T}(P)))}$ has the same decomposition into irreducible representations as the one above. Since $\pi = \bigoplus \pi_k$ is injective on $\mathcal{J}(\mathcal{T}^c(P))$, we have that $\pi(\mathcal{J}(\mathcal{T}^c(P))) = \oplus_{k\in\Omega}\mathcal{K}(\mathcal{F}_{P,k}) = \pi(\mathcal{J}(\mathcal{T}(P)))$, and by taking the inverse of the faithful *-representation π, we obtain $\mathcal{J}(\mathcal{T}(P)) = \mathcal{J}(\mathcal{T}^c(P))$.

Finally, by [42, Proposition 5.5] we have that $\mathcal{T}(P) = \mathcal{T}(P) + \mathcal{J}(\mathcal{T}(P)) = \mathcal{T}(P) + \mathcal{J}(\mathcal{T}^c(P)) = \mathcal{T}^c(P)$ so that $\mathcal{T}^\infty(P) \subseteq \mathcal{T}^c(P) = \mathcal{T}(P)$. ☐

We next wish to parametrize all irreducible representations of $\mathcal{T}(P)$. Under the identification $\mathcal{O}(P) \cong C(\mathbb{T}, M_d)$ and $\mathcal{J}(\mathcal{T}(P)) \cong \oplus_{k\in\Omega}\mathcal{K}(\mathcal{F}_{P,k})$ we have the following exact sequence

$$0 \to \oplus_{k\in\Omega}\mathcal{K}(\mathcal{F}_{P,k}) \to \mathcal{T}(P) \to C(\mathbb{T}, M_d) \to 0.$$

If $\rho : \mathcal{T}(P) \to B(\mathcal{H})$ is a unital representation, by the discussion preceding [7, Theorem 1.3.4] it decomposes uniquely into a central direct sum of representations $\rho = \rho_{\mathcal{J}(P)} \oplus \rho_{\mathcal{O}(P)}$, where $\rho_{\mathcal{J}(P)}$ is the unique extension to $\mathcal{T}(P)$ of the restriction of ρ to $\mathcal{J}(\mathcal{T}(P))$, and $\rho_{\mathcal{O}(P)}$ annihilates $\mathcal{J}(\mathcal{T}(P))$. Hence, the spectrum of $\mathcal{T}(P)$ decomposes into a disjoint union of the spectrum of $\mathcal{J}(\mathcal{T}(P)) \cong \oplus_{k\in\Omega}\mathcal{K}(\mathcal{F}_{P,k})$ and the spectrum of $\mathcal{O}(P) \cong C(\mathbb{T}, M_d)$.

For $\lambda \in \mathbb{T}$, we define $\mathrm{ev}_\lambda : C(\mathbb{T}, M_d) \to M_d$ given by $\mathrm{ev}_\lambda([f_{ij}]) = [f_{ij}(\lambda)]$. Since ev_λ has range M_d, we obtain that $\mathrm{ev}_\lambda \circ q$ is an irreducible representation of $\mathcal{T}(P)$ where $q : \mathcal{T}(P) \to \mathcal{O}(P)$ is the quotient map. Note that every $\mathrm{ev}_\lambda \circ q$ is a d dimensional representation.

Corollary 4.3.6. *Let P be an irreducible stochastic matrix over Ω of size d. Then the spectrum of $\mathcal{T}(P)$ is parametrized by d irreducible representations of infinite dimension, each unitarily equivalent to some π_k, and a torus $\mathbb{T}$ of irreducible representations of dimension d that annihilate $\mathcal{J}(\mathcal{T}(P))$, each unitarily equivalent to $\mathrm{ev}_\lambda \circ q$ for some $\lambda \in \mathbb{T}$.*

Proof. If ρ is an irreducible representation of $\mathcal{T}(P)$ that does not annihilate $\mathcal{J}(\mathcal{T}(P))$, we have by [7, Theorem 1.3.4] that $\rho|_{\mathcal{J}(\mathcal{T}(P))}$ is also irreducible. We use π^{-1} to obtain an irreducible representation $\rho \circ \pi^{-1}$ of $\pi(\mathcal{J}(\mathcal{T}(P)))$. Since $\pi(\mathcal{J}(\mathcal{T}(P)))$ is a C*-algebra of compact operators, by [7, Theorem 1.4.4] every irreducible representation of it is unitarily equivalent to some restriction to some $\mathcal{F}_{P,k}$. Pushing this back via π we obtain that ρ is unitarily equivalent to some π_k.

For the other part, if ρ does annihilate $\mathcal{J}(\mathcal{T}(P))$, it induces an irreducible representation of $\mathcal{O}(P) \cong C(\mathbb{T}, M_d)$ by taking the quotient by $\mathcal{J}(\mathcal{T}(P))$. Since the irreducible representations of $C(\mathbb{T})$ are just point evaluations, and since $C(\mathbb{T})$ is strongly Morita equivalent to $C(\mathbb{T}, M_d)$, we see that ρ must be unitarily equivalent to the composition $\mathrm{ev}_\lambda \circ q$ of an evaluation $\mathrm{ev}_\lambda : C(\mathbb{T}, M_d) \to M_d$ given by $\mathrm{ev}_\lambda([f_{ij}]) = [f_{ij}(\lambda)]$ and the natural quotient map $q : \mathcal{T}(P) \to \mathcal{O}(P)$.

Thus, the spectrum of $\mathcal{T}(P)$ is parametrized by d irreducible representations of infinite dimension, and a torus $\mathbb{T}$ of irreducible representations of dimension d. $\square$

Lemma 4.3.7. *Let P be an irreducible stochastic matrix over a finite set Ω, and let $\epsilon > 0$. There exists $m \geq 1$ and $M > 0$ such that for every $(i, j) \in Gr(P)$ we have*

$$(1 + \epsilon)p_j \geq T^{(1)*}_{E_{ij}} T^{(1)}_{E_{ij}} - M \cdot Q_{[0,m]}.$$

Proof. For $E_{k\ell} \in \mathrm{Arv}(P)_m$ and $m \geq 1$, by definition of $T^{(1)}_{E_{ij}}$, we see that

$$T^{(1)}_{E_{ij}}(E_{k\ell}) = \delta_{j,k}\sqrt{\frac{P^{(m)}_{k\ell}}{P^{(m+1)}_{i\ell}}}E_{i\ell} \text{ and } T^{(1)*}_{E_{ij}}(E_{k\ell}) = \delta_{i,k}\sqrt{\frac{P^{(m)}_{j\ell}}{P^{(m+1)}_{k\ell}}}E_{j\ell}.$$

So that

$$T_{E_{ij}}^{(1)*}T_{E_{ij}}^{(1)}(E_{k\ell}) = \delta_{j,k}\sqrt{\frac{P_{k\ell}^{(m)}}{P_{i\ell}^{(m+1)}}}T_{E_{ij}}^{(1)*}(E_{i\ell}) = \frac{P_{j\ell}^{(m)}}{P_{i\ell}^{(m+1)}}p_j(E_{k\ell}).$$

By Theorem 2.3.15, there exists m such that $\frac{P_{j\ell}^{(m)}}{P_{i\ell}^{(m+1)}} \leq 1+\epsilon$ for all $(i,j) \in Gr(P)$ and $\ell \in \Omega$ such that $(j,\ell) \in Gr(P^m)$. Hence, if we take $M = \|Q_{[0,m]}T_{E_{ij}}^{(1)*}T_{E_{ij}}^{(1)}\|$ it follows that $(1+\epsilon)p_j \geq T_{E_{ij}}^{(1)*}T_{E_{ij}}^{(1)} - M \cdot Q_{[0,m]}$ as required. □

We next show that representations annihilating $\mathcal{J}(P)$ have unique extension property when restricted to $\mathcal{T}_+(P)$.

Proposition 4.3.8. *Let P be a finite irreducible stochastic matrix over Ω, and let $\rho : \mathcal{T}(P) \to B(\mathcal{H})$ be a *-representation such that $\rho(\mathcal{J}(P)) = \{0\}$. Then $\rho|_{\mathcal{T}_+(P)}$ has UEP.*

Proof. Let $\widetilde{\rho} : \mathcal{T}_+(P) \to B(\mathcal{K})$ be a maximal dilation of $\rho|_{\mathcal{T}_+(P)}$ such that $\mathcal{H}$ is a subspace of $\mathcal{K}$, and let $\psi : \mathcal{T}(P) \to B(\mathcal{K})$ be its (unique) extension to a *-representation. Denote

$$\psi(p_i) = \begin{bmatrix} \rho(p_i) & X_i \\ Y_i & Z_i \end{bmatrix} \quad \text{and} \quad \psi(T_{E_{ij}}^{(1)}) = \begin{bmatrix} \rho(T_{E_{ij}}^{(1)}) & X_{ij} \\ Y_{ij} & Z_{ij} \end{bmatrix}.$$

First note that since p_i is a self-adjoint projection, we get that

$$\begin{bmatrix} \rho(p_i) & X_i \\ Y_i & Z_i \end{bmatrix} = \psi(p_i) = \psi(p_i)\psi(p_i)^* = \begin{bmatrix} \rho(p_i)\rho(p_i)^* + X_iX_i^* & * \\ * & * \end{bmatrix}.$$

So that by taking the $(1,1)$ compression, we obtain that $X_iX_i^* = 0$, so that $X_i = 0$. Now, since $\psi(p_i)$ is self-adjoint, we see that we must also have that $Y_i = 0$.

Next, for $(i,j) \in Gr(P^m)$, Suppose

$$\psi(S_{E_{ij}}^{(m)}) = \begin{bmatrix} \rho(S_{E_{ij}}^{(m)}) & X(m)_{ij} \\ Y(m)_{ij} & Z(m)_{ij} \end{bmatrix}.$$

Observe that for all $m \geq 1$, by the proof of [42, Proposition 5.5] we have that

$$0 \leq Q_{[0,m-1]} = Id - \sum_{(i,j)\in Gr(P^m)} S_{E_{ij}}^{(m)}S_{E_{ij}}^{(m)*}.$$

Hence, by applying ψ to this equation, we obtain that

$$0 \le \psi(Q_{[0,m-1]}) = Id - \sum_{(i,j)\in Gr(P^m)} \psi(S^{(m)}_{E_{ij}})\psi(S^{(m)}_{E_{ij}})^*.$$

Then by compressing to the $(1,1)$ corner we get

$$0 \le Id - \sum_{(i,j)\in Gr(P^m)} \left[\rho(S^{(m)}_{E_{ij}})\rho(S^{(m)}_{E_{ij}})^* + X(m)_{ij}X(m)^*_{ij}\right] = - \sum_{(i,j)\in Gr(P^m)} X(m)_{ij}X(m)^*_{ij},$$

where the last equality follows due to the fact that ρ annihilates $\mathcal{J}(P)$. Hence we must have that $X(m)_{ij} = 0$ for all $(i,j) \in Gr(P^m)$, so that the $(1,1)$ compression of $\psi(Q_{[0,m]})$ is 0, and if we specify $m = 1$, and note that $S^{(1)}_{E_{ij}} = \sqrt{P_{ij}} \cdot T^{(1)}_{E_{ij}}$, then the above also yields that $X_{ij} = 0$ for all $(i,j) \in Gr(P)$.

Next, let $\epsilon > 0$. By Lemma 4.3.7 there exists $m \ge 1$ and $M > 0$ such that for all $(i,j) \in Gr(P)$ we have that

$$(1+\epsilon)p_j \ge T^{(1)*}_{E_{ij}}T^{(1)}_{E_{ij}} - M \cdot Q_{[0,m]}$$

Hence,

$$(1+\epsilon)\psi(p_j) \ge \psi(T^{(1)}_{E_{ij}})^*\psi(T^{(1)}_{E_{ij}}) - M \cdot \psi(Q_{[0,m]})$$

By compressing to the $(1,1)$ corner, we obtain that

$$(1+\epsilon)\rho(p_j) \ge \rho(T^{(1)}_{E_{ij}})^*\rho(T^{(1)}_{E_{ij}}) + Y^*_{ij}Y_{ij}$$

but $\rho(T^{(1)}_{E_{ij}})^*\rho(T^{(1)}_{E_{ij}}) = \rho(W^{(1)*}_{E_{ij}}W^{(1)}_{E_{ij}}) = \rho(p_j)$, so for every $\epsilon > 0$ we have that $\epsilon \cdot \rho(p_j) \ge Y^*_{ij}Y_{ij}$. Hence we have that $Y_{ij} = 0$ for all $(i,j) \in Gr(P)$.

Since $\mathcal{T}(P)$ is generated by $\{p_i\}_{i\in\Omega}$ and $\{T^{(1)}_{E_{ij}}\}_{(i,j)\in Gr(P)}$, we must have that ψ has ρ as a direct summand, so that $\widetilde{\rho}$ is a trivial dilation of $\rho|_{\mathcal{T}(P)}$. Hence, $\rho|_{\mathcal{T}(P)}$ is maximal, and must then have the unique extension property. □

We next define a notion that will help us detect when an irreducible π_k is *not* a boundary representation for $\mathcal{T}(P)$.

Definition 4.3.9. *Let P be a finite irreducible stochastic matrix over Ω. A state $k \in \Omega$ is called* exclusive *if whenever for $i \in \Omega$ and $n \in \mathbb{N}$ we have $P^{(n)}_{ik} > 0$, then $P^{(n)}_{ik} = 1$. We denote by Ω_e the set of all exclusive states in Ω.*

One should think of exclusive states as those states k such that for any n for which i leads to k in n steps, it cannot lead anywhere else in n steps.

Lemma 4.3.10. *Let P be a finite irreducible t-periodic stochastic matrix over Ω, and $\Omega_0, ..., \Omega_{t-1}$ be a cyclic decomposition for P. Suppose that $k \in \Omega_0$ is a state.*

1. *$|\Omega_0| > 1$ if and only if $k \notin \Omega_e$. In this case, any state in Ω_0 is non-exclusive and there is an n_0 such that for any $n \geq n_0$ and $i, j \in \Omega_0$ we have $0 < P_{ij}^{(tn)} < 1$.*
2. *Assume $k \notin \Omega_e$ and $k \neq s \in \Omega$ is some different state. If there is $k \neq k' \in \Omega_0$ such that $P_{k's}^{(m)} > 0$ whenever $P_{ks}^{(m)} > 0$ for all $m \in \mathbb{N}$, then there exists $n \in \mathbb{N}$ such that $0 < P_{kk}^{(tn)} < 1$ and for all $m \in \mathbb{N}$ with $(k, s) \in Gr(P^m)$ we have $P_{kk}^{(tn)} P_{ks}^{(m)} < P_{ks}^{(tn+m)}$.*

Proof. We first prove (1). Suppose $|\Omega_0| > 1$ and let $k \in \Omega_0$. By item (2) of Theorem 2.3.14 there is n_0 such that for all $n \geq n_0$ we would have that $P_{ij}^{(tn)} > 0$ for all $i, j \in \Omega_0$ and $n \geq n_0$. In particular, the second part of item (1) holds.

Thus, for some $j \in \Omega_0$ we have that $P_{jj}^{(tn)}, P_{jk}^{(tn)} > 0$, and since the j-th row sums up to 1 we get that $0 < P_{jk}^{(tn)} < 1$, and we conclude that $k \notin \Omega_e$.

For the converse, suppose $k \notin \Omega_e$. We show that $|\Omega_0| > 1$. Let $k' \in \Omega$ and n_0 be so that $0 < P_{k'k}^{(n_0)} < 1$, and let m_0 be large enough so that $P_{kk'}^{(m_0)} > 0$. Then

$$P_{kk}^{(m_0+n_0)} = \sum_{j \in \Omega} P_{kj}^{(m_0)} P_{jk}^{(n_0)} < \sum_{j \in \Omega} P_{kj}^{(m_0)} = 1.$$

On the other hand,

$$P_{kk}^{(m_0+n_0)} = \sum_{j \in \Omega} P_{kj}^{(m_0)} P_{jk}^{(n_0)} \geq P_{kk'}^{(m_0)} P_{k'k}^{(n_0)} > 0.$$

So we see that $0 < P_{kk}^{(m_0+n_0)} < 1$. Since the k-th row sums up to 1, there must be an $i \in \Omega$ different from k such that $P_{ki}^{(m_0+n_0)} > 0$ and by definition of the cyclic decomposition we have that $i \in \Omega_0$. This shows that $|\Omega_0| > 1$.

We now prove (2). By item (1) we can find n_0 so that $0 < P_{ij}^{(tn)} < 1$ for all $i, j \in \Omega_0$ and $n \geq n_0$. Now fix $m \in \mathbb{N}$ with $P_{ks}^{(m)} > 0$, so that by assumption $P_{k's}^{(m)} > 0$. Then

$$P_{ks}^{(tn+m)} = \sum_{i \in \Omega} P_{ki}^{(tn)} P_{is}^{(m)} \geq P_{kk'}^{(tn)} P_{k's}^{(m)} + P_{kk}^{(tn)} P_{ks}^{(m)} > P_{kk}^{(tn)} P_{ks}^{(m)}. \qquad \square$$

Proposition 4.3.11. *Let P be a finite t-periodic irreducible matrix over Ω and $\Omega_0, ..., \Omega_{t-1}$ a cyclic decomposition for P. Let $k \in \Omega$.*

1. *If $k \in \Omega_0$ is non-exclusive and for any other non-exclusive $s \neq k$ there is some $k \neq k' \in \Omega_0$ such that $P^{(m)}_{k's} > 0$ whenever $P^{(m)}_{ks} > 0$, then π_k is a boundary representation.*

2. *If k is exclusive then π_k is not a boundary representation.*

Proof. (1): Assume k is non-exclusive. We use [12, Theorem 7.2] to show that π_k is a strongly peaking representation according to [12, Definition 7.1]. Since the irreducible representations of $\mathcal{T}(P)$ are given by Corollary 4.3.6, it suffices to find an element $T \in \mathcal{T}_+(P)$ such that $\|\pi_k(T)\| > \|(\mathrm{ev}_\lambda \circ q)(T)\|$ for any $\lambda \in \mathbb{T}$ and such that $\|\pi_k(T)\| > \|\pi_s(T)\|$ for any $k \neq s$.

Choose $T = T^{(n)}_{E_{kk}}$, and wait until prescribing n is necessary. Recall Notation 4.3.2, so that

$$\|\pi_k(T)\| \geq \|\pi(T^{(n)}_{E_{kk}})(e^{(0)}_{kk})\| = \|\frac{1}{P^{(n)}_{kk}} e^{(n)}_{kk}\| = \frac{1}{P^{(n)}_{kk}}.$$

On the other hand, $q(T^{(n)}_{E_{kk}}) = (z \mapsto z^m E_{kk})$ for $m \in \mathbb{N}$ satisfying $n = tm$, so that

$$\|(\mathrm{ev}_\lambda \circ q)(T)\| = \|\mathrm{ev}_\lambda(z \mapsto z^m E_{kk})\| = |\lambda^m| = 1.$$

So we see that $\|\pi_k(T)\| > \sup_{\lambda \in \mathbb{T}} \|(\mathrm{ev}_\lambda \circ q)(T)\|$.

Next, fix $s \in \Omega$ with $k \neq s$. Since $T^*T \in \mathcal{L}(\mathcal{F}_{\mathrm{Arv}(P)})$ sends $\mathrm{Arv}(P)_m$ to $\mathrm{Arv}(P)_m$, it is a finite-block diagonal operator, so we must have that $T^*T|_{\mathcal{F}_{P,s}} = (T^{(n)}_{E_{kk}})^*(T^{(n)}_{E_{kk}})|_{\mathcal{F}_{P,s}}$ is also finite-block diagonal. Denote $I(k,s) = \{ \ m \in \mathbb{N} \mid (k,s) \in Gr(P^m), \ m \geq 1 \ \}$, and note that since $T|_{\mathcal{F}_{P,s}}(\mathrm{Arv}(P)_{0,s}) = 0$, we have that

$$\|\pi_s(T)\|^2 = \|\pi_s(T^*T)\| = \|T^*T|_{\mathcal{F}_{P,s}}\| = \sup_{m \in \mathbb{N}} \|T^*T|_{\mathrm{Arv}(P)_{m,s}}\| =$$

$$\sup_{m \in I(k,s)} \|(T^{(n)}_{E_{kk}})^*(T^{(n)}_{E_{kk}})(e^{(m)}_{ks})\| = \sup_{m \in I(k,s)} \frac{P^{(m)}_{ks}}{P^{(n+m)}_{ks}} \|e^{(m)}_{ks}\| = \sup_{m \in I(k,s)} \frac{P^{(m)}_{ks}}{P^{(n+m)}_{ks}}.$$

By Theorem 2.3.15 we see that as $m \in I(k,s)$ goes to infinity, the fraction $\frac{P^{(m)}_{ks}}{P^{(n+m)}_{ks}}$ converges to the constant $\frac{\nu_s t}{\nu_s t} = 1$.

Hence, if $\sup_{m \in I(k,s)} \frac{P^{(m)}_{ks}}{P^{(n+m)}_{ks}} \leq 1$, as k is non-exclusive, we have that $P^{(n)}_{kk} < 1$ for large enough n, and so that $\|\pi_k(T)\| > \|\pi_s(T)\|$.

On the other hand, if $\sup_{m\in I(k,s)} \frac{P^{(m)}_{ks}}{P^{(n+m)}_{ks}} > 1$, then the supremum above is in fact a maximum, and s must be non-exclusive. By item (2) of Lemma 4.3.10 there is n large enough (which we now prescribe) so that $0 < P^{(n)}_{kk} < 1$ and $P^{(n)}_{kk} P^{(m)}_{ks} < P^{(n+m)}_{ks}$ for all $m \in I(k,s)$. Hence, we see that $\frac{1}{P^{(n)}_{kk}} > \frac{P^{(m)}_{ks}}{P^{(n+m)}_{ks}}$ for all $m \in I(k,s)$ so that still we obtain $\|\pi_k(T)\| > \|\pi_s(T)\|$.

To conclude, we have shown that

$$\|\pi_k(T)\| > \max\{\sup_{s\neq k}\{\|\pi_s(T)\|\}, \sup_{\lambda\in\mathcal{T}} \|(\mathrm{ev}_\lambda \circ q)(T)\|\},$$

so that by [12, Theorem 7.2] we have that π_k is a boundary representation.

(2): Suppose that k is exclusive. By the formula for $T^{(n)}_A$, we see that $\pi_k(T^{(n)}_A) = \pi_k(W^{(n)}_A)$. Indeed, this follows since any weights appearing in an application of $T^{(n)}_A$ to a k-th column of a matrix $B \in \mathrm{Arv}(P)_m$ arise only from entries of the k-th columns of P^n, which are either 0 or 1 by exclusivity assumption on k.

We will use the above to show that π_k is not strongly peaking anywhere by showing that it is not strongly peaking at any $\sum_{n=-N}^N [T^{(n)}_{ij}] \in M_s(\mathcal{T}_+(P)^* + \mathcal{T}_+(P))$ where each $T^{(n)}_{ij} \in \mathcal{T}_+(P)^* + \mathcal{T}_+(P)$ is of degree $n \in [-N,N]$ (which must then be either of the form $T^{(n)}_A$ or $T^{(n)*}_A$). We also denote by W_{ij} the element (which is either of the form $W^{(n)}_A$ or $W^{(n)*}_A$ respectively) satisfying $\pi_k(T^{(n)}_{ij}) = \pi_k(W^{(n)}_{ij})$ above.

We note that there exists m_0 such that for all $m \geq m_0$ we have that $(U^m)^* W^{(n)}_{ij} U^m = W^{(n)}_{ij}$ for all i,j and $n \in [-N,N]$. We then have that

$$\|\pi^{(s)}_k(\sum_{n=-N}^N [T^{(n)}_{ij}])\| = \|\sum_{n=-N}^N [\pi_k(T^{(n)}_{ij})]\| = \|\sum_{n=-N}^N [\pi_k(W^{(n)}_{ij})]\| =$$

$$\|\sum_{n=-N}^N [\pi_k(U^{m*} W^{(n)}_{ij} U^m)]\| \leq \|[U^{m*}(\sum_{n=-N}^N W^{(n)}_{ij})U^m]\| \leq \|[Q_{[m,\infty)}(\sum_{n=-N}^N W^{(n)}_{ij})Q_{[m,\infty)}]\|.$$

So we see that by Proposition 2.2.16

$$\|\pi^{(s)}_k(\sum_{n=-N}^N [T^{(n)}_{ij}])\| \leq \lim_{m\to\infty} \|[(\sum_{n=-N}^N W^{(n)}_{ij})Q_{[m,\infty)}]\| = \|[q(\sum_{n=-N}^N W^{(n)}_{ij})]\| = \|q^{(s)}(\sum_{n=-N}^N [T^{(n)}_{ij}])\|.$$

Since $q^{(s)}(\sum_{n=-N}^{N}[T_{ij}^{(n)}]) \in C(\mathbb{T}, M_d) \otimes M_s$, there exists $\lambda \in \mathbb{T}$ such that

$$\|\pi_k^{(s)}(\sum_{n=-N}^{N}[T_{ij}^{(n)}])\| \leq \|q^{(s)}(\sum_{n=-N}^{N}[T_{ij}^{(n)}])\| = \|(\mathrm{ev}_\lambda \circ q)^{(s)}(\sum_{n=-N}^{N}[T_{ij}^{(n)}])\|.$$

Since elements of the form $\sum_{n=-N}^{N}[T_{ij}^{(n)}]$ with $T_{ij}^{(n)}$ of degree n are dense inside $M_s(\mathcal{T}_+(P)^* + \mathcal{T}_+(P))$, we see that for any $[V_{ij}] \in M_s(\mathcal{T}_+(P)^* + \mathcal{T}_+(P))$ we have

$$\|[\pi_k(V_{ij})]\| \leq \sup_{\lambda \in \mathbb{T}} \|[(\mathrm{ev}_\lambda \circ q)(V_{ij})\|$$

so that π_k cannot be strongly peaking. By [12, Theorem 7.2] we see that π_k is not a boundary representation. □

Remark 4.3.12. It is clear that for exclusive $k \in \Omega$ the representation π_k is not boundary by item (2) of Proposition 4.3.11. Item (1) in Proposition 4.3.11 above provides a sufficient condition for π_k to be boundary when k is non-exclusive. We believe that this condition is not necessary, however we do not have examples to that effect.

We next introduce a class of stochastic matrices for which we can completely identify the non-commutative Choquet boundary of $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$ in terms of the matrix P.

Definition 4.3.13. *Let P be a finite t-periodic irreducible stochastic matrix over Ω. We say that P has the* multiple-arrival *property if whenever $k, s \in \Omega - \Omega_e$ are distinct such that whenever k leads to s in n steps, then there exists $k \neq k' \in \Omega$ such that k' leads to s in n steps.*

Corollary 4.3.14. *Let P be a finite irreducible stochastic matrix over Ω, and $k \in \Omega$. If P has the multiple-arrival property, then π_k is a boundary representation if and only if $k \in \Omega - \Omega_e$. Hence, the non-commutative Choquet boundary of $\mathcal{T}_+(P)$ is parametrized by a circle $\mathbb{T}$ of irreducible representations of dimension d, each unitarily equivalent to one of $\mathrm{ev}_\lambda \circ q$ for $\lambda \in \mathbb{T}$, and $|\Omega - \Omega_e|$ irreducible representations of infinite dimension, each unitarily equivalent to one of π_k for $k \in \Omega - \Omega_e$.*

Proof. This follows directly since if P has multiple-arrival, then the conditions of Proposition 4.3.11 item (1) are automatically satisfied for any non-exclusive $k \in \Omega$, and item (2) of Proposition 4.3.11 then gives the reverse implication. □

There is an easy class of examples which automatically has the multiple arrival property. Suppose that P is an irreducible t-periodic stochastic matrix with cyclic decomposition $\Omega_0, ..., \Omega_{t-1}$. Then we may write

$$\begin{bmatrix} 0 & P_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{t-2} \\ P_{t-1} & \cdots & 0 & 0 \end{bmatrix}$$

for rectangle stochastic matrices $P_0, ..., P_{t-1}$. If all entries of the matrices $P_0, ..., P_{t-1}$ are non-zero, then P is called *fully-supported*, and has the multiple-arrival property.

Suppose P is a finite irreducible stochastic matrix P over Ω of size d. We next discuss $C^*_e(\mathcal{T}_+(P))$ and its spectrum. Denote by Ω_b the set of states k for which π_k is a boundary representation, which is a subset of $\Omega - \Omega_e$. Since for all $k \in \Omega$ and $\lambda \in \mathbb{T}$ we have that $\operatorname{Ker} \pi_k \subseteq \mathcal{J}(P) \subseteq \operatorname{Ker}(\mathrm{ev}_\lambda \circ q)$, and since the intersection of kernels of all boundary representations is the Shilov ideal $\mathcal{J}_{\mathcal{T}_+(P)}$ of $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$, we must have that $\mathcal{J}_{\mathcal{T}_+(P)} = \pi^{-1}(\oplus_{k\in\Omega-\Omega_b}\mathcal{K}(\mathcal{F}_{P,k}))$ is the Shilov ideal, thought of as a subalgebra of $\pi^{-1}(\oplus_{k\in\Omega}\mathcal{K}(\mathcal{F}_{P,k})) = \mathcal{J}(P)$.

We hence get the following short exact sequence

$$0 \longrightarrow \oplus_{k\in\Omega_b}\mathcal{K}(\mathcal{F}_{P,k}) \longrightarrow C^*_e(\mathcal{T}_+(P)) \longrightarrow C(\mathbb{T}, M_d) \longrightarrow 0 \tag{4.1}$$

while we identify $q_e(\mathcal{J}(P)) \subseteq C^*_e(\mathcal{T}_+(P))$ with $\oplus_{k\in\Omega_b}\mathcal{K}(\mathcal{F}_{P,k})$, where $q_e : \mathcal{T}(P) \to C^*_e(\mathcal{T}_+(P))$ is the quotient map by the Shilov ideal, which is completely isometric on $\mathcal{T}_+(P)$.

If $\rho : C^*_e(\mathcal{T}_+(P)) \to B(\mathcal{H})$ is a unital *-representation, it decomposes uniquely into a central direct sum of representations $\rho = \rho_{q_e(\mathcal{J}(P))} \oplus \rho_{\mathcal{O}(P)}$, where $\rho_{q_e(\mathcal{J}(P))}$ is the unique extension to $C^*_e(\mathcal{T}_+(P))$ of the restriction of ρ to $q_e(\mathcal{J}(P))$, and $\rho_{\mathcal{O}(P)}$ annihilates $q_e(\mathcal{J}(P))$. Hence, the spectrum of $C^*_e(\mathcal{T}_+(P))$ decomposes into a disjoint union of the spectrum of $\oplus_{k\in\Omega_b}\mathcal{K}(\mathcal{F}_{P,k})$ and the spectrum of $C(\mathbb{T}, M_d)$. That is, the spectrum of $C^*_e(\mathcal{T}_+(P))$ is comprised of $|\Omega_b|$ irreducible representations of infinite dimension, and a torus $\mathbb{T}$ of irreducible representations of dimension d that annihilate $q_e(\mathcal{J}(P))$.

Theorem 4.3.15. *Suppose that P is a finite irreducible matrix over Ω. Then $\mathcal{T}_+(P)$ has the unique extension property in $C^*_e(\mathcal{T}_+(P))$ via q_e. Moreover, if P has multiple-arrival, the Shilov ideal for $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$ is given by*

$$\mathcal{J}_{\mathcal{T}_+(P)} = \bigcap_{k\in\Omega-\Omega_e} \{ \, T \in \mathcal{J}(P) \mid \pi_k(T) = 0 \, \}$$

*and is *-isomorphic via π to $\oplus_{k\in\Omega_e}\mathcal{K}(\mathcal{F}_{P,k})$*

Proof. Let $\rho : C^*_e(\mathcal{T}_+(P)) \to B(\mathcal{H})$ be a *-representation. By the above discussion, we may decompose it into a central direct sum of representations $\rho = \rho_{q_e(\mathcal{J}(P))} \oplus \rho_{\mathcal{O}(P)}$, where $\rho_{q_e(\mathcal{J}(P))}$ is the unique extension to $C^*_e(\mathcal{T}_+(P))$ of the restriction of ρ to $q_e(\mathcal{J}(P))$, and $\rho_{\mathcal{O}(P)}$ annihilates $q_e(\mathcal{J}(P))$.

By Proposition 4.3.8 we have that $\rho_{\mathcal{O}(P)} \circ q_e$ has the unique extension property when restricted to $\mathcal{T}_+(P)$, so that $\rho_{\mathcal{O}(P)}$ has unique extension property when restricted to $q_e(\mathcal{T}_+(P))$ by invariance of UEP. Next, since $\rho_{q_e(\mathcal{J}(P))} \circ q_e = \oplus_{k\in\Omega_b} n_k \cdot \pi_k$ is a direct sum of *-representations, with certain multiplicities n_k, that have the UEP when restricted to $\mathcal{T}_+(P)$, by [12, Theorem 4.4] we have that $\rho_{q_e(\mathcal{J}(P))} \circ q_e$ has UEP when restricted to $\mathcal{T}_+(P)$. Hence, again by invariance of UEP, $\rho_{q_e(\mathcal{J}(P))}$ has UEP when restricted to $q_e(\mathcal{T}_+(P))$. By another application of [12, Theorem 4.4] we obtain that $\rho = \rho_{q_e(\mathcal{J}(P))} \oplus \rho_{\mathcal{O}(P)}$ also has UEP when restricted to $q_e(\mathcal{T}_+(P))$, so that $\mathcal{T}_+(P)$, which is completely isometric to $q_e(\mathcal{T}_+(P))$ via q_e, has the unique extension property within $C^*_e(\mathcal{T}_+(P))$.

For the second part, by Corollary 4.3.14 we know that $\Omega_e = \Omega - \Omega_b$. Furthermore, by Proposition 4.3.8, we have that $\mathrm{ev}_\lambda \circ q$ is a boundary representation for $\mathcal{T}_+(P)$ for any $\lambda \in \mathbb{T}$, and that $\mathcal{J}(P) = \bigcap_{\lambda\in\mathbb{T}} \mathrm{Ker}(\mathrm{ev}_\lambda \circ q)$. By the discussion preceding the theorem, we get that

$$\mathcal{J}_{\mathcal{T}_+(P)} = \mathcal{J}(P) \cap \bigcap_{k\in\Omega_b} \mathrm{Ker}(\pi_k) = \bigcap_{k\in\Omega-\Omega_e} \{ \, T \in \mathcal{J}(P) \mid \pi_k(T) = 0 \, \}.$$

□

We now give equivalent conditions that guarantee that the C*-envelope of $\mathcal{T}_+(P)$ is either the Toeplitz algebra, or the Cuntz-Pimsner algebra.

Corollary 4.3.16. *Let P be a finite irreducible stochastic matrix of size d with multiple-arrival. Then we have that $C^*_e(\mathcal{T}_+(P)) \cong \mathcal{T}(P)$ if and only if $\Omega_e = \emptyset$.*

In particular, if P is aperiodic and of size $d \geq 2$ with multiple-arrival, we have that $C^*_e(\mathcal{T}_+(P)) \cong \mathcal{T}(P)$.

Proof. By Theorem 4.3.15 we see that if $\Omega_e = \emptyset$, then $\mathcal{J}_{\mathcal{T}_+(P)} = \{0\}$ and so $\mathcal{T}(P)$ is the C*-envelope of $\mathcal{T}_+(P)$.

Conversely, if $\mathcal{T}(P) \cong C^*_e(\mathcal{T}_+(P))$, then $C^*_e(\mathcal{T}_+(P))$ has d irreducible representations of infinite dimension, which can only occur if $|\Omega_b| = d$. Since P has multiple-arrival, we see by Corollary 4.3.14 that $\Omega_b = \Omega - \Omega_e$, and so that $\Omega_e = \emptyset$.

For the second part, aperiodicity guarantees that $\Omega = \Omega_0$ is the cyclic decomposition for P, and as $d \geq 2$ we have that $|\Omega| > 1$. Hence, by Lemma 4.3.10 we get that $\Omega_e = \emptyset$, so by the first part $\mathcal{T}(P) \cong C^*_e(\mathcal{T}_+(P))$. □

Corollary 4.3.17. *Let P be a finite t-periodic irreducible stochastic matrix of size d, and let $\Omega_0, ..., \Omega_{t-1}$ be a cyclic decomposition for P. The following are equivalent:*

1. *$\Omega = \Omega_e$.*
2. *$|\Omega_\ell| = 1$ for all $\ell \in \mathbb{Z}_t$, or equivalently $t = d$.*
3. *$P : C(\Omega) \to C(\Omega)$ is a *-homomorphism.*
4. *$C^*_e(\mathcal{T}_+(P)) \cong \mathcal{O}(P)$.*

Proof. (1) ⇒ (2): By item (1) of Lemma 4.3.10 we see that $|\Omega_\ell| = 1$ for all $\ell \in \mathbb{Z}_t$.

(2) ⇒ (3): If all Ω_ℓ are of size 1, we see that the cyclic decomposition for P yields that P is in fact a permutation matrix of a single-cycle permutation, and is hence a homomorphism.

(3) ⇒ (4): The Arveson-Stinespring construction of a subproduct system generally yields a product system when applied to a *-homomorphism (See [113, Theorem 2.2]). Hence, $\mathrm{Arv}(P)$ is a product system, and its tensor algebra is the tensor algebra of a single correspondence, so by [119, Proposition 2.8] this is also true for the Cuntz-Pimsner algebra in our case. By [76, Theorem 3.7] we have $C^*_e(\mathcal{T}_+(P)) \cong \mathcal{O}(P)$.

(4) ⇒ (1): Assume towards contradiction that there is $k \in \Omega - \Omega_e$. In this case, let n be so that $0 < P^{(n)}_{kk} < 1$, and observe that $\|q(T^{(n)}_{E_{kk}})\| = \|q(W^{(n)}_{E_{kk}})\| = 1$, while for $E_{kk} \in \mathrm{Arv}(P)_0$ we have

$$\|T^{(n)}_{E_{kk}}\| \geq \|T^{(n)}_{E_{kk}}(E^{(0)}_{kk})\| = \frac{1}{P^{(n)}_{kk}} \|E^{(n)}_{kk}\| = \frac{1}{P^{(n)}_{kk}}.$$

This means that $q : \mathcal{T}(P) \to \mathcal{O}(P)$ is not isometric on $\mathcal{T}_+(P)^* + \mathcal{T}_+(P)$, and in particular, not *completely* isometric on $\mathcal{T}_+(P)^* + \mathcal{T}_+(P)$. By [12, Theorem 7.2], there is a boundary representation for $\mathcal{T}_+(P)$ coming from an extension to $\mathcal{T}(P)$, of an element in the spectrum of $\mathcal{J}(P)$, which then must be equivalent to one of the π_k. This means that $C^*_e(\mathcal{T}_+(P))$ has an irreducible representation of infinite dimension, which is impossible since $C^*_e(\mathcal{T}_+(P)) \cong C(\mathbb{T}, M_d)$ only has irreducible representations of dimension d. □

Example 4.3.18. *We next give an example of* 3×3 *stochastic matrix for which* $C^*_e(\mathcal{T}_+(P))$, $\mathcal{T}(P)$ *and* $\mathcal{O}(P)$ *are pairwise non *-isomorphic. Let*

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

be over $\Omega = \{1,2,3\}$. *The matrix* P *is fully-supported and we see that states* 1 *and* 2 *are non-exclusive, while* 3 *is exclusive. Hence,* $\Omega_b = \Omega - \Omega_e \subsetneq \Omega$. *Therefore, the Shilov ideal* $\mathcal{J}_{\mathcal{T}_+(P)} \cong \mathcal{K}(\mathcal{F}_{P,3}) \ncong \oplus_{j\in\Omega}\mathcal{K}(\mathcal{F}_{P,j}) \cong \mathcal{J}(P)$. *This yields a quotient* $C^*_e(\mathcal{T}_+(P))$ *for which* $C^*_e(\mathcal{T}_+(P))$, $\mathcal{T}(P)$ *and* $\mathcal{O}(P)$ *are pairwise non *-isomorphic.*

Without the irreducibility assumption on P, it is easy to construct intermediary C*-envelopes from "extremal" C*-envelopes. Indeed, if for finite stochastic matrices P and Q of sizes at least 2 we have that $C^*_e(\mathcal{T}_+(P)) = \mathcal{T}(P)$, and $C^*_e(\mathcal{T}_+(Q)) = \mathcal{O}(Q)$, then $R = P\oplus Q$ is a finite stochastic matrix such that $C^*_e(\mathcal{T}_+(R)) = C^*_e(\mathcal{T}_+(P))\oplus C^*_e(\mathcal{T}_+(Q)) = \mathcal{T}(P)\oplus\mathcal{O}(Q)$, and one can similarly use representation theory to show that $C^*_e(\mathcal{T}_+(R))$ is non *-isomorphic to $\mathcal{T}(R)$ nor to $\mathcal{O}(R)$.

However, by Remark 2.3.19 we know that P is irreducible if and only if the subproduct system Arv(P) is minimal according to Definition 2.2.12. The above example then shows that even under the assumption of irreducibility of the matrix, which is equivalent to minimality of the associated subproduct system, up to *-isomorphism the C*-envelope may be distinct from both the Cuntz-Pimsner algebra, and the Toeplitz algebra.

4.4 Classification of C*-envelopes

In this section we compute the K-theory of, and classify up to *-isomorphism and stable isomorphisms, the C*-envelopes $C^*_e(\mathcal{T}_+(P))$ for a finite irreducible stochastic matrix P. Recall Notation 4.3.2. Let Ω_b be the collection of $k \in \Omega$ for which π_k is a boundary representation. We will henceforth identify $\mathcal{T}(P)$ with its image under $\pi : \mathcal{T}(P) \to B(\oplus_{k\in\Omega}\mathcal{F}_{P,k})$, where $\mathcal{F}_{P,k}$ are the invariant subspaces of π and $\pi_k : \mathcal{T}(P) \to B(\mathcal{F}_{P,k})$ the irreducible, pairwise non-unitarily equivalent representations given by restriction $\pi_k(T) = T|_{\mathcal{F}_{P,k}}$, for each $k \in \Omega$. Recall the short exact sequence from equation (4.1). We refer to [111] for the K–theory results used in this section.

We know from [111] that K_0 and K_1 are additive functors, and that for any $k \in \Omega$ we have $K_1(\mathcal{K}(\mathcal{F}_{P,k})) = \{0\}$, and $K_0(C(\mathbb{T}, M_d)) \cong K_0(\mathcal{K}(\mathcal{F}_{P,k})) \cong K_1(C(\mathbb{T}, M_d)) \cong \mathbb{Z}$. Hence, the six-term exact sequence of K-theory induced from the exact sequence of equation (4.1) yields

$$\begin{array}{ccccc} 0 & \longrightarrow & K_1(C^*_e(\mathcal{T}_+(P))) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \delta_1 \\ \mathbb{Z} & \longleftarrow & K_0(C^*_e(\mathcal{T}_+(P))) & \longleftarrow & \mathbb{Z}^{|\Omega_b|} \end{array} \tag{4.2}$$

Our first goal is to compute the index map $\delta_1 : K_1(C(\mathbb{T}, M_d)) \to K_0(\oplus_{k\in\Omega_b}\mathcal{K}(\mathcal{F}_{P,k}))$, which will then enable the computation of the K_0 and K_1 groups for $C^*_e(\mathcal{T}_+(P))$. It will suffice to compute the value of δ_1 on a generator of $K_1(C(\mathbb{T}, M_d)) \cong \mathbb{Z}$, and in our computations we will work with the unitary element $w := z \mapsto \mathrm{diag}(z, 1, ..., 1) \in C(\mathbb{T}, M_d)$, as $[w]_1$ is a generator for $K_1(C(\mathbb{T}, M_d)) \cong \mathbb{T}$.

Lemma 4.4.1. *Let P be a t-periodic irreducible stochastic matrix over $\Omega = \{1, ..., d\}$, with properly enumerated cyclic decomposition $\Omega_0, ..., \Omega_{t-1}$ such that $1 \in \Omega_0$ is the first element, and let $(U, (S_{ij})_{i,j\in\Omega})$ be its associated standard family.*

1. *For all $i \in \Omega$ we have that $(S_{ii} = p_i)_{i\in\Omega}$ is a family of pairwise orthogonal projections that commute with U.*
2. *The element $w := z \mapsto \mathrm{diag}(z, 1, ..., 1) \in C(\mathbb{T}, M_d)$ lifts to a partial isometry $V := US_{11} + S_{22} + ...S_{dd}$ inside $\mathcal{T}(P)$.*

Proof. (1): By definition, for any $m \in \mathbb{N}$ and $E_{jk} \in \mathrm{Arv}(P)_m$ we have that $S_{ii}(E_{jk}) = \mathrm{Adj}(P^m) * (E_{ii} \cdot E_{jk}) = \delta_{i,j}E_{jk} = p_i(E_{jk})$ so that $S_{ii} = p_i$. Next, note that

$$US_{ii}(E_{jk}) = \mathrm{Adj}(P^{m+r}) * (\delta_{ij}E_{jk}) = \delta_{ij}U(E_{jk}) = S_{ii}U(E_{jk}),$$

so that U and S_{ii} commute on the dense subset of $\mathcal{F}_{\mathrm{Arv}(P)}$, and hence commute.

(2): It is clear that w lifts to V inside $\mathcal{T}(P)$ since under the identification $C(\mathbb{T}) \otimes M_d \cong C(\mathbb{T}; M_d)$, the element $\overline{V}$ in the quotient is identified with w. Hence, we need only verify that V is a partial isometry. Indeed, since by item (1), U commutes with S_{11}, and since U is a partial isometry, we have that

$$VV^*V = S_{11}UU^*US_{11} + S_{22} + ...S_{dd} = V$$

so that V is also a partial isometry. □

Let v be the image of V under the C*-envelope quotient map $q_e : \mathcal{T}(P) \to C^*_e(\mathcal{T}_+(P))$. By Lemma 4.4.1 we know that V is a partial isometry that lifts w, and hence v is a partial isometry that lifts w. By item (ii) of [111, Proposition 9.2.5] we have that $1 - v^*v$ and $1 - vv^*$ are projections in $\oplus_{k\in\Omega_b}\mathcal{K}(\mathcal{F}_{P,k}) \cong q_e(\mathcal{J}(P))$ with

$$\delta_1([w]_1) = [1 - v^*v]_0 - [1 - vv^*]_0.$$

But due to the identification $K_0(\oplus_{k\in\Omega_b}\mathcal{K}(\mathcal{F}_{P,k})) \cong \oplus_{k\in\Omega_b}K_0(\mathcal{K}(\mathcal{F}_{P,k}))$, we obtain that

$$\delta_1([w]_1) = [1 - v^*v]_0 - [1 - vv^*]_0 = \Big([(1 - V^*V)|_{\mathcal{F}_{P,k}}] - [(1 - VV^*)|_{\mathcal{F}_{P,k}}]\Big)_{k\in\Omega_b} =$$

$$\Big(\dim \mathrm{Ker}(V|_{\mathcal{F}_{P,k}}) - \dim \mathrm{Ker}(V^*|_{\mathcal{F}_{P,k}})\Big)_{k\in\Omega_b} = \big(\mathrm{ind}(V|_{\mathcal{F}_{P,k}})\big)_{k\in\Omega_b},$$

where we are then left with computing the Fredholm indices of $V|_{\mathcal{F}_{P,k}}$ for $k \in \Omega_b$.

Proposition 4.4.2. *Let P be a t-periodic irreducible stochastic matrix over $\Omega = \{1, ..., d\}$. Suppose that $\Omega_0, ..., \Omega_{t-1}$ is a properly enumerated cyclic decomposition such that $s \in \Omega$ is its first element, and let $(U, (S_{ij})_{i,j\in\Omega})$ be its associated standard family. Let $V_s := S_{11} + ... + S_{s-1,s-1} + US_{ss} + S_{s+1,s+1} + ... + S_{dd}$. Then for every $k \in \Omega$ we have that* $\mathrm{ind}(V_s|_{\mathcal{F}_{P,k}}) = -1$.

Proof. Up to conjugating P with a permutation matrix, we may assume that $s = 1$ is the first element. For each state $k \in \Omega$, let $\ell = \sigma(k) = \sigma(k) - \sigma(1)$, and denote by

$$b_n = \begin{cases} 1 & : P_{1k}^{(nt+\ell)} > 0 \\ 0 & : P_{1k}^{(nt+\ell)} = 0 \end{cases}$$

where $P_{1k}^{(0)} = 1$ if $k = 1$, and is 0 otherwise. Recall Notation 4.3.2. Since $\mathcal{F}_{P,k} = \oplus_{n=0}^{\infty} \mathrm{Arv}(P)_{n,k}$, and since V shifts only the first rows of the matrix A in an element $A \otimes e_k \in \mathrm{Arv}(P)_{n,k}$, we have for all $n \in \mathbb{N}$ that

$$\dim \mathrm{Ker}\, V|_{\mathrm{Arv}(P)_{n,k}} = b_n - b_{n+1}b_n$$

and for all $n \geq 1$ that

$$\dim \mathrm{Ker}\, V^*|_{\mathrm{Arv}(P)_{n,k}} = b_{n+1} - b_{n+1}b_n$$

due to the support of elements in $\mathrm{Arv}(P)_{n,k}$. Note also that for $n = 0$, and we get

$$\dim \mathrm{Ker}\, V^*|_{Arv(P)_{0,k}} = \begin{cases} 0 & : k \neq 1 \\ 1 & : k = 1. \end{cases}$$

Hence, if we sum up dimensions, we obtain that

$$\dim \operatorname{Ker} V|_{\mathcal{F}_{P,k}} = \sum_{n=0}^{\infty} b_n - b_{n+1}b_n$$

and

$$\dim \operatorname{Ker} V^*|_{\mathcal{F}_{P,k}} = \begin{cases} \sum_{n=0}^{\infty} b_{n+1} - b_{n+1}b_n & : k \neq 1 \\ 1 + \sum_{n=0}^{\infty} b_{n+1} - b_{n+1}b_n & : k = 1. \end{cases}$$

Thus,

$$\operatorname{ind}(V|_{\mathcal{F}_{P,k}}) = \dim \operatorname{Ker} V|_{\mathcal{F}_{P,k}} - \dim \operatorname{Ker} V^*|_{\mathcal{F}_{P,k}}$$

$$= \begin{cases} \sum_{n=0}^{\infty} b_n - b_{n+1} & \text{if } k \neq 1 \\ -1 + \sum_{n=0}^{\infty} b_n - b_{n+1} & \text{if } k = 1 \end{cases} = \begin{cases} 0 - 1 & \text{if } k \neq 1 \\ -1 + 1 - 1 & \text{if } k = 1. \end{cases}$$

Hence, we see that in any case, $\operatorname{ind}(V|_{\mathcal{F}_{P,k}}) = -1$, as required. □

Corollary 4.4.3. *Let P be an irreducible stochastic matrix over finite Ω. Then the index map $\delta_1 : \mathbb{Z} \to \mathbb{Z}^{|\Omega_b|}$ is given by $\delta_1(n) = -(n, ..., n)$*

We then obtain the K-theory of $C_e^*(\mathcal{T}_+(P))$ in terms of $|\Omega_b|$.

Theorem 4.4.4. *Let P be a finite irreducible stochastic matrix over Ω. Then*

1. *If P has a non-exclusive state then*
$$K_0(C_e^*(\mathcal{T}_+(P))) \cong \mathbb{Z}^{|\Omega_b|} \quad \textit{and} \quad K_1(C_e^*(\mathcal{T}_+(P))) \cong \{0\}.$$
2. *If all states of P are exclusive then*
$$K_0(C_e^*(\mathcal{T}_+(P))) \cong \mathbb{Z} \quad \textit{and} \quad K_1(C_e^*(\mathcal{T}_+(P))) \cong \mathbb{Z}.$$

Proof. If all states of P are exclusive, then by Corollary 4.3.17 we have that $C_e^*(\mathcal{T}_+(P)) \cong C(\mathbb{T}, M_d)$ so that the K_0 and K_1 groups of $C_e^*(\mathcal{T}_+(P))$ must both be $\mathbb{Z}$.

Next, if P has a non-exclusive state, since δ_1 is injective, by exactness at $K_1(C(\mathbb{T}, M_d))$ in the six-term exact sequence of equation (4.2), we see that $K_1(C_e^*(\mathcal{T}_+(P))) = \{0\}$.

Since $\delta_1(1) = (-1, ..., -1)$, we see that the six-term exact sequence in equation (4.2) can be reduced to the single exact sequence

$$0 \leftarrow \mathbb{Z} \leftarrow K_0(C_e^*(\mathcal{T}_+(P))) \leftarrow \mathbb{Z}^{|\Omega_b|}/Sp_{\mathbb{Z}}\big((-1, ..., -1)\big) \leftarrow 0.$$

Since $\mathbb{Z}^{|\Omega_b|}/Sp_{\mathbb{Z}}\big((-1, ..., -1)\big) \cong \mathbb{Z}^{|\Omega_b|-1}$, we see that $K_0(C_e^*(\mathcal{T}_+(P))) \cong \mathbb{Z}^{|\Omega_b|}$, and the proof is complete. □

Corollary 4.4.5. *Let P be a finite irreducible stochastic matrix over Ω. Then conditions (1) through (4) of Corollary 4.3.17 are all equivalent to $K_1(C_e^*(\mathcal{T}_+(P))) \cong \mathbb{Z}$.*

We next turn to extension theory to extract the more refined structure of our C*-envelope up to $*$-isomorphism and stable isomorphism. For every finite irreducible stochastic matrix P over Ω^P, which has at least one non-exclusive state, let Ω_b^P be the (non-empty) set of indices $k \in \Omega^P$ such that $\pi_k : \mathcal{T}(P) \to B(\mathcal{F}_{P,k})$ is a boundary representation for $\mathcal{T}_+(P)$. We note that $C_e^*(\mathcal{T}_+(P))$ is Type I (equivalently GCR), being an extension of a CCR algebra by a CCR algebra. Thus, we may identify an irreducible representation with its kernel when discussing elements of the primitive ideal spectrum of $C_e^*(\mathcal{T}_+(P))$.

The analysis done around the exact sequence of equation (4.1) shows that the spectrum of $C_e^*(\mathcal{T}_+(P))$ as a set is comprised of $|\Omega_b^P|$ irreducible representations of infinite dimensions induced from π_k, which we still denote by $\pi_k : C_e^*(\mathcal{T}_+(P)) \to B(\mathcal{F}_{P,k})$ for $k \in \Omega_b^P$, and a torus $\mathbb{T}$ of irreducible representations of dimension $|\Omega^P|$ given by $\mathrm{ev}_\lambda \circ q$ for every $\lambda \in \mathbb{T}$, where $q : C_e^*(\mathcal{T}_+(P)) \to C(\mathbb{T}, M_{|\Omega^P|})$ is the quotient map. Moreover, we have the exact sequence

$$0 \longrightarrow \oplus_{k\in\Omega_b^P}\mathcal{K}(\mathcal{F}_{P,k}) \xrightarrow{\iota} C_e^*(\mathcal{T}_+(P)) \xrightarrow{q} C(\mathbb{T}, M_{|\Omega_P|}) \longrightarrow 0.$$

Since $\mathrm{Ker}\,\pi_k \subseteq \mathrm{Ker}(\mathrm{ev}_\lambda \circ q)$ for every $k \in \Omega_b^P$ and every $\lambda \in \mathbb{T}$, and $\mathrm{Ker}(ev_\lambda \circ q)$ is not a subset of any $\mathrm{Ker}(\mathrm{ev}_{\lambda'} \circ q)$ for $\lambda' \neq \lambda \in \mathbb{T}$, we see that for every $\lambda \in \mathbb{T}$, each $\mathrm{Ker}(\mathrm{ev}_\lambda \circ q)$ is a maximal element in the lattice $\mathrm{Prim}(C_e^*(\mathcal{T}_+(P)))$.

Notation 4.4.6. *For a finite irreducible stochastic matrix P, we denote from now on $\mathcal{K}_P := \oplus_{k\in\Omega_b^P}\mathcal{K}(\mathcal{F}_{P,k})$, $\mathcal{B}_P := C(\mathbb{T}, M_{|\Omega_P|})$ and $\mathcal{A}_P := C_e^*(\mathcal{T}_+(P))$. We will also denote $\mathcal{K}$ the compact operators on separable infinite dimensional Hilbert space.*

Let P and Q be irreducible stochastic matrices over finite sets Ω^P and Ω^Q respectively. Then we have the following exact sequences

$$0 \to \mathcal{K}_P \to \mathcal{A}_P \to \mathcal{B}_P \to 0 \ \text{ and } \ 0 \to \mathcal{K}_Q \to \mathcal{A}_Q \to \mathcal{B}_Q \to 0 \tag{4.3}$$

with Busby invariants η_P and η_Q, and the stabilized exact sequences

$$0 \to \mathcal{K}_P \otimes \mathcal{K} \to \mathcal{A}_P \otimes \mathcal{K} \to \mathcal{B}_P \otimes \mathcal{K} \to 0 \ \text{ and } \ 0 \to \mathcal{K}_Q \otimes \mathcal{K} \to \mathcal{A}_Q \otimes \mathcal{K} \to \mathcal{B}_Q \otimes \mathcal{K} \to 0 \tag{4.4}$$

with Busby invariants $\eta_P^{(\infty)}$ and $\eta_Q^{(\infty)}$ given by $\eta_P^{(\infty)}([T_{ij}]) = [\eta_P(T_{ij})]$ for $[T_{ij}] \in B_P \otimes \mathcal{K}$.

For $k \in \Omega_b$ denote by $\rho_k : \oplus_{\ell\in\Omega_b^P} B(\mathcal{F}_{P,\ell}) \to B(\mathcal{F}_{P,k})$ the restriction map, which then promotes to a restriction $\widetilde{\rho}_k : \oplus_{\ell\in\Omega_b^P}\mathcal{Q}(\mathcal{F}_{P,\ell}) \to \mathcal{Q}(\mathcal{F}_{P,k})$.

Proposition 4.4.7. *Let P and Q be finite irreducible stochastic matrices over Ω^P and Ω^Q respectively.*

1. *$C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are *-isomorphic if and only if there exists a *-isomorphism $\beta : C(\mathbb{T}, M_{|\Omega_P|}) \to C(\mathbb{T}, M_{|\Omega_Q|})$ and a bijection $\tau : \Omega_b^P \to \Omega_b^Q$ such that for all $k \in \Omega_b^P$ the extensions $\widetilde{\rho}_k \eta_P$ and $\widetilde{\rho}_{\tau(k)} \eta_Q \beta$ are strongly equivalent.*

2. *$C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are stably isomorphic if and only if there exists a *-isomorphism $\beta : C(\mathbb{T}, M_{|\Omega_P|}) \otimes \mathcal{K} \to C(\mathbb{T}, M_{|\Omega_Q|}) \otimes \mathcal{K}$ and a bijection $\tau : \Omega_b^P \to \Omega_b^Q$ such that for all $k \in \Omega_b^P$ the extensions $\widetilde{\rho}_k^{(\infty)} \eta_P^{(\infty)}$ and $\widetilde{\rho}_{\tau(k)}^{(\infty)} \eta_Q^{(\infty)} \beta$ are weakly equivalent.*

Proof. We first show (1). Suppose that $\alpha : \mathcal{A}_P \to \mathcal{A}_Q$ is a *-isomorphism. Let $\alpha_* : \mathrm{Prim}(\mathcal{A}_P) \to \mathrm{Prim}(\mathcal{A}_Q)$ be the induced lattice isomorphisms between the spectra. Since α_* must send maximal elements to maximal elements, we see that for $\lambda \in \mathbb{T}$ we have that α_* sends $\mathrm{Ker}(\mathrm{ev}_\lambda^P \circ q)$ to $\mathrm{Ker}(\mathrm{ev}_{\lambda'}^Q \circ q)$ for some $\lambda' \in \mathbb{T}$ in bijection. In particular, since $\mathcal{K}_P = \cap_{\lambda \in \mathbb{T}} \mathrm{Ker}(\mathrm{ev}_\lambda^P \circ q)$ and $\mathcal{K}_Q = \cap_{\lambda \in \mathbb{T}} \mathrm{Ker}(\mathrm{ev}_\lambda^Q \circ q)$, we see that $\alpha(\mathcal{K}_P) = \mathcal{K}_Q$. Hence $C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are *-isomorphic if and only if the exact sequences of equation (4.3) are isomorphic, which happens if and only if the restriction $\kappa := \alpha|_{\mathcal{K}_P} : \mathcal{K}_P \to \mathcal{K}_Q$ and the induced map β satisfy $\widetilde{\kappa} \eta_P = \eta_Q \beta$, where $\beta : \mathcal{B}_P \to \mathcal{B}_Q$ is the induced *-isomorphism from α between the quotients by $\mathcal{K}_P$ and by $\mathcal{K}_Q$.

So suppose $\widetilde{\kappa} \eta_P = \eta_Q \beta$ for κ and β as above. Since $\kappa : \oplus_{k \in \Omega_b^P} \mathcal{K}(\mathcal{F}_{P,k}) \to \oplus_{k \in \Omega_b^Q} \mathcal{K}(\mathcal{F}_{Q,k})$, there is a bijection $\tau : \Omega_b^P \to \Omega_b^Q$ and a unitaries $U_k : \mathcal{F}_{P,k} \to \mathcal{F}_{Q,\tau(k)}$ such that $\kappa|_{\mathcal{K}(\mathcal{F}_{P,k})} = \mathrm{Ad}_{U_k} : \mathcal{K}(\mathcal{F}_{P,k}) \to \mathcal{K}(\mathcal{F}_{Q,\tau(k)})$, so that

$$\widetilde{\rho}_{\tau(k)} \eta_Q \beta = \widetilde{\rho}_{\tau(k)} \widetilde{\kappa} \eta_P = \widetilde{\mathrm{Ad}_{U_k}} \widetilde{\rho}_k \eta_P.$$

For the converse, if U_k are unitaries implementing strong equivalence between $\widetilde{\rho}_{\tau(k)} \eta_Q \beta$ and $\widetilde{\mathrm{Ad}_{U_k}} \widetilde{\rho}_k \eta_P$, by setting $\kappa = \oplus_{k \in \Omega_P^b} \mathrm{Ad}_{U_k} : \oplus_{k \in \Omega_b^P} \mathcal{K}(\mathcal{F}_{P,k}) \to \oplus_{k \in \Omega_b^Q} \mathcal{K}(\mathcal{F}_{Q,k})$, we have that

$$\widetilde{\kappa} \eta_P = \oplus_{k \in \Omega_b^P} \mathrm{Ad}_{U_k} \widetilde{\rho}_k \eta_P = \oplus_{k \in \Omega_b^Q} \widetilde{\rho}_{\tau(k)} \eta_Q \beta = \eta_Q \beta.$$

Next, we show (2). Since stabilizing an algebra does not change its primitive ideal spectrum, the same argument as used in (1) shows that $C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are stably isomorphic if and only if the exact sequences in equation (4.4) are isomorphic, which happens if and only if there are *-isomorphisms $\kappa : \mathcal{K}_P \otimes \mathcal{K} \to \mathcal{K}_Q \otimes \mathcal{K}$ and $\beta : \mathcal{B}_P \otimes \mathcal{K} \to \mathcal{B}_Q \otimes \mathcal{K}$ such that $\widetilde{\kappa} \eta_P^{(\infty)} = \eta_Q^{(\infty)} \beta$. Then a similar argument to the one used for item (1) shows that this happens if and only if there is a bijection $\tau : \Omega_b^P \to \Omega_b^Q$ such that for all $k \in \Omega_b^P$ the extensions $\widetilde{\rho}_k^{(\infty)}\eta_P^{(\infty)}$ and $\widetilde{\rho}_{\tau(k)}^{(\infty)}\eta_Q^{(\infty)}\beta$ are *strongly* equivalent. Since these are non-unital extensions, this happens if and only if they are weakly equivalent. $\square$

For an irreducible finite stochastic matrix P over Ω^P with period t_P, and $k \in \Omega^P$. Let $\Omega_0, ..., \Omega_{t_P-1}$ be a cyclic decomposition for P. Then there exists m_0, such that for all $m \geq m_0$ we have

$$|\Omega_{\sigma(k)-m}| = |\{ \ i \in \Omega_{\sigma(k)-m} \mid P_{ik}^{(m)} > 0 \ \}|.$$

Indeed, fix $0 \leq \ell \leq t_P - 1$. By item (2) of Theorem 2.3.14 there is $n_0^{(\ell)}$ such that for all $n \geq n_0^{(\ell)}$ we have that $P_{ij}^{(nt_P+\ell)} > 0$ for $i, j \in \Omega^P$ with $\sigma(i) - \sigma(j) = \ell$. Hence, if we fix $j = k$, we see that

$$|\Omega_{\sigma(k)-(nt_P+\ell)}| = |\{ \ i \in \Omega_{\sigma(k)-(nt_P+\ell)} \mid P_{ik}^{(nt_P+\ell)} > 0 \ \}|.$$

Then simply take $m_0 = \max_\ell\{n_0^{(\ell)}t_P + \ell\}$ to obtain the desired claim above.

Definition 4.4.8. *Let P be a t-periodic finite irreducible stochastic matrix over Ω of size d, and $k \in \Omega$. Let $\Omega_0, ..., \Omega_{t-1}$ be a cyclic decomposition for P, so that $\sigma(k)$ is the unique index such that $k \in \Omega_{\sigma(k)}$. We define the k-th* column nullity *of P to be*

$$\mathcal{N}_P(k) = \sum_{m=1}^{\infty} |\{ \ i \in \Omega_{\sigma(k)-m} \mid P_{ik}^{(m)} = 0 \ \}|$$

where $\sigma(k) - m$ is taken as an element in the cyclic group $\mathbb{Z}_t$ of order t.

Put in other words, the column nullity of a state $k \in \Omega$ is the number of zeros in all k-th columns of iterations of P, that lie in the support of a cyclic decomposition for P. The above infinite sum is in fact always finite by the discussion preceding Definition 4.4.8 and is hence convergent. We note also that P is fully supported if and only if for all $k \in \Omega$ is we have $\mathcal{N}_P(k) = 0$.

For a finite irreducible stochastic matrix P, we find the element in $\mathrm{Ext}_s(C(\mathbb{T}) \otimes M_d)$ representing each extension $\eta_{P,k} := \widetilde{\rho}_k\eta_P$, for each $k \in \Omega_b^P$, appearing in Proposition 4.4.7. Note that the exact sequence corresponding to the extension $\eta_{P,k}$ is

$$0 \to \mathcal{K}(\mathcal{F}_{P,k}) \to \pi_k(\mathcal{T}(P)) \to C(\mathbb{T}, M_{|\Omega^P|}) \to 0.$$

Recall the computation of $\mathrm{Ext}_s(C(\mathbb{T}) \otimes M_d)$ and $\mathrm{Ext}_w(C(\mathbb{T}) \otimes M_d)$ preceding Proposition 4.2.5.

Proposition 4.4.9. *Let P be a finite irreducible stochastic matrix over Ω^P with period t_P, and let $\Omega_0, ..., \Omega_{t_P-1}$ be a properly enumerated cyclic decomposition for P. Then for each $k \in \Omega_b^P$, there exists n_0 large enough so that for all $n \geq n_0$ we have that $[j_*\eta_{P,k}]_s$ is identified with $0 \leq s < |\Omega^P|$ given by*

$$s \equiv \sum_{m=0}^{nt_P-1} |\{ i \in \Omega_{\sigma(k)-m} \mid P_{ik}^{(m)} > 0\}| \mod |\Omega^P|$$

and $[\iota_*\eta_{P,k}]_s$ is identified with $-|\Omega^P|$. In particular, $[\eta_{P,k}]_w = -1$.

Proof. To compute the class of $[j_*\eta_{P,k}]$, we apply the algorithm in Example 2.3.22 to $j_*\eta_{P,k}$. Let $\{\overline{S_{ij}}\}$ be the system of matrix units for $C(\mathbb{T}, M_{|\Omega^P|})$ associated to a properly enumerated cyclic decomposition $\Omega_0, ..., \Omega_{t_P-1}$, and let $1 \in \Omega$ be the first element in this enumeration. There then exists m_0 such that for all $m \geq m_0$ we have $|\Omega_{\sigma(k)-m}| = |\{ i \in \Omega_{\sigma(k)-m} \mid P_{ik}^{(m)} > 0 \}|$. We abuse notation for sake of brevity and write T instead of $\pi_k(T) = T|_{\mathcal{F}_{P,k}}$ for $T \in \mathcal{T}(P)$.

Following Example 2.3.22, lift each $\overline{S_{ii}}$ to $p_i \cdot Q_{[nt_P,\infty)} \in \pi_k(\mathcal{T}(P))$. Then for $j \neq 1$ we may lift each $\overline{S_{1j}}$ to $S_{1j}Q_{[nt_P,\infty)} \in \pi_k(\mathcal{T}(P))$ so that $e_{ij} := Q_{[nt_P,\infty)}S_{1j}^*S_{1i}Q_{[nt_P,\infty)}$ with $i,j \in \Omega^P$ is a system of matrix units for $pB(\mathcal{F}_{P,k})p$, and for all $n \in \mathbb{N}$ with $nt_P \geq m_0$ the projection

$$p = \sum_{i=1}^{|\Omega^P|} Q_{[nt_P,\infty)}S_{1i}^*S_{1i}Q_{[nt_P,\infty)}$$

has finite dimensional cokernel equal to the defect of $j_*\eta_{P,k}$ modulo $|\Omega^P|$.

Denote by $b_{ik}^{(m)}$ the indicator, which is 1 if and only if $P_{ik}^{(m)} > 0$ and 0 otherwise. Then, the dimension of the cokernel of p is congruent mod $|\Omega^P|$ to

$$\sum_{i=1}^{|\Omega^P|} \sum_{m=0}^{nt_P-1} b_{ik}^{(m)} = \sum_{m=0}^{nt_P-1} |\{ i \in \Omega_{\sigma(k)-m} \mid P_{ik}^{(m)} > 0 \}|$$

so we may take $n_0 = \lceil \frac{m_0}{t_P} \rceil$.

As for $\iota_*\eta_{P,k}$, a lift for $z \otimes I \in C(\mathbb{T}) \otimes I$ can be taken to be U^P, where U^P is the unitary associated to the properly enumerated cyclic decomposition $\Omega_0, ..., \Omega_{t_P-1}$ (restricted to $\mathcal{F}_{P,k}$). Then by Proposition 4.4.2 and the notation there, $\text{ind}(U^P) = \text{ind}(\Pi_{i\in\Omega}V_i) = -|\Omega^P|$. Finally, recall that the image $[\iota_*\eta_{P,k}]_s \in |\Omega^P| \cdot \mathbb{Z}$ is identified with $[\eta_{P,k}]_w \in \mathbb{Z}$ up to dividing by $|\Omega^P|$, so that $[\eta_{P,k}]_w = -1$. □

We now reach the two main results of this chapter, which classify stable isomorphism and *-isomorphism of C*-envelopes in terms of the underlying stochastic matrices and boundary representations supported on different copies of compact operator subalgebras.

Theorem 4.4.10. *Let P and Q be finite irreducible stochastic matrices over Ω^P and Ω^Q respectively. Then $|\Omega_b^P| = |\Omega_b^Q|$ if and only if $C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are stably isomorphic.*

Proof. If $C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are stably isomorphic, since K_0 and K_1 are stable functors, we must have that $|\Omega_b^P| = |\Omega_b^Q|$ by Theorem 4.4.4.

For the converse, suppose $\Omega_b := \Omega_b^P = \Omega_b^Q$. For $k \in \Omega_b$, denote by $\eta_{P,k}^{(|\Omega^Q|)}$ and $\eta_{Q,k}^{(|\Omega^P|)}$ the amplifications of these extensions to $C(\mathbb{T}) \otimes M_{|\Omega_P|} \otimes M_{|\Omega_Q|}$. By Proposition 4.4.9 we then have that $\eta_{P,k}^{(|\Omega^Q|)}$ and $\eta_{Q,k}^{(|\Omega^P|)}$ are weakly unitarily equivalent. Hence, $\eta_{P,k}^{(\infty)}$ and $\eta_{Q,k}^{(\infty)}$ are also weakly equivalent, so that by item (2) of Proposition 4.4.7 (with $\beta = Id$) we have that $C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are stably isomorphic. □

Theorem 4.4.11. *Let P and Q be finite irreducible stochastic matrices over Ω^P and Ω^Q respectively. Then $C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are $*$-isomorphic if and only if $d := |\Omega^P| = |\Omega^Q|$ and there is a bijection $\tau : \Omega_b^P \to \Omega_b^Q$ such that for all $k \in \Omega_b^P$ we have $\mathcal{N}_P(k) \equiv \mathcal{N}_Q(\tau(k)) \mod d$.*

Proof. Suppose $C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are $*$-isomorphic. By item (1) of Proposition 4.4.7 there is a $*$-isomorphism $\beta \in C(\mathbb{T}, M_{|\Omega^P|}) \to C(\mathbb{T}, M_{|\Omega^Q|})$ (so that $d := |\Omega^P| = |\Omega^Q|$) and a bijection $\tau : \Omega_b^P \to \Omega_b^Q$ such that $\eta_{P,k}$ and $\eta_{Q,\tau(k)}\beta$ are strongly equivalent. By Proposition 4.2.5 β_s is the identity on the second coordinate of $\mathrm{Ext}_s(C(\mathbb{T}) \otimes M_d) \cong d\mathbb{Z} \times \mathbb{Z}_d$. Hence, we see that $[j_*\eta_{P,k}] = [j_*\eta_{Q,\tau(k)}]$, so that $k \in \Omega_b^P$ and $\mathcal{N}_P(k) \equiv \mathcal{N}_Q(\tau(k)) \mod d$ by Proposition 4.4.9.

For the converse, suppose $\mathcal{N}_P(k) \equiv \mathcal{N}_Q(\tau(k)) \mod d$ for all $k \in \Omega_b^P$ via some bijection $\tau : \Omega_b^P \to \Omega_b^Q$, and that $|\Omega^P| = |\Omega^Q|$. We see by Proposition 4.4.9 that $j_*\eta_{P,k}$ and $j_*\eta_{Q,\tau(k)}$ are strongly equivalent. Again by Proposition 4.4.9 we have that $[\iota_*\eta_{P,k}]$ and $[\iota_*\eta_{Q,\tau(k)}]$ are represented by the numbers $-|\Omega^P|$ and $-|\Omega^Q|$ which are equal by assumption. Hence, we have that $\eta_{P,k}$ and $\eta_{Q,\tau(k)}$ are strongly equivalent. Thus, by item (1) of Proposition 4.4.7 (with $\beta = Id$) we have that $C_e^*(\mathcal{T}_+(P))$ and $C_e^*(\mathcal{T}_+(Q))$ are *-isomorphic. □

4.5 Comparison with Cuntz-Krieger algebras

It is interesting to try and compare these invariants with the one obtained from the Cuntz-Krieger C*-algebra of the graph of the stochastic matrix P. Given an irreducible graph matrix $A = (a_{ij})$ over Ω, where $a_{ij} \in \{0, 1\}$, in their first paper [28], Cuntz and Krieger defined a C*-algebra $\mathcal{O}_A$ generated by partial isometries $\{S_i\}_{i\in\Omega}$ with pairwise orthogonal ranges, satisfying the relations

$$S_i^* S_i = \sum_{j\in\Omega} a_{ij} \cdot S_j S_j^*.$$

For a stochastic matrix P, one has the $\{0, 1\}$-matrix $\mathrm{Adj}(P)$ representing the directed graph of P. Since the C*-correspondence $\mathrm{Arv}(P)_1$ is the graph C*-correspondence of the graph $\mathcal{Q}_P$, we get that the Cuntz-Pimsner algebra $\mathcal{O}(\mathrm{Arv}(P)_1)$ is $*$-isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{\mathrm{Adj}(P)}$. In particular, by [107, Remark 4.3] we see that $\mathcal{O}_{\mathrm{Adj}(P)}$ is nuclear.

In [27], Cuntz computed the K-theory of these C*-algebras. He showed that for finite $\{0, 1\}$ matrix A over Ω where every column and row is non-zero, the K_0 and K_1 groups of $\mathcal{O}_A$ are given as the cokernel and kernel of the map $I - A^t : \mathbb{Z}^\Omega \to \mathbb{Z}^\Omega$.

In the case where A is an irreducible finite matrix which is not a permutation matrix, Cuntz and Krieger establish in [28] that $\mathcal{O}_A$ is simple and purely infinite. Hence, for a finite irreducible stochastic matrix P which is not a permutation matrix, the Cuntz-Krieger algebra $\mathcal{O}_{\mathrm{Adj}(P)}$ is separable, unital, nuclear, simple and purely infinite, or in other words a Kirchberg algebra.

A famous classification theorem of Kirchberg and Phillips [103] then comes into play to show that for two finite irreducible stochastic matrices P and Q which are not permutation matrices, the Cuntz-Krieger algebras $\mathcal{O}_{\mathrm{Adj}(P)}$ and $\mathcal{O}_{\mathrm{Adj}(Q)}$ are $*$-isomorphic (or stably isomorphic) if and only if $(K_0(\mathcal{O}_{\mathrm{Adj}(P)}), [1_P]_0) \cong (K_0(\mathcal{O}_{\mathrm{Adj}(Q)}), [1_Q]_0)$ and $K_1(\mathcal{O}_{\mathrm{Adj}(P)}) \cong K_1(\mathcal{O}_{\mathrm{Adj}(Q)})$ (or $K_0(\mathcal{O}_{\mathrm{Adj}(P)}) \cong K_0(\mathcal{O}_{\mathrm{Adj}(Q)})$ and $K_1(\mathcal{O}_{\mathrm{Adj}(P)}) \cong K_1(\mathcal{O}_{\mathrm{Adj}(Q)})$ respectively). That is, the *-isomorphism and stable isomorphism class are completely determined by K-theory.

Example 4.5.1. *In this example, we will use the above to show that for a finite irreducible stochastic matrix P, the Cuntz-Krieger algebra $\mathcal{O}_{\mathrm{Adj}(P)}$ and the C*-envelope $C_e^*(\mathcal{T}_+(P))$ generally yield incomparable invariants for P. If we restrict to matrices P with multiple-arrival, we have that $\Omega_e = \Omega - \Omega_b$ and the invariant $C_e^*(\mathcal{T}_+(P))$ will only depend on the adjacency matrix $\mathrm{Adj}(P)$. Hence, we will only specify the $\{0, 1\}$ graph incidence matrices* *of three stochastic matrices P, Q, R with multiple arrival. Suppose the graph matrices for P, Q, R are given respectively by*

$$\mathrm{Adj}(P) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \ , \ \mathrm{Adj}(Q) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \ , \ \mathrm{Adj}(R) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

then P, Q and R have multiple-arrival, and it is clear that $\mathcal{N}_P(j) = \mathcal{N}_Q(j) = \mathcal{N}_R(j) = 0$ for $j = 1, 2$, and that $\mathcal{N}_P(3) = 0$. We also see that $\mathcal{N}_Q(3) = \mathcal{N}_R(3) = 1$, so that $C^*_e(\mathcal{T}_+(Q)) \cong C^*_e(\mathcal{T}_+(R))$. However, $\Omega^P_e = \{3\}$ whereas $\Omega^Q_e = \Omega^R_e = \emptyset$, and hence $C^*_e(\mathcal{T}_+(Q))$ is not stably isomorphic to $C^*_e(\mathcal{T}_+(P))$.

For the Cuntz-Krieger C^*-algebras the situation is reversed. The maps $I - \mathrm{Adj}(P)^t$, $I - \mathrm{Adj}(Q)^t$ and $I - \mathrm{Adj}(R)^t$ on $\mathbb{Z}^3$ determining K_0 and K_1 for the Cuntz-Krieger algebras are given respectively by the matrices

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \ , \ \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \ \textit{and} \ \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

Hence, we see that the K_1 groups for $\mathcal{O}_{\mathrm{Adj}(P)}$, $\mathcal{O}_{\mathrm{Adj}(Q)}$ and $\mathcal{O}_{\mathrm{Adj}(R)}$ are trivial, and that $\mathrm{Ran}(I - \mathrm{Adj}(P)^t) = \mathrm{Ran}(I - \mathrm{Adj}(Q)^t) = \mathbb{Z}^3$, so that $K_0(\mathcal{O}_{\mathrm{Adj}(P)}) = K_0(\mathcal{O}_{\mathrm{Adj}(Q)})$ are trivial. Hence, by the above mentioned result of Kirchberg and Phillips, we have that $\mathcal{O}_{\mathrm{Adj}(P)}$ is $*$-isomorphic to $\mathcal{O}_{\mathrm{Adj}(Q)}$. However, since $\mathrm{Ran}(I - \mathrm{Adj}(R)^t) \subsetneq \mathbb{Z}^3$, we see that the cokernel $K_0(\mathcal{O}_{\mathrm{Adj}(R)})$ is non-trivial, and hence $\mathcal{O}_{\mathrm{Adj}(R)}$ is not stably isomorphic to $\mathcal{O}_{\mathrm{Adj}(P)}$. Altogether, we obtain that

$$C^*_e(\mathcal{T}_+(P)) \not\sim C^*_e(\mathcal{T}_+(Q)) \cong C^*_e(\mathcal{T}_+(R)) \qquad \textit{and} \qquad \mathcal{O}_{\mathrm{Adj}(P)} \cong \mathcal{O}_{\mathrm{Adj}(Q)} \not\sim \mathcal{O}_{\mathrm{Adj}(R)}$$

where $\cong$ stands for $*$-isomorphism and $\sim$ stands for stable isomorphism. Note that just like the Cuntz-Krieger algebras, the C^*-envelope still loses considerable information about the tensor algebra, for instance, the graphs of P and Q are not isomorphic so by [42, Theorem 7.29] $\mathcal{T}_+(Q)$ and $\mathcal{T}_+(R)$ are not even algebraically isomorphic.

Chapter 5

Full Cuntz-Krieger dilations via non-commutative boundaries

5.1 Introduction

Perhaps the simplest dilation result in operator theory is the dilation of an isometry to a unitary. If $V \in B(\mathcal{H})$ is an isometry and $\Delta := I_{\mathcal{H}} - VV^*$, we may define a unitary U on $K := \mathcal{H} \oplus \mathcal{H}$ via

$$U := \begin{bmatrix} V & \Delta \\ 0 & V^* \end{bmatrix}$$

such that for any polynomial in a single variable $p \in \mathbb{C}[x]$ we have $p(V) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$ where $P_{\mathcal{H}}$ is the orthogonal projection onto the first summand of $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. It is easy to see that Toeplitz-Cuntz-Krieger families and Cuntz-Krieger families generalize the notions of an isometry and unitary respectively, by taking the graph with a single loop and a single vertex. In fact, one of our goals in this chapter is to generalize this dilation result to the free multivariable setting in the context of families of operators arising from directed graphs.

We will assume in this chapter that directed graphs $G = (V, E, s, r)$ are countable, meaning that both the sets V and E are countable. A directed graph is said to be *row-finite* if every vertex receives at most finitely many edges, and is *sourceless* if every vertex receives at least one edge. We refer to [106] for an excellent exposition on graph C*-algebras. Due to the universal properties of $\mathcal{T}(G)$ and $\mathcal{O}(G)$, we know that $*$-representations of $\mathcal{T}(G)$ are in bijection with TCK G-families, and $*$-representations of $\mathcal{O}(G)$ are in bijection with CK G-families. Hence, we will often pass freely between these two points of view.

In our context of dilation of an isometry to a unitary, in [105, Proposition 2.6] Popescu proves that for a countable set F and a row-isometry $V = (V_i)_{i \in F}$ on a space $\mathcal{H}$ there is a dilation to a row-unitary. In other words, this means that for any family of isometries $V_i : \mathcal{H} \to \mathcal{H}$ such that $\text{SOT-}\sum_{i \in F} V_i V_i^* \leq I_{\mathcal{H}}$ there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and isometries $U = (U_i)_{i \in F}$ on $\mathcal{K}$ such that $\text{SOT-}\sum_{i \in F} U_i U_i^* = I_{\mathcal{K}}$, and for any polynomial $p \in \mathbb{C}\langle x_i \rangle_{i \in F}$ in non-commuting variables, we have $p(V) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$ where $P_{\mathcal{H}}$ is the projection from $\mathcal{K}$ to $\mathcal{H}$. In terms of graphs, this means that for a graph with a single vertex and $|F|$ loops, dilation of a TCK family to a CK family is possible, with the extra SOT-convergence $\text{SOT-}\sum_{i \in F} U_i U_i^* = I_{\mathcal{K}}$ when F is infinite.

On the other hand, from [115, Theorem 5.4] we see that if G is row-finite and sourceless, then any TCK family has a CK dilation. More precisely for a row-finite sourceless graph $G = (V, E, s, r)$, if (P, S) is a TCK family on $\mathcal{H}$, then there exists a CK family (Q, T) on a larger space $\mathcal{K}$ such that for any polynomial $p \in \mathbb{C}\langle V, E \rangle$ in non-commuting variables we have $p(P, S) = P_{\mathcal{H}} p(Q, T)|_{\mathcal{H}}$.

In order to put both of these results in the same context, we make the following definition.

Definition 5.1.1. *Let (P, S) be a Cuntz-Krieger family for a countable directed graph G. We say that (P, S) is a* full Cuntz-Krieger family *if*

(CKF) $\quad \text{SOT-}\sum_{r(e)=v} S_e S_e^* = P_v$, *for every $v \in V$ with $r^{-1}(v) \neq \emptyset$.*

In this chapter, which is based on joint work with Guy Salomon [44], we will show that, in dilation theoretic terms, full CK families are the proper generalization of the notion of a unitary operator. More precisely, we will show that every TCK family has a full CK dilation, and that no non-trivial TCK dilations for full CK families are possible (See Corollary 5.2.7).

In [76], Katsoulis and Kribs improve on the work in [90] and [54, Theorem 5.3] and by using a tail-adding technique they show that the C*-envelope of a tensor algebra associated to a general C*-correspondence, is the Cuntz-Pimsner-Katsura algebra of the C*-correspondence. In particular, the C*-envelope of the tensor algebra $\mathcal{T}_+(G)$ is the Cuntz-Krieger algebra $\mathcal{O}(G)$ (this was also shown directly in [75]). We will provide an alternative proof for this fact on graph algebras in Theorem 5.2.9, avoiding any tail-adding techniques.

One of our main results is the classification of $*$-representation of $\mathcal{T}(G)$ that have the unique extension property when restricted to $\mathcal{T}_+(G)$. They turn out to coincide with those

$*$-representations that are associated with full Cuntz-Krieger families (see Theorem 5.2.5). This allows us to improve upon several known results and show that any TCK family dilates to a full CK family in the sense described above. Further applications of this result allows us to give a bijective correspondence between irreducible $*$-representations of $\mathcal{T}(G)$ that are not boundary, and "gap" TCK families of the graph G (see Corollary 5.2.6), and a characterization of the unique extension property of $\mathcal{T}_+(G)$ inside $\mathcal{O}(G)$ in terms of the graph G (see Theorem 5.2.9).

Trying to leverage our results to free products, we discuss some of the general theory of free products of operator algebras, and prove a joint unital completely positive extension theorem for free products of operator algebras amalgamated over any common unital C*-algebra (see Theorem 5.3.1). Complete injectivity of amalgamated free products of C*-algebras was shown by Armstrong, Dykema, Exel and Li [3], and we are able to use our joint extension result to generalize their result to free products of operator algebras amalgamated over any common C*-subalgebra (see Proposition 5.3.3). In [32, Theorem 5.3.21] a gap in the proof of [46, Theorem 3.1] was corrected, and it was shown that the amalgamated free product of C*-envelopes is a C*-cover for the amalgamated free product of operator algebras $\{\mathcal{A}_i\}_{i\in I}$. By [32, Theorem 5.3.21], this C*-cover turns out to be the C*-envelope when each $\mathcal{A}_i$ has the unique extension property inside its C*-envelope. Using Proposition 5.3.3, we are able to obtain these results as well.

Using complete injectivity and the results on free products, along with a stronger joint extension result [21, Theorem 3.1], we characterize representations with the unique extension property on amalgamated free products (see Proposition 5.3.4), and apply our results to free products of graph operator algebras. Free products of graph operator algebras have been investigated by Ara and Goodearl in [2] as C*-algebras associated to separated graphs, and by Duncan [47] as operator algebras associated to edge-colored directed graphs. We combine our results to prove that a full-CK dilation exists for any TCK family of a colored directed graph (See Corollary 5.4.2), and to show that the free product of Cuntz-Krieger algebras is a C*-cover for the free product of tensor graph algebras, which is the C*-envelope when all graphs involved are row-finite (see Theorem 5.4.4).

5.2 Full Cuntz-Krieger dilations

Let $G = (V, E, s, r)$ be a countable directed graph. We will abuse terminology and call associated $*$-representations of either $\mathcal{T}(G)$ or $\mathcal{O}(G)$ "Cuntz-Krieger" or "full Cuntz-Krieger" if their associated TCK families are such. A (universal) TCK or CK family generating $\mathcal{T}(G)$ or $\mathcal{O}(G)$ (respectively) will usually be denoted by lowercase letters (p, s).

There is a canonical $*$-representation of the Toeplitz-Cuntz-Krieger graph C*-algebra which we now describe. First, recall that a path in G is a sequence of edges $\lambda = \mu_n \cdots \mu_1$ such that $r(\mu_i) = s(\mu_{i+1})$, where we extend the range and source maps to apply for paths by specifying $r(\lambda) := r(\mu_n)$ and $s(\lambda) := s(\mu_1)$, and set $|\lambda| := n$ for the length of the path; vertices are considered as paths of length 0. We use $E^\bullet$ to denote the collection of all paths in G of finite length.

Let $\mathcal{H}_G := \ell^2(E^\bullet)$ be the Hilbert space with canonical standard orthonormal basis $\{\xi_\lambda\}_{\lambda \in E^\bullet}$, we define a Toeplitz-Cuntz-Krieger family (P, S) on $\mathcal{H}_G$ by specifying each operator on an orthonormal basis, that is, for each $v \in V$, $\mu \in E$ and $\lambda \in E^\bullet$ we define

$$P_v(\xi_\lambda) = \begin{cases} \xi_\lambda & \text{if } r(\lambda) = v \\ 0 & \text{if } r(\lambda) \neq v \end{cases} \quad \text{and} \quad S_e(\xi_\lambda) = \begin{cases} \xi_{e\lambda} & \text{if } r(\lambda) = s(e) \\ 0 & \text{if } r(\lambda) \neq s(e) \end{cases}.$$

For every $v \in V$, consider the subspace $\mathcal{H}_{G,v} := \ell^2(s^{-1}(v))$ with its orthonormal basis $\{\xi_\lambda\}_{s(\lambda)=v}$. Clearly, $\mathcal{H}_{G,v}$ is reducing for (P, S), so by the universal property of $\mathcal{T}(G)$ there exists a $*$-representation $\pi_v : \mathcal{T}(G) \to B(\mathcal{H}_{G,v})$ satisfying $\pi_v(p_w) = P_w|_{\mathcal{H}_{G,v}}$ for every $w \in V$ and $\pi_v(s_e) = S_e|_{\mathcal{H}_{G,v}}$ for every $e \in E$. The next proposition is easily verified, and we omit its proof.

Proposition 5.2.1. *Let $\pi_v : \mathcal{T}(G) \to B(\mathcal{H}_{G,v})$ be the $*$-representation described above. Then the following hold:*

(a) π_v is irreducible,

(b) for every $w \neq v$ we have SOT-$\sum_{r(e)=w} \pi_v(s_e s_e^*) = \pi_v(p_w)$, *and*

(c) $\pi_v(p_v) -$ SOT-$\sum_{r(e)=v} \pi_v(s_e s_e^*)$ *is a rank 1 projection.*

Toeplitz-Cuntz-Krieger families have the following useful version of the Wold decomposition. A slightly different Wold decomposition was given in [69, Section 2] by Jury and Kribs under the assumption that the graphs have no sinks. Here we give a self-contained, and slightly more general version that is tailored to our context. Let V_r the set of vertices $v \in V$ such that $r^{-1}(v) \neq \emptyset$. For a TCK family (Q, T) and a reducing subspace $\mathcal{K}$ for it, we denote $(Q, T)|_{\mathcal{K}} := (\{Q_v|_{\mathcal{K}}\}, \{T_e|_{\mathcal{K}}\})$.

Theorem 5.2.2 (Wold decomposition)**.** *Let (Q, T) be a Toeplitz-Cuntz-Krieger family on a Hilbert space $\mathcal{H}$. For every $v \in V_r$, denote by α_v the dimension of the space $W_v := (Q_v - \sum_{r(e)=v} T_e T_e^*)H$. Then (Q, T) is unitarily equivalent to*

$$\oplus_{v \in V_r} ((P, S)|_{\mathcal{H}_{G,v}})^{(\alpha_v)} \oplus (R, L)$$

where (R, L) is a full CK G-family. In addition, this representation is unique in the sense that if (Q, T) is unitarily equivalent to

$$\oplus_{v\in V_r}((P,S)|_{\mathcal{H}_{G,v}})^{(\alpha'_v)} \oplus (R', L')$$

where (R', L') is a full CK G-family, then $\alpha'_v = \alpha_v$ for every $v \in V_r$, and (R, L) is unitarily equivalent to (R', L').

Proof. Uniqueness follows by Proposition 5.2.1. Indeed, as $(P, S)|_{\mathcal{H}_{G,v}}$ cannot be unitarily equivalent to $(P, S)|_{\mathcal{H}_{G,w}}$ for $w \neq v$ nor to restrictions to reducing subspaces for either full CK families (R', L') or (R, L). Thus, we must have that $((P, S)|_{\mathcal{H}_{G,v}})^{(\alpha_v)}$ is unitarily equivalent to $((P, S)|_{\mathcal{H}_{G,v}})^{(\alpha'_v)}$ so that $\alpha_v = \alpha'_v$. Once this is established, restricting to the orthocomplement of the (reducing) subspaces associated with $\oplus_{v\in V_r}((P, S)|_{\mathcal{H}_{G,v}})^{(\alpha_v)}$ and $\oplus_{v\in V_r}((P, S)|_{\mathcal{H}_{G,v}})^{(\alpha'_v)}$, we obtain a unitary equivalence between (R, L) and (R', L').

As for existence, fix $v \in V_r$, and denote $W_v = (Q_v - \sum_{r(e)=v} T_e T_e^*)H$. Choose an orthonormal basis $\{\zeta_v^{(i)}\}$ for W_v, of cardinality α_v, and for every i set

$$\mathcal{H}_{v,i} := \text{span}\{T_\lambda \zeta_v^{(i)} : \lambda \in s^{-1}(v)\}.$$

We will show these subspace are reducing. Indeed, $\mathcal{H}_{v,i}$ is clearly invariant for the family (Q, T). As for co-invariance, note that $T_\mu^*(T_\lambda \zeta_v^{(i)})$ is either 0, a vector of the form $T_{\lambda'}\zeta_v^{(i)}$ for some path λ', or a vector of the form $T_{\mu'}^*\zeta_v^{(i)}$ for some path μ' with $|\mu'| \geq 1$. As the two first cases immediately imply that $T_\mu^*(T_\lambda \zeta_v^{(i)}) \in \mathcal{H}_{v,i}$, we need to deal only with the third case. To this end, write $\mu' = e_0\mu''$ for some edge $e_0 \in E$ and a path μ'' with $s(\mu'') = r(e_0)$. Note that if $e_0 \in s^{-1}(v)$, then $T_{e_0}^*(Q_v - \sum_{r(e)=v} T_e T_e^*) = T_{e_0}^* - T_{e_0}^* = 0$, and if not, $T_{e_0}^*(Q_v - \sum_{r(e)=v} T_e T_e^*) = 0 - 0 = 0$. Thus, in any case,

$$T_{\mu'}^*\zeta_v^{(i)} = T_{\mu'}^*(Q_v - \sum_{r(e)=v} T_e T_e^*)\zeta_v^{(i)} = T_{\mu''}^* T_{e_0}^*(Q_v - \sum_{r(e)=v} T_e T_e^*)\zeta_v^{(i)} = 0.$$

We next show simultaneously that for fixed $v \in V_r$ and $1 \leq i \leq \alpha_v$, the set $\{T_\lambda \zeta_v^{(i)}\}_{\lambda \in s^{-1}(v)}$ is an orthonormal family, and that the spaces $\mathcal{H}_{v,i}$ are pairwise orthogonal for all $v \in V_r$ and $1 \leq i \leq \alpha_v$. Our first step is to show that for two vertices $v, w \in V_r$, two indices $1 \leq i \leq \alpha_v$ and $1 \leq j \leq \alpha_w$, and two paths λ, μ in G, if $\langle T_\lambda \zeta_v^{(i)}, T_\mu \zeta_w^{(j)}\rangle \neq 0$ then we must have $\lambda = \mu$. Indeed,

$$\langle T_\lambda \zeta_v^{(i)}, T_\mu \zeta_w^{(j)}\rangle = \Big\langle \big((Q_w - \sum_{r(e)=w} T_e T_e^*) T_\mu^* T_\lambda (Q_v - \sum_{r(e)=v} T_e T_e^*)\big)\zeta_v^{(i)}, \zeta_w^{(j)}\Big\rangle.$$

For $T_\mu^* T_\lambda$ to be non-zero, it must be either of the form $T_{\lambda'}$ where $\lambda = \mu\lambda'$, or $T_{\mu'}^*$ where $\mu = \lambda\mu'$. We deal with the first case, and the second is proven similarly. So assume $\lambda = \mu\lambda'$. If $|\lambda'| = 0$, then $\lambda = \mu$. If $|\lambda'| \geq 1$, write $\lambda' = e_0\lambda''$. Then we have

$$(Q_w - \sum_{e \in r^{-1}(w)} T_e T_e^*) T_\mu^* T_\lambda = T_{\lambda'} - T_{e_0} T_{e_0}^* T_{\lambda'} = 0$$

which yields a contradiction. Thus, $\lambda = \mu$.

As a consequence of this, we see that $v = s(\lambda) = s(\mu) = w$. As T_λ is an isometry on $P_v\mathcal{H}$, the assumption $\langle T_\lambda \zeta_v^{(i)}, T_\lambda \zeta_v^{(j)} \rangle \neq 0$ yields $i = j$ as well. We therefore must have that the sets $\{T_\lambda \zeta_v^{(i)} : \lambda \in s^{-1}(v)\}$ are orthonormal bases for the pairwise orthogonal reducing subspaces $\mathcal{H}_{v,i}$.

We next define unitaries $U_{v,i} : \mathcal{H}_{v,i} \to \mathcal{H}_{G,v}$ by mapping an orthonormal basis to an orthonormal basis $U_{v,i} : T_\lambda \zeta_v^{(i)} \mapsto \xi_\lambda$. Clearly $U_{v,i}$ intertwines $(Q,T)|_{\mathcal{H}_{v,i}}$ and $(P,S)|_{\mathcal{H}_{G,v}}$. Denote by $\mathcal{K} = (\oplus_{v \in V_r} \mathcal{H}_{v,i})^\perp$, we then have that (Q,T) is unitarily equivalent to

$$\oplus_{v \in V_r} ((P,S)|_{H_{G,v}})^{(\alpha_v)} \oplus (Q,T)|_{\mathcal{K}}.$$

As a result $(R,L) := (Q,T)|_{\mathcal{K}}$ is a Toeplitz-Cuntz-Krieger family such that for any $v \in V_r$ we have $R_v = \text{SOT} - \sum_{r(e)=v} L_e L_e^*$. Hence, it is a full CK family. □

By rephrasing the previous proposition in terms of $*$-representations, we obtain the following corollary.

Corollary 5.2.3. *Let G be a directed graph, and let $\pi : \mathcal{T}(G) \to B(\mathcal{H})$ be a $*$-representation. Then there are multiplicities $\{\alpha_v\}_{v \in V_r}$ such that π is unitarily equivalent to the $*$-representation $\pi_s \oplus \pi_b$, where $\pi_s = \oplus_{v \in V_r} \pi_v^{(\alpha_v)}$ and π_b is a full CK representation. In addition, this representation is unique in the sense that if π is also unitarily equivalent to the $*$-representation $\pi_s' \oplus \pi_b'$, where $\pi_s' = \oplus_{v \in V_r} \pi_v^{(\alpha_v')}$ and π_b' is a full CK representation, then $\alpha_v' = \alpha_v$ for every $v \in V_r$, and π_b' is unitarily equivalent to π_b.*

We next characterize those $*$-representations which have the unique extension property with respect to $\mathcal{T}_+(G)$.

Definition 5.2.4. *Let $G = (V,E,s,r)$ be a directed graph, and let $\pi : \mathcal{T}(G) \to B(\mathcal{H})$ be a $*$-representation. We say that $v \in V_r$ is* singular *with respect to π (or simply that v is π-singular) if*

$$\text{SOT-} \sum_{r(e)=v} \pi(S_e S_e^*) \lneq \pi(P_v).$$

Note that π is a full CK representation of $\mathcal{T}(G)$, if and only if π has no singular vertices.

Theorem 5.2.5. *Suppose that $\pi : \mathcal{T}(G) \to B(\mathcal{H})$ is a $*$-representation. The restriction $\pi|_{\mathcal{T}_+(G)}$ has the unique extension property if and only if π is a full CK representation.*

Proof. Let (p, s) be a generating TCK family for $\mathcal{T}(G)$. Suppose G has a π-singular vertex v. If we assume towards contradiction that $\pi|_{\mathcal{T}_+(G)}$ has the unique extension property, then by [12, Proposition 4.4], so does the restriction of the infinite inflation $\pi^{(\infty)} : \mathcal{T}(G) \to B(\mathcal{H}^{(\infty)})$ to $\mathcal{T}_+(G)$. We therefore may assume without loss of generality that π has infinite multiplicity. We will arrive at a contradiction by showing that $\pi|_{\mathcal{T}_+(G)}$ is not maximal.

As v is π-singular, and π has infinite multiplicity, the projection $Q_v := \pi(p_v) - \text{SOT-}\sum_{r(e)=v} \pi(s_e s_e^*)$ is infinite dimensional. Thus, we may decompose $Q_v\mathcal{H} = \oplus_{r(e)=v}\mathcal{H}_e$ into infinite dimensional spaces $\mathcal{H}_e$ for each $e \in r^{-1}(v)$. We can then define for every $e \in r^{-1}(v)$ some isometry $W_e : P_{s(e)}\mathcal{H}_G \to \mathcal{H}_e$ where $\mathcal{H}_G$ is the Hilbert space $\ell^2(E^\bullet)$ and (P, S) is the associated TCK G-family. We moreover define a $*$-representation $\rho : \mathcal{T}(G) \to B(\mathcal{H} \oplus \mathcal{H}_G)$ by specifying a Toeplitz-Cuntz-Krieger G-family

$$\rho(p_v) = \begin{bmatrix} \pi(p_v) & 0 \\ 0 & P_v \end{bmatrix} \text{ for all } v \in V$$

and

$$\rho(s_e) = \begin{cases} \begin{bmatrix} \pi(s_e) & W_e \\ 0 & 0 \end{bmatrix} & \text{if } r(e) = v, \quad \text{and} \\ \\ \begin{bmatrix} \pi(s_e) & 0 \\ 0 & S_e \end{bmatrix} & \text{otherwise.} \end{cases}$$

We show this defines a Toeplitz-Cuntz-Krieger family. Clearly, we need to verify only those relations which involve edges in $r^{-1}(v)$. For every $e \in r^{-1}(v)$

$$\begin{aligned} \rho(s_e)^*\rho(s_e) &= \begin{bmatrix} \pi(s_e)^* & 0 \\ W_e^* & 0 \end{bmatrix} \cdot \begin{bmatrix} \pi(s_e) & W_e \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \pi(p_{s(e)}) & \pi(s_e)^*W_e \\ W_e^*\pi(s_e) & P_{s(e)} \end{bmatrix}. \end{aligned}$$

As the range of W_e is orthogonal to that of $\pi(s_e)$, we see that $\pi(s_e)^*W_e = W_e^*\pi(s_e) = 0$, so

$$\rho(s_e)^*\rho(s_e) = \rho(p_{s(e)}),$$

and condition (I) is verified. Next, for every finite subset $F \subseteq r^{-1}(v)$

$$\begin{aligned}\sum_{e\in F} \rho(s_e)\rho(s_e)^* &= \sum_{e\in F} \begin{bmatrix} \pi(s_e) & W_e \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \pi(s_e)^* & 0 \\ W_e^* & 0 \end{bmatrix} \\ &= \sum_{e\in F} \begin{bmatrix} \pi(s_e s_e^*) + W_e W_e^* & 0 \\ 0 & 0 \end{bmatrix} \\ &\leq \begin{bmatrix} \pi(p_v) & 0 \\ 0 & P_v \end{bmatrix} = \rho(p_v),\end{aligned}$$

where the inequality is true since $\{\pi(s_e s_e^*)\} \cup \{W_e W_e^*\}$ is a collection of pairwise orthogonal projections dominated by $\pi(p_v)$. We therefore have shown condition (TCK), and we conclude that $\rho|_{\mathcal{T}_+(G)}$ is a well-defined representation which dilates $\pi|_{\mathcal{T}_+(G)}$ non-trivially. Hence $\pi|_{\mathcal{T}_+(G)}$ is not maximal.

For the converse, suppose that G has no π-singular vertices. Let $\widetilde{\rho} : \mathcal{T}_+(G) \to B(\mathcal{K})$ be a maximal dilation of $\pi|_{\mathcal{T}_+(G)}$, and let $\rho : \mathcal{T}(G) \to B(\mathcal{K})$ be its extension to a $*$-representation. Denote

$$\rho(p_v) = \begin{bmatrix} \pi(p_v) & X_v \\ Y_v & Z_v \end{bmatrix} \quad \text{and} \quad \rho(s_e) = \begin{bmatrix} \pi(s_e) & X_e \\ Y_e & Z_e \end{bmatrix}$$

for all $v \in V$ and $e \in E$. We have that $X_v = 0$ and $Y_v = 0$ for all $v \in V$. Indeed, let $v \in V$, and $P : \mathcal{K} \to \mathcal{H}$ the orthogonal projection onto $\mathcal{H}$, then

$$\begin{aligned}P\rho(p_v)^*(1-P)\rho(p_v)P &= P\rho(p_v)P - P\rho(p_v)P\rho(p_v)P \\ &= \pi(p_v) - \pi(p_v)\pi(p_v) = 0.\end{aligned}$$

and the C*-identity implies $Y_v = (1-P)\rho(p_v)P = 0$. As $\rho(p_v)$ is self-adjoint, we have $X_v = 0$ as well.

Next, for all $e \in E$ we have $p_{s(e)} = s_e^* s_e$, so

$$\begin{bmatrix} \pi(p_{s(e)}) & 0 \\ 0 & * \end{bmatrix} = \rho(s_e^* s_e) = \rho(s_e)^*\rho(s_e) = \begin{bmatrix} \pi(s_e)^*\pi(s_e) + Y_e^* Y_e & * \\ * & * \end{bmatrix}$$

which implies $Y_e = 0$ for all $e \in E$.

Finally, let $e \in E$ and let $v = r(e)$. For every finite subset F of $r^{-1}(v)$, we have $p_v \geq \sum_{f\in F} s_f s_f^*$, so

$$\begin{aligned}\begin{bmatrix} \pi(p_v) & 0 \\ 0 & * \end{bmatrix} &= \rho(p_v) \geq \sum_{f\in F} \rho(s_f)\rho(s_f)^* \\ &= \sum_{f\in F} \begin{bmatrix} \pi(s_f)\pi(s_f)^* + X_f X_f^* & * \\ * & * \end{bmatrix}.\end{aligned}$$

In particular, by compressing this inequality to $\mathcal{H}$ we obtain

$$\sum_{f \in F} \pi(s_f)\pi(s_f)^* + X_f X_f^* \leq \pi(p_v)$$

for every finite subset F of $r^{-1}(v)$. Since v is not π-singular, we must have that

$$\sup_F \sum_{f \in F} \pi(s_f)\pi(s_f)^* = \text{SOT-}\sum_{r(f)=v} \pi(s_f)\pi(s_f)^* = \pi(p_v).$$

We therefore obtain that $X_f = 0$ for all $f \in r^{-1}(v)$, and in particular $X_e = 0$. Since $\mathcal{T}(G)$ is generated as a C*-algebra by $\mathcal{T}_+(G)$, we must have that ρ has π as a direct summand, and hence $\rho|_{\mathcal{T}_+(G)}$ is a trivial dilation of $\pi|_{\mathcal{T}_+(G)}$. $\square$

The previous theorem gives rise to two interesting corollaries. The first is a parametrization of those irreducible $*$-representations of $\mathcal{T}(G)$ which are not boundary representations with respect to $\mathcal{T}_+(G)$, and the second is the dilation of TCK families to full CK families.

Corollary 5.2.6. *For every vertex $v \in V_r$, the $*$-representation $\pi_v : \mathcal{T}(G) \to B(\mathcal{H}_{G,v})$ is the unique irreducible $*$-representation (up to unitary equivalence) for which v is π-singular, so that the irreducible $*$-representations of $\mathcal{T}(G)$ which are not boundary for $\mathcal{T}_+(G)$ are parametrized by V_r.*

Proof. If π is an irreducible $*$-representation that lacks the unique extension property on $\mathcal{T}_+(G)$, then by Theorem 5.2.5 there exists $v \in V_r$ which is π-singular. By the Wold decomposition (Corollary 5.2.3), up to a unitary equivalence, π must have π_v as a subrepresentation, and by irreducibility, π is unitarily equivalent to π_v. $\square$

Corollary 5.2.7. *Let $G = (V, E, s, r)$ be a countable directed graph, and (P, S) a TCK family on $\mathcal{H}$. Then there exists a full CK family (Q, T) on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, such that $f(P, S) = P_{\mathcal{H}} f(Q, T)|_{\mathcal{H}}$ for any polynomial $f \in \mathbb{C}\langle V, E\rangle$ in non-commuting variables.*

Proof. Let $\pi_{P,S} : \mathcal{T}(G) \to B(\mathcal{H})$ be the $*$-representation of $\mathcal{T}(G)$ associated to (P, S). By [45, Theorem 1.2] we can dilate $\pi_{P,S}|_{\mathcal{T}_+(G)}$ to a maximal representation $\tau : \mathcal{T}_+(G) \to B(\mathcal{K})$, and without loss of generality, $\mathcal{H}$ is a subspace of $\mathcal{K}$. Hence, τ is the restriction to $\mathcal{T}_+(G)$ of a $*$-representation $\rho : \mathcal{T}(G) \to B(\mathcal{K})$ such that $\rho|_{\mathcal{T}_+(G)}$ has the unique extension property. Let (Q, T) be the TCK family associated to τ. By Theorem 5.2.5 (Q, T) is a full CK family, and it dilates (P, S) in the sense that for every polynomial $f \in \mathbb{C}\langle V, E\rangle$ we have that $f(P, S) = P_{\mathcal{H}} f(Q, T)|_{\mathcal{H}}$. $\square$

Our next goal is to construct, for every directed graph G, faithful full CK representations of $\mathcal{O}(G)$. We do this by constructing certain universal CK families arising from backward-infinite paths.

Let $E^{\infty} = \{ \lambda \mid \lambda = e_1 e_2 e_3 \cdots, s(e_i) = r(e_{i+1}), \ e_i \in E \}$ be the collection of all backward infinite paths in G, and extend the range map to E^{∞} by setting $r(\lambda) = r(e_1)$ for $\lambda = e_1 e_2 e_3 \cdots \in E^{\infty}$. Let $E^{<\infty}$ be the collection of all finite paths, including paths of length 0, emanating from sources, and set $E^{\leq\infty} = E^{\infty} \cup E^{<\infty}$ as a disjoint union.

For each vertex $v \in V$ fix an element $\mu_v \in E^{\leq\infty}$ with $r(\mu_v) = v$. For $i \in \mathbb{N}$ define $\mu_{v,i} = e_{v,1} ... e_{v,i}$ as the i-th truncation of μ_v, where if $|\mu_v| \leq i$, $\mu_{v,i} = \mu_v$. Denote $\mathcal{H}_{v,i}$ the Hilbert space with the orthonormal basis $\{ \xi_{\lambda\mu_{v,i}^{-1}} \mid \lambda \in E^{\bullet}, \ s(\lambda) = v \}$ where $\lambda\mu_{v,i}^{-1}$ corresponds to the equivalence class of reduced products determined by λ and $\mu_{v,i}$. We set $\Gamma := \{ \lambda\mu_{v,i}^{-1} \mid \lambda \in E^{\bullet}, \ s(\lambda) = v, \ i \in \mathbb{N}, \ v \in V \}$, and let $\mathcal{H}_b := \ell^2(\Gamma)$ denote the Hilbert space with orthonormal basis $\{\xi_{\lambda\mu^{-1}}\}_{\lambda\mu^{-1} \in \Gamma}$, which is unitarily equivalent to $\oplus_{v\in V}\big[\bigvee_{i\in\mathbb{N}} \mathcal{H}_{v,i}\big]$, where $\mathcal{H}_{v,i}$ is identified as a subspace of $\mathcal{H}_{v,i+1}$ since $\lambda\mu_{v,i}^{-1}$ is identified with $\lambda e_{v,i+1}\mu_{v,i+1}^{-1}$ whenever $|\mu_v| > i$ and with $\lambda\mu_{v,i}^{-1}$ when $|\mu_v| \leq i$. We define a TCK family (Q,T) on $\mathcal{H}_b$ by specifying it on the orthonormal basis $\{\xi_{\lambda\mu^{-1}}\}_{\lambda\mu^{-1}\in\Gamma}$ by

$$Q_v(\xi_{\lambda\mu^{-1}}) = \begin{cases} \xi_{\lambda\mu^{-1}} & \text{if } r(\lambda) = v \\ 0 & \text{if } r(\lambda) \neq v \end{cases}, \quad T_e(\xi_{\lambda\mu^{-1}}) = \begin{cases} \xi_{e\lambda\mu^{-1}} & \text{if } r(\lambda) = s(e) \\ 0 & \text{if } r(\lambda) \neq s(e) \end{cases}.$$

It is easy to verify that (Q,T) is a full CK family. Indeed, $\xi_{\lambda\mu_{v,i}^{-1}}$, with with $|\lambda| \geq 1$ is in the range of T_e for $e \in E$ such that $\lambda = e\lambda'$, and each $\xi_{\mu_{v,i}^{-1}}$ where $s(\mu) = s(\mu_{v,i})$ is not a source is in the range of $T_{e_{v,i+1}}$ as $\mu_{v,i}^{-1}$ is identified with $e_{v,i+1}\mu_{v,i+1}^{-1}$.

By construction, each Q_v is non-zero for all $v \in V$, and we let ρ_∞ be the $*$-representation of $\mathcal{T}(G)$ associated to (Q,T) above. Moreover, by construction of ρ_∞, for each $z \in \mathbb{T}$ we get a well-defined unitary $U_z : \mathcal{H}_b \to \mathcal{H}_b$ by specifying $U_z(\xi_{\lambda\mu^{-1}}) = z^{|\mu|-|\lambda|}\xi_{\lambda\mu^{-1}}$. So we get a gauge action $\alpha : \mathbb{T} \to \mathrm{Aut}(C^*(Q,T))$ via $\alpha_z(A) = U_z A U_z^*$, so that $\alpha_z(Q_v) = Q_v$ and $\alpha_z(T_e) = zT_e$. Hence, by the gauge invariant uniqueness theorem ρ_∞ is injective.

Remark 5.2.8. The advantage of the above construction is that it produces a space which has a natural action of $\mathbb{T}$ on it. One may form a full CK family on $\ell^2(E^{\leq\infty})$ in a similar way, but this representation will fail to be injective when the graph has a vertex-simple cycle with no entry.

Let $\mathcal{J}(G)$ denote the kernel of the quotient $q : \mathcal{T}(G) \to \mathcal{O}(G)$. Evidently, $\mathcal{J}(G)$ is the ideal of $\mathcal{T}(G)$ generated by terms of the form $p_v - \sum_{r(e)=v} s_e s_e^*$ for vertices v with

$0 < |r^{-1}(v)| < \infty$. In [70, Theorem 3.3], Kakariadis showed that $\mathcal{T}_+(G)$ has the unique extension property in $\mathcal{O}(G)$ when G is row-finite. We provide the proof for this statement along with its converse, and the computation of the C*-envelope in the general case.

Theorem 5.2.9. *Let $G = (V, E, s, r)$ be a directed graph, and let $q : \mathcal{T}(G) \to \mathcal{O}(G)$ be the natural quotient map. Then q is completely isometric on $\mathcal{T}_+(G)$, and $C^*_e(\mathcal{T}_+(G)) \cong \mathcal{O}(G)$.*

Moreover, $\mathcal{T}_+(G)$ has the unique extension property in $\mathcal{O}(G)$ if and only if G is row-finite.

Proof. Let (p, s) be a TCK family such that its associated $*$-representation $\pi_{p,s}$ is faithful. By [45, Theorem 1.1] we know that $\pi_{p,s}|_{\mathcal{T}_+(G)}$ has a maximal dilation, so let $\tau : \mathcal{T}(G) \to B(\mathcal{K})$ be a $*$-representation such that $\tau|_{\mathcal{T}_+(G)}$ is the dilation of $\pi_{p,s}|_{\mathcal{T}_+(G)}$, so that it is completely isometric and with the unique extension property. By Theorem 5.2.5, τ is a full CK representation, and hence annihilates the Cuntz-Krieger ideal $\mathcal{J}(G)$. Hence, τ must factor through the quotient map $q : \mathcal{T}(G) \to \mathcal{O}(G)$ by $\mathcal{J}(G)$, and we have that q is completely isometric on $\mathcal{T}_+(G)$.

Next, we show that if G is row-finite then $\mathcal{T}_+(G)$ has the unique extension property in $\mathcal{O}(G)$ via q. By Theorem 5.2.5, we see that every $*$-representation of $\mathcal{T}(G)$ that annihilates $\mathcal{J}(G)$ has the unique extension property when restricted to $\mathcal{T}_+(G)$ inside $\mathcal{T}(G)$. Since q is completely isometric on $\mathcal{T}_+(G)$, by invariance of the unique extension property, we see that every $*$-representation of $\mathcal{O}(G)$ has unique extension property when restricted to $\mathcal{T}_+(G)$ inside $\mathcal{O}(G)$. By [45, Theorem 1.1] we know that the C*-envelope of $\mathcal{T}_+(G)$ is the image under the direct sum of all $*$-representations of $\mathcal{O}(G)$ with the unique extension property, so that $C^*_e(\mathcal{T}_+(G)) \cong \mathcal{O}(G)$ when G is row-finite, as all $*$-representations of $\mathcal{O}(G)$ have the unique extension property when restricted to $\mathcal{T}_+(G)$ inside $\mathcal{O}(G)$.

Otherwise, if G is not row-finite, we have that $\rho_\infty \circ q$ is a full CK representation with kernel $\mathcal{J}(G)$, and hence again by invariance of the unique extension property and Theorem 5.2.5 we have that ρ_∞ has the unique extension property on $\mathcal{T}_+(G)$. Hence, since ρ_∞ is faithful, we still have that $C^*_e(\mathcal{T}_+(G)) \cong \mathcal{O}(G)$.

For the converse of the second part of the statement, suppose that G is not row-finite, and let $v \in V$ be an infinite receiver. Then π_v annihilates $\mathcal{J}(G)$, so we may consider the induced $*$-representation $\dot{\pi}_v : \mathcal{O}(G) \to B(\mathcal{H}_{G,v})$. By Corollary 5.2.6 π_v does not have the unique extension property when restricted to $\mathcal{T}_+(G)$, so that by invariance of the unique extension property, $\dot{\pi}_v$ does not have the unique extension property when restricted to $\mathcal{T}_+(G)$. Thus, $\mathcal{T}_+(G)$ does not have the unique extension property in $\mathcal{O}(G)$. □

Remark 5.2.10. In Theorem 5.2.5, and Corollaries 5.2.6 and 5.2.7 we avoided the use of a uniqueness theorem. This is also true for the computation of the C*-envelope in Theorem

5.2.9 when G is row-finite. A uniqueness theorem was needed only for the computation of the C*-envelope when the graph is not row-finite.

5.3 Free products and unique extension

Consider the category of unital $\mathbb{C}$-algebras (with unital homomorphisms as morphisms). Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of unital $\mathbb{C}$-algebras and let $\mathcal{D}$ be a common unital subalgebra with $1_{\mathcal{A}_i} = 1_{\mathcal{D}}$, let $\iota_i : \mathcal{D} \to \mathcal{A}_i$ denote the natural embeddings. Pushouts in this category are known to exist, and are called *free product of* $\{\mathcal{A}_i\}_{i\in I}$ *amalgamated over the common subalgebra* $\mathcal{D}$, denoted by $\underset{\mathcal{D}}{*}\mathcal{A}_i$. We recall the details briefly.

We let $*\mathcal{A}_i$ (with no $\mathcal{D}$) denote the vector space spanned by formal expressions $a_1*\cdots*a_n$ where $a_1 \in \mathcal{A}_{i_1}, \dots, a_n \in \mathcal{A}_{i_n}$ such that $i_1 \neq i_2 \neq \cdots \neq i_n$ and $n \geq 1$, where this expression behaves multilinearly, and we define multiplication of two such expressions, where $b_1 \in \mathcal{A}_{j_1}, \dots, b_m \in \mathcal{A}_{j_m}$ with $j_1 \neq \cdots \neq j_m$ via

$$(a_1 * \cdots * a_n) \cdot (b_1 * \cdots * b_m) = \begin{cases} a_1 * \cdots * a_n * b_1 * \cdots * b_m, & \text{if } i_n \neq j_1, \\ a_1 * \cdots * (a_n \cdot b_1) * \cdots * b_m, & \text{if } i_n = j_1. \end{cases}$$

With this multiplication, $*\mathcal{A}_i$ becomes a $\mathbb{C}$-algebra generated by $\{\mathcal{A}_i\}_{i\in I}$. Next, we identify the different copies of $\mathcal{D}$ by taking a quotient by the ideal $\langle \iota_i(d) - \iota_j(d)\rangle_{i,j\in I, d\in\mathcal{D}}$. This quotient, which is denoted by $\underset{\mathcal{D}}{*}\mathcal{A}_i$, has the following universal property. If $\mathcal{B}$ is another unital $\mathbb{C}$-algebras with unital $\mathbb{C}$-homomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{B}$ which agree on $\mathcal{D}$, then there is a unital $\mathbb{C}$-homomorphism $\underset{\mathcal{D}}{*}\psi_i : \underset{\mathcal{D}}{*}\mathcal{A}_i \to \mathcal{B}$ extending each ψ_i on $\mathcal{A}_i$.

Next, we construct free products in the category of unital operator algebras with unital completely contractive homomorphisms. Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of unital operator algebras with $\mathcal{D}$ a common unital operator subalgebra with $1_{\mathcal{A}_i} = 1_{\mathcal{D}}$ for all $i \in I$, and let $\underset{\mathcal{D}}{*}\mathcal{A}_i$ denote the free product of $\{\mathcal{A}_i\}_{i\in I}$ amalgamated over $\mathcal{D}$ in the larger category of $\mathbb{C}$-algebras. We define matrix semi-norms

$$\|a\|_n := \sup \left\| \underset{\mathcal{D}}{*}\psi_i^{(n)}(a) \right\|_{B(\mathcal{H})}, \quad \forall n \in \mathbb{N},\ a \in M_n\left(\underset{\mathcal{D}}{*}\mathcal{A}_i\right)$$

where the supremum is taken over all families $\{\psi_i : \mathcal{A}_i \to B(\mathcal{H})\}_{i\in I}$ of unital completely contractive homomorphisms that agree on $\mathcal{D}$ and over a Hilbert space $\mathcal{H}$ of large enough cardinality. It follows that $\mathcal{J} = \{a \in \underset{\mathcal{D}}{*}\mathcal{A}_i : \|a\| = 0\}$ is a two-sided ideal, and we denote the norms on the quotient by $\|\cdot\|_n$ as well. The norms $\|\cdot\|_n$ then define an operator-algebraic structure on the completion $\hat{\ast}_{\mathcal{D}}\mathcal{A}_i$ of $\ast_{\mathcal{D}}\mathcal{A}_i/\mathcal{J}$, by the Blecher-Ruan-Sinclair theorem [20]. Furthermore, by construction there are unital completely contractive homomorphisms $\iota_j : \mathcal{A}_j \to \hat{\ast}_{\mathcal{D}}\mathcal{A}_i$. We will show in Corollary 5.3.2 that each ι_j is in fact completely isometric, so that each $\mathcal{A}_j$ can be thought of as an operator subalgebra of $\hat{\ast}_{\mathcal{D}}\mathcal{A}_i$ via ι_j.

The operator algebra $\hat{\ast}_{\mathcal{D}}\mathcal{A}_i$ is called the free product of $\{\mathcal{A}_i\}_{i\in I}$ amalgamated over the common operator subalgebra $\mathcal{D}$, and has the following universal property by construction: for any unital operator algebra $\mathcal{B}$ and $\psi_i : \mathcal{A}_i \to \mathcal{B}$ unital completely contractive homomorphisms which agree on $\mathcal{D}$, there exists a completely contractive homomorphism $\psi := \ast_{\mathcal{D}}\psi_i$ from $\hat{\ast}_{\mathcal{D}}\mathcal{A}_i$ into $\mathcal{B}$ such that $\psi_i = \psi \circ \iota$.

We now provide a joint completely contractive extension result for free products of operator algebras amalgamated over a common C*-subalgebra. Our proof is an adaptation of a proof given by Ozawa in [96, Theorem 15] in the case of amalgamation over the complex numbers. Recall that whenever $\mathcal{D}$ is a subalgebra of an operator algebra $\mathcal{A}$, a completely contractive map $\phi : \mathcal{A} \to B(\mathcal{H})$ is said to be a $\mathcal{D}$-*bimodule* map if $\phi(a_1 d a_2) = \phi(a_1)\phi(d)\phi(a_2)$ for every $a_1, a_2 \in \mathcal{A}$ and $d \in \mathcal{D}$. By [24, Proposition 1.5.7] the restriction of a completely contractive map ϕ to a C*-subalgebra $\mathcal{D}$ is multiplicative if and only if ϕ is a $\mathcal{D}$-bimodule map.

Theorem 5.3.1. *Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of unital operator algebras containing a common unital C*-algebra $\mathcal{D}$ with $1_{\mathcal{A}_i} = 1_{\mathcal{D}}$, and let $\phi_i : \mathcal{A}_i \to B(\mathcal{H})$ be unital completely contractive $\mathcal{D}$-bimodule maps that agree on $\mathcal{D}$. Then there exists a unital completely contractive map $\phi : \hat{\ast}_{\mathcal{D}}\mathcal{A}_i \to B(\mathcal{H})$ such that $\phi \circ \iota_i = \phi_i$ for all $i \in I$.*

Proof. We construct multiplicative dilations of ϕ_i which agree on $\mathcal{D}$, so that the compression of their free product to $\mathcal{H}$ yields a unital completely contractive joint extension ϕ as in the statement of the theorem.

First, we set $\mathcal{H}_1 := \mathcal{H}$ and $\phi_i^{(1)} := \phi_i$. By the Arveson-Stinespring dilation theorem we may dilate each of these to a completely contractive homomorphism from $\mathcal{A}_i$ to $B(\mathcal{H}_1 \oplus \mathcal{K}_1^{(i)})$. Denote by $\rho_i^{(1)}$ this dilation of ϕ_i, so that $\phi_i(a) = P_{\mathcal{H}_1}\rho_i^{(1)}(a)|_{\mathcal{H}_1}$. We note that since $\mathcal{D}$ is a C*-algebra, and each $\phi_i^{(1)}$ is multiplicative on $\mathcal{D}$, the space $\mathcal{K}_1^{(i)}$ is reducing for $\rho_i^{(1)}|_{\mathcal{D}}$, so that for all $d \in \mathcal{D}$ we have

$$\rho_i^{(1)}(d) = \phi_i^{(1)}(d) \oplus P_{\mathcal{K}_1^{(i)}}\rho_i^{(1)}(d)|_{\mathcal{K}_1^{(i)}}.$$

Now suppose we have a sequence of subspaces

$$\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots \subseteq \mathcal{H}_n$$

such that for all $1 \leq m \leq n$ we have unital completely contractive $\mathcal{D}$-bimodule maps $\phi_i^{(m)} : \mathcal{A}_i \to B(\mathcal{H}_m)$ that agree on $\mathcal{D}$, along with multiplicative unital completely contractive maps $\rho_i^{(m)} : \mathcal{A}_i \to B(\mathcal{H}_m \oplus \mathcal{K}_m^{(i)})$ that dilate each $\phi_i^{(m)}$, so that for all $d \in \mathcal{D}$ we have

$$\rho_i^{(m)}(d) = \phi_i^{(m)}(d) \oplus P_{\mathcal{K}_m^{(i)}} \rho_i^{(m)}(d)|_{\mathcal{K}_m^{(i)}}$$

and for every $j \in I$ the sequence of subspaces

$$\mathcal{H}_1 \oplus \mathcal{K}_1^{(j)} \subseteq \mathcal{H}_2 \oplus \mathcal{K}_2^{(j)} \subseteq \cdots \subseteq \mathcal{H}_m \oplus \mathcal{K}_m^{(j)}$$

is a sequence of reducing subspaces for $\rho_j^{(m)}$.

Denote by $\mathcal{H}_{n+1} = \mathcal{H}_n \oplus \bigoplus_{i\in I} \mathcal{K}_n^{(i)}$. Fix $i \in I$, and consider the map $\tau_i : \mathcal{D} \to B(\mathcal{K}_n^{(i)})$ given by $\tau_i(d) = P_{\mathcal{K}_n^{(i)}} \rho_i^{(n)}(d)|_{\mathcal{K}_n^{(i)}}$. By applying Arveson's extension theorem, followed by a restriction, for any $j \in I$ distinct from i, we may extend τ_i to a unital completely contractive map $\sigma_{ji} : \mathcal{A}_j \to B(\mathcal{K}_n^{(i)})$. We define for all $j \in I$,

$$\phi_j^{(n+1)} := \rho_j^{(n)} \oplus \bigoplus_{j\neq i \in I} \sigma_{ji} : \mathcal{A}_j \to B(\mathcal{H}_{n+1}), \tag{5.1}$$

so that $\mathcal{H}_n \oplus \mathcal{K}_n^{(j)}$ is a reducing subspace for $\phi_j^{(n+1)}$. We then have for every $d \in \mathcal{D}$ that

$$\phi_j^{(n+1)}(d) = \phi_j^{(n)}(d) \oplus \bigoplus_{i\in I} P_{\mathcal{K}_n^{(i)}} \rho_i^{(n)}(d)|_{\mathcal{K}_n^{(i)}}.$$

Hence, since the maps $\{\phi_i^{(n)}\}_{i\in I}$ all agree on $\mathcal{D}$, we have that the maps $\{\phi_i^{(n+1)}\}_{i\in I}$ all agree on $\mathcal{D}$.

We use Arveson's extension, Stinespring's theorem and the special form of $\phi_j^{(n+1)}$ in equation (5.1) to obtain a *multiplicative* unital completely contractive map $\rho_j^{(n+1)} : \mathcal{A}_j \to B(\mathcal{H}_{n+1} \oplus \mathcal{K}_{n+1}^{(j)})$ dilating $\phi_j^{(n+1)}$ such that each $\mathcal{H}_n \oplus \mathcal{K}_n^{(j)}$ is a reducing subspace for $\rho_j^{(n+1)}$. Hence, we then get that $\rho_j^{(n+1)}(a)|_{\mathcal{H}_m \oplus \mathcal{K}_m^{(j)}} = \rho_j^{(m)}(a)$ for all $1 \leq m \leq n$, and that for all $d \in \mathcal{D}$,

$$\rho_i^{(n+1)}(d) = \phi_i^{(n+1)}(d) \oplus P_{\mathcal{K}_{n+1}^{(j)}} \rho_i^{(n+1)}(d)|_{\mathcal{K}_{n+1}^{(j)}}.$$

Since for each $j \in I$ and $n \in \mathbb{N}$ we have $\mathcal{H}_n \subseteq \mathcal{H}_n \oplus \mathcal{K}_n^{(j)} \subseteq \mathcal{H}_{n+1}$, we may define a multiplicative unital completely contractive map $\rho_j : \mathcal{A}_j \to B(\mathcal{K})$ on the inductive limit of Hilbert spaces $\mathcal{K} = \bigvee_{n\in\mathbb{N}} \mathcal{H}_n$ by specifying $\rho_j(a)h = \rho_j^{(n)}(a)h$ for $h \in \mathcal{H}_n \oplus \mathcal{K}_n^{(j)}$. These maps then agree on $\mathcal{D}$, since for $h \in \mathcal{H}_n$ we have that

$$\begin{aligned}\rho_i(d)h &= \rho^{(n+1)}(d)h \\ &= \big(\phi_i^{(n+1)}(d) \oplus P_{\mathcal{K}_{n+1}^{(i)}} \rho_i^{(n+1)}(d)|_{\mathcal{K}_{n+1}^{(i)}}\big)h \\ &= \phi_i^{(n+1)}(d)h\end{aligned}$$

and as the maps $\{\phi_i^{(n+1)}\}_{i\in I}$ all agree on $\mathcal{D}$ and the union of $\mathcal{H}_n$ is dense in $\mathcal{K}$, we have that $\rho_j(d) = \rho_i(d)$ for all $i \neq j$ in I.

Hence, we may form the free product $\rho := \hat{*}_{\mathcal{D}} \rho_i : \hat{*}_{\mathcal{D}} \mathcal{A}_i \to B(\mathcal{K})$ which satisfies $\rho \circ \iota_i = \rho_i$, and the compression of ρ to $\mathcal{H}$ would yield a joint unital completely contractive extension ϕ as in the statement of the theorem. □

The following allows us to identify $\mathcal{A}_i$ as a unital operator subalgebra of $\hat{*}_{\mathcal{D}} \mathcal{A}_i$ via ι_i.

Corollary 5.3.2. *Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of unital operator algebras containing a common unital C*-algebra $\mathcal{D}$ with $1_{\mathcal{A}_i} = 1_{\mathcal{D}}$. Then for each $i \in I$ the map $\iota_i : \mathcal{A}_i \to \hat{*}_{\mathcal{D}} \mathcal{A}_i$ is completely isometric. Hence, $\hat{*}_{\mathcal{D}} \mathcal{A}_i$ is the pushout of $\{\mathcal{A}_i\}_{i\in I}$ by $\mathcal{D}$ in the category of unital operator algebras with unital completely contractive homomorphisms.*

Proof. Fix $j \in I$ and let $\phi_j : \mathcal{A}_j \to B(\mathcal{H})$ be a unital completely isometric homomorphism. We may then restrict it to $\mathcal{D}$ and use Arveson's extension theorem to extend to unital completely contractive maps $\phi_i : \mathcal{A}_i \to B(\mathcal{H})$ for $i \in I$ such that $i \neq j$. By Theorem 5.3.1 there is a joint unital completely contractive map $\phi : \hat{*}_{\mathcal{D}} \mathcal{A}_i \to B(\mathcal{H})$ which we may then dilate to a multiplicative map $\rho : \hat{*}_{\mathcal{D}} \mathcal{A}_i \to B(\mathcal{K})$. However, the compression of $\phi \circ \iota_j$ to $\mathcal{H}$ coincides with ϕ_j, which is a unital completely isometric map. Hence, ι_j is completely isometric. □

In [19, Section 4], Blecher and Paulsen prove the complete injectivity of the free product of operator algebras amalgamated over the complex numbers. We next prove this where the amalgamation is over any common C*-algebra. This generalizes [3, Proposition 2.2] due to Armstrong, Dykema, Exel and Li for free products of finitely many C*-algebras amalgamated over a common C*-algebra.

Proposition 5.3.3. *The free product of unital operator algebras amalgamated over a common C*-subalgebra is completely injective. That is, if* $\{\mathcal{A}_i\}_{i\in I}$ *and* $\{\mathcal{B}_i\}_{i\in I}$ *are two families of unital operator algebras containing a common C*-subalgebra* $\mathcal{D}$ *such that* $\mathcal{A}_i$ *is an operator subalgebra of* $\mathcal{B}_i$ *for every* $i \in I$ *and* $1_{\mathcal{A}_i} = 1_{\mathcal{B}_i} = 1_{\mathcal{D}}$*, then the inclusion* $\hat{\underset{\mathcal{D}}{*}}\mathcal{A}_i \subseteq \hat{\underset{\mathcal{D}}{*}}\mathcal{B}_i$ *is completely isometric.*

Proof. Denote $\hat{\mathcal{A}} := \hat{\underset{\mathcal{D}}{*}}\mathcal{A}_i$ and $\hat{\mathcal{B}} := \hat{\underset{\mathcal{D}}{*}}\mathcal{B}_i$. Let $\iota_i : \mathcal{B}_i \to \hat{\mathcal{B}}$ and $\kappa_i : \mathcal{A}_i \to \hat{\mathcal{A}}$ denote the natural completely isometric inclusions. Then the unital completely isometric homomorphisms $\iota_i|_{\mathcal{A}_i} : \mathcal{A}_i \to \hat{\mathcal{B}}$ agree on $\mathcal{D}$, so $\phi := \hat{\underset{\mathcal{D}}{*}}(\iota_i|_{\mathcal{A}_i}) : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ is a unital completely contractive homomorphism. Denote by $\|\cdot\|_{\hat{\mathcal{B}},n}$ and $\|\cdot\|_{\hat{\mathcal{A}},n}$ the n norms on $M_n(\hat{\mathcal{A}})$ and $M_n(\hat{\mathcal{B}})$ respectively.

Evidently, it would suffice to show that $\|A\|_{\hat{\mathcal{A}},n} \leq \|A\|_{\hat{\mathcal{B}},n}$ for every $A \in M_n(\hat{\mathcal{A}})$. To this end, represent $\hat{\mathcal{A}}$ completely isometrically as a unital subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. By Arveson's extension theorem the maps $\kappa_i : \mathcal{A}_i \to \hat{\mathcal{A}} \subseteq B(\mathcal{H})$ extend to unital completely contractive maps $\tilde{\kappa}_i : \mathcal{B}_i \to B(\mathcal{H})$ which agree on $\mathcal{D}$. By Theorem 5.3.1, there exists a unital completely contractive map $\psi : \hat{\mathcal{B}} \to B(\mathcal{H})$ such that $\psi(\iota_i(b_i)) = \tilde{\kappa}_i(b_i)$ for all $b_i \in \mathcal{B}_i$ which is completely isometric on $\hat{\mathcal{A}}$. Hence, by Stinespring's dilation theorem, we may dilate it to a unital completely contractive homomorphism $\hat{\psi} : \hat{\mathcal{B}} \to B(\mathcal{K})$. So for $A \in M_n(\hat{\mathcal{A}})$, we have

$$\|A\|_{\hat{\mathcal{A}},n} = \|\psi^{(n)}(A)\| \leq \|\hat{\psi}^{(n)}(A)\| \leq \|A\|_{\hat{\mathcal{B}},n}.$$

□

When $\{\mathcal{A}_i\}_{i\in I}$ are all non-unital with a common non-unital C*-algebra $\mathcal{D}$, we define their free product $\hat{\underset{\mathcal{D}}{*}}\mathcal{A}_i$ to be the operator algebra generated by the images of $\mathcal{A}_i$ inside the free product of their unitization $\hat{\underset{\mathcal{D}^1}{*}}\mathcal{A}_i^1$ amalgamated over the unitization $\mathcal{D}^1$. By Meyer's theorem and the proof of [3, Lemma 2.3], we similarly get that $\hat{\underset{\mathcal{D}^1}{*}}\mathcal{A}_i^1$ coincides with the unitization $(\hat{\underset{\mathcal{D}}{*}}\mathcal{A}_i)^1$. Using this, it follows that the non-unital free product shares the analogous pushout universal property and complete injectivity described above, just as in the unital case.

Using complete injectivity, for operator algebra $\{\mathcal{A}_i\}_{i\in I}$ with a common C*-subalgebra $\mathcal{D}$ we can freely identify $\hat{\underset{\mathcal{D}}{*}}\mathcal{A}_i$ as a subalgebra of $\hat{\underset{\mathcal{D}}{*}}\mathcal{B}_i$, where $\mathcal{B}_i$ is any C*-cover for $\mathcal{A}_i$. We henceforth abuse notations and denote $\underset{\mathcal{D}}{*}\mathcal{A}_i$ instead of $\hat{\underset{\mathcal{D}}{*}}\mathcal{A}_i$.

Complete injectivity for free products of not-necessarily unital operator algebras was used implicitly in [47] by Duncan to show that the free product of graph tensor algebras embeds inside the free product of associated Toeplitz and Cuntz-Krieger algebras. In [32, Theorem 5.3.20] Davidson, Kakariadis and Fuller filled a gap introduced by Duncan in [46, Section 3, Theorem 1], and proved his claim [32, Theorem 5.3.21]. Our complete injectivity result provides another way of addressing issues of this sort.

Next, we describe a joint unital completely positive extension in the context of free products of C*-algebras, with a special multiplicative property due to Boca.

Suppose $\{\mathcal{B}_i\}_{i\in I}$ is a family of unital C*-algebras containing a common C*-subalgebra $\mathcal{D}$ with $1_{\mathcal{B}_i} \in \mathcal{D}$, and let $E_i : \mathcal{B}_i \to \mathcal{D}$ be conditional expectations. Then $\mathcal{B}_i = \mathcal{D} \oplus \operatorname{Ker} E_i$ where the sum is in the $\mathcal{D}$-bimodule sense. Denote $\mathcal{B}_i^0 := \operatorname{Ker} E_i$. Then as $\mathcal{D}$-bimodules we have that

$$\mathop{*}_{\mathcal{D}} \mathcal{B}_i = \mathcal{D} \oplus \bigoplus_{n\geq 1} \bigoplus_{i_1\neq\cdots\neq i_n} \mathcal{B}_{i_1}^0 \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} \mathcal{B}_{i_n}^0.$$

In [21, Theorem 3.1] Boca shows that if $\phi_i : \mathcal{B}_i \to B(\mathcal{H})$ are $\mathcal{D}$-bimodule unital completely positive maps that agree on $\mathcal{D}$, then there is a $\mathcal{D}$-bimodule unital completely positive map $\phi : \mathop{*}_{\mathcal{D}} \mathcal{B}_i \to B(\mathcal{H})$ with the additional multiplicative property that for $b_1 \in \mathcal{B}_{i_1}^0, \dots, b_n \in \mathcal{B}_{i_n}^0$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ we have

$$\phi(b_1 * \cdots * b_n) = \phi_{i_1}(b_1) \cdots \phi_{i_n}(b_n).$$

Proposition 5.3.4. *Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of either all unital, or all non-unital operator-algebras, each generating a C*-algebra $\mathcal{B}_i$ and containing a common C*-algebra $\mathcal{D}$ (which has a common unit to $\mathcal{A}_i$ in the unital case).*

1. *If $\pi_i : \mathcal{B}_i \to B(\mathcal{H})$ are $*$-representations that agree on $\mathcal{D}$ such that $\pi_i|_{\mathcal{A}_i}$ has the unique extension property when restricted to $\mathcal{A}_i$, then the restriction of their free product $\mathop{*}_{\mathcal{D}} \pi_i$ to $\mathop{*}_{\mathcal{D}} \mathcal{A}_i$ has the unique extension property.*
2. *If additionally we have conditional expectations $E_i : \mathcal{B}_i \to \mathcal{D}$, such that $\mathcal{A}_i = \mathcal{D} \oplus (\mathcal{A}_i \cap \operatorname{Ker} E_i)$ as a $\mathcal{D}$-bimodule, then we have the converse. That is, if the restriction of a $*$-representation $\pi : \mathop{*}_{\mathcal{D}} \mathcal{B}_i \to B(\mathcal{H})$ to $\mathop{*}_{\mathcal{D}} \mathcal{A}_i$ has the unique extension property, then for every $i \in I$ the restrictions of the $*$-representations $\pi_i := \pi \circ \iota_i : \mathcal{B}_i \to B(\mathcal{H})$ to $\mathcal{A}_i$ has the unique extension property.*

Proof. We prove the non-unital case, where the unital case is similarly established.

To show (1), suppose that for every i the map $\pi_i|_{\mathcal{A}_i}$ has the unique extension property. Denote $\mathcal{A} = \underset{\mathcal{D}}{*}\mathcal{A}_i$ and $\pi := \underset{\mathcal{D}}{*}\pi_i$. Suppose τ is a completely contractive completely positive extension of $\pi|_{\mathcal{A}}$. Then each $\tau_i := \tau \circ \iota_i$ is a completely contractive completely positive extension of $\pi_i|_{\mathcal{A}_i}$, hence $\tau_i = \pi_i$. Since each $\mathcal{B}_i$ belongs to the multiplicative domain of τ, by [24, Proposition 1.5.7], the multiplicative domain of τ is a $*$-subalgebra of $\underset{\mathcal{D}}{*}\mathcal{B}_i$ generated by $\{\mathcal{B}_i\}_{i\in I}$ and hence must be equal to $\underset{\mathcal{D}}{*}\mathcal{B}_i$, so that $\tau = \pi$.

Next, to show (2), let $\tau_i : \mathcal{B}_i \to B(\mathcal{H})$ be a completely contractive completely positive extension of $\pi_i|_{\mathcal{A}_i}$. By [24, Proposition 1.5.7] we have that $\mathcal{D}$ is in the multiplicative domain of τ_i, and so τ_i must be a $\mathcal{D}$-bimodule map. By Meyer's theorem we may extend each τ_i to a unital completely positive map τ_i^1 on $\mathcal{B}_i^1$. By Boca's theorem there exists a unital $\mathcal{D}$-bimodule completely positive map $\tau^1 := \underset{\mathcal{D}^1}{*}\tau_i^1 : \underset{\mathcal{D}^1}{*}\mathcal{B}_i^1 \to B(\mathcal{H})$ such that $\tau^1 \circ \iota_i = \tau_i^1$ and such that for $b_1 \in \operatorname{Ker} E_{i_1}^1 = \operatorname{Ker} E_{i_1}, \dots, b_n \in \operatorname{Ker} E_{i_n}^1 = \operatorname{Ker} E_{i_n}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ one has

$$\tau(b_1 * \cdots * b_n) = \tau_{i_1}(b_1)\cdots\tau_{i_n}(b_n).$$

Hence, $\tau = \tau^1|_{\underset{\mathcal{D}}{*}\mathcal{B}_i}$ is a joint completely contractive completely positive extension of τ_i with the same multiplicative property as above. Now, since each $\mathcal{A}_i$ is generated as a $\mathcal{D}$-bimodule by $\mathcal{A}_i^0 := \mathcal{A}_i \cap \operatorname{Ker} E_i$ and $\mathcal{D}$, every monomial in $\{\mathcal{A}_i\}_{i\in I}$ can always be written as a polynomial in $\{\mathcal{A}_i^0\}_{i\in I}$ with coefficients in $\mathcal{D}$, so that τ and π must coincide on polynomials in $\{\mathcal{A}_i\}_{i\in I}$. Hence, we see that $\tau|_{\mathcal{A}} = \pi|_{\mathcal{A}}$. Since $\pi|_{\mathcal{A}}$ has the unique extension property we have that $\tau = \pi$, so that $\tau_i = \tau \circ \iota_i = \pi \circ \iota_i = \pi_i$. □

5.4 Full Cuntz-Krieger dilation for free families

In this section we will write $G = (V, E)$ for a directed graph, where the source and range maps are understood implicitly. Let $G = (V, E)$ be a directed graph. A function $c : E \to I$ is an I-coloring of edges of G, and we define an I-colored graph to be the triple (V, E, c). Given such a colored directed graph (V, E, c), we denote $E_i = c^{-1}(i)$.

Let $G = (V, E, c)$ be an I-colored directed graph and $\mathcal{H}$ a Hilbert space. A *Toeplitz-Cuntz-Krieger I-colored family* on $\mathcal{H}$ is a pair (P, S) comprised of a operators $P := \{P_v : v \in V\}$ on $\mathcal{H}$ and an I-tuple of sets of operators $S := \{S^{(i)}\}_{i\in I}$ on $\mathcal{H}$ with $S^{(i)} := \{S_e^{(i)} : e \in c^{-1}(i)\}$ such that each $(P, S^{(i)})$ is a TCK family for (V, E_i) for each $i \in I$.

We say that (P, S) is a *Cuntz-Krieger I-colored family* / *full Cuntz-Krieger I-colored family* if, in addition, each $(P, S^{(i)})$ is a a CK / full CK family for (V, E_i) for each $i \in I$ respectively.

From here on out, for a given set of vertices V, we set $\mathcal{V} := C_0(V)$. We will identify $\mathcal{V}$ with $C^*(\{P_v\})$ for some (all) TCK or CK I-colored families (P, S) where $P_v \neq 0$ for all $v \in V$ and any colored graph G with V as its set of vertices.

Given a colored directed graph $G = (V, E, c)$, let $G_i = (V, c^{-1}(i))$ be the graph of color $i \in I$. By compounding universal properties, it is easy to see that TCK I-colored families are in bijection with $*$-representations of the free product $\underset{\mathcal{V}}{*}\mathcal{T}(G_i)$ over $i \in I$ and that CK I-colored families are in bijection with $*$-representations of the free product $\underset{\mathcal{V}}{*}\mathcal{O}(G_i)$ over $i \in I$. We will call a $*$-representation of either $\underset{\mathcal{V}}{*}\mathcal{T}(G_i)$ or $\underset{\mathcal{V}}{*}\mathcal{O}(G_i)$ full Cuntz-Krieger if its associated TCK I-colored family is is a full CK I-colored family.

As we saw in item (2) of Proposition 5.3.4, the existence of conditional expectations to the common subalgebra is desirable, so as to apply results that require the use of Boca's Theorem. In [2], for the purpose of defining certain reduced free products of graph C*-algebras, it was shown that *faithful* conditional expectations exist from $\mathcal{O}(G) \to \mathcal{V}$ when G is row-finite. When G is not necessarily row-finite, we can build conditional expectations on the level of the *Toeplitz* algebras $\mathcal{T}(G)$ instead.

Using the left regular representation $\pi_\ell : \mathcal{T}(G) \to B(\mathcal{H}_G)$ given via the TCK family (P, S) as in the beginning of Section 5.2, we may define for each vertex $v \in V$ a norm one positive functional φ_v on $\mathcal{T}(G)$ by $\varphi_v(a) = \langle \pi_\ell(a)\xi_v, \xi_v \rangle$. Hence, we may define a contractive map $\Psi_V : \mathcal{T}(G) \to B(\mathcal{H}_G)$ by way of

$$\Psi_V(a) = \text{SOT-}\sum_{v \in V} \varphi_v(a) \cdot P_v.$$

Next, since each $a \in \mathcal{T}(G)$ is a norm limit of polynomials in $\{p_v\}_{v \in V}$ and $\{s_e\}_{e \in E}$, where (p, s) is a TCK family generating $\mathcal{T}(G)$, and since for each monomial $m \in \mathcal{T}(G)$ we have $\varphi_v(m)$ is non-zero for at most one vertex, we see the SOT-sum above is in fact a norm-convergent-sum for every $a \in \mathcal{T}(G)$. Hence, since the restriction of Ψ_v to $C^*(\{p_v\})$ is a $*$-homomorphism, the range of Ψ_V is the C*-algebra $C^*(P)$ generated by $\{P_v\}_{v \in V}$, which is isomorphic to $C^*(\{p_v\}) \subseteq \mathcal{T}(G)$ via the isomorphism θ mapping P_v to p_v. Hence, the composition $\Phi_V = \theta \circ \Psi_V$ is a contractive idempotent. By a theorem of Tomiyama [24, Theorem 1.5.10], we have that $\Phi_V : \mathcal{T}(G) \to \mathcal{T}(G)$ is a conditional expectation onto $C^*(\{p_v\}) \cong \mathcal{V}$.

We next characterize those representations of the free product of Toeplitz graph algebras with the unique extension property when restricted to the free product of graph tensor algebras. Recall that for countable directed graphs $\{G_i\}_{i \in I}$ on the same vertex set V, by Proposition 5.3.3, we have that the embedding of $\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)$ in $\underset{\mathcal{V}}{*}\mathcal{T}(G_i)$ is completely isometric, so we may identify the former as a subalgebra of the latter.

Proposition 5.4.1. *Let $\{G_i\}_{i \in I}$ be a collection of countable directed graphs on the same vertex set V, and let $\pi : \underset{\mathcal{V}}{*}\mathcal{T}(G_i) \to B(\mathcal{H})$ be a $*$-representation. Then $\pi|_{\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)}$ has the unique extension property if and only if for each $i \in I$ the $*$-representation $\pi_i := \pi|_{\mathcal{T}(G_i)}$ is full CK with respect to G_i.*

Proof. The above constructed conditional expectation satisfies the conditions of item (2) in Proposition 5.3.4. Hence, $\pi : \underset{\mathcal{V}}{*}\mathcal{T}(G_i) \to B(\mathcal{H})$ has the unique extension property if and only if each π_i has the unique extension property. Thus, by Theorem 5.2.5, this occurs if and only if each π_i is a full CK with respect to G_i. □

We apply Proposition 5.4.1 to draw a dilation result that generalizes Corollary 5.2.7.

Corollary 5.4.2. *Let $G = (V, E, c)$ be an I-colored directed graph, and let (P, S) be a TCK I-colored family on $\mathcal{H}$. Then there exists a full CK I-colored family (Q, T) on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, such that $f(P, S) = P_{\mathcal{H}} f(Q, T)|_{\mathcal{H}}$ for any polynomial $f \in \mathbb{C}\langle V, E\rangle$ in non-commuting variables.*

Proof. Let $\pi_{P,S} : \underset{\mathcal{V}}{*}\mathcal{T}(G_i) \to B(\mathcal{H})$ be the $*$-representation associated to (P, S). By [45, Theorem 1.2] we can dilate $\pi_{P,S}|_{\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)}$ to a maximal completely contractive homomorphism $\tau : \underset{\mathcal{V}}{*}\mathcal{T}_+(G_i) \to B(\mathcal{K})$. Without loss of generality, $\mathcal{H}$ is a subspace of $\mathcal{K}$. Let $\rho : \underset{\mathcal{V}}{*}\mathcal{T}(G_i) \to B(\mathcal{K})$ be its unique extension to a $*$-representation, and (Q, T) the associated TCK I-colored family of ρ. As τ has the unique extension property, by Proposition 5.4.1 we must have that (Q, T) is full CK I-colored family, and as τ dilates $\pi_{P,S}|_{\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)}$, we have that every polynomial $f \in \mathbb{C}\langle V, E\rangle$ in non-commuting variables must satisfy $f(P, S) = P_{\mathcal{H}} f(Q, T)|_{\mathcal{H}}$. □

The following result mirrors [115, Proposition 1.6] on the existence of maximal fully co-isometric summands, but when restricting to the context of directed graphs our result is more general as it requires no relations between families of different color.

Corollary 5.4.3. *Let $G = (V, E, c)$ be an I-colored directed graph, and let (P, S) be a TCK I-colored family on $\mathcal{H}$. Then there is a unique maximal common reducing subspace $\mathcal{K}$ for operators in (P, S) such that $(P, S)|_{\mathcal{K}}$ is a full CK I-colored family.*

Proof. Let $\pi_{P,S}$ be the $*$-representation associated to (P, S). By Proposition 2.1.3 there is a unique largest reducing subspace $\mathcal{K}$ for $\pi_{P,S}$ such that $\rho : \underset{\mathcal{V}}{*}\mathcal{T}(G_i) \to B(\mathcal{K})$ given by $\rho(b) = \pi_{P,S}(b)|_{\mathcal{K}}$ has the unique extension property when restricted to $\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)$. By Proposition 5.4.1, we see that the associated TCK I-colored family $(P,S)|_{\mathcal{K}}$ is in fact a full CK I-colored family, and $\mathcal{K}$ is a unique maximal common reducing subspace with this property. $\square$

Denote by $\mathcal{H}_V = \bigoplus_{v \in V} \mathcal{H}_v$ a Hilbert space direct sum of separable infinite dimensional Hilbert spaces $\mathcal{H}_v$. Then $\mathcal{V} \cong C^*(\{p_v\})$ can be represented on $\mathcal{H}_V$ by mapping p_v to the projection P_v onto $\mathcal{H}_v$, and if $\rho : \mathcal{T}(G_i) \to B(\mathcal{H})$ is a non-degenerate representation such that $\rho|_{C^*(\{p_v\})} : \mathcal{V} \cong C^*(\{p_v\}) \to B(\mathcal{H})$ where $\rho(p_v)\mathcal{H}$ is infinite dimensional for each $v \in V$, then ρ is unitarily equivalent to a representation on $\mathcal{H}_V$ where $\rho(p_v) = P_v$ is the projection onto $\mathcal{H}_v$.

As mentioned earlier, assuming complete injectivity and when all G_i are row-finite, Duncan [47, Proposition 4.4] explained how $\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)$ has the unique extension property in $\underset{\mathcal{V}}{*}\mathcal{O}(G_i)$. We next prove this while providing the converse.

Theorem 5.4.4. *Let $\{G_i\}_{i \in I}$ be a collection of countable directed graphs over the same vertex set V. Then the quotient map $q : \underset{\mathcal{V}}{*}\mathcal{T}(G_i) \to \underset{\mathcal{V}}{*}\mathcal{O}(G_i)$ is completely isometric on $\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)$. Hence $C^*_e(\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i))$ is a quotient of $\underset{\mathcal{V}}{*}\mathcal{O}(G_i)$.*

Moreover, we have that each G_i is row-finite if and only if $\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)$ has the unique extension property in $\underset{\mathcal{V}}{*}\mathcal{O}(G_i)$. In particular, in this case we have $C^*_e(\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)) \cong \underset{\mathcal{V}}{*}\mathcal{O}(G_i)$.

Proof. As $\mathcal{T}_+(G_i)$ can be identified as a subalgebra of $\mathcal{O}(G_i)$ via the image of the quotient map $q_i : \mathcal{T}(G_i) \to \mathcal{O}(G_i)$, by Proposition 5.3.3, we see that $\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)$ can be identified as a subalgebra of $\underset{\mathcal{V}}{*}\mathcal{O}(G_i)$ via the image of $q = \underset{\mathcal{V}}{*}q_i$.

For the second part, suppose each G_i is row-finite. Let $\pi : \underset{\mathcal{V}}{*}\mathcal{O}(G_i) \to B(\mathcal{H})$ be a $*$-representation. Then $\pi \circ q$ is a $*$-representation of $\underset{\mathcal{V}}{*}\mathcal{T}(G_i)$, and by invariance of the UEP it will suffice to show that $(\pi \circ q)|_{\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)}$ has the UEP. By Proposition 5.4.1, this happens if and only if $\pi_i \circ q_i$ is full CK. However, as each G_i is row-finite, Theorem 5.2.5 implies that each $\pi_i \circ q_i : \mathcal{T}(G_i) \to B(\mathcal{H})$ is full CK. Hence, $\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)$ has the unique extension property in $\underset{\mathcal{V}}{*}\mathcal{O}(G_i)$.

For the converse, assume one of $\{G_i\}_{i \in I}$ is not row-finite. We will construct a $*$-representation of $\underset{\mathcal{V}}{*}\mathcal{O}(G_i)$ that lacks the unique extension property when restricted to

$\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)$. Indeed, by Theorem 5.2.9 there is some $j \in I$ for which there is a CK representation $\rho_j : \mathcal{T}(G_j) \to B(\mathcal{H})$ that is not a full CK representation, and up to inflating $\mathcal{H}$ we may assume $\rho_j = \rho_j^{(\infty)}$. For $i \in I$ different from j, let $\rho_i : \mathcal{T}(G_i) \to B(\mathcal{H})$ be any representation annihilating the Cuntz-Krieger ideal $\mathcal{J}(G_i)$ for which $\rho_i(p_v) \neq 0$ for all $v \in V$. Again up to inflating $\mathcal{H}$ we may assume $\rho_i = \rho_i^{(\infty)}$. In this case for all $i \in I$ the representation ρ_i is unitarily equivalent to a representation on $\mathcal{H}_V$ where $\rho_i(p_v)$ is mapped to the projection P_v. In this case the free product $\underset{\mathcal{V}}{*}\rho_i$ is well-defined, and by Proposition 5.4.1 $\underset{\mathcal{V}}{*}\rho_i$ does not have the unique extension property when restricted to $\underset{\mathcal{V}}{*}\mathcal{T}_+(G_i)$ while still annihilating $\mathcal{J} = \langle \mathcal{J}(G_i) \rangle_{i \in I}$, so that it induces a representation $\underset{\mathcal{V}}{*}\dot{\rho}_i$ on $\underset{\mathcal{V}}{*}\mathcal{O}(G_i)$ that does not have the unique extension property. $\square$

Chapter 6

Dilations, inclusions of MCS, and completely positive maps

6.1 Introduction

This chapter is based on joint work with Davidson, Shalit and Solel in [31], and was inspired by a series of papers by Helton, Klep, McCullough and others on the advantages of using matrix convex sets when studying linear matrix inequalities (LMI). In particular, Helton, Klep and McCullough [63] showed that the matricial positivity domain of an LMI contains the information needed to determine minimal LMI up to unitary equivalence. We were particularly interested in a recent paper by these authors and Schweighofer [64] who dilate d-tuples of Hermitian matrices to commuting Hermitian matrices in order to obtain bounds on inclusions of the matrix cube inside other spectrahedra up to a scaling. The two central problems that attracted our attention are the following.

Problem 6.1.1. *Given two d-tuples of operators $A = (A_1, \dots, A_d) \in B(\mathcal{H})^d$ and $B = (B_1, \dots, B_d) \in B(\mathcal{K})^d$, determine whether there exists a unital completely positive (UCP) map $\phi : B(\mathcal{H}) \to B(\mathcal{K})$ such that $\phi(A_i) = B_i$ for all $i = 1, \dots, d$.*

Problem 6.1.2. *Given two matrix convex sets $\mathcal{S}$ and $\mathcal{T}$, and given that $\mathcal{S}_1 \subseteq \mathcal{T}_1$, determine whether $\mathcal{S} \subseteq r\mathcal{T}$ for some constant r.*

These problems were treated by Helton, Klep, McCullough, Schweighofer, and by others. Our goal is to approach these problems from a categorical perspective, and to sharpen, generalize and unify existing results. While Helton et. al. tend to deal with d-tuples of real Hermitian matrices, we have chosen to work in the context of d-tuples of matrices or operators on complex Hilbert spaces (it seems that with a little care our methods are applicable to the setting of symmetric matrices over the reals). Moreover we simultaneously consider self-adjoint and non-self-adjoint matrix convex sets.

Duality plays a central role in our work, but takes a somewhat different character. We find that a more natural object to associate to a d-tuple of matrices of bounded operators is the *matrix range* introduced by Arveson [6] in the early days of non-commutative dilation theory (see Subsection 2.4.1).

Matrix ranges are ideally suited to describe the possible images of a d-tuple under UCP maps. This was established by Arveson in the single-variable case, and easily extends to the multivariable situation. We use this to obtain, in Section 6.2, a functorial duality between the category of finite dimensional operator systems with UCP maps and closed and bounded matrix convex sets with matrix-affine maps. In particular, we obtain a complete description of when a d-tuple of operators can be mapped onto another by a UCP map, treating Problem 6.1.1.

The basic result is that there is a UCP map as in Problem 6.1.1 if and only if $\mathcal{W}(B) \subseteq \mathcal{W}(A)$, where $\mathcal{W}(A)$ and $\mathcal{W}(B)$ denote the matrix ranges of A and B, respectively (see Corollary 6.2.7). This turns out to be a very specific case of this much wider duality of categories (See Propositions 6.2.4 and 6.2.5, and Theorem 6.2.6).

In Section 6.3 we discuss minimal and maximal matrix convex sets determined by a convex set at the first level (in $\mathbb{C}^d$), along with polar duality in the sense of Effros and Winkler [49]. Minimal and maximal matrix convex sets correspond to minimal and maximal operator system structures on Archimedean order unit spaces as studied by [100], and we give a description for them in Theorem 6.3.1. We relate minimal and maximal matrix convex sets with duality, and show that duality of matrix convex sets essentially corresponds to duality of operator systems (See Theorem 6.3.11 and Corollary 6.3.12). Using duality, in Theorem 6.3.15 we relate Problem 6.1.1 back to free spectrahedra, and generalize results of many authors (See for instance [1, 62, 63, 66, 68]).

In Section 6.4 we figure out the extent to which a d-tuple of operators is determined by its matrix range. We show that a d-tuple A of compact operators can always be compressed to a minimal tuple that has the same matrix range. We characterize minimal tuples of compact operators in terms of their multiplicity and C*-envelope, and show that a minimal tuple of compact operators is determined by its matrix range up to unitary equivalence. We also treat the opposite case of a d-tuple of operators that generates a C*-algebra with no compact operators. When combining our approach with Voiculescu's Weyl-von Neumann Theorem, we show that under suitable conditions on the C*-envelope, the matrix range determines a d-tuple up to approximate unitary equivalence.

The remainder of the paper deals with Problem 6.1.2. A key ingredient is the construction of commuting normal dilations, following [64]. In [64, Theorem 1.1] it was established that *all* symmetric $m \times m$ matrices can be simultaneously dilateed up to a scale factor to a family $\mathcal{F}$ of commuting Hermitian contractions on a Hilbert space $\mathcal{H}$, in the sense that there is a constant r and an isometry $V : \mathbb{R}^m \to \mathcal{H}$ so that for every symmetric contraction $S \in M_m(\mathbb{R})$, there is some $T \in \mathcal{F}$ such that $rS = V^*TV$. In this result, it is crucial that m is fixed. We provide counterparts of this result that are independent of the ranks of the dilated operators.

In Section 6.5 we discuss the general problem of scaled inclusion as in Problem 6.1.2. We provide several useful equivalent conditions for solving this Problem with a scaling constant r in terms of minimal and maximal matrix convex sets and commuting dilations (See Theorems 6.5.3 and 6.5.4). As an application, we show that solving Problem 6.1.2 with constant r yields a uniform upper-bound of $2r - 1$ on the completely bounded norm of unital (not-necessarily completely) positive maps between associated operator systems (See Corollary 6.5.5).

Section 6.6 is devoted to finding concrete scales r for Problem 6.1.2. We show that for every d-tuple of contractive operators $X = (X_1, \ldots, X_d)$ on a Hilbert space $\mathcal{H}$, there is a normal commuting family of contractive operators $T = (T_1, \ldots, T_d)$ on a Hilbert space $\mathcal{K}$ and an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $\frac{1}{2d}X_i = V^*T_iV$ (or $\frac{1}{d}X_i = V^*T_iV$ for self-adjoints) for all i (See Corollary 6.6.5).

In the self-adjoint context we are able to provide variants of this dilation result under different symmetry conditions. In particular, if X lies in some matrix convex set $\mathcal{S}$, then under some symmetry conditions we can construct a commuting normal dilation T such that the spectrum $\sigma(T)$ of T is contained in $d \cdot \mathcal{S}_1$. This is used to obtain scaled inclusion results for new matrix convex sets.

For example, in Theorem 6.6.3 we show that if A and B are two d-tuples of self-adjoint operators, and if $\mathcal{D}_A^{sa}$ and $\mathcal{D}_B^{sa}$ denote the free self-adjoint spectrahedra determined by A and B, then under some symmetry assumptions on the set $\mathcal{D}_A^{sa}(1) \subseteq \mathbb{R}^d$, we have

$$\mathcal{D}_A^{sa}(1) \subseteq \mathcal{D}_B^{sa}(1) \Longrightarrow \mathcal{D}_A^{sa} \subseteq d\mathcal{D}_B^{sa}. \tag{6.1}$$

We provide a rich class of convex sets to which our dilation and inclusion results can be applied. We show that if K is a convex set in $\mathbb{R}^d$ that is invariant under the projection onto an isometric tight frame, then for self-adjoint matrix convex sets $\mathcal{S}$ and $\mathcal{T}$ such that $\mathcal{S}_1 = K$, we have the implication

$$K \subseteq \mathcal{T}_1 \Longrightarrow \mathcal{S} \subseteq d\mathcal{T}. \tag{6.2}$$

We then use this result to show that when K is the convex hull of a vertex-reflexive isometric tight frame (See Definition 6.6.13), invariance of K under projections onto the isometric tight frame defining K is automatic, so that the implication in equation (6.2) holds. Hence, not only can we obtain inclusion results as in equation (6.2) when the ground level K is the (hyper)-cube $[-1,1]^d$, we can in fact obtain inclusion results as in equation (6.2) when K is any real regular polytope (See Theorem 6.6.16).

In Section 6.7 we study inclusion problems where the ground floor K is the real unit ball $\overline{\mathbb{B}}_d$. We show there is a self-dual matrix convex set $\mathfrak{S} = (\mathfrak{S}_n)$ defined by

$$\mathfrak{S}_n = \big\{X \in (M_n)^d_{sa} : \big\| \sum_i X_i \otimes \overline{X}_i \big\| \leq 1\big\}.$$

We find that for all self-adjoint matrix convex sets $\mathcal{S}$ of d-dimensions,

$$\mathcal{S}_1 \subseteq \overline{\mathbb{B}}_d \Longrightarrow \mathcal{S} \subseteq \sqrt{d}\, \mathfrak{S}$$

and

$$\overline{\mathbb{B}}_d \subseteq \mathcal{S}_1 \Longrightarrow \mathfrak{S} \subseteq \sqrt{d}\, \mathcal{S}.$$

Moreover, the constant $\sqrt{d}$ is the optimal constant in both implications (see Theorem 6.7.7). In fact, in both implications one may replace $\mathfrak{S}$ with the matrix ball $\mathfrak{B} = \{X \in \cup(M_n)^d_{sa} : \sum_i X_i^2 \leq I\}$.

In [95] an operator system structure SOH(d) was constructed from Pisier's self-dual operator space by adding a unit. It was shown that the completely bounded norm of a complete order isomorphism between it and its dual, with the dual operator *space* structure, must be at least 2, and that 2 is realized by a complete order isomorphism that maps a certain canonical self-adjoint basis $\{I, H_1, ..., H_d\}$ to its dual basis. As a final application, in Section 6.7 we show that for $H = (H_1, ..., H_d)$, the matrix range $\mathcal{W}(H)$ is equal to $\mathfrak{S}$, and that SOH(d) is the unique operator system of dimension $d+1$ that has such a basis $\{I, T_1, ..., T_d\}$ for which $\mathcal{W}(T)$ is self-dual and closed under entry-wise conjugation (See Theorem 6.7.9).

Here is a brief overview of the organization of this chapter. This Chapter contains seven sections, including this introductory section. Section 6.2 provides the categorical duality between matrix convex sets and finite dimensional operator systems, and contains results on completely positive interpolation, connecting it to inclusion of matrix ranges. Polar duality together with minimal and maximal matrix convex sets determined by the first level are treated in Section 6.3, and the extent to which a d-tuple is determined by its matrix range is discussed in Section 6.4. Equivalent conditions for having a scaled inclusion result are obtained in Section 6.5, along with an application to operator systems. In Section 6.6, we establish our version of the dilation to commuting normal operators, and provide a rich class of examples for which our results can be applied. Finally, in Section 6.7 we deal with the construction of a self-dual matrix ball based on an inequality due to Haagerup, and obtain optimality for various matrix ball inclusions.

6.2 Operator systems and matrix convex sets

In this section we give a rendition of a functorial duality between finite dimensional operator systems and matrix convex sets as defined in subsection 2.4.1. In its most general form, this duality is attributed to Effros and Winkler [49]. However, it can be made simpler for finite dimensional operator systems, and has been used implicitly for them in the literature under various guises (See for instance [1, 55, 62, 63]). We were mainly inspired by the work of Arveson on matrix ranges in [6] and the work of Helton, Klep and McCullough on linear matrix inequalities in [63].

When $C = [c_{ij}] \in M_{d',d}(\mathbb{C})$, and $X = (X_1, ..., X_d) \in B(\mathcal{H})^d$, we will denote by $C(X) \in B(\mathcal{H})^{d'}$ the d'-tuple of operators given by $C(X) = (\sum_j c_{ij}X_j)$. This is the same as applying the matrix C to X thought of as a column vector.

When we have two matrix convex sets $\mathcal{S}$ in d-dimensions, and $\mathcal{T}$ in d' dimensions, an *n-matrix-affine* map $\phi : \mathcal{S} \to \mathcal{T}$ is a sequence of maps $\phi = \{ \phi^{(m)} : \mathcal{S}_m \to \mathcal{T}_m \}$ such that

1. For $A \in \mathcal{S}_{m_1}$ and $B \in \mathcal{S}_{m_2}$ we have $\phi^{(m_1+m_2)}(A \oplus B) = \phi^{(m_1)}(A) \oplus \phi^{(m_2)}(B)$ whenever $m_1 + m_2 \leq n$.
2. For any isometry $V \in M_{m_1,m_2}(\mathbb{C})$ and $X \in \mathcal{S}_{m_1}$ we have $\phi^{(m_1)}(V^*XV) = V^*\phi^{(m_2)}(X)V$ for $m_1, m_2 \leq n$.

We will say that a sequence $\phi := \{ \phi^{(m)} : \mathcal{S}_m \to \mathcal{T}_m \}_{m=1}^{\infty}$ is a *matrix-affine* map if ϕ is n-matrix-affine for all $n \in \mathbb{N}$.

Proposition 6.2.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be matrix convex sets in d and d' dimensions respectively, and let $\phi : \mathcal{S} \to \mathcal{T}$ be an n-matrix-affine map. Then there exists $C \in M_{d',d}(\mathbb{C})$ and a vector $c := (c_1, ..., c_{d'}) \in \mathbb{C}^{d'}$ such that $\phi^{(m)}(X) = C(X) + c \cdot I_{d'}$ for any $X \in \mathcal{S}_m$ and $m \leq n$. In particular, if ϕ is matrix affine, then $C \in M_{d',d}(\mathbb{C})$ and $c \in \mathbb{C}^{d'}$ above33 satisfy $\phi^{(m)}(X) = C(X) + c \cdot I_{d'}$ for any $X \in \mathcal{S}_m$ and $m \in \mathbb{N}$.*

Proof. When $\phi = \{\phi^{(m)}\}$ is n-matrix-affine, we see that ϕ_1 is an affine map, so there exists $C \in M_{d',d}(\mathbb{C})$ and $c \in \mathbb{C}^{d'}$ such that $\phi^{(1)}(x) = C(x) + c$. Hence, for any $X \in \mathcal{S}_m$ with $m \leq n$ and norm-one column vector $v \in \mathbb{C}^m$ we have that $v^*\phi^{(m)}(X)v = \phi^{(1)}(v^*Xv) = C(v^*Xv) + c = v^*(C(X) + c \cdot I_n)v$. Since this identity will also hold for the coordinate-wise adjoint, we see that the real and imaginary parts of each coordinate coincide, so that $\phi^{(m)}(X) = C(X) + c \cdot I_m$. □

We denote $\phi_{C,c} = \{\phi_{C,c}^{(m)}\}$ the matrix-affine map given by $\phi_{C,c}^{(m)}(X) = C(X) + cI_m$ for each $X \in \mathcal{S}_m$. Depending on $\mathcal{S}$, different $C \in M_{d,d'}(\mathbb{C})$ and $c \in \mathbb{C}^{d'}$ may yield the same matrix-affine map $\phi_{C,c}$.

Definition 6.2.2. *Let $\mathcal{V}$ be a finite dimensional operator system. A* choice of coordinates *for $\mathcal{V}$ is a d-tuple of operators $T = (T_1, ..., T_d)$ in $\mathcal{V}$ such that $I, T_1, ..., T_d$ is a $*$-spanning set for $\mathcal{V}$. We say that this choice is* self-adjoint *if each T_i is self-adjoint.*

When we have a choice of coordinates for an operator system $\mathcal{V}$, we may form the matrix range $\mathcal{W}(T)$. The following proposition shows that the choice of coordinates is irrelevant up to a matrix-affine bijection. We will say that two matrix convex sets $\mathcal{S}$ and $\mathcal{T}$ are *matrix-affine isomorphic* if there exists a matrix affine map $\phi : \mathcal{S} \to \mathcal{T}$ such that ϕ_n is bijective. It follows from this that $\phi^{-1} = \{\phi_n^{-1}\}$ is also a matrix affine map.

Proposition 6.2.3. *Let $\mathcal{V}$ be a finite dimensional operator system, and $T = (T_1, ..., T_d)$ and $T' = (T_1, ..., T_{d'})$ be two choices of coordinates for $\mathcal{V}$. Then $\mathcal{W}(T)$ and $\mathcal{W}(T')$ are matrix-affine isomorphic.*

Proof. We denote by I the matrix Archimedean order unit of $\mathcal{V}$. First we show that $\mathcal{W}(T)$ is always matrix-affine isomorphic to the self-adjoint matrix range $\mathcal{W}(\mathrm{Re}(T), \mathrm{Im}(T))$ where

$$(\mathrm{Re}(T), \mathrm{Im}(T)) = (\mathrm{Re}(T_1), ..., \mathrm{Re}(T_d), \mathrm{Im}(T_1), ..., \mathrm{Im}(T_d)).$$

Indeed, since $T_i = \mathrm{Re}(T_i) + i\,\mathrm{Im}(T_i)$, by specifying the $d \times 2d$ matrix $C = [c_{ij}]$ where

$$c_{ij} = \begin{cases} 1 & \text{if } j = i \\ i & \text{if } j = i + d \\ 0 & \text{if } j \neq i, i + d. \end{cases}$$

We get that $\phi_{C,0} : \mathcal{W}(\mathrm{Re}(T), \mathrm{Im}(T)) \to \mathcal{W}(T)$ is a *bijective* matrix-affine map. Indeed, as every UCP map preserves adjoints, if $X = (X_1, ..., X_d) \in \mathcal{W}(T)$, then $(\mathrm{Re}(X), \mathrm{Im}(X)) \in$

$\mathcal{W}(\text{Re}(T), \text{Im}(T))$ is the *unique* $2d$-tuple in $\mathcal{W}(\text{Re}(T), \text{Im}(T))$ that gets mapped to X via $\phi_{C,0}$.

Thus, we may assume henceforth that both T and T' are self-adjoint. In particular, not only are $\{I, T_1, ..., T_d\}$ and $\{I, T'_1, ..., T'_d\}$ $*$-spanning sets for $\mathcal{V}$, they are in fact *spanning* for $\mathcal{V}$.

In this case we may write $T_i = a_i \cdot I + \sum_{j=1}^{d'} a_{ij} T'_j$ for each $1 \leq i \leq d$. Hence, the matrix $A = [a_{ij}]$ and $a = (a_1, ...a_d)$ makes $\phi_{A,a}$ into a matrix-affine map, where a matrix-affine inverse for it is given by $\phi_{B,b}$ where $B = [b_{ji}]$ and $b = (b_1, ..., b_{d'})$ are given from the coefficients of some linear combinations $T'_j = b_j \cdot I + \sum_{i=1}^{d} b_{ji} T_i$. □

From this proof we see that some choices of coordinates are better than others, for instance, when $T = (T_1, ..., T_d)$ is a self-adjoint choice of coordinates, we get that $\mathcal{W}(T)$ is a self-adjoint matrix convex set.

Denote by $\mathbb{M} = \{M_n(\mathbb{C})\}$. Whenever we have a *closed and bounded* matrix convex set $\mathcal{S}$ in d-dimensions, we define a finite dimensional operator system associated to it. The $*$-vector space is given by

$$\mathbb{A}(\mathcal{S}) = \{ \ f : \mathcal{S} \to \mathbb{M} \mid f \text{ is matrix affine } \}.$$

It is then clear by Proposition 6.2.1 that the constant function $\mathbb{1}$ and the coordinate functions $z_i : \mathcal{S} \to \mathbb{M}$ given by $z_i(X_1, ..., X_d) = X_i$ are matrix-affine functions that are $*$-spanning for the $*$-vector space $\mathbb{A}(\mathcal{S})$. Denote by $\mathbb{M} \otimes M_k = \{M_n(\mathbb{C}) \otimes M_k(\mathbb{C})\}$. Then each element in $M_k(\mathbb{A}(\mathcal{S}))$ is identified with a function $F : \mathcal{S} \to \mathbb{M} \otimes M_k$ given by

$$F(z) = A_0 \otimes I_k + \sum_{i=1}^{d} A_i \otimes z_i + \sum_{i=1}^{d} B_i \otimes \overline{z_i}.$$

We define the cones

$$M_k(\mathbb{A}(\mathcal{S}))_+ = \{ \ F \in M_k(\mathbb{A}(\mathcal{S}))_{sa} \mid F(X) \geq 0 \text{ for } X \in \mathcal{S}_n \ \}.$$

It is easy to verify that $\{M_k(\mathbb{A}(\mathcal{S}))_+\}$ is a matrix ordering on $\mathbb{A}(\mathcal{S})$ and since $\mathcal{S}$ is bounded, we get that the constant function $\mathbb{1}$ is an Archimedean matrix order unit for this matrix ordering. Hence, $(\mathbb{A}(\mathcal{S}), \{M_k(\mathbb{A}(\mathcal{S}))_+\}, \mathbb{1})$ becomes a finite dimensional operator system.

Our next goal is to show that UCP maps of operator systems correspond to matrix affine maps between matrix ranges.

Proposition 6.2.4. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional operator systems, and $V = (V_1, ..., V_d)$, $W = (W_1, ..., W_{d'})$ are choices of coordinates for them.*

1. *If $\phi : \mathcal{V} \to \mathcal{W}$ is a unital n-positive map, then the map $\phi^* : \mathcal{W}(W) \to \mathcal{W}(V)$ given for any UCP map $\psi : \mathcal{W} \to M_m$ with $m \leq n$ by $\phi^*(\psi(W_1), ..., \psi(W_{d'})) = (\psi \circ \phi(V_1), ..., \psi \circ \phi(V_d))$ is an n-matrix-affine map. In particular, if ϕ is a unital completely positive map, then ϕ^* is a matrix-affine map.*

2. *If ϕ is a complete order isomorphism, then ϕ^* is a matrix-affine isomorphism.*

3. *If $d = d'$ and ϕ is a unital n-positive map such that $\phi(V_i) = W_i$, then $\phi^* = Id_{\mathcal{W}_n(W)}$ so that $\mathcal{W}_n(W) \subseteq \mathcal{W}_n(V)$. In particular, when ϕ is also a unital completely positive map, then $\mathcal{W}(W) \subseteq \mathcal{W}(V)$. If additionally ϕ is a complete order isomorphism, then $\mathcal{W}(W) = \mathcal{W}(V)$.*

Proof. It is easy to verify by definition that ϕ^* is n-matrix-affine, so that item (1) follows. Items (2) and (3) then also follow easily from item (1). □

To reverse the picture, we start with given matrix convex sets and ask for induced morphisms between their associated operator systems.

Proposition 6.2.5. *Let $\mathcal{S}$ and $\mathcal{T}$ be bounded and closed matrix convex sets in d and d' dimensions.*

1. *If $\phi : \mathcal{S} \to \mathcal{T}$ is an n-matrix-affine map, then the map $\phi_* : \mathbb{A}(\mathcal{T}) \to \mathbb{A}(\mathcal{S})$ given by $\phi_*(f) = f \circ \phi$ is a unital n-positive map. In particular, if ϕ is matrix-affine, then ϕ_* is unital completely positive.*

2. *If ϕ is a matrix-affine isomorphism, then ϕ_* is a complete order isomorphism.*

3. *Suppose $d = d'$ and that $z = (z_1, ..., z_d)$ and $w = (w_1, ..., w_d)$ are coordinate functions for $\mathbb{A}(\mathcal{S})$ and $\mathbb{A}(\mathcal{T})$ respectively. If $\phi = \phi_{I,0}$ is a matrix-affine map, then $\phi_*(z_i) = w_i$. If additionally ϕ is a matrix-affine isomorphism, then ϕ_* is a unital complete order isomorphism.*

Proof. We show (1), where items (2) and (3) follow easily from (1). It is clear that ϕ_* is unital, so let $F \in M_n(\mathbb{A}(\mathcal{T}))_+$. For $X \in \mathcal{T}_m$ and $m \leq n$, as F is self-adjoint, we have $F(X) = A_0 \otimes I_m + \text{Re}(\sum_{i=1}^d A_i \otimes X_i)$. Let $Y \in \mathcal{S}_m$. Applying ϕ_* we obtain

$$\phi_*(F)(Y) = F(\phi^{(m)}(Y)) = A_0 \otimes I_m + \text{Re}(\sum_{i=1}^d A_i \otimes \phi^{(m)}(Y)_i)$$

so as $\phi^{(m)}(Y) \in \mathcal{T}_m$ we get that $\phi_*(F)(Y) \geq 0$. Hence ϕ^* is n-positive. $\square$

Theorem 6.2.6. *Let $(\mathcal{V}, \{M_n(\mathcal{V})_+\}, e)$ be a finite dimensional operator system, and $\mathcal{S}$ a d-dimensional* bounded and closed *matrix convex set.*

1. *Suppose $T = (T_1, .., T_d)$ is some choice of coordinates for $\mathcal{V}$. Then the unital map $\varphi : \mathcal{V} \to \mathbb{A}(\mathcal{W}(T))$ given by*

$$\varphi(a_0 e + \sum_{i=1}^{d} a_i T_i + b_i T_i^*) = a_0 \mathbb{1} + \sum_{i=1}^{d} a_i z_i + b_i \overline{z_i}$$

   is a complete order isomorphism.

2. *There exists $T \in B(\mathcal{H})^d$ such that $\mathcal{S} = \mathcal{W}(T)$ and if $z = (z_1, .., z_d)$ are the coordinate functions for $\mathbb{A}(\mathcal{S})$, then we have $\mathcal{S} = \mathcal{W}(z)$.*

Proof. We start with item (1). Clearly φ is unital, so we need only show it is a complete order isomorphism, and the rest will follow. Suppose $A_0, ..., A_d \in M_n(\mathbb{C})$ are given, such that $A_0 \otimes e + \mathrm{Re}(\sum_{i=1}^{d} A_i \otimes T_i)$ is positive. Let $X \in \mathcal{W}(T)$. Then there is a UCP map $\psi : \mathcal{V} \to M_n$ such that $X = (\psi(T_1), ..., \psi(T_d))$ so we see that $F(z) = A_0 \otimes I_n + \mathrm{Re}(\sum_{i=1}^{d} A_i \otimes z_i)$ satisfies $F(X) = (I \otimes \psi)(A_0 \otimes e + \sum_{i=1}^{d} A_i \otimes T_i) \geq 0$ for any $X \in \mathcal{W}(T)$.

Conversely, let $F \in M_n(\mathbb{A}(\mathcal{W}(T)))$ be given by

$$F(z) = A_0 \otimes I_k + \sum_{i=1}^{d} A_i \otimes z_i + \sum_{i=1}^{d} B_i \otimes \overline{z_i}$$

and suppose it satisfies $F(X) \geq 0$ for all $X \in \mathcal{W}(T)$. Embed $\mathcal{V} \subseteq B(\mathcal{H})$ unitally for some Hilbert space $\mathcal{H}$ and take $\{P_\alpha\}$ to be an increasing net of projections onto finite dimensions converging SOT to $I_{\mathcal{H}}$. Then clearly $P_\alpha T P_\alpha = (P_\alpha T_1 P_\alpha, ..., P_\alpha T_d P_\alpha) \in \mathcal{W}(T)$ so that for any α we have

$$A_0 \otimes I_k + \sum_{i=1}^{d} A_i \otimes P_\alpha T_i P_\alpha + \sum_{i=1}^{d} B_i \otimes P_\alpha T_i^* P_\alpha = F(P_\alpha T P_\alpha) \geq 0.$$

By taking a limit through α we get that $A_0 \otimes I_k + \sum_{i=1}^{d} A_i \otimes T_i + \sum_{i=1}^{d} B_i \otimes T_i^*$ is positive.

Next we show item (2). For the first part, let $\{T^{(k)}\}_{k=1}^{\infty}$ be a dense sequence of points in $\mathcal{S}$, where each point appears infinitely many times, and consider $T = \oplus_k T^{(k)}$ acting on $\mathcal{H} = \oplus_k \mathbb{C}^{n_k}$, where $T^{(k)} \in \mathcal{S}_{n_k}$. As $\mathcal{S}$ is bounded, we have that T is a bounded operator. Clearly $\mathcal{S} \subseteq \mathcal{W}(T)$ since $\mathcal{W}(T)$ contains each $T^{(k)}$ and is closed. Denote by $\mathcal{V}_T$ the operator system generated by T. For the reverse inclusion, note that by our choice of T, the intersection of $C^*(\mathcal{V}_T)$ with the compacts $\mathcal{K}(\mathcal{H})$ is trivial. Hence, by Voiculescu's theorem (e.g., [30, Lemma II.5.2]), we have that if $\varphi \in UCP(\mathcal{V}_T, M_n)$, there is a sequence of isometries $V_m : \mathbb{C}^n \to \mathcal{H}$ such that

$$\|\phi(T_j) - V_m^* T_j V_m\| \xrightarrow{m\to\infty} 0 \ , \ j = 1, \dots, d.$$

But every V_m has the form $V_m = (V_m^{(k)})_{k=1}^{\infty}$ such that $V_m^{(k)} : \mathbb{C}^n \to \mathbb{C}^{n_k}$ and $\sum_k V_m^{(n_k)*} V_m^{(n_k)} = I_n$, so $\lim_{k\to\infty} \|V_m^{(k)}\| = 0$. Therefore

$$V_m^* T_j V_m = \sum_k V_m^{(k)*} T_j^{(k)} V_m^{(k)},$$

where the sequence converges in norm. For a large enough finite set $F \subseteq \mathbb{N}$, we have $\|\sum_{k\in F} V_m^{(k)*} V_m^{(k)} - I\| < 1$, and hence $K_F := \sum_{k\in F} V_m^{(k)*} V_m^{(k)}$ must be invertible. Then,

$$\sum_{k\in F} K_F^{-\frac{1}{2}*} V_m^{(k)*} V_m^{(k)} K_F^{-\frac{1}{2}} = I,$$

so that $\sum_{k\in F} K_F^{-\frac{1}{2}*} V_m^{(k)*} T_j^{(k)} V_m^{(k)} K_F^{-\frac{1}{2}}$ is a genuine matrix convex combination of points in $\mathcal{S}$, converging (as F grows) to $V_m^* T_j V_m$. It then follows that $V_m^* T V_m \in \mathcal{S}$ and so $\phi(T) \in \mathcal{S}$.

For the last assertion of item (2), we know by item (1) that the UCP map sending T_i to z_i from $\mathcal{V}_T$ to $\mathbb{A}(\mathcal{W}(T)) = \mathbb{A}(\mathcal{S})$ is a complete order isomorphism. Then by item (3) of Proposition 6.2.4, we see that $\mathcal{S} = \mathcal{W}(T) = \mathcal{W}(z)$ where $z = (z_1, ..., z_d)$ are the coordinates for $\mathbb{A}(\mathcal{S})$. □

Combining Theorem 6.2.6 and Propositions 6.2.4 and 6.2.6, we see that there is a contravariant duality between the category of finite dimensional operator systems with unital completely positive maps, and the category of closed and bounded matrix convex sets with matrix-affine maps.

We will denote by $\mathcal{V}_A$ the operator system generated by a d-tuple of operators $A = (A_1, ..., A_d)$ in $B(\mathcal{H})$. We next sketch some equivalent formulations for the inclusion $\mathcal{W}(B) \subseteq \mathcal{W}(A)$ for d-tuples $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$. We denote by $\mathcal{H}^{(\infty)} = \mathcal{H} \oplus \mathcal{H} \oplus ...$ and for $A \in B(\mathcal{H})$ we denote $A^{(\infty)} = A \oplus A \oplus ... \in B(\mathcal{H}^{(\infty)})$.

Corollary 6.2.7. *Let $A = (A_1, ..., A_d) \in B(\mathcal{H})$ and $B = (B_1, ..., B_d) \in B(\mathcal{K})$ be d-tuples of operators. Then $\mathcal{W}(B) \subseteq \mathcal{W}(A)$ if and only if there is a UCP map $\phi : \mathcal{V}_A \to \mathcal{V}_B$ such that $\phi(A_i) = B_i$ for all $i = 1, ..., d$.*

Theorem 6.2.8. *Let $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$ be d-tuples of operators on separable Hilbert spaces. Then $\mathcal{W}(B) \subseteq \mathcal{W}(A)$ if and only if there are isometries $V_n : \mathcal{K} \to \mathcal{H}^{(\infty)}$ such that*

$$\lim_{n\to\infty} \|B_i - V_n^* A_i^{(\infty)} V_n\| = 0, \text{ for } 1 \leq i \leq d.$$

Moreover the isometries may be chosen so that $B - V_n^* A^{(\infty)} V_n$ are in $\mathcal{K}(\mathcal{H})^d$. Moreover, if $C^*(\mathcal{V}_A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ we can replace $A^{(\infty)}$ with A in the above statement.

Proof. Suppose that $\mathcal{W}(B) \subseteq \mathcal{W}(A)$. Then by Corollary 6.2.7 we have a UCP map $\phi : \mathcal{V}_A \to \mathcal{V}_B$ sending A_i to B_i. By Arveson's extension theorem, there is a UCP extension $\widetilde{\phi} : C^*(\mathcal{V}_A) \to B(\mathcal{K})$. By Stinespring's dilation theorem, there is a separable Hilbert space $\mathcal{L}$, an isometry $V : \mathcal{K} \to \mathcal{L}$ and a *-representation $\pi : C^*(\mathcal{V}_A) \to B(\mathcal{L})$ such that $\widetilde{\phi}(T) = V^*\pi(T)V$. By Voiculescu's Weyl-von Neumann theorem (e.g., [30, Theorem II.5.3]), $\mathrm{id}^{(\infty)} \sim_{\mathcal{K}(\mathcal{H})} \mathrm{id}^{(\infty)} \oplus \pi$, and if $C^*(\mathcal{V}_A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, we can instead say that $\mathrm{id} \sim_{\mathcal{K}(\mathcal{H})} \mathrm{id} \oplus \pi$ see [30, Lemma II.5.2], where id is the identity representation of $C^*(\mathcal{V}_A)$. This means that there is a sequence of unitaries $U_n : \mathcal{H}^{(\infty)} \to \mathcal{H}^{(\infty)} \oplus \mathcal{L}$ so that

$$\lim_{n\to\infty} \|(T^{(\infty)} \oplus \pi(T)) - U_n T^{(\infty)} U_n^*\| = 0 \quad \text{for} \quad T \in C^*(\mathcal{V}_A),$$

and moreover the differences in the limit expression are all compact operators. Let J be the natural injection of $\mathcal{L}$ into $\mathcal{H}^{(\infty)} \oplus \mathcal{L}$. Then $V_n = U_n^* J V$ is a sequence of isometries of $\mathcal{K}$ into $\mathcal{H}^{(\infty)}$ such that

$$\begin{aligned}
&\lim_{n\to\infty} B_i - V_n^* A_i^{(\infty)} V_n \\
&\qquad = \lim_{n\to\infty} V^* J^* \big(A_i^{(\infty)} \oplus \pi(A_i)\big) J V - V^* J^* U_n A_i^{(\infty)} U_n^* J V \\
&\qquad = \lim_{n\to\infty} V^* J^* \big(A_i^{(\infty)} \oplus \pi(A_i) - U_n A_i^{(\infty)} U_n^*\big) J V = 0;
\end{aligned}$$

and the differences are all compact. The converse is straightforward. □

Motivated by a similar analysis of single operators in [29], we make the following definitions.

Definition 6.2.9. *Define the distance of a d-tuple $X = (X_1, \dots, X_d)$ in M_n^d (or $B(\mathcal{H})^d$) from a subset $\mathcal{W}_n \subseteq M_n^d$ by*

$$d_n(X, \mathcal{W}_n) = \inf_{W \in \mathcal{W}_n} \max_{1 \leq i \leq d} \|X_i - W_i\|.$$

Then define the measure of containment *of a matrix convex set $\mathcal{S}$ in another matrix convex set $\mathcal{T}$ by*

$$\delta_{\mathcal{T}}(\mathcal{S}) = \sup\{d_n(S, \mathcal{T}_n) : S \in \mathcal{S}_n,\ n \geq 1\}.$$

Also define the distance between two bounded matrix convex sets by

$$\delta(\mathcal{S}, \mathcal{T}) = \max\{\delta_{\mathcal{S}}(\mathcal{T}), \delta_{\mathcal{T}}(\mathcal{S})\}.$$

We may define the semi-metric on $B(\mathcal{H})^d$ by

$$\rho(S, T) = \delta(\mathcal{W}(S), \mathcal{W}(T)),\ \textit{for}\ S, T \in B(\mathcal{H})^d.$$

It is easy to see that ρ is symmetric and satisfies the triangle inequality. However distinct tuples can be at distance 0. This semi-metric is blind to multiplicity; that is, $\rho(T, T^{(\infty)}) = 0$. The following proposition which describes when this occurs is immediate from the definition and Corollary 6.2.7.

Proposition 6.2.10. *For $A, B \in B(\mathcal{H})^d$, the following are equivalent:*

1. $\rho(A, B) = 0$.
2. $\mathcal{W}(A) = \mathcal{W}(B)$.
3. *There is a completely isometric UCP map of $\mathcal{V}_A$ onto $\mathcal{V}_B$ sending A to B.*

We next proceed to prove an approximate version of Proposition 6.2.10.

Lemma 6.2.11. *Let $A \in B(\mathcal{H})^d$. Select a countable dense subset*

$$\{A^{(k)} = (A_1^{(k)}, \ldots, A_d^{(k)}) : k \geq 1\} \quad \textit{of} \quad \mathcal{W}(A)$$

and define $\tilde{A} = \bigoplus_{k \geq 1} A^{(k)}$. Then $\rho(A, \tilde{A}) = 0$. Moreover if $\tilde{A}'$ is defined by another dense subset $\{A'^{(k)} : k \geq 1\}$, then $\tilde{A}' \sim_{\mathcal{K}(\mathcal{H})} \tilde{A}$.

Proof. That $\mathcal{W}(\tilde{A}) = \mathcal{W}(A)$ was established in the proof of item (2) of Theorem 6.2.6, so $\rho(A, \tilde{A}) = 0$ (actually, in the proof of item (2) of Theorem 6.2.6 we had an infinite multiplicity version of $\tilde{A}$, but ρ is blind to multiplicity). For the second statement, we may assume that A is not a d-tuple of scalars, as that case is trivial. Thus $\mathcal{W}_n(A)$ is a convex set containing more than one point, and hence there are countably many of the $A^{(k)}$s and

$A'^{(k)}$s in each $\mathcal{W}_n(A)$. As both are dense, given $\epsilon > 0$, it is a routine combinatorial exercise to find a permutation π of $\mathbb{N}$ such that

$$\|A^{(k)} - A'^{(\pi(k))}\| < \epsilon \ \ \forall k \geq 1, \ \text{ and } \lim_{k\to\infty} \|A^{(k)} - A'^{(\pi(k))}\| = 0.$$

It follows that there is a unitary operator U_π implementing this permutation so that

$$\|\tilde{A} - U_\pi \tilde{A}' U_\pi^*\| < \epsilon \ \text{ and } \ \tilde{A} - U_\pi \tilde{A}' U_\pi^* \in \mathcal{K}(\mathcal{H}).$$

Thus $\tilde{A}' \sim_{\mathcal{K}(\mathcal{H})} \tilde{A}$. □

Theorem 6.2.12. *Let $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$ such that*

$$\delta_{\mathcal{W}(A)}(\mathcal{W}(B)) = r.$$

Then there is a UCP map ψ of $\mathcal{V}_A$ into $B(\mathcal{K})$ such that

$$\|\psi(A_i) - B_i\| \leq r \ \ \textit{for } 1 \leq i \leq d.$$

Proof. Following Lemma 6.2.11, let $\tilde{B} = \bigoplus_{k\geq 1} B^{(k)}$ be a block diagonal operator in $B(\tilde{\mathcal{K}})$ with $n \times n$ summands dense in $\mathcal{W}_n(B)$ for each $n \geq 1$. For each k, select $A^{(k)} \in \mathcal{W}_{n_k}(A)$ so that $\|B_i^{(k)} - A_i^{(k)}\| \leq r$ for all i. Let $\tilde{A} = \bigoplus_{k\geq 1} A^{(k)}$ in $B(\tilde{\mathcal{K}})$. Then by Proposition 6.2.10, $\rho(A \oplus \tilde{A}, A) = 0$. The map $\phi : B(\mathcal{H} \oplus \tilde{\mathcal{K}}) \to B(\tilde{\mathcal{K}})$ given by compression to $\tilde{\mathcal{K}}$ is a UCP map that takes $A \oplus \tilde{A}$ to $\tilde{A}$. Let ψ_1 be the completely isometric map of $\mathcal{V}_A$ onto $\mathcal{V}_{A\oplus\tilde{A}}$ and let ψ_2 be the completely isometric map of $\mathcal{V}_{\tilde{B}}$ onto $\mathcal{V}_B$ that take generators to generators. Then letting $\tilde{\psi}_2$ be the extension of ψ_2 to $B(\tilde{\mathcal{K}})$, $\psi = \tilde{\psi}_2 \phi \psi_1$ is the desired UCP map satisfying

$$\|\psi(A_i) - B_i\| \leq r \ \ \text{for } 1 \leq i \leq d.$$

□

6.3 Maximal, minimal and dual structures

In this section we will describe the smallest and largest matrix convex sets with a given ground floor. A d-tuple $X \in M_n^d$ is a compression of $A \in B(\mathcal{H})^d$ if there is an isometry $V : \mathbb{C}^n \to \mathcal{H}$ such that $X_i = V^* A_i V$ for $1 \leq i \leq d$. On the flip side, we will say that A dilates X when X is a compression of A. Recall that a d-tuple $N = (N_1, ..., N_d)$ will be called *normal* if it is comprised of normal commuting operators, and that $\sigma(N)$ denotes the joint spectrum of N.

Suppose $\mathcal{S}$ is a d-dimensional closed and bounded matrix convex set. We saw that $\mathbb{A}(\mathcal{S})$ is an operator system, but we may "forget" about the matrix ordering on $\mathcal{V} := \mathbb{A}(\mathcal{S})$ and see $(\mathbb{A}(\mathcal{S}), \mathbb{A}(\mathcal{S})_+, \mathbb{1})$ only as an Archimedean order unit space. Hence, there are minimal and maximal operator structures OMIN($\mathcal{V}$) and OMAX($\mathcal{V}$) on it as described in Subsection 2.4.2, such that the identity on $\mathcal{V}$ is a unital completely positive map from OMAX($\mathcal{V}$) into $\mathcal{V}$, and from $\mathcal{V}$ into OMIN($\mathcal{V}$). Using Proposition 6.2.4 we get two matrix convex sets $\mathcal{W}^{min}(\mathcal{S}_1)$, and $\mathcal{W}^{max}(\mathcal{S}_1)$ with ground level $\mathcal{S}_1$ such that $\mathcal{W}^{min}(\mathcal{S}_1) \subseteq \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{W}^{max}(\mathcal{S}_1)$. Hence, to any given ground level compact convex set $K \subseteq \mathbb{C}^d$ and a closed matrix convex set $\mathcal{S}$ (which is automatically bounded) with $\mathcal{S}_1 = K$, there are $\mathcal{W}^{min}(K)$, and $\mathcal{W}^{max}(K)$ such that

$$\mathcal{W}^{min}(K) \subseteq \mathcal{S} \subseteq \mathcal{W}^{max}(K).$$

We give useful descriptions of $\mathcal{W}^{min}(K)$ and $\mathcal{W}^{max}(K)$ in terms of K. When K is a compact subset of $\mathbb{C}^d$, it is the intersection of all half spaces containing K of the form

$$H(\alpha, a) = \{\ x \in \mathbb{C}^d \mid \operatorname{Re}(\sum_{i=1}^{d} \alpha_i x_i) \leq a\ \}$$

with $\alpha_i \in \mathbb{C}$ and $a \in \mathbb{R}$. Moreover, when $A \in B(\mathcal{H})^d$, then $\mathcal{W}_1(A) \subseteq H(\alpha, a)$ if and only if $\operatorname{Re}(\sum_{i=1}^{d} \alpha_i A_i) \leq aI$.

Theorem 6.3.1. *Let K be a compact convex subset of $\mathbb{C}^d$. Then*

$$\mathcal{W}_n^{min}(K) = \{\ X \in M_n^d \mid X \text{ dilates to } N \text{ normal and } \sigma(N) \subseteq K\ \}$$

and

$$\mathcal{W}_n^{max}(K) = \{\ X \in M_n^d \mid \operatorname{Re}(\sum_{i=1}^{d} \alpha_i X_i) \leq aI_n, \text{ when } K \subseteq H(\alpha, a)\ \}.$$

Proof. Clearly the right hand sides in both characterizations are matrix convex with ground level K, so that one inclusion in each characterization is always immediate.

For the other inclusion in the minimal characterization, let $X \in M_n^d$ be with normal dilation N such that $\sigma(N) \subseteq K$. By Theorem 2.4.6, $\mathcal{W}(N)$ is the smallest matrix convex set containing $\sigma(N)$, so that $\mathcal{W}(N) \subseteq \mathcal{W}^{min}(K)$, and we get $X \in \mathcal{W}(N) \subseteq \mathcal{W}^{min}(K)$.

For the other inclusion in the maximal characterization, if $X \in \mathcal{W}^{max}(K)$, and $H(\alpha, a)$ is a half-space containing K, since $\mathcal{W}(X) \subseteq \mathcal{W}^{max}(K)$ we have that $\mathcal{W}_1(X) \subseteq K$ so that $\operatorname{Re}(\sum_{i=1}^{d} \alpha_i X_i) \leq aI_n$ for any half space $H(\alpha, a)$ that contains K. Hence, we obtain the other inclusion for the maximal characterization. □

Corollary 6.3.2. *Let N be a normal d-tuple, and denote $K = \mathcal{W}_1(N)$. Then $\mathcal{W}(N) = \mathcal{W}^{min}(K)$.*

Proof. Clearly $\mathcal{W}^{min}(K) \subseteq \mathcal{W}(N)$. However, by Arveson's extension theorem we may extend any UCP map $\phi : \mathcal{V}_N \to M_n$ to a UCP map on $C^*(I, N)$, and then by Stinespring's dilation theorem it has the form $\phi(T) = V^*\pi(T)V$ for an isometry $V : \mathbb{C}^n \to \mathcal{H}$, and some unital $*$-representation $\pi : C^*(I, N) \to B(\mathcal{H})$. Hence, $\pi(N) = (\pi(N_1), ..., \pi(N_d))$ is a normal commuting tuple that dilates $\phi(N) = (\phi(N_1), ..., \phi(N_d))$ for any UCP map $\phi : \mathcal{V}_N \to M_n$, and we obtain the reverse inclusion. $\square$

Corollary 6.3.3. *Suppose $\mathcal{S}$ is a matrix convex set such that $\mathcal{S}_1 \subseteq r\overline{\mathbb{D}}^d$ where $\overline{\mathbb{D}}^d$ is the closed polydisc. Then $\mathcal{S}$ is bounded by $2r$ in the sense that $\|X_i\| \leq 2r$ for all $X \in \mathcal{S}$ and $1 \leq i \leq d$.*

Proof. By Proposition 2.4.2 we know that $\mathcal{S}$ is bounded. Since $\mathcal{S} \subseteq r\overline{\mathbb{D}}^d$ and

$$r\overline{\mathbb{D}}^d = \{ \ z = (z_1, ..., z_d) \in \mathbb{C}^d \ | \ \mathrm{Re}(\lambda z_i) \leq r, \ \forall \lambda \in \mathbb{T} \ \},$$

for every $X \in \mathcal{W}^{max}(r\mathbb{B}^d)$ we have that $\|X_i\| \leq \|\mathrm{Re}\, X_i\| + \|\mathrm{Im}\, X_i\| = \|\mathrm{Re}\, X_i\| + \|\mathrm{Re}(iX_i)\| \leq 2r$, and we get that $\mathcal{S} \subseteq \mathcal{W}^{max}(r\overline{\mathbb{D}}^d) \subseteq 2r\mathfrak{D}^{(d)}$ where $\mathfrak{D}^{(d)}$ is the matrix polydisc. Hence $\|X_i\| \leq 2r$ for every $X \in \mathcal{S}$ and $1 \leq i \leq d$. $\square$

Next, we will discuss dual matrix convex sets in the sense of Effros and Winkler [49]. Suppose $\mathcal{S}$ is a matrix convex set in d-dimensions. Then we define $\mathcal{S}^\circ$ by

$$\mathcal{S}_n^\circ = \{ \ X \in M_n^d \ | \ \mathrm{Re}(\sum_{i=1}^d A_i \otimes X_i) \leq I \ \forall A \in \mathcal{S} \ \}$$

and in case $\mathcal{S}$ is self-adjoint we define $\mathcal{S}^\bullet$ by

$$\mathcal{S}_n^\bullet = \{ \ X \in (M_n)_{sa}^d \ | \sum_{i=1}^d A_i \otimes X_i \leq I \ \forall A \in \mathcal{S} \ \}.$$

Then $\mathcal{S}^\circ$ and $\mathcal{S}^\bullet$ are closed matrix convex sets containing 0, and by [49, Lemma 5.1] $\mathcal{S}_n^\circ$ and $\mathcal{S}_n^\bullet$ are determined by all $A \in \mathcal{S}_n$ in level n. Moreover, by the Effros–Winkler bipolar theorem, we know that if $0 \in \mathcal{S}$, then $\mathcal{S}^{\circ\circ} = \mathcal{S}$. We also have this for self-adjoint matrix convex sets and self-adjoint polar duals. For a d-tuple $X = (X_1, ..., X_d)$ we denote $\mathrm{Re}(X) = (\mathrm{Re}(X_1), ..., \mathrm{Re}(X_d))$.

Proposition 6.3.4. *Let $\mathcal{S}$ be a closed self-adjoint matrix convex set. Then*

$$\mathcal{S}_n^\bullet = \{ \operatorname{Re}(X) \mid X \in \mathcal{S}_n^\circ \}.$$

Furthermore, if $0 \in \mathcal{S}$, we get that $\mathcal{S}^{\bullet\bullet} = \mathcal{S}$.

Proof. If $X \in \mathcal{S}^\bullet$ then $X = \operatorname{Re}(X)$ so that for any $A \in \mathcal{S}$ we have

$$\operatorname{Re}(\sum_{i=1}^d A_i \otimes X_i) = \sum_{i=1}^d \operatorname{Re}(X_i) \otimes A_i \leq I$$

and $X \in \mathcal{S}^\circ$. Conversely, as each $A \in \mathcal{S}$ is self-adjoint, the same equality above shows that $\operatorname{Re}(X) \in \mathcal{S}^\bullet$.

Next, assume that $0 \in \mathcal{S}$. Then clearly $\mathcal{S} \subseteq \mathcal{S}^{\bullet\bullet}$. Since we know that $\mathcal{S} = \mathcal{S}^{\circ\circ}$, it will suffice to show that $\mathcal{S}^{\bullet\bullet} \subseteq \mathcal{S}^{\circ\circ}$. So let $X \in \mathcal{S}_n^{\bullet\bullet}$. Then $X \in (\mathcal{S}^\bullet)^\circ \cap (M_n)_{sa}^d$ so that

$$\operatorname{Re}(\sum_{i=1}^d A_i \otimes X_i) = \sum_{i=1}^d A_i \otimes X_i \leq I$$

for all $A \in \mathcal{S}_n^\bullet$. So if $B \in \mathcal{S}_n^\circ$, we have $\operatorname{Re}(B) \in \mathcal{S}_n^\bullet$. Hence, as X is self-adjoint,

$$\operatorname{Re}(\sum_{i=1}^d B_i \otimes X_i) = \sum_{i=1}^d \operatorname{Re}(B_i) \otimes X_i \leq I$$

and we get that $X \in \mathcal{S}^{\circ\circ}$. □

We next show that matrix ranges and free operator spectrahedra are essentially duals of each other.

Proposition 6.3.5. *Let $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{H})_{sa}^d$. Then*

$$(\mathcal{W}(A) \cup \{0\})^\circ = \mathcal{W}(A)^\circ = \mathcal{D}_A \quad \text{and} \quad (\mathcal{W}(B) \cup \{0\})^\bullet = \mathcal{W}(B)^\bullet = \mathcal{D}_B^{sa}.$$

If furthermore we have $0 \in \mathcal{W}(A)$ and $0 \in \mathcal{W}(B)$ then

$$\mathcal{D}_A^\circ = \mathcal{W}(A) \quad \text{and} \quad (\mathcal{D}_B^{sa})^\bullet = \mathcal{W}(B).$$

Proof. We prove the self-adjoint claims, where the non-self-adjoint claims follow similarly by taking real parts in the appropriate places.

First note that by definition of the dual, we have that $(\mathcal{W}(B) \cup \{0\})^\bullet = \mathcal{W}(B)^\bullet$. Next, if $X \in \mathcal{D}_B^{sa}(n)$ and $\phi \in UCP(\mathcal{V}_B, M_n)$, then by applying $\phi \otimes id$ to the inequality $\sum_{i=1}^d B_i \otimes X_i \leq I$ we get that $\sum_{i=1}^d \phi(B_i) \otimes X_i \leq I$ so that $X \in \mathcal{W}(B)_n^\bullet$. Conversely, if $\sum_{i=1}^d \phi(B_i) \otimes X_i \leq I$ for all $\phi \in UCP(\mathcal{V}_B, M_n)$, by [49, Lemma 5.1] we know this occurs for all $\phi \in UCP(\mathcal{V}_B, M_k)$ and any $k \in \mathbb{N}$. By letting ϕ range over all finite dimensional compressions of $B(\mathcal{H})$, we get that $X \in \mathcal{D}_B^{sa}$. Finally, in case $0 \in \mathcal{W}(B)$, by using Proposition 6.3.4 we get that $(\mathcal{D}_B^{sa})^\bullet = \mathcal{W}(B)$. □

We will say that 0 is an interior point for a matrix convex set $\mathcal{S}$, and write $0 \in \text{int}(\mathcal{S})$, if there exists $\delta > 0$ such that if $X = (X_1, ..., X_d) \in M_n^d$ with $\|X_i\| < \delta$, then $X \in \mathcal{S}_n$. When $\mathcal{S}$ is self-adjoint, we will require that $X \in \mathcal{S}_n$ only for $X \in (M_n)_{sa}^d$ with $\|X_i\| < \delta$.

We then obtain the following relation between boundedness and 0 being interior for the dual.

Proposition 6.3.6. *Let $\mathcal{S}$ be a self-adjoint closed and bounded matrix convex set. Then the following are equivalent:*

1. $0 \in \text{int}(\mathcal{S})$.
2. $0 \in \text{int}(\mathcal{S}_1)$.
3. $\mathcal{S}^\bullet$ *is bounded.*
4. $\mathcal{S}_1^\bullet$ *is bounded.*

A similar statement holds without the assumption of self-adjointness, where the self-adjoint dual is replaced by the regular dual.

Proof. We provide the proof in the self-adjoint case where the proof for the non-self-adjoint case is done similarly. In this case, it is clear by Proposition 2.4.2 that (3) and (4) are equivalent, and that (1) implies (2).

We will show that (3) implies (2) and that (1) implies (4) to finish the proof. First note that by Theorem 6.2.6, as $\mathcal{S}$ is self-adjoint, we know that $\mathcal{S} = \mathcal{W}(A)$ for some $A \in B(\mathcal{H})_{sa}^d$.

Hence, assuming towards contradiction that $0 \notin \text{int}(\mathcal{W}_1(A))$, by Hahn-Banach theorem there are real numbers $a_1, ..., a_d$ not all zero, such that for every $x = (x_1, ..., x_d) \in \mathcal{W}_1(A)$

we have $\sum_{i=1}^d a_i x_i \geq 0$. Hence, for every $t < 0$ we would get that $\sum_{i=1}^d t a_i x_i \leq 0 < 1$ so that for every $t < 0$ we get that $(ta_1, ..., ta_d) \in \mathcal{D}_A^{sa}(1) = \mathcal{S}_1^\bullet$ which contradicts (3). This shows that (3) implies (2).

Finally, suppose there is $\delta > 0$ is such that whenever $\|X_i\| < \delta$ we have $X \in \mathcal{W}(A)$ for $1 \leq i \leq d$. Fix $1 \leq i \leq d$ and take $X^{(i)} = (X_1, ..., X_d)$ such that $X_j^{(i)} = 0$ if $i \neq j$ and $X_i^{(i)} = \frac{1}{2}\delta$. This way we have that $\pm X^{(i)} \in \mathcal{W}(A)$ for each $1 \leq i \leq d$. Now, for every $Y \in \mathcal{S}^\bullet = \mathcal{W}(A)^\bullet = \mathcal{D}_A^{sa}$ we have that

$$\frac{1}{2}\delta \otimes Y_i = \sum_{j=1}^d X_j^{(i)} \otimes Y_j \leq I.$$

However, $\|\frac{1}{2}\delta \otimes Y_i\| = \frac{1}{2}\delta\|Y_i\|$, so that $\|Y_i\| \leq \frac{2}{\delta}$, and therefore $\mathcal{S}^\bullet = \mathcal{D}_A^{sa}$ is bounded. Hence, we see that (1) implies (4). □

Using duality, we can now obtain a characterization of those matrix convex sets that are free operator spectrahedra, akin to item (2) of Theorem 6.2.6.

Corollary 6.3.7. *Let $\mathcal{S}$ be a closed self-adjoint matrix convex set. Then $\mathcal{S} = \mathcal{D}_B^{sa}$ for some $B \in B(\mathcal{H})_{sa}^d$ if and only if $0 \in \text{int}(\mathcal{S})$. A similar result holds for non-self-adjoint matrix convex sets.*

Proof. As a self-adjoint matrix convex set, it is clear that $0 \in \text{int}(\mathcal{D}_B^{sa})$. Conversely, by Proposition 6.3.6 we know that $\mathcal{S}^\bullet$ is a closed self-adjoint bounded set, and hence by Theorem 6.2.6 there exists $B \in B(\mathcal{H})_{sa}^d$ such that $\mathcal{S}^\bullet = \mathcal{W}(B)$. This way, we get that $\mathcal{S} = \mathcal{S}^{\bullet\bullet} = \mathcal{W}(A)^\bullet = \mathcal{D}_B^{sa}$ by the self-adjoint Effros-Winkler bipolar theorem and Proposition 6.3.5. □

If $K \subseteq \mathbb{R}^d$, we denote by K' the usual polar dual in the sense that

$$K' = \{x \in \mathbb{R}^d | \sum_{i=1}^d x_i y_i \leq 1 \text{ for } y \in K\}.$$

By [49, Lemma 5.1] we know that when $\mathcal{S}$ is a matrix convex set, then $\mathcal{S}_1^\bullet = \mathcal{S}_1'$. The following theorem is the matrix convex set version of [100, Theorem 4.8] in the operator systems world.

Theorem 6.3.8. *Let K be a compact convex set in $\mathbb{R}^d$. Then*

$$\mathcal{W}^{min}(K)^\bullet = \mathcal{W}^{max}(K').$$

If $0 \in K$, then also

$$\mathcal{W}^{max}(K)^\bullet = \mathcal{W}^{min}(K').$$

Proof. If X is a compression of a normal tuple N, then $\sum_{i=1}^d N_i \otimes Y_i \leq I$ implies that $\sum_{i=1}^d X_i \otimes Y_i \leq I$. By the spectral theorem, $\sum_{i=1}^d N_i \otimes Y_i \leq I$ is equivalent to the inequalities $\sum_{i=1}^d \alpha_i Y_i \leq I$ for $\alpha = (\alpha_1, ..., \alpha_d) \in \sigma(N)$. Therefore,

$$\mathcal{W}^{min}(K)^\bullet = \{ \ Y \in (M_n)_{sa}^d \mid \sum_{i=1}^d \alpha_i Y_i \leq I \text{ for } \alpha \in K \ \} = \mathcal{W}^{max}(K').$$

If $0 \in K$, by taking polar duals, we get that

$$\mathcal{W}^{min}(K) = \mathcal{W}^{max}(K')^\bullet.$$

Hence, by replacing K with K' we obtain the second equality. □

Next we wish to understand how polar duality behaves with respect to duality of operator systems. When $(\mathcal{V}, \{\mathcal{P}_n\}, e)$ is an operator system, we may define an operator system on the bidual matrix ordered space $(\mathcal{V}'', \{\mathcal{P}_n''\})$. Indeed, the functional $\hat{e} : \mathcal{V}' \to \mathbb{C}$ given by $\hat{e}(f) = f(e)$ for every $f \in \mathcal{V}'$ is an Archimedean matrix order unit, so we have that $(\mathcal{V}'', \{\mathcal{P}_n''\}, \hat{e})$ is an operator system in its own right. The canonical embedding of $\mathcal{V}$ inside $\mathcal{V}''$ is then a unital complete order embedding. Hence, when $\mathcal{V}$ is finite dimensional, we obtain a bidual theorem in the sense that $\mathcal{V}''$ is canonically unital completely order isomorphic to $\mathcal{V}$.

When $\mathcal{V}$ is finite dimensional, we also know by Theorem 2.4.10 that there is a strictly positive functional $\tau : \mathcal{V} \to \mathbb{C}$ that plays the role of an Archimedean matrix order unit for $(\mathcal{V}', \{\mathcal{P}_n'\})$.

Proposition 6.3.9. *Let $\mathcal{V}$ be a finite dimensional operator system. Let $\tau : \mathcal{V} \to \mathbb{C}$ be a unital positive functional, and suppose $T = (T_1, ..., T_d)$ is a self-adjoint choice of coordinates such that $T_i \in \operatorname{Ker} \tau$ and $\{T_1, ..., T_d\}$ is independent. Then τ is strictly positive if and only if $0 \in \operatorname{int}(\mathcal{W}(T))$.*

Proof. If we take $T = (T_1, ..., T_d)$ as above, τ is an indicator for the fact that $0 \in \mathcal{W}(T)$. Suppose that τ is strictly positive. As $\{T_1, .., T_d\}$ is independent, we see that no non-trivial linear combination of $T_1, ..., T_d$ can be positive nor can it be negative.

To show that $0 \in \operatorname{int}(\mathcal{W}(T))$, by Proposition 6.3.6 it will suffice to show that $\mathcal{D}_T^{sa}(1)$ is bounded. Assume towards contradiction that $\mathcal{D}_T^{sa}(1)$ is unbounded. As this set is convex, there is some non-zero vector $x = (x_1, ..., x_d) \in \mathcal{D}_T^{sa}(1) \subseteq \mathbb{R}^d$ such that $tx \in \mathcal{D}_T^{sa}(1)$ for all $t > 0$. This means that $t \sum_{i=1}^d x_i T_i \leq I$ for all $t > 0$. However, as no non-trivial linear combination of $T_1, ..., T_d$ can be positive, nor can it be negative, there always is a positive *and* negative element in $\sigma(\sum_{i=1}^d x_i T_i)$. Hence, for t large enough, the set $t\sigma(\sum_{i=1}^d x_i T_i)$ would contain an element greater than 1, contradicting the fact that $t \sum_{i=1}^d x_i T_i \leq I$ for all $t > 0$.

Conversely, if $0 \in \operatorname{int}(\mathcal{W}(T))$, suppose $P := a_0 I + \sum_{i=1}^d a_i T_i$ is positive in $\mathcal{V}$. If $a_0 = \tau(P) = 0$ then $\sum_{i=1}^d a_i T_i \geq 0$ would imply that $\sum_{i=1}^d a_i x_i \geq 0$ for arbitrarily small $x = (x_1, ..., x_d) \in \mathbb{R}^d$. As $\sum_{i=1}^d a_i x_i$ is linear in x, this is impossible unless $a_i = 0$ for all $1 \leq i \leq d$. Hence, τ is strictly positive. $\square$

Corollary 6.3.10. *Let $\mathcal{V}$ be a finite dimensional operator system. Then there is always a self-adjoint choice of coordinates $T = (T_1, ..., T_d)$ such that $0 \in \operatorname{int}(\mathcal{W}(T))$.*

Proof. By the Choi-Effros Theorem 2.4.10 there is a strictly positive linear functional $\tau : \mathcal{V} \to \mathbb{C}$. So choose a self-adjoint basis $T = (T_1, ..., T_d)$ for $\operatorname{Ker} \tau$ and apply Proposition 6.3.9. $\square$

For a closed and bounded matrix convex set $\mathcal{S}$, we know that $0 \in \operatorname{int}(\mathcal{S})$ if and only if $\mathcal{S}^\bullet$ is closed and *bounded*. Hence, when $0 \in \operatorname{int}(\mathcal{S})$, the matrix ordered space $\mathbb{A}(\mathcal{S}^\bullet)$ has $\mathbb{1}$ as an Archimedean matrix order unit, and becomes an operator system as in Section 6.2. Again because $0 \in \operatorname{int}(\mathcal{S})$, the choice of coordinates $z = (z_1, ..., z_d)$ of coordinate functions on $\mathcal{S}$ is independent, so $\{\mathbb{1}, z_1, ..., z_d\}$ is a basis for $\mathbb{A}(\mathcal{S})$. We will denote by $\mathbb{1}'$ the unital positive map on $\mathbb{A}(\mathcal{S})$ given by $\mathbb{1}'(\mathbb{1}) = 1$ and $\mathbb{1}'(z_i) = 0$ for each $1 \leq i \leq d$. We will also denote by $w = (w_1, ..., w_d)$ the (independent) coordinate functions on $\mathcal{S}^\bullet$.

In linear algebra, it is usually a standard operation to take the dual basis of a given finite basis. In trying to relate finite dimensional operator systems to their duals via matrix convex set duality, it is perhaps a bit surprising that the more natural thing to do is to map a choice of coordinates for affine functions on the dual to the *minus* of its dual choice of coordinates. This is made precise in the following theorem.

Theorem 6.3.11. *Let $\mathcal{S}$ be a* self-adjoint *closed and bounded matrix convex set of d-dimensions with $0 \in \text{int}(\mathcal{S})$. Then the map $\Psi : \mathbb{A}(\mathcal{S}^\bullet) \to \mathbb{A}(\mathcal{S})'$, given by $\Psi(\mathbb{1}) = \mathbb{1}'$ and $\Psi(w_i)$ given by $\Psi(w_i)(\mathbb{1}) = 0$ and $\Psi(w_i)(z_j) = -\delta_{ij}$, extends to a complete order isomorphism.*

In particular, we see that $\mathbb{1}'$ is an Archimedean matrix order unit for $\mathbb{A}(\mathcal{S})'$, so that $(\mathbb{A}(\mathcal{S})', \{(M_n(\mathbb{A}(\mathcal{S}))_+)'\}, \mathbb{1}')$ is an operator system, and Ψ is a unital complete order isomorphism.

Proof. Since $0 \in \text{int}(\mathcal{S})$, we have that $\{\mathbb{1}, -z_1, ..., -z_d\}$ is a basis for $\mathbb{A}(\mathcal{S})$. We denote its dual basis by $\{\mathbb{1}', \Psi(w_1), ..., \Psi(w_d)\}$, which is a basis for $\mathcal{V}'$, so that Ψ is a well-defined linear map. Now suppose $F \in M_n(\mathbb{A}(\mathcal{S}^\bullet))_+$ is given by

$$F(w) = A_0 \otimes \mathbb{1} - \sum_{i=1}^{d} A_i \otimes w_i.$$

We wish to show that $\Psi^{(n)}(F)$ satisfies $\Psi^{(n)}(F)(G) \geq 0$ for any $G \in M_n(\mathbb{A}(\mathcal{S}))_+$. So write

$$G(z) = C_0 \otimes \mathbb{1} - \sum_{i=1}^{d} C_i \otimes z_i.$$

We know that $G(X) \geq 0$ for any $X \in \mathcal{S}$, and as $0 \in \mathcal{S}$ we get that $C_0 \geq 0$. Suppose first that C_0 is invertible. In this case, the function $G'(z) = I_n \otimes \mathbb{1} - \sum_{i=1}^{d} C_0^{-1/2} C_i C_0^{-1/2} \otimes z_i$ satisfies $G'(X) \geq 0$ for any $X \in \mathcal{S}$. Hence, $C' = (C_0^{-1/2} C_1 C_0^{-1/2}, ..., C_0^{-1/2} C_d C_0^{-1/2})$ is in $\mathcal{S}^\bullet$. Now since F is positive, we know that $F(C') \geq 0$, which means that $A_0 \otimes I_n - \sum_{i=1}^{d} A_i \otimes C_0^{-1/2} C_i C_0^{-1/2} \geq 0$. By multiplying with $C_0^{1/2}$ on both sides we obtain that

$$\Psi^{(n)}(F)(G) = A_0 \otimes C_0 - \sum_{i=1}^{d} A_i \otimes C_i \geq 0.$$

If C_0 is not invertible, we may define $G_\epsilon(z) = (C_0 + \epsilon I_n) \otimes \mathbb{1} - \sum_{i=1}^{d} C_i \otimes z_i$ which would satisfy $\psi^{(n)}(F)(G_\epsilon) \geq 0$ as $C_0 + \epsilon I_n$ is invertible. By taking ϵ to 0, we would obtain that $\Psi^{(n)}(F)(G) \geq 0$ still.

Conversely, if $\psi^{(n)}(F)(G) \geq 0$ for all $G \in M_n(\mathbb{A}(\mathcal{S}))_+$, for any element $C \in \mathcal{S}^\bullet$ we may define $G_C(z) = I_n \otimes \mathbb{1} - \sum_{i=1}^{d} C_i \otimes z_i$ which is in $M_n(\mathbb{A}(\mathcal{S}))_+$ by our choice of $C \in \mathcal{S}^\bullet$. But then for any $C \in \mathcal{S}^\bullet$

$$F(C) = A_0 \otimes \mathbb{1} - \sum_{i=1}^{d} A_i \otimes C_i = \Psi^{(n)}(F)(G_C) \geq 0$$

so that F is in $M_n(\mathbb{A}(\mathcal{S}^\bullet))_+$. Hence Ψ is a complete order embedding. In particular, Ψ is injective.

To show that Ψ is surjective, let $g \in \mathbb{A}(\mathcal{S})'$. Then g is completely determined by its values on z_i and $\mathbb{1}$. Hence, if we denote $a_0 = f(\mathbb{1})$ and $a_i = -g(z_i)$ we may form the matrix affine map $f(w) = a_0 \cdot \mathbb{1} - \sum_{i=1}^d a_i \cdot w_i$ to get that $\Psi(f) = g$. Hence, Ψ is a complete order isomorphism. □

As a corollary to Theorem 6.3.11, we obtain the a description of dual matrix convex sets in terms of matrix ranges. When $T = (T_1, ..., T_d)$ is a self-adjoint choice of coordinates for a $d+1$ dimensional operator system $\mathcal{V}$ with $0 \in \text{int}(\mathcal{W}(T))$, the positive unital functional $\tau \in \mathcal{V}'$ indicating that $0 \in \mathcal{W}(T)$ is strictly positive and satisfies $\tau(T_i) = 0$ for all $1 \le i \le d$. Since $\{I, -T_1, ..., -T_d\}$ is a basis for $\mathcal{V}$, we may form its dual basis $\{\tau, f_1, ..., f_d\}$ for $\mathcal{V}'$, and we call $f = (f_1, ..., f_d)$ the *dual choice of coordinates* to T, which is also self-adjoint and satisfies $0 \in \text{int}(\mathcal{W}(f))$.

Corollary 6.3.12. *Let $\mathcal{V}$ be a $(d+1)$-dimensional operator system, and $T = (T_1, ..., T_d)$ some self-adjoint choice of coordinates for $\mathcal{V}$ with $0 \in \text{int}(\mathcal{W}(T))$. Let $f = (f_1, ..., f_d)$ be the dual choice of coordinates in $\mathcal{V}'$ for T. Then $\mathcal{W}(T)^\bullet = \mathcal{W}(f)$.*

Proof. By Theorem 6.2.6 we see that $\mathcal{V}$ is completely order isomorphic to $\mathbb{A}(\mathcal{W}(T))$, so that by Theorem 6.3.11 we have a complete order isomorphism between $\mathbb{A}(\mathcal{W}(T)^\bullet)$ and $\mathcal{V}'$ that maps the coordinates $w = (w_1, ..., w_d)$ to the dual choice of coordinates $f = (f_1, ..., f_d)$. By using Proposition 6.2.4 we obtain that $\mathcal{W}(T)^\bullet = \mathcal{W}(f)$. □

In the literature, the following corollary is cited as part of the Choi-Effros theorem on the existence of Archimedean matrix order units on the dual matrix ordered space.

Corollary 6.3.13. *Let $\mathcal{V}$ be an operator system of dimension $d+1$ with matrix cones $\{\mathcal{P}_n\}$, and $\tau : \mathcal{V} \to \mathbb{C}$ a positive functional. Then τ is a strictly positive functional if and only if τ is an Archimedean matrix order unit for $(\mathcal{V}', \{\mathcal{P}_n'\})$.*

Proof. Without loss of generality, we may normalize τ to be unital. Let $T = (T_1, ..., T_d)$ be a self-adjoint choice of coordinates from $\text{Ker}\, \tau$ such that $\{T_1, ..., T_d\}$ are independent. For the forward implication, by Theorem 6.2.6 we may identify $\mathcal{V}$ with $\mathbb{A}(\mathcal{W}(T))$ where τ is identified with $\mathbb{1}'$. By Proposition 6.3.9 we have that $0 \in \text{int}(\mathcal{W}(T))$ so that Theorem 6.3.11 we get that $\mathbb{1}'$ is an Archimedean matrix order unit. As τ is identified with $\mathbb{1}'$, we are done.

Conversely, since $\{I, T_1, ..., T_d\}$ is a self-adjoint basis for $\mathcal{V}$, we may form its dual self-adjoint basis $\{\tau, f_1, ..., f_d\}$. Since τ is an order unit, there exists $r_i > 0$ such that $\pm f_i \leq r \cdot \tau$. We show that τ is strictly positive. Let $V \in \mathcal{V}_+$ be some non-zero positive, and write it as $V = a_0 I + \sum_{i=1}^d a_i T_i$ for $a_i \in \mathbb{R}$. In particular $\tau(V) = a_0 \geq 0$. If we assume towards contradiction that $a_0 = \tau(V) = 0$, since $\pm f_i \leq r \cdot \tau$ we also get

$$0 \leq (r \cdot \tau \pm f_i)(V) = \pm a_i.$$

Hence, $a_i = 0$ for all $0 \leq i \leq d$, so that $V = 0$ in contradiction. This means that $\tau(V) = a_0 > 0$, so that τ is strictly positive. □

We next obtain uniqueness of the dual operator system structure up to unital complete order isomorphism. Alternatively, when two matrix convex sets contain 0 in their interior and are matrix-affine isomorphic, we get that their polar duals are also matrix-affine isomorphic.

Corollary 6.3.14. *Let $\mathcal{S}$ and $\mathcal{T}$ be two closed and bounded matrix convex sets in d-dimensions, containing 0 in their interiors. If $\mathcal{S}$ and $\mathcal{T}$ are matrix-affine isomorphic, then $\mathcal{S}^\bullet$ and $\mathcal{T}^\bullet$ are matrix-affine isomorphic.*

Hence, for an operator system $\mathcal{V}$ of dimension $d+1$ with order cones $\{\mathcal{P}_n\}$ and two Archimedean matrix order units τ and κ for $(\mathcal{V}', \{\mathcal{P}'_n\})$, we have that $(\mathcal{V}', \{\mathcal{P}'_n\}, \tau)$ and $(\mathcal{V}', \{\mathcal{P}'_n\}, \kappa)$ are unital completely order isomorphic.

Proof. Let $\mathcal{S}$ and $\mathcal{T}$ be matrix-affine isomorphic matrix convex sets as in the statement. By categorical duality, this isomorphism promotes to a unital complete order isomorphisms ϕ between $\mathbb{A}(\mathcal{S})$ and $\mathbb{A}(\mathcal{T})$. The induced dual map $\phi' : \mathbb{A}(\mathcal{T})' \to \mathbb{A}(\mathcal{S})'$ given by $\phi'(f) = f \circ \phi$ maps $\mathbb{1}'_{\mathcal{T}}$ to $\mathbb{1}'_{\mathcal{S}}$, and is also a complete order isomorphism by Proposition 2.4.9. Hence, ϕ' is a *unital* complete order isomorphism. By Theorem 6.3.11 we have unital complete order isomorphisms $\mathbb{A}(\mathcal{S}^\bullet) \cong \mathbb{A}(\mathcal{S})'$ and $\mathbb{A}(\mathcal{T}^\bullet) \cong \mathbb{A}(\mathcal{T})'$. Since $\mathbb{A}(\mathcal{T})'$ and $\mathbb{A}(\mathcal{S})'$ are unital completely order isomorphic, by categorical duality again, we get that $\mathcal{S}^\bullet$ and $\mathcal{T}^\bullet$ are matrix-affine isomorphic.

For the second part, we may assume, perhaps after applying normalization maps on $\mathcal{V}$ and $\mathcal{W}$, that τ and κ are unital. Let $T = (T_1, ..., T_d)$ be a self-adjoint choice of coordinates in $\operatorname{Ker} \tau$, and $T' = (T'_1, ..., T'_d)$ a self-adjoint choice of coordinates in $\operatorname{Ker} \kappa$. By Theorem 6.2.6 we have that $\mathbb{A}(\mathcal{W}(T))$ and $\mathbb{A}(\mathcal{W}(T'))$ are identified with $\mathcal{V}$ where τ is identified with $\mathbb{1}'_{\mathcal{W}(T)}$ and κ with $\mathbb{1}'_{\mathcal{W}(T')}$, so by Proposition 6.2.3 we have a matrix-affine isomorphism between $\mathcal{W}(T)$ and $\mathcal{W}(T')$. By Corollary 6.3.13 both κ and τ are strictly positive, so by Proposition 6.3.9 we see that 0 is in the interior of both $\mathcal{W}(T)$ and $\mathcal{W}(T')$. Hence, by what we have already shown, we have a unital complete order isomorphism between $\mathbb{A}(\mathcal{W}(T'))'$ and $\mathbb{A}(\mathcal{W}(T))'$, which, by our identifications, yields a unital complete order isomorphism between $(\mathcal{V}', \{\mathcal{P}'_n\}, \tau)$ and $(\mathcal{V}', \{\mathcal{P}'_n\}, \kappa)$. □

We next sketch a generalization of [63, Theorem 3.5] that yields interpolation of UCP maps in terms of free operator spectrahedra.

Theorem 6.3.15. *Let $A = (A_1, ..., A_d)$ and $B = (B_1, ..., B_d)$ be two d-tuples of operators. Suppose there is a positive unital map $\tau : \mathcal{V}_A \to \mathbb{C}$ such that $A_i \in \operatorname{Ker}\tau$. Then*

1. *For a given $n \in \mathbb{N}$, there is a unital n-positive map $\phi : \mathcal{V}_A \to \mathcal{V}_B$ mapping A_i to B_i if and only if $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$.*
2. *There is a UCP $\phi : \mathcal{V}_A \to \mathcal{V}_B$ mapping A_i to B_i if and only if $\mathcal{D}_A \subseteq \mathcal{D}_B$.*
3. *There is a unital complete order isomorphism $\phi : \mathcal{V}_A \to \mathcal{V}_B$ mapping A_i to B_i if and only if $\mathcal{D}_A = \mathcal{D}_B$.*

A similar result holds in the self-adjoint context.

Proof. The existence of $\tau : \mathcal{V}_A \to \mathbb{C}$ for which $\tau(A_i) = 0$ is equivalent to $0 \in \mathcal{W}(A)$. Hence, from (the proof of) Proposition 6.3.5 we know that $\mathcal{D}_A(n)^\circ_n = \mathcal{W}_n(A)$ and $\mathcal{W}_n(A)^\circ = \mathcal{D}_A(n)$ for each $n \in \mathbb{N}$, and similarly for B. Hence, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ if and only if $\mathcal{W}_n(B) \subseteq \mathcal{W}_n(A)$ for each n, and the theorem follows from Corollary 6.2.7. □

Example 6.3.16 ($0 \in \mathcal{W}(A)$ is a necessary condition)**.** *Suppose that $A = (I, I, \dots, I)$. Then $\mathcal{W}(A)$ contains only tuples of identity matrices, and not 0. Now $\mathcal{D}^{sa}_A = \{X \in (M_n)^d_{sa} : \sum X_i \leq I\}$. If we define B to be the d-tuple $\oplus_{n=d}^\infty((1 - 1/n), \dots, (1 - d/n))$, then $\mathcal{D}^{sa}_B = \mathcal{D}^{sa}_A$, but there is no UCP map (actually, no linear map) sending A to B. The same example shows that also in the non-self-adjoint case the condition $0 \in \mathcal{W}(A)$ is necessary for the inclusion $\mathcal{D}_A \subseteq \mathcal{D}_B$ to imply the existence of a UCP map sending A to B.*

6.4 Minimal defining d-tuple

In this section, we wish to understand $\mathcal{W}(T)$ as an invariant for $T \in B(\mathcal{H})^d$. We will see that in many cases, up to minimality assumptions regarding $\mathcal{W}(T)$, the matrix ranges determines the d-tuple T up to unitary / approximate unitary equivalence. Up to using polar duality, this yields improved results to those that appeared for LMI domains in [63, Theorem 3.12] (See also [123, Theorem 1.2]).

Definition 6.4.1. *A tuple* $A = (A_1, ..., A_d) \in B(\mathcal{H})^d$ *is said to be* minimal defining *if there is no nontrivial reducing subspace* $\mathcal{H}_0 \subseteq \mathcal{H}$ *such that* $\widetilde{A} = (A_1|_{\mathcal{H}_0}, ..., A_d|_{\mathcal{H}_0})$ *satisfies* $\mathcal{W}(A) = \mathcal{W}(\widetilde{A})$.

In other words, A is minimal if $\mathcal{V}_A$ is not unitally completely isometrically isomorphic to $\mathcal{V}_{\widetilde{A}}$ for any direct summand $\widetilde{A}$ of A.

Proposition 6.4.2. *If* $A = (A_1, ..., A_d) \in B(\mathcal{H})^d$*, then* A *is minimal if and only if there are no two nontrivial orthogonal reducing subspaces* $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{H}$ *such that* $\mathcal{W}(A\big|_{\mathcal{K}_1}) \subseteq \mathcal{W}(A\big|_{\mathcal{K}_2})$.

Proof. Suppose that there are two nontrivial orthogonal reducing subspaces $\mathcal{K}_1, \mathcal{K}_2$ in $\mathcal{H}$. Let $A^{(i)} = (A_1|_{\mathcal{K}_i}, ..., A_d|_{\mathcal{K}_i})$ $(i = 1, 2)$ and suppose that $\mathcal{W}(A^{(1)}) \subseteq \mathcal{W}(A^{(2)})$. Set $\mathcal{H}_0 = \mathcal{K}_1^{\perp}$, and denote $\widetilde{A} = (A_1|_{\mathcal{H}_0}, ..., A_d|_{\mathcal{H}_0})$. It always holds that $\mathcal{W}(\widetilde{A}) \subseteq \mathcal{W}(A)$, we show the converse. The assumption $\mathcal{W}(A^{(1)}) \subseteq \mathcal{W}(A^{(2)})$ is equivalent to the existence of a UCP map ψ mapping $A^{(2)}$ to $A^{(1)}$. Define a map $\phi : S_{\widetilde{A}} \to S_A$, by setting $\phi(\widetilde{A}_j) = A_j$. Since $A = A^{(1)} \oplus \widetilde{A} = \psi(P_{\mathcal{K}_2}\widetilde{A}P_{\mathcal{K}_2}) \oplus \widetilde{A}$, ϕ is a UCP map. Thus $\mathcal{W}(A) \subseteq \mathcal{W}(\widetilde{A})$. Hence $\mathcal{W}(A) = \mathcal{W}(\widetilde{A})$, so A is not minimal.

For the converse, suppose that A is not minimal. Then there exists a nontrivial reducing subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ such that $\widetilde{A} = (A_1|_{\mathcal{H}_0}, ..., A_d|_{\mathcal{H}_0})$ satisfies $\mathcal{W}(A) = \mathcal{W}(\widetilde{A})$, therefore there is a unital completely isometric map ϕ mapping $\widetilde{A}$ to A. By compressing to the subspace $\mathcal{H}_0^{\perp}$, we get that $P_{\mathcal{H}_0}^{\perp}AP_{\mathcal{H}_0}^{\perp}$ is orthogonal to $\widetilde{A}$ and is also the image of $\widetilde{A}$ under a UCP map. ∎

In the case A is a d-tuple of compact operators, $C^*(A)$ is a C*-subalgebra of compact operators, and every minimal reducing subspace $\mathcal{H}_\lambda$ of $\mathcal{H}$ gives rise to an irreducible representation $\pi_\lambda : C^*(A) \to B(\mathcal{H})$ by restriction $\pi_\lambda(T) = T|_{\mathcal{H}_\lambda}$. So for each unitary equivalence class of irreducible representations ζ, we can pick $\pi_\zeta : C^*(A) \to B(\mathcal{H}_\zeta)$, an irreducible subrepresentation of the identity representation, which must be among $\{\pi_\lambda\}$, as $C^*(A)$ is a subalgebra of compact operators. Hence, the direct sum $\oplus\pi_\zeta : C^*(A) \to B(\oplus\mathcal{H}_\zeta)$ is a faithful representation of $C^*(A)$, and if we denote $\mathcal{H}_0 := \oplus\mathcal{H}_\zeta \subseteq \mathcal{H}$, then $\widetilde{A} := A|_{\mathcal{H}_0}$ certainly satisfies $\mathcal{W}(A) = \mathcal{W}(\widetilde{A})$.

Proposition 6.4.3. *Let* $A = (A_1, ..., A_d) \in \mathcal{K}(\mathcal{H})^d$ *be a* d*-tuple of compact operators. Then* A *is minimal defining if and only if*

1. *The identity representation of* $C^*(A)$ *is multiplicity free, and;*

2. The Shilov ideal of $\mathcal{V}_A$ inside $C^*(I,A)$ is trivial.

Proof. Suppose that A is minimal. We first show that the identity representation of $C^*(A)$ is multiplicity free. For otherwise $\mathcal{H}_0^\perp \neq \{0\}$, where we use the notation set before the proposition. This means that we can find two orthogonal subspaces $\mathcal{H}_{\lambda_1} \subseteq \mathcal{H}_0$ and $\mathcal{H}_{\lambda_2} \subseteq \mathcal{H}_0^\perp$ such that the restrictions π_{λ_1} and π_{λ_2} are unitarily equivalent. However, as A is minimal, by Proposition 6.4.2, we see that this is impossible. Hence, we must have that $\mathcal{H} = \oplus \mathcal{H}_\zeta$ where $\{\pi_\zeta\} = \{\pi_\lambda\}$ are mutually inequivalent irreducible *-representations, and in fact $C^*(A) = \oplus_\zeta \mathcal{K}(\mathcal{H}_\zeta)$ inside $B(\mathcal{H})$.

Next, we show that $C^*(I,A)$ is the C*-envelope of $\mathcal{V}_A$. If not, then there is some ζ for which $\mathcal{K}(\mathcal{H}_\zeta)$ is a subset of the Shilov ideal of $\mathcal{V}_A$ inside $C^*(I,A)$. Denote by $\widetilde{A} = (A_1|_{\mathcal{H}_\zeta^\perp}, ..., A_d|_{\mathcal{H}_\zeta^\perp})$.

By the universal property of the C*-envelope, there is a *-surjection

$$\rho : C^*(I,A) \to C^*(I,A)/\mathcal{K}(\mathcal{H}_\zeta) \cong C^*(\mathcal{V}_{\widetilde{A}})$$

which is completely isometric on $\mathcal{V}_A$. Thus, by restricting ρ to $\mathcal{V}_A$, we obtain a unital complete isometry $\rho|_{\mathcal{V}_A} : \mathcal{V}_A \to \mathcal{V}_{\widetilde{A}}$, so that $\mathcal{W}(A) = \mathcal{W}(\widetilde{A})$ in contradiction to minimality.

Now suppose that A is not minimal, so there is a nontrivial reducing subspace $\mathcal{H}_1$ such that $\mathcal{W}(A) = \mathcal{W}(\widetilde{A})$, where $\widetilde{A} = A|_{\mathcal{H}_1}$. We will show that if the identity representation of $C^*(A)$ is multiplicity free, then the Shilov ideal of $\mathcal{V}_A$ in $C^*(I,A)$ is not trivial. The multiplicity free assumption means that $\mathcal{H}_1 = \oplus_{\zeta \in \Lambda} \mathcal{H}_\zeta$ must be a direct sum of $\mathcal{H}_\zeta$ for some subset Λ of equivalence classes of irreducible representations of $C^*(A)$. This means that $P_{\mathcal{H}_1}^\perp C^*(A) = C^*(A) P_{\mathcal{H}_1}^\perp$ is a two sided ideal inside $C^*(A)$. Denote $P := P_{\mathcal{H}_1}$. Then for $S = (S_{ij}) \in M_n(\mathcal{V}_A)$, and $T = (T_{ij})(I_n \otimes P^\perp) \in M_n(C^*(I,A)P^\perp)$, we have that

$$\|S + T\| = \|S(I_n \otimes P) \oplus (S+T)(I_n \otimes P^\perp)\| \geq \|S(I_n \otimes P)\| = \|S\|,$$

where the last equality holds as the map sending A_i to $A_i|_{\mathcal{H}_1}$, which we identify with $A_i P$, is completely isometric. Hence, the map induced $\mathcal{V}_A \to C^*(I,A)/C^*(I,A)P^\perp$ is completely isometric, so that $C^*(I,A)P^\perp$ is contained in the Shilov ideal of $\mathcal{V}_A$. Therefore the Shilov ideal of $\mathcal{V}_A$ in $C^*(I,A)$ is not trivial. □

Corollary 6.4.4. *Let $A = (A_1, ..., A_d) \in \mathcal{K}(\mathcal{H})^d$ be a d-tuple of compact operators. Then there is a reducing subspace $\mathcal{H}_0$ such that $\widetilde{A} = (A_1|_{\mathcal{H}_0}, ..., A_d|_{\mathcal{H}_0})$ is minimal, and $\mathcal{W}(A) = \mathcal{W}(\widetilde{A})$.*

Proof. Let $\mathcal{H}_0 := \oplus \mathcal{H}_{\zeta \in \Lambda} \subseteq \mathcal{H}$ be a direct sum of a maximal set of inequivalent nondegenerate subrepresentations of the identity representation of $C^*(A)$, as in the discussion preceding Proposition 6.4.3. Write $\widetilde{A} = A|_{\mathcal{H}_0}$. By construction we know that $\mathcal{W}(A) = \mathcal{W}(\widetilde{A})$, and $C^*(\widetilde{A})$ as well as every compression of $C^*(\widetilde{A})$ to an invariant subspace is multiplicity free. If the Shilov ideal of $\mathcal{V}_{\widetilde{A}}$ in $C^*(I, \widetilde{A})$ is trivial then we are done.

For any ideal J in $C^*(I, \widetilde{A})$, J is the direct sum $\oplus_{\zeta \in \Lambda_1} \mathcal{K}(\mathcal{H}_\zeta)$ for a subset $\Lambda_1 \subseteq \Lambda$. Let $\mathcal{H}_1 = \oplus_{\zeta \in \Lambda \setminus \Lambda_1} \mathcal{H}_\zeta$. The quotient map $C^*(I, \widetilde{A}) \to C^*(I, \widetilde{A})/J$ is completely isometric on $\mathcal{V}_{\widetilde{A}}$ if and only if $\mathcal{W}(\widetilde{A}) = \mathcal{W}(\widetilde{A}\big|_{\mathcal{H}_1})$. Therefore if we let J be the Shilov ideal of $\mathcal{V}_{\widetilde{A}}$ in $C^*(I, \widetilde{A})$, then $A\big|_{\mathcal{H}_1}$ (in place of $A\big|_{\mathcal{H}_0}$) is the required matrix minimal compression of A. □

The following should be compared with [6, Theorem 2.4.3].

Theorem 6.4.5. *Let A and B be two minimal d-tuples of operators in $\mathcal{K}(\mathcal{H})$. Then $\mathcal{W}(A) = \mathcal{W}(B)$ if and only if A and B are unitarily equivalent.*

Proof. There is only one direction to prove, so assume that $\mathcal{W}(A) = \mathcal{W}(B)$. Then there is a unital completely isometric isomorphism ϕ from $\mathcal{V}_A$ to $\mathcal{V}_B$. This map extends to a $*$-isomorphism π between the respective C*-envelopes, which by minimality are $C^*(I, A)$ and $C^*(I, B)$. We therefore have a $*$-isomorphism $\pi : C^*(A) \to C^*(B)$. We show that this $*$-isomorphism must be unitarily implemented. Indeed, by the representation theory of C*-algebras of compact operators, $\pi = \oplus_i \pi_i$ is (up to unitary equivalence) the direct sum of irreducible subrepresentations of the identity representation of $C^*(A)$. Every subrepresentation of $\mathrm{id}_{C^*(A)}$ appears at most once, since π is multiplicity free. Moreover, every subrepresentation of $\mathrm{id}_{C^*(A)}$ appears at least once, because the Shilov boundary is trivial. □

Example 6.4.6. *In general, a non-compact d-tuple of operators does not always have a minimal subspace as in Corollary 6.4.4. Let $(\lambda_i)_{i \in \mathbb{N}}$ be a dense subset of* distinct *numbers on the circle $\mathbb{T}$. Define the diagonal unitary operator T on $\ell^2(\mathbb{N})$ by $T(e_i) = \lambda_i e_i$. Then T is certainly normal, but has no* minimal *reducing subspace $L \subseteq \ell^2(\mathbb{N})$ for which $\mathcal{W}(T) = \mathcal{W}(T|_L)$.*

Indeed, if L is a reducing subspace for T, then the projection P_L onto it belongs to the von-Neumann algebra $W^*(T)$ generated by T, since $W^*(T) = \ell^\infty(\mathbb{N})$ is maximal abelian, and is hence equal to its own commutant inside $\mathcal{B}(\ell^2(\mathbb{N}))$. Thus, P_L commutes with P_i, where P_i is the projection onto $\mathrm{Sp}\{e_i\}$ *for each $i \in \mathbb{N}$. Hence, for a fixed $i \in \mathbb{N}$, we either have $P_L(e_i) = e_i$ or $P_L(e_i) = 0$. Hence, we establish that $L = \mathrm{Sp}\{e_i | i \in \Lambda\}$ for some subset $\Lambda \subseteq \mathbb{N}$.*

Since $\mathcal{W}(T) = \mathcal{W}(T|_L)$, we must have that $\sigma(T) = \sigma(T|_L)$, so that $(\lambda_i)_{i\in\Lambda}$ must still be dense in $\mathbb{T}$. But this is impossible because then we have that $T|_L$ has a reducing subspace $L' \subsetneq L$ such that $\mathcal{W}(T|_L) = \mathcal{W}(T|_{L'})$. So $T|_L$ cannot be minimal.

This example also has the property that there are representations of $C^*(T)$ which are not unitarily equivalent, but are approximately unitarily equivalent, such as M_z in $\mathcal{B}(L^2(\mathbb{T}))$. As $M_z \sim_{\mathcal{K}(\mathcal{H})} T$, we have $\mathcal{W}(M_z) = \mathcal{W}(T)$. It also does not have a minimal subspace. Nor is any restriction of T to a reducing subspace unitarily equivalent to any restriction of M_z to any reducing subspace.

This example shows the limits of possibility, but also shines a light on a reasonable resolution.

Theorem 6.4.7. *Let A and B be d-tuples of operators on a separable Hilbert space $\mathcal{H}$ such that*

1. *$C^*(I, A) = C^*_e(\mathcal{V}_A)$ and $C^*(I, B) = C^*_e(\mathcal{V}_B)$, and*
2. *$C^*(A) \cap \mathcal{K}(\mathcal{H}) = \{0\} = C^*(B) \cap \mathcal{K}(\mathcal{H})$.*

Then $A \sim_{\mathcal{K}(\mathcal{H})} B$ if and only if $\mathcal{W}(A) = \mathcal{W}(B)$.

Proof. One direction is trivial, so assume that $\mathcal{W}(A) = \mathcal{W}(B)$. Then there is a completely isometric map ϕ of $\mathcal{V}_A$ onto $\mathcal{V}_B$ such that $\phi(A) = B$. Hence by the universal property of the C*-envelope, perhaps after restriction, there is a $*$-isomorphism $\tilde{\phi}$ of $C^*_e(A)$ onto $C^*_e(B)$ extending ϕ. By (1), this yields a $*$-isomorphism $\hat{\phi}$ of $C^*(A)$ onto $C^*(B)$. Finally by (2) and Voiculescu's Theorem (see [30, Theorem II.5.8]), $\hat{\phi}$ is implemented by an approximate unitary equivalence. Thus $A \sim_{\mathcal{K}(\mathcal{H})} B$. □

6.5 Dilation and scaled inclusion

We show that once we have a dilation of a d-tuple of matrices to a commuting normal d-tuple, then we can choose our dilation to be on a finite dimensional space.

Theorem 6.5.1. *Let $X = (X_1, \dots, X_d) \in M_n^d$ for which there exists a commuting d-tuple $T = (T_1, \dots, T_d)$ of normal operators on a Hilbert space H and an isometry $V : \mathbb{C}^n \to H$ such that $X_i = V^*T_iV$. Then there is an integer $m \le 2n^3(d + 1) + 1$, a d-tuple $Y = (Y_1, \dots, Y_d)$ of commuting normal operators on $\mathbb{C}^m$ satisfying $\sigma(Y) \subseteq \sigma(T)$, and an isometry $W : \mathbb{C}^n \to \mathbb{C}^m$ such that $X_i = W^*Y_iW$ for all $i = 1, \dots, d$.*

Proof. The proof is based on some ideas from [26]. Suppose that $T = (T_1, ..., T_d)$ and $V : \mathbb{C}^n \to H$ are as in the statement of the theorem. Let E_T be the joint spectral measure for T. We may then write $T_i = \int_{\sigma(T)} z_i dE_T(z)$, where $\sigma(T)$ is the joint spectrum of T, identified as a subset of $\mathbb{C}^d$. For all $i = 1, \dots, d$,

$$X_i = V^* \left(\int_{\sigma(T)} z_i dE_T(z) \right) V = \int_{\sigma(T)} z_i d(V^* E_T V)(z)$$

and $V^* E_T V$ is a positive operator valued measure on $\sigma(T) \subseteq \mathbb{C}^d$ with values in $M_n(\mathbb{C})$. Now, the space $\mathrm{Sp}\{z_1, \dots, z_d\}$ of linear functions on $\sigma(T)$ is finite dimensional, and one therefore expects to have a finite sequence of points $w^{(1)}, \dots, w^{(M)} \in \sigma(T)$ and positive-definite matrices $A_1, \dots, A_M$ in $M_n(\mathbb{C})$ such that $\sum_{j=1}^M A_j = I_n$ and

$$\int_{\sigma(T)} f(z) d(V^* E_T V)(z) = \sum_{j=1}^{M} f(w^{(j)}) A_j, \tag{6.3}$$

for every $f \in \mathrm{Sp}\{z_1, \dots, z_d\}$. Indeed, by [26, Theorem 4.7] and the dimension estimates in the proof for it, when applied to the collection of functions $\{z \mapsto z_i\}_{i=1}^d$, we have $M = 2n^2(d+1)+1$ points $w^{(1)}, \dots, w^{(M)} \in \sigma(T)$ and positive-definite matrices $A_1, \dots, A_M$ in $M_n(\mathbb{C})$ such that $\sum_{j=1}^M A_j = I_n$ so that (6.3) holds. In particular,

$$\int_{\sigma(T)} z_i d(V^* E_T V)(z) = \sum_{j=1}^{M} w_i^{(j)} A_j$$

for $i = 1, \dots, d$.

The sequence $A_1, \dots, A_M$ can be considered as a positive operator valued measure on the set $\{w^{(1)}, \dots, w^{(M)}\}$. By Naimark's dilation theorem, this measure dilates to a spectral measure E on the set $\{w^{(1)}, \dots, w^{(M)}\}$ with values in $M_m(\mathbb{C})$ where $m \le nM$ (the bound on the dimension m on which the spectral measure E acts follows from the proof of Naimark's theorem via Stinespring's theorem — see Chapter 4 of [99]). That is, there exist M pairwise orthogonal projections $E_1, ..., E_M$ on $\mathbb{C}^m$ such that $\sum E_j = I_m$, and an isometry $W : \mathbb{C}^n \to \mathbb{C}^m$ such that $A_j = W^* E_j W$ for $j = 1, \dots, d$.

We now construct the dilation Y by defining $Y_i = \sum_{j=1}^M w_i^{(j)} E_j$. Thus, $Y = (Y_1, ..., Y_d)$ is a commuting normal d-tuple and by construction $\sigma(Y) = \{w^{(1)}, \dots, w^{(M)}\} \subseteq \sigma(T)$. Moreover,

$$W^* Y_i W = \sum_{j=1}^{M} w_i^{(j)} A_j = \int_{\sigma(T)} z_i d(V^* E_T V)(z) = X_i,$$

for $i = 1, \dots, d$. Thus, Y is a commuting normal dilation for X on a space of dimension at most $nM = 2n^3(d+1)+1$ with $\sigma(Y) \subseteq \sigma(T)$. $\square$

Remark 6.5.2. One of the main results, [64, Theorem 1.1] of Helton, Klep McCullough and Schweighofer, is that there is a constant $\vartheta(n)$ such that every d-tuple of symmetric $n \times n$ contractive matrices $(X_1, \dots, X_d)$, there is a d-tuple $(T_1, \dots, T_d)$ of commuting self-adjoint contractions on a Hilbert space H and an isometry $V : \mathbb{R}^d \to H$ such that

$$\vartheta(n) X_i = V^* T_i V \ , \ i = 1, \dots, d. \tag{6.4}$$

A significant amount of effort in [64] was dedicated to the determination of the optimal value of $\vartheta(n)$. In fact [64, Theorem 1.1] is stronger, in that the dilation actually works for *all* $n \times n$ symmetric matrices simultaneously. It is therefore not surprising that the dilation Hilbert space H in that theorem must be infinite dimensional. It is natural to ask whether if one begins with a fixed d-tuple of real symmetric matrices, can one obtain equation (6.4) with the commuting tuple of contractions T acting on a *finite* dimensional space H. The method of Theorem 6.5.1 shows that this can be done, with the constant unchanged, and with control on the dimension of H.

We obtain two characterizations of scaled dilation in terms of minimal and maximal matrix convex set inclusion and in terms of minimal and maximal operator system structures (cf. [64, Proposition 2.1 & Theorem 8.4]).

Theorem 6.5.3. *Let $\mathcal{V}$ be a finite dimensional operator system, $A = (A_1, ..., A_d)$ a choice of coordinates for $\mathcal{V}$ and $r \geq 1$. Denote $K = \mathcal{W}_1(A)$. The following are equivalent*

1. *For all $X \in \mathcal{W}(A)$ there exists $N = (N_1, ..., N_d) \in \mathcal{W}(A)$ commuting normal d-tuple such that rN dilates X,*
2. *$\mathcal{W}(A) \subseteq r\mathcal{W}^{min}(K)$,*
3. *For any closed and bounded matrix convex set $\mathcal{T}$ in d-dimensions we have*

$$\mathcal{W}_1(A) \subseteq \mathcal{T}_1 \implies \mathcal{W}(A) \subseteq r\mathcal{T}.$$

4. *The* unital *bijection* $\mathrm{id}_r : \mathrm{OMIN}(\mathcal{V}) \to \mathcal{V}$ *given by* $\mathrm{id}_r(A_i) = \frac{1}{r} A_i$ *is a well-defined completely positive map.*

Proof. (2) implies (3) since whenever $K = \mathcal{S}_1 \subseteq \mathcal{T}_1$, we have that $\mathcal{W}^{min}(K) \subseteq \mathcal{W}^{min}(\mathcal{T}_1) \subseteq \mathcal{T}$, so that $\mathcal{W}(A) \subseteq r\mathcal{W}^{min}(K) \subseteq r\mathcal{T}$. Conversely, (3) implies (2) since we can take $\mathcal{T} = \mathcal{W}^{min}(K)$ to obtain that $\mathcal{W}(A) \subseteq r\mathcal{W}^{min}(K)$.

(1) implies (2) because whenever $X \in \mathcal{W}(A)$ is such that rN dilates X for $N \in \mathcal{W}(A)$ a normal d-tuple, by Corollary 6.3.2 we have that $\sigma(N) \subseteq \mathcal{W}_1(N) \subseteq \mathcal{W}_1(A)$; so $N \in \mathcal{W}^{min}(K)$ and thus $X \in r\mathcal{W}^{min}(K)$ by matrix convexity.

We next show that (2) implies (1). Indeed, suppose that $X \in \mathcal{W}(A)$, so that by the inclusion (2) there is a normal commuting d-tuple N on some Hilbert space $\mathcal{H}$ with $\sigma(N) \subseteq K$ so that rN dilates X. By Theorem 6.5.1, we can choose $\mathcal{H} \cong \mathbb{C}^m$ to be finite dimensional. By Theorem 2.4.6 and Corollary 6.3.2 we know that $N \in \mathcal{W}(N) = \mathcal{W}^{min}(\mathrm{conv}(\sigma(N))) \subseteq \mathcal{W}^{min}(K) \subseteq \mathcal{W}(A)$, so that $N \in \mathcal{W}(A)$ shows that (1) holds.

Finally, by Corollary 6.2.7 it is clear that (2) and (4) are equivalent. □

By similar arguments, and by using Corollary 6.2.7, we also obtain a version of the above theorem for maximal structures.

Theorem 6.5.4. *Let $\mathcal{V}$ be a finite dimensional operator system, $A = (A_1, ..., A_d)$ a choice of coordinates for $\mathcal{V}$ and $r \geq 1$. Denote $K = \mathcal{W}_1(A)$. The following are equivalent*

1. $\mathcal{W}^{max}(K) \subseteq r\mathcal{W}(A)$,
2. *For any closed and bounded matrix convex set $\mathcal{T}$ in d-dimensions we have*
$$\mathcal{T}_1 \subseteq \mathcal{W}_1(A) \implies \mathcal{T} \subseteq r\mathcal{W}(A).$$
3. *The unital bijection $\mathrm{id}_r : \mathcal{V} \to \mathrm{OMAX}(\mathcal{V})$ given by $\mathrm{id}_r(A_i) = \frac{1}{r}A_i$ is a well-defined completely positive map.*
4. *For any finite dimensional operator system $\mathcal{W}$ and any unital positive map $\psi : \mathcal{V} \to \mathcal{W}$, the unital map $\psi_r : \mathcal{V} \to \mathcal{W}$ given by $\psi_r(A_i) = \frac{1}{r}\psi(A_i)$ is completely positive.*

As an application of the above, when a constant r works in one of the above two theorems, we obtain an upper bounds of $2r - 1$ for the completely bounded norms of any unital positive map from some finite dimensional operator system into $\mathcal{V}$, or from $\mathcal{V}$ to some finite dimensional operator system (See [100, Section 5]).

Corollary 6.5.5. *Let $\mathcal{V}$ be a finite dimensional operator system, let $A = (A_1, ..., A_d)$ be a choice of coordinates for $\mathcal{V}$ such that $0 \in \mathcal{W}(A)$. Denote $K = \mathcal{W}_1(A)$ and let $r \geq 1$.*

1. *Suppose that $\mathcal{W}(A) \subseteq r\mathcal{W}^{min}(K)$. Then for any finite dimensional operator system $\mathcal{W}$ and any* unital *positive map $\psi : \mathcal{W} \to \mathcal{V}$ we have $\|\psi\|_{cb} \leq 2r - 1$.*

2. *Suppose that $\mathcal{W}^{max}(K) \subseteq r\mathcal{W}(A)$. Then for any finite dimensional operator system $\mathcal{W}$ and any* unital *positive map $\psi : \mathcal{V} \to \mathcal{W}$ we have $\|\psi\|_{cb} \leq 2r - 1$.*

Proof. We do the proof for (1), where the proof for (2) is done similarly. By Theorem 6.5.3 (or Theorem 6.5.4 for maximal structures) we have that the unital map $\mathrm{id}_r : \mathrm{OMIN}(\mathcal{V}) \to \mathcal{V}$ given by $\mathrm{id}_r(A_i) = \frac{1}{r}A_i$ is a well-defined completely positive map. Since $0 \in \mathcal{W}(A)$, there is a UCP map $\tau : \mathrm{OMIN}(\mathcal{V}) \to \mathbb{C}$ such that $\tau(A_i) = 0$ for each $1 \leq i \leq d$. It is then clear that for $X \in \mathrm{OMIN}(\mathcal{V})$ we have

$$X = r \cdot \mathrm{id}_r(X) - (r-1)\tau(X) \otimes I$$

Hence, the identity map $\mathrm{id} : \mathcal{V} \to \mathrm{OMIN}(\mathcal{V})$ decomposes into a difference of two completely positive maps so that $\|\mathrm{id}\|_{cb} \leq \|r \cdot \mathrm{id}_r + (r-1)I\| = 2r-1$. By appealing to [100, Proposition 5.4] (or to [100, Proposition 5.5] for maximal structures) the proof is now complete. □

6.6 Scaled inclusion results given symmetry

The main point of this section is to obtain the inclusion $\mathcal{W}^{max}(K)$ in $d\mathcal{W}^{min}(K)$ for as many compact convex sets K in $\mathbb{R}^d$ as we can, while providing concrete corresponding dilation theorems.

Our dilation methods include the dilation constructed in [64, Proposition 14.1] and provide new examples for which such dilation results can be obtained.

Let $\mathcal{S}$ be a self-adjoint matrix convex set. Recall that for a $d \times d$ matrix $C = [c_{ij}]$, we define the set $(C\mathcal{S})_n = \{C(X) : X \in \mathcal{S}_n\}$ where $C(X) = (\sum_j c_{ij}X_j)$.

Definition 6.6.1. *Let $r \geq 1$. We say that a convex set $K \subseteq \mathbb{C}^d$ is* r-symmetric *if there are rank one real $d \times d$ matrices $\mathcal{C} := \{C^{(m)}\}_{m=1}^k$ such that $I_d \in \mathrm{conv}\, \mathcal{C}$ and $C^{(m)}K \subseteq rK$ for all $1 \leq m \leq k$. Analogously, we will say that a matrix convex set $\mathcal{S}$ of d dimensions is* r-symmetric *if $I_d \in \mathrm{conv}\, \mathcal{C}$ and $C^{(m)}\mathcal{S}_n \subseteq r\mathcal{S}_n$ for all $n \in \mathbb{N}$.*

We will need the following dilation result to establish our main scaled inclusion results.

Theorem 6.6.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be self-adjoint matrix convex sets. Assume that there is a k-tuple of rank-one real $d \times d$ matrices $\mathcal{C} = \{C^{(m)}\}_{m=1}^k$ such that $I_d \in \mathrm{conv}\, \mathcal{C}$ and such that $C^{(m)}\mathcal{S} \subseteq \mathcal{T}$ for all $1 \leq m \leq k$. Then for every $X \in \mathcal{S}$ there is a d-tuple $T = (T_1, \ldots, T_d)$ of self-adjoint matrices such that*

(1) $\{T_1, \ldots, T_d\}$ *is a commuting family of operators,*

(2) $T \in \mathcal{T}$*,*

(3) T *is a dilation for* X*.*

Proof. Consider X as a tuple of operators on a Hilbert space H, and suppose $C^{(m)} = [c_{ij}^{(m)}]$. Write $K = H \otimes \mathbb{C}^k$ and define d^2 diagonal, self-adjoint matrices $S_{i,j}$, $1 \leq i, j \leq d$, by

$$S_{i,j} = \text{diag}(c_{i,j}^{(1)}, \ldots, c_{i,j}^{(k)}). \tag{6.5}$$

For every $1 \leq i \leq d$, let

$$T_i = \sum_{j=1}^{d} X_j \otimes S_{i,j} \in \mathcal{B}(K). \tag{6.6}$$

We shall now verify (1)-(3). For (1), we fix i, n and compute

$$T_i T_s - T_s T_i = \sum_{j,t} X_j X_t \otimes (S_{i,j} S_{s,t} - S_{s,j} S_{i,t}).$$

But the (p,p) coordinate of the (diagonal) matrix $S_{i,j}S_{s,t} - S_{s,j}S_{i,t}$ is $c_{i,j}^{(p)} c_{s,t}^{(p)} - c_{s,j}^{(p)} c_{i,t}^{(p)}$. Since $C^{(p)}$ has rank one, the last expression is 0 and, thus, $T_i T_s = T_s T_i$, proving (1).

To prove (3), recall that $I_d \in \text{conv}\{C^{(1)}, \ldots, C^{(k)}\}$. Thus there are nonnegative real numbers $\beta_1, \ldots, \beta_k$ whose sum is 1 and $I_d = \sum_{m=1}^{k} \beta_m C^{(m)}$. Set $v = \sum_{m=1}^{k} \sqrt{\beta_m} e_m$ where $\{e_m\}$ is the standard basis of $\mathbb{C}^k$. Then, $||v|| = 1$ and for $1 \leq i, j \leq d$, $\langle S_{i,j} v, v \rangle = \sum_{m=1}^{k} \beta_m c_{i,j}^{(m)} = \delta_{i,j}$. Define an isometry $V : \mathcal{H} \to \mathcal{K} = \mathcal{H} \otimes \mathbb{C}^k$ by $Vh = h \otimes v$. Then since $V^*(X \otimes S_{ij})V = \delta_{ij} X$, we obtain

$$V^* T_i V = \sum_j V^*(X_j \otimes S_{ij}) V = X_i.$$

To prove (2), rewrite $T_i = \sum_{j=1}^{d} X_j \otimes S_{i,j}$ as a direct sum (over m) of operators of the form $Y_i^{(m)} = \sum_{j=1}^{d} c_{i,j}^{(m)} X_j$. By matrix convexity, it suffices to show that $Y^{(m)} := (Y_1^{(m)}, \ldots, Y_d^{(m)}) \in \mathcal{T}$. But $Y^{(m)}$ is obtained from X by left multiplication by $C^{(m)}$. The assumption $C^{(m)} \mathcal{S} \subseteq \mathcal{T}$ (and the fact that $X \in \mathcal{S}$), implies that $T \in \mathcal{T}$ and (2) follows. $\square$

We then get the following main theorem that is the main machine that we apply to get scaled inclusion results.

Theorem 6.6.3. *Let $\mathcal{S}$ be a self-adjoint matrix convex set. Assume that $\mathcal{S}_1$ is r-symmetric for $r \geq 1$. Then for every other self-adjoint matrix convex set $\mathcal{T}$, we have*

$$\mathcal{S}_1 \subseteq \mathcal{T}_1 \Longrightarrow \mathcal{S} \subseteq r\mathcal{T}.$$

Proof. If $\mathcal{S}_1$ is r-symmetric, then $\mathcal{W}^{max}(\mathcal{S}_1)$ is r-symmetric as a matrix convex set, since it is defined by the same linear inequalities. If we take in Theorem 6.6.2 the matrix convex set $r \cdot \mathcal{W}^{min}(\mathcal{S}_1)$, then r-symmetry of $\mathcal{W}^{max}(\mathcal{S}_1)$ yields that $\mathcal{W}^{max}(\mathcal{S}_1) \subseteq r \cdot \mathcal{W}^{min}(\mathcal{S}_1)$. Hence, by maximality, $\mathcal{S} \subseteq \mathcal{W}^{max}(\mathcal{S}_1) \subseteq r \cdot \mathcal{W}^{min}(\mathcal{S}_1) \subseteq r\mathcal{T}$. □

As a corollary to the proof of Theorem 6.6.2 we get the following, which will allow us to get scaled inclusion results for matrix convex sets with ground floor the polyball.

Corollary 6.6.4. *Let $X \in B(H)^d_{sa}$, and let $C^{(1)}, \dots, C^{(k)}$ be a k-tuple of real $d \times d$ rank one matrices such that $I_d \in \operatorname{conv}\{C^{(1)}, \dots, C^{(k)}\}$. Then X can be dilated to a commuting tuple of self-adjoint operators $T = (T_1, \dots, T_d)$ such that*

$$\sigma(T) \subseteq \bigcup_{p=1}^{k} C^{(p)} \mathcal{W}_1(X).$$

Proof. Given $X \in B(H)^d_{sa}$, construct the dilation $T = (T_1, \dots, T_d)$ as in the proof of Theorem 6.6.2. Then T_i is the direct sum of operators $Y_i^{(m)} \in B(\mathcal{H})_{sa}$ of the form $Y_i^{(m)} = \sum_{j=1}^d c_{i,j}^{(m)} X_j$. We will show that $\sigma(Y^{(m)}) \subseteq C^{(m)} \mathcal{W}_1(X)$ for all m. Since $\sigma(N) \subseteq \mathcal{W}_1(N)$ for every normal tuple N, it suffices to show that $\mathcal{W}_1(Y^{(m)}) \subseteq C^{(m)} \mathcal{W}_1(X)$.

If ϕ is a state on $B(\mathcal{H})$, then $\phi(Y_i^{(m)}) = \sum_{j=1}^d c_{i,j}^{(m)} \phi(X_j)$, so that

$$(\phi(Y_1^{(m)}), \dots, \phi(Y_d^{(m)})) = C^{(m)} \left(\phi(X_1), \dots, \phi(X_d)\right).$$

Therefore $\mathcal{W}_1(Y^{(m)}) \subseteq C^{(m)} \mathcal{W}_1(X)$, as required. □

Corollary 6.6.5. *For all d we have that,*

$$\mathcal{W}^{max}(\overline{\mathbb{D}}^d) \subseteq 2d \mathcal{W}^{min}(\overline{\mathbb{D}}^d).$$

Consequently, for every d-tuple of contractions $A_1, \dots, A_d$ on a Hilbert space $\mathcal{H}$ there exists a Hilbert space $\mathcal{K}$, an isometry $V : \mathcal{H} \to \mathcal{K}$, and d commuting normal operators $T_1, \dots, T_d$ satisfying $\|T_i\| \leq 2d$ for $i = 1, \dots, d$ such that

$$A_i = V^* T_i V \quad , \quad i = 1, \dots, d.$$

A similar result holds in the self-adjoint context, where $2d$ is replaced by d and $\overline{\mathbb{D}}^d$ is replaced by $[-1, 1]^d$.

Proof. Let $A = (A_1, ..., A_d) \in \mathcal{W}^{max}(\overline{\mathbb{D}}^d)$, and denote

$$X = (\mathrm{Re}(A_1), \mathrm{Im}(A_1), \ldots, \mathrm{Re}(A_d), \mathrm{Im}(A_d)).$$

Then

$$\mathcal{W}_1(X) \subseteq \{(\alpha_1, \alpha_2, \ldots, \alpha_{2d}) : |\alpha_{2j-1} + i\alpha_{2j}| \leq 1, 1 \leq j \leq d\} = \overline{\mathbb{D}}^d \subseteq \mathbb{R}^{2d}$$

where $\mathbb{D}$ is the unit disc in $\mathbb{R}^2$.

For every $1 \leq m \leq 2d$, write e_m for the m-th element of the standard basis of $\mathbb{R}^{2d}$. Then $e_m e_m^*$ is the projection onto $\mathbb{R}e_m$. For every such m write $C^{(m)} = 2de_m e_m^*$. Then C is a $2d$-tuple of real $2d \times 2d$ rank one matrices such that $I_{2d} \in \mathrm{conv}\{C^{(1)}, \ldots, C^{(2d)}\}$. It is easy to check that $C^{(m)}\mathcal{W}_1(X) \subseteq 2d\overline{\mathbb{D}}^d$ for every $1 \leq m \leq 2d$. By Corollary 6.6.4 we get that X can be dilated to a $2d$-tuple of commuting self-adjoint operators $Y = (Y_1, \ldots, Y_{2d})$ with $\sigma(Y) \subseteq 2d\overline{\mathbb{D}}^d$.

We now define $T_j := Y_{2j-1} + iY_{2j}$ for all $1 \leq j \leq d$, and it remains to show that $\sigma(T) \subseteq 2d\overline{\mathbb{D}}^d$ where now $\mathbb{D}$ is the unit disc in $\mathbb{C}$. For this, note that since $\sigma(Y) \subseteq 2d\overline{\mathbb{D}}^d$, we can write Y_m as a diagonal matrices $\mathrm{diag}(\alpha_k^{(m)})$ such that, for every k we have $(\alpha_k^{(1)}, \ldots, \alpha_k^{(2d)}) \in 2d\overline{\mathbb{D}}^d \subseteq \mathbb{R}^{2d}$. But, then, for every $1 \leq j \leq d$ we get that $(\alpha_k^{(2j-1)}, \alpha_k^{(2j)}) \in 2d\overline{\mathbb{D}}$. Since $T_j = \mathrm{diag}(\alpha_k^{(2j-2)} + i\alpha_k^{(2j)})$, we have $\sigma(T) \subseteq 2d\overline{\mathbb{D}}^d$ and $||T_j|| \leq 2d$ as required. □

Example 6.6.6 (A non-scalable example)**.** *In this example we show that the conclusion of Theorem 6.6.3 can sometimes fail. Let* $T = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$*. Then*

$$\mathcal{W}_1(T) = \overline{\mathbb{D}}_1(1) = \{\ z \in \mathbb{C} \mid |z - 1| \leq 1\ \},$$

is a closed disc with center 1 *containing* 0 *on the boundary. Let* $N = M_{1+z}$ *be the multiplication operator on* $L^2(\mathbb{T})$*, so that* $\sigma(N) = 1 + \mathbb{T}$*. We see that* $\mathcal{W}_1(T) = \mathcal{W}_1(N)$*. By Corollary 6.3.2 we have* $\mathcal{W}(N) = \mathcal{W}^{min}(\overline{\mathbb{D}}_1(1))$*. Thus* $\mathcal{W}(T) \supseteq \mathcal{W}(N)$*. However, there is no* $r > 0$ *such that* $\mathcal{W}(T) \subseteq r\mathcal{W}(N)$*. To see this, we show that there is no UCP map* $\phi : \mathcal{V}_N \to \mathcal{V}_T$ *sending* N *to* rT *for any* $r > 0$*, and invoke the duality between closed and bounded matrix convex sets and finite dimensional operator systems.*

If there was such a UCP map, let $U = M_z = N - 1$*, then* $\phi(U) = rT - I$*. But* $||rT - I|| > 1$ *so this is impossible. Indeed, observe that*

$$\|rT - I\| = \left\| \begin{pmatrix} r-1 & 2r \\ 0 & r-1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 2r & 1-r \\ 1-r & 0 \end{pmatrix} \right\|.$$

But this is equal to the largest root of $t^2 - 2rt - (1-r)^2 = 0$*. Substituting* $u = t - 1$ *to get* $u^2 + 2(1-r)u - r^2 = 0$*, this equation must now have a positive root, so that* $\|rT - I\| > 1$*.*

To get an example involving self-adjoint matrix convex set, simply take matrix ranges of (the same) self-adjoint choice of coordinates for $\mathcal{V}_N$ *and* $\mathcal{V}_T$*. In particular, by Theorem 6.6.3 we see that* $\overline{D}_1(1)$ *is not* r*-symmetric for any* $r > 0$*.*

We next apply Theorem 6.6.3 to as many examples as possible, so as to cover [64, Proposition 14.1] and provide many new scaled inclusion results. We begin with some illuminating examples.

Example 6.6.7. *Fix* $d = 2$*. In* $\mathbb{R}^2$ *draw two straight lines that pass through the origin and are not parallel. Call them* L_1, L_2*. On each line draw two points (different from the origin), say* T_1, T_3 *on* L_1 *and* T_2, T_4 *on* L_2 *such that the origin lies in the intervals* $[T_1, T_3]$ *and* $[T_2, T_4]$*. Write*

$$P_m = \mathrm{conv}\{T_i : 1 \leq i \leq 4\}.$$

Now, through T_1 *and* T_3 *(on* L_1*) draw straight lines parallel to* L_2*. Similarly, through* T_2 *and* T_4 *draw lines parallel to* L_1*. These 4 lines form a parallelogram, call it* P_M*. Clearly,* $P_m \subseteq P_M$*. Both* P_m *and* P_M *are given by four linear inequalities and hence define matrix convex sets* $\mathcal{W}^{max}(P_m)$ *and* $\mathcal{W}^{max}(P_M)$ *that are in fact free matrix spectrahedra.*

Write q_i *for the projection of* $\mathbb{R}^2$ *onto* L_i *(parallel to the other line) and set* $C^{(i)} = 2q_i$*,* $i = 1, 2$*. Note that both are real rank one matrices and* $q_1 + q_2 = I_2$*. If* E *is any convex set between* P_M *and* P_m*, then* E *is* 2*-symmetric, because each* q_i *maps* P_M *into* P_m*.*

Example 6.6.8 (The regular simplex)**.** *Here is an example of a convex set in* $\mathbb{R}^3$ *for which is not invariant under projecting onto an orthonormal basis, but to which we may still apply Theorem 6.6.3. The regular* 3-simplex Δ_3 *in* $\mathbb{R}^3$ *is the convex hull of* $v_1 = (1, 1, 1)$*,* $v_2 = (1, -1, -1)$*,* $v_3 = (-1, 1, -1)$ *and* $v_4 = (-1, -1, 1)$*, and is not invariant under projecting onto any orthonormal basis.*

Take $C^{(m)} = v_m v_m^*$ *for* $m = 1, \ldots, 4$*. Then* Δ_3 *is* 3*-symmetric, since* $\frac{1}{3}C^{(m)}$ *is the orthogonal projection onto* v_m*, and one sees that* Δ_3 *is invariant under that. One computes directly that* $\sum_{m=1}^{4} C^{(m)} = 4I$*, thus* $I \in \mathrm{conv}\{C^{(m)} : 1 \leq m \leq 4\}$ *so Theorem 6.6.3 is applicable with* $r = 3 = d$*.*

Remark 6.6.9. In [55, Theorem 4.7], Thom, Netzer and Fritz show that the simplex satisfies the conclusion of Theorem 6.6.3 with constant $r = 1$, and that the simplex in each dimension is the essentially the unique convex set K such that $\mathcal{W}^{min}(K) = \mathcal{W}^{max}(K)$.

In general, given a convex set $K \subseteq \mathbb{R}^d$, and an orthonormal basis $\mathcal{B} = \{e_1, ..., e_d\}$, it is easy to verify whether or not K is invariant under the projections $e_i e_i^*$ for $1 \leq i \leq d$. If it is invariant, by either applying Theorem 6.6.3 or [64, Proposition 14.1], one obtains a scaled dilation result with $r = d$. In general however, it is much harder to actually find or disprove the existence of an orthonormal basis that leaves K invariant. Instead of this, we turn to tight frames, which are often used to *define* many convex polytopes $K \subseteq \mathbb{R}^d$. We will show that for matrix convex sets with ground level invariant under a uniform tight frame, or a convex polytope generated by a uniform tight frame, Theorem 6.6.3 is applicable to obtain a scaled inclusion result.

A set of vectors $\Phi := \{v_1, ..., v_N\} \subseteq \mathbb{R}^d$ (without repetition) is called a *tight frame* if there is a constant $\sigma > 0$ such that for all $x \in \mathbb{R}^d$ we have

$$\sum_i |\langle x, v_i \rangle|^2 = \sigma \|x\|^2;$$

this condition is equivalent to $\sum_i v_i v_i^* = \sigma I$.

When all the vectors v_i are of the same length ℓ, we call Φ a *uniform* tight frame, and it turns out that in this case we have $\sigma = \ell^2 \cdot \frac{N}{d}$ (See [110, Lemma 2.1]).

Every tight frame Φ gives rise to a finite subgroup of isometric symmetries given by

$$\mathrm{Sym}(\Phi) := \{U \in \mathcal{U}(\mathbb{R}^d) | U\Phi = \Phi\},$$

where $\mathcal{U}(\mathbb{R}^d)$ denotes the unitary group on $\mathbb{R}^d$. We can then turn this construction around, and define uniform tight frames from finite subgroups of $\mathcal{U}(\mathbb{R}^d)$. This will provide us with an abundance of examples. By [117, Theorem 6.3], for a finite irreducible subgroup G and a non-zero vector $v \in \mathbb{R}^d$, the set $\Phi = \{gv\}_{g \in G}$ is a uniform tight frame, and $K = \mathrm{conv}\,\Phi$ is invariant under G.

Proposition 6.6.10. *Let $K \subseteq \mathbb{R}^d$ be a closed convex set, and $\{v_1, ..., v_N\}$ be a uniform tight frame in $\mathbb{R}^d$ with vectors of length 1 such that $v_i v_i^*(K) \subseteq K$. Then $\mathcal{W}^{max}(K) \subseteq d\mathcal{W}^{min}(K)$.*

Proof. By prescribing $C^{(i)} = d \cdot v_i v_i^*$ and using $\sigma = \frac{N}{d}$, we have that

$$\sum_i C^{(i)} = \sum_i d v_i v_i^* = d\sigma \cdot I = N \cdot I$$

so that

$$I = \frac{1}{N} \sum_{i=1}^{N} C^{(i)} \in \mathrm{conv}\{C^{(i)}\} \text{ and } \frac{1}{d} C^{(i)}(K) = v_i v_i^*(K) \subseteq K.$$

Therefore K is d-symmetric. By Theorem 6.6.3, and Theorem 6.5.3 we see that

$$\mathcal{W}^{max}(K) \subseteq d\mathcal{W}^{min}(K). \qquad \square$$

We obtain the following consequence of Corollary 6.5.5 (See also [100, Proposition 5.5]).

Corollary 6.6.11. *Let $\mathcal{V}$ be a finite dimensional operator system, and suppose that $T = (T_1, ..., T_d)$ is a self-adjoint choice of coordinates such that $0 \in \mathcal{W}(T)$. Suppose $\{v_1, ..., v_N\}$ is a uniform tight frame in $\mathbb{R}^d$ with vectors of length 1 such that $v_i v_i^*(\mathcal{W}_1(T)) \subseteq \mathcal{W}_1(T)$. Then for any finite dimensional operator system $\mathcal{W}$ and any unital positive map ψ, either from $\mathcal{W}$ to $\mathcal{V}$ or from $\mathcal{V}$ to $\mathcal{W}$, is completely bounded with $\|\psi\|_{cb} \leq 2d - 1$.*

Our goal in the remainder of this section is to find classes of uniform tight frames for which the conditions of Proposition 6.6.10 hold with $K = \operatorname{conv}\Phi$. We begin with the following simple condition.

Corollary 6.6.12. *Let $\Phi = \{v_1, ..., v_N\} \subseteq \mathbb{R}^d$ be a uniform tight frame with vectors of length 1 and $K = \operatorname{conv}\Phi$. If $-\Phi = \Phi$, then*

$$\mathcal{W}^{max}(K) \subseteq d\mathcal{W}^{min}(K).$$

Proof. We need only verify that $v_i v_i^*(v_j) \in K$ for every $1 \leq i, j \leq N$. Note that

$$v_i v_i^*(v_j) = \langle v_j, v_i \rangle v_i.$$

Hence, if $\langle v_j, v_i \rangle \geq 0$ then $v_i v_i^*(v_j)$ is simply a rescaling of v_i by a non-negative constant, and is hence in $\operatorname{conv}\Phi$, and if $\langle v_j, v_i \rangle \leq 0$, then $v_i v_i^*(v_j)$ is a rescaling of $-v_i$ by a non-negative constant and is hence in $\operatorname{conv}\Phi$. In either case we have that $v_i v_i^*(v_j) \in K$ so that by Proposition 6.6.10 we have

$$\mathcal{W}^{max}(K) \subseteq d\mathcal{W}^{min}(K). \qquad \square$$

The assumption $\Phi = -\Phi$ is rather restrictive (consider Example 6.6.8). For the purpose of exhibiting a class of uniform tight frames for which the invariance condition $v_i v_i^*(K) \subseteq K$ in Proposition 6.6.10 is automatic, we bring forth the following definition. For a tight frame Φ, denote by

$$\operatorname{Stab}(v) = \{U \in \operatorname{Sym}(\Phi) : Uv = v\},$$

the stabilizer subgroup of $\operatorname{Sym}(\Phi)$ of symmetries that fix the vector $v \in \mathbb{R}^d$.

Definition 6.6.13. *Let* $\Phi = \{v_1, ..., v_N\}$ *be a tight frame. We say that* Φ *is* vertex reflexive *if* $\mathrm{Stab}(v_i)$ *fixes a subspace of dimension exactly one, namely* $\mathrm{Sp}\{v_i\}$*, for every* $1 \le i \le N$.

If $\Phi = \{v_1, ..., v_N\}$ is a uniform tight frame, then no v_i can be a convex combination of $\Phi \setminus \{v_i\}$. Every element of $\mathrm{Sym}(\Phi)$ must leave the barycenter $\frac{1}{N}\sum_{i=1}^N v_i$ invariant, and thus, when Φ is a vertex reflexive uniform tight frame we must have that $\frac{1}{N}\sum_{i=1}^N v_i = 0$. Since the barycenter is relatively interior to $\mathrm{conv}\,\Phi$ inside the linear subspace spanned by Φ, from the fact that that Φ is a tight frame, this subspace must be all of $\mathbb{R}^d$, and hence 0 is interior to $\mathrm{conv}(\Phi)$. We obtain that the vectors $\{v_1, ..., v_N\}$ must comprise the vertices of a d-dimensional polytope.

We say that a face F of a polytope K is *m-dimensional*, if m is the minimal dimension of an affine subspace containing F. For a uniform tight frame Φ, we must have that every element of $\mathrm{Sym}(\Phi)$ maps m-dimensional faces to m-dimensional faces.

For a computational method for constructing many examples of vertex reflexive uniform tight frames (satisfying the additional requirement that $\mathrm{Sym}(\Phi)$ is irreducible and transitive), see [22].

Proposition 6.6.14. *Let* $\Phi = \{v_1, ..., v_N\}$ *be a vertex reflexive uniform tight frame of vectors of length* 1*, and let* $K := \mathrm{conv}(\Phi)$ *be the d-dimensional convex polytope generated by* Φ*. Then* $v_i v_i^*(K) \subseteq K$ *for all* $1 \le i \le N$.

Proof. Fix $1 \le i \le N$. Let α be maximal such that $-\alpha v_i \in K$, and let F be a face of K of minimal dimension m such that $-\alpha v_i \in F$. Since $0 \in \mathrm{int}(K)$ we see that $\alpha > 0$.

We first claim that every element $g \in \mathrm{Stab}(v_i)$ must leave F invariant. Indeed, if not, $g(F)$ must be an m-dimensional face with $-\alpha v_i \in g(F)$ which is different from F. Since $g(F) \cap F$ must be a face of dimension strictly less than m, we arrive at a contradiction to the definition of F.

Since $F = \mathrm{conv}\{v_{i_1}, ..., v_{i_p}\}$ is left invariant under $\mathrm{Stab}(v_i)$, we may restrict each element $g \in \mathrm{Stab}(v_i)$ to the subspace $W = \mathrm{Sp}(\{v_i\} \cup F)$. Within W, since every $g \in \mathrm{Stab}(v_i)$ maps F to itself, it must then map the affine subspace A generated by F inside W, to itself, and hence must map the normal of A (again inside W) to itself. But even within the subspace W, we still have that $\mathrm{Stab}(v_i)$ fixes a subspace of dimension exactly one, and hence the normal of A in W can be chosen to be v_i. In other words, v_i is perpendicular to $-\alpha v_i - v_{i_j}$ for any $1 \le j \le p$. This means that

$$v_i v_i^*(v_{i_j}) = \langle v_i, v_{i_j} \rangle v_i = \langle v_i, -\alpha v_i \rangle v_i = -\alpha v_i \in K$$

and $-\alpha = \langle v_i, v_{i_j} \rangle$ is the cosine of the angle between v_i and v_{i_j} for all $1 \leq j \leq p$. We note that by maximality of α, we have for each $1 \leq k \leq N$ that the angle between v_i and v_k is at most $\arccos(-\alpha)$.

Hence, for all $1 \leq k \leq N$ we have that $v_i v_i^*(v_k) = \langle v_i, v_k \rangle v_i$ is a convex combination of $-\alpha v_i$ and v_i and is hence in K. As $v_i v_i^*$ is linear, we see that $v_i v_i^*(K) \subseteq K$ as required. $\square$

Combining Propositions 6.6.10 and 6.6.14, we obtain the following.

Theorem 6.6.15. *Let $\Phi \subseteq \mathbb{R}^d$ be a vertex reflexive uniform tight frame, and let $K = \operatorname{conv}(\Phi)$ be the convex polytope it generates. Then*

$$\mathcal{W}^{max}(K) \subseteq d\mathcal{W}^{max}(K).$$

As a consequence of Theorem 6.6.15 and [22, Theorem 5.4], we obtain our results for any convex regular polytope (See [22, Definition 5.1]).

Corollary 6.6.16. *Let $K = \operatorname{conv}\{v_1, ..., v_N\}$ be a convex regular* real *polytope according to [22, Definition 5.3]. Then $\mathcal{W}^{max}(K) \subseteq d\mathcal{W}^{min}(K)$.*

Example 6.6.17. *In [22], a class of frames called* highly symmetric *frames was studied. A* highly symmetric tight frame *is a vertex reflexive tight frame for which* $\operatorname{Sym}(\Phi)$ *is also transitive and irreducible (and is then automatically uniform). The class of highly symmetric frames was shown to be rich, yet tractable.*

We will now construct an example of a vertex reflexive uniform tight frame Θ for which $\operatorname{Sym}(\Theta)$ *is not irreducible, not transitive, and for which* no *vector $u \in \Theta$ satisfies $-u \in \Theta$. Thus Theorem 6.6.15 applies, while Corollaries 6.6.12 and 6.6.16 do not.*

Let $G = S_5$ act on $(e_1+...+e_5)^\perp$ inside $\mathbb{R}^5$, where $\{e_1, ..., e_5\}$ is the standard orthonormal basis, and S_5 acts by permutation matrices. Take the vector $\phi := 3w_2 = (3, 3, -2, -2, -2)$. Then by [22, Example 4] the frame $\Phi_2 := g\phi_{g \in S_5}$ is a vertex-reflexive uniform tight frame comprised of 10 *distinct vectors, and by construction we see that for all $v \in \Phi_2$, we have $-v \notin \Phi_2$.*

Hence, let $\Phi = \{v_1, ..., v_{10}\}$ be the unit-norm vertex-reflexive tight frame in $\mathbb{R}^4$ that is identified with the normalization of Φ_2 inside $(e_1 + ... + e_5)^\perp \subseteq \mathbb{R}^5$. So we still have $-v \notin \Phi$ for all $v \in \Phi$.

We then take the vertex-reflexive unit-norm tight frame of the pentagon inside $\mathbb{R}^2$,

$$\Psi := \{(\cos(2\pi k/5), \sin(2\pi k/5))\}_{k=1}^5$$

Which has 5 distinct elements, and satisfies $-w \notin \Psi$ for all $w \in \Psi$. Then define

$$\Theta = (\Phi \oplus 0_2) \cup (0_4 \oplus \Psi)$$

where $\Phi \oplus 0_2$ is adding two zeroes on the right to each vector in Φ and $0_4 \oplus \Psi$ is adding four zeroes on the left to each vector in Ψ. Each $u \in \Theta$ then satisfies $-u \notin \Theta$. We know by [110, Lemma 2.1] that

$$\sum_{v \in \Phi \oplus 0_2} vv^* = \frac{10}{4} P \quad \textit{and} \quad \sum_{w \in 0_4 \oplus \Psi} ww^* = \frac{5}{2} Q$$

Where P and Q are the orthogonal projections onto $\mathbb{R}^4$ and $\mathbb{R}^2$ respectively, that sum to the identity I on $\mathbb{R}^6 = \mathbb{R}^4 \oplus \mathbb{R}^2$. Therefore,

$$\sum_{u \in \Theta} uu^* = \frac{5}{2}(P + Q) = \frac{5}{2} I.$$

Thus Θ is a unit-norm tight frame in $\mathbb{R}^6$. Since $\mathrm{Sym}(\Phi)$ and $\mathrm{Sym}(\Psi)$ fix $\mathbb{R}^4$ and $\mathbb{R}^2$ in the decomposition of $\mathbb{R}^6$ above, and Φ and Ψ are vertex-reflexive, in their respective spaces, we have that their union Θ is also a vertex-reflexive unit-norm tight frame.

Since elements of Φ and Ψ are always perpendicular when identified as elements of $\mathbb{R}^6$, no symmetry of Θ can map an element of Φ to an element of Ψ. Indeed, from the construction of Φ_2, an element v_i from Φ has no element perpendicular to it from Φ, so that in Θ, there are only 5 elements perpendicular to v_i: those of Ψ. On the other hand, an element w_j of Ψ has exactly 10 elements of Θ perpendicular to it: those of Φ. Thus, no $v_i \in \Phi$ can be mapped to any $w_j \in \Psi$ via by a symmetry of Θ, and $\mathrm{Sym}(\Theta)$ is not transitive. Hence, elements of $\mathrm{Sym}(\Theta)$ can only permute elements of Φ among themselves, and elements of Ψ among themselves. Thus

$$\mathrm{Sym}(\Theta) = \mathrm{Sym}(\Phi) \oplus \mathrm{Sym}(\Psi),$$

so that $\mathrm{Sym}(\Theta)$ is reducible.

6.7 Matrix balls and optimality

Our goal in this section is two-fold. We show that the constant $r = d$ is optimal for scaled inclusion of matrix convex sets of dimension d as in the conclusion of Theorem 6.6.3. We then obtain an array of different matrix convex sets with ground floor the closed unit ball $\overline{\mathbb{B}} = \overline{\mathbb{B}}_d$ inside $\mathbb{R}^d$, including a *self-dual* such matrix ball $\mathfrak{S}$, which we show is essentially the unique self-dual self-adjoint matrix convex set of dimension d.

Lemma 6.7.1. *For every d, there exist d self-adjoint $2^{d-1} \times 2^{d-1}$ matrices $B_1, \ldots, B_d$ such that for all $v \in \mathbb{R}^d$, $\|v\| = 1$,*

$$\sum v_i B_i \leq I,$$

and such that d is an eigenvalue of $\sum_{i=1}^d B_i \otimes B_i$. Hence if $\sum_{i=1}^d B_i \otimes B_i \leq \rho I$, then $\rho \geq d$.

Proof. The proof is by induction. For $d = 1$ we take $B_1 = [1]$. Suppose that $d \geq 1$, and let $B_1, \ldots, B_d$ be self-adjoint $2^{d-1} \times 2^{d-1}$ matrices as in the statement of the lemma. We will construct self-adjoint $2^d \times 2^d$ matrices $B'_1, \ldots, B'_{d+1}$ as required.

Let

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define

$$B'_i = E_1 \otimes B_i \ , \quad \text{for } i = 1, \ldots, d,$$

and

$$B'_{d+1} = E_2 \otimes I_{2^{d-1}}.$$

The matrices $B'_1, \ldots, B'_{d+1}$ are self-adjoint. For a unit vector $(v_1, \ldots, v_{d+1})$, we compute

$$I - \sum_{i=1}^{d+1} v_i B'_i = \begin{pmatrix} (1 - v_{d+1})I & -\sum_{i=1}^d v_i B_i \\ -\sum_{i=1}^d v_i B_i & (1 + v_{d+1})I \end{pmatrix}$$

By [99, Lemma 3.1], this matrix is positive semidefinite if and only if

$$\Big(\sum_{i=1}^d v_i B_i\Big)^2 \leq (1 - v_{d+1}^2)I.$$

By the inductive hypothesis,

$$\Big(\sum_{i=1}^d v_i B_i\Big)^2 \leq \Big(\sum_{i=1}^d v_i^2\Big)I \leq (1 - v_{d+1}^2)I.$$

Therefore $\sum_{i=1}^{d+1} v_i B'_i \leq I$.

It remains to show that $d+1$ is an eigenvalue of $\sum_{i=1}^{d+1} B'_i \otimes B'_i$. We write $T = \sum_{i=1}^d B_i \otimes B_i$ and examine the operator

$$\sum_{i=1}^{d+1} B'_i \otimes B'_i \simeq \begin{pmatrix} I & 0 & 0 & T \\ 0 & -I & T & 0 \\ 0 & T & -I & 0 \\ T & 0 & 0 & I \end{pmatrix}.$$

Now, if x is an eigenvector of T corresponding to d, then $(x,0,0,x)^t$ is an eigenvector of $\sum_{i=1}^{d+1} B_i' \otimes B_i'$ corresponding to the eigenvalue $d+1$. $\square$

The next example shows that $r = d$ is sharp in the conclusion of Theorem 6.6.3.

Example 6.7.2. *We construct tuples of operators A and B of self-adjoint matrices, such that the implication in Theorem 6.6.3 holds with a constant d but with no smaller constant. In this example the tuple A consists of operators on an infinite dimensional space, but by taking a sufficiently large finite dimensional corner of A, we get sharpness in the finite dimensional case as well.*

Let $\{v^{(n)}\}$ be a dense sequence of points on the unit sphere of $\mathbb{R}^d$. Let A be the d-tuple of diagonal operators such that nth element on the diagonal of A_j is the jth coordinate $v_j^{(n)}$ of $v^{(n)}$. Then

$$\mathcal{D}_A^{sa} = \{X \in M_n(\mathbb{C})_{sa}^d : \sum X_j v_j \leq I \text{ for all } v \in \mathbb{R}^d, \ \|v\| = 1\}.$$

Observe that $\mathcal{D}_A^{sa} = \mathcal{W}^{max}(\overline{\mathbb{B}}_d)$, and in particular $\mathcal{D}_A^{sa}(1) = \overline{\mathbb{B}}_d$ is the unit ball of $\mathbb{R}^d$, which is invariant under projections onto any uniform tight frame. Hence, we get from Theorem 6.6.10 that for any matrix convex set $\mathcal{T}$ we have that

$$\mathcal{D}_A^{sa}(1) \subseteq \mathcal{T}_1 \implies \mathcal{D}_A^{sa} \subseteq d\mathcal{T}.$$

Let B be as in Lemma 6.7.1. For every unit vector $v \in \mathbb{R}^d$, $\sum v_i B_i \leq I$. Thus $\mathcal{D}_A^{sa}(1) \subseteq \mathcal{D}_B^{sa}(1)$ and $B \in \mathcal{D}_A^{sa} \subseteq d\mathcal{D}_B^{sa}$. On the other hand, $B \notin \rho\mathcal{D}_B^{sa}$ for any $\rho < d$, since $\sum_i B_i \otimes B_i$ has an eigenvalue equal to d. Thus $\mathcal{D}_A^{sa} \nsubseteq \rho\mathcal{D}_B^{sa}$ for any $\rho < d$.

Recall that the (d-dimensional self-adjoint) *matrix ball* is defined to be

$$\mathfrak{B} = \mathfrak{B}^{(d)} = \{X \in M_n(\mathbb{C})_{sa}^d : \sum_{j=1}^{d} X_j^2 \leq I\}.$$

We also introduce another "matrix ball" which will turn out to conform more naturally to duality of matrix convex sets. Recall that the transpose is a linear map from $A \in \mathcal{B}(H)$ to $\mathcal{B}(H^*)$ is given by $A^t f = f \circ A$. Define the *conjugate* of $A \in \mathcal{B}(H)$ to be $\overline{A} := (A^*)^t$. It is called the conjugate because if $A = [a_{ij}]$ belongs to M_n, then $\bar{A} = [\overline{a_{ij}}]$. Haagerup [59, Lemma 2.4] established an important identity for the spatial tensor product by observing that if H and K are Hilbert spaces, then a spatial tensor product $A \otimes B$ on $H \otimes K$ of two operators $A \in B(H)$ and $B \in B(K)$ can be represented as an operator on the Hilbert-Schmidt operators $\mathcal{S}_2(K, H)$ from K into H, which is canonically isomorphic to the Hilbert space $H \otimes K^*$. Indeed, the operator $A \otimes \overline{B}$ is unitarily equivalent to the operator $u \to AuB^*$ in $\mathcal{B}(\mathcal{S}_2(K, H))$. Haagerup shows that (in the spatial tensor norm)

$$\big\| \sum A_i \otimes \overline{B_i} \big\| \le \big\| \sum A_i \otimes \overline{A_i} \big\|^{1/2} \big\| \sum B_i \otimes \overline{B_i} \big\|^{1/2}.$$

Proposition 6.7.3. *For each $d \in \mathbb{N}$ denote $\mathfrak{S} = \{\mathfrak{S}_n\}$*

$$\mathfrak{S}_n = \mathfrak{S}_n^{(d)} = \{X \in M_n(\mathbb{C})^d_{sa} : \big\| \sum_{j=1}^d X_j \otimes \overline{X_j} \big\| \le 1\}.$$

Then, $\mathfrak{S}$ is a closed matrix convex set.

Proof. Clearly $\mathfrak{S}$ is closed. Observe that $\mathfrak{S} = -\mathfrak{S}$ and is invariant under the entrywise conjugation map sending A to $\overline{A}$. In particular, the norm condition is equivalent to the two inequalities

$$\pm \sum_{j=1}^d X_j \otimes \overline{X_j} \le I.$$

Haagerup's inequality shows that if $X \in \mathfrak{S}_n$ and $Y \in \mathfrak{S}_m$, then $X \oplus Y \in \mathfrak{S}_{n+m}$. It is also routine to show that if $A \in M_{m,n}$ is an isometry, and $X \in \mathfrak{S}_n$, then $AXA^* \in \mathfrak{S}_m$. So $\mathfrak{S}$ is matrix convex. □

Observe that $\mathfrak{B} = -\mathfrak{B}$ and is also closed under entrywise conjugation, and that entrywise conjugation is isometric. A self-adjoint d-tuple X belongs to $\mathfrak{B}$ exactly when

$$\|X\| = \|[X_1 \ X_2 \ \dots \ X_d]\| = \big\| \sum_{j=1}^d X_j^2 \big\|^{1/2} \le 1.$$

Therefore

$$\begin{aligned} \big\| \textstyle\sum_{j=1}^d X_j \otimes \overline{X_j} \big\| &\le \|X \otimes I\| \, \|I \otimes \overline{X}\| \\ &= \big\| \textstyle\sum_{j=1}^d X_j^2 \otimes I \big\|^{1/2} \ \big\| \textstyle\sum_{j=1}^d I \otimes \overline{X_j}^2 \big\|^{1/2} \\ &= \|X\| \, \|\overline{X}\| \le 1. \end{aligned}$$

It follows that $\mathfrak{B} \subseteq \mathfrak{S}$.

Clearly $\mathfrak{B}(1) = \mathfrak{S}(1) = \overline{\mathbb{B}}$. Thus we have

$$\mathcal{W}^{min}(\overline{\mathbb{B}}) \subseteq \mathfrak{B} \subseteq \mathfrak{S} \subseteq \mathcal{W}^{max}(\overline{\mathbb{B}}).$$

It is natural to ask about the precise place these matrix balls take in this inequality.

Proposition 6.7.4. $\mathfrak{S}^{(d)}$ *is the unique* self-dual *self-adjoint matrix convex set* $\mathcal{S}$ *of dimension* d *that is invariant under entrywise conjugation.*

Proof. Note that $\mathfrak{S}$ is closed under entrywise conjugation and $\mathfrak{S} = -\mathfrak{S}$. Haagerup's inequality shows that for X and Y in $\mathfrak{S}$ we have that

$$\left\| \sum X_i \otimes \overline{Y_i} \right\| \leq 1.$$

Since $\overline{Y}$ also belongs to $\mathfrak{S}$, we deduce that $I \geq \sum_{i=1}^{d} X_i \otimes Y_i$, so that $\mathfrak{S}^{\bullet} \supseteq \mathfrak{S}$.

Conversely, take $Y \in \mathfrak{S}^{\bullet}$. As $\mathfrak{S}_1 = \mathbb{B}_d$, it is clear that $0 \in \text{int}(\mathfrak{S})$ so that

$$r_0 = \sup\{r : rY \in \mathfrak{S}\}$$

is positive. If $r_0 \geq 1$, then $Y \in \mathfrak{S}$ and we are done. If however $r_0 < 1$, as $\mathfrak{S} = -\mathfrak{S}$, we have $\pm\overline{Y} \in \mathfrak{S}^{\bullet}$, so that $I \pm \sum_{i=1}^{d} Y_i \otimes \overline{Y}_i \geq 0$. This implies that

$$\left\| \sum r_0 Y_i \otimes \overline{Y_i} \right\| \leq 1.$$

However, this shows that $\sqrt{r_0}Y$ belongs to $\mathfrak{S}$. This is impossible since, as $\sqrt{r_0} > r_0$, we get a contradiction to the definition of r_0.

For uniqueness, if $\mathcal{S}$ is a self-dual self-adjoint matrix convex set of dimension d, then $\mathcal{S}_1 = \overline{\mathbb{B}_d}$. Indeed, for every $x \in \mathcal{S}_1$, we have that $\langle x, x \rangle \leq 1$, thus $\mathcal{S}_1 \subseteq \overline{\mathbb{B}_d}$. It follows from self-duality of $\mathcal{S}$ that $\mathcal{S}_1 = \mathcal{S}_1^{\bullet} = \mathcal{S}_1' \supseteq \overline{\mathbb{B}}_d$ so that $\mathcal{S}_1 = \overline{\mathbb{B}}_d$. In particular $0 \in \text{int}(\mathcal{S})$.

We now show that self-dual matrix convex sets $\mathcal{S}$ are closed under taking minuses. By categorical duality, it will suffice to show that the *unital* map id_- that sends z_i to $-z_i$ is a complete order automorphism of $\mathbb{A}(\mathcal{S})$, where $z = (z_1, ..., z_d)$ are the d-tuple of self-adjoint coordinate functions on $\mathcal{S}$. Since $\mathcal{S} = \mathcal{W}(z)$, we have that $\mathbb{A}(\mathcal{W}(z)) = \mathbb{A}(\mathcal{W}(z)^{\bullet})$ is an operator system. Since $\mathcal{S}$ is d-dimensional and $0 \in \text{int}(\mathcal{S})$, we see that $\{\mathbb{1}, -z_1, ..., -z_d\}$ is a basis for $\mathbb{A}(\mathcal{S})$. Hence, we may take its dual basis $\{\mathbb{1}', f_1, ..., f_d\}$ for $\mathbb{A}(\mathcal{S})'$ so that $f = (f_1, ..., f_d)$ is the dual choice of coordinates to $z = (z_1, ..., z_d)$. By Theorem 6.3.11, item (1) of Theorem 6.2.6 and Corollary 6.3.12 we see that id_- is the composition of unital complete order isomorphisms

$$\mathbb{A}(\mathcal{S}) = \mathbb{A}(\mathcal{S}^{\bullet}) \cong \mathbb{A}(\mathcal{S})' \cong \mathbb{A}(\mathcal{W}(f)) = \mathbb{A}(\mathcal{W}(z)^{\bullet}) = \mathbb{A}(\mathcal{W}(z)) = \mathbb{A}(\mathcal{S})$$

and is hence a unital complete order isomorphism. Hence, we have that $\mathcal{S} = -\mathcal{S}$.

Next, if we assume $\mathcal{S}$ is also closed under entrywise conjugation, for $X \in \mathcal{S}$ we have that $\pm\overline{X} \in \mathcal{S}$. Hence, by self-duality we have

$$\pm \sum_{i=1}^{d} X \otimes \overline{X} \leq I,$$

which is equivalent to $\|\sum_{i=1}^{d} X_i \otimes \overline{X}_i\| \leq 1$. Thus, we see that $\mathcal{S} \subseteq \mathfrak{S}$. Applying polar duality and self-duality, we find that $\mathfrak{S} \subseteq \mathcal{S}$, so that $\mathcal{S} = \mathfrak{S}$. □

Justified by the above proposition, we will call $\mathfrak{S}$ the *self-dual matrix ball*. We then obtain the following immediate consequence.

Corollary 6.7.5.
$$\mathcal{W}^{min}(\overline{\mathbb{B}}) \subsetneq \mathfrak{B} \subsetneq \mathfrak{S} = \mathfrak{S}^{\bullet} \subsetneq \mathfrak{B}^{\bullet} \subsetneq \mathcal{W}^{max}(\overline{\mathbb{B}})$$
for $d > 1$.

Proof. For the first proper containment, let

$$X_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \ , \ X_2 = \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{3}{4} & 0 \end{pmatrix}.$$

One verifies $X \in \mathfrak{B}$, but by [63, Example 3.1], X is not contained in some other spectrahedron $\mathcal{D}_B^{sa}$ with $\mathcal{D}_B^{sa}(1) = \overline{\mathbb{B}}$ where $B = (B_1, B_2) = (E_1, E_2)$ are as in Lemma 6.7.1. In particular, X is not in $\mathcal{W}^{min}(\overline{\mathbb{B}})$. By adding zeroes if necessary, we obtain this for any dimension $d \geq 3$.

Next, we show that $\mathfrak{B} \neq \mathfrak{B}^{\bullet}$ foor $d > 1$. Indeed, for $i = 1, \ldots, d$, let B_i be the matrix on $\mathbb{C}^{d+1}$ that switches between e_1 and e_{i+1} and sends all other basis vectors to 0. Then $\mathfrak{B} = \mathcal{D}_B^{sa}$ (recall Example 2.4.4), thus $B \in \mathfrak{B}^{\bullet}$. On the other hand,

$$\sum_j B_j^2 = I + (d-1)e_1 e_1^*,$$

where $e_1 e_1^*$ denotes the orthogonal projection onto the first basis vector. Therefore, B is not in $\mathfrak{B}$ so that $\mathfrak{B}^{\bullet} \neq \mathfrak{B}$. Thus, the two middle proper containments hold.

For the last proper containment, as $\mathfrak{B} \subseteq \mathfrak{S}$, and as $\mathfrak{S}$ *is* self-dual, we have $\mathfrak{B} \subsetneq \mathfrak{S}$. By duality and Theorem 6.3.8, we obtain $\mathfrak{S} \subsetneq \mathfrak{B}^{\bullet} \subsetneq \mathcal{W}^{min}(\overline{\mathbb{B}})^{\bullet} = \mathcal{W}^{max}(\overline{\mathbb{B}})$. □

By analogy to the matrix cube / polyball problem, we ask for which constant r does the following implication hold:

$$\overline{\mathbb{B}} \subseteq \mathcal{D}_A(1) \Longrightarrow \mathfrak{S} \subseteq r\mathcal{D}_A. \tag{6.7}$$

Note that this is not in perfect analogy with the matrix cube problem, because $\mathfrak{C} = \mathcal{W}^{max}([-1,1]^d)$, whereas $\mathfrak{S}$ is somewhere near the 'center' of the range of matrix convex sets with first level equal to $\overline{\mathbb{B}}$. We already completely solved the problem for $\mathcal{W}^{max}(\overline{\mathbb{B}})$ above in Proposition 6.6.10 and Example 6.7.2. We know by Proposition 6.6.10 that $r = d$ works in equation (6.7), but we will do better in this case. Since $\mathfrak{S} \neq \mathcal{W}^{max}(\overline{\mathbb{B}})$, we also ask for a constant r such that

$$\mathcal{D}_A(1) \subseteq \overline{\mathbb{B}} \Longrightarrow \mathcal{D}_A \subseteq r\mathfrak{S}. \tag{6.8}$$

Remark 6.7.6. In a recent revision of the paper [64] (that appeared after we obtained the results of this section), results similar to those in this section were obtained using different methods. It is worth noting that [64] treats four sets which they call matrix balls: $\mathfrak{B}^{\max}$, which is what we denote by $\mathcal{W}^{min}(\overline{\mathbb{B}})$; $\mathfrak{B}^{\min}$, which is what we denote by $\mathcal{W}^{max}(\overline{\mathbb{B}})$; $\mathfrak{B}^{\mathrm{oh}}$, which is what we denote by $\mathfrak{B}$; and finally, $\mathfrak{B}^{\mathrm{spin}}$, which is a certain free spectrahedra with $\mathfrak{B}^{\mathrm{spin}}(1) = \overline{\mathbb{B}}$ which we do not discuss. (However, the spin matrices have arisen in Example 6.7.2.)

Theorem 6.7.7. *Let $\mathcal{S}$ be a self-adjoint matrix convex set. Then*

$$\mathcal{S}_1 \subseteq \overline{\mathbb{B}} \Longrightarrow \mathcal{S} \subseteq \sqrt{d}\,\mathfrak{B} \subseteq \sqrt{d}\,\mathfrak{S}$$

and

$$\overline{\mathbb{B}} \subseteq \mathcal{S}_1 \Longrightarrow \mathfrak{S} \subseteq \mathfrak{B}^\bullet \subseteq \sqrt{d}\,\mathcal{S}.$$

Moreover, the constant $\sqrt{d}$ is the optimal constant in both implications regarding $\mathfrak{S}$.

Proof. Suppose that $\mathcal{S}_1 \subseteq \overline{\mathbb{B}}$ and that $X \in \mathcal{S}$. Then $X \in \mathcal{W}^{max}(\overline{\mathbb{B}})$, so $\sum_j a_j X_j \leq I$ for all $a \in \overline{\mathbb{B}}$. In particular $\pm X_j \leq I$, so equivalently $X_j^2 \leq I$ for all j. Thus $\sum_j X_j^2 \leq dI$, meaning that $\frac{1}{\sqrt{d}} X \in \mathfrak{B} \subseteq \mathfrak{S}$, as required.

To obtain the second implication we use polar duality. If $\overline{\mathbb{B}} \subseteq \mathcal{S}_1$, then $\mathcal{S}_1^\bullet \subseteq \overline{\mathbb{B}}$, so by the first implication

$$\mathcal{S}^\bullet \subseteq \sqrt{d}\,\mathfrak{B}.$$

Applying the polar dual again, we obtain

$$\frac{1}{\sqrt{d}}\mathfrak{S} \subseteq \frac{1}{\sqrt{d}}\mathfrak{B}^\bullet = (\sqrt{d}\mathfrak{B})^\bullet \subseteq \mathcal{S}^{\bullet\bullet} = \mathcal{S}.$$

So we get that

$$\mathfrak{S} \subseteq \mathfrak{B}^{\bullet} \subseteq \sqrt{d}\mathcal{S}.$$

We next verify that the first inclusion implication is sharp with $\mathfrak{S}$. Indeed, by Lemma 6.7.1, there is a d-tuple of real Hermitian matrices B in $\mathcal{W}^{max}(\overline{\mathbb{B}})$ such that $\|\sum B_i \otimes B_i\| = d$. Since B is real, we have $B = \overline{B}$. Therefore it is clear that $r = 1/\sqrt{d}$ is the largest constant so that $rB \in \mathfrak{S}$. The second inclusion implication is also sharp with $\mathfrak{S}$ by duality. □

By Theorem 6.7.7 $\mathfrak{B} \subseteq \mathfrak{S} \subseteq \sqrt{d}\mathcal{W}^{min}(\overline{\mathbb{B}})$, so we obtain the following corresponding dilation result.

Corollary 6.7.8. *For $X = (X_1, ..., X_d) \in (M_n)^d_{sa}$, if $\|\sum_{j=1}^d X_j \otimes \overline{X_j}\| \leq 1$ (in particular, if $\sum_{j=1}^d X_j^2 \leq I$), then there exists $T = (T_1, ..., T_d)$ commuting self-adjoint matrices such that $\sigma(T) \subseteq \overline{\mathbb{B}}$ and $\sqrt{d}T$ dilates X.*

In [95], an operator system structure on Pisier's self-dual operator space was obtained by adding a unit I. For each natural number $d \in \mathbb{N}$ an operator system SOH(d) of dimension $d+1$ was introduced. This operator system has a natural self-adjoint choice of coordinates $H = (H_1, ..., H_d)$, so that an element $A_0 \otimes I + \sum_{i=1}^d A_i \otimes H_i$ is positive if and only if $-A_0 \otimes \overline{A_0} \leq \sum_{i=1}^d A_i \otimes \overline{A_i} \leq A_0 \otimes \overline{A_0}$ and if and only if $A_0 \otimes I - \sum_{i=1}^d A_i \otimes H_i$ is positive.

It was shown in [95] that an identification between a self-adjoint basis of SOH(d) and its dual basis for SOH(d)$'$ yields a complete order isomorphism of *matrix ordered spaces*. However, in [95] the dual *operator space* complete norm structure was put on SOH(d)$'$, as opposed to a complete norm induced from an operator system structure on SOH(d)$'$. With the dual operator space complete norm structure on SOH(d)$'$, it was shown in [95] that the smallest completely bounded norm of a complete order isomorphism between SOH(d) and SOH(d)$'$ is 2.

Our approach here is to put a complete norm structure on the dual that is induced from an operator system structure on the dual, through the existence of an Archimedean matrix order unit for the dual matrix ordered space given to us by the theorem of Choi and Effros. As a final application, we show that $\mathbb{A}(\mathfrak{S}^{(d)})$ is unitally complete order isomorphic to SOH(d) and that it is the unique operator system with a self-adjoint choice of coordinates T such that $\mathcal{W}(T)$ is self-dual and closed under entrywise conjugation.

Theorem 6.7.9. *Let $\mathcal{V}$ be a $(d+1)$-dimensional operator system. Suppose $\mathcal{V}$ has a self-adjoint choice of coordinates $T = (T_1, ..., T_d)$ such that $\mathcal{W}(T)$ is self-dual and closed under* *entrywise conjugation. Then $\mathcal{V}$ is unitally completely order isomorphic to $\mathbb{A}(\mathfrak{S}^{(d)})$. In particular* SOH($d$) *and* $\mathbb{A}(\mathfrak{S}^{(d)})$ *are unitally completely order isomorphic.*

Proof. Since $\mathcal{W}(T)$ is self-dual and closed under entrywise conjugation, by Proposition 6.7.4 we have that $\mathcal{W}(T) = \mathfrak{S}^{(d)}$. By Theorem 6.2.6 we know that $\mathcal{V}$ is unital complete order isomorphic to $\mathbb{A}(\mathcal{W}(T)) = \mathbb{A}(\mathfrak{S}^{(d)})$.

For the remaining part of the theorem, we need only show that the basis $\{I, H_1, ..., H_d\}$ of SOH(d) given in [95], the matrix range $\mathcal{W}(H)$ is self-dual and closed under entrywise conjugation.

We first show that $\mathcal{W}(H)$ is self-dual. By [95, Theorem 3.4] the map sending the basis $\{I, H_1, ..., H_d\}$ to the dual basis $\{\delta_0, \delta_1, ..., \delta_d\}$ is a complete order isomorphism, so our strategy is to apply Corollary 6.3.12 to conclude that $\mathcal{W}(H) = \mathcal{W}(\delta) = \mathcal{W}(H)^\bullet$. Hence, we need only make sure that $0 \in \text{int}(\mathcal{W}(H))$. However, by Proposition 6.3.9 this is equivalent to showing that δ_0 is strictly positive, so we show this. Indeed, by [95, Proposition 3.3], if $P := A_0 \otimes H_0 + \sum_{i=1}^d A_i \otimes H_i$ is positive in $M_n(\text{SOH}(d))$, then A_0 is positive, A_i are self-adjoint, and we have that $-A_0 \otimes \overline{A_0} \leq \sum_{i=1}^d A_i \otimes \overline{A_i} \leq A_0 \otimes \overline{A_0}$. Hence, if $A_0 = \delta_0(P) = 0$, we must have that $\sum_{i=1}^d A_i \otimes \overline{A_i} = 0$. By the discussion preceding [95, Proposition 3.3], we see that $\| \sum_{i=0}^d A_i \otimes H_i \| = \| \sum_{i=0}^d A_i \otimes \overline{A_i} \|$. Hence, if $A_0 = 0$ then $\sum_{i=0}^d A_i \otimes H_i = 0$. This means that δ_0 is strictly positive, so that by Corollary 6.3.12 together with Proposition 6.3.9 we see that $\mathcal{W}(H)$ is self-dual.

Next, since $\mathcal{W}(H)$ is self-dual, we see that $\mathcal{W}(H) = \mathcal{W}(H)^\bullet = \mathcal{D}_H^{sa}$. Hence, $A \in \mathcal{W}(H) = \mathcal{D}_H^{sa}$ if and only if $I \otimes I \geq \sum_{i=1}^d A_i \otimes H_i$. By [95, Proposition 3.3] this occurs if and only if $I \otimes I \geq \sum_{i=1}^d \overline{A_i} \otimes H_i$, so that $\mathcal{W}(H)$ is closed under entrywise conjugation. By Proposition 6.7.4 we see that $\mathcal{W}(H) = \mathfrak{S}$, and by categorical duality we have that SOH(d) and $\mathbb{A}(\mathcal{W}(H)) = \mathbb{A}(\mathfrak{S})$ are unitally completely order isomorphic. □

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