Hamilton Paths
in
Generalized Petersen Graphs

by

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AUTHOR'S DECLARATION FOR ELECTRONIC SUBMISSION OF A THESIS

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by examiners.

I understand that my thesis may be made electronically available to the public.

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Thanks to my supervisor, Bruce, for his patience and optimism. Thanks also to my partner, Jim, for his love and understanding. I am also indebted to the Natural Sciences and Engineering Research Council of Canada for financial assistance.

Abstract

This thesis puts forward the conjecture that for $n > 3k$ with $k > 2$, the generalized Petersen graph, $GP(n,k)$ is Hamilton-laceable if n is even and k is odd, and it is Hamilton-connected otherwise. We take the first step in the proof of this conjecture by proving the case $n = 3k+1$ and $k \geq 1$. We do this mainly by means of an induction which takes us from $GP(3k+1,k)$ to $GP(3(k+2)+1,k+2)$. The induction takes the form of mapping a Hamilton path in the smaller graph piecewise to the larger graph and then inserting subpaths we call *rotors* to obtain a Hamilton path in the larger graph.

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Chapter 1

Introduction

Why are Hamiltonian properties of generalized Petersen graphs important?

Lovász [3] has conjectured that all Cayley graphs are Hamiltonian. Recently, Yusheng Qin [6] proved that certain classes of Cayley graphs are Hamiltonian by using a subset of generators of the Cayley graph to induce certain isomorphic generalized Petersen Graphs. The proof subsequently pieces together Hamilton paths in each copy of the generalized Petersen graphs to create a Hamilton cycle in the original Cayley graph. For this proof, it was helpful to know that the generalized Petersen graph obtained was Hamilton-connected, so that one would not have to delve into the details of how Hamilton paths arise in these subgraphs.

Alspach [1] determined precisely which generalized Petersen graphs are Hamiltonian. It is natural to ask which are Hamilton-connected. It is the purpose of this thesis to analyze a partial result along these lines and to suggest a method for extending the partial result. Before we do that, we will need some definitions.

For positive integers n and k, the *generalized Petersen graph* $GP(n,k)$ (introduced by Watkins [7]) has vertex set $\{i|i \in \mathbf{Z}_n\} \cup \{i'|i \in \mathbf{Z}_n\}$ and three types of edges:

Spoke Edges: $\langle i, i' \rangle$.

Rim Edges: $\langle i, i+1 \rangle$.

Hub Edges: $\langle i', (i+k)' \rangle$.

For $i \in \mathbf{Z}_n$, i is a *rim* vertex while i' is the adjacent *hub* vertex. All arithmetic is modulo n (in $\mathbf{Z}_n$.)

If $2k \not\equiv 0 \pmod n$, then $GP(n,k)$ is cubic. The Petersen graph is $GP(5,2)$.

Now we introduce some basic isomorphisms of the generalized Petersen graphs for later reference. Let $T : \mathbf{Z}_n \to \mathbf{Z}_n$ be the translation function defined by $T(i) = i+1$. Extended in the obvious way to the vertices and edges of the generalized Petersen graph $GP(n,k)$, T is an automorphism of $GP(n,k)$. Similarly, let $R : \mathbf{Z}_n \to \mathbf{Z}_n$ be the reflection defined by $R(i) = n - i$. As with T, R extends to an automorphism of $GP(n,k)$ and gives an isomorphism between $GP(n,k)$ and $GP(n, n-k)$. Thus we may assume $k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Let k^{-1} be the multiplicative inverse of $k \pmod n$, if it exists. For $i \in \mathbf{Z}_n$, let $j = ik^{-1}$. Then $j+1 = (i+k)k^{-1}$ and $(i+1)k^{-1} = j + k^{-1}$. Thus, $F : \mathbf{Z}_n \to \mathbf{Z}_n$ defined by $F(i) = ik^{-1}$ takes the rim edge $\langle i, i+1 \rangle$ of $GP(n,k)$ to the hub edge $\langle j, j+k^{-1} \rangle$ of $GP(n,k^{-1})$ and the hub edge $\langle i, i+k \rangle$ of $GP(n,k)$ to the rim edge $\langle j, j+1 \rangle$ of $GP(n,k^{-1})$. Hence, F is an isomorphism between $GP(n,k)$ and $GP(n,k^{-1})$.

1.1 Hamilton Paths

A u–v *Hamilton path* is a path which contains all the vertices of the graph and has u and v as ends. We will say that the Hamilton path *joins u and v*.

A graph is *Hamilton-connected* if it has a u-v Hamilton path for all pairs of vertices u and v. Similarly a bipartite graph is *Hamilton-laceable* if it has a u-v Hamilton path for all pairs of vertices u and v, where u belongs to one set of the bipartition, and v to the other.

Now $GP(n,k)$ is bipartite precisely when n is even and k is odd.

We make the following conjecture:

Conjecture 1 *For $n > 3k$ and $k > 2$:*

1. *When n is even and k is odd, $GP(n,k)$ is Hamilton-laceable.*

2. *For all other combinations of parities for n and k, $GP(n,k)$ is Hamilton-connected.*

Some examples demonstrate why we believe this to be true:

- $GP(10,3)$ is Hamilton-laceable and $GP(13,4)$ is Hamilton-connected.

- $GP(11,3)$ and $GP(14,4)$ are Hamilton-connected.

- In $GP(8,2)$ there is no Hamilton path from $0'$ to $4'$, but $GP(12,3)$ is Hamilton-laceable and $GP(16,4)$ is Hamilton-connected.

- In $GP(6,3)$ there is no Hamilton path from $0'$ to $1'$, and in $GP(12,4)$ there is no path from 0 to 2, but $GP(12,3)$ is Hamilton-laceable and $GP(16,4)$ is Hamilton-connected.

This conjecture naturally raise the question: what about $n = 3k$? Well, $GP(3,1)$, $GP(9,3)$ and $GP(15,5)$ are Hamilton-connected, but $GP(6,2)$ and $GP(12,4)$ are not. Specifically, $GP(6,2)$ is missing a 0–2 and a 0–4 Hamilton path, and $GP(12,4)$ is missing a 0–2, 0-4, 0–6, 0-8 and a 0-10 Hamilton path. In this thesis, we prove the conjecture when $n = 3k + 1$.

The existence of the translation automorphism T of $GP(n,k)$ implies that, for a given $w \in \mathbf{Z}_n$, a u–v Hamilton path for one pair of rim vertices $\{u, v\}$ satisfying $v - u = w$ yields a Hamilton path having any other pair of rim vertices as endpoints whenever that pair's labels differ by the same amount w. For this reason, we define the *spread* $\delta(u, v)$ of the rim vertices $u, v \in \mathbf{Z}_n$ as $v - u \pmod n$. We similarly define the spread for other pairs of vertices as: $\delta(u, v') = \delta(u', v') = v - u$. We are now able to say that the *spread of a path* is the spread of its ends.

Also, applying the reflection automorphism R we determine that it is enough to find one u–v Hamilton path for each value of $\delta(u, v)$ ranging from 1 to $\lfloor n/2 \rfloor$ to show that such paths exist for all pairs of rim vertices. A similar argument applies to the bipartite case, when only odd values of $\delta(u, v)$ need to be considered.

Similarly, only $\lfloor n/2 \rfloor$ pairs of vertices among hub vertex pairs and another $\lfloor n/2 \rfloor + 1$ of rim-hub vertex pairs need to be considered to demonstrate Hamilton-connectedness. For a bipartite graph $GP(n,k)$ we need only consider even values (including 0) up to the same upper limit of the spread for rim-hub pairs and odd values in the same range for hub-hub pairs to show that the graph is Hamilton-connected.

Chapter 2 introduces rotors. We use these rotors to prove that if k is even, then every pair of rim vertices in $GP(3k+1, k)$ are joined by a Hamilton path and any rim vertex and any hub are joined by a Hamilton path. Chapter 3 introduces two new rotors to deal with the hub to hub Hamilton paths. Finally, in Chapter 4, we show that, when k is odd, $GP(3k+1, k)$ is Hamilton-laceable.

Chapter 2

Rotors

In this chapter we prepare the groundwork for the proof of Conjecture 1 in the case $n = 3k+1$ and prove that if k is even, then, for every pair of vertices in $GP(3k+1, k)$ at least one of which is a rim vertex, there is a Hamilton path joining the pair. The proof is by induction on k. The inductive step consists of an operation which we call *inserting a rotor*.

First we stipulate some preconditions necessary for the insertion of a rotor. We observe that these preconditions are identical for the two different rotors. Next, we show that the preconditions for the insertion of either rotor are preserved. That is to say, that we can then again insert either rotor. Insertion of a rotor provides the inductive step which takes us from a Hamilton path in $GP(3k+1, k)$ to a Hamilton path in $GP(3(k+2)+1, k+2)$. In order to reach all paths we will add the operation of *reflection*.

2.1 Alternative Point of View

In the introduction, we observed that:

$$GP(n, n-k) \cong GP(n, k) \cong GP(n, k^{-1}),$$

where $kk^{-1} \equiv 1 \pmod{n}$. Now, $k^{-1} \equiv -3 \pmod{3k+1}$. Thus, multiplying the number part of each label by -3, interchanging rim and hub vertices, and applying the reflection isomorphism, we obtain:

$$GP(3k+1, k) \cong GP(3k+1, -3) \cong GP(3k+1, 3).$$

The following theorem has been proved by Liu [2] in his thesis:

Theorem 2 *1. $GP(n, 3)$ is Hamilton-connected if and only if n is odd and $n \neq 5$.*

2. $GP(n, 3)$ is Hamilton-laceable if and only if n is even and $n \neq 6$.

Thus our problem has been solved. However, we feel that our way of solving the problem introduces a new method, which is worthwhile for its own sake. In particular, we are using an induction which changes n and k simultaneously. We believe that has not been done previously.

2.2 Inserting Rotors

Here we formally introduce the notion of *inserting a rotor*. In this chapter, two rotors will be introduced, which we call an A-rotor and a B-rotor. First we present the A-rotor, but before we do that we need some notation:

Definition 3 *For* $a,\ b \in \mathbf{Z}_n$, *the* interval $[a, b]$ *is the set of vertices* $\{a, a+1, a+2, \ldots, b-2, b-1, b\} \cup \{a', (a+1)', (a+2)', \ldots, (b-2)', (b-1)', b'\}$ *of* $GP(n, k)$. *A hub edge* $\langle u', (u+k)' \rangle$ crosses *an interval* $[a, b]$ *if* $a, b \in [u, u+k]$. *Similarly, a rim edge* $\langle u, u+1 \rangle$ crosses *the interval* $[u, u+1]$.

Now we state what it means for a Hamilton path to be *rotor-ready*.

Definition 4 *A Hamilton path* P *in* $GP(3k+1, k)$ *is* rotor-ready at i *if the following conditions hold:*

(a) P *includes the edge* $\langle i, i' \rangle$;

(b) P *does not include any of*

$$\langle i, i+1 \rangle, \langle i', (i-k)' \rangle, \langle i+k, i+k+1 \rangle, \textit{and } \langle i-k, i-k-1 \rangle; \ \textit{and}$$

(c) $i - k$ *is an end vertex of* P.

Consider the following injective function $f_{i,k}$:

Definition 5 *For* $i \in \mathbf{Z}_{3k+1}$:

$$\begin{array}{rcll} f_{i,k} : \mathbf{Z}_{3k+1} & \rightarrow & \mathbf{Z}_{3(k+2)+1} \textit{ is defined by:} & \\ v & \mapsto & v, & \text{for } v \in [i-k, i]; \\ v & \mapsto & v+2, & \text{for } v \in [i+1, i+k]; \\ v & \mapsto & v+4, & \text{for } v \in [i+k+1, i-k-1]. \end{array}$$

The domain of $f_{i,k}$ *is extended to the vertices and edges of* $GP(3k+1, k)$ *by: for* $u \in \mathbf{Z}_{3k+1}$, $f_{i,k}(u') = (f_{i,k}(u))'$, *and, for* $u, v \in V(GP(3k+1, k))$, $f_{i,k}(\langle u, v \rangle) = \langle f_{i,k}(u), f_{i,k}(v) \rangle$. *The intervals* $[i, i+1]$, $[i+k, i+k+1]$, *and* $[i-k-1, i-k]$ *are* gaps *of* $f_{i,k}$.

Since i and k will usually be fixed, we will sometimes write f for $f_{i,k}$. Notice that the numbers of rim vertices in the three intervals of the domain of $f_{i,k}$ are $k+1$, k, and k, respectively.

First, we prove the following elementary fact.

Lemma 6 *Let k be a positive integer, and let $i \in \mathbf{Z}_{3k+1}$. Let $f_{i,k}$ be as in Definition 5.*
(a) If $j \in \mathbf{Z}_{3k+1}$, and $j \notin \{i, i+k, i-k-1\}$ then $f_{i,k}(\langle j, j+1 \rangle)$ is an edge of $GP(3(k+2)+1, k+2)$.
(b) If $j \in \mathbf{Z}_{3k+1}$, then $f_{i,k}(\langle j, j' \rangle)$ is an edge of $GP(3(k+2)+1, k+2)$.
(c) If $j \in \mathbf{Z}_{3k+1}$, and $j \neq i-k$, then $f_{i,k}(\langle j', (j+k)' \rangle)$ is an edge of $GP(3(k+2)+1, k+2)$.

Proof: Let $H = GP(3(k+2)+1, k+2)$ and let $I_1 = [i-k, i]$, $I_2 = [i+1, i+k]$, and $I_3 = [i+k+1, i-k-1]$. Fix $r \in \{1, 2, 3\}$ and let $j \in I_r$.
(a) Since $j \notin \{i, i+k, i-k-1\}$, we have $j+1 \in I_r$. So, if $f_{i,k}(j) = t$, then $f_{i,k}(j+1) = t+1$. Therefore $f_{i,k}(\langle j, j+1 \rangle)$ is a rim edge of H.
(b) This is obviously so.
(c) Now $j \neq i-k$ and $j \in I_r$ for some $r \in \{1, 2, 3\}$ imply that $j+k \in I_s$ with $s \equiv r+1 \pmod 3$. Let $f(j') = s'$ and let $f((j+k)') = t'$. If $r = 1$ or $r = 2$, then $t = s+k+2$, so $\langle s', t' \rangle$ is an edge of $GP(3(k+2)+1, k+2)$, as required. If $r = 3$, then $s = j+4$ and $t \equiv j+k \pmod{3k+1}$. That is, $t = j+k-(3k+1) = j-2k-1$. Since $s+k+2 \equiv j+4+k+2 \equiv j+k+6 \equiv j+k+6-(3(k+2)+1) \equiv j-2k-1 \pmod{3(k+2)+1}$, we see that $t \equiv s+k+2 \pmod{3(k+2)+1}$, and again $\langle s', t' \rangle$ is in $GP(3(k+2)+1, k+2)$ as required. ■

This leads us to the following:

Proposition 7 *If the Hamilton path P is rotor-ready at i, then $f_{i,k}(P)$ is a path of length $3k$ in $GP(3(k+2)+1, k+2)$.*

Proof: Let $H = GP(3(k+2)+1, k+2)$. Lemma 6 above shows that

$$\langle u, v\rangle \in E(P) \Longrightarrow \langle f(u), f(v)\rangle \in E(H).$$

Observe that $f_{i,k}$ is a one-to-one function in each of the intervals, and the intervals are disjoint, as are their images. Extended, it is a one-to-one function on the vertices and edges of $GP(3k+1, k)$. In particular, if a vertex of P has degree one in the subgraph P of $GP(3k+1, k)$, then it has degree one in the subgraph $f(P)$ of $GP(3(k+2)+1, k+2)$. A similar statement applies if the vertex has degree two in P. Thus $f(P)$ has two vertices of degree one and $3k-1$ vertices of degree two, just like P. Since it has the same number of edges as P, we conclude that it must be a path of the same length as P, namely $3k$. ■

2.2.1 Rotor A

We are now ready to introduce our first rotor.

Definition 8 *An* $A(i, k+2)$-rotor *is the path*

$$i, i+1, i+2, (i+2)', (i+2+k+2)', i+2+k+2, i+2+k+1,$$
$$(i+2+k+1)', (i+1)', (i-k-1)', i-k-1, i-k-2, (i-k-2)', i'.$$

We will often just say A to indicate the $A(i, k+2)$-rotor when the values of i and k are clear from the context. This definition leads immediately to the following:

Proposition 9 *The union* $(f_{i,k}(P) - \langle i, i' \rangle) \cup A(i, k+2)$ *is a Hamilton path in* H *whose ends are the ends of* $f_{i,k}(P)$.

Proof: Let $Q = (f_{i,k}(P) - \langle i, i' \rangle) \cup A(i, k+2)$. Note that $V(A(i, k+2)) \cap V(f_{i,k}(P)) = \{i, i'\}$ and $E(A(i, k+2)) \cap E(f_{i,k}(P)) = \emptyset$. Since $f_{i,k}(P)$ is a path containing the edge $\langle i, i' \rangle$, $f_{i,k} - \langle i, i' \rangle$ is the union of two paths, one having $i = f_{i,k}(i)$ as an end and the other having $i' = f_{i,k}(i')$ as an end. Thus Q is a path. Since $|V(Q)| = |V(f_{i,k}(P))| + 12$, $|V(Q)| = |V(H)|$, so Q is a Hamilton path in H whose ends are the ends of $f_{i,k}(P)$. ■

Next we show that we can repeat the process of inserting rotor A.

Proposition 10 *If* P *is rotor-ready at* i*, then the path obtained from* P *by the insertion of rotor* $A(i, k+2)$ *is rotor-ready at* $i+2$.

Proof: Let Q be the path obtained from P by the insertion of the rotor $A(i, k+2)$. We prove this for each of the excluded edges first.

1. $\langle i+2, i+3 \rangle \notin E(Q)$:

$$\left.\begin{array}{l} \langle i+1, i+2 \rangle \in E(A) \\ \langle i+2, (i+2)' \rangle \in E(A) \end{array}\right\} \text{ imply } \langle i+2, i+3 \rangle \notin E(Q)$$

2. $\langle (i+2)', (i-k)' \rangle \notin E(Q)$:

$$\left.\begin{array}{l} \langle i+2, (i+2)' \rangle \in E(A) \\ \langle (i+2)', (i+2+k+2)' \rangle \in E(A) \end{array}\right\} \text{ imply } \langle (i+2)', (i-k)' \rangle \notin E(Q).$$

3. $\langle i+2+k+2, i+2+k+3 \rangle \notin E(Q)$:

$$\left.\begin{array}{l} \langle i+k+4, i+k+3 \rangle \in E(A) \\ \langle i+k+4, (i+k+4)' \rangle \in E(A) \end{array}\right\} \text{ imply } \langle i+k+4, i+k+5 \rangle \notin E(Q).$$

4. $\langle i-k, i-k-1\rangle \notin E(Q)$:

$$\left.\begin{array}{l}\langle (i-k-1)', i-k-1\rangle \in E(A)\\ \langle i-k-1, i-k-2\rangle \in E(A)\end{array}\right\} \text{ imply } \langle i-k, i-k-1\rangle \notin E(Q).$$

Finally, $\langle i+2, (i+2)'\rangle \in E(A)$, and $i-k$ is an end of Q since $i-k$ is an end of P and $f_{i,k}(i-k) = i-k$. ■

Now we have all the facts we need about rotor A.

2.2.2 Rotor B

Here we introduce our second rotor.

Definition 11 *A* $B(i, k+2)$-rotor *in* $GP(3(k+2)+1, k+2)$ *is the path*

$$i-k, i-k-1, (i-k-1)', (i+1)', i+1, i+2, (i+2)', (i+2+k+2)',$$
$$i+2+k+2, i+1+k+2, (i+k+3)', (i-k-2)', i-k-2.$$

When the values of i and k are clear from the context, we will simply write rotor B. This definition, similar to the case of the $A(i, k+2)$-rotor leads immediately to the following:

Proposition 12 *The union* $B \cup f_{i,k}(P)$ *is a Hamilton path in* $GP(3(k+2)+1, k+2)$ *having ends* $i-k-2$ *and the end of* $f_{i,k}(P)$ *other than* $i-k$.

Proof: Let $H = GP(3(k+2)+1, k+2)$ and let $f_{i,k} = f$. Let $S = B \cup f(P)$. Firstly, $V(B) \cap f(V(P)) = \{i-k\}$. Secondly, the edge of B incident with the vertex $i-k$ does not duplicate any edge of $f(P)$ since $\langle i-k, i-k-1\rangle$ is an edge of B but $[i-k, i-k-1]$ is a gap of f and thus $\langle i-k, i-k-1\rangle$ does not lie in the domain of f or the image, $f(P)$ in H. Furthermore B has 12 edges, and S is therefore a Hamilton path in H since $|E(S)| = |E(P)| + 12 = |V(G)| - 1 + 12 = |V(H)| - 1$. ■

Next, we show that we may again insert either rotor in the path S in H.

Proposition 13 *The path S in $GP(3(k+2)+1, k+2)$, obtained from the path P in $G = GP(3k+1, k)$ by inserting a $B(i, k+2)$-rotor, is rotor-ready at i.*

Proof: Let $H = GP(3(k+2)+1, k+2)$ First of all, the end $i-k$ of P becomes the end $i-k-2$ of S. Secondly, $f(\langle i, i' \rangle) = \langle i, i' \rangle$ since this edge lies in I_1. Thirdly, the forbidden edges come about as follows:

1. $\langle i, i+1 \rangle \notin E(S)$:

$$\left.\begin{array}{l} \langle i+1, (i+1)' \rangle \in E(B) \\ \langle i+1, i+2 \rangle \in E(B) \end{array}\right\} \text{ imply } \langle i, i+1 \rangle \notin E(S).$$

2. $\langle i', (i-k-2)' \rangle \notin E(S)$:

$$\left.\begin{array}{l} \langle (i+k+3)', (i-k-2)' \rangle \in E(B) \\ \langle (i-k-2)', i-k-2 \rangle \in E(B) \end{array}\right\} \text{ imply } \langle i', (i-k-2)' \rangle \notin E(S).$$

3. $\langle i+k+2, i+k+3 \rangle \notin E(S)$:

$$\left.\begin{array}{l} \langle i+k+4, i+k+3 \rangle \in E(B) \\ \langle i+k+3, (i+k+3)' \rangle \in E(B) \end{array}\right\} \text{ imply } \langle i+k+2, i+k+3 \rangle \notin E(S).$$

4. $\langle i-k-2, i-k-3 \rangle \notin E(S)$.

$$\left.\begin{array}{l} i-k-2 \text{ is an end of } S, \\ \langle i-k-2, (i-k-2)' \rangle \in E(B) \end{array}\right\} \text{ imply } \langle i-k-2, i-k-3 \rangle \notin E(S).$$

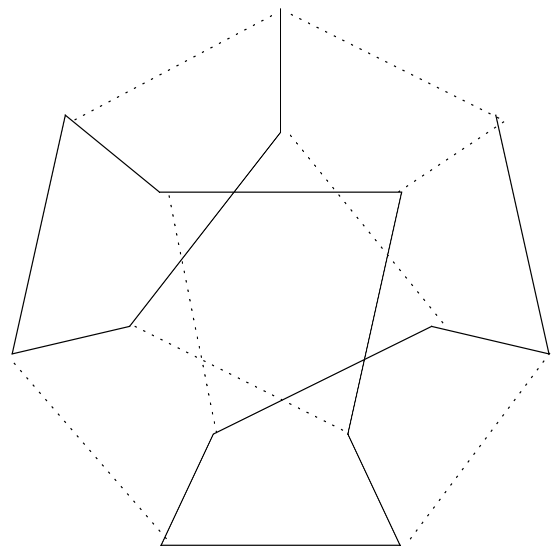

Figure 2.1: A path in $GP(7,2)$ that is rotor-ready at $i = 2$.

This concludes the proof of Proposition 13. ■

Now we know everything we need to about the B-rotor.

We illustrate the process of inserting rotors into a Hamilton path in Figures 2.1, 2.2, 2.3, and 2.4.

2.3 The Inductive Step

In Section 1.1, it was observed that we need only find Hamilton paths for $\lfloor n/2 \rfloor$ pairs of rim vertices to show that such paths exist for all pairs of rim vertices. What does this mean in terms of $GP(3k+1, k)$ for even k? Once again, let u and v be end vertices of the path. Then we need only consider the following range for $v - u$:

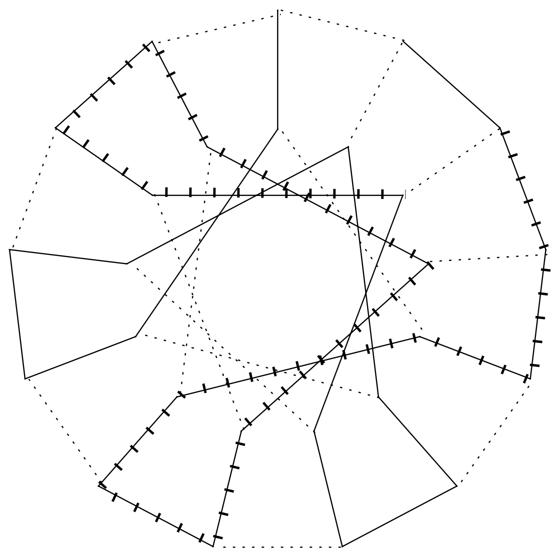

Figure 2.2: Inserting an A-rotor into the path of Figure 2.1.

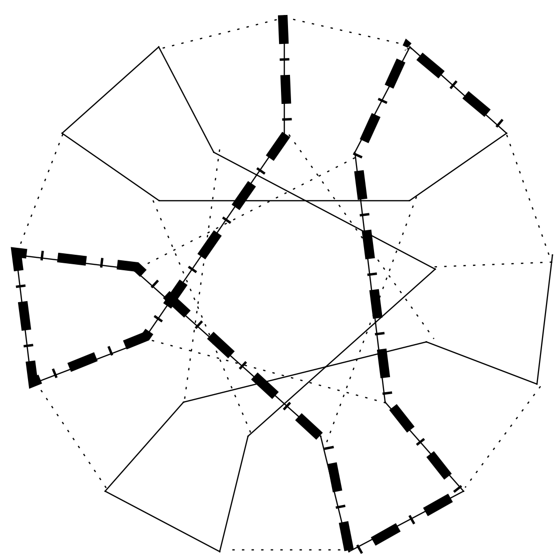

Figure 2.3: Inserting a B-rotor into the path of Figure 2.1.

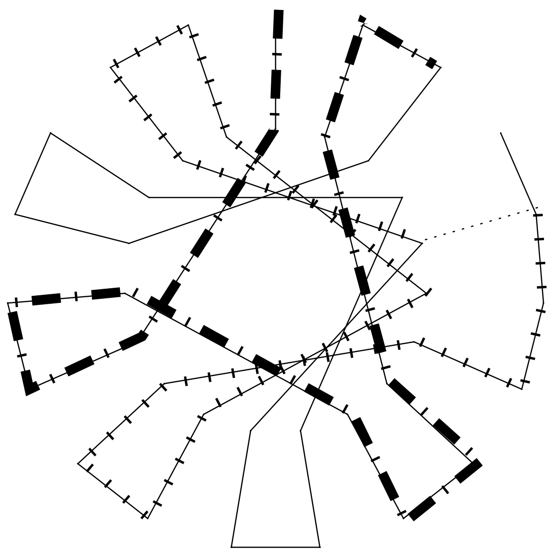

Figure 2.4: Inserting both rotors into the path of Figure 2.1.

$$1 \le v - u \le \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{3k+1}{2} \right\rfloor = \frac{3k}{2}.$$

The last equality holds since k is even.

For each value of $v - u$ between 1 and $\left\lfloor \frac{n}{2} \right\rfloor$, we will show the existence of a u–v Hamilton path for one pair $u, v \in \mathbf{Z}_n$. As already mentioned, this will suffice to prove the existence of a u–v Hamilton path for all pairs $u, v \in \mathbf{Z}_n$. Recall that the quantity $\delta(u, v) = v - u$ is the *spread* of the pair u, v.

Similarly, later in this work, we show the existence of u–v' and u'–v' Hamilton paths for nearly the same range of the spread. This will then prove that $GP(3k + 1, k)$ is Hamilton-connected for all $k \ge 2$.

We start with u–v paths.

Let $G = GP(3k+1, k)$ and $H = GP(3(k+2)+1, k+2)$ As the domain of

$f = f_{i,k}$, $V(G)$ is split into three intervals. For the inductive step, we apply f before inserting a rotor. For the first part of this proof, we will show that a vertex of $V(G)$ is mapped to an *equivalent* vertex of $V(H)$. By an equivalent vertex we mean a vertex where the action of f is identical in the next step.

We defined the action of f on its three intervals in Definition (5). However, this is not enough. We have yet to identify i, u, and v for the next inductive step. These may not correspond exactly to their images under f, depending upon which rotor we choose to insert.

For the A rotor, we replace i by $i+2$ for the next inductive step. We will denote the replacement of vertices by their equivalents for the next inductive step as the action of a function, $\mathcal{A}$, in the case of the A rotor, or $\mathcal{B}$, in the case of the B rotor. Thus $\mathcal{A}$ maps intervals in $V(G)$ to equivalent intervals in $V(H)$ as follows:

$$\begin{aligned} \mathcal{A}: \quad & [i-k, i] && \mapsto && [i-k, i+2] \\ & [i+1, i+k] && \mapsto && [i+3, i+k+4] \\ & [i+k+1, i-k-1] && \mapsto && [i+k+5, i-k-1] \end{aligned}$$

Comparing these to the actions of f on the same intervals of $V(G)$, we see that the equivalent intervals under $\mathcal{A}$ in $V(H)$ contain the images of the same intervals in $V(G)$ under the action of f. So we see that f maps a vertex in a specified interval of its domain to a vertex in an equivalent interval in its co-domain.

Furthermore $\mathcal{A}$ acts as follows on i, u, and v:

$$\begin{aligned} \mathcal{A}: \quad & i && \mapsto && i+2 \\ & u && \mapsto && f(u) \\ & v && \mapsto && f(v) \end{aligned}$$

So that these three are mapped to equivalent vertices. Specifically, i is the right end of the interval $[i-k, i]$ and $\mathcal{A}(i) = i+2$ is the right end of the equivalent interval $\mathcal{A}([i-k, i]) = [i-k, i+2]$.

The above shows that the ends of the path, u and v, are mapped via $\mathcal{A}$ into an equivalent interval no matter how often $\mathcal{A}$ is applied. Thus the spread of the path is changed by the same amount each time that an A-rotor is inserted. More specifically, the change in spread of the ends of the path is completely determined by the initial intervals of f the ends of the original path fall in. As a matter of fact, if $v \in I_r$, then $\mathcal{A}$ changes the spread by $2(r-1)$.

For the B-rotor, intervals are mapped as follows:

$$\begin{aligned} \mathcal{B}: \quad & [i-k, i] && \mapsto && [i-(k+2), i]; \\ & [i+1, i+k] && \mapsto && [i+1, i+(k+2)]; \\ & [i+k+1, i-k-1] && \mapsto && [i+(k+2)+1, i-(k+2)-1]. \end{aligned}$$

One end of the original Hamilton path in $GP(3k+1, k)$, $u = i-k$, and

$$\begin{aligned} \mathcal{B}: \quad i & \mapsto i; \\ u & \mapsto u-2; \\ v & \mapsto f(v). \end{aligned}$$

So we see that, as was the case for $\mathcal{A}$, the three vertices are mapped into equivalent vertices.

As with the A-rotor, the initial intervals of f the ends of the original path occupy are all that is needed to completely determine the effect of inserting a B-rotor on the spread of the ends of the path. In this case, if $v \in I_r$ then $\mathcal{B}$ changes the spread by $2r$.

Besides inserting rotors, we also include the operation of reflection in two different ways.

In the introduction, we defined the reflection $R(u) = n - u$ in $\mathbf{Z}_n$. Define R analogously for hub vertices and for edges. This changes the spread $v - u$ of the u–v path as follows:

$$R(v) - R(u) \equiv (n - v) - (n - u) \equiv -(v - u) \equiv n - (v - u) \pmod{n}.$$

Now we determine what R does to forbidden edges. We let $n - i = j$.

$$\begin{array}{rlcl} R: & \langle i, i+1\rangle & \mapsto & \langle j-1, j\rangle \\ & \langle (i-k)', i'\rangle & \mapsto & \langle j', (j+k)'\rangle \\ & \langle i+k, i+k+1\rangle & \mapsto & \langle j-k-1, j-k\rangle \\ & \langle i-k-1, i-k\rangle & \mapsto & \langle j+k, j+k+1\rangle \end{array}$$

The action of R on the intervals is as follows:

$$\begin{array}{rlcl} R: & [i-k, i] & \mapsto & [j, j+k] \\ & [i+1, i+k] & \mapsto & [j-k, j-1] \\ & [i+k+1, i-k-1] & \mapsto & [j+k+1, j-k-1] \end{array}$$

In general, we will refer to an interval I_r of f, and its reflected counterpart, $R(I_r)$, generically as I_r.

How does R affect the action of f? After reflecting with R, the graph is identical if we count the labels in reverse order. But we wish to maintain the same labelling of the graph, only to reflect the paths. Thus we first reflect, then apply f, then reflect back (R is idempotent). Thus, f becomes $R \circ f \circ R$, with an added twist. The function f takes us from its domain to its co-domain, where n has increased by 6. Namely, $R \circ f \circ R(w) = (n+6) - f(n-w)$.

We list here the original and reflected action of f on the three intervals:

$$\begin{array}{rcc} & f & R \circ f \circ R \\ I_1: & w \mapsto w & w \mapsto w+6 \\ I_2: & w \mapsto w+2 & w \mapsto w+4 \\ I_3: & w \mapsto w+4 & w \mapsto w+2 \end{array}$$

$R(B)$ is inserted in the opposite sense to B. Thus $R \circ \mathcal{B} : u \mapsto R \circ f \circ R(u) + 2$. The reflected end is $u = j + k$.

The spread is also affected by R. The change in the spread is now replaced by six minus the change in spread for the unreflected case. Thus, if $v \in I_r$ then $R \circ \mathcal{A}$ changes the spread by $2(4 - r)$ and $R \circ \mathcal{B}$ changes the spread by $2(3 - r)$.

2.4 Rim to Rim Paths

In this section, we construct Hamilton paths for all rim-rim pairs in $GP(3k+1, k)$ for even k. We split our constructions into several cases depending upon the spread δ. We start the construction with given paths in $GP(7, 2)$ and construct paths from these for $GP(3k + 1, k)$ with even k greater than 2.

We start with a Hamilton path in $GP(7, 2)$ and no rotors. The addition of a rotor inctreases k by 2. Thus, letting a denote the number of A-rotors inserted, and letting b denote the number of B-rotors inserted, the total numbers of rotors inserted is $a + b = \frac{k-2}{2} = \frac{k}{2} - 1$.

Case 1: δ is odd, and $1 \leq \delta \leq k - 1$. This case was precisely illustrated in Figures 2.1, 2.2, 2.3, and 2.4. In $GP(7, 2)$, P_1 is the following 0–1 Hamilton path:

$$0, 0', 5', 5, 6, 6', 1', 3', 3, 4, 4', 2', 2, 1.$$

The initial insertion point for rotors is $i = 2$. The forbidden edges can easily be verified:

$$\begin{aligned}
\langle i, i+1 \rangle &= \langle 2, 3 \rangle \notin E(P_1); \\
\langle (i-k)', i' \rangle &= \langle 0', 2' \rangle \notin E(P_1); \\
\langle i+k, i+k+1 \rangle &= \langle 4, 5 \rangle \notin E(P_1); \\
\langle i-k-1, i-k \rangle &= \langle 6, 0 \rangle \notin E(P_1).
\end{aligned}$$

Furthermore, $u = i - k = 0$ is an end of P_1. Then $v = 1$. Since $u, v \in [0, 2] = [i-k, i]$, $f(u) = u$ and $f(v) = v$.

Now the rotors act as follows:

$$\begin{aligned}
&\mathcal{A} : u \mapsto f(u) = u,\ v \mapsto f(v) = v,\ \delta(\mathcal{A}(u), \mathcal{A}(v)) = \delta(u, v); \\
&\mathcal{B} : u \mapsto u - 2,\ v \mapsto f(v) = v,\ \delta(\mathcal{B}(u), \mathcal{B}(v)) = \delta(u, v) + 2.
\end{aligned}$$

Setting $b = \frac{\delta-1}{2}$ and $a = \frac{k}{2} - 1 - b$, after inserting a A-rotors and b B-rotors, we get a Hamilton path with spread δ in $GP(3k+1, k)$.

Case 2: δ is even and $2 \leq \delta \leq k$. Let P_2 be the following 0–2 Hamilton path in $GP(7, 2)$:

$$0, 0', 2', 4', 4, 3, 3', 5', 5, 6, 6', 1', 1, 2.$$

Reflected rotors will initially be inserted at $j = 5$. Only reflected rotors will be inserted. The forbidden edges are:

$$\begin{aligned}
\langle j-1, j \rangle &= \langle 4, 5 \rangle \notin E(P_2); \\
\langle j', (j+k)' \rangle &= \langle 5', 0' \rangle \notin E(P_2);
\end{aligned}$$

$$\begin{aligned}\langle j-k-1, j-k\rangle &= \langle 2,3\rangle \notin E(P_2);\\ \langle j+k, j+k+1\rangle &= \langle 0,1\rangle \notin E(P_2).\end{aligned}$$

Furthermore $u = j + k = 0$ is an end of P_2, and $v = 2$ is the other end. Also, $u \in [5, 0] = I_1$ and $v \in [1, 2] = I_3$. The action of the reflected rotors on the ends of the path are as below:

$$R \circ \mathcal{A} : u \mapsto u+6,\ v \mapsto v+2,\ \delta(R \circ \mathcal{A}(u), R \circ \mathcal{A}(v)) = \delta(u, v) + 2;$$
$$R \circ \mathcal{B} : u \mapsto u+6+2,\ v \mapsto v+2,\ \delta(R \circ \mathcal{B}(u), R \circ \mathcal{B}(v)) = \delta(u, v).$$

Setting $a = \frac{\delta}{2} - 1$, and $b = \frac{k}{2} - 1 - a$, after inserting a $R(A)$ and b $R(B)$ rotors we obtain a Hamilton path with spread δ in $GP(3k+1, k)$.

Case 3: δ is an integer with $k+1 \le \delta \le 3k/2$. Let P_3 be the following 0–3 Hamilton path in $GP(7, 2)$:

$$0, 0', 2', 2, 1, 1', 3', 5', 5, 6, 6', 4', 4, 3.$$

The point of insertion is $j = 5$ and we use reflected rotors only. The forbidden edges are thus the same as for P_2 and may easily be checked. The ends of P_3 are $u = j + k = 0$ and $v = 3$. Once again $u \in [5, 0] = I_1$, but $v \in [3, 4] = I_2$. The action of the reflected rotors on the ends of P_3 is as follows:

$$R \circ \mathcal{A} : u \mapsto u+6,\ v \mapsto v+4,\ \delta(R \circ \mathcal{A}(u), R \circ \mathcal{A}(v)) = \delta(u, v) + 4;$$
$$R \circ \mathcal{B} : u \mapsto u+6+2,\ v \mapsto v+4,\ \delta(R \circ \mathcal{B}(u), R \circ \mathcal{B}(v)) = \delta(u, v) + 2.$$

After inserting a $R(A)$ rotors and b $R(B)$ rotors, the spread, $\delta = 3+2b+4a = -1 + 2(k-b)$, the latter since $a + b = \frac{k}{2} - 1$.

Sub-case 3(*i*): δ is odd. Setting $b = k - \frac{\delta+1}{2}$ and $a = \frac{k}{2} - 1 - b$, after inserting a $R(A)$-rotors and b $R(B)$-rotors we obtain a Hamilton path with spread δ in $GP(3k+1, k)$.

Sub-case 3(*ii*): δ is even. Setting $b = \frac{\delta-k}{2} - 1$, and $a = \frac{k}{2} - 1 - b$, after inserting a $R(A)$-rotors and b $R(B)$-rotors and reflecting the path so obtained we get a Hamilton path with spread δ in $GP(3k+1,k)$.

Summing up, for Case 3 the spread takes on one of the $\frac{k}{2}$ integers in the range from $k+1$ to $\frac{3k}{2}$. Finally, all the spreads of all the paths generated from P_1, P_2, and P_3 cover the range from 1 to $\frac{3k}{2}$, inclusively. As already stated, this shows that, for every pair of rim vertices, there is a Hamilton path having that pair as ends.

2.5 Rim to Hub Paths

In this section we show that there is a Hamilton path between all rim-hub pairs of vertices in $GP(3k+1,k)$ for even k. To do this, we will once again be using induction on k. We start the base cases by exhibiting four rotor-ready paths in $GP(7,2)$ and one such path in $GP(13,4)$. We then insert the same two rotors as for the rim-rim pairs of vertices.

We must now cover spreads, $v - u = \delta$, over all integers from 0 to $\frac{3k}{2}$, inclusively to obtain a u–v' Hamilton path for all possible pairs (u, v').

We accomplish our goal by doing a case analysis.

Case 1: δ is even and $0 \le \delta \le k-2$. Consider the following 0–0′ Hamilton path in $GP(7,2)$:

$$P_4 : 0, 1, 2, 2', 4', 4, 3, 3', 1', 6', 6, 5, 5', 0'.$$

The initial insertion point for both rotors is $i = 2$. Observe that P_4 in $GP(7,2)$ is rotor-ready. It satisfies all the same conditions as P_1 in Section 2.4 except that $v = 1$ is replaced by $v' = 0'$. So both u and v lie in I_1 and the effect on the spread of inserting each rotor is the same as for P_1 in Section 2.4. Setting $b = \delta/2$, and $a = \frac{k}{2} - 1 - b$, after inserting a A-rotors and b B-rotors, we obtain a rim-hub Hamilton path with spread δ in $GP(3k+1, k)$.

Case 2: $\delta = k - 1$. Consider the following 0–3′ Hamilton path in $GP(13, 4)$:

$$P_5 : 0, 0', 4', 4, 3, 2, 1, 1', 10', 10, 9, 9', 5', 5, 6, 6', 2', 11', 11, 12, 12', 8', 8, 7, 7', 3'.$$

The path P_5 is reflected-rotor-ready at $j = 9$. Insert a $R(A)$-rotors. The spread $\delta = 3 + 2a$ becomes $k - 1$.

Case 3: $\delta = k$. Consider the following 0–2′ Hamilton path in $GP(7, 2)$:

$$P_6 : 0, 0', 5', 5, 6, 6', 4', 4, 3, 3', 1', 1, 2, 2'.$$

Again, we insert no A-rotors and $\frac{k}{2} - 1$ B-rotors with the same initial insertion point, $i = 2$ and the same analysis as for P_1. The spread $\delta = 2 + 2b$, with $b = \frac{k}{2} - 1$, becomes k.

Case 4: $k + 1 \le \delta \le \frac{3k}{2}$. Consider the following 0–3′ Hamilton path in $GP(7, 2)$:

$$P_7 : 0, 0', 5', 5, 6, 6', 1', 1, 2, 2', 4', 4, 3, 3'.$$

Once again we insert either rotor initially at $i = 2$. However, $v = 3$ now falls in I_2, and $u = 0$ still lies in I_1. The analysis that shows that P_7 is rotor-ready is the same as for P_1. The action of the rotors on the ends of the path are as follows:

$$\mathcal{A} : u \mapsto u,\ v \mapsto v + 2,\ \delta(\mathcal{A}(u), \mathcal{A}(v)) = \delta(u, v) + 2;$$
$$\mathcal{B} : u \mapsto u - 2,\ v \mapsto v + 2,\ \delta(\mathcal{B}(u), \mathcal{B}(v)) = \delta(u, v) + 4.$$

Thus $\delta = 1 + k + 2b$, with $0 \le b \le \frac{k}{2} - 1$. Now we consider even and odd values of the spread for this case separately:

Sub-case 4*(i)*: δ is odd. Setting $b = (\delta - k - 1)/2$ and $a = \frac{k}{2} - 1 - b$, after inserting a A-rotors and b B-rotors, we obtain a Hamilton path with spread δ in $GP(3k + 1, k)$.

Sub-case 4*(ii)*: δ is even. Setting $b = k - (\delta(u, v)/2)$, and $a = \frac{k}{2} - 1 - b$, after inserting a A-rotors and b B-rotors and reflecting the resultant path, we obtain a Hamilton path with spread δ in $GP(3k + 1, k)$.

Case 5: δ is odd and $1 \leq \delta \leq k - 3$. Consider the following 0–1′ Hamilton path in $GP(13, 4)$: Remembering that we start in $GP(13, 4)$, we have

$$P_8 : 0, 0', 4', 4, 3, 3', 7', 11', 11, 12, 12', 8', 8, 7, 6, 5, 5', 9', 9, 10, 10', 6', 2', 2, 1, 1'.$$

The path P_8 is reflected-rotor-ready at $j = 9$. Insert $a = (\delta - 1)/2$ R(A)-rotors and $b = \frac{k-4}{2} - a$ R(B)-rotors to obtain the desired δ.

This concludes the construction for rim to hub Hamilton paths.

Chapter 3

More Rotors

In the following sections, we will show the existence of hub-hub Hamilton paths for all such possible pairings.

3.1 Primed Rotors

Unfortunately, the rotors already presented with their reflections do not suffice to build up all the paths we need. For this reason, we introduce the primed rotors, each closely related to the unprimed version. First we define what it means for a Hamilton path P to be *primed-rotor-ready*.

Definition 14 *A Hamilton path P, in $GP(3k+1, k)$, is* primed-rotor-ready *at rim vertex $i \in \mathbf{Z}_n$ if:*

1. $\langle i', (i-k)' \rangle \in E(P)$.
2. $\langle i-1, i \rangle$ *and* $\langle i, i+1 \rangle \in E(P)$.
3. $\langle i-k, i-k-1 \rangle$ *and* $\langle i+k, i+k+1 \rangle \notin E(P)$.

4. The hub vertex i' is an end vertex of P.

Next, consider the following injective function $g_{i,k}$:

Definition 15 *For $i \in \mathbf{Z}_{3k+1}$:*

$$\begin{array}{rcll} g_{i,k} : \mathbf{Z}_{3k+1} & \to & \mathbf{Z}_{3(k+2)+1} & \\ v & \mapsto & v, & \text{for } v \in [i-k, i-1] = J_1 \\ v & \mapsto & v+2, & \text{for } v \in [i, i+k] = J_2 \\ v & \mapsto & v+4, & \text{for } v \in [i+k+1, i-k-1] = J_3. \end{array}$$

When it is clear what is meant, we will omit the subscripts on g. We extend the domain of g to all vertices of $G = GP(3k+1, k)$ and edges of P similarly to what we did with f. The main difference is now that the ends of $g(\langle i-1, i\rangle)$ would lie in two intervals, J_1 and J_2. We solve this problem by redefining g on this edge so that $g(\langle i-1, i\rangle) = \langle i-1, i\rangle$. This leads us immediately to the following:

Proposition 16 *If P is a Hamilton path that is primed-rotor-ready at i, then $g_{i,k}(P)$ has two components which are paths in $GP(3(k+2)+1, k+2)$, one having $(i+2)'$ as an end, the other having $g(w')$ as an end, where w' is the end of P other than i'. The other ends are i and $i+2$.*

The proof is similar to the proof of Proposition 7 and we do not give it here. Now we are ready to introduce our primed rotors.

3.1.1 Rotor A'

Here we introduce the first of our primed rotors.

Definition 17 *The $A'(i, k+2)$-rotor consists of the following two paths:*

$$i, i+1, i+2; \textit{and}$$

$$(i+2)', (i+k+4)', i+k+4, i+k+3, (i+k+3)', (i+1)', (i-k-1)',$$
$$i-k-1, i-k-2, (i-k-2)', i'.$$

We insert this rotor in $GP(3(k+2)+1, k+2)$ thereby connecting the components of $g(P)$ to form a Hamilton path. The following is easy to prove:

Proposition 18 *If P is primed-rotor-ready at i in $GP(3k+1, k)$, then, after inserting the $A'(i, k+2)$-rotor, the resultant path $g_{i,k}(P) \cup A(i, k+2)$ in $GP(3(k+2)+1, k+2)$ is primed-rotor-ready at vertex i.*

Defining the effect of inserting an A'-rotor as that of a function $\mathcal{A}'$, we obtain:

$$\begin{aligned} \mathcal{A}' : i &\mapsto i; \\ i' &\mapsto i'; \\ u &\mapsto g(u); \end{aligned}$$

where u is the other end of the path P other than i'. If $u \in J_r$ then $\mathcal{A}'$ changes the spread by $2(r-1)$.

3.1.2 Rotor B'

We now introduce the other primed rotor.

Definition 19 *The $B'(i, k+2)$-rotor is the following path:*

$$i, i', (i-k-2)', i-k-2, i-k-1, (i-k-1)', (i+k+4)', i+k+4,$$
$$i+k+3, (i+k+3)', (i+1)', i+1, i+2.$$

Once again, we insert this rotor in $GP(3(k+2), k+2)$ connecting the components of $g_{i,k}(P)$ into a Hamilton path. The following is immediately clear:

Proposition 20 *If P is primed-rotor-ready at i in $GP(3k+1, k)$, then, after inserting the $B'(i, k+2)$-rotor, the resultant path $g_{i,k}(P) \cup B(i, k+2)$ in $GP(3(k+2), k+2)$ is primed-rotor-ready at vertex $i+2$.*

Once more, defining the effect of inserting the rotor as that of the action of a function $\mathcal{B}'$, we obtain:

$$\begin{aligned} \mathcal{B}' : i &\mapsto i+2; \\ i' &\mapsto (i+2)'; \\ u &\mapsto g(u); \end{aligned}$$

where, once again u is the end of the path P other than i'. If $u \in J_r$, then $\mathcal{B}'$ changes the spread by $2r$.

Now we have all the facts we need about the primed rotors. These rotors are illustrated in Figures 3.1 and 3.2.

3.2 Hub to Hub Paths

In this section we show the existence of all Hamilton paths in $GP(3k+1, k)$ for even k starting and ending on hub vertices. To do this we construct paths having spreads δ in the range 1 to $\frac{3k}{2}$, inclusively. We accomplish our task by a case analysis.

Case 1: δ is odd and $1 \le \delta \le k-1$. This case is illustrated in Figures 3.1 and 3.2. Consider the following hub-hub Hamilton path in $GP(7, 2)$:

$$0', 2', 4', 4, 5, 5', 3', 3, 2, 1, 0, 6, 6', 1'.$$

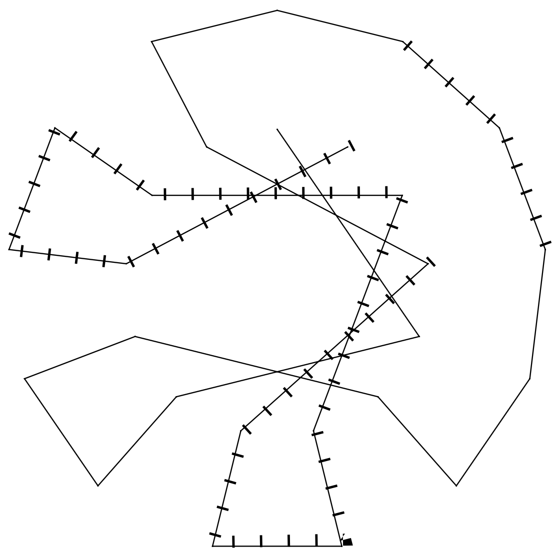

Figure 3.1: The A'-rotor.

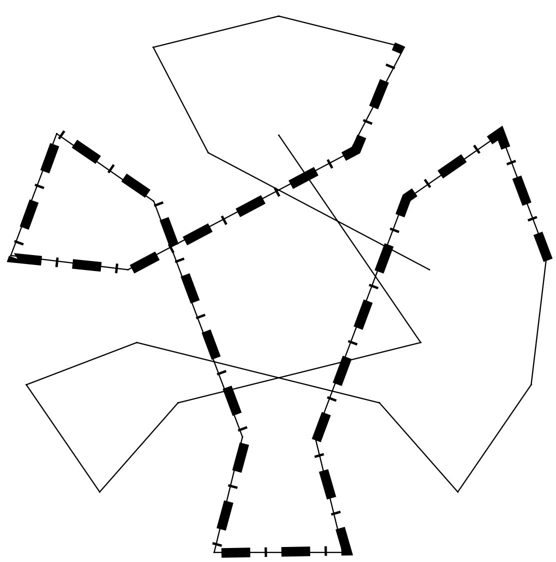

Figure 3.2: The B'-rotor.

This path is primed-rotor-ready at vertex 1. The same analysis as in the previous section then shows that adding an A' does not change the spread, and adding a B'-rotor adds two to the spread. The result follows.

Case 2: δ is even and $2 \leq \delta \leq k-2$. Consider the following hub-hub Hamilton path in $GP(13,4)$:

$$0', 9', 9, 10, 11, 11', 7', 3', 12', 12, 0, 1, 2, 3, 4, 4', 8', 8, 7, 6, 5, 5', 1', 10', 6', 2'.$$

The given path is primed-rotor-ready at 0. We see that the A'-rotor adds two to the spread, and that the B'-rotor leaves the spread unchanged. The result follows.

Case 3: $\delta = k$. We insert reflected A'-rotors starting at vertex 4 to the following $0'$–$4'$ Hamilton path in $GP(13,4)$:

$$0', 0, 1, 1', 5', 9', 9, 10, 10', 6', 6, 5, 4, 3, 2, 2', 11', 11, 12, 12', 3', 7', 7, 8, 8', 4'.$$

Inserting the reflected rotor adds two to the spread. The result follows.

Case 4: $\delta = k+2$. We insert reflected A'-rotors starting at vertex 0 to the following $0'$–$7'$ Hamilton path in $GP(13,4)$:

$$0', 4', 4, 5, 6, 7, 8, 8', 12', 3', 3, 2, 2', 6', 10', 10, 9, 9', 5', 1', 1, 0, 12, 11, 11', 7'.$$

This adds four to the spread. Now we reflect the graph to obtain the desired spread.

Case 5(*i*): δ is odd and $k+1 \leq \delta \leq 3k/2$. Consider the following $0'$–$5'$ Hamilton path in $GP(13,4)$

$$0', 0, 12, 11, 10, 10', 1', 1, 2, 3, 3', 12', 8', 4', 4, 5, 6, 6', 2', 11', 7', 7, 8, 9, 9', 5'.$$

The path is reflected-primed-rotor-ready at 5. Thus we add reflected A'-rotors to increase the spread by four and reflected B'- rotors to increase the spread by two. Setting $b = k - (\delta - 1)/2$ and $a = \frac{k}{2} - 2 - b$, after inserting a A'-rotors and b B'-rotors, we obtain a Hamilton path with spread δ.

Case 5(*ii*): δ is even and $k + 4 \leq \delta \leq 3k/2$. This case occurs only if $k \geq 8$. Setting $b = (\delta - k)/2$ and $a = \frac{k}{2} - 2 - b$, after inserting a A'-rotors and b B'-rotors into the same path as Case $5(i)$, and reflecting the resultant path, we obtain a Hamilton path with spread δ in $GP(3k + 1, k)$.

Chapter 4

From $GP(3k+1, k)$ to $GP(3(k+2)+1, k+2)$ for odd k

This chapter is much like the last one, except here we consider odd values of k only. When k is odd, $GP(3k+1, k)$ is bipartite with equal-sized partite sets. For this reason we need only consider Hamilton paths starting in one partite set and ending in the other.

It turns out that the method of inserting rotors works for these cases as well.

4.1 Rim to Rim Hamilton Paths

The spread will be an odd integer between 1 and $(3k+1)/2$. We consider two cases:

Case 1: The spread is an odd integer between 1 and $k-2$. Consider the following 0–1 Hamilton path in $GP(10, 3)$:

$$0, 0', 7', 7, 8, 9, 9', 2', 5', 8', 1', 4', 4, 5, 6, 6', 3', 3, 2, 1.$$

The initial insertion point is $i = 3$. Inserting an A-rotor doesn't change the spread, and inserting a B-rotor adds two to the spread. Inserting a combination of $(k - 3)/2$ A and B rotors yields the desired spread.

Case 2: The spread is an odd integer between k and $(3k+1)/2$ with k being at least 5. Consider the following 0–3 Hamilton path in $GP(10, 3)$:

$$0, 0', 7', 4', 4, 5, 5', 8', 1', 1, 2, 2', 9', 9, 8, 7, 6, 6', 3', 3.$$

The initial insertion point for reflected rotors is $j = 6$. The A-rotor is simply the original one, reflected. However, the B- rotor, shown in Figure 4.1, is a modified one. If $j+k+1$ is an end of the original path, the reflected modified B-rotor is the following path in $GP(3(k + 2) + 1, k + 2)$:

$$\begin{gathered} j + k + 3, j + k + 2, (j + k + 2)', (j - k - 3)', j - k - 3, j - k - 4, \\ (j - k - 4)', (j - 2)', j - 2, j - 1, (j - 1)', (j + k + 1)', j + k + 1. \end{gathered}$$

The next insertion point for either reflected rotor is $j - 2$. Now, inserting the reflected A-rotor increases the spread by two, and inserting the reflected modified B-rotor increases the spread by four. So, insert a combination of $(k - 3)/2$ reflected rotors, of which up to $\lfloor (k + 1)/4 \rfloor$ may be reflected modified B-rotors, to obtain the desired spread.

In this way, we may obtain any desired rim-rim Hamilton path with odd spread.

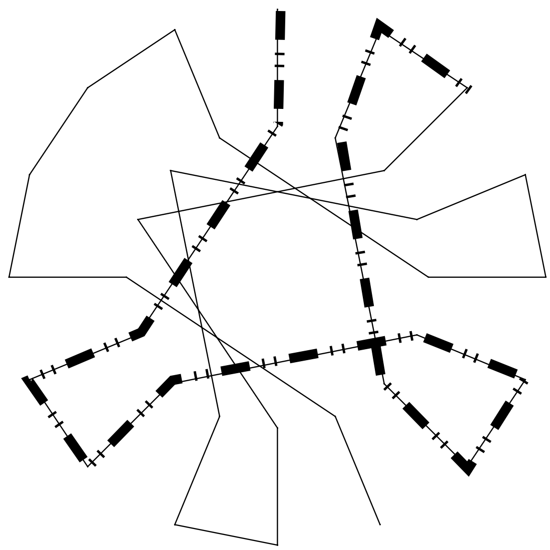

Figure 4.1: The reflected modified B-rotor.

4.2 Rim to Hub Hamilton Paths

For odd k, a rim-hub path has even spread $\delta \in [0, (3k+1)/2]$.

Case 1: $\delta = 0$. The following sequence of vertices is a Hamilton cycle in $GP(3k+1, k)$ for any odd k:

$$0', 0, 1, 1', (k+1)', k+1, k+2, (k+2)', (2k+2)', \ldots$$
$$\ldots (qk+q)', qk+q, qk+q+1, (qk+q+1)', ((q+1)k+(q+1))', \ldots$$
$$\ldots (\tfrac{3k+1}{2}k + \tfrac{3k+1}{2})'.$$

To verify this claim, observe that this sequence contains 4 edges between $(qk+q)'$ and $((q+1)k+(q+1))'$. Therefore, there are $4\frac{(3k+1)}{2} = 2(3k+1)$ edges, the right number, in this sequence. Next, observe that none of the vertices, except the first and last are repeated in the sequence. If they were, then the congruence:

$$dk + d \equiv c \pmod{3k+1}$$

would have more than one solution for d in the range $[1, (3k+1)/2]$ for each of the following values of c: 0, 1, and -1. Since $\text{GCD}(3k+1, k+1) = 2$ for odd k, this is certainly not the case.

Removing the edge $\{0, 0'\}$ from this cycle leaves a 0–0′ Hamilton path.

Case 2: $2 \leq \delta \leq k-3$. Consider the following 0–2′ Hamilton path in $GP(16, 5)$:

$0, 0', 11', 11, 12, 12', 7', 7, 6, 6', 1', 1, 2, 3, 4, 5, 5', 10', 10, 9, 8, 8', 3', 14', 9', 4', 15', 15, 14, 13, 13', 2'.$

It is rotor-ready at $i = 5$. Inserting an A-rotor does not change the spread. Inserting a B-rotor adds two to the spread. The result follows.

Case 3: $\delta = k - 1$. Consider the following 0–4′ Hamilton path in $GP(16, 5)$:

$0, 0', 5', 5, 4, 3, 2, 1, 1', 12', 12, 11, 11', 6', 6, 7, 7', 2', 13', 13, 14, 15, 15', 10', 10, 9, 8, 8', 3', 14', 9', 4'.$

The path is reflected-rotor-ready at $j = 11$. We insert $\frac{k-5}{2}$ $R(A)$-rotors, each one increasing the spread by two, yielding the desired Hamilton path. The result follows.

Case 4: $k + 1 \leq \delta \leq (3k+1)/2$. Consider the following 0–6′ Hamilton path in $GP(16, 5)$:

$0, 0', 11', 11, 12, 12', 1', 1, 2, 3, 4, 5, 5', 10', 10, 9, 8, 8', 3', 14', 9', 4', 15', 15, 14, 13, 13', 2', 7', 7, 6, 6'.$

This path is rotor-ready at $i = 5$. Inserting an A-rotor adds two to the spread. Inserting a B-rotor adds four to the spread. The result follows.

Now we have found all the necessary rim-hub Hamilton paths.

4.3 Hub to Hub Hamilton Paths

Now we need to consider odd values for the spread only.

Case 1: $1 \leq \delta \leq k - 2$. Consider the following $0'$–$1'$ Hamilton path in $GP(10, 3)$:

$$0', 0, 1, 2, 2', 5', 5, 6, 7, 7', 4', 4, 3, 3', 6', 9', 9, 8, 8', 1'.$$

The path is primed-rotor-ready at $i = 1$. Inserting an A'-rotor doesn't change the spread. Inserting a B'-rotor adds two to the spread. The result follows.

Case 2: $\delta = k$. Reflecting the Hamilton cycle of Case 1 in Section 4.2 and deleting the edge $\langle 0', k' \rangle$ yields a $0'$–k' Hamilton path.

Case 3: $k+2 \leq \delta \leq (3k+1)/2$. Consider the following $0'$–$5'$ Hamilton path in $GP(10, 3)$:

$$0', 7', 7, 8, 8', 1', 4', 4, 5, 6, 6', 3', 3, 2, 1, 0, 9, 9', 2', 5'.$$

This path is primed-rotor-ready at $i = 0$. Inserting an A'-rotor adds four to the spread. Inserting a B'-rotor adds two to the spread. The result follows.

This completes our proof that $GP(3k + 1, k + 1)$ is Hamilton-laceable for odd k.

Chapter 5

Conclusion

The main fact we have proven about generalized Petersen graphs is as follows.

Theorem 21 *For $k \geq 4$, $GP(3k+1, k)$ is Hamilton-connected if k is even, and Hamilton-laceable if k is odd.*

In our research, we have actually verified all the small cases by computer search to show that the restriction $k \geq 4$ can be removed.

What comes next? We now make a suggestion that would bring us closer to proving Conjecture 1. Mavraganis [4] introduced the notion of *inserting a 2s-pack* and used $s = 6$ to take the inductive step from a Hamilton path in $GP(n, 2)$ to one in $GP(n+6, 2)$. By trial and error, we have found that from a Hamilton path in $GP(n, 3)$ we can similarly obtain one in $GP(n+12, 3)$ by inserting a $2s$-pack with $s = 12$. Furthermore, using a value of $s = 20$, we obtain a Hamilton path in $GP(n+20, 4)$ from one in $GP(n, 4)$. To generalize this notion, we postulate that a Hamilton path in $GP(n, k)$ can be extended to a Hamilton path in $GP(n+k(k+1), k)$ by inserting a $2k(k+1)$-pack.

Using this procedure, for a fixed value of k, we would only need to show the existence of the requisite Hamilton paths in $k(k+1)$ base cases as well as the existence of the $2s$-packs to prove by induction on n that $GP(n,k)$ is Hamilton-connected or Hamilton-laceable. In this thesis, we have shown the existence of the required Hamilton paths for one of those base cases for every value of k.

We hope that investigation of the cases $GP(3k+2,k)$ and $GP(3k+3,k)$ will shed light on Conjecture 1. It is not currently clear if some generalization of rotors alone or a combination of rotors and $(2s)$-packs will be required to prove Conjecture 1. General progress on either would be welcome.

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