Growth estimates and Phragmén-Lindelöf principles for half space problems

by

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Abstract

Pointwise growth estimates for the n-dimensional ($n \geq 2$) half space Dirichlet and Neumann Poisson integrals are given. They are shown to be the best estimates possible within the class of functions for which the integrals converge. A modified Poisson kernel can be formed by subtracting M terms from the Fourier expansion (in Gegenbauer polynomials) of the Poisson kernel. With Dirichlet data, the resulting modified Poisson integral satisfies $u(x) = o(|x|^{M+1} \sec^{n-1} \theta)$ $(x \to \infty, x_n > 0)$ where θ is the angle between x and the normal to the half space $x_n = 0$. Here the data is $f : \mathbb{R}^{n-1} \to \mathbb{R}$ and satisfies $\int_{\mathbb{R}^{n-1}} |f(y)|(|y|^{M+n} + 1)^{-1} dy < \infty$. Thus, a convergent modified Poisson integral can be defined for any polynomial data. Similar estimates are obtained for the half space Neumann problem and for λ-harmonic functions in a half space. The modified kernels of Finkelstein and Scheinberg (1975) are used to write modified Poisson integrals that give a classical solution to the half space Dirichlet problem for any continuous data. Growth estimates are obtained for these solutions. When $n = 2$ the Neumann kernel is logarithmic and different estimates are obtained. A key feature of all the above solutions is that they can have angular and radial blow-up as x approaches the boundary at infinity. When $\int_{\mathbb{R}^{n-1}} |f(y)|(|y|^p + 1) dy < \infty$ for some $p > 0$ a similar type of modified kernel is used to give the asymptotic expansion of the Poisson integral as $|x| \to \infty$. Using the Henstock–Kurzweil integral, growth estimates for conditionally convergent Poisson integrals are also given.

A Phragmén-Lindelöf principle that takes into account the above angular blow-up is proved. This is done by means of barriers on cusped sub-domains of the half space. This gives an extension of the Phragmén-Lindelöf principles of Wolf (1939) and Yoshida (1981) and leads to a uniqueness theorem. Uniqueness is also proven directly using a spherical harmonics expansion.

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Chapter 1

Introduction

1.1 Overview

It will first be necessary to introduce some notation. For x in the half space $\Pi_+ = \{x \in \mathbb{R}^n | x_n > 0\}$ ($n \geq 2$) let $y \in \mathbb{R}^{n-1}$ be identified with the projection of x onto the hyperplane $x_n = 0$, which will be denoted as $\partial\Pi_+$. Let θ be the angle between x and $\hat{e}_n$, i.e., $x_n = |x| \cos\theta$, $|y| = |x| \sin\theta$ and $0 \leq \theta < \pi/2$ when $x \in \Pi_+$. We will write $x = \sum_{i=1}^{n-1} y_i \hat{e}_i + x_n \hat{e}_n$ where $\hat{e}_i$ is the i$^{\text{th}}$ unit coordinate vector and $\hat{e}_n$ is normal to $\partial\Pi_+$. Unit vectors will be denoted with a caret, $\hat{x} = x/|x|$ for $x \neq 0$. Let $B_r(a)$ be the open ball in $\mathbb{R}^{n-1}$ with centre $a \in \mathbb{R}^{n-1}$ and radius $r > 0$. It's surface element is written dS_{n-2}. When $a = 0$ we write B_r. The volume of the unit n-ball is $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$. When integrating over regions in $\mathbb{R}^{n-1}$ the integration variable is written y' and the angle between y' and y (for fixed y) is θ'. When $n = 2$, we take $\theta' = 0$ or π according as y' and x_1 are on the same or opposite side of the origin. Equivalently, $\cos\theta' = \mathrm{sgn}(\mathrm{x}_1\, \mathrm{y}')$.

Of central importance will be the boundary value problem

$$u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+) \tag{1.1}$$

$$\Delta u = 0, \quad x \in \Pi_+ \tag{1.2}$$

$$u = f, \quad x \in \partial\Pi_+ \tag{1.3}$$

for Laplace's equation ($\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$). Here f is a given continuous function on $\mathbb{R}^{n-1}$. Such a function u is called a classical solution, i.e., u is twice continuously differentiable in the open half space and continuous in its closure. The Poisson kernel is

$$K(x, y') = \frac{2x_n}{n\omega_n}\left[|y' - y|^2 + x_n^2\right]^{-\frac{n}{2}} \tag{1.4}$$

and defines the Dirichlet Poisson integral

$$D[f](x) = \int_{\mathbb{R}^{n-1}} K(x, y')\, f(y')\, dy'. \tag{1.5}$$

When the normal derivative of u is specified on the boundary we have the Neumann problem, i.e., replace (1.3) with

$$\frac{\partial u}{\partial x_n} = -f, \qquad x \in \Pi_+$$

and demand that $u \in C^2(\Pi_+) \cap C^1(\overline{\Pi}_+)$.

This thesis arose from the following observation. There are various proofs in the literature that if f is bounded and continuous then the Poisson integral of f is the unique bounded solution to the half space problem (1.1)–(1.3) (see [7]). However, f need not be bounded for the Poisson integral to exist. The Poisson integral exists on Π_+ if

$$\int_{\mathbb{R}^{n-1}} \frac{|f(y')|\, dy'}{|y'|^n + 1} < \infty. \tag{1.6}$$

See [23]. But then $D[f]$ need not be bounded. This leads to the questions

(i) Under (1.6) what is the best estimate for the growth of the Poisson integral?

(i) What condition will ensure uniqueness for the half space problem and be compatible with the Poisson integral?

(iii) What type of solutions does the half space problem admit when (1.6) fails?

In order for (1.1)–(1.3) to have a unique solution, some additional condition on u is necessary. Otherwise, any harmonic function that vanishes when $x_n = 0$ could be added to u. For example, $v(x) = x_n$ is harmonic and vanishes on $\partial\Pi_+$. So if u is a solution of (1.1)–(1.3) then $u + cv$ is as well, for any constant c. And, if f is an entire function of the complex variable $z = x_1 + ix_2$ with only real coefficients in its power series, then $v(x_1, x_2) = \mathcal{I}m[f(z)]$ is harmonic in $\mathbb{R}^2$ and is zero on $x_2 = 0$. Witness $f(z) = z^2 \exp(-z^2)$. Then

$$v(x_1, x_2) = \exp(x_2^2 - x_1^2)[(x_2^2 - x_1^2)\sin(2x_1x_2) - 2x_1x_2\cos(2x_1x_2)]$$

and $\Delta v = 0$, $v(x_1, 0) = 0$.

To ensure uniqueness, a growth condition is applied that will rule out examples like the two functions v in the preceding paragraph, although it need not be so stringent as to demand that u be bounded. A classical result is that if $u(x) = o(|x|)$ as $|x| \to \infty$, i.e., $u(x)/|x| \to 0$, then any solution to (1.1)–(1.3) is unique. See [55]. It is shown in Proposition 2.2.1 that if f is bounded then $D[f]$ is bounded and if f has compact support then $D[f](x) = O(x_n|x|^{-n})$ ($x \in \Pi_+$, $|x| \to \infty$), i.e., $|x|^n D[f](x)/x_n$ is bounded. However, under (1.6) the best estimate is in general $D[f](x) = o(|x| \sec^{n-1}\theta)$ (Corollary 2.2.1). The order relation is interpreted as $\mu(r)/r \to 0$ as $r \to \infty$ where $\mu(r)$ is the supremum of $|D[f](x)| \cos^{n-1}\theta$ over $x \in \Pi_+$, $|x| = r$, i.e., uniform in $\hat{x}$ as $|x| \to \infty$. This growth estimate predicts that $|x|^{-1}D[f](x) \to 0$ uniformly in $\hat{x}$ as $|x| \to \infty$ if $0 \le \theta \le \theta_0 < \pi/2$ (x in a closed cone intersecting $\overline{\Pi}_+$ at $\{0\}$). But, $|x|^{-1}D[f](x)$ can be unbounded if $\theta \to \pi/2$ as $|x| \to \infty$. It will be proven that the growth condition is sharp, using the following new definition. A growth condition ω is said to be sharp if given any function $\psi = o(\omega)$ and any sequence $\{x^{(i)}\} \in \Pi_+$ with

$|x^{(i)}| \to \infty$, we can find data f so that the solution corresponding to f is not *little oh* of ψ on this sequence (see Definition 3.3.1 and Theorem 3.3.1 below). Much of the literature on the half space problem deals with data f that is bounded, of compact support or in an L^p space (see [62]). We will drop these assumptions and be guided by the integrability condition (1.6).

Uniqueness for the half space problem follows immediately from a Phragmén–Lindelöf Principle, see Corollary 7.3.1. A classical result is that if u is subharmonic in Π_+ ($\Delta u \geq 0$), $u \leq 0$ on $\partial\Pi_+$ and $u(x) = o(|x|)$ ($x \in \Pi_+$, $|x| \to \infty$) then $u \leq 0$ in Π_+. In Chapter 7 we will prove a more general version of this Phragmén–Lindelöf Principle that is compatible with the growth condition $D[f](x) = o(|x|^{n-1}\sec^{n-1}\theta)$. This will give a uniqueness theorem for (1.1)–(1.3) under a growth condition that allows u to be the Poisson integral of any function for which (1.6) holds.

Another uniqueness theorem, also in accord with $u(x) = o(|x|^{n-1}\sec^{n-1}\theta)$, is proven in Chapter 6 by expanding a harmonic function in spherical harmonics.

The Poisson kernel has the expansion

$$K(x,y') = \frac{2x_n}{n\omega}\sum_{m=0}^{\infty}\frac{|x|^m}{|y'|^{m+n}}C_m^{n/2}(\sin\theta\cos\theta') \tag{1.7}$$

where $C_m^{n/2}$ is a Gegenbauer polynomial. The series converges for $|y'| > |x|$. Each term in the series is a harmonic function of x. If (1.6) fails but $\int_{\mathbf{R}^{n-1}} |f(y')|(|y'|^{M+n} + 1)^{-1}\,dy'$ converges for a positive integer M then the Poisson integral can be modified by subtracting the first M terms in (1.7) from $K(x,y')$. The new kernel, $K_M(x,y')$, will be of order $|y'|^{-(M+n)}$ as $|y'| \to \infty$ and can be used to define a solution to (1.1)–(1.3) when f satisfies the weaker integral condition above. This new Poisson integral has growth $o(|x|^{M+1}\sec^{n-1}\theta)$. This is proven sharp in Theorem 3.3.1. The proof is complicated because K_M is no longer positive like K was. It is thus difficult to obtain a lower bound on the modified Poisson integral. A crucial tool in the proof is an integral representation of the modified kernel, see Lemma 3.3.1. A type of Riesz kernel,

$$K(\lambda, x, y') = \left[|y' - y|^2 + x_n^2\right]^{-\lambda}, \tag{1.8}$$

is used to write solutions to both the Dirichlet and Neumann problems. This kernel is modified as above and estimates are obtained in Theorem 3.3.1.

In Chapter 4, various other results on modified kernels are collected.

The expansion

$$K(x, y') = \frac{2x_n}{n\omega} \sum_{m=0}^{\infty} \frac{|y'|^m}{|x|^{m+n}} C_m^{n/2}(\sin\theta\cos\theta') \tag{1.9}$$

(valid for $|x| > |y'|$) is used to define a modified kernel that gives asymptotic expansions of $D[f]$ as $|x| \to \infty$ in the case that $\int_{\mathbf{R}^{n-1}} |f(y')|(|y'|^{M-1} + 1)\, dy' < \infty$ for a positive integer M.

It is shown that if $f \geq 0$ and $\int_{\mathbf{R}^{n-1}} f(y')(|y'|^n + 1)^{-1}\, dy' = +\infty$ then there are no positive solutions to (1.1)–(1.3). Note that the modified kernels that can be used to solve (1.1)–(1.3) are not positive.

Modified Neumann integrals are represented as integrals over modified Dirichlet integrals. A relation of this type is of particular importance in determining the growth in the $n = 2$ Neumann case.

M. Finkelstein and S. Scheinberg have shown in [22] that for the modified Poisson kernel with M terms removed, if M is allowed to be a function of the integration variable y' then for any continuous function f it is possible to construct a modified Poisson integral for which $\int_{\mathbf{R}^{n-1}} K_{M(y')}(x, y')\, f(y')\, dy'$ is a classical solution of (1.1)–(1.3). If $M(y')$ is a given function taking values in the natural numbers then this defines an integral condition that determines a class of functions for which the modified Poisson integral with $M(y')$ is convergent. We establish a growth condition for such solutions. And, if f is a given function, we give an algorithm for choosing M so that the modified Poisson integral of f converges. See §4.3.

When $n = 2$, a solution of the half plane Neumann problem is

$$u(x) = -\frac{1}{2\pi} \int_{\xi=-\infty}^{\infty} f(\xi) \log\left(\xi^2 - 2\xi r \cos\phi + r^2\right) d\xi. \tag{1.10}$$

Here polar coordinates are used, $x_1 = r\cos\phi$ and $x_2 = r\sin\phi$, with $0 < \phi < \pi$. The function u satisfies $\Delta u = 0$ for $x_2 > 0$ and $-\partial u/\partial x_2 = f$ when $x_2 = 0$. The integrability condition is now

$$\int_{\xi=-\infty}^{\infty} |f(\xi)| \log(\xi^2 + 2)\, d\xi < \infty.$$

Under this condition we have the surprising estimate $u(x) = o(\log(1 - |\cos\phi|)/\log r)$, Theorem 5.4.1. This is for an odd function. When f is not odd a term proportional to $\log r$ must be added. In order to obtain this estimate, the Neumann solution is written as an integral over a Dirichlet solution. Modified kernels are developed in this case as well.

The Lebesgue integral has powerful advantages over the Riemann integral. It can integrate unbounded, nowhere continuous functions over unbounded domains. And, if each f_n is a measurable function then so are $\limsup f_n$ and $\liminf f_n$ (and $\lim f_n$ if this exists). However, Lebesgue integrals must be absolutely convergent. This poses a problem for the Poisson integral. For example, if $n = 2$ and $f(\xi) = \xi \sin \xi$ then the Poisson integral of $|f|$ diverges but the Poisson integral of f exists as a conditionally convergent improper Riemann integral (and is calculated in §8.2). If f is changed to zero on the rational numbers then even this last integral fails to converge. These problems are overcome by using the Henstock–Kurzweil integral (also called gauge or generalised Riemann integral). This integral reduces to the Lebesgue integral in the case of absolute convergence but allows conditional convergence. We can then handle the Dirichlet problem with data for which the Riemann or Lebesgue integral does not exist. Without absolute convergence, a replacement for the Dominated Convergence Theorem must be found. This is done for the case at hand. A strengthened version of Abel's test for uniform convergence is stated but not proved. The growth estimate for the Poisson integral with conditional convergence turns out to be the same as the estimate for the Lebesgue integral.

It is not clear when the first reference to the half space problem was. Certainly such potential integrals were known to George Green by the 1820's, [31]. The justification for studying such an ancient problem that has been tackled by so many mathematicians is that, first of all, the half space Laplace equation is not fully understood. We present new growth estimates for solutions given by the Poisson integral, the modified Poisson integral and conditionally convergent integrals. In addition, these estimates are proved to be the best possible in a strong sense. Second, this is an important equation. Understanding Laplace's equation and other equations with constant coefficients is a crucial first step in understanding more general elliptic equations. A typical method in the study of such equations is to "freeze" the coefficients at a point and study the corresponding constant coefficient equation at that point. Thus, for elliptic equations we must have a solid theory of constant coefficient equations. As far as nonlinear equations go, consider the remarks of N. V. Krylov at the 1986 International Congress ([43], p.1103),

> One can say that a good linear theory breeds a good nonlinear theory in contradiction with the known claim that "linearity breeds contempt."

Many of the results here are in the papers [61] and [64] but as these arose from this thesis no particular reference will be made to them.

1.2 Mathematical preliminaries

All of the integrals appearing here are Lebesgue integrals, except in Chapter 8 where the Henstock–Kurzweil integral is used. We will distinguish between *measurable* and *integrable* functions. If $E \subset \mathbf{R}^n$ is a (Lebesgue) measurable set and the function f maps E to the extended real numbers then f is *measurable* if the set $\{x \in E \mid f(x) > \alpha\}$ is measurable for each $\alpha \in \mathbf{R}$. The function f is *integrable* over E if $\int_E |f| < \infty$. And, f is *locally integrable* if $\int_A |f| < \infty$ for each compact set $A \subset E$. Thus, the function $x \mapsto x^{-1}$ is measurable and locally integrable over the interval $(0,1)$ but not integrable over $(0,1)$. Note that for a function to be integrable, it is required that the positive and negative parts be separately integrable, i.e., the Lebesgue integral does not admit conditional convergence.

When integrating functions that depend on a parameter, it will be important to know when it is valid to interchange limit and integration operations. For the Lebesgue integral the most useful convergence test is Lebesgue's Dominated Convergence Theorem.

Dominated Convergence Theorem *Suppose $\{f_m\}$ is a sequence of measurable functions on the measurable set E and g is integrable over E such that $|f_m(x)| \leq g(x)$ on E. If $f_m(x) \to f(x)$ almost everywhere on E then*

$$\lim_{m\to\infty} \int_E f_m = \int_E f. \tag{1.11}$$

See, for example, [58].

For Laplace's equation (and elliptic differential equations as a whole), the well–posed problems are the boundary value problems, where some combination of the function and its normal derivative are specified on the boundary. There are three basic types of boundary. The theory of elliptic differential equations in bounded domains is a mature field. For existence proofs within the space of Hölder continuous functions, see [27] or [42]. A second type of boundary is that of the exterior problem, for which the domain has compact complement. Specifying the value of the solution at infinity gives uniqueness. Integral equation methods can be used to reduce the problem to the finite boundary. In the case of Laplace's equation for $n \geq 3$, it is known that bounded harmonic functions defined in an exterior region have a limit at infinity. Thus, assigning this limit gives uniqueness and letting the limit value run through $\mathbb{R}$ gives all solutions satisfying the inner boundary condition ([53]). We will be concerned with the case of an unbounded region whose boundary is also unbounded. The archetype example is the half space, which we will study exclusively.

The following properties of Laplace's equation will be important. Most of them have some sort of analogue in the case of an elliptic equation. General references are [7] and [27].

Mean value property If u is continuous in a domain $\Omega \subset \mathbb{R}^n$ then u is harmonic if and only if u satisfies the mean value property,

$$u(x) = \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(x)} u \, dS_{n-1}, \tag{1.12}$$

for every ball $B_R(x)$ properly contained in Ω.

Schwarz reflection principle Suppose $\Omega \subset \mathbb{R}^n$ is symmetric about $\partial\Pi_+$. If u is continuous on $\Omega \cap \overline{\Pi}_+$, u is harmonic on $\Omega \cap \Pi_+$ and $u = 0$ on $\Omega \cap \partial\Pi_+$, then the odd extension of u across $\partial\Pi_+$ is harmonic in Ω.

Weak maximum principle Suppose $\Delta u \geq 0$ in a bounded domain Ω and that $u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+)$. Then the maximum of u is achieved on $\partial\Omega$,

$$\sup_{\Omega} u = \sup_{\partial\Omega} u. \tag{1.13}$$

If u is not assumed to be continuous in $\overline{\Omega}$ then the conclusion is

$$\sup_{\Omega} u = \limsup_{x\in\Omega, x\to\partial\Omega} u. \tag{1.14}$$

This is also true for elliptic operators. See [27].

Laplace's equation admits the *fundamental solution*

$$E(x) = \begin{cases} \frac{-1}{(n-2)n\omega_n}|x|^{2-n}, & n > 2 \\ \frac{1}{2\pi}\log|x|, & n = 2. \end{cases} \tag{1.15}$$

It satisfies $\Delta E(x - x') = \delta(x - x')$ when x and x' are in $\mathbb{R}^n$ and δ is the Dirac distribution. A Green function can be defined for the half space Dirichlet problem by taking an odd reflection across $\partial\Pi_+$, $G(x, x') = E(x, x') - E(x, x^*)$, where x^* is the reflection of x across $x_n = 0$. If $x = y + x_n\hat{e}_n$ then $x^* = y - x_n\hat{e}_n$ where $y \cdot \hat{e}_n = 0$. Write $x' = y' + x'_n\hat{e}_n$ as above. The Poisson integral is now given by

$$D[f](x) = \int_{\mathbb{R}^{n-1}} f(y') \left.\frac{\partial G(x, x')}{\partial x'_n}\right|_{x'_n=0} dy' \tag{1.16}$$

$$= \frac{2x_n}{n\omega_n} \int_{\mathbb{R}^{n-1}} f(y') \left[|y' - y|^2 + x_n^2\right]^{-\frac{n}{2}} dy'. \tag{1.17}$$

For the Neumann problem with $-\partial u/\partial x_n = f$ on $\partial\Pi_+$, take $G(x, x') = E(x, x') + E(x, x^*)$ and then

$$N[f](x) = -\int_{\mathbb{R}^{n-1}} f(y') G(x, y')\, dy' \tag{1.18}$$

$$= \frac{2x_n}{(n-2)n\omega_n} \int_{\mathbb{R}^{n-1}} f(y') \left[|y' - y|^2 + x_n^2\right]^{-\frac{(n-2)}{2}} dy'. \tag{1.19}$$

These formulas can also be derived by taking an $(n-1)$–fold Fourier transform in the variables orthogonal to $\hat{e}_n$.

Gamma function For the Gamma function we will need Stirling's approximation

$$\Gamma(z) = \sqrt{\tfrac{2\pi}{z}}\, e^{z(\log z - 1)} \left[1 + O\left(\frac{1}{z}\right)\right] \qquad \text{as } z \to \infty. \tag{1.20}$$

For fixed a and b, this leads to

$$\frac{\Gamma(a+z)}{\Gamma(b+z)} = z^{a-b} \left[1 + O\left(\frac{1}{z}\right)\right] \qquad \text{as } z \to \infty. \tag{1.21}$$

Chapter 2

The half space Dirichlet problem

2.1 The Poisson integral

The classical half space Dirichlet problem is to find u satisfying

$$u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+) \tag{2.1}$$

$$\Delta u = 0, \quad x \in \Pi_+ \tag{2.2}$$

$$u = f, \quad x \in \partial\Pi_+ \tag{2.3}$$

where f is a given continuous function on $\mathbb{R}^{n-1}$. The Dirichlet Poisson integral is defined by

$$D[f](x) = \int_{\mathbb{R}^{n-1}} K(x, y')\, f(y')\, dy' \tag{2.4}$$

where the Poisson kernel is

$$K(x, y') = \frac{2x_n}{n\omega_n} \left[|y' - y|^2 + x_n^2\right]^{-\frac{n}{2}}. \tag{2.5}$$

The integral will exist if

$$\int_{\mathbb{R}^{n-1}} \frac{|f(y')|\, dy'}{|y'|^n + 1} < \infty. \tag{2.6}$$

Since the kernel satisfies the mean value property for harmonic functions, $u = D[f]$ will then define a harmonic function in Π_+. If f is continuous then $u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+)$ and satisfies (2.3) ([7], Exercise 16 of Chapter 7 and [23]).

It is a classical result that if $u = o(|x|)$ then any solution to (2.1)–(2.3) is unique ([24]). However, we show below that the Poisson integral behaves as $o(|x| \sec^{n-1}\theta)$ when $|x| \to \infty$ in Π_+ (recall the notation $x_n = |x|\cos\theta$, $|y| = \sin\theta$). It is thus desirable to have a uniqueness theorem that allows this behaviour. Such a theorem will be presented in Chapters 6 and 7.

2.2 Growth estimates

We first present some basic estimates for $D[f]$.

Proposition 2.2.1 *(i) If f is bounded and measurable then $D[f]$ is bounded.*
(ii) If $f \in L^1$ with compact support then $D[f](x) = O(x_n|x|^{-n})$ $(x \in \Pi_+,\ |x| \to \infty)$.

Proof. (i) It is shown in [7] (p. 128) that $D[1](x) = 1$. Therefore, if $|f(y)| \le M$ then $|D[f](x)| \le M\, D[1](x) = M$.

(ii) Suppose the support of f is contained in B_R. Let $|x| \ge 2R$. Then

$$\begin{aligned}
|D[f](x)| &\le \frac{2x_n}{n\omega_n}\int_{B_R}\frac{|f(y')|\,dy'}{(|y'-y|^2+x_n^2)^{n/2}} \\
&\le \frac{2x_n}{n\omega_n}\int_{B_R}\frac{|f(y')|\,dy'}{(|x|-|y'|)^n} \\
&\le \frac{2x_n}{n\omega_n}\int_{B_R}\frac{|f(y')|\,dy'}{(|x|-R)^n} \\
&\le \frac{2^{n+1}x_n|x|^{-n}}{n\omega_n}\int_{B_R}|f(y')|\,dy'. \quad \blacksquare
\end{aligned}$$

In what follows, we will be primarily concerned with data f satisfying (2.6), or subsequent convergence conditions, with no assumption on the boundedness or support of f.

When u is a solution of (2.2), (2.3), various estimates on the L^p norm of u and of u_{x_n} (where $u_{x_n}(y) = u(x)$) are given in [7] and [13]. However, as we are concerned with pointwise behaviour of u we give the following estimate of $|u|$. (See [7] for estimates when u is in a harmonic Hardy space.)

Theorem 2.2.1 *Let $a > 1$, $0 < b < a+n-1$ or $a = 1$, $0 < b \le n$. If f is measurable such that $\int_{\mathbb{R}_{n-1}} |f(y')|^a(|y'|^b + 1)^{-1}\,dy' < \infty$ then (2.6) holds and $u = D[f]$ satisfies $u = o(|x|^{(b-n+1)/a}\sec^{(n-1)/a}\theta)$ $(x \in \Pi_+, |x| \to \infty)$.*

Proof: Let $0 < \alpha \le n/2$ and p, q Hölder conjugate exponents ($p^{-1} + q^{-1} = 1$, $p \ge 1$). The Poisson kernel, (2.5), may be written

$$K(x,y') = \frac{2x_n}{n\omega_n}(1-\sin\theta\,\cos\theta')^{-\alpha} \times \left[\frac{(|y'|-|x|)^2}{1-\sin\theta\cos\theta'} + 2|y'|\,|x|\right]^{-\alpha} \left[|y'-y|^2 + x_n^2\right]^{-(\frac{n}{2}-\alpha)} \tag{2.7}$$

$$\le \frac{2^{\alpha+1}x_n(1+\sin\theta)^\alpha}{n\omega_n\cos^{2\alpha}\theta}\,(|y'|+|x|)^{-2\alpha}\,\left[|y'-y|^2 + x_n^2\right]^{-(\frac{n}{2}-\alpha)}. \tag{2.8}$$

Let $|x| \ge 1$. For $p > 1$, $\alpha < n/2$, the Hölder inequality gives

$$\int_{\mathbb{R}^{n-1}} K(x,y')\,|f(y')|dy' \le \frac{2^{2\alpha+1}}{n\omega_n}|x|\sec^{2\alpha-1}\theta\, I_1^{\frac{1}{p}}\, I_2^{\frac{1}{q}}, \tag{2.9}$$

where

$$I_1 = \int_{\mathbb{R}^{n-1}} \frac{|f(y')|^p\,dy'}{(|y'|+|x|)^{2\alpha p}} \tag{2.10}$$

$$\le 2\int_{\mathbb{R}^{n-1}} \frac{|f(y')|^p\,dy'}{|y'|^{2\alpha p} + |x|^{2\alpha p}} \tag{2.11}$$

and

$$I_2 = \int_{\mathbb{R}^{n-1}} \left[|y'-y|^2 + x_n^2\right]^{-q(\frac{n}{2}-\alpha)}\,dy'. \tag{2.12}$$

To evaluate I_2, introduce spherical coordinates centred on y, i.e., $\rho = |y' - y|$. Then

$$\begin{aligned} I_2 &= \int_{\rho=0}^{\infty} [\rho^2 + x_n^2]^{-q(\frac{n}{2}-\alpha)} \int_{\partial B_\rho} dS_{n-2}\, d\rho \\ &= (n-1)\omega_{n-1} \int_{\rho=0}^{\infty} [\rho^2 + x_n^2]^{-q(\frac{n}{2}-\alpha)} \rho^{n-2}\, d\rho \\ &= (n-1)\, \omega_{n-1} x_n^{n-1-q(n-2\alpha)} \int_{\rho=0}^{\infty} [\rho^2 + 1]^{-q(n/2-\alpha)} \rho^{n-2} d\rho. \end{aligned} \tag{2.13}$$

This integral converges whenever $n - q(n-2\alpha) < 1$ or $2\alpha p < p + n - 1$.

When $p = 1$ ($\alpha < n/2$), (2.9) holds with $I_2^{\frac{1}{q}}$ replaced by

$$\sup_{y' \in \mathbb{R}^{n-1}} \left[|y'-y|^2 + x_n^2\right]^{-(\frac{n}{2}-\alpha)} = x_n^{-(n-2\alpha)}.$$

And, if $\alpha = n/2$, (2.9) holds with $I_2 = 1$.

Now, put $a = p$, $b = 2\alpha p$. Hence, (2.6) holds and $u = D[f]$ exists on Π_+. Furthermore, by dominated convergence and (2.11), $I_1 \to 0$ as $|x| \to \infty$. The theorem follows by putting (2.11) and (2.13) into (2.9). ∎

Corollary 2.2.1 *If* (2.6) *holds, then* $u = D[f](x) = o(|x| \sec^{n-1}\theta)$.

Proof. Let $a = 1$, $b = n$. ∎

Despite the crude appearance of the estimate in (2.8), it will be shown in Chapter 3 (Theorem 3.3.1) that for $a = 1$ this leads to the best estimate possible for $u = D[f]$ under (2.6).

Remark 2.2.1 *(i)* Corollary 2.2.1 with $n = 2$ was obtained by F. Wolf [65] and D. Siegel [60].

(ii) If $a \geq 1$ and $f \in L^a$ then the Hölder inequality shows that (2.6) holds and $|u(x)| \leq c_{n,a}\|f\|_a x_n^{-(n-1)/a}$. The above constant is given in terms of the beta function,

$$c_{n,a} = \frac{2}{n\omega_n}\left[\frac{(n-1)\,\omega_{n-1}}{2}B\left(\frac{n-1}{2}, \frac{n}{2(a-1)} + \frac{1}{2}\right)\right]^{1-\frac{1}{a}} \quad \text{when } a > 1$$

and $c_{n,1} = 2/(n\omega_n)$. It is obtained by evaluating (2.13) in the case $\alpha = 0$ ([19], 1.5.2). In [7], Theorem 7.11, an inequality of the same form is derived by a different method.
(iii) The estimate in the above corollary may not hold if we allow principal value integrals. For example, if f is odd and $n = 2$ then

$$D[f](x_1, x_2) = \frac{x_2}{\pi}\mathcal{PV}\int_{\xi=-\infty}^{\infty} \frac{f(\xi)\,d\xi}{(\xi - x_1)^2 + x_2^2} \tag{2.14}$$

$$= \frac{x_2}{\pi}\lim_{N\to\infty}\int_{\xi=-N}^{N} \frac{f(\xi)\,d\xi}{(\xi - x_1)^2 + x_2^2} \tag{2.15}$$

$$= \frac{x_2}{\pi}\lim_{N\to\infty}\int_{\xi=0}^{N} f(\xi)\left[\frac{1}{(\xi - x_1)^2 + x_2^2} - \frac{1}{(\xi + x_1)^2 + x_2^2}\right] d\xi \tag{2.16}$$

$$= \frac{4x_1x_2}{\pi}\int_{\xi=0}^{\infty} \frac{f(\xi)\,\xi\,d\xi}{[(\xi - x_1)^2 + x_2^2][(\xi + x_1)^2 + x_2^2]}. \tag{2.17}$$

Use polar coordinates $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, where $r = |x|$ and $0 < \phi < \pi$ for $x \in \Pi_+$. This last integral will exist on Π_+ if $\int_{\xi=0}^{\infty} |f(\xi)|(\xi^3 + 1)^{-1}\,d\xi < \infty$. Under this integrability condition, the same method as in the proof of Theorem 2.2.1 gives $D[f](x) = o(r^2\cos\phi\csc\phi)$. Indeed, if $f(\xi) = \xi$ then the residue calculus applied to (2.17) readily shows that $D[f](x_1, x_2) = x_1 \neq o(r\csc\phi)$.

When f is majorised by a radial function a better estimate of $|u|$ is possible.

Proposition 2.2.2 *If $|f(y)| \leq F(|y|)$ for F such that $\int_{\rho=0}^{\infty} F(\rho)(\rho^2 + 1)^{-1}\,d\rho < \infty$ then $u(x) = D[f](x) = o(|x|\sec\theta)$.*

Proof. From (2.7) and (2.8) (and the binomial theorem),

$$K(x,y') \leq \frac{2^{\frac{n}{2}+1}}{n\omega} x_n (1-\sin\theta\,\cos\theta')^{-\frac{n}{2}} (|y'|^n + |y'|^{n-2}|x|^2)^{-1}.$$

We then have

$$|u(x)| \leq \frac{2^{\frac{n}{2}+1} x_n}{n\omega} \int\limits_{\rho=0}^{\infty} \frac{F(\rho)\,d\rho}{\rho^2+|x|^2} I_3,$$

where

$$I_3 = \int_{\partial B_1} (1-\sin\theta\,\cos\theta')^{-\frac{n}{2}}\,dS_{n-2}.$$

The integral I_3 is singular when $\theta = \pi/2$ ($x_n = 0$). To determine the nature of the singularity we use the method of spherical means [38] to write

$$I_3 = (n-2)\,\omega_{n-2} \int\limits_{\phi=0}^{\pi} (1+\sin\theta\,\cos\phi)^{-\frac{n}{2}} \sin^{n-3}\phi\,d\phi.$$

Using the substitution $1-2t = \cos\phi$, an integral representation of the hypergeometric function and quadratic and linear transformations ([19], 2.12.1, 2.11.4, 2.9.2) we have

$$I_3 = \frac{2\sqrt{\pi}\,\omega_{n-2}\Gamma(\frac{n}{2})\,{}_2F_1(a,b;c;\sin^2\theta)}{\Gamma(\frac{n}{2}-\frac{1}{2})\cos^2\theta},$$

where $a = n/4-1$, $b = n/4-1/2$ and $c = n/2-1/2$. The hypergeometric function, ${}_2F_1$, (with these a, b, c) is bounded above (and below) by positive constants so that

$$|u(x)| \leq \frac{A_n|x|}{\cos\theta} \int\limits_{\rho=0}^{\infty} \frac{F(\rho)\,d\rho}{\rho^2+|x|^2},$$

where A_n is a positive constant. As $|x| \to \infty$ we have $u(x) = o(|x|\sec\theta)$. ■

The radial term in $o(|x|\sec^{n-1}\theta)$ and $o(|x|\sec\theta)$ of Proposition 2.2.2 and Corollary 2.2.1 cannot be replaced by any positive function that is *little oh* of $|x|$. This will be proved with the help of the following lemma.

Lemma 2.2.1 *Given a function ψ that is bounded and positive on $[0,\infty)$ with limit zero at infinity, there exists $\Psi \in C^1([0,\infty))$ such that $\lim_{r\to\infty}\Psi(r) = 0$, $\Psi' \leq 0$ and $\Psi(r) \geq \psi(r)$ for $r \geq 0$.*

Proof. Let $\psi_1(r) = \sup_{t\geq r} \psi(t)$. Then ψ_1 is positive, decreasing, bounded and majorises ψ (i.e. $\psi_1(r) \geq \psi(r)$).

Let $\psi_2(r) = \psi_1(\lfloor r \rfloor)$ where $\lfloor r \rfloor$ is the integer part of r. Then ψ_2 is a decreasing step function with steps only at the positive integers.

We can now find a C^1 majorant by using a cubic spline. For each $n \geq 1$, we require $p_n(x) = a_n x^3 + b_n x^2 + c_n x + d_n$ to satisfy

$$p_n(n) = \psi_2(n-1) \qquad p_n(n+1) = \psi_2(n)$$

$$p_n'(n) = 0 = p_n'(n+1).$$

This gives the system

$$\begin{aligned}
\psi_2(n-1) &= a_n n^3 + b_n n^2 + c_n n + d_n \\
\psi_2(n) &= a_n(n+1)^3 + b_n(n+1)^2 + c_n(n+1) + d_n \\
0 &= 3a_n n^2 + 2b_n n + c_n \\
0 &= 3a_n(n+1)^2 + 2b_n(n+1) + c_n.
\end{aligned}$$

For a solution, we need

$$\Delta = \begin{vmatrix} n^3 & n^2 & n & 1 \\ 3n^2 & 2n & 1 & 0 \\ (n+1)^3 & (n+1)^2 & n+1 & 1 \\ 3(n+1)^2 & 2(n+1) & 1 & 0 \end{vmatrix} \neq 0.$$

Let

$$\begin{aligned}
A(x_1, x_2, x_3, x_4) &= \begin{vmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \end{vmatrix} \\
&= (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)
\end{aligned}$$

be a Vandermonde determinant. Then, with subscripts on A denoting partial differentiation, $A_{24}(s,s,t,t) = (s-t)^4$ and $\Delta = A_{24}(n,n,n+1,n+1) = 1$. Hence, we can solve for a_n, b_n, c_n, d_n uniquely. Since cubic polynomials have at most one point of inflection we know that $p_n'(t) \leq 0$ for $n \leq t \leq n+1$. We can now let $\Psi(r) = p_n(r)$ for $1 \leq n \leq r \leq n+1$ and $\Psi(r) = \psi_2(r)$ for $0 \leq r \leq 1$. Thus, Ψ is the required majorant. ■

Proposition 2.2.3 *Let ψ be a bounded positive function on $[0,\infty)$ that tends to zero at infinity. For any fixed $0 \leq \theta_0 < \pi/2$ there is a function f satisfying (2.6) such that $D[f](x) \geq |x|\psi(|x|)$ for all $x \in \Pi_+$ with $0 \leq \theta \leq \theta_0$.*

Proof: By Lemma 2.2.1 we can assume $\psi \in C^1([0,\infty))$ and ψ is decreasing. Let $f(y) = -c_n|y|^2\psi'(|y|)$ where $c_n = 2^{n-1}\sec\theta_0\, n\omega_n/((n-1)\omega_{n-1})$. Then (2.6) reduces to

$$\begin{aligned} -\int_{\rho=0}^{\infty} \frac{\rho^2\psi'(\rho)\rho^{n-2}\,d\rho}{\rho^n+1} &\leq -\int_{\rho=0}^{\infty} \psi'(\rho)\,d\rho \\ &= \psi(0) \\ &< \infty. \end{aligned}$$

And,

$$\begin{aligned} D[f](x) &\geq -\frac{2x_n}{n\omega_n}c_n\int_{\partial B_1} d\omega_{n-1}\int_{\rho=0}^{\infty}\frac{\rho^2\psi'(\rho)\rho^{n-2}\,d\rho}{(\rho+|x|)^n} \\ &\geq -\frac{2|x|\cos\theta\, c_n(n-1)\omega_{n-1}}{n\omega_n}\int_{\rho=|x|}^{\infty}\frac{\rho^2\psi'(\rho)\rho^{n-2}\,d\rho}{(\rho+|x|)^n} \\ &\geq -|x|\int_{\rho=|x|}^{\infty}\psi'(\rho)\,d\rho \\ &= |x|\psi(|x|). \quad ■ \end{aligned}$$

Hence, in any closed cone in Π_+ with $0 \leq \theta \leq \theta_0 < \pi/2$ we can have $D[f](x)\,|x|^{-1}$ tend to zero arbitrarily slowly as $|x| \to \infty$. However, we cannot have $\theta_0 = \pi/2$ in Proposition 2.2.3. For example,

$$g(y) = \frac{|y|}{\log(|y|+2)} = o(|y|) \qquad \text{as } |y| \to \infty$$

but $D[g](x)$ diverges. Since g is continuous, in order for there to be a function f such that $D[f](x) \geq g(x)$ for all x in $\overline{\Pi}_+$ we would need $f(y) \geq g(y)$ for all $y \in \mathbb{R}^{n-1}$. But then the integral in (2.6) would diverge.

The following example will show that the estimate on the Poisson integral in the above corollary is sharp in the sense that if we try to use the growth condition $D[f](x) = o(|x|^\beta \sec^\gamma \theta)$ then this may fail to be true for some f satisfying the integrability condition (2.6) if $\beta + \gamma < n$, $\gamma > 0$, or $\beta + \gamma = n$, $\gamma \leq 0$. Define continuous data, f, to be zero except on a sequence of balls along the x_1-axis,

$$f(y) = \begin{cases} f_i(1 - \frac{1}{r_i}\,|y - a_i\hat{e}_1|), & y \in B_{r_i}(a_i\hat{e}_1) \subset \mathbb{R}^{n-1} \\ 0, & \text{otherwise,} \end{cases}$$

where f_i, a_i and r_i are sequences of positive real numbers such that $a_i \to \infty$, $r_i < 1$ and the $B_{r_i}(a_i\hat{e}_1)$ are disjoint. If $u = D[f]$ then (2.6) is equivalent to convergence of the series

$$\sum_{i=1}^{\infty} \frac{f_i r_i^{n-1}}{a_i^n}. \tag{2.18}$$

We can write u as the superposition of translates of the solution to the normalised problem

$$\Delta\tilde{u} = 0, \qquad x \in \Pi_+ \tag{2.19}$$

$$\tilde{u} = \begin{cases} 1 - |y|, & x \in B_1 \subset \mathbb{R}^{n-1} \\ 0; & x_n = 0,\ x \notin B_1. \end{cases} \tag{2.20}$$

Thus, since $\widetilde{u} \geq 0$,

$$u(x) = \sum_{i=1}^{\infty} f_i \widetilde{u}\left(\frac{x - a_i \hat{e}_1}{r_i}\right) \tag{2.21}$$

$$\geq f_i \widetilde{u}\left(\frac{x - a_i \hat{e}_1}{r_i}\right). \tag{2.22}$$

Consider the sequence $x^{(m)} = a_m \hat{e}_1 + r_m \hat{e}_n$. We now show that if $\beta + \gamma < n$, $\gamma > 0$, or $\beta + \gamma = n$, $\gamma \leq 0$, then $u(x)|x|^{-\beta} \cos^{\gamma} \theta \not\to 0$ along this sequence. Put $a_i = e^i$, $f_i = e^{ni}$, $r_i = i^{-2}$. Then the series (2.18) converges and yet

$$\frac{(x_n^{(m)})^{\gamma}\, u(x^{(m)})}{|x^{(m)}|^{\beta+\gamma}} \geq \frac{r_m^{\gamma}\, f_m\, \widetilde{u}(\hat{e}_n)}{(a_m^2 + r_m^2)^{(\beta+\gamma)/2}} \tag{2.23}$$

$$= \frac{m^{-2\gamma}\, e^{mn}\, \widetilde{u}(\hat{e}_n)}{(e^{2m} + m^{-4})^{(\beta+\gamma)/2}} \tag{2.24}$$

$$\not\to 0 \quad \text{as} \quad m \to \infty. \tag{2.25}$$

Taking $\psi(y) = \frac{1}{\log(|y|+2)}$ in Proposition 2.2.3 shows that we need $\beta \geq 1$. However, the above example does not rule out a growth condition such as $o(|x|^3 \sec^{n-2} \theta)$. In the next chapter we will prove that the estimate in $o(|x| \sec^{n-1} \theta)$ is indeed the best possible.

Chapter 3

Modified kernels

3.1 Kernels for the half space

For $\lambda > 0$ ($\lambda \in \mathbb{R}$) and $y' \in \mathbb{R}^{n-1}$ define the kernel

$$K(\lambda, x, y') = \left[|y' - y|^2 + x_n^2\right]^{-\lambda}. \tag{3.1}$$

The Poisson integrals for the half space problem $\Delta u = 0$ ($x \in \Pi_+$) with Dirichlet and Neumann data $f \colon \mathbb{R}^{n-1} \to \mathbb{R}$ on $\partial\Pi_+$ are, respectively,

$$D[f](x) = \alpha_n x_n \int\limits_{\mathbb{R}^{n-1}} f(y') K\left(\tfrac{n}{2}, x, y'\right)\, dy' \quad (n \geq 2) \tag{3.2}$$

and

$$N[f](x) = \frac{\alpha_n}{n-2} \int\limits_{\mathbb{R}^{n-1}} f(y') K\left(\tfrac{n-2}{2}, x, y'\right)\, dy' \quad (n \geq 3). \tag{3.3}$$

Here $\alpha_n = 2/(n\omega_n)$ and $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$ is the volume of the unit n-ball . When $n = 2$, N has a logarithmic kernel. This case is dealt with in Chapter 5.

The functions defined by (3.2) and (3.3) will be harmonic in Π_+ if

$$\int\limits_{\mathbb{R}^{n-1}} \frac{|f(y')|\, dy'}{|y'|^{2\lambda} + 1} < \infty \tag{3.4}$$

with $\lambda = n/2$ and $(n-2)/2$, respectively ([23], Theorem 6). If f is continuous then convergence of the appropriate integral in (3.4) is sufficient for (3.2) or (3.3) to be a classical solution of the respective Dirichlet or Neumann problem on Π_+ (cf. Corollary 3.3.1, 3.3.2). Notice that since Π_+ is unbounded the integral of f over $\mathbf{R}^{n-1}$ need not vanish for $N[f]$ to be a solution of the Neumann problem. If Ω is a bounded region with C^1 boundary and u is harmonic in Ω with normal derivative equal to f on $\partial\Omega$ then the Divergence Theorem gives $\int_{\partial\Omega} f = 0$. See, for example, [27]. However, the Divergence Theorem does not apply to the unbounded region $\mathbf{R}^{n-1}$.

When the integral in (3.4) diverges but

$$\int_{\mathbf{R}^{n-1}} \frac{|f(y')|\, dy'}{|y'|^{M+2\lambda}+1} < \infty \tag{3.5}$$

for a positive integer M we can use the modified kernel

$$K_M(\lambda, x, y') = K(\lambda, x, y') - \sum_{m=0}^{M-1} \frac{|x|^m}{|y'|^{m+2\lambda}} C_m^\lambda(\sin\theta\cos\theta') \tag{3.6}$$

(defined for $|y'| > 0$) where $0 \le \theta' \le \pi$ is the angle between y and y', i.e., $y \cdot y' = |y'|\,|x|\sin\theta\cos\theta'$ and $K_0 = K$. If $y = 0$ or $y' = 0$ we take $\theta' = \pi/2$.

In (3.6) the first M terms of the asymptotic expansion of K in inverse powers of $|y'|$ are removed. The coefficients are in terms of Gegenbauer polynomials, C_m^λ, most of whose properties used herein are derived in [63].

When $n = 2$, we take $\theta' = 0$ or π according as y' and x_1 are on the same or opposite side of the origin. Equivalently, $\cos\theta' = \mathrm{sgn}(x_1 y')$. Or, we may write $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, where $r = |x|$ and $0 < \phi < \pi$ for $x \in \Pi$. And then, using $C_m^1(\cos\phi) = \sin[(m+1)\phi]\csc\phi$,

$$K_M(1, x_1, x_2, \xi) = \left[(\xi - x_1)^2 + x_2^2\right]^{-1} - \sum_{m=1}^{M} \frac{r^{m-1}\sin(m\phi)}{\xi^{m+1}\sin\phi} \tag{3.7}$$

for the modified Dirichlet kernel. This formula follows by taking the imaginary part of the geometric series $\sum_{m=0}^{\infty} s^m e^{im\phi}$ $(s, \phi \in \mathbf{R})$.

When $\lambda = 1/2$, as for $n = 3$ in the Neumann case, the expansion is in Legendre polynomials via $C_m^{1/2}(t) = P_m(t)$.

Let $w\colon \mathbf{R}^{n-1} \to [0,1]$ be continuous so that $w(y) \equiv 0$ for $0 \le |y| \le 1$ and $w(y) \equiv 1$ for $|y| \ge 2$. Define modified Dirichlet and Neumann integrals

$$D_M[f](x) = \alpha_n x_n \int\limits_{\mathbf{R}^{n-1}} f(y')K_M\left(\tfrac{n}{2}, x, y'\right)\, dy' \quad (n \ge 2) \tag{3.8}$$

$$N_M[f](x) = \frac{\alpha_n}{n-2} \int\limits_{\mathbf{R}^{n-1}} f(y')K_M\left(\tfrac{n-2}{2}, x, y'\right)\, dy' \quad (n \ge 3). \tag{3.9}$$

Then $u(x) = D_M[wf](x) + D[(1-w)f](x)$ and $v(x) = N_M[wf](x) + N[(1-w)f](x)$ are respective solutions of the classical half space Dirichlet and Neumann problems. The Dirichlet version of K_M appears in [5], [61] and [67], with inspiration from [22]. The Neumann version is discussed by Gardiner ([25]) and Armitage ([6]). Modified kernels with radial data are considered in [52]. When $n = 2$, we write

$$D_M[f](x_1, x_2) = \frac{x_2}{\pi} \int\limits_{\xi=-\infty}^{\infty} f(\xi)\, K_M\left(\tfrac{n}{2}, x_1, x_2, \xi\right) d\xi. \tag{3.10}$$

In this chapter we give growth estimates for u and v under (3.5) and prove they are sharp. This is done in Theorem 3.3.1 by first defining

$$F_{\lambda,M}[f](x) = \int\limits_{|y'|>1} f(y')K_M(\lambda, x, y')\, dy' \tag{3.11}$$

and proving that

$$F_{\lambda,M}[f](x) = o(|x|^M \sec^{2\lambda}\theta) \quad \text{as} \quad |x| \to \infty \quad \text{with} \quad x \in \Pi_+. \tag{3.12}$$

The order relation is interpreted as $\mu(r)/r \to 0$ as $r \to \infty$ where $\mu(r)$ is the supremum of $|F_{\lambda,M}[f](x)| \cos^{2\lambda}\theta$ over $x \in \Pi_+$, $|x| = r$. A growth condition ω is said to be sharp if given any function $\psi = o(\omega)$ and any sequence $\{x_i\} \in \Pi_+$ with $|x_i| \to \infty$, we can find data f so that the solution corresponding to f is not little *oh* of ψ on this sequence (see Definition 3.3.1 below). The sharpness proof is complicated by the fact that the modified kernels are not of one sign. For each x_i, regions in $\mathbf{R}^{n-1}$ are determined where the kernel is of one sign. Data is then chosen so that the contribution from integrating where the sign of the kernel is not known is cancelled out and the main contribution comes from integrating over a neighbourhood of the singularity of the kernel. This proof makes up a substantial portion of the thesis. Note that condition (3.5) is necessary and sufficient for $F_{\lambda,M}[f](x)$ to exist as a Lebesgue integral on Π_+. See Proposition 3.4.1 below.

3.2 First type of modified kernel

The expansion (3.6) arises from the generating function for Gegenbauer polynomials ([63], 4.7.23)

$$(1-2tz+z^2)^{-\lambda} = \sum_{m=0}^{\infty} z^m C_m^\lambda(t), \quad \lambda > 0, \tag{3.13}$$

where $C_m^\lambda(t) = m! \frac{\partial^m}{\partial z^m}(1-2tz+z^2)^{-\lambda}\Big|_{z=0}$. If $-1 \le t \le 1$ the series converges absolutely for $|z| < 1$ (the left side of (3.13) is singular at $z = t \pm i\sqrt{1-t^2}$). The majorisation and derivative formulas

$$|C_m^\lambda(t)| \le C_m^\lambda(1) = \binom{2\lambda+m-1}{m} = \frac{\Gamma(2\lambda+m)}{\Gamma(2\lambda)\,\Gamma(m+1)} \tag{3.14}$$

$$\frac{d}{dt}C_m^\lambda(t) = 2\lambda\, C_{m-1}^{\lambda+1}(t) \tag{3.15}$$

are proved in [63] (4.7.3, 7.33.1, 4.7.14). Hence, the series in (3.13) converges if $|z| < 1$, uniformly for $-1 \le t \le 1$ and the same can be said for all of its derivatives with respect to z and t. From the definition above and Faà di Bruno's formula for the m^{th} derivative of a composite function ([1], p. 823) it can be seen that $C_m^\lambda(t)$ is a polynomial in t of degree m. And,

$$C_0^\lambda(t) = 1, \quad C_1^\lambda(t) = 2\lambda t. \tag{3.16}$$

A proof of the following lemma is hinted at in [5], [25] and [67] by reference to a more general result on axial polynomials in [44] (Theorem 2). However, we give a simple direct proof.

Lemma 3.2.1 *For $m = 0, 1, 2, 3, \ldots$ the functions $h^{(0)}_{m+1}(x) = x_n|x|^m\, C^{n/2}_m(\Theta)(n \geq 2)$ and $h^{(1)}_m(x) = |x|^m C^{(n-2)/2}_m(\Theta)$ $(n \geq 3)$ are homogeneous harmonic polynomials of degree $m+1$ and m, respectively, where $\Theta = \sin\theta\cos\theta'$.*

Proof. Using (3.13) we obtain the expansion of the fundamental solution of Laplace's equation

$$|x - x'|^{2-n} = \sum_{m=0}^{\infty} \frac{|x|^m}{|x'|^{m+n-2}} C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}'), \quad n \geq 3. \tag{3.17}$$

If $x' \neq 0$ this series converges for $|x| < |x'|$ and defines a harmonic function. Each term is homogeneous in x of degree m and it is clear from (3.16) that the first two terms are harmonic. Given x, take x' such that $|x'| > |x|$. Differentiating termwise in x gives

$$\Delta|x - x'|^{2-n} = 0 = \sum_{m=2}^{\infty} |x'|^{-(m+n-2)} \Delta\left(|x|^m C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}')\right). \tag{3.18}$$

Each term $\Delta\left(|x|^m C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}')\right)$ is homogeneous of degree $m-2$, hence, by the linear independence of homogeneous functions, $|x|^m C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}')$ is harmonic on $\mathbb{R}^n$ for each $m \geq 0$. Every harmonic function can be uniquely written as a sum of homogeneous harmonic polynomials so $|x|^m\, C^{(n-2)/2}_m(\hat{x}\cdot\hat{x}')$ is a homogeneous harmonic polynomial of degree m ([7], 1.26, 1.27).

Now set $x'_n = 0$, then $\hat{x}\cdot\hat{x}' = y\cdot y'/(|x|\,|y'|) = \sin\theta\ \hat{y}\cdot\hat{y}' = \Theta$. Hence, the Neumann half space expansion is

$$\left[|y' - y|^2 + x_n^2\right]^{-(n-2)/2} = \sum_{m=0}^{\infty} \frac{|x|^m}{|y'|^{m+n-2}} C^{(n-2)/2}_m(\Theta), \quad n \geq 3, \tag{3.19}$$

and each term in the series is a homogeneous harmonic polynomial of degree m.

For the Dirichlet expansion differentiate (3.17) with respect to x'_n, use (3.15) and (3.16), and set $x'_n = 0$. Then for $n \geq 3$

$$x_n\left[|y'-y|^2 + x_n^2\right]^{-n/2} = \sum_{m=0}^{\infty} \frac{x_n|x|^m}{|y'|^{m+n}} C_m^{n/2}(\Theta), \tag{3.20}$$

each term in the series is a homogeneous harmonic polynomial of degree $m+1$.

When $n = 2$, use $C_m^1(\cos\phi) = \sin[(m+1)\phi]\csc\phi$. Then from (3.17) we recover the trigonometric expansion

$$\frac{x_2}{(\xi - x_1)^2 + x_2^2} = \sum_{m=1}^{\infty} \frac{r^m \sin(m\phi)}{\xi^{m+1}}, \quad r < |\xi|, \tag{3.21}$$

where we have written $r = |x|$, and $\phi = \pi/2 - \theta$ to conform with the usual polar coordinates ($x_1 = r\cos\phi$, $x_2 = r\sin\phi$). Each $r^m \sin(m\phi)$ is a homogeneous harmonic polynomial of degree m. ■

Remark 3.2.1 When $x_n = 0$, $h_m^{(0)}$ and $\partial h_m^{(1)}/\partial x_n$ vanish. The spherical harmonics of degree m are the restriction of the homogeneous harmonic polynomials to the unit sphere. If we write $Y_m^{(0)}(\hat{x}) = h_m^{(0)}(\hat{x})$ and $Y_m^{(1)}(\hat{x}) = h_m^{(1)}(\hat{x})$ then $h_m^{(0)}(x) = |x|^m Y_m^{(0)}(\hat{x})$ and $h_m^{(1)}(x) = |x|^m Y_m^{(1)}(\hat{x})$. The functions $|x|^{-(m+n-2)} Y_m^{(0)}(\hat{x})$ and $|x|^{-(m+n-2)} Y_m^{(1)}(\hat{x})$ are harmonic for $|x| > 0$ (interchange x and x' in (3.17) and (3.18)).

3.3 A sharp growth estimate

In equation (2.8) of the proof of Theorem 2.2.1, the Poisson kernel was factored to obtain a growth estimate for $u(x) = D[f](x)$. This will be done again to derive a growth estimate for $F_{\lambda,M}$. As well, we will prove the estimate is sharp, i.e., the best possible under (3.5). We will use a lemma and the following definition.

Definition 3.3.1 *Let $\omega : \Pi_+ \to (0,\infty)$ then ω is a sharp growth condition for $F_{\lambda,M}$ if* *(i)* $F_{\lambda,M}[f](x) = o(\omega(x))$ $(x \in \Pi_+,\ |x| \to \infty)$, *for all* f *satisfying* (3.5)

(ii) *If* $\psi\colon \Pi_+ \to (0,\infty)$ *and* $\psi(x) = o(\omega(x))$ *then for any sequence* $\{x^{(i)}\}$ *in* Π_+ *such that* $|x^{(i)}| \to \infty$ *as* $i \to \infty$ *there exists a continuous function* f *satisfying* (3.5) *with* $\lim_{i\to\infty} F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \neq 0$.

Note that it is essential that the limit condition on $F_{\lambda,M}[f]/\psi$ be checked on all paths to infinity. For example, $\omega_1(x) = |x|$ and $\omega_2(x) = |x| \sec\theta$ agree on all radial paths but allow very different behaviour on paths approaching $\partial\Pi_+$.

Let

$$\Phi_\pm(\Theta,\zeta) = MC_M^\lambda(\Theta) \pm (2\lambda + M - 1)C_{M-1}^\lambda(\Theta)\,\zeta \quad \text{and} \quad \Theta = \sin\theta\cos\theta'. \tag{3.22}$$

Lemma 3.3.1 *For* $M \geq 1$, *the modified kernel has the integral representation*

$$K_M(\lambda, x, y') = K(\lambda, x, y') \int_{\zeta=0}^{|x|/|y'|} (1 - 2\Theta\zeta + \zeta^2)^{\lambda-1}\Phi_-(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta. \tag{3.23}$$

Proof. The Gegenbauer polynomials satisfy the recurrence relation

$$(m+2)\,C_{m+2}^\lambda(t) - 2(\lambda + m + 1)\,t\,C_{m+1}^\lambda(t) + (2\lambda + m)\,C_m^\lambda(t) = 0. \tag{3.24}$$

Let $S_{M-1}(z) = \sum_{m=0}^{M-1} z^m C_m^\lambda(t)$. Following the method in [63] used to derive (3.13), multiply (3.24) by z^{m+1} and sum from $m = 0$ to $m = M - 1$:

$$\begin{aligned} 0 &= \sum_{m=2}^{M+1} m z^{m-1} C_m^\lambda(t) - 2\lambda t \sum_{m=1}^{M} z^m C_m^\lambda(t) - 2tz \sum_{m=1}^{M} m z^{m-1} C_m^\lambda(t) \\ &\quad + 2\lambda z \sum_{m=0}^{M-1} z^m C_m^\lambda(t) + z^2 \sum_{m=1}^{M-1} m z^{m-1} C_m^\lambda(t) \\ &= \left(1 - 2tz + z^2\right) S'_{M-1}(z) - 2\lambda(t - z)S_{M-1}(z) + Mz^{M-1}C_M^\lambda(t) \\ &\quad - (2\lambda + M - 1)\,z^M C_{M-1}^\lambda(t), \end{aligned} \tag{3.25}$$

using (3.24), $C_0^\lambda(t) = 1$ and $C_1^\lambda(t) = 2\lambda t$. Solving the above differential equation with the initial condition $S_{M-1}(0) = 1$, we obtain

$$S_{M-1}(z) = (1-2tz+z^2)^{-\lambda}\left\{\int_{\zeta=0}^{z}(1-2t\zeta+\zeta^2)^{\lambda-1}\left[(2\lambda+M-1)\zeta^M C_{M-1}^\lambda(t) - M\zeta^{M-1}C_M^\lambda(t)\right]d\zeta+1\right\}. \tag{3.26}$$

Letting $z = |x|/|y'|$ and $t = \sin\theta\cos\theta'$, (3.22) gives (3.23). ■

When λ is an integer the integrand in (3.23) is a polynomial and the integral can be evaluated (without integrating by parts M times!) to a polynomial in $|x|/|y'|$. This polynomial is of degree $2\lambda + M - 1$ with no terms of degree less than M. The coefficients are functions of Θ. And, when $n = 2$ and $\lambda = 1$, (3.23) is the modified Dirichlet kernel and is written

$$\begin{aligned}\mathcal{D}_M(r,\phi,\xi) &= \frac{1}{\xi^2-2\xi r\cos\phi+r^2}\int_{\zeta=0}^{r/\xi}\frac{(M\sin[(M+1)\phi]-(M+1)\sin(M\phi)\zeta)\,\zeta^{M-1}\,d\zeta}{\sin\phi}\\ &= \frac{\sin[(M+1)\phi]\,(r/\xi)^M-\sin(M\phi)\,(r/\xi)^{M+1}}{(\xi^2-2\xi r\cos\phi+r^2)\sin\phi} \qquad (3.27)\\ &= \frac{r^M(\xi\sin[(M+1)\phi]-r\sin(M\phi))}{\xi^{M+1}(\xi^2-2\xi r\cos\phi+r^2)\sin\phi}. \qquad (3.28)\end{aligned}$$

This appears in [60].

Use of (3.23) allows us to prove

Theorem 3.3.1 *Let* $\lambda > 0$ *and* f *be measurable so that* (3.5) *holds for integer* $M \geq 0$. *Then* $F_{\lambda,M}[f](x) = o\left(|x|^M \sec^{2\lambda}\theta\right)$ $(x \in \Pi^+, |x| \to \infty)$ *and the order relation is sharp in the above sense.*

The proof is based on the idea of using "spikey" data as in the example at the end of the previous chapter. Now, however, things are more complicated since the modified kernel is not of one sign as was the original Poisson kernel. Regions must be found where the modified kernel is of one sign. The proof is quite long but has been broken down into digestible pieces as detailed below. Of crucial importance is the integral form of the modified kernel, given in Lemma 3.3.1.

Step I It is shown that $F_{\lambda,M}[f] = o(|x|^M \sec^{2\lambda}\theta)$ for any measurable function f satisfying the integrability condition (3.5). In (3.23), the original kernel K is estimated as was done in Theorem 2.2.1 ($\alpha = n/2$). The function $\Phi(\Theta,\zeta)$ has a simple zero precisely where $1-2\theta\zeta+\zeta^2$ vanishes, at $\Theta = \zeta = 1$. So the ratio $\Phi(\Theta,\zeta)/\sqrt{1-2\theta\zeta+\zeta^2}$ is bounded and the integrand in (3.23) is continuous for $\lambda \geq 1/2$ and unbounded but integrable when $0 < \lambda < 1/2$. In either case, elementary approximations lead to an upper bound for $|K_M|$ on which the Dominated Convergence Theorem can be used to prove (3.12).

Step II The order estimate $o(|x|^M \sec^{2\lambda}\theta)$ is now proven to be sharp, first for given sequences which have a subsequence $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ that stays bounded away from the $\hat{e}_1$ axis of $\partial\Pi_+$ by an angle θ_0 ($0 \leq \theta_0 < \pi/2$). On such a sequence the growth condition reduces to $o(|x|^M)$. A region $\Omega_1 \subset \mathbb{R}^{n-1}$ is found on which Φ_- and hence K_M are of one sign. Due to the parity of C_m^λ about zero (C_m^λ is even if m is even and odd if m is odd) it follows that $\Phi_-(\Theta,\zeta)$ will be of one sign if $|\Theta|$ is small enough. Since $\Theta = \sin\theta\,\cos\theta'$, this is accomplished by restricting θ' to lie near $\pi/2$. And, Ω_1 is taken as the region between two cones, both of which have an opening angle of nearly $\pi/2$ from the $\hat{e}_1$ axis. For $y' \in \Omega_1$, the combination $(-1)^{\lceil M/2\rceil}\Phi_-(\Theta,\zeta)$ is strictly positive when $\zeta > 0$. (When $x \in \mathbb{R}$, the *ceiling* of x, $\lceil x\rceil$, is x if $x \in \mathbb{Z}$ and is the next largest integer if $x \notin \mathbb{Z}$.) A lower bound on $(-1)^{\lceil M/2\rceil}K_M$ is now obtained, equation (3.43). Data is then chosen that has support in Ω_1 and is large on a sequence of unit half balls along the $\hat{e}_2$ axis. (This is an axis orthogonal to $\hat{e}_1$. Something slightly different is done when $n = 2$.) Sharpness of the growth estimate for this special type of sequence now follows from an argument similar to that in the example at the end of Chapter 2.

Step III Now considered are sequences with a subsequence $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ that approaches the boundary at the $\hat{e}_1$ axis. Again, a region is found where Φ_- is of one sign. On the sequence, we have $\sin\theta \to 1$ so taking θ' near 0 makes Θ nearly equal to 1. In this case then, the kernel $K(\lambda, x, y')$ will be singular for $|y'| = |x|$ and $\Theta \to 1$. Hence, in (3.6) it will dominate the Gegenbauer terms subtracted from it. A region $\Omega_2 \subset \mathbb{R}^{n-1}$ is defined to be the portion of a cone with $|y'| > 1$ and axis along $\hat{e}_1$. This region is shown in Figure 3.1, drawn for $n = 3$ (so that y and y' are in $\mathbb{R}^2$). The opening angle θ' is taken small enough so that when $y_1' > 0$ and $|y'|$ is near $|x|$, $|x|/A < |y'| < A|x|$ for a constant $A > 1$, we have $K_M > 0$, i.e., near the singularity of K. The modified kernel is also positive for large values of $|y'|$ in Ω_2 but changes sign when $y_1' > 0$ and $1 < |y'| < |x|/A$ (the region $\Omega_>$ in Figure 3.1). And, due to the parity of C_m^λ, the modified kernel is one sign when $y' \in \Omega_2$ with $y_1' < 0$. Data is chosen to have support within Ω_2 on a sequence of balls along the $\hat{e}_1$ axis. When y_1' is positive, $f(y')$ is positive and when y_1' is negative, fK_M is positive. Contributions to $\int_{|y'|>1} f(y')K_M(\lambda, x, y')\,dy'$ are now known to be positive except when integrating over $\Omega_>$. But f is chosen so that if the reflection of y' across the $y_1' = 0$ hyperplane is denoted y^*, then if $y_1' > 0$ we have $f(y^*) = (-1)^M A_\lambda f(y')$, where $A_\lambda > 1$ is a constant. The data is given a "super odd" or "super even" extension from $y_1' > 0$ to $y_1' < 0$, according as M is even or odd. This allows the contribution from integrating over $\Omega_>$, where fK_M is not of one sign, to be balanced out by the contribution from integrating over the reflection of $\Omega_>$ to $y_1' < 0$, where fK_M is positive. The contribution to $\int_{|y'|>1} f(y')K_M(\lambda, x, y')\,dy'$ from integrating near the singularity of K_M, i.e., over Ω_2, produces a lower bound for $F_{\lambda,M}[f]$ from which it follows that $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \not\to 0$, where ψ and $x^{(i)}$ are given in the theorem. Note that all the Ω regions defined here depend on $|x|$.

Step IV The special case of sequences $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ considered in II and III is shown to be applicable to general sequences in Π_+. Since ∂B_+ is compact, for any sequence $r_i\hat{r}_i$ in Π_+, the sequence $\{\hat{r}_i\}$ has a limit point $\hat{r}_0 \in \partial\overline{B}_+$. This direction is then rotated to correspond to $\hat{y}_1$.

Proof. Write $s = |x|/|y'|$. Throughout the proof $d_1, d_2, \ldots, d_9$ will be positive constants (depending on λ and M).

Step I First suppose $M \geq 1$.

In [63] (4.7.27) for $M \geq 2$ we have

$$MC_M^\lambda(t) = (2\lambda + M - 1)\,t\,C_{M-1}^\lambda(t) - 2\lambda\,(1-t^2)\,C_{M-2}^{\lambda+1}(t). \tag{3.29}$$

With reference to (3.22) and (3.14) we can write

$$\begin{aligned} \frac{|\Phi_-(\Theta,\zeta)|}{\sqrt{1-2\Theta\zeta+\zeta^2}} &= \frac{\left|(2\lambda+M-1)(\Theta-\zeta)C_{M-1}^\lambda(\Theta) - 2\lambda(1-\Theta^2)C_{M-2}^{\lambda+1}(\Theta)\right|}{\sqrt{(\Theta-\zeta)^2+(1-\Theta^2)}} && (3.30)\\ &\leq (2\lambda+M-1)\binom{2\lambda+M-2}{M-1} + 2\lambda\binom{2\lambda+M-1}{M-2} \\ &= 2\lambda\binom{2\lambda+M}{M-1} && (3.31) \end{aligned}$$

for $M \geq 2$. If we define $C_{-m}^\lambda = 0$ for $m = 1, 2, 3, \ldots$ and use the fact that $C_0^\lambda(\Theta) = 1$ and $C_1^\lambda(\Theta) = 2\lambda\Theta$ then (3.29) and (3.31) still hold when $M = 1$. Hence, (3.23) and (3.31) give

$$|K_M(\lambda, x, y')| \leq d_1\, K(\lambda, x, y')\, s^{M-1} \int_{\zeta=0}^{s} (1 - 2\Theta\zeta + \zeta^2)^{\lambda - \frac{1}{2}}\, d\zeta. \tag{3.32}$$

For $M \geq 0$ and $\lambda \geq \frac{1}{2}$ the integrand in (3.32) is continuous and $|\Theta| \leq 1$ so $(1 - 2\Theta\zeta + \zeta^2) \leq (1+s)^2$. Therefore,

$$|K_M(\lambda, x, y')| \leq d_1\, K(\lambda, x, y')\, s^M\, (1+s)^{2\lambda-1}. \tag{3.33}$$

The estimate

$$|K(\lambda, x, y')| \leq 2^{2\lambda} \sec^{2\lambda}\theta\, (|x| + |y'|)^{-2\lambda} \tag{3.34}$$

is in (2.8) of the proof of Theorem 2.2.1. Hence,

$$|K_M(\lambda, x, y')| \le d_2\, s^M \sec^{2\lambda}\theta\, |y'|^{-2\lambda}(1+s)^{-1}. \tag{3.35}$$

$$\le d_2\, s^M \sec^{2\lambda}\theta\, |y'|^{-2\lambda}. \tag{3.36}$$

Multiply (3.35) by $|f(y')|$ and integrate $y' \in \mathbf{R}^{n-1}$, $|y'| > 1$. Letting $|x| \to \infty$, the Dominated Convergence Theorem gives (3.12)

When $0 < \lambda < \frac{1}{2}$ the integrand in (3.32) can be singular. In this case

$$\int_{\zeta=0}^{s} (1 - 2\Theta\zeta + \zeta^2)^{\lambda-\frac{1}{2}} d\zeta \le \int_{\zeta=0}^{s} |1-\zeta|^{2\lambda-1}\, d\zeta$$

$$= \frac{1}{2\lambda} \begin{cases} 1-(1-s)^{2\lambda}, & 0 \le s \le 1 \\ 1+(s-1)^{2\lambda}, & s \ge 1 \end{cases}$$

$$\le \left(\frac{1}{\lambda}\right) \min(s, s^{2\lambda}). \tag{3.37}$$

And,

$$|K_M(\lambda, x, y')| \le d_3\, s^M \sec^{2\lambda}\theta\, (|x| + |y'|)^{-2\lambda} \tag{3.38}$$

so (3.12) holds for $0 < \lambda < \frac{1}{2}$ as well.

Step II We now prove sharpness. Given any sequence $\{x^{(i)}\}$ in Π_+ with $|x^{(i)}| \to \infty$ and any function $\psi(x) = o\left(|x|^M \sec^{2\lambda}\theta\right)$ we find a continuous function f satisfying (3.5) for which $\lim_{i\to\infty} F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \ne 0$.

Note that (3.23) may be written

$$K_M(\lambda, x, y') = K(\lambda, x, y')\, s^M \int_{\zeta=0}^{1} (1 - 2\Theta s\zeta + s^2\zeta^2)^{\lambda-1}\, \Phi_-(\Theta, s\zeta)\, \zeta^{M-1}\, d\zeta. \tag{3.39}$$

Suppose first that $\{x^{(i)}\}$ has a subsequence $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$, $i \ge 1$, where $b_i > 0$ and $0 \le a_i \le b_i \tan\theta_0$ for some $0 \le \theta_0 < \pi/2$. Then $0 \le \sin\theta = a_i/\sqrt{a_i^2 + b_i^2} \le$

$\sin\theta_0 < 1$. Since $\psi(x) = o(|x|^M \sec^{2\lambda}\theta)$ and $1 \le \sec\theta = \sqrt{a_i^2 + b_i^2}/b_i \le \sec\theta_0 < \infty$ we also have $\psi(x) = o(|x|^M)$. We may assume that $\tilde{x}^{(i)}$ have been chosen so that $\psi(\tilde{x}^{(i)}) \le |\tilde{x}^{(i)}|^M / i^2$, $i \ge 1$.

Now find a region $\Omega_1 \subset \mathbb{R}^{n-1}$ in which $\Phi_-(\Theta, s\zeta)$ is of one sign. Consider $n \ge 3$ and $M \ge 1$. Let β_1 be the smallest positive root of $\{C_M^\lambda, C_{M-1}^\lambda\}$. And, $C_m^\lambda(\Theta)$ is a polynomial in Θ of degree m with m simple zeroes in $(-1, 1)$. If $M = 1$, take $\beta_1 = 1$. So $0 < \beta_1 \le 1$. Now, C_m^λ is even or odd about the origin according as m is even or odd ([63], 4.7.4) and $(-1)^m C_{2m}^\lambda(0) > 0$ ([20], 10.9.19). Therefore, for any $0 \le \theta \le \pi/2$, $C_M^\lambda(\sin\theta\cos\theta')$ and $C_{M-1}^\lambda(\sin\theta\cos\theta')$ are each of one sign for $\arccos(\beta_1) \le \theta' \le \pi/2$ or $\pi/2 \le \theta' \le \pi - \arccos(\beta_1)$. Write $M = 2\mu + \varepsilon_0$ where ε_0 is 0 or 1. From (3.15) we see that if $0 < t < \beta_1$ then $\mathrm{sgn}(\mathrm{C}_{2\mu+1}^\lambda(\mathrm{t})) = \mathrm{sgn}(\mathrm{C}_{2\mu}^{\lambda+1}(\mathrm{t})) = (-1)^\mu$ and if $-\beta_1 < t < 0$ then $\mathrm{sgn}(\mathrm{C}_{2\mu+1}^\lambda(\mathrm{t})) = -\mathrm{sgn}(\mathrm{C}_{2\mu}^{\lambda+1}(\mathrm{t})) = (-1)^{\mu+1}$. Let

$$\begin{aligned}\Omega_1(\hat{y}) \quad = \quad & \Big\{ y' \in \mathbb{R}^{n-1} \Big| \arccos(\beta_1/2) \le \theta' \le \arccos(\beta_1/3) \quad \text{if } M \text{ is even} \\ & \text{and} \quad \arccos(\beta_1/3) \le \theta' \le \pi - \arccos(\beta_1/2) \quad \text{if } M \text{ is odd} \Big\}. \end{aligned} \tag{3.40}$$

Then, since C_M^λ and C_{M-1}^λ have no common roots, there exists a positive constant d_4 such that

$$(-1)^{\mu+\varepsilon_0} \Phi_-(\Theta, s\zeta) \ge d_4, \tag{3.41}$$

whenever $0 \le \theta \le \pi/2$, $y' \in \Omega_1(\hat{y})$, $0 \le \zeta \le 1$, $s \ge 0$. In (3.40), θ' is restricted to lie in a smaller region than $\arccos\beta_1 \le \theta' \le \pi/2$ so that $(-1)^{\mu+\varepsilon_0}\Phi_-$ will be strictly positive for $y' \in \Omega_1$.

From (3.39) we will need the estimate,

$$\begin{aligned}(1 - 2\Theta s\zeta + s^2\zeta^2)^{\lambda-1} \quad & \ge \quad \begin{cases} (1+s)^{2(\lambda-1)}, & 0 \le \lambda \le 1 \\ ((s\zeta - \sin\theta_0)^2 + \cos^2\theta_0)^{\lambda-1}, & \lambda \ge 1 \end{cases} \\ & \ge \quad (1+s)^{-2}\cos^{2|\lambda-1|}\theta_0. \end{aligned} \tag{3.42}$$

These give

$$(-1)^{\mu+\varepsilon_0} K_M(\lambda, x, y') \ge \frac{d_5\, K(\lambda, x, y')\, s^M}{(1+s)^2}, \tag{3.43}$$

whenever $y' \in \Omega_1$.

If $M = 0$ then (3.42) and (3.43) hold and we can take $\Omega_1 = \mathbf{R}^{n-1}$.

Let

$$f(y') = \begin{cases} (-1)^{\mu+e_0} f_i \Big[1 - |y' - c_i\hat{e}_2|\Big] |y'_1|; & y' \in B_1(c_i\hat{e}_2), (-1)^M y'_1 \geq 0 \\ 0, & \text{otherwise}, \end{cases} \tag{3.44}$$

where $c_i := |\tilde{x}^{(i)}| = \sqrt{a_i^2 + b_i^2}$ and the constants f_i are defined in (3.46) below. Then $f : \mathbf{R}^{n-1} \to \mathbf{R}$, has support in a sequence of half balls along the $\hat{e}_2$ axis and is continuous. The factor $[1 - |y' - c_i\hat{e}_2|]|y'_1|$ makes f vanish on the perimeter of the i^{th} half ball. Without loss of generality we may assume $c_i \to \infty$ monotonically so that the $B_1(c_i\hat{e}_2)$ are disjoint, $\text{supp}(f) \subset \Omega_1$ and $c_i \geq 2$ (otherwise, take an appropriate subsequence of $\{\tilde{x}^{(i)}\}$).

Now, for any $j \geq 1$,

$$F_{\lambda,M}[f](\tilde{x}^{(j)}) \geq \int\limits_{B_1(c_j\hat{e}_2)} f(y') K_M(\lambda, \tilde{x}^{(j)}, y')\, dy'.$$

When $y' \in B_1(c_j\hat{e}_2)$ we have $s = |\tilde{x}^{(j)}| / |y'| \leq c_j/(c_j - 1) \leq 2$ and
$s \geq c_j/(c_j + 1) \geq 2/3$. And,

$$\begin{aligned} K(\lambda, \tilde{x}^{(j)}, y') &\geq \Big[(|y'| + a_j)^2 + b_j^2\Big]^{-\lambda} \\ &\geq \Big[(c_j + 1 + a_j)^2 + b_j^2\Big]^{-\lambda} \\ &\geq (7c_j^2)^{-\lambda}. \end{aligned}$$

Thus, using (3.43),

$$\begin{aligned} F_{\lambda,M}[f](\tilde{x}^{(j)}) &\geq \frac{d_6\, f_j}{|\tilde{x}^{(j)}|^{2\lambda}} \int\limits_{B_1} (1 - |y'|)\, |y'_1|\, dy' \\ &= d_7^{-1}\, f_j\, |\tilde{x}^{(j)}|^{-2\lambda}. \end{aligned} \tag{3.45}$$

Let

$$f_i = d_7\, \psi(\tilde{x}^{(i)})\, |\tilde{x}^{(i)}|^{2\lambda}. \tag{3.46}$$

Then $f_i/c_i^{M+2\lambda} \leq (d_7 i^2)^{-1}$ and $\sum_{i=1}^{\infty} f_i c_i^{-(M+2\lambda)} < \infty$ so (3.5) holds. And, on $\{\tilde{x}^{(i)}\}$ we have $F_{\lambda,M}[f](\tilde{x}^{(j)}) \geq \psi(\tilde{x}^{(j)})$ for each $j \geq 1$ so $\limsup_{i\to\infty} F_{\lambda,M}[f](x^{(i)}/\psi(x^{(i)})) \geq 1$ and $\lim_{i\to\infty} F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \neq 0$.

When $n = 2$, write $x_1 = r\cos\phi$, $x_2 = r\sin\phi$. Then in place of (3.23) we have $F_{\lambda,M}[f](x) = \int_{-\infty}^{\infty} f(\xi)\, K_M(\lambda, x, \xi)\, d\xi$ where

$$K_M(\lambda,x,\xi) = K(\lambda,x,\xi)\left(\frac{r}{\xi}\right)^M \int\limits_{\zeta=0}^{1} \left(1 - 2\frac{r\zeta}{\xi}\cos\phi + \frac{r^2\zeta^2}{\xi^2}\right)^{\lambda-1} \Phi_-\left(\cos\phi, \frac{r\zeta}{\xi}\right)\zeta^{M-1}\, d\zeta.$$

If $0 \leq \theta \leq \theta_0 < \pi/2$ then $0 < \phi_0 \leq \phi \leq \pi - \phi_0 < \pi$ where $\phi_0 = \pi/2 - \theta_0$.

Let t_i, $1 \leq i \leq q$, be the roots of $C_M^\lambda \circ \cos$ and $C_{M-1}^\lambda \circ \cos$ in $[\phi_0, \pi - \phi_0]$, ordered by size. We then have the partition $\phi_0 = t_0 \leq t_1 < t_2 < \cdots < t_{q-1} < t_q \leq t_{q+1} = \pi - \phi_0$. In each interval $[t_i, t_{i+1}]$, $0 \leq i \leq q$, $C_M^\lambda \circ \cos$ and $C_{M-1}^\lambda \circ \cos$ are each of one sign. If ϕ_0 is a root, we omit the singleton $\{t_1\}$, similarly with $\pi - \phi_0$.

For any sequence $\phi_i \in [\phi_0, \pi - \phi_0]$, $i \geq 1$, there is a subsequence $\{\tilde{\phi}_i\}$ in one of the above intervals $[t_j, t_{j+1}]$. If $C_M^\lambda(\cos\tilde{\phi}_i)$ and $C_{M-1}^\lambda(\cos\tilde{\phi}_i)$ are of the same sign, take $\Omega_1 = \{\xi \in \mathbb{R} | \xi < 0\}$ and $\Omega_1 = \{\xi \in \mathbb{R} | \xi > 0\}$ if they are of opposite sign. Then $(-1)^{\mu_0}\Phi_-(\cos\tilde{\phi}_i, r\zeta/\xi) \geq 0$ for $i \geq 1$, $\xi \in \Omega_1$, where $(-1)^{\mu_0} = \mathrm{sgn}(C_M^\lambda(\cos\tilde{\phi}_i))$ ($\mu_0 = 0$ or 1). Since C_M^λ and C_{M-1}^λ have no common zeroes there is a subsequence $\{\check{\phi}_i\}$ of $\{\tilde{\phi}_i\}$ such that either $C_M^\lambda(\cos\check{\phi}_i)$ or $C_{M-1}^\lambda(\cos\check{\phi}_i)$ is bounded away from zero for all $i \geq 1$. Hence, there is a positive constant $\check{d}_5$ such that $(-1)^{\mu_0}\Phi_-(\cos\check{\phi}_i, \check{r}_i\zeta/\xi) \geq \check{d}_5(\check{r}_i\zeta/|\xi|)^{\mu_1}$ for $i \geq 1$. Here $\check{x}^{(i)} = \check{r}_i\cos\check{\phi}_i\,\hat{e}_1 + \check{r}_i\sin\check{\phi}_i\,\hat{e}_2$ is a sub-subsequence of the given sequence $\{x^{(i)}\}$ and μ_1 is 0 or 1. We now proceed in a similar manner to the case $n \geq 3$ given above.

Step III In the previous argument $0 \leq \theta_0 < \pi/2$ was arbitrary so now suppose that given the sequence $\{x^{(i)}\}$ there is a subsequence $\tilde{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ such that $\sin\theta_0 \leq \sin\theta = a_i/\sqrt{a_i^2 + b_i^2} < 1$. Since $0 < b_i \leq a_i\cot\theta_0$ we may assume $0 < b_i \leq a_i/2$ and that $a_i \to \infty$ monotonically.

Find a region $\Omega_2 \subset \mathbb{R}^{n-1}$ on which K_M is of one sign. Let $M \geq 1$ and let $x = x_1\hat{e}_1 + x_n\hat{e}_n \in \{\tilde{x}^{(i)}\}$. Let

$$\frac{1}{A} \leq s \leq A, \quad \frac{1}{A} \leq \Theta \leq 1 \tag{3.47}$$

and take $1 < A < 2$ close enough to 1 so that $K_M(\lambda, x, y')$, $C_M^\lambda(\Theta)$ and $C_{M-1}^\lambda(\Theta)$ are positive. From (3.1) and (3.6), $K_M(\lambda, x, y') \geq |y'|^{-2\lambda}\Big[(1-2A^{-2}+A^2)^{-\lambda} - \gamma_{\lambda,M}^{-\lambda}\Big]$, where $\gamma_{\lambda,M} = \left(\sum_{m=0}^{M-1} 2^m C_m^\lambda(1)\right)^{-1/\lambda}$ $(0 < \gamma_{\lambda,M} < \infty)$. Note that $A > 1$ implies $1-2A^{-2}+A^2 > 0$. Now, $K_M > 0$ if $A^4+(1-\gamma_{\lambda,M})A^2-2 < 0$. Let $r_0 > 1$ be the largest root of this quartic. Let β_2 be the largest zero of $C_M^{\min(1,\lambda)}$. Then $\cos(\pi/(M+1)) \leq \beta_2 \leq \cos(\pi/(2M))$ ([63], 6.21.7). Hence, if $1 < A < \min(2, r_0, \sec(\pi/(2M))$ and s and Θ are as in (3.47) then $K_M(\lambda, x, y') > 0$, $C_M^\lambda(\Theta) > 0$ and $C_{M-1}^\lambda(\Theta) > 0$ ([63], 6.21.3).

To satisfy $\Theta \geq 1/A$ (3.47) we will restrict x and y' so that $\sin\theta \geq 1/\sqrt{A}$ and $\cos\theta' \geq 1/\sqrt{A}$. First, take $\theta_0 = \arcsin\left(\sqrt{A/(2A-1)}\right)$ then $\sin\theta \geq \sin\theta_0 = \sqrt{A/(2A-1)} \geq 1/\sqrt{A}$. And, since $y = x_1\hat{e}_1$, we have $\cos\theta' = (y \cdot y')/(|y|\,|y'|) = \hat{e}_1 \cdot y'\,|y'|^{-1}$ for $y' \neq 0$. Let

$$\Omega_2(\hat{y}) = \left\{y' \in \mathbb{R}^{n-1} \big|\, |y'| > 1,\ 1/\sqrt{A} < \cos\theta' \leq 1\right\}, \tag{3.48}$$

a portion of a cone with axis along $\hat{e}_1$. If $y' \in \Omega_2$ then $\cos\theta' \geq 1/\sqrt{A}$. If $n = 2$, take $\Omega_2 = \{\xi \in \mathbb{R} | \xi > 1\}$. See Figure 3.1.

Define $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$f(y') = \begin{cases} f_i\left(1 - \frac{1}{b_i}|y' - a_i\hat{e}_1|\right), y' \in B_{b_i}(a_i\hat{e}_1) & \text{for some} \quad i \geq 1 \\ (-1)^M A_\lambda f_i\left(1 - \frac{1}{b_i}|y' + a_i\hat{e}_1|\right), y' \in B_{b_i}(-a_i\hat{e}_1) & \text{for some} \quad i \geq 1 \\ 0, \quad \text{otherwise,} \end{cases} \tag{3.49}$$

where $A_\lambda \geq 1$ is given in (3.51) and f_i in (3.57). By taking an appropriate subsequence of $\{\tilde{x}^{(i)}\}$ we may assume the balls $B_{b_i}(a_i\hat{e}_1)$ are disjoint ($a_{i+1} \geq 3a_i$ suffices). The condition $\sin\theta_0 = \sqrt{A/(2A-1)}$ ensures that each $B_{b_i}(a_i\hat{e}_1) \subset \Omega_2$. Then f is

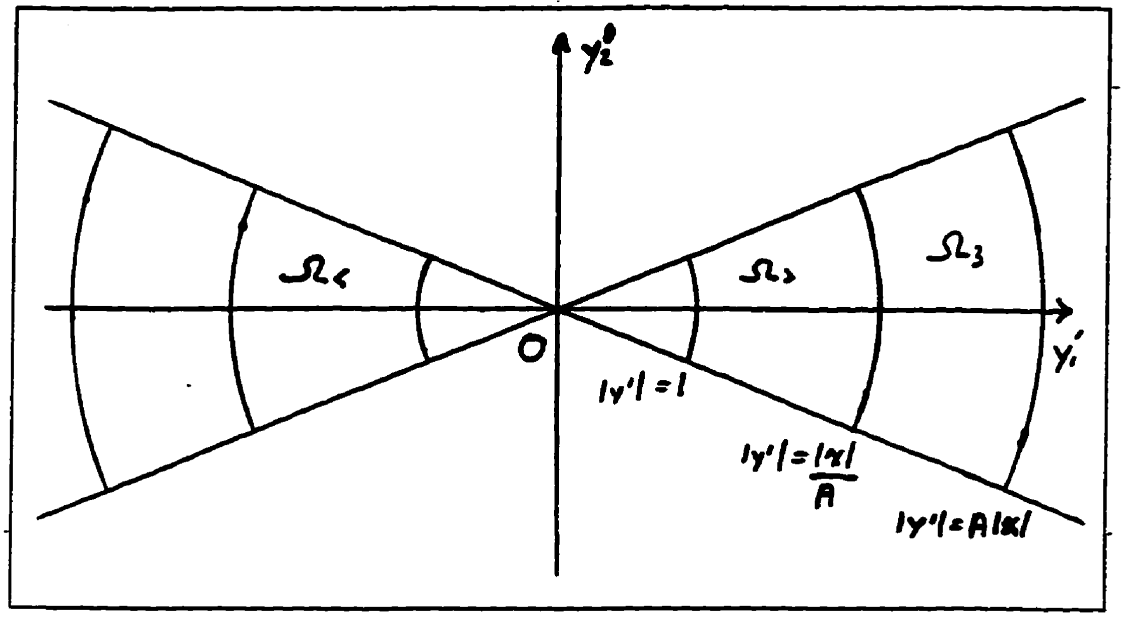

Figure 3.1: The regions Ω_2, Ω_3 and $\Omega_{\gtrless}$

continuous, has support on a sequence of balls along the $\hat{e}_1$ axis and is non-negative for $y_1' \geq 0$.

With $y' \in \Omega_2$ such that $y_1' > 0$ and x as above (preceding (3.47)), $A^{-1} \leq \Theta = \sin\theta\cos\theta' \leq 1$. So $C_M^\lambda(\Theta)$, $C_{M-1}^\lambda(\Theta) > 0$. As a function of s, with fixed Θ as in (3.47), the integral in (3.23) is zero when $s = 0$, is an increasing function of s for $0 < s < MC_M^\lambda(\Theta)/[(2\lambda+M-1)C_{M-1}^\lambda(\Theta)]$ (where it has a maximum) and decreases for larger values of s. And, we know from the analysis following (3.47) that this integral is positive at $s = A$. Hence, $K_M(\lambda, x, y') > 0$ for $0 < s \leq A$ (with $y' \in \Omega_2$, $y_1' > 0$).

If $y' \in \Omega_2$ and $y_1' < 0$ then $\Theta < 0$. Since $C_m^\lambda(-t) = (-1)^m C_m^\lambda(t)$ we have $\Phi_-(\Theta, \zeta) = (-1)^M \Phi_+(|\Theta|, \zeta)$ and $\mathrm{sgn}(K_M(\lambda, x, y')) = (-1)^M$. From (3.49), $f(y')K_M(\lambda, x, y') \geq 0$.

Define

$$\Omega_{\gtrless} = \{y' \in \Omega_2 | y_1' \gtrless 0, \quad s > A\}. \tag{3.50}$$

See Figure 3.1. If $x \in \{\bar{x}^{(i)}\}$ then $f(y')K_M(\lambda, x, y') \geq 0$ for $y' \in \Omega_2$ except possibly for $y' \in \Omega_>$. By taking A_λ large enough we can ensure $\int\limits_{\Omega_> \cup \Omega_<} f(y')K_M(\lambda, x, y')\, dy' \geq 0$. Indeed, let y^* be the reflection of y' in the hyperplane $y_1' = 0$ and θ^* the angle between y^* and y. Then $y^* \in \Omega_<$ if and only if $y' \in \Omega_>$.

If $\lambda \geq 1$ and $y^* \in \Omega_<$ then, as in (3.22), $\Theta^* := \sin\theta \cos\theta^* = -\Theta$. Then, using (3.23) and (3.49),

$$\begin{aligned} f(y^*)K_M(\lambda, x, y^*) &= A_\lambda f(y') \Big[|y'|^2 + 2\Theta |y'|\,|x| + |x|^2\Big]^{-\lambda} \int\limits_{\zeta=0}^{s} \frac{\Phi_+(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta}{(1+2\Theta\zeta+\zeta^2)^{1-\lambda}} \\ &\geq A_\lambda f(y')|x|^{-2\lambda}(1+A^{-1})^{-2\lambda} \int\limits_{\zeta=0}^{s} (1+\zeta^2)^{\lambda-1}\Phi_+(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta. \end{aligned}$$

And,

$$\begin{aligned} f(y')K_M(\lambda, x, y') &= f(y') \Big[|y'|^2 - 2\Theta |y'|\,|x| + |x|^2\Big]^{-\lambda} \int\limits_{\zeta=0}^{s} \frac{\Phi_-(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta}{(1-2\Theta\zeta+\zeta^2)^{1-\lambda}} \\ &\geq -f(y')|x|^{-2\lambda}(1-A^{-1})^{-2\lambda} \int\limits_{\zeta=0}^{s} (1+\zeta^2)^{\lambda-1}\,\Phi_+(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta. \end{aligned}$$

Therefore,

$$\int\limits_{\Omega_< \cup \Omega_>} f(y')K_M(\lambda, x, y')\, dy' \geq 0 \quad \text{if} \quad A_\lambda \geq (A+1)^{2\lambda}(A-1)^{-2\lambda}.$$

If $0 < \lambda < 1$ and $y^* \in \Omega_<$ then

$$\begin{aligned} f(y^*)K_M(\lambda, x, y^*) &\geq A_\lambda\, f(y')\,(|x| + |y'|)^{-2\lambda}\,(1+s)^{2\lambda-2} \int\limits_{\zeta=0}^{s} \Phi_+(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta \\ &\geq A_\lambda\, f(y')\,|x|^{-2\lambda}\,\big(1+A^{-1}\big)^{-2\lambda}\, s^{M+2\lambda-1}\,\frac{(2\lambda+M-1)\,C_{M-1}^{\lambda}(\Theta)}{4(M+1)}. \end{aligned}$$

If $0 < \lambda < 1/2$ then, using (3.31) and (3.37),

$$f(y')K_M(\lambda, x, y') \geq -f(y')(|x| - |y'|)^{-2\lambda} 2\lambda \binom{2\lambda + M}{M-1} \frac{s^{M+2\lambda-1}}{2\lambda}.$$

And, if $1/2 \leq \lambda < 1$,

$$f(y')K_M(\lambda, x, y') \geq -f(y')(|x| - |y'|)^{-2\lambda} 2\lambda \binom{2\lambda + M}{M-1} \frac{s^M}{M} \left(1 + s^2\right)^{\lambda - \frac{1}{2}}.$$

Hence, for $0 < \lambda < 1$,

$$f(y')K_M(\lambda, x, y') \geq -2\sqrt{2} f(y')|x|^{-2\lambda} \left(1 - A^{-1}\right)^{-2\lambda} s^{M+2\lambda-1} \binom{2\lambda + M}{M-1}.$$

And, $\int\limits_{\Omega_< \cup \Omega_>} f(y')K_M(\lambda, x, y')\, dy' \geq 0$ if

$$A_\lambda \geq \frac{8\sqrt{2}\,(M+1)}{2\lambda + M - 1} \binom{2\lambda + M}{M-1} \left(\frac{A+1}{A-1}\right)^{2\lambda} \left[\min_{A^{-1} \leq t \leq 1} C^\lambda_{M-1}(t)\right]^{-1}.$$

Hence, for $\lambda > 0$, $x \in \{\tilde{x}^{(i)}\}$, if we take

$$A_\lambda \geq \left(\frac{A+1}{A-1}\right)^{2\lambda} \max\left(1, \frac{8\sqrt{2}\,(M+1)}{2\lambda + M - 1} \binom{2\lambda + M}{M-1} \left[\min_{A^{-1} \leq t \leq 1} C^\lambda_{M-1}(t)\right]^{-1}\right) \tag{3.51}$$

then $F_{\lambda,M}[f](x) \geq \int\limits_{\Omega_3} f(y')K_M(\lambda, x, y')\, dy'$, where

$$\Omega_3 = \left\{y' \in \Omega_2 \middle| y'_1 > 0,\ A^{-1} < s < A\right\}. \tag{3.52}$$

See Figure 3.1. Note that if $x = a_i\hat{e}_1 + b_i\hat{e}_n$ then $B_{b_i}(a_i\hat{e}_1) \subset \Omega_3$ if $a_i - |x|/A \geq b_i$ and $A|x| - a_i \geq b_i$. Since $a_i = |x|\sin\theta$, $b_i = |x|\cos\theta$ and $\theta_0 \leq \theta < \pi/2$, these conditions are satisfied if $\frac{1}{2}[\pi - \arcsin(1 - A^{-2})] \leq \theta_0 < \pi/2$, i.e., by taking θ_0 close enough to $\pi/2$.

From (3.23),

$$K_M(\lambda, x, y') = K(\lambda, x, y')\, s^M \int\limits_{\zeta=0}^{1} (1 - 2\Theta s\zeta + s^2\zeta^2)^{\lambda-1}\, \Phi_-(\Theta, s\zeta)\, \zeta^{M-1}\, d\zeta, \tag{3.53}$$

which is strictly positive on $\overline{\Omega}_3$ ((3.47) and following). And, $K(\lambda, x, y')$ is positive but singular at $s = \Theta = 1$. Using (3.14), the integral (3.53) above reduces to

$$\frac{\Gamma(2\lambda + M)}{\Gamma(2\lambda)\,\Gamma(M)} \int_{\zeta=0}^{1} (1-\zeta)^{2\lambda-1}\zeta^{M-1}\,d\zeta > 0$$

at $s = \Theta = 1$. The integral in (3.53) is a strictly positive continuous function of s and Θ when the conditions in (3.47) are satisfied. Hence, it must be bounded below by a positive constant, say d_8, i.e.,

$$K_M(\lambda, x, y') \geq d_8\, K(\lambda, x, y')\, s^M \quad \text{for} \quad y' \in \Omega_3,\ \sin\theta \geq \sin\theta_0. \tag{3.54}$$

If $M = 0$ we can dispense with the sets $\Omega_2, \Omega_3, \Omega_<$ and $\Omega_>$. In (3.49), $A_\lambda = 1$ and f is extended as an even function. Then (3.54) holds for $x \in \Pi_+$ with $d_8 = 1$.

For $M \geq 0$ each element of the sequence $\bar{x}^{(i)} = a_i\hat{e}_1 + b_i\hat{e}_n$ satisfies $\sin\theta \geq \sin\theta_0$ so, using (3.49) and (3.54)

$$\begin{aligned}
F_{\lambda,M}[f](\bar{x}^{(j)}) &\geq d_8|\bar{x}^{(j)}|^M \int_{\Omega_3} f(y')K(\lambda, \bar{x}^{(j)}, y')\,|y'|^{-M}\,dy' \\
&\geq d_8(a_j^2 + b_j^2)^{M/2} f_j \int_{B_{b_j}(a_j\hat{e}_1)} \frac{(1 - |y' - a_j\hat{e}_1|\, b_j^{-1})}{(|y' - a_j\hat{e}_1|^2 + b_j^2)^\lambda}\,|y'|^{-M}\,dy' \\
&\geq \frac{d_8(a_j^2 + b_j^2)^{M/2}\, f_j\, b_j^{n-1}}{(a_j + b_j)^M\, b_j^{2\lambda}} \int_{B_1} (1 - |y'|)\,(|y'|^2 + 1)^{-\lambda}\,dy' \\
&\geq d_9\, f_j\, b_j^{n-1-2\lambda}
\end{aligned} \tag{3.55}$$

where $d_9 = d_8\, 2^{-M/2}(n-1)w_{n-1} \int_{\rho=0}^{1} (1-\rho)(\rho^2+1)^{-\lambda}\,\rho^{n-2}\,d\rho.$

Note that (3.5) holds if and only if

$$\sum_{i=1}^{\infty} \frac{f_i b_i^{n-1}}{a_i^{M+2\lambda}} < \infty. \tag{3.56}$$

Now suppose $\psi\colon \mathbb{R}^{n-1} \to (0,\infty)$ such that $\psi(x) = o(|x|^M \sec^{2\lambda}\theta)$. On the sequence $\bar{x}^{(j)} = a_j\hat{e}_1 + b_j\hat{e}_n$, $|x|^M \sec^{2\lambda}\theta = |x|^{M+2\lambda}x_n^{-2\lambda} = (a_j^2 + b_j^2)^{M/2+\lambda}\, b_j^{-2\lambda}$ and $\psi(\bar{x}^{(j)}) =$

$o(a_j^{M+2\lambda} b_j^{-2\lambda})$ (since $0 < b_i \le a_i/2$). We may assume that $\{\tilde{x}^{(i)}\}$ has been chosen so that $\psi(\tilde{x}^{(i)}) \le a_i^{M+2\lambda} b_i^{-2\lambda} i^{-2}$ for $i \ge 1$. Let

$$f_i = d_9^{-1} \psi(a_i \hat{e}_1 + b_i \hat{e}_n) b_i^{2\lambda - n + 1}, \quad i \ge 1. \tag{3.57}$$

Then (3.56) is satisfied and $F_{\lambda,M}[f](\tilde{x}^{(j)}) \ge \psi(\tilde{x}^{(j)})$ so $\lim F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \ne 0$ and $F_{\lambda,M}[f](x) \ne o(|x|^M \sec^{2\lambda}\theta)$. Hence, the order relation $F_{\lambda,M}[f](x) = o(|x|^M \sec^{2\lambda}\theta)$ is sharp for $x \in \Pi_+$ of form $x = x_1\hat{e}_1 + x_n\hat{e}_n$.

Step IV For $n = 2$ this completes the proof. For $n \ge 3$ we now remove this restriction on x. For any sequence $\{x^{(i)}\}$ in Π_+, we can write $x^{(i)} = |x^{(i)}|\,\hat{x}^{(i)}$ where $\hat{x}^{(i)} \in \partial B_1^+ = \{x \in \mathbb{R}^n \,|\, |x| = 1,\ x_n > 0\}$. Then $\{\hat{x}^{(i)}\}$ must have a limit point, say $\hat{s}_0$, in the compact set $\overline{\partial B_1^+}$. Let θ_0 be the angle between $\hat{s}_0$ and $\hat{e}_n$.

If $0 < \theta_0 < \pi/2$ then let $\hat{s}_1$ be in the direction of the projection of $\hat{s}_0$ onto $\partial\Pi_+$ and let $\hat{s}_2 \in \partial\Pi_+$ be any unit vector orthogonal to $\hat{s}_1$. For any $\delta > 0$ there is a subsequence $\tilde{x}_\delta^{(i)} = \tilde{x}^{(i)} + \delta_i \hat{t}_i$ where $\tilde{x}^{(i)} = a_i\hat{s}_1 + b_i\hat{e}_n$, $c_i = |\tilde{x}^{(i)}| \to \infty$ monotonically, $b_i > 0$, $\{\hat{s}_1, \hat{e}_n, \hat{t}_i\}$ is orthonormal, each $\hat{t}_i \in \partial\Pi_+$ and $0 \le \delta_i \le \delta$. We can now try to repeat the first part of the sharpness proof, beginning with (3.39). Then $\hat{s}_1$ and $\hat{s}_2$ play the roles $\hat{e}_1$ and $\hat{e}_2$ did before, except that we now have perturbations by δ_i.

Let $x \in \{\tilde{x}_\delta^{(i)}\}$. Let η_i be the angle between $a_i\hat{s}_1 + \delta_i\hat{t}_i$ and $\hat{s}_1$. Without loss of generality $a_i \ge 1$. We have $0 \le \eta_i = \arctan(\delta_i/a_i) \le \delta$. Hence, we can replace (3.40) with the narrower cone

$$\begin{aligned} \Omega_1'(\hat{y}) \;=\; \Big\{ y' \in \mathbb{R}^{n-1} \Big| & \arccos(\beta_1/2) + \delta \le \theta_1' \le \arccos(\beta_1/3) - \delta \quad \text{if } M \text{ is even} \\ & \text{and } \arccos(\beta_1/3) + \delta \le \theta_1' \le \pi - \arccos(\beta_1/2) - \delta \ \text{ if } M \text{ is odd} \Big\} \end{aligned} \tag{3.58}$$

where θ_1' is the angle between y' and $a_i\hat{s}_1$. (Take $2\delta < \arccos(\beta_1/3) - \arccos(\beta_1/2)$.) For any $x \in \{\tilde{x}_\delta^{(i)}\}$, if $y' \in \Omega_1'$ then $|\cos\theta'| \le \beta_1/2$ and (3.41) holds.

If $\hat{s}_0 = \hat{e}_n$ ($\theta_0 = 0$) then take $a_i \equiv 0$. Let $\hat{s}_1 = \hat{e}_1$ and $\hat{s}_2 = \hat{e}_2$. For any $x \in \{\tilde{x}_\delta^{(i)}\}$ and $y' \in \Omega_1'$ we have $0 \le \theta \le \delta$ and so $0 \le \sin\theta \le \sin\delta$. Therefore, $|\Theta| = |\sin\theta\cos\theta'| \le \delta \le \beta_1/2$ for small enough δ. And, (3.41) holds.

Now, for $0 \leq \theta_0 < \pi/2$, replace (3.44) with

$$f(y') = \begin{cases} (-1)^{\mu+e_0} f_i\Big[1 - |y' - c_i\hat{e}_\delta|\Big] y'_\delta ; & y' \in B_1(c_i\hat{e}_\delta),\ y'_\delta \geq 0 \\ 0, & \text{otherwise,} \end{cases} \tag{3.59}$$

where $y'_\delta = y' \cdot ((-1)^M \cos\delta\, \hat{s}_1 - \sin\delta\, \hat{s}_2)$. We align the half balls of the support of f along the unit vector $\hat{e}_\delta$ in the direction $(-1)^M \sin\delta\, \hat{s}_1 + \cos\delta\, \hat{s}_2$ so that $B_1(c_i\hat{e}_\delta) \subset \Omega'_1$.

The rest of the proof for this case follows without serious change, through (3.46).

If $\hat{s}_0 \in \partial\Pi_+$ $(\theta_0 = \pi/2)$ then $\hat{s}_1 = \hat{s}_0$ and a subsequence approaches the boundary. As before, for any $\delta > 0$ there is a subsequence of form $\bar{x}_\delta^{(i)} = \bar{x}^{(i)} + \delta_i\hat{t}_i$ where $\bar{x}^{(i)} = a_i\hat{s}_1 + b_i\hat{e}_n$, $0 < b_i \leq a_i, a_i \geq 1, a_i \to \infty$ monotonically, $b_i/a_i \to 0, \{\hat{s}_1, \hat{e}_n, \hat{t}_i\}$ is orthonormal and $0 \leq \delta_i \leq \delta$. Follow the second part of the sharpness proof, from (3.47).

Let

$$B_\delta = \min\left(\frac{A}{(1+\delta\sqrt{A})^2},\ A - \delta,\ \frac{A}{1+\delta A}\right)$$

then $B_\delta < A$. And, $B_\delta > 1$ if

$$\begin{aligned} 0 \ < \ \delta \ &< \ \min\Big((\sqrt{A}-1)/\sqrt{A},\ A-1,\ (A-1)/A\Big) \\ &= \ (\sqrt{A}-1)/\sqrt{A}. \end{aligned}$$

Without loss of generality, take A satisfying the conditions following (3.47) and $0 < \delta < (A-1)/A < 1/2$.

For each $j \geq 1$, let θ' be the angle between y' and $a_j\hat{s}_1$ and θ'_δ the angle between y' and $a_j\hat{s}_1 + \delta_j\hat{t}_j$. We have $\theta' - \delta \leq \theta'_\delta \leq \theta' + \delta$ so replace (3.48) with

$$\Omega'_2(\hat{y}) = \left\{ y' \in \mathbf{R}^{n-1} \big| |y'| > 1,\ 0 \leq \theta'_\delta < \arccos(1/\sqrt{B_\delta}) - \delta \right\}. \tag{3.60}$$

For each $j \geq 1$, let θ be the angle between $\bar{x}^{(j)}$ and $\hat{e}_n$ and θ_δ the angle between $\bar{x}_\delta^{(j)}$ and $\hat{e}_n$. Then

$$\sin\theta_\delta = \frac{|a_j\hat{s}_1 + \delta_j\hat{t}_j|}{|\bar{x}^{(j)} + \delta_j\hat{t}_j|} \geq \frac{a_j - \delta}{|\bar{x}^{(j)}| + \delta}.$$

For large enough $|\tilde{x}^{(j)}|$, we have $\sin\theta_\delta \geq a_j/|\tilde{x}^{(j)}| - \delta = \sin\theta - \delta$. It follows from the first component of the definition of B_δ that $\sin\theta \geq 1/\sqrt{B_\delta}$ implies $\sin\theta_\delta \geq 1/\sqrt{A}$.

Write $s = |\tilde{x}^{(j)}|/|y'|$, $s_\delta = |\tilde{x}_\delta^{(j)}|/|y'|$. Then for $y' \in \Omega_2'$, $s - \delta \leq s_\delta \leq s + \delta$. From the second and third components in the definition of B_δ, $1/B_\delta \leq s \leq B_\delta$ implies $1/A \leq s_\delta \leq A$. Hence, we can replace Ω_2 with Ω_2' and carry out the sharpness proof for $a_j\hat{s}_1 + b_j\hat{e}_n$ with the following changes. In (3.49), replace $a_i\hat{e}_1$ with $a_i\hat{s}_1 + \delta_i\hat{t}_i$. In (3.50) and (3.52), replace y_1' with $y'\cdot\hat{s}_1$. The rest of the proof, through (3.57), follows with minor changes. ■

The growth estimate on $F_{\lambda,M}[f]$ gives estimates for the solutions of the half space Dirichlet and Neumann problems. The modified kernel introduces a singularity at the origin of the integration space. To avoid integrating f there, a continuous cutoff function that vanishes in a neighbourhood of the origin is used.

Corollary 3.3.1 *Let $w : \mathbb{R}^{n-1} \to [0,1]$ be continuous such that $w(y) \equiv 0$ when $|y| \leq 1$ and $w(y) \equiv 1$ when $|y| \geq 2$. Let f be continuous on $\mathbb{R}^{n-1}$ and satisfy* (3.5) *with $\lambda = n/2 \quad (n \geq 2)$. The function $u(x) = D_M[wf](x) + D[(1-w)f](x)$ satisfies*

$$u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+) \tag{3.61}$$

$$\Delta u = 0, \quad x \in \Pi_+ \tag{3.62}$$

$$u = f, \quad x \in \partial\Pi_+ \tag{3.63}$$

$$u(x) = o\big(|x|^{M+1}\sec^{n-1}\theta\big); \quad x \in \Pi_+, \quad |x| \to \infty. \tag{3.64}$$

Proof. That u is a classical solution, (3.61), (3.62), (3.63), is contained in Corollary 2 of [67]. To prove (3.64), note that $D_M[wf](x) = \alpha_n x_n F_{\frac{n}{2},M}[wf](x) = o\big(|x|^{M+1}\sec^{n-1}\theta\big)$

by the Theorem. And,

$$|D[(1-w)f](x)| \leq \alpha_n x_n \int\limits_{|y'|<2} |f(y')| \, (|x|-2)^{-n} \, dy'$$
$$\leq \alpha_n 2^n x_n |x|^{-n} \quad \text{if} \quad |x| \geq 4,$$

so (3.64) is satisfied. ■

Corollary 3.3.2 *Let f and w be as in Corollary 3.3.1 such that* (3.5) *holds with* $\lambda = (n-2)/2 \quad (n \geq 3)$. *Then* $v(x) = N_M[wf](x) + N[(1-w)f](x)$ *satisfies* (3.62) *and*

$$v \in C^2(\Pi_+) \cap C^1(\overline{\Pi}_+) \tag{3.65}$$

$$\frac{\partial v}{\partial x_n} = -f, \quad x \in \partial\Pi_+ \tag{3.66}$$

$$v(x) = o\left(|x|^M \sec^{n-2}\theta\right); \quad x \in \Pi_+, \quad |x| \to \infty. \tag{3.67}$$

Proof. The growth estimate (3.67) follows from the Theorem:

$$N_M[wf](x) = \frac{\alpha_n}{n-2} F_{\frac{n-2}{2},M}[wf](x).$$

And,

$$|N[(1-w)f](x)| \leq \frac{\alpha_n}{n-2} \int\limits_{|y'|<2} |f(y)| \, (|x|-2)^{2-n} \, dy'$$
$$\leq \alpha_n (n-2)^{-1} 2^{n-2} |x|^{2-n} \quad \text{if} \quad |x| \geq 4.$$

Theorem 1 of [25] shows (3.62), (3.65) and (3.66) hold. ■

Remark 3.3.1 In Corollary 3.3.1, the solution to (3.61)–(3.64) is unique if $M = 0$ and if $M \geq 1$ it is unique to the addition of a harmonic polynomial of degree M vanishing on $\partial\Pi_+$ (see Theorem 6.4.1). Similarly, in Corollary 3.3.2, if $M = 0$ the solution to (3.62), (3.65)–(3.67) is unique and if $M \geq 1$ it is unique to the addition of a harmonic polynomial $p(x)$ of degree $M-1$ that is even about $x_n = 0$. See Theorem 6.4.2 below.

Remark 3.3.2 If $f(y')\,|y'|^{n-2-M}$ is integrable at the origin then we can use $u(x) = D_M[f](x)$ and $v(x) = N_M[f](x)$ in Corollaries 3.3.1 and 3.3.2, respectively.

Corollary 3.3.3 *If* $\omega : \Pi_+ \to (0,\infty)$ *then* ω *is a sharp growth condition for* $F_{\lambda,M}$ *if and only if there are constants* $0 < S < T < \infty$ *and* $N > 0$ *such that* $S \leq |x|^{-M}\cos^{2\lambda}\theta\,\omega(x) \leq T$ *for all* $x \in \Pi_+$ *with* $|x| > N$.

Proof: Throughout the proof f will satisfy (3.5) and $|x|$, $|x^{(i)}| > N$.

Suppose S and T exist as above. Then

$$|F_{\lambda,M}[f](x)|\,/\omega(x) \leq |F_{\lambda,M}[f](x)|\,S^{-1}|x|^{-M}\cos^{2\lambda}\theta \to 0$$

so $F_{\lambda,M}[f] = o(\omega)$.

Let $\psi : \Pi_+ \to (0,\infty)$ with $\psi = o(\omega)$ then $\psi(x) = o(|x|^M \sec^{2\lambda}\theta)$. Given $\{x^{(i)}\}$ in Π_+ take f as in the proof of the Theorem. Then $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \not\to 0$. Hence, ω is sharp.

Now suppose ω is sharp. If $\chi(x) := |x|^{-M}\cos^{2\lambda}\theta\,\omega(x)$ is unbounded then there is a sequence $\{x^{(i)}\}$ on which $\chi(x^{(i)}) \to \infty$. Let

$$\psi(x) = \begin{cases} |x|^M \sec^{2\lambda}\theta & \text{on} \quad \{x^{(i)}\} \\ \omega(x)/|x|, & \text{otherwise,} \end{cases}$$

then $\psi = o(\omega)$ but for any f we have $F_{\lambda,M}[f](x^{(i)})/\psi(x^{(i)}) \to 0$, since $F_{\lambda,M}[f](x) = o(|x|^M \sec^{2\lambda}\theta)$. This contradicts the assumption that ω was sharp (Definition 4.2.1, (ii)). Hence T exists as above.

If $\chi \to 0$ on some sequence $\{x^{(i)}\}$ then take f such that

$$\limsup_{i\to\infty} F_{\lambda,M}[f](x^{(i)})|x^{(i)}|^{-M}\cos^{2\lambda}\theta_i \geq 1 \quad (\cos\theta_i = |x^{(i)}|/x_n^{(i)}). \tag{3.68}$$

Then

$$\limsup_{i\to\infty} \frac{F_{\lambda,M}[f](x^{(i)})}{\omega(x^{(i)})} = \limsup_{i\to\infty} \frac{F_{\lambda,M}[f](x^{(i)})}{|x^{(i)}|^M\sec^{2\lambda}\theta_i}\frac{|x^{(i)}|^M\sec^{2\lambda}\theta_i}{\omega(x^{(i)})} = \infty,$$

which contradicts the sharpness assumption (i) of Definition 4.2.1. Hence, S exists as above. ■

Remark 3.3.3 The angular blow up predicted for $F_{\lambda,M}$ as $|x| \to \infty$ can be expected to occur only as x approaches $\partial\Pi_+$ within a thin or rarefied set. See [3], [21], [51] and references therein.

3.4 The integrability condition

It is shown in [23] that condition (3.5) with $M = 0$ and $\lambda = n/2$ is necessary and sufficient for the Poisson integral to exist on Π_+. Using the estimates in the proof of Theorem 3.3.1 we now show that $F_{\lambda,M}[f]$ exists on Π_+ whenever (3.5) holds. Here we are considering absolute integrability. Conditionally convergent integrals will be discussed in Chapter 8.

Proposition 3.4.1 *If f is measurable on $\mathbf{R}^{n-1}$ then $F_{\lambda,M}[f]$ exists on Π_+ if and only if* (3.5) *holds.*

Proof. Let $x \in \Pi_+$ and $|y'| \geq \max(1, 2|x|)$. Then using (3.6) and (3.13),

$$K_M(\lambda, x, y') = |y'|^{-2\lambda} \sum_{m=M}^{\infty} \left(\frac{|x|}{|y'|}\right)^m C_m^\lambda(\Theta). \tag{3.69}$$

Then from (3.14),

$$\begin{aligned} |K_M(\lambda, x, y')| &\leq |y'|^{-2\lambda} s^M \sum_{m=0}^{\infty} s^m |C_{m+M}^\lambda(\Theta)| & (3.70) \\ &\leq \frac{|x|^M}{|y'|^{M+2\lambda}} \sum_{m=0}^{\infty} s^m \binom{2\lambda + m + M - 1}{m + M}. & (3.71) \end{aligned}$$

Let $\Lambda = \lceil 2\lambda \rceil$. Then

$$\begin{aligned} \binom{2\lambda + m + M - 1}{m + M} &\leq \binom{\Lambda + m + M - 1}{m + M} & (3.72) \\ &= \begin{cases} \frac{(\Lambda+m+M-1)(\Lambda+m+M-2)\cdots(m+M+1)}{(\Lambda-1)!}, & \Lambda \geq 2 \\ 1, & \Lambda = 1 \end{cases} \\ &\leq \frac{(\Lambda + m + M - 1)^{\Lambda - 1}}{(\Lambda - 1)!} & (3.73) \end{aligned}$$

holds for $\lambda > 0$. Therefore,

$$\begin{aligned} |K_M(\lambda, x, y')| &\leq \frac{|x|^M}{|y'|^{M+2\lambda}} \sum_{m=0}^{\infty} 2^{-m} (\Lambda + m + M - 1)^{2\lambda} & (3.74) \\ &\leq \frac{k_0\, |x|^M}{|y'|^{M+2\lambda}} & (3.75) \end{aligned}$$

where k_0 depends only on M and λ. It follows that

$$|F_{\lambda,M}[f](x)| \leq k_0 |x|^M \int_{|y'|>1} |f(y')|\, |y'|^{-(M+2\lambda)}\, dy'. \tag{3.76}$$

So (3.5) is sufficient to define $F_{\lambda,M}[f]$ on Π_+.

To prove the necessity of (3.5), suppose f is given and $F_{\lambda,M}[f]$ exists on Π_+. With no loss of generality, $f(y') \geq 0$. There is a finite sequence of points $y^{(1)}, y^{(2)}, \cdots, y^{(L)} \in \mathbb{R}^{n-1}$ such that $\Omega_1(y^{(1)}) \cup \Omega_1(y^{(2)}) \cup \cdots \cup \Omega_1(y^{(L)}) = \mathbb{R}^{n-1}$. See (3.40) for the definition of Ω_1. Each set $\Omega_1(y^{(\ell)})$ is the region between two cones of fixed angular opening.

The intersection of $\Omega_1(y^{(\ell)})$ with the unit sphere of $\mathbf{R}^{n-1}$ is a zone, i.e., if $\hat{e}$ is the unit vector $y'/|y'|$ then $\hat{v}$ is in the zone of y' if $\beta_1/3 \leq \hat{e}\cdot\hat{v} \leq \beta_1/2$. The number $0 < \beta_1 \leq 1$ is given before (3.39). Let $\tilde{\Omega}_1(y')$ be the interior of the intersection of $\Omega_1(y')$ with the unit sphere. Then $\cup_{|y'|=1}\tilde{\Omega}_1(y') = \partial B_1$, the unit sphere of $\mathbf{R}^{n-1}$. Since ∂B_1 is compact this open covering has a finite subcovering as proclaimed above.

By taking unions and intersections, we have measurable sets $E_\ell \subset \Omega_1(y^{(\ell)})$, $1 \leq \ell \leq L$, that are disjoint and whose union is $\mathbf{R}^{n-1} \setminus B_1$. Let $x_n > 0$ be fixed and let $x^{(\ell)} = y^{(\ell)} + x_n\hat{e}_n$. Define $\theta^{(\ell)}$ by $\sin\theta^{(\ell)} = |y^{(\ell)}|/|x^{(\ell)}|$ then $0 \leq \sin\theta^{(\ell)} \leq \sin\theta_0 < 1$ for some fixed θ_0 as discussed following (3.39). With the above definitions, $K_M(\lambda, x^{(\ell)}, y')$ is positive for each $1 \leq \ell \leq L$ and $y' \in \Omega_1(y^{(\ell)})$ (and hence positive for all $y' \in E_\ell$). Since $F_{\lambda,M}[f]$ exists on Π_+ we have

$$|F_{\lambda,M}[f](x^{(\ell)})| = \sum_{j=1}^{L}\left|\int_{B_j} f(y')K_M(\lambda, x^{(\ell)}, y')\,dy'\right| < \infty. \tag{3.77}$$

So, $\int_{B_j} f(y')K_M(\lambda, x^{(\ell)}, y')\,dy' < \infty$ for each $1 \leq \ell \leq L$. Then, from (3.41), (3.42) and (3.43),

$$\begin{aligned} \infty &> k_1^{(\ell)} \int_{B_\ell} \frac{f(y')K_M(\lambda, x^{(\ell)}, y')\,dy'}{|y'|^M} && (3.78)\\ &\geq k_2^{(\ell)} \int_{B_\ell} f(y')|y'|^{-(M+2\lambda)}\,dy' && (3.79)\\ &\geq k^* \int_{B_\ell} f(y')|y'|^{-(M+2\lambda)}\,dy', && (3.80)\end{aligned}$$

where $k_1^{(\ell)}$ and $k_2^{(\ell)}$ are positive and depend on n, λ, M and x. And, $k^* = \min_{1\leq\ell\leq L} k_2^{(\ell)}$. Now, summing (3.80) over ℓ from 1 to L shows (3.5) holds. ■

Other properties of modified kernels are dealt with in the next chapter.

Chapter 4

Further results on modified kernels

In this chapter we extend the definition of the modified kernel K_M by allowing M to be an integer-valued function of y'. There are also results on the non-existence of positive solutions to a Dirichlet problem and representations of Neumann solutions in terms of Dirichlet solutions. Another type of modified kernel is introduced and used to give asymptotic expansions of solutions to half space problems.

4.1 A Dirichlet problem without positive solutions

The modified kernels K_M, introduced in (3.6), are not of one sign. In the proof of Theorem 3.3.1 we had $\mathrm{sgn}(\mathrm{K_M}(\lambda, \mathrm{x}, \mathrm{y}')) = -\mathrm{sgn}(\mathrm{C}_\mathrm{m}^\lambda(\sin\theta\cos\theta'))$, where $m \leq M - 1$ is the largest integer such that $C_m^\lambda(\sin\theta\cos\theta') \neq 0$. As $|y'| \to \infty$, (3.22) and (3.23) show that $\mathrm{sgn}(\mathrm{K_M}(\lambda, \mathrm{x}, \mathrm{y}')) = \mathrm{sgn}(\mathrm{C}_\mathrm{M}^\lambda(\sin\theta\cos\theta'))$ if $C_M^\lambda(\sin\theta\cos\theta') \neq 0$ and $-\mathrm{sgn}(\mathrm{C}_{\mathrm{M}-1}^\lambda(\sin\theta\cos\theta'))$ otherwise. And, for small enough x_n, if y' is close enough to y then $[|y'-y|^2 + x_n^2]^{-\lambda}$ will dominate all other terms in (3.6) and $K_M(\lambda, x, y')$ will be positive.

We have the following result about solutions to the Dirichlet problem.

Theorem 4.1.1 *If $f \geq 0$ such that $\int_{\mathbb{R}^{n-1}} f(y')\left(|y'|^n + 1\right)^{-1} dy' = \infty$ then there are no positive solutions to* (2.1)–(2.3).

Proof: Introduce a cutoff function, ξ_N, such that

$$\xi_N = \begin{cases} 1, & |x| \leq N \\ 0, & |x| \geq N+1, \end{cases}$$

$0 \leq \xi_N \leq 1$ and ξ_N is continuous.

Suppose $u \geq 0$ and satisfies (2.1)–(2.3). Let $u_N = D[f\xi_N]$. Given $\epsilon > 0$, we claim that $u \geq u_N - \epsilon$ on $\partial B_\rho \cap \Pi_+$ for large enough ρ. Indeed, we have $\Delta u_N = 0$ in Π_+, $u_N = f\xi_N$ on $\partial\Pi_+$ and $u_N = O(x_n|x|^{-1})$ as $|x| \to \infty$ (since $f\xi_N$ has compact support, Proposition 2.2.1). So, $u \geq u_N$ on $\partial\Pi_+$ and $|u_N| < \epsilon$ on $\partial B_\rho \cap \Pi_+$ for large enough ρ. Therefore, $u \geq u_N - \epsilon$ on ∂B_ρ^+ ($\partial B_\rho^+ = \{x \in \mathbb{R}^n \,|\, |x| = \rho,\, x_n > 0\}$). Since ϵ is arbitrary, $u \geq u_N$ on ∂B_ρ^+. By the weak maximum principle, $u \geq u_N$ in B_ρ^+. But,

$$\begin{aligned} u_N(x) &\geq \int_{|y'| \leq N} K(x, y') f(y')\, dy' \\ &\to \infty \quad \text{as } N \to \infty. \end{aligned}$$

Hence, there can be no such u. ■

The above theorem can also be deduced from the general representation of non-negative harmonic functions on Π_+. The representation theorem states that all positive harmonic functions on Π_+ are of the form $u(x) = D[\mu](x) + c\,x_n$ where μ is a positive Borel measure on $\mathbb{R}^{n-1}$ and $c \geq 0$ is a constant ([7], Theorem 7.24). Replacing μ by the continuous function f gives Theorem 4.1.1. See item (ii) in §9.1.

4.2 Representation of the Neumann solution

The modified kernel $K_M(\lambda, x, y')$ satisfies a differential-difference equation for the derivative with respect to $\theta, |x|, y_i, |y|, x_n, |y'|, y_i'$ and θ', relating the derivative to

$K_M(\lambda+1,x,y')$, $K_{M-1}(\lambda+1,x,y')$ and $K_{M-2}(\lambda+1,x,y')$. The integration of these equations give representations of the modified Neumann integral in terms of the modified Dirichlet integral.

Proposition 4.2.1 *Let $n \geq 3$, $M \geq 0$, $\lambda > 0$, $x \in \Pi_+$ and $y' \in \mathbb{R}^{n-1}$. Use the convention that $K_m = K$ if $m \leq 0$. Then*

(i) $\dfrac{\partial K_M}{\partial \theta}(\lambda,x,y') = 2\lambda\, x_n\, \hat{y}\cdot y'\, K_{M-1}(\lambda+1,x,y')$

(ii) $\dfrac{\partial K_M}{\partial |x|}(\lambda,x,y') = 2\lambda\,[\sin\theta\, \hat{y}\cdot y'\, K_{M-1}(\lambda+1,x,y') - |x| K_{M-2}(\lambda+1,x,y')]$

(iii) $\dfrac{\partial K_M}{\partial y_i}(\lambda,x,y') = 2\lambda\,[y_i' K_{M-1}(\lambda+1,x,y') - y_i K_{M-2}(\lambda+1,x,y')]\,,\ 1 \leq i \leq n-1$

(iv) $\dfrac{\partial K_M}{\partial |y|}(\lambda,x,y') = 2\lambda\,[\hat{y}\cdot y'\, K_{M-1}(\lambda+1,x,y') - |y| K_{M-2}(\lambda+1,x,y')]$

(v) $\dfrac{\partial K_M}{\partial x_n}(\lambda,x,y') = -2\lambda\, x_n\, K_{M-2}(\lambda+1,x,y')$

(vi) $\dfrac{\partial K_M}{\partial |y'|}(\lambda,x,y') = 2\lambda\,[y\cdot\hat{y}'\, K_{M-1}(\lambda+1,x,y') - |y'| K_M(\lambda+1,x,y')]$

(vii) $\dfrac{\partial K_M}{\partial y_i'}(\lambda,x,y') = 2\lambda\,[y_i\, K_{M-1}(\lambda+1,x,y') - y_i' K_M(\lambda+1,x,y')]\,,\ 1 \leq i \leq n-1$

(viii) $\dfrac{\partial K_M}{\partial \theta'}(\lambda,x,y') = -2\lambda\,|y|\,|y'|\sin\theta'\, K_{M-1}(\lambda+1,x,y')$.

The proofs rest on the identities

$$\frac{d}{dt}C_m^{\lambda}(t) = 2\lambda\, C_{m-1}^{\lambda+1}(t) \tag{4.1}$$

$$C_m^{\lambda}(t) \equiv 0 \quad \text{for} \quad m < 0, \quad C_0^{\lambda}(t) = 1 \tag{4.2}$$

$$mC_m^{\lambda}(t) = 2\lambda[t\, C_{m-1}^{\lambda+1}(t) - C_{m-2}^{\lambda+1}(t)] \tag{4.3}$$

$$(m+2\lambda)C_m^{\lambda}(t) = 2\lambda[C_m^{\lambda+1}(t) - t\, C_{m-1}^{\lambda+1}(t)] \tag{4.4}$$

([63],4.7.28). In (iv) x_n is fixed. If θ is held constant for the differentiation then

$$\frac{\partial K_M}{\partial |y|}(\lambda, x, y') = 2\lambda\left[\hat{y}\cdot y'\, K_{M-1}(\lambda+1, x, y') - |x| \csc\theta\, K_{M-2}(\lambda+1, x, y')\right].$$

This leads to a similar change in (iv) of Proposition 4.2.2.

Proof of (i): From (3.1), (3.6), (4.1) and (4.2)

$$\begin{aligned}\frac{\partial K_M}{\partial \theta}(\lambda, x, y') &= 2\lambda\, |y'|\, |x| \cos\theta \cos\theta' K(\lambda+1, x, y') \\ &\qquad - 2\lambda \sum_{m=1}^{M-1} |x|^m |y'|^{-(m+2\lambda)} C_{m-1}^{\lambda+1}(\Theta) \cos\theta \cos\theta' \\ &= 2\lambda\, x_n\, |y'| \cos\theta' K_{M-1}(\lambda+1, x, y'). \quad \blacksquare\end{aligned}$$

The other proofs follow in a similar manner from (4.1)–(4.4) or by differentiating (3.23). Note that $\hat{y}$ and $\Theta = \sin\theta \cos\theta' = \sin\theta\, \hat{y}\cdot\hat{y}'$ are independent of $|x|$ and that $\tan\theta = |y|/x_n$ so $\partial\theta/\partial y_i = y_i x_n/(|x|^2|y|)$ and $\partial\theta/\partial x_n = -\sin\theta/|x|$.

Integrating (i) through (iv) above and setting $\lambda = (n-2)/2$ we obtain

Proposition 4.2.2 *Let f be measurable with the origin not in the closure of its support and satisfy (3.5) with $\lambda = (n-2)/2$ $(n \geq 3)$. Let $M \geq 0$, $\lambda > 0$, $x \in \Pi_+$ and adopt the convention that $D_m = D$ for $m \leq 0$. Then the following are equal to $N_M[f](x)$*

(i) $\displaystyle\int_{t=\theta_0}^{\theta} D_{M-1}[f_{\hat{y}}](x(t))\, dt + N_M[f](x(\theta_0)), \quad 0 \leq \theta_0 \leq \tfrac{\pi}{2}$

(ii) $\displaystyle\tan\theta \int_{t=r_0}^{|x|} D_{M-1}[f_{\hat{y}}](t\hat{y})\tfrac{dt}{t} - \sec\theta \int_{t=r_0}^{|x|} D_{M-2}[f](t\hat{x})\, dt + N_M[f](r_0\hat{x}), \quad r_0 \geq 0$

(iii) $\displaystyle\frac{1}{x_n}\int_{t=t_i}^{y_i} D_{M-1}[f_{\hat{e}_i}](\check{x}_i(t))\, dt - \frac{1}{x_n}\int_{t=t_i}^{y_i} D_{M-2}[f](\check{x}_i(t))\, t\, dt$

$$+N_M[f](\check{x}_i(t_i)); \quad t_i \in \mathbb{R}, 1 \leq i \leq n-1$$

$$(iv)\ \frac{1}{x_n}\int_{t=\rho}^{|y|} D_{M-1}[f_{\hat{y}}](t\hat{y}+x_n\hat{e}_n)\,dt - \frac{1}{x_n}\int_{t=\rho}^{|y|} D_{M-2}[f](t\hat{y}+x_n\hat{e}_n)\,t\,dt$$

$$+N_M[f](\rho\hat{y}+x_n\hat{e}_n), \quad \rho \geq 0$$

$$(v)\ -\int_{t=t_n}^{x_n} D_{M-2}[f](\check{x}_n(t))\,dt + N_M[f](\check{x}_n(t_n)), \quad t_n \geq 0.$$

The following notation has been introduced. In (i)–(iv), if z_1 and z_2 are in $\mathbb{R}^{n-1}$ then $f_{z_1}(z_2) = z_1 \cdot z_2\, f(z_2)$. In (i), if $0 \leq s \leq \pi/2$ then $x(s)$ indicates x with the polar angle θ replaced by s, i.e., $x(s) = y(s) + x_n(s)\hat{e}_n$, where $y(s) = |x| \sin s\, \hat{y}$, $x_n(s) = |x| \cos s$, $x(\theta) = x$ and $y(\theta) = y$. Note that $|x|$ and $\hat{y}$ are independent of θ. In (iii) and (v), if $x = \sum_{j=1}^{n} x_j\hat{e}_j$ then $\check{x}_i(t) = \sum_{j\neq i} x_j\hat{e}_j + t\hat{e}_i$ $(1 \leq i \leq n)$.

Proof: Integrate each of (i)–(v) in Proposition 4.2.1 with respect to the relevant variable and set $\lambda = (n-2)/2$. Multiply by $((n-2)/2)f(y')$ and integrate $y' \in \mathbb{R}^{n-1}$. Because of (3.5) the integrals $D_{M-2}[|f|](x)$ and $D_{M-1}[|f_{\hat{y}}|](x)$ converge to continuous functions on $\overline{\Pi}_+$. The same is true for each modified Dirichlet integral in (i)–(v). Fubini's Theorem now justifies the interchange of orders of integration. ∎

Remark 4.2.1 We can relax the condition that f be continuous if we refrain from evaluating $N_M[f]$ on $\partial\Pi_+$. This requires taking $0 \leq \theta_0 < \pi/2$, $r_0 > 0$, $\rho > 0$ and $t_n > 0$. We can dispense with the restriction on the support of f if $f(y')\,|y'|^{-(M-3+2\lambda)}$ is integrable at the origin. When $M = 0$ there is no restriction on the support of f.

We can use Proposition 4.2.2(i) to confirm the growth estimate (3.67). If f satisfies (3.5) with $\lambda = (n-2)/2$, $n \geq 3$, and if $0 \leq \theta_0 < \pi/2$ then (3.36) gives $N_M[f](x(\theta_0)) = o(|x|^M)$. From Corollary 3.3.1, $D_{M-1}[f_{\hat{y}}](x) = o(|x|^M \sec^{n-1}\theta)$. Integrating over θ and using (i) of Proposition 4.2.2,

$$N_M[f](x) = o(|x|^M \sec^{n-2}\theta) + o(|x|^M) = o(|x|^M \sec^{n-2}\theta),$$

in agreement with Corollary 3.3.2.

4.3 Variable M

By taking M a fixed positive integer we were able to construct the modified kernel K_M and this allowed us to write a modified Poisson integral that converged for data f satisfying

$$\int_{\mathbb{R}^{n-1}} \frac{|f(y')|\,dy'}{|y'|^{M+2\lambda}+1} < \infty. \tag{4.5}$$

In particular, this includes any polynomial data. In [22], Finkelstein and Scheinberg prove that if M is allowed to be an integer-valued function on $\mathbb{R}^{n-1}$,

$$M\colon \mathbb{R}^{n-1} \to \{0,1,2,\cdots\}, \tag{4.6}$$

then for any continuous function f on $\mathbb{R}^{n-1}$ there exists a function M of the above type so that $D_M[f]$ is a classical solution to the half space Dirichlet problem, (2.1)–(2.3). The kernel K_M is defined as in (3.6) with $M = M(y')$, and D_M and $F_{\lambda,M}$ as in (3.9) and (3.11). It follows that if f is any measurable function on $\mathbb{R}^{n-1}$ then there is an M as in (4.6) such that $F_{\lambda,M}[f]$ exists on Π_+. With this new type of kernel, the number of terms subtracted from $[|y-y'|^2+x_n^2]^{-\lambda}$ in (3.6) can vary with y' to compensate for growth of $f(y')$.

The previous estimate

$$K(\lambda,x,y') = [|y-y'|^2+x_n^2]^{-\lambda} \le \frac{2^{2\lambda}\sec^{2\lambda}\theta}{(|y'|+|x|)^{2\lambda}} \tag{4.7}$$

from (2.8) of Theorem 2.2.1 (with $\alpha = \lambda$ in the first two terms of (2.7) and $\alpha = n/2$ in the third) can be used again to estimate $F_{\lambda,M}$. There are two approaches. First, if M is a given function, determine an integral condition on f (analogous to (4.5)) under which $F_{\lambda,M}[f]$ exists on Π_+. Also, find a growth estimate for $F_{\lambda,M}[f]$ and prove it is sharp. The second approach is to assume f is a given function. The problem is then to find the "smallest" M for which $F_{\lambda,M}[f]$ exists on Π_+.

Start off with a given function M as in (4.6). The modified kernel is

$$K_{M(y')}(\lambda, x, y') = K(\lambda, x, y') - \sum_{m=0}^{M(y')-1} \frac{|x|^m}{|y'|^{m+2\lambda}} C_m^\lambda(\sin\theta\cos\theta') \tag{4.8}$$

$$= K(\lambda, x, y') \int_{\zeta=0}^{s} (1 - 2\Theta\zeta + \zeta^2)^{\lambda-1} \Phi_-(\Theta, \zeta)\, \zeta^{M(y')-1}\, d\zeta, \tag{4.9}$$

where $\Theta = \sin\theta\cos\theta'$, $s = |x|/|y'|$ and Φ_- is given in (3.22). To estimate $F_{\lambda,M}[f]$, we need to prove an inequality like (3.35) but now taking into account the dependence of M on y'.

Let

$$I_1 = \int_{\zeta=0}^{s} (1 - 2\Theta\zeta + \zeta^2)^{\lambda-1} \Phi_-(\Theta, \zeta)\, \zeta^{M-1}\, d\zeta \tag{4.10}$$

and estimate the integrand. As in (3.30),

$$\frac{|\Phi_-(\Theta, \zeta)|}{\sqrt{1 - 2\Theta\zeta + \zeta^2}} = \frac{\left|(2\lambda + M - 1)(\Theta - \zeta) C_{M-1}^\lambda(\Theta) - 2\lambda(1 - \Theta^2) C_{M-2}^{\lambda+1}(\Theta)\right|}{\sqrt{(\Theta - \zeta)^2 + (1 - \Theta^2)}} \tag{4.11}$$

$$\leq (2\lambda + M - 1) C_{M-1}^\lambda(1) + \max_{|\Theta|\leq 1} \left[2\lambda\sqrt{1 - \Theta^2}\, \left|C_{M-2}^{\lambda+1}(\Theta)\right|\right] \tag{4.12}$$

$$= \frac{\Gamma(2\lambda + M)}{\Gamma(2\lambda)\Gamma(M)} + 2\lambda \max_{0\leq\eta\leq\pi/2} \left[\sin\eta\; C_{M-2}^{\lambda+1}(\cos\eta)\right]. \tag{4.13}$$

The maximum of $C_{M-2}^{\lambda+1}(\cos\eta)$ occurs at $\eta = 0$ which is where $\sin\eta = 0$. To determine the maximum of this product we use the integral representation

$$C_m^\lambda(\cos\eta) = \frac{2^{1-\lambda}\,\Gamma(2\lambda + m)}{\Gamma^2(\lambda)\, m!} \sin^{1-2\lambda}\eta \int_{t=0}^{\eta} \frac{\cos[(\lambda + m)t]\, dt}{(\cos t - \cos\eta)^{1-\lambda}}, \tag{4.14}$$

valid for $\lambda > 0$ ([49], p.224). For $M \geq 2$ it follows from integration by parts that

$$\left|\sin\eta\; C^{\lambda+1}_{M-2}(\cos\eta)\right| \tag{4.15}$$

$$= \frac{\Gamma(2\lambda+M)\,\sin^{-2\lambda}\eta}{2^{\lambda}\,\Gamma^{2}(\lambda+1)\,\Gamma(M-1)}\,\frac{\lambda}{\lambda+M-1}\left|\int\limits_{t=0}^{\eta}\sin[(\lambda+M-1)t](\cos t-\cos\eta)^{\lambda-1}\sin t\,dt\right|$$

$$\leq \frac{\Gamma(2\lambda+M)}{2^{\lambda}\,\Gamma^{2}(\lambda+1)\,\Gamma(M-1)\,(\lambda+M-1)}\left(\frac{1-\cos\eta}{\sin^{2}\eta}\right)^{\lambda} \tag{4.16}$$

$$\leq \frac{\Gamma(2\lambda+M)}{2^{\lambda}\,\Gamma^{2}(\lambda+1)\,\Gamma(M-1)\,(\lambda+M-1)}\qquad (0\leq\eta\leq\frac{\pi}{2}). \tag{4.17}$$

From (4.13), if $M \geq 2$ then for $\lambda > 0$

$$\frac{|\Phi_{-}(\Theta,\zeta)|}{\sqrt{1-2\Theta\zeta+\zeta^{2}}} \leq \frac{\Gamma(2\lambda+M)}{\Gamma(2\lambda)\,\Gamma(M)}+\frac{2^{1-\lambda}\,\lambda\,\Gamma(2\lambda+M)}{\Gamma^{2}(\lambda+1)\,\Gamma(M-1)\,(\lambda+M-1)} \tag{4.18}$$

$$\sim \left[\frac{1}{\Gamma(2\lambda)}+\frac{2^{1-\lambda}\,\lambda}{\Gamma^{2}(\lambda+1)}\right]M^{2\lambda}\qquad (M\to\infty). \tag{4.19}$$

The last step is by Stirling's approximation, (1.21). Therefore,

$$\frac{|\Phi_{-}(\Theta,\zeta)|}{\sqrt{1-2\Theta\zeta+\zeta^{2}}} \leq a_{\lambda}M^{2\lambda} \tag{4.20}$$

where a_λ is a positive constant that depends only on λ. When $M = 1$ this result still holds since $C^{\lambda}_{m} = 0$ for $m < 0$ and $C^{\lambda}_{0} = 1$.

Now, from (4.10),

$$|I_1| \leq a_{\lambda}M^{2\lambda}\int\limits_{\zeta=0}^{s}(1-2\Theta\zeta+\zeta^{2})^{\lambda-1/2}\,\zeta^{M-1}\,d\zeta. \tag{4.21}$$

If $\lambda \geq 1/2$ then

$$|I_1| \leq a_{\lambda}M^{2\lambda}(1+s)^{2\lambda-1}\int\limits_{\zeta=0}^{s}\zeta^{M-1}\,d\zeta \tag{4.22}$$

$$= a_{\lambda}M^{2\lambda-1}(1+s)^{2\lambda-1}s^{M}. \tag{4.23}$$

Hence, (3.34) and the above results lead to the estimate

$$|K_M(\lambda,x,y')| \le \frac{b_\lambda\, s^M \sec^{2\lambda}\theta\, M^{2\lambda-1}}{|y'|^{2\lambda}(1+s)} \tag{4.24}$$

(b_λ depends only on λ). Hence, $|F_{\lambda,M}[f](x)| < \infty$ if

$$\int\limits_{|y'|>1} \frac{|f(y')|\,|x|^M\, M^{2\lambda-1}\,dy'}{|y'|^{M+2\lambda}} < \infty. \tag{4.25}$$

When $0 < \lambda < 1/2$, (4.9) becomes

$$|K_M(\lambda,x,y')| \le K(\lambda,x,y')\, a_\lambda\, M^{2\lambda} I_2, \tag{4.26}$$

where

$$I_2 = \int\limits_{\zeta=0}^{s} |1-\zeta|^{2\lambda-1}\,\zeta^{M-1}\,d\zeta \tag{4.27}$$

$$= s^M \int\limits_{\zeta=0}^{1} |1-s\zeta|^{2\lambda-1}\,\zeta^{M-1}\,d\zeta. \tag{4.28}$$

If $0 \le s \le 1$, then

$$I_2 \le s^M \int\limits_{\zeta=0}^{1} (1-\zeta)^{2\lambda-1}\zeta^{M-1}\,d\zeta \tag{4.29}$$

$$= \frac{s^M\,\Gamma(2\lambda)\,\Gamma(M)}{\Gamma(2\lambda+M)} \tag{4.30}$$

$$\sim \frac{s^M\,\Gamma(2\lambda)}{M^{2\lambda}} \quad (M\to\infty,\ \text{Stirling's approximation}). \tag{4.31}$$

If $M = 1$ and $s \ge 1$ then from (3.37),

$$I_2 \le \frac{s^{2\lambda}}{\lambda}. \tag{4.32}$$

If $M \ge 2$ and $s \ge 1 + M^{-1}$ then write

$$\begin{aligned}
I_2 &= \int_{\zeta=0}^{1-M^{-1}} (1-\zeta)^{2\lambda-1}\zeta^{M-1}\,d\zeta + \int_{\zeta=1-M^{-1}}^{1+M^{-1}} |1-\zeta|^{2\lambda-1}\zeta^{M-1}\,d\zeta \\
&\quad + \int_{\zeta=1+M^{-1}}^{s} (\zeta-1)^{2\lambda-1}\zeta^{M-1}\,d\zeta \\
&\le M^{1-2\lambda}\int_{\zeta=0}^{1-M^{-1}} \zeta^{M-1}\,d\zeta + (1+M^{-1})^{M-1}\int_{\zeta=1-M^{-1}}^{1+M^{-1}} |1-\zeta|^{2\lambda-1}\,d\zeta \\
&\quad + M^{1-2\lambda}\int_{\zeta=1+M^{-1}}^{s} \zeta^{M-1}\,d\zeta \\
&= M^{-2\lambda}(1-M^{-1})^M + \frac{(1+M^{-1})^{M-1}M^{-2\lambda}}{\lambda} + M^{-2\lambda}\left[s^M - (1+M^{-1})^M\right] \\
&\le \frac{s^M}{\lambda M^{2\lambda}}(e\,s^{-M}+\lambda) \\
&\le \frac{c_\lambda s^M}{M^{2\lambda}}, \quad \text{where } c_\lambda \text{ depends only on } \lambda. \qquad (4.33)
\end{aligned}$$

And, if $M \ge 2$ and $1 \le s \le 1 + M^{-1}$, then write

$$\begin{aligned}
I_2 &= \int_{\zeta=0}^{1-M^{-1}} (1-\zeta)^{2\lambda-1}\zeta^{M-1}\,d\zeta + \int_{\zeta=1-M^{-1}}^{s} |1-\zeta|^{2\lambda-1}\zeta^{M-1}\,d\zeta \\
&\le M^{1-2\lambda}\int_{\zeta=0}^{1-M^{-1}} \zeta^{M-1}\,d\zeta + s^{M-1}\int_{\zeta=1-M^{-1}}^{s} |1-\zeta|^{2\lambda-1}\,d\zeta \\
&= M^{-2\lambda}(1-M^{-1})^M + \frac{s^{M-1}}{2\lambda}\left[(s-1)^{2\lambda} + M^{-2\lambda}\right] \\
&\le \frac{s^M}{\lambda M^{2\lambda}}\left[\frac{\lambda s^{-M}}{e} + \frac{(s-1)^{2\lambda}M^{2\lambda}}{2s} + \frac{1}{2s}\right] \\
&\le \frac{d_\lambda s^M}{M^{2\lambda}}, \quad \text{where } d_\lambda \text{ depends only on } \lambda. \qquad (4.34)
\end{aligned}$$

The four estimates (4.31)–(4.34) give

$$|I_2| \le \frac{d_\lambda s^M}{M^{2\lambda}}, \text{ valid for } s \ge 0 \text{ and } M \ge 1. \qquad (4.35)$$

From (3.34), (4.26) and (4.28),

$$|K_M(\lambda, x, y')| \leq \frac{e_\lambda \, |x|^M \sec^{2\lambda}\theta}{|y'|^M(|y'|+|x|)^{2\lambda}}, \tag{4.36}$$

where e_λ depends only on λ. Hence, $|F_{\lambda,M}[f](x)| < \infty$ if

$$\int\limits_{|y'|>1} \frac{|f(y')|\,|x|^M\,dy'}{|y'|^{M+2\lambda}} < \infty \quad (0 < \lambda \leq \frac{1}{2}). \tag{4.37}$$

Theorem 4.3.1 *In order for $F_{\lambda,M}[f]$ to exist on Π_+, a necessary and sufficient condition is*

$$\int\limits_{|y'|>1} \frac{|f(y')|\,e^{\alpha M}\,dy'}{|y'|^{M+2\lambda}} < \infty. \tag{4.38}$$

for each $\alpha > 0$.

Proof. Let $\alpha > 0$ be given. Let μ be $2\lambda-1$ or 2λ, according as $\lambda \geq 1/2$ or $0 < \lambda \leq 1/2$. Then

$$|x|^M M^\mu \geq e^{\alpha M} \tag{4.39}$$

if $|x| \geq e^\alpha M^{-\mu/M}$. And, $M^{-\mu/M} \leq 1$ so if $|x| \geq e^\alpha$ then $|x|^M M^\mu \geq e^{\alpha M}$ and convergence in (4.25) or (4.37) implies convergence in (4.38).

Now suppose $x \in \Pi_+$ is given. Then

$$e^{\alpha M} \geq |x|^M M^\mu \tag{4.40}$$

if $\alpha \geq \log|x| + (\mu \log M)/M$. Since $(\log M)/M \leq e^{-1}$, if we take $\alpha = \max(\log|x| + \mu/e, 0)$ then $e^{\alpha M} \geq |x|^M M^\mu$ and convergence in (4.38) implies convergence in (4.25) or (4.37). ■

Theorem 4.3.2 *Given $M\colon \mathbb{R}^{n-1} \to \{0, 1, 2, \cdots\}$ and a measurable function f defined on $\mathbb{R}^{n-1}$, if (4.38) holds then $F_{\lambda,M}[f](x) = o(G(|x|)\sec^{2\lambda})$ ($x \in \Pi_+$, $|x| \to \infty$), where*

$$G(|x|) = \int\limits_{|y'|>1} |f(y')|\,|x|^M M^\mu |y'|^{-(M+2\lambda)}\,dy'. \tag{4.41}$$

The parameter μ is taken as $2\lambda - 1$ when $\lambda \geq 1/2$ and 0 when $0 < \lambda \leq 1/2$.

Proof: Apply the Dominated Convergence Theorem to (4.24) and (4.36). ■

Determining the behaviour of $G(|x|)$ as $|x| \to \infty$ is an interesting asymptotics problem. Assuming something about $M(y')$ as $|y'| \to \infty$ will probably be necessary. This point will not be pursued further here.

Now consider the situation where f is a given function on $\mathbb{R}^{n-1}$. We wish to determine $M(y')$ so that $F_{\lambda,M}[f]$ converges on Π_+. Hence, we want $|f(y')|e^{\alpha M}|y'|^{-(M+2\lambda)}$ to be integrable over $|y'| \geq 1$ for each $\alpha > 0$. For any $y' \in \mathbb{R}^{n-1} \setminus B_1$ such that $|f(y')| \leq A|y'|^B$ where $A \geq 0$ and $B < M_0 + 2\lambda - n + 1$ are constants and $M_0 \geq 0$ is an integer, we can take $M(y') = M_0$. In particular, if $|f(y')| \leq 1$ then let

$$M(y') = \begin{cases} \max(0, \lceil n - 2\lambda - 1 \rceil) & \text{if } 2\lambda \notin \mathbb{Z} \\ \max(0, n - 2\lambda) & \text{if } 2\lambda \in \mathbb{Z}. \end{cases} \tag{4.42}$$

Hence, in choosing $M(y')$ we really only need to consider those y' for which $|f(y')| \geq 1$.

Let $I > 0$ be a measurable function such that $\int_{|y'|>1} I(y')\,dy' < \infty$. Suppose $|f(y')| \geq 1$. Let $|f(y')|e^{\alpha M(y')}|y'|^{-(M(y')+2\lambda)} \leq I(y')$, $|y'| > e^{\alpha}$ and solve for M:

$$\log|f| + \alpha M \leq \log I + (M + 2\lambda)\log|y'| \tag{4.43}$$

yields

$$(\log|y'| - \alpha)M \geq \log|f| - \log I - 2\lambda \log|y'|. \tag{4.44}$$

Therefore,

$$M \geq \frac{\log|f| - \log I - 2\lambda\log|y'|}{\log|y'| - \alpha} \tag{4.45}$$

$$= \left[\frac{\log|f| - \log I - 2\lambda\log|y'|}{\log|y'|}\right]\left[1 + \frac{\alpha}{\log|y'|} + \cdots\right]. \tag{4.46}$$

This says what M must be in order for (4.38) to hold.

Theorem 4.3.3 *Given functions f, I and h locally integrable on $\mathbb{R}^{n-1} \setminus B_1$ such that I is positive and integrable over $\mathbb{R}^{n-1} \setminus B_1$ and $h(y') \to \infty$ as $|y'| \to \infty$ then* (4.38) *holds if M is defined as follows. For those $y' \in \mathbb{R}^{n-1} \setminus B_1$ such that $|y'| \leq 2$ or $|f(y')| \leq 1$ let*

$$M(y') = \begin{cases} \max(0, \lceil n - 2\lambda - 1 \rceil) & \text{if } 2\lambda \notin \mathbb{Z} \\ \max(0, n - 2\lambda) & \text{if } 2\lambda \in \mathbb{Z}. \end{cases}$$

And, if $|y'| > 2$ so that $|f(y')| > 1$ then let $M(y') = \max(0, \lceil M_1(y') \rceil)$ where

$$M_1(y') = \begin{cases} c & \text{if } \frac{\log|f(y')| - \log I(y')}{\log|y'|} - 2\lambda \leq c \\ \left(1 + \frac{h(y')}{\log|y'|}\right)\left(\frac{\log|f(y')| - \log I(y')}{\log|y'|} - 2\lambda\right), & \text{otherwise.} \end{cases} \tag{4.47}$$

Here $c \geq 0$ is any constant.

Proof. By the preceding remarks we need only consider $|y'| > 2$ where $|f(y')| \geq 1$.

Let $c \geq 0$ be a constant and let $E_f \subset \mathbb{R}^{n-1} \setminus B_2$ be the set where $|f(y')| \geq 1$ and $\frac{\log|f(y')| - \log I(y')}{\log|y'|} - 2\lambda \leq c$. Let $M_1(y') = c$ on E_f. For $y' \in E_f$ we have

$$\frac{|f(y')|\, e^{\alpha M_1(y')}}{|y'|^{M_1(y')+2\lambda}} \leq I(y')\, e^{\alpha c} \tag{4.48}$$

and

$$\int_{E_f} \frac{|f(y')|\, e^{\alpha M_1(y')}\, dy'}{|y'|^{M_1(y')+2\lambda}} < \infty \qquad \text{for each} \qquad \alpha > 0. \tag{4.49}$$

If $|y'| > 2$ so that $|f(y')| > 1$ and

$$\frac{\log|f(y')| - \log I(y')}{\log|y'|} - 2\lambda > c, \tag{4.50}$$

then take

$$M_1(y') = \left(1 + \frac{h(y')}{\log|y'|}\right)\left(\frac{\log|f(y')| - \log I(y')}{\log|y'|} - 2\lambda\right). \tag{4.51}$$

We now have

$$\begin{aligned}
&\frac{|f(y')|\, e^{\alpha M_1(y')}}{|y'|^{M_1(y')+2\lambda}} \\
&= \frac{|f(y')| \exp\left[\alpha\left(1+\frac{h(y')}{\log|y'|}\right)\left(\frac{\log|f(y')|-\log I(y')}{\log|y'|}-2\lambda\right)\right]}{|y'|^{2\lambda}\exp\left[\left(1+\frac{h(y')}{\log|y'|}\right)(\log|f(y')|-\log I(y')-2\lambda\log|y'|)\right]} \\
&= I(y')\exp\left\{\left[\alpha\left(1+\frac{h(y')}{\log|y'|}\right)-h(y')\right]\left(\frac{\log|f(y')|-\log I(y')}{\log|y'|}-2\lambda\right)\right\} \\
&\le I(y') \qquad \text{for large enough} \qquad |y'|. \qquad (4.52)
\end{aligned}$$

Hence, (4.38) holds with $M = M_1$.

The function $M \mapsto e^{\alpha M}|y'|^{-M}$ has derivative $e^{\alpha M}|y'|^{-M}(\alpha - \log|y'|)$. It is a decreasing function of M if $|y'| \geq e^{\alpha}$. Therefore, (4.38) still holds when we let $M(y') = \max(0, \lceil M_1 \rceil)$. ■

Erratic pointwise behaviour of f need not lead to erratic behaviour of M. The function I can be chosen so that $\log|f(y')| - \log I(y')$ is continuous. For example, we could write $I = fg$ where g is continuous and chosen so that fg is integrable over $|y'| > 1$.

The condition $h(y') \to \infty$ cannot in general be omitted. For suppose $0 \leq h(y') \leq h_0$ where h_0 is a positive constant. Let $\alpha \geq 2h_0$. Then for $|y'| > 1$ it follows that

$$\alpha\left(1+\frac{h(y')}{\log|y'|}\right) - h(y') \geq 2h_0 - h_0 = h_0. \qquad (4.53)$$

Let $f(y') = e^{|y'|} \geq 1$ and $I(y') = |y'|^{-n}$. Then $\int_{|y'|>1} I(y')\,dy' < \infty$ but

$$\frac{|f(y')|\, e^{\alpha M_1(y')}}{|y'|^{M_1(y')+2\lambda}} \geq |y'|^{-n}\exp\left[h_0\left(\frac{|y'|}{\log|y'|}+n-2\lambda\right)\right], \qquad (4.54)$$

which is not integrable over $|y'| > 1$. So the integral in (4.38) diverges.

And, if I is not integrable over $|y'| > 1$ then (4.38) may fail to hold. Let $c \geq 0$ be a constant. Given $h(y') \to \infty$, let $f(y') = \exp[h(y')|y'|/\log|y'| + |y'|]$ and $I(y') =$

$\exp[h(y')|y'|/\log|y'|]$. Then $I(y') \geq 1$ for $|y'| > 1$ and I is not integrable. And,

$$\begin{aligned} \frac{\log|f(y')| - \log I(y')}{\log|y'|} - 2\lambda &= \frac{|y'|}{\log|y'|} - 2\lambda \\ &> c \qquad \text{for large enough } |y'|. \end{aligned}$$

But, with M_1 given by (4.47) we have

$$\begin{aligned} &\frac{|f(y')|\, e^{\alpha M_1(y')}}{|y'|^{M_1(y')+2\lambda}} \\ &= \exp\left\{\left[\frac{h(y')|y'|}{\log|y'|} + |y'|\right] + \left[\alpha\left(1 + \frac{h(y')}{\log|y'|}\right) - h(y')\right]\left(\frac{|y'|}{\log|y'|} - 2\lambda\right)\right\} \\ &\geq \exp\left(|y'| + 2\lambda h(y')\right) \\ &\to \infty \qquad \text{as } |y'| \to \infty. \end{aligned} \tag{4.55}$$

So (4.38) does not hold.

When f is a polynomial (or of polynomial order) M can be taken as a constant.

Corollary 4.3.1 *When f is of polynomial order, $|f(y')| \leq A|y'|^N$ for large enough $|y'|$, it suffices to take*

$$M(y') = \begin{cases} \max(0, N - \lfloor 2\lambda \rfloor + n - 1) & \text{if } 2\lambda \notin \mathbb{Z} \\ \max(0, N - 2\lambda + n) & \text{if } 2\lambda \in \mathbb{Z}. \end{cases} \tag{4.56}$$

Proof: Let $I(y') = |y'|^{1-n} \log^{-\beta}|y'|$ where $\beta > 1$ is a constant. Let h be as in the theorem with $h(y') = o(\log|y'|)$. Then for y' such that $|y'| > 1$ and $|f(y'| \geq 1$, we have

$$M_1(y') = \left(1 + \frac{h(y')}{\log|y'|}\right)\left(\frac{\log|f(y')| - \log I(y')}{\log|y'|} - 2\lambda\right) \tag{4.57}$$

$$\leq \left(1 + \frac{h(y')}{\log|y'|}\right)\left(\frac{\log A + \beta \log\log|y'|}{\log|y'|} + N - 2\lambda + n - 1\right) \tag{4.58}$$

$$\sim N - 2\lambda + n - 1 \qquad \text{as} \qquad |y'| \to \infty. \tag{4.59}$$

If 2λ is not an integer then for large enough $|y'|$ we have $M_1(y') \leq N - \lfloor 2\lambda \rfloor + n - 1$. And, we can take

$$M(y') = \begin{cases} \max(0, N - \lfloor 2\lambda \rfloor + n - 1) & \text{if } \; 2\lambda \notin \mathbb{Z} \\ \max(0, N - 2\lambda + n) & \text{if } \; 2\lambda \in \mathbb{Z} \end{cases} \tag{4.60}$$

for all $y' \in \mathbb{R}^{n-1} \setminus B_1$. ■

Example. Suppose $\int_{\mathbb{R}^{n-1}} \frac{|f(y')|\,dy'}{|y'|^{M_0+2\lambda}+1} < \infty$ for a constant $M_0 \geq 0$. Let $I(y') = |f(y')|\,|y'|^{-(M_0+2\lambda)}$. Then

$$\frac{\log|f(y')| - \log I(y')}{\log|y'|} - 2\lambda = M_0. \tag{4.61}$$

Take $c = M_0$. Then $M_1 = M_0$ and $M = \lceil M_0 \rceil$ and the theorem gives the expected value for M. Note that if $h(y') = o(\log|y'|)$ then, when $|f(y')| \geq 1$, $M_1(y') = [1 + h(y')/\log|y'|]M_0$, which is asymptotic to M_0.

We conclude this section with an example of data with greater than polynomial growth.

Example. If β and γ are positive constants and $f(y') = \exp(\beta|y'|^\gamma)$, let $I(y') = |y'|^{-n}$. Then for any positive constant c,

$$\begin{aligned} \frac{\log|f(y')| - \log I(y')}{\log|y'|} - 2\lambda &= \frac{\beta|y'|^\gamma + n\log|y'|}{\log|y'|} - 2\lambda \\ &= \frac{\beta|y'|^\gamma}{\log|y'|} + n - 2\lambda \\ &> c \qquad \text{for large enough } |y'|. \end{aligned}$$

Let $h(y') = o(\log|y'|)$. Then,

$$\begin{aligned} M_1(y') &= \left(1 + \frac{h(y')}{\log|y'|}\right)\left(\frac{\beta|y'|^\gamma}{\log|y'|} + n - 2\lambda\right) \\ &\sim \frac{\beta|y'|^\gamma}{\log|y'|} \qquad \text{as } |y'| \to \infty. \end{aligned}$$

4.4 Second type of modified kernel

Using the generating function (3.13) with $z = |y'|/|x|$, $t = \Theta = \sin\theta\cos\theta'$, we can define a second type of modified kernel

$$\tilde{K}_M(\lambda, x, y') = K(\lambda, x, y') - \sum_{m=0}^{M-1} \frac{|y'|^m}{|x|^{m+2\lambda}} C_m^{\lambda}(\Theta), \tag{4.62}$$

defined for $|x| > 0$ and $M \geq 1$. The convergence condition corresponding to (3.5) is now

$$\int_{\mathbf{R}^{n-1}} |f(y')|(|y'|^{M-1} + 1)\, dy' < \infty. \tag{4.63}$$

If (4.63) is satisfied, define

$$\tilde{F}_{\lambda,M}[f](x) = \int_{\mathbf{R}^{n-1}} f(y')\tilde{K}_M(\lambda, x, y')\, dy'. \tag{4.64}$$

Define $\tilde{D}_M$ and $\tilde{N}_M$ in terms of $\tilde{F}_{\lambda,M}$ as in (3.2) and (3.3). Each $x_n|x|^{-(m+n)}C_m^{n/2}(\Theta)$ in the kernel $\tilde{D}_M$ is harmonic in $\mathbf{R}^n\backslash\{0\}$ (3.2.1). Similarly with $|x|^{-(m+n-2)}C_m^{(n-2/2)}(\Theta)$ in the kernel $\tilde{N}_M$. Hence, $\tilde{D}_M[f]$ and $\tilde{N}_M[f]$ are harmonic in Π_+. Results similar to Propositions 4.2.1 and 4.2.2 hold for $\tilde{K}_M$, $\tilde{D}_M$ and $\tilde{N}_M$.

However, $\tilde{D}_M[f]$ is not continuous on $\overline{\Pi}_+$. Since (4.63) implies (2.6) (with $\lambda = n/2$) the unmodified Poisson integral $D[f]$ is continuous for $x_n \geq 0$ if f is continuous. Hence,

$$\tilde{D}_M[f](x) = D[f](x) - \alpha_n x_n \sum_{m=0}^{M-1} |x|^{-(m+n)} \int_{\mathbf{R}^{n-1}} |y'|^m f(y') C_m^{n/2}(\Theta)\, dy'$$

and $\tilde{D}_M[f]$ is continuous for $x_n \geq 0$, $x \neq 0$. Similar remarks apply to the Neumann case. We will work with $\tilde{D}_M$ and $\tilde{N}_M$ only in the limit $|x| \to \infty$.

Growth estimates for $\tilde{F}_{\lambda,M}$ are similar to those for $F_{\lambda,M}$.

Theorem 4.4.1 *If* (4.63) *holds for measurable* f *then*

$$\tilde{F}_{\lambda,M}[f](x) = o(|x|^{-(M+2\lambda-1)}\sec^{2\lambda}\theta) \quad (x \in \Pi_+,\ |x| \to \infty)$$

and this estimate is sharp in the sense of Definition 3.3.1.

Proof: Throughout the proof d_1 and d_2 will be positive constants (depending on λ and M). In (3.23) replace $|x|/|y'|$ by $|y'|/|x|$ and in the proof of Theorem 3.3.1 let $p = |y'|/|x|$.

If $\lambda \geq \frac{1}{2}$ then (3.32) and (3.34) give

$$\begin{aligned} |\tilde{K}_M(\lambda, x, y')| &\leq \frac{d_1 p^{M-1}\sec^{2\lambda}\theta\,[(1+p)^{2\lambda}-1]}{(|x|+|y'|)^{2\lambda}} \\ &\leq \frac{d_1|y'|^{M-1}\sec^{2\lambda}\theta[(1+p)^{2\lambda}-1]}{|x|^{M+2\lambda-1}(1+p)^{2\lambda}}. \end{aligned} \tag{4.65}$$

Since $(1+p)^{2\lambda}-1 \to 0$ as $p \to 0$, integrating (4.65) and noting (4.63), dominated convergence gives

$$\int\limits_{\mathbf{R}^{n-1}} f(y')\tilde{K}_M(\lambda, x, y')\,dy' = o(|x|^{-(M+2\lambda-1)}\sec^{2\lambda}\theta) \quad (x \in \Pi_+,\ |x| \to \infty).$$

If $0 < \lambda < \frac{1}{2}$ then (3.32), (3.34) and (3.37) give

$$\begin{aligned} |\tilde{K}_M(\lambda, x, y')| &\leq d_5\, K(\lambda, x, y')\, p^{M-1} \int\limits_{\zeta=0}^{p} |1-\zeta|^{2\lambda-1}\, d\zeta \\ &\leq \frac{d_6\, p^{M+2\lambda-1}\sec^{2\lambda}\theta}{(|x|+|y'|)^{2\lambda}}, \end{aligned}$$

from which $\tilde{F}_{\lambda,M}[f](x) = o(|x|^{-(M+2\lambda-1)}\sec^{2\lambda}\theta)$.

To prove this sharp, interchange $|x|$ and $|y'|$ in the proof of Theorem 3.3.1 and proceed in a similar manner. ∎

The modified kernel furnishes an asymptotic expansion of $D[f]$ and $N[f]$.

Theorem 4.4.2 *Let f be measurable such that* (4.63) *holds for a positive integer M. Then, as $x \to \infty$ in Π_+*

$$(i)\ D[f](x) = \sum_{m=0}^{M-1} |x|^{-(m+n-1)} Y^{(0)}_{m+1}(\hat{x}) + o(|x|^{-(M+n-2)} \sec^{n-1}\theta) \quad (n \geq 2)$$

$$(ii)\ N[f](x) = \sum_{m=0}^{M-1} |x|^{-(m+n-2)} Y^{(1)}_{m}(\hat{x}) + o(|x|^{-(M+n-3)} \sec^{n-2}\theta) \quad (n \geq 3)$$

where $Y^{(0)}_m$ is given by (4.66) *below and is a spherical harmonic of degree m that vanishes on $\partial\Pi_+$ and $Y^{(1)}_m$ is given by* (4.67) *below and is a spherical harmonic of degree m whose normal derivative vanishes on $\partial\Pi_+$. The data f can be chosen so that simultaneously the leading order term* ($m = 0$) *does not vanish and the order relation is sharp (in the sense of Definition 3.3.1).*

Proof. To prove (i) use (4.62) with $\lambda = n/2$, f as in the Theorem and $|x| > 0$,

$$D[f](x) = \alpha_n x_n \int_{\mathbf{R}^{n-1}} f(y') \sum_{m=0}^{M-1} \frac{|y'|^m}{|x|^{m+n}} C^{n/2}_m(\Theta)\, dy' + \tilde{D}_M[f](x).$$

Now (i) follows from Theorem 4.4.1 and the definition

$$Y^{(0)}_{m+1}(\hat{x}) = \alpha_n \cos\theta \int_{\mathbf{R}^{n-1}} f(y')\, |y'|^m\, C^{n/2}_m(\sin\theta\, \hat{y}\cdot\hat{y}')\, dy'. \tag{4.66}$$

Clearly $Y^{(0)}_{m+1}$ vanishes when $\theta = \pi/2$. It is a spherical harmonic of degree $m+1$ by Remark 3.2.1.

Given $\psi(x) = o(|x|^{-(M+n-2)} \sec^{n-1}\theta)$ take f as in Theorem 4.4.1 so that $\tilde{D}_M[f](x) = o(|x|^{-(M+n-2)} \sec^{n-1}\theta)$ is sharp. In particular, f can be taken to be positive for $y_1 > 0$ with a super-even extension (M even) or super-odd extension (M odd) to $y_1 < 0$ ($A_\lambda > 1$ in the proof of Theorem 3.3.1 (3.51)). The leading order term in (i) is with

$m = 0$, $Y_1^{(0)}(\hat{x}) = \alpha_n \cos\theta \int_{\mathbb{R}^{n-1}} f(y')\,dy'$. With f as above this spherical harmonic does not vanish if $0 \le \theta < \pi/2$. With the definition

$$Y_m^{(1)}(\hat{x}) = \frac{\alpha_n}{n-2} \int_{\mathbb{R}^{n-1}} f(y')\,|y'|^m\, C_m^{(n-2)/2}(\Theta)\,dy', \tag{4.67}$$

the proof of (ii) is similar. ■

Remark 4.4.1 If $\int_{\mathbb{R}^{n-1}} |f(y')|\,dy' < \infty$ and $\int_{\mathbb{R}^{n-1}} f(y')\,dy' = 0$ then (4.63) holds with $M = 1$ and the leading behaviour is $D[f](x) = o(|x|^{-(n-1)} \sec^{n-1}\theta)$ and $N[f](x) = o(|x|^{-(n-2)} \sec^{n-2}\theta)$.

The addition formula for Gegenbauer polynomials can be used to separate the θ dependence in (i) and (ii). First write

$$C_m^{n/2}(\sin\theta\, \hat{y}\cdot\hat{y}') = \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \gamma_{n,m,\ell}(\theta)\, C_{m-2\ell}^{(n-1)/2}(\hat{y}\cdot\hat{y}'),$$

where

$$\gamma_{n,m,\ell}(\theta) = \frac{(n-2)!\,(-1)^\ell\, 4^{m-2\ell}\,(2\ell)!\,(n+2m-4\ell-1)\,\Gamma\left(\frac{n}{2}+m-2\ell\right)\,\Gamma\left(\frac{n}{2}+m-\ell\right)}{\Gamma^2(n/2)\,\ell!\,(n+2m-2\ell-1)!} \times$$
$$\times \sin^{m-2\ell}\theta\, C_{2\ell}^{n/2+m-2\ell}(\cos\theta)$$

([20] 10.9.34†, 10.9.19). Then (4.66) becomes

$$Y_{m+1}^{(0)}(\hat{x}) = \alpha_n \cos\theta \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \gamma_{n,m,\ell}(\theta)\, \delta_{n,m,\ell}(\hat{y}),$$

where

$$\delta_{n,m,\ell}(\hat{y}) = \int_{\mathbb{R}^{n-1}} f(y')\,|y'|^m\, C_{m-2\ell}^{(n-1)/2}(\hat{y}\cdot\hat{y}')\,dy'$$

and is independent of $|x|$ and θ.

†The first term in the sum over m in this formula should read 2^{2m}.

A similar separation of $|x|$, θ and $\hat{y}$ dependence in (ii) is given by

$$Y_m^{(1)}(\hat{x}) = \frac{\alpha_n}{n-2} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \gamma_{n-2,m,\ell}(\theta)\, \delta_{n-2,m,\ell}(\hat{y}) \quad (n \geq 3).$$

The function defined by $Z_m(\hat{y}_1, \hat{y}_2) = C_m^{n/2}(\hat{y}_1 \cdot \hat{y}_2)$ is known as a zonal harmonic of degree m with pole $\hat{y}_1 \in \partial B_1$, evaluated at $\hat{y}_2 \in \partial B_1$ (see [7], Chapter 5).

If the integral in (4.63) converges for all $M \geq 1$, letting $M \to \infty$ in Theorem 4.4.2 will give asymptotic series for $D_M[f]$ and $N_M[f]$. As the following example shows, these series will not in general be convergent.

Example. Let $f(y) = \exp(-|y|)$ and let $d\omega_{n-1}$ be surface measure on the unit ball of $\mathbb{R}^{n-1}$. Then for $n \geq 3$, (4.67) becomes

$$\begin{aligned} Y_m^{(1)}(\hat{x}) &= \frac{\alpha_n}{n-2} \int_{\rho=0}^{\infty} e^{-\rho} \rho^{m+n-2}\, d\rho \int_{\partial B_1} C_m^{(n-2)/2}(\sin\theta\, \hat{y} \cdot \hat{y}')\, d\omega_{n-1} \\ &= \frac{\alpha_n}{n-2} (m+n-2)!\, (n-2)\, \omega_{n-2}\, I_{n,m}^{(1)}(\theta), \end{aligned}$$

where

$$I_{n,m}^{(1)}(\theta) = \int_{\phi=0}^{\pi} C_m^{(n-2)/2}(\sin\theta\cos\phi) \sin^{n-3}\phi\, d\phi$$

and the surface integral was evaluated by spherical means [38]. The integral $I_{n,m}^{(1)}(\theta)$ is known ([30] 7.323.2, together with [20] 10.9.19),

$$I_{n,m}^{(1)}(\theta) = \begin{cases} \frac{2^{n-3}\Gamma(n/2-1)\,(-1)^k\,(2k)!\,\Gamma(k+n/2-1)\, C_{2k}^{(n-2)/2}(\cos\theta)}{k!\,\Gamma(2k+n-2)}, & m = 2k \\ 0, \quad m \quad \text{odd}. \end{cases}$$

And,

$$Y_{2k}^{(1)}(\hat{x}) = \frac{2^{n-2}\Gamma(n/2-1)\,(-1)^k\,(2k)!\,\Gamma(k+n/2)\, C_{2k}^{(n-2)/2}(\cos\theta)}{\pi k!}$$

$(Y_{2k+1}^{(1)}(\hat{x}) = 0)$. As $k \to \infty$,

$$C_{2k}^{(n-2)/2}(\cos\theta) \sim \frac{2\,(n/2+2k-2)!\, \sin[n\pi/4 - (n/2+2k-1)\,\theta]}{(n/2-2)!\,(2k)!\,(2\sin\theta)^{n/2-1}}$$

([63] 8.4.13), so Stirling's approximation shows that for fixed x

$$\sum_{m=0}^{M-1} |x|^{-m} Y_m^{(1)}(\hat{x}) \quad \text{diverges as} \quad M \to \infty.$$

With the Dirichlet expansion we have from (4.66)

$$Y_{m+1}^{(0)}(\hat{x}) = \alpha_n (m+n-2)!\,(n-2)\,\omega_{n-2}\, I_{n,m}^{(0)}(\theta),$$

where

$$I_{n,m}^{(0)}(\theta) = \cos\theta \int_{\phi=0}^{\pi} C_m^{n/2}(\sin\theta\cos\phi)\sin^{n-3}\phi\, d\phi.$$

If $n \geq 5$ then (3.15) and integration by parts give

$$I_{n,m}^{(0)}(\theta) = \frac{1}{(n-2)\sin\theta}\frac{d}{d\theta} I_{n-2,m+2}^{(1)}(\theta)$$

and

$$\sum_{m=0}^{M-1} |x|^{-m} Y_m^{(1)}(\hat{x}) \quad \text{diverges as} \quad M \to \infty. \tag{4.68}$$

When $n = 2$, we use

$$C_m^1(\cos\phi) = \frac{\sin[(m+1)\phi]}{\sin\phi}$$

and replace θ by $\pi/2 - \phi$ (see the end of the proof of Lemma 3.2.1). With Dirichlet data $f(\xi) = \exp(-|\xi|)$ we have $Y_{m+1}^{(0)}(\hat{x}) = 2\,m!\sin[(m+1)\phi]/\pi$, if m is even and again

$$\sum_{m=0}^{M-1} \frac{m!\sin[(m+1)\theta]}{r^m} \quad \text{diverges as} \quad M \to \infty$$

for fixed $x = r\cos\theta\,\hat{e}_1 + r\sin\theta\,\hat{e}_2$.

When $n = 3$, $I_{3,m}^{(0)}(\theta)$ can be evaluated in terms of Legendre polynomials (since $C_m^{1/2}(t) = P_m(t)$) and when $n = 4$, $I_{4,m}^{(0)}(\theta)$ can be evaluated in terms of trigonometric functions. In both cases, the conclusion of (4.68) remains valid.

Chapter 5

The logarithmic kernel

5.1 Kernels for the half space Neumann problem

When $n = 2$, the half plane Neumann problem is

$$u \in C^2(\Pi_+) \cap C^1(\overline{\Pi}_+) \tag{5.1}$$

$$\Delta u = 0, \quad x_2 > 0 \tag{5.2}$$

$$\frac{\partial v}{\partial x_2} = -f, \quad x_2 = 0, \tag{5.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Polar coordinates are $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, and Π_+ is the set $x_2 > 0$, $x_1 \in \mathbb{R}$ or $r > 0$, $0 < \phi < \pi$. The angle ϕ is measured from the positive x_1 axis on $\partial\Pi_+$, whereas θ from chapters 3, 4 and 5 was measured from the normal ($|\pi/2 - \phi| = \theta$).

Provided

$$\int_{\xi=-\infty}^{\infty} |f(\xi)| \log(\xi^2 + 2)\, d\xi < \infty, \tag{5.4}$$

a solution to (5.1)–(5.3) is

$$N[f](r,\phi) = \frac{1}{2\pi} \int_{\xi=-\infty}^{\infty} f(\xi)\mathcal{N}(r,\phi,\xi)\, d\xi \tag{5.5}$$

where the Neumann kernel is

$$\mathcal{N}(r,\phi,\xi) = -\log\left(\xi^2 - 2\xi r\cos\phi + r^2\right) \tag{5.6}$$

$$= -\log\left[(\xi - x_1)^2 + x_2^2\right]. \tag{5.7}$$

Since the domain Π_+ is unbounded there is no further integrability condition on f.

As with $K(\lambda, x, y') = [|y'-y|^2 + x_n^2]^{-\lambda}$, (3.1), the Neumann kernel diverges to $+\infty$ at its singularity $\xi = x_1$, $x_2 = 0$. Because the kernel in (5.6) is not of one sign and also diverges as $r \to \infty$, it is more difficult to find the growth of (5.5) under (5.4). This will be done by writing Neumann solutions as integrals of Dirichlet solutions, along the lines of Proposition 4.2.2.

When the integral in (5.4) diverges but

$$\int_{\xi=-\infty}^{\infty} \frac{|f(\xi)|\,d\xi}{|\xi|^M + 1} < \infty, \tag{5.8}$$

for a positive integer M the modified Neumann kernel is of use. This is defined as in (3.6) from the generating function

$$\log\left(1 - 2z\cos t + z^2\right) = -2\sum_{m=1}^{\infty} \frac{z^m \cos(mt)}{m}; \qquad |z| < 1, \quad t \in \mathbb{R}. \tag{5.9}$$

This formula may be obtained from the Taylor expansion

$$\log(1-z) = \sum_{m=1}^{\infty} \frac{z^m}{m}, \quad |z| < 1, \tag{5.10}$$

by writing

$$\log(1 - 2z\cos t + z^2) = \log(1 - ze^{it}) + \log\left(1 - ze^{-it}\right). \tag{5.11}$$

See, for example, [18], p.27. In equation (5.9), put $z = r/\xi$ and $t = \phi$. The modified kernel is

$$\mathcal{N}_M(r,\phi,\xi) = \begin{cases} -\log\left(\frac{\xi^2 - 2\xi r\cos\phi + r^2}{\xi^2}\right), & M = 1 \\ -\log\left(\frac{\xi^2 - 2\xi r\cos\phi + r^2}{\xi^2}\right) - 2\sum_{m=1}^{M-1} \frac{r^m \cos(m\phi)}{m\xi^m}, & M \geq 2. \end{cases} \tag{5.12}$$

And $\mathcal{N}_0$ is defined to be $\mathcal{N}$. Let $N_M[f]$ be defined by

$$N_M[f](r,\phi) = \frac{1}{2\pi}\int\limits_{|\xi|>1} f(\xi)\mathcal{N}_M(r,\phi,\xi)\,d\xi + \frac{1}{2\pi}\int\limits_{|\xi|<1} f(\xi)\mathcal{N}(r,\phi,\xi)\,d\xi \tag{5.13}$$

and let ω be a continuous function as in Corollary 3.3.1. Then $u(r,\phi) = N_M[\omega f](r,\phi)+ N_M[(1-\omega)f](r,\phi)$ is a solution of (5.1)–(5.3) for each $M \geq 0$. The major goal of this chapter is to provide a sharp estimate (Definition 3.3.1) of (5.13) under (5.4) $(M = 0)$ or (5.8) $(M \geq 1)$.

In Chapters 3 and 4 we had an expansion of K_M in Gegenbauer polynomials, C_m^λ, with $\lambda > 0$, (3.6). Because of the limit ([63], 4.7.8)

$$\begin{aligned} \lim_{\lambda\to\infty} \tfrac{1}{\lambda} C_m^\lambda(\cos t) &= C_m^0(\cos t) && (5.14) \\ &= \begin{cases} \frac{2}{m}\cos(mt), & m \geq 1 \\ 1, \quad m = 0, \end{cases} && (5.15) \end{aligned}$$

$\mathcal{N}_M$ is in some sense the limit of K_M as $\lambda \to 0$. Many of the results in this chapter will closely parallel those of Chapters 3 and 4.

5.2 Integral form of the modified kernel

To derive an integral representation of $\mathcal{N}_M$, let

$$S_{M-1}(s) = \sum_{m=1}^{M-1} \frac{s^m \cos(m\phi)}{m}, \quad M \geq 2. \tag{5.16}$$

Lemma 5.2.1 *If $M \geq 1$ then*

$$\mathcal{N}_M(r,\theta,\xi) = 2\int\limits_{\zeta=0}^{r/\xi} \frac{(\cos(M\phi) - \zeta\cos[(M-1)\phi])\,\zeta^{M-1}\,d\zeta}{1 - 2\zeta\cos\phi + \zeta^2}. \tag{5.17}$$

Proof. First suppose $M \geq 2$. Multiply the identity

$$\cos[(m+2)\phi] - 2\cos\phi\cos[(m+1)\phi] + \cos(m\phi) = 0 \tag{5.18}$$

by s^{m+1} and sum from $m = 1$ to $M - 1$. As in Lemma 3.3.1,

$$\begin{aligned} 0 &= \sum_{m=3}^{M+1} s^{m-1}\cos(m\phi) - 2s\cos\phi\sum_{m=2}^{M} s^{m-1}\cos(m\phi) + s^2\sum_{m=1}^{M-1} s^{m-1}\cos(m\phi) \\ &= \left(1 - 2s\cos\phi + s^2\right) S'_{M-1}(s) + s^{M-1}\cos(M\phi) - s^M\cos[(M-1)\phi] + s - \cos\phi. \end{aligned}$$

This differential equation is solved subject to $S_{M-1}(0) = 0$. The solution is

$$S_{M-1}(s) = -\int_{\zeta=0}^{s} \frac{(\cos(M\phi) - \zeta\cos[(M-1)\phi])\,\zeta^{M-1}\,d\zeta}{1 - 2\zeta\cos\phi + \zeta^2} - \int_{\zeta=0}^{s} \frac{(\zeta - \cos\phi)\;d\zeta}{1 - 2\zeta\cos\phi + \zeta^2}.$$

The second integral is $-\frac{1}{2}\log(1 - 2s\cos\phi + s^2)$. So (5.17) follows on putting $s = r/\xi$ in the above equation.

When $M = 1$, direct integration of (5.17) gives (5.12). ■

5.3 Integral representations of Neumann solutions

In the present notation, the modified Dirichlet integral is

$$D_M[f](r,\phi) = \frac{r\sin\phi}{\pi}\int_{\xi=-\infty}^{\infty} f(\xi)\mathcal{D}_M(r,\phi,\xi)\,d\xi \tag{5.19}$$

where (from (3.7) and (3.28))

$$\mathcal{D}_M(r,\phi,\xi) = \frac{1}{\xi^2 - 2\xi r\cos\phi + r^2} - \sum_{m=1}^{M} \frac{r^{m-1}\sin(m\phi)}{\xi^{m+1}\sin\phi}. \tag{5.20}$$

In order to work with both Cartesian and polar coordinates we will abuse notation by writing $D_M[f](x_1, x_2) = D_M[f](r,\phi)$ where $x_1 = r\cos\phi$ and $x_2 = r\sin\phi$. Similarly with N_M, $\mathcal{D}_M$ and $\mathcal{N}_M$.

Each derivative of $\mathcal{N}_M$ with respect to x_1, x_2, r or ϕ is a linear combination of $\mathcal{D}_M$, $\mathcal{D}_{M-1}$ and $\mathcal{D}_{M-2}$.

Proposition 5.3.1 *Let $M \geq 0$. Use the convention that $\mathcal{D}_m = \mathcal{D}$ if $m \leq 0$. Then*

(i) $\dfrac{\partial \mathcal{N}_M}{\partial \phi}(r,\phi,\xi) = -2\xi r \sin\phi \mathcal{D}_{M-1}(r,\phi,\xi)$

(ii) $\dfrac{\partial \mathcal{N}_M}{\partial r}(r,\phi,\xi) = 2\xi r \cos\phi \mathcal{D}_{M-1}(r,\phi,\xi) - 2r\mathcal{D}_{M-2}(r,\phi,\xi)$

(iii) $\dfrac{\partial \mathcal{N}_M}{\partial x_1}(x_1,x_2,\xi) = 2\xi \mathcal{D}_{M-1}(x_1,x_2,\xi) - 2x_1\mathcal{D}_{M-2}(x_1,x_2,\xi)$

(iv) $\dfrac{\partial \mathcal{N}_M}{\partial x_2}(x_1,x_2,\xi) = -2x_2\mathcal{D}_{M-2}(x_1,x_2,\xi)$.

Proof of (ii): Differentiating (5.17) with respect to r gives

$$\begin{aligned}\frac{\partial \mathcal{N}_M}{\partial r}(r,\phi,\xi) &= \frac{2\left(\cos(M\phi) - \frac{r}{\xi}\cos[(M-1)\phi]\right)\left(\frac{r}{\xi}\right)^{M-1}}{\xi\left(1 - 2\frac{r}{\xi}\cos\phi + (\frac{r}{\xi})^2\right)} &(5.21)\\ &= \frac{2r^{M-1}\left(\xi\cos(M\phi) - r\cos[(M-1)\phi]\right)}{\xi^{M-2}\left(\xi^2 - 2\xi r\cos\phi + r^2\right)}. &(5.22)\end{aligned}$$

Using the identity

$$\cos(m\phi) = \frac{\sin(m\phi)\cos\phi - \sin[(m-1)\phi]}{\sin\phi} \tag{5.23}$$

with $m = M$ and $M-1$,

$$\begin{aligned}\frac{\partial \mathcal{N}_M}{\partial r}(r,\phi,\xi) &= \frac{2r^{M-1}\left(\xi\sin(M\phi) - r\sin[(M-1)\phi]\right)\cos\phi}{\xi^{M-2}\left(\xi^2 - 2\xi r\cos\phi + r^2\right)\sin\phi}\\ &\quad - \frac{2r^{M-1}\left(\xi\sin[(M-1)\phi] - r\sin[(M-2)\phi]\right)}{\xi^{M-2}\left(\xi^2 - 2\xi r\cos\phi + r^2\right)} &(5.24)\\ &= 2\xi\cos\phi\mathcal{D}_{M-1}(r,\phi,\xi) - 2r\mathcal{D}_{M-2}(r,\phi,\xi). \quad \blacksquare &(5.25)\end{aligned}$$

The other proofs are done in a like manner by differentiating (5.17) or (5.12). Or, let $\lambda \to 0$ in Proposition 4.1 ((iii) and (iv) there reduce to (iii) in the present case).

Integrating (i)–(iv) above gives

Proposition 5.3.2 *Let $M \geq 0$. Let f be a measurable function so that*

$$\int_{\xi=-\infty}^{\infty} |f(\xi)|\xi^{-M}\, d\xi < \infty. \tag{5.26}$$

Use the convention that $D_m = D$ for $m \leq 0$. Let $x = x_1\hat{e}_1 + x_2\hat{e}_2 \in \Pi_+$ ($x_1 = r\cos\phi$, $x_2 = r\sin\phi$). Then

(i) $$N_M[f](r,\phi) = \int_{t=\phi}^{\phi_0} D_{M-1}[\iota f](r,t)\, dt + N_M[f](r,\phi_0), \quad 0 \leq \phi_0 \leq \pi$$

(ii) $$N_M[f](r,\phi) = \cot\phi \int_{t=r_0}^{r} D_{M-1}[\iota f](t,\phi)\,\frac{dt}{t} - \csc\phi \int_{t=r_0}^{r} D_{M-2}[f](t,\phi)\, dt$$
$$+N_M[f](r_0,\phi), \quad r_0 \geq 0$$

(iii) $$N_M[f](x_1,x_2) = \tfrac{1}{x_2} \int_{t=t_1}^{x_1} D_{M-1}[\iota f](t,x_2)\, dt - \frac{1}{x_2}\int_{t=t_1}^{x_1} D_{M-2}[f](t,x_2)\, t\, dt$$
$$+N_M[f](t_1,x_2), \quad t_1 \in \mathbb{R}$$

(iv) $$N_M[f](x_1,x_2) = -\int_{t=t_2}^{x_2} D_{M-2}[f](x_1,t)\, dt + N_M[f](x_1,t_2), \quad t_2 \geq 0.$$

The function $\iota\colon \mathbb{R} \to \mathbb{R}$ is the identity and ιf is interpreted as pointwise multiplication of ι and f, i.e., $(\iota f)(t) = tf(t)$.

Proof. Integrate each of (i)–(iv) in Proposition 5.3.1 with respect to the relevant variable. The condition (5.26) ensures all the Dirichlet integrals converge to continuous functions on $\overline{\Pi}_+$. Fubini's Theorem allows interchange of orders of integration. ■

5.4 Growth estimates for the logarithmic kernel

It is difficult to estimate $N_M[f]$ directly. But, (i) of Proposition 5.3.2 writes $N_M[f]$ in terms of $D_{M-1}[\iota f]$ and this Dirichlet integral can be estimated using Theorem 3.3.1.

First consider $N[f]$ under

$$\int_{\xi=-\infty}^{\infty} |f(\xi)| \log(\xi^2+2)\, d\xi < \infty. \tag{5.27}$$

Part (i) of Proposition 5.3.2 gives

$$N[f](r,\phi) = \int_{t=\phi}^{\pi/2} D[\iota f](r,t)\, dt + N[f](r,\pi/2). \tag{5.28}$$

Each part above can be estimated separately.

Lemma 5.4.1 *If f satisfies* (5.27) *then*

$$N[f](r,\pi/2) = \begin{cases} -\frac{\log r}{\pi} \int_{\xi=-\infty}^{\infty} f(\xi)\, d\xi + o(1) \ \textit{as}\ r \to \infty, & \textit{if } f \textit{ is not odd.} \\ 0 \quad \textit{for all} \quad r, & \textit{if } f \textit{ is odd} \end{cases}$$

If f is not odd, then the term $o(1)$ in the case of odd f is sharp in that given any bounded positive function ψ on $[0,\infty)$, with $\psi(r) \to 0$ as $r \to \infty$, there exists a function f satisfying (5.27) *such that*

$$\left| N[f](r,\pi/2) + \frac{\log r}{\pi} \int_{\xi=-\infty}^{\infty} f(\xi)\, d\xi \right| \geq \psi(r) \qquad \textit{for all} \qquad r \geq 1. \tag{5.29}$$

Proof. First note that if (5.27) holds then

$$\int_{\xi=-\infty}^{\infty} |f(\xi)|\, d\xi < \infty. \tag{5.30}$$

We have

$$N[f](r,\pi/2) = -\frac{1}{2\pi}\int_{\xi=-\infty}^{\infty} f(\xi)\log\left(\xi^2+r^2\right)\,d\xi \tag{5.31}$$

$$= -\frac{\log r}{\pi}\int_{\xi=-\infty}^{\infty} f(\xi)\,d\xi - \frac{1}{2\pi}\int_{\xi=-\infty}^{\infty} f(\xi)\log\left(\frac{\xi^2}{r^2}+1\right)\,d\xi \tag{5.32}$$

$$= -\frac{\log r}{\pi}\int_{\xi=-\infty}^{\infty} f(\xi)\,d\xi + o(1) \quad (r\to\infty). \tag{5.33}$$

The last line above is by the Dominated Convergence Theorem.

And, $N[f](r,\pi/2)=0$ for all r if f is odd.

To prove the sharpness of

$$N[f](r,\pi/2) + \frac{\log r}{\pi}\int_{\xi=-\infty}^{\infty} f(\xi)\,d\xi = o(1) \quad (r\to\infty) \tag{5.34}$$

when f is not odd, use the method in Proposition 2.2.3. Given a bounded function ψ with $\psi(r)\to 0$ as $r\to\infty$, Lemma 2.2.1 allows us to assume ψ is decreasing, $\psi\in C^1([0,\infty))$ and $\psi'(0)=0$. Let

$$f(r) = \begin{cases} -\frac{2\pi}{\log 2}\psi'(r), & r\ge 0 \\ 0, & r\le 0. \end{cases} \tag{5.35}$$

Then

$$\left|N[f](r,\pi/2) + \frac{\log r}{\pi}\int_{\xi=-\infty}^{\infty} f(\xi)\,d\xi\right| = \frac{1}{2\pi}\int_{\xi=-\infty}^{\infty} f(\xi)\log\left(\frac{\xi^2}{r^2}+1\right)\,d\xi \tag{5.36}$$

$$= -\frac{1}{\log 2}\int_{\xi=0}^{\infty} \psi'(\xi)\log\left(\frac{\xi^2}{r^2}+1\right)\,d\xi \tag{5.37}$$

$$\ge -\frac{1}{\log 2}\int_{\xi=r}^{\infty} \psi'(\xi)\log\left(\frac{\xi^2}{r^2}+1\right)\,d\xi \tag{5.38}$$

$$\ge -\int_{\xi=r}^{\infty} \psi'(\xi)\,d\xi \tag{5.39}$$

$$= \psi(r). \quad \blacksquare \tag{5.40}$$

Note that convergence in (5.27) implies the existence of $D[\iota f]$ in (5.28). The growth estimate in Theorem 2.2.1 can be integrated over ϕ. Put $a = b = 1$ and $n = 2$ in Theorem 2.2.1. Since

$$\int_{t=\phi}^{\pi/2} \frac{dt}{\sin t} = -\left[\log\left(\csc t - \cot t\right)\right]_{t=\phi}^{\pi/2} \tag{5.41}$$

$$= \log(\csc\phi - \cot\phi) \tag{5.42}$$

it follows that

$$\int_{t=\phi}^{\pi/2} D[\iota f](r,t)\,dt = o\left(|\log(\csc\phi - \cot\phi)|\right) \qquad \text{as} \qquad r \to \infty. \tag{5.43}$$

The function $|\log(\csc\phi - \cot\phi)|$ is symmetric about $\phi = \pi/2$ since

$$\csc\phi - \cot\phi \sim \begin{cases} \frac{\phi}{2}, & \phi \to 0^+ \\ 1, & \phi \to \frac{\pi}{2} \\ \frac{2}{\pi-\phi}, & \phi \to \pi^-. \end{cases} \tag{5.44}$$

The estimate in (5.43) can be improved because it fails to take into account the logarithm factor in (5.27).

To accurately determine the growth of $D[\iota f]$ under (5.27), write

$$D[\iota f](r,\phi) = \frac{r\sin\phi}{\pi}(I_1 + I_2 + I_3) \tag{5.45}$$

where

$$I_1 = \int_{\xi=-\epsilon}^{\epsilon} p_1(\xi) f(\xi)\,d\xi \tag{5.46}$$

$$I_2 = \int_{\epsilon<|\xi|<N} p_1(\xi) f(\xi)\,d\xi \tag{5.47}$$

$$I_3 = \int_{|\xi|>N} p_1(\xi) f(\xi)\,d\xi \tag{5.48}$$

and

$$p_1(\xi) = \frac{\xi}{\xi^2 - 2\xi r \cos\phi + r^2}. \tag{5.49}$$

The number $N = N(r) > e$ will be determined in Lemma 5.4.4.

Lemma 5.4.2 *If* (5.27) *holds then* $I_1 = O(r^{-2})$ *as* $x \to \infty$ *in* Π_+.

Proof. Let $r \geq e + 1$. Then from (5.46),

$$|I_1| \leq \int_{\xi=-e}^{e} \frac{|\xi f(\xi)|\, d\xi}{(r - |\xi|)^2} \tag{5.50}$$

$$\leq \frac{1}{(r-e)^2} \int_{\xi=-e}^{e} |\xi f(\xi)|\, d\xi \tag{5.51}$$

$$\leq \frac{(e+1)^2}{r^2} \int_{\xi=-e}^{e} |\xi f(\xi)|\, d\xi. \quad \blacksquare \tag{5.52}$$

Remark 5.4.1 A complete asymptotic expansion of I_1 as $r \to \infty$ may be obtained by expanding in Legendre polynomials ($\lambda = 1/2$ in (3.13)).

Lemma 5.4.3 *If* (5.27) *holds then* $I_3 = o(1/[(1 - |\cos\phi|) r \log r])$ *as* $x \to \infty$ *in* Π_+.

Proof. Let

$$p_2(\xi) = \frac{\xi}{\log \xi\, (\xi^2 - 2\xi r \cos\phi + r^2)}, \tag{5.53}$$

$$F(\xi) = \int_{t=\xi}^{\infty} f(t) \log t\, dt \tag{5.54}$$

and

$$F_1(\xi) = \int_{t=\xi}^{\infty} |f(t)| \log t\, dt. \tag{5.55}$$

Then $p_2(\xi)$, $F(\xi)$ and $F_1(\xi) \to 0$ as $\xi \to \infty$. Let $I_4 = \int_{\xi=N}^{\infty} f(\xi) \log \xi \, p_2(\xi) \, d\xi$ and integrate by parts:

$$I_4 = \int_{\xi=N}^{\infty} f(\xi) \log \xi \, p_2(\xi) \, d\xi \tag{5.56}$$

$$= F(N) p_2(N) + \int_{\xi=N}^{\infty} F(\xi) \, p_2'(\xi) \, d\xi. \tag{5.57}$$

And,

$$|I_4(\xi)| = F_1(N) p_2(N) + F_1(N) \int_{\xi=N}^{\infty} |p_2'(\xi)| \, d\xi. \tag{5.58}$$

The last integral above can be evaluated if the roots of p_2' are known. We have

$$p_2'(\xi) = \frac{(\log \xi - 1) r^2 + 2\xi r \cos\phi - (\log \xi + 1)\xi^2}{\log^2 \xi \, (\xi^2 - 2\xi r \cos\phi + r^2)^2} \tag{5.59}$$

$$= \frac{(r^2 - \xi^2) \log \xi - (\xi^2 - 2\xi r \cos\phi + r^2)}{\log^2 \xi \, (\xi^2 - 2\xi r \cos\phi + r^2)^2}. \tag{5.60}$$

The transcendental equation $p_2'(\xi) = 0$ cannot be solved directly. However, the quadratic in (5.59) can be factored for $r = r(\xi, \phi)$, viz.

$$r(\xi, \phi) = \xi \left(\frac{-\cos\phi + \sqrt{\log^2 \xi - \sin^2 \phi}}{\log \xi - 1} \right). \tag{5.61}$$

the function $\xi \mapsto \xi/(\log \xi - 1)$ has derivative $(\log \xi - 2)(\log \xi - 1)^{-1}$ and is an increasing function of ξ for $\xi \geq e^2$. Consequently, $r(\xi, \phi)$ is an increasing function of ξ ($\xi \geq e^2$) and (5.61) has a unique solution for ξ in terms of r (for each fixed ϕ). This equation can be solved approximately by noting the asymptotic behaviour as $\xi \to \infty$:

$$\begin{aligned} r(\xi, \phi) &= \frac{\xi}{\log \xi} \left\{ -\cos\phi + \log \xi \left[1 - \frac{\sin^2 \phi}{2 \log^2 \xi} + O\left(\log^{-4} \xi\right) \right] \right. \\ &\qquad \left. \left[1 + \log^{-1} \xi + O\left(\log^{-2} \xi\right) \right] \right\} \end{aligned} \tag{5.62}$$

$$= \xi \left[1 + \frac{1 - \cos\phi}{\log \xi} + O\left(\log^{-2} \xi\right) \right] \quad (\xi \to \infty), \tag{5.63}$$

i.e., $\lim_{\xi \to \infty} r(\xi, \phi)/\xi = 1^+$. Hence, we can write

$$\frac{r}{\xi} = 1 - E(r, \phi) \tag{5.64}$$

where $E(r,\phi) = o(1)$ $(x \in \Pi_+, r \to \infty)$ and $0 \le E(r,\phi) < 1$ for large enough r. Put $\xi/r = 1 - E(r,\phi)$ in (5.60). Then

$$
\begin{aligned}
0 &= (2E - E^2)\,[\log(1-E) + \log r] - 2(1-\cos\phi) + 2(1-\cos\phi)E - E^2 \\
&= 2E\log r - 2(1-\cos\phi) - E^2\log r + R(E) \qquad (5.65)
\end{aligned}
$$

where

$$
\begin{aligned}
R(E) &= (2E - E^2)\log(1-E) + 2(1-\cos\phi)E + E^2 \qquad (5.66)\\
&\sim 2(1-\cos\phi)E - 3E^2 \quad (x \in \Pi_+, r \to \infty). \qquad (5.67)
\end{aligned}
$$

Treating (5.65) as a quadratic in E,

$$
\begin{aligned}
E &= 1 - \sqrt{1 - 2(1-\cos\phi)/\log r + R(E)/\log r} \qquad (5.68)\\
&= \frac{2(1-\cos\phi)/\log r - R(E)/\log r}{1 + \sqrt{1 - 2(1-\cos\phi)/\log r + R(E)/\log r}}. \qquad (5.69)
\end{aligned}
$$

Since $R(E(r,\phi)) = o(1)$ as $x \to \infty$, we have $E(r,\phi) \sim (1-\cos\phi)/\log r$ as $x \to \infty$.

From (5.60),

$$
p_2'(\xi) \begin{cases} > 0 & \text{for} \quad N \le \xi < r - rE(r,\phi) \\ = 0 & \text{for} \quad \xi = r - rE(r,\phi) \\ < 0 & \text{for} \quad \xi > r - rE(r,\phi) \end{cases} \qquad (5.70)
$$

and

$$
\begin{aligned}
\int_{\xi=N}^{\infty} |p_2'(\xi)|\,d\xi &= [p_2(\xi)]_{\xi=N}^{r(1-E)} - [p_2(\xi)]_{\xi=r(1-E)}^{\infty} \qquad (5.71)\\
&= 2p_2(r(1-E)) - p_2(N). \qquad (5.72)
\end{aligned}
$$

Therefore, from (5.58),

$$
\begin{aligned}
|I_4| &\le 2p_2(r(1-E))\,F_1(N) \qquad (5.73)\\
&= \frac{2(r - rE)\,F_1(N)}{(\log(1-E) + \log r)\,[2(1-\cos\phi) - 2(1-\cos\phi)E + E^2]\,r^2}. \qquad (5.74)
\end{aligned}
$$

As $x \to \infty$ in Π_+,

$$|I_4| \le \frac{(1-E)\,F_1(N)}{(1-\cos\phi)\,r\,\log r\left[1+O\left(\frac{E}{\log r}\right)\right]\left[1+O(E)+O\left(\frac{E^2}{1-\cos\phi}\right)\right]} \tag{5.75}$$

$$= \frac{F_1(N)}{(1-\cos\phi)\,r\,\log r}\left[1+O\left(\frac{1-\cos\phi}{\log r}\right)+O\left(\frac{1}{\log^2 r}\right)\right]. \tag{5.76}$$

And, $F_1(N) \to 0$ as $N \to \infty$, so given $\epsilon_1 > 0$ there is $N_1 > e$ such that $N > N_1$ implies $F_1(N) < \epsilon_1$. Therefore, $I_4 = o\left(\frac{1}{(1-\cos\phi) r \log r}\right)$ as $x \to \infty$ in Π_+.

For the integral $\int_{\xi=-\infty}^{-N} f(\xi)\,\log\xi\,p_2(\xi)\,d\xi$ we have the same results with $\phi \mapsto \pi-\phi$. The lemma now follows. ■

Lemma 5.4.4 *If* (5.27) *holds then* $I_2 = o(1/(r\log r))$ *as* $x \to \infty$ *in* Π_+.

Proof. Let $p_3(\xi) = |\xi|/\log|\xi|$ then

$$p_3'(\xi) = \frac{\operatorname{sgn}(\xi)\,(\log|\xi|-1)}{\log^2\xi} \tag{5.77}$$

$$\begin{cases} \ge 0 & \text{for} \quad \xi \ge e \\ \le 0 & \text{for} \quad 0 < \xi \le e. \end{cases} \tag{5.78}$$

Therefore, $p_3(\xi) \le p_3(N)$ for $e \le |\xi| \le N$. Let $r > N$ then

$$|I_2| \le \int_{e<|\xi|<N} \frac{p_3(\xi)\,|f(\xi)|\,\log|\xi|\,d\xi}{(r-|\xi|)^2} \tag{5.79}$$

$$\le \frac{N}{(r-N)^2\log N}\int_{e<|\xi|<N} |f(\xi)|\,\log|\xi|\,d\xi. \tag{5.80}$$

In Lemma 5.4.3 we had $N > N_1 > e$. Now, given $\epsilon_2 > 0$ we will show that $N/[(r-N)^2\log N] < \epsilon_2/(r\log r)$ for appropriately large r and N. We can take $\epsilon_2 \le \min(1, 25/(4N_1))$. Let $r \ge 25\epsilon_2^{-2}$. Then

$$\frac{\epsilon_2 r}{4} \ge \frac{25}{4\epsilon_2} \ge \max\left(N_1, \frac{5}{\epsilon_2}\right) \tag{5.81}$$

and we can take

$$\max\left(N_1, \frac{\epsilon_2 r}{5}\right) \le N \le \frac{\epsilon_2 r}{4}. \tag{5.82}$$

Then

$$\frac{N}{(r-N)^2 \log N} \le \frac{\epsilon_2 r}{4\left(1-\frac{\epsilon_2}{4}\right)^2 r^2 \log\left(\frac{\epsilon_2 r}{5}\right)} \tag{5.83}$$

$$\le \frac{\epsilon_2}{4\left(1-\frac{1}{4}\right)^2 r \log\left(\sqrt{r}\right)} \tag{5.84}$$

$$= \frac{8\epsilon_2}{9r\log r} \tag{5.85}$$

$$< \frac{\epsilon_2}{r\log r}. \tag{5.86}$$

Now replace ϵ_2 by

$$\frac{\epsilon_2}{\max\left(1, \int\limits_{\epsilon<|\xi|<N} |f(\xi)| \log|\xi|\, d\xi\right)} \tag{5.87}$$

then $|I_2| \le \epsilon_2/(r\log r)$ for $r \ge 25\epsilon_2^{-2}$ and N given by (5.82). ∎

Lemmas (5.4.1)–(5.4.4) can now be combined for an estimate of $N[f]$.

Theorem 5.4.1 *Let* (5.27) *hold for a measurable function* f. *If* f *is odd then*

$$N[f](r,\phi) = o\left(\frac{\log(1-|\cos\phi|)}{\log r}\right) \qquad \text{as } x \to \infty \text{ in } \Pi_+. \tag{5.88}$$

If f *is not odd, write* $N[f](r,\phi) = u(r,\phi) + v(r)$ *where* $u(r,\phi) = N[f](r,\phi) - N[f](r,\pi/2)$ *and* $v(r) = N[f](r,\pi/2)$. *Then*

$$u(r,\phi) = o\left(\frac{\log(1-|\cos\phi|)}{\log r}\right) \text{ and } v(r) = -\frac{\log r}{\pi}\int_{-\infty}^{\infty} f(\xi)\, d\xi + o(1) \text{ as } x \to \infty \text{ in } \Pi_+. \tag{5.89}$$

The estimates $o\left(\frac{\log(1-|\cos\phi|)}{\log r}\right)$ *are sharp in the sense of Definition 3.13. And, the estimate* $V(r) := v(r) + \frac{\log r}{\pi}\int_{-\infty}^{\infty} f = o(1)$ *is sharp in that, for any positive function* $\psi(r) = o(r)$ *as* $r \to \infty$, *there is a function* f *satisfying* (5.27) *so that* $|V(r)|/\psi(r) \not\to 0$ *as* $r \to \infty$.

Note that $o\left(\frac{\log(1-|\cos\phi|)}{\log r}\right)$ and $o(1)$ are not comparable. Let $\Psi(r,\phi) = \log\csc\phi/\log r$ and suppose $0 < \phi \le \phi_0 < \pi/2$. Then $u = o(\Psi)$ if and only if $u(r,\phi) = o(\log(1 - |\cos\phi|)/\log r)$, since $\log(1-|\cos\phi|)$ and $\log\csc\phi$ have essentially the same behaviour except at $\phi = \pi/2$ (Their ratio is bounded if ϕ is kept away from $\pi/2$.). On the curve $r(\phi) = \csc\phi$ we have $\Psi(r,\phi) = 1$. On the curve $r(\phi) = \log\csc\phi$ we have $\Psi(r,\phi) = \log\csc\phi/\log\log\csc\phi \to \infty$ as $r \to \infty$. And, on the curve $r(\phi) = constant$ we have $\Psi(r,\phi) = O(1/\log r)$. Hence, neither $o\left(\frac{\log(1-|\cos\phi|)}{\log r}\right)$ nor $o(1)$ is dominant.

Proof: From Lemmas 5.4.2–5.4.4 and equations (5.46)–(5.48), $I_1 + I_2 + I_3 = o(1/[(1 - |\cos\phi|)r\log r])$ as $x \to \infty$. Now look at (5.28), (5.45) and Lemma 5.4.1. If f is odd then

$$N[f](r,\phi) = o\left(\frac{1}{\log r}\int_{t=\phi}^{\pi/2}\frac{\sin t\,dt}{1-|\cos t|}\right). \tag{5.90}$$

And,

$$\left|\int_{t=\phi}^{\pi/2}\frac{\sin t\,dt}{1-|\cos\phi|}\right| = -\log\left(1-|\cos t|\right). \tag{5.91}$$

This gives (5.88).

When f is not odd, the contribution from $N[f](r,\pi/2)$ (Lemma 5.4.1) must be added for (5.89).

To prove (5.88) is sharp a lower bound on $N[f]$ is needed. Let f be an odd function such that $f \ge 0$ on $[0,\infty)$, $f \le 0$ on $(-\infty, 0]$ and $\int_{\xi=-\infty}^{\infty} |f(\xi)|\log(\xi^2+2)\,d\xi < \infty$.

First suppose that $0 < \phi \le \pi/6$ so that $\cos\phi \ge 0$. Then $\xi^2 r^{-2} - 2\xi r^{-1}\cos\phi + 1 \ge 1$

whenever $\xi \geq 2r\cos\phi$ or $\xi \leq 0$. Now use the fact that f is odd to write

$$\begin{aligned}
N[f](r,\phi) &= -\frac{\log r}{\pi}\int_{\xi=-\infty}^{\infty} f(\xi)\,d\xi - \frac{1}{2\pi}\int_{|\xi|>2r\cos\phi} f(\xi)\log\left(\frac{\xi^2}{r^2}-2\frac{\xi}{r}\cos\phi+1\right)d\xi \\
&\quad -\frac{1}{2\pi}\int_{|\xi|<2r\cos\phi} f(\xi)\log\left(\frac{\xi^2}{r^2}-2\frac{\xi}{r}\cos\phi+1\right)d\xi \\
&= \frac{1}{2\pi}\int_{\xi=2r\cos\phi}^{\infty} f(\xi)\log\left(\frac{\frac{\xi^2}{r^2}+2\frac{\xi}{r}\cos\phi+1}{\frac{\xi^2}{r^2}-2\frac{\xi}{r}\cos\phi+1}\right)d\xi \\
&\quad -\frac{1}{2\pi}\left\{\int_{\xi=-2r\cos\phi}^{0}+\int_{\xi=0}^{2r\cos\phi}\right\} f(\xi)\log\left(\frac{\xi^2}{r^2}-2\frac{\xi}{r}\cos\phi+1\right)d\xi \\
&\geq -\frac{1}{2\pi}\int_{\xi=0}^{2r\cos\phi} f(\xi)\log\left(\frac{\xi^2}{r^2}-2\frac{\xi}{r}\cos\phi+1\right)d\xi.
\end{aligned}$$

Since $0<\phi\leq\pi/6$ if follows that $0\leq\cos\phi-\sin\phi\leq\cos\phi+\sin\phi\leq 2\cos\phi$. Let $F(r,\phi)$ be the minimum of $f(\xi)$ over $r(\cos\phi-\sin\phi)\leq\xi\leq r(\cos\phi+\sin\phi)$. Then

$$N[f](r,\phi)\geq F(r,\phi)\left(-\frac{1}{2\pi}\right)\int_{\xi=r(\cos\phi-\sin\phi)}^{r(\cos\phi+\sin\phi)}\log\left[\left(\frac{\xi}{r}-\cos\phi\right)^2+\sin^2\phi\right]d\xi. \tag{5.92}$$

With the change of variable $t=\frac{\xi}{r}-\cos\phi$ this becomes

$$\begin{aligned}
N[f](r,\phi) &\geq F(r,\phi)\left(-\frac{r}{\pi}\right)\int_{t=0}^{\sin\phi}\log\left(t^2+\sin^2\phi\right)dt \\
&= F(r,\phi)\left(-\frac{r}{\pi}\right)\left[t\log\left(t^2+\sin^2\phi\right)-2t+2\sin\phi\arctan\left(\frac{t}{\sin\phi}\right)\right]_{t=0}^{\sin\phi} \\
&= F(r,\phi)\left(-\frac{r\sin\phi}{\pi}\right)\left[2\log(\sin\phi)+\log 2-2+\pi/2\right].
\end{aligned}$$

Finally, since $\log(\sin\phi)\leq\log(1/2)<2-\pi/2-\log 2$ when $0<\phi\leq\pi/6$, we have the lower bound

$$N[f](r,\phi)\geq\frac{F(r,\phi)\,r\sin\phi\,\log(1-|\cos\phi|)}{\pi}. \tag{5.93}$$

Now, given any sequence $\{x^{(i)}\}$ in Π_+ and any function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(x) = o(\log(1 - |\cos\phi|)/\log r)$ ($x \to \infty$ in Π_+), to prove sharpness of (5.88) we have to show that $N[f]/\psi \not\to 0$ on some subsequence of $\{x^{(i)}\}$. With no loss of generality, it may be assumed that $|x^{(i)}| \to \infty$ monotonically (take a subsequence of $\{x^{(i)}\}$ if necessary).

First suppose there is a subsequence $\tilde{x}^{(i)} = r_i \cos\phi_i\, \hat{e}_1 + r_i \sin\phi_i\, \hat{e}_2$ with $0 < \phi_i \leq \pi/6$. Using (5.93), it suffices to show that

$$F(r_i \cos\phi_i, r_i \sin\phi_i) \geq \frac{\pi}{i^2\, r_i \sin\phi_i \log r_i}. \tag{5.94}$$

Let

$$f(\xi) = \begin{cases} \frac{f_i}{r_i \sin\phi_i}\left(|\xi - r_i\cos\phi_i| + 1.5\, r_i \sin\phi_i\right), & r_i \sin\phi_i \leq |\xi - r_i \cos\phi_i| \leq 1.5\, r_i \sin\phi_i \\ f_i, & |\xi - r_i\cos\phi_i| \leq r_i \sin\phi_i \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $0 < \phi_i \leq \pi/6$, we have $r_i(\cos\phi_i + 1.5 \sin\phi_i) \leq r_i(\sqrt{3} + 1.5)/2$ and $r_{i+1}(\cos\phi_{i+1} - 1.5 \sin\phi_{i+1}) \geq r_{i+1}(\sqrt{3} - 1.5)/2$. The intervals $(r_i \cos\phi_i, r_i \sin\phi_i)$ are disjoint if $r_{i+1} \geq (\sqrt{3} + 1.5)(\sqrt{3} - 1.5)^{-1} r_i$. This can always be arranged by taking a sparse enough subsequence.

From the definition of f, we have $F(r_i \cos\phi_i, r_i \sin\phi_i) = f_i$. So if we define $f_i = \pi(r_i \sin\phi_i \log r_i\, i^2)^{-1}$ then (5.94) is satisfied. It remains to show (5.27) holds. That condition is equivalent to convergence of the series

$$\sum_{i=1}^{\infty} f_i\, r_i \sin\phi_i \log(r_i \cos\phi_i). \tag{5.95}$$

And with f_i as above, this series becomes

$$\pi \sum_{i=1}^{\infty} \frac{\log(r_i \cos\phi_i)}{i^2 \log r_i} \leq \pi \sum_{i=1}^{\infty} i^{-2} < \infty. \tag{5.96}$$

This establishes sharpness in (5.88) when f is odd and there is a subsequence of $\{x^{(i)}\}$ in the sector $0 < \phi \leq \pi/6$.

Now suppose that f is odd and that each element of $\{x^{(i)}\}$ is in the sector $\pi/6 \leq \phi \leq \pi/2$. We have

$$\begin{aligned}
\xi^2 - 2\xi r_i \cos\phi_i + r_i^2 &= (\xi - r_i\cos\phi)^2 + r_i^2 \sin^2\phi && (5.97)\\
&\geq \frac{r_i^2}{4} && (5.98)\\
&\geq 1 \qquad \text{if } r_i \geq 2. && (5.99)
\end{aligned}$$

We can assume that $r_1 \geq 2$ and $\{r_i\}$ is an increasing sequence. Then $\log(\xi^2 - 2\xi r_i \cos\phi_i + r_i^2) \geq 0$ for each $\xi \in \mathbb{R}$. Let

$$f(\xi) = \begin{cases} -f_i(1 - |\xi - r_i|), & r_i - 1 \leq \xi \leq r_i + 1 \text{ for some } i \geq 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.100}$$

where $f_i > 0$ is a sequence given below. We now have

$$\begin{aligned}
N[f](r_i, \phi_i) &= -\sum_{i=1}^{\infty} \frac{1}{2\pi} \int_{\xi=r_i-1}^{r_i+1} f(\xi) \log(\xi^2 - 2\xi r_i \cos\phi_i + r_i^2)\, d\xi && (5.101)\\
&\leq \frac{f_i}{\pi} \int_{\xi=r_i-1}^{r_i+1} (1 - |\xi - r_i|) \log r_i \, d\xi && (5.102)\\
&= \frac{f_i \log r_i}{\pi}. && (5.103)
\end{aligned}$$

We can assume we have a subsequence $\tilde{x}^{(i)} = r_i \cos\phi_i\, \hat{e}_1 + r_i \sin\phi_i\, \hat{e}_2$ such that

$$|\psi(\tilde{x}^{(i)})| \leq \frac{|\log(1 - \cos\phi_i)|}{i^2 \log r_i}. \tag{5.104}$$

Let $f_i = |\psi(\tilde{x}^{(i)})| / \log r_i$. Condition (5.4) is satisfied if and only if

$$\sum_{i=1}^{\infty} f_i \log r_i < \infty. \tag{5.105}$$

With f_i as above we have

$$\begin{aligned}
f_i \log r_i &\leq \frac{\pi |\log(1 - \cos\phi)|}{i^2} && (5.106)\\
&\leq -\frac{\pi \log(1 - \sqrt{3}/2)}{i^2} && (5.107)
\end{aligned}$$

and (5.105) holds. Also, $N[f](r_i; \phi_i)/|\psi(\bar{x}^{(i)})| \geq 1$. Therefore, (5.88) is sharp when f is odd. By symmetry this holds for $0 < \phi < \pi$.

When f is not odd, take f as above and add a positive function satisfying (5.35) in Lemma 5.4.1. The resulting function gives the desired sharpness. ■

Since Π_+ is unbounded, $\int_{\xi=-\infty}^{\infty} f(\xi)\, d\xi$ need not vanish for $N[f]$ to be a solution of (5.1)– (5.3). However, as shown below, $\int_{\xi=-\infty}^{\infty} f(\xi) \log(\xi^2 + r^2)\, d\xi$ is zero (for all r) precisely when f is odd. This means that v in (5.89) is not identically zero.

Lemma 5.4.5 *If* (5.27) *holds for continuous f then $N[f](r, \pi/2) = 0$ for all $r \geq 0$ if and only if f is an odd function.*

Proof: If f is odd then $N[f](r, \pi/2) = 0$ since it has an even kernel.

If $N[f](r, \pi/2) = 0$ for all $r \geq 0$ let $g(\xi) = f(\xi) + f(-\xi)$ if $\xi \geq 0$ and $g(\xi) = 0$ if $\xi < 0$. We have

$$f(0) = -\frac{\partial N[f]}{\partial r}(r, \pi/2) = 0,$$

so g is continuous. And,

$$N[f](r, \pi/2) = 0 = \frac{1}{2\pi} \int_{\xi=0}^{\infty} g(\xi) \log\left(\xi^2 + r^2\right)\, d\xi. \tag{5.108}$$

Let $u(r, \phi) = N[g](r, \phi)$ and consider u as a function in the left quarter plane $r > 0$, $\pi/2 < \phi < \pi$. Since $u(r, \pi/2) = 0$, u can be extended across the y-axis to $0 < \phi < \pi/2$ as an odd function (Schwarz Reflection Principle, see §1.2). Define

$$v(r, \phi) = \begin{cases} -u(r, \pi - \phi), & 0 < \phi \leq \pi/2 \\ u(r, \phi), & \pi/2 \leq \phi < \pi. \end{cases} \tag{5.109}$$

Then

$$v \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+) \tag{5.110}$$

$$\Delta v = 0, \quad x \in \Pi_+ \tag{5.111}$$

$$\frac{\partial v}{\partial n} = 0, \quad x \in \partial\Pi_+ \tag{5.112}$$

$$v = o(r \csc \phi) \quad (x \in \Pi_+,\ r \to \infty). \tag{5.113}$$

Theorem 6.4.2 (with $N = 0$) now says $v = N[0] = 0$. But u and v agree on the left quarter plane so $u = 0$ on Π_+. By the same theorem, $g = 0$ and f is odd. ∎

Remark 5.4.2 If f is assumed only to be measurable so that (5.27) holds and $\int_{\xi=0}^{1}(f(\xi) + f(-\xi))\xi^{\alpha}\,d\xi$ converges for some $-1 < \alpha < 0$ (f being bounded at the origin suffices) and if $N[f](r, \pi/2)$ is only known to be zero for r in an open subset of $\mathbb{R}_+$, we can still obtain the conclusion that f is odd. Define g as above. The function $N[f]$ is real analytic on Π_+ ([7], page 21). It follows that $N[f](r, \pi/2)$ is a real analytic function of r for $r > 0$. Since it vanishes on an open set it vanishes for all $r \in \mathbb{R}$. Differentiate under the integral sign and write

$$\frac{d}{dr}N[g](r, \pi/2) = \frac{r}{\pi}\int_{\xi=0}^{\infty}\frac{g(\xi)\,d\xi}{\xi^2 + r^2} \tag{5.114}$$

$$= \frac{1}{\pi}\int_{\xi=0}^{\infty}\frac{g(r\xi)\,d\xi}{\xi^2 + 1} \tag{5.115}$$

$$= 0 \quad \text{for } r > 0. \tag{5.116}$$

The Mellin transform of g,

$$\mathcal{M}[g](s) = \int_{\xi=0}^{\infty} g(\xi)\xi^{s-1}\,d\xi := G(s), \tag{5.117}$$

is analytic in the strip $0 < \mathcal{R}e(s) \leq 1 + \alpha$. Let $h(\xi) = (\xi^2 + 1)^{-1}$. The Mellin transform is

$$H(s) = \int_{\xi=0}^{\infty}\frac{\xi^{s-1}\,d\xi}{\xi^2 + 1} \tag{5.118}$$

$$= \frac{1}{2}\int_{\xi=0}^{\infty}\frac{\xi^{s/2-1}\,d\xi}{\xi + 1} \tag{5.119}$$

$$= \frac{\Gamma(s/2)\,\Gamma(1 - s/2)}{2} \qquad ([19], 1.5.2) \tag{5.120}$$

$$= \frac{\pi}{2\sin(\pi s/2)} \qquad ([19], 1.2.6) \tag{5.121}$$

$$\tag{5.122}$$

and is analytic in the strip $0 < \mathcal{R}e(s) < 2$. The convolution of g and h is

$$g * h(r) = \int_{\xi=0}^{\infty} g(r\xi)h(\xi)\, d\xi \tag{5.123}$$

and has Mellin transform

$$\mathcal{M}[g * h](s) = G(s)H(1 - s) \tag{5.124}$$

(see, for example, [15]). The convolution is analytic in the common strip $1 + \alpha < \mathcal{R}e(s) < 1$. But from (5.115), $g*h(r) = 0$ for $r > 0$. Therefore, $G(s)H(1-s)$ vanishes identically for $1 + \alpha < \mathcal{R}e(s) < 1$. And, $H(1 - s) = H(s) = \pi \sec(\pi s/2)/2 \neq 0$ in this strip. Hence, G vanishes identically there and so g vanishes almost everywhere for $r > 0$. Therefore, f is odd (almost everywhere).

5.5 The modified kernel

The modified Neumann integral, $N_M[f]$, can be estimated using the integral representation of the kernel given in part (i) of Proposition 5.3.1.

Theorem 5.5.1 *Let $M \geq 1$ be an integer. If* (5.8) *holds for a measurable function f then*

$$N_M[f](r, \phi) = o\left(r^M \log(1 - |\cos\phi|)\right) + o\left(r^M\right) \qquad (x \in \Pi_+,\ |x| \to \infty). \tag{5.125}$$

The estimate is sharp in the sense of Theorem 5.4.1.

Proof. Use (i) of Proposition 5.3.2 to write

$$N_M[f](r, \phi) = \int_{t=\phi}^{\pi/2} D_{M-1}[\iota f](r, t)\, dt + N_M[f](r, \pi/2). \tag{5.126}$$

An upper bound on $|D_M[f]|$ was obtained in (3.35) of Theorem 3.23. And, $|D_{M-1}[\iota f]|$ will have the same upper bound. It follows that $D_{M-1}[\iota f](r,\phi) = o(r^M \log(1 - |\cos\phi|))$.

The final term in (5.126) is

$$N_M[f](r,\pi/2) = \frac{(-1)^M}{2\pi} \int\limits_{|\xi|>1} f(\xi) \int\limits_{\zeta=0}^{r/\xi} \frac{(1+\zeta)\,\zeta^{M-1}\,d\zeta}{1+\zeta^2}\,d\xi. \tag{5.127}$$

We have

$$\frac{1+\zeta}{1+\zeta^2} = 1 + \frac{\zeta(1-\zeta)}{1+\zeta^2}, \tag{5.128}$$

so

$$1 \leq \frac{1+\zeta}{1+\zeta^2} \leq 2 \qquad \text{for } 0 \leq \zeta \leq 1. \tag{5.129}$$

And,

$$\frac{\zeta+\zeta^2}{1+\zeta^2} = 1 + \frac{\zeta-1}{1+\zeta^2}, \tag{5.130}$$

so

$$\frac{1}{\zeta} \leq \frac{1+\zeta}{1+\zeta^2} \leq \frac{2}{\zeta} \qquad \text{for } \zeta \geq 1. \tag{5.131}$$

Let

$$I = \int\limits_{\zeta=0}^{r/\xi} \frac{(1+\zeta)\,\zeta^{M-1}\,d\zeta}{1+\zeta^2}. \tag{5.132}$$

If $r/|\xi| \leq 1$ then

$$|I| \leq 2 \int\limits_{\zeta=0}^{r/|\xi|} \zeta^{M-1}\,d\zeta = \frac{2}{M}\left(\frac{r}{|\xi|}\right)^M. \tag{5.133}$$

If $r/|\xi| \geq 1$ then

$$|I| \leq 2\int_{\zeta=0}^{1} \zeta^{M-1}\,d\zeta + 2\int_{\zeta=1}^{r/|\xi|} \zeta^{M-2}\,d\zeta \tag{5.134}$$

$$= \begin{cases} \frac{2}{M} + \frac{2}{M-1}\left[(r/|\xi|)^{M-1} - 1\right], & M \geq 2 \\ 2 + 2\log(r/|\xi|), & M = 1. \end{cases} \tag{5.135}$$

$$\leq \begin{cases} \frac{2}{M-1}(r/|\xi|)^{M-1}, & M \geq 2 \\ 2 + 2\log(r/|\xi|), & M = 1. \end{cases} \tag{5.136}$$

The preliminary results above will now lead to an estimate of $N_M[f](r, \pi/2)$. Let $r > 1$. From (5.133), we have

$$\left|\int_{\xi=r}^{\infty} f(\xi) I\,d\xi\right| \leq \frac{2r^M}{M}\int_{\xi=r}^{\infty} \frac{|f(\xi)|}{\xi^M}\,d\xi \tag{5.137}$$

$$= o(r^M) \qquad (r \to \infty). \tag{5.138}$$

Let $F(\xi) = \int_{t=\xi}^{\infty} |f(t)|t^{-M}\,dt$. Then $F(\xi) = o(1)$ as $\xi \to \infty$. And, given $\epsilon > 0$, there is a number $N \geq 1$ depending on ϵ so that $F(\nu) \leq \epsilon/2$ for all $\nu \geq N$. Let $r \geq N$ and $M \geq 2$. Then, using (5.136),

$$\left|\int_{\xi=N}^{r} f(\xi) I\,d\xi\right| \leq \frac{2r^{M-1}}{M-1}\int_{\xi=N}^{r} \frac{|f(\xi)|\,d\xi}{\xi^{M-1}} \tag{5.139}$$

$$\leq \frac{2}{M-1} r^{M-1} F(N) \tag{5.140}$$

$$\leq \epsilon r^M. \tag{5.141}$$

And,

$$\left|\int_{\xi=1}^{N} f(\xi) I\,d\xi\right| \leq \frac{2r^{M-1}}{M-1}\int_{\xi=1}^{N} \frac{|f(\xi)|\,d\xi}{\xi^{M-1}} \tag{5.142}$$

$$\leq 2Nr^{M-1}F(1) \tag{5.143}$$

$$\leq \epsilon r^M \qquad \text{if } r \geq 2NF(1)/\epsilon. \tag{5.144}$$

Hence, $N_M[f](r,\pi/2) = o(r^M)$ for $M \geq 2$. (Using a similar construction for $\xi < -1$.)

When $M = 1$,

$$\left| \int_{\xi=N}^{r} f(\xi) I \, d\xi \right| \leq I_1 + I_2, \tag{5.145}$$

where

$$\begin{aligned} I_1 &= 2 \int_{\xi=N}^{r} |f(\xi)| \, d\xi && (5.146) \\ &\leq 2r \int_{\xi=N}^{r} |f(\xi)| \frac{d\xi}{\xi} && (5.147) \\ &\leq 2rF(N) && (5.148) \\ &\leq \epsilon r. && (5.149) \end{aligned}$$

And,

$$\begin{aligned} I_2 &= 2 \int_{\xi=N}^{r} |f(\xi)| \log(r/\xi) \, d\xi && (5.150) \\ &\leq 2rF(N) \sup_{t \geq 1} \frac{\log t}{t} && (5.151) \\ &\leq \epsilon r/e. && (5.152) \end{aligned}$$

As well,

$$\left| \int_{\xi=1}^{N} f(\xi) I \, d\xi \right| \leq I_3 + I_4, \tag{5.153}$$

where

$$\begin{aligned} I_3 &= 2 \int_{\xi=1}^{N} |f(\xi)| \, d\xi && (5.154) \\ &\leq 2NF(1) && (5.155) \\ &\leq \epsilon r \qquad \text{if } r \geq 2Nf(1)/\epsilon. && (5.156) \end{aligned}$$

And,

$$I_4 = 2 \int_{\xi=1}^{N} |f(\xi)| \log(r/\xi)\, d\xi. \tag{5.157}$$

Since $\log r/r \to 0$ as $r \to \infty$, we can always find r (depending on ϵ) such that $\log r/r \leq \epsilon/(2NF(1))$. With such an r,

$$\begin{aligned} I_4 &\leq 2N \log r \int_{\xi=1}^{N} |f(\xi)| \frac{d\xi}{\xi} &(5.158)\\ &\leq 2NF(1)\log r &(5.159)\\ &\leq \epsilon r. &(5.160)\end{aligned}$$

(If $F(1) = 0$ then (5.160) holds for any $r \geq 1$.) Hence, $N_M[f](r, \pi/2) = o(r^M)$ for all $M \geq 1$.

To prove the estimate $N_M[f](r, \pi/2) = o(r^M)$ is sharp, suppose ψ is any bounded positive function that tends to zero at infinity. By Lemma 2.2.1 we can assume ψ is continuously differentiable and decreasing. Let

$$f(\xi) = \begin{cases} 2\pi M(-1)^{M+1}\psi'(\xi)\,\xi^M, & \xi \geq 1 \\ 0, & \xi \leq 1. \end{cases} \tag{5.161}$$

From (5.129) and (5.132) we have $I \geq M^{-1}(r/\xi)^M$ for $\xi \geq r$. Thus,

$$\begin{aligned} N_M[f](r, \pi/2) &= \frac{1}{2\pi} \int_{\xi=-\infty}^{\infty} f(\xi) I\, d\xi &(5.162)\\ &\geq -r^M \int_{\xi=r}^{\infty} \psi'(\xi)\, d\xi &(5.163)\\ &= r^M \psi(\xi), &(5.164)\end{aligned}$$

giving sharpness as in Theorem 5.4.1.

The proof that $\int_{t=\phi}^{\pi/2} D_M[f](r,t)\, dt = o(r^M \log(1 - |\cos\phi|))$ is sharp is similar to the previous sharpness proofs in Theorems 3.3.1 and 5.4.1. ∎

Chapter 6

Uniqueness and spherical harmonics

6.1 Introduction

The classical Phragmén–Lindelöf Principle(see the next chapter) ensures uniqueness to (2.1)– (2.3) under the growth condition $u = o(|x|)$ as $|x| \to \infty$ in Π_+. However, if f satisfies (2.6) then $u = D[f]$ is a solution even though $f(y)$ needn't be $o(|y|)$ for the Poisson integral to exist. And, as we have seen above, existence of the Poisson integral does not imply any *a priori* pointwise behaviour of u on $\partial\Pi_+$. We now establish a theorem that guarantees a unique solution to (2.1)–(2.3) with a growth condition compatible with any data f satisfying (2.6). It gives uniqueness to a harmonic polynomial of degree N when f satisfies

$$\int_{\mathbb{R}^{n-1}} \frac{|f(y')|\,dy'}{|y'|^{N+n}+1} < \infty \quad \text{for some } N \geq 1. \tag{6.1}$$

6.2 Spherical harmonics and homogeneous harmonic polynomials

In this section we will list some relevant facts about homogeneous harmonic polynomials and spherical harmonics. General references are [7], [14], [20] and [62]. The basic properties of spherical harmonics are derived from first principles in [59]. Group symmetry properties are discussed in [32].

Let $\mathcal{P}_k$ denote the set of homogeneous harmonic polynomials of degree k; $h \in \mathcal{P}_k$ if h is a polynomial of degree k such that

$$h(tx) = t^k h(x); \quad x \in \mathbb{R}^n,\ t \geq 0 \tag{6.2}$$

$$\Delta h = 0 \text{ in } \mathbb{R}^n. \tag{6.3}$$

It will be convenient to define $\mathcal{P}_k = \{0\}$ for $k < 0$. Due to (6.2), an element of $\mathcal{P}_k$ is determined by its values on the unit sphere ($h(x) = h(|x|\hat{x}) = |x|\, h(\hat{x})$).

The spherical harmonics of degree k are the restriction of elements of $\mathcal{P}_k$ to the unit sphere. We write $Y(\hat{x}) = h(\hat{x})$ for $h \in \mathcal{P}_k$ and define

$$\mathcal{Y}_k = \left\{ Y : S_{n-1} \to \mathbb{R} \,\middle|\, Y(\hat{x}) = h(\hat{x}) \text{ for some } h \in \mathcal{P}_k \right\}. \tag{6.4}$$

There is a one-to-one correspondence between $\mathcal{P}_k$ and $\mathcal{Y}_k$; $h(x) = |x|^k Y(\hat{x})$. The elements of $\mathcal{Y}_k$ are analytic functions on the unit sphere S_{n-1}. If we write r for $|x|$ then the Laplacian in $\mathbb{R}^n$ can be written

$$\Delta = r^{1-n} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Lambda$$

which defines Λ as a differential operator on functions twice differentiable on S_{n-1}. We call Λ the Laplace–Beltrami operator. The spherical harmonics are eigenfunctions of Λ, i.e., $\Lambda Y + k(k+n-2)Y = 0$ for $Y \in \mathcal{Y}_k$.

With the usual pointwise definitions of scalar multiplication and vector addition $\mathcal{Y}_k$ is a vector space of dimension

$$d(n,k) = \binom{n+k-1}{n-1}.$$

When $n = 2$, $d(2,k) = 2$ and $\mathcal{Y}_k$ is the span of $\{sin(k\phi), cos(k\phi)\}$. Equivalently, there are two linearly independent homogeneous harmonic polynomials of each degree in two variables. These are listed up to degree five in Table 6.1 below. For ease of reading we have written $x_1 = x = r\cos\phi$ and $x_2 = y = r\sin\phi$.

$\mathcal{P}_0$	1	
$\mathcal{P}_1$	x	y
$\mathcal{P}_2$	$x^2 - y^2$	$2xy$
$\mathcal{P}_3$	$x^3 - 3xy^2$	$3x^2y - y^3$
$\mathcal{P}_4$	$x^4 - 6x^2y^2 + y^4$	$4xy(x^2 - y^2)$
$\mathcal{P}_5$	$x^5 - 10x^3y^2 + 5xy^4$	$y^5 - 10x^2y^3 + 5x^4y$

Table 6.1: Homogeneous harmonic polynomials in two variables

The elements of $\mathcal{P}_k$ have been chosen so that the corresponding elements of $\mathcal{Y}_k$ are $\cos(k\phi)$ and $\sin(k\phi)$. For example, $x^3 - 3xy^2 = r^3(\cos^3\phi - 3\cos\phi\sin^2\phi)$ and

$$\begin{aligned}\cos(3\phi) &= \cos(2\phi)\cos\phi - \sin(2\phi)\sin\phi \\ &= (\cos^2\phi - \sin^2\phi)\cos\phi - 2\cos\phi\sin^2\phi \\ &= \cos^3\phi - 3\cos\phi\sin^2\phi.\end{aligned}$$

When $n = 3$ we write $x_1 = |x|\sin\theta\cos\phi$, $x_2 = |x|\sin\theta\sin\phi$ and $x_3 = |x|\cos\theta$. A basis for $\mathcal{Y}_k$ is usually taken as

$$Y_k^m(\hat{x}) = P_k^{|m|}(\cos\theta)\, e^{im\phi},$$

where $-k \le m \le k$ and P_k^m is the associated Legendre function. Alternatively, the set $\{1, P_k^m(\cos\theta)\cos(m\phi), P_k^m(\cos\theta)\sin(m\phi)\}_{m=1}^k$ is a real basis for $\mathcal{Y}_k$.

Explicit formulæ for spherical harmonics are given in [20]. For $n > 3$ they are represented as products of Gegenbauer polynomials and trigonometric functions.

Let f and g be (real) L^2 functions on the unit sphere and $d\omega_{n-1}$ surface measure on ∂B_1. Under the inner product

$$\langle f, g\rangle = \int_{\partial B_1} fg \, d\omega_{n-1}$$

the spaces $\mathcal{Y}_k$ are orthogonal, i.e., if $f \in \mathcal{Y}_k$, $g \in \mathcal{Y}_\ell$ and $k \neq \ell$ then $\langle f, g\rangle = 0$. Each $\mathcal{Y}_k$ is a vector subspace of the set of L^2 functions on the unit sphere S_{n-1} and for any $f \in L^2(S_{n-1})$ we have the expansion $f = \sum f_k$ where $f_k \in \mathcal{Y}_k$ and convergence is in the norm induced by the above inner product. Hence, the Hilbert space $L^2(S_{n-1})$ is the direct sum of $\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \cdots$ and we write

$$L^2(S_{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{Y}_k. \tag{6.5}$$

There is a corresponding result for $\mathcal{P}_k$. A strengthened form of this expansion will be used in the proof of the uniqueness theorem (Theorem 6.4.1).

6.3 Three lemmas

The following lemmas will be useful.

Lemma A: *Fix $1 \leq i \leq n$. For each $j \geq 0$, if $H_j \in \mathcal{P}_j$ then there exists $\lambda_j \in \mathbb{R}$ such that $\widehat{H}_{j+1}(x) := x_i H_j(x) - \lambda_j |x|^2 \frac{\partial H_j}{\partial x_i}$ is in $\mathcal{P}_{j+1}$.*

A proof is given in [14], p.534. It is based on Euler's theorem for homogeneous functions, namely, $x \cdot \nabla H(x) = k\, H(x)$ where H is a homogenous function of degree k.

Our uniqueness proof depends on the fact that, on the unit sphere, the product of any polynomial and a spherical harmonic has a finite expansion in terms of spherical harmonics. To prove this we use three lemmas.

Lemma 6.3.1 *If $h \in \mathcal{P}_k$ $(k \geq 0)$ and $p \geq 0$ is an integer, then there are $h_j \in \mathcal{P}_j$ such that*

$$x_i^p\, h(x) = \sum_{\ell=0}^{p} |x|^{2\ell}\, h_{k+p-2\ell}(x), \tag{6.6}$$

where i is a fixed integer, $1 \leq i \leq n$.

Proof. The proof is by induction on p.

If $p = 0$ the result is immediate.

If (6.6) holds for $0 \leq p \leq q$ then

$$x_i^{q+1} h(x) = x_i \sum_{\ell=0}^{q} |x|^{2\ell} H_{k+q-2\ell}(x) \qquad (H_j \in \mathcal{P}_j). \tag{6.7}$$

Writing $\lambda_j = [n + 2(j-1)]^{-1}$ for $j \geq 1$ and $\lambda_j = 0$ for $j \leq 0$, the function $\widehat{H}_{j+1}(x) := x_i\, H_j(x) - \lambda_j |x|^2 \frac{\partial H_j}{\partial x_i}$ is in $\mathcal{P}_{j+1}$ (Lemma A). Since $\frac{\partial H_j}{\partial x_i} \in \mathcal{P}_{j-1}$, (6.7) may be written

$$\begin{aligned} x_i^{q+1}\, h(x) &= \sum_{\ell=0}^{q} |x|^{2\ell} \left(\widehat{H}_{k+q-2\ell+1}(x) + \lambda_{k+q-2\ell}\, |x|^2 \frac{\partial H_{k+q-2\ell}}{\partial x_i} \right) \\ &= \sum_{\ell=0}^{q} |x|^{2\ell} \widehat{H}_{k+q-2\ell+1}(x) + \sum_{\ell=1}^{q+1} |x|^{2\ell} \lambda_{k+q-2\ell+2} \frac{\partial H_{k+q-2\ell+2}}{\partial x_i} \\ &= \sum_{\ell=0}^{q+1} |x|^{2\ell}\, \widehat{\widehat{H}}_{k+q-2\ell+1}(x) \qquad (\text{for some } \widehat{\widehat{H}}_j \in \mathcal{P}_j) \end{aligned}$$

and the result follows. ∎

The case $p = 1$ is given in [62], p. 226, Lemma 3.4.

The spherical harmonics of degree k are the restriction of elements of $\mathcal{P}_k$ to the unit sphere. Lemma 6.3.1 with $i = n$ may be written

$$\cos^p \theta\, Y_k(\hat{x}) = \sum_{\ell=0}^{p} \widehat{Y}_{k+p-2\ell}(\hat{x}) \qquad \text{where } Y_j,\ \widehat{Y}_j \in \mathcal{Y}_j. \tag{6.8}$$

Lemma 6.3.2 *If $h \in \mathcal{P}_k$ $(k \geq 0)$ and $r_i \geq 0$ are integers, then there are $h_j \in \mathcal{P}_j$ such that*

$$x_n^{p_n} x_{n-1}^{p_{n-1}} \cdots x_1^{p_1} = \sum_{\ell=0}^{p} |x|^{2\ell} h_{k+p-2\ell}(x), \tag{6.9}$$

where $p = \sum_{i=1}^{n} p_i$.

Proof. The case when all the r_i are zero except one is covered by Lemma 6.3.1. Use the convention that for any m-tuple σ, $h_k^{(\sigma)} \in \mathcal{P}_k$ $(1 \leq m \leq n)$. Hence, repeatedly using Lemma 6.3.1,

$$\begin{aligned} x_n^{p_n} \cdots x_1^{p_1} h(x) &= x_n^{p_n} \cdots x_2^{p_2} \sum_{\ell_1=0}^{p_1} |x|^{2\ell_1} h_{k+p_1-2\ell_1}^{(\ell_1)}(x) \\ &= x_n^{p_n} \cdots x_3^{p_3} \sum_{\ell_1=0}^{p_1} |x|^{2\ell_1} \sum_{\ell_2=0}^{p_2} |x|^{2\ell_2} h_{k+p_1+p_2-2\ell_1-2\ell_2}^{(\ell_1,\ell_2)}(x) \\ &\;\vdots \\ &= \sum_{\ell_1=0}^{p_1} |x|^{2\ell_1} \sum_{\ell_2=0}^{p_2} |x|^{2\ell_2} \cdots \sum_{\ell_n=0}^{p_n} |x|^{2\ell_n} h_{k+p-2\ell}^{(\ell_1,\cdots,\ell_n)}(x), \end{aligned} \tag{6.10}$$

where we have written $\ell = \ell_1 + \ell_2 + \cdots + \ell_n$. Collecting together homogeneous polynomials of the same degree gives the combinatorial sum

$$\begin{aligned} x_n^{p_n} \cdots x_1^{p_1} h(x) &= h_{k+p}^{(0,0,\cdots,0)}(x) + |x|^2 \left(h_{k+p-2}^{(1,0,\cdots,0)}(x) + h_{k+p-2}^{(0,1,0,\cdots,0)}(x) + \cdots + h_{k+p-2}^{(0,\cdots,0,1)}(x) \right) \\ &\quad + |x|^4 \left(h_{k+p-4}^{(1,1,0,\cdots,0)}(x) + h_{k+p-4}^{(1,0,1,0,\cdots,0)}(x) + \cdots + h_{k+p-4}^{(0,\cdots,0,2)}(x) \right) \\ &\quad + \cdots + |x|^{2p} h_{k-p}^{(p_1,\cdots,p_n)}(x) \\ &= \sum_{\ell=0}^{p} |x|^{2\ell} \sum_{\ell_1+\ell_2+\cdots+\ell_n=\ell} h_{k+p-2\ell}^{(\ell_1,\cdots,\ell_n)}(x) \\ &= \sum_{\ell=0}^{p} |x|^{2\ell} h_{k+p-2\ell}(x), \quad \text{where } h_j \in \mathcal{P}_j. \quad \blacksquare \end{aligned}$$

Example. Let $n = 2$ and $h(x,y) = x^2 - y^2 \in \mathcal{P}_2$. Then

$$\begin{aligned} xy^2 h(x,y) &= \sum_{\ell=0}^{2} r^{2\ell} h_{5-2\ell}(x,y) \\ &= h_5(x,y) + (x^2+y^2) h_3(x,y) + (x^2+y^2)^2 h_1(x,y). \end{aligned}$$

A calculation shows

$$\begin{aligned} h_5(x,y) &= -\frac{1}{8}\left(x^5 - 10x^3y^2 + 5xy^4\right) \\ h_3(x,y) &= \frac{1}{8}\left(x^3 - 3xy^2\right) \\ h_1(x,y) &= 0. \end{aligned}$$

Lemma 6.3.3 *Let P be a polynomial of degree p and $h \in \mathcal{P}_k$. Then there are $Y_j \in \mathcal{Y}_j$ such that*

$$P(\hat{x})\,h(\hat{x}) = \sum_{\ell=0}^{p} Y_{k+q-2\ell}(\hat{x}). \tag{6.11}$$

Proof. First consider $x_n^{r_n} x_{n-1}^{r_{n-1}} \cdots x_1^{r_1}$, where $r_i \geq 0$, $\sum_{i=1}^{n} r_i = r \leq p$. By Lemma 6.3.2, $x_n^{r_n} \cdots x_1^{r_1}\, h(x) = \sum_{\ell=0}^{p} |x|^{2\ell} h_{k+p-2\ell}(x)$, where $h_j \in \mathcal{P}_j$. Restricting x to the unit sphere via the map $x \mapsto \hat{x} = x/|x|$ gives $\tilde{x}_n^{r_n} \cdots \tilde{x}_1^{r_1}\, h(x) = \sum_{\ell=0}^{p} Y_{k+p-2\ell}(\hat{x})$, where $Y_j \in \mathcal{Y}_j$ and $\tilde{x}_i = x_i/|x|$. Since $P(\hat{x})$ is a linear combination of terms like $\tilde{x}_n^{r_n} \cdots \tilde{x}_1^{r_1}$ the result follows. ■

In the example above, when x is restricted to the unit sphere, we have $x = r\cos\phi \mapsto \cos\phi$ and $y = r\sin\phi \mapsto \sin\phi$, and (6.11) expresses a trigonometric identity.

6.4 A uniqueness theorem

We are now in a position to prove the following uniqueness theorem.

Theorem 6.4.1 *If $N \geq 0$ ($N \in \mathbb{Z}$), P a polynomial of degree p and f a continuous function on $\mathbb{R}^{n-1}$ then any solution to*

$$u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+) \tag{6.12}$$

$$\Delta u = 0, \quad x \in \Pi_+ \tag{6.13}$$

$$u = f, \quad x \in \partial\Pi_+ \tag{6.14}$$

$$u = o\left(\tfrac{|x|^{N+1}}{P(\hat{x})}\right) \quad (x \in \Pi_+,\ |x| \to \infty) \tag{6.15}$$

is unique to the addition of a harmonic polynomial of degree N that vanishes on $\partial\Pi_+$.

Proof. Let v be a solution of the corresponding homogeneous problem ($f = 0$). It is equivalent to prove that $v \in \mathcal{P}_N$ and $v = 0$ on $\partial\Pi_+$. By the Schwarz reflection principle any such v must be harmonic in $\mathbf{R}^n$. The spherical harmonics expansion theorem ([14], p. 535) gives

$$v(x) = \sum_{k=1}^{\infty} |x|^k Y_k^{(0)}(\hat{x}), \tag{6.16}$$

where we will write $Y_k^{(i)} \in \mathcal{Y}_k$ and $Y_k^{(0)}$ vanish on $\partial\Pi_+ \cap \partial B_1$.

Using Lemma 6.3.3 we have

$$P(\hat{x})\, Y_k^{(0)}(\hat{x}) = \sum_{\ell=0}^{p} Y_{k+p-2\ell}^{(k)}(\hat{x}) \tag{6.17}$$

and

$$P(\hat{x})\, v(x) = \sum_{k=1}^{\infty} |x|^k \sum_{\ell=0}^{p} Y_{k+p-2\ell}^{(k)}(\hat{x}). \tag{6.18}$$

Let $j \in \mathbb{Z}_+$ and $0 \leq m \leq p$. The series in (6.16) converges uniformly on compact sets and so may be integrated over the unit sphere term by term. With δ_{ab} the Kronecker delta, orthogonality of spherical harmonics gives

$$\begin{aligned} &\int_{\partial B_1} Y_{j+p-2m}^{(j)}(\hat{x}) P(\hat{x})\, v(|x|\hat{x})\, dS_{n-1} \\ &\qquad = \sum_{k=1}^{\infty} |x|^k \sum_{\ell=0}^{p} \delta_{j+p-2m,k+p-2\ell} \int_{\partial B_1} Y_{k+p-2\ell}^{(j)}(\hat{x})\, Y_{k+p-2\ell}^{(k)}(\hat{x})\, dS_{n-1}. \end{aligned} \tag{6.19}$$

The notation $v(|x|\hat{x})$ indicates $|x|$ remains fixed for the integration. The condition $j+p-2m = k+p-2\ell$ is satisfied by only a finite number of $k \in \mathbb{Z}_+$, $0 \leq \ell \leq p$. The right member of (6.19) is then a polynomial in $|x|$ with no constant term. Integrating the order relation (6.15),

$$\begin{aligned} \int_{\partial B_1} Y_{j+p-2m}^{(j)}(\hat{x}) P(\hat{x})\, v(|x|\hat{x})\, dS_{n-1} &= o\left(|x|^{N+1} \int_{\partial B_1} Y_{j+p-2m}^{(j)}(\hat{x})\, dS_{n-1}\right) \\ &= o\left(|x|^{N+1}\right) \qquad (|x| \to \infty), \end{aligned} \tag{6.20}$$

shows the coefficient of $|x|^j$ in (6.19) vanishes when $j > N$, i.e., $\| Y^{(j)}_{j+p-2m} \|^2 = 0$. From (6.17), $Y^{(0)}_k \equiv 0$ for $k > N$. Hence, by (6.16), $v(x) \equiv 0$ if $N = 0$ and if $N \geq 1$,

$$v(x) = \sum_{k=1}^{N} |x|^k Y^{(0)}_k(\hat{x}) \in \mathcal{P}_N.$$

The theorem follows. ■

Corollary 6.4.1 *If* (2.6) *holds for continuous function f then $u = D[f]$ gives the unique solution to the Dirichlet problem* (2.1)–(2.3) *that satisfies the growth condition* $u = o(|x| \sec^{n-1}\theta) \quad (x \in \Pi_+, |x| \to \infty)$.

Proof. Use Corollary 2.2.1 and put $N = 0$, $p = n-1$ in Theorem 6.4.1. ■

There is a corresponding result for the Neumann problem.

Theorem 6.4.2 *If $N \geq 0$ ($N \in \mathbb{Z}$), P a polynomial of degree p and g a continuous function on $\mathbb{R}^{n-1}$ then any solution to*

$$u \in C^2(\Pi_+) \cap C^1(\overline{\Pi}_+) \tag{6.21}$$

$$\Delta u = 0, \quad x \in \Pi_+ \tag{6.22}$$

$$\tfrac{\partial u}{\partial x_n} = -g, \quad x \in \partial\Pi_+ \tag{6.23}$$

$$u = o\Big(\tfrac{|x|^N}{P(\hat{x})}\Big) \quad (x \in \Pi_+, |x| \to \infty) \tag{6.24}$$

is unique to the addition of a harmonic polynomial of degree $N-1$ whose normal derivative vanishes on $\partial\Pi_+$.

Proof. The proof is similar to that above. Let v be a solution to the homogeneous problem. Extend v to $x_n < 0$ as an even function, $v(x) = \sum_{k=0}^{\infty} |x|^k Y^{(0)}_k(\hat{x})$, where now $\partial Y^{(0)}_k(\hat{x})/\partial\theta = 0$ on $\partial\Pi_+ \cap \partial B_1$. Proceeding as before, (6.20) becomes

$$\begin{aligned}\int_{\partial B_1} Y^{(j)}_{j+p-2m}(\hat{x}) P(\hat{x})\, v(|x|\hat{x})\, dS_{n-1} &= o\left(|x|^N \int_{\partial B_1} Y^{(j)}_{j+p-2m}(\hat{x})\, dS_{n-1}\right) \\ &= o\left(|x|^N\right) \qquad (|x| \to \infty). \end{aligned} \tag{6.25}$$

It follows that $Y_k^{(0)} \equiv 0$ for $k \geq N$. Thus $v(x) \equiv 0$ if $N = 0$ and if $N \geq 1$,

$$v(x) = \sum_{k=0}^{N-1} |x|^k Y_k^{(0)}(\hat{x}) \in \mathcal{P}_N. \quad \blacksquare$$

If $n > 2$ then $N[f] = o(\sec^{n-2}\theta)$. Therefore, the Poisson integral $u = N[f]$ gives the unique solution to the Neumann problem under the growth condition $u(x) = o(\sec^{n-2}\theta)$, since in Theorem 6.4.2 we can take $N = 0$ and the degree of P at least $n-2$. If $n = 2$ and f is odd then $N[f] = o(\log(1 - |\cos\phi|)/\log r)$. And, since we can take $N = 0$ we have uniqueness of $u = N[f]$ under $o(\csc\phi)$. (The lowest degree odd polynomial is $g(\xi) = \xi$, which fails this growth condition.) However, when f is not odd then $N[f] = O(\log r)$ and we must take $N = 1$. In this case, the growth condition $u = o(r/\sin\phi)$ does not give a unique solution. The solution will be $u = N[f] + c$ where c is any constant. Growth estimates for the $n = 2$ Neumann case were given in Theorem 5.4.1.

In the next chapter a uniqueness theorem will be derived using barrier functions.

Chapter 7

A Phragmén–Lindelöf Principle

7.1 Phragmén–Lindelöf Principles

The classical Phragmén–Lindelöf Principle of complex analysis gives an estimate for an analytic function in a sector based on its boundary behaviour and growth at infinity.

Theorem A (Phragmén and Lindelöf): *Let $0 < \alpha \le \pi$ and let K_α be the sector $0 < |\arg z| \le \alpha$. If f is analytic in K_α such that*

$$\limsup_{z \in K_\alpha, z \to z_0} |f(z)| \le 1 \qquad \text{for each } z_0 \in \partial K_\alpha, \tag{7.1}$$

$$\limsup_{r \to \infty} \frac{\log \mu(r)}{r^{\pi/(2\alpha)}} = 0 \qquad \text{where } \mu(r) = \sup_{|z|=r, z \in K_\alpha} |f(z)|, \tag{7.2}$$

then $|f(z)| \le 1$ for all $z \in K_\alpha$.

See [54] for the original reference.

A similar result holds for subharmonic functions in cones of $\mathbb{R}^n$.

Theorem B: *Let $0 < \alpha \leq \pi$ and let K_α be the cone $\{x \in \mathbf{R}^n \mid 0 \leq \theta < \alpha\}$. If $u \in C^2(K_\alpha)$ such that*

$$\Delta u \geq 0 \quad \text{in } K_\alpha \tag{7.3}$$

$$\limsup_{x \in K_\alpha,\ x \to x_0} u(x) \leq 0 \quad \text{for each } x_0 \in \partial K_\alpha \tag{7.4}$$

$$\limsup_{r \to \infty} \frac{\mu(r)}{r^\rho} \leq 0 \qquad \text{where } \mu(r) = \sup_{|x|=r, x \in K_\alpha} |u(x)|, \tag{7.5}$$

then $u \leq 0$ in K_α.

Here ρ is the smallest positive root of $\rho(\rho+n-2) = \lambda_1$ where $\lambda_1 > 0$ is the smallest eigenvalue for the problem $\Delta v + \lambda v = 0$ in K_α with $v = 0$ on $\partial K_\alpha \cap \partial B_1$. For the half space, $\alpha = \pi/2$ and $\rho = 1$. See [17]. The first theorem is a special case of the second since if f is analytic then $\log|f(z)|$ is a subharmonic function of the two variables $x_1 = \mathcal{R}e(z)$ and $x_2 = \mathcal{I}m(z)$. There are many other types of Phragmén–Lindelöf Principles, for different differential equations, for different regions and with different types of growth conditions. The works by M. R. Essen [21], W. K. Hayman and P. B. Kenney [33], V. A. Kondrat'ev and E. M. Landis [40] and P. Koosis [41] survey the literature. As will be seen below, a Phragmén–Lindelöf Principle immediately leads to a uniqueness theorem.

In Chapter 2 we had the growth estimate $D[f](x) = o(|x| \sec^{n-1}\theta)$ (Corollary 2.2.1) and this was proven to be sharp in Chapter 3, Theorem 3.3.1. Hence, the $\sec^{n-1}\theta$ term cannot be dropped and the Poisson integralwill not in general satisfy the premise of the Phragmén–Lindelöf Principle ($\alpha = \pi/2$ in this case). And yet we know from the spherical harmonic expansion that the Poisson integralgives the unique solution to the Dirichlet problem under a growth condition compatible with $o(|x| \sec^{n-1}\theta)$. See (6.15) and Corollary 6.4.1. Thus it is desirable to prove a Phragmén–Lindelöf Principle that allows divergence at the boundary. For the half plane, F. Wolf has the following result, [66].

Theorem C: *Let Π_+ be the half plane $x_2 > 0$ of $\mathbf{R}^2$. Let f be a non-negative* *measurable function on* $(0,\pi)$ *with* $\int_{\phi=0}^{\pi} \log^+ f(\phi)\, d\phi < \infty$. *If* $u \in C^2(\Pi_+)$ *such that*

$$\Delta u \geq 0 \quad in\ \Pi_+ \tag{7.6}$$

$$\limsup_{x\in\Pi_+,\ x\to x_0} u(x) \leq 0 \quad for\ any\ x_0 \in \partial\Pi_+ \tag{7.7}$$

$$\limsup_{r\to\infty} \frac{\mu(r)}{r} \leq 0 \qquad where\ \mu(r) = \sup_{|x|=r, x\in\Pi_+} e^{-f(\phi)}|u(x)|, \tag{7.8}$$

then $u \leq 0$ *in* Π_+.

The function $\log^+ t$ is $\log t$ for $t \geq 1$ and 0 for $0 < t \leq 1$. In this theorem the growth condition allows the angular function to be singular for any value in $[0,\phi]$ provided its logarithm is still L^1. This includes the growth condition in (6.15). The proof of Wolf's theorem depends on construction of a conformal map and so is specific to $n = 2$. It will be the aim of this chapter to develop a Phragmén–Lindelöf Principle in Π_+ of $\mathbb{R}^n$ that allows angular blow up compatible with the estimate on the Poisson integral. This will be done using barriers.

7.2 Barriers in the plane

In order to prove a Phragmén–Lindelöf Principle we will need two lemmas.

Lemma 7.2.1 *For any decreasing, measurable function* $m_0 : (0,1] \to (0,\infty)$ *with* $\int_{t=0}^{1} m_0(t)\, dt < \infty$ *and any positive number* A *there is a function* $m : (0,1] \to (0,\infty)$ *majorising* m_0 *such that*

(i) $m(t) \geq m_0(t) - A\log t$

(ii) m *is* C^2 *on* $(0,1)$

(iii) $m' < 0$

(iv) $-m'(t)\, t \geq A$

(v) $\displaystyle\int_{t=0}^{1} m(t)\,dt < \infty$

(vi) $-t^3 m'(t)$ *is increasing*

Proof. Start with (iii), (v) and (vi). Let $t = \xi^{-1/2}$ then $\frac{d}{dt} = -2t^{-3}\frac{d}{d\xi}$. Write $m_1(\xi) = m_0(t)$. Condition (iii) becomes $m_1'(\xi) > 0$. And,

$$\frac{d}{dt}\left(-t^3\, m_0'(t)\right) = -2t^{-3}\,\frac{d}{d\xi}\left(2m_1'(\xi)\right). \tag{7.9}$$

So (vi) is equivalent to $m_1''(\xi) \le 0$ or m_1 is concave.

Since $2\,dt = -\xi^{-3/2}\,d\xi$ we require $\int_{\xi=1}^{\infty} m_1(\xi)\xi^{-3/2}\,d\xi < \infty$ in place of condition (v). With m_0 as given in the lemma, the function $m_1 : [1,\infty) \to (0,\infty)$ is increasing and satisfies the above integral condition. Hence, $m_1(\xi) = o(\sqrt{\xi})$ as $\xi \to \infty$. Let

$$\begin{aligned} V \;=\; & \left\{h : [1,\infty) \to [0,\infty) \mid h \in C^2\left((1,\infty)\right), h'(\xi) \ge 0, h''(\xi) \le 0 \right. \\ & \left. \text{and } h(\xi) \ge m_1(\xi) \text{ for } \xi > 1\right\}. \end{aligned} \tag{7.10}$$

The map $\xi \mapsto a + b\sqrt{\xi}$ is in V for large enough a and b so V is not empty. Let $m_2(\xi) = \inf\limits_{h\in V} h(\xi)$. Then $m_2(\xi) : [1,\infty) \to [0,\infty)$ and is a concave, increasing majorant of m_1. It is piecewise linear where it does not agree with m_1. If $1 \le \xi_1 \le \xi_2$ then for all $\epsilon > 0$ there is a function h^* in V (depending on ϵ and ξ_2) so that

$$0 \le h^*(\xi_2) - m_2(\xi_2) < \epsilon. \tag{7.11}$$

We have

$$m_2(\xi_1) \le h^*(\xi_1) \le h^*(\xi_2) \le m_2(\xi_2) + \epsilon. \tag{7.12}$$

And, $\epsilon > 0$ was arbitrary so $m_2(\xi_1) \le m_2(\xi_2)$. Therefore, m_2 is increasing (in the wide sense) on $[1,\infty)$. To prove m_2 is concave we have to show that

$$m_2(\lambda s + (1-\lambda)t) \ge \lambda\, m_2(s) + (1-\lambda) m_2(t)$$

for all s, $t \geq 1$ and all $0 < \lambda < 1$. Suppose $1 \leq s \leq t = \xi_2$. Using (7.11) we have

$$\begin{aligned} m_2(\lambda s + (1-\lambda)t) &> h^*(\lambda s + (1-\lambda)t) - \epsilon & (7.13) \\ &\geq \lambda h^*(s) + (1-\lambda)h^*(t) - \epsilon & (7.14) \\ &\geq \lambda(m_2(s) - \epsilon) + (1-\lambda)(m_2(t) - \epsilon) - \epsilon & (7.15) \\ &= \lambda m_2(s) + (1-\lambda)m_2(t) - 2\epsilon. & (7.16) \end{aligned}$$

Since $\epsilon > 0$ is arbitrary it follows that m_2 is concave on $[1, \infty)$.

Now show that $\int_{\xi=1}^{\infty} m_2(\xi)\xi^{-3/2}\, d\xi < \infty$. Let $\{\xi_n\}_{n=0}^{\infty}$ be the set of points such that $\xi_0 = 1$; $\xi_{2n} < \xi_{2n+1}$; $m_2(\xi_{2n}) = m_1(\xi_{2n})$ and $m_2(\xi_{2n+1}) = m_1(\xi_{2n+1})$; for all ξ satisfying $\xi_{2n} < \xi < \xi_{2n+1}$ we have $m_2(\xi) > m_1(\xi)$. Let $\Omega \subset [1, \infty)$ be the set of points $\omega \notin \{\xi_n\}$ such that $m_2(\omega) = m_1(\omega)$. If a sequence of points in $\{\xi_n\}$ has a finite limit then it will be impossible to label ξ_n such that $\xi_{2n} < \xi_{2m}$ whenever $n < m$.

Let V_n be the semi-infinite strip

$$V_n = \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi > \xi_{2n+1}, \eta_{2n} < \eta < \eta_{2n+1}\} \tag{7.17}$$

and Δ_n the triangle with vertices (ξ_{2n}, η_{2n}), (ξ_{2n+1}, η_{2n}) and $(\xi_{2n+1}, \eta_{2n+1})$. We have written $\eta_k = m_1(\xi_k)$ for $k = 0, 1, 2, \cdots$. The line containing (ξ_{2n}, η_{2n}) and $(\xi_{2n+1}, \eta_{2n+1})$ has equation

$$\eta - \eta_{2n} = \left(\frac{\eta_{2n+1} - \eta_{2n}}{\xi_{2n+1} - \xi_{2n}}\right)(\xi - \xi_{2n}) \tag{7.18}$$

or, $\eta = a_n\xi + b_n$ where

$$a_n = (\eta_{2n+1} - \eta_{2n})/(\xi_{2n+1} - \xi_{2n}) \text{ and } b_n = \eta_{2n} - a_n\xi_{2n}. \tag{7.19}$$

Let σ be measure in the $\xi\eta$-plane with weight $\xi^{-3/2}$. By construction, $m_2(\xi) \leq a_n\xi + b_n$ on (ξ_{2n}, ξ_{2n+1}). Therefore,

$$\begin{aligned}
\sigma(\Delta_n) &= \int_{\xi=\xi_{2n}}^{\xi_{2n+1}} \int_{\eta=\eta_{2n}}^{a_n\xi+b_n} \xi^{-3/2}\, d\eta d\xi && (7.20)\\
&= \int_{\xi=\xi_{2n}}^{\xi_{2n+1}} (a_n\xi + b_n - \eta_{2n})\, \xi^{-3/2}\, d\xi && (7.21)\\
&= \left[2a_n\xi^{1/2} - 2(b_n - \eta_{2n})\xi^{-1/2}\right]_{\xi=\xi_{2n+1}}^{\xi_{2n+1}} && (7.22)\\
&= 2a_n\left(\sqrt{\xi_{2n+1}} - \sqrt{\xi_{2n}}\right) - 2(b_n\eta_{2n})\left(\frac{1}{\sqrt{\xi_{2n+1}}} - \frac{1}{\sqrt{\xi_{2n}}}\right). && (7.23)
\end{aligned}$$

And,

$$\begin{aligned}
\iint_{V_n} d\sigma &= \int_{\xi=\xi_{2n+1}}^{\infty} \xi^{-3/2}\, d\xi \int_{\eta=\xi_{2n}}^{\xi_{2n+1}} d\eta && (7.24)\\
&= \frac{2(\eta_{2n+1} - \eta_{2n})}{\sqrt{\xi_{2n+1}}}. && (7.25)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\sigma(\Delta_n)}{\sigma(V_n)} &= \frac{(\eta_{2n+1} - \eta_{2n})\left[\frac{\sqrt{\xi_{2n+1}}-\sqrt{\xi_{2n}}}{\xi_{2n+1}-\xi_{2n}} + \frac{\xi_{2n}}{(\xi_{2n+1}-\xi_{2n})}\left(\frac{1}{\sqrt{\xi_{2n+1}}} - \frac{1}{\sqrt{\xi_{2n}}}\right)\right]}{\frac{\eta_{2n+1}-\eta_{2n}}{\sqrt{\xi_{2n+1}}}} && (7.26)\\
&= \frac{\xi_{2n+1} - 2\sqrt{\xi_{2n}\xi_{2n+1}} + \xi_{2n}}{\xi_{2n+1} - \xi_{2n}} && (7.27)\\
&= \frac{\sqrt{\xi_{2n+1}} - \sqrt{\xi_{2n}}}{\sqrt{\xi_{2n+1}} + \sqrt{\xi_{2n}}} && (7.28)\\
&< 1. && (7.29)
\end{aligned}$$

Now,

$$\sum_{n=1}^{\infty} \sigma(V_n) \leq \int_{\xi=1}^{\infty} m_1(\xi)\xi^{-3/2}\, d\xi < \infty. \qquad (7.30)$$

We then have

$$\begin{aligned}
\sigma(m_2) &\leq \sum_{n=1}^{\infty} \sigma(V_n) + \sigma(\Delta_n) + \int_{\Omega} m_1(\xi)\, \xi^{-3/2}\, d\xi && (7.31) \\
&\leq 2\sum_{n=1}^{\infty} \sigma(V_n) + \int_{\Omega} m_1(\xi)\, \xi^{-3/2}\, d\xi && (7.32) \\
&< \infty. && (7.33)
\end{aligned}$$

In order to smooth out m_2 we will need to extend its domain to $[0, \infty)$. Suppose ξ_{2m} is not a limit point of Ω (for some fixed $m \geq 0$). Then m_2 is a linear function on $[\xi_{2m}, \xi_{2m+1}]$, i.e., $m_2(\xi) = a_m\xi + b_m$ where a_m and b_m are given in (7.19). Since $m_0(1) \geq 0$ it follows that $m_2(1) \geq 0$ and there is a number $M \geq 0$ such that if

$$m_3(\xi) = \begin{cases} M + a_m\xi + b_m, & 0 \leq \xi \leq \xi_{2m+1} \\ M + m_2(\xi), & \xi \geq \xi_{2m+1} \end{cases}$$

then $m_3(\xi) \geq 0$ for $\xi \geq 0$. And, m_3 is concave and increasing on $[0, \infty)$.

We can mollify m_3 with a convolution. Let $\Phi : [0,1] \to [0,1]$ be a C^2 function with the properties $\Phi(t) = \Phi(1-t)$ (even about $t = 1/2$), Φ is increasing on $(0, 1/2)$, $\Phi(t) = 0$ for $t \leq 0$ or $t \geq 1$, $\Phi'(0) = \Phi'(1) = 0$ and $\int_{t=0}^{1} \Phi(t)\, dt = 1$. Let

$$\begin{aligned}
m_4(\xi) &= \int_{u=1}^{\xi} \int_{v=u-1}^{u} m_3'(v)\, \Phi(u-v)\, dv\, du \;+\; m_3(1) && (7.34) \\
&\geq m_3(1). && (7.35)
\end{aligned}$$

Since m_3 is increasing, m_3' exists almost everywhere and m_4 is C^2 on $[1, \infty)$. And, $m_3' \geq 0$. Now,

$$\begin{aligned}
m_4'(\xi) &= \int_{v=\xi-1}^{\xi} m_3'(v)\, \Phi(\xi - v)\, dv && (7.36) \\
&\geq m_3'(\xi^-) \int_{v=\xi-1}^{\xi} \Phi(\xi - v)\, dv && (7.37) \\
&= m_3'(\xi^-). && (7.38)
\end{aligned}$$

The last line follows from the change of variables $\xi - v \mapsto v$. Since m_3' may not exist everywhere, we define

$$m_3'(\xi^{\pm}) = \lim_{t \to \xi^{\pm}} m_3'(t). \tag{7.39}$$

It now follows from (7.35) and (7.38) that $m_4(\xi) \geq m_3(\xi)$ for $\xi \geq 1$.

And, m_4 is concave, for

$$m_4''(\xi) = \int_{v=\xi-1}^{\xi} m_3'(v)\, \Phi'(\xi - v)\, dv \tag{7.40}$$

$$= \int_{s=0}^{1} m_3'(\xi - s)\, \Phi'(s)\, ds \tag{7.41}$$

$$= \int_{s=0}^{1/2} m_3'(\xi - s)\, \Phi'(s)\, ds + \int_{s=0}^{1/2} m_3'(\xi - 1 + s)\, \Phi'(1 - s)\, ds \tag{7.42}$$

$$= \int_{s=0}^{1/2} [m_3'(\xi - s) - m_3'(\xi - 1 + s)]\ \Phi'(s)\, ds. \tag{7.43}$$

The last line follows from the fact that Φ is even about $1/2$ implies Φ' is odd about $1/2$. When $0 \leq s \leq 1/2$ we have $\xi - s \geq \xi - 1 + s$. Since m_3 is concave and $\Phi'(s) \geq 0$ $(0 \leq s \leq 1/2)$ (7.43) shows that $m_4''(\xi) \leq 0$.

The function m_4 also has finite measure with respect to σ. We have

$$m_4'(\xi) \leq m_3'\left((\xi - 1)^+\right) \int_{v=\xi-1}^{\xi} \Phi(\xi - v)\, dv \tag{7.44}$$

$$= m_3'\left((\xi - 1)^+\right) \tag{7.45}$$

$$\leq m_3'\left((\xi - 1)^-\right). \tag{7.46}$$

And, $m_4(1) = m_3(1) = m_3(0) + (m_3(1) - m_3(0))$ so $m_4(\xi) \leq m_3(\xi - 1) + m_3(1) - m_3(0)$. Therefore,

$$\sigma(m_4) \leq \int_{\xi=1}^{\infty} [m_3(\xi) + m_3(1) - m_3(0)]\, \xi^{-3/2}\, d\xi \tag{7.47}$$

$$< \infty. \tag{7.48}$$

If we now transform back to the interval $(0,1)$ via $\xi = t^{-2}$ and then define $m_5(t) = m_4(\xi)$ then m_5 majorises m_0 and satisfies (ii), (iii), (v) and (vi).

Finally, let $m_6(t) = m_5(t) - A\log t$. Then on $(0,1)$, $m_6(t) \geq m_5(t) \geq m_0(t)$ and $m_6 \in C^2((0,1))$, so (i) and (ii) hold. We have $m_6'(t) = m_5'(t) - A/t < m_5'(t) \leq 0$, which gives (iii). As well, $-m_6'(t)\,t = -m_5'(t)\,t + A \geq A$, giving (iv). To demonstrate (vi), write

$$\int_{t=0}^{1} m_6'(t)\,dt = \int_{t=0}^{1} m_5'(t)\,dt - A\int_{t=0}^{1} \log t\,dt \tag{7.49}$$

$$= \sigma(m_4) + A \tag{7.50}$$

$$< \infty. \tag{7.51}$$

For (vi),

$$\frac{d}{dt}\left(-t^3\,m_6'(t)\right) = \frac{d}{dt}\left(-t^3\,m_5'(t)\right) + \frac{d}{dt}\left(3At^2\right) \tag{7.52}$$

$$= -4t^{-3}\,m_4''(\xi) + 6At \tag{7.53}$$

$$> 0. \tag{7.54}$$

Hence, m_6 is the desired majorant. ■

Lemma 7.2.2 *Let m_0 be any decreasing measurable function, $m_0 : (0,1) \to [0,\infty)$, with $\int_{t=0}^{1} m_0(t)\,dt < \infty$. For each $\rho > 0$ there is a barrier function $\psi_\rho : (-\rho,\rho) \to [0,\infty)$ and a set $T_\rho \subset \Pi_+$ satisfying*

$$\psi_\rho \in C^2(T_\rho) \cap C^0(\overline{T}_\rho \setminus S) \tag{7.55}$$

$$\Delta\psi_\rho \leq 0, \qquad x \in T_\rho \tag{7.56}$$

$$\psi_\rho(x) \geq r\,e^{m_0(\sin\phi)}, \qquad x \in \partial^+ T_\rho \tag{7.57}$$

$$\psi_\rho \geq 0, \qquad x \in \overline{T}_\rho \setminus S \tag{7.58}$$

$$\lim_{\rho\to\infty} \psi_\rho(x_1,x_2) < \infty \qquad \textit{for each } (x_1,x_2) \in T_\rho. \tag{7.59}$$

The boundary of T_ρ is piecewise smooth. The set S is the intersection of T_ρ and the x_1-axis,

$$S = \{(x_1, x_2) \in \mathbf{R}^2 \mid -\rho \le x_1 \le \rho,\ x_2 = 0\} \tag{7.60}$$

and $\partial^+ T_\rho = \partial T_\rho \cap \Pi_+$.

Proof. The proof is partly based on the construction in Lemma II of [11]. Let $m \ge m_0$ be the majorant in Lemma 7.2.1 with $A = 3$. We will be referring to the properties (i)–(vi) of Lemma 7.2.1.

There is a number $0 < \lambda \le 2/3$ such that $-\int_{t=0}^{\lambda} m'(t)\,dt = 2$. Indeed, let $I(\mu) = -\int_{t=0}^{\mu} m'(t)\,dt$. Then $I(0) = 0$ and by (iv), $I(2/3) \ge 2$. Due to the continuity of I, the number λ exists as above.

Let

$$g(t) = \frac{d}{dt} m\left(\frac{1}{g(t)}\right) \qquad \text{and} \qquad g(-1) = \lambda^{-1} \tag{7.61}$$

define g on $[-1, 1)$. We have

$$dt = m'\left(\frac{1}{g}\right) \frac{1}{g}\, d\left(\frac{1}{g(t)}\right) \tag{7.62}$$

so that

$$\int_{s=-1}^{t} ds \;=\; t + 1 \;=\; \int_{s=-1}^{t} m'\left(\frac{1}{g}\right) \frac{1}{g}\, d\left(\frac{1}{g(s)}\right) \tag{7.63}$$

$$= \; - \int_{u=1/g(t)}^{\lambda} m'(u) u \, du. \tag{7.64}$$

The left side is an increasing function of t which says that g is an increasing function (recall $m' < 0$).

The domain of g is $[-1,1)$. If $\lim\limits_{t\to t_0^-} g(t) = +\infty$ for some $-1 < t_0 < 1$ then as $t \to t_0^-$ we have

$$-\int_{u=1/g(t)}^{\lambda} m'(u)u\,du \;\to\; -\int_{u=0}^{\lambda} m'(u)u\,du = 2 \tag{7.65}$$

but $\lim\limits_{t\to t_0^-}(t+1) = t_0 + 1 < 2$. Therefore, $t_0 = 1$.

Now, from (7.61),

$$1 = -m'\left(\frac{1}{g}\right)\frac{1}{g^3}g' \tag{7.66}$$

and $1/g(t)$ is a decreasing function of t so (vi) shows g' is an increasing function. Therefore, g is a positive, increasing, convex function on $[-1,1)$ with $g(t) \to +\infty$ as $t \to 1^-$.

Let

$$f(s) = \exp\left(\int_{t=-1}^{s} g(t)\,dt\right) \qquad \text{for } -1 \le s < 1. \tag{7.67}$$

Then on $(-1,1)$,

$$\begin{aligned} f' &> 0, && (7.68)\\ f' &= fg > 0, && (7.69)\\ f'' &= fg^2 + fg' > 0. && (7.70) \end{aligned}$$

Let

$$\psi(x_1,x_2) = f(x_1)\left(2x_2 - x_2^3\, g^2(x_1)\right). \tag{7.71}$$

Then $\psi(x_1,x_2) \ge 0$ when $-1 \le x_1 < 1$ and $0 \le x_2 \le \sqrt{2}/g(x_1)$. And,

$$\Delta\psi(x_1,x_2) = 2f''(x_1)x_2 - \left(f(x_1)g^2(x_1)\right)'' - 6f(x_1)g^2(x_1)x_2. \tag{7.72}$$

Both f and g are increasing and convex so fg^2 is convex and, using (7.70) and (7.66),

$$\begin{aligned}
\Delta\psi &\leq \left[2f''(x_1) - 6f(x_1)g^2(x_1)\right] x_2 &\quad (7.73)\\
&= 2f(x_1)g^2(x_1)\left[1 + g'g^{-2} - 3\right] x_2 &\quad (7.74)\\
&= 2f(x_1)g^2(x_1)\left[-\frac{g(x_1)}{m'(1/g(x_1))} - 2\right] x_2 &\quad (7.75)\\
&\leq 0. &\quad (7.76)
\end{aligned}$$

The last inequality is from (iv) of Lemma 7.2.1.

For $\rho > 0$, let

$$\psi_\rho(x_1, x_2) = 5e^{m(\lambda)}\rho\left[\psi\left(\frac{x_1}{\rho}, \frac{x_2}{\rho}\right) + \psi\left(-\frac{x_1}{\rho}, \frac{x_2}{\rho}\right)\right]. \qquad (7.77)$$

This will be the form of the desired barrier function. We have $\psi_\rho \geq 0$ in the set

$$\left\{|x_1| < \rho,\ 0 \leq x_2 \leq \rho \min\left(1/g\left(\frac{x_1}{\rho}\right), 1/g\left(\frac{-x_1}{\rho}\right)\right)\right\} \qquad (7.78)$$

and ψ_ρ is C^2 in its interior. From (7.76), $\Delta\psi_\rho \leq 0$. We have $0 < \lambda < g(0) < \infty$ and $f(0) = \exp(\int_{t=-1}^{0} g(t)\,dt) < \infty$ so

$$\lim_{\rho\to\infty} \psi_\rho(x_1, x_2) = 2f(0)x_2 < \infty \text{ for each } (x_1, x_2) \text{ in the above set.} \qquad (7.79)$$

Define T_ρ to be a subset of the set in (7.78). Let

$$T_\rho = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 < \rho\left(\frac{3}{\sqrt{5}}\right)\min\left(1/g\left(\frac{x_1}{\rho}\right), 1/g\left(\frac{-x_1}{\rho}\right)\right)\right\}. \qquad (7.80)$$

(See Figure 7.1.) With these definitions, (7.55), (7.56), (7.58) and (7.59) are satisfied. It remains to show that the choice of g in (7.66) will give us (7.57). Note that ψ_ρ is singular at $(\pm\rho, 0)$ and that in general T_ρ will have a cusp at these points and at $(0, 3\rho/(\sqrt{5}g(0)))$.

The boundary of T_ρ consists of the three smooth arcs $\{x_2 = 3\rho/(\sqrt{5}g(|x_1|/\rho))$, $|x_1| < \rho\}$ (which make up ∂^+T_ρ) and the segment $[-\rho, \rho]$ of the x_1-axis. Consider one

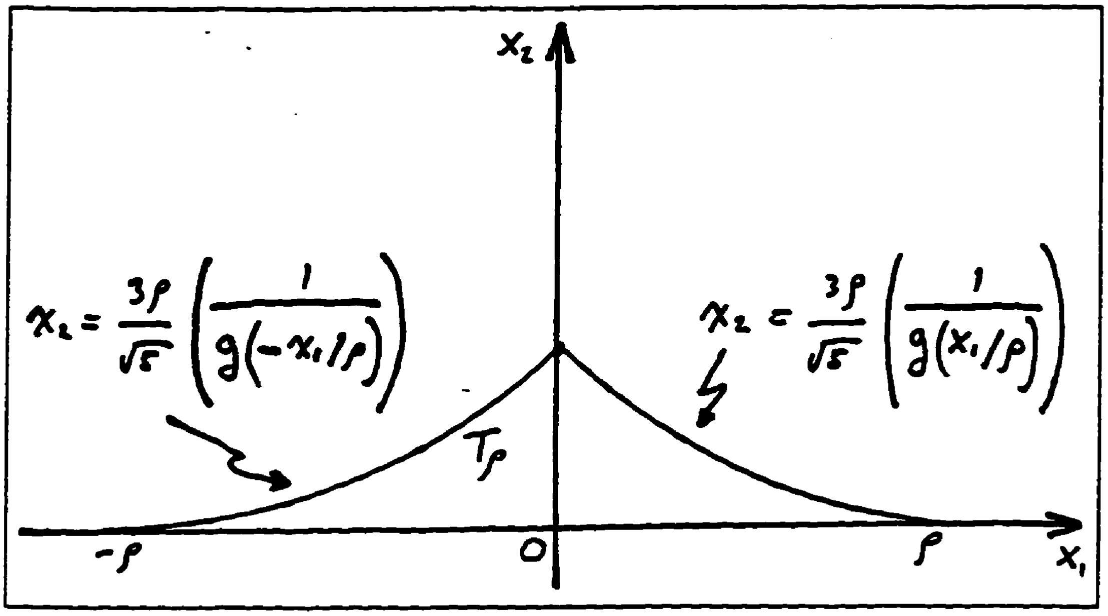

Figure 7.1: The cusped region T_ρ

half of $\partial^+ T_\rho$. On, $x_2 = 3\rho/(\sqrt{5}g(x_1/\rho))$, $0 \le x_1 < \rho$, we have

$$\frac{x_1}{\rho} = g^{-1}\left(\frac{3\rho}{\sqrt{5}\,x_2}\right). \tag{7.81}$$

Since g is strictly increasing, it has an inverse function g^{-1}. On the curve $x_2 = 3\rho/(\sqrt{5}g(x_1/\rho))$, (7.71) and (7.77) show that

$$\begin{aligned} \psi_\rho(x_1, x_2) &\ge 5e^{m(\lambda)} f\left(\frac{x_1}{\rho}\right)\left(2 - \tfrac{9}{5}\right) x_2 & (7.82)\\ &= e^{m(\lambda)} f\left(g^{-1}\left(\frac{3\rho}{\sqrt{5}\,x_2}\right)\right) x_2 & (7.83)\\ &\ge e^{m(\lambda)}\left(f \circ g^{-1}\right)(\csc\phi)\, r \sin\phi. & (7.84) \end{aligned}$$

The last line is entailed from the following observations. First, the functions f, g and g^{-1} are all increasing. With polar coordinates $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, if $(x_1, x_2) \in \partial^+ T_\rho$ then $r \le \sqrt{1 + (3/(\sqrt{5}\,g(0)))^2}\,\rho$. We have $g(0) > \lambda^{-1} \ge 3/2$ (second paragraph of this proof) so that $r < \sqrt{1 + 4/5}\,\rho = 3\rho/\sqrt{5}$. Putting $r < 3\rho/\sqrt{5}$ in (7.69) gives (7.70).

Now, on $x_2 = 3\rho/(\sqrt{5}\, g(x_1/\rho))$, (7.67), (7.84) and (7.61) give

$$\psi_\rho(x_1, x_2) \geq \exp\left(m(\lambda) + \int_{t=-1}^{g^{-1}(\csc\phi)} g(t)\, dt \right) r \sin\phi \tag{7.85}$$

$$= r \sin\phi \exp\left(m(\lambda) + \int_{t=-1}^{g^{-1}(\csc\phi)} dm\left(\frac{1}{g(t)}\right) \right) \tag{7.86}$$

$$= r \sin\phi \exp\left[m(\sin\phi) \right]. \tag{7.87}$$

But using (i) of Lemma 7.2.1, $\exp[m(\sin\phi)] \geq \exp(m_0(\sin\phi)) \csc\phi$ so (7.58) is satisfied and ψ_ρ is the required barrier on T_ρ. ∎

7.3 Phragmén–Lindelöf Principle

These lemmas allow us to prove the following Phragmén–Lindelöf Principle in Π_+. The barrier in Lemma 7.2.2 is extended to n-dimensions and then the Weak Maximum Principle is used.

Theorem 7.3.1 *Let m_0 be any decreasing measurable function, $m_0 : (0,1) \to [0, \infty)$, with $\int_{t=0}^{1} m_0(t)\, dt < \infty$. If $u \in C^2(\Pi_+)$ such that*

$$\Delta u \geq 0 \quad \textit{in } \Pi_+ \tag{7.88}$$

$$\limsup_{x \in \Pi_+,\ x \to x_0} u(x) \leq 0 \quad \textit{for any } x_0 \in \partial \Pi_+ \tag{7.89}$$

$$u = o\left(|x| e^{m_0(\cos\theta)} \right) \quad \textit{as } |x| \to \infty \textit{ in } \Pi_+ \tag{7.90}$$

then $u \leq 0$ in Π_+.

Proof. It suffices to prove the theorem for a function that is larger than the given m_0. In Lemma 7.2.1 let $m \geq m_0$ be the majorant with properties (i)–(vi). Let ψ_ρ be as in Lemma 7.2.2, (7.77), with f given in (7.67), g in (7.66) and ψ in (7.71). Extend T_ρ to be a subset of $\mathbf{R}^n$. Let

$$T_\rho = \left\{x \in \mathbf{R}^n \mid 0 < x_n < \rho\left(\frac{3}{\sqrt{5}}\right) \min_{1 \le i \le n-1}\left(1/g\left(\frac{|x_i|}{\rho}\right)\right); |x_i| < \rho,\ 1 \le i \le n-1\right\}.$$

When $n = 2$, this is the same as in Figure7.1. When $n = 3$, $\overline{T}_\rho$ is a square-based pyramid with base corners $(\pm\rho, \pm\rho, 0)$ and apex $(0, 0, 3\rho\lambda/\sqrt{5})$. The sides, however, may be curved and join the base at a cusp. Write $\partial^+ T_\rho = \partial T_\rho \cap \Pi_+$ and

$$S = \{x \in \overline{T}_\rho \mid x_n = 0,\ |x_i| = \rho \text{ for some } 1 \le i \le n-1\}. \tag{7.91}$$

Note that T_ρ expands to become Π_+ as $\rho \to \infty$. A barrier function is a solution $\Psi_\rho \in C^2(T_\rho) \cap C^0(\overline{T}_\rho \setminus S)$ of

$$\Delta\Psi_\rho \le 0, \quad x \in T_\rho \tag{7.92}$$

$$\Psi_\rho(x) \ge |x| e^{m(\sec\theta)}, \quad x \in \partial^+ T_\rho \tag{7.93}$$

$$\Psi_\rho \ge 0, \quad x \in \overline{T}_\rho \backslash S \tag{7.94}$$

(Ψ_ρ is not defined on S).

Define Ψ_ρ by writing

$$\Psi_\rho(x) = \sum_{i=1}^{n-1} \psi_\rho(x_i, x_n). \tag{7.95}$$

Since ψ_ρ was superharmonic as a function of two variables, Ψ_ρ is superharmonic as a function of n variables in T_ρ. And Ψ_ρ is singular only on S so it is a continuous function in $\overline{T}_\rho \setminus S$. Each ψ_ρ is non-negative in T_ρ and although ψ_ρ does not have a limit as $x \to x_0 \in S$ it is true that as $x \to x_0$ in T_ρ, $\liminf \psi_\rho(x) \ge 0$ for any $x_0 \in S$. Also, ψ_ρ vanishes when $x_n = 0$, $|x_j| < \rho$ $(1 \le j \le n-1)$. To show Ψ_ρ is a barrier function we need only prove (7.93).

Let $x \in T_\rho$ so that $x_n = 3\rho/(\sqrt{5}\, g(x_1/\rho))$, i.e., on the face through $x_1 = \rho$. Consider two right triangles, one with vertices at $P = x = (x_1, \cdots, x_n)$, $Q = y =$

$(x_1, \cdots, x_{n-1}, 0)$ and $R = (0, x_2, x_3, \cdots, x_{n-1}, 0)$, the second with vertices at O (the origin), P and Q. It will be convenient to use n-tuple notation for elements of $\mathbb{R}^n$. Let $r_1 = |PR|$ and θ_1 be the angle PR and the normal to $\partial^+ T_\rho$ through R. As usual, θ will be the angle between OP and the normal at O. We have angles $\angle OPQ = \angle PQR = \pi/2$. Therefore, $x_n^2 = |OP|^2 - |OQ|^2 = |x|^2 - |y|^2 = |PR|^2 - |QR|^2 = r_1^2 - x_1^2$. And, $r_1^2 = x_n^2 + x_1^2 \geq |x|^2 \cos^2\theta$ so that $r_1 \geq |x| \cos\theta$. Also, $\tan\theta = |y|/x_n$ and $\tan\theta_1 = x_1/x_n$ so $\tan\theta_1 = x_1 \tan\theta/|y|$. If x is on the face though $x_1 = \rho$ then the minimum of $x_1/|y|$ occurs when $x_1 = (\rho, 0, \cdots, 0)$ and $y = (\rho, \rho, \cdots, \rho, 0)$ Therefore, $\tan\theta_1 \geq \rho \tan\theta/\sqrt{(n-1)\rho^2} = \tan\theta/\sqrt{n-1}$. And, $\sec^2\theta_1 = 1 + \tan^2\theta_1 \geq 1 + \tan^2\theta/(n-1) \geq \sec^2\theta/(n-1)$ so that $\cos\theta_1 \leq \sqrt{n-1}\cos\theta$. The maximum of $x_1/|y|$ is 1 so $\cos\theta_1 \geq \cos\theta$.

In f, g and ψ let $m \mapsto m(t/\sqrt{n-1})$. It is readily verified that $m(t/\sqrt{n-1})$ satisfies (i)–(vi) of Lemma 7.2.1 (with $m_0 \mapsto m$). Take $A = 3$ in that lemma. Use (7.87) and the bounds on r_1 and $\cos\theta_1$ above to write

$$\psi_\rho(x_1, x_n) \geq r_1 \cos\theta_1 e^{m(\cos\theta_1/\sqrt{n-1})} \tag{7.96}$$

$$\geq |x| \cos^2\theta e^{m(\cos\theta)} \tag{7.97}$$

$$\geq |x| e^{m_0(\cos\theta)}. \tag{7.98}$$

Notice that $\sin\phi_+ = \cos\theta_1$. By symmetry, $\psi_\rho(x_i, x_n) \geq |x| e^{m_0(\cos\theta)}$ when $x \in T_\rho$ on the face through $x_i = \rho$ or $x_i = -\rho$ $(1 \leq i \leq n-1)$. Therefore, if $x \in T_\rho$ on the face through $x_j = \pm\rho$ (for some $1 \leq j \leq n-1$) then $\Psi_\rho(x) \geq \psi_\rho(x_j, x_n) |x| e^{m_0(\cos\theta)}$. Hence, Ψ_ρ is a barrier.

Now, let $\epsilon > 0$. Since $u = o(|x| e^{m_0(\cos\theta)})$ it follows that $u \leq \epsilon \Psi_\rho$ on $\partial^+ T_\rho$ for sufficiently large ρ. Write $w = u - \epsilon \Psi_\rho$. With ρ as above,

$$\Delta w \geq 0, \qquad x \in T_\rho \tag{7.99}$$

$$w \leq 0, \qquad x \in \partial^+ T_\rho \tag{7.100}$$

$$\limsup_{x \in T_\rho, x \to x_0} w \leq 0 \qquad \text{for any } x_0 \in \overline{T}_\rho \cap \partial\Pi_+. \tag{7.101}$$

Note that (7.101) holds in particular when x_0 is in the singular set S (on an edge in the $x_n = 0$ hyperplane). For as $x \to x_0$ in T_ρ

$$\limsup w(x) \leq \limsup u(x) - \epsilon \liminf \psi_\rho(x) \tag{7.102}$$

$$\leq 0. \tag{7.103}$$

The Weak Maximum Principle, §1.2, applied to w shows that $w \leq 0$ in T_ρ. Finally, given $x \in \Pi_+$, let ρ be large enough so that $x \in T_\rho$. Then, using (7.71), (7.77) and (7.79)

$$0 \geq \lim_{\rho\to\infty} [u(x) - \epsilon\psi_\rho(x)] \tag{7.104}$$

$$= u(x) - 20e^{m(\lambda)}(n-1)x_n\,\epsilon \tag{7.105}$$

and ϵ was arbitrary so $u(x) \leq 0$. Hence, $u \leq 0$ in Π_+. ■

Remark 7.3.1 *Condition* (7.90) *may be replaced with the weaker condition*

$$\limsup_{r\to\infty} \left\{ \sup_{\substack{|x|=r \\ x\in\Pi_+}} \left[u(x)|x|^{-1}e^{-m_0(\cos\theta)}\right] \right\} \leq 0. \tag{7.106}$$

Also, if $u \in C^0(\overline{\Pi}_+)$ *then* (7.89) *may be replaced by* $u \leq 0$ *on* $\partial\Pi_+$.

A Phragmén–Lindelöf Principle leads to a uniqueness theorem.

Corollary 7.3.1 *Let* m_0 *be as in the theorem and* $f : \mathbb{R}^{n-1} \to \mathbb{R}$ *be a continuous function satisfying* $\int_{\mathbb{R}^{n-1}} |f(y')| \left(|y'|^n + 1\right)^{-1} dy' < \infty$. *Then any solution to*

$$u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+) \tag{7.107}$$

$$\Delta u = 0 \qquad \text{in } \Pi_+ \tag{7.108}$$

$$u = f \qquad \text{on } \partial\Pi_+ \tag{7.109}$$

$$u = o\left(|x|e^{m_0(\cos\theta)}\right) \qquad (x \in \Pi_+,\ |x| \to \infty) \tag{7.110}$$

is unique and is given by $u = D[f]$.

Proof. By the remark above, (7.89) may be replaced with the condition $u \leq 0$ on $\partial\Pi_+$.

If u_1 and u_2 satisfy (7.107)–(7.110) then $v = u_1 - u_2$ satisfies the premises of the theorem. Hence, $v \leq 0$ in $\overline{\Pi}_+$. The same is true for the function $-v$. Therefore, $v = 0$ and $u_1 = u_2$, giving uniqueness.

Let $m_0(t) = (n-1)\log\sec t$. Then (7.110) becomes $u = o(|x|\sec^{n-1}\theta)$. Corollary 2.2.1 now shows $D[f]$ is the unique solution. ■

7.4 Evolution of a barrier function

The definition of the barrier in Lemma 7.2.2 ((7.61), (7.67) and (7.71)) may seem rather mystical but is the end result of a reasonable chain of thought. In this section we endeavour to give some explanation for the choice of ψ.

Barriers are often constructed on balls, for example, [26] and [55]. Suppose T_ρ in Lemma 7.2.2 was the semicircle $\{x \in \mathbb{R}^2 \mid |x| < \rho, 0 < \phi < \pi\}$. Try to solve (7.56)–(7.58) with equality. The Poisson integral for a circle of radius ρ is

$$u(r,\phi) = \frac{\rho^2 - r^2}{2\pi} \int_{t=0}^{2\pi} \frac{h(t)\,dt}{r^2 - 2r\rho\cos(t-\phi) + \rho^2}. \tag{7.111}$$

The function u is harmonic in the unit disc with boundary values $u(\rho,\phi) = h(\phi)$ (if h is continuous). For the semicircle problem

$$\Delta u = 0, \qquad r < \rho \tag{7.112}$$

$$u(\rho,\phi) = \rho\, h(\phi), \qquad 0 < \phi < \pi \tag{7.113}$$

$$u(r,0) = u(r,\pi) = 0 \tag{7.114}$$

take an odd reflection of h across the x_1-axis, $h(2\pi - \phi) = -h(\phi)$. Then

$$\begin{aligned} u(r,\phi) & \\ &= \frac{(\rho^2 - r^2)\rho}{2\pi} \int_{t=0}^{\pi} h(t) \left(\frac{1}{r^2 - 2r\rho\cos(t-\phi) + \rho^2} - \frac{1}{r^2 - 2r\rho\cos(t+\phi) + \rho^2} \right) dt \\ &= \frac{2}{\pi} (\rho^2 - r^2)\rho^2 r \sin\phi \int_{t=0}^{\pi} \frac{\sin t\, dt}{(r^2 - 2r\rho\cos(t-\phi) + \rho^2)(r^2 - 2r\rho\cos(t+\phi) + \rho^2)} \end{aligned}$$

is the solution to (7.112)–(7.114). If u is to be a barrier function for the upper semicircle of radius ρ then h should be positive and singular at the points $(x_1, x_2) = (\pm\rho, 0)$. Near $t = 0$ the kernel for u behaves like

$$\frac{\sin t}{(r^2 - 2r\rho\cos(t-\phi) + \rho^2)(r^2 - 2r\rho\cos(t+\phi) + \rho^2)} \sim \frac{\sin t}{(r^2 - 2r\rho\cos(\phi) + \rho^2)^2}.$$

It is necessary that $\int_{t=0}^{\pi} h(t) \sin t\, dt$ converge. This is a much more stringent condition on h than (v) imposes on m in Lemma 7.2.1 If $H(\phi) = \exp(m(\sin\phi))$ in (7.57) then $\int_{t=0}^{\pi} \log H(t)\, dt$ must converge, i.e., a much weaker condition than that on h above.

A similar result is obtained for the half ball in $\mathbb{R}^n$. In this case

$$u(x) = \frac{\rho^2 - |x|^2}{n\omega} \int_{\partial B_1} \frac{h(y')\, dy'}{|y' - y|^2} \tag{7.115}$$

where h is a function on the unit ball of $\mathbb{R}^{n-1}$.

In the semicircle problem, the region T_ρ meets the x_1-axis at right angles. By making this angle more acute it is possible to fabricate a barrier function with higher growth. The function

$$u_k(r,\phi) = \frac{\sin(k\phi)}{r^k} \qquad (k > 0) \tag{7.116}$$

is harmonic on $\mathbb{R}^2 \setminus \{0\}$, singular at the origin and positive in the sector $0 < \phi < \pi/k$. As k is increased the growth at the origin increases and the width of this sector decreases. A barrier can be made by adding two translated versions of u_k. Let

$$r_\pm = \sqrt{(\rho \mp x_1)^2 + x_2^2} \qquad \text{and} \qquad \phi_\pm = \arctan\left(\frac{x_2}{\rho \mp x_1}\right). \tag{7.117}$$

The function

$$\psi_\rho(x_1, x_2) = \rho^{k+1}\left[\frac{\sin(k\phi_+)}{r_+^k} + \frac{\sin(k\phi_-)}{r_-^k}\right] \tag{7.118}$$

is a barrier in the triangle with vertices $(\pm\rho, 0)$ and $(0, \rho\pi/(2k))$. The sides make an angle $\pi/(2k)$ with the x_1-axis. The angular growth function can now be taken as $h(\phi) = \csc^k \phi$, allowing arbitrary power growth in $1/\phi$ as $\phi \to 0^+$. Notice that $\log h(\phi) = k \log \csc \phi$ is integrable over $(0, \pi)$ in accordance with the condition on m_0 in Lemma 7.2.2. Full details are in [61].

If T_ρ is allowed to have a cusp at the x_1-axis then a barrier can have growth exceeding power growth at $(\pm\rho, 0)$. This can be achieved by summing the functions u_k. Let

$$\begin{aligned} U_k(r,\phi) &= \sum_{\ell=0}^{\infty} \frac{\rho^{k\ell} u_{k\ell}(r,\phi)}{\ell!} && (7.119)\\ &= \mathcal{I}m\left[\sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(\frac{\rho^k e^{ik\phi}}{r^k}\right)^{\ell}\right] && (7.120)\\ &= \mathcal{I}m\left\{\exp\left[\frac{\rho^k \cos(k\phi)}{r^k}\right]\exp\left[\frac{i\rho^k \sin(k\phi)}{r^k}\right]\right\} && (7.121)\\ &= \exp\left[\frac{\rho^k \cos(k\phi)}{r^k}\right]\sin\left[\frac{\rho^k \sin(k\phi)}{r^k}\right]. && (7.122) \end{aligned}$$

The function U_k is harmonic in $\mathbb{R}^2 \setminus \{0\}$ and vanishes when $\phi = 0$. It is positive for

$$2l\pi < \frac{\rho^k \sin(k\phi)}{r^k} < (2l+1)\pi \quad (l \in \mathbb{Z}). \tag{7.123}$$

Let $k > 1$ and $l = 0$ then $U_k(r,\phi) > 0$ for $r > \rho[\sin(k\phi)/\pi]^{1/k}$ and $0 < \phi < \pi/k$. This region has a cusp at the origin. Let $\gamma_\pm$ be the curves

$$r_\pm = \frac{\rho}{2}\left[\frac{\sin(k\phi_\pm)}{\pi}\right]^{1/k} \tag{7.124}$$

where $r_\pm$ and $\phi_\pm$ are defined in (7.117). we can define T_ρ to be the region bounded by γ_-, γ_+ and the portion of the x_1-axis between $x_1 = -\rho$ and $x_1 = \rho$. There are now cusps at $(\pm\rho, 0)$. And, letting

$$\psi_\rho(x_1, x_2) = \rho\left[U_k(r_+, \phi_+) + U_k(r_-, \phi_-)\right] \tag{7.125}$$

gives a barrier in T_ρ. The boundary behaviour is

$$\psi_\rho(x_1, x_2) \geq A\,r \exp\left(B \csc^{\frac{k}{k+1}} \phi\right) \qquad \text{on} \qquad \partial^+ T_\rho. \tag{7.126}$$

Here A and B are positive constants.

We do not include all the details of this argument as Lemma 7.2.2 includes this result. The purpose here is to motivate the choice of barrier in that lemma.

The function corresponding to m_0 in (7.57) is $\csc^{\frac{k}{k+1}} \phi$, which is integrable over $(0, \pi)$. However, as $k \to \infty$ this function approaches $\csc \phi$ which fails this integrability condition. This is an indication that $\int_{t=0}^{\pi} m(t)\, dt < \infty$ is the correct integrability condition.

The final form of the barrier in Lemma 7.2.2 can now be derived. The barrier is to be singular at $(\pm\rho, 0)$ and T_ρ has a cusp at these points so ϕ must tend to zero as one approaches the singularity. As $\phi \to 0$, we have $x_1 \sim r$ and $x_2 \sim r\phi$. In this same limit,

$$u_k \sim e^{-k \log x_1} \sin\left(\frac{k\, x_2}{x_1}\right) \tag{7.127}$$

$$U_k \sim e^{\rho^k x_1^{-k}} \sin\left(\frac{\rho^k k\, x_2}{x_1^{k+1}}\right). \tag{7.128}$$

Both of these functions are of the form

$$\exp\left(\int_{t=x_1}^{c} f(t)\, dt\right) \sin\left(f(x_1)\, x_2\right) \tag{7.129}$$

where c is a constant (U_k has been scaled by ρ). For small x_2, this is approximately

$$\exp\left(\int_{t=x_1}^{c} f(t)\, dt\right) \left[f(x_1)\, x_2 - \frac{1}{6} f^3(x_1)\, x_2^3\right] \tag{7.130}$$

and this is similar to (7.71).

7.5 Yoshida's Phragmén–Lindelöf Principle

H. Yoshida has a stronger version of the n-dimensional Phragmén–Lindelöf Principle ([69], Corollary 3) . It allows angular blow up as with Wolf's result. To discuss the theorem the following notation is needed. His results are for a cone but we simplify to the case of a half space. If $f \geq 0$ is a measurable function on $\partial^+ B_1 = \partial B_1 \cap \Pi_+$ then let

$$S_f(t) = \left\{ \hat{x} \in \partial^+ B_1 \mid f(\hat{x}) \geq t \right\}. \tag{7.131}$$

For each $t \geq 0$, $S_f(t)$ is a subset of the upper unit ball $\partial^+ B_1$. Its surface area is designated $|S_f(t)|$, which is a decreasing function of t with $|S_f(0)| = n\omega_n/2$. Denote its inverse by T_f. If there is more than one t giving the same value $|S_f(t)|$ then one of them is chosen arbitrarily as the value of $T_f(|S_f(t)|)$. Also, if there is a value $t = t_0$ such that the left and right limits of $|S_f|$ are different at t_0, that is

$$\lim_{t \to t_0^-} |S_f(t)| > \lim_{t \to t_0^+} |S_f(t)|, \tag{7.132}$$

then define $|S_f(t_0)| = \lim_{t \to t_0^-} |S_f(t)|$ and $T_f(s) = t$ for every s satisfying $|S_f(t)| > s \geq \lim_{t \to t_0^+} |S_f(t)|$. Then T_f is a decreasing function on the interval $[0, n\omega_n/2]$. It now follows that $|S_f(t)|$ is the one-dimensional measure of the set $\{s \in [0, n\omega_n/2] \mid T_f(s) \geq t\}$.

Theorem D (Yoshida's Phragmén–Lindelöf Principle): *Let u be subharmonic in Π_+ such that*

$$\limsup_{x \to x_0, x \in \Pi_+} u(x) \leq 0 \qquad \textit{for each } x_0 \in \partial^+ T_\rho. \tag{7.133}$$

Let $f : \partial^+ B_1 \to [0, \infty)$ be measurable so that

$$\int_{s=0}^{n\omega_n/2} s^{-(n-2)/(n-1)} \log^+ T_f(s)\, ds < +\infty \tag{7.134}$$

and $u(x) \le \epsilon|x|f(\hat{x})$ *for any* $\epsilon > 0$ *and* $x = |x|\hat{x} \in \Pi_+$ *such that* $|x| > R(\epsilon)$, *where* $R(\epsilon)$ *is a constant depending only on* ϵ. *Then* $u(x) \le 0$ *for all* $x \in \Pi_+$.

The function $\log^+ s$ is defined to be $\log s$ if $s \ge 1$ and 0 if $0 < s \le 1$. Notice that the theorem applies to subharmonic functions,i.e., u is less than its mean value. See §1.2 in Chapter 1.

When $n = 2$, the term $n\omega_n/2$ becomes π, the exponent on s in (7.134) vanishes and condition (7.134) reduces to Wolf's L^1 condition.

The Phragmén–Lindelöf Principle is a corollary of a theorem that replaces a growth condition in $\mathbf{R}^n$ dependent on angle with a purely radial one.

Theorem E (Yoshida): *Let* u *be subharmonic in* $\mathbf{R}^n$ *and let* $f \ge 0$ *be a measurable function on* ∂B_1 *such that*

$$\int_{s=0}^{n\omega_n} s^{-(n-2)/(n-1)} \log^+ T_f(s)\, ds < +\infty. \tag{7.135}$$

Given $\epsilon > 0$, *if there are constants* $\mu \ge 0$ *and* $R_1(\epsilon)$ *such that* $u(x) \le \epsilon|x|^\mu f(\hat{x})$ *for all* $x \in \mathbf{R}^n$ *with* $|x| > R_1(\epsilon)$ *then there constants* A *and* $R_2(\epsilon)$ *such that*

$$u(x) \le A\epsilon|x|^\mu, \tag{7.136}$$

where R_1 *and* R_2 *depend on* ϵ *only.*

The proof of Theorem E depends, ultimately, on the Mean Value Theorem. Theorem D then follows as a corollary on appeal to the angle-independent Phragmén–Lindelöf Principle of Deny and Lelong, [17].

Yoshida gives an example to show that Theorem D is sharp in the sense that if the exponent $(n-2)/(n-1)$ on s in (7.135) is replaced by any number greater than $(n-2)/(n-1)$ then there are functions f and u satisfying the premises of the theorem and yet $u(x)$ can be positive in Π_+. However, in the example given the function f is singular only in the $\hat{e}_n$ direction, i.e., as θ approaches 0. For the ball $B_R(0)$ we have surface measure given by $dS_n = R\sin^{n-2}\theta\, dS_{n-1}$ where dS_{n-1} is surface measure on the ball of radius R in $\mathbf{R}^{n-1}$ and is independent of θ. Note that for $n > 2$, in the half space, $\sin^{n-2}\theta$ vanishes only as θ approaches 0. Since $\sin^{n-2}\theta \to 0$ as $f \to \infty$ this aids the convergence of $|S_f(t)|$ when integrating near $\theta = 0$.

Suppose f is not singular in the direction $\hat{e}_n$. In this scenario $\sin^{n-2}\theta$ is bounded away from 0 for θ bounded away from 0. Thus we do not get this interplay between $\sin^{n-2}\theta$ and f at the singularity of f and $\sin^{n-2}\theta$ is not a convergence factor in $|S_f(t)|$ as it was in Yoshida's example. Similar remarks apply to the other angular variables in $\hat{e}_n$. Our Theorem 7.3.1 applies only when u has a growth condition as in (7.89) whose angular portion depends only on θ. We state a result comparing Theorem 7.3.1 and Theorem D in this case.

Proposition 7.5.1 *Suppose m_0 satisfies the conditions of Theorem 7.3.1. Let $f = \exp(m_0(\cos\theta))$. Then the condition $\int_{t=0}^{1} m_0(t)\, dt < \infty$ in Theorem 7.3.1 is less restrictive than the condition*

$$\int_{s=0}^{n\omega_n/2} s^{-(n-2)/(n-1)} \log^+ T_f(s)\, ds < +\infty \tag{7.137}$$

in Theorem D if $n > 2$. If $n = 2$ the conditions are the same.

Proof. Let m be a decreasing integrable function on $(0,1]$. For $\hat{x} \in \partial^+ B_1$, let $f(\hat{x}) = \exp(m(\cos\theta))$. Then f is increasing for $\theta \in [0, \pi/2)$. For $t \geq 0$, the set $S_f(t)$ is $\{\hat{x} \in \partial^+ B_1 \mid f(\hat{x}) \geq t\}$. As usual, θ is the azimuthal angle of $\hat{x} \in \partial^+ B_1$. We have $S_f(t) = \partial^+ B_1$ if $0 \leq t \leq e^{m(1)}$. If $t \geq e^{m(1)}$ then $\hat{x} \in S_f(t)$ if and only if $m(\cos\theta) \geq \log t$, i.e.,

$$\theta = \arccos\left(m^{-1}(\log t)\right) \tag{7.138}$$

$$= (m \circ \cos)^{-1}(\log t). \tag{7.139}$$

And, if $0 < \theta_0 < \pi/2$ then

$$\int\limits_{\substack{\partial^+ B_1 \\ \theta > \theta_0}} dS_n = \int\limits_{\partial^+ B_1^{n-1}} dS_{n-1} \int\limits_{\theta=\theta_0}^{\pi/2} \sin^{n-2}\theta \, d\theta = (n-1)\omega_{n-1} \int\limits_{\theta=\theta_0}^{\pi/2} \sin^{n-2}\theta \, d\theta, \tag{7.140}$$

where B_1^{n-1} is the unit ball in $\mathbf{R}^{n-1}$. These results give

$$|S_f(t)| = \begin{cases} \frac{n\omega_n}{2}, & \text{if } 0 \le t \le e^{m(1)} \\ (n-1)\omega_{n-1} \int\limits_{\theta=\theta_0}^{\pi/2} \sin^{n-2}\theta \, d\theta, & \text{if } t \ge e^{m(1)}. \end{cases} \tag{7.141}$$

We have written $\theta_0 = (m \circ \cos)^{-1}(\log t)$. Now,

$$\int\limits_{\theta=\theta_0}^{\pi/2} \sin^{n-2}\theta \, d\theta \sim \frac{\pi}{2} - \theta_0 \qquad \text{as } \theta_0 \to \frac{\pi}{2}. \tag{7.142}$$

So, $|S_f(t)| \sim (n-1)\omega_{n-1}[\pi/2 - (m \circ \cos)^{-1}(\log t)]$ as $t \to \infty$ (since $(m \circ \cos)^{-1}(s) \to \pi/2$ as $s \to \infty$). It now follows that

$$T_f(s) \sim \exp\left\{(m \circ \cos)\left[\frac{\pi}{2} - \frac{s}{(n-1)\omega_{n-1}}\right]\right\} \qquad \text{as } s \to 0^+ \tag{7.143}$$

$$= \exp\left\{(m \circ \sin)\left[\frac{s}{(n-1)\omega_{n-1}}\right]\right\} \tag{7.144}$$

$$\sim \exp\left\{m\left[\frac{s}{(n-1)\omega_{n-1}}\right]\right\} \qquad \text{as } s \to 0^+. \tag{7.145}$$

Now,

$$\int\limits_{s=0}^{n\omega_n/2} s^{-(n-2)/(n-1)} \log^+ T_f(s) \, ds < \infty \tag{7.146}$$

if and only if $s^{-(n-2)/(n-1)} m(s)$ is integrable at the origin. However, the condition on m in Theorem 7.3.1 says that $m(s)$ must be integrable at the origin. Since

$$\frac{n-2}{n-1} = 1 - \frac{1}{n-1} \tag{7.147}$$

$$\ge \frac{1}{2} \qquad \text{for } n \ge 3, \tag{7.148}$$

Theorem D puts a more stringent integrability condition on f than does Theorem 7.3.1. When $n = 2$ these two conditions are the same. ■

Wolf's proof (see below) shows a way to create a barrier ψ_ρ and region T_ρ capable of proving Theorem D in the case that f depends only on θ. A barrier could be constructed by the method we employed in the proof of Theorem 7.3.1 (although Wolf did not use a barrier in his proof). This barrier would be more complicated than the one we have constructed. In proving Theorem 7.3.1, the planar barrier was used to build an n-dimensional barrier and eventually prove the n-dimensional version of the Phragmén–Lindelöf Principle in the case of only θ dependence in the growth condition. It may be possible to use this technique with the more general barrier from the Wolf proof and strengthen Yoshida's Theorem D to the case where m is (essentially) bounded at the origin and positive but otherwise need satisfy only $\int_{t=0}^{1} m(t)\,dt < \infty$. Equivalently, the function f in Theorem D would satisfy $\int_{s=0}^{n\omega_n/2} \log^+ T_f(s)\,ds < +\infty$, that is, the term $s^{-(n-2)/(n-1)}$ can be dropped from the integrand. As seen above, this allows the function f to be more singular so that the condition $u(x) \le \epsilon|x|f(\hat{x})$ is less constraining and yet we obtain the same conclusion.

7.6 Wolf's Phragmén–Lindelöf Principle

The proof of Theorem C is based on constructing a bounded region D in Π_+ that has a cusp at angles where $m(\sin\phi)$ is singular. An explicit conformal map is produced that maps D to the unit disc and allows one to apply the Mean Value Theorem in a disc. Our barrier approach is different from both the work of Yoshida and Wolf because it depends on the Weak Maximum Principle.

Wolf's set D is star-like from the origin and $\overline{D}$ intersects $\overline{\Pi}_+$ in a segment of the x_1-axis containing the origin. It can have a countable number of cusps, at angles between 0 and π, whereas our set T_ρ had at most three cusps and only the two on the x_1-axis were significant. The difference is that Wolf allows $f(\phi)$ to be singular anywhere in $[0, \pi]$ so long as it is integrable and we allowed $m(\sin\phi)$ to be singular only at 0 and π. To extend our result to cover this more general case we would need to construct a new region T_ρ with attendant barrier function ψ_ρ that could be singular at any angle. A possible scheme is the following.

The function ψ_ρ, (7.77), was an odd function of x_2 and vanished when $x_2 = 0$. Suppose T_ρ is extended to negative values of x_2 by a reflection across the x_1-axis. If ψ_ρ were extended to negative values of x_2 as an even function then it would be positive and continuous in the extended version of T_ρ. However, it would not be C^1 across the x_1-axis as its x_2 derivative would have a jump discontinuity at $x_2 = 0$. Consider the Green function for this new region. Write the upper boundary of T_ρ as the curve $x_2 = c_\rho(x_1)$. See (7.80) for the explicit formula. Our reflected version of T_ρ is defined by $|x_2| < c_\rho(x_1)$, for $|x_1| < \rho$. Let (x_1, x_2) be a point in this set. Write the Green function with source point (ξ, η) in the above set as $G(x_1, x_2; \xi, \eta)$. Then

$$\int_{\eta=-c_\rho(x_1)}^{c_\rho(x_1)} G_2(x_1, x_2; x_1, \eta)\, d\eta \tag{7.149}$$

will also have a jump discontinuity at the point $(x_1, 0)$. The subscript denotes partial derivative. It may be possible to add this to ψ_ρ and obtain a C^2 superharmonic function. This idea is due to Beurling ([11]). We would then have an even function with a cusp at $(\rho, 0)$. Rotation about the origin would lead to a region with a cusp at any desired angle. Summing over such functions might produce a barrier able to reproduce Wolf's Theorem C. This could then be extended to n-dimensions as in 7.3.1.

Chapter 8

Conditional convergence

8.1 Conditionally convergent integrals

All of the real integrals appearing so far have been Lebesgue integrals. For a Lebesgue integral to exist it must be absolutely convergent. If a function $f:\mathbb{R} \to \mathbb{R}$ is oscillatory it may happen that the integral of $|f|$ over $\mathbb{R}$ diverges but $\int_{\mathbb{R}} f$ exists as a conditionally convergent integral. If f were continuous (or continuous except on a set of measure zero) this would be the improper Riemann integral $\lim_{L_1,L_2\to\infty} \int_{\xi=-L_1}^{L_2} f(\xi)\,d\xi$. By allowing conditionally convergent integrals we will extend the validity of the Poisson integral to a wider class of functions than was allowed in Chapter 2.

A more general integral than that of Lebesgue or Riemann is the Henstock–Kurzweil integral (also called gauge, generalised Riemann or Riemann complete). This new integral arose from the following defect of the Lebesgue and Riemann integrals. In the Fundamental Theorem of calculus, in order for $\int_a^b f'(x)\,dx = f(a) - f(b)$, it must be assumed that the derivative of f is integrable. In the Lebesgue case this amounts to assuming $f = F'$ almost everywhere on $[a, b]$ for some absolutely continuous function F. The search for a theory that included the "calculus integral" led A. Denjoy (1912) and O. Perron (1914) to new formulations of the integral. (See [34] for original references and a history of Henstock–Kurzweil integration.) Because every differentiable function was the indefinite integral of its derivative, the new integral has been called the *Riemann complete* integral. The integrals of Denjoy and Perron are discussed in [58] and [29]. Unfortunately, the new definitions were rather unwieldy and it required tremendous effort to develop even simple results like integration by parts.

A major breakthrough came 40 years later when J. Kurzweil (1960, [45]) and R. Henstock (1961, [36]) independently gave a powerful new formulation of the integral in terms of Riemann sums. The definition is simple, requiring no measure theory. For a function $f : [a,b] \to \mathbb{R}$, f is *Henstock–Kurzweil integrable*, $\int_a^b f = I$, if and only if

For all $\epsilon > 0$ *there is a function* $\delta : [a,b] \to (0,\infty)$ *such that whenever a* tagged division $\{\xi_i, [x_{i-1}, x_1]\}_{i=1}^n$ *given by*

$$a = x_0 < x_1 < \cdots < x_n = b \qquad \textit{and} \qquad \xi_i \in [x_{i-1}, x_i] \;\; \textit{for each} \;\; i = 1, \cdots, n \tag{8.1}$$

satisfies

$$[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \qquad \textit{for} \qquad i = 1, 2, \cdots, n \tag{8.2}$$

we have

$$\left| \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) - I \right| < \epsilon. \tag{8.3}$$

The *tag* for $[x_{i-1}, x_1]$ is ξ_i. The division is said to be "δ–fine" when (8.2) holds. The definition is easily extended to $\mathbb{R}$, without the need for "improper" integrals. If $b = +\infty$ then we take $\xi_n = +\infty$ and define $f(+\infty) = 0$ and $f(\xi_n)(x_n - x_{n-1}) = 0$. Similarly if $a = -\infty$. There is an extension to $\mathbb{R}^n$ as well. Usual properties such as linearity, integration on sets other than intervals, integration by parts, change of variable, etc. can be proven quite readily. Expository accounts of the Henstock–Kurzweil integral are in [16], [46] and [50].

The integrals of Denjoy and Perron are in fact equivalent to the above definition. The difference between the Henstock–Kurzweil and Riemann integral is that δ, the *gauge*, can now be a function rather than a constant. If f oscillates wildly near x_0 then δ is taken to be small near x_0 and this forces the interval $[x_{i-1}, x_i]$ containing x_0 to be small. A function is Riemann integrable if and only if δ can be taken to be a constant. It is remarkable that the class of Henstock–Kurzweil integrable functions includes the Lebesgue integrable functions. In fact, the Henstock–Kurzweil integral reduces to the Lebesgue integral whenever we have absolute integrability. Thus, we immediately have all the L^p results. Henstock has since shown, [34], that the Henstock–Kurzweil integral can encompass integration over more general sets than $\mathbb{R}^n$ (division spaces) and includes Feynman and Wiener integrals, etc. Because it has wider applicability and is easier to define than the Lebesgue integral, there is a movement to replace the Lebesgue integral with the Henstock–Kurzweil integral in the undergraduate curriculum. In the recent article [8], Bartle presents this case.

One of the important properties of the Lebesgue integral is that integrals over unbounded domains or with unbounded integrand are handled with no special procedures such as must be done with the Riemann integral. Despite its apparent similarity with the Riemann definition, the Henstock–Kurzweil integral has this same property. The function $\sin(x)/x$ is integrable over $[1, \infty]$. And, if $f(x) = x^2 \cos(1/x^2)$ for $x > 0$ and $f(0) = 0$ then $\int_0^1 f' = f(1)$ for the Henstock–Kurzweil integral but f' is neither Riemann nor Lebesgue integrable over $[0, 1]$. The Riemann integral of f' fails to exist since f' is not bounded and the Lebesgue integral does not exist since $|f'|$ is not integrable over $[0, 1]$. Note that we have the improper Riemann integral

$$\lim_{\epsilon \to 0^+} \int_{x=\epsilon}^{1} f'(x)\, dx = \lim_{\epsilon \to 0^+} [f(1) - f(\epsilon)] = f(1). \tag{8.4}$$

However, if f' is changed to be zero on the rationals then this improper Riemann integral no longer exists but the Henstock–Kurzweil integral is unchanged.

With conditionally convergent integrals, the Dominated Convergence Theorem no longer applies. In order to take a limit under the integral sign of an integral that depends on a parameter, some condition other than dominated convergence is needed. There has been considerable work in this direction and it remains an active area of research. Recently, R. Bartle has given necessary and sufficient conditions under which the limit and integration can be interchanged for Henstock–Kurzweil integrals. (See [9] and, for an addendum, [28].) These are, unfortunately, not easy conditions to use in practise. For the Poisson integral, we will develop an *ad hoc* proof of the validity of taking limits under the integral sign. This will be done using the Second Mean Value Theorem for Henstock–Kurzweil integrals.

Second Mean Value Theorem: *Suppose f and g are real-valued functions defined on $[a,b]$, g is monotonic and $\int_a^b f$ exists. Then there exists c in $[a,b]$ such that*

$$\int_a^b fg = g(a)\int_a^c f + g(b)\int_c^b f. \tag{8.5}$$

Note that g is bounded since we are saying $g(a)$ and $g(b)$ are in $\mathbb{R}$. The monotonicity of g then implies g is of bounded variation which shows the existence of $\int_a^b fg$. A proof for bounded intervals is given in [50]. In general, the theorem does not apply on unbounded intervals. However, since g is bounded it has a limit at a and b ($-\infty \leq a < b \leq +\infty$). The theorem then holds with $g(a)$ and $g(b)$ replaced by their respective limiting values. This can be seen by using a change of variables, say $t \mapsto \tan t$, which transforms $[a,b]$ to a finite interval. See [50], p.64, for the change of variables formula in Henstock–Kurzweil integration.

8.2 A conditionally convergent Poisson integral

A simple example will illustrate the importance of allowing conditionally convergent integrals. Consider the function $f(\xi) = \xi\cos\xi$. Let $n = 2$. The Poisson integral of

$|f|$ diverges but $D[f]$ converges conditionally. The same is true with $g(\xi) = \xi \sin \xi$. Both $D[f]$ and $D[g]$ can be evaluated using the residue calculus. Let $w(z) = ze^{iz} = u(x_1, x_2) + i\, v(x_1, x_2)$ where

$$u(x_1, x_2) = \mathcal{R}e\,[w(z)] = e^{-x_2}(x_1 \cos x_1 - x_2 \sin x_1) \tag{8.6}$$

$$v(x_1, x_2) = \mathcal{I}m\,[w(z)] = e^{-x_2}(x_1 \sin x_1 + x_2 \cos x_1) \tag{8.7}$$

and $z = x_1 + i\, x_2$. The function w is analytic so u and v are harmonic. For $R > 0$ let γ_R be the interval $[-R, R]$ of the x_1-axis and let Γ_R be the semicircle $\{Re^{i\phi} \mid 0 < \phi < \pi\}$. Then $C_R = \gamma_R \cup \Gamma_R$ is a simple closed curve in $\overline{\Pi}_+$. Let $z \in \Pi_+$. We have

$$\frac{x_2}{(\zeta - x_1)^2 + x_2^2} = \frac{1}{2i}\left(\frac{1}{\zeta - x_1 - ix_2} - \frac{1}{\zeta - x_1 + ix_2}\right). \tag{8.8}$$

The Cauchy integral representation gives

$$\frac{x_2}{\pi}\int_{C_R} \frac{w(\zeta)\, d\zeta}{(\zeta - x_1)^2 + x_2^2} = \frac{1}{2\pi i}\int_{C_R} \frac{w(\zeta)\, d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{C_R} \frac{w(\zeta)\, d\zeta}{\zeta - \overline{z}} \tag{8.9}$$

$$= w(z) \tag{8.10}$$

$$= ze^{iz} \tag{8.11}$$

since z is inside C_R and $\overline{z}$ is not (or, $\frac{w(\zeta)}{\zeta - z}$ has residue $w(z)$ inside C_R).

An easy estimate now shows

$$\int_{\Gamma_R} \frac{w(\zeta)\, d\zeta}{(\zeta - x_1)^2 + x_2^2} \to 0 \text{ as } R \to \infty. \tag{8.12}$$

Letting $R \to \infty$ in (8.11) and taking real and imaginary parts gives

$$D[f](x_1, x_2) = \frac{x_2}{2\pi}\int_{\xi=-\infty}^{\infty} \frac{\xi \cos \xi\, d\xi}{(\xi - x_1)^2 + x_2^2} = e^{-x_2}(x_1 \cos x_1 - x_2 \sin x_1) \tag{8.13}$$

$$D[g](x_1, x_2) = \frac{x_2}{2\pi}\int_{\xi=-\infty}^{\infty} \frac{\xi \sin \xi\, d\xi}{(\xi - x_1)^2 + x_2^2} = e^{-x_2}(x_1 \sin x_1 + x_2 \cos x_1). \tag{8.14}$$

Note that $u(x_1,0) = D[f](x_1,0) = f(x_1) = x_1 \cos x_1$ and $v(x_1,0) = D[g](x_1,0) = g(x_1) = x_1 \sin x_1$. Thus u and v are classical solutions to the half plane Dirichlet problem with boundary data f and g, respectively.

To see what growth condition v satisfies use polar coordinates and write (abusing notation again)

$$\begin{aligned} v(r,\phi) &= e^{-r\sin\phi}\, r\,[\cos\phi\sin(r\cos\phi) + \sin\phi\cos(r\cos\phi)] && (8.15)\\ &= e^{-r\sin\phi}\, r\sin(\phi + r\cos\phi). && (8.16) \end{aligned}$$

We have

$$\left|\frac{v(r,\phi)}{r}\right| = e^{-r\sin\phi}\,|\sin(\phi + r\cos\phi)| \le 1 \qquad (8.17)$$

so $v(r,\phi) = O(r)$. And, $v(r,\phi) \ne o(r)$. Let $x^{(i)}$ be a sequence with components $x_1^{(j)} = \pi/2 + 2j\pi$ and $x_2^{(j)} = j^{-1}$. On this sequence v approaches $|x^{(j)}|$ so $v(r,\phi) \ne o(r)$.

However, let $h(\phi)$ be any bounded positive function on $(0,\pi)$ that tends to 0 as ϕ tends to 0 or π. Then $v(r,\phi) = o(r/h(\phi))$ as $x \to \infty$ in Π_+. To see this, let $M > 0$ be a bound on h. Given $0 < \epsilon < M$, we show that taking r large enough gives$|v(r,\phi)h(\phi)/r| < \epsilon$. There is no loss of generality in assuming $0 < \phi \le \pi/2$. Let $H(r,\phi) = e^{-r\sin\phi}h(\phi)$. By (8.16) it suffices to show $H(r,\phi) \to 0$. For $x \in \Pi_+$ with $r > \log(M/\epsilon)$, either $r\sin\phi > \log(M/\epsilon)$, in which case $H(r,\phi) < \epsilon$, or $r\sin\phi \le \log(M/\epsilon)$. In this instance we have $0 < \phi \le (\pi/(2r))\log(M/\epsilon)$. Because h has limit 0 as $\phi \to 0$, there is a function $\delta : (0,M] \to (0,\pi/2]$ such that $0 < h(\phi) < \epsilon$ whenever $0 < \phi < \delta(\epsilon)$. To make $H(r,\phi) < \epsilon$, take $r > (\pi/(2\delta(\epsilon)))\log(M/\epsilon)$.

The same order relation holds for $u = D[g]$. These functions exhibit the most mild angular blow up possible. Note also that u and v have exponential decay along every ray from the origin but are still not $o(r)$. Of course, the exponential decay is not uniform in ϕ. This example can be extended to n-dimensions by writing $U(x) = u(x_1, x_n) = D[F](x)$ where now $F(y) = f(x_1)$. As above, the integral can be evaluated explicitly.

8.3 Growth estimates

Conditionally convergent Poisson integrals have many of the same properties as their absolutely convergent counterparts. The growth estimates are the same and they are classical solutions of the Dirichlet problem when the data is continuous. Proving the growth estimate requires interchanging limits and integration. If $\int_{\xi=-\infty}^{\infty} |f(\xi)|(\xi^2+1)^{-1}\,d\xi < \infty$ then $\int_{\xi=-\infty}^{\infty} f(\xi)(\xi^2+r^2)^{-1}\,d\xi \to 0$ as $r \to \infty$. This depends on the Dominated Convergence Theorem which is no longer applicable. Instead we have the following lemma.

Lemma 8.3.1 *If $\int_{\xi=-\infty}^{\infty} f(\xi)(\xi^2+1)^{-1}\,d\xi$ converges then*

$$\lim_{r\to\infty} \int_{\xi=-\infty}^{\infty} \frac{f(\xi)\,d\xi}{\xi^2+r^2} = 0. \tag{8.18}$$

Proof: Write

$$\int_{\xi=-\infty}^{\infty} \frac{f(\xi)\,d\xi}{\xi^2+r^2} = \int_{\xi=-\infty}^{\infty} \frac{f(\xi)}{\xi^2+1} g(\xi)\,d\xi, \tag{8.19}$$

where $g(\xi) = (\xi^2+1)/(\xi^2+r^2)$. We have $g'(\xi) = 2\xi(r^2-1)(\xi^2+r^2)^{-2}$ so $0 \le g(\xi) \le 1$ and g is an increasing function of $|\xi|$ for $r > 1$. By the remark above, the Second Mean Value Theorem applies on any subinterval of $[-\infty, 0]$ or $[0, +\infty]$ with $g(\pm\infty) = 1$.

From the hypothesis, the function $f(\xi)(\xi^2+1)^{-1}$ is integrable over any subinterval of the real line and $\lim_{N\to\infty} \int_{\xi=N}^{\infty} f(\xi)(\xi^2+1)^{-1}\,d\xi = 0$. Hence, given $\epsilon > 0$ there is $N(\epsilon) > 0$ such that $|\int_{\xi=N}^{\infty} f(\xi)(\xi^2+1)^{-1}\,d\xi| < \epsilon/3$. Now apply the Second Mean Value Theorem. There exist numbers c_1 and c_2 such that $0 \le c_1 \le N \le c_2 \le +\infty$ and

$$\int_{\xi=0}^{\infty} \frac{f(\xi)g(\xi)\,d\xi}{\xi^2+1} = g(0)\int_{\xi=0}^{c_1} \frac{f(\xi)\,d\xi}{\xi^2+1} + g(N)\int_{\xi=c_1}^{c_2} \frac{f(\xi)\,d\xi}{\xi^2+1} + g(\infty)\int_{\xi=c_2}^{\infty} \frac{f(\xi)\,d\xi}{\xi^2+1}. \tag{8.20}$$

Let M be the supremum of $|\int_{\xi=\xi_1}^{\xi_2} f(\xi)(\xi^2+1)^{-1}\,d\xi|$ over $0 \le \xi_1 \le \xi_2 \le +\infty$. Then

$$\left| \int_{\xi=0}^{\infty} \frac{f(\xi)g(\xi)\,d\xi}{\xi^2+1} \right| \le \frac{M}{r^2} + \left(\frac{N^2+1}{N^2+r^2}\right) M + \frac{\epsilon}{3}. \tag{8.21}$$

Taking $r \ge \max(\sqrt{3M/\epsilon}, |N^2 - 3(N^2+1)M/\epsilon|^{1/2})$ completes the proof. ■

As in the absolutely convergent case, the conditionally convergent Poisson integral has the following properties.

Proposition 8.3.1 *If $D[f]$ exists for one value of x in Π_+ then it exists on all of Π_+.*

Proof. Suppose $(x_0, y_0) \in \Pi_+$ and $D[f](x_0, y_0)$ converges. Let $(x_1, x_2) \in \Pi_+$ and write $h(\xi) = (\xi^2 - 2\xi x_0 + r_0^2)/(\xi^2 - 2\xi x_1 + r^2)$, where $r_0 = (x_0^2 + y_0^2)^{1/2}$. Then

$$D[f](x_1, x_2) = \frac{x_2}{\pi} \int_{\xi=-\infty}^{\infty} \frac{f(\xi)}{(\xi - x_0)^2 + y_0^2} h(\xi)\,d\xi. \tag{8.22}$$

Also,

$$h'(\xi) = \frac{2\left[(x_0 - x_1)\xi^2 - (r_0^2 - r^2)\xi + x_1 r_0^2 - x_0 r^2\right]}{(\xi^2 - 2\xi x_1 + r^2)^2}. \tag{8.23}$$

If h' has no real roots then h is monotonic. Otherwise, let $\xi_- \le \xi_+$ be the roots of h' and write

$$h_1(\xi) = \begin{cases} h(\xi) - h(\xi_-), & \xi \le \xi_- \\ 0, & \xi \ge \xi_- \end{cases}$$

$$h_2(\xi) = \begin{cases} h(\xi_-), & \xi \le \xi_- \\ h(\xi), & \xi_- \le \xi \le \xi_+ \\ h(\xi_+), & \xi \ge \xi_+ \end{cases}$$

$$h_3(\xi) = \begin{cases} 0, & \xi \le \xi_+ \\ h(\xi) - h(\xi_+), & \xi \ge \xi_+. \end{cases}$$

Each function h_i is bounded, continuous and monotonic. And, $h = h_1 + h_2 + h_3$.

Writing

$$D[f](x_1, x_2) = \frac{x_2}{\pi}\left\{\int_{-\infty}^{\infty} f_0 h_1 + \int_{-\infty}^{\infty} f_0 h_2 + \int_{-\infty}^{\infty} f_0 h_3\right\}, \tag{8.24}$$

the proposition follows since the product of an integrable function and one of bounded variation is integrable. We have written f_0 for $f(\xi)/(\xi^2 - 2\xi x_0 + r^2)$. ■

Proposition 8.3.2 *If $\int_{\xi=-\infty}^{\infty} f(\xi)(\xi^2+1)^{-1}\,d\xi$ converges for a continuous function f then $u = D[f]$ is a solution of*

$$u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+) \tag{8.25}$$

$$\Delta u = 0, \quad x \in \Pi_+ \tag{8.26}$$

$$u = f, \quad x \in \partial\Pi_+. \tag{8.27}$$

A proof is given for the Dirichlet problem in a disc in [10]. We skip the proof and instead concentrate on proving a growth estimate for $D[f]$.

Theorem 8.3.1 *If $\int_{\xi=-\infty}^{\infty} f(\xi)(\xi^2+1)^{-1}\,d\xi$ converges then $D[f](r,\phi) = o(r\csc\phi)$ as $x \to \infty$ in Π_+.*

Proof. Write

$$u(r,\phi) = D[f](r,\phi) = \frac{r\sin\phi}{\pi}\int_{\xi=-\infty}^{\infty} \frac{f(\xi)\,d\xi}{\xi^2 - 2\xi r\cos\phi + r^2}. \tag{8.28}$$

Then

$$\frac{\pi\, u(r,\phi)\sin\phi}{r} = \sin^2\phi \int_{\xi=-\infty}^{\infty} \frac{f(\xi)}{\xi^2+r^2}\,\frac{\xi^2+r^2}{\xi^2 - 2\xi r\cos\phi + r^2}\,d\xi \tag{8.29}$$

$$= \sin^2\phi \int_{\xi=-\infty}^{\infty} \frac{f(\xi)}{\xi^2+r^2}\left[1 + \frac{2\xi r\cos\phi}{\xi^2 - 2\xi r\cos\phi + r^2}\right]d\xi \tag{8.30}$$

$$= \sin^2\phi \int_{\xi=-\infty}^{\infty} \frac{f(\xi)\,d\xi}{\xi^2+r^2} + \int_{\xi=-\infty}^{\infty} \frac{f(\xi)\,g(\xi)\,d\xi}{\xi^2+r^2}, \tag{8.31}$$

where

$$g(\xi) = \frac{2\,\xi\, r\cos\phi\,\sin^2\phi}{\xi^2 - 2\,\xi\, r\cos\phi + r^2}. \tag{8.32}$$

The first integral in (8.31) goes to 0 as $r \to \infty$ by Lemma 8.3.1. Now show the second integral in (8.31) is $o(1)$.

With no loss of generality, take $0 < \phi \le \pi/2$. When $\xi \ge 0$ we have

$$\begin{aligned} 0 \le g(\xi) &\le \frac{(1-\cos\phi)(1+\cos\phi)2\,\xi\, r\cos\phi}{(1-\cos\phi)\left[\frac{(\xi-r)^2}{1-\cos\phi} + 2\xi r\right]} && (8.33) \\ &\le \frac{8\xi r}{(\xi+r)^2} && (8.34) \\ &\le 2. && (8.35) \end{aligned}$$

And,

$$\begin{aligned} g'(\xi) &= \frac{2r\cos\phi\,\sin^2\phi\,(r^2-\xi^2)}{(\xi^2 - 2\,\xi\, r\cos\phi + r^2)^2} && (8.36) \\ &\le 0 \qquad \text{for } \xi \ge r. && (8.37) \end{aligned}$$

Therefore, the maximum of g for $\xi \ge 0$ occurs at $\xi = r$ and is $g(r) = \cos\phi\,(1+\cos\phi)$.

Let

$$p(\xi) = \begin{cases} 0, & 0 \le \xi \le r \\ g(\xi) - \cos\phi\,(1+\cos\phi), & \xi \ge r \end{cases} \tag{8.38}$$

and

$$q(\xi) = \begin{cases} g(\xi), & 0 \le \xi \le r \\ \cos\phi\,(1+\cos\phi), & \xi \ge r. \end{cases} \tag{8.39}$$

The function p is continuous and decreasing and q is continuous and increasing. And, $p(\xi) + q(\xi) = g(\xi)$. Now, write

$$\int_{\xi=0}^{\infty} \frac{f(\xi)\,g(\xi)\,d\xi}{\xi^2+r^2} = \int_{\xi=0}^{\infty} \frac{f(\xi)\,p(\xi)\,d\xi}{\xi^2+r^2} + \int_{\xi=0}^{\infty} \frac{f(\xi)\,q(\xi)\,d\xi}{\xi^2+r^2}. \tag{8.40}$$

Applying the Second Mean Value Theorem as in Lemma 8.3.1 shows that all three integrals in (8.40) tend to zero as r tends to infinity. The same holds true when integrating over the negative real line. The theorem follows. ∎

8.4 Abel's test

Several authors have investigated conditions under which the operations of limit and integration can be interchanged when integrating sequences of functions, i.e., $\lim \int f_n = \int f$ where $\lim f_n = f$. The criteria obtained so far are mostly based on a special type of absolute continuity of the indefinite integral of f_n and are not easy to apply in practise. See [35], [46] and [48]. With the Riemann integral, we can justify taking a limit under an integral sign by proving $\int f_n$ converges uniformly in n. Useful sufficient tests for uniform convergence in the case of conditional convergence are those of Dirichlet and Abel. The conjectured form of Abel's test for Henstock–Kurzweil functions is the following.

Abel's test: *Let $\{f_k\}$ and $\{g_k\}$ be sequences of Henstock–Kurzweil integrable functions on $[a, b]$ $(-\infty \leq a < b \leq +\infty)$. If $\int_{t=a}^{b} f_k(t)\, dt$ converges uniformly for $k \geq 1$, if g_k is uniformly bounded on $[a, b]$ and if $g_k(t)$ is a monotonic function of t for each fixed $k \geq 1$, then $\int_{t=a}^{b} f_k(t) g_k(t)\, dt$ converges uniformly for $k \geq 1$.*

Dirichlet's test is similar. Abel's test is proved in [57] when all functions in the sequences $\{f_k\}$ and $\{g_k\}$ are continuous. The proof does not carry over in an obvious way to Henstock–Kurzweil integrals. Note that we were able to employ the Second Mean Value Theorem in Lemma 8.3.1 because $g(\xi) = (\xi^2+1)/(\xi^2+r^2) \to 1$ as $\xi \to \infty$ for each value of r except $r = \infty$. Since there was only one point where we did not have $g(\xi) \to 1$ pointwise it was easy to isolate this one bad point by integrating in a neighbourhood of infinity. The general case where we have almost everywhere convergence requires more care. We will leave the proof of Abel's test as a goal for the future.

Chapter 9

Further work and conclusions

9.1 Further work

There are several straightforward additions to the work included here.

(i) The method of Theorem 2.2.1 will certainly apply to derivatives of $D[f]$ and we can find estimates under the condition $\int_{\mathbf{R}_{n-1}} |f(y')|^a(|y'|^b + 1)^{-1}\, dy' < \infty$. Similarly with $F_{\lambda,M}[f]$ and its derivatives, once we use $|K_M| \leq d_1\, K\, s^M\,(1+s)^{2\lambda-1}$, $s = |x|/|y'|$, (3.33). This is for $\lambda \geq 1/2$. There was a similar estimate on $|K_M|$ when $0 < \lambda \leq 1/2$.

(ii) The estimates carry through unchanged if we replace the Lebesgue measurable data f by a Borel measure.

(iii) The Poisson integral for the cone

$$K_\alpha = \{x \in \mathbf{R}^n \mid 0 \leq \theta < \alpha\} \qquad (0 < \alpha < \pi) \tag{9.1}$$

can be calculated using the Mellin transform ([15], p. 212). We can obtain estimates as before. The function $r^{-m\pi/\alpha}\sin(m\pi\theta/\alpha)$ is harmonic in $\mathbf{R}^2 \setminus \{0\}$ and vanishes on the boundary of K_α. It can take the place of $r^{-m}\sin(m\theta)$ in the derivation of a barrier

(§7.4) and we can prove a Phragmén–Lindelöf Principle in the cone. Yoshida states his results in [69] for a cone.

(iv) Suppose g is a function on $(0,\infty)$. Then for $u(x) = g(x_n)\int_{\mathbb{R}^{n-1}} f(y')K(\lambda, x, y')\,dy'$ we have the sharp estimate $u(x) = o(g(x_n)\sec^{2\lambda}\theta)$ (under (3.4)). It may happen that u is the solution of a half space boundary value problem. In Cartesian coordinates, the differential operator for such a problem would have to have coefficients that depended only on $|y|$ and x_n. It should be possible to find all such linear operators. For example, there are solutions of form $x_n^{\alpha}|x|^{\beta}$ (and hence of the above type) for the generalised Weinstein equation

$$\Delta u + \frac{p}{x_n}\frac{\partial u}{\partial x_n} + \frac{q}{x_n^2}u = 0, \tag{9.2}$$

where p and q are parameters. Many references to this equation are given in [4] and [47].

(v) With all of the results obtained for $F_{\lambda,M}[f]$, there are analogues for the logarithmic kernel (necessary and sufficient conditions for existence of the modified kernel, M a function of y', expansions at infinity, etc.).

(vi) In the Robin problem, a linear combination of f and its normal derivative are specified on the boundary. There is a solution integral ([27]) and we can repeat the calculations that were done for the Poisson integral.

(vii) The integral representation, (3.23),

$$K_M(\lambda, x, y') = K(\lambda, x, y') \int_{\zeta=0}^{|x|/|y'|} (1 - 2\Theta\zeta + \zeta^2)^{\lambda-1}\Phi_-(\Theta,\zeta)\,\zeta^{M-1}\,d\zeta \tag{9.3}$$

can be used to *define* the modified kernel K_M. Then K_M will be defined for any complex number M and, in particular, for $M \notin \mathbb{Z}$. The Gegenbauer functions are defined through hypergeometric functions when $M \notin \mathbb{Z}$. (They won't be polynomials anymore.) The following result is easy to prove.

Proposition 9.1.1 *Let K_M be defined for $M > 0$ by (9.3) and $K_0 = K$. Then*

$$K_M(\lambda, x, y') = K_{M-1}(\lambda, x, y') - \frac{|x|^{M-1}}{|y'|^{M+2\lambda-1}} C^{\lambda}_{M-1}(\sin\theta\cos\theta') \tag{9.4}$$

holds for $M \geq 1$.

An induction argument now shows that if $M \geq 1$ then

$$K_M(\lambda, x, y') = K_{M_0}(\lambda, x, y') - \sum_{m=0}^{\lfloor M \rfloor - 1} \frac{|x|^{m+M_0}}{|y'|^{m+M_0+2\lambda}} C^{\lambda}_{m+M_0}(\sin\theta\cos\theta'), \tag{9.5}$$

where $0 \leq M_0 = M - \lfloor M \rfloor < 1$. So, for $M \geq 0$, K_M is determined by $K_{\lfloor M \rfloor}$. The growth properties of $F_{\lambda,M}[f]$ will be the same as when M was an integer. These integrals are harmonic but will not take on correct boundary values on $\partial\Pi_+$. The modified kernels are rather like the functions $r^{\pm\alpha}\sin(\alpha\phi)$ and $r^{\pm\alpha}\cos(\alpha\phi)$ in $\mathbb{R}^2$ when $\alpha \in \mathbb{R}$. They may be useful in cones. Further investigation is called for.

(viii) When M is a function of y' we had the growth estimate, Theorem 4.3.2,

$$F_{\lambda,M}[f](x) = o(G(|x|)\sec^{2\lambda}\theta) \qquad (x \in \Pi_+,\ |x| \to \infty), \tag{9.6}$$

where

$$G(|x|) = \int\limits_{|y'|>1} |f(y')|\,|x|^M M^{\mu} |y'|^{-(M+2\lambda)}\,dy' \tag{9.7}$$

and $\mu = 2\lambda - 1$ when $\lambda \geq 1/2$ and $\mu = 0$ when $0 < \lambda \leq 1/2$. It would be interesting to determine the behaviour of $G(x)$ as $|x| \to \infty$. Some further assumption on M would almost certainly be necessary, such as M monotonic increasing.

(ix) Taking distributional data continues the theme of extending the validity of the Poisson integral. For example, if δ is the Dirac distribution then

$$D[\delta](x) = \delta(K(x, y')) = K(x, 0) = \frac{2x_n}{n\omega_n|x|^n}. \tag{9.8}$$

This function is harmonic in $\mathbf{R}^n \setminus \{0\}$ and vanishes on $\partial\Pi_+$ (only tangentially at the origin). In $\mathbf{R}^2$, taking data as derivatives of δ gives the family of solutions $r^{-m}\sin(m\phi)$ for $m = 0, 1, 2, \cdots$. Notice that each function satisfies the Poisson integral growth condition $o(r/\sin\phi)$.

Suppose f is a continuous function on $\mathbf{R}$ and g is a monotonic differentiable function on $\mathbf{R}$ with one real root. Then

$$(\delta \circ g)(f) = \frac{f(g^{-1}(0))}{g'(g^{-1}(0))}. \tag{9.9}$$

This formula appears, for example, in [39]. Now, let $g(t) = 1/t$. Proceeding formally, we have

$$(\delta \circ g)(f) = \int_{|t|>1} \delta\left(\frac{1}{t}\right) f(t)\,dt \tag{9.10}$$

$$= \int_{t=-1}^{1} \delta(t) f\left(\frac{1}{t}\right) \frac{dt}{t^2} \tag{9.11}$$

$$= \lim_{t\to 0} t^{-2} f\left(\frac{1}{t}\right). \tag{9.12}$$

If we now let f be the Poisson kernel (with x_1 and x_2 as parameters) then

$$(\delta \circ g)(K) = \lim_{t\to 0} \frac{x_2}{\pi t^2[(x_1 - 1/t)^2 + x_2^2]} \tag{9.13}$$

$$= \frac{x_2}{\pi}. \tag{9.14}$$

Carrying out a similar calculation with the derivative of δ gives

$$(\delta' \circ g)(K) = \frac{2x_1x_2}{\pi}. \tag{9.15}$$

In a similar manner we can generate the harmonic polynomials that vanish on $\partial\Pi_+$. Notice that neither of the above functions satisfies the Poisson integralgrowth condition. Consider $\mu = \delta \circ g$ as a measure. Then for each interval $I_i = [i, i+1] \cup [-(i+1), -i]$ we have $\mu(I_i) = 0$ so $\sum_{i=0}^{\infty} \mu(I_i) = 0$. But, $\cup_{i=0}^{\infty} I_i = \mathbf{R}$ and $\mu(\mathbf{R}) = +\infty$. Therefore, μ is not countably additive. There is more to be done here!

(x) The spherical harmonics expansion has an analogue for other equations and regions. What made this expansion particularly appealing was that the Fourier series (in Gegenbauer polynomials) was also in ascending powers of $|x|$. This will not be the case for other operators or regions but uniqueness theorems may still be obtainable.

(xi) We can look at conditionally convergent integrals when $n > 2$ and also try to prove Abel's test.

More nebulous extensions are as follows.

(i) The barrier method of proving the Phragmén–Lindelöf Principle has been extended to uniformly elliptic operators by D. Gilbarg and E. Hopf in [26] and [37]. A barrier for the Laplacian capable of proving the Phragmén–Lindelöf Principle with growth $o(r)$ in the half plane (i.e., without angular blow up at the boundary) is

$$\psi_\rho(x_1, x_2) = \frac{2\rho}{\pi}\left[\arctan\left(\frac{x_2}{\rho - x_2}\right) + \arctan\left(\frac{x_2}{\rho + x_2}\right)\right]. \tag{9.16}$$

Compare with (7.118). Gilbarg and Hopf then defined a barrier for the elliptic equation by writing $\Psi = h(\psi_\rho)$. They were able to find a function h that would make Ψ the desired elliptic barrier by solving a differential inequality for h. Being able to do this depended on the simple form of ψ. It may now be possible to take the barrier from Theorem 7.3.1 and repeat this procedure to prove a Phragmén–Lindelöf Principle for uniformly elliptic operators with blow up allowed at the boundary. It was pointed out before that Wolf has a barrier for angular blow up but it seems too complicated for this task. However, the barrier from Theorem 7.3.1 may be simple enough to carry out the method of Gilbarg and Hopf. In fact, this was the original motivation for the barrier constructions in Chapter 7.

(ii) We have considered only pointwise growth, both for our estimates of the Poisson integraland in the Phragmén–Lindelöf Principle. A next step would be to investigate some sort of mean growth for the Poisson integral. In [67], Yoshida has used the Nevanlinna norm,

$$\mathcal{N}[u](|x|) = \int_{\partial^+ B_1} u(\hat{x}|x|) \cos\theta \, d\omega_{n-1}, \tag{9.17}$$

to obtain a Phragmén–Lindelöf Principle. And, in [56], A.Yu. Rashkovskii also obtains Phragmén–Lindelöf Principles with an integral growth condition.

(iii) There are some very general representation formulas for solutions of elliptic problems. For example, [12] and [2]. The techniques used here might have some applicability in determining growth for these elliptic problems.

9.2 Conclusions

The goals of this thesis were to obtain growth estimates for the Poisson integral, to extend the validity of this integral and to consider uniqueness for the half space problem.

By using the half space kernel $K(\lambda, x, y')$, we were able to estimate the Dirichlet and Neumann solutions under the most general convergence conditions for which these integrals converge. The technique was application of the Dominated Convergence Theorem after some algebraic manipulation of the kernel. An integral representation of the modified kernels led to a growth estimate here as well. Also, for any given continuous function it is possible to construct a convergent modified Poisson integral. A growth estimate was found in this case. An important new definition of sharpness for a growth condition was introduced. This allowed us to prove that our growth estimates were the best possible. This portion of the thesis is largely independent of existing theory, our pointwise estimates being under more general conditions than considered by most authors. There is good reason to believe that techniques used here will be applicable to other equations with explicitly known solutions.

The construction of a barrier function proceeded by first considering the basic singular solution $u(r,\phi) = r^{-m}\sin(m\phi)$ in the plane. The barrier was singular at the boundary of the half space and gave a Phragmén–Lindelöf Principle that allowed angular blow up in accord with the growth estimate for the Poisson integral. An outcome of this was a uniqueness theorem that showed the Poisson integralwas the unique solution under a growth condition that was not unduly constraining. The spherical harmonics expansion gave a similar result. Methods for extending these results to other elliptic equations have been proposed.

The applicability of the Poisson integralwas further increased by considering it as a Henstock–Kurzweil integral. The growth estimate was the same as for absolutely convergent integrals.

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