On the Crossing Numbers of Complete Graphs

by

Shengjun Pan

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2006

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis we prove two main results. The Triangle Conjecture asserts that the convex hull of any optimal rectilinear drawing of K_n must be a triangle (for $n \geqslant 3$). We prove that, for the larger class of pseudolinear drawings, the outer face must be a triangle. The other main result is the next step toward Guy's Conjecture that the crossing number of K_n is $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$. We show that the conjecture is true for $n = 11, 12$; previously the conjecture was known to be true for $n \leqslant 10$. We also prove several minor results.

Acknowledgements

Special thanks to my supervisor, Bruce Richter, for all the help with my thesis. I learned how to do research from him, from which I will benefit in my future study. I also learned how to smile and laugh from him, from which I will benefit in my whole life. Also thanks to Jim Geelen, Chris Calzonetti, Graeme Kemkes, Bill Cunningham, Oswin Aichholzer, Gelasio Salazar for providing various help. Thanks to all my friends for their encouragement. I would like also thank my parents for understanding my choice of studying abroad.

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Chapter 1

Introduction

This thesis is concerned with aspects of the rectilinear crossing number and the crossing number of the complete graph K_n in the plane. In particular, we consider the number of extreme vertices of an optimal rectilinear drawing of K_n and we use a computer to calculate the crossing number of K_{11}, which naturally implies the crossing number of K_{12}.

A *graph* $G = (V, E)$ consists a pair of finite sets V and E, where each element in E is an unordered pair of elements in V. Each element $v \in V$ is called a *vertex* of G. Each element $e = \{x, y\} \in E$ is called an *edge* of G, and x, y are *endpoints* of e.

We also use $V(G)$ and $E(G)$ to denote V and E when more than one graph is concerned. For simplicity, we also use xy to represent the edge $\{x, y\}$.

Although a graph is not a geometrical object, it is very natural to represent a graph in the plane as points and simple curves between them. A *drawing* D of a graph G in the plane is the union of the images of $|E| + 1$ injective maps

$$\begin{aligned} \varphi_V : &\quad V \to \mathbb{R}^2, \\ \varphi_e : &\quad (0,1) \to \mathbb{R}^2,\ e \in E, \end{aligned}$$

such that

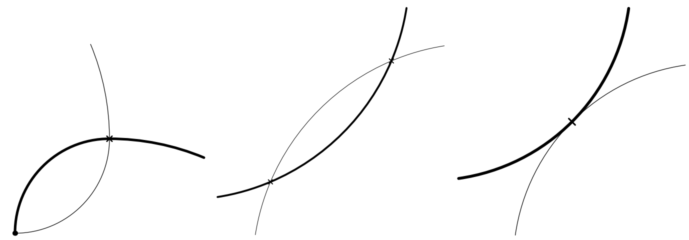


Figure 1.1: Forbidden crossings in a good drawing

1) for each $e = xy \in E$, φ_e is an embedding, in the topological sense, for which

$$\lim_{t \to 0+} \varphi_e(t) = x, \lim_{t \to 1-} \varphi_e(t) = y, \text{ and}$$

2) for any $v \in V$ and any $e \in E$, $\varphi(v) \notin \varphi_e((0, 1))$.

The *crossing number* of a graph G, denoted as $cr(G)$, is the minimum number of edge crossings over all drawings of G in the plane. A graph G is *planar* if and only if $cr(G) = 0$. Garey and Johnson [15] showed that determining the crossing number is an NP-complete problem.

We also use $cr(D)$ to denote the number of crossings in a drawing D.

A *rectilinear drawing* , or *straight line drawing*, is a drawing where the representation $\varphi_e((0, 1))$ of each edge $e \in E(G)$ is a straight-line segment (excluding two endpoints) in the plane. The *rectilinear crossing number* of a graph G, denoted as $\overline{cr}(G)$, is the minimum number of edge crossings over all rectilinear drawings of G in the plane. Evidently, $\overline{cr}(G) \geqslant cr(G)$.

A drawing is *good* if

1) any two edges incident to the same vertex don't cross each other,

2) any two edges cross at most once, and

3) and any two edges are not tangent to each other.

Figure 1.1 shows the three types of crossings forbidden in a good drawing. Obviously, any rectilinear drawing is a good drawing.

1.1 Crossing number

The origin of crossing number is the "Turán's brick factory problem" in 1944, first introduced by Paul Turán [33]. The problem, mathematically, is to find the crossing number of the complete bipartite graph $K_{m,n}$.

For a general graph G, there is a lower bound on $cr(G)$:

Theorem 1.1.1 (Leighton [23], Ajtai, Chvátal, Newborn and Szemerédi [6]). *For any graph G with n vertices and $e \geqslant cn$ edges,*

$$cr(G) \geqslant \frac{c-3}{c^3}\frac{e^3}{n^2},$$

where the constant factor $(c-3)/c^3$ achieves its maximum at $c = 4.5$.

More work in crossing numbers has focused on particular graphs. We will introduce the progress on $cr(K_{m,n})$, $cr(K_n)$ and $\overline{cr}(K_n)$.

Zarankiewicz [36] conjectured that:

Conjecture 1 (Zarankiewicz's Conjecture). *For any positive integers m and n,*

$$cr(K_{m,n}) = Z(m,n),$$

where

$$Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

As described by Richter and Thomassen in [28], a drawing of $K_{m,n}$ with $Z(m,n)$ crossings can be obtained as follows: place m vertices along the x-axis, with $\lfloor m/2 \rfloor$ on the positive half and $\lceil m/2 \rceil$ on the negative half. Place n vertices along the y-axis, with $\lfloor n/2 \rfloor$ on the positive half and $\lceil n/2 \rceil$ on the negative half. Draw a straight segment from each vertex on the x-axis to each vertex on the y-axis. This such drawing has $Z(m,n)$ crossings and so implies that $Z(m,n)$ is an upper bound for $cr(K_{m,n})$.

Zarankiewicz's Conjecture has been verified, for $\min\{m,n\} \leqslant 6$, by Kleitman [21] and, for the special cases $7 \leqslant m \leqslant 8$, $7 \leqslant n \leqslant 10$, by Woodall [35].

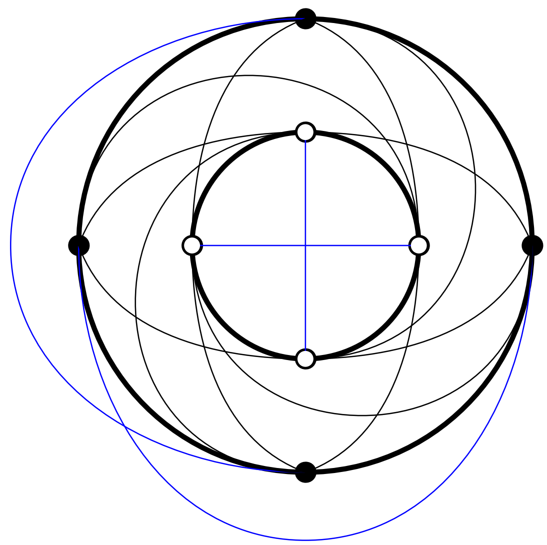


Figure 1.2: A cylindrical drawing of K_8

de Klerk, Maharry, Pasechnik, Richter and Salazar [12] showed that, for $m \geqslant 9$,

$$\lim_{n\to\infty} \frac{cr(K_{m,n})}{Z(m,n)} \geqslant \alpha \frac{m}{m-1},$$

where $\alpha = 0.83$. A recent paper [13] improved α to 0.8594.

Guy [18] initiated the hunt for $cr(K_n)$. He conjectured that:

Conjecture 2 (Guy's Conjecture)**.** *For any positive integer n,*

$$cr(K_n) = Z(n),$$

where

$$Z(n) = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

He gave proofs for $n \leqslant 10$ in [19].

Richter and Thomassen [28] gave a detailed description of a *cylindrical drawing* of K_n, for each even n, with exactly $Z(n)$ crossings. The drawing in Figure 1.2 is a cylindrical drawing of K_8. Deleting any vertex and all the edges incident to this vertex from a cylindrical drawing of K_n, n even, gives a drawing of K_{n-1} with $Z(n-1)$ crossings. Hence, for any n, $Z(n)$ is an upper bound for $cr(K_n)$.

In the same paper they proved that, for any $n \geqslant 3$,

$$\frac{cr(K_{2n})}{\binom{2n}{4}} \geqslant \frac{cr(K_{n,n})}{\binom{n}{2}^2}.$$

They also showed that, as $n \to \infty$, $cr(K_n)/\binom{n}{4}$ and $cr(K_{n,n})/\binom{n}{2}^2$ are nondecreasing and are at most 1, which implies that both $\lim_{n\to\infty} cr(K_n)/Z(n)$ and $\lim_{n\to\infty} cr(K_{n,n})/Z(n,n)$ exist, and

$$\lim_{n\to\infty} \frac{cr(K_n)}{Z(n)} \geqslant \lim_{n\to\infty} \frac{cr(K_{n,n})}{Z(n,n)}.$$

This relation between the crossing numbers of complete graphs and complete bipartite graphs can be used to give an asymptotic lower bound for $cr(K_n)$:

Theorem 1.1.2.

$$\lim_{n\to\infty} \frac{cr(K_n)}{Z(n)} \geqslant \lim_{n\to\infty} \frac{cr(K_{n,n})}{Z(n,n)} \geqslant \alpha = 0.8594.$$

Proof. First, we use an analogous technique in [28] to prove that, for any positive integers M, N, m, n with $M \geqslant m, N \geqslant n$,

$$\frac{cr(K_{M,N})}{\binom{M}{2}\binom{N}{2}} \geqslant \frac{cr(K_{m,n})}{\binom{m}{2}\binom{n}{2}}. \tag{1.1}$$

Let X, Y be the bipartite sets of $K_{M,N}$ with $|X| = M, |Y| = N$. Let D be an optimal drawing of $K_{M,N}$. Then D contains

$$\binom{M}{m}\binom{N}{n}$$

drawings of $K_{m,n}$ with m vertices in one bipartite set chosen from X, and n vertices in the other bipartite set chosen from Y. Note that each crossing in D is also a crossing in

$$\binom{M-2}{m-2}\binom{N-2}{n-2}$$

drawings of $K_{m,n}$. Therefore

$$cr(K_{M,N}) \geqslant \frac{cr(K_{m,n}) \cdot \binom{M}{m}\binom{N}{n}}{\binom{M-2}{m-2}\binom{N-2}{n-2}},$$

which can be rewritten as (1.1).

Now, fixing m and letting $M = m$ in (1.1), we obtain that, for $N \geqslant n$,

$$\frac{cr(K_{m,N})}{\binom{N}{2}} \geqslant \frac{cr(K_{m,n})}{\binom{n}{2}}.$$

Obviously $cr(K_{m,n}) \leqslant \binom{m}{2}\binom{n}{2}$, so $\lim\limits_{n\to\infty} cr(K_{m,n})/\binom{n}{2}$ exists.

Fixing m and letting $N = n$ in (1.1), we obtain that, for $N \geqslant n$,

$$\frac{cr(K_{n,n})}{\binom{n}{2}^2} \geqslant \frac{cr(K_{m,n})}{\binom{m}{2}\binom{n}{2}},$$

which implies that

$$\lim_{n\to\infty} \frac{cr(K_{n,n})}{\binom{n}{2}^2} \geqslant \lim_{n\to\infty} \frac{cr(K_{m,n})}{\binom{m}{2}\binom{n}{2}}.$$

Therefore,

$$\lim_{n\to\infty} \frac{cr(K_{n,n})}{Z(n,n)} \geqslant \lim_{n\to\infty} \frac{cr(K_{m,n})}{Z(m,n)} \cdot \frac{2\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor}{\binom{m}{2}}.$$

As mentioned before, for $m \geqslant 9$,

$$\lim_{n\to\infty} \frac{cr(K_{m,n})}{Z(m,n)} \geqslant \alpha \frac{m}{m-1}.$$

Thus we have

$$\lim_{n\to\infty} \frac{cr(K_{n,n})}{Z(n,n)} \geqslant \alpha \frac{m}{m-1} \cdot \frac{2\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor}{\binom{m}{2}}.$$

Let $m \to \infty$. Then

$$\lim_{n\to\infty} \frac{cr(K_{n,n})}{Z(n,n)} \geqslant \alpha.$$

Finally,

$$\lim_{n\to\infty} \frac{cr(K_n)}{Z(n)} \geqslant \lim_{n\to\infty} \frac{cr(K_{n,n})}{Z(n,n)}$$
$$\geqslant \alpha = 0.8594.$$

1.2 Rectilinear crossing number of K_n

Unlike the crossing number of K_n, for general n, there is no conjectured formula or conjectured optimal drawings for $\overline{cr}(K_n)$. Guy [19] gave $\overline{cr}(K_n)$ for $n \leqslant 9$. Except for $n = 8$ where $\overline{cr}(K_8) = 19$ and $cr(K_8) = 18$, for any $n \leqslant 9$, $\overline{cr}(K_n) = cr(K_n)$. Brodsky, Durocher and Gethner [9] proved that $\overline{cr}(K_{10}) = 62$, which is greater than $cr(K_{10}) = 60$.

The latest progress on $\overline{cr}(K_n)$ is made by Aichholzer, Aurenhammer and Krasser [3, 4]. By enumerating *abstract order types* on computers, they determined $\overline{cr}(K_n)$ for n up to 17.

The rectilinear crossing number has a surprising connection with *Sylvester's four-point problem* (Sylvester 1865). Let R be an open set in the plane with a finite Lebesgue measure. Sylvester's four-point problem asks for the probability $q(R)$ that four points which are chosen independently uniformly at random in R form a convex quadrilateral. Let $q_* = \inf_R q(R)$. In 1994, more than 100 years after Sylvester, Scheinerman and Wilf [29] proved that

$$q_* = \lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}}.$$

1.2.1 Constructions for upper bounds

To date, searching for better upper bounds for $\overline{cr}(K_n)$ has been based on constructions of rectilinear drawings with few crossings. Singer suggests a recursive construction [30, 34] of rectilinear drawings D_n of K_n, where $n = 3^k$. For $k = 1$, draw K_n as a triangle. To construct D_n, $n = 3^k$, we first *flatten* the constructed drawing $D_{n/3}$, so

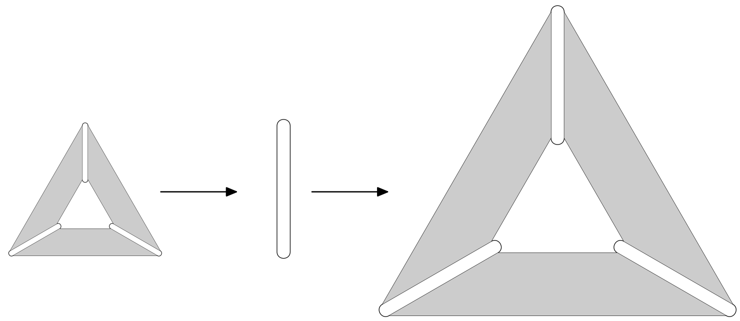


Figure 1.3: Recursive construction

that the vertices of $D_{n/3}$ are *almost on a line*. Then take three copies of flattened $D_{n/3}$ in triangular positions and add new edges, as shown in Figure 1.3. Then

$$cr(D_n) = 3cr(D_{n/3}) + \Delta_n,$$

where Δ_n is the number of new crossings. A routine calculation shows that

$$cr(D_n) = \frac{5}{312}n^4 - \frac{1}{8}n^3 + \frac{7}{24}n^2 - \frac{19}{104}n.$$

An argument entirely analogous to that in [28] by Richter and Thomassen shows that $\overline{cr}(K_n)/\binom{n}{4}$ is nondecreasing and is at most 1. Therefore

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}}$$

exists. Thus the recursive construction implies that

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \leqslant \lim_{n\to\infty} \frac{cr(D_n)}{\binom{n}{4}} \approx 0.3846\,.$$

Brodsky, Durocher and Gethner [10] improved the recursive construction by *sliding* the three copies of *flattened* $D_{n/3}$, as shown Figure 1.4, where a is defined in terms of n, and represents the number of vertices that still dock on the same side of the other two copies of flattened $D_{n/3}$. Each copy of flattened $D_{n/3}$ is not actually broken, only translated; it is drawn as two parts for counting. Then by recalculating the number of

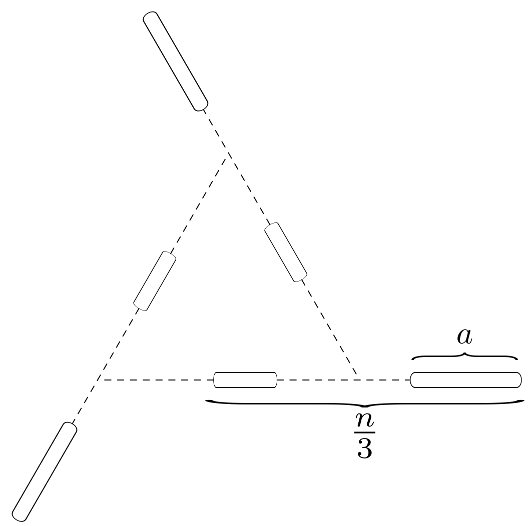


Figure 1.4: Sliding flattened $D_{n/3}$'s

new crossings, we have the number of crossings in D_n:

$$cr(D_n, a) = \frac{139}{4212}n^4 - \frac{23}{216}n^3 a + \frac{1}{6}n^2 a^2 + O(n^3).$$

This is minimized when $a = a_0 = 23n/72 - 1/24$. Calculations show that

$$cr(D_n, a_0) = \frac{6467}{404352}n^4 + O(n^3).$$

Thus

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \leqslant \lim_{n\to\infty} \frac{cr(D_n, a_0)}{\binom{n}{4}} \approx 0.3838\,.$$

With assistance of computers, Aichholzer *et al.* [3] found many rectilinear drawings of K_n with few crossings by enumerating *abstract order types.* Following Singer's recursive construction [30], as discussed in Subsection 1.2.1, by choosing a drawing D_{39} with 29737 crossing, instead of K_3, as the base case, they obtained that

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \leqslant 0.380891\,,$$

which already improves the previous results without sliding flattened drawings, which has been done in [10].

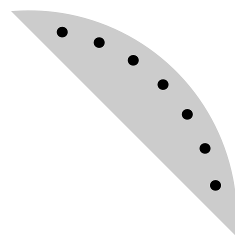


Figure 1.5: A lens

They made further improvements by using *lens replacement*. The basic idea is that, given a rectilinear drawing D_n of K_n, we replace each vertex with a configuration of c points, properly placed, to get a rectilinear drawing D_{nc} of K_{nc}. Then

$$\lim_{c\to\infty} \frac{\overline{cr}(D_{nc})}{\binom{nc}{4}}$$

is an upper bound for $\lim\limits_{n\to\infty} \overline{cr}(K_n)/\binom{n}{4}$.

Given a rectilinear drawing D_n, n is even, replace each vertex v_i with a point configuration C_i, $i = 1, 2, \cdots, n$. For each C_i, choose two parallel lines as close as possible, such that C_i is contained in the strip consisting of the two lines and the region between them. Denote such a strip as $\sigma(C_i)$. We say that a replacement satisfies the *halving property* if, for each C_i, the strip $\sigma(C_i)$ avoids all configurations but C_i and halves them into two groups of sizes $n/2$ and $n/2 - 1$.

A *lens* is a configuration of points on a (small portion of a) semicircle, as shown in Figure 1.5. They showed that

Theorem 1.2.1 (Theorem 5 in [3])**.** *Given a rectilinear drawing D_n, for even n and fixed c, lenses (of proper orientations) are the best choice as configurations for replacement if each configuration has size c and satisfies the halving property, as shown in Figure 1.6.*

By choosing a drawing D_{36} with 21191 crossings and letting $c \to \infty$, they obtained that:

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \leqslant 0.380858\,,$$

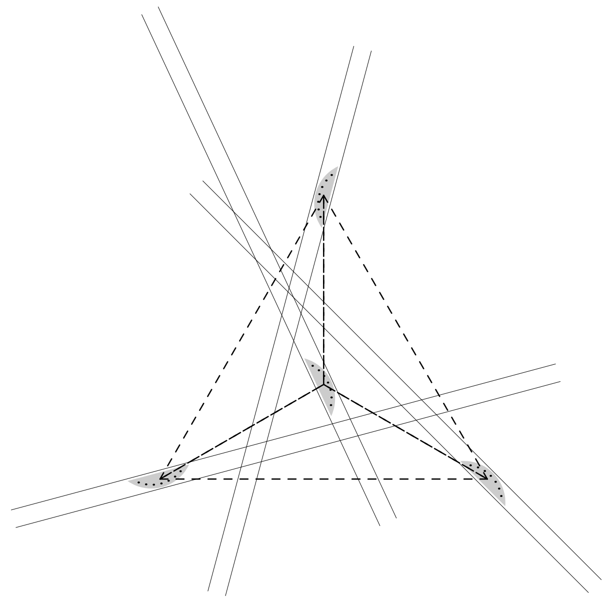


Figure 1.6: Lens replacement

which is slightly better than the bound from recursive replacement in the drawing D_{39}.

Further improvements are made by choosing different sizes of lenses as configurations. As they pointed out, even for the drawing D_{36}, it is not easy to optimize the problem with 35 variables (36 lens sizes with fixed sum). To reach tractability, they fixed certain multiplicative factors between the sizes, which results a problem with 8 variables. This gave an additional improvement:

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \leqslant 0.380739\,.$$

In [3] they suggested that it might be well possible to gain further improvement by splitting of lenses, e.g., the method of sliding flattened drawings as done in [10].

In a recent paper Aichholzer and Krasser [4], using the same idea, gave a new upper bound 0.38058. They are working on the realizability of an abstract order type (see [3]) for $n = 96$ points, which would give a bound of 0.38047.

1.2.2 Computing lower bounds

Circular sequences can be used to encode point sets in the plane, which gives a way to estimate $\overline{cr}(K_n)$ by using circular sequences. A *circular sequence* $\Pi = \left(\pi_0, \pi_1, \cdots, \pi_{\binom{n}{2}}\right)$ on n elements is a sequence of permutations of the set $\{1, 2, \cdots, n\}$, where

$$\pi_0 = (1, 2, \cdots, n), \pi_{\binom{n}{2}} = (n, n-1, \cdots, 1)$$

and any two successive permutations differ in exactly two adjacent positions.

More details of the connection between $\overline{cr}(K_n)$ and circular sequences are explained in Chapter 3.

By manipulating circular sequences, Lovász, Vesztergombi, Wagner and Welzl [24] proved that

$$\overline{cr}(K_n) \geqslant \left(\frac{3}{8} + 10^{-5}\right)\binom{n}{4} + O(n^3).$$

Since

$$\lim_{n\to\infty} \frac{cr(K_n)}{\binom{n}{4}} = \frac{3}{8} \lim_{n\to\infty} \frac{cr(K_n)}{Z(n)} \leqslant \frac{3}{8},$$

the asymptotic lower bound Lovász *et al.* gave implies that, for n large enough, $\overline{cr}(K_n)$ is different from $cr(K_n)$.

Based on the same idea of circular sequences, by modeling and solving a digraph optimization problem, Balogh and Salazar [8] improved the asymptotic lower bound to

$$0.37553\binom{n}{4} + O(n^3),$$

which is the largest lower bound known to date. Balogh, Leaños and Salazar have shown that, for $n \geq 10$, $\overline{cr}(K_n) > cr(K_n)$ (personal communication).

1.2.3 Triangle Conjecture

It has been conjectured (for example, see [9]) that:

Conjecture 3 (Triangle Conjecture)**.** *For $n \geqslant 3$, the convex hull of any optimal rectilinear drawing of K_n must be a triangle.*

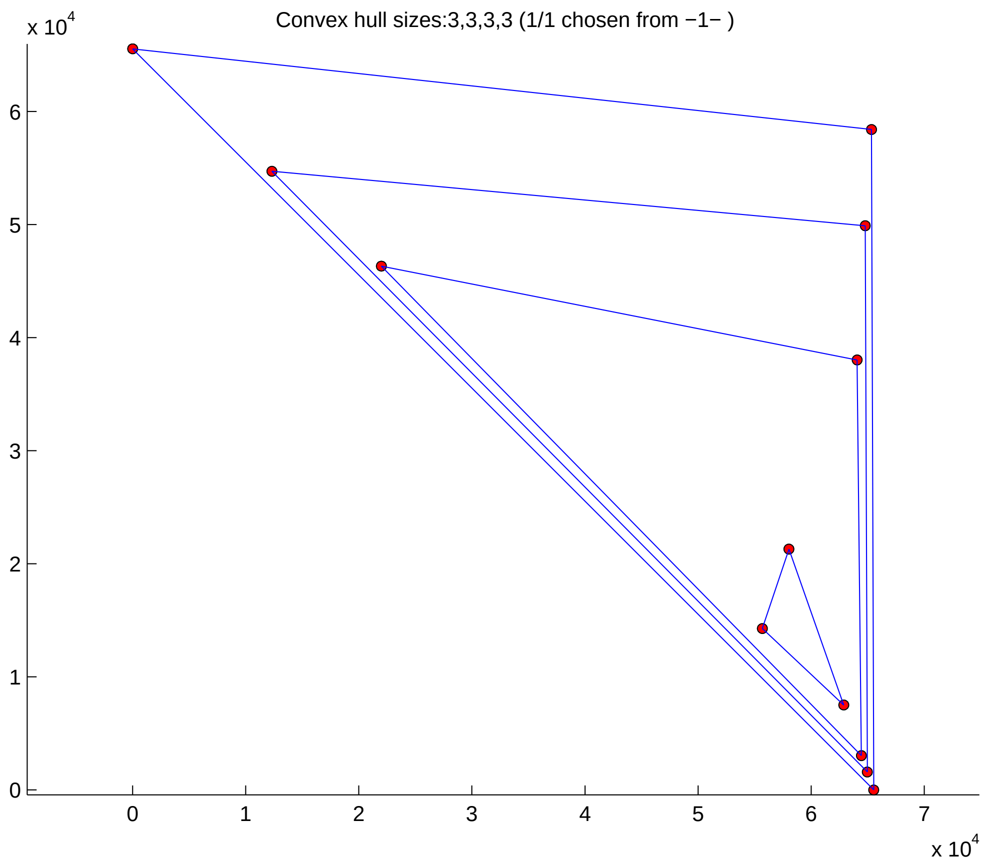


Figure 1.7: An optimal rectilinear drawing of K_{12} with edges partially drawn

Recently, a proof was announced by Aichholzer, Orden, and Ramos [5]. Before the announcement, Aichholzer and Krasser [4] had found all the optimal rectilinear drawings of K_n, for n up to 16. They verified that the convex hull of any such drawing is indeed a triangle.

An optimal rectilinear drawing of K_{12} is shown in Figure 1.7 with only some of the edges drawn. The coordinates of the vertices are obtained from the web page [2].

1.3 Applications of crossing numbers

Székely [32] gave a very nice introduction to various applications of crossing numbers.

When a graph is not planar, it is natural to ask "how far" a non-planar graph is from being a planar graph. Crossing number is a good concept for this measurement.

In computer science, Leighton's interest in crossing numbers was motivated by VLSI. In [23] he used the crossing number to set lower bound for the VLSI layout area of the graph. Before this paper, the relevance of crossing number for engineering was well known already [11].

Székely [31] used Theorem 1.1.1 to give a new proof for the Szemerédi-Trotter theorem, which tells how many incidences can be among n points and m straight lines in the plane. In the same paper he also gave simple proofs, by using the same idea of crossing numbers, to two classic Erdős problems.

Crossing numbers have other applications in discrete geometry, number theory, extremal graph theory and etc. Please refer to Székely [32] for a more detailed introduction to the applications of crossing numbers.

1.4 Contents in this thesis

In Chapter 2, we will first give a rough asymptotic upper bound on the number of extreme vertices in any optimal rectilinear drawing of K_n; a better upper bound will be obtained in Chapter 3. Then we will give a new proof that, for $n \leqslant 16$, the Triangle Conjecture holds. In the last section we will introduce an isomorphic transformation of a drawing of K_n we found when attempting to prove the Triangle Conjecture.

In Chapter 3, we will introduce circular sequences, which are closely related to the rectilinear crossing number of K_n. We will prove that the circular sequence version of the Triangle Conjecture is true! Then we will use circular sequences to get a better asymptotic upper bound on the number of extreme vertices in any optimal rectilinear drawing of K_n.

In Chapter 4, we will give an algorithm for generating new good drawings from a set of known good drawings. We use the algorithm to search for all the optimal drawings of K_{11}, which leads to a proof of Guy's Conjecture for $n = 11, 12$. Thanks

to Jim Geelen for providing the opt.math server for running codes. Thanks to Chris Calzonetti and Graeme Kemkes for help with computing and programming problems.

In Appendix A, we will prove that, for any $n \geqslant 4$, the planar graph of any good drawing of K_n is 3-connected, which has an application in Chapter 4. This result also settles affirmatively an open problem in [10] by Brodsky, Durocher and Gethner. We will further prove that, for $n \geqslant 5$, the planar graph of any good drawing of K_n is 4-connected; they are obviously not 5-connected.

Chapter 2

Warm Up

The Triangle Conjecture says that the convex hull of any optimal rectilinear drawing of K_n must be a triangle. In this chapter we give some natural attempts towards this conjecture. Although they do not lead to substantial results, they may still have value in future work.

In the first section, we estimate how big the convex hull is. In the second section, we give a lower bound on the number of crossings in any rectilinear drawing which does not have a triangular convex hull. In the last section, we introduce an isomorphic transformation found when we tried to adjust a rectilinear drawing of K_n.

2.1 Number of extreme vertices

Let D be an optimal rectilinear drawing of K_n. In this section we prove, by transforming D into a general drawing, that when n is large enough there are fewer than $0.3033n$ extreme vertices, i.e., vertices on the boundary of the convex hull. In Chapter 3 a better upper bound $0.1912n$ is obtained by using circular sequences.

Definition 2.1.1. *Let D be a rectilinear drawing of K_n with vertices in general position, i.e., no three vertices are collinear. Adopting the notations in [9], the* first hull *of D is the convex hull, denoted by $H_1(D)$. The i*-th hull, *denoted by $H_i(D)$, is the*

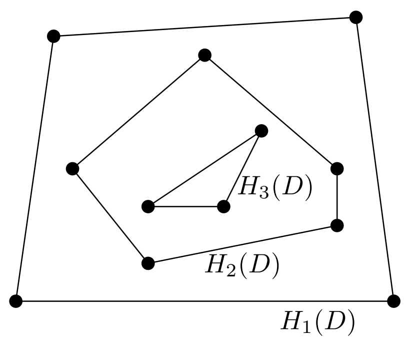


Figure 2.1: Hulls of a drawing

convex hull of the sub-drawing induced by all the vertices strictly inside $H_{i-1}(D)$ *(see Figure 2.1). Each i-th hull is called a* convex layer *of* D.

Let $\partial_i D$ *be all the vertices on the boundary of* $H_i(D)$*, and let* $h_i(D) := |\partial_i D|$*. An* extreme vertex *is a vertex in* $\partial_1 D$.

Definition 2.1.2. *Let* v *be a vertex in a drawing of a graph* G*. We say* v *is* responsible *for* m *crossings, or has* responsibility m*, if there are* m *crossings in total on the edges incident to* v.

Definition 2.1.3. *In a rectilinear drawing* D *of* K_n*, a* diagonal *is an edge which lies in the interior of* $H_1(D)$ *with both endpoints in* $\partial_1(D)$.

Lemma 2.1.1. *Let* D *be a rectilinear drawing of* K_n*. Let* $k = h_1(D)$*. Then each vertex inside* $H_1(D)$ *is responsible for at least*

$$p(k) := \frac{1}{4}k\left(\left\lfloor\frac{k-1}{2}\right\rfloor\left\lfloor\frac{k-3}{2}\right\rfloor + \left\lceil\frac{k-1}{2}\right\rceil\left\lceil\frac{k-3}{2}\right\rceil\right)$$

crossings with the diagonals.

Proof. Let v be a vertex inside $H_1(D)$. Fix $u \in \partial_1 D$. The edges uw, $w \in \partial_1 D$, triangulate the interior of $H_1(D)$. Therefore there exist $u_1, u_2 \in \partial_1 D$ such that v is inside the triangle $\triangle uu_1u_2$ (see Figure 2.2). Let k_1, k_2 be the numbers of vertices in $H_1(D)$ on each side of the line through u and v. The number of crossings for which v

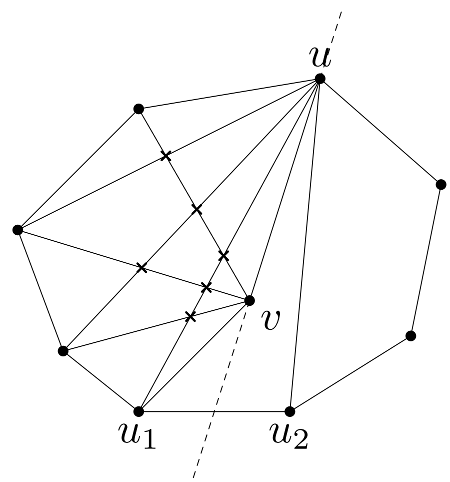


Figure 2.2: Crossings on the diagonals which v is responsible for

is responsible on the diagonals incident to u is

$$\begin{aligned} cr(u) &= (0+1+2+\cdots+k_1-1)+(0+1+2+\cdots+k_2-1) \\ &= \frac{1}{2}k_1(k_1-1)+\frac{1}{2}k_2(k_2-1) \\ &= \frac{1}{2}\left((k_1^2+k_2^2)-(k_1+k_2)\right) \\ &= \frac{1}{2}\left(\frac{1}{2}(k_1+k_2)^2+\frac{1}{2}(k_1-k_2)^2-(k_1+k_2)\right). \end{aligned}$$

Note that $k_1+k_2=k-1$. To get a lower bound, $(k_1-k_2)^2$ is minimized when $k_1=\lfloor (k-1)/2\rfloor, k_2=\lceil (k-1)/2\rceil$ or vice versa. Then we have a lower bound

$$\frac{1}{2}\left(\left\lfloor\frac{k-1}{2}\right\rfloor\left\lfloor\frac{k-3}{2}\right\rfloor+\left\lceil\frac{k-1}{2}\right\rceil\left\lceil\frac{k-3}{2}\right\rceil\right).$$

Let u range over all the vertices in $\partial_1(D)$. Then each such crossing is counted twice. Thus the total number of crossings on the diagonals which v is responsible for is

$$\frac{1}{2}\sum_{u\in\partial_1(D)} cr(u) \geqslant \frac{1}{4}k\left(\left\lfloor\frac{k-1}{2}\right\rfloor\left\lfloor\frac{k-3}{2}\right\rfloor+\left\lceil\frac{k-1}{2}\right\rceil\left\lceil\frac{k-3}{2}\right\rceil\right).$$

□

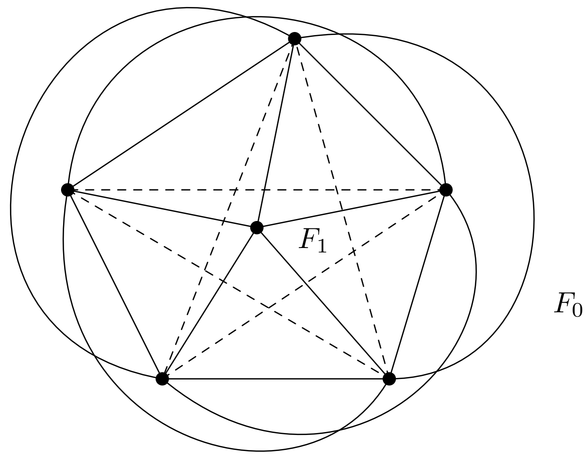


Figure 2.3: Flipping the diagonals

Lemma 2.1.2. *Let $D_{n,k}$ be a rectilinear drawing of K_n with $h_1(D) = k$. Then*

$$cr(D_{n,k}) \geqslant cr(K_n) + (n-k)p(k),$$

where $p(k)$ is as defined in Lemma 2.1.1.

Before we prove Lemma 2.1.2, we note that Guy [19] proved that the boundary of a drawing which is both rectilinear and optimal, in the sense of general crossing number, must be a triangle. He proved it by turning all the diagonals from inside to outside of the boundary, which creates a new drawing with strictly fewer crossings. We use a similar argument here.

Proof of Lemma 2.1.2. Let F_0 be the infinite face of $D_{n,k}$, and let F_1 be the interior of $H_1(D)$. At first flip the diagonals, i.e., move all the diagonals from F_1 to F_0, as shown in Figure 2.3. This may be done as follows:

1) Delete a point p in F_1 without affecting the drawing, i.e., $p \in F_1$, and $p \notin D_{n,k}$.

2) Topologically there is a homeomorphism f from $F_1 \setminus \{p\}$ to F_0 keeping ∂F_0 fixed. Replace all the diagonals with their images under f.

Let D' be the new drawing. Since each inner vertex is responsible for at least $p(k)$ crossings on the diagonals,

$$\begin{aligned} cr(D_{n,k}) &\geqslant cr(D') + (n-k)p(k) \\ &\geqslant cr(K_n) + (n-k)p(k). \end{aligned}$$

□

Theorem 2.1.3. *For fixed n, let*

$$e_n = \max\{h_1(D) \,|\, D \textit{ is an optimal rectilinear drawing of } K_n\}.$$

Then

$$\overline{\lim_{n\to\infty}} \frac{e_n}{n} < 0.3033.$$

Proof. Let D_n be an optimal rectilinear drawing of K_n with e_n extreme vertices. By Lemma 2.1.2,

$$cr(D_n) \geqslant cr(K_n) + (n-e_n)p(e_n).$$

Since D_n is optimal, $cr(D_n) = \overline{cr}(K_n)$. So

$$\overline{cr}(K_n) \geqslant cr(K_n) + (n-e_n)p(e_n). \tag{2.1}$$

In Chapter 1 on Page 5 we mentioned that

$$\lim_{n\to\infty} \frac{cr(K_n)}{Z(n)} \geqslant \alpha = 0.8594. \tag{2.2}$$

Aichholzer and Krasser [4] proved that

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \leqslant \beta = 0.38058\,. \tag{2.3}$$

Then (2.1), (2.2) and (2.3) together give

$$\begin{aligned} \alpha Z(n) + (n-e_n)p(e_n) &\leqslant \beta\binom{n}{4}, \text{ as } n \to \infty, \\ \Longrightarrow \quad \frac{\alpha}{64} + \frac{1}{8}\lambda^3(1-\lambda) &\leqslant \frac{\beta}{24}, \text{ where } \lambda := \overline{\lim_{n\to\infty}} \frac{e_n}{n}, \\ \Longrightarrow \quad \lambda^4 - \lambda^3 + \left(\frac{\beta}{3} - \frac{\alpha}{8}\right) &\geqslant 0. \end{aligned}$$

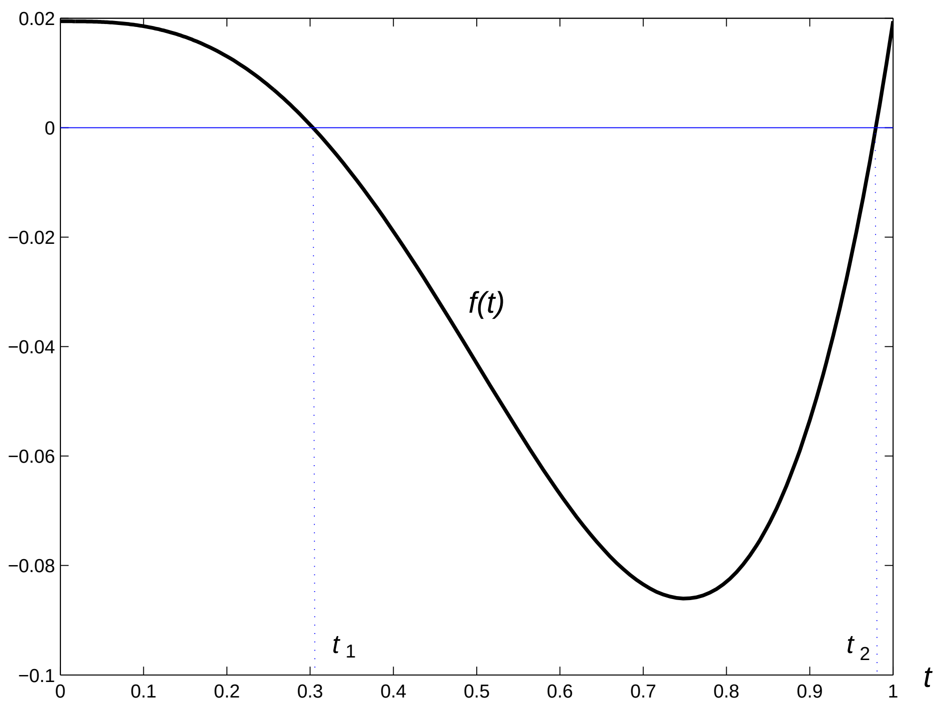


Figure 2.4: $f(t) = t^4 - t^3 + \left(\frac{\beta}{3} - \frac{\alpha}{8}\right)$

Let

$$f(t) = t^4 - t^3 + \left(\frac{\beta}{3} - \frac{\alpha}{8}\right).$$

We used MATLAB to solve the equation $f(t) = 0$. There are two solutions

$$t_1 = 0.303278\ldots,\ t_2 = 0.979306\ldots,$$

in the interval $[0, 1]$, as shown in Figure 2.4. Note that $f(0) > 0$, hence

$$0 \leqslant \lambda \leqslant t_1,\ \text{or}\ t_2 \leqslant \lambda \leqslant 1.$$

On the other hand, if we only consider the crossings on the diagonals,

$$cr(D_n) \geqslant \binom{e_n}{4}.$$

So

$$\begin{aligned}\binom{e_n}{4} &\leqslant cr(D_n)\\ &= \overline{cr}(K_n)\\ &\leqslant \beta\binom{n}{4}, \text{ as } n\to\infty.\end{aligned}$$

Therefore,

$$\lambda = \overline{\lim_{n\to\infty}}\frac{e_n}{n} \leqslant \sqrt[4]{\beta} = 0.7854\cdots.$$

Thus it is impossible that $\lambda \geqslant t_2$. Hence

$$\overline{\lim_{n\to\infty}}\frac{e_n}{n} = \lambda \leqslant t_1 < 0.3033,$$

as claimed. □

2.2 Recursive lower bounds

The Triangle Conjecture says that the convex hull of any optimal rectilinear drawing of K_n must be a triangle. With assistance of computers, Aichholzer and Krasser [4] determined $\overline{cr}(K_n)$ for n up to 17. They also found, for n up to 16, all the optimal rectilinear drawings of K_n, which are available as binary files on Aichholzer's web page [2]. They claimed that the convex hulls of these optimal rectilinear drawings are all triangles, which supports the Triangle Conjecture.

In this section, for $n \leqslant 16$, we use only the rectilinear crossing numbers of K_n, without using their optimal drawings, to verify the Triangle Conjecture. To do this, we first give a lower bound, in a recursive form, on the number of crossings in any rectilinear drawing which does not have a triangular convex hull.

Definition 2.2.1. *For any integers n and k with $3 \leqslant k \leqslant n$, we let:*

$$\begin{aligned}\mathcal{D}_{n,k} &:= \{D:\ D \text{ is a rectilinear drawing of } K_n,\ \text{and } h_1(D)=k\};\\ \mathcal{D}_{n,\geqslant k} &:= \cup_{i=k}^{n}\mathcal{D}_{n,i};\text{ and}\\ cr(\mathcal{D}) &:= \min_{D\in\mathcal{D}} cr(D),\ \text{ where } \mathcal{D} \text{ is a set of drawings.}\end{aligned}$$

Lemma 2.2.1. *For any integers n and k with $3 \leqslant k < n$ and $n \geqslant 5$,*

$$cr(\mathcal{D}_{n,k}) \geqslant \frac{(n-k)\, cr(\mathcal{D}_{n-1,k}) + k\, cr(\mathcal{D}_{n-1,\geqslant k-1})}{n-4}$$

Proof. Let $D \in \mathcal{D}_{n,k}$ be such that $cr(D) = cr(\mathcal{D}_{n,k})$. Consider all the sub-drawings D' of K_{n-1} in D, by removing one vertex v and all the edges incident to v.

1) If v is inside $H_1(D)$, then $D' \in \mathcal{D}_{n-1,k}$. There are $n-k$ such sub-drawings.

2) If $v \in \partial_1(D)$, then there are at least $k-1$ vertices in $\partial_1(D')$, hence $D' \in \mathcal{D}_{n,\geqslant k-1}$. There are k such sub-drawings.

Note that each crossing of D is a crossing in $n-4$ sub-drawings of K_{n-1} in D. Hence

$$\begin{aligned} cr(\mathcal{D}_{n,k}) &= cr(D) \\ &= \frac{\sum_{D'} cr(D')}{n-4}, \text{where } D' \text{ ranges over all the drawings of } K_{n-1} \text{ in } D, \\ &\geqslant \frac{(n-k)\, cr(\mathcal{D}_{n-1,k}) + k\, cr(\mathcal{D}_{n-1,\geqslant k-1})}{n-4}. \end{aligned}$$

□

Theorem 2.2.2. *Let $n \geqslant 5$. Then*

$$cr(\mathcal{D}_{n,\geqslant 4}) \geqslant \frac{(n-2)\, cr(\mathcal{D}_{n-1,\geqslant 4}) + 2\overline{cr}(K_{n-1})}{n-4}$$

Proof. Since a rectilinear drawing of K_n with n extreme vertices has the maximum number of crossings, for any k with $3 \leqslant k \leqslant n$,

$$cr(\mathcal{D}_{n,k}) \leqslant cr(\mathcal{D}_{n,n}).$$

Then

$$cr(\mathcal{D}_{n,\geqslant 4}) = cr(\cup_{k=4}^{n} \mathcal{D}_{n,k}) = cr(\cup_{k=4}^{n-1} \mathcal{D}_{n,k}).$$

For $5 \leqslant k \leqslant n-1$,

$$\begin{aligned} cr(\mathcal{D}_{n-1,k}) &\geqslant cr(\mathcal{D}_{n-1,\geqslant 4}), \\ cr(\mathcal{D}_{n-1,\geqslant k-1}) &\geqslant cr(\mathcal{D}_{n-1,\geqslant 4}). \end{aligned}$$

Hence by Lemma 2.2.1

$$\begin{aligned} cr(\mathcal{D}_{n,k}) &\geqslant \frac{(n-k)\,cr(\mathcal{D}_{n-1,k}) + k\,cr(\mathcal{D}_{n-1,\geqslant k-1})}{n-4} \\ &\geqslant \frac{n\,cr(\mathcal{D}_{n-1,\geqslant 4})}{n-4} \\ &= \frac{(n-2)cr(\mathcal{D}_{n-1,\geqslant 4}) + 2cr(\mathcal{D}_{n-1,\geqslant 4})}{n-4} \\ &\geqslant \frac{(n-2)cr(\mathcal{D}_{n-1,\geqslant 4}) + 2\overline{cr}(K_{n-1})}{n-4}. \end{aligned} \tag{2.4}$$

For $k = 4$, let $D \in \mathcal{D}_{n,4}$ be such that $cr(D) = cr(\mathcal{D}_{n,4})$. Let v_1, v_2, v_3, v_4 be the four vertices in $\partial_1(D)$, where v_1v_3, v_2v_4 are diagonals. Let $D - v_i$ be the sub-drawings of K_{n-1} in D by removing v_i and all edges incident to v_i, for $i = 1, 2, 3, 4$ respectively. Since $n \geqslant 5$, there is at least one vertex v inside $H_1(D)$. Then v is either inside the triangles $\triangle v_1v_2v_4$ or $\triangle v_3v_2v_4$. So $h_1(D - v_3) > 3$ or $h_1(D - v_1) > 3$, as shown in Figure 2.5. Similarly $h_1(D - v_2) > 3$ or $h_1(D - v_4) > 3$. Hence there are at most 2 sub-drawings of K_{n-1} which have triangular convex hulls. So

$$\begin{aligned} cr(\mathcal{D}_{n,4}) &\geqslant \frac{(n-2)\,cr(\mathcal{D}_{n-1,4}) + 2\,cr(\mathcal{D}_{n-1,\geqslant 3})}{n-4} \\ &\geqslant \frac{(n-2)\,cr(\mathcal{D}_{n-1,\geqslant 4}) + 2\,\overline{cr}(K_{n-1})}{n-4}. \end{aligned} \tag{2.5}$$

Combine (2.4) and (2.5), then we have the desired inequality. □

We use Theorem 2.2.2 and the rectilinear crossing numbers of K_n, for $n \geqslant 16$, obtained by Aichholzer and Krasser [4] to compute a lower bound for $cr(\mathcal{D}_{n,\geqslant 4})$, as shown in Table 2.1. The results in the table imply that

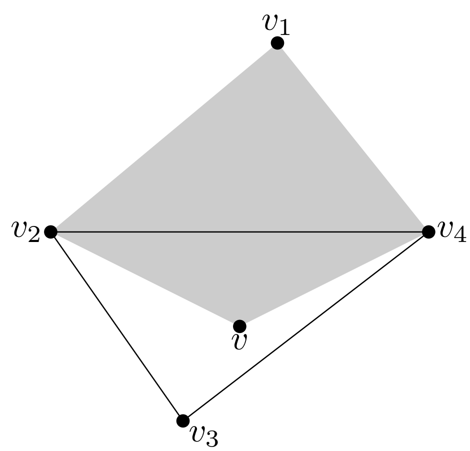


Figure 2.5: Either $h_1(D - v_3) > 3$ or $h_1(D - v_1) > 3$

n	$\overline{cr}(K_n)$	$cr(D_{n,\geqslant 4})$
4	0	1
5	1	3
6	3	7
7	9	$14 \to 15^*$
8	19	27
9	36	46
10	62	74

n	$\overline{cr}(K_n)$	$cr(D_{n,\geqslant 4})$
11	102	114
12	153	168
13	229	241
14	324	335
15	447	455
16	603	606
17	798	792

Table 2.1: Lower bounds for $cr(\mathcal{D}_{n,\geqslant 4})$

*Due to the parity argument in [22] by Kleitman, for each odd n, the numbers of crossings in any good drawings of K_n have the same parity.

n	non-isomorphic drawings	$h_1(D), h_2(D), \cdots$
3	1	3
4	1	3, 1
5	1	3, 2
6	1	3, 3
7	3	3, 3, 1
		3, 4
8	2	3, 3, 2
9	10	3, 3, 3
10	2	3, 3, 4
11	374	3, 3, 3, 2
		3, 3, 4, 1
		3, 3, 5

n	non-isomorphic drawings	$h_1(D), h_2(D), \cdots$
12	1	3, 3, 3, 3
13	4534	3, 3, 3, 4
		3, 3, 3, 3, 1
14	20	3, 3, 3, 4, 1
		3, 3, 3, 5
15	16001	3, 3, 3, 4, 2
		3, 3, 3, 5, 1
		3, 3, 3, 3, 3
		3, 3, 3, 6
16	36	3, 3, 3, 3, 4
		3, 3, 3, 3, 3, 1

Table 2.2: Size sequences of convex layers

Corollary 2.2.3. *Any optimal rectilinear drawing of K_n must have a triangular convex hull, for n up to 16.*

By using MATLAB to analyze each optimal drawing D from Aichholzer's web page [2], we can determine all the possible size sequences of convex layers, i.e., $h_1(D), h_2(D), \cdots$. Results from our calculation are shown in Table 2.2.

As a conclusion of this section, we pose a conjecture concerning the convex layers:

Conjecture 4. *For $n \geqslant 3$, in any optimal rectilinear drawing there are at least $\lfloor n/3 \rfloor - 1$ convex layers, of which the first $\lfloor n/3 \rfloor - 2$ must be triangles.*

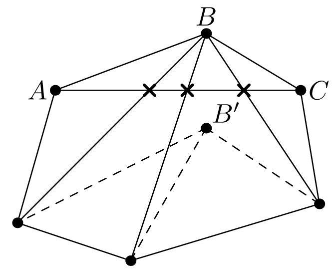


Figure 2.6: Pulling B to B' across AC

2.3 An isomorphic transformation

The following problem is a special case of the Triangle Conjecture.

Problem. *Let D be a rectilinear drawing of K_n with at least 4 extreme vertices, and A, B, C be three consecutive extreme vertices, i.e., AB and BC are two sides of the outer face. If there is no vertex inside the triangle $\triangle ABC$, can we prove that D is not optimal?*

One possible way is to transform D into another rectilinear drawing with strictly fewer crossings. We tried to pull vertex B across the line AC without increasing the number of crossings, as shown in Figure 2.6. Then at least $n-3$ crossings disappear. However, it is very difficult to estimate the number of new crossings.

The *pulling* is more likely to succeed if B is much closer to AC, compared with other vertices. This gives us the idea of pulling A, B, C away from the other vertices. In this section, we give an isomorphic transformation to pull one extreme vertex, or two consecutive extreme vertices, away from the other vertices.

Lemma 2.3.1. *Let D be a rectilinear drawing D of graph G. Given a point K in the infinite face of D, there is a transformation T of the extended plane such that $T(K) = \infty$ and $T(D)$ is a rectilinear drawing isomorphic to D.*

Proof. Suppose D is in the plane π. Let S be a hemisphere with the south pole

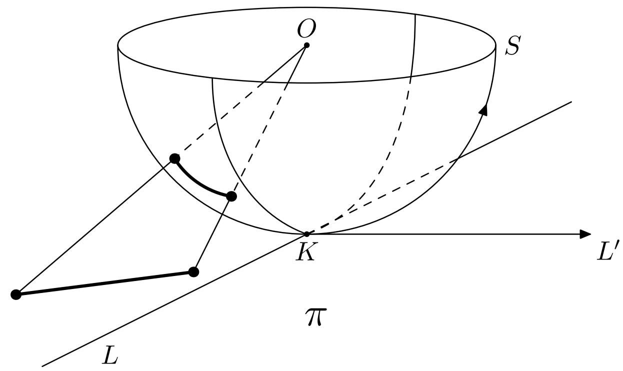


Figure 2.7: Transformation: projection and rotation

tangent to π, so that K is on the south pole, as shown in Figure 2.7.

Let $f : \pi \to S$ be the projection, such that, for any point $P \in \pi$, $f(P) \in S$ is the intersection of S and the line OP, where O is the center of S. Then the image of each straight segment in π is an arc on a great semicircle of S.

In the plane π, since K is in the outer face of D, we can draw a line L through K such that D is entirely on one side of L. Then in S, $f(K) = K$, $f(L)$ is a great semicircle, and $f(D)$ is still on one side of $f(L)$.

Let $L' \in \pi$ be the ray on the other side of L, with endpoint K and perpendicular to L. Suppose L' and D are on different sides of L. Rotate K and $f(D)$ around O along $f(L')$, until $f(K)$ is on the boundary of S. Let $r(\cdot)$ be the rotation. Then $r \circ f(D)$ is still completely in S, with only $r \circ f(K)$ on the boundary of S.

Since the rotation is around O, after rotation each arc in $f(D)$ is still on some great semicircle. Then if we project $r \circ f(D)$ back to the plane π, the image $f^{-1} \circ r \circ f(D)$ is still a rectilinear drawing of graph G, while $f^{-1} \circ r \circ f(K) = \infty$. $\square$

If a rectilinear drawing of D is between two lines L_1, L_2 in the plane, apply Lemma 2.3.1 by taking the intersection $L_1 \cap L_2$ (if any) as K and choosing L, L' properly. Then we have:

Corollary 2.3.2. *Let D be a rectilinear drawing of graph G. Let L_1, L_2 be two lines*

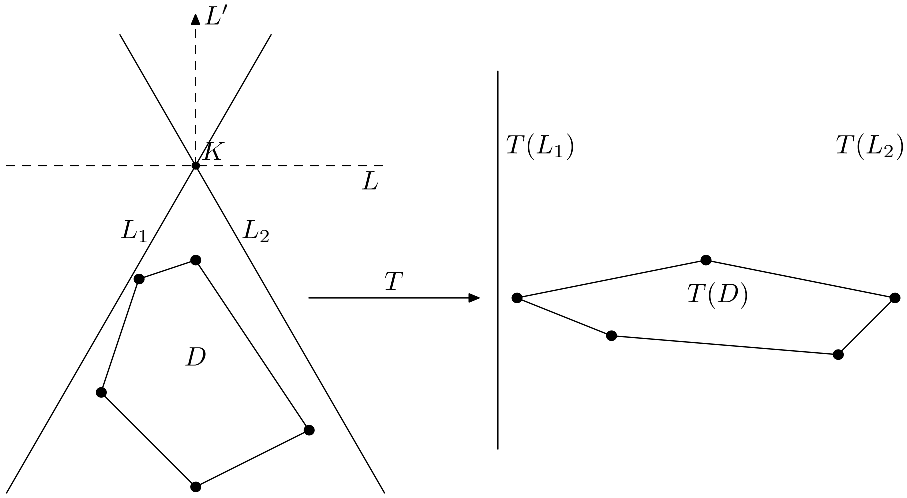


Figure 2.8: Transform two lines to parallel lines

not going through the interior of the convex hull of D, with their intersection K in the outer face of D. Then there is a transformation T of the plane, such that

1) $T(D)$ is a rectilinear drawing isomorphic to D;

2) $T(L_i)$, $i = 1, 2$, are two parallel lines;

3) $T(D)$ is between the two lines $T(L_i)$, $i = 1, 2$;

as shown in Figure 2.8.

Theorem 2.3.3. *Let D be a rectilinear drawing of K_n and P be an extreme vertex, i.e., $P \in \partial_1 D$. Then there is a rectilinear drawing D' isomorphic to D and a line L, such that the projections of the other vertices of D on L has the same order as the cyclic order around P, as shown in Figure 2.9.*

Proof. Let Q_1, Q_2 be the two extreme vertices next to P on the boundary of the convex hull of D. Let U be a small neighborhood of P, such that, for any $P' \in U$, all the vertices other than P have the same cyclic order around P' as around P. Choose P' such that P is contained in the triangle $\triangle P'Q_1Q_2$, as shown in Figure 2.10. The existence of U and P' is obvious.

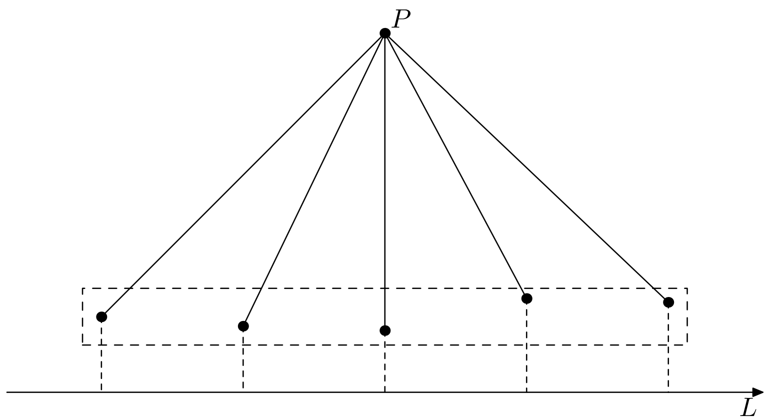


Figure 2.9: Projective order and cyclic order

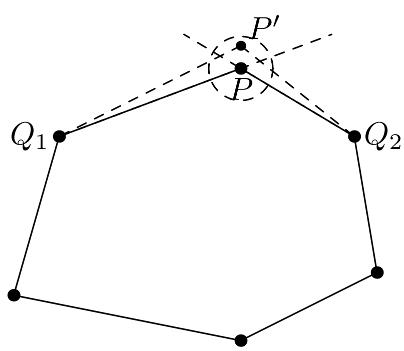


Figure 2.10: Choosing point P'

Apply Corollary 2.3.2 by taking P' as K, and the lines $P'Q_i$ as L_i, $i = 1, 2$. Then we have a transformation T, such that $T(D)$ is the desired drawing, and L can be any line perpendicular to $T(P'Q_1)$ or $T(P'Q_2)$, where $T(P'Q_1)$ is parallel to $T(P'Q_2)$. □

Remarks.

1) Theorem 2.3.3 is a mathematical description of the fact that, given a rectilinear drawing D of K_n and an extreme vertex P, D can be *stretched* isomorphically such that all the vertices other than P are *almost on a line*, as shown in Figure 2.9. The line L in Theorem 2.3.3 represents such a line. Note that in the transformed drawing, P can be pulled away from the other vertices as far as we want, while the drawing remains isomorphic.

2) Given a rectilinear drawing D of K_n and two extreme vertex P_1, P_2 next to each other on the boundary of the convex hull of D, by using a similar argument, we can prove that D can be transformed isomorphically such that, in the transformed drawing, P_1, P_2 can be pulled away from the other vertices as far away as we want, while the drawing remains isomorphic.

Chapter 3

Circular Sequences

The concept of circular sequences was introduced by Goodman and Pollack [16]. Circular sequences are combinatorial structures, which can be used to encode point sets in the plane. There is a close relationship between the number of "convex quadrilaterals" in circular sequences and the rectilinear crossing number of K_n; the latest asymptotic lower bound on the rectilinear crossing number of K_n is obtained by using circular sequences.

In this chapter the first three sections introduce circular sequences and prove that any optimal circular sequence has exactly 3 extreme points. The last two sections give an improved asymptotic upper bound on the number of extreme vertices in optimal rectilinear drawings of K_n.

3.1 Definitions

In this section we introduce circular sequences and their extreme points.

Definition 3.1.1. *A* circular sequence $\Pi = \left(\pi_0, \pi_1, \cdots, \pi_{\binom{n}{2}}\right)$ *on* n *elements is a sequence of permutations of the set* $\{1, 2, \cdots, n\}$*, where*

$$\pi_0 = (1, 2, \cdots, n), \pi_{\binom{n}{2}} = (n, n-1, \cdots, 1)$$

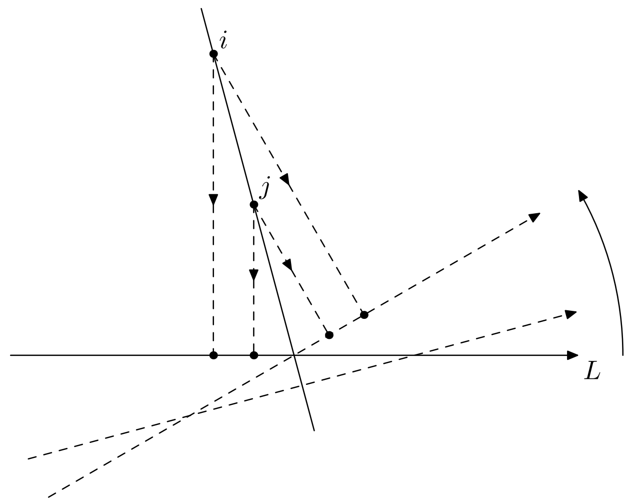


Figure 3.1: Encoding a point set in the plane

and any two successive permutations differ in exactly two adjacent positions.

Circular sequences can be used to encode point sets in the plane. Let S be a finite point set in the plane in general position, i.e., no three points are collinear and no two lines spanned by points in S are parallel. Let L be a directed line which is not orthogonal to any line spanned by points in S. Label the points in S as $\{1, 2, \cdots, n\}$. Without loss of generality, we may assume that the projective order of the points on L is $(1, 2, \cdots, n)$. Now we rotate L, say counterclockwise, by up to $180°$. The projective order remains the same until some line spanned by two points in S, denoted as i and j, is perpendicular to L. As soon as L passes over this position the projective order is updated by switching i and j, as shown in Figure 3.1. Each time when the projective order is updated, we write down the new order. In the end we have a circular sequence Π.

Note that every pair of elements $\{i, j\}$ in $\{1, 2, \cdots, n\}$ switches exactly once in any circular sequence. We can also represent a circular sequence as a series of switches.

Definition 3.1.2. *An i-*switch *in a permutation is a switch between elements in positions i and $i + 1$, or positions $n - i$ and $n - i + 1$.*

A *k-set* of a finte point set S in the plane is a subset $T \subseteq S$ with cardinality k

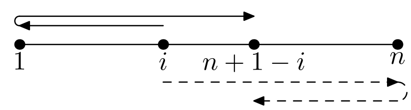


Figure 3.2: Movement of an extreme point

such that T can be separated from $S \setminus T$ by a line. Let Π be the circular sequence for encoding S with the associated directed line L. We have the following observation:

Observation 1: Let T be a k-set of S with the separating line ij, $i, j \in S$. Then when L passes over the position perpendicular to line ij, the corresponding switch (i, j), or (j, i), in S is a k-switch. Hence there is a natural one to one correspondence between k-sets of S and k-switches of Π.

Definition 3.1.3. *Let* Π *be a circular sequence on* n *elements and let* m *be a number in* $\{1, 2, \cdots, n\}$. *If* m *appears at position 1 or* n *in some permutation of* Π, *then* m *is an* extreme point *of* Π.

It is easy to see that for a circular sequence Π encoding a point set S, the projection of any extreme point of S, i.e., a point in S which is on the boundary of the convex hull of S, is an extreme point in Π.

In any circular sequence on n elements, each element must be in $n-1$ switches. So 1 moves all the way to n, and n moves all the way to 1. For any other extreme point i, $1 < i < n$, the only possibility for i to move from position i to position $n+1-i$ by exactly $n-1$ switches is to move all the way to 1 or n, then turn around and move all the way to $n+1-i$, as shown in Figure 3.2.

Definition 3.1.4. *Let* Π *be a circular sequence* Π *on* n *elements. For any* $\{a, b, c, d\} \subseteq \{1, 2, \cdots, n\}$, *we have a naturally induced circular sequence* Π' *on 4 elements. If each of* $\{a, b, c, d\}$ *is an extreme point of* Π', *then* Π' *is* convex *and* $\{a, b, c, d\}$ *is a* convex quadrilateral *of* Π. *Otherwise* Π' *is* concave *and* $\{a, b, c, d\}$ *is a* concave quadrilateral *of* Π.

For a circular sequence encoding a point set S, we have the following observation:

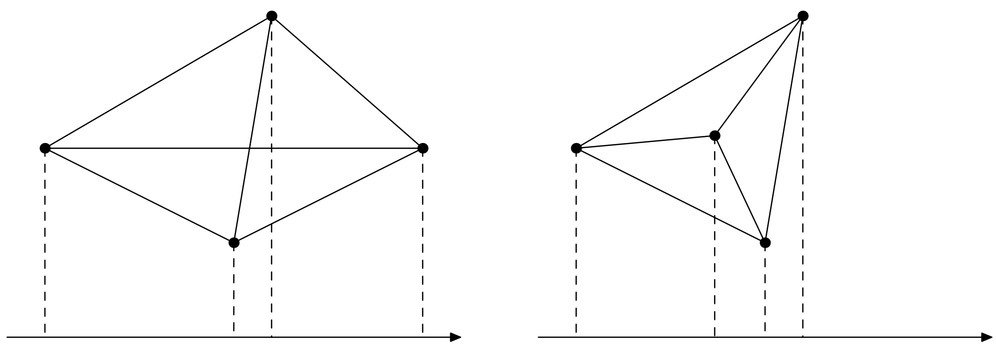


Figure 3.3: Correspondences

Observation 2: Any convex (concave) quadrilateral with vertices in S produce a convex (concave) quadrilateral in Π, as shown in Figure 3.3. On the other hand, Goodman and Pollack [16] proved that every circular sequence on $n \leqslant 4$ elements encodes a point set with size n, i.e., the circular sequence is geometrically realizable. Hence there is a natural one-to-one correspondence between convex (concave) quadrilaterals in S and convex (concave) quadrilaterals in Π.

3.2 Sequence crossing number

In this section we introduce the crossing number of a circular sequence, and its connection to the rectilinear crossing number of K_n.

Any finite point set in the plane, with size at least 3 and no three points collinear, has at least 3 extreme points. An analogous argument holds for a circular sequence.

Lemma 3.2.1. *In any circular sequence* Π *on* $n \geqslant 3$ *elements, there are at least 3 extreme points.*

Proof. Suppose for the first time 1 moves, the permutations are:

$$\cdots \to (1, a, \cdots, b) \to (a, 1, \cdots, b) \to \cdots$$

Then $1, a, b$ are all extreme points. □

In any four point set S in the plane with no three points collinear, the complementary set of any 2-set is a 2-set. The number of *unordered* pairs of 2-set and its complementary set is 2 when the convex hull of S is a quadrilateral, and 3 when the convex hull is a triangle. We have a similar counting result on a circular sequence.

Lemma 3.2.2. *In any circular sequence* Π *on* 4 *elements, the number of* 2*-switches is two when* Π *is convex, and three when* Π *is concave.*

Proof. Since $n = 4$ there are only 1- and 2-switches. Obviously, any extreme point of Π is in one 2-switch and two 1-switches and any non-extreme point is in three 2-switches and no 1-switch.

If Π is convex, then there are 4 extreme points, so the number of 2-switches is $(4 \times 1)/2 = 2$.

If Π is concave, by Lemma 3.2.1 there are exactly three extreme points. So the number of 2-switches is $(1 \times 3 + 3 \times 1)/2 = 3$. □

Definition 3.2.1. *The* sequence crossing number *of a circular sequence* Π *is the number of convex quadrilaterals in* Π*, denoted as* $cr(\Pi)$*.*

Let D be a rectilinear drawing of K_n, with vertex set S with encoding circular sequence Π. Then by **Observation 2** we have

$$cr(D) = cr(\Pi).$$

Thus

$$\begin{aligned} \overline{cr}(K_n) &= \min\{cr(D) : D \text{ is a rectilinear drawing of } K_n\} \\ &\geqslant \min\{cr(\Pi_n) \mid \Pi_n \in \mathcal{C}_n\}. \end{aligned} \tag{3.1}$$

Inequality (3.1) implies that any lower bound for $cr(\Pi_n)$, $\Pi_n \in \mathcal{C}_n$, is a lower bound for $cr(K_n)$.

If, for any circular sequence Π, we could find a point set S in the plane such that Π encodes S, then $cr(K_n) = \min\{cr(\Pi_n) \mid \Pi_n \in \mathcal{C}_n\}$. However, Goodman and Pollack

[16] proved that not every circular sequence is geometrically realizable. They gave an example for $n = 5$.

The sequence crossing number of can be calculated from numbers of j-switches, where $j \leqslant n/2$.

Theorem 3.2.3. *Let e_j be the number of j-switches in a circular sequence Π on n elements. Then*

$$cr(\Pi) = \sum_{j=1}^{\lfloor n/2 \rfloor} e_j \left(\frac{n}{2} - j\right)^2 - \frac{3}{4}\binom{n}{3}.$$

We adapt the proof of Theorem 8 of Lovász *et al.* [24], which is for circular sequences arising from finite point sets in the plane.

Proof. Denote the number of convex quadrilaterals by $\square$ (this is also $cr(\Pi)$), and the number of concave quadrilaterals by $\triangle$. Let us count the total number of 2-switches in induced circular sequences by all 4-sets $\{a, b, c, d\}$. By Lemma 3.2.2 each $\{a, b, c, d\}$ has a contribution 2 to the counting if convex, 3 if concave. On the other hand, for each j-switch of Π, $1 \leqslant j \leqslant \lfloor n/2 \rfloor$, there are

$$(j-1)(n-j-1)$$

possibilities for this j-switch to be extended to a 2-switch in an induced circular sequence on 4 elements. Hence we have

$$2\square + 3\triangle = \sum_{j=1}^{\lfloor n/2 \rfloor} e_j (j-1)(n-j-1).$$

Clearly we have

$$\square + \triangle = \binom{n}{4}.$$

Together we have

$$\square = 3\binom{n}{4} - \sum_{j=1}^{\lfloor n/2 \rfloor} e_j (j-1)(n-j-1).$$

Multiplying both sides of

$$\sum_{j=1}^{\lfloor n/2 \rfloor} e_j = n(n-1)/2$$

by $(n-2)(n-3)/4$ yields

$$3\binom{n}{4} = \sum_{j=1}^{\lfloor n/2 \rfloor} e_j \frac{(n-2)(n-3)}{4}.$$

Thus

$$\begin{aligned} \square &= \sum_{j=1}^{\lfloor n/2 \rfloor} e_j \frac{(n-2)(n-3)}{4} - \sum_{j=1}^{\lfloor n/2 \rfloor} e_j (j-1)(n-j-1) \\ &= \sum_{j=1}^{\lfloor n/2 \rfloor} e_j \left(\frac{n}{2} - j\right)^2 - \frac{3}{4}\binom{n}{3}, \end{aligned}$$

as expected. □

3.3 Number of extreme points

In this section we prove that any optimal circular sequence, on at least 3 elements, has exactly 3 extreme points.

Definition 3.3.1. *Let $\mathcal{C}_n$ be the set of circular sequences on n elements. A circular sequence $\Pi \in \mathcal{C}_n$ is* optimal *if*

$$cr(\Pi) = \min\{cr(\Pi_n) \mid \Pi_n \in \mathcal{C}_n\}.$$

Lemma 3.3.1. *Let i, j be two extreme points of Π. If $1 < i < j \leqslant n/2 + 1$ and i, j both start by moving to the left, then Π is not optimal.*

Proof. Recall that, for any extreme point p, $1 < p < n$, p must move all the way to 1 or n, then turn back and move all the way to $n + 1 - p$.

We construct a new circular sequence Π' as follows:
By the moving of an extreme point, i must switch with $i - 1$ before i reaches position 1. When they switch, we skip this switch and let i follow the path of $i - 1$ and vice versa. In the end, $i - 1$ must be at position $n + 1 - i$ and i at position $n + 1 - (i - 1)$. We append a new switch, i switching with $i - 1$. In this way we have a new circular

sequence Π'.

Suppose $i-1$ is at position p and i is at position $p+1$ when they switch in Π, where $1 \leqslant p \leqslant i-1$. Then we can see that Π' has one more $i-1$ switch but one less p-switch than Π. By Theorem 3.2.3

$$cr(\Pi) - cr(\Pi') = \left(\frac{n}{2} - p\right)^2 - \left(\frac{n}{2} - (i-1)\right)^2 \geqslant 0,$$

since $p \leqslant i-1 \leqslant n/2$. Hence $cr(\Pi') \leqslant cr(\Pi)$.

Note that in Π' the extremity of i has been passed on to $i-1$, and $i-1$ still starts by moving to the left. We call this process a *passing extremity* operation. By repeating this process until the extremity is passed to 2, we have a new circular sequence Π', where $1, 2, j, n$ are extreme points. We can now pass the extremity of j to 3 to obtain a circular sequence Π'' where $1, 2, 3, n$ are extreme points and $2, 3$ both start by moving to the left. Moreover,

$$cr(\Pi'') \leqslant cr(\Pi') \leqslant cr(\Pi)$$

Claim: Π'' is not optimal.

If the claim is true, then Π cannot be optimal, which completes the proof.

Proof of Claim. We represent each circular sequence as a series of switches. Denote the switch of p and q as (p, q) if p was in the lower position before switching. Since $2, 3$ have to start by moving to the left, the order of switches among $1, 2, 3$ must be

$$(1, 2), (1, 3), (2, 3).$$

The permutations of Π'' would look like those in the left column in Table 3.1. The construction of the new circular sequence Π_0 in the right column can be interpreted in this way:

In Π'',

1) Replace $(1, 2)$ by $(2, 3)$.

2) Replace $(1, 3)$ by two consecutive switches $(1, 3)$ and $(1, 2)$.

$\Pi'' =$		$\Pi_0 =$
$\vdots$		$\vdots$
$\underline{1, 2}, 3, \cdots$		$1, \underline{2, 3}, \cdots$
$2, 1, 3, \cdots$		$1, 3, 2, \cdots$
$\vdots$		$\vdots$
$2, \underline{1, 3}, \cdots$	$\longrightarrow$	$\underline{1, 3}, 2, \cdots$
$2, 3, 1, \cdots$		$3, \underline{1, 2}, \cdots$
$\vdots$		$3, 2, 1, \cdots$
$\underline{2, 3}, \cdots, 1, \cdots$		$\vdots$
$3, 2, \cdots, 1, \cdots$		$3, 2, \cdots, 1, \cdots$
$\vdots$		$\vdots$

Table 3.1: Construction of a new circular sequence

3) Remove $(2, 3)$.

4) Keep all the other switches unchanged.

Note that Π_0 has one more 2-switch but one less 1-switch than Π''. Hence

$$cr(\Pi'') - cr(\Pi_0) = \left(\frac{n}{2} - 1\right)^2 - \left(\frac{n}{2} - 2\right)^2 > 0,$$

where the last strict inequality requires that $n \geqslant 4$, which is implied by $1 < i < j \leqslant n/2 + 1$. Hence Π'' can't be optimal. This proves the claim and the lemma. $\square$

Theorem 3.3.2. *Let $n \geqslant 3$. Any optimal circular sequence Π on n elements has exactly 3 extreme points.*

Before proving Theorem 3.3.2, we introduce two operations to construct new circular sequences.

Locally Reversing and Relabelling (LRR). Given a circular sequence on n elements, construct a new circular sequence by

1) replacing each permutation $(i_1, i_2, \cdots, i_n)$ with $(i_n, \cdots, i_2, i_1)$, and then

2) in each permutation replacing number m with $n+1-m$, where $1 \leqslant m \leqslant n$.

Globally Reversing and Relabelling (GRR). Given a circular sequence on n elements, construct a new circular sequence by

1) reversing the order of the permutations, i.e., the k-th permutation becomes the $\left(\binom{n}{2}+2-k\right)$-th, for all $1 \leqslant k \leqslant \binom{n}{2}+1$, and then,

2) in all permutations replacing number m by $n+1-m$, where $1 \leqslant m \leqslant n$.

Let Π be a circular sequence on n elements, Π' is constructed by LRR, and Π'' is constructed by GRR.

1) Both Π' and Π'' have the same number of p-switches as Π, for all $p \in \{1, 2, \cdots, \lfloor n/2 \rfloor\}$, so
$$cr(\Pi') = cr(\Pi'') = cr(\Pi),$$
by Theorem 3.2.3.

2) for any p, $1 < p < n$, p is an extreme point in Π if and only if $n+1-p$ is an extreme point in Π'. When p is an extreme point in Π, the moving direction of $n+1-p$ in its first switch in Π' is opposite to that of p in Π.

3) for any p, $1 < p < n$, p is an extreme point in Π if and only if $n+1-p$ is an extreme point in Π''. When p is an extreme point in Π, the moving direction of $n+1-p$ in its first switch in Π'' is the same as that of p in Π.

Proof of Theorem 3.3.2. Suppose Π has at least 4 extreme points $1, i, j, n$, and $1 < i < j < n$. In the first 8 cases, we treat all the cases in which i and j are in the same half of the interval $[1, n]$.

Case 1: $1 < i < j \leqslant n/2+1$ and i, j both start by moving to the left. By Lemma

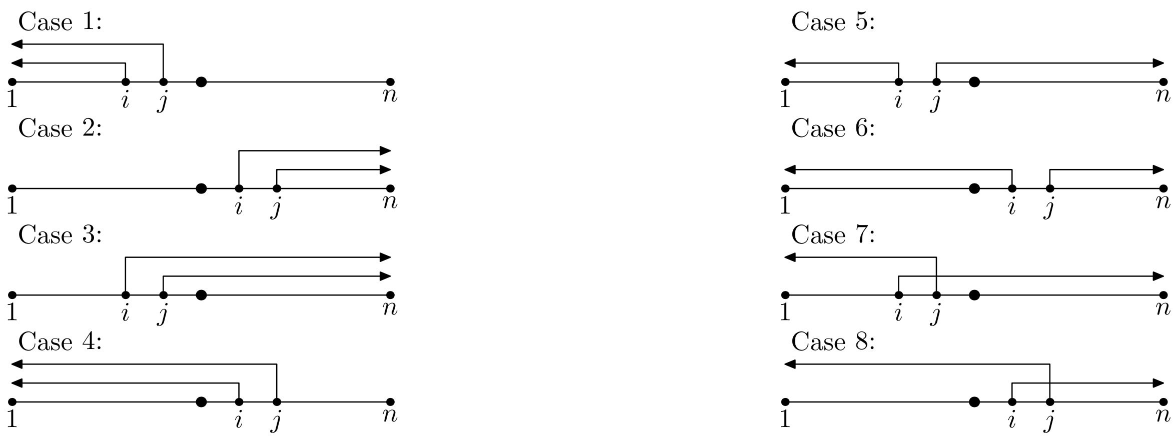


Figure 3.4: Cases 1-8 in the proof of Theorem 3.3.2

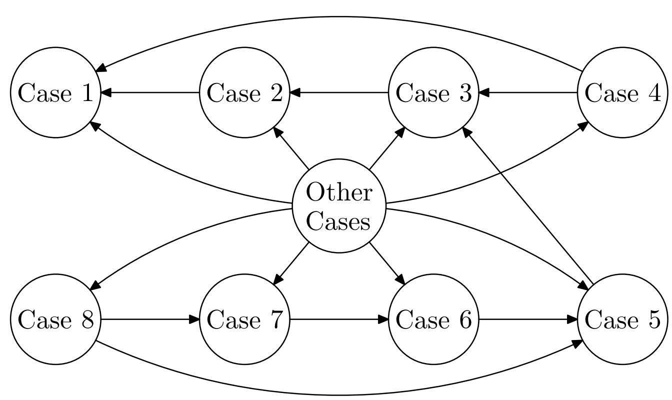


Figure 3.5: Hierarchy in the proof of Theorem 3.3.2

3.3.1, Π is not optimal.

Case 2: $n/2 - 1 \leqslant i < j < n$ and i, j both start by moving to the right. Construct a new circular sequence Π' by LRR; then $cr(\Pi') = cr(\Pi)$. In Π', $n + 1 - j$ and $n + 1 - i$ are extreme points, and they both start by moving to the left. Let $i' = n + 1 - j$, $j' = n + 1 - i$. Then $1 < i' < j' \leqslant n/2 + 1$. Hence Case 2 reduces to Case 1.

Case 3: $1 < i < j \leqslant n/2 + 1$ and i, j both start by moving to the right. Construct a new circular sequence Π' by GRR. Then $cr(\Pi') = cr(\Pi)$. In Π', $n+1-j$ and $n + 1 - i$ are extreme points, and $n + 1 - j$ both start by moving to the right. Let $i' = n+1-j$, $j' = n+1-i$. Then $n/2-1 \leqslant i' < j' < n$. Hence Case 3 reduces to Case 2.

Case 4: $n/2 - 1 \leqslant i < j < n$ and i, j both start by moving to the left. Constructing a new circular sequence by LRR, then Case 4 reduces to Case 3. Alternatively, constructing a new circular sequence by GRR, Case 4 reduces to Case 1.

Case 5: $1 < i < j \leqslant n/2 + 1$, where i starts by moving to the left and j starts by moving to the right. By using the *passing extremity* operation defined in Lemma 3.3.1, we can construct a new circular sequence Π' with $cr(\Pi') \leqslant cr(\Pi)$, where 2 and j are extreme points and 2 starts by moving to the left. We construct a second new circular Π'' from Π' as follows: When 1 and 2 switch in Π, we skip this switch and let 1 follow the path of 2 and vice versa. In the end we switch 1 and 2. It's easy to see that $cr(\Pi'') = cr(\Pi')$. In Π'' 2 remains as an extreme point but starts by moving to the right. By Case 3, Π'' is not optimal, so neither is Π.

Case 6: $n/2 - 1 \leqslant i < j < n$, where i starts by moving to the left and j starts by moving to the right. Constructing a new circular sequence by LRR, Case 6 reduces to Case 5.

Case 7: $1 < i < j \leqslant n/2 + 1$, where i starts by moving to the right and j starts by moving to the left. Construct a new circular sequence by GRR, Case 7 reduces to Case 6.

Case 8: $n/2 - 1 \leqslant i < j < n$, and i starts by moving to the right and j starts by moving to the left. Constructing a new circular sequence by LRR, Case 8 reduces to Case 7. Alternatively, constructing a new circular sequence by GRR, Case 8 reduces to Case 5.

Other Cases: Generally, let $\Pi = \left(\pi_0, \pi_1, \cdots, \pi_{\binom{n}{2}}\right)$. Since the number 1 is in exactly two 1-switches, at least one of the switches $(1, i), (1, j), (1, n)$ is not a 1-switch. Let $(1, p)$ be such a switch. Let the permutation right before the switch $(1, p)$ occurs be

$$\pi_t = (a_1, a_2, \cdots, a_{k-1}, a_k = 1, a_{k+1} = p, a_{k+2}, \cdots, a_n),$$

where $0 < t < \binom{n}{2}$, $1 < k < n - 1$. Then the next permutation is

$$\pi_{t+1} = (a_1, a_2, \cdots, a_{k-1}, p, 1, a_{k+2}, \cdots, a_n).$$

Construct a new circular sequence Π' as follows:

1) For each t', $1 \leqslant t' < t$, reverse the permutation $\pi_{t'}$, i.e., replace $\pi_{t'} = (a'_1, a'_2, \cdots, a'_n)$ with $\overleftarrow{\pi_{t'}} = (a'_n, \cdots, a'_2, a'_1)$.

2) Construct a sequence of permutations:

$$S = \left(\pi_t, \pi_{t+1}, \cdots, \pi_{\binom{n}{2}}, \overleftarrow{\pi_1}, \overleftarrow{\pi_2}, \cdots, \overleftarrow{\pi_{t-1}}, \overleftarrow{\pi_t}\right).$$

Then in S any two successive permutations differ in exactly two adjacent positions.

3) Define a function

$$f: \{1, 2, \cdots, n\} \longrightarrow \{1, 2, \cdots, n\},$$

such that $f(a_{k'}) = k'$, for each k', $1 \leqslant k' \leqslant n$. Replace each number m in any permutation in S with $f(m)$. Then we have a circular sequence Π', in which k and $k+1$ are extreme points.

Note that, for any $k' \leqslant n/2$, Π and Π' have the same number of k'-switches. Hence $cr(\Pi) = cr(\Pi')$. If $k \leqslant n/2$, then $1 < k < k+1 \leqslant n/2+1$. If $k > n/2$, then $n/2 - 1 < k < k+1 < n$. So Π' always reduces to one of Cases 1-8.

Finally we come to the conclusion that Π is not optimal if it has more than 3 extreme points. Hence it must have exactly 3 extreme points. □

In our proof of Lemma 3.3.1 the sequences constructed with *passing extremity* operation may not be geometrically realizable. Thus Theorem 3.3.2 does NOT lead to a proof of the Triangle Conjecture.

However, in another paper [17] Goodman and Pollack proved that every circular sequence is realizable by a *pseudolinear drawing*. A *pseudoline* is a simple closed curve in the projective plane $\mathbb{P}^2$ which does not disconnect $\mathbb{P}^2$. Given a good drawing D of K_n (not necessarily rectilinear) in the plane $\mathbb{R}^2$, let C be a disk which contains D. By identifying antipodal points on the boundary of C and discarding $\mathbb{R}^2 \backslash C$ we may regard D lying in $\mathbb{P}^2$. If each edge can be extended to a pseudoline, such that each pair of the pseudolines intersect exactly once, then D is a *pseudolinear drawing*.

The *pseudolinear crossing number* of K_n, denoted as $\widetilde{cr}(K_n)$, is the minimum number of edge crossings over all pseudolinear drawings of K_n. A pseudolinear drawing of K_n with $\widetilde{cr}(K_n)$ crossings is said to be *optimal*. Therefore our result on the circular sequence implies that any optimal pseudolinear drawing of K_n, for $n \geqslant 3$, has exactly three extreme vertices.

The study on extreme points of circular sequences was motivated by the Triangle Conjecture on rectilinear drawings of K_n. Although our proof on the pseudolinear drawings does not imply the Triangle Conjecture, the recently announced proof of the Triangle Conjecture by Aichholzer *et al.* [5] does not seem to imply our result either.

3.4 Extreme vertices of a drawing

Let e_j be the number of j-switches in a circular sequence Π on n elements, and E_j be the number of ($\leqslant j$)-switches, i.e. $E_j = e_1 + e_2 + \cdots + e_j$. Lovász *et al.* [24] proved the following result:

Theorem 3.4.1.

$$cr(\Pi) = \sum_{1\leqslant j<n/2} e_j\left(\frac{n}{2}-j\right)^2 - \frac{3}{4}\binom{n}{3} = \sum_{1\leqslant j<n/2} E_j(n-2j-1) + O(n^3), \tag{3.2}$$

$$E_j \geqslant 3\binom{j+1}{2} \textit{ for any } j < n/2, \textit{ and} \tag{3.3}$$

$$cr(\Pi) > \left(\frac{3}{8}+10^{-5}\right)\binom{n}{4} + O(n^3). \tag{3.4}$$

The inequality (3.4), together with Inequality (3.1) in Section 3.2, implies that

$$\overline{cr}(K_n) \geqslant \left(\frac{3}{8}+10^{-5}\right)\binom{n}{4} + O(n^3).$$

Since $\frac{3}{8}\binom{n}{4}$ is an upper bound for $cr(K_n)$, for n large enough,

$$\overline{cr}(K_n) \neq cr(K_n).$$

Let

$$\begin{aligned} k_1(n) &:= (1/162)\left(-71+71n+\sqrt{19n^2-38n+19}\right), \text{ and} \\ F(j,n) &:= \left(2-\frac{1}{s(k,n)}\right)k^2 - \left(\frac{(s(k,n)-1)^2}{s(k,n)}\right)k(n-2k-1) \\ &\quad + \left(\frac{s(k,n)^4-7s(k,n)^2+12s(k,n)-6}{12s(k,n)}\right)(n-2k-1)^2, \end{aligned}$$

where

$$s(k,n) := \left\lfloor \frac{1}{2}\left(1+\sqrt{\frac{1+6\left(\frac{k}{n}\right)-\left(\frac{9}{n}\right)}{1-2\left(\frac{k}{n}\right)-\left(\frac{1}{n}\right)}}\right)\right\rfloor$$

In [8] Balogh and Salazar improved the asymptotic lower bound as follows:

Theorem 3.4.2. *For each $j < n/2$*

$$E_j \geqslant F(j,n) + O(n), \tag{3.5}$$

for any $j > k_1(n) \approx 0.465178n + O(\sqrt{n})$

$$F(j,n) > 3\binom{j+1}{2}, \tag{3.6}$$

and

$$\begin{aligned} cr(\Pi) \geqslant & \sum_{j=1}^{\lfloor k_1(n)\rfloor} (n-2j-1)\cdot 3\binom{j+1}{2} \\ & + \sum_{\lfloor k_1(n)\rfloor+1\leqslant j<n/2} (n-2j-1)F(j,n) \\ \approx & \ 0.37553\binom{n}{4} + O(n^3) \end{aligned} \tag{3.7}$$

Now we can estimate E_j, given the number of extreme points in a circular sequence.

Lemma 3.4.3. *Let $\Pi \in \mathcal{C}_n$ with k extreme points. Then, for any $j < n/2-1$,*

$$E_j \geqslant kj - \frac{3}{4}j^2 + \frac{1}{2}j\,.$$

Hence

$$E_j \geqslant 3\binom{j+1}{2},\ \textit{for any } j \leqslant \frac{4}{9}(k-1).$$

Proof. For any $j < n/2$, let:

P_1:=$\{i$ extreme point$\,|\, i \leqslant j+1$ and it starts by moving to the left$\}$;

P_2:=$\{i$ extreme point$\,|\, i \geqslant n-j$ and it starts by moving to the right$\}$;

Q_1:=$\{i$ extreme point$\,|\, i \leqslant j+1$ and it starts by moving to the right$\}$; and

Q_2:=$\{i$ extreme point $| \, i \geqslant n-j$ and it starts by moving to the left$\}$.

Then

$$E_j \;\geqslant\; \frac{1}{2}(p_1+1)p_1 + \frac{1}{2}(p_2+1)p_2 + j(k-p)$$

where $p_1 = |P_1|$, $p_2 = |P_2|$, $p = |P_1 \cup P_2|$.

Obviously $p_1 + p_2 = p$, and $\frac{1}{2}(p_1+1)p_1 + \frac{1}{2}(p_2+1)p_2$ achieves its minimum when $p_1 = p_2 = p/2$. Hence

$$E_j \geqslant (p/2+1)p/2 + j(k-p)$$

By using GRR we have a new sequence Π' with $e'_i = e_i$ hence $E'_i = E_i$ for all $i < n/2$. We define P'_1, P'_2, Q'_1, Q'_2 for Π' corresponding to P_1, P_2, Q_1, Q_2. Let

$$p' := |P'_1 \cup P'_2|, \;\; q' = |Q'_1 \cup Q'_2|,$$

then $p' = q$, $\;q' = p$. Similarly we also have

$$\begin{aligned} E'_j \;&\geqslant\; (p'/2+1)p'/2 + j(k-p') \\ &=\; (q/2+1)q/2 + j(k-q). \end{aligned}$$

Hence

$$E_j = E'_j \geqslant (q/2+1)q/2 + j(k-q).$$

Thus

$$\begin{aligned} 2E_j \;&\geqslant\; [(p/2+1)p/2 + j(k-p)] + [(q/2+1)q/2 + j(k-q)] \\ &=\; \frac{1}{4}(p^2+q^2) - \left(j - \frac{1}{2}\right)(p+q) + 2kj \\ &\geqslant\; \frac{1}{8}(p+q)^2 - \left(j - \frac{1}{2}\right)(p+q) + 2kj \\ &=\; \frac{1}{8}[p+q-(4j-2)]^2 + \left(2kj - 2j^2 + 2j - \frac{1}{2}\right). \end{aligned}$$

Obviously

$$p + q \leqslant 2j < 4j - 2,$$

so

$$\begin{aligned} 2E_j &\geqslant \frac{1}{8}\left[2j-(4j-2)\right]^2+\left(2kj-2j^2+2j-\frac{1}{2}\right) \\ &= 2kj-\frac{3}{2}j^2+j\,, \end{aligned}$$

which gives

$$E_j \geqslant kj-\frac{3}{4}j^2+\frac{1}{2}j.$$

□

Theorem 3.4.4. *For fixed n, let*

$$e_n=\max\{h_1(D)\,|\,D \textit{ is an optimal rectilinear drawing of } K_n\}.$$

Then

$$\overline{\lim_{n\to\infty}}\frac{e_n}{n}<0.1912.$$

Proof. Let D_n be an optimal rectilinear drawing of K_n with e_n extreme vertices. Let $\Pi\in\mathcal{C}_n$ be a circular sequence for encoding D_n. By Lemma 3.4.3,

$$\begin{aligned} \overline{cr}(K_n) &= \overline{cr}(D_n) \\ &= cr(\Pi) \\ &= \sum_{j=1}^{\lfloor 4(e_n-1)/9\rfloor}(n-2j-1)E_j+\sum_{j=\lfloor 4(e_n-1)/9\rfloor+1}^{\lfloor k_1(n)\rfloor}(n-2j-1)E_j \\ &\qquad +\sum_{j=\lfloor k_1(n)\rfloor+1}^{<n/2}(n-2j-1)E_j \\ &\geqslant \sum_{j=1}^{\lfloor 4(e_n-1)/9\rfloor}(n-2j-1)\left(e_nj-\frac{3}{4}j^2+\frac{1}{2}j\right) \\ &\quad +\sum_{j=\lfloor 4(e_n-1)/9\rfloor+1}^{\lfloor k_1(n)\rfloor}(n-2j-1)\cdot 3\binom{j+1}{2} \\ &\quad +\sum_{j=\lfloor k_1(n)\rfloor+1}^{<n/2}(n-2j-1)F(j,n) \end{aligned}$$

$$\begin{aligned} = & \sum_{j=1}^{\lfloor 4(e_n-1)/9 \rfloor} (n-2j-1)\left(e_n j - \frac{3}{4}j^2 + \frac{1}{2}j - 3\binom{j+1}{2}\right) \\ & + \sum_{j=1}^{\lfloor 4(e_n-1)/9 \rfloor} (n-2j-1)\cdot 3\binom{j+1}{2} \\ & + \sum_{j=\lfloor 4(e_n-1)/9 \rfloor+1}^{<n/2} (n-2j-1)F(j,n) \end{aligned}$$

$$\begin{aligned} \geqslant & \underbrace{\sum_{j=1}^{\lfloor 4(e_n-1)/9 \rfloor} \left(e_n j - \frac{3}{4}j^2 + \frac{1}{2}j - 3\binom{j+1}{2}\right)(n-2j-1)}_{\text{part (I)}} \\ & +0.37553\binom{n}{4} + O(n^3), \end{aligned}$$

where the last inequality is from (3.7) in Theorem 3.4.2. Note that

$$\begin{aligned} \text{part (I)} &= -\frac{95}{4374}\,e_n + \frac{65}{486}\,n - \frac{11}{162}\,e_n n + \frac{62}{729}\,{e_n}^2 + \frac{8}{243}\,{e_n}^3 n \\ & \quad -\frac{8}{81}\,{e_n}^2 n + \frac{56}{2187}\,{e_n}^3 - \frac{32}{2187}\,{e_n}^4 - \frac{325}{4374} + O(n^3) \\ &= \frac{8}{243}e_n^3 n - \frac{32}{2187}e_n^4 + O(n^3). \end{aligned}$$

As mentioned earlier, Aichholzer and Krasser [4] proved that

$$\lim_{n\to\infty} \frac{\overline{cr}(K_n)}{\binom{n}{4}} \leqslant \beta,\ \ \beta = 0.38058\,.$$

Hence, for sufficiently large n,

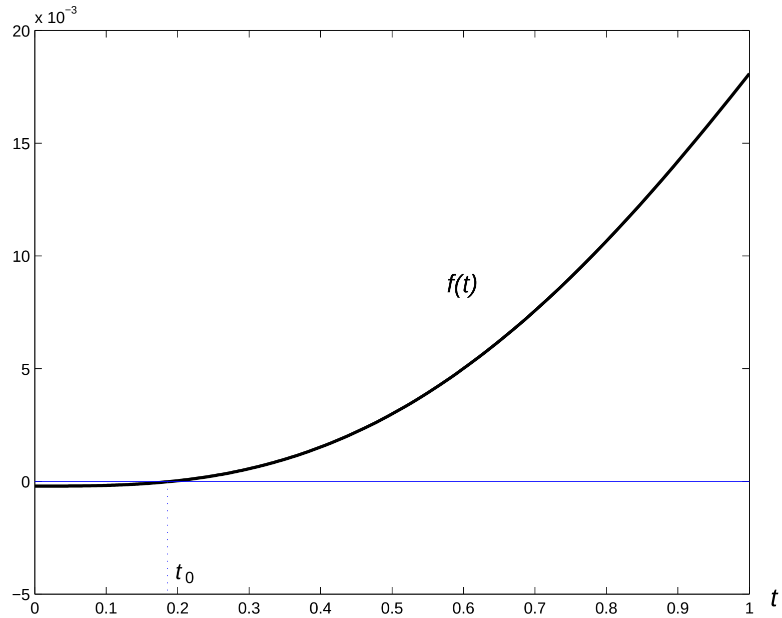
$$\frac{8}{243}e_n^3 n - \frac{32}{2187}e_n^4 + 0.37553\binom{n}{4} \leqslant 0.38058\binom{n}{4}.$$

Let $\lambda = \overline{\lim_{n\to\infty}}\,\frac{e_n}{n}$. Then

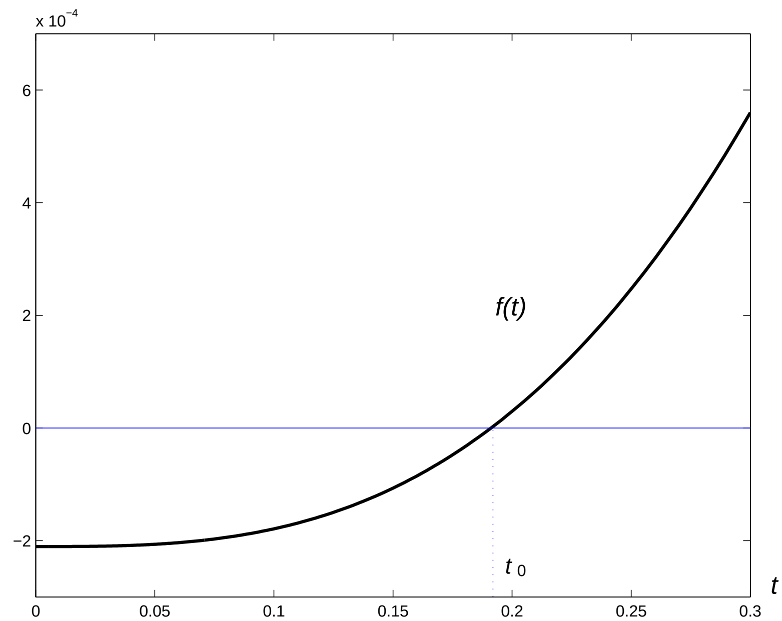
$$\frac{8}{243}\lambda^3 - \frac{32}{2187}\lambda^4 \leqslant (0.38058 - 0.37553)\frac{1}{24}.$$

Let

$$f(t) = \frac{8}{243}t^3 - \frac{32}{2187}t^4 - (0.38058 - 0.37553)\frac{1}{24}.$$



$0 \leqslant t \leqslant 1$



$0 \leqslant t \leqslant 0.3$

Figure 3.6: $f(t) = \frac{8}{243}t^3 - \frac{32}{2187}t^4 - (\beta - 0.37553)\frac{1}{24}$

Use MATLAB to solve the equation $f(t) = 0$. There is a unique solution $t_0 = 0.1911547\ldots$ in the interval $[0, 1]$, as shown in Figure 3.6. Hence either $0 \leqslant \lambda \leqslant t_0$ or $t_0 \leqslant \lambda \leqslant 1$. But $f(0) < 0$, so

$$\lambda \leqslant t_0 < 0.1912.$$

$\square$

Chapter 4

Crossing Number of K_{11}

Guy [19] proved that, for $n = 3, 4, 5, 6, 7, 8$, the total number of non-isomorphic optimal drawings of K_n is

$$1, 1, 1, 1, 1, 1, 5, 3$$

respectively. In this chapter we give an algorithm to find all the optimal drawings of K_9 and K_{10}. Also we use the algorithm to prove that $cr(K_{11}) = 100$, which implies $cr(K_{12}) = 150$.

4.1 Theory

In this section we develop the simple counting properties we shall exploit in showing by computer that $cr(K_{11}) = 100$. The main point of this section is to show that any good drawing of K_{11} with fewer than 100 crossings contains an optimal drawing of K_9.

Lemma 4.1.1. *For any $n \geqslant 5$*

$$cr(K_n) \geqslant \left\lceil \frac{n}{n-4} \cdot cr(K_{n-1}) \right\rceil .$$

Proof. Let D_n be an optimal drawing of K_n, i.e., $cr(D_n) = cr(K_n)$. There are n copies of sub-drawings of K_{n-1} in D_n obtained by deleting a single vertex from D_n.

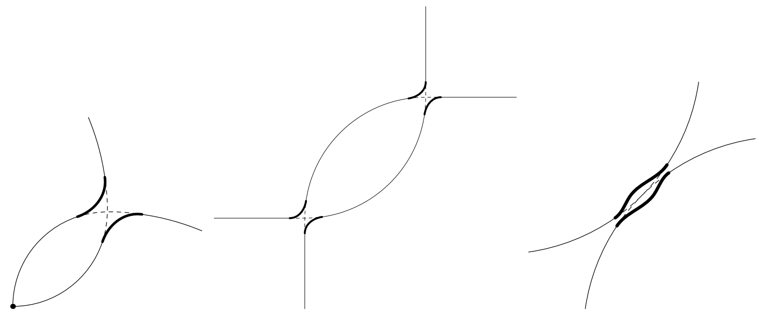


Figure 4.1: Modifying a drawing

Each crossing of D_n occurs in $\binom{n-4}{n-1-4} = n-4$ of these, so $(n-4)cr(D_n) \geqslant n\,cr(K_{n-1})$.□

A simple calculation shows that, for any even n, $\left\lceil \frac{n}{n-4} \cdot Z(n-1) \right\rceil = Z(n)$. Then we have

Corollary 4.1.2. *If, for some odd n, $cr(K_n) = Z(n)$, then $cr(K_{n+1}) = Z(n+1)$.*

Recall that a good drawing is a drawing such that:

1) any two edges incident to the same vertex don't cross each other;

2) any two edges cross at most once; and

3) and any two edges are not tangent to each other.

Lemma 4.1.3. *Any optimal drawing of K_n is a good drawing.*

Proof. Let c be a crossing in an optimal drawing D of K_n. Note that any optimal drawing of K_n must have at most $\binom{n}{4}$ therefore finitely many crossings. Then we can find a small neighborhood U of c, such that inside U the two edges which create c are drawn as two simple curves. Furthermore, we may assume that other than these two curves there is no part of D inside U.

If c is one of the forbidden crossings in Figure 1.1, as shown in Figure 4.1, we can always replace the two curves inside U with two non-intersecting curves. The new geometrical object D' is still a drawing of K_n, and D' has exactly one or two fewer crossings than D does. Thus D cannot be optimal, which is a contradiction. □

The following is an important result from Kleitman[22]:

Lemma 4.1.4 (parity)**.** *For any odd n, the number of crossings in any good drawing of K_n has the same parity as $Z(n)$.*

Since $Z(11) = 100$, $cr(K_{11})$ must be even by Lemma 4.1.4. By Lemma 4.1.1 $cr(K_{11}) \geqslant 95$, so $cr(K_{11}) \in \{96, 98, 100\}$. Recall that the responsibility of a vertex v in a drawing of a graph G is the number of crossings on the edges incident to v.

Lemma 4.1.5. *Let D be a good drawing of a graph G with n vertices. Then there is a vertex v of G so that v has responsibility at least $\lceil 4cr(D)/n \rceil$. Thus*

$$cr(D - v) \leqslant cr(D) - \lceil 4cr(D)/n \rceil .$$

Here $D - v$ denotes the drawing of $G - v$ by removing v in D and all the curves representing the edges incident to v.

Proof. In a good drawing each crossing is created by two edges which have no common endpoints. Hence the total responsibility of all the vertices is $4cr(D)$. Thus by the Pigeonhole Principle there is a vertex v with responsibility at least $\lceil 4cr(D)/n \rceil$, which yields the second conclusion. □

Theorem 4.1.6 (containing)**.**

1) *For $n \leqslant 8$, any optimal drawing of K_n contains an optimal drawing of K_{n-1}.*

2) *Any optimal drawing of K_9 contains a good drawing of K_8 with at most 20 crossings. Any good drawing of K_8 with at most 20 crossings contains an optimal drawing of K_7.*

n	4	5	6	7	8
$cr(K_n) = Z(n)$	0	1	3	9	18
$cr(K_n) - \lceil 4cr(K_n)/n \rceil$	0	0	1	3	9

Table 4.1: Calculation for $n \leqslant 8$

3) Any good drawing of K_{11} with fewer than 100 crossings contains a good drawing of K_{10} with at most 62 crossings. Any good drawing of K_{10} with at most 62 crossings contains an optimal drawing of K_9.

Proof.

1) By Lemma 4.1.5, any optimal drawing D_n of K_n contains a good drawing of K_{n-1} with at most $cr(K_n) - \lceil 4cr(K_n)/n \rceil$ crossings. For $n \leqslant 8$, we have the calculation results in Table 4.1. Hence we can see that, for $n \leqslant 8$,

$$cr(K_n) - \lceil 4cr(K_n)/n \rceil = cr(K_{n-1})\,.$$

Thus the resulting drawing of K_{n-1} contained in D_n must be optimal.

2) For $n = 9$ let D_9 be an optimal drawing of K_9. By Lemma 4.1.5, D_9 contains a good drawing D_8 of K_8 with at most $36 - \lceil 4 \cdot 36/9 \rceil = 20$ crossings. Since $cr(K_8) = 18$, $cr(D_8) = 18, 19$ or 20.
By using Lemma 4.1.5 again, we know that D_8 contains a good drawing D_7 of K_7 with at most $cr(D_8) - \lceil 4cr(D_8)/8 \rceil$ crossings. As shown in Table 4.2, $cr(D_7) \leqslant 10$. By the parity argument in Lemma 4.1.4 $cr(D_7) \leqslant 9$. Since $cr(K_7) = 9$, we must have $cr(D_7) = 9$, i.e., D_7 is optimal.

3) For $n = 11$, let D_{11} be a good drawing of K_{11} with less than 100 crossings. By the parity argument in Lemma 4.1.4, $cr(K_{11}) \in \{96, 98\}$. Similarly we can prove that D_{11} must contain a good drawing D_{10} of K_{10} with at most 62 crossings, which must contain an optimal drawing of K_9. The calculation is shown in Tables 4.3

18	19	20
$\lceil 18 \cdot 4/8 \rceil = 9$ $18 - 9 = 9$	$\lceil 19 \cdot 4/8 \rceil = 10$ $19 - 10 = 9$	$\lceil 20 \cdot 4/8 \rceil = 10$ $20 - 10 = 10$ $\rightarrow 9$

Table 4.2: Calculation for $n = 8$

96	98
$\lceil 96 \cdot 4/11 \rceil = 35$ $96 - 35 = 61$	$\lceil 98 \cdot 4/11 \rceil = 36$ $98 - 36 = 62$

Table 4.3: Calculation for $n = 11$

and 4.4. □

4.2 Idea

In this section, we define some notations and describe our main idea of proving $cr(K_{11}) = 100$ by computer.

Definition 4.2.1. *For a set $\mathcal{D}$ of good drawings,*

$$cr(\mathcal{D}) := \min\{cr(D) \mid D \in \mathcal{D}\}.$$

60	61	62
$\lceil 60 \cdot 4/10 \rceil = 24$ $60 - 24 = 36$	$\lceil 61 \cdot 4/10 \rceil = 25$ $61 - 25 = 36$	$\lceil 62 \cdot 4/11 \rceil = 25$ $62 - 25 = 37$ $\rightarrow 36$

Table 4.4: Calculation for $n = 10$

Definition 4.2.2. *Let $D_F + v$ be the set of all good drawings obtained by inserting a new vertex v in a face F of a good drawing D, and drawing new edges from v to all vertices of D.*

Definition 4.2.3. *Let $\mathcal{D}$ be a set of good drawings of K_n. Then $\mathcal{D} + v$ is the set of all good drawings D of K_{n+1} so that deleting some vertex from D leaves a drawing in $\mathcal{D}$.*

Obviously, $\mathcal{D} + v = \bigcup_{D \in \mathcal{D}} \left(\bigcup_{F \in \mathcal{F}(D)} D_F + v \right)$, where $\mathcal{F}(D)$ denotes the set of faces of a drawing D. For simplicity, we use $D + v$ to denote $\{D\} + v$ for a single drawing D.

Definition 4.2.4. *For any drawing D of a graph G, let G_D be the graph with vertices $V(G_D) = V(G) \cup \{crossings\}$, where the edges are the components of $D \backslash V(G_D)$ and v is incident to e if and only if v is in the closure of e.*

We say G_D is the planar graph *of D, and any vertex in $V(G)$ is a* non-crossing (vertex) *of G_D, and any crossing of D is a* crossing (vertex) *of G_D.*
The dual graph *of D is the planar dual graph of G_D.*

Given a good drawing D of K_n with vertices v_i, $i = 1, 2, \cdots, n$, and a face $F \in \mathcal{F}(D)$, let d_i be the minimum distance in the dual graph from $F \in \mathcal{F}(D)$ to the faces incident to v_i. Then $cr(D) + \sum_{i=1}^{n} d_i$ is a lower bound for $cr(D_F + v)$. Therefore

$$cr(\mathcal{D} + v) \geqslant Lb[\mathcal{D}] := \min\{cr(D) + \sum_{i=1}^{n} d_i \mid D \in \mathcal{D}, F \in \mathcal{F}(D)\}. \tag{4.1}$$

Then, by Part (3) of Theorem 4.1.6 we have

Corollary 4.2.1 (main)**.** *Let $\mathcal{D}_n^c$ be the set of all good drawings of K_n with c crossings and let $\mathcal{D}_n^{\leqslant c}$ be the set of all good drawings of K_n with at most c crossings. If $Lb\left[\mathcal{D}_{10}^{\leqslant 62}\right] > 98$, then $cr(K_{11}) = 100$.*

4.3 Main algorithm

In this section we present an algorithm, which is the main algorithm for extending good drawings of K_n to good drawings of K_{n+1}. We shall prove its validity and give the results from our code. Our code is available at [1].

Let D be a good drawing of K_n. To obtain a good drawing of K_{n+1}, we put a new vertex v_0 in a face F of D, and draw new edges from v_0 to each vertex of D. Each such new edge corresponds to a sequence of adjacent faces in D, i.e., a path in the dual graph of G_D. We define a *face path* as follows:

Definition 4.3.1. *A* face path (P, v) *of a drawing D is a path P in the dual graph of D with an endpoint v in $V(D)$, where the last face on P is incident to v, i.e. v is on the boundary of the last face.*
The length of a face path (P, v) *is the length of P.*

We also denote the face path as P when the endpoint v is known from context.

Each face path (P, v) corresponds to a new edge drawn from a new vertex in the starting face of P to v by following the faces on P.

Definition 4.3.2. *Two face paths (P_1, v_1) and (P_2, v_2)* cross each other *if we can NOT draw the corresponding new edges E_1, E_2 without crossing each other.*

Algorithm 1.

Input:

- a set $\mathcal{D} = \{D^1, D^2, \cdots, D^t\}$ of good drawings of K_n ($n \geqslant 4$);
- an integer $\delta \in \{0, 1, 2\}$.

Output:

- $Lb\,[\mathcal{D}]$;
- if $n \leqslant 9$, all the good drawings in $\mathcal{D} + v$ with at most $cr(K_{n+1}) + \delta$ crossings.

Procedure:

1) For each drawing $D^i, i = 1, 2, \cdots, t$, find $c^i := Lb\,[\{D\}]$. Let $c = \min\{c^1, c^2, \cdots, c^t\}$;
2) Output c as $Lb\,[\mathcal{D}]$.
3) If $n = 10$, exit;
4) Set $i \leftarrow 1$;
5) Let $F_1, F_2, \cdots, F_m$ be all the faces of D^i. Set $j \leftarrow 1$;
6) Set $k \leftarrow 1$;
7) Find the minimum length d_{jk} of face paths (P, v_k) which start from F_j with endpoint v_k. Then d_{jk} is the least number of new crossings we must have if drawing an edge from a new vertex in F_j to v_k;
8) Find the set S_{jk} of face paths (P, v_k) starting from F_j with endpoint v_k and with length at most $d_{jk} + \delta$, where each path P in the dual graph does not cross any edge in D twice and does not cross any edge incident to v_k;
9) If $k < n$, set $k \leftarrow k + 1$ and go to Step 7;
10) There are $|S_{j1}| \cdot |S_{j2}| \cdot \ldots \cdot |S_{jn}|$ combinations of face paths $(P_1, P_2, \cdots, P_n)$, where $(P_k, v_k) \in S_{jk}$, $k = 1, 2, \cdots, n$. Choose the first combination $(P_1, P_2, \cdots, P_n)$;
11) Let Δc be the sum of the lengths of $P_k, k = 1, 2, \cdots, n$;
12) If $\Delta c > cr(K_{n+1}) + \delta - cr(D^i)$, choose the next combination (if any) and go to Step 11;

13) If there exists $k \neq k'$ such that P_k and $P_{k'}$ *cross* each other, choose the next combination (if any) and go to Step 11;

14) Generate a new drawing D' by inserting a new vertex v in face F_j and drawing the new edges corresponding to face paths P_k for each $k = 1, 2, \cdots, n$;

15) Add D' to the output list. If required, check if D' is isomorphic to any drawing already in the output list;

16) If $j < m$, set $j \leftarrow j + 1$ and go to Step 6;

17) If $i < t$, set $i \leftarrow i + 1$ and go to Step 5;

18) Exit.

4.3.1 The proof

In this subsection we prove that Algorithm 1 does give the drawings we want.

Theorem 4.3.1. *Algorithm 1 is valid.*

Proof. It is clear that each new drawing in the output list is as required. Let D' be any good drawing in $\mathcal{D} + v$ with at most $cr(K_{n+1}) + \delta$ crossings. We have to prove that D' can be generated by Algorithm 1. To do this, it is sufficient to verify Step 8.

By definition, D' is obtained by adding a new vertex v to drawing $D \in \mathcal{D}$ in some face F and drawing new edges from v to each vertex v_i of D, $i = 1, 2, \cdots, n$. Here each new edge from v to v_i in D corresponds to a WALK in the dual graph G_D of D from F to some face incident to v_i.

In Algorithm 1, Step 8 only searches for all such PATHS. To see why finding paths is sufficient, suppose P is such a walk from F to a face F_i incident to v_i. If P is not a path, then there are repeated vertices on P, i.e., P goes through some face F_0 twice. D' is a good drawing, so P does not cross the same edge of D when it goes back to F_0. Hence P does not cross any side of F_0 twice. Furthermore, by Corollary A.1.2 in

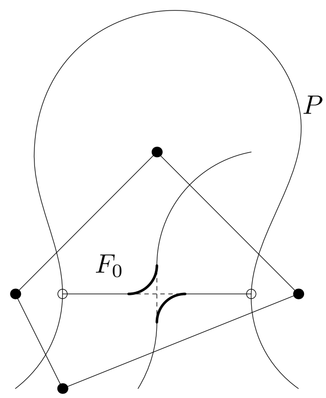


Figure 4.2: Modifying the shortcut

Appendix A the dual graph of G_D is simple, hence P goes back to F_0 by crossing a different side of F_0. Thus, after leaving F_0, P has to go through at least 2 other faces before going back to F_0. By removing from P these *redundant* faces and one copy of F_0 we have a new walk P' in the dual graph with length at least 3 less than P.

Now modify the edge from v to v_i by drawing a *shortcut* inside face F_0; the modified edge is corresponding to walk P'. If the shortcut crosses some other edge from v to v_j in face F_0, as shown in Figure 4.2 we can modify the curves to remove all such *internal* crossings in the same way as we did in the proof of Lemma 4.1.3. Then we have a new drawing D'' with at least 3 fewer crossings than D'. Thus, for $\delta \leqslant 2$,

$$cr(D'') \leqslant cr(D') - 3 \leqslant cr(K_{n+1}) + \delta - 3 < cr(K_{n+1})$$

, a contradiction. Hence P has to be a path. This validates the algorithm. □

4.3.2 Application and results

In this subsection, we present some particular applications of Algorithm 1 to our problem of determining $cr(K_{11})$.

1) Let $\mathcal{D}_n^*$ be the set of all optimal drawings of K_n. By Part (1) of Theorem 4.1.6, for $n \leqslant 8$, we have $\mathcal{D}_n^* \subseteq \mathcal{D}_{n-1}^* + v$. Hence

$$cr(\mathcal{D}_{n-1}^* + v) = cr(\mathcal{D}_n^*) = cr(K_n).$$

Thus by inputting $(\mathcal{D}, \delta) = (\mathcal{D}_{n-1}^*, 0)$ in Algorithm 1 we obtain $\mathcal{D}_n^*$. With the initial input $(\mathcal{D}_4^*, 0)$, by using Algorithm 1 iteratively we can find all the optimal drawings in $\mathcal{D}_5^*, \cdots, \mathcal{D}_8^*$.

Note that, in the proof of Part (1) of Theorem 4.1.6, $cr(K_n) = Z(n)$ is not necessary; $cr(K_{n-1}) = Z(n-1)$ and $cr(K_n) \leqslant Z(n)$ are sufficient. The results from our code show that, for $n = 5, 6, 7, 8$,

$$Lb\left[\mathcal{D}_{n-1}^*\right] = Z(n).$$

We may regard this as an algorithmic proof of Guy's Conjecture for $n \leqslant 8$. Alternatively, we may view this as confirming the correctness of our code. Furthermore, the number of non-isomorphic drawings found by our code is $1, 1, 5, 3$, respectively, which is exactly the same as in Guy [19].

2) By Part (2) of Theorem 4.1.6, in Algorithm 1

- with input $(\mathcal{D}_7^*, 2)$ we get all the good drawings of K_8 with at most 20 crossings, i.e., $\mathcal{D}_8^{\leqslant 20} = \mathcal{D}_8^{18} \cup \mathcal{D}_8^{19} \cup \mathcal{D}_8^{20}$,
- then with input $(\mathcal{D}_8^{\leqslant 20}, 0)$ we get $\mathcal{D}_9^*$,
- and with input $(\mathcal{D}_9^*, 2)$ we get $\mathcal{D}_{10}^{\leqslant 62} = \mathcal{D}_{10}^{60} \cup \mathcal{D}_{10}^{61} \cup \mathcal{D}_{10}^{62}$.

The results from our code show that there are **3080** optimal drawings of K_9 and there are **5679** optimal drawings of K_{10}, up to isomorphism.
Also the results show that

$$Lb\left[\mathcal{D}_8^{\leqslant 20}\right] = Z(9),$$

$$Lb\left[\mathcal{D}_9^*\right] = Z(10).$$

Similarly, we may regard this as an algorithmic proof of Guy's Conjecture for $n = 9, 10$.

3) With input $(\mathcal{D}_{10}^{\leqslant 62}, 0)$, the results from our code show that

$$Lb\left[\mathcal{D}_{10}^{\leqslant 62}\right] = 100 > 98.$$

Hence by Corollary 4.2.1 $cr(K_n) = Z(n)$ is true for $n = 11$. Moreover, by Corollary 4.1.2 we have $cr(K_{12}) = Z(12) = 150$.

Practically there are numerous good drawings in $\mathcal{D}_{10}^{\leqslant 62}$ and it would take lots of disk space to save them and very long time to check isomorphism. We actually first input $(\mathcal{D}_9^*, 2)$. Each time when the algorithm finds a good drawing $D_{10} \in \mathcal{D}_{10}^{\leqslant 62}$, instead of outputting D_{10}, we directly calculated $Lb\left[\{D_{10}\}\right]$ as we did in Step 1 of Algorithm 1. Finally we have $Lb\left[\mathcal{D}_{10}^{\leqslant 62}\right]$. Then we input $(\mathcal{D}_9^*, 0)$ to generate all the optimal drawings of K_{10}.

4) From the results of our code, we found that there are many optimal drawings of K_9 which generate no optimal drawing of K_{10}. It was known that not every optimal drawing of K_{n+1} contains an optimal drawing of K_n (see Guy [19] for an example). Now we also know that not every optimal drawing of K_n is contained in an optimal drawing of K_{n+1}.

4.4 Subroutines

In this section, we expand on the subroutines used in Algorithm 1. In particular,

1) how to find the minimum number of crossings in Step 1;
2) how to find all the face paths in Step 8;
3) how to determine if two face paths cross in Step 13; and
4) how to determine if two drawings are isomorphic in Step 15.

4.4.1 Minimum number of crossings in new drawings

For a good drawing D of K_n with vertices $v_i, i = 1, 2, \cdots, n$, and a face $F \in \mathcal{F}(D)$, let d_i be the minimum distance in the dual graph of D from F to the faces incident to v_i. The following algorithm finds $Lb\left[D_F\right] := \min\{cr(D) + \sum_{i=1}^{n} d_i \mid D \in \mathcal{D}\}$.

Algorithm 2.

Input: a good drawing D of K_n and face F.

Output: $Lb\,[D_F]$.

Procedure:

1) Set $i \leftarrow 1$;

2) Let $F_i^1, F_i^2, \cdots, F_i^{n-1}$ be the faces incident to vertex v_i;

3) Find distances d_{ij} from F to F_i^j, $j = 1, 2, \cdots, n-1$, in the dual graph of D. Let $d_i = \min_{1 \leqslant j \leqslant n-1} d_{ij}$;

4) If $i < n$, set $i \leftarrow i + 1$ and go to Step 2;

5) Output $cr(D) + \sum_{i=1}^{n} d_i$ as $Lb\,[D_F]$.

Theorem 4.4.1. *Algorithm 2 is valid, i.e.,*

$$cr(D) + \sum_{i=1}^{n} d_i = Lb\,[D_F]\,.$$

Proof. We need to prove that for any drawing $D' \in D_F + v$, $cr(D') \geqslant cr(D) + \sum_{i=1}^{n} d_i$. Let $D' \in D_F + v$ and let v be a new vertex in F. Note that any edge E drawn from v to v_i, $i = 1, 2, \cdots, n$, corresponds to a face path (P, v_i) in G_D starting from F. Each edge on path P corresponds to a new crossing created by E and some edge in D. Hence the total number of new crossings created by E is exactly the length of path P, which is at least d_i by the definition of d_i. Thus $cr(D') \geqslant cr(D) + \sum_{i=1}^{n} d_i$. □

Application. Algorithm 2 can be used to find c^i, the minimum number of crossings in Step 1 of Algorithm 1, where $\min\{c^1, c^2, \cdots, c^t\}$ is output as $Lb\,[\mathcal{D}]$.

4.4.2 Paths with restriction on endpoints and lengths

A generalized problem of finding all the paths in Step 8 of Algorithm 1 can be stated as:

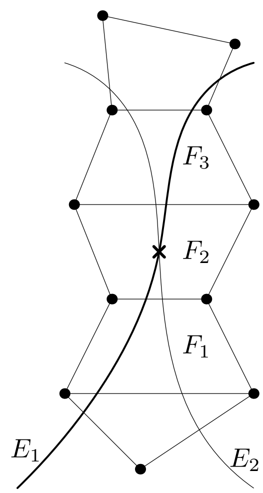


Figure 4.3: Two face paths cross

Problem: *Given one vertex v_1 and a vertex set S in graph G, let d be the distance from v_1 to S, i.e., $d := \min_{u \in S} d(v_1, u)$. We want to find all the paths from v_1 to S with length at most $d + \delta, \delta \in \{0, 1, 2\}$.*

One possible choice is to use the algorithm for searching for the *k-shortest paths*. There are several such algorithms, see [20] for example. We could input a large k, and stop as soon as we find the first path with length $> d + \delta$.

For convenience we didn't use the k-shortest paths algorithm. In our code we used depth-first search to find all the paths, i.e., find all the paths starting from v_1 with length in $[d, d + \delta]$ and only select those with the other endpoint in S.

4.4.3 Determine if two face paths cross

Given two face paths (P_1, v_1) and (P_2, v_2) in a drawing D which has a simple dual graph, we want to know if we can draw the corresponding new edges E_1, E_2 without crossing each other. One necessary condition is, that E_1, E_2 don't *have to* cross in each common face, which can be determined easily. However, this is not sufficient. As shown in Figure 4.3, E_1 and E_2 don't have to cross in each face F_i, $i = 1, 2, 3$. However they have to cross in one of them.

For two face paths not to cross each other, we need to revise the condition. Let $F_i \to F_{i+1} \to \cdots \to F_j$ be any common sub-path of (P_1, v_1) and (P_2, v_2). Let $F' = \left(\cup_{t=i}^{j} F_t\right) \bigcup \left(\cup_{t=i}^{j-1} int(\partial F_t \cap \partial F_{t+1})\right)$, i.e., F' is the combined region of faces $F_i, \cdots, F_j$. We require that E_1, E_2 don't have to cross in F', which is both necessary and sufficient.

4.4.4 Determine isomorphism of two drawings

Finally we discuss how to show two drawings are isomorphic.

Theorem 4.4.2. *Given two drawings D^1, D^2 of K_n, $n \geqslant 4$, then D^1 and D^2 are isomorphic if and only if G_{D^1} and G_{D^2} are isomorphic.*

Proof. By Corollary A.1.2 in Appendix A, G_{D^1}, G_{D^2} are 3-connected, and obviously simple. According to Whitney's Theorem (e.g., see Theorem 4.3.2 in [14], page 96), if a planar graph is simple and 3-connected, it has a unique drawing up to isomorphism. Hence we only need to determine if G_{D^1}, G_{D^2} are isomorphic. □

Thus the problem of drawing isomorphism boils down to graph isomorphism. In our code we used **nauty** to determine graph isomorphism.

nauty (**n**o **aut**omorphisms, **y**es?), by Brendan D. McKay, is a set of very efficient procedures written in C for determining the automorphism group of a vertex-colored graph. It is also able to produce a canonically labelled isomorph of the graph, which can be used to assist in isomorphism testing. Please see [26] for more details.

Appendix A

Connectivity

We prove that, for any $n \geqslant 4$, the planar graph of any good drawing of K_n is 3-connected, which has an application in the proof of Algorithm 1 for generating new drawings in Chapter 4. This also settles affirmatively the second open problem in Brodsky *et al.* [10], i.e., whether the planar graph of any rectilinear drawing of K_n is necessarily 3-connected.

We further prove that, for $n \geqslant 5$, the planar graph of any good drawing of K_n is 4-connected.

A.1 3-connectivity

In this section we prove by contradiction that, for any $n \geqslant 4$, the planar graph of any good drawing of K_n is 3-connected.

Theorem A.1.1. *The planar graph G_D of any good drawing D of $K_n (n \geqslant 4)$ is 3-connected.*

Proof. For $n = 4$, let the vertices of D be $v_i, i = 0, 1, 2, 3$. Then v_1, v_2, v_3 induce a good drawing of K_3. There is only one good drawing of K_3, which is *triangular* and has two faces. Let v_0 be in face F. As shown in Figure A.1, the edge v_0v_1 is be drawn either entirely inside F, or crosses only v_2v_3; there is no other possibility. Similar

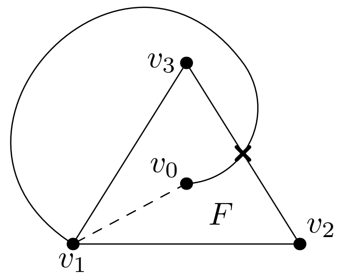


Figure A.1: Two possibilities of drawing edge v_0v_1

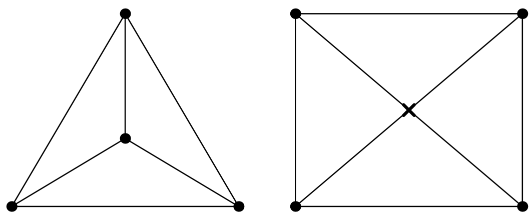


Figure A.2: Good drawings of K_4

arguments hold for the edges v_0v_2, v_0v_3. Then it is easy to see that there are only two good drawing of K_4, up to isomorphism, as shown in Figure A.2. The planar graph of either drawing is 3-connected.

For $n \geqslant 5$ to obtain contradiction, we suppose there is a separating set $S \subseteq V(G_D)$, $|S| \leqslant 2$. Then there is a partition C_1, C_2 of $V(G_D) \setminus S$ into nonempty sets so there is no edge of G_D between C_1 and C_2.

Now let m_i be the number of non-crossings in C_i, $i = 1, 2$, and m_0 be the number of non-crossings in S. Then $m_1 + m_2 + m_0 = n$.

First, we prove that, for $i = 1, 2$, $m_i > 0$. Suppose on the contrary $m_1 = 0$. Since C_1 is not empty, there is a crossing $v \in C_1$. Then there are four internally disjoint paths from v to four non-crossings $u_j \in S \cup C_2, j = 1, 2, 3, 4$, as shown in Figure A.3. Hence each of these paths goes through a vertex in S. However, $|S| \leqslant 2$, a contradiction. So $m_1 \neq 0$. Similarly $m_2 \neq 0$.

Let $u_1, u_2, \cdots, u_{m_1}$ be non-crossings in C_1, and $v_1, v_2, \cdots, v_{m_2}$ be non-crossings in C_2. Then, for each $j \in \{1, 2, \cdots, m_1\}$ and $k \in \{1, 2, \cdots, m_2\}$, there is $\{u_j, v_k\}$-path

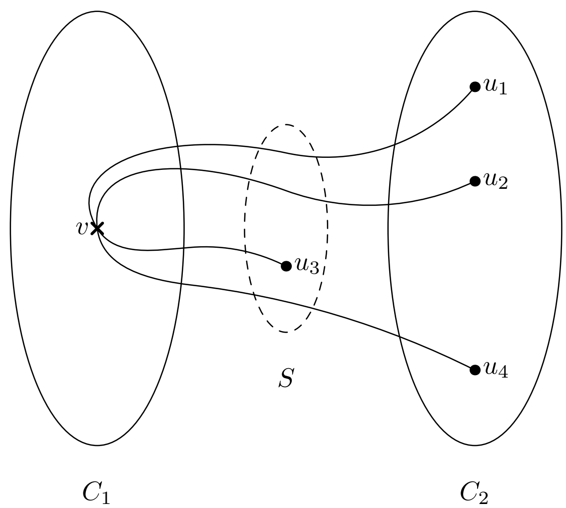


Figure A.3: No non-crossings in C_1

P_{jk} in G_D going through only crossings. Since D is a good drawing, $P_{j1}, P_{j2}, \cdots, P_{jm_2}$ are internally disjoint for any fixed j. So

$$m_2 \leqslant \text{number of crossings in } S = |S| - m_0 \leqslant 2 - m_0$$

Similarly $m_1 \leqslant 2 - m_0$. Thus

$$\begin{aligned} n &= m_1 + m_2 + m_0 \\ &\leqslant (2 - m_0) + (2 - m_0) + m_0 \\ &= 4 - m_0, \end{aligned}$$

which implies $m_0 \leqslant 4 - n \leqslant 4 - 5 < 0$, a contradiction. □

Since every optimal drawing is a good drawing by Lemma 4.1.3, Theorem A.1.1 implies that the planar graph of any optimal drawing of K_n, for $n \geqslant 4$, is 3-connected.

It is well known that the dual graph of any simple and 3-connected graph is simple and 3-connected (for example, see Theorem 2.6.7 in [27], page 46). Hence we have

Corollary A.1.2. *The dual graph of any good drawing of K_n, $n \geqslant 4$, is simple and 3-connected.*

A.2 4-connectivity

Let D be any good drawing of K_n, where $n \geqslant 5$. For $n \geqslant 5$, K_n is not planar. Then the planar graph G_D of D has at least one crossing vertex, which has degree 4. Thus G_D is at most 4-connected. In this section we prove that G_D *is* 4-connected.

Theorem A.2.1. *The planar graph G_D of any good drawing D of $K_n(n \geqslant 5)$ is 4-connected.*

Proof. To obtain a contradiction, suppose there is a separating set $S \subseteq V(G_D)$, $|S| \leqslant 3$. Then there is a partition C_1, C_2 of $V(G_D) \setminus S$ into nonempty sets so there is no edge of G_D between C_1 and C_2.

Let $m_i = |C_i|, i = 1, 2$, $m_0 = |S|$. As in the proof of Theorem A.1.1, we can prove that $m_i > 0, i = 1, 2$.

If $m_0 \geqslant 1$, let $w \in S$ be a non-crossing. By removing w in D and all the edges incident to w, we have a new good drawing D' of K_{n-1}, and $S' = S - w$ is a separating set of $G_{D'}$. Then $|S'| \leqslant 2$ and $n - 1 \geqslant 4$, contradicting Theorem A.1.1.
Hence we may assume $m_0 = 0$, and

$$m_i \leqslant |S| \leqslant 3, \ i = 1, 2$$

Note that $m_1 + m_2 = n \geqslant 5$. Without loss of generality, assume $m_1 \leqslant m_2$. Then $m_1 \in \{2, 3\}, m_2 = 3$ and $|S| = 3$. Let v_1, v_2 be non-crossings in C_1 and u_1, u_2, u_3 be non-crossings in C_2. Then for each $j \in \{1, 2\}, k \in \{1, 2, 3\}$ there is a $\{v_j, u_k\}$-path P_{jk} in G_D going through only crossings. Note that:

(disjoint argument) *since D is a good drawing, P_{j1}, P_{j2}, P_{j3} are internally disjoint for any fixed j, and P_{1k}, P_{2k} are internally disjoint for any fixed k.*

Let $S = \{w_1, w_2, w_3\}$. Without loss of generality assume that $w_1 \in P_{12} \cap P_{21}$, $w_2 \in P_{11}$ and $w_3 \in P_{22}$. Then v_1, v_2, u_1, u_2 induce a good drawing of K_4, which has a unique

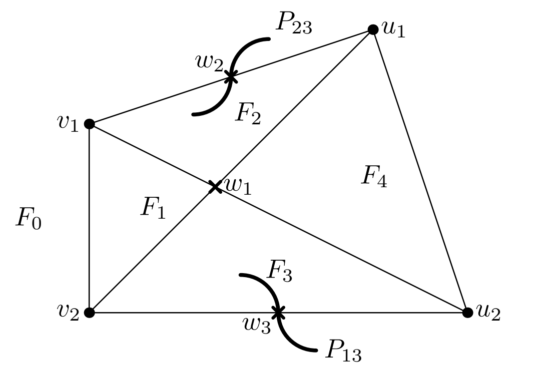


Figure A.4: An induced good drawing of K_4

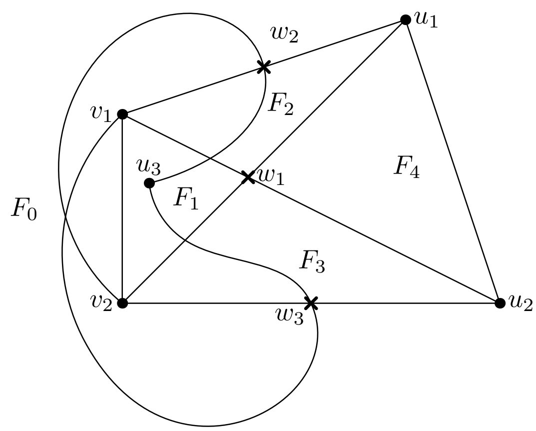


Figure A.5: $u_3 \notin F_1$

crossing w_1, and four inner *triangular* regions $F_1 = \triangle v_1v_2w_1, F_2 = \triangle v_1w_1u_1, F_3 = \triangle w_1v_2u_2, F_4 = \triangle w_1u_2u_1$, and an outer region F_0, as shown in Figure A.4. Since P_{11}, P_{13} are internally disjoint, $w_2 \notin P_{13}$, hence $w_3 \in P_{13}$. Similarly $w_2 \in P_{23}$. Since we don't allow tangent edges in a good drawing, P_{13} has parts in both F_3 and F_0, while P_{23} has parts in both F_2 and F_0.

Consider the position of u_3.

1) Suppose $u_3 \in F_1$. As shown in Figure A.5, by the disjoint argument P_{13} and P_{23} can only be drawn as:

$$P_{13} \; : \; v_1 \to F_0 \to w_3 \to F_3 \to F_1 \to u_3,$$

$$P_{23} \; : \; v_2 \to F_0 \to w_2 \to F_2 \to F_1 \to u_3.$$

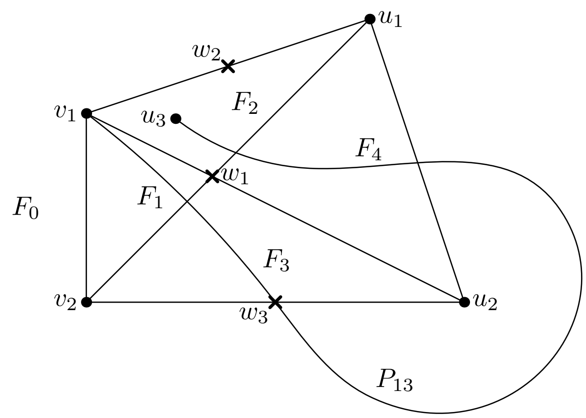


Figure A.6: $u_3 \notin F_2$

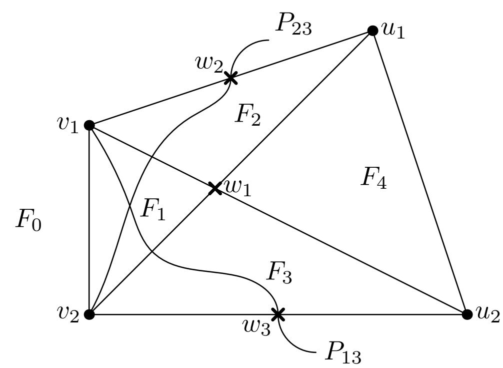


Figure A.7: $u_3 \notin F_0 \cup F_4$

Then P_{13} and P_{23} have to cross each other in face F_0, a contradiction.

2) Suppose $u_3 \in F_2$. As shown in Figure A.6, by the disjoint argument P_{13} can only be drawn as:

$$P_{13} \quad : \quad v_1 \to F_1 \to F_3 \to w_3 \to F_0 \to F_4 \to F_2 \to u_3.$$

Then P_{13} and P_{21} have to cross each other at least twice, a contradiction.

3) Similarly $u_3 \in F_3$ leads to a contradiction.

4) Suppose $u_3 \in F_0$ or $u_3 \in F_4$. As shown in Figure A.7, by the disjoint argument the sub-path of P_{13} from v_1 to w_3 and the sub-path of P_{23} from v_2 to w_2 can only

be drawn as:

$$P_{13}|_{v_1 \to w_3} \quad : \quad v_1 \to F_1 \to F_3 \to w_3$$
$$P_{23}|_{v_2 \to w_2} \quad : \quad v_2 \to F_1 \to F_2 \to w_2$$

Then P_{13} and P_{23} have to cross each other inside F_1, a contradiction. □

Since every optimal drawing is a good drawing by Lemma 4.1.3, Theorem A.2.1 implies that the planar graph of any optimal drawing of K_n, for $n \geqslant 5$, is 4-connected.

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