Properties of Two-Dimensional Words

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Combinatorics on words in one dimension is a well-studied subfield of theoretical computer science with its origins in the early 20th century. However, the closely-related study of two-dimensional words is not as popular, even though many results seem naturally extendable from the one-dimensional case. This thesis investigates various properties of these two-dimensional words.

In the early 1960s, Roger Lyndon and Marcel-Paul Schützenberger developed two famous results on conditions where nontrivial prefixes and suffixes of a one-dimensional word are identical and on conditions where two one-dimensional words commute. Here, the theorems of Lyndon and Schützenberger are extended in the one-dimensional case to include a number of additional equivalent conditions. One such condition is shown to be equivalent to the defect theorem from formal languages and coding theory. The same theorems of Lyndon and Schützenberger are then generalized to the two-dimensional case.

The study of two-dimensional words continues by considering primitivity and periodicity in two dimensions, where a method is developed to enumerate two-dimensional primitive words. An efficient computer algorithm is presented to assist with checking the property of primitivity in two dimensions. Finally, borders in both one and two dimensions are considered, with some results being proved and others being offered as suggestions for future work. Another efficient algorithm is presented to assist with checking whether a two-dimensional word is bordered.

The thesis concludes with a selection of open problems and an appendix containing extensive data related to one such open problem.

Keywords: bordered word, combinatorics on words, formal language theory, Lyndon–Schützenberger theorem, periodic word, primitive word, two-dimensional word

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List of Symbols

$\emptyset$	Empty set
ϵ	Empty word
Σ	Finite alphabet
Σ^*	Set of all finite one-dimensional words over Σ
Σ^+	Set of all finite one-dimensional words over Σ, excluding the empty word
$\Sigma^{m \times n}$	Set of all finite two-dimensional words over Σ of dimension $m \times n$
a	Symbol in Σ
$p, q, r, \ldots$	One-dimensional word
$A, B, C, \ldots$	Two-dimensional word
L	Language
$|L|$	Cardinality or size of a language
$|w|$	Length of a one-dimensional word w
$|A|$	Number of symbols in a two-dimensional word A
A_{RM}	Two-dimensional word A in row-majorized representation
A_{CM}	Two-dimensional word A in column-majorized representation
$\ominus$	Horizontal concatenation of two-dimensional words
$\oplus$	Vertical concatenation of two-dimensional words

Chapter 1

Introduction

1.1 Overview of Combinatorics on Words

Combinatorics on words is a field spanning both computer science and mathematics that focuses on combinatorial properties applied to formal languages. Its origins were largely algebraic, and were focused on finding patterns within sequences of symbols. However, the field has since expanded to other aspects of the mathematical sciences, including algorithms, complexity, coding theory, and even physics (particularly, symbolic dynamics).

The study of combinatorics on words originated in the early 1900s with the work of Axel Thue, who published two papers on patterns and repetitions within words [76]. However, the importance of Thue's work was not recognized until several decades later, when research in this area became very popular. The first book on combinatorics on words appeared in 1983 [68], and it was written by "M. Lothaire", a collective pseudonym for a group of researchers.

Combinatorics on words is closely related to the study of formal languages, due to the fact that both fields investigate various properties of the same mathematical object: words. Some of the more popular research areas in combinatorics on words include structures within words (e.g., words that contain or avoid certain patterns); types of words (e.g., infinite words); collections of words—also known as languages—and their properties (e.g., regularity, context-freeness, etc.); representing words and languages visually (e.g., using automata); and connections between words and mathematics (e.g., to group theory).

For more details on the origins of the study of combinatorics on words, Berstel and Perrin have published a comprehensive survey of the field's history [19].

1.2 Contributions of the Thesis

This thesis presents the following results. In Sections 3.1 and 3.2, we extend the well-known theorems of Lyndon and Schützenberger in the one-dimensional case to include a number of additional equivalent conditions. We show that one condition is equivalent to the defect theorem, which states that if a set of n words is nontrivially related, then they can be expressed as products of at most $n - 1$ words. We then generalize the same theorems of Lyndon and Schützenberger to the two-dimensional case in Sections 4.1, 4.2, and 4.3. In Chapter 5, we investigate the properties of primitivity and periodicity in two dimensions. We present a formula and a linear-time algorithm to assist with enumerating two-dimensional primitive words and to check primitivity in two dimensions. Chapter 6 considers borders in both one and two dimensions. We present a result on enumerating one-dimensional bordered words as polynomials and discuss a generalization of bordered and unbordered words in two dimensions. We also provide another linear-time algorithm to assist with checking whether a given two-dimensional word is bordered. A variety of open problems are included in Chapter 7.

Some of the results in this thesis first appeared in the author's paper with Gamard, Richomme, and Shallit [42]. These results, and the relevant sections of the paper where they first appeared, are listed in a footnote at the beginning of their respective chapters.

1.3 Chapter Outlines

This thesis is divided into seven chapters, which discuss the following topics:

1 Introduction
The present chapter aims to provide the reader with a brief history of combinatorics on words and an overview of the material contained within this thesis.

2 Background and Related Work
This chapter contains the various definitions required to understand the material presented in later chapters. We give an overview of previous work relating to this thesis, and we prove some important theorems that will be used in later chapters.

3 One-Dimensional Lyndon–Schützenberger Theorems
This chapter takes a closer look at two well-known theorems of Lyndon and Schützenberger. We present extensions of these theorems. We show that each theorem admits additional conditions and that these new conditions are equivalent to one another.

4 Two-Dimensional Lyndon–Schützenberger Theorems

This chapter builds upon the work in the previous chapter by extending the Lyndon–Schützenberger theorems to the two-dimensional case. We develop two-dimensional equivalents of each condition in the original theorems. We study the second Lyndon–Schützenberger theorem for two-dimensional words in both the overlapping and bordered case.

5 Two-Dimensional Primitivity and Periodicity

This chapter considers the properties of primitivity and periodicity in two-dimensional words. We review properties of one-dimensional primitive words. We present a formula to enumerate all two-dimensional primitive words of a given dimension, and we develop an efficient algorithm for verifying the primitivity of a two-dimensional word and computing its primitive root.

6 Two-Dimensional Borders

This chapter considers borders in two-dimensional words. We review properties of one-dimensional bordered words and prove a small result on the enumeration of one-dimensional bordered words as polynomials. We present a discussion on techniques for enumerating two-dimensional bordered words, and we develop an efficient algorithm for verifying that a two-dimensional word is bordered. An appendix containing data relating to this chapter is included at the end of the thesis.

7 Conclusions

This chapter summarizes the results of the thesis and offers suggestions for future work relating to this research.

Chapter 2

Background and Related Work

2.1 Preliminaries

In this section, we introduce some basic terminology and definitions.

We begin with terminology relating to combinatorics of words in one dimension. This terminology is standard (see, for example, Hopcroft and Ullman's text [56]) and so it shall not be discussed here in great detail. Let $\Sigma = \{0, \ldots, k-1\}$ represent some nonempty finite k-ary *alphabet*. A *word* $w = a_0 \cdots a_{n-1}$ over Σ (sometimes also called a *string*) is composed of *symbols* $a_0, \ldots, a_{n-1} \in \Sigma$. The *length* of this word w is denoted $|w| = n$. If $|w| = 0$, then we say that w is the *empty word* and we denote it by the symbol ϵ. If we can decompose a word w into the form $w = xyz$, then we say that y is a *subword* of w. A *language* L over Σ is a set of words over Σ. Languages may be finite or infinite. There exist two special languages: the set of all words over Σ including the empty word, denoted Σ^*, and the set of all nonempty words over Σ, denoted Σ^+.

We often wish to apply operations to words. Two words w and x can be combined, or *concatenated*, by joining the last symbol of w to the first symbol of x. We denote concatenation by placing the two word variables side by side, like wx. If $w = a_0 \cdots a_{n-1}$ is a word, then $w^m = \underbrace{(a_0 \cdots a_{n-1}) \cdots (a_0 \cdots a_{n-1})}_{m}$ is the word w concatenated with itself m times. The word w^m is called a *power* of w.

Example 2.1. Let $w = \mathtt{ha}$. Then w concatenated with itself gives the word $w^2 = \mathtt{haha}$, which is also a second power of w.

We conclude by defining some properties of words. We say that a word w is *periodic* if it can be written as a power of some smaller word z repeated $e \geq 2$ times, and we write $w = z^e$ if this is the case.

If a word w is not periodic, then we say that it is *primitive*. We observe the following property of primitive words, with the proof based on that given in Shallit's text [68, 80]:

Proposition 2.2. *For any nonempty word w, the smallest word z such that $w = z^e$ for some integer $e \geq 1$ is primitive. Moreover, the word z is unique.*

Proof. To prove primitivity, take the largest integer $1 \leq e \leq |w|$ such that $w = z^e$ has a solution. If the word z were not primitive, then we could write $z = y^f$ for some word y and integer $f \geq 2$. Therefore, we would have $w = (y^f)^e$ where $fe > e$; a contradiction.

The proof of uniqueness is left for Section 2.3. □

We call the unique word z obtained from Proposition 2.2 the *primitive root* of w.

Example 2.3. The word $v = \texttt{bird}$ is primitive. The word $w = \texttt{dodo}$ is periodic, since we can write $w = \texttt{dodo} = (\texttt{do})^2$. The primitive root of w is $z = \texttt{do}$.

Given two words $w, x \in \Sigma^*$, if there exists a word $y \in \Sigma^*$ such that $w = xy$ (respectively, $w = yx$), then we say that x is a *prefix* (respectively, a *suffix*) of w. Observe that, in both cases, we allow y to be empty. If $y \neq \epsilon$, then we say that x is a *proper prefix* (respectively, a *proper suffix*).

Example 2.4. Let $w = \texttt{haystack}$. Some prefixes of w include the words $\texttt{hay}$, $\texttt{hays}$, and $\texttt{haystack}$. Some suffixes of w include the words $\texttt{tack}$, $\texttt{stack}$, and $\texttt{haystack}$. The first two words in each list are also proper prefixes or proper suffixes, respectively.

If we can write a word w as $w = axaxa$, where $x \in \Sigma^*$ and $a \in \Sigma$, then we say that w is an *overlap*. This is because the subword axa in w is overlapping itself. Similarly, if w can be written as $w = xyx$, where $x \in \Sigma^+$ and $y \in \Sigma^*$, then we say that w is *bordered*. This is because the subword x acts as a "border" surrounding the overall word w.

Example 2.5. Let $w = \texttt{alfalfa}$. This word contains an overlap; namely, $w = axaxa$ where $a = \texttt{a}$ and $x = \texttt{lf}$, so the subword $\texttt{alfa}$ overlaps itself. This word is also bordered; namely, $w = xyx$ where $x = \texttt{a}$ and $y = \texttt{lfalf}$.

We now move on to terminology relating to combinatorics of words in two dimensions. The terminology given here is common in most of the literature on the topic [43, 75]. The definition of an alphabet does not change, as we require no special properties of alphabets in two dimensions. However, we do require a two-dimensional analogue for the notion of a *word*.

Definition 2.6 (Two-dimensional word). A two-dimensional word

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,n-1} \end{bmatrix}$$

is a map from $\{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$ to an alphabet Σ.

In this way, one may think of a two-dimensional word as being similar to an array or a matrix. By convention, we take $A[0, 0]$ to be the upper-left corner of the two-dimensional word A. We also say that $A[i..j, k..l]$ is the two-dimensional subword of A contained within rows i to j and within columns k to l.

We also give a handy notation for discussing *sets* of two-dimensional words.

Definition 2.7 (Set of two-dimensional words). The set of two-dimensional words $\Sigma^{m \times n}$ contains all two-dimensional words of dimension $m \times n$ over Σ.

Note that, occasionally, two-dimensional words may also be referred to as "pictures" or "figures". The latter term is often used when a number of rectangular two-dimensional words are combined to form a non-rectangular two-dimensional word. For more details on two-dimensional words, see the survey papers by Giammarresi and Restivo [43] and Morita [75].

We can generalize the notion of word length by saying that if $A \in \Sigma^{m \times n}$, then $|A| = mn$. The *length* of a two-dimensional word is therefore the number of symbols within that word. Although the one-dimensional empty word was the unique one-dimensional word of length zero, we cannot simply define the two-dimensional analogue to be the two-dimensional word of dimension 0×0. Although such a word is indeed empty, it is not unique. (Consider, for instance, two-dimensional words of dimension 1×0 or 0×2.) Therefore, we generalize the notion of the empty word to two dimensions by saying that *any* two-dimensional word with at least one dimension being equal to zero is "empty".

Certain operations may also be applied to two-dimensional words, with the proper generalization. We begin by discussing how to *concatenate* pairs of two-dimensional words. In one dimension, we could only concatenate words w and x side-by-side, which made the operation straightforward; in two dimensions, however, we must distinguish between concatenation in the horizontal direction and concatenation in the vertical direction and, furthermore, we must have a matching of dimensions in that direction.

Definition 2.8 (Concatenation of two-dimensional words). Let $A \in \Sigma^{m_1 \times n_1}$ and $B \in \Sigma^{m_2 \times n_2}$. If $n_1 = n_2 = n$, then the horizontal concatenation of A and B, denoted $A \ominus B$, is the $(m_1 + m_2) \times n$ two-dimensional word produced by placing B underneath A. Likewise, if $m_1 = m_2 = m$, then the vertical concatenation of A and B, denoted $A \oplus B$, is the $m \times (n_1 + n_2)$ two-dimensional word produced by placing B to the right of A.

Example 2.9. Given two-dimensional words

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \ B = \begin{bmatrix} 7 & 8 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 3 \\ 6 \end{bmatrix},$$

the horizontal concatenation of A with B and the vertical concatenation of A with C are, respectively,

$$A \ominus B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \text{ and } A \oplus C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Note that we cannot concatenate A with B vertically, nor can we concatenate A with C horizontally, since we do not have a matching of dimensions in those respective directions.

We may also generalize the notion of *powers* by considering repeated concatenations of two-dimensional words.

Definition 2.10 (Power of a two-dimensional word). Let $A \in \Sigma^{m \times n}$. The $p \times q$ power of A, written $A^{p \times q}$, is the $pm \times qn$ two-dimensional word with the property that $A^{p \times q}[i, j] = A[i \bmod p, j \bmod q]$.

Example 2.11. Given the two-dimensional word $A = \begin{bmatrix} 4 & 6 \end{bmatrix}$, which is evidently of dimension 1×2, the 2×3 power of A is

$$A^{2 \times 3} = \begin{bmatrix} 4 & 6 & 4 & 6 & 4 & 6 \\ 4 & 6 & 4 & 6 & 4 & 6 \end{bmatrix}.$$

Take care not to confuse the notation for powers, written $A^{p \times q}$, with the notation for a set of two-dimensional words, written $\Sigma^{m \times n}$. The latter will always be denoted using the Greek letter Σ, while the former may use any capital Latin letter.

We can define *periodicity* and *primitivity* for two-dimensional words as well. The properties are defined similarly to those in one dimension.

Definition 2.12 (Two-dimensional periodic word)**.** A two-dimensional word A is periodic if it can be written as $A = B^{p \times q}$ with either $p \geq 2$ or $q \geq 2$.

Definition 2.13 (Two-dimensional primitive word)**.** A two-dimensional word A is primitive if it is not periodic.

Example 2.14. The two-dimensional word $A = \begin{bmatrix} 2 & 4 \end{bmatrix}$ is primitive. On the other hand,

$$B = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

is periodic, since we can write $B = A^{2 \times 1}$.

Two-dimensional words may also contain *prefixes* or *suffixes*.

Definition 2.15 (Prefix/suffix of a two-dimensional word)**.** A two-dimensional word A has a prefix (respectively, a suffix) B if there exists a possibly empty two-dimensional word C such that $A = B \ominus C$ or $A = B \oplus C$ (respectively, $A = C \ominus B$ or $A = C \oplus B$).

The definition of a *proper prefix* or *proper suffix* of a two-dimensional word follows naturally from the above definition and from the one-dimensional case. A two-dimensional word B is a proper prefix or suffix of another two-dimensional word A if and only if B is a prefix or suffix of A and $B \neq A$ (that is, if and only if C is nonempty).

Example 2.16. The two-dimensional word

$$A = \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{bmatrix}$$

has, for example, prefixes

$$B_1 = \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}, \quad \text{and } B_3 = \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{bmatrix},$$

of which B_1 and B_2 are proper prefixes of A. By contrast, the two-dimensional word

$$B_4 = \begin{bmatrix} 9 & 6 \\ 8 & 5 \end{bmatrix}$$

is not a prefix of A because, by the definition of a prefix, we must have a matching in at least one dimension.

If the concatenation operation is $\ominus$, then we say that the two-dimensional word A has a (proper) prefix or suffix in the horizontal direction. Likewise, if the concatenation operation is $\oplus$, then we say that A has a (proper) prefix or suffix in the vertical direction.

We can consider the notions of overlap and border with respect to two-dimensional words. Intuitively speaking, *overlap* suggests we have some matching of symbols in one dimension, while *border* suggests we have some matching of symbols in both dimensions. We begin with the definition of two-dimensional overlapping words.

Definition 2.17 (Overlap of two-dimensional words). A pair of two-dimensional words A and B overlap if there exists a two-dimensional word C such that C is a suffix of A and a prefix of B or vice versa.

The notion of both horizontal and vertical overlap are illustrated in Figure 2.1.

Example 2.18. The 2 two-dimensional words

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$$

share a horizontal overlap $C_1 = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ and a vertical overlap $C_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(a) Horizontal overlap

(b) Vertical overlap

Figure 2.1: Illustrations of overlap

In a paper of Anselmo, Giammarresi, and Madonia [8], the authors refer to horizontal overlap as "h-slide overlap" and to vertical overlap as "v-slide overlap".

We now turn to the definition of a two-dimensional bordered word, which is, intuitively speaking, a combination of the two types of overlap. In another paper [7], Anselmo, Giammarresi, and Madonia state informally that a two-dimensional word is bordered "when we can find the same rectangular portion at two opposite corners" of the word. This notion, although similar in name, is not the same as our notion of border below. The paper of Anselmo et al. considers a border to be only an overlap of corners, whereas here we consider a border to be similar to a picture frame. In this way, we have an overlap from both the top-left and bottom-right corners to both the top-right and bottom-left corners.

Definition 2.19 (Border of a two-dimensional word)**.** A two-dimensional word A is bordered if there exist a nonempty two-dimensional word Q and possibly empty two-dimensional words R, S, and T such that

$$A = (Q \oplus R \oplus Q) \ominus (S \oplus T \oplus S) \ominus (Q \oplus R \oplus Q).$$

Informally, a two-dimensional bordered word has the subword Q appearing in each corner and the subword T in the center, with subwords R and S completing the frame around T. The notion of a two-dimensional border is illustrated in Figure 2.2.

Example 2.20. The two-dimensional word

$$A = \begin{bmatrix} 7 & 4 & 1 & 7 & 4 \\ 6 & 8 & 0 & 6 & 8 \\ 3 & 2 & 9 & 3 & 2 \\ 7 & 4 & 1 & 7 & 4 \\ 6 & 8 & 0 & 6 & 8 \end{bmatrix}$$

is bordered, with $Q = \begin{bmatrix} 7 & 4 \\ 6 & 8 \end{bmatrix}$, $R = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $S = \begin{bmatrix} 3 & 2 \end{bmatrix}$, and $T = \begin{bmatrix} 9 \end{bmatrix}$.

$$A = \begin{array}{|c|c|c|} \hline Q & R & Q \\ \hline S & T & S \\ \hline Q & R & Q \\ \hline \end{array}$$

Figure 2.2: Subwords of a two-dimensional bordered word A

2.2 Previous Work

The literature features a good deal of previous work on two-dimensional words, their properties, and algorithms. This section gives a brief summary of the existing body of knowledge relating to two-dimensional words.

General. There are a number of good survey articles on two-dimensional words and languages, such as those by Giammarresi and Restivo [43] and Morita [75].

Much of the early work on two-dimensional words related to polyominoes and tilings in the plane [44, 45, 46]. Work in this area later extended the notion of codes to two dimensions. These objects were called "picture codes" or "pictures" [26].

Origins. No research on combinatorics on words is complete without referencing the seminal papers of Thue in the early 20th century. In Thue's 1906 paper [83], he discusses the first notions of primitivity by studying repetitions in finite words and by constructing squarefree words over three and four symbols. In Thue's 1912 paper [84], he again deals with repetitions in finite words, but here he studies so-called "irreducible" words with a focus on such words over two and three symbols.

Bordered words. Anselmo, Giammarresi, and Madonia generalized unbordered words to two dimensions in their paper [7]. They also devised a method to construct unbordered two-dimensional words of a given dimension, as well as to construct "quasi-unbordered" two-dimensional words. (Informally, a two-dimensional word is quasi-unbordered if it has no border on its right-hand side only.)

Anselmo, Jonoska, and Madonia study the so-called "framed" and "unframed" two-dimensional words, which appears to be somewhat similar to the "picture frame" characterization of a border given in the previous section [9].

Holub and Shallit studied borders and periods in random words [55]. They showed that the asymptotic probability that a random word has a border of length at most k is constant both in k and in the alphabet size l, and that there exists a recurrence to determine these constants. They also give the probability of finding an unbordered random word, or a random word with border length l.

A "bifix" is a subword that is both a prefix and a suffix of some word. The notion of bifixes is identical to that of borders, so a good amount of work on bordered words may

be found in the literature on bifixes. Unger introduced bifixes in 1960 [87], and Nielsen devised a systematic way of generating all bifix-free sequences over some alphabet, and of recursively enumerating these sequences [77].

Bajic, Stojanovic, and Lindner introduced "cross-bifix-free" words [12], which generalizes bifixes to the case where, given two words in a language, no prefix of one word is a suffix of the other. Bajic later gave a method of constructing cross-bifix-free words [11]. Bilotta, Pergola, and Pinzani developed a similar method for binary cross-bifix-free words of a fixed length, as well as a method to find nonexpandable subsets of such words [21]. Chee, Kiah, and Purkayastha developed a method similar to Bajic for non-binary cross-bifix-free words [28].

Bifixes and cross-bifixes were extended to the two-dimensional case by Barcucci, Bernini, Bilotta, and Pinzani [14]. In their paper, the authors developed a method of constructing all two-dimensional words with these properties. Notably, they also coined the terms "bibifix" and "cross-bibifix" for two-dimensional bifix and cross-bifix, respectively.

Overlap in words. Blackburn's paper [23] provides a method of constructing non-overlapping codes which, as it turns out, are equivalent to cross-bifix-free codes. The work in this paper was extended by Barcucci, Bernini, Bilotta, and Pinzani [15], who generalized Blackburn's method to two dimensions and provided both an enumeration method and a generating function for two-dimensional non-overlapping words. Anselmo, Giammarresi, and Madonia [8] later solved an open problem given by Barcucci et al. by presenting a method to construct a non-expandable set of two-dimensional non-overlapping words.

1D pattern matching. The bibliography on one-dimensional pattern matching is vast. Aside from the Knuth–Morris–Pratt pattern matching algorithm, which will be mentioned in a later chapter, one-dimensional pattern matching techniques will not be discussed in great detail in this thesis.

For a comprehensive overview of the literature on this topic, see Beebe and Salomon's bibliography [16].

2D pattern matching. Two-dimensional pattern matching was first studied by Bird [22] and by Baker [13], who both independently extended the Knuth–Morris–Pratt algorithm to two dimensions. Their extensions only worked for *exact* pattern matching, and the algorithms run in $O(n^2 \log |\Sigma|)$ time for a bounded alphabet or $O(n^2 \log(m))$ time for an unbounded alphabet.

Amir and Landau presented a method for *approximate* pattern matching in two dimensions [5]. Their algorithm runs in $O(n^d(dk + k^2))$ time on a serial processor.

Amir, Landau, and Vishkin defined the notion of pattern matching with scaling, or detecting patterns in a two-dimensional word that are scaled to natural multiples of the size of the word [6]. They gave an algorithm for this problem that runs in $O(n^2 \log |\Sigma|)$ time. Amir and Farach later studied pattern matching in two-dimensional words where the patterns are square [4], and Idury and Schäffer gave an algorithm for pattern matching in two dimensions where each pattern is rectangular [59].

The first deterministic algorithm to solve the problem of two-dimensional exact pattern matching was given by Amir, Benson, and Farach [3]. Their algorithm runs in $O(n^2)$ time for text processing with a $O(n)$ matching step. The algorithm is alphabet independent for the text processing step, but not for the pattern processing step. Around the same time, Galil and Park gave a similar algorithm that is alphabet independent in both steps and runs in $O(m^2 + n^2)$ time [38].

A survey of two-dimensional pattern matching algorithms can be found in Chapter 12 of Crochemore and Rytter's text [34].

Periodic words. The periodicity lemma of Fine and Wilf [37], which we shall discuss briefly in the next section, is fundamental to the study of periodic words. This lemma shows that the longest word with periods p and q, but not with period $\gcd(p, q)$, is of length $p + q - \gcd(p, q) - 1$.

Holub extended Fine and Wilf's periodicity lemma to words with multiple periods [52, 53]. Holub shows that the longest word with periods $p_1, p_2, \ldots, p_n$, but not with period $\gcd(p_1, p_2, \ldots, p_n)$, can be determined algorithmically. Interestingly, he also shows that this longest word must be a palindrome. In a later paper [54], Holub gave a concise description of an algorithm originally published by Tijdeman and Zamboni [85, 86].

Ehrenfeucht and Silberger studied the connection between repetitions in one-dimensional words and unbordered segments of those words [36, 81]. Silberger later studied the enumeration of one-dimensional unbordered words [82].

Guibas and Odlyzko investigated the property of periodicity in one-dimensional words [47, 48]; their work arose from previous investigations into pattern matching algorithms, and how knowing the structure of a periodic string could affect the performance of such an algorithm.

Amir and Benson studied periodicity in two dimensions in a series of papers [1, 2]. Crochemore, Iliopoulos, and Korda [33] and, more recently, Gamard and Richomme [40, 41], considered quasiperiodicity in two dimensions. Régnier and Rostami studied the more general case of periodicity and space covering in d dimensions [78].

Periodicity has been studied in random words as well. See the paper of Holub and Shallit [55] referenced in the earlier discussion on bordered words.

Primitive words. Bacquey studied primitive roots in two-dimensional biperiodic infinite words [10]. Bacquey showed that all two-dimensional words with two periods contain at least one primitive root. Moreover, there are at most two ordered pairs of positive integers (m, n) where every primitive root is of dimension $m \times n$, and every $m \times n$ rectangular pattern is a primitive root of the word.

Other topics. Marcus and Sokol [72] considered two-dimensional Lyndon words. A Lyndon word is a word that is lexicographically strictly smaller than its circular shifts.

2.3 Important Theorems

In this section, we outline a few fundamental theorems that we will see in later chapters.

We begin with Levi's lemma, named for the German mathematician Friedrich Wilhelm Levi [67]. This rather basic result is not provided out of interest, but instead because some future theorems use this lemma as an intermediate step.

Lemma 2.21 (Levi's lemma). *Let $u, v, x, y \in \Sigma^*$ and suppose that $uv = xy$. If $|u| \geq |x|$, then there exists $t \in \Sigma^*$ such that $u = xt$ and $y = tv$. If $|u| < |x|$, then there exists $t \in \Sigma^+$ such that $x = ut$ and $v = ty$.*

Proof. Immediate. $\square$

We proceed to state two famous theorems due to the American mathematician Roger Lyndon and the French mathematician Marcel-Paul Schützenberger. These theorems, as well as their generalizations and extensions, will be the focus of the next two chapters of this thesis.

The first theorem of Lyndon and Schützenberger tells us under what conditions a word has a nontrivial proper prefix and suffix that are identical. Here, we reproduce the

statement and proof of the first theorem from the paper of Lyndon and Schützenberger [69], albeit slightly modified to fit the context of combinatorics on words rather than the authors' original context of elements in a free group.

Theorem 2.22 (First Lyndon–Schützenberger theorem). *Let $x, y, z \in \Sigma^+$. If $xy = yz$, then there exist $u \in \Sigma^+$, $v \in \Sigma^*$, and an integer $e \geq 0$ such that $x = uv$, $z = vu$, and $y = (uv)^e u = u(vu)^e$.*

Proof. Suppose $|x| \geq |y|$. Then, by Levi's lemma, there exists a word $v \in \Sigma^*$ such that $x = yv$ and $z = vy$. Take $u = y$ and $e = 0$ to complete the proof for this case.

Otherwise, $|x| < |y|$. Then, by Levi's lemma once again, there exists a word $y' \in \Sigma^+$ such that $y = xy'$. Rewrite our expression as $x(xy') = (xy')z$ and observe that $xy' = y'z$. Since x is nonempty, we know that $|y'| < |y|$. The result follows by an induction on the length of y. $\qquad\square$

The second theorem of Lyndon and Schützenberger essentially states a number of conditions under which two words x and y commute. In their paper, Lyndon and Schützenberger present a number of equivalent results separately, but here we combine these results into one theorem. Again, we reproduce a restated version of the results and their associated proofs from the authors' paper [69]. The directions $(1) \Rightarrow (2)$ and $(3) \Rightarrow (2)$ are given by Lyndon and Schützenberger; the remaining directions are provided for completeness. (Note that it is possible to prove this theorem using the directions $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$, but the proof given here remains true to the original paper.)

Theorem 2.23 (Second Lyndon–Schützenberger theorem). *Let $x, y \in \Sigma^+$. Then the following conditions are equivalent:*

(1) $xy = yx$;

(2) *There exist $z \in \Sigma^+$ and integers $i, j \geq 1$ such that $x = z^i$ and $y = z^j$;*

(3) *There exist integers $k, l \geq 1$ such that $x^k = y^l$.*

Proof. $(1) \Rightarrow (2)$: We proceed by induction on the length of xy. If $|xy| = 2$, then $|x| = |y| = 1$, so take $z = x = y$ and $i = j = 1$. Otherwise, assume our statement is true for $|xy| < n$ and, without loss of generality, assume $|x| \geq |y|$. Applying Lemma 2.21 to x and y with $u = y$ and $v = x$ produces the relation $x = yt = ty$ for some word $t \in \Sigma^*$. If $|x| = |y|$, then we can take $z = x = y$ and $i = j = 1$ as before. Otherwise, $|ty| < |xy| = n$,

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so by induction there exist a word $z \in \Sigma^+$ and integers $i, j \geq 1$ such that $t = z^i$ and $y = z^j$. It follows that $x = z^{i+j}$.

$(2) \Rightarrow (1)$: If $x = z^i$ and $y = z^j$, then $xy = z^i z^j = z^{i+j} = z^j z^i = yx$.

$(2) \Rightarrow (3)$: If $x = z^i$ and $y = z^j$, then take $k = j$ and $l = i$ to get $x^k = (z^i)^k = (z^i)^j = z^{ij} = (z^j)^i = (z^j)^l = y^l$.

$(3) \Rightarrow (2)$: If $k = 1$ or $l = 1$, then the proof is immediate. Otherwise, we have that $|x^k| = |y^l| \geq |x| + |y|$. This implies that x^k and y^l share a prefix of length $\geq |x| + |y|$. By concatenating x to the left of x^k and y^l, we see that x^{k+1} and xy^l also share a prefix of length $\geq |x| + |y|$. Similarly, concatenating y to the left of x^k and y^l gives yx^k and y^{l+1}, which also share a prefix of length $\geq |x| + |y|$. Since x^k and y^l share this prefix, then the same is true for the pairs x^{k+1} and y^{l+1}, as well as for the pairs x^{k+1} and xy^l and the pairs yx^k and y^{l+1}. Thus, by comparing xy^l and yx^k, we see that they too share a prefix of length $\geq |x| + |y|$, and hence $xy = yx$. By the equivalence of conditions (1) and (2), there exist a word z and integers $i, j \geq 1$ such that $x = z^i$ and $y = z^j$. $\square$

Now that we have these characterizations of commutative words due to the second Lyndon–Schützenberger theorem, we can prove a variety of results. For example, recall the proof of Proposition 2.2. To prove the uniqueness of the primitive root z where $w = z^e$ for $e \geq 1$, suppose to the contrary that z is not unique. Then we would have $w = z^e = y^f$ for two primitive words z and y and integers $e, f \geq 1$. By the second Lyndon–Schützenberger theorem, there exists a word x where $|x| \geq 1$ such that $z = x^k$ and $y = x^l$ for integers $k, l \geq 1$. However, since z and y are primitive, we must have that $k = l = 1$. But then $z = y = x$, so $e = f$ and w must only have one unique representation; a contradiction. This completes the uniqueness portion of the proof.

As a motivation for the rest of this thesis, we consider a theorem in combinatorics on words that is well-studied and has already been generalized to two dimensions. The Fine–Wilf theorem [37], named for the American mathematicians Nathan Fine and Herbert Wilf, relates the length of a word to the word's possible periods. What follows is the one-dimensional version of the theorem with the proof omitted.

Theorem 2.24 (Fine–Wilf theorem). *Let $w \in \Sigma^+$. If w has periods p and q, and $|w| \geq p + q - \gcd(p, q)$, then w also has period $\gcd(p, q)$.*

Proof. Omitted. $\square$

A good deal of work has been written about this theorem since its introduction. Castelli, Mignosi, and Restivo generalized this theorem to three periods [27] and, later, Justin generalized it to arbitrary periods [60]. Constantinescu and Ilie provide a more complete solution

and better bounds for the case involving arbitrary periods [30]. The same researchers, as well as Karhumäki, Puzynina, and Saarela, generalized the Fine–Wilf theorem to *abelian-equivalent* words, or words containing the same symbols up to rearrangement [31, 63]. A number of researchers extended the Fine–Wilf theorem to the case of *partial* words, or words that are a partial mapping into some alphabet [18, 24, 25], and to the case of *pseudo-repetitions* in words, a notion that appears often in the field of bioinformatics [35, 64, 70]. Tijdeman and Zamboni gave an algorithm to compute the maximal word w of length n according to the Fine–Wilf theorem [85, 86]. Lastly, Mignosi, Restivo, and Silva generalized the Fine–Wilf theorem to two dimensions [73].

The Fine–Wilf theorem stands as an example of how results in combinatorics on words can be generalized and expanded upon in many ways, which is, in a sense, the overarching theme of this thesis.

Chapter 3

One-Dimensional Lyndon–Schützenberger Theorems

3.1   One-Dimensional Extension of the First Theorem

In this section, we present and prove an extension of the first Lyndon–Schützenberger theorem. For the time being, we shall restrict ourselves to one dimension. In Sections 4.1 and 4.2, we shall generalize this theorem to two dimensions under the assumptions that our two-dimensional words are overlapping or bordered, respectively.

Why would we want an extended version of the theorem? For one, additional conditions would give more versatile methods of identifying when two words obey the properties given by the theorem. Instead of having to potentially rewrite words or replace variables to fit the original conditions, additional conditions would provide us with more options to identify words satisfying the theorem. In addition, the extended version allows us to draw connections between this theorem and related concepts in formal language theory; as we will see, one of the additional conditions has been studied for quite some time, yet has never been connected to the Lyndon–Schützenberger theorem.

Recall Theorem 2.22 from Chapter 2. The following theorem is a more general extension of Theorem 2.22 which introduces multiple new equivalent conditions for a word to have identical nontrivial prefixes and suffixes. Note that conditions (1) and (2) are restatements

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The results in Section 3.2 of this chapter are based on the work found in Section 1 of the author's paper with Gamard, Richomme, and Shallit [42].

of the original conditions, and the "if-and-only-if" argument of the original theorem has been replaced by a chain of implications.

Following a suggestion by Shallit [79], we shall begin by proving the implication $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (6)$. The implication $(3) \Rightarrow (5) \Rightarrow (7)$ follows a similar argument, so the proofs for conditions (5) and (7) are omitted. Finally, we conclude the proof using the implication $((6) \text{ and } (7)) \Rightarrow (1)$.

Theorem 3.1. *Let $y \in \Sigma^+$. Then the following conditions are equivalent:*

(1) *There exists $p \in \Sigma^+$ such that p is both a proper prefix and a proper suffix of y;*

(2) *There exist $u \in \Sigma^+$, $v \in \Sigma^*$, and an integer $e \geq 1$ such that $y = (uv)^e u$;*

(3) *There exist $s \in \Sigma^+$ and $t \in \Sigma^*$ such that $y = sts$;*

(4) *There exist $q \in \Sigma^+$ and $r \in \Sigma^*$ such that qr is a proper prefix of y and $qry = yrq$;*

(5) *There exist $q' \in \Sigma^+$ and $r' \in \Sigma^*$ such that $r'q'$ is a proper suffix of y and $q'r'y = yr'q'$;*

(6) *There exist $x \in \Sigma^+$, $w \in \Sigma^*$, and an integer $i \geq 2$ such that x is a proper prefix of y and $yw = x^i$;*

(7) *There exist $x' \in \Sigma^+$, $w' \in \Sigma^*$, and an integer $j \geq 2$ such that x' is a proper suffix of y and $w'y = x'^j$.*

Proof. $(1) \Rightarrow (2)$: Suppose that the nonempty word p is both a proper prefix and a proper suffix of y; that is, there exist nonempty words c and d such that $y = pc = dp$. We prove by induction on the length of p.

If $|p| = 1$, then by the conditions imposed on p, we get $y = pqp$ for some possibly empty word q. Take $u = p$, $v = q$, and $e = 1$ to get $y = (uv)^e u = (pq)^1 p = pqp$.

Now suppose $|p| > 1$. Write $y = pc = dp$ for some words c and d. We consider two cases.

- If $|c| \geq |p|$, then there exists a possibly empty word q such that $d = pq$ and $c = qp$. Take $u = p$, $v = q$, and $e = 1$ to get $y = (uv)^e u = (pq)^1 p = pqp$.

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- If $|c| < |p|$, then there exists a nonempty word q such that $p = qc = dq$. Since $0 < |q| = |p| - |d| < |p|$, we can apply induction on the equality $p = qc = dq$. This gives us a nonempty word u, a possibly empty word v, and an integer $e \geq 1$ such that $c = vu$, $d = uv$, and $p = (uv)^e u$. Hence, $y = pc = (uv)^e uvu = (uv)^{e+1} u$.

(2) $\Rightarrow$ (3): We know that $y = (uv)^e u = uv(uv)^{e-1} u$ for some $e \geq 1$. Take $s = u$ and $t = v(uv)^{e-1}$. Since u is nonempty, then s is nonempty, and $y = sts$.

(3) $\Rightarrow$ (4): We know that $y = sts$. Take $q = s$ and $r = t$. Then $qry = ststs = yrq$. Since s is nonempty, then q is nonempty, and so qr is a proper prefix of y.

(4) $\Rightarrow$ (6): Suppose $uvy = yvu$ for words u, v, and y. Multiply by v on the right to get $(uv)(yv) = (yv)(uv)$. Then, by Theorem 2.23, there exists some nonempty word x such that $uv = x^j$ and $yv = x^i$ for some integers $j \geq 1$ and $i \geq 1$. Take $z = v$ to get $yz = x^i$ as required. Now, using the fact that uv is a proper prefix of y, we get

$$|x^i| = |yz| \geq |y| > |uv| = |x^j|.$$

Since $|x^i| > |x^j|$, we see that $i > j \geq 1$, so $i \geq 2$ as required.

((6) and (7)) $\Rightarrow$ (1): Given $yw = x^i$, choose $m < i$ such that $m|x| < |y| \leq (m+1)|y|$. Since x is a proper prefix of y, we must have $m \geq 1$. Rewrite y as $y = x^m p$ for some word p. Since y is a prefix of x^i, it follows that p is a prefix of x. Therefore, p is a proper prefix of y. We know that p is nonempty because $|y| > m|x|$. Therefore, in conjunction with condition (7), p is both a proper prefix and a proper suffix of y. $\qquad\square$

3.2 One-Dimensional Extension of the Second Theorem

In this section, we present and prove an extension of the second Lyndon–Schützenberger theorem in a similar fashion to Section 3.1. Again, we restrict ourselves to the one-dimensional case here. In Section 4.3, we shall generalize this theorem to two dimensions.

Recall Theorem 2.23 from Chapter 2. The following theorem is a more general extension of Theorem 2.23 which introduces two additional equivalent conditions for words x and y to commute.

Theorem 3.2. *Let $x, y \in \Sigma^+$. Then the following conditions are equivalent:*

(1) $xy = yx$;

(2) *There exist $z \in \Sigma^+$ and integers $i, j \geq 1$ such that $x = z^i$ and $y = z^j$;*

(3) *There exist integers $k, l \geq 1$ such that $x^k = y^l$;*

(4) *There exist integers $r, s \geq 1$ such that $x^r y^s = y^s x^r$;*

(5) $x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset$.

Proof. The proof of the equivalence of conditions (1), (2), and (3) was presented in the proof of Theorem 2.23. Here, we shall demonstrate the equivalence of conditions (3), (4), (5), and (1), thus creating a chain of implications.

(3) $\Rightarrow$ (4): If $x^k = y^l$, then we immediately have $x^r y^s = y^s x^r$ with $r = k$ and $s = l$.

(4) $\Rightarrow$ (5): Let $z = x^r y^s$. Then by (4) we have $z = y^s x^r$. So $z = x x^{r-1} y^s$ and $z = y y^{s-1} x^r$. Thus $z \in x\{x, y\}^*$ and $z \in y\{x, y\}^*$. So $x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset$.

(5) $\Rightarrow$ (1): By induction on the length of $|xy|$. The base case is $|xy| = 2$. More generally, if $|x| = |y|$ then clearly (5) implies $x = y$ and so (1) holds. Otherwise without loss of generality $|x| < |y|$. Suppose $z \in x\{x, y\}^*$ and $z \in y\{x, y\}^*$. Then x is a proper prefix of y, so write $y = xw$ for a nonempty word w. Then z has prefix xx and also prefix xw. Thus $x^{-1}z \in x\{x, w\}^*$ and $x^{-1}z \in w\{x, w\}^*$, where by $x^{-1}z$ we mean remove the prefix x from z. So $x\{x, w\}^* \cap w\{x, w\}^* \neq \emptyset$, so by induction (1) holds for x and w, so $xw = wx$. Then $yx = (xw)x = x(wx) = xy$. $\qquad\square$

Before we conclude this chapter, we require one more definition. We say that a finite set of words X is a *code* if every word $w \in X$ has a unique factorization in X^*; that is, $u_0 \cdots u_{m-1} = v_0 \cdots v_{n-1}$ with $u_i, v_j \in X$ for all $i, j \geq 0$ implies $m = n$ and $u_i = v_i$ for all $0 \leq i < m$.

We note that condition (5) is essentially equivalent to the well-known "defect theorem", which is a property of a finite set of words that is a non-code; that is, a finite set of words where not all elements of the set have a unique factorization. Intuitively, the defect theorem says that we can represent n non-codewords by taking the product of at most $n - 1$ codewords. We omit the proof here.

Theorem 3.3 (Defect theorem). *If $X \subseteq \Sigma^*$ is a finite non-code, then there exists a code $Y \subseteq \Sigma^*$ such that $X \subseteq Y^*$ and $|Y| \leq |X| - 1$.*

Proof. Omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

The defect theorem has unclear origins and is sometimes considered folklore. Berstel, Perrin, Perrot, and Restivo give an early account of the defect theorem in their paper [20]. Huova explores the defect theorem applied to two dimensions in a paper [57] and in her PhD thesis [58]. Karhumäki, Maňuch, and Plandowski, as well as Maňuch alone, studied a bi-infinite version of the defect theorem [61, 62, 71]. Moczurad studied the defect theorem applied to two-dimensional planar figures [74]. For more information on the defect theorem, see Theorem 1.2.5 and Corollary 1.2.6 in Lothaire's text [68] or the survey article by Harju and Karhumäki [51].

Chapter 4

Two-Dimensional Lyndon–Schützenberger Theorems

4.1 Overlapping Two-Dimensional Generalization of the First Theorem

We now proceed to generalize our extended versions of the Lyndon–Schützenberger theorems to two-dimensional words. In this and the following section, we will focus on the first Lyndon–Schützenberger theorem. These generalizations apply to every condition of Theorem 3.1.

Recall the definition of two-dimensional overlap from Chapter 2: a pair of two-dimensional words A and B overlap if there exists a two-dimensional word C such that C is a suffix of A and a prefix of B or vice versa.

The following theorem is written in the context of vertical overlap. It can easily be adapted to the case of horizontal overlap by considering columns of each two-dimensional word instead of rows.

Theorem 4.1. *Let $Y \in \Sigma^{m \times n}$ be a nonempty two-dimensional word. Then the following are equivalent:*

(1) *There exists a nonempty two-dimensional word P such that P is both a proper prefix and suffix of Y in the vertical direction;*

The results in Section 4.3 of this chapter are based on the work found in Section 3 of the author's paper with Gamard, Richomme, and Shallit [42].

(2) *There exist a nonempty two-dimensional word U, a possibly empty two-dimensional word V, and an integer $e \geq 1$ such that $Y = (U \oplus V)^{e \times 1} \oplus U$;*

(3) *There exist a nonempty two-dimensional word S and a possibly empty two-dimensional word T such that $Y = S \oplus T \oplus S$;*

(4) *There exist a nonempty two-dimensional word Q and a possibly empty two-dimensional word R such that $Q \oplus R$ is a proper prefix of Y and $Q \oplus R \oplus Y = Y \oplus R \oplus Q$;*

(5) *There exist a nonempty two-dimensional word Q' and a possibly empty two-dimensional word R' such that $R' \oplus Q'$ is a proper suffix of Y and $Q' \oplus R' \oplus Y = Y \oplus R' \oplus Q'$;*

(6) *There exist a nonempty two-dimensional word X, a possibly empty two-dimensional word W, and an integer $i \geq 2$ such that $Y \oplus W = X^{i \times 1}$;*

(7) *There exist a nonempty two-dimensional word X', a possibly empty two-dimensional word W', and an integer $j \geq 2$ such that $W' \oplus Y = X'^{j \times 1}$.*

Proof. Follows from the proof of the one-dimensional version of the first Lyndon–Schützenberger theorem. Apply each direction of the one-dimensional proof to each row of the two-dimensional words. $\square$

It is evident that this generalization of the theorem is not very interesting, since it essentially amounts to applying the one-dimensional version of the theorem to some set of one-dimensional words concatenated together. Thus, we will not dwell on this generalization for too long. Fortunately, we have another property of two-dimensional words similar to overlap for which we can apply our generalized theorem.

4.2 Bordered Two-Dimensional Generalization of the First Theorem

Recall the definition of a two-dimensional border from Chapter 2: a two-dimensional word A is bordered if there exist a nonempty two-dimensional word Q and possibly empty two-dimensional words R, S, and T such that

$$A = (Q \oplus R \oplus Q) \ominus (S \oplus T \oplus S) \ominus (Q \oplus R \oplus Q).$$

The following theorem is written in the context of a "picture frame" border, as discussed in Chapter 2. Similar to the proof of Theorem 3.1, we shall begin by proving the implication $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. The implication $(4) \Rightarrow (6)$ follows a similar argument, so it is omitted. Finally, we conclude the proof using the implication $((5)$ and $(6)) \Rightarrow (1)$. Observe that there is one fewer condition here than in the one-dimensional case since condition (4) here essentially consolidates properties of conditions (4) and (5) of Theorem 3.1.

Theorem 4.2. *Let $A \in \Sigma^{m \times n}$ be a nonempty two-dimensional bordered word. Then the following are equivalent:*

(1) *There exist nonempty two-dimensional words P_1 and P_2 such that P_1 is both a proper prefix and suffix of A in the horizontal direction and P_2 is both a proper prefix and suffix of A in the vertical direction;*

(2) *There exist nonempty two-dimensional words U_1 and U_2, possibly empty two-dimensional words V_1 and V_2, and integers $e, f \geq 1$ such that $A = (U_1 \ominus V_1)^e \ominus U_1 = (U_2 \oplus V_2)^f \oplus U_2$;*

(3) *There exist nonempty two-dimensional words S_1 and S_2 and possibly empty two-dimensional words T_1 and T_2 such that $A = S_1 \ominus T_1 \ominus S_1 = S_2 \oplus T_2 \oplus S_2$;*

(4) *There exist nonempty two-dimensional words U_1 and U_2 and possibly empty two-dimensional words V_1 and V_2 such that $U_1 \ominus V_1 \ominus A = A \ominus V_1 \ominus U_1$ and $U_2 \oplus V_2 \oplus A = A \oplus V_2 \oplus U_2$.*

(5) *There exist nonempty two-dimensional words X_1 and X_2, which are proper prefixes of A in the horizontal and vertical directions, respectively, possibly empty two-dimensional words Z_1 and Z_2, and integers $i_1, i_2 \geq 2$ such that $A \ominus Z_1 = X_1^{i_1 \times 1}$ and $A \oplus Z_2 = X_2^{1 \times i_2}$.*

(6) *There exist nonempty two-dimensional words R_1 and R_2, which are proper suffixes of A in the horizontal and vertical directions, respectively, possibly empty two-dimensional words W_1 and W_2, and integers $j_1, j_2 \geq 2$ such that $W_1 \ominus A = R_1^{j_1 \times 1}$ and $W_2 \oplus A = R_2^{1 \times j_2}$.*

Proof. $(1) \Rightarrow (2)$: We prove the claim about P_1 by induction. The claim about P_2 follows by transposing A and applying the same proof with appropriate substitutions.

If $P_1 \in \Sigma^{1 \times n}$, then we have $A = P_1 \ominus K \ominus P_1$ for some possibly empty two-dimensional word K. Let $U_1 = P_1$, $V_1 = K$, and $e = 1$ to get $A = (P_1 \ominus K)^1 \ominus P_1 = (U_1 \ominus V_1)^e \ominus U_1$.

Suppose the claim is true for some proper prefix and suffix with $m > 1$ rows. We show it is true for $m+1$ rows. Write $A = P_1 \ominus O = M \ominus P_1$ for some nonempty two-dimensional words O and M. We consider two cases:

- If $\#_{\text{rows}}(M) \geq \#_{\text{rows}}(P_1)$, then there exists a "middle" two-dimensional word N such that $M = P_1 \ominus N$ and $O = N \ominus P_1$. Take $U_1 = P_1$, $V_1 = N$, and $e = 1$ to get $A = (P_1 \ominus N)^1 \ominus P_1 = (U_1 \ominus V_1)^e \ominus U_1$.

- If $\#_{\text{rows}}(M) < \#_{\text{rows}}(P_1)$, then again there exists a "middle" two-dimensional word N such that $P_1 = M \ominus N = N \ominus O$. We know that

$$0 < \#_{\text{rows}}(N) = (\#_{\text{rows}}(P_1) - \#_{\text{rows}}(M)) < \#_{\text{rows}}(P_1),$$

  so N is a proper prefix and suffix of P_1. Apply the inductive hypothesis to N and P_1 to get a nonempty two-dimensional word U_1, a possibly empty two-dimensional word V_1, and an integer $e \geq 1$ such that

$$M = U_1 \ominus V_1,$$
$$O = V_1 \ominus U_1,$$
$$N = (U_1 \ominus V_1)^e \ominus U_1.$$

It follows that

$$\begin{aligned} A &= M \ominus N \ominus O \\ &= U_1 \ominus V_1 \ominus (U_1 \ominus V_1)^e \ominus U_1 \ominus V_1 \ominus U_1 \\ &= (U_1 \ominus V_1)^{e+2} \ominus U_1. \end{aligned}$$

$(2) \Rightarrow (3)$: Since

$$A = (U_1 \ominus V_1)^e \ominus U_1 = U_1 \ominus V_1 \ominus (U_1 \ominus V_1)^{(e-1)} \ominus U_1$$

and

$$A = (U_2 \oplus V_2)^f \oplus U_2 = U_2 \oplus V_2 \oplus (U_2 \oplus V_2)^{(f-1)} \oplus U_2,$$

take $S_1 = U_1$, $T_1 = V_1 \ominus (U_1 \ominus V_1)^{(e-1)}$, $S_2 = U_2$, and $T_2 = V_2 \oplus (U_2 \oplus V_2)^{(f-1)}$. Since U_1 and U_2 are nonempty, then S_1 and S_2 are nonempty, and $A = S_1 \ominus T_1 \ominus S_1 = S_2 \oplus T_2 \oplus S_2$.

$(3) \Rightarrow (4)$: We know that $A = S_1 \ominus T_1 \ominus S_1 = S_2 \oplus T_2 \oplus S_2$. Take $U_1 = S_1$, $U_2 = S_2$, $V_1 = T_1$, and $V_2 = T_2$. Then

$$U_1 \ominus V_1 \ominus A = S_1 \ominus T_1 \ominus S_1 \ominus T_1 \ominus S_1 = A \ominus V_1 \ominus U_1$$

and
$$U_2 \oplus V_2 \oplus A = S_2 \oplus T_2 \oplus S_2 \oplus T_2 \oplus S_2 = A \oplus V_2 \oplus U_2.$$

(4) $\Rightarrow$ (5): Suppose $U_1 \ominus V_1 \ominus A = A \ominus V_1 \ominus U_1$ and $U_2 \oplus V_2 \oplus A = A \oplus V_2 \oplus U_2$ for two-dimensional words U_1, U_2, V_1, and V_2. Concatenate V_1 to the bottom of A and U_1 and V_2 to the right of A and U_1 respectively to get

$$(U_1 \ominus V_1) \ominus (A \ominus V_1) = (A \ominus V_1) \ominus (U_1 \ominus V_1)$$

and

$$(U_2 \oplus V_2) \oplus (A \oplus V_2) = (A \oplus V_2) \oplus (U_2 \oplus V_2).$$

Then, by the two-dimensional generalization of the second Lyndon–Schützenberger theorem, there exist nonempty two-dimensional words X_1 and X_2 and positive integers i_1, i_2, j_1, and j_2 such that $U_1 \ominus V_1 = X_1^{j_1 \times 1}$, $A \ominus V_1 = X_1^{i_1 \times 1}$, $U_2 \oplus V_2 = X_2^{1 \times j_2}$, and $A \oplus V_2 = X_2^{1 \times i_2}$. Take $Z_1 = V_1$ and $Z_2 = V_2$ to get $A \ominus Z_1 = X_1^{i_1 \times 1}$ and $A \oplus Z_2 = X_2^{1 \times i_2}$ as required. Now, using the fact that $U_1 \ominus V_1$ is a proper prefix of A, we get

$$\#_{\text{rows}}(X_1^{i_1 \times 1}) = \#_{\text{rows}}(A \ominus Z_1) \geq \#_{\text{rows}}(A) > \#_{\text{rows}}(U_1 \ominus V_1) = \#_{\text{rows}}(X_1^{j_1 \times 1})$$

and we arrive at a similar result when we consider the columns of each word. Since $\#_{\text{rows}}(X_1^{i_1 \times 1}) > \#_{\text{rows}}(X_1^{j_1 \times 1})$ (and likewise for the columns of X_2), we see that $i_1 > j_1 \geq 1$ and $i_2 > j_2 \geq 1$, so $i_1, i_2 \geq 2$ as required.

((5) and (6)) $\Rightarrow$ (1): Given $A \ominus Z_1 = X_1^{i_1 \times 1}$, choose $m_1 < i_1$ such that

$$\#_{\text{rows}}(X_1^{m_1 \times 1}) < \#_{\text{rows}}(A) \leq \#_{\text{rows}}(A^{(m_1+1) \times 1}).$$

Since X_1 is a proper prefix of A in the horizontal direction, we must have $m_1 \geq 1$. Rewrite A as $A = X_1^{m_1 \times 1} \ominus P_1$. Since A is a prefix of $X_1^{i_1 \times 1}$ in the horizontal direction, it follows that P_1 is a prefix of X_1 in the horizontal direction. Therefore, P_1 is a proper prefix of A in the horizontal direction. We know that P_1 is nonempty because $\#_{\text{rows}}(A) > \#_{\text{rows}}(X_1^{m_1 \times 1})$. Therefore, in conjunction with condition (6), P_1 is both a proper prefix and proper suffix of A in the horizontal direction. A similar argument follows on the two-dimensional word X_2 to show that there exists some P_2 that is a proper prefix (and, in conjunction with condition (6), a proper suffix) of A in the vertical direction. $\qquad\square$

4.3 Two-Dimensional Generalization of the Second Theorem

Recall Theorem 3.2 from Chapter 3. The following theorem is a generalization of conditions (2), (3), and (4) of Theorem 3.2 to two-dimensional words.

Theorem 4.3. *Let A and B be nonempty two-dimensional words. Then the following three conditions are equivalent:*

(1) *There exist positive integers p_1, p_2, q_1, q_2 such that $A^{p_1 \times q_1} = B^{p_2 \times q_2}$.*

(2) *There exist a nonempty two-dimensional word C and positive integers r_1, r_2, s_1, s_2 such that $A = C^{r_1 \times s_1}$ and $B = C^{r_2 \times s_2}$.*

(3) *There exist positive integers t_1, t_2, u_1, u_2 such that $A^{t_1 \times u_1} \circ B^{t_2 \times u_2} = B^{t_2 \times u_2} \circ A^{t_1 \times u_1}$ where $\circ$ can be either $\oplus$ or $\ominus$.*

Proof. $(1) \Rightarrow (2)$: Let $A \in \Sigma^{m_1 \times n_1}$ and $B \in \Sigma^{m_2 \times n_2}$ be two-dimensional words such that $A^{p_1 \times q_1} = B^{p_2 \times q_2}$. By dimensional considerations, we have $m_1 p_1 = m_2 p_2$ and $n_1 q_1 = n_2 q_2$. Let $P = A^{p_1 \times 1}$ and $Q = B^{p_2 \times 1}$. We have $P^{1 \times q_1} = Q^{1 \times q_2}$.

Taking P and Q to be two-dimensional words over $\Sigma^{m_1 p_1 \times 1}$ and considering horizontal concatenation, we can write $P^{q_1} = Q^{q_2}$. By Theorem 3.2, there exist a word $R \in \Sigma^{m_1 p_1 \times 1}$ and integers s_1, s_2 such that $P = R^{1 \times s_1}$ and $Q = R^{1 \times s_2}$.

Let r denote the number of columns of R, let $S = A[0 \ldots m_1 - 1, 0 \ldots r - 1]$, and let $T = B[0 \ldots m_2 - 1, 0 \ldots r - 1]$. Observe that $A = S^{1 \times s_1}$ and $B = T^{1 \times s_2}$. Considering the first r columns of P and Q, we see that $S^{p_1 \times 1} = T^{p_2 \times 1}$.

Taking S and T to be two-dimensional words over $\Sigma^{1 \times r}$ and considering vertical concatenation, we can write $S^{p_1} = T^{p_2}$. Again, by Theorem 3.2, there exist a word $C \in \Sigma^{1 \times r}$ and integers r_1, r_2 such that $S = C^{r_1 \times 1}$ and $T = C^{r_2 \times 1}$.

Therefore, $A = C^{r_1 \times s_1}$ and $B = C^{r_2 \times s_2}$.

$(2) \Rightarrow (3)$: Without loss of generality, assume that the concatenation operation is $\oplus$; that is, we are considering vertical concatenation. Recall that $A = C^{r_1 \times s_1}$ and $B = C^{r_2 \times s_2}$. Let $t_1 = r_2$, $t_2 = r_1$, $u_1 = s_2$, and $u_2 = s_1$. Then we have

$$
\begin{aligned}
A^{t_1 \times u_1} \oplus B^{t_2 \times u_2} &= C^{r_1 t_1 \times s_1 u_1} \oplus C^{r_2 t_2 \times s_2 u_2} \\
&= C^{r_1 t_1 \times (s_1 u_1 + s_2 u_2)} \qquad\qquad \text{(Observe that } r_1 t_1 = r_2 t_2) \\
&= C^{r_2 t_2 \times s_2 u_2} \oplus C^{r_1 t_1 \times s_1 u_1} \\
&= B^{t_2 \times u_2} \oplus A^{t_1 \times u_1}.
\end{aligned}
$$

$(3) \Rightarrow (1)$: Without loss of generality, assume that the concatenation operation is $\oplus$; that is, we are considering vertical concatenation. Assume that there exist positive integers $t_1, t_2, u_1,$ and u_2 such that

$$
A^{t_1 \times u_1} \oplus B^{t_2 \times u_2} = B^{t_2 \times u_2} \oplus A^{t_1 \times u_1}.
$$

We sketch the proof that we can concatenate multiple copies of A and B while maintaining this equality. Observe that concatenating $A^{t_1 \times u_1}$ to the left of each side of the above equation gives

$$A^{t_1 \times u_1} \oplus A^{t_1 \times u_1} \oplus B^{t_2 \times u_2} = A^{t_1 \times u_1} \oplus B^{t_2 \times u_2} \oplus A^{t_1 \times u_1}$$
$$= B^{t_2 \times u_2} \oplus A^{t_1 \times u_1} \oplus A^{t_1 \times u_1},$$

so $A^{t_1 \times 2u_1} \oplus B^{t_2 \times u_2} = B^{t_2 \times u_2} \oplus A^{t_1 \times 2u_1}$. A similar result follows if we concatenate $B^{t_2 \times u_2}$ instead. Repeating these concatenations i times for $A^{t_1 \times u_1}$ and j times for $B^{t_2 \times u_2}$ reveals that

$$A^{t_1 \times iu_1} \oplus B^{t_2 \times ju_2} = B^{t_2 \times ju_2} \oplus A^{t_1 \times iu_1}.$$

Let $A \in \Sigma^{m_1 \times n_1}$ and $B \in \Sigma^{m_2 \times n_2}$. Now choose $i = u_2 n_2$ and $j = u_1 n_2$ to get $iu_1 n_1 = ju_2 n_2$. Then, by considering the first $iu_1 n_1$ columns of the two-dimensional word defined in the above equation, we get $A^{t_1 \times iu_1} = B^{t_2 \times ju_2}$. Hence, we may take $p_1 = t_1$, $q_1 = iu_1$, $p_2 = t_2$, and $q_2 = ju_2$. $\qquad \square$

Note that generalizing condition (1) of Theorem 3.2 requires considering two-dimensional words with the same number of rows or same number of columns. Hence, the next result is a direct consequence of Theorem 4.3.

Corollary 4.4. *Let A, B be nonempty two-dimensional words. Then*

(a) *if A and B have the same number of rows, $A \oplus B = B \oplus A$ if and only there exist a nonempty two-dimensional word C and integers $e, f \geq 1$ such that $A = C^{1 \times e}$ and $B = C^{1 \times f}$;*

(b) *if A and B have the same number of columns , $A \ominus B = B \ominus A$ if and only there exist a nonempty two-dimensional word C and integers $e, f \geq 1$ such that $A = C^{e \times 1}$ and $B = C^{f \times 1}$.*

As another consequence of Theorem 4.3, we get the following result (which is itself an alternative proof of Lemma 3.3 in a paper by Gamard and Richomme [41]):

Corollary 4.5. *Let A be a nonempty two-dimensional word. Then there exists a unique two-dimensional primitive word C and positive integers i, j such that $A = C^{i \times j}$.*

Remark. In contrast to Corollary 4.5, Bacquey has shown that two-dimensional biperiodic infinite words can have two distinct primitive roots [10].

Chapter 5

Two-Dimensional Primitivity and Periodicity

5.1 Enumerating One-Dimensional Primitive Words

We now shift our attention from the Lyndon–Schützenberger theorems to focus on the properties of primitivity and periodicity in two-dimensional words. In this section, we begin by reviewing techniques to enumerate primitive words in the one-dimensional case.

There is a well-known formula to enumerate the number of one-dimensional primitive words of length n over a k-letter alphabet:

$$\psi_k(n) = \sum_{d|n} \mu(d) k^{n/d},$$

where μ is the Möbius function, introduced by the German mathematician August Ferdinand Möbius and defined as:

$$\mu(n) = \begin{cases} (-1)^t, & \text{if } n \text{ is a square-free positive integer with } t \text{ prime divisors;} \\ 0, & \text{if } n \text{ is divisible by a square greater than 1.} \end{cases}$$

For more details on this enumeration formula, see, for example, Section 16.3 of Hardy and Wright's text [50] or Section 1.3 of Lothaire's text [68].

Most of the results in this chapter are based on the work found in Sections 6 and 7 of the author's paper with Gamard, Richomme, and Shallit [42].

We recall an important property of the sum of the Möbius function $\mu(d)$. The proof of this lemma is based on the proof of Theorem 263 in Hardy and Wright's text [50].

Lemma 5.1. *For any positive integer n,*

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases}$$

Proof. Let k denote the number of distinct prime divisors of n. Rewrite n in terms of its prime decomposition; that is, $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$. For $k \geq 1$, we have

$$\sum_{d|n} \mu(d) = 1 + \sum_{1 \leq i \leq k} \mu(p_i) + \sum_{1 \leq i < j \leq k} \mu(p_i p_j) + \cdots$$

$$= 1 - k + \binom{k}{2} - \binom{k}{3} + \cdots$$

$$= (1 - 1)^k$$

$$= 0.$$

Otherwise, if $n = 1$, then $\mu(n) = 1$. $\qquad\square$

We also recall the Möbius inversion formula.

Lemma 5.2. *Given two functions f and g, if*

$$g(n) = \sum_{d|n} f(d)$$

for all integers $n \geq 1$, then

$$f(n) = \sum_{d|n} \mu(n/d) g(d) = \sum_{d|n} \mu(d) g(n/d).$$

Proof. Observe that we can rewrite the rightmost expression as

$$\sum_{d|n} \mu(d) g(n/d) = \sum_{d|n} \mu(d) \sum_{c|(n/d)} f(c)$$

$$= \sum_{cd|n} \mu(d) f(c)$$

$$= \sum_{c|n} f(c) \sum_{d|(n/c)} \mu(d),$$

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and, by Lemma 5.1, the rightmost sum is 1 when $n/c = 1$; that is, when $n = c$. Otherwise, the rightmost sum is 0. Therefore, the expression reduces to $f(n)$ as required. $\qquad\square$

5.2   Enumerating Two-Dimensional Primitive Words

In this section, we use the results from Section 5.1 to generalize the equation for enumerating one-dimensional primitive words to the case of two-dimensional primitive words in the following manner:

Theorem 5.3. *Let $\psi_k(m, n)$ denote the number of two-dimensional primitive words of dimension $m \times n$ over a k-letter alphabet. Then*

$$\psi_k(m, n) = \sum_{d_1 \mid m} \sum_{d_2 \mid n} \mu(d_1)\mu(d_2) k^{mn/(d_1 d_2)}.$$

Proof. We will use Lemmas 5.1 and 5.2 to prove our generalized formula.

Let $g_k(m, n) = k^{mn}$; this function counts the number of two-dimensional words of dimension $m \times n$ over a k-letter alphabet. By Corollary 4.5, each of these words has a unique primitive root of dimension $d_1 \times d_2$, where $d_1 \mid m$ and $d_2 \mid n$. Therefore, $g_k(m, n) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} \psi_k(d_1, d_2)$. Using Lemma 5.2, we get

$$\sum_{\substack{d_1 \mid m \\ d_2 \mid n}} \mu(d_1)\mu(d_2) \, g_k\left(\frac{m}{d_1}, \frac{n}{d_2}\right) = \sum_{d_1 \mid m} \mu(d_1) \sum_{d_2 \mid n} \mu(d_2) \, g_k\left(\frac{m}{d_1}, \frac{n}{d_2}\right)$$

$$= \sum_{d_1 \mid m} \mu(d_1) \sum_{d_2 \mid n} \mu(d_2) \sum_{\substack{c_1 \mid m/d_1 \\ c_2 \mid n/d_2}} \psi_k(c_1, c_2)$$

$$= \sum_{c_1 d_1 \mid m} \mu(d_1) \sum_{c_2 d_2 \mid n} \mu(d_2) \, \psi_k(c_1, c_2)$$

$$= \sum_{c_1 \mid m} \sum_{c_2 \mid n} \psi_k(c_1, c_2) \sum_{\substack{d_1 \mid m/c_1 \\ d_2 \mid n/c_2}} \mu(d_1)\mu(d_2).$$

Let $r = m/c_1$ and $s = n/c_2$. By Lemma 5.1, the last sum in the above expression is 1 if $r = 1$ and $s = 1$; that is, if $c_1 = m$ and $c_2 = n$. Otherwise, the last sum is 0. Therefore, the sum reduces to $\psi_k(m, n)$ as required. $\qquad\square$

	1	2	3	4	5	6	7
1	2	2	6	12	30	54	126
2	2	10	54	228	990	3966	16254
3	6	54	498	4020	32730	261522	2097018
4	12	228	4020	65040	1047540	16768860	268419060
5	30	990	32730	1047540	33554370	1073708010	34359738210
6	54	3966	261522	16768860	1073708010	68718945018	4398044397642
7	126	16254	2097018	268419060	34359738210	4398044397642	562949953421058

Table 5.1: Values of $\psi_2(m, n)$

To illustrate the growth of the function ψ, Table 5.1 gives the first few values of $\psi_2(m, n)$.

We take a brief digression now to investigate a curious relationship. Given a triangle T with nonzero area, we construct its *pedal triangle T'* by selecting a point P within T, drawing perpendicular lines from P to each side of T forming three points Q, R, and S, and connecting these points. We can iterate this process of constructing pedal triangles, and we denote the ith iteration by T^i. We say that the pedal triangle formed from $T^{(n-1)}$ is the nth pedal triangle of T, and if T is similar to its nth pedal triangle, then we say that T has period n.

A calculation reveals that $\psi_2(2, n)$ is equal to the number of pedal triangles with period n. Presently, it is not known whether there exists a connection between pedal triangles of period n and two-dimensional primitive words of dimension $2 \times n$ over a two-letter alphabet, but the fact that the two enumerations agree suggests some sort of bijection between these objects. For more details on pedal triangles, see the paper of Vályi [88] and the paper of Kingston and Synge [65].

5.3   Checking if a Two-Dimensional Word is Primitive

In this section, we take a computational view towards primitivity in two-dimensional words by proving that the property of two-dimensional primitivity can be checked in linear time and by giving an efficient algorithm to test the primitivity of a given two-dimensional word.

We start with a useful lemma that we will require for the proof of the main theorem. This lemma gives us a method of determining the dimensions of the primitive root of some two-dimensional word A using only the lengths of the one-dimensional primitive roots of each row and column of A.

Lemma 5.4. *Let A be an $m \times n$ two-dimensional word. Let the primitive root of row i of A be denoted by r_i and the primitive root of column j of A be denoted by c_j. Then the primitive root of A has dimension $p \times q$, where $q = \mathrm{lcm}(|r_0|, |r_1|, \ldots, |r_{m-1}|)$ and $p = \mathrm{lcm}(|c_0|, |c_1|, \ldots, |c_{n-1}|)$.*

Proof. Let P be the primitive root of the word A, where P has dimension $m' \times n'$. Then the row $A[i, 0..n-1]$ is periodic with period n'. But, since the primitive root of $A[i, 0..n-1]$ is of length r_i, we know that $|r_i|$ divides n'. It follows that q divides n', where $q = \mathrm{lcm}(|r_0|, |r_1|, \ldots, |r_{m-1}|)$.

Now, suppose $n' \neq q$. Then, since q divides n', we must have that $n'/q > 1$. Let $Q = P[0..m'-1, 0..q-1]$. Then $Q^{1 \times (n'/q)} = P$, which contradicts our hypothesis that P is primitive. It follows that $n' = q$ as claimed.

Applying the same argument to the columns proves the claim about p. $\qquad\square$

With this lemma, we can state the main result of this section.

Theorem 5.5. *Given an $m \times n$ two-dimensional word A, we can verify whether this word is primitive and compute the primitive root of A in $O(mn)$ time for fixed alphabet size.*

Proof. It is a well-known fact that a one-dimensional word u is primitive if and only if u is not an interior factor of its square uu [29]; that is, u is not a factor of the word $u_F u_L$, where u_F is u with the first letter removed and u_L is u with the last letter removed.

We can test whether u is a factor of $u_F u_L$ using a linear-time string matching algorithm such as the Knuth–Morris–Pratt algorithm [66]. If the algorithm returns no match, then u is indeed primitive. Furthermore, if u is not primitive, then the length of its primitive root is given by the index (starting at position 1) of the first match of u in $u_F u_L$.

We assume that there exists an algorithm 1DPrimitiveRoot to obtain the primitive root of a given one-dimensional word in this manner.

We use Lemma 5.4 as the basis for Algorithm 5.6, which computes the primitive root of a two-dimensional word. The algorithm takes as input a two-dimensional word A of dimension $m \times n$ and produces as output the primitive root C of A and its dimensions. The correctness of the algorithm follows from Lemma 5.4, and the running time is $O(mn)$. $\quad\square$

Algorithm 5.6: Computing the primitive root of A

1: **procedure** 2DPRIMITIVEROOT(A, m, n)
2:     **for** $0 \leq i < m$ **do**   ▷ compute primitive root of each row
3:         $r_i \leftarrow$ 1DPRIMITIVEROOT$(A[i,\ 0..n-1])$
4:     $q \leftarrow \mathrm{lcm}(|r_0|, |r_1|, \ldots, |r_{m-1}|)$ ▷ compute lcm of each primitive root length
5:     **for** $0 \leq j < n$ **do**   ▷ compute primitive root of each column
6:         $c_j \leftarrow$ 1DPRIMITIVEROOT$(A[0..m-1, j])$
7:     $p \leftarrow \mathrm{lcm}(|c_0|, |c_1|, \ldots, |c_{n-1}|)$ ▷ compute lcm of each primitive root length
8:     **for** $0 \leq i < p$ **do**
9:         **for** $0 \leq j < q$ **do**
10:             $C[i, j] \leftarrow A[i, j]$
11:     **return** (C, p, q)

Remark. One might suspect that it is easy to reduce two-dimensional primitivity to one-dimensional primitivity by considering the two-dimensional word A as a one-dimensional word constructed by taking the elements of A in row-major or column-major order. However, the natural conjectures that A is primitive if and only if

(a) either its corresponding row-majorized or column-majorized word is primitive, or

(b) both its row-majorized and column-majorized words are primitive,

both fail. For example, condition (a) fails because the two-dimensional word

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

is not two-dimensional primitive, even though its row-majorized word $A_{\mathrm{RM}} = [00][11]$ is one-dimensional primitive. Similarly, condition (b) fails because the two-dimensional word

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

is two-dimensional primitive and its column-majorized word $B_{\mathrm{CM}} = [01][10][01]$ is primitive, but its row-majorized word $B_{\mathrm{RM}} = [010][101]$ is not.

Chapter 6

Two-Dimensional Borders

6.1 Enumerating One-Dimensional Bordered Words

We conclude with a brief investigation of the property of one- and two-dimensional bordered words. We begin with a review of some results relating to borders in one dimension.

Recall the definition of a one-dimensional bordered word from Chapter 2. A one-dimensional word $w \in \Sigma^+$ is bordered if it can be written as $w = xyx$, where $x \in \Sigma^+$ and $y \in \Sigma^*$. Similarly, we say that a one-dimensional word w is *unbordered* if we cannot write it as $w = xyx$.

The number of one-dimensional unbordered words of length n over an alphabet of size k, denoted here by $u_k(n)$, satisfies the following recurrence relation:

$$u_k(n) = \begin{cases} k, & \text{if } n = 1; \\ k(k-1), & \text{if } n = 2; \\ k \cdot u_k(n-1), & \text{if } n \geq 3 \text{ is odd}; \\ k \cdot u_k(n-1) - u_k(n/2), & \text{if } n \geq 4 \text{ is even}. \end{cases}$$

Harborth [49] studied this relation for the case where $k = 2$ and Nielsen [77] did the same for the general case. This particular formulation was given by Holub and Shallit [55].

As a consequence of this recurrence relation, the number of one-dimensional bordered words of length n over an alphabet of size k, denoted here by $b_k(n)$, is therefore

$$b_k(n) = k^n - u_k(n),$$

where the k^n term comes from the fact that there are a total of k^n ways to form a one-dimensional word of length n using k symbols.

As an aside, we can consider the value $u_k(n)$ to be a polynomial in the size of the alphabet k. If we do this, then we can derive an interesting result about the lowest order term of this polynomial. We begin with a few intermediate results.

Lemma 6.1. *Let $\ell_2(n)$ denote the length of the binary representation of some number $n \in \mathbb{N}$. Then $\ell_2(2i + 1) = \ell_2(2i) = \ell_2(i) + 1$ for all $i \geq 1$.*

Proof. The first equality comes from the fact that decrementing an odd binary number by one does not affect the length of the bit string. The second equality comes from the fact that dividing an even binary number by two reduces the length of the bit string by one, necessitating the additive term. $\square$

Lemma 6.2. *Let $s_2(n)$ denote the sum of the bits in the binary representation of some number $n \in \mathbb{N}$. Then $s_2(2i) = s_2(i)$ and $s_2(2i + 1) = s_2(2i) + 1$ for all $i \geq 1$.*

Proof. The first equality comes from the fact that dividing an even binary number by two removes the least significant 0 bit, but does not remove any 1 bits, leaving the sum unaffected. The second equality comes from the fact that decrementing an odd binary string by one replaces the least significant 1 bit with a 0 bit, necessitating the additive term. $\square$

Note the following special cases of Lemmas 6.1 and 6.2: $\ell_2(0) = 1$ since $|0| = 1$, and $s_2(0) = 0$.

Lemma 6.3. *For all $i \geq 1$, we have $s_2(i) < s_2(2i - 1) + 1$.*

Proof. Observe that every number of the form $2i - 1$, where $i \geq 1$, has a binary representation of the form $(i - 1)_2 1$, where $(n)_2$ denotes the number n represented in base 2. Thus, we have that $s_2(2i - 1) = s_2(i - 1) + 1$ and therefore $s_2(2i - 1) + 1 = s_2(i - 1) + 2$. However, we also have that

$$s_2(i) \leq s_2(i - 1) + 1$$
$$s_2(i) + 1 \leq s_2(i - 1) + 2$$
$$s_2(i) + 1 \leq s_2(2i - 1) + 1$$
$$s_2(i) < s_2(2i - 1) + 1.$$

Therefore, $s_2(i) < s_2(2i - 1) + 1$ for all $i \geq 1$. $\square$

With these lemmas, we can now present the main result.

Theorem 6.4. *The lowest order term of the polynomial $u_k(n)$ is*

$$-(-1)^{\ell_2(n)} \cdot k^{s_2(n)},$$

where $\ell_2(n)$ and $s_2(n)$ follow the definitions from Lemmas 6.1 and 6.2.

Proof. We prove via induction. For the base cases, consider $n = \{1, 2\}$. Then, for $n = 1$, we have $\ell_2(1) = 1$, $s_2(1) = 1$, and

$$\begin{aligned} u_k(1) &= k \\ &= -(-1)^1 \cdot k^1, \end{aligned}$$

and for $n = 2$, we have $\ell_2(2) = 2$, $s_2(2) = 1$, and

$$\begin{aligned} u_k(2) &= k \cdot (k-1) \\ &= k^2 - k, \end{aligned}$$

where the lowest order term of $u_k(2)$ is evidently $-(-1)^2 \cdot k^1$.

Now, assume the theorem holds for some $n \in \mathbb{N}$. We will show that it holds for $n+1$. We split this inductive step into two cases.

- If $n + 1$ is odd, then it is of the form $2i + 1$ for some $i \geq 1$ and we have

$$\begin{aligned} u_k(2i+1) &= k \cdot u_k(2i) \\ &= k \cdot (-(-1)^{\ell_2(2i)} \cdot k^{s_2(2i)} + \text{higher-order terms}) \text{ (by the inductive hypothesis)} \\ &= k \cdot (-(-1)^{\ell_2(2i+1)} \cdot k^{s_2(2i)} + \cdots) \text{ (by Lemma 6.1)} \\ &= -(-1)^{\ell_2(2i+1)} \cdot k^{s_2(2i)+1} + \cdots \\ &= -(-1)^{\ell_2(2i+1)} \cdot k^{s_2(2i+1)} + \cdots \text{ (by Lemma 6.2)} \end{aligned}$$

as required.

- If $n + 1$ is even, then it is of the form $2i$ for some $i \geq 2$ and we have $u_k(2i) = k \cdot u_k(2i-1) - u_k(i)$, so the lowest order term could appear in either $k \cdot u_k(2i-1)$ or $u_k(i)$. However, by induction, we know that the lowest order term of $k \cdot u_k(2i-1)$ is of degree $s_2(2i-1) + 1$, while the lowest order term of $u_k(i)$ is of degree $s_2(i)$.

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Therefore, by Lemma 6.3, the lowest order term of $u_k(2i)$ will be found in $u_k(i)$, and so we have

$$
\begin{aligned}
u_k(2i) &= k \cdot u_k(2i-1) - u_k(i) \\
&= \text{higher-order terms} - (-(-1)^{\ell_2(i)} \cdot k^{s_2(i)}) \text{ (by the inductive hypothesis)} \\
&= \cdots - (-(-1)^{\ell_2(i)} \cdot k^{s_2(2i)}) \text{ (by Lemma 6.2)} \\
&= \cdots + (-(-1)^{\ell_2(i)+1} \cdot k^{s_2(2i)}) \\
&= \cdots + (-(-1)^{\ell_2(2i)} \cdot k^{s_2(2i)}) \text{ (by Lemma 6.1)}
\end{aligned}
$$

as required.

In either case, the inductive hypothesis holds, and so the theorem holds for all $n \in \mathbb{N}$. $\qquad\square$

6.2   Enumerating Two-Dimensional Bordered Words

We now progress to a discussion of borders in the two-dimensional case. Recall the definition of a two-dimensional bordered word from Chapter 2. A two-dimensional word A is bordered if there exist a nonempty two-dimensional word Q and possibly empty two-dimensional words R, S, and T such that

$$
A = (Q \oplus R \oplus Q) \ominus (S \oplus T \oplus S) \ominus (Q \oplus R \oplus Q).
$$

Similarly, we say that a two-dimensional word A is *unbordered* if we cannot write it in the form above.

A natural question to ask at this point is whether a recurrence relation similar to that for $u_k(n)$ exists for the set of two-dimensional unbordered words. That is, given an alphabet of size k and two dimensions m and n, is there a way to find $U_k(m, n)$, the number of two-dimensional unbordered words of dimension $m \times n$? If there were, then we could also easily find $B_k(m, n)$, the number of two-dimensional bordered words of dimension $m \times n$, by using the formula

$$
B_k(m, n) = k^{mn} - U_k(m, n).
$$

Again, the k^{mn} term comes from the fact that there are a total of k^{mn} ways to form a two-dimensional word of dimension $m \times n$ using k symbols.

Since we have an efficient method of enumerating two-dimensional primitive words from Chapter 5 which uses results on one-dimensional primitive words, we could use a similar

technique to enumerate two-dimensional unbordered words which takes advantage of the known properties of one-dimensional bordered words presented at the beginning of this chapter. Before we discuss this technique, however, we must take a brief digression to mention a connection between borders and periods in words.

Recall that a one-dimensional word w has period p if $w[i] = w[i+p]$ for all i. If a word has multiple periods, then by convention, *the* period is usually taken to mean the smallest period of the word. We make the following observation:

Proposition 6.5. *Let $1 \leq p < n$. A one-dimensional word w of length n has period p if and only if w has a border of length $n - p$.*

Proof. ($\Rightarrow$): If w is periodic with period p, then $w[1..(n-p)] = w[(p+1)..n]$. This subword of w is of length $n - p$ and it occurs at the beginning and the end of w, so it is a border.

($\Leftarrow$): If w has a border of length $n - p$, say x, then we can write $w = xy = zx$ for some words y and z. Then $x = w[1..(n - p)] = w[(p + 1)..(n - p)]$, so w is periodic with period p. $\qquad\square$

Note that if $p = n$, then our proposition does not apply since it would imply w has a "border" of length zero, which is not a valid border length. Thus, we get the following result as a corollary of Proposition 6.5.

Corollary 6.6. *A one-dimensional word w of length n has no period shorter than n if and only if w is unbordered.*

It is easy to see that Proposition 6.5 also allows for borders to overlap. If a one-dimensional word has a border that overlaps itself, then it must have another border of shorter length, so we can restrict the length of borders that we consider within a word. Therefore, given a one-dimensional bordered word w of length n, we will only consider borders of w that are of length $1 \leq l \leq \lfloor n/2 \rfloor$. Consequently, this means we will only consider periods of w of length $\lceil n/2 \rceil \leq p \leq n - 1$.

Now that we have seen the connection between borders and periods, we can continue discussing the technique for enumerating two-dimensional unbordered words. With this technique, we begin by taking all two-dimensional bordered words A of dimension $m \times n$ over an alphabet of size k and treating each column of A as a "symbol" from an alphabet of size k^m. For example, if

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 3 & 4 & 5 & 3 & 4 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix},$$

then the equivalent column-majorized word would be

$$A_{\text{CM}} = [030][141][252][030][141].$$

Observe that A is bordered if and only if A_{CM} is bordered. Moreover, if A is bordered, then each "symbol" of A_{CM} is bordered and each "symbol" has the same border length. Since each "symbol" is of length m, then the border length l is bounded by $1 \le l \le \lfloor m/2 \rfloor$ and, similarly, the period length p is bounded by $\lceil m/2 \rceil \le p \le m - 1$ by our earlier remark. We can then use the inclusion-exclusion principle over each period length applied to these one-dimensional bordered words to determine the number of two-dimensional unbordered words.

Using the inclusion-exclusion principle, our goal is to consider all possible period lengths and to find the most general word having each of these periods. After finding such a word, we can then determine how many choices we have for choosing the remaining free symbols, or the symbols that are not determined by any of the periods of the word.

This technique is similar to finding the *autocorrelation* of a one-dimensional word w of length n. An autocorrelation is a bit vector of length n whose ith entry is equal to 1 if w is periodic with period i and 0 otherwise. Guibas and Odlyzko discuss the problem of enumerating autocorrelations of words [47, 48].

We illustrate this technique with some examples over a two-letter alphabet.

Example 6.7. Consider the case where $m = 3$. Then, by our observation, the only period length we must consider is 2. Given a word of length 3, specifying 2 symbols in that word fixes the remaining symbol. Removing this symbol from the word and considering each possible pair of remaining symbols as being members of an alphabet of 4 "symbols", we get that

$$\begin{aligned} U_2(3, n) &= 2^{3n} - b_{2^2}(n) \\ &= 2^{3n} - b_4(n), \end{aligned}$$

where $m = 3$, $n > 1$, and $b_k(n)$ follows the earlier definition.

Example 6.8. Consider the case where $m = 4$. Then the possible period lengths are 2 or 3. We follow the same process as before, where this time we subtract $b_4(n)$ and $b_8(n)$ from our enumeration. However, doing so means we have double-counted, since a word could have both periods 2 and 3, in which case it must have a shortest period of 1. Therefore, we add $b_2(n)$ to our enumeration, resulting in the formula

$$\begin{aligned} U_2(4, n) &= 2^{4n} + b_{2^1}(n) - b_{2^2}(n) - b_{2^3}(n) \\ &= 2^{4n} + b_2(n) - b_4(n) - b_8(n). \end{aligned}$$

Shallit and Cummings give the following method for calculating the coefficients of $U_k(m, n)$ [79]. Given an integer m and a set of periods P, use an algorithm—for example, the algorithm given by Holub [54]—to find the most general word of length m having all periods $p \in P$. Such a word will contain symbols from an alphabet of size m. Now, consider all possible nonempty subsets of the periods p in the range $\lceil m/2 \rceil \le p \le m - 1$ and, for each possible subset S, find the most general word w of length m having at least those periods $p \in S$ as well as the period m. Then, starting with the initial polynomial $P(x) = 0$, add a coefficient of the form $(-1)^a x^b$ to $P(x)$ for each pair (S, w), where $a = |S|$ and b is the number of distinct elements in w. This can be expressed as a sum over all subsets S with c_i representing the coefficient of x^i, giving the expression

$$U_k(m, n) = k^{mn} + \sum_{i \in \mathcal{P}(S)} c_i b_{2^i}(n).$$

Example 6.9. Let $m = 5$. The possible sets of periods for m are $P = \{3, 4\}$. The nonempty subsets of P are $\{3\}$, $\{4\}$, and $\{3, 4\}$. We consider each subset individually.

- For the subset $\{3\}$, the most general word of length 5 with period 3 is `12312`.

- For the subset $\{4\}$, the most general word of length 5 with period 4 is `12341`.

- For the subset $\{3, 4\}$, the most general word of length 5 with periods 3 and 4 is `11211`.

This gives the polynomial $P(x) = -x^3 - x^4 + x^2$, from which we get the result

$$\begin{aligned}
U_2(5, n) &= 2^{5n} - b_{2^3}(n) - b_{2^4}(n) + b_{2^2}(n) \\
&= 2^{5n} - b_8(n) - b_{16}(n) + b_4(n).
\end{aligned}$$

To illustrate the growth of the function U, Table 6.1 gives the first few values of $U_2(m, n)$. Observe that the row corresponding to $m = 1$ (and, similarly, the column corresponding to $n = 1$) simply enumerates the one-dimensional unbordered words of length n (similarly, of length m), and so the formula for $U_2(m, n)$ is not needed in that case. Appendix A features tables of values that list coefficients and alphabet sizes to calculate $U_2(m, n)$ for various values of m.

	1	2	3	4	5	6	7
1	2	2	4	6	12	20	40
2	2	14	60	246	1004	4052	16296
3	4	60	496	4020	32464	260880	2092096
4	6	246	4020	64902	1043748	16739196	268136088
5	12	1004	32464	1043748	33480816	1072592432	34341627328
6	20	4052	260880	16739196	1072592432	68683762160	4396921430976
7	40	16296	2092096	268136088	34341627328	4396921430976	562879053495808

Table 6.1: Values of $U_2(m, n)$

6.3   Checking if a Two-Dimensional Word is Bordered

In this section, we take a computational view towards two-dimensional bordered words by presenting an algorithm to detect whether a given two-dimensional word is bordered. There is an obvious polynomial-time algorithm for checking this property; all we must do is check every possible border size. However, we can do better. As we will see in this section, we can check this property in linear time by exploiting an observation we made earlier.

We begin by providing a lemma that we will require for our algorithm. This lemma, as stated by Crochemore and Rytter in Chapter 13 of their text [34], gives a linear time bound for finding all periods of a one-dimensional word. In their text, the authors give a pseudocode description of an algorithm for finding these periods. This result was first published in two papers approximately a decade apart [32, 39]; the proof is based on the exposition given in the former paper.

Lemma 6.10. *The periods of a one-dimensional word w can be computed in $O(|w|)$ time, with a constant amount of space in addition to w.*

Proof. Recall from Proposition 6.5 that there exists a one-to-one correspondence between periods and borders of a one-dimensional word. Linear-time string matching algorithms such as the Knuth–Morris–Pratt algorithm [66] allow us to efficiently preprocess the borders of all prefixes of a word w. From this, we can compute the periods of w. This computation requires between $|w|$ and $2|w|$ comparisons and $O(|w|)$ additional space. ☐

Using Lemma 6.10, we can state the main result of this section.

Theorem 6.11. *Given an $m \times n$ two-dimensional word A, we can verify whether this word is bordered and compute the dimension of the largest border in $O(mn)$ time for fixed alphabet size.*

Proof. If some two-dimensional word A has a largest border of k rows and l columns, then this algorithm will return (k, l). Observe that if l is the largest border in the vertical direction, then each row has a border of length l. Moreover, l must be the largest such border for all rows; if not, then there would exist a larger border l' and A would therefore have a largest border of l' columns, contradicting the fact that the largest border of A had l columns. Since l is the largest border of each row, then $n - l$ is the smallest period common to all rows by Proposition 6.5. Therefore, the algorithm will return $n - (n - l) = l$ as one of the values of the pair. A similar argument applies to the other value k.

Now, if this algorithm returns (k, l), then the given two-dimensional word A has a largest border of k rows and l columns. For if not, then the algorithm would return a larger pair (k', l'), where $m - e' = k'$ and $n - d' = l'$ for some smaller periods e' and d'. But this contradicts the fact that e and d were taken to be the smallest periods common to all Q_j and P_i respectively.

Lemma 6.10 asserts the existence of a linear-time algorithm for obtaining all periods of a one-dimensional word. We assume there exists an algorithm 1DPERIOD to obtain the periods of a given one-dimensional word in this manner. Furthermore, we assume this algorithm returns the periods as a bit vector; that is, it returns a one-dimensional vector P where the ith bit of the vector is 1 if a period of length i exists in the word and 0 otherwise. Observe that this period-finding algorithm need only search for periods p of length $\lceil n/2 \rceil \leq p \leq n - 1$ in line 3 and periods q of length $\lceil m/2 \rceil \leq q \leq m - 1$ in line 9. This is due to our earlier remark on the bounds on the lengths of periods.

We use Proposition 6.5 and Lemma 6.10 as the bases for Algorithm 6.12, which computes the dimension of the largest border of a two-dimensional word. The algorithm takes as input a two-dimensional word A of dimension $m \times n$ and produces as output the dimension of the upper-left part of the border, from which the entire border can be calculated. The correctness of the algorithm follows from this proof, and the running time is $O(mn)$. $\qquad \square$

Algorithm 6.12: Computing the border of A

```
 1: procedure 2DBORDER(A, m, n)
 2:     for 0 ≤ i < m do
 3:         P_i ← 1DPERIOD(A[i, 0..n − 1])          ▷ compute periods of rows as bit vector
 4:         P ← P ∩ P_i                                              ▷ intersect bit vectors
 5:     if P = ∅ then
 6:         return "unbordered"                            ▷ if P is empty, A is unbordered
 7:     d ← smallest period common to all P_i
 8:     for 0 ≤ j < n do
 9:         Q_j ← 1DPERIOD(A[0..m − 1, j])    ▷ compute periods of columns as bit vector
10:         Q ← Q ∩ Q_i                                             ▷ intersect bit vectors
11:     if Q = ∅ then
12:         return "unbordered"                            ▷ if Q is empty, A is unbordered
13:     e ← smallest period common to all Q_j
14:     return (m − e, n − d)
```

Chapter 7

Conclusions

In this thesis, we studied a number of properties and results relating to two-dimensional words. We extended the well-known theorems of Lyndon and Schützenberger in the one-dimensional case to include a number of additional equivalent conditions and further generalized these theorems to the two-dimensional case. We showed that one new condition of the first Lyndon–Schützenberger theorem was related to the defect theorem for formal languages. We discussed two-dimensional primitive and periodic words. In particular, we proved a formula for the enumeration of such words similar to that of the one-dimensional case, and we presented computer methods to verify the primitivity of a two-dimensional word in linear time. We discussed one- and two-dimensional borders and presented results on the enumeration of one-dimensional bordered words and their representation as polynomials. In the two-dimensional case, borders were also considered, with results being proved that were similar to those in the previous chapter. Finally, following this chapter, we provide a large set of data relating to two-dimensional bordered words to assist in future work on this topic.

Throughout the process of writing this thesis, a number of questions arose that would serve as excellent topics for future work relating to this research. We provide a summary of these questions here.

- Bacquey poses the following problem in his paper on two-dimensional biperiodic infinite words [10]: for the d-dimensional generalization of an infinite word, show that there are at most $d!$ distinct primitive roots. Bacquey showed that this $d!$ bound was tight for $d \leq 2$, but did not provide a proof of correctness or tightness for $d > 2$.

- Can we generalize properties of words (e.g., overlaps, borders) to words of dimension greater than 2? What would be a "correct" characterization for these properties?

- Can we generalize either of the Lyndon–Schützenberger theorems or the Fine–Wilf theorem to words of dimension greater than 2?

- What is the connection (if any) between the number of two-dimensional primitive words of dimension $2 \times n$ over a two-letter alphabet, $\psi_2(2, n)$, and the number of pedal triangles with period n?

- Does there exist a recurrence relation for the enumeration of all two-dimensional unbordered words of dimension $m \times n$ over a k-letter alphabet, similar to the one-dimensional case?

- Is there a better or more efficient method for enumerating all two-dimensional unbordered words of dimension $m \times n$ over a k-letter alphabet?

- What is the general formula for $U_k(m, n)$ as defined in Chapter 6?

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Appendix A

Data for the Enumeration of Two-Dimensional Unbordered Words

In each of the following tables, the value above the line indicates the dimension m, and each pair of values (c, k) below the line indicates the coefficient c corresponding to the given alphabet size k. We assume that all two-dimensional words are over a two-letter alphabet. For example, taking $m = 4$, we get that

$$U_2(4, n) = 2^{4n} + b_2(n) - b_4(n) - b_8(n),$$

where $U_2(m, n)$ represents the number of two-dimensional unbordered words of dimension $m \times n$ over a two-letter alphabet and $b_k(n)$ represents the number of one-dimensional bordered words of length n over a k-letter alphabet. See Chapter 6 for more details.

2	
-1	2

3	
-1	4

4	
1	2
-1	4
-1	8

5	
1	4
-1	8
-1	16

6	
1	4
-1	16
-1	32

56

7	
1	8
−1	32
−1	64

8	
−1	2
1	4
1	8
−1	64
−1	128

9	
−1	4
1	8
1	16
−1	128
−1	256

10	
−1	4
2	16
−1	256
−1	512

11	
−1	8
2	32
−1	512
−1	1024

12	
−1	4
1	32
1	64
−1	1024
−1	2048

13	
−1	8
1	64
1	128
−1	2048
−1	4096

14	
−1	8
−1	16
1	32
1	64
1	256
−1	4096
−1	8192

15	
−1	16
−1	32
1	64
1	128
1	512
−1	8192
−1	16384

16	
1	2
−1	4
−1	8
−1	32
2	128
1	1024
−1	16384
−1	32768

17	
1	4
−1	8
−1	16
−1	64
2	256
1	2048
−1	32768
−1	65536

19	
1	8
−2	32
−1	64
1	512
1	1024
1	8192
−1	131072
−1	262144

20	
1	4
−1	16
−2	64
−1	128
1	256
1	512
1	2048
1	16384
−1	262144
−1	524288

18	
1	4
−2	16
−1	32
1	256
1	512
1	4096
−1	65536
−1	131072