Detectability of Singularly Perturbed Systems

by

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AUTHOR'S DECLARATION FOR ELECTRONIC SUBMISSION OF A THESIS

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Detectability of Singularly Perturbed Systems

A form of detectability, known as the input-output-to-state stability property, for singularly perturbed systems is examined in this work.

This work extends the result of Christofides & Teel [5] wherein they presented a notion of total stability for input-to-state stability with respect to singular perturbations. Analyzing singularly perturbed systems with outputs we show that if the boundary layer system is uniformly globally asymptotically stable and the reduced system is input-output-to-state stable with respect to disturbances, then these properties continue to hold, up to an arbitrarily small offset, for initial conditions in an arbitrarily large compact set and sufficiently small singular perturbation parameter over the time interval for which disturbances, their derivatives, and outputs remain in an arbitrarily large compact set.

An application of the result is presented where we analyze the stability of a circuit with a nonlinear element through the measurement of only one of the variables of interest.

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Chapter 1

Introduction

What is feedback control? Feedback control is all around us: from the cruise control in our cars to the thermostats in our homes. It affects our daily lives in how fiscal policies are managed and how funding is allocated to health care and education. It is used in complex technological devices such as guided missiles and appears in biological processes such as blood sugar regulation.

Having given all these instances in which feedback control appears, one asks: *so what is it*? Feedback control is the study of systems with *feedback* and how they may be manipulated or *controlled* to achieve a desired result.

Feedback is the idea that we can use present or past information and *send, or feed, it back* to alter the future behaviour of the system. An example would be that when driving, if you have turned too far left, your eyes will tell you that you have gone too far left, at which point your brain reacts telling your hands to move the steering wheel towards the right to correct the mistake.

By control, we mean an *input* which will affect the system to give a desired result or behaviour. In the previous example of turning a car, the steering wheel (or rather your hand turning the steering wheel) would be the control or input. In certain cases, there are undesired inputs or *disturbances* acting on the system. In the study of feedback systems, these can be treated as another input to the system, except that rather than being an aid, they are a hindrance towards the desired behaviour. In the previous example of turning a car, cross-wind could be considered as a disturbance input.

A brief history...

Feedback control is an engineering discipline. As such, its progress is closely tied to the practical problems that needed to be solved during the different eras of human history. A quick overview of the history of feedback control taken from Chapter 1 of [11] is given below.

In Antiquity, the Greeks designed a water clock which operated by measuring the amount of water that entered a tank. There was a mechanism, called a float regulator, which maintained the flow rate at a fixed value, hence allowing accurate measurement of time. The float regulator's purpose was to maintain the water level in a tank at a constant depth, which in turn maintained a constant flow of water into a second tank which acted as the clock. The float regulator operated by blocking the flow of water into a full first tank from an external source, a water reservoir for example, until some of the water flowed out into the second tank at which time it would then allow water to fill the first tank hence maintaining the water level of the first tank at an almost constant depth. During the dark ages, various Arab engineers such as the three brothers Musa, Al-Jazari, and Ibn al-Sa'ati used float regulators for water clocks and other applications.

The Industrial Revolution (circa 1700-1900) in Europe was largely due to the introduction of prime movers, or self-driven machines. It was marked by the invention of advanced grain mills, furnaces, boilers, and the steam engine. These devices could not be adequately regulated by hand, and so there arose a new requirement for automatic control systems.

The design of feedback control systems up through the Industrial Revolution was by trial-and-error together with a great deal of engineering intuition. One such design was Watt's flyball governor which was used to maintain the pressure of his steam engine at a fixed level. The governor is a part of a machine by which the velocity of the machine is kept nearly constant, notwithstanding variations in the driving-power of the resistance. If the pressure increased then the arms of the governor would contract, and so expand the aperture of the steam-valve hence letting out the extra steam. As the pressure dropped, the arms would expand outwards hence contracting the aperture of the steam-valve. It was only in 1868 that J.C. Maxwell used differential equations to analyze the stability of Watt's flyball governor [14]. Maxwell proved that the system would be stable if and only if the real part of all the roots of the characteristic equation of the linear ODE model were negative. With the work of Maxwell, the mathematical framework for control systems was firmly established and *Control Theory* as a discipline was born.

At Bell Telephone Laboratories during the 1920's and 1930's, the frequency domain approaches developed by P.-S. de Laplace (1749-1827), J. Fourier (1768-1830), A.L. Cauchy (1789-1857) and others were explored and used in communication systems with major contributions coming from Black, Nyquist and Bode. These tools were formalized and widely used, considered to be the best way to solve design control problems, until about 1960 and they are known as Classical Control Theory tools.

Around 1960 in the Soviet Union there was a great deal of activity in nonlinear controls design. Following the lead of Lyapunov (1893), attention was focused on time-domain techniques with advances provided by Ivachenko (1948), Tsypkin (1955), and Popov (1961). With the help of these advances, it was possible for the Soviets to launch the first satellite, Sputnik, in 1957.

The launch of Sputnik brought about the Space Race and caused the United States to pour a lot of money and effort into automatic controls design with the goal of launching rockets into space and landing a man on the moon. The problem was that the frequency-domain approach used at the time is tailored for use in linear time-invariant systems. This approach is at its best when dealing with single-input/single-output systems, for the graphical techniques are inconvenient when applied to systems with multiple inputs and outputs. Unfortunately, it is not possible to design good control systems for advanced nonlinear multi-variable systems, such as those arising in aerospace applications, using the assumption of linearity and treating the single-input/single-output transmission pairs one at a time. Due to the failure of the frequency-domain techniques, it was clear that a return to the time-domain techniques which were based on differential equations was needed.

It is quite remarkable that in almost exactly 1960, major developments occurred independently on several fronts in the theory of communication and control. In 1960, Draper invented a navigation system. Bellman (1957), Pontryagin (1958) and Kalman (1960) made advances in optimal control (i.e. the problem of finding the optimum input such that a particular control system achieves a particular state with minimum cost). Kalman also advanced the field of estimation theory by introducing the Kalman filter. As well as publicizing the work of Lyapunov in the U.S., Kalman also introduced the *state-space representation* in which the system dynamics of a linear time-invariant system was written in terms of matrices and tools from linear algebra could be used to solve them. Mathematically we mean that the dynamics of the state vector $x \in \mathbb{R}^n$ are expressed as

$$\dot{x} = Ax + Bu$$

A, B being matrices of appropriate sizes and $u \in \mathbb{R}^m$ being a control input vector. This state-space approach and the theory derived from it is known as Modern Control Theory.

Mathematical Description

For a general nonlinear time-invariant forced system (i.e. system with inputs) we write

$$\dot{x} = f(x, u) \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is an essentially locally bounded input, and f is a continuous function locally Lipschitz in x uniformly in u so that there exists a unique solution to initial value problem $x(0) = \xi, x(t)$ satisfying (1.1).

Often times in applications, we do not have access to the complete state information but only some measurements (called *outputs*) which give us partial knowledge of the state. For example, in a turbine engine, the behaviour of the turbine varies as a function of temperature. However, since most turbines operate at high temperatures (e.g. about 1000°C) it is not very cost effective to incorporate a temperature sensor. Such a system can be modeled by the equations:

$$\dot{x} = f(x, u), \qquad y = k(x) \tag{1.2}$$

where k is some function, often a projection, and y is called the output signal. In cases like this, one can ask: what can be said about the state given only knowledge of the output y and the input u? The notion of *detectability* deals with estimates of the behaviour of the state x given only the output y and input u, where the output y provides us with only partial knowledge of the state x.

The results in this thesis are based around the general framework provided by the input to state stability (ISS) property, which was introduced in [22]. This is a notion of stability for systems of the type (1.1). Roughly speaking, a system is ISS if it is globally asymptotically stable in the absence of disturbances and the effect of disturbances on that stability is proportional to their magnitude.

Generalizing the concept of ISS to systems with outputs we have a notion of detectability called input-output to state stability (IOSS). For systems of the type (1.2) which satisfy the IOSS property we have an upper bound on the asymptotic size of the state given the knowledge of the input disturbances u and output y. Effectively it means that for small input disturbances and small output signals the whole state is “nearly” asymptotically stable.

This thesis looks at singularly perturbed systems which often occur when modeling systems with two time scales. A simple example would be a second order system where one of the roots of the characteristic polynomial is a small negative number and the other is a large negative number:

$$x(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}, \quad 0 < r_1 << r_2.$$

Since $e^{-r_2 t} \to 0$ very quickly, we could simply ignore it by setting the second solution to 0 and achieve as approximate solution

$$x(t) \approx c_1 e^{-r_1 t}$$

which would still give us a good indication on how the actual system behaves. The study of singularly perturbed systems deals with the questions of after removing the faster time scale behaviour and examining the simpler system, whether or not any of the properties that the simpler system exhibits carry over to the full system.

Problem history & formulation

In [20], Sontag examined the system

$$\dot{x} = f(x, u)$$

and proved that if the system with $u(t) \equiv 0$ is globally asymptotically stable, then for "small" u relative to initial conditions, the system still preserves a type of stability, that is, it is ISS.

Using this same idea, Christofides & Teel [5] extended this result for systems with two inputs:

$$\dot{x} = f(x, u_1, u_2).$$

That is, given that the system with $u_2(t) \equiv 0$ is ISS with respect to u_1, then for "small" u_2 relative to the initial conditions and u_1 the system is ISS with respect to u_1 and u_2.

Using this result they went on to prove a stability result for singularly perturbed systems of the form

$$\begin{aligned} \dot{x} &= f(x, z, u, \epsilon) \\ \epsilon\dot{z} &= g(x, z, u, \epsilon) \end{aligned} \tag{1.3}$$

with ϵ being a small perturbation parameter which is treated as a second input to the system.

The result states that if the system

$$\dot{x} = f(x, h(x, u), u, 0)$$

with $z = h(x, u)$ being the solution of

$$0 = g(x, z, u, 0)$$

is ISS, then under certain assumptions, for "small" ϵ, the system is input-to-state practically stable (ISpS), that is, the ISS bound holds with an additional positive constant.

This thesis examines the case for which an output has been added to the system (1.3), which results in the system (1.4) given below.

$$\begin{aligned} \dot{x} &= f(x, z, u, \epsilon) \\ \epsilon\dot{z} &= g(x, z, u, \epsilon) \\ y &= k(x) \end{aligned} \tag{1.4}$$

The question that is raised is that of the behaviour of this system knowing that the simpler system

$$\dot{x} = f(x, h(x, u), u, 0) \tag{1.5}$$

is IOSS. The answer is a positive one in that the behaviour of the x subsystem in (1.4) under certain conditions has the property that it is IOSpS, that is the IOSS bound holds with an additional positive constant.

Outline

This thesis will deal with recovering an IOSS property, called input-output to state practical stability (IOSpS) from a singularly perturbed system of the form (1.4) whose reduced form (1.5) has the IOSS property and under which (1.4) satisfies a few other conditions.

The results in this work will be presented as follows:

- Chapter 2 will give a review of some basic concepts of linear control theory and stability analysis for differential equations, as well as some basic definitions and notation which will be used in the thesis. It will conclude by formalizing the concepts of ISS, IOSS and perturbed systems presented above.

- Chapter 3 will present the main result of the thesis and the lemmas required to prove it. The methodology of the proof of the result is based on the one used by Christofides & Teel in [5].

- Chapter 4 will give an example of a tunnel diode circuit treating one of the components of the circuit whose value is small as the perturbation parameter and demonstrating the use of the theorem when analyzing the behaviour of that circuit.

- Chapter 5 will conclude with a discussion of other types of conjectures on stability and comment on the usefulness of the main result of this thesis.

Chapter 2

Notation, Basic Concepts & Definitions

The following definitions and theorems will be used in the main body of the text. They are stated here for reference and can be found in standard texts on real analysis (e.g. [19], [9]), control theory (e.g. [15], [2], [4], [21]) and singular perturbation theory (e.g. [8], [16], [18]).

2.1 Basic Definitions

Definition 2.1 *The* essential supremum *of a function g on a set A is defined as:*

$$\operatorname*{ess\,sup}_{x \in A} g(x) = \inf_{Z \subset A, \mu(Z)=0} \left\{ \sup_{x \in A \setminus Z} g(x) \right\}$$

where $\mu(Z)$ is the Lebesgue measure of the set Z.

Definition 2.2 *We say a function $\delta_1(\epsilon)$ is of order $\delta_2(\epsilon)$, $\delta_1(\epsilon) = O(\delta_2(\epsilon))$, if there exist positive constants k and c such that*

$$|\delta_1(\epsilon)| \leq k|\delta_2(\epsilon)|, \quad \forall\, |\epsilon| < c$$

Definition 2.3 *A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and is zero at zero. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded.*

Definition 2.4 *A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{L}$ if it is continuous, decreasing and $\lim_{s\to\infty} \sigma(s) = 0$.*

Definition 2.5 *A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity. For the purposes of this thesis, it is assumed that for any class $\mathcal{KL}$ function $\beta(s, 0) \geq s$.*

Remark 2.1 *As with norms, there is a triangle inequality for $\mathcal{K}$ functions. For a function $\alpha \in \mathcal{K}$ we have that for any nonnegative real numbers s, t*

$$\alpha(s + t) \leq \max\{\alpha(2s), \alpha(2t)\} \leq \alpha(2s) + \alpha(2t) \tag{2.1}$$

2.2 Notation

- $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$, and I denotes the identity matrix.
- For a signal $u(t)$ defined on $[0, T)$, where T can be infinite, and for each $\tau \in [0, T), u_\tau$ is a signal defined on $[0, T)$ given by

$$u_\tau(t) = \begin{cases} u(t) & t \in [0, \tau] \\ 0 & t \in (\tau, T) \end{cases}$$

- For a signal $u(t)$ defined on $[0, T)$ and for each $\rho \in [0, T), u^\rho$ is a signal defined on $[0, T)$ given by

$$u^\rho(t) = \begin{cases} 0 & t \in [0, \rho) \\ u(t) & t \in [\rho, T) \end{cases}$$

- For any measurable (with respect to the Lebesgue measure) function $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $\|\theta\|$ denotes $\text{ess sup}_{x \in \mathbb{R}_{\geq 0}} |\theta(t)|$, whereas $\|\theta\|_{[0,t]}$ denotes $\|\theta_t\|$.
- For a real $n \times m$ matrix A, $\|A\|$ denotes the operator norm defined by

$$\|A\| := \max_{|\mu|=1} |A\mu|$$

- We denote by $\mathcal{M}_{\mathcal{D}}$ the set of all measurable functions $u : \mathbb{R}_{\geq 0} \to \mathcal{D}$, where D is a compact set.

2.3 Linear System Theory

This section gives a brief overview of linear system theory. Some results which are used later are stated here. They are taken from standard texts in linear control theory (e.g. [15], [21], [2], [4]).

Recall from linear systems theory that for the Linear Time Invariant (LTI) system

$$\dot{x}(t) = Ax(t) \tag{2.2}$$

with state vector $x \in \mathbb{R}^n$ and constant matrix $A \in \mathbb{R}^{n\times n}$ we have the following definition and result:

Definition 2.6 *The matrix A is said to be* stable *or* Hurwitz *if all of its eigenvalues have negative real part.*

Proposition 2.1 *All solutions $x(t)$ of (2.2) have the property that $\lim_{t\to\infty} x(t) = 0$ if and only if A is Hurwitz.*

In the study of control of linear time invariant systems we have the general system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with state vector $x \in \mathbb{R}^n$, input vector u whose values will belong to some set $\mathbb{U} \in \mathbb{R}^m$, output vector $y \in \mathbb{R}^r$ and constant real matrices A, B, C, D of appropriate dimensions. If $\mathbb{U}$ is not specified, we will assume that $\mathbb{U} = \mathbb{R}^m$.

A *control* or *input* will be a measurable and locally essentially bounded function $u : \mathcal{I} \to \mathbb{R}^m$, where $\mathcal{I}$ is a subinterval of $\mathbb{R}$ which contains the origin, such that $u(t) \in \mathbb{U}$ for almost all $t \in \mathcal{I}$. Whenever the domain $\mathcal{I}$ of an input is not specified, we assume $\mathcal{I} = \mathbb{R}_{\geq 0}$.

To simplify the analysis, we will assume that the output y does not depend directly on the input u, but only on the state x, that is, the matrix $D = 0$. This reduces the previous system to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned} \tag{2.3}$$

Often times in applications, the output y is a measurement of some of the state variables x and hence does not depend directly on the inputs u, therefore the assumption that $D = 0$ is valid in many cases.

When studying such systems, two main concepts which are the dual of one another emerge:

Definition 2.7 *A system (2.3) with* $u \equiv 0$ *(or pair (A,C)) is* observable *if any initial condition* $x(0) = x_0$ *can be determined by knowledge of the output* $y(t)$ *over any interval* $[0, T]$ *where* $T > 0$.

Definition 2.8 *A system (2.3) (or pair (A,B)) is* controllable *if for any initial state* x_0*, final state* x_f*, and time T, there exists a measurable essentially bounded input* $u(\cdot)$ *so that* $x(T) = x_f$.

Intuitively, observability means that from knowledge of the output $y(t)$ of the system with zero input over some time interval it is possible to determine or *observe* the previous values of the state $x(t)\, \forall t \in [0, T]$. As a natural dual, controllability means that given a start value of the state $x(0)$, it is possible with some input $u(t)$ defined on $[0, T]$ to move or *control* the state to any arbitrary end point $x(T)$.

Note that the property of controllability is only dependent on the matrices (A, B) as the output y has no role in moving the state, and that the property of observability is only dependent on the matrices (A, C) as due to linearity the output can be divided into two independent parts:

$$y(x; x_0, u) = y(x; x_0, 0) + y(x; 0, u)$$

where the second parameter x_0 indicates the initial condition and the third u indicates the input. It is fairly straightforward to compute the value of the signal $y(x; 0, u)$. All that remains given the signal $y(x; x_0, u)$ is to find the value of x_0 from this information. As $y(x; x_0, 0)$ is independent of the input signal, we see that the property of controllability is only dependent on the properties of the matrix pair (A,C).

Certain direct tests exists to verify if a pair (A, B) is controllable or (A, C) is observable.

Theorem 2.1 *The following statements are equivalent:*

(i) The pair (A, C) *is observable*

(ii) The observability matrix $Q = \begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix}$ *has rank n*

Theorem 2.2 *The following statements are equivalent:*

(i) The pair (A, B) *is controllable*

(ii) The controllability matrix $P = [B\, AB\, \cdots\, A^{n-1}B]$ *has rank n*

In the case that the system is not observable (or controllable), it is possible to decouple the observable part and unobservable part (similarly for controllability) by a linear transformation called *Kalman decomposition*, as follows.

In the case that $\text{rank}(Q) = r_q < n$, the state vector can be decomposed into two orthogonal subspaces $\Sigma = \Sigma_1 \oplus \Sigma_2$, with Σ_1 being the subspace spanned by the columns of Q. Σ_1 is called the observable subspace as it is spanned by the vectors of the observability matrix. Select r_q linearly independent vectors from Σ_1 and $n - r_q$ vectors which are linearly independent from the r_q previously chosen and from each other. Let these vectors form the columns of a transformation matrix $V = [V_1 \quad V_2]$. Transforming the original state variables by $x = Vv$ we then get:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} V_1^{-1}AV_1 & 0 \\ V_2^{-1}AV_1 & V_2^{-1}AV_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} V_1^{-1}B \\ V_2^{-1}B \end{bmatrix} u$$

and

$$y = [CV_1 \quad 0] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

which is called the *Kalman Observability Decomposition.*

In an analogous fashion, the *Kalman Controllability Decomposition* is done by choosing a basis set from the columns of P (these form the columns of a matrix T_1) and, augmenting vectors which form the columns of a matrix T_2 so that the matrix $T = [T_1 \quad T_2]$ is invertible. Then the transformation $x = Tw$ leads to the desired form

$$\dot{w} = T^{-1}ATw + T^{-1}Bu \quad \text{and} \quad y = CTw$$

Partitioning these equations according to the dimensions of Σ_1 and Σ_2 gives

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} T_1^{-1}AT_1 & T_1^{-1}AT_2 \\ 0 & T_2^{-1}AT_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} T_1^{-1}B \\ 0 \end{bmatrix} u$$

and

$$y = [CT_1 \quad CT_2]w$$

From this decomposition, it is possible to define the concept central to this thesis: the notion of detectability (at least for linear systems).

Definition 2.9 *A linear time-invariant system is said to be* detectable *if its unobservable part is stable, that is to say if the matrix $V_2^{-1}AV_2$ is Hurwitz, where V_2 is the matrix as defined from the Kalman Observability Decomposition.*

Simply speaking, after performing the Kalman Observability Decomposition we are left with two subsystems: an observable system and an unobservable system. A system is detectable if the unobservable subsystem is stable. That is, the state of the unobservable subsystem will tend to zero as t tends to infinity. As the other subsystem is observable, this implies that it is possible to determine the value of the initial state $x_{ob}(0) = x_0$ given the output $y(t)$ over some time interval

$[0, T]$, $T > 0$. In the case that $y(t) \equiv 0$ for $t \in [0, T]$, this implies that $x_{ob}(0) = 0$, in fact, it means that $x_{ob}(t) \equiv 0$ for $t \in [0, T]$.

Summarizing, we have that for linear detectable systems

$$u \equiv 0,\ y \equiv 0 \implies x \to 0 \text{ as } t \to \infty. \tag{2.4}$$

which defines the notion of *zero-detectability*.

More formally we say:

Definition 2.10 *A linear time-invariant system is said to be* zero-detectable *if for input* $u \equiv 0$ *and output* $y \equiv 0$ *then the state* x *will asymptotically tend to* 0 *i.e.* $lim_{t\to\infty} x(t) = 0$.

Remark 2.2 *A linear time-invariant system is* detectable *if and only if it is* zero-detectable*.*

As in the case that controllability is a dual property to observability, it is possible to define a dual property to detectability called *stabilizability*. It will not be used in the thesis and is given below for the sake of completeness.

Definition 2.11 *A linear time-invariant system is said to be* stabilizable *if its uncontrollable part is stable, that is to say if the matrix* $T_2^{-1} A T_2$ *is Hurwitz, where* T_2 *is the matrix as defined from the Kalman Controllability Decomposition.*

2.4 Differential Equations

For the control system described by a forced differential equation

$$\dot{x}(t) = f(x(t), u(t)),$$

the state x being a vector in $\mathbb{R}^n$, the inputs u are to be taken in some set $\mathbb{U} \subseteq \mathbb{R}^m$, we will assume that the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and is locally Lipschitz in x uniformly in u.

By the assumptions on f, given any input u defined on an interval $\mathcal{I}$ and any $\xi \in \mathbb{R}^n$, there exists a unique maximal solution to the initial value problem $\dot{x} = f(x, u), x(0) = \xi$. This solution is defined on some maximal open subinterval of $\mathcal{I}$. We denote the solution by $x(t, \xi, u)$.

A system is *forward complete* if each $\xi \in \mathbb{R}^n$ and each input u defined on $\mathbb{R}_{\geq 0}$ produce a solution $x(t, \xi, u)$ which is defined for all $t \geq 0$.

2.4.1 Stability

Certain definitions and useful results with regards to stability of differential equations are given below. They are taken from [7] and [6] and the reader is directed there for a more complete treatment of the subject.

The autonomous system

$$\dot{x}(t) = f(x(t)) \tag{2.5}$$

with initial condition $x(0) = \xi$, $f : \mathbb{R}^n \to \mathbb{R}^n$ being locally Lipschitz is known to have a unique solution over some time interval $t \in [0, t_\xi^{max})$. We denote this solution as $x(t, \xi)$.

An *equilibrium point* is a point x_{eq} such that

$$f(x_{eq}) = 0.$$

For systems (2.5) which are forward complete with an equilibrium point at the origin $x = 0$ we have the following definitions:

Definition 2.12 *The origin is* stable *for (2.5) if for each $\epsilon > 0$ there exists a $\delta > 0$ so that*

$$|\xi| < \delta \implies \quad |x(t, \xi)| < \epsilon, \qquad \forall\, t \geq 0.$$

Definition 2.13 *We say that the origin is* globally attractive *for (2.5) if for all $\xi \in \mathbb{R}^n$,*

$$\lim_{t \to \infty} x(t, \xi) = 0.$$

Remark 2.3 We next give the definition of uniform global attractivity. As the reader may know, the notions of global attractivity and uniform global attractivity are equivalent in the case of autonomous systems with $u \equiv 0$. We give separate definitions to allow the generalization to systems with inputs where these are distinct properties.

Definition 2.14 *We say that the origin is* uniformly globally attractive *for (2.5) if for each* $\kappa > 0$ *and* $\epsilon > 0$*, there exists a time* $T = T_{\kappa,\epsilon}$ *such that*

$$|\xi| \leq \kappa \implies \quad |x(t,\xi)| \leq \epsilon, \qquad \forall\, t \geq T.$$

Definition 2.15 *We say that the origin is* globally asymptotically stable *(GAS) for (2.5) if it is stable and globally attractive, and is* uniformly globally asymptotically stable *(UGAS) if it is stable and uniformly globally attractive.*

Proposition 2.2 *System (2.5) is UGAS if and only if there exists a* $\mathcal{KL}$ *function* β *so that for each* $\xi \in \mathbb{R}^n$*,*

$$|x(t,\xi)| \leq \beta(|\xi|, t) \quad \forall\, t \geq 0.$$

A partial proof, the sufficiency part, is given below. The reader is directed to [7] to find the necessity part. (Sufficiency) Assume there exists a $\mathcal{KL}$ function β so that for each $\xi \in \mathbb{R}^n$,

$$|x(t,\xi)| \leq \beta(|\xi|, t) \quad \forall\, t \geq 0.$$

Let $\kappa, \epsilon > 0$. For any ξ such that $|\xi| < \kappa$ we have

$$|x(t,\xi)| \leq \beta(\kappa, t) \quad \forall\, t \geq 0.$$

As $\beta(\kappa, t)$ is a decreasing function in t, there exists a time $T = T(\epsilon, \xi)$ such that

$$|x(t,\xi)| \leq \beta(\kappa, t) \leq \epsilon \quad \forall\, t \geq T,$$

hence the origin is uniformly globally attractive by Definition 2.14.

To show that the origin is stable, we have that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\beta(\delta, 0) < \epsilon$, as $\beta(s, 0)$ is a continuous function and $\beta(0,0) = 0$ hence for any $\xi < \delta, |x(t,\xi)| < \epsilon\, \forall t \geq 0$. Therefore by Definition 2.15, the origin is UGAS.

The necessity part of the proof, though not difficult, is quite technical as one constructs a β function using the definitions of uniform attractivity and stability. $\square$

The following definition is a useful concept especially when used in conjunction with Lyapunov functions which are to be discussed in the following section. The definition is needed as part of Theorem 2.7 in section 2.5

Definition 2.16 *The* region of attraction *of the origin, denoted by R_A is defined by*

$$R_A = \{\xi \in D \,|\, x(t,\xi) \text{ is defined } \forall t \geq 0 \text{ and } x(t,\xi) \to 0 \text{ as } t \to \infty\}$$

Lyapunov functions

Often times, it is very difficult to show that a nonlinear system is stable using the definition of stability. In fact, it is often difficult simply to find an upper bound for the trajectories of the system. One possibility would be to use some auxiliary measure which would give an indication of the magnitude of the state, for example using the energy in a mechanical system to give an indication of the position and velocity:

$$\begin{aligned} E &= \text{Kinetic Energy} + \text{Potential Energy} \\ &= \frac{1}{2}mv^2 + mgh, (h \geq 0) \end{aligned}$$

m being the constant mass of the object and g the gravitational acceleration (assumed constant). So when the energy $E = 0$ we know that both $v = 0$ and $h = 0$.

Generalizing this idea, we define a *Lyapunov function* as a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \text{ in } \mathbb{R}^n \backslash \{0\}$$

$$\nabla V(x) \cdot f(x) \leq 0 \text{ in } \mathbb{R}^n$$

Using this function we have the following theorem:

Theorem 2.3 *Let $x = 0$ be an equilibrium point for (2.5). If there exists a Lyapunov function V then $x = 0$ is stable.*

Moreover, if

$$\nabla V(x) \cdot f(x) < 0 \text{ in } \mathbb{R}^n \backslash \{0\}$$

then $x = 0$ is globally asymptotically stable.

2.4.2 Input-to-State Stability (ISS)

Recall that a forced differential equation is modeled by the system

$$\dot{x}(t) = f(x(t), u(t)). \tag{2.6}$$

In the case when $u \equiv 0$ we have defined various notions of stability of the origin. Suppose that $u \neq 0$. How would one go about defining stability of the origin in this case? One particular way to formulate this question, would be to ask that for the system (2.6) the equilibrium point $x = 0$ has the property that

$$\text{“}u \text{ small”} \implies \text{“}x \text{ small” as } t \to \infty.$$

That is, for small input disturbances the system has an asymptotically small state.

Suppose that for linear systems

$$x = Ax + Bu,$$

A and B being matrices of appropriate sizes we wish to find an explicit bound on the state. Solving this system explicitly:

$$x(t, \xi, u) = e^{At}\xi + \int_0^t e^{A(t-s)}Bu(s)ds.$$

If the matrix A is stable then for any initial state ξ there exists positive constants $M = M(|\xi|, A)$ and $k = k(A)$ such that

$$|e^{At}\xi| \leq Me^{-kt}, \qquad t \geq 0.$$

Which in turn gives a bound on the state x for any time $t \geq 0$:

$$\begin{aligned} \implies |x(t)| &\leq Me^{-kt} + \|u\|_{[0,t]} \int_0^t e^{-k(t-s)}\|B\|ds \\ &\leq Me^{-kt} + \|u\|_{[0,t]} \frac{1}{-k}(e^{-kt} - 1)\|B\|. \end{aligned}$$

And so it is possible to bound the state x by

$$|x(t)| \leq \beta(|\xi|, t) + a\|u\|_{[0,t]}, \qquad t \geq 0. \tag{2.7}$$

The function $\beta(s,t)$ here is defined as $\beta(s,t) := M(s)e^{-kt}$ and $a \in \mathbb{R}$ is defined as $a = \|B\|/k$ where $\|B\|$ is the matrix norm of B.

Do we have the same bound for stable nonlinear systems? Unfortunately, the answer is no as the following example shows. Consider the scalar system

$$\dot{x} = -x^3 + u. \tag{2.8}$$

With this system it is impossible to find a single linear gain with respect to u which will satisfy (2.7) for any u. To show this, we assume that there such a linear gain $a \geq 0$ and show that this gain cannot hold for all constant inputs. Using this linear gain, the asymptotic value of the state $x_{ss}(t) = \lim_{t\to\infty} x(t)$ is bounded by

$$|x_{ss}(t)| \quad \leq \quad a\|u\|_{[0,t]} \tag{2.9}$$

Here we have three possibilities for a: If $0 \leq a < 1$, then the constant input $u \equiv 1$ which yields steady-state solution $x_{ss}(t) = 1$ will violate the bound (2.9).

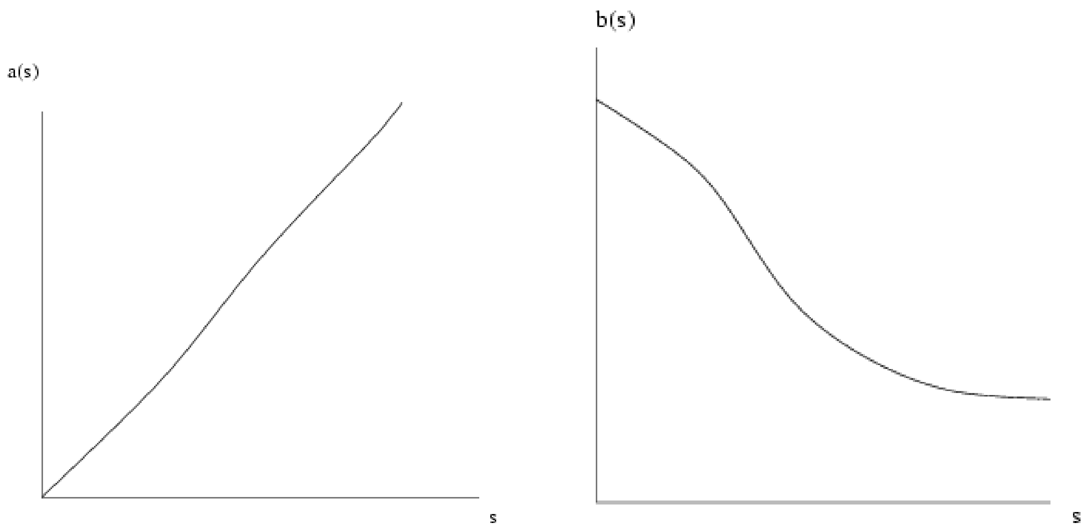
If $a = 1$, then the bound will not hold for the constant input $u \equiv \frac{1}{2}$ as the system's steady-state solution will be $x_{ss}(t) \equiv \sqrt[3]{1/2} > \frac{1}{2} \equiv u$.

Finally, in the case that $a > 1$, then using as input $u \equiv \frac{1}{a^3}$ we have steady-state solution $x_{ss}(t) \equiv \sqrt[3]{1/a^3} = 1/a$ and so the bound (2.9) gives

$$\begin{aligned} |x_{ss}(t)| &\leq a\|u\|_{[0,t]} \\ \frac{1}{a} &\leq a \cdot \frac{1}{a^3} \\ &\leq \frac{1}{a^2} \end{aligned}$$

which is false as $a > 1$ and so $a^2 > a$. Therefore, for any linear gain a, there exists an input u such that the bound (2.9) does not hold and so the gain a does not hold for all inputs u.

However, suppose that instead of using a linear gain on the input u to bound the state we used a nonlinear gain, say a strictly increasing function of the form of Figure 2.1(a). This is what is known as a $\mathcal{K}$ function (see Section 2.1 for formal definition). In a similar manner, to capture the decaying asymptotic behaviour of an asymptotically stable nonlinear system we introduce the concept of an $\mathcal{L}$ function as illustrated in Figure 2.1(b).



(a) $\mathcal{K}$ function (b) $\mathcal{L}$ function

Figure 2.1: Example of a $\mathcal{K}$ and $\mathcal{L}$ function

These classes of functions are useful in the study of dynamical systems and were first introduced by Hahn[6] in 1967.

Using these classes of functions one can bound the trajectory of system (2.8) by

$$|x(t)| \leq \beta(\xi, t) + \gamma(\|u\|_{[0,t]})$$

where β and γ are some $\mathcal{KL}$ and $\mathcal{K}$ functions. The class $\mathcal{KL}$ and $\mathcal{K}$ functions were formally defined in Section 2.1. The reader is directed there to find them.

The concept of defining stability of systems with inputs through a bound on the norm of the state using nonlinear gains on the initial conditions and input was first introduced in 1989 by Sontag [20]. More formally we say,

Definition 2.17 *: A system (2.6) is said to be input-to-state stable (or ISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(0)$ and any essentially bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq 0$ and satisfies*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_{[0,t]}) \qquad \forall t \geq 0. \tag{2.10}$$

Remark 2.4 *Since the systems of the type (2.6) are causal with respect to the inputs $u(\cdot)$, any bound on the magnitude of the trajectory that includes a supremum norm of an input can be given equivalently with the supremum taken over $[0, \infty)$.*

We mean that (2.10) holds if and only if

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_{[0,\infty)}) \qquad \forall t \geq 0. \tag{2.11}$$

Proof:It is obvious that a trajectory which is bounded by (2.10) is also bounded by (2.11) as $\|u\|_{[0,t]} \leq \|u\|_{[0,\infty)}$ for any $t \leq \infty$. To prove the converse, we have by causality, that at time t the trajectory $x(t)$ cannot depend on future values of u. Hence the state (and its bound) is dependent only on $u(\tau)$ for $\tau \in [0, t]$. □

Since, with $u(t) \equiv 0$, (2.10) reduces to

$$|x(t)| \leq \beta(|x(0)|, t)$$

input-to-state stability of (2.6) implies that the origin of the unforced system is globally asymptotically stable. However the converse is not true. Consider the scalar system

$$\dot{x} = -x + u^2 x^2.$$

Then for $u \equiv 0$ the system is GAS, however for $u \equiv 1$ the system is unstable for any initial condition $x(0) > 1$. Therefore, the trajectory is not bounded and thus the system cannot be ISS.

Though the definition of an ISS system gives us a bound on the state, it is often quite difficult to determine if a system of the type (2.10) is ISS by this definition. As in the case of stability analysis for autonomous systems with $u \equiv 0$, one can use Lyapunov-like functions to determine if a system is ISS.

Definition 2.18 *[7]: A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an* ISS-Lyapunov function *for system (2.10) if there exist $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$, and a $\mathcal{K}$ function ρ, such that*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$\nabla V(x) \cdot f(x,u) \leq -\alpha_3(|x|) + \rho(|u|)$$

holds for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

From the following theorem, we have a result that links the ISS-Lyapunov function and the ISS property of a system. The proof can be found in [23].

Theorem 2.4 *A system of the form (2.6) is ISS if and only if it admits an ISS-Lyapunov function.*

2.4.3 (Zero-)Detectability

Consider the system (2.6) with output $y \in \mathbb{R}^r$

$$\dot{x}(t) = f(x(t), u(t)), \qquad y = k(x) \tag{2.12}$$

with $k : \mathbb{R}^n \to \mathbb{R}^r$ being a locally Lipschitz function.

For the system (2.12), given any $u(\cdot)$ and any $\xi \in \mathbb{R}^n$, there exists a unique solution of the initial value problem:

$$\dot{x} = f(x,u), \quad x(0) = \xi.$$

Such a solution is defined over some open interval $(t^{min}_{\xi,u}, t^{max}_{\xi,u})$ where $t^{min}_{\xi,u} < 0 < t^{max}_{\xi,u}$ and is denoted as $x(\cdot, \xi, u)$. We also write $y(t,\xi,u) := k(x(t,\xi,u))$ for all ξ, u, and each $t \in (t^{min}_{\xi,u}, t^{max}_{\xi,u})$.

In many practical applications, the system (2.12) is not observable or the whole state x cannot be measured for all time instances t either due to infeasibility or cost constraints. However, it is sometimes still possible to determine the behaviour of the system with only partial information on the state.

For linear systems, if a system is zero-detectable then it is also detectable. However for nonlinear systems this is not the case and in fact the notion of *detectability* is not even well defined for nonlinear systems. If it is only the size of the state and not its exact value which is of importance to us, then the notion of zero-detectability is sufficient. As this is the case in this body of work, henceforth, we will simply say "detectability" when we really mean "zero-detectability".

Recall from Definition 2.10, we have that for linear time-invariant system zero-detectability means that

$$y(t) \equiv 0, u(t) \equiv 0 \implies x(t) \to 0 \quad \text{as} \quad t \to \infty.$$

This definition can also be used as a definition of zero-detectability of nonlinear systems. If however, we wished to define a more robust form of zero-detectability, that is

$$\text{"}y(t)\,\text{small"}, \text{"}u(t)\,\text{small"} \implies \text{"}x\,\text{small"} \quad \text{as} \quad t \to \infty.$$

A possible way of doing so, called Input-Output-to-State-Stability or IOSS for short was first introduced by Sontag & Wang in 1997 [24]:

Definition 2.19 *: A system of type (2.12) is said to be input-output-to-state stable (IOSS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that the bound:*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_1(\|u\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]})$$

holds for any initial state $x(0) \in \mathbb{R}^n$, control $u(\cdot)$, and time $t \in [0, t^{max}_{\xi,u})$.

As in the case of ISS, we also have a Lyapunov-like characterization of IOSS:

Definition 2.20 *: A C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an IOSS-Lyapunov function for system (2.12) if there exist $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha$ and $\mathcal{K}$ functions σ_1, σ_2 such that*

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|)$$

$$\nabla V(\xi) \cdot f(\xi, u) \leq -\alpha(|\xi|) + \sigma_1(|u|) + \sigma_2(|k(\xi)|)$$

holds for all $\xi \in \mathbb{R}^n$ *and all control values* $u \in \mathbb{R}^p$.

From [10] we have the following theorem relating the two:

Theorem 2.5 *A system* $\dot{x} = f(x,u), y = k(x)$ *is IOSS if and only if it admits an IOSS Lyapunov function.*

It is possible to extend the definition of IOSS to a slightly weaker form of detectability called IOSpS:

Definition 2.21 *: A system of type (2.12) is said to be input-output-to-state practically stable (IOSpS) if there exist functions* $\beta \in \mathcal{KL}$ *and* $\gamma_1, \gamma_2 \in \mathcal{K}$ *and a nonnegative real number* C *such that the bound:*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_1(\|u\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]}) + C$$

holds for any initial state $x(0) \in \mathbb{R}^n$*, control* $u(\cdot)$*, and time* $t \in [0, t^{max}_{\xi,u})$.

2.4.4 Unboundedness Observability

Definition 2.22 *We say that the system (2.12) has the* unboundedness observability property *(or just "UO") if, for each initial state* ξ *and bounded control* u *such that* $\bar{t} = t^{max}_{\xi,u} < \infty$*, necessarily* $\limsup_{t \nearrow \bar{t}} |y(t, \xi, u)| = +\infty$. *(*$t^{max}_{\xi,u}$ *is as defined in section 2.4.3)*

In other words, it is possible to "observe" any unboundedness of the state.

Proposition 2.3 *A system which is IOSS has the UO property.*

Proof: This is easily shown as for any bounded input u and bounded output y, then by the definition of IOSS, the state is bounded. If the state blows up at some finite time $\bar{t}$ with a bounded input u, then y must also blow up.

There are certain known results in regards to systems which have the UO property. Two such results will be stated below, the proofs of which can found in [1]. The first gives a Lyapunov-like characterization and the second gives a bound on the state of systems which satisfy the UO property.

Theorem 2.6 *: The system (2.12) has the unboundedness observability property if and only if there exist a proper and C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and σ_1, σ_2 functions of class $\mathcal{K}_\infty$ such that*

$$\nabla V(\xi) \cdot f(\xi, u) \quad \leq \quad V(\xi) + \sigma_1(|u|) + \sigma_2(|k(\xi)|)$$

holds $\forall \xi \in \mathbb{R}^n, \quad \forall u \in \mathbb{R}^p$

Proposition 2.4 *: The system (2.12) has the UO property if and only if there exist $\mathcal{K}$ function $\chi_1, \chi_2, \chi_3, \chi_4$ and a constant c such that*

$$|x(t, \xi, u)| \leq \chi_1(t) + \chi_2(|\xi|) + \chi_3(\|u\|_{[0,t]}) + \chi_4(\|y\|_{[0,t]}) + c$$

holds for all $\xi \in \mathbb{R}^n$, all bounded input signals u, and all $t \in [0, t^{max}_{\xi,u})$.

We will demonstrate a similar result: for inputs in a fixed compact set, the UO property implies that the state is bounded by some functions of time, initial value and output. The proof is almost identical to the one provided in [1] for the above result for which all bounded inputs are allowed.

Bounded reachable sets

The first result which we prove will be a critical step in our constructions; it shows that for UO systems of the form (2.12) the set of states reachable from any compact set, in bounded time and using bounded controls, is bounded, provided that the outputs remain bounded. When there are no outputs ($k = 0$), this fact amounts to the statement that the set of reachable states from compact sets in bounded time and using bounded controls is bounded, a fact proved in [12]; we shall prove the result by a reduction to that special case. (Note that when there are also no controls and we have just a differential equation $\dot{x} = f(x)$, the statement is an easy consequence of continuous dependence of solutions on initial conditions.)

Fix $d \geq 0$, then for any nonnegative real numbers η, ρ, T and τ, and each state $\xi \in \mathbb{R}^n$, we let:

$$\begin{aligned} \mathcal{U}_d(\xi, \eta, \tau) &:= \{u \mid \|u\|_{[0,\tau)} \leq d, t_{\xi,u}^{max} \geq \tau, \textit{ and } |y(t, \xi, u)| \leq \eta \quad \forall t \in [0, \tau]\}, \\ R_d(\xi, \eta, \tau) &:= \{x(\tau, \xi, u) \mid u \in \mathcal{U}_d(\xi, \eta, \tau)\} \end{aligned}$$

and

$$\mathcal{R}_d^{\leq T}(\rho, \eta) := \bigcup_{|\xi| \leq \rho, \tau \in [0,T]} R_d(\xi, \eta, \tau)$$

Note that a state ζ belongs to the reachable set $\mathcal{R}_d^{\leq T}(\rho, \eta)$ if and only if there is some initial state ξ with $|\xi| \leq \rho$, some time $\tau \leq T$, and some input u bounded by d such that $\zeta = x(\tau, \xi, u)$, where the solution $x(\cdot, \xi, u)$ is defined on the interval $[0, \tau]$ and has $|y(t, \xi, u)| \leq \eta$ for all $t \in [0, \tau]$.

Observe that $\mathcal{U}_d(\xi, \eta, \tau)$ is non-decreasing with η, so $R_d(\xi, \eta, \tau)$ is too. Then by definition, the sets $\mathcal{R}_d^{\leq T}(\rho, \eta)$ are nondecreasing in T, ρ, and η and thus the function

$$\gamma_d(T, \rho, \eta) := \sup\{|\zeta| : \zeta \in \mathcal{R}_d^{\leq T}(\rho, \eta)\} \tag{2.13}$$

(possibly taking infinite values) is nondecreasing separately on each of the variables T, ρ, η.

To prove the following lemma it will be necessary to use Proposition 5.1 from [12]. It is stated below for reference.

Proposition 2.5 *Assume that the system*

$$\dot{x}(t) = f(x(t), u(t))$$

with $x \in \mathbb{R}^n$ *and* $u \in \mathcal{M}_{\mathcal{D}}$ *is forward complete. Then for any compact subset* K *of* $\mathbb{R}^n$ *and any* $T > 0$*, the set* $\overline{\mathcal{R}^{\leq T}(K)}$ *is compact* ($\bar{S}$ *denotes the closure of* S *for any subset* S *of* $\mathbb{R}^n$*).*

Lemma 2.1 *If system (2.12) is UO; then for each fixed* $d \geq 0$, $\gamma_d(T, \rho, \eta) < \infty$ *for all* T, ρ, η.

Proof: The idea of the proof is this: since we are interested in sets of states which can be reached with output bounded by η, the dynamics of the system in the part of the state space where the outputs become larger than η do not affect the value of γ_d; thus, we modify the dynamics for those states, using a procedure motivated by an "output injection" construction often used in control theory. The modified system will be forward complete, and previously known results will be then applicable.

The idea of the "output injection" construction consists of creating an auxiliary system which is forward complete by setting the velocities ($\dot{x}$) of the system to 0 for large outputs. For this auxiliary system the states will be bounded so long as the outputs stay small and therefore it will be forward complete.

Fix $d \geq 0$, and take any T, ρ, η. We start by picking any smooth function $\phi_\eta : \mathbb{R} \to [0, 1]$ with the following properties:

$$\phi_\eta(r) = \begin{cases} 1 & \text{if} \quad r \leq \eta, \\ 0 & \text{iff} \quad r > \eta + 1. \end{cases}$$

Next, we introduce the following auxiliary system:

$$\dot{x} = f(x, u)\phi_\eta(|k(x)|), \qquad y = k(x) \tag{2.14}$$

Observe that the function $f(x, u)\phi_\eta(|k(x)|)$ is still locally Lipschitz because k is such. The set $\mathcal{R}_d^{\leq T}(\rho, \eta)$ for this new system is equal to the respective one defined for the original system. So, if we prove that system (2.14) is forward complete, then Proposition 2.5 will give that $\mathcal{R}_d^{\leq T}(\rho, \eta)$ is bounded, since that reference states that the reachable sets for forward complete systems (in bounded time, starting from a compact set, and using bounded controls) are bounded. Suppose by way of contradiction that system (2.14) is not forward complete, and pick an initial condition ξ and an input v such that the maximal solution of

$$\dot{z} = f(z, v)\phi_\eta(|k(z)|), \qquad z(0) = \xi \tag{2.15}$$

has

$$|z(s)| \to \infty \quad as \quad s \nearrow S < \infty \tag{2.16}$$

We claim that $|k(z(s))| \leq \eta + 1$ for all $s \in [0, S)$ so that $\phi_\eta(|k(z(s))|) > 0$ for all $s \in [0, S)$. If this were not the case, then there would be some $s_0 \in [0, S)$ so that $\zeta_0 := z(s_0)$ has $|k(\zeta_0)| > \eta + 1$. Hence ζ_0 is an equilibrium point as $\phi_\eta(|k(\zeta_0)|) = 0$. But $\hat{z} \equiv \zeta_0$ is a solution of (2.15) because ζ_0 is an equilibrium so $\dot{z} \equiv 0 \implies \hat{z}(s) = \zeta_0 \quad \forall s \in [0, S)$, and hence by uniqueness we have that $\hat{z} = z$, and thus z is bounded as $z = \hat{z} \equiv \zeta_0$ which is bounded, contradicting (2.16). We conclude that $\phi_\eta(|k(z(s))|) > 0$ for all $s \in [0, S)$. So, the function

$$\phi(s) := \int_0^s \phi_\eta(|k(z(\tau))|)d\tau$$

is strictly increasing, and maps $[0, S)$ onto an interval $[0, T)$ (with, in fact, $T \leq S$, because $\phi_\eta \leq 1$ everywhere). We let $x(t) := z(\phi^{-1}(t))$ for all $t \in [0, T)$. This is an absolutely continuous function, and it satisfies $\dot{x} = f(x, u)$ *a.e.* on $[0, T)$, where u is the input $u(t) = v(\phi^{-1}(t))$. Note that $x(0) = \xi$ and (2.16) says that $x(t) \to \infty$ as $t \nearrow T$, so $T = t^{max}_{\xi,u}$. The unboundedness observability property says then that $y(t, \xi, u)$ is unbounded on $[0, T)$. But $y(t, \xi, u) = k(x(t)) = k(z(s))$, where $s = \phi^{-1}(t)$, and we already proved that $|k(z(s))| \leq \eta + 1$, so we arrived at a contradiction. Therefore system (2.14) is forward complete and by Proposition 2.5 we have that $\mathcal{R}_d^{\leq T}(\rho, \eta)$ is bounded. By the definition of $\gamma_d(T, \rho, \eta)$ we have that $\gamma_d(T, \rho, \eta) < \infty$ for all T, ρ, η and the lemma is proved. □

Bounds on states

In order to keep notations simple, if the initial state ξ and input u are clear from the context, we use the convention that when we write "y", or "$y_{[0,t]}$" as above, we mean the output function $y(\cdot, \xi, u)$, or its restriction to the interval $[0, t]$, respectively.

Proposition 2.6 *If system (2.12) is UO, then given $d > 0$ and $T \in (0, t^{max}_{\xi,u}]$, there exists $\chi_1, \chi_2, \chi_3 \in \mathcal{K}$ so that the following holds for all $\xi \in \mathbb{R}^n$ and all u such that $\|u\|_{[0,T]} < d$:*

$$|x(t, \xi, u)| \leq \chi_1(t) + \chi_2(|\xi|) + \chi_3(\|y\|_{[0,t]}) \qquad \forall t \in [0, T). \tag{2.17}$$

Proof: Let $d > 0$ and $T \in (0, t^{max}_{\xi,u}]$ be given. Assume the system (2.12) is UO. Defining $\gamma_d(T, \rho, \eta)$ as in (2.13) we have by Lemma 2.4.4, $\gamma_d(t, \rho, \eta) < \infty$ for all t, ρ, η. Pick any $\xi \in \mathbb{R}^n$, input signal u such that $\|u\|_{[0,T]} < d$, and $t \in [0,T)$ and let $\rho := |\xi|$, and $\eta := \|y_{[0,t]}\|_\infty$. Then $x(t,\xi,u) \in \mathcal{R}^{\leq T}(\rho,\eta)$, so

$$|x(t,\xi,u)| \leq \gamma_d(t,\rho,\eta) \leq \chi(t) + \chi(\rho) + \chi(\eta)$$

where $\chi(r) := \gamma_d(r,r,r)$. The function $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is nondecreasing, because γ_d is nondecreasing in each variable, as remarked earlier. Thus, there exists a function $\tilde{\chi} \in \mathcal{K}_\infty$ such that $\chi(r) \leq \tilde{\chi}$ for all r, and therefore (2.17) is valid with all $\chi_i = \tilde{\chi}, i = 1,2,3$. □

2.5 Singularly Perturbed Systems

In mathematics, science and engineering one is often confronted with the task of dealing with a complex model whose solution is difficult or time consuming to find. In many instances, we simplify the model by ignoring "small" parameters (i.e. setting their values to 0) and we analyze the simplified model. If the solution to the original model is desired, one possibility is to write out the solution as a series solution in terms of one of the "small" parameters using the simplified solution as the zeroth-order term. The study of recovering the solution or behaviour of a more complex model by way of examining a simplified version of the model is known as the field of *Perturbation Theory* as one "perturbs" the simplified model to retrieve the original model. It has already been studied in depth and there are many books on the subject (e.g. [3],[16],[18]).

Consider for example the DE which models an unforced mass-spring system:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0, \qquad x(0) = 1,\, \dot{x}(0) = \zeta\omega_0.$$

Under the condition that $\zeta < 1$, it is underdamped and has as solution $x(t) = e^{-\gamma t}\cos(\omega t)$ where $\gamma = \zeta\omega_0$ and $\omega = \omega_0\sqrt{1-\zeta^2}$.

Suppose we model the restoring force of the spring more accurately by including third order effects, so that $F = \omega_0^2 x + \epsilon\omega_0^2 x^3$ where ϵ is some small value, say $\epsilon << 1$. The resulting DE is:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x + \epsilon\omega_0^2 x^3 = 0, \qquad x(0) = 1,\, \dot{x}(0) = \zeta\omega_0$$

which is nonlinear and difficult to solve. We can, however, naively assume a series solution in ϵ of the form

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots$$

and try to solve for $x(t)$.

We first set $\epsilon = 0$ which reduces the DE to the simpler linear spring-mass system and we have that $x(t) = x_0(t) = e^{-\gamma t}\cos(\omega t)$.

Now, substitute $x(t) = x_0(t) + \epsilon x_1(t)$ into the DE and using our knowledge of $x_0(t)$, solve for $x_1(t)$. This yields:

$$\ddot{x}_0 + \epsilon\ddot{x}_1 + 2\zeta\omega_0(\dot{x}_0 + \epsilon\dot{x}_1) + \omega_0^2(x_0 + \epsilon x_1) + \omega_0^2\epsilon(x_0 + \epsilon x_1)^3 = 0$$

Expanding and collecting all ϵ coefficients results in:

$$\begin{aligned}
\ddot{x}_1 + 2\zeta\omega_0\dot{x}_1 + \omega_0^2 x_1 + \omega_0^2 x_0^3 &= 0 \\
\ddot{x}_1 + 2\zeta\omega_0\dot{x}_1 + \omega_0^2 x_1 &= -\omega_0^2 e^{-3\gamma t}\cos^3(\omega t) \\
&= \frac{-\omega_0^2}{4} e^{-3\gamma t}[3\cos(\omega t) + \cos(3\omega t)]
\end{aligned}$$

which is simply a forced oscillator in terms of x_1. It should be noted that the initial conditions for x_1 are not the same as those of x. In fact, substituting the values for $t = 0$ we get as initial conditions for x_1:

$$\begin{aligned}
x(0) = x_0(0) + \epsilon x_1(0) \quad &\Longrightarrow \quad x_1(0) = 0 \\
\dot{x}(0) = \dot{x}_0(0) + \epsilon\dot{x}_1(0) \quad &\Longrightarrow \quad \dot{x}_1(0) = 0
\end{aligned}$$

Solving for x_1 in the above DE under these initial conditions yields:

$$x_1(t) = \frac{1}{2(2+3\zeta^2)}\left[\sqrt{1+\gamma^2}e^{-\gamma t}\cos(\omega t + \delta_1) - \frac{3}{4}e^{-3\gamma t}\cos(\omega t) - \frac{1}{4}e^{-3\gamma t}\cos(3\omega t)\right]$$

where $\delta_1 = \arctan(2\gamma)$. In the same manner one can continue this process to find $x_2(t), x_3(t)$, etc.

A caveat: there is no way to know if the series solution generated will converge and in fact in most cases it does not [16].

A more specific branch of *Perturbation Theory* studies the behaviour of systems where the small parameter ϵ introduces a second time scale to the system. One example would be:

$$\begin{aligned}\dot{x} &= ax + bz \\ \epsilon\dot{z} &= cz + dx\end{aligned}$$

with x, z being the scalar variables of interest and a, b, c, d being real nonzero constants. Setting $\epsilon = 0$ the system reduces to

$$\begin{aligned}\dot{x} &= ax + bz \\ 0 &= cz + dx\end{aligned}$$

Solving for z and substituting into the first equation yields,

$$\begin{aligned}\implies z &= -dx/c \\ \implies \dot{x} &= (a - bd/c)x\end{aligned}$$

where we have lost the dynamics of the variable z and how it would affect the whole system, however we have now simplified the analysis of the system.

A natural question arises when we do such a thing: does doing so preserve the behaviour of the original system? Or rather, can we say anything about the original system given knowledge of the behaviour of the simplified or *reduced* system.

The study of such questions is known as the field of *Singular Perturbation Systems* as at the point $\epsilon = 0$ the system has a singularity. It has also received much attention with material to be found in [8], [7], [13], etc.

2.5.1 The Standard Singular Perturbation Model

The singular perturbation model of finite-dimensional dynamic systems has already been extensively studied in the mathematical literature by Tikhonov (1948, 1952), Levinson (1950), Vasil'eva

(1963), Wasow (1965), Hoppensteadt (1967, 1971), O'Malley (1971), and others. This model is in state model form with the derivatives of some of the states being multiplied by a small positive parameter ϵ; that is,

$$\dot{x} = f(x, z, \epsilon) \tag{2.18}$$

$$\epsilon\dot{z} = g(x, z, \epsilon) \tag{2.19}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$ denote vectors of state variables, and ϵ is a small positive parameter. The functions f and g are C^1 on $\mathbb{R}^n \times \mathbb{R}^p \times [0, \epsilon_0)$ for some $\epsilon_0 > 0$.

Using the standard procedure to analyze these types of systems, the original system will be broken into two subsystems which are easier to analyze. These two subsystems, called the reduced-order system and boundary layer system, each have a different time scale. Hence the name two-time scale decomposition [7].

Setting $\epsilon = 0$ reduces the dimension of the state equation from $p + n$ to n as the differential equation (2.19) reduces to:

$$0 = g(x, z, 0) \tag{2.20}$$

We say that the model (2.18)-(2.19) is in *standard form*, if (2.20) has $k \geq 1$ isolated real roots

$$z = h_i(x), \quad i = 1, 2, ..., k \tag{2.21}$$

for each $x \in \mathbb{R}^n$. This assumption ensures that a well-defined n-dimensional reduced model will correspond to each root of (2.20). To obtain the ith reduced model, we substitute (2.21) into (2.18), at $\epsilon = 0$, to obtain

$$\dot{x} = f(x, h(x), 0) \tag{2.22}$$

where we have dropped the subscript i from h. It will be clear from the context which root of (2.18) we will be using. The variables (t, x) are defined over $(t, x) \in [0, t_{red}^{max}] \times D_x, D_x \subset \mathbb{R}^n$ a domain. t_{red}^{max} is the maximum time for which the solution $x(t)$ of the system (2.22) exists. This model is sometimes called a *quasi-steady-state model*, because z, whose velocity $\dot{z} = g/\epsilon$ can be large when ϵ is small and $g \neq 0$, may rapidly converge to a root of (2.20), which is the equilibrium of (2.19). This model (2.22) is also known as the *slow* or *reduced model*.

The following example from [8] of a singularly perturbed system describes an armature-controlled DC motor. It can be modeled by the second-order state equation

$$\begin{aligned} J\frac{d\omega}{dt} &= ki \\ L\frac{di}{dt} &= -k\omega - Ri + u \end{aligned}$$

where i, u, R, and L are the armature current, voltage, resistance, and inductance, J is the moment of inertia, ω is the angular speed, and ki and $k\omega$ are, respectively the torque and the back electromotive force (e.m.f.) developed with constant excitation flux ϕ.

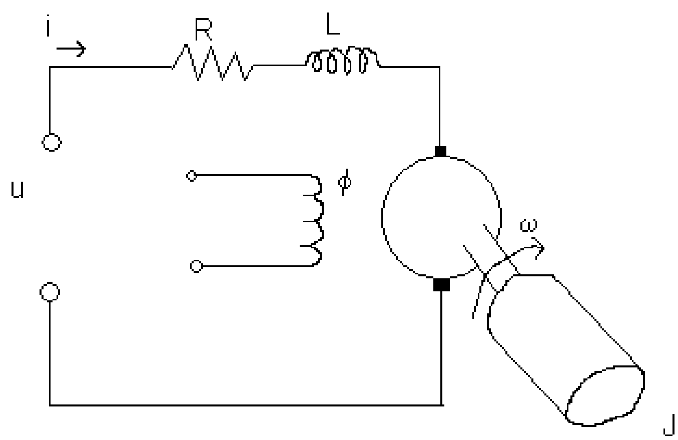


Figure 2.2: Armature controlled DC motor

The first state equation is a mechanical torque equation, and the second one is an equation for the electric transient in the armature circuit. Typically, L is "small" and can play the role of our parameter ϵ. This means that, with $\omega = x$ and $i = z$, the motor's model is in the standard form of (2.18)-(2.19) whenever $R \neq 0$. Neglecting L, we solve

$$0 = -k\omega - Ri + u$$

to obtain (the unique root)

$$i = \frac{u - k\omega}{R}$$

and substitute it into the torque equation to get the slow model:

$$J\dot{\omega} = -\frac{k^2}{R}\omega + \frac{k}{R}u$$

which is the commonly used first-order model of the DC motor.

Normally however, it is preferable to choose the perturbation parameter ϵ as a dimensionless quantity and to that end we non-dimensionalize the system first and extract the resulting dimensionless quantity ϵ. It is apparent that using different scaling factors one would end up with different values of ϵ. By a judicious scaling choice the perturbation parameter ϵ results in a "small" quantity i.e. $\epsilon << 1$.

When performing the analysis of such systems, it is more convenient to perform the change of variables

$$w_i = z - h_i(x)$$

that shifts the quasi-steady-state of z to the origin. Furthermore we scale the time variable by $\tau = t/\epsilon$ in the w dynamics (dropping the i index) to get

$$\begin{aligned} \dot{x} &= f(x, w + h(x), \epsilon) \\ \frac{dw}{d\tau} &= g(x, w + h(x), \epsilon) - \epsilon \frac{\partial h}{\partial x} f(x, w + h(x), \epsilon) \end{aligned} \tag{2.23}$$

The variables t and x in the previous equation will be slowly varying since, in the τ time scale they are given by

$$t = \epsilon\tau, \qquad x = x(t, \epsilon) = x(\epsilon\tau, \epsilon)$$

Setting $\epsilon = 0$ freezes these values to $t = 0$ and $x = x(0,0)$ and reduces (2.23) to the autonomous system

$$\frac{dw}{d\tau} = g(x, w + h(x), 0) \tag{2.24}$$

We call (2.24) the *boundary-layer system* or *boundary-layer model*.

To be able to properly state one of the following theorems it will be necessary to introduce a new concept.

Definition 2.23 *The equilibrium point $w = 0$ of the boundary-layer system (2.24) is* exponentially stable, *uniformly in $(t, x) \in [0, t_{red}^{max}] \times D_x$, if there exist positive constants k, γ, and ρ_0 such that the solutions of (2.24) satisfy*

$$|w(\tau)| \leq k|w(0)|e^{-\gamma\tau}, \quad \forall |w(0)| < \rho_0, \forall (t, x) \in [0, t_{red}^{max}] \times D_x, \forall \tau \geq 0.$$

Some main results using the standard perturbation model found in [7] are given below:

Theorem 2.7 *(Tikhonov's Theorem) Consider the initial value problem*

$$\dot{x} = f(x, z, \epsilon), \quad x(0) = \xi(\epsilon) \tag{2.25}$$

$$\epsilon\dot{z} = g(x, z, \epsilon), \quad z(0) = \eta(\epsilon) \tag{2.26}$$

where $\xi(\epsilon)$ and $\eta(\epsilon)$ depend smoothly on ϵ and let $z = h(x)$ be an isolated root of (2.20). Assume the following conditions are satisfied for all

$$[t, x, z - h(x), \epsilon] \in [0, t_{red}^{max}] \times D_x \times D_w \times [0, \epsilon_0]$$

for some $t_{red}^{max}, \epsilon_0 > 0$, domains $D_x \subset \mathbb{R}^n$ and $D_w \subset \mathbb{R}^m$, in which D_x is convex and D_w contains the origin:

- *The functions f, g, their first partial derivatives with respect to (x, z, ϵ), and the first partial derivative of g with respect to t are continuous; the function $h(t, x)$ and the Jacobian $[\partial g(t, x, z, 0)/\partial z]$ have continuous first partial derivatives with respect to their arguments; the initial data $\xi(\epsilon)$ and $\eta(\epsilon)$ are smooth functions of ϵ.*

- *The reduced problem*

$$\dot{x} = f(x, h(x), 0), \quad x(t_0) = \xi(0)$$

has a unique solution $\bar{x}(t) \in S$*, for* $t \in [0, t_{red}^{max}]$*, where* S *is a compact subset of* D_x*.* t_{red}^{max} *is the maximum time for which the unique solution* $\bar{x}(t)$ *exists.*

- *The origin is an exponentially stable equilibrium point of the boundary-layer model (2.24), uniformly in* (t, x)*; let* $\mathcal{R}_w \subset D_w$ *be the region of attraction of*

$$\frac{dw}{d\tau} = g(\xi(0), w + h(\xi(0)), 0), \quad w(0) = \eta(0) - h(\xi(0)) \tag{2.27}$$

and Ω_y *be a compact subset of* $\mathcal{R}_w$*.*

Then, there exists a positive constant $\epsilon^* \leq \epsilon_0$ *such that for all* $\eta(0) - h(\xi(0)) \in \Omega_w$ *and* $0 < \epsilon < \epsilon^*$*, the singular perturbation problem of (2.25) and (2.26) has a unique solution* $x(t, \epsilon), z(t, \epsilon)$ *on* $[0, t_{red}^{max}]$*, and*

$$x(t, \epsilon) - \bar{x}(t) = O(\epsilon)$$

$$z(t, \epsilon) - h(\bar{x}(t)) - \hat{w}(t/\epsilon) = O(\epsilon)$$

hold uniformly for $t \in [0, t_{red}^{max}]$*, where* $\hat{w}(\tau)$ *is the solution of the boundary-layer model (2.27). Moreover, given any* $t_b > 0$*, there is* $\epsilon^{**} \leq \epsilon^*$ *such that*

$$z(t, \epsilon) - h(t, \bar{x}(t)) = O(\epsilon)$$

holds uniformly for $t \in [t_b, t_{red}^{max}]$ *whenever* $\epsilon < \epsilon^{**}$*.*

From Saberi & Khalil [7] we have a result on stability using Lyapunov analysis:

Consider the autonomous singularly perturbed system

$$\dot{x} = f(x, z)$$

$$\epsilon \dot{z} = g(x, z)$$

and assume that the origin $(x = 0, z = 0)$ is an isolated equilibrium point and the functions f and g are locally Lipschitz in a domain that contains the origin.

Let $z = h(x)$ be an isolated root of

$$0 = g(x, z)$$

defined for all $x \in D_x \in \mathbb{R}^n$, where D_x is a domain that contains $x = 0$. Suppose $h(0) = 0$. As before we do a change of variable

$$w = z - h(x)$$

to shift the equilibrium of the boundary-layer model to the origin giving us as transformed system

$$\begin{aligned} \dot{x} &= f(x, w + h(x)) \\ \epsilon\dot{w} &= g(x, w + h(x)) - \epsilon \frac{\partial h}{\partial x} f(x, w + h(x)) \end{aligned}$$

Theorem 2.8 *For the above system, assume there are Lyapunov functions $V(x)$ and $W(x, w)$ that satisfy the following conditions for all $(x, w) \in D_x \times D_w$, D_x as defined above and $D_w \subset \mathbb{R}^m$ is a domain that contains $w = 0$:*

$$\frac{\partial V}{\partial x} f(x, h(x)) \leq -\alpha_1 \psi_1^2(x)$$

$$\frac{\partial V}{\partial x} [f(x, w + h(x)) - f(x, h(x))] \leq \beta_1 \psi_1^2(x) \psi_2^2(w)$$

$$\frac{\partial W}{\partial w} g(x, w + h(x)) \leq -\alpha_2 \psi_2^2(w)$$

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x, w + h(x)) \leq \beta_2 \psi_1(x) \psi_2(w) + \gamma \psi_2^2(w)$$

$$W_1(w) \leq W(x, w) \leq W_2(w)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$, and γ are some nonnegative constants and ψ_1, ψ_2, W_1 and W_2 are some positive definite functions. That is $\psi_1(0) = 0$ and $\psi_1(x) > 0$ for all $x \in D_x \backslash \{0\}$ and similarly for the other functions.

Then, the origin of that system is asymptotically stable for all $0 < \epsilon < \epsilon^*$. ϵ^* *being defined as*

$$\epsilon^* := \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \beta_1\beta_2}$$

Moreover, the function $\nu(x, w)$ *defined as*

$$\nu(x, w) = (1 - d)V(x) + dW(x, w), \quad 0 < d < 1$$

for any $0 < d < 1$ *is a Lyapunov function for* $\epsilon \in (0, \epsilon_d)$. ϵ_d *defined as*

$$\epsilon_d := \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \frac{1}{4d(1-d)}[(1 - d)\beta_1 + d\beta_2]^2}$$

A similar stability result which in a way generalizes the above theorem for systems with inputs was proved by Christofides and Teel [5]:

Theorem 2.9 *Consider the singularly perturbed system with inputs:*

$$\begin{aligned} \dot{x}(t) &= f(x(t), z(t), \theta(t), \epsilon) \\ \epsilon\dot{z}(t) &= g(x(t), z(t), \theta(t), \epsilon) \end{aligned} \tag{2.28}$$

where $x \in \mathbb{R}^n$ *and* $z \in \mathbb{R}^p$ *denote the states,* $\theta \in \mathbb{R}^q$ *denotes the input vector, and* ϵ *is a small positive parameter. The functions* f *and* g *are locally Lipschitz on* $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times [0, \bar{\epsilon})$ *for some* $\bar{\epsilon}$*. Then under the following assumptions*

- $\theta(t)$ *is an absolutely continuous function.*
- *The algebraic equation* $g(x, z_s, \theta, 0) = 0$ *possesses isolated roots*

  $$z_s = h_i(x, \theta)$$

  with the properties that h *and its partial derivatives* $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial \theta}$ *are locally Lipschitz with respect to* (x, θ).

- *The reduced system*

$$\dot{x} = f(x, h(x,\theta), \theta, 0)$$

  is ISS with Lyapunov gain γ.

- *The equilibrium* $w = 0$ *of the boundary system*

$$\frac{dw}{d\tau} = g(x, h(x,\theta) + w, \theta, 0)$$

  is globally asymptotically stable, uniformly in $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$.

we have that there exist functions β_x, β_w *of class* $\mathcal{KL}$*, such that given a pair of positive real numbers* (δ, d) *there is an* $\epsilon^* > 0$ *such that if* $\max\{|x(0)|, |w(0)|, \|\theta\|_{[0,\infty)}, \|\dot{\theta}\|_{[0,\infty)}\} \leq \delta$ *and* $\epsilon \in (0, \epsilon^*]$*, then for all* $t \geq 0$

$$\begin{aligned} |x(t)| &\leq \beta_x(|x(0)|, t) + \gamma(\|\theta\|_{[0,t]}) + d \\ |w(t/\epsilon)| &\leq \beta_w(|w(0)|, t/\epsilon) + d \end{aligned}$$

Essentially this theorem states that given that the reduced system is ISS then under some other mild assumptions we have that the perturbed system is ISS modulo a constant. Intuitively we note that the smaller d is the smaller ϵ^* will be as the bound on x approaches an ISS bound similar to the reduced system (for which $\epsilon = 0$). Inversely, the smaller δ is, the larger ϵ^* will be, as for small δ the states will start very near their equilibrium values and the inputs will be very small and so the states will not drift very far. Thus the bounds will still be valid for larger ϵ.

One should note that when using theorems in perturbation theory, the estimate of the upper bound for ϵ is usually conservative.

Chapter 3

Detectability of a singularly perturbed system

Having introduced the concept of a singularly perturbed system, we will now set out to do as Christofides and Teel did in [5] for stability. The paper [5] gives a theorem which, as stated in Section 2.5, gives a form of stability of a singularly perturbed system given that the reduced system is ISS and that certain other conditions are met. In a similar manner, this work will show that a form of detectability is achieved for a singularly perturbed system given that the reduced system is IOSS and that certain other conditions are met.

To prove their theorem, Christofides and Teel needed three smaller stability results with regards to systems with two inputs, where the second input was added to an ISS system with one input. The main result was then proved treating the perturbation parameter ϵ as a second input to the reduced ISS system.

In a similar manner, this work will extend those three lemmas to systems with outputs to reach three detectability results which will then be used to prove the main result.

The first step towards proving the lemmas is to introduce a system with two inputs. Consider the system

$$\begin{aligned} \dot{x} &= f(x, u_1, u_2) \\ y &= k(x) \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $u_1 \in \mathbb{R}^p, u_2 \in \mathbb{R}^q$ are two different inputs, and $y \in \mathbb{R}^r$ is the output. For any initial condition $x(0)$ and inputs u_1, u_2, we denote the maximal time for which the solution $x(t)$ exists as $t^{max}_{x_0,(u_1,u_2)}$ where the first subscript denotes the initial condition and the second denotes the inputs.

We next define the notion of IOSS for systems for two inputs and its Lyapunov-like characterization. These definitions are merely extensions of the definitions for systems with one input and an output (2.19)-(2.20).

Definition 3.1 *A system of type (3.1) is said to be input-output-to-state stable (IOSS) with Lyapunov gains* $(\gamma_1, \gamma_2, \gamma_y)$ *if there exist functions* $\beta \in \mathcal{KL}$ *and* $\gamma_1, \gamma_2, \gamma_y \in \mathcal{K}$ *such that the bound:*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_1(\|u_1\|_{[0,t]}) + \gamma_2(\|u_2\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]})$$

holds for any initial state $x(0) \in \mathbb{R}^n$*, controls* $u_1(\cdot), u_2(\cdot)$*, and time* $t \in [0, t^{max}_{x_0,(u_1,u_2)})$.

Definition 3.2 *: A* C^1 *function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is an IOSS-Lyapunov function for system (2.12) if there exist* $\mathcal{K}_\infty$ *functions* $\alpha_1, \alpha_2, \alpha$ *and* $\mathcal{K}$ *functions* $\sigma_1, \sigma_2, \sigma_y$ *such that*

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|)$$

$$\nabla V(\xi) \cdot f(\xi, u) \leq -\alpha(|\xi|) + \sigma_1(|u_1|) + \sigma_2(|u_2|) + \sigma_y(|k(\xi)|)$$

holds for all $\xi \in \mathbb{R}^n$ *and all control values* $u_1, u_2 \in \mathbb{R}^p$.

As the two previous definitions are merely extensions to their single input systems counterparts, the theorem that relates the definition and its Lyapunov function still holds:

Theorem 3.1 *The system (3.1) is IOSS if and only if it admits an IOSS Lyapunov function.*

The following lemma will be used in the proof of Lemma 3.2.

Lemma 3.1 *: Suppose a function f is locally Lipschitz on some open set $X \subseteq \mathbb{R}^n$. Then there exists a function $L(r) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ so that for any closed ball of radius r centered at the origin, $\bar{B}_r \subset X$*

$$|f(\eta) - f(\xi)| \leq L(r)|\eta - \xi| \quad \forall \eta, \xi \in \bar{B}_r$$

Proof: For a fixed radius r, by the local Lipschitz property of f we have that for each point $x \in \bar{B}_r$ there exists a neighborhood O_x which contains x and an L_i such that

$$|f(\eta) - f(\xi)| \leq L_x|\eta - \xi| \qquad \forall \eta, \xi \in O_x.$$

Since $\bar{B}_r$ is compact (closed and bounded in $\mathbb{R}^n$), it can be covered by a finite number of such neighborhoods $i = \{1, 2, \ldots, k\}$. Let $L_r := \max\{L_1, L_2, \ldots, L_k\}$. Define the function

$$L(r) = \begin{cases} L_r & , r > 0 \\ 0 & , r = 0 \end{cases}$$

and the proof is complete. □

3.1 Maintaining IOSS under an additional input

Lemma 3.2 basically says that given a Lyapunov characterization of a system with one bounded input, then for any second bounded input, it is possible to scale it with respect to the state such that we maintain that Lyapunov characterization with respect to two inputs.

The following lemma is stated in terms of a continuous function $K : \mathbb{R}^n \to \mathbb{R}$. For certain choices of K, this will give results on particular properties of the system in question.

Lemma 3.2 *: Assume that for the system (3.1) with* $u_2(t) \equiv 0$ *there exists a smooth, proper and positive definite function* $V : \mathbb{R}^n \to \mathbb{R}$*,* α_1, α_2 *functions of class* $\mathcal{K}_\infty$*, and a continuous function* $K : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\nabla V(x(t)) \cdot f(x(t), u_1(t), 0) \leq K(x(t)) + \alpha_1(|u_1(t)|) + \alpha_2(|y(t)|)$$

is satisfied for all possible trajectories of $x(t)$*. Then there exists a nonincreasing continuous function* $b(s) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$*, functions* $\alpha_{u_1}, \alpha_{u_2} \in \mathcal{K}_\infty$ *such that* $0 < b(s) \leq 1$ *and* $b(s) \equiv 1$ *in a neighborhood of the origin which satisfies the following: with* $B(x) := b(|x|)I_{q\times q}$ *a* $q \times q$ *matrix of smooth functions invertible for all* $x \in \mathbb{R}^n$ *the system*

$$\begin{aligned} \dot{x}(t) &= f(x(t), u_1(t), B(x(t))u_2(t)) \\ y &= k(x(t)) \end{aligned} \tag{3.2}$$

satisfies the following inequality for all trajectories $x(t)$*:*

$$\nabla V(x(t)) \cdot f(x(t), u_1(t), B(x(t)u_2(t)) \leq K(x(t)) + \alpha_{u_1}(|u_1(t)|) + \alpha_2(|y(t)|) + \alpha_{u_2}(|u_2(t)|) \tag{3.3}$$

Note: $B(x) \equiv I_{q\times q}$ *in a neighborhood of the origin,*

Proof: From the hypothesis of the lemma there exists a function V such that a solution $x(t)$ of the system (3.2) with $u_2(t) \equiv 0$ satisfies

$$\nabla V(x) \cdot f(x, u_1, 0) \leq K(x) + \alpha_1(|u_1|) + \alpha_2(|y|), \tag{3.4}$$

where $\alpha_i \in \mathcal{K}_\infty$ for $i = 1, 2$.

Instead of looking at (3.2) we look at the system (3.1) with the second input renamed as $\tilde{u}_2$.

$$\begin{aligned} \dot{x}(t) &= f(x(t), u_1(t), \tilde{u}_2(t)) \\ y &= k(x(t)) \end{aligned} \tag{3.5}$$

Computing the time derivative of V along the trajectories of (3.5) we get, for all $t \geq 0$

$$\begin{aligned} \nabla V(x(t)) \cdot f(x(t), u_1(t), \tilde{u}_2(t)) &= \nabla V(x(t)) \cdot f(x(t), u_1(t), 0) + \\ &\quad \nabla V(x(t)) \cdot [f(x(t), u_1(t), \tilde{u}_2(t)) - f(x(t), u_1(t), 0)]. \end{aligned} \tag{3.6}$$

Claim: It is possible to find a constant $L > 0$ and a $\mathcal{K}_\infty$ function $\Psi_1(\cdot)$ such that

$$\begin{aligned} \nabla V(x) \cdot [f(x, u_1, \tilde{u}_2) - f(x, u_1, 0)] &\leq (L + \Psi_1(\max\{|x|, |u_1|, |\tilde{u}_2|\}))|\tilde{u}_2| \\ &\quad \forall (x, u_1, \tilde{u}_2) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \end{aligned} \tag{3.7}$$

Proof of Claim: Consider the closed ball of radius r with respect to the Euclidean norm centered at the origin in $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$,

$$\bar{B}_r := \{(x, u_1, \tilde{u}_2) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q | \quad |x|^2 + |u_1|^2 + |\tilde{u}_2|^2 \leq r^2\}.$$

Since V is smooth, there exists a nondecreasing function $M_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as

$$M_1(r) = \sup_{x \in \bar{B}_r} |\nabla V(x)| \quad \forall r \in \mathbb{R}_{\geq 0}$$

Also, since $\bar{B}_r$ is compact (closed and bounded), and as f is locally Lipschitz with respect to $\tilde{u}_2$, then by Lemma 3.1 there exists a function $M_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending on r such that

$$|f(x, u_1, \eta) - f(x, u_1, 0)| \leq M_2(r)|\eta| \quad \forall (x, u_1, \eta) \in \bar{B}_r$$

Choose $M_2(r)$ such that it is increasing with r. Then, defining the function $\hat{M} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as

$$\hat{M}(r) := M_1(r) M_2(r)$$

we have that $\hat{M}$ is well-defined and nondecreasing. Hence, it is possible to bound it by a sum of a constant $L > 0$ and some class $\mathcal{K}_\infty$ function $\Psi_1(\cdot)$

$$\hat{M}(r) \leq L + \Psi_1(r).$$

Hence

$$\begin{aligned}\nabla V(x)\cdot[f(x,u_1,\tilde{u}_2)-f(x,u_1,0)] &\le (L+\Psi_1(\max\{|x|,|u_1|,|\tilde{u}_2|\}))|\tilde{u}_2| \\ &\qquad \forall (x,u_1,\tilde{u}_2)\in\mathbb{R}^n\times\mathbb{R}^p\times\mathbb{R}^q\end{aligned}$$

as desired and the claim is proved. □

Using (3.7) we have from (3.6),

$$\begin{aligned}\nabla V(x)\cdot f(x,u_1,\tilde{u}_2) &\le \nabla V(x)\cdot f(x,u_1,0) \\ &\quad +\nabla V(x)\cdot[f(x,u_1,\tilde{u}_2)-f(x,u_1,0)] \\ &\le K(x)+\alpha_1(|u_1|)+\alpha_2(|k(x)|)+ \\ &\quad |\tilde{u}_2|(L+\Psi_1(\max\{|x|,|u_1|,|\tilde{u}_2|\}))\end{aligned}$$

Let $b(s):\mathbb{R}_{\ge 0}\to\mathbb{R}$ be a smooth function that satisfies $b(s)\equiv 1$ in a neighborhood of the origin (i.e. there exists a $\delta_1>0$ such that $b(s)\equiv 1$, $\forall s\in[0,\delta_1)$) and moreover is chosen so that the inequality

$$0<b(s)\le\min\left\{\frac{1}{\Psi_1(s)},1\right\} \tag{3.8}$$

holds for all $s\in\mathbb{R}_{\ge 0}$.

Define $B(x):=b(|x|)I_{q\times q}$ and for any initial condition ξ and any input u_1 and u_2 to system (3.2) let $x(t)=x(t,\xi,u_1,u_2)$. Then defining the signal $\tilde{u}_2(t):=B(x(t))u_2(t)$, $x(t)$ is also the solution of (3.5) and therefore its output $y=k(x)$ is also the same as the output of (3.2).

Now using the fact that $b(|x|)\le 1$

$$\begin{aligned}\nabla V(x)\cdot f(x,u_1,B(x)u_2) &\le K(x)+\alpha_1(|u_1|)+\alpha_2(|k(x)|)+ \\ &\quad b(|x|)|u_2|(L+\Psi_1(\max\{|x|,|u_1|,b(|x|)|u_2|\})) \\ &\le K(x)+\alpha_1(|u_1|)+\alpha_2(|k(x)|)+ \\ &\quad b(|x|)|u_2|(L+\Psi_1(\max\{|x|,|u_1|,|u_2|\})).\end{aligned} \tag{3.9}$$

For each time t consider the three possible cases:

Case 1: $|u_2| \geq \max\{|u_1|, |x|\}$

Then (3.9) reduces to

$$\nabla V(x) \cdot f(x, u_1, B(x)u_2) \leq K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + b(|x|)|u_2|(L + \Psi_1(|u_2|))$$

Again, since $b(|x|) \leq 1$

$$\nabla V(x) \cdot f(x, u_1, B(x)u_2) \leq K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + |u_2|(L + \Psi_1(|u_2|))$$

Let $\Psi_2(s) := s(L + \Psi_1(s))$ hence $\Psi_2(s)$ is a $\mathcal{K}_\infty$ function ending up with

$$\nabla V(x) \cdot f(x, u_1, B(x)u_2) \leq K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + \Psi_2(|u_2|)$$

This bound will be used at the end of the proof.

Case 2: $|u_1| \geq \max\{|u_2|, |x|\}$

In this case, we can write (3.9) as

$$\begin{aligned}
\nabla V(x) \cdot f(x, u_1, B(x)u_2) &\leq K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + b(|x|)|u_2|(L + \Psi_1(|u_1|)) \\
&\leq K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + |u_2|(L + \Psi_1(|u_1|)) \\
&= K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + |u_2|L + |u_2|\Psi_1(|u_1|) \\
&\leq K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + |u_2|L + |u_1|\Psi_1(|u_1|) \\
&\leq K(x) + \Psi_3(|u_1|) + \Psi_4(|u_2|) + \alpha_2(|k(x)|)
\end{aligned}$$

where $\Psi_3(s) := \alpha_1(s) + s\Psi_1(s)$ and $\Psi_4(s) := Ls$ are both of class $\mathcal{K}_\infty$, again this is a bound which will be used later.

Case 3: $|x| \geq \max\{|u_1|, |u_2|\}$

Using the fact that $b(|x|) \leq 1$ and $b(|x|)\Psi_1(|x|) \leq 1$, inequality (3.9) is in this case:

$$\begin{aligned}
\nabla V(x) \cdot f(x, u_1, B(x)u_2) &\leq K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + b(|x|)|u_2|(L + \Psi_1(|x|)) \\
&\leq K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + |u_2|(L + 1) \\
&\leq K(x) + \alpha_1(|u_1|) + \alpha_2(|k(x)|) + \Psi_5(|u_2|)
\end{aligned}$$

Where we have defined $\Psi_5(s)$ as $\Psi_5(s) := (L+1)s$.

Hence defining a $\mathcal{K}_\infty$ function Ψ_6 as

$$\Psi_6(s) := \max\{\Psi_2(s), \Psi_4(s), \Psi_5(s)\}$$

the derivative of V along the trajectories is bounded by:

$$\nabla V(x) \cdot f(x, u_1, B(x)u_2) \leq K(x) + \Psi_3(|u_1|) + \Psi_6(|u_2|) + \alpha_2(|k(x)|)$$

as required with $\alpha_{u_1} = \Psi_3$ and $\alpha_{u_2} = \Psi_6$ and hence the lemma is proved. □

Corollary 3.1 *: Assume that the system (3.1) with $u_2(t) \equiv 0$ is IOSS with Lyapunov gains $\tilde{\gamma}_{u_1}, \gamma_y$, then there exists a nonincreasing continuous function $b(s) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, functions $\gamma_{u_1}, \gamma_{u_2} \in \mathcal{K}_\infty$ such that $0 < b(s) \leq 1$ and $b(s) \equiv 1$ in a neighborhood of the origin which satisfies the following: with $B(x) := b(|x|)I_{q\times q}$ a $q \times q$ matrix of smooth functions invertible for all $x \in \mathbb{R}^n$ the system*

$$\begin{aligned} \dot{x}(t) &= f(x(t), u_1(t), B(x(t))u_2(t)) \\ y &= k(x(t)) \end{aligned} \tag{3.10}$$

is IOSS with Lyapunov gains $(\gamma_{u_1}, \gamma_{u_2}, \gamma_y)$.

Proof: From the hypothesis of the corollary and Theorem 3.1 there exists a smooth, proper and positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ and an $\alpha_x \in \mathcal{K}_\infty$ such that

$$\nabla V(x) \cdot f(x, u_1, 0) \leq -\alpha_x(|x|) + \tilde{\gamma}_{u_1}(|u_1|) + \gamma_y(|k(x)|) \quad \forall (x, u_1) \in \mathbb{R}^n \times \mathbb{R}^p$$

Then applying lemma 3.2 with $K(x) = -\alpha_x(|x|)$ there exists a function $b(s)$, functions $\gamma_{u_1}, \gamma_{u_2} \in \mathcal{K}$ so that:

$$\nabla V(x) \cdot f(x, u_1, B(x)u_2) \leq -\alpha_x(|x|) + \gamma_{u_1}(|u_1|) + \gamma_{u_2}(|u_2|) + \alpha_2(|k(x)|) \quad \forall (x, u_1, u_2) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q.$$

This is the Lyapunov characterization of an IOSS system. Hence by Theorem 3.1 the system (3.10) is IOSS and the corollary is proved. □

3.2 Extending the IOSS bound on the state for a system influenced by an additional input

The next lemma gives us a result on a bound on the state x of system (3.1). If a system is IOSS with respect to one input then when adding a second input, it is possible to find a bound on the state x which depends on the initial condition, the two inputs, and the output. This bound holds for all time such that the second output remains "small" relative to the initial condition, the first input and the output.

Lemma 3.3 *: Assume that (3.1) with $u_2(t) \equiv 0$ is IOSS with Lyapunov gains $(\tilde{\gamma}_{u_1}, \gamma_y)$. Then there exists a function $\beta \in \mathcal{KL}$, a continuous nonincreasing function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\sigma(s) \leq 1\, \forall s \in \mathbb{R}_{\geq 0}$ and functions $\gamma_{u_1}, \gamma_{u_2}$ of class $\mathcal{K}$ such that for each $x_0 \in \mathbb{R}^n$ and each pair of essentially bounded inputs $u_1(\cdot), u_2(\cdot)$, the solution of (3.1) $x(t)$ satisfies*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|_{[0,t]}) + \gamma_{u_2}(\|u_2\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) \qquad \forall t \in [0, \bar{t})$$

where
$\bar{t} = \min\{t^*, t^{max}_{x_0,(u_1,u_2)}\}$*, t^* being the maximum time for which*
$\|u_2\|_{[0,t^*]} \leq \sigma(\max\{|x(0)|, \|u_1\|_{[0,t^*]}, \|y\|_{[0,t^*]}\})$ *holds.*

Proof: Referring to (3.1), Corollary 3.1 says that there exists a $q \times q$ matrix $B(x) = b(|x|)I_{q\times q}$ (where $b(s)$ is a nonincreasing function and satisfies $0 < b(s) \leq 1$, $\forall s \in \mathbb{R}_{\geq 0}$) and functions γ_{u_1}, γ_v of class $\mathcal{K}_\infty$ such that

$$\begin{aligned} \dot{x} &= f(x, u_1, B(x)v) \\ y &= k(x) \end{aligned} \qquad (3.11)$$

is IOSS with Lyapunov gain $(\gamma_{u_1}, \gamma_v, \gamma_y)$.

From the definition of IOSS, this implies that for given essentially bounded u_1, v, trajectories of (3.11) satisfy:

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|_{[0,t]}) + \gamma_v(\|v\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) \qquad \forall t \in [0, t^{max}_{x_0,(u_1,u_2)}) \qquad (3.12)$$

Fix inputs u_1, u_2 and initial condition $x(0)$ to (3.1). Define the signal $v(t) := B^{-1}(x(t))u_2(t)$, where $x(t)$ is the solution of (3.1) and note that the solution of (3.11) with initial condition $x(0)$, inputs u, v is also $x(t)$. Therefore the bound (3.12) on x for system (3.11) also holds for system (3.1) with this choice of $v(\cdot)$:

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|_{[0,t]}) + \gamma_v(\|B^{-1}(x)\|_{[0,t]}\|u_2\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) \qquad \forall t \in [0, t^{max}_{x_0,(u_1,u_2)}) \tag{3.13}$$

Since $b(s)$ is a nonincreasing function with $0 < b(s) \leq 1$ it is possible to find an $L \geq 1$ and $\Psi_2 \in \mathcal{K}_\infty$ such that

$$\frac{1}{b(\beta(q,0) + \gamma_{u_1}(r) + \gamma_v(1) + \gamma_y(s))} \leq L + \Psi_2(\max\{q, r, s\}) \tag{3.14}$$

for all $q, r, s \in \mathbb{R}_{\geq 0}$.

Define the continuous nonincreasing function $\sigma(\cdot)$ as $\sigma(s) := (2(L + \Psi_2(s)))^{-2}$. Note that since $L \geq 1$, $\sigma(s) \leq 1 \quad \forall s \in \mathbb{R}_{\geq 0}$. Choose $\bar{t}$ such that $\|u_2\|_{[0,\bar{t}]} \leq \sigma(\max\{|x(0)|, \|u_1\|_{[0,\bar{t}]}, \|y\|_{[0,\bar{t}]}\})$. We will show that

$$\frac{\|u_2\|^{\frac{1}{2}}_{[0,\bar{t}]}}{b(|x(t)|)} \leq 1 \, \forall t \in [0, \bar{t}). \tag{3.15}$$

To attain a contradiction, suppose that (3.15) fails for some choice of $\bar{t}$. Let $t' \in (0, \bar{t})$ be the maximal time such that for all $t \in [0, t')$

$$\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\|x\|_{[0,t']})} \leq 1 \tag{3.16}$$

We first note that at $t = 0$ we have using the fact that $\beta(s, 0) \geq s$ and $1/b(s)$ is nondecreasing that

$$\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(|x(0)|)} \leq \frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\beta(|x(0)|, 0))}.$$

Using (3.14) and $\|u_2\|_{[0,\bar{t}]} \leq \sigma(\max\{|x(0)|, \|u_1\|_{[0,\bar{t}]}, \|y\|_{[0,\bar{t}]}\}) \leq \sigma(|x(0)|)$ as $\sigma(\cdot)$ is nonincreasing

$$\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\beta(|x(0)|,0))} \leq (L + \Psi_2(|x(0)|)) \cdot \sigma^{1/2}(|x(0)|).$$

Combining the last two inequalities and using the definition of $\sigma(\cdot)$ we have

$$\begin{aligned}\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(|x(0)|)} &\leq (L + \Psi_2(|x(0)|)) \cdot \frac{1}{2(L + \Psi_2(|x(0)|))} \\ &= 1/2 < 1.\end{aligned}$$

Then the existence of t' follows from the continuity of x.

Then at time t', using (3.13), and the fact that $1/b(s)$ is nondecreasing we have:

$$\begin{aligned}\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\|x\|_{[0,t']})} &\leq \frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\beta(|x(0)|,0) + \gamma_{u_1}(\|u_1\|_{[0,t']}) + \gamma_v(\|B^{-1}(x)\|_{[0,t']}\|u_2\|_{[0,t']}) + \gamma_y(\|y\|_{[0,t']}))} \\ &= \frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\beta(|x(0)|,0) + \gamma_{u_1}(\|u_1\|_{[0,t']}) + \gamma_v(\|\frac{1}{b(x)}\|_{[0,t']}\|u_2\|^{1/2}_{[0,t']}\|u_2\|^{1/2}_{[0,t']}) + \gamma_y(\|y\|_{[0,t']}))}.\end{aligned}$$

As $\|1/b(s)\| = 1/b(\|s\|)$

$$\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\|x\|_{[0,t']})} \leq \frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\beta(|x(0)|,0) + \gamma_{u_1}(\|u_1\|_{[0,t']}) + \gamma_v(\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\|x\|_{[0,t']})} \cdot \|u_2\|^{1/2}_{[0,t']}) + \gamma_y(\|y\|_{[0,t']}))}.$$

Now using the fact that $\|u_2\|_{[0,t]} \leq \sigma(\max\{|x(0)|, \|u_1\|_{[0,t]}, \|y\|_{[0,t]}\})\, \forall t \in [0,\bar{t}]$, $t' < \bar{t}$ and $\sigma(s) \leq 1\, \forall s \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\|x\|_{[0,t']})} &\leq \frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\beta(|x(0)|,0) + \gamma_{u_1}(\|u_1\|_{[0,t']}) + \gamma_v(\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\|x\|_{[0,t']})} \cdot 1^{1/2}) + \gamma_y(\|y\|_{[0,t']}))} \\ &\leq \frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\beta(|x(0)|,0) + \gamma_{u_1}(\|u_1\|_{[0,t']}) + \gamma_v(1) + \gamma_y(\|y\|_{[0,t']}))} \quad \text{by (3.16)} \\ &\leq (L + \Psi_2(\max\{|x(0)|, \|u_1\|_{[0,t']}, \|y\|_{[0,t']}\})) \cdot \\ &\quad \sigma^{1/2}(\max\{|x(0)|, \|u_1\|_{[0,t']}, \|y\|_{[0,t']}\}) \qquad \text{by (3.14) and the definition of } \sigma(\cdot) \\ &\leq 1/2 < 1,\end{aligned}$$

a contradiction as we assumed that t' was the maximal time for which $\frac{\|u_2\|^{\frac{1}{2}}_{[0,t']}}{b(\|x\|_{[0,t']})} \leq 1$, therefore $t' = \bar{t}$., that is (3.15) holds.

So for any $t \in [0, \bar{t})$ we have from (3.13)

$$\begin{aligned} |x(t)| &\leq \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|_{[0,t]}) + \gamma_v(\|B^{-1}(x)\|_{[0,t]}\|u_2\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) \\ &= \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|_{[0,t]}) + \gamma_v(\frac{1}{b(\|x\|_{[0,t]})}\|u_2\|^{1/2}_{[0,t]}\|u_2\|^{1/2}_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) \\ &\leq \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|_{[0,t]}) + \gamma_v(\|u_2\|^{1/2}_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) \\ &= \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|_{[0,t]}) + \gamma_{u_2}(\|u_2\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) \end{aligned}$$

where $\gamma_{u_2}(s) = \gamma_v(s^{1/2})$ and the lemma is proved. □

As a special case of Lemma 2 of [5] when system (3.1) with $u_2(t) \equiv 0, y(t) \equiv 0$ is GAS uniformly in u_1 (i.e. it is ISS with Lyapunov gain $\gamma_{u_1} = 0$) we have the following result:

Proposition 3.1 *If the system (3.1) with $u_2(t) \equiv 0, y(t) \equiv 0$ is GAS uniformly in u_1 then $\exists$ a function $\beta \in \mathcal{KL}$, a continuous nonincreasing function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\sigma(s) \leq 1 \, \forall s \in \mathbb{R}_{\geq 0}$ and a function γ_{u_2} of class $\mathcal{K}$ such that for each $x_0 \in \mathbb{R}^n$, each essentially bounded inputs $u_1(\cdot), u_2(\cdot)$, the solution of (3.1) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_{u_2}(\|u_2\|_{[0,t]}) \qquad \forall t \in [0, t^*)$$

where t^* is the maximum time for which $\|u_2\|_{[0,t^*]} \leq \sigma(\max\{|x(0)|, \|u_1\|_{[0,t^*]}\})$ holds.

3.3 Maintaining a form of IOSS when ignoring initial transient input behaviour of IOSS systems

If we assume that system (3.1) has the UO property with inputs $u_1, u_2 \in \mathcal{M}_{\mathcal{D}}$, where $\mathcal{M}_{\mathcal{D}}$ is the compact set such that for a given constant D, time $\hat{t}$, $\max\{\|u_1\|_{[0,\hat{t}]}, \|u_2\|_{[0,\hat{t}]}\} < D$, then Proposition 2.6 gives the following bound on the state for any $t' \in [0, \min\{\hat{t}, t^{max}_{x_0,(u_1,u_2)}\})$:

$$|x(t)| \leq \chi(t) + \chi(|x(0)|) + \chi(\|y\|_{[0,t]}), \quad \forall t \in [0, t') \tag{3.17}$$

for some $\chi \in \mathcal{K}$, $\chi(s) > s$.

This allows us to prove the following lemma which states that given a bound on the state with respect to the initial condition, the two inputs, and the output up to some time, then it is possible to find a similar bound which depends on the initial condition, the two inputs when ignoring an initial time interval, the output and a constant. This is desirable in the case of singularly perturbed systems as the state variable of the *fast system* can be considered an input to the *slow system*. As the fast system converges to its equilibrium very rapidly (hence its name), being able to absorb its value for an initial time interval into a constant gives a better grasp on the size of the state of the slow system.

Lemma 3.4 *Referring to (3.1), assume that the system satisfies (3.17) for some* χ *and that there exists functions* $\beta \in \mathcal{KL}, \gamma_{u_1}, \gamma_{u_2}, \gamma_y \in \mathcal{K}$ *and a continuous nonincreasing function* $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *with* $\sigma(\cdot) \leq 1$ *such that for each* $x_0 \in \mathbb{R}^n$*, and each essentially bounded inputs* $u_1(\cdot)$*,* $u_2(\cdot)$

$$|x(t)| \quad \leq \quad \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|_{[0,t]}) + \gamma_{u_2}(\|u_2\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}),\ \forall t \in [0, \bar{t}') \tag{3.18}$$

holds, where $\bar{t}' = \min\{t^{max}_{x_0,(u_1,u_2)}, t'\}$*,* t' *being the maximum time for which* $\|u_2\|_{[0,t']} \leq \sigma(\max\{|x(0)|, \|u_1\|_{[0,t']}\})$.

Then for each pair of positive real numbers $(\bar{\delta}, \bar{d})$*, there exist functions* $\bar{\beta} \in \mathcal{KL}$ *and* $\bar{\gamma}_y \in \mathcal{K}$ *and a positive real number* ρ^* *such that for each* $\rho \in (0, \rho^*]$*, if* $\max\{|x(0)|, \|u_1\|_{[0,t']}, \|u_2\|_{[0,t']}\} \leq \bar{\delta}$*, then the solution of (3.1) satisfies*

$$|x(t)| \leq \bar{\beta}(|x(0)|, t) + \gamma_{u_1}(\|u_1^\rho\|_{[0,t]}) + \gamma_{u_2}(\|u_2^\rho\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + \bar{d} \tag{3.19}$$

for each $t \in [0, \bar{t})$*, where* $\bar{t} = \min\{t^*, t^{max}_{x_0,(u_1,u_2)}\}$*,* t^* *being the maximum time such that* $\|u_2\|_{[0,t^*]} \leq \sigma(\max\{2(\chi(|x(0)|) + \bar{d}, \|u_1\|_{[0,t^*]}, 2\chi(\|y\|_{[0,t^*]})\})$ *holds.*

Note: For a fixed input u_2, if $\|u_2\|_{[0,t']} = \sigma(\max\{|x(0)|, \|u_1\|_{[0,t']}\})$ then quite possibly $\|u_2\|_{[0,t']} > \sigma(\max\{2(\chi(|x(0)|) + \bar{d}), \|u_1\|_{[0,t']}, 2\chi(\|y\|_{[0,t']})\})$ as $2(\chi(s) + \bar{d}) > s$ and $\sigma(\cdot)$ is nonincreasing. This implies that $t^* \le t'$, therefore (3.19) holds over a possibly smaller time interval than (3.18). *Proof*: Let $\bar{d}, \bar{\delta}$ be given. Fix $x(0), u_1$, and u_2 so that $\max\{|x(0)|, \|u_1\|_{[0,t']}, \|u_2\|_{[0,t']}\} \le \bar{\delta}$. Hence (3.17) holds for some $\chi \in \mathcal{K}$ with $D = \bar{\delta}$ and $\hat{t} = t'$.

Now observe that for fixed $s, \bar{T} \ge 0$ the difference $\beta(2(\chi(s)+\chi(\bar{T})), t-\bar{T}) - \beta(2\chi(s), t)$ tends to 0 as $t \to \infty$. From this fact, we have that for the pair of positive real numbers $(\bar{\delta}, \bar{d})$, there exists a positive real number $T = T(\bar{\delta}, \bar{d})$ sufficiently large (without loss of generality let $T \ge 1$) such that

$$\beta(2(\chi(s)+\chi(1)), t-1) - \beta(2\chi(s), t) \le \bar{d} \qquad \forall s \in [0, \bar{\delta}], \forall t \ge T. \tag{3.20}$$

Define $0 < \rho^* \le 1$ such that

$$\beta(\chi(s), 0) - \beta(\chi(s), t) + \chi(t) \le \bar{d} \qquad \forall s \in [0, \bar{\delta}], \forall t \in [0, \rho^*] \tag{3.21}$$

and

$$\beta(2(\chi(s)+\chi(\rho^*)), t-\rho^*) - \beta(2\chi(s), t) \le \bar{d} \qquad \forall s \in [0, \bar{\delta}], \forall t \in [0, T] \tag{3.22}$$

Set $\bar{t} = \min\{t^*, t^{max}_{x_0,(u_1,u_2)}\}$, t^* being the maximum time such that $\|u_2\|_{[0,t^*]} \le \sigma(\max\{2(\chi(|x(0)|) + \bar{d}, \|u_1\|_{[0,t^*]}, 2\chi(\|y\|_{[0,t^*]})\})$ holds. As $\sigma(\cdot)$ is a nonincreasing function and $2(\chi(|x(0)|)+\bar{d} > |x(0)|$, we have that $\|u_2\|_{[0,\bar{t}]} \le \|u_2\|_{[0,\bar{t}']}$, hence $\bar{t} \le \bar{t}'$. Now we will show that (3.19) holds for each $\rho \in (0, \rho^*]$ and $t \in [0, \bar{t})$ by examining the intervals $[0, \rho^*]$, $[\rho^*, T]$, and $[T, \bar{t})$ separately. If $\bar{t} < \rho^*$ or $\bar{t} < T$ then the proof is the same only simpler. Define a function $\bar{\beta} \in \mathcal{KL}$ as $\bar{\beta}(s,t) = \beta(2\chi(s), t) \ge \beta(\chi(s), t)$ and a function $\bar{\gamma}_y \in \mathcal{K}$ as $\bar{\gamma}_y(s) = \gamma_y(s) + \beta(2\chi(s), 0) \ge \chi(s)$.

From (3.17) using (3.21), we get for all $t \in [0, \rho^*]$:

$$\begin{aligned} |x(t)| &\le \chi(|x(0)|) + \chi(t) + \chi(\|y\|_{[0,t]}) \\ &\le \beta(\chi(|x(0)|, 0) + \chi(t) + \chi(\|y\|_{[0,t]}) \\ &= \beta(\chi(|x(0)|), t) + [\beta(\chi(|x(0)|), 0) - \beta(\chi(|x(0)|), t) + \chi(t)] + \\ &\quad \chi(\|y\|_{[0,t]}) \\ &\le \bar{\beta}(|x(0)|, t) + \bar{d} + \bar{\gamma}_y(\|y\|_{[0,t]}). \end{aligned}$$

Thus (3.19) holds for all $t \in [0, \rho^*]$.

For the remaining two intervals, from (3.17), using (3.21) which gives us that $\chi(\rho^*) \leq \bar{d}$ we have

$$\begin{aligned} |x(\rho^*)| &\leq \chi(|x(0)|) + \chi(\rho^*) + \chi(\|y\|_{[0,\rho^*]}) \\ &\leq \chi(|x(0)|) + \bar{d} + \chi(\|y\|_{[0,\rho^*]}) \\ &\leq \max\{2(\chi(|x(0)|) + \bar{d}), 2\chi(\|y\|_{[0,\rho^*]})\}. \end{aligned} \tag{3.23}$$

By the definition of $\bar{t}$ we have that $\forall t \in [0, \bar{t}), u_2$ satisfies the following bound:

$$\begin{aligned} \|u_2\|_{[0,t]} &\leq \sigma(\max\{2(\chi(|x(0)|) + \bar{d}), \|u_1\|_{[0,t]}, 2\chi(\|y\|_{[0,t]})\}) \\ &\leq \sigma(\max\{|x(\rho^*)|, \|u_1\|_{[0,t]}\}) \end{aligned}$$

where the second inequality stems from (3.23) and the fact that $\sigma(\cdot)$ is nondecreasing.

Using this fact and time invariance, we have that since $\|u_2\|_{[0,\bar{t}]} \leq \sigma(\max\{|x(\rho^*)|, \|u_1\|_{[0,\bar{t}]}\})$, then we have by (3.18) applied at $x_0 = x(\rho^*)$ that for all $t \in [\rho^*, \bar{t})$

$$\begin{aligned} |x(t)| &\leq \beta(|x(\rho^*)|, t - \rho^*) + \gamma_{u_1}(\|u_1^{\rho^*}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho^*}\|_{[0,t]}) + \gamma_y(\|y^{\rho^*}\|_{[0,t]}) \\ &= \beta(2\chi(|x(0)|), t) + [\beta(|x(\rho^*)|, t - \rho^*) - \beta(2\chi(|x(0)|), t)] + \\ &\quad \gamma_{u_1}(\|u_1^{\rho^*}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho^*}\|_{[0,t]}) + \gamma_y(\|y^{\rho^*}\|_{[0,t]}) \end{aligned} \tag{3.24}$$

using (3.17),

$$\begin{aligned} |x(t)| &\leq \beta(2\chi(|x(0)|), t) + \\ &\quad [\beta(\chi(|x(0)|) + \chi(\rho^*) + \chi(\|y\|_{[0,\rho^*]}), t - \rho^*) - \beta(2\chi(|x(0)|), t)] + \\ &\quad \gamma_{u_1}(\|u_1^{\rho^*}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho^*}\|_{[0,t]}) + \gamma_y(\|y^{\rho^*}\|_{[0,t]}). \end{aligned}$$

using the triangle inequality for $\mathcal{K}$ functions (2.1), we have

$$\begin{aligned} |x(t)| \leq\ & \beta(2\chi(|x(0)|), t) + \\ & [\beta(2(\chi(|x(0)|) + \chi(\rho^*)), t - \rho^*) - \beta(2\chi(|x(0)|), t)] + \\ & \beta(2\chi(\|y\|_{[0,\rho^*]}), t - \rho^*) + \gamma_y(\|y^{\rho^*}\|_{[0,t]}) + \\ & \gamma_{u_1}(\|u_1^{\rho^*}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho^*}\|_{[0,t]}) \end{aligned}$$

Using our definitions of $\bar{\beta}(s,t)$ and $\bar{\gamma}_y(s)$ simplifies the inequality to:

$$\begin{aligned} |x(t)| \leq\ & \bar{\beta}(|x(0)|, t) + \\ & [\beta(2(\chi(|x(0)|) + \chi(\rho^*)), t - \rho^*) - \beta(2\chi(|x(0)|), t)] + \\ & \bar{\gamma}_y(\|y\|_{[0,t]}) + \\ & \gamma_{u_1}(\|u_1^{\rho^*}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho^*}\|_{[0,t]}) \end{aligned} \tag{3.25}$$

Then for $t \in [\rho^*, T)$ since $|x(0)| < \bar{\delta}$ we can use (3.22) giving as bound for all $t \in [\rho^*, T)$

$$|x(t)| \leq \bar{\beta}(|x(0)|, t) + \bar{d} + \gamma_{u_1}(\|u_1^{\rho^*}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho^*}\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]})$$

for any $\rho \leq \rho^*$,

$$|x(t)| \leq \bar{\beta}(|x(0)|, t) + \gamma_{u_1}(\|u_1^{\rho}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho}\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + \bar{d} \qquad \forall t \in [\rho^*, T)$$

since $\| \cdot^{\rho^*} \| \leq \| \cdot^{\rho} \|$.

Finally for the last time interval, $t \in [T, \bar{t})$ from (3.25) we have for any such t

$$\begin{aligned} |x(t)| \leq\ & \bar{\beta}(|x(0)|, t) + \\ & [\beta(2(\chi(|x(0)|) + \chi(\rho^*)), t - \rho^*) - \beta(2\chi(|x(0)|), t)] + \\ & \gamma_{u_1}(\|u_1^{\rho^*}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho^*}\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}). \end{aligned}$$

Using the fact that $\beta(2(\chi(s) + \chi(\rho^*)), t - \rho^*) \leq \beta(2(\chi(s) + \chi(1)), t - 1)$ as $\rho^* \leq 1$, we have:

$$\begin{aligned} |x(t)| \leq\ & \bar{\beta}(|x(0)|, t) + \\ & [\beta(2(\chi(|x(0)|) + \chi(1)), t - 1) - \beta(2\chi(|x(0)|), t)] + \\ & \gamma_{u_1}(\|u_1^{\rho^*}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho^*}\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) \end{aligned}$$

and finally using (3.20) we have for all $t \in [T, \bar{t})$:

$$\begin{aligned} |x(t)| &\leq \bar{\beta}(|x(0)|, t) + \gamma_{u_1}(\|u_1^{\rho^*}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho^*}\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + \bar{d} \\ &\leq \bar{\beta}(|x(0)|, t) + \bar{\gamma}_{u_1}(\|u_1^{\rho}\|_{[0,t]}) + \bar{\gamma}_{u_2}(\|u_2^{\rho}\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + \bar{d}, \text{ for any } \rho \leq \rho^* \end{aligned}$$

Hence over all intervals, for $t \in [0, \bar{t})$ the state is bounded by

$$|x(t)| \leq \bar{\beta}(|x(0)|, t) + \gamma_{u_1}(\|u_1^{\rho}\|_{[0,t]}) + \gamma_{u_2}(\|u_2^{\rho}\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + \bar{d}$$

and the lemma is proved. □

3.4 Main Result

Having proven the previous three lemmas on detectability of systems with outputs under an additional input we are now ready to prove the main result of detectability of singularly perturbed systems with inputs.

Recall from Section 2.5.1 that the standard model for singularly perturbed systems is

$$\begin{aligned} \dot{x} &= f(x, z, \epsilon) \\ \epsilon\dot{z} &= g(x, z, \epsilon) \end{aligned}$$

If we extend this model to consider singularly perturbed systems with an input $\theta(\cdot)$ and an output y of the "slow state" x we get:

$$\begin{aligned} \dot{x} &= f(x, z, \theta(t), \epsilon), \qquad y = k(x) \\ \epsilon\dot{z} &= g(x, z, \theta(t), \epsilon) \end{aligned} \tag{3.26}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$ denote vectors of state variables, $\theta \in \mathbb{R}^q$ denotes the vector of the disturbances, $y \in \mathbb{R}^r$ denotes the vector of outputs, and ϵ is a small positive parameter. The functions f and g are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times [0, \bar{\epsilon})$ for some $\bar{\epsilon} > 0$ and the function k is locally Lipschitz on $\mathbb{R}^n$. In what follows, for simplicity, we will suppress the time-dependence in the notation of the vector of input variables $\theta(t)$.

Using the standard procedure outlined in Section 2.5.1 to analyze these types of systems, the original system will be broken into two subsystems which are easier to analyze. Recall that these two subsystems are called the reduced-order system and boundary layer system, each have a different time scale.

The *slow* or *reduced system* is found by first setting $\epsilon = 0$ and solving for z in the g system. Setting $\epsilon = 0$ gives us:

$$\dot{x} = f(x, z_s, \theta(t), 0), \qquad y = k(x) \tag{3.27}$$

$$0 = g(x, z_s, \theta(t), 0) \tag{3.28}$$

where z_s, denotes a quasi steady state for the fast state vector z. We require that the singularly perturbed system in (3.26) is in *standard form*:

Assumption 3.1 *: The algebraic equation* $g(x, z_s, \theta, 0) = 0$ *possesses an isolated root*

$$z_s = h(x, \theta) \tag{3.29}$$

such that h *and its partial derivatives are locally Lipschitz.*

Substituting z_s from (3.29) into (3.27) we get

$$\dot{x} = f(x, h(x, \theta), \theta(t), 0), \qquad y = k(x) \tag{3.30}$$

The dynamical system in (3.30) is called the *reduced system* or *slow system.* The standard technique used in perturbation theory is to identify some property that the slow system (3.30) has and to identify if the perturbed system (3.26) has the same property. We will assume that the reduced system is IOSS and attempt to show that the perturbed still has that property.

Assumption 3.2 *: The reduced system in (3.30) is IOSS with Lyapunov gains* $(\gamma_\theta, \gamma_y)$*. By Proposition 2.3, this implies that it has the UO property.*

The inherent two-time scale behaviour of the system of (3.26) is analyzed by defining a fast time scale

$$\tau = \frac{t}{\epsilon}$$

and the new coordinate $w := z - h(x, \theta)$. In the (x, w) coordinates and with respect to the τ time scale, the system of (3.26) takes the form

$$\begin{aligned}
\frac{\partial x}{\partial \tau} &= \epsilon f(x, h(x, \theta) + w, \theta, \epsilon) \\
\frac{\partial w}{\partial \tau} &= g(x, h(x, \theta) + w, \theta, \epsilon) - \epsilon \Big[\frac{\partial h}{\partial x} f(x, h(x, \theta) + w, \theta, \epsilon) + \frac{\partial h}{\partial \theta}\dot{\theta}\Big]
\end{aligned} \tag{3.31}$$

Setting ϵ equal to zero, the following locally Lipschitz system is obtained:

$$\frac{\partial w}{\partial \tau} = g(x, h(x, \theta) + w, \theta, 0) \tag{3.32}$$

Here, x and θ are to be thought as input vectors. This system is called the *boundary layer system* or *fast system.*

The assumption on the fast system will be required to be able to show that the perturbed system (3.26) maintains a form of detectability.

Assumption 3.3 *: The equilibrium $w = 0$ of the boundary layer system in (3.32) is GAS, uniformly in $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$.*

The main result is given below.

Theorem 3.2 *: Consider the singularly perturbed system in (3.26) and suppose Assumptions 3.1-3.3 hold and that $\theta(t)$ is absolutely continuous. Define $w := z - h(x, \theta)$ and let $(\gamma_\theta, \gamma_y)$ be the gains given in Assumption 3.2. Then there exist functions β_x, β_w of class $\mathcal{KL}$, and for each pair of positive numbers (δ, d), there is an $\epsilon^* > 0$ such that if $\epsilon \in (0, \epsilon^*]$,then for all initial states $x(0), w(0)$ with $\max\{|x(0)|, |w(0)|\} \leq \delta$ the solutions of (3.26) satisfy*

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) + d \tag{3.33}$$

$$|w(t/\epsilon)| \leq \beta_w(|w(0)|, t/\epsilon) + d \tag{3.34}$$

for all $t \in [0, t^{max}_{(|x(0)|,|w(0)|),(\theta,\epsilon)})$ such that $\max\{\|\theta\|_{[0,t]}, \|\dot{\theta}\|_{[0,t]}, \|y\|_{[0,t]}\} \leq \delta$ where $t^{max}_{(|x(0)|,|w(0)|),(\theta,\epsilon)}$ is the maximum time for which $x(t), w(t)$ are defined.

Intuitively one would guess that as $d \to 0$ then $\epsilon^* \to 0$ as the more restrictive we want the bounds on the state to be, the closer we wish to retrieve the IOSS bound that the reduced system has. This in turn will only hold if the system is very close in nature to the reduced system, meaning for ϵ very small. This is indeed the case as will be seen in the proof, the construction of ϵ^* depends on d and that for smaller values of d we will have a smaller value of ϵ^*.

With regards to the relationship of δ and ϵ^*, we would expect an inverse relationship, as for small δ the initial values of the states will be very close to their equilibrium values so even under a perturbation, they will not need to travel far to reach their equilibrium values. Meaning that those trajectories are more robust to perturbations hence the bound will still hold for larger ϵ^*.

One should also note that the result only holds over the time interval that the output remains "small" (i.e. $\|y\|_{[0,t]} \leq \delta$). This is because for large outputs the reduced state variable x can be large. As the reduced and boundary systems are coupled, a large state x may cause the boundary state w to grow hence violating the bound for w in (3.34). The proof of the theorem is given below.

Proof: We analyze (3.26) in the (x, w) coordinates. In the τ time scale ($\tau = t/\epsilon$), the w dynamics are governed by

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= g(x, h(x,\theta) + w, \theta, \epsilon) - \epsilon\Big[\frac{\partial h}{\partial x} f(x, h(x,\theta) + w, \theta, \epsilon) + \frac{\partial h}{\partial \theta}\dot{\theta}\Big] \\ &= F(x, w, \theta, \dot{\theta}, \epsilon) \end{aligned} \tag{3.35}$$

where we have defined

$$F(x, w, \theta, \dot{\theta}, \epsilon) := g(x, h(x,\theta) + w, \theta, \epsilon) - \epsilon\Big[\frac{\partial h}{\partial x} f(x, h(x,\theta) + w, \theta, \epsilon) + \frac{\partial h}{\partial \theta}\dot{\theta}\Big]$$

Note that F is locally Lipschitz with respect to $x, w, \theta, \dot{\theta}$ and ϵ (g is locally Lipschitz with respect to x, θ, ϵ the partial derivatives of h are locally Lipschitz with respect to x, θ, and $\dot{\theta}$ is bounded since $\theta(t)$ is absolutely continuous).

We first start out by proving that if we treat x and θ as constant inputs in (3.35) then that system is ISS with respect to ϵ. We then introduce an auxiliary system in the x coordinates and using Lemma 3.2 show that this auxiliary system can be made IOSS with respect to inputs $\tilde{w}, \epsilon$ and output y for any given input $\tilde{w}$ and any value $\epsilon > 0$ by mapping the input $\tilde{w}$ through a scaling function $b(\cdot)$. From this, we relate the input $\tilde{w}$ with fast state variable w by treating it as an input to the slow system and using this relation we show by contradiction that for small enough ϵ if the initial states $(x(0), w(0))$, input disturbance, its derivative $(\theta, \dot{\theta})$ and the output y are bounded then the states x, w in (3.26) are bounded by (3.33) and (3.34) for all time t such that the solution exists and the output remains bounded.

Claim 1: There exists a function β_w of class $\mathcal{KL}$, a continuous nonincreasing function $\sigma_w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \sigma_w(s) \leq 1, \forall s \in \mathbb{R}_{\geq 0}$ and a function γ_{ϵ_w} of class $\mathcal{K}$ such that the solution of (3.35) with an initial state $w(0) \in \mathbb{R}^p$, a fixed bounded continuous signal $x(\cdot)$, an input θ and a $\epsilon > 0$ exists for each $t \geq 0$ and satisfies:

$$|w(t/\epsilon)| \leq \beta_w(|w(0)|, t/\epsilon) + \gamma_{\epsilon_w}(\epsilon) \quad \forall t \in [0, \bar{t}) \tag{3.36}$$

where $\bar{t}$ is the maximum time that $\epsilon \leq \sigma_w(\max\{|w(0)|, \|\theta\|_{[0,\bar{t}]}, \|\dot{\theta}\|_{[0,\bar{t}]}, \|x\|_{[0,\bar{t}]}\})$ holds.

Proof of Claim 1: In (3.35), set $\epsilon = 0$ giving

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= F(x, w, \theta, \dot{\theta}, 0) \\ &= g(x, h(x, \theta) + w, \theta, 0) \end{aligned} \tag{3.37}$$

this being the boundary layer system (3.32) of the full system (3.26).

By Assumption 3.3, for system (3.37) the point $w = 0$ is GAS uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Then using Proposition 3.1, let the vector $(x, \theta, \dot{\theta}) = u_1$, Note that

$$\begin{aligned} \|u_1\|_{[0,t]} &= \operatorname*{ess\,sup}_{\tau \in [0,t]} |u(\tau)| \\ &= \operatorname*{ess\,sup}_{\tau \in [0,t]} \sqrt{|x(\tau)|^2 + |\theta(\tau)|^2 + |\dot{\theta}(\tau)|^2} \\ &\leq \operatorname*{ess\,sup}_{\tau \in [0,t]} \sqrt{3} \max\{|x(\tau)|, |\theta(\tau)|, |\dot{\theta}(\tau)|\} \\ &= \sqrt{3} \max\{\|x\|_{[0,t]}, \|\theta\|_{[0,t]}, \|\dot{\theta}\|_{[0,t]}\}. \end{aligned} \tag{3.38}$$

Treating ϵ as u_2 we have that there exists a function β_w of class $\mathcal{KL}$, a continuous nonincreasing function $\bar{\sigma_w} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \bar{\sigma_w}(s) \leq 1, \forall s \in \mathbb{R}_{\geq 0}$ and a function γ_{ϵ_w} of class $\mathcal{K}$ such that the solution of (3.35) with initial state $w(0)$ exists for each $t \geq 0$ and satisfies:

$$|w(t/\epsilon)| \leq \beta_w(|w(0)|, t/\epsilon) + \gamma_{\epsilon_w}(\epsilon) \quad \forall t \in [0, t')$$

where t' is the maximum time that $\epsilon \leq \sigma_w(\max\{|w(0)|, \|u_1\|_{[0,\bar{t}]}\})$ holds. Define $\bar{t}$ to be the maximum time such that $\epsilon \leq \bar{\sigma_w}(\sqrt{3}\max\{|w(0)|, \|\theta\|_{[0,\bar{t}]}, \|\dot{\theta}\|_{[0,\bar{t}]}, \|x\|_{[0,\bar{t}]}\})$ holds. Note that $\bar{t} \leq t'$ using (3.38) and the fact that $\bar{\sigma_w}$ is a nonincreasing function. Define $\sigma_w(s) := \bar{\sigma_w}(\sqrt{3}s)$, $\sigma_w(\cdot)$ is a nonincreasing function such that $\sigma_w(\cdot) \leq 1$ and the claim is proved.

Now analyzing the x dynamics of system (3.26) with the substitution $w := h(x,\theta) - z$ giving us:

$$\dot{x} = f(x, h(x,\theta) + w, \theta, \epsilon), \qquad y = k(x) \tag{3.39}$$

Let $\delta, d > 0$ be given.

Claim 2: For the system (3.39), there exists continuous nonincreasing functions $b : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, b(s) \leq 1$, $\sigma_x : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \sigma_x(s) \leq 1$, a $\mathcal{K}$ function χ with $\chi(s) > s$, and functions $\beta_x \in \mathcal{KL}$, $\gamma_\theta, \gamma_{\tilde{w}}, \gamma_{\epsilon_x}, \gamma_y \in \mathcal{K}$ such that given a fixed bounded continuous trajectory $\tilde{w}(\cdot)$, a fixed input θ, a fixed value for $\epsilon > 0$, and an initial condition $x(0)$ such that $\max\{|x(0)|, \|\theta\|_{[0,\infty)}, \|\dot{\theta}\|_{[0,\infty)}\} < \delta$ the solution to the system

$$\dot{x} = f(x, h(x,\theta) + b(|x|)\tilde{w}, \theta, \epsilon), \qquad y = k(x) \tag{3.40}$$

exists and satisfies:

$$|x(t)| \leq \chi(t) + \chi(|x(0)|) + \chi(\|y\|_{[0,t]}), \quad \forall t \in [0, \bar{T}) \tag{3.41}$$

and

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \gamma_{\tilde{w}}(\|\tilde{w}\|_{[0,t]}) + \gamma_{\epsilon_x}(\epsilon) + \gamma_y(\|y\|_{[0,t]}) \qquad \forall t \in [0, \bar{T}) \tag{3.42}$$

where $\bar{T} = \min\{t^*, t^{max}_{x(0),(\tilde{w},\theta,\epsilon)}\}$, t^* being the maximum time for which

$$\epsilon \;\; \leq \;\; \sigma_x(\max\{|x(0)|, \|\theta\|_{[0,t^*]}, \|\tilde{w}\|_{[0,t^*]}, \|y\|_{[0,t^*]}\}) \tag{3.43}$$

holds.

Proof of Claim 2: First set $\epsilon = 0$ in (3.39) giving the system, relabeling the input w as $\tilde{w}$,

$$\dot{x} = f(x, h(x,\theta) + \tilde{w}, \theta, 0), \qquad y = k(x). \tag{3.44}$$

When $\tilde{w} \equiv 0$, system (3.44) is equivalent to the reduced system (3.30). Then using Assumption 3.2, system (3.44) with $\tilde{w} \equiv 0$ is IOSS with Lyapunov gains $(\tilde{\gamma}_\theta, \gamma_y)$. By Corollary 3.1, we have that there exist a nonincreasing function $b(s) : \mathbb{R}_{\geq 0} \to \mathbb{R}, 0 < b(s) \leq 1$ and functions $\gamma_\theta, \gamma_{\tilde{w}1}$ of class $\mathcal{K}$ such that the system

$$\dot{x} = f(x, h(x, \theta) + b(|x|)\tilde{w}, \theta, 0), \qquad y = k(x) \tag{3.45}$$

is IOSS with gain $(\gamma_\theta, \gamma_{\tilde{w}1}, \gamma_y)$ with respect to inputs $\theta, \tilde{w}$, and output y.

From system (3.45), if we allow ϵ to be something other than 0 we have:

$$\dot{x} = f(x, h(x, \theta) + b(|x|)\tilde{w}, \theta, \epsilon), \qquad y = k(x). \tag{3.46}$$

Treating $(\theta, \tilde{w})$ as u_1 and ϵ as u_2, Lemma 3.3 states that there exists functions $\beta_x \in \mathcal{KL}$, $\gamma_\theta, \gamma_{\tilde{w}}, \gamma_{\epsilon_x}, \gamma_y \in \mathcal{K}$ and a continuous nonincreasing function $\bar{\sigma_x} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \bar{\sigma_x}(s) \leq 1, \forall s \in \mathbb{R}_{\geq 0}$ such that all trajectories of (3.46) will satisfy the following bound:

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \gamma_{\tilde{w}}(\|\tilde{w}\|_{[0,t]}) + \gamma_{\epsilon_x}(\epsilon) + \gamma_y(\|y\|_{[0,t]}) \qquad \forall t \in [0, T') \tag{3.47}$$

where $T' = \min\{t', t^{max}_{x(0),(\tilde{w},\theta,\epsilon)}\}$, $t^{max}_{x(0),(\tilde{w},\theta,\epsilon)}$ being the maximum time that a particular solution of (3.46) exists and t' being the maximum time for which

$$\epsilon \leq \bar{\sigma_x}(\max\{|x(0)|, \|u_1\|_{[0,t^*]}, \|y\|_{[0,t^*]}\}).$$

We define t^* to be the maximum time that

$$\epsilon \quad \leq \quad \bar{\sigma_x}(\sqrt{2}\max\{|x(0)|, \|\theta\|_{[0,t^*]}, \|\tilde{w}\|_{[0,t^*]}, \|y\|_{[0,t^*]}\})$$

holds. Using the fact that

$$\begin{aligned}
\|u_1\|_{[0,t]} &= \operatorname*{ess\,sup}_{\tau\in[0,t]} |u(\tau)| \\
&= \operatorname*{ess\,sup}_{\tau\in[0,t]} \sqrt{|\tilde{w}(\tau)|^2 + |\theta(\tau)|^2} \\
&\leq \operatorname*{ess\,sup}_{\tau\in[0,t]} \sqrt{2}\max\{|\tilde{w}(\tau)|, |\theta(\tau)|\} \\
&= \sqrt{2}\max\{\|\tilde{w}\|_{[0,t]}, \|\theta\|_{[0,t]}\}
\end{aligned}$$

and that $\bar{\sigma_x}(\cdot)$ is a decreasing function, we have that $t^* \leq t'$. Defining $T^* = \min\{t^*, T'\}$ and $\sigma_x(s) = \bar{\sigma_x}(\sqrt{2}s)$ we have that that (3.47) holds for all $t \in [0, T^*)$ hence proving (3.42).

So long as (3.40) satisfies (3.42) then by definition it is IOSS with gains $(\gamma_\theta, \gamma_{\tilde{w}}, \gamma_{\epsilon_x}, \gamma_y)$. From Proposition 2.3, this implies that it also has the UO property. Since the system (3.40) is UO, given any M, for bounded inputs $\theta, \tilde{w}, \epsilon$ dominated by M, Proposition 2.6 states that there exists a function $\chi \in \mathcal{K}$, depending on M, with $\chi(s) > s$ such that the solution of (3.40) with such inputs $\theta, \tilde{w}, \epsilon$ satisfies

$$|x(t)| \leq \chi(t) + \chi(|x(0)|) + \chi(\|y\|_{[0,t]}), \quad \forall t \in [0, \bar{T})$$

hence proving (3.41) and the claim is proved.

Now, given $x(0), w(0), \theta(\cdot)$, and ϵ such that $\max\{|x(0)|, |w(0)|\} < \delta$ we wish to show that the solution of (3.26) is bounded by (3.33) and (3.34) for all $t \in [0, T^*)$ where $T^* =$ $\min\{t^{max}_{(|x(0)|,|w(0)|),(\theta,\epsilon)}, t^*\}$, t^* being the maximum time such that $\max\{\|\theta\|_{[0,t]}, \|\dot{\theta}\|_{[0,t]}, \|y\|_{[0,t]}\} \leq \delta$.

Claim 2 states that the solution to the system

$$\dot{x} = f(x, h(x,\theta) + b(|x|)\tilde{w}, \theta, \epsilon), \qquad y = k(x) \tag{3.48}$$

given a fixed bounded continuous trajectory $\tilde{w}(\cdot)$, a fixed input θ, a fixed value for $\epsilon > 0$, and an initial condition $x(0)$ such that $|x(0)| < \delta$ exists and satisfies:

$$|x(t)| \leq \chi(t) + \chi(|x(0)|) + \chi(\|y\|_{[0,t]}), \quad \forall t \in [0, \bar{T})$$

and

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \gamma_{\tilde{w}}(\|\tilde{w}\|_{[0,t]}) + \gamma_{\epsilon_x}(\epsilon) + \gamma_y(\|y\|_{[0,t]}) \qquad \forall t \in [0, \bar{T})$$

where $\bar{T} = \min\{t^*, t^{max}_{x(0),(\tilde{w},\theta,\epsilon)}\}$, t^* being the maximum time for which

$$\epsilon \quad \leq \quad \sigma_x(\max\{|x(0)|, \|\theta\|_{[0,t^*]}, \|\tilde{w}\|_{[0,t^*]}, \|y\|_{[0,t^*]}\})$$

holds.

Define $\tilde{w}(t) := \frac{1}{b(|x(t)|)} w(t/\epsilon)$. Then the trajectories of system (3.48) and (3.39) are identical so bounds on the state of the former will hold for the latter.

Choose $\delta_x \in \mathbb{R}_{\geq 0}$ satisfying

$$\delta_x > \bar{\beta}_x(\delta, 0) + \gamma_\theta(\delta) + \bar{\gamma}_y(\delta) + d$$

where $\bar{\beta}_x(s,t) = \beta_x(2\chi(s), t)$ and $\bar{\gamma}_y(s) = \gamma_y(s) + \beta_x(2\chi(s), 0)$. Note that $\delta_x > \delta$ as $\chi(s) > s$ and $\beta_x(s, 0) > s$.

Define $[0, T), 0 < T \leq T^*$, to be the maximum interval in which $\|x\|_{[0,t]} < \delta_x$ and $\max\{\|\theta\|_{[0,t]}, \|\dot{\theta}\|_{[0,t]}, \|y\|_{[0,t]}\} \leq \delta \quad \forall t \in [0, T)$. To show by contradiction that $T = T^*$ for ϵ sufficiently small, suppose $T < T^*$. In the case that $T^* = 0$, then $T = T^*$ and we have that the bounds (3.33) and (3.34) are not guaranteed to hold for any time.

Define $\epsilon_1 := \sigma_w(\delta_x)$. Then for all $t \in [0, T), |x(t)| < \delta_x$ hence is bounded and

$$\begin{aligned}
\epsilon_1 &= \sigma_w(\delta_x) \\
&= \sigma_w(\max\{\delta, \delta_x\}) \\
&\leq \sigma_w(\max\{|w(0)|, \|\theta\|_{[0,t]}, \|\dot{\theta}\|_{[0,t]}, \|x\|_{[0,t]}\}
\end{aligned}$$

as σ_w is a nonincreasing function. We have by Claim 1 that if the given fixed ϵ is less than ϵ_1, then there exists functions $\beta_w \in \mathcal{KL}$ and $\gamma_{\epsilon_w} \in \mathcal{K}$ so that w is bounded by

$$|w(t/\epsilon)| \leq \beta_w(|w(0)|, t/\epsilon) + \gamma_{\epsilon_w}(\epsilon) \quad \forall t \in [0, T) \tag{3.49}$$

Next we show that $\tilde{w}$ is bounded so that the bounds of Claim 2 hold for all $t \in [0, T)$. $\tilde{w}$ is bounded by:

$$\begin{aligned}
|\tilde{w}(t)| &= \frac{1}{b(|x(t)|)} w(t/\epsilon) \\
&\leq \frac{1}{b(\delta_x)} [\beta_w(\delta, 0) + \gamma_{\epsilon_w}(\epsilon_1)] =: \delta_{\tilde{w}} \quad \forall t \in [0, T)
\end{aligned}$$

as $|x(t)| < \delta_x$ for all $t \in [0, T)$, $1/b(s)$ is nondecreasing and $w(t/\epsilon)$ is bounded above by (3.49). Note that $\delta_{\tilde{w}} \geq \delta$ as $1/b(s) \geq 1$ and $\beta_w(\delta, 0) \geq \delta$.

Define $\epsilon_2 := \sigma_x(\delta_{\tilde{w}})$ and notice that

$$\begin{aligned}\epsilon_2 &= \sigma_x(\delta_{\tilde{w}}) \\ &= \sigma_x(\max\{\delta, \delta_{\tilde{w}}\}) \\ &\leq \sigma_x(\max\{|x(0)|, \|\theta\|_{[0,T]}, \|\tilde{w}\|_{[0,T]}, \|y\|_{[0,T]}\}).\end{aligned}$$

So the result of Claim 2 holds for all $t \in [0, T)$ as $\tilde{w}$ is bounded on that time interval, wherein we assume $\epsilon \leq \epsilon_2$.

For the system (3.40), let the vector $(\theta, \tilde{w})$ play the role of u_1 and ϵ play the role of u_2. It follows from Lemma 3.4, using (3.43) and (3.47) that there exists a positive real number $\rho < T$ such that if

$$\epsilon \leq \sigma_x(\max\{2(\chi(|x(0)| + d/2), \|\theta\|_{[0,T]}, \|\tilde{w}\|_{[0,T]}, 2\chi(\|y\|_{[0,T]})\}) \tag{3.50}$$

then the solution of (3.39) with $x(0) = x_0$ satisfies

$$|x(t)| \leq \bar{\beta}(|x(0)|, t) + \gamma_\theta(\|\theta^\rho\|_{[0,t]}) + \gamma_{\tilde{w}}(\|\tilde{w}^\rho\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + \gamma_{\epsilon_x}(\epsilon) + d/2 \tag{3.51}$$

where $\bar{\beta}(s,t) = \beta_x(2\chi(s), t)$ and $\bar{\gamma}_y(s) = \gamma_y(s) + \beta_x(2\chi(s), 0)$. So defining

$$\epsilon_3 := \sigma_x(\max\{2(\chi(\delta) + d/2), \delta, \delta_{\tilde{w}}, 2\chi(\delta)\})$$

and using the fact that σ_x is a decreasing function we have that if $\epsilon \leq \epsilon_3$, (3.50) is satisfied hence the bound (3.51) will hold for all $t \in [0, T)$.

Combining (3.49) and (3.51), if $\epsilon \leq \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$ then for all $t \in [0, T)$

$$\begin{aligned}|x(t)| &\leq \bar{\beta}_x(|x(0)|, t) + \gamma_\theta(\|\theta^\rho\|_{[0,t]}) + d/2 + \bar{\gamma}_y(\|y\|_{[0,t]}) \\ &\quad + \gamma_{\tilde{w}}\Big(\frac{1}{b(\|x_t^\rho\|)}\Big[\beta_w(|w(0)|, \rho/\epsilon) + \gamma_{\epsilon_w}(\epsilon)\Big]\Big) + \gamma_{\epsilon_x}(\epsilon) \\ &\leq \bar{\beta}_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + d/2 + \bar{\gamma}_y(\|y\|_{[0,t]}) \\ &\quad + \gamma_{\tilde{w}}\Big(\frac{1}{b(\delta_x)}\Big[\beta_w(\delta, \rho/\epsilon) + \gamma_{\epsilon_w}(\epsilon)\Big]\Big) + \gamma_{\epsilon_x}(\epsilon)\end{aligned}$$

Since the last two terms converge to 0 as $\epsilon \to 0$ there exists an $\epsilon_4 > 0$ such that if $\epsilon \leq \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, then $\forall t \in [0, T)$

$$|x(t)| \leq \bar{\beta}_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + d.$$

Then, $|x(t)|$ is bounded by

$$|x(t)| \leq \bar{\beta}_x(\delta, 0) + \gamma_\theta(\delta) + \bar{\gamma}_y(\delta) + d < \delta_x \quad \forall t \in [0, T)$$

As shown in Figure 3.1, using the above bound for $|x(t)| \quad \forall t \in [0, T)$, from the assumptions that T is finite and that x is continuous in t, there must exist some positive real number τ such that $\|x_t\| < \delta_x, \quad \forall t \in [0, T + \tau)$. This contradicts the definition of T. Hence $T = \bar{T}$ and

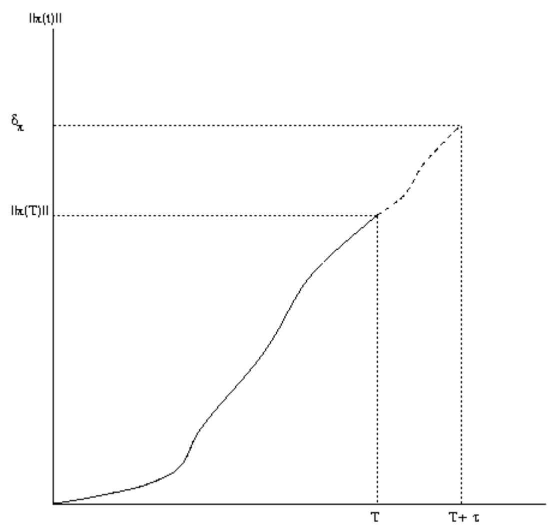


Figure 3.1: Visualization of contradiction argument with regards to maximum state size

$$|x(t)| \leq \bar{\beta}(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + d \tag{3.52}$$

holds for all $t \in [0, \bar{T})$. Finally letting ϵ_5 be such that $\gamma_{\epsilon_w}(\epsilon) \leq d$ for all $\epsilon \in (0, \epsilon_5]$, it follows that both (3.52) and

$$|w(t/\epsilon)| \leq \beta_w(|w(0)|, t/\epsilon) + d$$

hold for any fixed $\epsilon \in (0, \epsilon^*]$ where $\epsilon^* = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\}$ and the theorem is proved. $\square$

Chapter 4

Application

4.1 Diodes

4.1.1 General Diodes

A semiconductor diode consists of a PN (Positively doped-Negatively doped) junction and has two terminals, an anode(+) and a cathode(-). By convention current flows from anode to cathode within the diode. A diode and schematic representation are shown below.

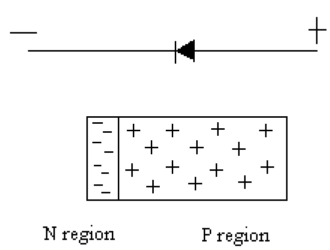


Figure 4.1: Diode and schematic representation

An ideal diode acts as a switch. When a switch is closed the current is allowed to flow. When it is open, the current is stopped.

However, the diode has an additional property: it is unidirectional, i.e. current flows in only one direction (anode to cathode internally). When a forward voltage is applied, the diode conducts; and when a reverse voltage is applied, there is no conduction. A mechanical analogy is a ratchet, which allows motion in one direction only.

An ideal diode current-voltage relationship, called the i-v characteristic, would be:

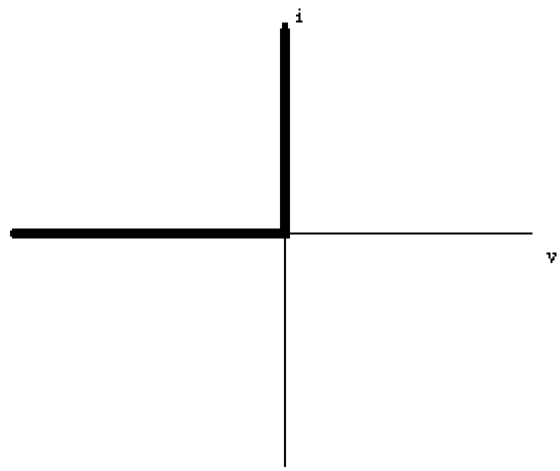


Figure 4.2: Ideal diode i-v characteristics

However, a typical model of a diode characteristic is more like the following:

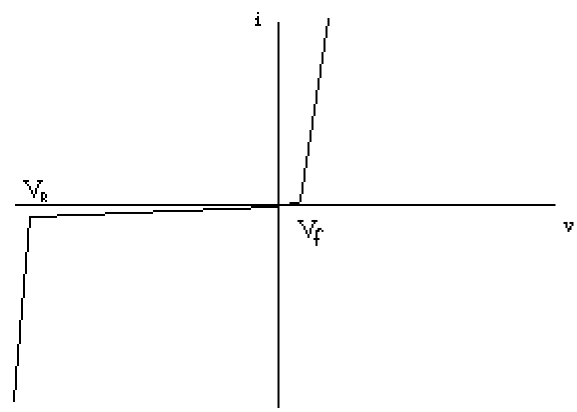


Figure 4.3: Typical diode i-v characteristics

There are a few things that should be pointed out: first there is a minimum forward bias or *threshold voltage* V_f which must be reached before the diode will allow current to go through. Second, there is something called the *breakdown voltage* V_R beyond which the diode will allow conduction in the reverse direction. Typically this is about 50 times the value of the forward threshold. Third, even before reaching the breakdown voltage, the diode will allow a very small amount of current to flow in the reverse direction. This is known as the *leakage current*.

4.1.2 Tunnel Diodes

A special type of diode which has certain interesting characteristics is the tunnel diode. A tunnel diode operates upon a quantum mechanic principle known as “tunneling” wherein certain electrons can tunnel through the intrinsic energy barrier of the material and hence some current can pass through such a diode even at low voltages. In reality, there is a certain amount of tunneling that occurs even in the regular diode, however it is quite minimal. It is possible to build a diode by enhancing this tunneling effect which results in what is known as a tunnel diode.

Referring to the curves below, we superimpose the tunneling characteristic upon a conventional P-N junction:

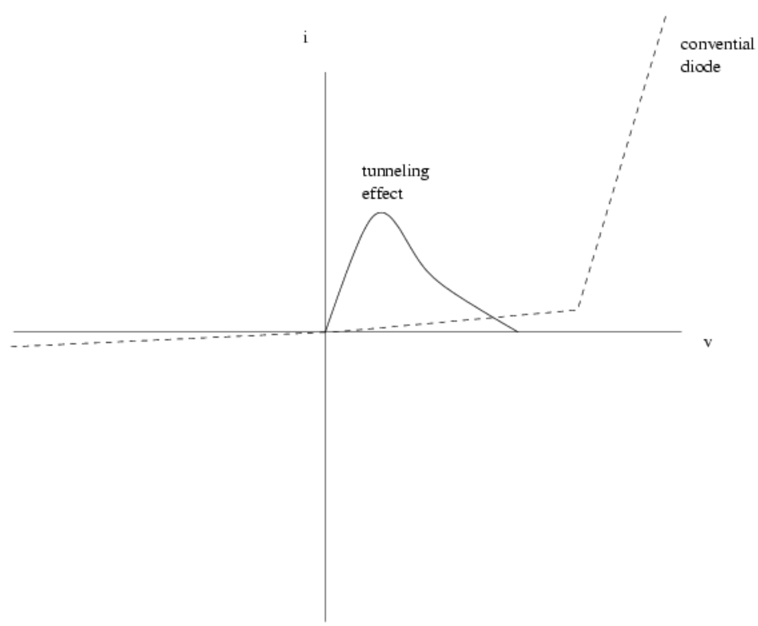


Figure 4.4: Construction of the tunnel diode i-v characteristics

This results in a typical tunnel diode i-v characteristic as shown in Figure 4.5. This diode behaves differently than a conventional diode in that for small positive voltages, the current is proportional to the voltage, at a certain voltage, the current starts to decrease, and upon reaching the threshold voltage, it then behaves like a conventional diode.

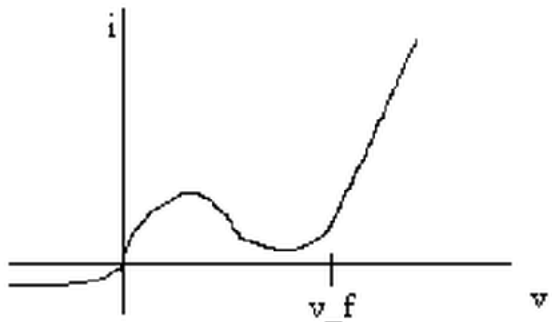


Figure 4.5: Tunnel diode i-v characteristics (for positive v)

The fact that a tunnel diode has a region of negative resistance (negative slope in the i-v characteristic curve) can be exploited to build oscillators and microwave amplifiers [17].

4.2 Example Circuit

An example circuit with a tunnel diode is given below in Figure 4.6 where the tunnel diode is represented by the box with a downward arrow. This circuit is a modified version of a circuit found in [8] given there as an example of stability analysis of singularly perturbed systems.

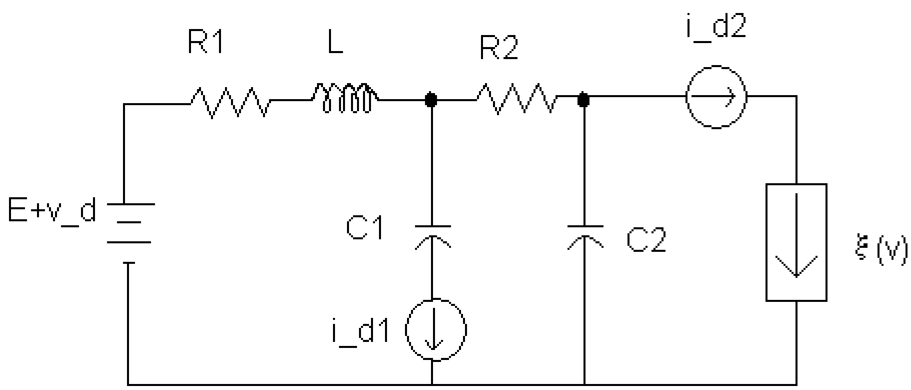


Figure 4.6: Example circuit with a tunnel diode

Using Kirchhoff's Current Law at nodes v_1 & v_2 and Kirchhoff's Voltage Law around the closed loop formed by the voltage source E and the C_1 capacitor we get:

$$\begin{aligned} v_2(t) &= -L\frac{di(t)}{dt} - R_1 i(t) + E + v_d(t) \\ i(t) &= C_1\frac{dv_2(t)}{dt} + \frac{v_2(t) - v_1(t)}{R_2} + i_{d1}(t) \\ \frac{v_2(t) - v_1(t)}{R_2} &= C_2\frac{dv_1(t)}{dt} + \xi(v_1(t)) + i_{d2}(t) \end{aligned}$$

where $\xi(v)$ is a function which models the i-v characteristic of a tunnel diode and would look like Figure 4.5 and v_d, i_{d1}, i_{d2} model disturbances or exogenous inputs. Note that $\xi(v)$ is a nonlinear function of voltage.

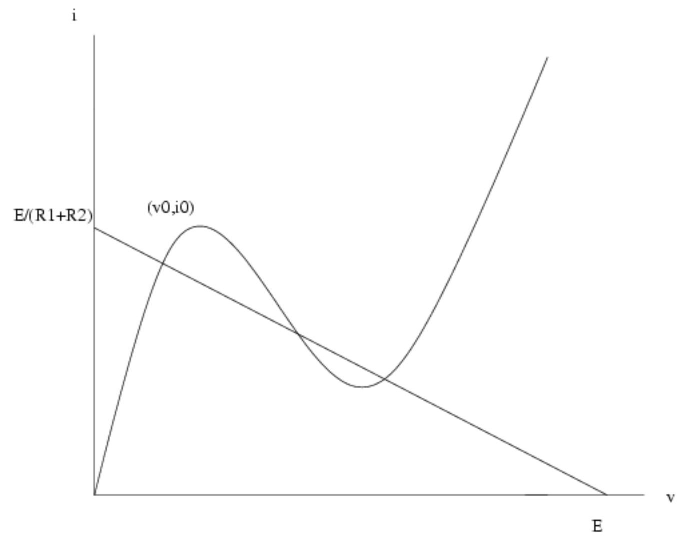


Figure 4.7: i-v characteristics of a tunnel diode vs. circuit resistance i-v characteristics

We find the equilibrium points by assuming that at equilibrium the capacitors act like open circuits, hence not allowing any current to pass through them, and that the inductor acts like a short circuit, hence having a $0V$ drop across it and allowing all current to pass. By doing so and ignoring the disturbances, we have that the total resistance in the circuit excluding the tunnel diode is $R_1 + R_2$. We call the i-v characteristics of this total resistance the *load line* of the circuit as this is the resistance the extra element or load, being the tunnel diode, sees looking into the circuit. By superimposing both i-v characteristics as done in Figure 4.7 and finding the points of intersection, we find the equilibrium points.

From the graph we see that there are possibly three equilibrium points with $v_d \equiv i_{d1} \equiv i_{d2} \equiv 0$, given appropriate choices of E, R_1 and R_2. At the very least, there is always going to be at least one equilibrium point. Denote one of these equilibrium points as $(v_1, v_2, i) = (v_{01}, v_{02}, i_0)$.

Nondimensionalizing the system first by rescaling the time variable with $t' = t/C_2R_2$:

$$\begin{aligned}
\frac{L}{C_2R_2}\dot{i} &= -v_2 - R_1 i + E + v_d && (4.1)\\
\frac{C_1}{C_2R_2}\dot{v}_2 &= i + \frac{v_1 - v_2}{R_2} - i_{d1} && (4.2)\\
\frac{1}{R_2}\dot{v}_1 &= \frac{v_2 - v_1}{R_2} - \xi(v_1) - i_{d2} && (4.3)
\end{aligned}$$

where the overdot refers to differentiation with respect to t'.

Then shifting the equilibrium point to the origin with the choice $x_1 = (v_1 - v_{01})/v_{01}, x_2 = (v_2 - v_{02})/v_{02}, z = (i - i_0)/i_0$ and $\epsilon = \frac{L/R_1}{C_2R_2} = \frac{L}{C_2R_1R_2}$ in (4.1) yields:

$$\begin{aligned}
R_1\epsilon i_0 \dot{z} &= -(v_{02}x_2 + v_{02}) - R_1(zi_0 + i_0) + E + v_d\\
\epsilon\dot{z} &= -z - \frac{v_{02}}{R_1 i_0}x_2 + \frac{E - v_{02} - i_0R_1}{i_0R_1} + \frac{v_d}{i_0R_1}
\end{aligned}$$

Noting that by definition of v_{02} and i_0, $E = v_{02} + i_0R_1$, defining $a = \frac{R_1 i_0}{v_{02}}$ and $\theta_3(t) = \frac{v_d(t)}{i_0R_1}$ simplifies the equation to

$$\epsilon\dot{z} = -z - \frac{1}{a}x_2 + \theta_3$$

Similarly for (4.2),

$$\begin{aligned}
\frac{C_1}{C_2R_2}v_{02}\dot{x}_2 &= (zi_0 + i_0) + \frac{(v_{01}x_1 + v_{01}) - (v_{02}x_2 + v_{02})}{R_2} - i_{d1}\\
\frac{C_1}{C_2}v_{02}\dot{x}_2 &= i_0R_2z + v_{01}x_1 - v_{02}x_2 + (v_{01} - v_{02} + i_0R_2) - i_{d1}R_2
\end{aligned}$$

Now, using the fact that $v_{02} = v_{01} + i_0R_2$, defining $k = C_2/C_1$ and $b = i_0R_2/v_{01}$ yields:

$$\begin{aligned}
\dot{x}_2 &= k\frac{i_0R_2}{v_{01} + i_0R_2}z + k\frac{v_{01}}{v_{01} + i_0R_2}x_1 - kx_2 - k\frac{R_2}{v_{02}}i_{d1}\\
\dot{x}_2 &= k\frac{b}{1+b}z + k\frac{1}{1+b}x_1 - kx_2 + \theta_2
\end{aligned}$$

where $\theta_2(t) = -k\frac{R_2}{v_{02}}i_{d1}(t)$.

Finally, transforming (4.3)

$$\begin{aligned} v_{01}\dot{x}_1 &= (v_{02}x_2 + v_{02}) - (v_{01}x_1 + v_{01}) - R_2\xi(v_{01}x_1 + v_{01}) - R_2 i_{d2} \\ \dot{x}_1 &= \frac{v_{02}}{v_{01}}x_2 - x_1 - \frac{R_2}{v_{01}}\xi(v_{01}x_1 + v_{01}) + \frac{v_{02} - v_{01}}{v_{01}} - \frac{R_2}{v_{01}}i_d 2 \end{aligned} \tag{4.4}$$

Again, using the fact that $v_{02} = v_{01} + i_0 R_2$ and the fact that $\xi(v_{01}) = i_0$, we have that defining $\eta(\cdot)$ as

$$\eta(x_1) = \frac{R_2}{v_{01}}[\xi(v_{01}x_1 + v_{01}) - \xi(v_{01})]$$

equation (4.4) reduces to

$$\begin{aligned} \dot{x}_1 &= \frac{v_{01} + i_0 R_2}{v_{01}}x_2 - x_1 - \eta(x_1) - \frac{R_2}{v_{01}}i_{d2} \\ \dot{x}_1 &= (1+b)x_2 - x_1 - \eta(x_1) + \theta_1 \end{aligned}$$

where we have used the fact that $b = i_0 R_2 / v_{01}$ and defined $\theta_1(t) = -\frac{R_2}{v_{01}} i_{d2}(t)$.

Summarizing we have:

$$\dot{x}_1 = -x_1 - \eta(x_1) + (1+b)x_2 + \theta_1 \tag{4.5}$$

$$\dot{x}_2 = k(\frac{1}{1+b})x_1 - kx_2 + k(\frac{b}{1+b})z + \theta_2 \tag{4.6}$$

$$\epsilon\dot{z} = -\frac{1}{a}x_2 - z + \theta_3 \tag{4.7}$$

with $k = C_2/C_1, a = i_0 R_1 / v_{02}, b = i_0 R_2 / v_{01}, \theta_i \ (i = 1, 2, 3)$ the normalized disturbances, and $\eta(\cdot)$ defined as

$$\eta(x_1) = \frac{R_2}{v_{01}}[\xi(x_1 v_{01} + v_{01}) - \xi(v_{01})].$$

In most commercial electronic device with circuits, physical circuit board space is an extremely valuable commodity. As such, it is often impossible to allocate board space for voltage or current measurement points. This renders the task of circuit verification very difficult. It would be of great help to know the behaviour of some of the circuits without having to insert measurement test points at every possible physical location. If we assume that the circuit of Figure 4.6 is part of a larger electronic device and that it is only possible to have a single measurement point for the circuit, we can choose that value to be the voltage across the tunnel diode v and perform a stability analysis of the circuit. We normalize the measurement so that the output function is $y = x_1 = (v_1 - v_{01})/v_{01}$. We perform a local stability analysis about one of the equilibrium points.

Performing perturbation analysis, setting $\epsilon = 0$ in equation (4.7) yields:

$$0 = -\frac{1}{a}x_2 - z + \theta_3$$

Solving for z gives:

$$z_R = h(x, \theta) = -\frac{1}{a}x_2 + \theta_3 \tag{4.8}$$

which has an isolated root for z, hence Assumption 3.1 is satisfied. Substituting the value of the root z_R for z in Equations (4.5) and (4.6) gives us the reduced system:

$$\begin{aligned} \dot{x}_1 &= -x_1 - \eta(x_1) + (1+b)x_2 + \theta_1 && (4.9) \\ \dot{x}_2 &= k(\frac{1}{1+b})x_1 - \frac{k}{r}x_2 + \theta_4 && (4.10) \\ y &= x_1 && (4.11) \end{aligned}$$

where $\theta_4 = \theta_2 + \frac{kb}{1+b}\theta_3$, $r = \frac{R_1}{R_2+R_1}$ and we have used the fact that since $v_{02} = v_{01} + i_0R_2$ and $a = i_0R_1/v_{02}, b = i_0R_2/v_{01}$, then

$$a = \frac{R_1}{R_2} \cdot \frac{b}{1+b}$$

Examining the differential equation for x_2 while considering x_1 and θ_4 as inputs we get the following bound on x_2:

$$\begin{aligned} x_2(t) &= x_2(0)e^{-k_r t} + \int_0^t \left[k(\frac{1}{1+b})x_1(s) + \theta_4(s)\right] e^{k_r(s-t)} ds \\ |x_2(t)| &\leq |x_2(0)|e^{-k_r t} + \left[k(\frac{1}{1+b})\|x_1\|_{[0,t]} + \|\theta_4\|_{[0,t]}\right] \int_0^t e^{k_r(s-t)} ds \\ &= |x_2(0)|e^{-k_r t} + \left[k(\frac{1}{1+b})\|x_1\|_{[0,t]} + \|\theta_4\|_{[0,t]}\right] \frac{1}{k_r}(1 - e^{-k_r t}) \\ &\leq |x_2(0)|e^{-k_r t} + \frac{r}{1+b}\|x_1\|_{[0,t]} + \frac{1}{k_r}\|\theta_4\|_{[0,t]} \end{aligned}$$

where $k_r = \frac{k}{r}$. Hence a bound on the state $x = (x_1, x_2)$ of the reduced system can be given as:

$$\begin{aligned}
|x(t)| &= |(x_1, 0) + (0, x_2)| \\
&\leq |(x_1, 0)| + |(0, x_2)| \qquad \text{(triangle inequality)} \\
&= |x_1(t)| + |x_2(t)| \\
&\leq \|x_1\|_{[0,t]} + |x_2(0)|e^{-k_r t} + \frac{r}{1+b}\|x_1\|_{[0,t]} + \frac{1}{k_r}\|\theta_4\|_{[0,t]} \\
&= |x_2(0)|e^{-k_r t} + \frac{1}{k_r}\|\theta_2 + \frac{kb}{1+b}\theta_3\|_{[0,t]} + \left(\frac{r}{1+b} + 1\right)\|y\|_{[0,t]} \\
&\leq |x(0)|e^{-k_r t} + \frac{1}{k_r}\left(1 + \frac{kb}{1+b}\right)\|\theta\|_{[0,t]} + \\
&\quad \left(\frac{r}{1+b} + 1\right)\|y\|_{[0,t]} \\
&= \beta(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) \qquad (4.12)
\end{aligned}$$

where $\theta = (\theta_1, \theta_2, \theta_3)$ and since $k_r > 0$, $\beta(s,t) = se^{-k_r t}, \gamma_\theta(s) = \frac{1+b+kb}{k_r(1+b)}s$ and $\gamma_y(s) = (\frac{r}{1+b} + 1)s$. Hence the reduced system is IOSS, satisfying Assumption 3.2.

Performing the substitution $w = z - h(x,\theta) = z + \frac{1}{a}x_2 - \theta_3$ and rescaling time with $\tau = t/\epsilon$ in Equation (4.7), assuming that x and θ are constant with respect to τ give the boundary layer system:

$$\frac{\partial w}{\partial \tau} = -w \qquad (4.13)$$

Hence the boundary layer is GAS uniformly in (x, θ) satisfying Assumption 3.3.

Since all the assumptions are satisfied, if we assume that the disturbances are absolutely continuous then by Theorem 3.2 we have that there exists $\mathcal{KL}$ functions β_x, β_w, a $\mathcal{K}$ function $\bar{\gamma}_y$ such that for each pair of positive numbers (δ, d), there is an $\epsilon^* > 0$ such that if $\max\{|x(0)|, |w(0)|, \|\theta\|, \|\dot{\theta}\|\} \leq \delta$ and $\epsilon \in (0, \epsilon^*]$ then

$$\begin{aligned}
|x(t)| &\leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + d \\
|w(t/\epsilon)| &\leq \beta_w(|w(0)|, t/\epsilon) + d
\end{aligned}$$

for all $t \in [0, t_{max})$ which satisfy $\|y\|_{[0,t]} \leq \delta$ where t_{max} is the maximum time for which $x(t)$ is defined. Note γ_θ is as defined above in (4.12).

From this result, we know that so long as the measured voltage across the tunnel diode is small, that is $|v_1 - v_{01}| \leq v_{01}\delta$ then for a choice of components as to guarantee that $\epsilon = \frac{L}{C_2 R_1 R_2}$ is sufficiently small we have that the circuit is asymptotically stable up to a constant. This is a wonderful result as we have managed to guarantee the stability of three circuit variables from the measurement of one. Stability is of course a desired result as we do not want the circuit to cause the voltage and/or current to "blow up" and saturate or damage any other circuits whose inputs might be dependent on the the voltages or currents of the circuit of Figure 4.6. Of course, one runs into the problem that it is not a trivial matter to find the exact value of ϵ^* which will guarantee stability of the singularly perturbed system as is demonstrated in the proof of the theorem.

4.3 Numerical Example

If for the circuit of Figure 4.6 the values of R_1, R_2 were $400, 600\Omega$ respectively, C_1, C_2 being $10, 20\mu F$ and $L = 1mH$, then we would have $\epsilon = 1/24 = 0.042 << 1$.

Modeling the curve in Figure 4.7 as a cubic polynomial, a possible function would be:

$$\xi(v) = \begin{cases} 125v^3 - 80v^2 + 15v\,mA & v \geq 0 \\ 0 & v < 0 \end{cases}$$

With $E = 1V$ we would then have 3 equilibrium points for v_{01}: 0.1283V, 0.2V, 0.3117V. If we choose to normalize against $v_{01} = 0.3117V$ then we have $i_0 = 0.6883mA$ and $v_{02} = 0.7247V$. Solving for the constants in the normalized equations we get $a = i_0 R_1/v_{02} = 0.3799$, $b = i_0 R_2/v_{01} = 1.3249$ and $k = C_2/C_1 = 20/10 = 2$. The function η is

$$\eta(x_1) = \begin{cases} 7.286x_1^3 + 6.898x_1^2 + 0.9371x_1 + 1.582 \times 10^{-7} & x_1 \geq -1 \\ -1.324 & x_1 < -1 \end{cases}$$

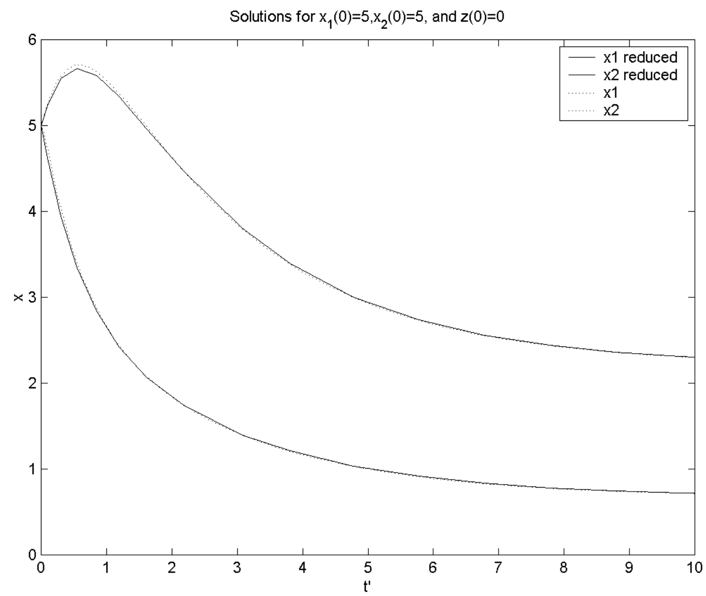


Figure 4.8: Comparison of trajectories of slow state variables vs. their reduced model counterparts

Suppose that the disturbances are sinusoids with the form

$$\begin{aligned} \theta_1(t) &= \sin(t) \\ \theta_2(t) &= \cos(t) \\ \theta_3(t) &= \sin(t + \pi/4) \end{aligned}$$

so that $\|\theta_i\|_{[0,t]} \leq 1$ and that $\|\dot{\theta}_i\|_{[0,t]} \leq 1$ for $i = 1, 2, 3$.

Plotting the graphs of the reduced system variables x_1 and x_2 from (4.9) and (4.10) against their counterparts from the actual system of (4.5) and (4.6) for $\epsilon = 1/24$ we see as shown in Figure 4.8 that without any disturbances the actual system stays close to the (undisturbed) reduced system as predicted by perturbation theory.

Figure 4.9 shows that even under different initial conditions where $z_0 \neq h(x_0, 0)$, the undisturbed boundary system converges towards its zero (that is $z = h(x, 0)$) as expected by perturbation theory.

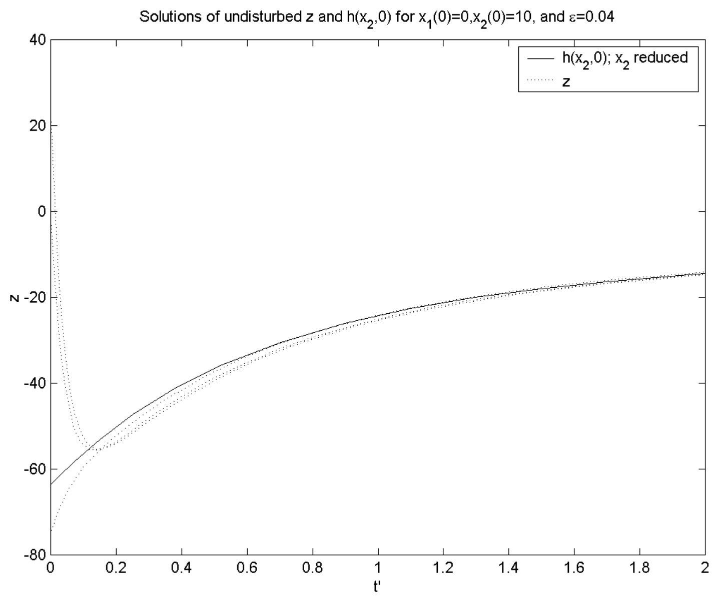


Figure 4.9: Different boundary state trajectories z compared with the zero trajectory z_s

From the theoretical discussion above (4.12), for the reduced system, substituting the numerical values we have $r = R_1/(R_1 + R_2) = 400/(400 + 600) = 0.4$ so $k_r = k/r = 2/0.4 = 5$ and the bounds for the reduced system are

$$\beta(s, t) = se^{-k_r t} = se^{-5t}$$

$$\begin{aligned}
\gamma_\theta(s) &= \frac{1}{k_r}(1 + \frac{kb}{1+b})s \\
&= 0.2(1 + \frac{2 \times 1.3249}{1 + 1.3249})s \\
&= 0.428s
\end{aligned}$$

$$\gamma_y(s) = \left(\frac{r}{1+b} + 1\right) s = (0.4/(1 + 1.3249) + 1)s = 1.17s$$

As a result of Claim 2 in the proof of the theorem, it is given that the (x_1, x_2) system is UO for $\epsilon < \epsilon^*$, $|w(0)| < \delta$ and as such there exits a $\mathcal{K}$ function $\chi(s)$ (which depends on the bound on

θ) such that:

$$|x(t)| \leq \chi(t) + \chi(|x(0)|) + \chi(\|y\|_{[0,t]}) \qquad \text{over some time interval } [0,t)$$

Since it is very difficult to find such a χ as we cannot explicitly solve the nonlinear system, we will make certain approximations to try to come up with one. First, assume $w = 0$ and $\epsilon = 0$ to reduce the full system to the reduced system. The bound on x is then the one of the reduced system which is

$$|x(t)| \leq |x(0)|e^{-5t} + 0.428\|\theta\|_{[0,t]} + 1.17\|y\|_{[0,t]}.$$

When there is no disturbance we have as bound

$$|x(t)| \leq |x(0)|e^{-5t} + 1.17\|y\|_{[0,t]}.$$

If we assume that the system is at equilibrium ($x = 0$), then under a sinusoidal input (which is what we have here as $\theta = (\sin(t), \cos(t), \sin(t + \pi/4))$ a linear system will have as forced response a sinusoidal output of the same frequency. If we assume that this nonlinear system behaves in the same way, then the state might be bounded by:

$$|x(t)| \leq 2\sin(t), \qquad \text{for } |x(0)| = 0$$

If we assume superposition, then the state would be bounded by

$$\begin{aligned} |x(t)| &\leq |x(0)|e^{-5t} + 2\sin(t) + 1.17\|y\|_{[0,t]} \\ &\leq |x(0)| + 2\sin(t) + 1.17\|y\|_{[0,t]} \end{aligned}$$

If we wish the state to be bounded by a function χ of class $\mathcal{K}$ such that

$$|x(t)| \leq \chi(|x(0)|) + \chi(t) + \chi(\|y\|_{[0,t]})$$

then a possible choice would be

$$\chi(s) = \begin{cases} 2\sin s & 0 < s < \pi/2 \\ 2 + 1.17(s - \pi/2) & s \geq \pi/2 \end{cases}$$

Then assuming that this choice of $\chi(\cdot)$ function satisfies the bound

$$|x(t)| \leq \chi(|x(0)|) + \chi(t) + \chi(\|y\|_{[0,t]}) \quad \forall t \in [0, t_{max})$$

then taking the result from the previous section, we have that given a pair of positive real numbers (δ, d) there is an $\epsilon^* > 0$ such that if $\max\{|x(0)|, |w(0)|, \|\theta\|, \|\dot{\theta}\|\} \leq \delta$ and $\epsilon \in (0, \epsilon^*]$,then

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + d \tag{4.14}$$

$$|w(t/\epsilon)| \leq \beta_w(|w(0)|, t/\epsilon) + d \tag{4.15}$$

for all $t \in [0, t_{max})$ which satisfy $\|y\|_{[0,t]} \leq \delta$ where t_{max} is the maximum time for which $x(t)$ is defined.

Working out the gains we have $\beta_x(s,t) = 2\chi(s)e^{-5t}, \gamma_\theta(s) = 0.428s, \bar{\gamma}_y(s) = 1.17s + 2\chi(s)$. We do not know what the $\mathcal{KL}$ function $\beta_w(s,t)$ is. We merely know that $\beta_w(s,t)$ dominates $se^{-t/\epsilon}$, which is the response of the boundary layer system.

The following analysis with the choice of $\delta = 10$ and $d = 1$, demonstrates that the state are indeed bounded as stated by the theorem.

Due to causality, it is also possible to restart the bound at a new initial time t_0. That is for the bound

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + d$$

then at time $t = t_0$, the system still satisfies the bound so restarting the system with new initial conditions $x_0 = x(t_0)$ gives as bound in terms of t_0:

$$|x(t)| \leq \beta_x(|x(t_0)|, t - t_0) + \gamma_\theta(\|\theta\|_{[t_0,t]}) + \bar{\gamma}_y(\|y\|_{[t_0,t]}) + d \quad \forall t \in [t_0, t_{max})$$

If we do this for every time instance with regards to the output, this has the effect of looking merely at the instantaneous value of $|y(t)|$ as opposed to $\|y\|_{[0,t]}$. This gives as bound on the state

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[t_0,t]}) + \bar{\gamma}_y(|y(t)|) + d \quad \forall t \in [0, t_{max})$$

As shown in Figure 4.10, we first observe that if we turn off the disturbances that for different initial conditions that the solution is indeed below both the bound, and in fact is also the bound using $|y(t)|$ as opposed to $\|y\|_{[0,t]}$.

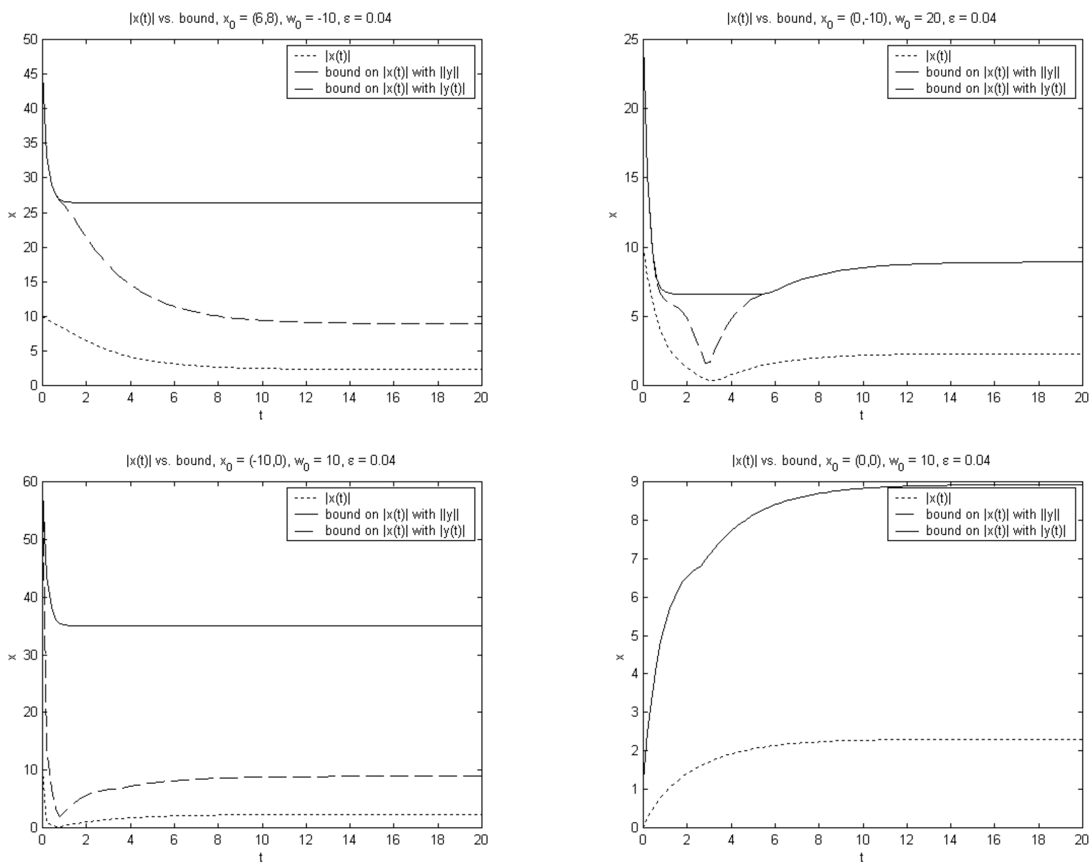


Figure 4.10: Comparing the size of x trajectories with upper bound under different initial conditions without disturbances

Indeed, even after turning on the disturbances, the solutions for those four particular initial conditions are still bounded by both the bound that uses $|y(t)|$ and the one that uses $\|y\|_{[0,t]}$ as demonstrated below in Figure 4.11. This of course does not guarantee that the bounds will hold for all initial conditions with $\max\{|x(0)|, |w(0)|\} < \delta = 10$.

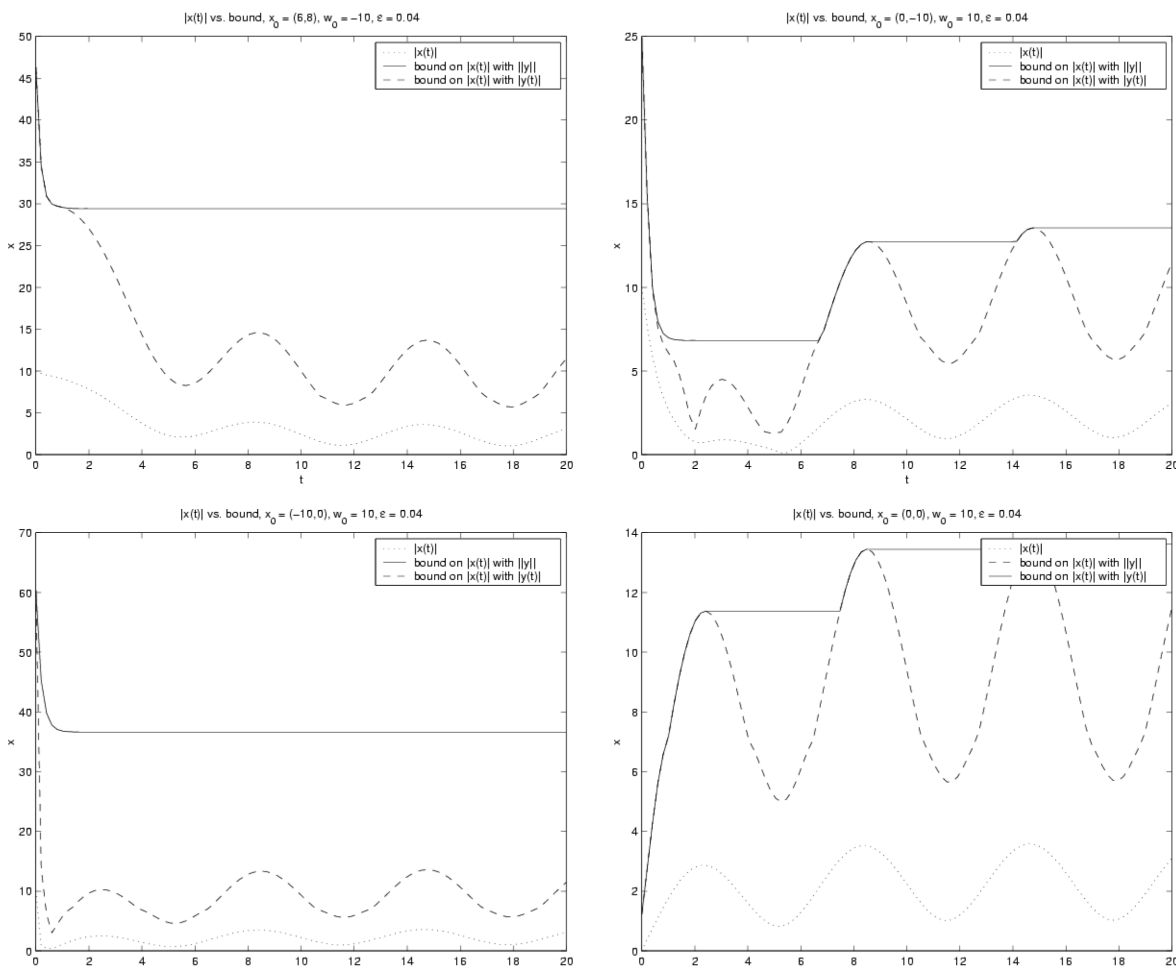


Figure 4.11: Comparing the size of x trajectories with upper bound under different initial conditions

Note that the size of the state ($|x(t)|$) is closest to the bound in the case for which $x_1(0) = 0$.

This makes sense as the output $y = x_1$ so for larger x_1 we have a larger output hence a larger upper bound.

Figure 4.12 shows that the system transformed into the w coordinates does indeed asymptotically tend to some value less than our choice of $d = 1$. Hence for a particular initial condition we have that the bound (4.15) holds for some β_w

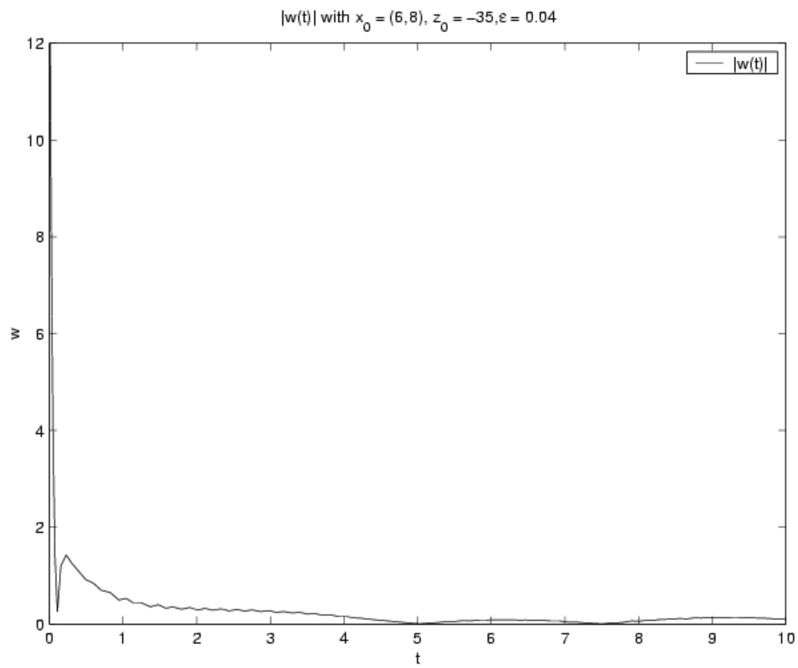


Figure 4.12: w trajectory for a particular initial condition

Finally, if we use larger values for ϵ with the most restrictive bound $x_1(0) = 0$ as shown in Figure 4.13 we notice that for $\epsilon = 10$ the bound using the instantaneous value of the output $|y(t)|$ does not hold for all times due to the fact that the bound decays much faster than the actual trajectory and that the output $y = x_1$ has no knowledge of this.

It is only when choosing a much larger value of $\epsilon = 1000$ that the bound on the state using the essential supremum of the output $\|y\|_{[0,t]}$ fails and only for small amounts of time when the state reaches its peak.

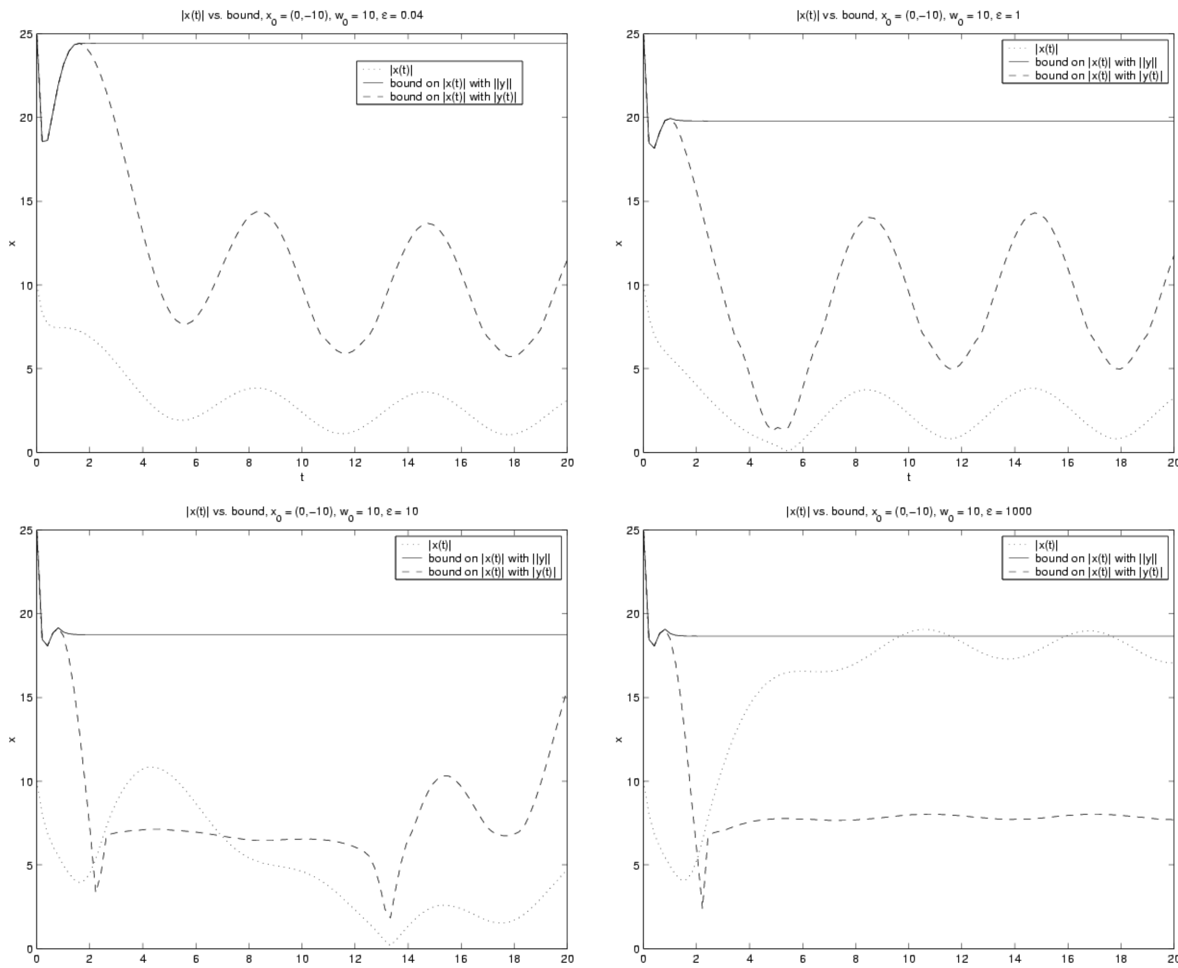


Figure 4.13: Comparing the size of x trajectories with upper bound under different values of ϵ

As a final note, one should mention that though the construction of the theorem was used to find the $\mathcal{K}$ and $\mathcal{KL}$ functions for the bound on $|x(t)|$, it is quite impractical to find the value of ϵ^*, the largest value of ϵ for which the bounds would hold, as it involves finding the decreasing functions $0 < \sigma(s) < 1$ and taking smaller and smaller values of ϵ until one of the terms in the bound which appears during the proof of the theorem is less than $d/2$. The difficulties outlined in finding the function χ give a minor indication of how hard it would be to calculate the value of ϵ^*. This is why it was not calculated for this numerical example.

Chapter 5

Conclusion

Throughout this thesis we have seen that even with limited information it is still possible to gain some insight into the behaviour of a system.

For nonlinear systems, it is a much more difficult task than for linear systems and a particular form of detectability called IOSS was introduced. Using this concept it was possible to prove that for a singularly perturbed system whose reduced system had the IOSS property, under certain assumptions the perturbed system still maintained a detectability property.

Using techniques similar to the ones in the proof, it might also be possible to prove other stability results with regards to singularly perturbed systems. One such possibility would be to look at a weaker form of Input to State Stability called integral-Input to State Stability (iISS) defined by the existence of $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that the state $x(t)$ satisfies

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \gamma(|u(s)|)\, ds.$$

In the case that $\gamma(|u|) = |u|^2$ and $\alpha(|x|) = a|x|^2$ this is the same as looking at $\mathcal{L}_2$-to-$\mathcal{L}_2$ stability. We can therefore think of iISS as a generalization of $\mathcal{L}_2$-to-$\mathcal{L}_2$ to nonlinear systems in the same way that ISS is a generalization of $\mathcal{L}_\infty$-to-$\mathcal{L}_2$ stability for linear systems.

Another possibility would be to look at a form of external stability called Input to Output

Stability (IOS) for which the system with outputs

$$\dot{x} = f(x,u) \quad y = k(x) \tag{5.1}$$

has outputs which satisfy the bound

$$|y(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_{[0,t]})$$

for all solutions (for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$).

Both these concepts have Lyapunov-like characterizations so it should be possible to prove an equivalent version of Lemma 3.2. However, for the IOS case it would be impossible to prove an equivalent version of Lemma 3.3 as the proof relies on using the fact that the state is bounded up to some time T of which we have no knowledge. If however we required that system (5.1) had the bounded-input bounded-state property then it might be possible to recover this result.

It should also be noted that though it should be possible to extract the greatest value of ϵ^* for which the result holds, in practice it is quite difficult to find this value as it requires us to find certain bounding functions and to evaluate them at certain points. It would be much better from a design standpoint if the value of ϵ^* could be found by some simple algebraic formula as this way it could easily be determined up to what value of ϵ our approximation is valid.

In the end, we were able to show that for singularly perturbed systems, it is possible to recover a detectability property given that the reduced system is IOSS. For complex systems, after proper rescaling, if we can treat one of the "small" parameters as a perturbation parameter, setting it to zero, it might be possible to analyze this simpler, reduced system. If it can be shown that the reduced system has the IOSS property and hence is detectable, then the theorem, though limited in its practical use, enables us to know that the more complex system is also detectable (being IOSpS) for small enough values of the perturbation parameter. Hence it is possible to determine if a very complex system is stable by first reducing the order of the system, then analyzing the stability of the reduced system by observing enough variables so that the reduced system is detectable. This is a tremendous cost and time savings as it is not necessary to measure all the state variables of the system and so serves as an invaluable tool when dealing with large systems.

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