THE CYCLE SPACES OF AN INFINITE GRAPH

by

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AUTHOR'S DECLARATION FOR ELECTRONIC SUBMISSION OF A THESIS

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The edge space of a finite graph $G = (V, E)$ over a field $\mathbb{F}$ is simply an assignment of field elements to the edges of the graph. The edge space can equally be thought of us an $|E|$-dimensional vector space over $\mathbb{F}$. The cycle space and bond space are the subspaces of the edge space generated by the cycle and bonds of the graph respectively. It is easy to prove that the cycle space and bond space are orthogonal complements.

Unfortunately many of the basic results in finite dimensional vector spaces no longer hold in infinite dimensions. Therefore extending the cycle and bond spaces to infinite graphs is not at all a trivial exercise.

This thesis is mainly concerned with the algebraic properties of the cycle and bond spaces of a locally finite, infinite graph. Our approach is to first topologize and then compactify the graph. This allows us to enrich the set of cycles to include infinite cycles. We introduce two cycle spaces and three bond spaces of a locally finite graph and determine the orthogonality relations between them. We also determine the sum of two of these spaces, and derive a version of the Edge Tripartition Theorem.

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Chapter 1

Introduction

1.1 An Outline

Graph theory has always been closely linked with topology and algebra. Perhaps the best illustration of this point is in the theory of cycle spaces. Identify a subgraph of a graph $G = (V, E)$ with its characteristic vector in an $|E|$-dimensional vector space over $\mathbb{GF}(2)$. The space of all such vectors is called the *edge space* of G. One can then look at subspaces of the edge space generated by certain classes of subgraphs.

The most useful to this point is the edge space generated by the cycles of G, which is called the *cycle space* of G. The cycle space has been mainly used in determining planarity conditions of finite graphs. For example, Mac Lane (1937, see [5]) determined that a graph is planar if and only if the cycle space has a *simple* generating set (a set such that any edge appears in at most 2 members of the set). Tutte (1963, see [5]) later used cycle spaces to show that a 3-connected graph is planar if and only if each edge lies in at most 2 *peripheral* cycles (the induced, non-separating cycles).

One can also rephrase the Four Colour Theorem in terms of cycle spaces: If G is planar and bridgeless, then G is the union of two elements of the cycle space.

The edge space can be generalized to be taken over arbitrary fields. The members of the cycle space are then precisely the circulations (or flows) of the graph, a topic which has far ranging applications in combinatorial optimization and graph theory itself.

Recently Diestel and Kühn [6, 7], answering a question of Richter, developed a theory of cycle spaces in infinite graphs. Their approach begins by topologizing a graph by viewing the graph as a simplicial 1-complex. In the locally finite case they then compactify the graph (using the Freudenthal compactification) and view a cycle as any image of a circle (i.e. S^1). This introduces the possibility of "infinite" cycles.

Their approach, while successful in generalizing some of the planarity results to infinite graphs, also gives counterintuitive results. For example there may exist "cycles" contained in the set of vertices and certain infinite paths may be admitted as cycles.

Richter and Vella [11] approached the cycle space in a slightly different way. They give a graph a topology which is not Hausdorff but nevertheless allows for a combinatorial description of infinite cycles and trees. This approach provides a more intuitive view of the cycle and bond spaces of a graph and is the starting point of this thesis.

We study the edge space of locally finite, infinite graphs over arbitrary fields. We begin by defining the topology of Richter and Vella on the graph. We then briefly study compactifications of graphs before turning toward the cycle space. In fact in turns out that we can well define two cycle spaces and three bond spaces and determine all orthogonality relations between them. Since we are working in infinite dimensions, many of the standard results in linear algebra may not be applicable to these spaces and so these results are not always trivial.

We are also able to derive a tripartition of the edges of a locally finite, infinite graph which generalizes a result of finite graphs.

This thesis is organized as follows. The remainder of this chapter gives a brief introduction to infinite graphs. The second chapter serves as a reference for the theory of cycle spaces of finite graphs. The third chapter presents a discussion of infinite dimensional vector spaces and so is vital to a study of the edge space of an infinite graph. The fourth chapter defines the topology we are using and how we can apply it to locally finite, infinite graphs. We also examine the set of compactifications of a graph, which turns out to be partially ordered. The fifth chapter defines what we mean by cycle and bond spaces and proves the orthogonality relations between them. Finally we end with some brief thoughts on the difficulty of extending our results to more general infinite graphs.

1.2 Infinite Graphs

This section introduces some of the basic concepts of infinite graphs. Since many of the concepts of finite graphs are exactly the same for infinite graphs, we only state those which need modification or clarification. It should be assumed that any graph theoretic term which is not defined here is the same as for finite graphs.

An infinite graph $G = (V, E)$ is a graph defined in the usual way but where V is an infinite set. A graph is *locally finite* if every vertex has finite degree. A graph is (graph theoretically) *connected* if there exists a path of finite length between any two vertices.

Proposition 1.1 *Every connected graph in which every vertex has countable degree has countably many vertices.*

Proof. Let v be an arbitrary vertex. Then the set of vertices V_k at distance k is a countable set. Since every vertex of the graph is in V_k for some k it follows that $|V(G)|$ is countable, being the countable union of countable sets. ∎

A *ray* $R = (v_0, v_1, \ldots)$ is a sequence of infinitely many distinct vertices such that v_i is adjacent to v_{i+1} for all $i \geq 0$. A ray is also called a *1-way infinite path*. A *double ray* or a *2-way infinite path* consists of two rays $R = (v_0, v_1, \ldots)$ and $R' = (v_0, v_1', \ldots)$ such that $R \cap R' = \{v_0\}$. A *tail* of a ray R is simply a subray of R.

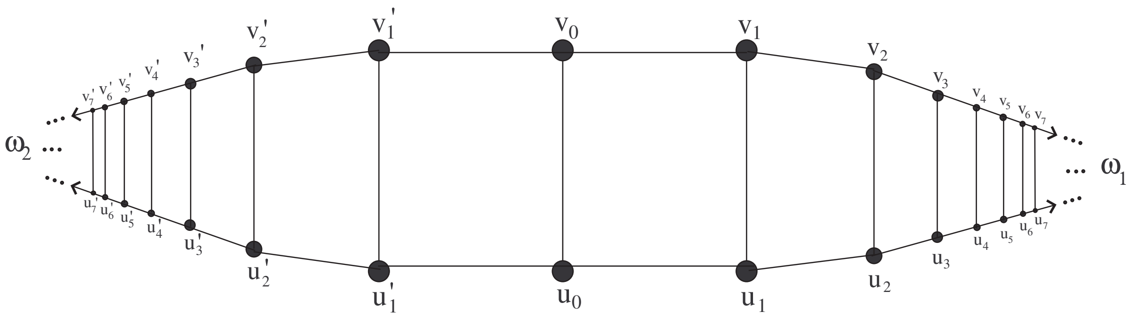

Figure 1.1: A 2-way Infinite Ladder

An instructive example of a locally finite infinite graph is the 2-way infinite ladder shown in Figure 1.1. The sequence $R = (v_0, v_1 \ldots)$ is a ray while $R' = (\ldots, v_2', v_1', v_0, v_1, v_2, \ldots)$ is a double ray.

Two rays are said to be *equivalent* if they cannot be separated by a finite set of vertices. This definition gives an equivalence relation on the set of rays and we call each equivalence class an *end*. In Figure 1.1, the rays $(u_0, u_1, u_2, \ldots)$ and $(u_1, v_1, v_2, u_2, \ldots, u_i, v_i, v_{i+1}, u_{i+1}, \ldots)$ are equivalent to R and so are in the same end, which is there denoted by ω_1. One may think of an end as an extra "point at infinity".

A graph together with its set of ends is the *Freudenthal compactification* of the graph. We will discuss compactifications further in Chapter 4.

A basic result that we will need is the following (see [5]):

Lemma 1.2 (König's Infinity Lemma) *Let G be an infinite graph and let $V_0, V_1, \ldots$ be a sequence of disjoint non-empty finite vertex sets that partition $V(G)$. Suppose that for each V_i with $i \geq 1$ every $v \in V_i$ has a neighbour $f(v)$ in V_{i-1}. Then G contains a ray $\{v_0, v_1, \ldots\}$ with $v_i \in V_i$ for all $i \geq 0$.*

Proof. Let $\mathcal{P}$ be the set of all finite paths of the form $v, f(v), f(f(v)), \ldots$ that end in V_0. Since $\mathcal{P}$ is infinite and V_0 is finite, there must be a subset $\mathcal{P}_0$ of $\mathcal{P}$ consisting of infinitely many paths ending at the same vertex in $v_0 \in V_0$. Of the paths in $\mathcal{P}_0$

there is a subset $\mathcal{P}_1$ consisting of infinitely many paths that go through the same vertex $v_1 \in V_1$. Continuing in this way we get a sequence $v_0, v_1, \ldots$ which is infinite by virtue of the fact that at the i^{th} step we have an infinite set of paths beginning with $v_0, v_1, \ldots v_i$. $\blacksquare$

We have only given an extremely terse introduction to infinite graph theory. The subject is in fact rich with deep results and we refer the reader to [5] for an excellent introduction to the subject.

Chapter 2

The Cycle Space of a Finite Graph

The material presented in this chapter is mainly based on Bondy and Murty [2],Godsil and Royle [10] and Rosenstiehl and Read [13]. See Biggs [1] for a more advanced discussion. The definitions and theorems are presented here solely as a reminder and a reference to the reader.

2.1 The Basics

In this section we introduce the edge space and the cycle and bond spaces of a finite graph.

Let $\mathbb{F}$ be a field and let $G = (V, E)$ be a connected finite graph with an arbitrary orientation on its edges. The set of functions $\{f : E(G) \to \mathbb{F}\}$ is called the *edge space* of G over $\mathbb{F}$. Since the edge space is isomorphic as a vector space to the vector space $\mathbb{F}^{|E|}$, we denote the edge space simply by $\mathbb{F}^E$. We always equip the edge space with a vector product defined by $u^T v = \sum_{e \in E} u(e)v(e)$ for $u, v \in \mathbb{F}^E$.

Note that although in general the spaces we will introduce in this section differ slightly depending on the given orientation of the edges, the results all hold independently of the orientation. Therefore when we talk about, for example, the "cycle space" we mean the cycle space with respect to the given orientation.

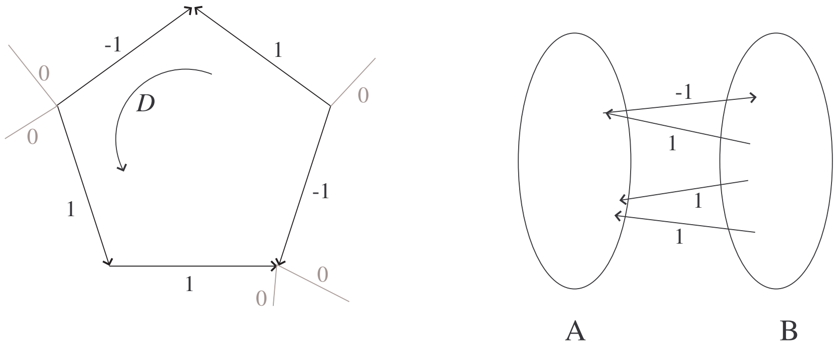

Figure 2.1: A cycle and a bond labelled by their corresponding edge space elements.

For each cycle C in G we may define an element c of the edge space (which we also call a cycle) in the following way. First pick a direction of traversal D of C. Then

$$c(e) = \begin{cases} 1 & \text{if } e \in C \text{ and the orientation of } e \text{ agrees with } D \\ -1 & \text{if } e \in C \text{ and the orientation of } e \text{ disagrees with } D \\ 0 & \text{if } e \notin C. \end{cases}$$

An example of a cycle and its corresponding element in the edge space is given on the left of Figure 2.1.

The subspace $\mathcal{Z}(G)$ of $\mathbb{F}^E$ generated by all linear combinations of cycles is called the *cycle space* of G.

Given a spanning tree of G it is clear that any edge $e \notin T$ generates a unique cycle in the subgraph $T \cup \{e\}$, called a *fundamental cycle* of G. The corresponding edge space cycle, where we adopt the convention that the direction of traversal always agrees with the orientation of e, is also called a fundamental cycle.

Dual to the notion of the cycle space is the *bond space* (also called the *cut space*). Let (A, B) be a partition of $V(G)$. Let $\delta(A)$ be the set of edges with one end in A and the other end in B. We define an element b of $\mathbb{F}^E$ (which we also call a bond, or a cut, with vertex partition (A, B)) by

$$b(e) = \begin{cases} 1 & \text{if } e \in \delta(A) \text{ and if the head of } e \text{ is in } A \\ -1 & \text{if } e \in \delta(A) \text{ and if the tail of } e \text{ is in } A \\ 0 & \text{if } e \notin \delta(A). \end{cases}$$

An example of a bond and its corresponding element in the edge space is given on the right of Figure 2.1. The bond space $\mathcal{B}(G)$ is the set of all linear combinations of bonds of G. Let T be a spanning tree of G and let $e \in T$. Then $T \setminus \{e\}$ partitions $V(G)$ into two parts A and B. The corresponding cut $\delta(A)$ is called a *fundamental bond* of G. Note that e is the unique tree edge in the fundamental bond.

To simplify notation somewhat, we will usually denote the cycle and bond spaces of a graph G by $\mathcal{Z}$ and $\mathcal{B}$ if no confusion occurs as to which graph G is under discussion.

We now take a slightly different viewpoint of the cycle and bond spaces.

A *circulation* is an element c of the edge space such that for each vertex in G we have

$$\sum_{e \in \delta^{\text{in}}(v)} c(e) \quad = \quad \sum_{e \in \delta^{\text{out}}(v)} c(e) \tag{2.1}$$

where $\delta^{\text{in}}(v)$ (respectively $\delta^{\text{out}}(v)$) is the set of edges whose head (resp. tail) is v. In other words, for each vertex the "flow" in equals the "flow" out. Note that in general if c_1 and c_2 are circulations then so is $c_1 + c_2$.

Theorem 2.1 *The set of circulations of a connected finite graph G is precisely its cycle space.*

Proof. Let $\mathcal{C}$ denote the set of circulations. Since each cycle is a circulation by definition of cycle, we have $\mathcal{Z} \subseteq \mathcal{C}$.

Let f be a circulation. Let T be a spanning tree of G. Consider the circulation $z = f - \sum_{e \notin T} f(e) Z_{(T,e)}$ where $Z_{(T,e)}$ is the fundamental cycle corresponding to an edge $e \notin T$. Since there exists a unique fundamental cycle corresponding to e it follows that the support of z (the set of edges for which z has a non-zero value) is contained in T.

We claim that there can be no circulation with a non-zero edge value on an acyclic graph. This is easily seen by noting that every acyclic graph H has a vertex v with degree 1. Then $\delta^{\text{in}}(v) + \delta^{\text{out}}(v) = 1$ implies that the edge incident with v has value zero in any circulation of H. The graph $H \setminus \{e\}$ is still acyclic and so one can apply an inductive argument to show that every edge must have value zero in any circulation of H.

Therefore we have $z \equiv 0$ and so $f = \sum_{e \notin T} f(e) Z_{(T,e)}$. Hence $\mathcal{C} \subseteq \mathcal{Z}$. ∎

Given a subspace $\mathcal{U}$ of $\mathbb{F}^E$, the *orthogonal complement* $\mathcal{U}^{\perp}$ is defined as the set of vectors u such that $u^T v = 0$ for all $v \in \mathcal{U}$.

Theorem 2.2 *Let G be a finite connected graph. Then $\mathcal{Z} = \mathcal{B}^{\perp}$.*

Proof. We first show that any cycle is orthogonal to every bond. So let c be a cycle of G and let b be a bond of G. It is easy to see that c and b must have an even number of edges in common, say $e_1, e_2, \ldots, e_{2k}$ where we have ordered the edges so that in a traversal of c, e_{i+1} is the first edge encountered after e_i for every i. Suppose that the orientations of both e_i and e_{i+1} agree with respect to the traversal of c, i.e. we have $c(e_i) = c(e_{i+1})$. Then we must have $b(e_i) = -b(e_{i+1})$. Thus $c(e_i)b(e_i) + c(e_{i+1})b(e_{i+1}) = 0$. On the other hand if we have $c(e_i) = -c(e_{i+1})$ then we must have $b(e_i) = b(e_{i+1})$ and so again $c(e_i)b(e_i) + c(e_{i+1})b(e_{i+1}) = 0$. It follows that $z^T b = 0$ and so $\mathcal{Z} \subseteq \mathcal{B}^{\perp}$.

On the other hand, let $z \in \mathcal{B}^{\perp}$. Then in particular for each bond $\delta(v)$ with vertex partition $(v, V(G) \setminus \{v\})$ we have that Equation 2.1 is satisfied by z. Hence $z \in \mathcal{Z}$ and the result now follows. ∎

Another way to look at the bond space is to assign *potentials* (elements of $\mathbb{F}$) to every vertex. That is, let $q : V(G) \to \mathbb{F}$ be a function on the vertices of G. The *potential difference* of q is the function $p : E(G) \to \mathbb{F}$ defined by $p(vu) = q(v) - q(u)$ where v is the head of the edge vu.

Another point of view is to consider a potential difference as a linear combination of vertex cuts (in the edge space), where a vertex cut $\delta(v)$ of $v \in V(G)$ is the cut with vertex partition $(v, V(G) \setminus \{v\})$. Here, we take the potential at vertex v to be the coefficient of $\delta(v)$ in the linear combination.

Theorem 2.3 *Let G be a connected finite graph. Then the bond space of G is the set of all potential differences.*

Proof. Let $\mathcal{P}$ be the set of all potential differences. Then as noted above, $\mathcal{P} \subseteq \mathcal{B}$. On the other hand let $\delta(A)$ be a bond whose vertex partition is (A, B). Define the potential q by $q(v) = 1$ for all $v \in A$ and $q(v) = 0$ for all $v \in B$. Then $\delta(X)$ is the potential difference of q since every edge not in $\delta(X)$ has potential difference zero. Thus $b \in \mathcal{P}$ and hence we see that $\mathcal{B} \subseteq \mathcal{P}$. ∎

Therefore we have that the set of vertex cuts generates the bond space.

Fix a vertex $v_0 \in V(G)$. Define a potential q by $q(v_0) = 1$ and $q(v) = 0$ for $v \neq v_0$. Define another potential q' by $q'(v_0) = 0$ and $q'(v) = 1$ for $v \neq v_0$. If p and p' are the potential differences of q and q' respectively then it is easy to see that $p(e) = -p'(e)$ for every $e \in E(G)$. Since p is simply the vertex cut $\delta(v_0)$ and p is a linear combination of vertex cuts of vertices in $V(G) \setminus \{v\}$ it follows that $\delta(v_0)$ is linearly dependent on the set $\{\delta(v) \mid v \in V(G) \setminus \{v\}\}$. Therefore $\dim(B) \leq |V(G)| - 1$.

Another description of the cycle and bond spaces is via the incidence matrix. Recall that the incidence matrix of an oriented graph $G = (V, E)$ is the $|V| \times |E|$ matrix D, rows indexed by the vertices and columns by the edges, defined as

$$D[v, e] = \begin{cases} 1 & \text{if vertex } v \text{ is the head of edge } e \\ -1 & \text{if vertex } v \text{ is the tail of edge } e \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.3 is then simply saying that the bond space of a graph is the row space of D and the cycle space is the null space of its incidence matrix.

Now let G be a connected graph and let T be any spanning tree of G. We noted above that there exists a unique fundamental bond for every $e \in T$. Hence the set of $|V(G)| - 1$ fundamental bonds is linearly independent, and therefore a basis of $\mathcal{B}$ and we have $\dim(B) = |V(G)| - 1$. Furthermore, by the rank-nullity theorem, we get $\dim Z = |E(G)| - |V(G)| + 1$. Since each edge $e \notin T$ is in a unique fundamental cycle it follows that the set of $|E(G)| - |V(G)| + 1$ fundamental cycles is linearly independent and thus forms a basis for $\mathcal{Z}$.

As a final note for this section, although we have mainly considered connected finite graphs, all the results extend in a straightforward manner to arbitrary finite graphs. In this case, the dimension of the bond space of a graph G is $|V(G)| - c$ where c is the number of components of G and the dimension of the cycle space is $|E(G)| - |V(G)| + c$.

2.2 The Edge Tripartition Theorem

In this section we consider the edge space over $\mathbb{GF}(2)$ of a finite graph G . In this case the orientation on the edges is redundant and so we need not bother introducing one. An element of the edge space corresponds to a subgraph of G. In particular the cycle space is simply the set of all even subgraphs (i.e. subgraphs where each vertex has even degree). Addition of two elements of the edge space is simply the symmetric difference of the two subgraphs.

Now we know from linear algebra that since $\mathcal{Z} = \mathcal{B}^{\perp}$ we have $\mathcal{Z} + \mathcal{B} = (\mathcal{Z} \cap \mathcal{B})^{\perp}$. The elements of $\mathcal{Z} \cap \mathcal{B}$ are called *bicycles*.

A nice consequence of the above sum relation is as follows. Suppose B is a bicycle of G. Then $B^T b = 0$ for any cut b. Thus B induces an even subgraph of G. Now consider the set of edges $E \in \mathbb{GF}(2)^E$ (E is the all 1's vector). Then $E^T B = 0$ since

B is even and thus $E \in (\mathcal{Z} \cap \mathcal{B})^{\perp}$. i.e. $E = z + b$ for some even subgraph z and some cut b. In other words:

Corollary 2.4 *Let G be a finite graph. Then there exists a cut b of G with vertex partition (A, B) such that the two subgraphs induced by A and B are even (we allow one of A or B to be empty).*

Let e denote both an edge in $E(G)$ and its corresponding characteristic vector in $\mathbb{GF}(2)^E$. Suppose e is contained in some bicycle B. Then $e^T B \neq 0$ and so $e \notin (\mathcal{Z} \cap \mathcal{B})^{\perp} = \mathcal{Z} + \mathcal{B}$. On the other hand, if e is not contained in any bicycle, then $e^T B = 0$ for every bicycle B. Hence $e \in (\mathcal{Z} \cap \mathcal{B})^{\perp} = \mathcal{Z} + \mathcal{B}$.

Now suppose we are in the latter case and assume that $e = z + b = z' + b'$ where $z, z' \in \mathcal{Z}$ and $b, b' \in \mathcal{B}$. Then $z + z' = b + b'$ and so $z + z'$ is a bicycle. Now e does not lie in a bicycle so it follows that either e lies in both z and z' (and not in b and b') or e lies in both b and b' (and not in z and z'). The point is that in every representation of e as the symmetric difference of a cut and an even subgraph, e either always lies in the cut or always lies in the even subgraph.

We have just proved the following theorem (see [13]).

Theorem 2.5 (Edge Tripartition Theorem) *Let G be a finite graph. Then for every edge e exactly one of the following three situations holds for e:*

1. *e is contained in some bicycle,*

2. *There exists a cut b such that $b + e$ is an even subgraph, or*

3. *There exists an even subgraph H such that $H + e$ is a cut.* ∎

The Edge Tripartition Theorem has found use in knot theory. The original paper in fact was used to solve an old conjecture of Gauss concerning the sequence of crossing points in a closed curve. The bicycle space can also be used to derive certain parity results. For example let $t(G)$ denote the number of spanning trees of G. Chen [3] (for

$char(\mathbb{F}) = 2$) and Shank [15] (for general finite fields with characteristic p) showed that there exists a bicycle on G if and only if p divides $t(G)$.

Chapter 3

On Infinite Dimensional Vector Spaces

The first section of this chapter is a discussion of some of the differences between finite and infinite dimensional vector spaces. We also introduce the thin span of an infinite dimensional vector space which will play an important role in later chapters. Section 3.2 introduces a new type of vector product that we will need in order to discuss orthogonality of subspaces. Section 3.3 is devoted to one of the main results of this thesis. Here we generalize to certain infinite dimensional spaces $\mathcal{U}$ and $\mathcal{V}$, the orthogonality relation $\mathcal{U} + \mathcal{V} = (\mathcal{U}^{\perp} \cap \mathcal{V}^{\perp})^{\perp}$, which holds for finite dimensional spaces but fails for general infinite dimensional spaces.

3.1 Finite vs. Infinite Dimensional Vector Spaces

The core of this thesis will be a study of the basic properties of the cycle space of a locally finite infinite graph. The difficulty in extending the finite theory is that many of the fundamental properties for finite dimensional spaces no longer hold in infinite dimensions.

Let $\mathcal{W}$ be a vector space (finite or infinite dimensional) over a field $\mathbb{F}$. We equip

$\mathcal{W}$ with a notion of a vector product $\langle \cdot, \cdot \rangle : \mathcal{W} \times \mathcal{W} \to \mathbb{F}$ (a *symmetric bilinear form*) that satisfies the following:

- $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{W}$,
- $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in \mathcal{W}$,
- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for all $x, y \in \mathcal{W}, \lambda \in \mathbb{F}$.

For example the vector product we used with the edge space in Chapter 2 is a symmetric bilinear form. Two vectors in $\mathcal{W}$ are said to be *orthogonal* if their product is zero. The *orthogonal complement* of a subspace $\mathcal{U}$ is the set $\mathcal{U}^\perp$ of all vectors in $\mathcal{W}$ that are orthogonal to every vector in $\mathcal{U}$. Note that the term "complement" here is somewhat of a misnomer as in general we do not always have $\mathcal{U} + \mathcal{U}^\perp = \mathcal{W}$, where $\mathcal{U} + \mathcal{U}^\perp = \{u + v \mid u \in \mathcal{U},\, v \in \mathcal{U}^\perp\}$.

In any case, if $\mathcal{W}$ is **finite** dimensional the following properties always hold for subspaces $\mathcal{V}$ and $\mathcal{U}$ of $\mathcal{W}$:

Fact 1. $\mathcal{V}^{\perp^\perp} = \mathcal{V}$;

Fact 2. $\mathcal{V}^\perp = \mathcal{U}$ if and only if $\mathcal{V} = \mathcal{U}^\perp$;

Fact 3. $(\mathcal{V} + \mathcal{U})^\perp = \mathcal{V}^\perp \cap \mathcal{U}^\perp$;

Fact 4. $\mathcal{V}^\perp + \mathcal{U}^\perp = (\mathcal{V} \cap \mathcal{U})^\perp$;

Fact 5. If $\mathcal{U} = \mathcal{V}^\perp$ in (3) and (4), then the sum is a direct sum if and only if $\mathcal{U} \cap \mathcal{U}^\perp = \{0\}$.

Now suppose that $\mathcal{W}$ is an **infinite** dimensional vector space over a field $\mathbb{F}$.

It is easy to verify that $\mathcal{U} \subseteq \mathcal{U}^{\perp^\perp}$ for any subspace $\mathcal{U} \subseteq \mathcal{W}$. Much of the cause of our trouble (and the justification for this thesis!) is that the converse implication is not necessarily true. For example suppose that $\mathcal{W}$ is the set of all sequences in $\mathbb{R}$ such that the sequence $\{x_i\}_{i \in I}$ is in $\mathcal{W}$ if and only if the series $\sum_{i \in I} x_i$ is absolutely convergent.

Then we can define a symmetric bilinear form by $\langle \{x_i\}_{i\in I}, \{y_j\}_{j\in J}\rangle = \sum_{i\in I} x_i \sum_{j\in J} y_j$. Now if $\mathcal{U}$ is the subspace consisting of all finite sequences then clearly $\mathcal{U}^\perp = \{0\}$, but then $\mathcal{U}^{\perp^\perp} = \mathcal{W}$.

Now Facts (1) and (2) are easily seen to be equivalent, and thus (2) does not necessarily hold in infinite dimensions. Fact (3) in fact does hold in arbitrary dimensions.

Proposition 3.1 *Let $\mathcal{W}$ be a vector space equipped with a symmetric bilinear form $\langle \cdot, \cdot, \rangle$ and let $\mathcal{U}$ and $\mathcal{V}$ be subspaces of $\mathcal{W}$. Then*

$$(\mathcal{V} + \mathcal{U})^\perp = \mathcal{V}^\perp \cap \mathcal{U}^\perp.$$

Proof. If $x \in \mathcal{V}^\perp \cap \mathcal{U}^\perp$ then $\langle x, v + u\rangle = \langle x, v\rangle + \langle x, u\rangle = 0$. On the other hand if $x \in (\mathcal{V}+\mathcal{U})^\perp$ then $\langle x, v+u\rangle = 0$ for all $v \in V$ and $u \in \mathcal{U}$. In particular if we set $v = 0$ then $\langle x, u\rangle = 0$ for all $u \in \mathcal{U}$, i.e. $x \in \mathcal{U}^\perp$. Similarly $x \in \mathcal{V}^\perp$ and so $x \in \mathcal{U}^\perp \cap \mathcal{V}^\perp$. ∎

A proof of Fact (4) for the spaces that we consider in the sequel is one of the core results of this thesis. For this, we will need the following elementary result from linear algebra.

Lemma 3.2 (Fredholm's Theorem (see [14])) *Let $\mathbb{F}^m$ be the m-dimensional vector space over the field $\mathbb{F}$ equipped with the standard inner product. Let A be an $m \times n$ matrix with entries in $\mathbb{F}$ and let $b \in \mathbb{F}^m$. Then exactly one of the following holds:*

1. *There exists an $x \in \mathbb{F}^n$ such that $Ax = b$, or*
2. *There exists a $y \in \mathbb{F}^m$ such that $y^T A = 0$ but $y^T b \neq 0$.*

Proof. Suppose (1) and (2) hold simultaneously. Then $0 = y^T Ax = y^T b \neq 0$, a contradiction.

Now suppose that (1) does not hold, that is, there does not exist an x such that $Ax = b$. If C is the column space of A then this is equivalent to saying that $b \notin C = (C^\perp)^\perp$ so there exists a $y \in C^\perp$ such that $y^T b \neq 0$. But $y \in C^\perp$ implies that $y^T A = 0$. ∎

We need to expand somewhat our definition of a subspace. Consider the vector space $\mathbb{F}^\alpha$, where α is an ordinal number. We can think of $\mathbb{F}^\alpha$ either as the set of functions $\{f : \alpha \to \mathbb{F}\}$ or simply as the set of column vectors indexed by α. But if $u \in \mathbb{F}^\alpha$ then we write u_i instead of $u(i)$ for $i \in \alpha$. A set $U \subseteq \mathbb{F}^\alpha$ is called *thin* if for all $i \in \alpha$, there exists only finitely many $u \in U$ such that $u_i \neq 0$.

It makes sense to take infinite linear combinations of elements of distinct elements of a thin set U as each $i \in \alpha$ will appear in only finitely many terms. The *thin span* of a thin set U is the set of all linear combinations (finite or infinite) of distinct elements of U. A *thin subspace* $\mathcal{U}$ of $\mathbb{F}^\alpha$ is the thin span of a thin set $U \subset \mathbb{F}^\alpha$.

3.2 A Quasi-Bilinear Form

Some of the basic theorems for cycle spaces of finite graphs are concerned with orthogonality relations, e.g. the standard inner product of a cycle and a cut is always zero where we view a cycle and a cut as elements in certain finite dimensional subspaces. Since we wish to generalize these results to certain infinite dimensional spaces over arbitrary fields it is necessary to define a vector product for these spaces. It will turn out that these vector spaces are in fact isomorphic to $\mathbb{F}^\alpha$ where α is either a finite ordinal or the first transfinite ordinal, i.e., the order type of the natural numbers. Hence we always assume that our vector space has a countable generating set.

Since we are generalizing we require that such a vector product coincide with the standard inner product if the ordinal α is finite. A first attempt is to define the product "$\circ$" of two vectors v and w to be the "sum" $\sum_i v_i w_i$. Of course in general such a product is ill-defined, especially for finite fields. We thus have to make a compromise. Our vector product will not be a symmetric bilinear form yet it will, in a certain sense, "often" act as a symmetric bilinear form.

The essential idea is that for any vector $v \in \mathbb{F}^\alpha$ we restrict the vectors that can be "multiplied" with v. For example if we are working over the reals than we can say that v and w are multipliable if and only if the series $\sum_{i\in\alpha} v_i w_i$ is absolutely convergent. On the other hand we can just as well say that v and w are multipliable if and only if they have only finitely many nonzero components in common.

The notion of convergence of series is captured by what we call a *caste*, which is the subject of Definition 3.3.

Given a field $\mathbb{F}$ we let $\mathcal{MP}(\mathbb{F})$ be the set of multisets with elements in $\mathbb{F}$. Note that a member of $\mathcal{MP}(\mathbb{F})$ may have infinite cardinality even if $\mathbb{F}$ is finite.

Definition 3.3 *Let $\mathbb{F}$ be a field. A* caste of convergent series of $\mathbb{F}$ *(or simply a* caste *of $\mathbb{F}$) is a set $\mathcal{C} \subseteq \mathcal{MP}(\mathbb{F})$ together with a function $\Sigma : \mathcal{C} \to \mathbb{F}$ that satisfy the following:*

1. *The set $\mathcal{O}$ of infinitely many zeroes is in $\mathcal{C}$ and $\Sigma(\mathcal{O}) = 0$.*
2. *If $M \in \mathcal{MP}(\mathbb{F})$ has finite cardinality then $M \in \mathcal{C}$ and $\Sigma(M) = \sum_{x\in M} x$.*
3. *If $M \in \mathcal{C}$ and $N \subseteq M$ then $N \in \mathcal{C}$.*
4. *If M and N are in $\mathcal{C}$ then $M \cup N \in \mathcal{C}$ and $\Sigma(M \cup N) = \Sigma(M) + \Sigma(N)$.*
5. *If $\alpha \in \mathbb{F}$ and $M \in \mathcal{C}$ then $\alpha M := \{\alpha m \mid m \in M\} \in \mathcal{C}$ and $\Sigma(\alpha M) = \alpha\Sigma(M)$.*
6. *Let $M \in \mathcal{MP}$ and suppose that for every $m \in M$ we partition m into two parts, i.e., x_m and y_m such that $m = x_m + y_m$. Furthermore suppose that M_x and M_y, the set of multisets containing all the x_m and y_m respectively are both in $\mathcal{C}$. Then $\Sigma(M) = \Sigma(M_x) + \Sigma(M_y)$.*

A note on notation: following Property (2) we will often write $\sum_{x\in M} x$ to mean $\Sigma(M)$ regardless of the cardinality of M and we will usually omit writing the function Σ when declaring that a field has a caste, i.e. we simply say that $\mathbb{F}$ has the caste $\mathcal{C}$.

An easy consequence of the axioms is that no member of a caste of $\mathbb{F}$ may have infinitely many x's for any $x \in \mathbb{F} \setminus \{0\}$. This follows after noting that the set $M = \{x, x, \ldots\}$ of infinitely many x's can not be in $\mathbb{C}$ since we have $M = M \cup \{x\}$ so that

$\Sigma(M) = x + \Sigma(M)$, which is clearly impossible in a field. Thus the only possible caste of a finite field is the family of all finite multisets of the field. The interesting case is when the field has infinitely many elements.

In general when the field has characteristic zero, there are really only two possibilities for a caste: the set of all finite sums, and the set of absolutely convergent series. If the infinite field has characteristic $p \neq 0$ then there are still only two possibilities for the caste: the finite series and the convergent series.

Now we define our vector product.

Definition 3.4 *Let $\mathbb{F}^\alpha$ be a vector space over a field $\mathbb{F}$ and let $(\mathcal{C}, \Sigma)$ be a caste of $\mathbb{F}$. For two vectors x and y in $\mathbb{F}^\alpha$, define the multiset $M_{(x,y)} = \{x_i y_i | i \in \alpha\}$.*

The circle product *with respect to $\mathcal{C}$ is the function $\circ_{\mathcal{C}} : \mathcal{V} \times \mathcal{V} \to \mathbb{F} \cup \{\infty\}$ defined by*

$$\circ_{\mathcal{C}}(x, y) = \begin{cases} \Sigma(M_{(x,y)}) & \text{if } M_{(x,y)} \in \mathcal{C} \\ \infty & \text{otherwise.} \end{cases}$$

As with most binary operations we will always write $x \circ_{\mathcal{C}} y$ for $\circ_{\mathcal{C}}(x, y)$. Also, when $\mathcal{C}$ is fixed or understood then we shall simply write $x \circ y$. Given two vectors x and y we say that the circle product $x \circ_{\mathcal{C}} y$ is well-defined for x and y if $M_{(x,y)} \in \mathcal{C}$, (equivalently if $\Sigma(M_{(x,y)}) \in \mathbb{F}$). In this case, the two vectors are *orthogonal* if $x \circ_{\mathcal{C}} y = 0$.

The *support* of a vector $x \in \mathbb{F}^\alpha$ is the set $\mathrm{supp}(x) \subseteq \alpha$ of non-zero components of x. The vector x has *finite support* if it has only finitely many non-zero components. Note that if x has finite support then $x \circ y$ is well defined for any vector y.

Lemma 3.5 *Let x be a vector in $\mathbb{F}^\omega$ and let $\mathcal{C}$ be a caste for $\mathbb{F}$. Let $Y = \{y^1, y^2, \ldots, y^k\}$ be a finite set of vectors such that $x \circ y^j$ is well-defined for each $y^j \in Y$. Then $x \circ \sum_{i=1}^k y^i$ is well defined and*

$$x \circ \sum_{i=1}^{k} y^i = \sum_{i=1}^{k} (x \circ y^i).$$

Proof. We only prove the theorem for $k = 2$ since the result for general k follows inductively. Suppose $x \circ y$ and $x \circ z$ are well-defined. Now $M_{(x,y+z)} = \{x_i y_i + x_i z_i \mid i \in \omega\}$ and so by Property 5 of Definition 3.3 of caste we have that $\Sigma(M_{(x,y+z)}) = \Sigma(M_{(x,y)}) + \Sigma(M_{(x,z)})$, i.e. $x \circ (y+z) = x \circ y + x \circ z$. Since $\Sigma(M_{(x,y+z)}) \in \mathbb{F}$ it follows that $x \circ (y+z)$ is well-defined. ∎

The circle product is a *quasi-bilinear form* in the sense that if $x \circ y$ is well-defined for two vectors x and y then it satisfies the properties of a bilinear form.

Lemma 3.6 *Let x be a vector in $\mathbb{F}^\alpha$ with finite support and let $\mathcal{Y} = \{y^i \mid i \in I\}$ be a thin family of vectors in $\mathbb{F}^\alpha$. Then*

$$x \circ \sum_{i \in I} y^i = \sum_{i \in I} x \circ y^i.$$

Proof. Since x has finite support all circle products in the theorem statement are well-defined. Now $M_{(x, \sum_{i\in I} y^i)} = \{x_j (\sum_{i \in I} y^i_j) \mid j \in \alpha\}$. Since $\mathcal{Y}$ is a thin family $\sum_{i\in I} y^i_j$ is a finite sum for each $j \in \alpha$. Since there are only finitely many nonzero x_j it follows that $M_{(x, \sum_{i\in I} y^i)}$ is a finite multiset and the result follows from Property (2) in Definition 3.3. ∎

For a set of vectors U in a vector space $\mathbb{F}^\alpha$ over $\mathbb{F}$ with caste $\mathcal{C}$ define the *orthogonal complement* of U to be the set $U^\perp = \{v \in \mathbb{F}^\omega \mid v \circ_{\mathcal{C}} x = 0 \ \forall x \in U\}$.

The following lemma is proved in the standard way.

Lemma 3.7 *If $U \subseteq V \subseteq \mathbb{F}^\omega$ then $V^\perp \subseteq U^\perp$.* ∎

3.3 Solving a System of Infinitely Many Linear Equations

This section presents one of the main results of this thesis. Throughout this section it will turn out that whenever we wish to take the circle products of vectors x and

y, at least one of them will have finite support. Thus we will omit any mention of caste and simply write $x^T y$ for their circle product in order to conform to standard notation in linear algebra.

Theorem 3.8 *Let $\mathbb{F}^\alpha$ be a countable-dimensional vector space over a field $\mathbb{F}$. Let $\mathcal{V}$ and $\mathcal{U}$ be thin subspaces of $\mathbb{F}^\alpha$. Then*

$$\mathcal{U} + \mathcal{V} = (\mathcal{U}^\perp \cap \mathcal{V}^\perp)^\perp.$$

Proof. From Proposition 3.1 we know that $\mathcal{U} + \mathcal{V} \subseteq (\mathcal{U} + \mathcal{V})^{\perp^\perp}) = (\mathcal{U}^\perp \cap \mathcal{V}^\perp)^\perp$.

Let U and V be thin generating sets of $\mathcal{U}$ and $\mathcal{V}$ respectively and suppose that $b \in (\mathcal{U}^\perp \cap \mathcal{V}^\perp)^\perp$. Construct an infinite matrix A whose columns consist of the vectors in U and V. Then we wish to solve the infinite matrix equation $Ax = b$. Let us arrange the columns of A so that for each row the finitely many non-zero entries are at a finite distance from the left. This is accomplished as follows. First arrange the k_1 columns with non-zero entries in their first rows to be the first k_1 columns of A. Define $c(1) = k_1$. Then for every i there exists finitely many (say k_i) columns after the $c(i-1)^{\text{th}}$ column with non-zero entries. Set these columns to be the next k_i columns of A. Let $c(i) = c(i-1) + k_i$. Note that $c(i) \geq c(i-1)$.

Given i let A_i denote the $i \times c(i)$ submatrix of A obtained by taking the first i rows and first $c(i)$ columns. Let b_i be the finite vector consisting of the first i entries of b. Then we claim that the finite linear system $A_i x = b$ is consistent. For suppose there is no solution. Then by Theorem 3.2 there exists a $y \in \mathbb{F}^i$ such that $yA = 0$ and $yb \neq 0$. Consider $\tilde{y} \in \mathbb{F}^\alpha$ defined by taking the first i entries in $\tilde{y}$ to be the first i entries in y respectively, and all other entries to be zero. Then $\tilde{y}^T c = 0$ for every column c of A and so $\tilde{y} \in (\mathcal{U}^\perp \cap \mathcal{V}^\perp)$. But $\tilde{y}b \neq 0$, a contradiction.

Note that we distinguish between the i^{th} column of A and the member of $U \cup V$ which the column represents. We will be performing some slight reordering of the columns later on in this proof (solely for convenience), however when we write column i we always mean the i^{th} column in the current ordering of the columns.

Each component (or variable) x_c of x in a solution to $Ax = b$ corresponds to the column of A indexed by $c \in U \cup V$. When we reorder the columns of A we also reorder the variables of x to match the new ordering of the columns of A.

Finally for the remainder of this proof when we perform column reordering and elementary row operations on A_i, we really mean that we are performing those same operations on A. Considering A_i is simply a device to (hopefully) make the discussion slightly more intuitive and to aid the readability of the proof.

First consider A_1, which of course is just a $1 \times c(1)$ row. Multiply this row by an appropriate scalar so that the entry in column $c(1)$ is 1. Call the 1×1 submatrix consisting of this entry J_1 and let B_1 be the $1 \times c(1) - 1$ submatrix consisting of the rest of A_1. Here we allow a matrix to have zero rows or zero columns.

If the column in J_1 represents $c_1 \in U \cup V$ then we say that x_{c_1} is a dependent variable, depending on the variables corresponding to the columns in B_1, which we call the independent variables (even those whose entry is zero). These names come from the fact that any values assigned to the independent variables determines the value of x_{c_1}.

Now suppose that we have reduced A_{i-1} by row reductions and column reordering to the form

$$\begin{bmatrix} B_{i-1} & J_{i-1} \end{bmatrix}, \tag{3.1}$$

where J_{i-1} is a matrix with $i-1$ rows such that each column has precisely one 1, and each row has at most one 1. Furthermore if a row of J_{i-1} is all zeros, then the same row in B_{i-1} is all zeros.

Consider A_i. Then the submatrix consisting of first $i-1$ rows of the first $c(i-1)$ columns is A_{i-1} so we can reduce this submatrix to Form 3.1 to get a matrix of the form

$$\begin{bmatrix} B_{i-1} & J_{i-1} & \mathbf{0} \\ & a_i & \end{bmatrix},$$

where $\mathbf{0}$ is a $[c(i) - c(i-1)] \times [i-1]$ all zeros matrix and a_i is the last row of A_i.

There are two cases to consider in deciding how to proceed.

Case 1: $c(i) - c(i-1) > 0$. In this case all entries of a_i in columns $c(i-1)+1, \ldots, c(i)$ are non-zero by our original ordering of the columns. First eliminate (by elementary row reductions) all non-zero entries in a_i in the columns indexed by J_{i-1}. Then multiply a_i by a scalar to make the entry in column $c(i)$ equal to 1.

Let B_i consist of the i rows of the columns in B_{i-1} and the columns $c(i-1)+1, \ldots, c(i)-1$. Reorder the columns of A_i so that these last $c(i) - c(i-1) - 2$ come immediately after the columns of B_i.

Let J_i be the i rows of the columns in J_{i-1} together with column $c(i)$. Note that J_i satisfies the conditions on it which we claimed above. If column $c(i)$ represents $c_i \in U \cup V$ then x_{c_i} is now a dependent variable, depending on all the entries in row i of B_i.

Case 2: $c(i) = c(i-1)$. In this case we again eliminate all non-zero entries in a_i in the columns corresponding to the columns of J. If this results in a_i becoming an all zero row then we simply add the row to B_{i-1} and J_{i-1} and move on. Otherwise there exists a last column $p(i)$ in $B(i)$ with a non-zero entry. Multiply this row by an appropriate scalar to change the entry in $p(i)$ to 1.

Now we can eliminate all entries in column $p(i)$ above row i to zero. In other words, this column is now dependent. The crucial point about this entire process is that every dependent variable in J_{i-1} with a non-zero entry in column $p(i)$ has its independent variables reduced by one, and in such a way that it now only depends on earlier variables (earlier in the sense of columns closer to the left). Now all we do is move column $p(i)$ so that it is the column directly before the columns of J_{i-1}. And we let J_i be the columns of J_{i-1} together with the column that we moved.

Now every variable in x either becomes dependent after adding some row, or it stays independent forever. Therefore every dependent variable x_c always depends on finitely many independent variables, and furthermore after some finite amount of iterations, the variables upon which x_c is dependent remain fixed forever. ∎

Theorem 3.8 immediately gives us the following generalization of Fredholm's Theorem.

Corollary 3.9 *Let A be a a matrix over a field $\mathbb{F}$ with countably many rows and columns. Furthermore suppose that every row of A has finitely many non-zero entries. Let b be a column vector over $\mathbb{F}$ whose set of rows has the same cardinality as the set of rows of A. Then precisely one of the following two possibilities occur:*

1. *There exists an x such that Ax is well defined and $Ax = b$, or*

2. *There exists a y with finite support such that $y^T A = 0$ and $y^T b \neq 0$.*

Proof. Suppose both (1) and (2) occur. Since y and $y^T A$ have finite support it is a straightforward calculation to show that $(y^T A)x = y^T(Ax)$ is a well-defined equation. But the left hand side equals zero, while the right hand side is not zero (since $Ax = b$) and we have a contradiction.

If (1) does not occur then there does not exist an x such that $Ax = b$. By the proof of Theorem 3.8 we must have that there exists an i such that $A_i x = b_i$ has no solution. Hence by Fredholm's Theorem there is a y such that $y^T A = 0$ and $y^T b \neq 0$. The extension of y to $\tilde{y}$ by taking $\tilde{y}_j = y_j$ for $1 \leq j \leq i$ and $\tilde{y}_j = 0$ for $j > i$ has finite support and is such that $y^T A = 0$ and $y^T b \neq 0$. ∎

Chapter 4

Topology and Graphs

In this chapter we follow Vella [16] and Vella and Richter [11] by giving a graph a topology which captures the combinatorial nature of the graph. Section 4.1 defines this topology and derives some basic properties. In order to study the cycle and bond spaces of a locally finite infinite graph, we then compactify the graph (viewed as a topological space). It turns out that there can be many compactifications of a graph, and the set of all compactifications can be partially ordered. Section 4.2 looks at the compactifications of a graph. This section comprises new results only in the sense that it is a specialization to graph theory of the theory of compactifications in general topological spaces. Finally, Sections 4.3 and 4.4 discuss what we mean by a spanning tree, a cycle and a bond of an infinite graph graph viewed as a compactified topological space. The topology we define here and its related results (except Section 4.2) is developed much more completely in [16].

4.1 A Topology on a Graph

We begin this chapter by topologizing a graph in a nonstandard way.

Applying topology to graph theory usually begins by viewing a graph as a simplicial 1-complex, i.e. we take each edge to be homeomorphic to the unit interval $(0, 1)$

and we give the graph the identification topology. A basic open set at a vertex v is the union of all half open intervals $[v, \epsilon)$, one for each edge containing v in the graph. We call this topology the *classical topology* of the graph.

The classical topology of a graph certainly has nice topological properties, in particular the topology is normal (i.e., T_4). The trade-off is that we have lost much of the combinatorial structure of the graph. For example each graph cycle is homeomorphic to a circle. But then we cannot recover the number of vertices of the cycle since there is no way to topologically distinguish points of a circle as there exists a homeomorphism of the circle swapping any two given points. Thus we are forced to somehow record the combinatorial meaning of a graph in some other way.

The fact of the matter is that a graph is a purely combinatorial object. Embedding the graph is usually only a convenient visualization device. Therefore it seems to be in some sense unfitting to study combinatorial and algebraic properties of a graph (such as the cycle space) using a topology which is intrinsically based on the visualization of the graph, rather than the combinatorial definition.

So we must make a choice: use the classical topology at the expense of losing much of the combinatorial meaning of the graph, or define a new topology which keeps the combinatorial structure of the graph at the expense of losing nice topological properties. This thesis makes the latter choice.

Definition 4.1 *Given a graph $G = (V, E)$ define the* natural topology *on the point set $V \cup E$ by taking as the basic open sets the singletons $\{e\}$ for every edge e and the sets $N(v) = \{v\} \cup \{e \in E | v \in e\}$ for each vertex v.*

Note that the closure of an edge is simply the edge and its incident vertices. Substructures of a graph such as paths, trees and minors have their natural analogues in the natural topology. For example, a graph H is a (graph theoretic) minor of a graph G if and only if H is the image of a subgraph of G under a monotone map. (A *monotone map* is a continuous function between two topological spaces such that the inverse image of every point is connected).

Since the topological and combinatorial descriptions of a graph both give the essential structure of the graph we henceforth use the term "graph" to mean either description.

Taking the natural topology has some disadvantages. The big one is that since singletons may be open the natural topology is not even T_1. However if the graph is locally finite, it is at least *weakly Hausdorff*:

Definition 4.2 *Let X be a topological space and let x and y be any two distinct points in X. If there exists open sets U_x and U_y with $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y$ is finite then we say that X is* weakly Hausdorff.

In fact one can develop (as is done in [16]) a set of "weak" separation axioms in parallel to the usual separation axioms.

4.2 Compactifications

This section outlines a theory of compactifications of infinite graphs. Most of the topological results are specializations of known results, for example see Willard [8] and Engelking [17].

Roughly speaking, a compactification of an infinite graph is the graph together with "points at infinity". More precisely a "point at infinity" (an *end*) is an equivalence class of rays. An example that we have already seen is the Freudenthal compactification in which two rays are equivalent if they cannot be separated by the removal of finitely many vertices. The Freudenthal compactification will in a certain sense (defined later) be the "maximum" compactification. On the other hand we can simply declare all rays are equivalent, i.e., all rays converge to the same end. This is the *Alexandroff compactification*, or *1-point compactification*, and is the "least" compactification. Figure 4.1 gives the possible compactifications of a triple ray (three rays connected at one vertex). The lattice-like structure of the figure is intentionally suggestive.

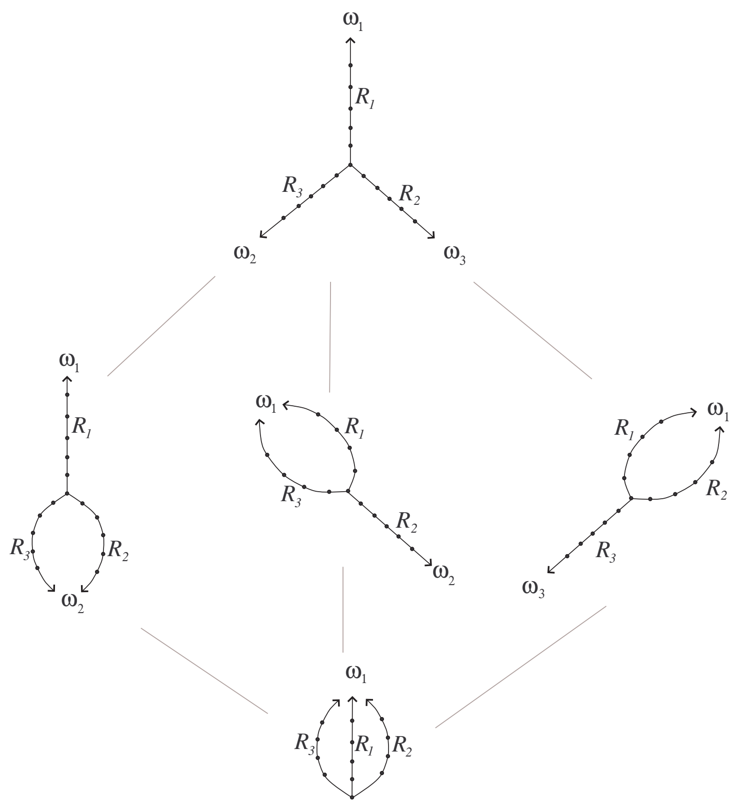

Figure 4.1: Possible Compactifications of a triple ray

Definition 4.3 *Two rays are* Halin equivalent *if there exists infinitely many disjoint paths between the two rays or equivalently, if no finite set of vertices separates the two rays.*

Let G be a locally finite graph and let $\Omega(G)$ be a set of points disjoint from $V(G) \cup E(G)$ such that there exists a bijection between $\Omega(G)$ and the set of equivalence classes of Halin equivalent rays. Each element of $\Omega(G)$ is called an *end* and we identify an end with its corresponding equivalence class of Halin equivalent rays.

For any finite set S of vertices in G let C be a component of the graph $G - S$. Then for any end ω either a tail of every ray in ω is in C or no ray of ω is in C. Let $\Omega(C) \subseteq \Omega(G)$ be the set of all ends that satisfy the former condition. If $\omega \in \Omega(C)$ we simply say that C contains ω.

Define a topology on $V(G) \cup E(G) \cup \Omega(G)$ by taking the natural topology on the points of G and a basic open set of $\omega \in \Omega(G)$ defined as follows. For any finite set of vertices S let C be the component containing ω. Then a basic open set of ω is the set $C \cup \Omega(C) \cup \delta(C)$, where $\delta(C)$ is the set of edges with one end in C and the other end in S. The *Freudenthal compactification* of G is the space $\phi G = V(G) \cup E(G) \cup \Omega(G)$ equipped with this topology.

Now for any two ends $\omega_1, \omega_2 \in \Omega(G)$ there exists a finite set C whose removal from the graph leaves ω_1 and ω_2 in two different components. Thus there exist disjoint open sets containing ω_1 and ω_2 respectively. We say that $\Omega(G)$ is a *Hausdorff set*:

Definition 4.4 *Given a subset A of a topological space X, we say that A is a* Hausdorff set *if for any two points of $x, y \in A$ there exist disjoint open sets $U_x \subseteq X$ and $U_y \subseteq X$ containing x and y respectively.*

Note that $\overline{G} = \phi G$, i.e. G is embedded in ϕG as a dense subset. The justification of the term compactification is given by the following lemma.

Lemma 4.5 *The Freudenthal compactification of a connected, locally finite graph is compact and weakly Hausdorff.*

Proof. Since G is weakly Hausdorff we need only worry about the ends. Let $\omega \in \Omega(G)$ be an end. If $x \in V \cup U$ then two open sets containing x and ω respectively can easily be found. So suppose ω' is another end of G. Let the ray R be in ω and the ray R' be in R'. Since $\omega \neq \omega'$ there exists a finite set of vertices that separates R and R', i.e. in $G - S$ there exist distinct components C and C' containing ω and ω' respectively. Then $C \cup \Omega(C)$ and $C' \cup \Omega(C')$ are two open sets containing ω and ω' with finite (in fact empty) intersection. Hence ϕG is weakly Hausdorff.

Now let $\mathcal{O}$ be an open cover of ϕG. Let $v_0 \in V(G)$ be a fixed vertex and let D_n denote the set of vertices at distance n from v_0. Let $S_n = \bigcup_{i=0}^{n-1} D_i$, and for every $v \in D_n$, let $C(v)$ be the component of $G - S_n$ containing v. Finally let $\overline{C(v)}$ be the closure of $C(v)$ in ϕG. The idea of the proof is to show that there exists some n such that, for every $v \in D_n$, there exists an open set $O(v) \in \mathcal{O}$ containing $\overline{C(v)}$. Then these open sets together with a finite subcover of $G[S_n]$ (which exists as S_n is finite) form a finite subcover of ϕG.

Suppose no such n exists. Then, for every n, there exists a non-empty set V_n consisting of those vertices $v \in D_n$ such that no $O \in \mathcal{O}$ contains $\overline{C(v)}$. Now if $v \in V_n$ then, for every neighbour $u \in D_{n-1}$ of v, we have $C(v) \subseteq C(u)$. But then we must have $u \in V_{n-1}$. Hence we may apply Konig's Infinity Lemma 1.2 to find a ray $R = \{v_0, v_1, \ldots\}$ with $v_i \in V_i$ for every i.

Now let ω be the end containing R and let $O \in \mathcal{O}$ be an open set containing ω. Since O is open, O contains a basic neighbourhood of ω, i.e. there exists a finite set S of vertices such that the component C of $G - S$ together with the ends of C is contained in O. Choose n large enough so that S_n contains both S and all its neighbours. Let $v_n \in R$. Then $\overline{C(v_n)}$ lies inside $\phi G - S$ and contains the tail of R starting at v_n. Hence $\overline{C(v_n)} \subseteq C \subseteq O$, a contradiction. ∎

We now give a definition of a general compactification of a locally finite graph. Let G be a locally finite graph. Let $\Omega_\phi(G)$ be the set of ends of the Freudenthal compactification.

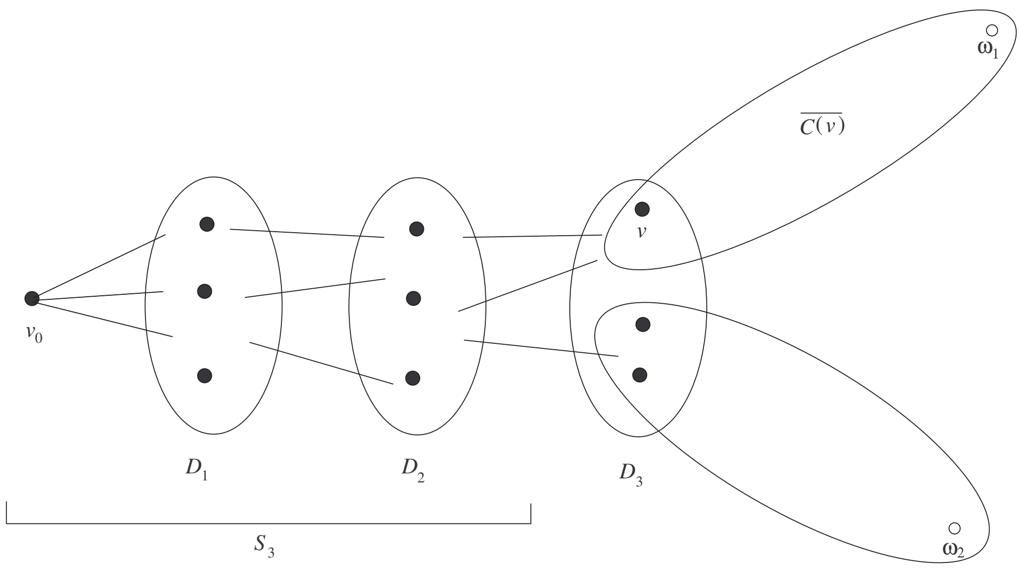

Figure 4.2: Finding a finite subcover

Definition 4.6 *A compactification of a locally finite graph G is a set $cG = G \cup \Omega_c(G)$ together with a continuous, surjective map $c : \phi G \to cG$ such that: $\Omega_c(G)$ is a Hausdorff set, c is the identity on all points of G and $c(\Omega_\phi(G)) = \Omega_c(G)$.*

So a compactification of a graph can be thought of us identifying its Halin ends in some prescribed way. It is theoretically possible to be more liberal in defining a compactification, for example by letting two rays converge to a unique end if they share a tail. However even in a simple example such as an infinite ladder, we get infinitely many ends! To avoid such an unwieldy situation we content ourselves to restricting an end to contain any ray provided that all Halin equivalent rays are contained in the same end.

The condition that $\Omega_c(G)$ is a Hausdorff set is included as it is needed in the proof of Theorem 4.8. We feel that it may be possible to replace this condition with a weaker one, or none at all, but have not yet been able to do so.

We can partially order the set of compactifications to form the *lattice of compactifications* in the following way.

Definition 4.7 *Let G be a locally finite graph and let c_1G and c_2G be two compactifications of G. We write $c_1G \leq c_2G$ if there exists a continuous, surjective mapping $f : c_2G \to c_1G$ such that $f \circ c_2 = c_1$. Here the symbol "$\circ$" is function composition.*

Two compactifications c_1G and c_2G are equivalent *if there exists a homeomorphism $f : c_1X \to c_2X$ such that $f \circ c_1(x) = c_2(x)$ for every $x \in G$.*

Evidently if cG is a compactification of a locally finite graph then $cG \leq \phi G$. On the other hand it is easy to see that if ωG is the compactification (up to equivalence) such that $\Omega_\omega(G)$ consists of a single point then $\omega G \leq cG$. This is the Alexandroff compactification of G.

Theorem 4.8 *The set of compactifications of G is partially ordered by the "$\leq$" relation.*

Proof. Reflexivity and transitivity are simple to prove so we only show antisymmetry. Suppose we have compactifications c_1G and c_2G such that $c_1G \leq c_2G$ and $c_2G \leq c_1G$. Then there exist continuous, surjective $f_1 : c_1G \to c_2G$ and $f_2 : c_2G \to c_1G$ such that $f_1 \circ c_1 = c_2$ and $f_2 \circ c_2 = c_1$.

We aim to show that f_1 is a homeomorphism. Thus, we are done if we can show that $f_1 \circ f_2(x) = x$ for all $x \in c_2G$ and $f_2 \circ f_1(x) = x$ for all $x \in c_1G$.

Note that $f_2 \circ f_1 \circ c_1 = c_1$ and hence $f_2 \circ f_1(x) = x$ for all $x \in c_1(G) = G$.

Now consider the set $A = \{x \in c_1G | f_2 \circ f_1(x) = x\}$. We claim that A is closed. Consider $A^C = \{x \in c_1G | f_2 \circ f_1(x) \neq x\}$, the complement of A in c_1G. Since $G \subseteq A$, we have that $A^C \subseteq \Omega_{c_1}(G)$. Let $x \in A^C$ so that $f_2 \circ f_1(x) \neq x$. Since $x \in \Omega_{c_1}(G)$ and $\Omega_{c_1}(G)$ is a Hausdorff set we can find disjoint open sets U_1 and U_2 such that $x \in U_1$ and $f_2 \circ f_1(x) \in U_2$.

Consider $U = (f_2 \circ f_1)^{-1}(U_2) \cap U_1$. Then clearly U is an open set containing x. Furthermore suppose $y \in U$. Then $y \in U_1$ and $f_2 \circ f_1(y) \in U_2$ and so $y \in A^C$ since U_1 and U_2 are disjoint. Hence $U \subseteq A^C$ and so A^C is open, i.e. A is closed.

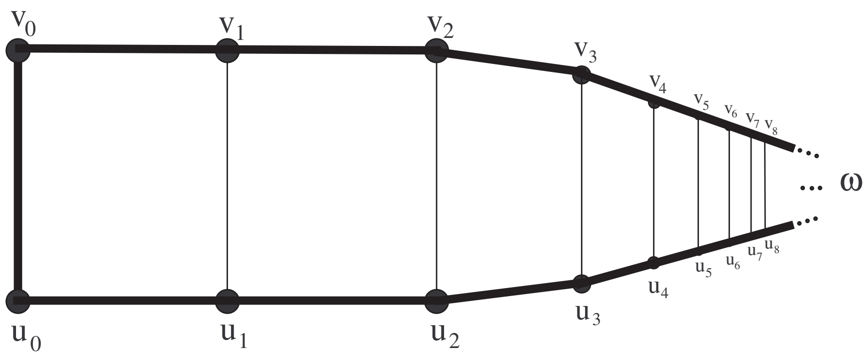

Figure 4.3: 1-way infinite ladder with proposed spanning tree edges darkened.

But we have that $c_1(G) \subseteq A$ and hence $A = c_1G$ since $c_1G = \overline{c_1(G)} \subseteq A$. Thus $f_2 \circ f_1$ is the identity on c_1G. Similarly one can show that $f_1 \circ f_2$ is the identity on c_2G. It now follows that f_1 is a homeomorphism and hence c_1G and c_2G are equivalent. ∎

4.3 Spanning Trees

Spanning trees will play a fundamental role in our study of cycle spaces. The first task is to precisely define what we mean by "spanning tree". In classical graph theory a spanning tree of a finite graph is defined to be a connected acyclic subgraph containing every vertex of the graph. An obvious property of any finite tree is that the removal of an edge disconnects the tree.

In extending the definition of a spanning tree to infinite graphs we must be careful. As an example, consider the compactified 1-way infinite ladder $\mathcal{L}_1$ in Figure 4.3. Let T be the subgraph containing the darkened edges. Certainly T is connected and contains each vertex in the graph. But is T acyclic? If a cycle is a closed finite walk then T is obviously acyclic. However a cycle in a finite graph can also be characterized in other ways. For example, a finite connected graph is a cycle if and

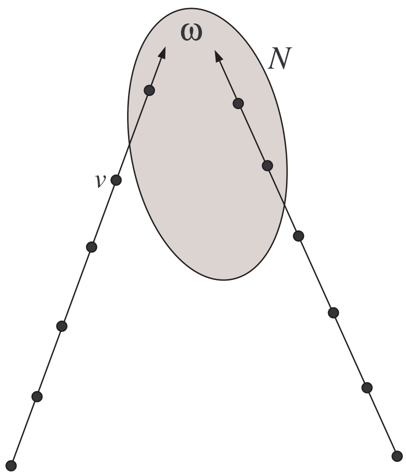

Figure 4.4: A compactified graph which is not graph theoretically connected but is topologically connected.

only if the removal of any edge leaves the graph connected, whereas the removal of any two edges disconnects the graph.

The key point in this last characterization is the term "connected". Consider a compactified graph cG consisting solely of two disjoint rays converging to the same end ω. See Figure 4.4 for an illustration of cG. Now G is not graph theoretically connected; however cG is topologically connected. For suppose H and K are two nonempty disjoint open sets whose union is cG. Then without loss of generality $\omega \in H$. Since H is open there must exist a basic open neighbourhood N of ω contained in H. Then N contains a proper subray of at least one of the rays. But then K consists of one or two finite paths and so is not open as K does not contain a basic neighbourhood of the vertex in the ray nearest to N, e.g. see the vertex v in Figure 4.4. This contradicts the fact that K is open.

Definition 4.9 *A* cycle *is a* ***topologically*** *connected compactified graph* $cG = (V, E, \Omega)$ *such that, for each edge* $e \in cG$, $cG \setminus \{e\}$ *is connected, but for any two distinct edges* $e, f \in cG$, $cG \setminus \{e, f\}$ *is disconnected.*

Thus in Figure 4.3, T is a cycle according to Definition 4.9. For the remainder of this thesis we will always (unless otherwise noted) use the term cycle in the sense of Definition 4.9. We can now define what we mean by a spanning tree.

Definition 4.10 *A* spanning tree *T of a compactified graph $cG = (V, E, \Omega)$ is a maximal acyclic topologically connected subset of E such that $V \cup \Omega \subset \overline{T}$. Equivalently, T is a spanning tree if the subspace $\overline{T}$ is topologically connected but the removal of any edge in T disconnects $\overline{T}$.*

The fact that a spanning tree always exists will be the focus of the remainder of this section. Recall that the natural topology of the graph is weakly Hausdorff. Since a compact Hausdorff space is normal it is natural to wonder if there is also a "weak" analogue for normality and if the same sort of theorem holds. In fact it does.

Definition 4.11 *Let X be a topological space and let C and D be disjoint and closed in X. If there exist open sets $U_C \supseteq C$ and $U_D \supseteq D$ such that $U_C \cap U_D$ is finite then we say that X is* weakly normal*.*

Lemma 4.12 *Any compact weakly Hausdorff topological space is weakly normal.*

Proof. Let X be a compact weakly Hausdorff space and let C and D be disjoint and closed in X. If one of C and D is empty then the theorem is trivial so suppose neither is empty.

We first show that for every $c \in C$ there exist open sets U_c and V_D containing c and D respectively such that U_c and V_D have finite intersection. One could say we are showing that X is *weakly regular*.

For every $d \in D$ let Y_d and Z_d be open sets such that $c \in Y_d$ and $d \in Z_d$ and $Y_d \cap Z_d$ is finite. Then $\bigcup_{d \in D} Z_d$ is an open covering of D and hence has a finite subcovering, $Z_{d_1}, \ldots, Z_{d_k}$. Then $V_D = \bigcup_{i=1}^{k} Z_{d_i}$ and $U_c = \bigcap_{i=1}^{k} Y_{d_i}$ are open sets containing D and c respectively and $V_D \cap U_c$ is finite as desired.

So for each c let U_c and V_D^c be open sets containing c and D respectively where $U_c \cap V_D^c$ is finite. Since $\bigcup_{c \in C} U_c$ is an open covering of C there exists a finite subcovering, say $U_{c_1}, \ldots, U_{c_m}$. Let $U = \bigcup_{i=1}^{k} U_{c_i}$ and $V = \bigcap_{i=1}^{k} V_D^{c_i}$. Then U and V are open sets containing C and D such that $U \cap V$ is finite. ∎

Theorem 4.13 *Let X be a connected compact weakly Hausdorff space and let $A \subseteq X$. Then there exists a minimal closed and connected subset C of X such that $A \subset C$.*

Proof. Let $\mathcal{C}$ be a chain of closed connected subsets of X, each containing A, and ordered by inclusion. Let $\hat{C} = \bigcap_{C \in \mathcal{C}} C$. Then $\hat{C}$ is closed and contains A. We wish to show that $\hat{C}$ is connected.

Suppose that $\hat{C}$ is disconnected, i.e. suppose there exist disjoint, closed and non-empty sets H and K such that $\hat{C} = H \cup K$. Since X is weakly normal by Lemma 4.12, we can find open sets $U_H \supseteq H$ and $U_K \supseteq K$ such that $U_H \cap U_K$ is a finite set.

Since H and K are closed, we may assume that $H \cap U_K$ and $K \cap U_H$ are both empty as otherwise take, for example, $U'_K = U_K \cap H^C$, where H^C is the complement of H. It then follows that $U_H \cap U_K \cap \hat{C} = \emptyset$. Since $U_H \cap U_K$ is finite, there must exist a $C_1 \in \mathcal{C}$ such that $U_H \cap U_K \cap C_1 = \emptyset$.

Now suppose $C_1 \not\subseteq U_H \cup U_K$. If $x \in C_1 - (U_H \cup U_K)$ then $x \notin \hat{C}$ and so there is a $C_x \in \mathcal{C}$ with $x \notin C_x$. Since C_x is closed there is an open set U_x containing x and disjoint from C_x. The union of all such U_x is a covering of $C_1 - (U_1 \cup U_2)$ and so there exists a finite subcover, say $\{U_{x_1}, U_{x_2}, \ldots, U_{x_k}\}$. For each x_i, $U_{x_i} \cap C_{x_i} = \emptyset$. Since k is finite there is a $C_2 \in \mathcal{C}$ disjoint from each $U_{x_i}, 1 \leq i \leq k$. But then $C_2 \subseteq C_1$ and C_2 is disjoint from $C_1 - (U_H \cup U_K)$ and so $C_2 \subseteq U_H \cup U_K$.

Therefore there exists a $C_2 \in \mathcal{C}$ such that $C_2 \subseteq C_1$ and $C_2 \subseteq U_H \cup U_K$. It follows that $(U_H \cap C_2, U_K \cap C_2)$ is a separation of C_2, but $C_2 \in \mathcal{C}$ and so is connected by definition. We have therefore arrived at the contradiction. ∎

The main result of this section is the following:

Corollary 4.14 *Let $cG = (V, E, \Omega)$ be a compactification of a connected locally finite graph. Then there is a subset T of cG such that $(cG \setminus E) \cup T$ is connected, but for all $e \in T$, $(cG \setminus E) \cup (T \setminus \{e\})$ is not. In other words, T is a spanning tree.*

Proof. Apply Theorem 4.13 with $X = cG$ and $A = V \cup \Omega$. Then we take $T = C \cap E$ (so $C = \overline{T}$). ∎

4.4 Cycles and Bonds

Throughout the remainder of this chapter we let $G = (V, E)$ be a connected, locally finite graph with compactification $cG = (V, E, \Omega(G))$.

Definition 4.15 *Let A and B partition $V(G) \cup \Omega(G)$ into two disjoint, closed non-empty sets. Then the set of edges $\delta(A)$ with exactly one end in A is called a* bond *(also sometimes called a* cut*).*

It turns out that as in finite graphs, the cycles and bonds of a graph are dual to each other. Since we allow cycles with infinitely many edges, we may imagine there to be bonds with infinitely many edges. This is (fortunately, as we will see) not true.

Theorem 4.16 *Let G be a locally finite graph and let cG be a compactification of G. Then every bond in cG is finite.*

Proof. Suppose that A and B are two disjoint, nonempty, closed sets that partition $V(G) \cup \Omega(G)$. Since the topology on cG is weakly normal there exist open sets U_A and U_B containing A and B respectively and such that $U_A \cap U_B$ is finite. But $\delta(A) \subseteq U_A \cap U_B$ and hence $\delta(A)$ is finite. ∎

Let T be a spanning tree for cG and let e be an edge not in T with endpoints u and v. We can apply Theorem 4.13 to $\overline{T}$ with $A = \{u, v\}$ to obtain a minimal closed and connected subset $P_{u,v}$ of T containing u and v. One can think of $P_{u,v}$ as a (possibly infinite) path from u to v. It is not difficult to see that $P_{u,v} \cup e := Z_{(T,e)}$ is a cycle in T and hence a cycle in cG. Furthermore there exists precisely one such cycle in $T \cup \{e\}$. We call such a $Z_{(T,e)}$ a *fundamental cycle* of T.

Dual to the notion of a fundamental cycle is a fundamental bond. Let e be an edge in T with endpoints u and v. Then since T is minimally connected $\overline{T} \setminus \{e\}$ contains two disjoint, non-empty closed components T_u and T_v. The set $\delta(T_u) := B_{(T,e)}$ is called a *fundamental bond* of T. Note that e is the only edge of T in $B_{(T,e)}$.

It is not difficult to check that given a spanning tree T of cG, an edge $e \in E \cap T$ and an edge $f \in cG \setminus T$ we have that $e \in \mathrm{supp}(Z_{(T,f)})$ if and only if $f \in \mathrm{supp}(B_{(T,e)})$.

A family of subsets $\mathcal{F}$ of G is called *thin* if each element of $V \cup E$ appears in only finitely members of $\mathcal{F}$.

Theorem 4.17 *Let cG be a compactified locally finite graph with spanning tree T. Then the family of fundamental cycles of T is thin.*

Proof. As noted above, if $e \notin T$ there exists precisely one fundamental cycle containing e. On the other hand, if $e \in Z_{(T,f)}$ for some edge f we have that $f \in B_{(T,e)}$. But all bonds are finite and so there are only finitely many f with this property. Hence e is in finitely many fundamental cycles. ∎

In general the family of fundamental bonds of T is not thin. As an example take G to be the 2-way infinite path with the Alexandroff compactification. Then a spanning tree of T of G is the set $E(G) \setminus \{e\}$ for an arbitrary edge $e \in E(G)$, and every fundamental bond contains e.

Chapter 5

The Cycle Spaces of an Infinite Graph

This chapter comprises the core of this thesis. We generalize results of Chapter 2 to locally finite infinite graphs. The first section translates the cycles and bonds of a graph (in the sense of Section 4.4) into members of the edge space over a field $\mathbb{F}$ of the graph. We then proceed to introduce two cycle spaces and two bond spaces. Richter and Vella [11] considered only two of these (i.e. the thin cycle space and the finite bond space). Section 5.2 gives orthogonality relations between these cycle and bond spaces, although we will see that depending on $\mathbb{F}$ we might have to introduce a third bond space to complete the orthogonality theory. Section 5.3 is a short discussion of a fourth type of bond space introduced in [11]. We show that in the case of locally finite graphs, it is a redundant space. Section 5.4 gives some simple consequences of the orthogonality relations including a version of the Edge Tripartition Theorem.

5.1 The Cycle and Bond Spaces

We begin by introducing the edge space of a graph over a field $\mathbb{F}$. We then move on to translating the cycles and bonds of Section 4.4 into members of the edge space and we then define several cycle and bond spaces.

Throughout this section we let $G = (V, E)$ be a locally finite graph and cG a compactification of G. We also orient the edges of G in some fixed but arbitrary manner.

Definition 5.1 *Let $\mathbb{F}$ be a field. The* edge space *of a graph G is the set of functions $\mathbb{F}^E := \{f : E(G) \to \mathbb{F}\}$. Equivalently we can view the edge space as an $|E|$ dimensional vector space over $\mathbb{F}$ indexed by the edges of G.*

Let C be a cycle in cG. We associate a member $z \in \mathbb{F}^E$ (which we also call a cycle) with C as follows. First choose a direction of traversal D for C. In this context, D is a cyclic ordering of the edges of C such that if (w, x, y, z) is a cyclic subsequence, then x and z separate w and y. The existence of such a D is guaranteed by a result in [11]. We then define z as follows:

$$z(e) = \begin{cases} 1 & \text{if } e \in C \text{ and the orientation of } e \text{ agrees with } D \\ -1 & \text{if } e \in C \text{ and the orientation of } e \text{ disagrees with } D \\ 0 & \text{if } e \notin C. \end{cases}$$

We will not usually explicitly distinguish between a cycle and its corresponding member of the edge space. The context will always make clear which meaning we intend.

We adopt the convention that for a fundamental cycle $Z_{(T,e)}$ of some tree T, the direction of traversal always agrees with the orientation of e.

Definition 5.2 *Let cG be a compactified locally finite graph. Let T be a spanning tree of cG. We define the* thin cycle space *$\mathcal{Z}_t$ to be the thin span of all fundamental cycles of T. The* finite cycle space *$\mathcal{Z}_f$ is the set of all finite linear combinations of fundamental cycles of T such that $z \in \mathcal{Z}_f$ only if z has finite support.*

The fact that both cycle spaces are independent of the choice of T follows from the next two results. A *circulation* is an element $c \in \mathbb{F}^E$ such that

$$\sum_{e \in \delta^{\text{in}}(X)} c(e) = \sum_{e \in \delta^{\text{out}}(X)} c(e) \tag{5.1}$$

holds for every bond $\delta(X)$, where $\delta^{\text{in}}(X)$ is the set of edges in $\delta(X)$ whose head is in X and $\delta^{\text{out}}(X)$ is the set of edges in $\delta(X)$ whose tail is in X. Note the slight difference between the definition here and the definition of a circulation for finite graphs.

Theorem 5.3 *Let cG be a compactified, locally finite graph. The thin cycle space of cG is precisely the set of all circulations of cG (cf. Theorem 2.1).*

Proof. Denote the set of all circulations of cG by $\mathcal{C}_t$.

Every cycle is a circulation by the definition of cycle in $\mathbb{F}^E$. Let $z = \sum_{c \in Z} \alpha_c c$ be a thin linear combination of a thin set of cycles Z. Let $\delta(X)$ be a bond of cG. Since $\delta(X)$ is finite there exist only finitely many cycles $\{c_1, c_2, \ldots c_k\}$ in Z containing edges of $\delta(X)$. Clearly the finite sum of circulations is a circulation, and so $z' = \sum_{i=1}^{k} \alpha_c c_i$ is a circulation and therefore satisfies equation (5.1) for $\delta(X)$. Since $z'(e) = z(e)$ for all $e \in \delta(X)$ it follows that for any bond $\delta(X)$, z satisfies equation (5.1) and hence z is a circulation. Therefore $\mathcal{Z}_t \subseteq \mathcal{C}_t$.

Now let $c \in \mathcal{C}_t$. Consider the element $z = c - \sum_{e \notin T} c(e) Z_{(T,e)}$. Since a sum of circulations is a circulation, we have that $z \in \mathcal{C}_t$. As there exists a unique fundamental cycle for each edge $e \notin T$ it follows that the support of z is contained in the edges of T. But every fundamental bond has exactly one edge of T, and hence we must have $z(e) = 0$ for every edge $e \in T$ in order to satisfy the definition of a circulation. Hence $z = 0$ and so $c = \sum_{e \notin T} c(e) C_e$. Hence $c \in \mathcal{Z}_t$ and so $\mathcal{Z}_t = \mathcal{C}_t$. ∎

The reason behind the naming of the finite cycle space should become apparent with the next result.

Corollary 5.4 *Let cG be a compactified connected locally finite graph. Then every finite cycle is in $\mathcal{Z}_f$. Hence $\mathcal{Z}_f$ is the set of all finite linear combinations of finite cycles and at the same time is the set of all circulations with finite support.*

Proof. The proof closely follows the proof of Theorem 5.3. So let $\mathcal{C}_f$ be the set of all circulations with finite support. Let T be a spanning tree for cG. Let c be a finite cycle and consider the edge space member $c - \sum_{e \notin T} c(e) Z_{(T,e)} \in \mathcal{C}_t = \mathcal{Z}_t$. By reasoning similar to the proof of Theorem 5.3 it follows that $c - \sum_{e \notin T} c(e) Z_{(T,e)} = 0$ and hence $c = \sum_{e \notin T} c(e) Z_{(T,e)} = \sum_{e \notin T : c(e) \neq 0} Z_{(T,e)}$, which is a finite linear combination of fundamental cycles. Since c has finite support it follows that $c \in \mathcal{Z}_f$. ∎

Dual to the cycle spaces are the bond spaces. Given a bond $\delta(A)$ of a compactified, locally finite graph cG we define an element of $b \in \mathbb{F}^E$ as follows.

$$b(e) = \begin{cases} 1 & \text{if } e \in \delta(A) \text{ and if the head of } e \text{ is in } A \\ -1 & \text{if } e \in \delta(A) \text{ and if the tail of } e \text{ is in } A \\ 0 & \text{if } e \notin \delta(A). \end{cases}$$

Definition 5.5 *The* finite bond space $\mathcal{B}_f$ *is the set generated by all finite linear combinations of bonds of G. The* thin bond space $\mathcal{B}_t$ *is the set generated by (possibly infinite) linear combinations of thin families of bonds.*

We see in the next section that we can take as a basis for $\mathcal{B}_t$ the set of vertex cuts. We will also need one more bond space $\mathcal{B}_w$ which fits between $\mathcal{B}_f$ and $\mathcal{B}_t$ but which will be defined at a more convenient time.

5.2 Orthogonality Relations

In this section we explain in what sense cycle and bond spaces are duals. We begin with the following easy fact.

Lemma 5.6 *Let cG be a compactified locally finite graph. Let $\mathbb{F}$ be a field with caste $\mathcal{C}$ and let the edge space be $\mathbb{F}^E$. Let z be a cycle in cG and b a bond in $\mathbb{F}^E$. Then $z \circ b = 0$.*

Proof. Since b has finite support $z \circ b$ is well defined. A cycle and a bond always have an even number of edges in common by a result in [11]. If $\mathrm{supp}(b) \cap \mathrm{supp}(z) = \emptyset$ then $z \circ b = 0$ trivially. Otherwise let e_1 be an edge in both z and b. Follow z in one direction until we get to the next edge e_2 in $\mathrm{supp}(b) \cap \mathrm{supp}(z)$. If $z(e_1) = z(e_2)$ then we must have $b(e_1) = -b(e_2)$. Conversely if $z(e_1) = -z(e_2)$ then we must have $b(e_1) = b(e_2)$. In either case we have that $z(e_1)b(e_1) + z(e_2)b(e_2) = 0$. Since $|\mathrm{supp}(b) \cap \mathrm{supp}(z)|$ is finite and even it follows that z and b are orthogonal. ∎

We are now ready to prove the orthogonality relations between our cycle and bond spaces. Since $A = B^\perp$ does not in general imply that $B = A^\perp$ we will need to prove both equalities (if possible).

Theorem 5.7 *Let cG be a compactified locally finite graph. Let $\mathbb{F}$ be a field with caste $\mathcal{C}$ and let the edge space be $\mathbb{F}^E$. Then $\mathcal{B}_f^\perp = \mathcal{Z}_t$.*

Proof. Fix a spanning tree T of cG. Let $z \in \mathcal{Z}_t$. Then we know that $z = \sum_{e \notin T} z(e) Z_{(T,e)}$. But then for any $b \in \mathcal{B}_f$ we have by Lemma 3.6 that $z \circ b = \sum_{e \notin T} z(e)(Z_{(T,e)} \circ b) = 0$. Hence $\mathcal{Z}_t \subseteq \mathcal{B}_f^\perp$.

Now let $a \in \mathcal{B}_f^\perp$. Consider $z = a - \sum_{e \notin T} a(e) Z_{(T,e)}$. We aim to show that $z = 0$ so suppose not. First note that for any $b \in \mathcal{B}_f$ we have $z \circ b = a \circ b - \sum_{e \notin T} a(e)(Z_{(T,e)} \circ b) = 0$ and so $z \in \mathcal{B}_f^\perp$. Also we clearly have $\mathrm{supp}(z) \subseteq T$. Since by assumption $z \neq 0$ there exists an edge $e \in T \cap \mathrm{supp}(z)$ and so $B_{(T,e)} \circ z \neq 0$, a contradiction. Thus $z = 0$ and so $\mathcal{B}_t^\perp = \mathcal{Z}_t$. ∎

Theorem 5.8 *Let cG be a compactified locally finite graph. Let $\mathbb{F}$ be a field with caste $\mathcal{C}$ and let the edge space be $\mathbb{F}^E$. Then $\mathcal{B}_t^\perp = \mathcal{Z}_f$.*

Proof. If $z \in \mathcal{Z}_f$ then by Theorem 5.4 we can write z as a finite linear combination of finite cycles. Thus if $b \in \mathcal{B}_t$ it follows that $b \circ z = 0$ and so $z \in \mathcal{B}_t^{\perp}$.

Now suppose $z \in \mathcal{B}_t^{\perp}$. Since $\mathcal{B}_t^{\perp} \subseteq \mathcal{B}_f^{\perp} = \mathcal{Z}_t$, it suffices to show that z has finite support, by Theorem 5.4. Suppose z has infinite support.

Let $W = \{v_1, v_2, \ldots\}$ be an infinite set of pairwise non-adjacent vertices such that for each v_i there exists an $e_i \in \delta(v_i) \cap \operatorname{supp}(z)$. Consider $b = \sum_i (-1)^s z(e_i)^{-1} \delta(v_i)$, where $s = 0$ if $e_i \in \delta^{\text{out}}(v_i)$ and $s = 1$ if $e_i \in \delta^{\text{in}}(v_i)$. Then we should have $z \circ b = 0$ but we have $M_{(z,b)} = \{z_e b_e \mid e \in E\} \supseteq \{1, 1, \ldots\}$ and thus $M_{(x,y)} \notin \mathcal{C}$. Therefore $z \circ b$ is not even well defined, let alone equal to zero, and so we have a contradiction. Hence the support of z must be finite. ∎

Theorem 5.9 *Let cG be a compactified locally finite graph. Let $\mathbb{F}$ be a field with caste $\mathcal{C}$ and let the edge space be $\mathbb{F}^E$. Then $\mathcal{B}_t = \mathcal{Z}_f^{\perp}$.*

Proof. We immediately have (by Theorem 5.8) $\mathcal{B}_t \subseteq \mathcal{B}_t^{\perp^{\perp}} = \mathcal{Z}_f^{\perp}$.

Now let $b \in \mathcal{Z}_f^{\perp}$. We will show that we can write b as a sum of vertex cuts. So $b' = \sum_{v \in V(G)} \alpha_v \delta(v)$ and our task is to define the α_v's so that $b = b'$.

Let v_0 be an arbitrary vertex and construct a (graph-theoretic) breadth first search spanning tree T rooted at v_0. We begin by setting $\alpha_{v_0} = 0$. Let v be any other vertex of G. Then there exists a unique vertex w closer to v_0 in T. If the head of edge vw is v then set $\alpha_v = \alpha_w + b(vw)$. If v is the tail of edge vw then set $\alpha_v = -(\alpha_w) + b(vw)$. It follows that for every edge vw in T (say with head at v), $b'(vw) = \alpha_v - \alpha_w = b(vw)$ as required.

Let $w_k w_0$ be any edge of $E(G) \setminus E(T)$ directed from w_k to w_0. Then there exists a unique finite cycle $z = \{w_0, w_1, w_2, \ldots, w_{k-1}, w_k\}$ in $T \cup e$. For definiteness pick the direction of traversal for z so that it agrees with the orientation of $w_0 w_k$. Since z is orthogonal to b we have

$$0 = z \circ b = z(w_0 w_1) b(w_0 w_1) + z(w_1 w_2) b(w_1 w_2) + \ldots z(w_{k-1} w_k) b(w_{k-1} w_k) + b(w_k w_0).$$

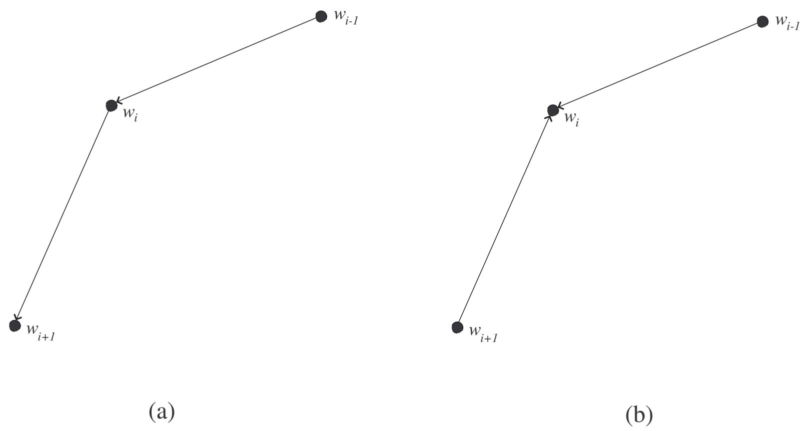

Figure 5.1: Two possible sections of cycle z containing w_i.

Thus

$$b(w_k w_0) = -(z(w_0w_1)b(w_0w_1) + z(w_1w_2)b(w_1w_2) + \dots z(w_{k-1}w_k)b(w_{k-1}w_k)).$$

In fact the right hand side reduces exactly to $\alpha_{w_0} - \alpha_{w_k} = b'(w_0w_k)$. To see this note that for every $1 \leq i \leq k-1$, α_{w_i} appears only in $b(w_{i-1}w_i)$ and $b(w_iw_{i+1})$. There are four cases to consider depending on the orientations of the edges $w_{i-1}w_i$ and w_iw_{i+1}. However need only closely examine cases (a) and (b) in Figure 5.1, as the other two are algebraically identical to one of these up to a factor of -1.

Case (a) Here we have $z(w_{i-1}w_i) = z(w_iw_{i+1}) = 1$. Therefore

$$z(w_{i-1}w_i)b(w_{i-1}w_i) + z(w_iw_{i+1})b(w_iw_{i+1}) = (\alpha_{w_i} - \alpha_{w_{i-1}}) + (\alpha_{w_{i+1}} - \alpha_{w_i}).$$

Case (b) Here $z(w_{i-1}w_i) = -z(w_iw_{i+1}) = 1$. Therefore

$$z(w_{i-1}w_i)b(w_{i-1}w_i) + z(w_iw_{i+1})b(w_iw_{i+1}) = (\alpha_{w_i} - \alpha_{w_{i-1}}) - (\alpha_{w_i} - \alpha_{w_{i+1}}).$$

In both cases the terms involving α_{w_i} cancel out and so we have

$$\alpha_{w_0} - \alpha_{w_k} = b'(w_0w_k) = b(w_0w_k),$$

as desired. ∎

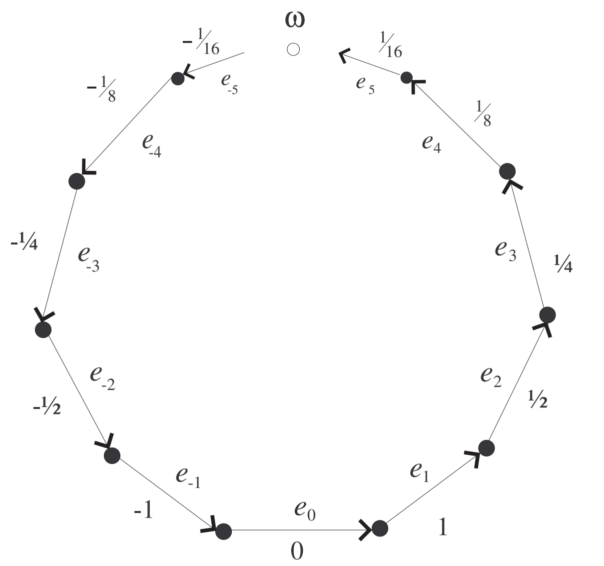

Figure 5.2: 2-way infinite path with 1-point compactification.

It is curious that since $\mathcal{Z}_f$ is clearly independent of the compactification of G so is $\mathcal{B}_t$. This despite the fact that there can be a bond in a compactification cG which is not a bond in a compactification $c'G \leq cG$.

Deriving the orthogonal complement of $\mathcal{Z}_t$ is where things get interesting. The pattern so far would imply that we should have $\mathcal{Z}_t^{\perp} = \mathcal{B}_f$. When the compactified graph has finitely many bridges and $\mathbb{F}$ is finite this will indeed be the case. However, consider the case when $\mathbb{F} = \mathbb{R}$, and the caste of $\mathbb{R}$ is the family of all absolutely convergent series. Let us look at the 2-way infinite path with the 1-point compactification as shown in Figure 5.2.

Here we have assigned an element $b \in \mathbb{R}^E$ defined by $b(e_i) = \frac{1}{2^i}$ for $i \geq 1$ and $b(e_i) = -\frac{1}{2^i}$ for $i \leq -1$. Since there is essentially only one cycle $z \in \mathbb{R}^E$ (i.e. the constant vector) it follows that $b \circ z = 0$ and so $b \in \mathcal{Z}_t^{\perp}$. Recall that $B_{(T,e)}$ denote the fundamental bond containing edge $e \in T$ for some spanning tree T.

Definition 5.10 *Let cG be a compactified locally finite graph. Let $\mathbb{F}$ be a field with* *caste $\mathcal{C}$ and let the edge space be $\mathbb{F}^E$. Let T be a spanning tree for cG. The* extended finite bond space *$\mathcal{B}_w$ is defined as follows. An element b is in $\mathcal{B}_w$ if and only if $b = \sum_{e \in F} \alpha_e B_{(T,e)}$ where $F \subset T$ is such that $(\bigcup_{e \in F} supp(B_{(T,e)})) \setminus T$ is finite and $\{\alpha_e \mid e \in T,\ |supp(B_{(T,e)})| > 1\} \in \mathcal{C}$.*

Our next result shows that the extended finite bond space does not depend on the choice of spanning tree. More importantly, we find that $\mathcal{B}_w$ is the orthogonal complement of $\mathcal{Z}_t$.

Theorem 5.11 *Let cG be a compactified locally finite graph. Let $\mathbb{F}$ be a field with caste $\mathcal{C}$ and let the edge space be $\mathbb{F}^E$. Then $\mathcal{Z}_t^\perp = \mathcal{B}_w$.*

Proof. In this proof we write $x(e)$ for the e^{th} entry of $x \in \mathbb{F}^E$.

Let $b = \sum_{e \in F} b_e B_{(T,e)} \in \mathcal{B}_w$ and let $z = \sum_{e \notin T} z_e Z_{(T,e)} \in \mathcal{Z}_t$. First we show that $b \circ z$ is well defined and equal to zero. Now $M_{(b,z)} = \{b(g)z(g) \mid g \in E(G)\} = \{b(g)z(g) \mid g \in T\} \cup \{b(g)z(g) \mid g \in E(G) \setminus T\}$ so we simply need to show that the latter two multisets are in $\mathcal{C}$, and then that $\Sigma(M_{(z,b)})$ is zero.

We automatically have $L = \{z(g)b(g) \mid g \in E(G) \setminus T\} \in \mathcal{C}$ since there are only finitely many g such that $b(g)$ is nonzero (by definition of $\mathcal{B}_w$).

Consider the set

$$\{z(g)b(g) \mid g \in T\} = \{z(g)b(g) \mid g \in T,\ |B_{(T,g)}| > 1\} \cup \{0 \mid g \in T,\ |B_{(T,g)}| = 1\},$$

where the equality follows since $z(g) = 0$ if $|B_{(T,g)}| = 1$.

Let $N = \{z(g)b(g) \mid g \in T,\ |B_{(T,g)}| > 1\}$.

If $g \in T$ we have

$$\begin{aligned} z(g) &= \sum_{e \notin T} z_e Z_{(T,e)}(g) \\ &= \sum_{e \notin T : g \in \mathrm{supp}(Z_{(T,e)})} z_e Z_{(T,e)}(g) \\ &= \sum_{e \notin T : e \in \mathrm{supp}(B_{(T,g)})} z_e Z_{(T,e)}(g) \\ &= \sum_{e \in \mathrm{supp}(B_{(T,g)}) \setminus T} z_e Z_{(T,e)}(g). \end{aligned} \tag{5.2}$$

Now $\bigcup_{g \in F} \mathrm{supp}(B_{(T,g)}) \setminus T$ is finite, so there are only finitely many edges $e_1, e_2, \ldots, e_k$ that give a nonzero value for z_e in the sum of (5.2) for any $g \in T$. Hence

$$\begin{aligned} N &= \bigcup_{i=1}^{k} \{ b(g) z(e_i) B_{(T,g)}(e_i) \mid g \in T,\ |B_{(T,g)}| > 1 \} \\ &= \bigcup_{i=1}^{k} z(e_i) \{ b(g) B_{(T,g)}(e_i) \mid g \in T,\ |B_{(T,g)}| > 1 \} \end{aligned}$$

is in $\mathcal{C}$ since it is a finite union of members of $\mathcal{C}$.

Hence $b \circ z$ is well-defined and

$$\begin{aligned} \Sigma(M_{(z,b)}) &= \Sigma(N) + \Sigma(L) \\ &= \sum_{g \in T} z(g) b(g) + \sum_{g \notin T} z(g) b(g) \\ &= \sum_{g \in F} z(g) b(g) + \sum_{g \notin T} \left(z(g) \sum_{e \in F} b_e B_{(T,e)}(g) \right) \\ &= \sum_{g \in F} z(g) b(g) + \sum_{e \in F} b_e \left(\sum_{g \notin T} z(g) B_{(T,e)}(g) \right) \qquad (5.3) \\ &= \sum_{e \in F} b_e \left(\sum_{g \in E(G)} z(g) B_{(T,e)}(g) \right) \\ &= \sum_{e \in F} b_e (z \circ B_{(T,e)}) \\ &= 0, \end{aligned}$$

where Equation (5.3) follows since the sum on the right is finite. Hence $b \in \mathcal{Z}_t^{\perp}$.

To show the other containment let $b \in \mathcal{Z}_t^{\perp}$. First we show that

$$\left(\bigcup_{e \in T \cap \operatorname{supp}(b)} \operatorname{supp}(\delta(B_{(T,e)}))\right) \setminus T$$

is finite. Suppose it is infinite.

We form two infinite sets of edges $K = \{e_1, e_2, \ldots\}$ and $K' = \{e'_1, e'_2, \ldots\}$ as follows. Pick $e_1 \in T \cap \operatorname{supp}(b)$ such that $B_{(T,e_1)}$ has an edge $e'_1 \notin T$. For $i \geq 1$ pick $e_i \in T \cap \operatorname{supp}(b)$ such that $B_{(T,e_i)}$ has an edge e'_i not in $T \cup \bigcup_{1 \leq j < i} B_{e_j}$. Now recursively define

$$\beta_i = b(e_i)^{-1} - \sum_{j<i : e'_j \in B_{(T,e_i)} \setminus T} \beta_j Z_{(T,e'_j)}(e_i).$$

Finally let $z = \sum_{i=1}^{\infty} \beta_i Z_{(T,e'_i)} \in \mathcal{Z}_t$.

Then

$$\begin{aligned} z \circ b &= \sum_{g \in E(G)} z(g) b(g) \\ &= \sum_{g \in E(G)} \sum_{e'_j \in K'} \beta_i Z_{(T,e'_j)}(g) b(g). \end{aligned}$$

Consider the term corresponding to $g = e_i$:

$$\begin{aligned} \sum_{e'_j \in K'} \beta_j Z_{(T,e'_j)}(e_i) b(e_i) &= \beta_i Z_{(T,e'_i)}(e_i) b(e_i) + \sum_{e'_j \in (K' \setminus e'_i)} \beta_j Z_{(T,e'_j)}(e_i) b(e_i) \\ &= \left[b(e_i)^{-1} - \sum_{j<i : e'_j \in B_{(T,e_i)} \setminus T} \beta_j Z_{(T,e'_j)}(e_i) \right] b(e_i) \\ &\quad + \sum_{e'_j \in (K' \setminus e'_i)} \beta_j Z_{(T,e'_j)}(e_i) b(e_i) \\ &= 1, \end{aligned}$$

where the last equality follows since the two sums are equal by the definition of K'. But then $z \circ b$ contains infinitely many ones, a contradiction since we should have $z \circ b = 0$. Hence $\left(\bigcup_{e \in T \cap \operatorname{supp}(b)} \operatorname{supp}(B_{(T,e)}))\right) \setminus T$ is finite.

Now consider $A = b - \sum_{e \in T \cap \mathrm{supp}(b)} b(e) B_{(T,e)} \in \mathcal{Z}_t^{\perp}$. Since $\mathrm{supp}(A) \subseteq E(G) \setminus T$ we must have that $A = 0$ as otherwise there exists a fundamental cycle not orthogonal to A. Hence $b = \sum_{e \in T \cap \mathrm{supp}(b)} b(e) B_e \in \mathcal{Z}_t^{\perp}$.

Furthermore, for each edge $g \notin T$, we must have that the set $N_g = \{b(e)B_{(T,e)}(g) | e \in T\}$ is in $\mathcal{C}$. But since there are only finitely many $g \notin T$ such that there exists an $e \in T$ with $g \in B_{(T,e)}$ it follows that the union of the finitely many N_g's is also in $\mathcal{C}$ and hence $\{b(e) | e \in T, |B_T, e| > 1\} \in \mathcal{C}$. Hence $b \in \mathcal{B}_w$. ∎

Corollary 5.12 *Let cG be a compactified locally finite graph. Let $\mathbb{F}$ be a field with caste $\mathcal{C}$ and let the edge space be $\mathbb{F}^E$. Then the following two properties of $\mathcal{B}_w$ hold:*

1. $\mathcal{B}_f \subseteq \mathcal{B}_w \subseteq \mathcal{B}_t$
2. $\mathcal{B}_w^{\perp} = \mathcal{Z}_t$

Proof. The first property follows since we trivially have $\mathcal{B}_f \subseteq \mathcal{B}_w$ and $\mathcal{B}_w = \mathcal{Z}_t^{\perp} \subseteq \mathcal{Z}_f^{\perp} = \mathcal{B}_t$. For the second property we immediately have from the first property that $\mathcal{B}_w^{\perp} \subseteq \mathcal{B}_f^{\perp} = \mathcal{Z}_t \subseteq Z_t^{\perp^{\perp}} = \mathcal{B}_w^{\perp}$. ∎

In summary we have the following relations between the various cycle and bond spaces of a compactified locally finite, finitely connected graph:

1. $\mathcal{Z}_f \subseteq \mathcal{Z}_t$
2. $\mathcal{B}_f \subseteq \mathcal{B}_w \subseteq \mathcal{B}_t$
3. $\mathcal{B}_t^{\perp} = \mathcal{Z}_f$
4. $\mathcal{Z}_f^{\perp} = \mathcal{B}_t$
5. $\mathcal{B}_f^{\perp} = \mathcal{B}_w^{\perp} = \mathcal{Z}_t$
6. $\mathcal{Z}_t^{\perp} = \mathcal{B}_w$.

The final question which we have not attempted to solve yet is what is the subspace A such that $A^{\perp} = \mathcal{B}_f$? Such an A must be contained in $\mathcal{Z}_t$ since $A \subseteq A^{\perp^{\perp}} = \mathcal{B}_f^{\perp} = \mathcal{Z}_t$.

5.3 The Strong Bond Space

In [11], Vella and Richter introduce another bond space, the *strong bond space* $\mathcal{B}_s$. The purpose of this short section is to show that this bond space is in fact redundant for locally finite graphs.

Let G be a locally finite, connected graph with compactification cG. Let $\mathbb{F}$ be a field with caste $\mathcal{C}$ so that the edge space of G is $\mathbb{F}^E$.

Let $\mathcal{S} = \{S \subseteq \mathbb{F}^E \mid \mathcal{B}_f \subseteq S,\ S \text{ is closed under sums of thin families}\}$. So $S \in \mathcal{S}$ if it contains the finite bond space and whenever F is a thin set of vectors in $\mathcal{S}$, the thin span of F is also in S.

The *strong bond space* is the set $\mathcal{B}_s = \bigcap_{S\in\mathcal{S}} S$. Note that $\mathcal{B}_f \subseteq \mathcal{B}_t \subseteq \mathcal{B}_s$ and hence $\mathcal{B}_s^\perp \subseteq \mathcal{B}_t^\perp \subseteq \mathcal{B}_f^\perp$.

Theorem 5.13 *Let cG be a compactified locally finite graph. Let $\mathbb{F}$ be a field with caste $\mathcal{C}$ and let the edge space be $\mathbb{F}^E$. Then $\mathcal{B}_s^\perp = \mathcal{B}_t^\perp$. Furthermore, $\mathcal{B}_s = \mathcal{B}_t$.*

Proof. We already have that $\mathcal{B}_s^\perp \subseteq \mathcal{B}_t^\perp = \mathcal{Z}_f$. So we only need the reverse containment. Now for any $S \in \mathcal{S}$ we have $S^\perp \subseteq \mathcal{B}_s^\perp$. Thus we need to show that for every $z \in \mathcal{Z}_f$ there exists some $S \in \mathcal{S}$ such that $z \in S^\perp$. Consider the set $S = \{z\}^\perp$. Then we know that $\mathcal{B}_t \subseteq S$ so we simply need to show that $\{z\}^\perp$ is closed under sums of thin families. But if $\sum \alpha_b b$ is a sum of a thin family then $z \circ (\sum \alpha_b b) = 0$ by Lemma 3.6 since z has finite support and $b \in \{z\}^\perp$. Therefore $S \subseteq \{z\}^\perp$ and it follows that $\mathcal{Z}_f = \mathcal{B}_s^\perp$.

Now we have $\mathcal{B}_s \subseteq \mathcal{B}_s^{\perp\perp} = \mathcal{B}_t^{\perp\perp} = \mathcal{Z}_f^\perp = \mathcal{B}_t$ and hence $\mathcal{B}_s = \mathcal{B}_t$. ∎

5.4 An Edge Tripartition Theorem

This section details some easy consequences of the main theorems of this thesis. The first result should probably be expected.

Theorem 5.14 *Let G be a locally finite, connected graph with compactifications c_1G and c_2G such that $c_1G \leq c_2G$. Let $\mathbb{F}$ be a field with caste $\mathcal{C}$ and let the edge space of G be $\mathbb{F}^E$. Then Let $\mathcal{Z}_t^i, \mathcal{B}_f^i$, etc, be the thin cycle space, finite bond space, etc, in c_iG. Then we have:*

1. $\mathcal{Z}_f^2 = \mathcal{Z}_f^1$,
2. $\mathcal{B}_f^1 \subseteq \mathcal{B}_f^2$,
3. $\mathcal{Z}_t^2 \subseteq \mathcal{Z}_t^1$,
4. $\mathcal{B}_w^1 \subseteq \mathcal{B}_w^2$, *and*
5. $\mathcal{B}_t^1 = \mathcal{B}_t^2$.

Proof. Clearly the finite cycle space is independent of the compactification of G. Therefore (1) and (5) follow from Theorem 5.8 and Theorem 5.9

A cut in c_1G is clearly a cut of c_2G and hence $\mathcal{B}_f^1 \subseteq \mathcal{B}_f^2$ where $\mathcal{B}_f^i$ is the finite bond space of c_iG. Hence $(\mathcal{B}_f^2)^\perp \subseteq (\mathcal{B}_f^2)^\perp$ and so $\mathcal{Z}_t^1 \subseteq \mathcal{Z}_t^2$. The inclusion in (4) follows from (3) by using Theorem 5.11 and Lemma 3.7. ∎

We again point out the most surprising result of our investigations: the thin bond space is independent of the compactification of G!

The second result is a direct application of Proposition 3.1 since the same proof holds with the circle product.

Theorem 5.15 *Let G be a locally finite, finitely connected graph with compactification cG and let $\mathbb{F}$ be a field with caste $\mathcal{C}$. Then we have:*

- $(\mathcal{Z}_t + \mathcal{B}_t)^\perp = \mathcal{Z}_f \cap \mathcal{B}_f$,
- $(\mathcal{Z}_f + \mathcal{B}_f)^\perp = \mathcal{Z}_t \cap \mathcal{B}_t$,
- $(\mathcal{Z}_t + \mathcal{B}_f)^\perp = \mathcal{Z}_t \cap \mathcal{B}_w$, *and*

- $(\mathcal{Z}_f + \mathcal{B}_t)^\perp = \mathcal{Z}_f \cap \mathcal{B}_t$.

Proof. The proofs follow directly from Proposition 3.1 after noting that $\mathcal{Z}_f \cap \mathcal{B}_w = \mathcal{Z}_f \cap \mathcal{B}_f$ by virtue of the finiteness of the elements in $\mathcal{Z}_f \cap \mathcal{B}_w$. ∎

The following result was the impetus to prove Theorem 3.8.

Theorem 5.16 *Let G be a locally finite, connected graph with compactification cG and let $\mathbb{F}$ be a field with caste $\mathcal{C}$. Let $\mathbb{F}^E$ be the edge space of G. Then*

$$\mathcal{Z}_t + \mathcal{B}_t = (\mathcal{Z}_f \cap \mathcal{B}_f)^\perp.$$

Proof. Since $\mathcal{Z}_t$ and $\mathcal{B}_t$ are the thin spans of thin generating sets we can apply Theorem 3.8. ∎

We can use Theorem 5.16 to prove an infinite version of the Edge Tripartition Theorem. Let G be a locally finite, connected graph with compactification cG with edge space over the field $\mathbb{GF}(2)$. Call the members of the set $\mathcal{Z}_f \cap \mathcal{B}_f$ the *finite bicycles* of G. Then the proof of the finite Edge Tripartition Theorem carries over directly, except we replace "even subgraph" with "2-edge connected even subgraph" (since we need to exclude the case of there being an end in the subgraph containing a single ray, as otherwise the subgraph is not in $\mathcal{Z}_t$). Here we mean 2-edge connected in the sense of topologically connected, that is, a graph is 2-edge connected if the removal of any 2 edges disconnects the graph as a topological space.

Theorem 5.17 (Edge Tripartition Theorem for Finite Bicycles (cf. Theorem 2.5)) *Let G be a compactified, locally finite, finitely connected graph. Then for every edge e in G exactly one of the following three situations holds for e:*

1. *e is contained in a finite bicycle,*

2. *There exists a cut b such that $b + e$ is a 2-edge connected even subgraph, or*

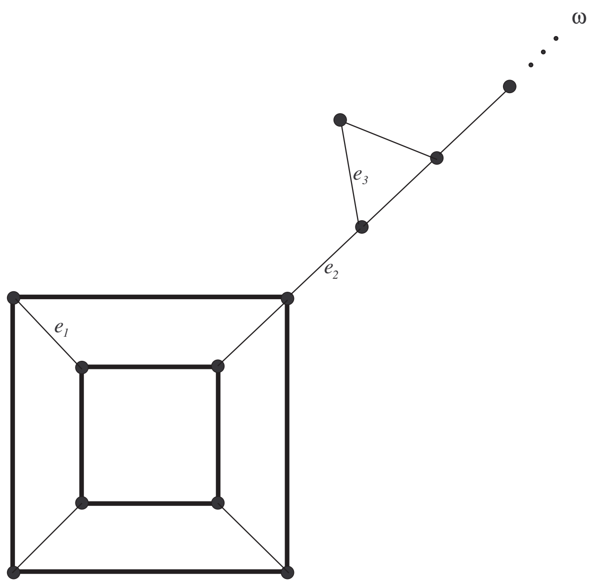

Figure 5.3: Examples of edge types in Theorem 5.17.

3. There exists a 2-edge connected even subgraph H such that $H + e$ is a cut. ∎

We call an edge *type i* if it falls into case i in Theorem 5.17.

As an example consider the (admittedly contrived) graph in Figure 5.3 with the Freudenthal compactification. The darkened edges form a finite bicycle. The reader can verify that edge e_1 is type 1, edge e_2 is type 2 (the empty graph is in $\mathcal{Z}_t$!) and e_3 is type 3.

In some sense Theorem 5.17 does not have the right feel to it. Why should we restrict ourselves to finite bicycles? For example, in the 2-way infinite ladder, the two outer infinite cycles form a subgraph that is in the thin cycle space (obviously) as well as in the thin bond space (it is the thin sum of the cut induced by every second rung).

Let $\mathcal{Z}_t \cap \mathcal{B}_t$ be called the *thin bicycle space* . Theorem 5.17 immediately implies that $(\mathcal{Z}_t \cap \mathcal{B}_t)^{\perp} \subseteq \mathcal{Z}_t + \mathcal{B}_t$. The reverse inclusion seems to be the difficult case (if true) since members of the thin bicycle space and members of $\mathcal{Z}_t + \mathcal{B}_t$ both could have infinite support, in which case determining their $\circ$-product could be impossible (at least in the framework we have presented).

Another guess could be that $\mathcal{Z}_f+\mathcal{B}_f = (\mathcal{Z}_t \cap \mathcal{B}_t)^{\perp}$. We have the inclusion $\mathcal{Z}_f+\mathcal{B}_f \subseteq (\mathcal{Z}_t \cap \mathcal{B}_t)^{\perp}$. But it is possible that $(\mathcal{Z}_t \cap \mathcal{B}_t)$ has only finite elements (as in Figure 5.3) and so a member of $(\mathcal{Z}_t \cap \mathcal{B}_t)^{\perp}$ can have infinite support.

Of course there are several other sets of the form $\mathcal{B}_x + \mathcal{Z}_y$ which we could investigate. It seems that Theorem 3.8 could be somehow applied except we would need to prove further information about a solution x to $Ax = b$, e.g. that the variables in x corresponding to, say the fundamental cycles, are non-zero only finitely often. This seems to be an unlikely consequence of the proof of Theorem 3.8.

Chapter 6

Conclusion and Future Research

The investigations in this thesis can be broken down into three main parts. The first is a study into the solvability of an infinite system of finite linear equations with infinitely many variables. We showed that such a system can always be solved in such a way that each variable can be found in a finite (although undetermined) amount of time.

The second focus of study was the lattice of compactifications of a locally finite graph. We showed that the set of compactifications of a locally finite graph indeed forms a lattice. The results in this part of the thesis were only new in the sense of their application to infinite graphs.

The final part and the main focus of this thesis was the study of the different cycle spaces and their orthogonal complements. We defined two cycle spaces and three bond spaces and determined the orthogonal complement relations between them. We were able to find a tripartition of the edges of a locally finite, infinite graph, although the tripartition is not quite as satisfying as it could possibly be.

Throughout this thesis we have almost always considered only locally finite graphs. This is due to the fact that in this case, it makes sense to define a vector product as often one of the vectors we wish to multiply has finite support. Therefore the logical next step would be to attempt to generalize our results to graphs with vertices

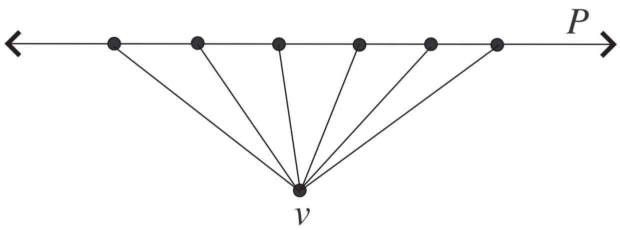

Figure 6.1: A Non-locally Finite Graph.

of infinite degree. However this presents some problems, not only algebraically, but topologically as well.

One immediate problem is that our topology may no longer be weakly Hausdorff. Another is that if there is a vertex of infinite degree, the graph and its Halin ends may no longer be a compact space! Yet another problem is that there may exist a (topological) spanning tree whose fundamental cycles do not form a thin set.

For example consider the 2-way infinite path P plus a vertex v such that v is adjacent to every vertex in P. See Figure 6.1 for an illustration.

Take as a spanning tree the set of edges in P together with a single edge between P and v. Then it is easy to see that this spanning tree can produce infinitely many fundamental cycles using the edge between P and v.

On the other hand if we take as a spanning tree the set of all edges between P and v (i.e. the "star" with centre v) then the fundamental cycles clearly do form a thin set.

Thus it is not clear which spanning trees we should allow in this context. However Diestel and Kühn [7] have derived a characterization of the spanning trees that generate the cycle space, however they use the classical topology extended to the ends in a certain way. Therefore it would be interesting to see how their results extend to the natural topology of Chapter 4.

In any case bonds may no longer be finite (as the "star" example above shows) so it is unclear how the orthogonality results of Chapter 5 extend to graphs with vertices of infinite degree.

Another avenue of research is to study the extended finite bond space $\mathcal{B}_w$. Although we know that $\mathcal{B}_w = \mathcal{B}_f$ when the edge space is over a finite field, it would be pleasing to have a much more aesthetic description of $\mathcal{B}_w$ over other fields.

It seems clear that we have barely scratched the surface of the study of cycle spaces of infinite graphs. We are sure that deep results await anyone who dares venture further into this infinite space.

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