Stochastic Perishable Inventory Systems: Dual-Balancing and Look-Ahead Approaches

by

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A thesis
presented to the University Of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Applied Science
in
Management Sciences

Waterloo, Ontario, Canada, 2016

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We study a single-item, multi-period, stochastic perishable inventory problem under both backlogging and lost-sales circumstances, with and without an order capacity constraint in each period.

We first model the problem as a dynamic program and then develop two heuristics namely, Dual-Balancing (DB) and Look-Ahead (LA) policies, to approximate the optimal inventory level at the beginning of each period. To characterize the holding and backlog cost functions under the proposed polices, we introduce a truncated marginal holding cost for the marginal cost accounting scheme. Our numerical examples demonstrate that both DB and LA policies have a possible worst-case performance guarantee of two in perishable inventory systems under different assumptions, and the LA policy significantly outperforms the DB policy in most situations.

We also analyze the target inventory level in each period (the inventory level at the beginning of each period) under different policies. We observe that the target inventory level under the LA policy is not larger than the optimal one in each period in systems without an order capacity constraint.

Acknowledgements

First and foremost, I would like to express my sincere gratitude to my supervisor Dr. Hossein Abouee Mehrizi for his constant encouragement, professional instructions and patience. Without his help, the completion of this thesis would not have been possible. I feel very lucky to have been a student of such an insightful professor. I also thank Dr. Bissan Ghaddar and Dr. Fatma Gzara for their time and patience in reviewing my thesis.

Secondly, I want to thank all my friends at Waterloo for the great time. Yangzi and Haoran have also helped me a lot.

Finally, my thanks would go to my beloved Mum, Dad and Xiaohan, who have always been helping and supporting me out of difficulties.

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1 Introduction

Perishable products are very common in the real life. The main difference between perishable and non-perishable products is that the former must be consumed within a relatively short period of time before they become outdated, while the latter one can be kept in inventory for a long time, until they are used to satisfied demand. For example, food products, such as fresh fruit and dairy product, usually have a short quality guarantee period. Besides, blood products are also perishable and need to be used in a short time. For example, blood platelets are very precious and must be used within six-days [1], and blood plasma also deteriorates with time [2]. If perishable products are not consumed in time, outdating can lead to a significant financial loss. For example, about 321,000 units of apheresis platelets outdated in US in 2011, accounting for 12.8% of the total processed/produced amount [3]. Another report indicates that the average outdating rate of perishable products for retailers and distributors was around 1.21% in 2007 [4]. On the other hand, approximately 50% of the causes of outdating across a value chain can be mitigated effectively through improved planning practices [4]. These facts indicate the importance of developing efficient inventory management policies for perishable products.

In this study, we study a stochastic periodic-review inventory system over a finite planning horizon with a perishable product, which has a fixed lifetime. Intuitively, the lifetime that we consider needs to be no longer than the length of the planning horizon. Also, if the lifetime is 1 period, there will be no excess inventory left at the end of each period, no matter whether all demand has been satisfied or not, and we can solve the problem for one period at a time, separately. Therefore, in our study, we assume that the lifetime of the products is longer than 1 period.

We assume that products in the inventory are used based on the first-ordered first-consumed (FOFC) policy to simplify the modeling of the problem. This is a common

assumption in the vast majority of the literature that products in the inventory are used based on the FOFC policy [5], which is quite reasonable in practice. For example, grocery managers usually put the oldest products in the most convenient place on shelves for customers, in order to reduce the outdating rate. The FOFC policy is also the foundation of many algorithms and policies for perishable inventory problems. For example, it is a critical assumption for the marginal cost accounting scheme that matches the available products and demand in perishable systems [6], [7].

In general, there are two different ways to treat unmet demand at the end of each period, either backlog them (systems with backlogs) or lose them (systems with lost-sales). In backlogging problems, unmet demand is backlogged and should be satisfied in the next period while in lost-sales problems, unmet demand is lost. This difference impacts the period state transition and the amount of demand at the beginning of each period, which is zero in lost-sales problems and the number of backlogs in backlogging problems. Order capacity is another feature in the inventory problems, which can limit the order quantity in each period and, thus, affect the ordering policy and the total cost of the system. In this research, we consider both backlogging and lost-sales problems, with and without an order capacity constraint in each period.

The optimal inventory policy for the multi-period inventory systems with a perishable product can be characterized using a dynamic program. However, the dimension of the dynamic program depends on the length of the lifetime, and the optimal order quantity depends on both the age distribution of the on-hand inventory and the time of the current period. Therefore, the dynamic problem usually has a very large number of state variables, action spaces and outcome spaces [1], and the computation of the program is usually intractable due to the "curse-of-dimensionality". Thus, many efforts were put into developing and examining effective and efficient heuristics to approximate the optimal policy. For example, the Look-Ahead (LA)

policy has been proposed for systems without perishable products [8] and the Dual-Balancing (DB) policy has been applied to both problems with and without perishable products [6], [7]. In this study, we first propose algorithms based on both DB and LA policies for the inventory control of the perishable products with limited order capacity. Then, we investigate the performance of the proposed algorithms using numerical examples.

The main contributions of this thesis are summarized as follows.

Algorithms. We use the marginal cost accounting scheme and nested structure to extend DB and LA policies for perishable inventory problems with different modeling features. When there is no order capacity constraint, the marginal cost accounting was presented in the literature [7], [9], and a nested structure was used to model problems with perishable products [7]. We extend the marginal cost accounting scheme and nested structure to perishable problems with an order capacity constraint in each period.

Worst-case Performance Based on Numerical Examples. For a perishable inventory system without an order capacity constraint, the DB policy was proved to have a worst-case performance guarantee of two in the literature [10]. In this study, based on our numerical examples, we observe that the expected total costs under the LA policy in perishable systems without an order capacity constraint, as well as both DB and LA policies in systems with an order capacity constraint, are less than twice the optimal expected total cost.

Performance of LA and DB Policies. We compare performance of both DB and LA policies in perishable inventory systems under different assumptions. We observe that in perishable inventory systems without an order capacity constraint, the LA policy has much better performance, on average, than the DB policy, under both backlogging and lost-sales assumptions. In problems with an order capacity constraint, while the LA policy outperforms the DB policy under the backlogging assumption, it does not always outperform the DB policy under the lost-sales assumption.

Truncated marginal holding cost. We introduce a new truncated marginal holding cost for the marginal cost accounting scheme. We show that with the truncated marginal holding cost, we do not need to consider the impact of one decision on the holding cost in all following periods. We also examine the effectiveness of this truncated marginal holding cost.

The rest of this thesis is organized as follows. In Chapter 2, we review the literature related to the stochastic periodic-review perishable inventory systems and the DB and LA policies. In Chapter 3, we model a perishable inventory problem with both backlogs and lost-sales, with and without an order capacity constraint. In Chapter 4, we provide some insights into the perishable inventory systems. In Chapter 5, we conclude our results and present some possible directions for the future study.

2 Literature Review

Stochastic periodic-review perishable inventory systems have attracted many researchers since 1960's. At early stages, the focus of the studies was on developing optimal policies using dynamic programs. The fundamental characterization of the optimal ordering policy with a two-period lifetime is provided by Nahmias and Pierskalla [11]. Then, Nahmias [12] and Fries [13] independently study the optimal policy for the general lifetime problem, focusing on backlogging and lost-sales problems, respectively. Nahmias [12] uses a per-unit outdating cost, ordering cost, holding cost, and per-unit per-period shortage cost, which is a "standard" cost structure [14]. They show that, due to perishability, the structure of the optimal policy is quite complex. Also, the computation of the optimal policy using a dynamic problem is tractable only if the lifetime of the product is short. This complexity of the optimal policy is reinforced by Cohen [15], who derives an explicit closed-form solution for a two-period problem, and discusses procedures to obtain the solution for the *m*-period case.

Many researchers put effort into developing heuristic policies for both backlogging and lost-sales models, which would be easier to define and implement while remaining close to the optimal policy [14]. Brodheim et al. [16] evaluate a class of inventory policies and obtain some measures for the inventory system, such as the probability of shortage, as well as the easily computable bounds for the policies. Nahmias [17] proposes three heuristics, including the critical number and linear approximations, and compares their performance using a simulation model.

Nahmias [18] constructs a bound on the outdating cost which is only a function of the total on-hand inventory, and develops a myopic policy by substituting the outdating cost with the bound. Then, Nahmias [19] compares the two models developed by Nahmias [12] and Fries [13] with the myopic policy, and shows that when the remaining periods in the horizon is more than

the lifetime, the optimal policy in Fries [13] is the same as a discounted version of the one characterized in Nahmias [12]. Nahmias [20] uses a bounded expected outdating function and a refined transfer function to approximate the problem, focusing on a two-period problem, which yields to a one-dimensional state variable. For problems with lost-sales, Nandakumar and Morton [21] derive upper and lower bounds for the order quantity in each period, and examine Nahmias' approximations with a weighted average of the upper and lower bounds. The results show that all heuristics have good performance. Cooper [22] considers a fixed-critical number ordering policy and derives new families of bounds on the expected number of outdates per period. The performance of the critical-number policies is close to the optimal policy based on his numerical study. Recently, Chen et al. [23] propose two heuristic policies for joint inventory and pricing control problems with both continuous and discrete demand, and both backlogging and lost-sales cases. They indicate some monotonicity properties of the optimal policy and identify bounds for the optimal order-up-to inventory level. Li et al. [24] analyze the optimal solution-structure of a two-period lifetime problem and develop a base-stock/list-price heuristic policy.

In addition to the above literature focusing on periodic-review perishable inventory systems, many other aspects of perishable inventory problems were considered as well. We refer readers to Kempf et al. [14] for a comprehensive literature review.

Recently, Levi et al. [6] utilize a marginal cost accounting scheme and cost balancing idea to develop an approximate policy for stochastic periodic-review inventory problems, and propose a Dual-Balancing as well as a randomized Dual-Balancing algorithms. They show that both the Dual-Balancing and randomized Dual-Balancing policies have a worst-case performance guarantee of two. Then, Levi et al. [25] apply the marginal cost accounting scheme and cost balancing idea to a stochastic inventory problem with an order capacity constraint. For lost-sales problems, Levi et al. [10] prove that both the Dual-Balancing policy and randomized

Dual-Balancing policy have a worst-case performance guarantee of two. They also consider a capacitated model using forced lost-sales cost and prove that the Dual-Balancing policy has a worst-case performance guarantee of two, even when there is a capacity constraint on ordering in each period. The marginal cost accounting scheme and Dual-Balancing policy have been also used in perishable inventory systems. Chao et al. [7] develop the first approximation algorithm for a periodic-review perishable inventory system with a perishable product for both backlogging and lost-sales models. They propose a proportional-balancing policy and a dual-balancing policy with discounted marginal holding cost, and indicate that both policies have a computational complexity of O(mT), which is very efficient compared to the optimal policy. Zhang et al. [9] develop a marginal-cost Dual-Balancing policy for a periodic-review, fixed-lifetime perishable inventory control problem, and prove that it has a worst-case performance guarantee of two. This result works for both backlogging and lost-sales problems due to a zero lead time. Then, they prove that a myopic policy under the marginal cost accounting scheme provides a lower bound on the optimal ordering quantity, based on a given on-hand inventory.

Another approach related to our study is the Look-Ahead optimization approach. Truong [8] studies a single-item, multi-period stochastic inventory problem and shows that the Look-Ahead policy has a worst-case performance guarantee of two. Also, the Look-Ahead policy, on average, performs well within 3.9% of the optimal policy, compared to 19.7% for the Dual-Balancing policy, based on the numerical study.

3 Models and Methodologies

3.1 Perishable Inventory Systems

We consider a periodic-review stochastic perishable inventory problem, in a finite planning horizon with T periods, where each period is denoted by t, t=1,…, T. We distinguish a random variable and its realization by using capital letters and lower case letters, respectively. The demand in period t is denoted by D^t, t=1,…, T, which is a random variable.

At the beginning of each period t, an information set f_t can be observed, containing the information we can have at that time. The information set has two parts: realized demands d^1,…, d^{t-1}, and all other information w^1,…, w^t related to the problem in each period. Thus, the information set f_t is one specific realization of the random vector F_t=(D^1,…, D^{t-1}; W^1,…, W^t). Furthermore, when f_t is observed, the future demands will have a known conditional joint distribution, including the upcoming current demand D^t, which means $\mathrm{E}[D^s \mid f_t]$ is well-defined and finite for each period s, s=t,…, T. When we consider different policies, the information set f_t and the conditional joint distribution is independent of a specific policy [6].

Let lead time be 0, so the units ordered in one period can arrive and become available to meet demand in the same period before demand occurs. We apply the first-ordered first-consumed (FOFC) policy to calculate the on-hand inventory at the end of each period. This is a common assumption in perishable problems, because units in perishable problem have a lifetime, and those having a shorter remaining lifetime are usually used to satisfy customer demand prior to those arriving later. The lifetime of the product is denoted by m, m=2,…, T. This means that after one unit enters the system, it can stay for at most m periods if it has not been used to meet customer demand by the end of the m-th periods. For example, suppose there is no on-hand inventory at the beginning of period t, and q^t units are ordered. These q^t units

arrive in the current period t and are used to meet the demand d^t. Let m equal to 3 and assume that the inventory level at the beginning of period t is zero. Then, if $q^t > d^t + d^{t+1} + d^{t+2}$, there will be still some excess units of q^t at the end of period t+2. But, they cannot be retained in the system because they have been staying for 3 periods, t, t+1 and t+2. As a result, the remaining $q^t - (d^t + d^{t+1} + d^{t+2})$ units will perish and leave the system at the end of period t+2. On the other hand, if $q^t < d^t + d^{t+1} + d^{t+2}$, all q^t units have been used before they reach their lifetime and perish. Some new units may arrive in period t+1 and t+2, but they need to be kept in the system before all q^t units arriving in period t have been consumed, due to the FOFC policy.

At the beginning of period t, the on-hand inventory consists of m-1 types of units, whose remaining lifetime is from m-1 periods to 1 period. Let $x^{t,i}$ represent the number of units having a remaining lifetime of i periods, i=1,…, m-1, in the inventory at the beginning of period t. Then, the total inventory at the beginning of period t is $X^t = \sum_{i=1}^{m-1} x^{t,i}$. Let non-negative integer q^t denote the number of units ordered in period t. Because lead time is 0, the q^t units will arrive and become available in current period t before demand occurs, with a lifetime of m periods. So in period t, the total inventory used to meet demand is $Y^t = X^t + q^t$, which is the base-stock level in period t.

As we discussed in Section 1, two general features of inventory problems in the literature are whether excess demand is backlogged or lost, and whether there is a limit on the ordering level in each period. In problems without an ordering capacity constraint, the number of ordered units can be any non-negative integer, but in problems with an ordering capacity constraint **u**, the order quantity cannot surpass the capacity limit.

In a periodic-review stochastic perishable inventory control problem, several types of cost are incurred. A per-unit ordering cost $\hat{c}^t$ occurs when one unit is ordered in period t. A per-unit holding cost is denoted by $\hat{h}^t$, which occurs at the end of period t for each excess unit that

is still in inventory, representing a cost of keeping this unit from period t to t+1. These excess units can be held in inventory for a long time if the demands in the following periods are not high enough to consume all available units. Also, because one unit can stay in the system for at most m periods, the holding cost incurred by one order occurs in at most m-1 following periods. If the demand in period t cannot be satisfied immediately, there will be a penalty, which is $\hat{b}^t$ per unit of unsatisfied demand. This cost has different meanings in the backlogging and lost-sales problems. In a backlogging problem, the unsatisfied demand is backlogged in the system and can be eliminated by ordering more in the following period. Therefore, this penalty means the compensation to the unsatisfied demand and will exist in only one period, if there is no ordering capacity constraint. Systems with an ordering capacity constraint in each period will be discussed in more detail later. In a lost-sales problem, the unfulfilled demand will leave the system at the end of the period. Thus, the penalty is the loss of value caused by the unsatisfied demand and will occur once. For simplicity, the per-unit penalty in both types of problems are denoted by $\hat{b}^t$. The fourth cost in a perishable problem is a per-unit outdating cost $\hat{o}^t$. It occurs when a unit has been in the system for m periods and perishes, before been used to meet demand.

3.1.1 System Evolution in Each Period

4.1.2.1 Systems with Backlogs and No Ordering Capacity Constraint

In this section, we discuss a complete process of a perishable inventory system with backlogs, integer demand and order quantity, and no ordering capacity constraint.

1. At the beginning of period t, t=1,…, T, information set f_t and on-hand inventory $x^{t,i}$, i=1,…, m-1, are observed, where

$$\begin{cases} x^{t,i}=0,\ i=1,\ \ldots,m-1 & t=1 \\ x^{t,m-1}<0,\ x^{t,i}=0,\ i=1,\ \ldots,m-2 & t\in\{2,\cdots,T\},\ \text{means backlogs in period } t-1 \\ x^{t,i}\geq 0,\ i=1,\ \ldots,m-1 & t\in\{2,\cdots,T\},\ \text{means no backlogs in period } t-1 \end{cases}.$$

Therefore, the total inventory level is $X^t=\sum_{i=1}^{m-1}x^{t,i}$. The conditional joint distribution of

upcoming demand D^t is known based on the information set f_t.

2. The system orders q^t units, and they arrive and become available, having a lifetime of m periods. In systems without an ordering capacity constraint, the feasible range of q^t is $\{0,\cdots,\infty\}$. The total cost of ordering is $\hat{c}^t q^t$. Also, the on-hand inventory level increases from X^t to $Y^t=X^t+q^t$.

3. Demand d^t occurs, and the on-hand inventory is used to meet this demand. The $x^{t,1}$ units having only 1-period lifetime, are consumed first. If $x^{t,1}>d^t$, the remaining $x^{t,1}$-d^t units will perish and an outdating cost of $\hat{o}^t(x^{t,1}-d^t)$ occurs. Otherwise, the $x^{t,2}$ units having 2-period lifetime are used to meet the excess d^t-$x^{t,1}$ units of demand, and so on, until all inventory is consumed. If $d^t>Y^t$, the demand cannot be satisfied completely by the current inventory. The remaining d^t-Y^t units of demand becomes backlogs that need to be met in period t+1, with a backlogging penalty of $\hat{b}^t(d^t-Y^t)$. If $d^t<Y^t$, the remaining Y^t-d^t units result in a holding cost of $\hat{h}^t(Y^t-d^t)$.

4. At the end of period t, the remaining lifetime of the excess units of the on-hand inventory decreases by 1 period. Note that if $x^{t,i}<\left(d^t-\sum_{j=1}^{i-1}x^{t,j}\right)^+$, where $x^+=\max\{x,0\}$, all $x^{t,i}$ units will be used to meet demand. Otherwise, $x^{t,i}-\left(d^t-\sum_{j=1}^{i-1}x^{t,j}\right)^+$ units remain for period t+1, becoming $x^{t+1,i-1}$, i=2,…, m-1. In period t, the q^t units have a lifetime of m periods, and they will meet the excess demand after all X^t have been used. Therefore, in a backlogging problem, if $q^t<\left(d^t-\sum_{j=1}^{m-1}x^{t,j}\right)^+$, backlogs occur and the number of backlogs is $\left|q^t-\left(d^t-\sum_{j=1}^{m-1}x^{t,j}\right)^+\right|$. These backlogs will retain in the system for period t+1 as a negative inventory. So the new inventory level at the beginning of period t+1 can be calculated using (similar to Eq. (1) in [7])

$$
\begin{aligned}
x^{t+1,i} &= \left(x^{t,i+1} - \left(d^t - \sum_{j=1}^{i} x^{t,j} \right)^+ \right)^+ , i \in \{1,\cdots,m-2\}, \\
x^{t+1,m-1} &= q^t - \left(d^t - \sum_{j=1}^{m-1} x^{t,j} \right)^+ .
\end{aligned} \tag{1}
$$

Then, d^t will be added into f_{t+1}, which will be the given information for period t+1, as well as $x^{t+1,i}$, i=1, …, m-1.

4.1.2.2 Systems with Lost-Sales and No Ordering Capacity Constraint

The process of a lost-sales problem without an order capacity constraint can be considered as follows.

1. At the beginning of period t, t=1, …, T, information set f_t and on-hand inventory $x^{t,i}$ are observed, where

$$
\begin{cases} x^{t,i} = 0, i = 1, \ \ldots, m-1 & t = 1 \\ x^{t,i} \geq 0, i = 1, \ \ldots, m-1 & t \in \{2,\cdots,T\} \end{cases}.
$$

   The total inventory level is $X^t = \sum_{i=1}^{m-1} x^{t,i}$. The conditional joint distribution of the upcoming demand D^t is known based on the information set f_t.

2. The system orders q^t units, and they arrive and become available, having a lifetime of m periods. In systems without an order capacity constraint, the feasible range of q^t is $\{0,\cdots,\infty\}$. The total cost of ordering is $\hat{c}^t q^t$. Also, the on-hand inventory level increases from X^t to Y^t=X^t+q^t.

3. Demand d^t occurs, and the on-hand inventory is used to meet this demand. The $x^{t,1}$ units having only 1-period lifetime are consumed first. If $x^{t,1}$>d^t, the remaining $x^{t,1}$-d^t units will perish and an outdating cost of $\hat{o}^t(x^{t,1} - d^t)$ occurs. Otherwise, the $x^{t,2}$ units having 2-period lifetime are used to meet the excess d^t-$x^{t,1}$ units of demand, and so on, until all inventory is consumed. If d^t>Y^t, the demand cannot be satisfied completely by the current inventory. The remaining d^t-Y^t units of demand will be lost and leave the system, with a

lost-sales penalty of $\hat{b}^t(d^t - Y^t)$. If $d^t<Y^t$, the remaining Y^t-d^t units result in a holding cost of $\hat{h}^t(Y^t - d^t)$.

4. At the end of period *t*, the remaining lifetime of the excess units of the on-hand inventory decreases by 1 period. Note that if $x^{t,i} < \left(d^t - \sum_{j=1}^{i-1} x^{t,j}\right)^+$, all $x^{t,i}$ units will be used to meet demand. Otherwise, $x^{t,i} - \left(d^t - \sum_{j=1}^{i-1} x^{t,j}\right)^+$ units remain for period t+1, becoming $x^{t+1,i-1}$, i=2,…, m-1. In period *t*, the q^t units have a lifetime of *m* periods, and they will meet the excess demand after all X^t have been used. Therefore, in a lost-sales problem, if $q^t < \left(d^t - \sum_{j=1}^{m-1} x^{t,j}\right)^+$, the excess demand will leave the system. If $q^t > \left(d^t - \sum_{j=1}^{m-1} x^{t,j}\right)^+$, the $q^t - \left(d^t - \sum_{j=1}^{m-1} x^{t,j}\right)^+$ remaining units will become $x^{t+1,m-1}$. So the new inventory level at the beginning of period t+1 can be calculated using (similar to Eq. (18) in [7])

$$\begin{aligned} x^{t+1,i} &= \left(x^{t,i+1} - \left(d^t - \sum_{j=1}^{i} x^{t,j} \right)^+ \right)^+, i \in \{1,\cdots,m-2\}, \\ x^{t+1,m-1} &= \left(q^t - \left(d^t - \sum_{j=1}^{m-1} x^{t,j} \right)^+ \right)^+. \end{aligned} \tag{2}$$

4.1.2.3 Systems with Backlogs and an Ordering Capacity Constraint

In a backlogging problem with an order capacity constraint, the order quantity is limited and the complete process in period *t* can be listed as follows.

1. At the beginning of period *t*, t=1, …, *T*, information set f_t and on-hand inventory $x^{t,i}$, i=1, …, m-1, are observed, where

$$\begin{cases} x^{t,i} = 0, i = 1, \ \ldots, m-1 & t = 1 \\ x^{t,m-1} < 0, x^{t,i} = 0, i = 1, \ \ldots, m-2 & t \in \{2,\cdots,T\}, \text{means backlogs in period } t-1 \\ x^{t,i} \geq 0, i = 1, \ \ldots, m-1 & t \in \{2,\cdots,T\}, \text{means no backlogs in period } t-1 \end{cases}.$$

The total inventory level is $X^t = \sum_{i=1}^{m-1} x^{t,i}$ and the conditional joint distribution of upcoming demand D^t is known based on the information set f_t.

2. The system orders q^t units, and they arrive and become available, having a lifetime of *m*

periods. In a system with an order capacity constraint **u**, the feasible range of q^t is $\{0,\cdots,u^t\}$. The total cost of ordering is $\hat{c}^t q^t$. Also, the on-hand inventory level increases from X^t to $Y^t=X^t+q^t$.

3. Demand d^t occurs, and the on-hand inventory is used to meet this demand. The $x^{t,1}$ units having only 1-period lifetime are consumed first. If $x^{t,1}>d^t$, the remaining $x^{t,1}-d^t$ units will perish and an outdating cost of $\hat{o}^t(x^{t,1}-d^t)$ occurs. Otherwise, the $x^{t,2}$ units having 2-period lifetime are used to meet the excess $d^t-x^{t,1}$ units of demand, and so on, until all inventory is consumed. If $d^t>Y^t$, the demand cannot be satisfied completely using the current inventory. Therefore, the remaining d^t-Y^t units of demand becomes backlogs that need to be met in period t+1, and a backlogging penalty of $\hat{b}^t(d^t-Y^t)$ occurs. If $d^t<Y^t$, the remaining Y^t-d^t units result in a holding cost of $\hat{h}^t(Y^t-d^t)$.

4. At the end of period t, the remaining lifetime of the excess units of the on-hand inventory decreases by 1 period. Note that if $x^{t,i}<\left(d^t-\sum_{j=1}^{i-1}x^{t,j}\right)^+$, all $x^{t,i}$ units will be used to meet demand. Otherwise, $x^{t,i}-\left(d^t-\sum_{j=1}^{i-1}x^{t,j}\right)^+$ units remain for period t+1, becoming $x^{t+1,i-1}$, i=2, …, m-1. In period t, the q^t units have a lifetime of m periods, and they will meet the excess demand after all X^t have been used. Therefore, in a backlogging problem, if $q^t<\left(d^t-\sum_{j=1}^{m-1}x^{t,j}\right)^+$, backlogs occur and the number of backlogs is $\left|q^t-\left(d^t-\sum_{j=1}^{m-1}x^{t,j}\right)^+\right|$. These backlogs will retain in the system for period t+1 as a negative inventory. So the new inventory level at the beginning of period t+1 can be calculated using (similar to Eq. (1))

$$x^{t+1,i}=\left(x^{t,i+1}-\left(d^t-\sum_{j=1}^{i}x^{t,j}\right)^+\right)^+,\ i\in\{1,\cdots,m-2\},$$

$$x^{t+1,m-1}=q^t-\left(d^t-\sum_{j=1}^{m-1}x^{t,j}\right)^+.$$

4.1.2.4 Systems with Lost-Sales and an Ordering Capacity Constraint

The process of a lost-sales problem with an order capacity constraint can be considered as follows.

1. At the beginning of period t, t=1, …, T, information set f_t and on-hand inventory $x^{t,i}$ are observed, where

$$\begin{cases} x^{t,i}=0, i=1, \ldots, m-1 & t=1 \\ x^{t,i}\geq 0, i=1, \ldots, m-1 & t\in\{2,\cdots,T\} \end{cases}.$$

   Then total inventory level is $X^t=\sum_{i=1}^{m-1} x^{t,i}$ and the conditional joint distribution of the upcoming demand D^t is known based on the information set f_t.

2. The system orders q^t units, and they arrive and become available, having a lifetime of m periods. In a system with an order capacity constraint **u**, the feasible range of q^t is $\{0,\cdots,u^t\}$. The total cost of ordering is $\hat{c}^t q^t$. Also, the on-hand inventory level increases from X^t to $Y^t=X^t+q^t$.

3. Demand d^t occurs, and the on-hand inventory is used to meet this demand. The $x^{t,1}$ units having only 1-period lifetime are consumed first. If $x^{t,1}>d^t$, the remaining $x^{t,1}$-d^t units will perish and an outdating cost of $\hat{o}^t(x^{t,1}-d^t)$ occurs. Otherwise, the $x^{t,2}$ units having 2-period lifetime are used to meet the excess d^t-$x^{t,1}$ units of demand, and so on, until all inventory is consumed. If $d^t>Y^t$, the demand cannot be satisfied completely using the current inventory. Therefore, the remaining d^t-Y^t units of demand will be lost and leave the system, with a lost-sales penalty of $\hat{b}^t(d^t-Y^t)$. If $d^t<Y^t$, the remaining Y^t-d^t units result in a holding cost of $\hat{h}^t(Y^t-d^t)$.

4. At the end of period t, the remaining lifetime of the excess units of the on-hand inventory decreases by 1 period. Note that if $x^{t,i}<\left(d^t-\sum_{j=1}^{i-1}x^{t,j}\right)^+$, all $x^{t,i}$ units will be used to meet demand. Otherwise, $x^{t,i}-\left(d^t-\sum_{j=1}^{i-1}x^{t,j}\right)^+$ units remain for period t+1, becoming $x^{t+1,i-1}$,

i=2, …, m-1. In period t, the q^t units have a lifetime of m periods, and they will meet the excess demand after all X^t have been used. Therefore, in a lost-sales problem, if $q^t < \left(d^t - \sum_{j=1}^{m-1} x^{t,j}\right)^+$, the excess demand will leave the system. If $q^t > \left(d^t - \sum_{j=1}^{m-1} x^{t,j}\right)^+$, the $q^t - \left(d^t - \sum_{j=1}^{m-1} x^{t,j}\right)^+$ remaining units of q^t will become $x^{t+1,m-1}$. Therefore, the new inventory level at the beginning of period t+1 can be calculated using (similar to Eq. (2))

$$x^{t+1,i} = \left(x^{t,i+1} - \left(d^t - \sum_{j=1}^{i} x^{t,j} \right)^+ \right)^+, i \in \{1, \cdots, m-2\},$$

$$x^{t+1,m-1} = \left(q^t - \left(d^t - \sum_{j=1}^{m-1} x^{t,j} \right)^+ \right)^+.$$

3.1.2 Objective

We consider a finite horizon problem with T periods. The total cost occurring in each period includes holding, backlogging/lost-sales penalty, and outdating costs. At the beginning of period t, the information set f_t and on-hand inventory $x^{t,i}$, i=1, …, m-1 are observed, and order quantity q^t is determined, which causes an ordering cost of $\hat{c}^t q^t$. The holding cost occurs when there are some units left after fulfilling the demand, which means that Y^t-d^t>0. Therefore, the holding cost in period t is $\hat{h}^t (Y^t - D^t)^+$. If Y^t-d^t<0, not all demands are satisfied, and a backlogging/lost-sales penalty of $\hat{b}^t (D^t - Y^t)^+$ occurs. Also, in each period, perishable units cause an outdating cost of $\hat{o}^t (x^{t,1} - D^t)^+$. Let α, $0 < \alpha \le 1$, denote the discount factor. Then, because demand in each period is uncertain, the expected total cost in period t under policy P can be obtained using

$$S^t(\mathrm{P}) = \alpha^{t-1} \mathrm{E}\left[\hat{c}^t q^t + \hat{h}^t (Y^t - D^t)^+ + \hat{b}^t (D^t - Y^t)^+ + \hat{o}^t (x^{t,1} - D^t)^+ \mid f_t \right],$$

where $Y^t = \sum_{i=1}^{m-1} x^{t,i} + q^t$.

At the end of T periods, some units may remain in the system, which can be sold in a salvage market. Let $\hat{v}$ denote the per-unit salvage value. Then, the total salvage value at the

end of T periods is $\alpha^T \hat{v} \sum_{i=1}^{m-1} x^{T+1,i}$. The objective is to find the optimal policy based on the information set f_t and on-hand inventory $x^{t,i}$, i=1, …, m-1, that minimizes the expected total cost over T periods (similar to Eq. (2) in [7]):

$$\begin{aligned} & C(\mathrm{P}) \\ & = \sum_{t=1}^{T} S^t(\mathrm{P}) - \mathrm{E}\left[\alpha^T \hat{v} \sum_{i=1}^{m-1} x^{T+1,i} \right] \\ & = \mathrm{E}\left[\sum_{t=1}^{T} \alpha^{t-1} \left(\hat{c}^t q^t + \hat{h}^t (Y^t - D^t)^+ + \hat{b}^t (D^t - Y^t)^+ + \hat{o}^t (x^{t,1} - D^t)^+ \right) - \alpha^T \hat{v} \sum_{i=1}^{m-1} x^{T+1,i} \right]. \end{aligned} \tag{3}$$

3.1.3 Cost Transformation

Suppose that all per-unit costs are constant over the T periods and denote $\hat{c}$, $\hat{h}$, $\hat{b}$ and $\hat{o}$ as the constant per-unit ordering cost, holding cost, backlogging/lost-sales penalty and outdating cost. Then,

$$\hat{c} = \hat{c}^t, \hat{h} = \hat{h}^t, \hat{b} = \hat{b}^t, \hat{o} = \hat{o}^t, t \in \{1, \cdots, T\}.$$

A cost transformation can be carried out to make ordering cost c=0. Let salvage value $\hat{v}$ be equal to $\hat{c}$, and set non-negative cost parameters to c=0, $h = \hat{h} + (1-\alpha)\hat{c}$, $b = \hat{b} - (1-\alpha)\hat{c}$ and $o = \hat{o} + \alpha\hat{c}$. Then, it has been proved that the expected total cost described in (3) can be rewritten as (see Proposition 1 in [7])

$$C(\mathrm{P}) = \mathrm{E}\left[\sum_{t=1}^{T} \alpha^{t-1} \left(h(Y^t - D^t)^+ + b(D^t - Y^t)^+ + o(x^{t,1} - D^t)^+ \right) \right] + \sum_{t=1}^{T} \alpha^{t-1} \hat{c} \mathrm{E}\left[D^t \right].$$

Note that $\sum_{t=1}^{T} \alpha^{t-1} \hat{c} \mathrm{E}\left[D^t \right]$ is a constant and does not depend on the inventory policy. To simplify the notation, we let α=1 and only consider the expected cost per period and expected total cost as:

$$\begin{aligned} & \bar{S}^t(\mathrm{P}) = \mathrm{E}\left[\left(h(Y^t - D^t)^+ + b(D^t - Y^t)^+ + o(x^{t,1} - D^t)^+ \right) \middle| f_t \right], \\ & \bar{C}(\mathrm{P}) = \sum_{t=1}^{T} \bar{S}^t(\mathrm{P}) = \mathrm{E}\left[\sum_{t=1}^{T} \left(h(Y^t - D^t)^+ + b(D^t - Y^t)^+ + o(x^{t,1} - D^t)^+ \right) \middle| f_t \right]. \end{aligned}$$

3.2 Dynamic Programming Approach

Dynamic programming approach is usually used to describe a multi-period stochastic inventory control problem, which is defined recursively and determines the order quantity q^t in period t to minimize the expected total cost over the time horizon. Note that the state at the beginning of period t consists of the information set f_t and the on-hand inventory $x^{t,i}$, i=1, …, m-1, and the expected cost in period t is

$$\bar{S}^t(\mathrm{P}) = \mathrm{E}\left[\left(h(Y^t - D^t)^+ + b(D^t - Y^t)^+ + o(x^{t,1} - D^t)^+\right) \mid f_t\right].$$

Let $V_t(x^{t,1}, \cdots, x^{t,m-1}, f_t)$ denote the optimal expected cost over periods t to T, and suppose that $V_j(x^{j,1}, \cdots, x^{j,m-1}, f_j)$, j=t+1,…, T, have been calculated. The Bellman formulation for calculating the optimal expected cost in period t for the perishable inventory problem without an order capacity constraint is

$$\begin{aligned}
&V_t(x^{t,1}, \cdots, x^{t,m-1}, f_t) \\
&= \min_{q^t \in \{0,\cdots,\infty\}} \left\{\bar{S}^t + \mathrm{E}\left[V_{t+1}(x^{t+1,1}, \cdots, x^{t+1,m-1}, F_{t+1}) \mid f_t\right]\right\} \\
&= \min_{q^t \in \{0,\cdots,\infty\}} \left\{\begin{array}{l} \mathrm{E}\left[h(Y^t - D^t)^+ + b(D^t - Y^t)^+ + o(x^{t,1} - D^t)^+ \mid f_t\right] \\ +\mathrm{E}\left[V_{t+1}(x^{t+1,1}, \cdots, x^{t+1,m-1}, F_{t+1}) \mid f_t\right] \end{array}\right\},
\end{aligned}$$

and for the perishable inventory problem with an order capacity constraint is

$$\begin{aligned}
&V_t(x^{t,1}, \cdots, x^{t,m-1}, f_t) \\
&= \min_{q^t \in \{0,\cdots,u^t\}} \left\{\bar{S}^t + \mathrm{E}\left[V_{t+1}(x^{t+1,1}, \cdots, x^{t+1,m-1}, F_{t+1}) \mid f_t\right]\right\} \\
&= \min_{q^t \in \{0,\cdots,u^t\}} \left\{\begin{array}{l} \mathrm{E}\left[h(Y^t - D^t)^+ + b(D^t - Y^t)^+ + o(x^{t,1} - D^t)^+ \mid f_t\right] \\ +\mathrm{E}\left[V_{t+1}(x^{t+1,1}, \cdots, x^{t+1,m-1}, F_{t+1}) \mid f_t\right] \end{array}\right\}.
\end{aligned}$$

Considering Eq. (1) and (2), the state transition equations for a problem with backlogs are

$$\begin{aligned}
x^{t+1,i} &= \left(x^{t,i+1} - \left(D^t - \sum_{j=1}^{i} x^{t,j}\right)^+\right)^+, i \in \{1, \cdots, m-2\}, \\
x^{t+1,m-1} &= q^t - \left(D^t - \sum_{j=1}^{m-1} x^{t,j}\right)^+,
\end{aligned}$$

and for a problem with lost-sales are

$$x^{t+1,i} = \left(x^{t,i+1} - \left(d^t - \sum_{j=1}^{i} x^{t,j} \right)^+ \right)^+, i \in \{1, \cdots, m-2\},$$

$$x^{t+1,m-1} = \left(q^t - \left(d^t - \sum_{j=1}^{m-1} x^{t,j} \right)^+ \right)^+.$$

Due to the curse-of-dimensionality, solving the dynamic program discussed above to obtain the optimal order quantity in each period is not straightforward [6]. Therefore, in the next section, we propose two simple efficient policies based on the marginal cost accounting scheme to approximate the optimal inventory level at the beginning of each period.

3.3 Heuristic Policies

3.3.1 Marginal Cost Accounting

In the marginal cost accounting scheme, the cost incurred by a decision is not considered periodically. Instead, the total cost incurred by one decision over all the periods from when the decision is made to the end of the planning horizon is calculated. There are three different types of costs caused by ordering q^t units in period t, which are the holding, backlogging/lost-sales penalty, and outdating costs. When they occur, the number of periods they will last for depends on the specific model assumptions. Also, note that the cost occurring in one period may contain the marginal costs caused by decisions made in previous periods. In this section, we first characterize these marginal costs.

Recall that the FOFC policy is applied here, meaning that units ordered earlier should be used to satisfy demand prior to those ordered later. One important feature about using the marginal cost accounting is that once one decision is made, the costs caused by it from the current period to the end of the planning horizon are only affected by the demand, but not the decisions in the subsequent periods. This feature ensures that the expected total costs associated to different decisions can be obtained before the decision is made, so we can compare the

different decisions and their corresponding expected total costs, and choose the best one.

3.3.1.1 Systems with Backlogs and No Ordering Capacity Constraint

a) Marginal holding cost accounting

Holding cost occurs when some units have not been consumed at the end of one period. Let the order quantity in period t be q^t. Because these q^t units have a lifetime of m periods in a perishable problem when they become available, they can stay in the system for at most m periods, from period t to $t+m-1$, if they are not used to meet demand. Therefore, in the marginal holding cost accounting, the holding cost incurred by these q^t units may occurs at most from period t to period $t+m-1$. In order to calculate the marginal holding cost incurred by these q^t units, we need to know how many units of these q^t units can retain in the system at the end of each period.

Recall that at the beginning of period t, the on-hand available inventory is $X^t = \sum_{i=1}^{m-1} x^{t,i}$. These X^t units need to be consumed prior to q^t due to the FOFC policy. Therefore, whether q^t units will be used in period t depends on the sign of $D^t - X^t$. If $D^t - X^t > 0$, the on-hand inventory is not enough to meet demand in period t, thus, some units of q^t are needed. If $D^t - X^t \le 0$, all q^t units will be kept in the system for the next period. Therefore, the remaining units of q^t at the end of period t is

$$\left(q^t - \left(D^t - \sum_{i=1}^{m-1} x^{t,i} \right)^+ \right)^+ .$$

In period $t+1$, after unused units of $x^{t,1}$, if there are any, have perished, the number of unmet D^t is $(D^t - x^{t,1})^+$, which will be met by $\sum_{i=2}^{m-1} x^{t,i}$. Then, D^{t+1} occurs and this will also be met by $\sum_{i=2}^{m-1} x^{t,i}$ before q^t is used. Therefore, the remaining units of q^t at the end of period $t+1$ is

$$\left(q^t - \left(D^{t+1} + \left(D^t - x^{t,1} \right)^+ - \sum_{i=2}^{m-1} x^{t,i} \right)^+ \right)^+ .$$

The number of remaining units of q^t at the end of period $t+2, \ldots, t+m-1$ can be calculated

similarly. For the simplicity of expression, a nested structure is used to describe these results (see e.g., [7]). Let $B^t(\mathbf{x}^t, i)$ denote the outstanding demand in period $t+i-1$ after the units having a lifetime of i periods or less have been depleted. Then, (similar to Eq. (5) in [7])

$$B^t(\mathbf{x}^t, i) := \begin{cases} 0 & i = 0 \\ \left(D^{t+i-1} + B^t(\mathbf{x}^t, i-1) - x^{t,i}\right)^+ & i \in \{1, \cdots, m-1\} \end{cases}. \qquad (4)$$

Therefore, the number of remaining units of q^t at the end of period j, $j=t,\ldots, t+m-1$, can be calculated, using

$$\left(q^t - \left(D^j + B^t\left(\mathbf{x}^t, j-t\right) - \sum_{i=j-t+1}^{m-1} x^{t,i} \right)^+ \right)^+ .$$

Considering that there are T periods in the horizon, the total marginal holding cost over period t to $t+m-1$ incurred by ordering q^t units in period t can be defined as (similar to Eq. (4) in [7])

$$H^t(q^t) := h \sum_{j=t}^{(t+m-1)\wedge T} \left(q^t - \left(D^j + B^t(\mathbf{x}^t, j-t) - \sum_{i=j-t+1}^{m-1} x^{t,i} \right)^+ \right)^+ ,$$

where $x \wedge y = \min\{x, y\}$.

In practical situations, a truncated marginal holding cost can be calculated. Instead of adding all holding cost incurred by q^t from period t to $(t+m-1)\wedge T$, the truncated marginal holding cost considers the holding cost from period t to $t+l-1$, $l=1, \ldots, m$. Let $\tilde{H}^{t,l}(q^t)$, $l=1, \ldots, m$, denote the truncated marginal holding cost , which represents the marginal holding cost incurred by q^t over periods t to $(t+m-1)\wedge T$.

When $l=1$, only the marginal holding cost in period t is considered. Thus, the truncated marginal holding cost incurred by q^t in period t with $l=1$ is

$$\tilde{H}^{t,1}(q^t) = h\left(q^t - \left(D^t - \sum_{i=1}^{m-1} x^{t,i} \right)^+ \right)^+ .$$

When $l=2$ and $t<T$, $\tilde{H}^{t,2}(q^t)$ can be obtained by adding the marginal holding cost in period t

and t+1, which is

$$\begin{aligned}
&\tilde{H}^{t,2}(q^t)\\
&= h\sum_{j=t}^{(t+1)\wedge T}\left(q^t-\left(D^j+B^t(\mathbf{x}^t,j-t)-\sum_{i=2}^{m-1}x^{t,i}\right)^+\right)^+\\
&= h\left(q^t-\left(D^t-\sum_{i=1}^{m-1}x^{t,i}\right)^+\right)^+ + h\left(q^t-\left(D^{t+1}+\left(D^t-x^{t,1}\right)^+-\sum_{i=2}^{m-1}x^{t,i}\right)^+\right)^+.
\end{aligned}$$

If t=T, we only need to consider the marginal holding cost in period T, and it is easy to see that in period T, no matter what l is, the truncated marginal holding cost will be

$$\tilde{H}^{T,l}(q^T)=h\left(q^T-\left(D^T-\sum_{i=1}^{m-1}x^{T,i}\right)^+\right)^+.$$

For l=3,…, m-1, the truncated marginal holding cost can be obtained using a similar method, and when l=m, the truncated marginal holding cost is exactly equal to the marginal holding cost. In general, the truncated marginal holding cost is

$$\tilde{H}^{t,l}(q^t):=h\sum_{j=t}^{(t+l-1)\wedge T}\left(q^t-\left(D^j+B^t(\mathbf{x}^t,j-t)-\sum_{i=j-t+1}^{m-1}x^{t,i}\right)^+\right)^+,\ l\in\{1,\cdots,m\}.$$

An intuitive explanation for the truncated marginal holding cost is that because the ordering cost is constant, ordering more in one period for the future demand will cost more than ordering in future periods. Therefore, there is no motivation to order more in the current period for the future demand unless the variability of demand is high. Hereafter, we will use $\tilde{H}^{t,l}(q^t)$, l=1,…, m, instead of $H^t(q^t)$.

b) Marginal backlogging penalty accounting

The marginal backlogging penalty caused by ordering q^t units only occurs in period t in a backlogging problem without an order capacity constraint. Before the demand is observed, the available on-hand inventory that can be used to meet the demand is $\sum_{i=1}^{m-1}x^{t,i}+q^t$. Therefore, the unsatisfied demand is

$$\left(D^t - \sum_{i=1}^{m-1} x^{t,i} - q^t \right)^+ ,$$

and the marginal backlogging penalty is (similar to Eq. (7) in [7])

$$\Pi^t(q^t) := b\left(D^t - \sum_{i=1}^{m-1} x^{t,i} - q^t \right)^+ . \qquad (5)$$

c) Marginal outdating cost accounting

The marginal outdating cost incurred by ordering q^t in period t occurs in period $t+m-1$ if these q^t units have not been consumed completely by the end of period $t+m-1$. This outdating cost can only be incurred by orders placed from period t to $T-m+1$ due to the m-period lifetime. The units arriving in period $T-m+2$ to T will have not reached their lifetime of m periods by the end of period T, so the outdating cost for these orders is 0.

For the q^t units ordered in period t, $t=1,\ldots, T-m+1$, they will be used to meet demand only when all $\sum_{i=1}^{m-1} x^{t,i}$ have been used up. Note that $x^{t,i}$ units have a lifetime of i periods, so they will perish and leave the system at the end of period $t+i-1$ if they have not been used by then. Therefore, the nested structure in the marginal holding cost is also needed to calculate the outdating cost. With the same definition of $B^t(\mathbf{x}^t, i)$ given in the marginal holding cost, the marginal outdating cost can be obtained using (similar to eq. (6) in [7])

$$O^t(q^t) := o\left(q^t - D^{t+m-1} - B^t(\mathbf{x}^t, m-1)\right)^+, t \in \{1, \cdots, T-m+1\} . \qquad (6)$$

3.3.1.2 Systems with Lost-Sales and No Ordering Capacity Constraint

a) Marginal holding cost accounting

The difference between systems with backlog and lost-sales is the way the excess demand is treated. However, holding cost only occurs when excess inventory exists. In period t, if the on-hand inventory $\sum_{i=1}^{m-1} x^{t,i} + q^t$ is larger than d^t, there will be no excess demand at the end of the period. Therefore, the truncated marginal holding cost in lost-sales problems without an

order capacity constraint is

$$\tilde{H}^{t,l}(q^t) := h \sum_{j=t}^{(t+l-1)\wedge T} \left(q^t - \left(D^j + B^t(\mathbf{x}^t, j-t) - \sum_{i=j-t+1}^{m-1} x^{t,i} \right)^+ \right)^+, l \in \{1, \cdots, m\}.$$

b) Marginal lost-sales penalty accounting

When the demand in period *t* is not satisfied completely, the excess part is lost at the end of period *t* and leads to a lost-sales penalty. This lost-sales penalty only occurs once at the end of period *t*. Although in a backlogging problem this excess demand retain in the system, the calculation of the amount of the excess demand is the same for both systems with backlogs and lost-sales, which is

$$\left(D^t - \sum_{i=1}^{m-1} x^{t,i} - q^t \right)^+.$$

Therefore, the marginal lost-sales penalty for ordering q^t units in period *t* is (see Eq. (5))

$$\Pi^t(q^t) := b\left(D^t - \sum_{i=1}^{m-1} x^{t,i} - q^t \right)^+.$$

c) Marginal outdating cost accounting

The marginal outdating cost incurred by ordering q^t in period *t* occurs in period *t*+*m*-1 if these q^t units have not been consumed completely by the end of period *t*+*m*-1. If an outdating cost occurs, the calculation is the same as the one given in Section 3.3.1.1, which is (see Eq. (6))

$$O^t(q^t) := o\left(q^t - D^{t+m-1} - B^t(\mathbf{x}^t, m-1) \right)^+, t \in \{1, \cdots, T-m+1\}.$$

3.3.1.3 Systems with Backlogs and an Ordering Capacity Constraint

a) Marginal holding cost accounting

For any given order quantity at the beginning of period *t*, q^t, the available inventory at the beginning of period *t* is $\sum_{i=1}^{m-1} x^{t,i} + q^t$. Therefore, the truncated marginal holding cost, which is based on the available inventory and observed demand, follows the discussion in Section

3.3.1.1 and can be obtained using

$$\tilde{H}^{t,l}(q^t) := h\sum_{j=t}^{(t+l-1)\wedge T}\left(q^t - \left(D^j + B^t(\mathbf{x}^t, j-t) - \sum_{i=j-t+1}^{m-1} x^{t,i}\right)^+\right)^+, l \in \{1, \cdots, m\}.$$

b) Marginal backlogging penalty accounting

To obtain the marginal backlogging penalty in systems with an ordering capacity constraint, a forced shortage is introduced in [25]. Note that if the order capacity in period t, u^t, is less than the order quantity q_P^t under policy P in a problem without an ordering capacity constraint, the order capacity makes q^t to be less than q_P^t. Therefore, the number of backlogs that should be considered to calculate the backlogging penalty is not $D^t - \left(\sum_{i=1}^{m-1} x^{t,i} + q^t\right)$, because the order quantity q^t cannot reach q_P^t due to the capacity constraint. Also, the penalty of the unsatisfied demand beyond u^t cannot be calculated as the backlogging penalty caused by the decision of ordering q^t units. Let $\bar{q}^t$ denote the slack capacity in period t, where $\bar{q}^t = u^t - q^t$, $0 \le \bar{q}^t \le u^t$. In period t, we use a forced shortage, which is the number of units that could have been ordered more to avoid the backlogging penalty, to calculate the cost of the backlogs, instead of comparing the available inventory and demand directly. Let $W^{t,s}$ denote the forced shortage, which occurs in period s, $s=t,\ldots, T$, because of the ordering decision in period t. Then, when $s=t$, we get

$$W^{t,t} = \min\left\{\bar{q}^t, \left(D^t - \sum_{i=1}^{m-1} x^{t,i} - q^t\right)^+\right\}. \qquad (7)$$

If all demand in period t is satisfied, or we order u^t units in period t, $W^{t,t}$ will be 0.

If backlog occurs in period t, the capacity u^{t+1} should be considered to see whether the sum of the backlogs and new demand d^{t+1} in period $t+1$ can be satisfied by ordering q^{t+1} units under the order capacity u^{t+1}. If so, no backlog will exist at the end of period $t+1$. If not, backlogs from period t should be satisfied with a higher priority due to the FOFC policy. After $x^{t,1}$ units have

perished at the end of period t, the remaining demand is $\left(D^t - x^{t,1}\right)^+$, and the inventory is $\sum_{i=2}^{m-1} x^{t,i} + q^t$. Suppose that u^{t+1} units are ordered in period t+1. Then, the available inventory increases to $\sum_{i=2}^{m-1} x^{t,i} + q^t + u^{t+1}$. After demand D^{t+1} is observed, the total demand in period t+1 increases to $D^{t+1} + \left(D^t - x^{t,1}\right)^+$. Comparing the available inventory and demand, if $D^{t+1} + \left(D^t - x^{t,1}\right)^+ \le \sum_{i=2}^{m-1} x^{t,i} + q^t + u^{t+1}$, ordering q^t units in period t causes no backlogging penalty in period t+1. Otherwise, ordering q^t units in period t is not enough to meet the cumulative demand $D^{[t,t+1]}$, where $D^{[t,s]} = \sum_{i=t}^{s} D^i$, s=t,…, T, even though the units of the maximum feasible order quantity u^{t+1} are ordered in period t+1. Note that $\left(D^{t+1} + \left(D^t - x^{t,1}\right)^+ - \left(\sum_{i=2}^{m-1} x^{t,i} + q^t + u^{t+1}\right)\right)^+$ is the shortage of inventory in period t+1. Similar to period t, we compare this value with the slack capacity $\bar{q}^t$ to determine the forced shortage caused by ordering q^t units, which is

$$W^{t,t+1} = \min\left\{\bar{q}^t, \left(D^{t+1} + \left(D^t - x^{t,1}\right)^+ - \left(\sum_{i=2}^{m-1} x^{t,i} + q^t + u^{t+1}\right)\right)^+\right\}.$$

Because u^{t+1} is the maximum feasible order quantity in period t+1, the forced shortage $W^{t,t+1}$ using u^{t+1} is not larger than that using other order quantities under the ordering capacity constraint in period t+1.

The analysis for the subsequent periods to period t+1 is similar. We define a new vector $\tilde{\mathbf{x}}^t$, which consists of units that are already in the inventory in period t and that will come into the inventory, as

$$\tilde{\mathbf{x}}^t := \left\{x^{t,1}, \cdots, x^{t,m-1}; q^t; u^{t+1}, \cdots, u^T\right\},$$

where t=1,…, T-1. If t=T, no u^t units will be ordered by the end of T periods, so we have

$$\tilde{\mathbf{x}}^T := \left\{x^{T,1}, \cdots, x^{T,m-1}; q^T\right\}.$$

Note that $\tilde{\mathbf{x}}^t$ consists of inventory $x^{t,i}$, q^t and order capacity u^j, for j=t+1,...,T. Therefore, there are T-t+m components in total. The orders of q^t and u^j, j=t+1,...,T, units are placed from period

t to T, from when they will have a lifetime of m periods. We can consider the periods from t to the period when the q^t and u^j, $j=t+1,...,T$, units are ordered as an extra lifetime for these units. For example, consider period t and assume that u^{t+1} units will be ordered in period $t+1$. The one period from t to $t+1$ can be considered as an extra lifetime for the u^{t+1} units. Thus, in period t, we can think that the u^{t+1} units have a lifetime of $m+1$ periods. Therefore, generally, in period t, the $\tilde{x}^{t,I}$ units, $I=1,\ldots,T-t+m$, have a lifetime from 1 period to $T-t+m$ periods. But, when they come into the inventory and become available, they all have a lifetime of only m periods. Then, we modify the definition of $B^t(\mathbf{x}^t,i)$ in Eq. (4) to $\tilde{B}^t(\tilde{\mathbf{x}}^t,I)$, using $\tilde{\mathbf{x}}^t$ and I, as

$$\tilde{B}^t(\tilde{\mathbf{x}}^t,I) := \begin{cases} 0 & I=0 \\ \left(D^{t+I-1}+\tilde{B}^t(\tilde{\mathbf{x}}^t,I-1)-\tilde{x}^{t,I}\right)^+ & I\in\{1,\cdots,T-t+m\} \end{cases}.$$

Note that $\tilde{B}^t(\tilde{\mathbf{x}}^t,I)$ denotes the outstanding demand in period $t+I-1$ after the units having a lifetime of I periods or less have been depleted, similar to $B^t(\mathbf{x}^t,i)$. The difference between $\tilde{B}^t(\tilde{\mathbf{x}}^t,I)$ and $B^t(\mathbf{x}^t,i)$ is that the former considers order quantities placed in periods $t+1,\ldots,T$. Therefore, a general expression for the forced shortage in period s, $s=t,\ldots,T$, which is caused by ordering q^t units in period t is

$$W^{t,s}=\min\left\{\overline{q}^t,\left(D^s+\tilde{B}^t(\tilde{\mathbf{x}}^t,s-t)-\sum_{I=s-t+1}^{s-t+m}\tilde{x}^{t,I}\right)^+\right\},\ s\in\{t,\cdots,T\},$$

and the corresponding backlogging penalty in the problem with an order capacity constraint is

$$\tilde{\Pi}^t(q^t) := b\sum_{s=t}^{T}W^{t,s}.$$

c) Marginal outdating cost accounting

In a backlogging problem with an order capacity constraint, the order capacity does not impact the calculation of the marginal outdating cost. If the order quantity is smaller than that in a problem without an order capacity constraint, the probability of outdating decreases. Therefore, the outdating cost can be calculated using (see Eq. (6))

$$O^t(q^t) := o\left(q^t - D^{t+m-1} - B^t(\mathbf{x}^t, m-1)\right)^+, t \in \{1, \cdots, T-m+1\}.$$

3.3.1.4 Systems with Lost-Sales and an Ordering Capacity Constraint

a) Marginal holding cost accounting

The derivation of the truncated marginal holding cost is based on the available inventory at the beginning of a period and observed demand during that period. Therefore, following the discussion in Section 3.3.1.1, we get

$$\tilde{H}^{t,l}(q^t) := h \sum_{j=t}^{(t+l-1)\wedge T} \left(q^t - \left(D^j + B^t(\mathbf{x}^t, j-t) - \sum_{i=j-t+1}^{m-1} x^{t,i} \right)^+ \right)^+, l \in \{1, \cdots, m\}.$$

b) Marginal lost-sales penalty accounting

In a lost-sales problem with an order capacity constraint, similar to Section 3.3.1.3, the excess demand in period *t* cannot be used to calculate the lost-sales penalty incurred by ordering q^t units in period *t* directly. This is because this excess demand may surpass the difference between q^t and the order capacity u^t. Therefore, the forced shortage is needed to obtain the marginal lost-sales penalty. Similar to the backlogging problem with an ordering capacity constraint, if we always order at the capacity level in period *s*, *s*=*t*+1,…, *T*, and unsatisfied demand occurs in a period, this lost-sales penalty is associated to the shortage in period *t*.

In period *t*, the excess demand is $\left(D^t - \sum_{i=1}^{m-1} x^{t,i} - q^t\right)^+$. We compare this excess demand with the slack capacity $\bar{q}^t = u^t - q^t$, $0 \le \bar{q}^t \le u^t$, to determine the exact lost-sales quantity that is caused by ordering q^t units in period *t*, which is (see Eq. (7))

$$W^{t,t} = \min\left\{\bar{q}^t, \left(D^t - \sum_{i=1}^{m-1} x^{t,i} - q^t\right)^+\right\}.$$

Because the excess demand will be lost in a lost-sales problem, so in period *t*+1 the only demand that needs to be met is D^{t+1}. Note that the unused units of $x^{t,1}$ will perish at the end of period *t*, and should be deducted from the inventory. Therefore, at the end of period *t*, after $x^{t,1}$ units have perished, the excess inventory is

$$\left(\sum_{i=1}^{m-1} x^{t,i} + q^t - D^t\right)^+ - \left(x^{t,1} - D^t\right)^+.$$

Therefore, if we order u^{t+1} units in period t+1 before D^{t+1} is observed, the inventory level increases to

$$\left(\sum_{i=1}^{m-1} x^{t,i} + q^t - D^t\right)^+ - \left(x^{t,1} - D^t\right)^+ + u^{t+1},$$

and the shortage in period t+1 is

$$\left(D^{t+1} - \left(\sum_{i=1}^{m-1} x^{t,i} + q^t - D^t\right)^+ + \left(x^{t,1} - D^t\right)^+ - u^{t+1}\right)^+.$$

Similar to period t, the forced shortage in period t+1 caused by ordering q^t units in period t is

$$W^{t,t+1} = \min\left\{\overline{q}^t, \left(D^{t+1} - \left(\sum_{i=1}^{m-1} x^{t,i} + q^t - D^t\right)^+ + \left(x^{t,1} - D^t\right)^+ - u^{t+1}\right)^+\right\}.$$

In order to calculate the forced shortage in each period, we use a nested structure as well. Recall that the vector $\tilde{\mathbf{x}}^t$ defined in Section 3.3.1.3 is

$$\tilde{\mathbf{x}}^t := \begin{cases} \left\{x^{t,1}, \cdots, x^{t,m-1}; q^t; u^{t+1}, \cdots, u^T\right\} & t \in \{1, \cdots, T-1\} \\ \left\{x^{T,1}, \cdots, x^{T,m-1}; q^T\right\} & t = T \end{cases}.$$

We define $G(\tilde{\mathbf{x}}^t, I)$ as

$$G(\tilde{\mathbf{x}}^t, I) := \begin{cases} 0 & I = 0 \\ \left(\tilde{x}^{t,I} - \left(-G(\tilde{\mathbf{x}}^t, I-1)\right)^+\right)^+ - D^{t+I-1} & I \in \{1, \cdots, T-t+m\} \end{cases}.$$

$G(\tilde{\mathbf{x}}^t, I)$ is the remaining quantity of $\tilde{x}^{t,I}$ units after meeting demand D^{t+I-1} and before they perish in period $t+I-1$. If $G(\tilde{\mathbf{x}}^t, I) > 0$, the excess units will perish. If $G(\tilde{\mathbf{x}}^t, I) < 0$, the excess demand will be met by $\tilde{x}^{t,I+1}$. Therefore, when we calculate $G(\tilde{\mathbf{x}}^t, I+1)$, we need to deduct the part of $\tilde{x}^{t,I+1}$ units which have been used in period $t+I$ first, and the remaining available part of $\tilde{x}^{t,I+1}$ used for satisfying D^{t+I} is $\left(\tilde{x}^{t,I+1} - \left(-G(\tilde{\mathbf{x}}^t, I)\right)^+\right)^+$.

At the beginning of period t, we have a known vector $\tilde{\mathbf{x}}^t$. Then, at the beginning of period

s, s=t,…, T, we define the available inventory $AI^{t,s}$, before the units ordered at the beginning of the period arrive, as

$$AI^{t,s}=\begin{cases}\sum_{i=1}^{m-1}\tilde{x}^{t,i} & s=t\\ \left(AI^{t,s-1}+\tilde{x}^{t,s-t+m-1}-D^{s-1}\right)^{+}-\left(G(\tilde{\mathbf{x}}^{t},s-t)\right)^{+} & s\in\{t+1,\cdots,T\}\end{cases}.$$

Note that $\left(AI^{t,s-1}+\tilde{x}^{t,s-t+m-1}-D^{s-1}\right)^{+}$ is the excess inventory after meeting the demand D^{s-1}. Then, we need to check whether all $\tilde{x}^{t,s-t}$ have been used. If so, $G(\tilde{\mathbf{x}}^{t},s-t)\le 0$ and the consumption of $\tilde{x}^{t,s-t+1}$ is captured by $\left(AI^{t,s-1}+\tilde{x}^{t,s-t+m-1}-D^{s-1}\right)^{+}$. If not, $G(\tilde{\mathbf{x}}^{t},s-t)>0$ and we deduct this part from the remaining inventory.

Therefore, the forced shortage occurs in period s, s=t,…, T, which results from ordering q^t units in period t is

$$W^{t,s}=\min\left\{\bar{q}^{t},\left(D^{s}-AI^{t,s}-\tilde{x}^{t,s-t+m}\right)^{+}\right\},\ s\in\{t,\cdots,T\},$$

and the total lost-sales penalty incurred by ordering q^t units in period t is

$$\tilde{\Pi}^{t}(q^{t}):=b\sum_{s=t}^{T}W^{t,s}.$$

c) Marginal outdating cost accounting

Similar to Eq. (6), the order capacity does not impact the calculation of the marginal outdating cost in a backlogging problem, which is

$$O^{t}(q^{t}):=o\left(q^{t}-D^{t+m-1}-B^{t}(\mathbf{x}^{t},m-1)\right)^{+},\ t\in\{1,\cdots,T-m+1\}.$$

In summary, the difference between problems with backlogs and lost-sales is the inventory state transition equations. In a backlogging problem, after the inventory is transferred from period t to t+1, $x^{t+1,m-1}$ can be negative if backlog occurs, while in a lost-sales problem, $x^{t+1,m-1}$ is non-negative. Moreover, the difference between problems with and without a capacity constraint is the marginal backlogging/lost-sales penalty, and the number of periods that the marginal backlogging/lost-sales penalty will last for.

3.3.2 Dual-Balancing and Look-Ahead Approaches

3.3.2.1 Dual-Balancing Approach

The main idea behind the Dual-Balancing algorithm is that the different types of marginal costs caused by an ordering decision can be separated into two categories. In the first one, the cost increases when additional units are ordered, while in the second one, the cost decreases when ordering more units. The order quantity that makes the sum of costs in the first category equal to that in the second category is considered as the order quantity under the Dual-Balancing policy [6]. If the order quantity and demand are continuous, this selected amount can be obtained directly by balancing the two types of costs. We first discuss problems with continuous order quantity and demand, where $q^t \in R$ and $q^t \geq 0$, and then explain how the algorithm can be extended for the problems with integer-valued order quantity and demand.

Recall that in a problem without an order capacity constraint, the definitions of the truncated marginal holding, backlogging/lost-sales penalty, and outdating costs are

$$\tilde{H}^{t,l}(q^t) := h \sum_{j=t}^{(t+l-1)\wedge T} \left(q^t - \left(D^j + B^t(\mathbf{x}^t, j-t) - \sum_{i=j-t+1}^{m-1} x^{t,i} \right)^+ \right)^+, l \in \{1, \cdots, m\},$$

$$\Pi^t(q^t) := b\left(D^t - \sum_{i=1}^{m-1} x^{t,i} - q^t \right)^+,$$

$$O^t(q^t) := o\left(q^t - D^{t+m-1} - B^t(\mathbf{x}^t, m-1) \right)^+.$$

Note that in period t, when q^t=0, $\mathrm{E}\left[\tilde{H}^{t,l}(0)\right] = \mathrm{E}\left[O^t(0)\right] = 0$, and both $\tilde{H}^{t,l}\left(q^t\right)$ and $O^t\left(q^t\right)$ increase in q^t. Therefore, $\mathrm{E}\left[\tilde{H}^{t,l}\left(q^t\right)\right] + \mathrm{E}\left[O^t\left(q^t\right)\right]$ is also an increasing function in q^t, with $\mathrm{E}\left[\tilde{H}^{t,l}(0)\right] + \mathrm{E}\left[O^t(0)\right] = 0$. On the other hand, $\mathrm{E}\left[\Pi^t\left(q^t\right)\right]$ has a positive value when q^t=0, and decreases to 0 as q^t goes to infinity. Therefore, we can find a well-defined $q^t_{DB,l}$ for each l letting

$$\mathrm{E}\left[\tilde{H}^{t,l}\left(q^t_{DB,l}\right)\right] + \mathrm{E}\left[O^t\left(q^t_{DB,l}\right)\right] = \mathrm{E}\left[\Pi^t\left(q^t_{DB,l}\right)\right], l \in \{1, \cdots, m\}, \text{for } q^t_{DB,l} \geq 0.$$

For problems with integer-valued order quantity and demand, a randomized Dual-

Balancing policy was proposed [6]. First, we calculate q_{DB}^t that balances $\mathrm{E}\left[\Pi^t\left(q_{DB}^t\right)\right]$ and $\mathrm{E}\left[\tilde{H}^{t,l}\left(q_{DB}^t\right)\right]+\mathrm{E}\left[O^t\left(q_{DB}^t\right)\right]$. If this q_{DB}^t it is not an integer, there are two consecutive integers, $q_{DB}^{t,1}$ and $q_{DB}^{t,2}=q_{DB}^{t,1}+1$, which satisfy $q_{DB}^{t,1}<q_{DB}^t<q_{DB}^{t,2}$. Define λ, $0<\lambda<1$, such that $q_{DB}^t=\lambda q_{DB}^{t,1}+(1-\lambda)q_{DB}^{t,2}$ [6]. So, when we need to select an integer order quantity instead of q_{DB}^t, we pick $q_{DB}^{t,1}$ with probability λ and $q_{DB}^{t,2}$ with probability 1-λ [6]. This randomized DB policy will be applied in this study to make order quantities integer.

In a backlogging/lost-sales problem with an ordering capacity constraint, the definitions of the marginal holding and outdating costs are the same as those in a problem without an order capacity constraint. The backlogging/lost-sales penalty here is $\tilde{\Pi}^t(q^t):=b\sum_{s=t}^{T}W^{t,s}$, where

$$W^{t,s}=\min\left\{\bar{q}^t,\left(D^s+\tilde{B}^t(\tilde{\mathbf{x}}^t,s-t)-\sum_{I=s-t+1}^{s-t+m}\tilde{x}^{t,I}\right)^+\right\},\ s\in\{t,\cdots,T\}$$

in backlogging problems, and

$$W^{t,s}=\min\left\{\bar{q}^t,\left(D^s-AI^{t,s}-\tilde{x}^{t,s-t+m}\right)^+\right\},\ s\in\{t,\cdots,T\}$$

in lost-sales problems. Similar to $\mathrm{E}\left[\Pi^t\left(q^t\right)\right]$, $\mathrm{E}\left[\tilde{\Pi}^t\left(q^t\right)\right]$ has a positive value when $q^t=0$, and decreases to 0 when $q^t=u^t$. Therefore, we can also find a well-defined $q_{DB,l}^t$ for each l letting

$$\mathrm{E}\left[\tilde{H}^{t,l}\left(q_{DB,l}^t\right)\right]+\mathrm{E}\left[O^t\left(q_{DB,l}^t\right)\right]=\mathrm{E}\left[\tilde{\Pi}^t\left(q_{DB,l}^t\right)\right],\ l\in\{1,\cdots,m\},\ \text{for } q_{DB,l}^t\geq 0.$$

Furthermore, the randomized DB policy holds in backlogging/lost-sales problems with a capacity constraint as well.

3.3.2.2 Look-Ahead Approach

In Look-Ahead algorithm, the integer ordering amount $q_{LA,l}^t$ minimizing the total marginal costs in period t is desired where

$$q_{LA,l}^t=\arg\min_{q_{LA,l}^t\in\{0,\cdots,\infty\}}\mathrm{E}\left[\tilde{H}^{t,l}(q_{LA,l}^t)+\Pi^t(q_{LA,l}^t)+O^t(q_{LA,l}^t)\mid f_t\right],\ l\in\{1,\cdots,m\},\qquad(8)$$

in problems without an ordering capacity constraint, and

$$q_{LA,l}^t = \arg \min_{q_{LA,l}^t \in \{0,\cdots,u^t\}} \mathrm{E}\left[\tilde{H}^{t,l}(q_{LA,l}^t) + \tilde{\Pi}^t(q_{LA,l}^t) + O^t(q_{LA,l}^t) \mid f_t \right], l \in \{1,\cdots, m\}, \qquad (9)$$

in problems with an ordering capacity constraint.

Based on the definitions, $\tilde{H}^{t,l}(q^t)$, $O^t(q^t)$, $\Pi^t(q^t)$ and $\tilde{\Pi}^t(q^t)$ are all convex functions (see [7], [6], [25]). Therefore, the following two functions are both convex in problems with and without a capacity constraint,

$$A(q_{LA,l}^t) = \tilde{H}^{t,l}(q_{LA,l}^t) + \Pi^t(q_{LA,l}^t) + O^t(q_{LA,l}^t), q_{LA,l}^t \in \{0,\cdots,\infty\}, l \in \{1,\cdots, m\}$$

and

$$\tilde{A}(q_{LA,l}^t) = \tilde{H}^{t,l}(q_{LA,l}^t) + \tilde{\Pi}^t(q_{LA,l}^t) + O^t(q_{LA,l}^t), q_{LA,l}^t \in \{0,\cdots,u^t\}, l \in \{1,\cdots, m\}.$$

Besides, $A(0)$, $\tilde{A}(0)$ and $\tilde{A}(u^t)$ are all positive, and $A(q^t)$ goes to infinity as q^t increases, so the $q_{LA,l}^t$ satisfying (8) or (9) is well-defined .

3.4 Algorithms

3.4.1 Dynamic Programming Algorithm

We apply a backward recursion algorithm to solve the dynamic program given in Section 3.2.

1. Calculate the range of the on-hand inventory $x^{t,i}$, i=1, …, m-1, for each period t. One period state in period t is a combination of possible values of $x^{t,i}$, i=1, …, m-1.
2. Find the order quantity in period T that leads to the optimal expected period cost $\bar{S}^T = V_T$ for each period state.
3. In period t, t=T-1, T-2, …, 1, for each period state, find the order quantity optimizing the expected total cost from current period t to the last period T, which is $V_t = V_{t+1} + \bar{S}^t$. Then, we built the outcome and action spaces already.
4. Consider the inventory problem forward from period 1 and calculate the optimal expected

total cost.

3.4.2 Dual-Balancing Algorithm

The DB policy is applied in each period from period 1. The algorithm of the DB policy used in our study is as follows.

1. In period *t*, find two continuous integer $q_l^{t,1}$ and $q_l^{t,1}$ satisfying

$$\begin{cases} \mathrm{E}\left[\tilde{H}^{t,l}\left(q_l^{t,1}\right)\right]+\mathrm{E}\left[O^t\left(q_l^{t,1}\right)\right]\le \mathrm{E}\left[\tilde{\Pi}^t\left(q_l^{t,1}\right)\right] \\ \mathrm{E}\left[\tilde{H}^{t,l}\left(q_l^{t,2}\right)\right]+\mathrm{E}\left[O^t\left(q_l^{t,2}\right)\right]\ge \mathrm{E}\left[\tilde{\Pi}^t\left(q_l^{t,2}\right)\right] \end{cases},$$

for each *l*, *l*=1,…,*m*, respectively. Thus, the desired order quantity q_l^t under the DB policy is in the range of $[q_l^{t,1}, q_l^{t,2}]$.

2. Let $q_l^{\prime t,1}=q_l^{t,1}$, $q_l^{\prime t,11}=q_l^{t,2}$. Divide the range of $[q_l^{\prime t,1}, q_l^{\prime t,11}]$ into 10 equal sections by 9 numbers $q_l^{\prime t,2},\ldots,q_l^{\prime t,10}$ that satisfy $q_l^{\prime t,i+1}=q_l^{\prime t,i}+0.1$, *i*=1,…,10. Then, find two continuous $q_l^{\prime t,a}$ and $q_l^{\prime t,a+1}$, *a*=1,…,10, satisfying

$$\begin{cases} \mathrm{E}\left[\tilde{H}^{t,l}\left(q_l^{\prime t,a}\right)\right]+\mathrm{E}\left[O^t\left(q_l^{\prime t,a}\right)\right]\le \mathrm{E}\left[\tilde{\Pi}^t\left(q_l^{\prime t,a}\right)\right] \\ \mathrm{E}\left[\tilde{H}^{t,l}\left(q_l^{\prime t,a+1}\right)\right]+\mathrm{E}\left[O^t\left(q_l^{\prime t,a+1}\right)\right]\ge \mathrm{E}\left[\tilde{\Pi}^t\left(q_l^{\prime t,a+1}\right)\right] \end{cases},$$

for each *l*, *l*=1,…,*m*, respectively. Thus, the desired order quantity q_l^t under the DB policy is in the range of $[q_l^{\prime t,a}, q_l^{\prime t,a+1}]$.

3. We assume that the two curves representing the backlogging/lost-sales penalty and the sum of holding and outdating costs are approximately linear on $[q_l^{\prime t,a}, q_l^{\prime t,a+1}]$. Therefore, the intersection of these two curves can be easily obtained. This intersection is the order quantity q_l^t under the DB policy in period *t*.

4. Consider the inventory problem forward from period 1 and calculate the expected total cost under the DB policy with q_l^t obtained through steps 1 to 3 in each period *t*.

3.4.3 Look-Ahead Algorithm

The LA policy is applied in each period from period 1. The algorithm of the LA policy used in our study is as follows.

1. In period *t*, the desired order quantity q_l^t under the LA policy can be found directly through calculating the expected total marginal cost with different order quantity and selecting the smallest one.
2. Consider the inventory problem forward from period 1 and calculate the expected total cost under the LA policy with q_l^t obtained through step 1 in each period *t*.

4 Numerical Study

In the previous section, we discussed four periodic-review stochastic perishable inventory control problems based on whether unmet demand in each period is backlogged or lost, and whether there is a capacity constraint in each period or not.

The numerical study aims to evaluate the performance of the two heuristic policies in comparison with the optimal policy, and examine the impact of different parameters on the performance of the proposed policies for each problem.

We first consider a finite-horizon with 6 periods. In order to determine the target inventory level at the beginning of each period, we apply DP, DB and LA algorithms to find the order quantity q^t in period t, t=1,…, 6, based on the on-hand inventory X^t. Then, the target inventory level in period t is set to $Y^t=X^t+q^t$. Second, we use this Y^t to calculate the expected total cost over the 6 periods to compare the performance of DB and LA with the optimal policy. Let C^{DB}, C^{LA} and C^* represent the expected total cost obtained under DB, LA and DP policies, respectively. Then, the relative gap between the total cost under the optimal policy and policy P is defined as

$$\rho^{\mathrm{P}} = \frac{C^{\mathrm{P}} - C^*}{C^*} \times 100\%,\ \mathrm{P} \in \{\mathrm{DB},\ \mathrm{LA}\}.$$

In the numerical study, we focus on integer-valued demand and order quantity. Three different discrete distributions, including a discrete uniform distribution, a binomial distribution and a specific discrete distribution, are considered to capture the demand arrival in each period. The discrete uniform distribution and binomial distribution used in the numerical examples are $U\{1,8\}$ and $B(8,\frac{1}{2})$. The specific discrete distribution gets values $\{1,2,4,8\}$ with corresponding probabilities $\left\{\frac{1}{8},\frac{2}{8},\frac{3}{8},\frac{2}{8}\right\}$. In the following discussion, we use Uniform, Binomial and Distribution-3 to denote the above three distributions, respectively.

In order to examine the relationship between cost parameters and the policy performance, we vary the cost parameters as given in Table 1. Note that we assume $h \leq b$ which means the cost of generating one unit of backlog or lost-sale is at least as high as keeping one unit of the product in stock for one period.

Table 1. Per-Unit Cost Combinations

Fixed cost	Modified costs
b=10	$h \in \{0.1, 0.5, 1, 2, 5\}$ $o \in \{1, 5, 10, 20\}$
h=2	$o \in \{1, 5, 10, 20\}$ $b \in \{2, 5, 10, 20\}$
o=10	$b \in \{2, 5, 10, 20\}$ $h \in \{0.1, 0.5, 1, 2, 5\}$

In the problems with the capacity constraint, we assume that $\mathbf{u} = \{8, 4, 6, 5, 8, 5\}$.

Some other system parameters are specified as follows. The lifetime of the units is m=3, and we vary l from 1 to 3 in the truncated marginal holding cost.

Then, we apply the DP, DB and LA algorithms for a perishable inventory system on a finite-horizon with 8 periods without an ordering capacity constraint using the cost parameters given in Table 1, to investigate the impact of T on the performance of the algorithms. Other parameters are the same as those in the problem on a 6-period planning horizon. In the following Sections 4.1 and 4.2, we analyze numerical results of the 6-period problem first, and then illustrate the results of the 8-period problem.

All algorithms were programmed in Matlab and all experiments were done on a system with 3.4GHz CPU.

4.1 Systems with Backlogs and No Ordering Capacity Constraint

4.1.1 Performance of DB and LA Policies

It has been proved that in a perishable inventory problem with backlogs and no ordering

capacity constraint, the DB policy has a worst-case performance guarantee of two when demands in different periods are independent and stochastically non-decreasing [7]. This means that for each instance of the problem, the expected total cost of the DB policy is at most twice the expected total cost of the DP policy. However, the performance of the LA policy has not been investigated in the literature for perishable inventory problems. In systems that units do not perish, Truong [8] shows that the LA policy outperforms the DB policy significantly.

In this research, we examine the performance of both DB and LA policies. Table 2 shows the mean and maximum of ρ^{DB} and ρ^{LA} for different values of l, with T=6.

Table 2. ρ^{LA} and ρ^{DB} with Backlogs and No Capacity Constraint (%, T=6)

Policy	l	Uniform		Binomial		Distribution-3	
		Mean	Max	Mean	Max	Mean	Max
LA	1	2.09	29.28	2.31	8.18	2.09	5.65
	2	3.31	22.40	2.77	17.96	3.01	10.86
	3	3.13	13.02	2.44	16.29	3.30	15.91
DB	1	22.33	71.96	15.52	48.50	28.63	76.39
	2	25.30	89.19	17.36	44.91	32.03	87.96
	3	25.87	87.68	18.86	60.25	32.40	89.43

From Table 2, we observe that ρ^{DB} and ρ^{LA} are less than 100% in the columns indicating the maximum relative gap. This observation demonstrates that the expected total costs under both DB and LA policies are less than twice the optimal expected total cost. Therefore, based our numerical examples, the performance of the DB policy is consistent with the worst-case performance guarantee stated in previous studies, and for the average performance of all numerical examples that we checked, the LA policy, which is investigated in perishable inventory problems for the first time based on our best knowledge currently, is much better than the DB policy in the same circumstance.

Moreover, we observe that the average of ρ^{LA} is much smaller than the average of ρ^{DB}. This shows that the expected total cost under the LA policy is much closer to the optimal

expected total cost than that under the DB policy. This result is similar to the one in inventory problems without perishable products considered in Truong [8]. Therefore, in inventory problems with backlogs and no ordering capacity constraint, the LA policy, on average, outperforms the DB policy significantly.

Comparing the mean of ρ^{LA} and ρ^{DB} for different values of l, we observe that both ρ^{LA} and ρ^{DB} have the smallest average value when l=1. In a problem without a capacity constraint, the order quantity can be any non-negative integer. Therefore, intuitively, we do not need to consider the demand in the following periods a new order is placed. For l=1, the holding cost in the following periods incurred by the order in the current period does not need to be considered. Therefore, the most important goal is to ensure that the on-hand inventory is enough to meet the demand in the current period. Therefore, the target inventory level policy and the expected total cost are the closest to the optimal solution when l=1. In perishable inventory problems with backlog and no ordering capacity constraint, applying the truncated marginal holding cost with l=1 would obtain the best approximation for the optimal policy.

Similar result can be obtained for 8-period problem, as listed in Table 3.

Table 3. ρ^{LA} and ρ^{DB} with Backlogs and No Capacity Constraint (%, T=8)

Policy	l	Uniform		Binomial		Distribution-3	
		Mean	Max	Mean	Max	Mean	Max
LA	1	3.19	16.27	3.58	14.42	3.18	20.09
	2	3.77	10.61	3.67	14.73	5.18	26.82
	3	4.14	13.23	5.51	19.58	6.47	22.95
DB	1	23.76	79.54	16.51	57.42	30.33	84.84
	2	25.76	87.83	17.49	46.24	33.90	95.16
	3	27.98	76.48	21.93	79.62	34.01	82.12

We observe that all ρ^{DB} and ρ^{LA} are less than 100% in the columns indicating the maximum relative gap, which demonstrates that the expected total costs under both DB and LA policies are less than twice the optimal expected total cost. Second, the LA policy significantly

outperforms the DB policy for all three demand distributions. Third, both ρ^{LA} and ρ^{DB} have the smallest average value when l=1.

Next, we examine how the cost parameters affect the performance of the DB policy. Figures 1 and 2 illustrate how h, b affect ρ^{DB} when the demand in each period follows Uniform and Binomial distributions, respectively (l=3, T=6). We observe that ρ^{DB} tends to be larger as the per-unit backlogging penalty b increases, while it tends to be smaller as the per-unit holding cost h increases.

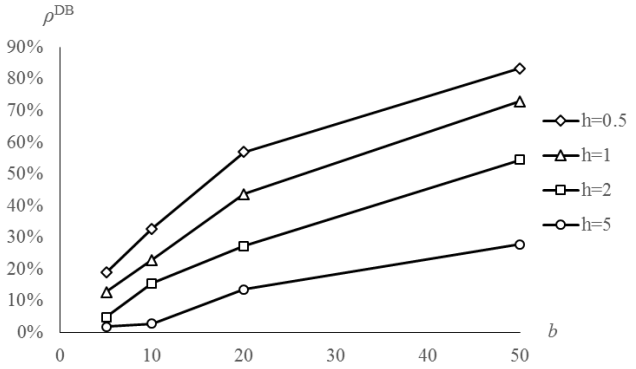

Figure 1. ρ^{DB} for Uniform Demand

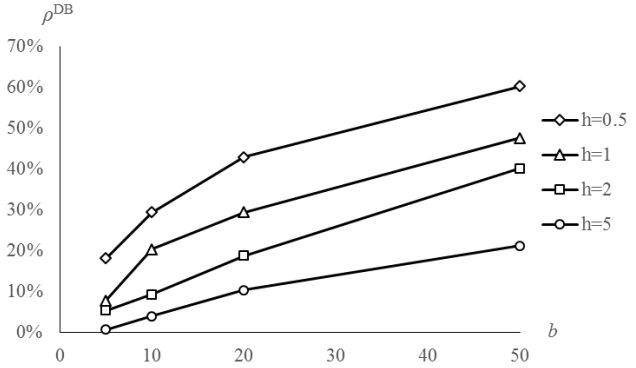

Figure 2. ρ^{DB} for Binomial Demand

Figure 3 and 4 show ρ^{DB} for different h and o, based on our numerical examples. We observe that ρ^{DB} is decreasing in both the per-unit holding cost h and the per-unit outdating cost o, except for h=0.1 and o=20. Intuitively, the number of units incurring outdating cost is much smaller than that incurring holding cost. Thus, the effect of the per-unit outdating cost o on the expected total cost and ρ^{DB} is smaller than that of the per-unit holding cost h. For example, in

Figure 5, when o is small, ρ^{DB} is decreasing in h. But when o=20 and h is small, the effect of o surpasses that of h. As a result, ρ^{DB} with o=20 and h=0.1 is smaller than that with o=20 and h=0.5. We also notice that the effect of h and o varies for different demand distributions, which have different variances. With a demand having a lower variance, such as the binomial distribution in this research, the range of ρ^{DB} is much smaller for both h and o.

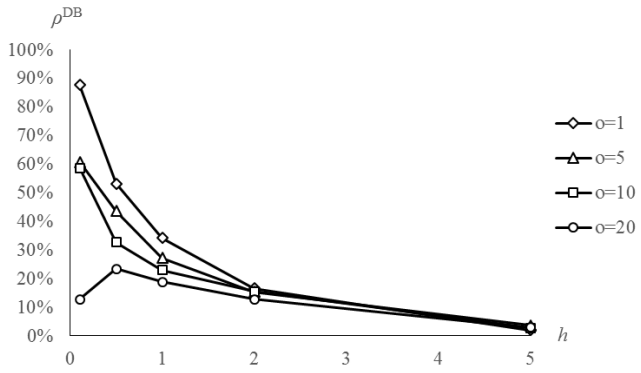

Figure 3. ρ^{DB} for Uniform Demand

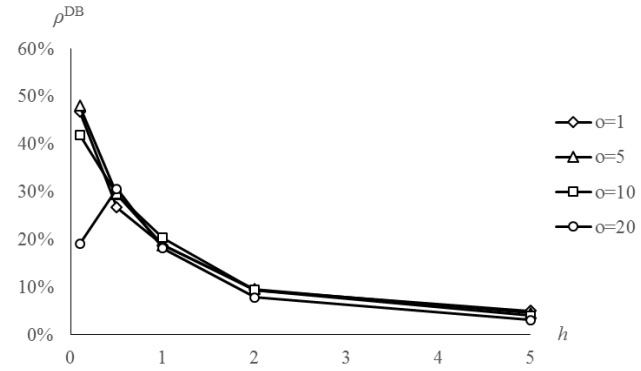

Figure 4. ρ^{DB} for Binomial Demand

In summary, based on our numerical examples, increasing h or decreasing b both lead to a smaller ρ^{DB}, which means that larger h and smaller b can improve the performance of the DB policy. Besides, the effect of o is not obvious when o is small. However, for large o, the DB policy performs well and results in a smaller ρ^{DB}.

4.1.2 Optimal and Heuristic Policies

Table 4 shows the average optimal target inventory level over 6 periods for Uniform

distribution and b=10.

Table 4. Average Optimal Target Inventory Level (T=6)

h	o	Optimal Y^t
0.1	1, 2, 5, 10	8
	20	7.48
0.2	2	8
0.25	5	8
0.4	2	8
0.5	1, 5, 10	8
	20	7.44
1	0.5, 1, 2, 2.5, 5	8
	10	7.44
	20	7.35
2	1, 5, 10, 20	7
2.5	5	7
4	2, 10, 20, 40	6
5	1, 5, 10, 20	6
10	5, 20, 25, 50, 100	4

It indicates that the average optimal target inventory level decreases in h and o, respectively. When the per-unit holding and outdating cost are significantly smaller than the per-unit backlogging penalty, keeping more inventory to avoid backlogs will cost less. Therefore, from our observation, when $h \leq 1$ and $o \leq 10$ (except h=1 and o=10), the average optimal target inventory level is equal to the maximum of demand, which can result in zero backlog. The motivation of holding high level of inventory reduces as the holding and outdating cost increase. We also observe that the effect of the per-unit outdating cost is much less than that of the holding cost, similar to Section 4.1.1. When the holding cost is small, like h=0.1, 0.5, the outdating cost needs to be 20 to reduce the target inventory level. The similar optimal inventory level results are obtained for Binomial and Distribution-3 distributions. They have the same monotone patterns for the target inventory level.

Table 5 shows the average optimal target inventory level in 8-period problem for the

Uniform distribution. We observe the same trend observed for the average optimal target inventory level for *h* and *o*.

Table 5. Average Optimal Target Inventory Level (*T*=8)

h	*o*	Optimal Y^t
0.1	1, 5, 10	8
	20	7.93
0.5	1, 5, 10	8
	20	7.41
1	1, 5	8
	10	7.39
	20	7.27
2	1, 5, 10, 20	7
5	1, 5, 10, 20	6

Then, we examine the ordering policy for the optimal and LA policies in a problem with backlogs. In our numerical examples, we observe that Y^t under the LA policy with *l*=3 is lower than the optimal Y^t. Some examples for Uniform distribution are listed in Table 6.

Table 6. Target Inventory Level under LA and Optimal Policies (*T*=6)

Cost Combinations			Policies	Target Inventory Level in Each Period					
h	*b*	*o*		1	2	3	4	5	6
0.1	10	20	OPT	8	8	8	7.376	8	8
			LA	7	7.246	7.306	7.304	8	8
1	10	20	OPT	7	7	7.059	7.016	8	8
			LA	7	7	7	7	8	8
2.5	10	5	OPT	7	7	7	7	7	7
			LA	6	6	6.031	6.038	7	7
5	10	1	OPT	6	6	6	6	6	6
			LA	5	5	5.018	5.015	5.011	6

The same result can be observed for the other cost parameters with Uniform, Binomial and Distribution-3 distributions as well. Furthermore, the total target inventory level over 6 periods under the LA policy (*l*=3) is obviously less than the optimal total target inventory level.

In 8-period problem, the target inventory level under the LA policy in each period is also

lower than the optimal target inventory level. Some numerical examples for the Uniform distribution are listed in Table 7, and the same result can be observed for other cost parameters with Uniform, Binomial and Distribution-3 distributions as well.

Table 7. Target Inventory Level under LA and Optimal Policies (T=8)

Cost Combinations			Policies	Target Inventory Level in Each Period							
h	b	o		1	2	3	4	5	6	7	8
0.1	10	20	OPT	8	8	8	8	8	7.411	8	8
			LA	7	7.250	7.318	7.288	7.298	7.350	8	8
1	10	20	OPT	7	7	7.051	7.037	7.057	7.014	8	8
			LA	7	7	7	7	7	7	8	8
2	10	20	OPT	7	7	7	7	7	7	7	7
			LA	6	6.123	6.123	6.151	6.137	6.123	7	7
5	10	1	OPT	6	6	6	6	6	6	6	6
			LA	5	5	5.019	5.012	5.010	5.008	5.011	6

Next, we observe that there is no obvious consistent relationship between the target inventory level Y^t under the LA and DB policies with l=3. As discussed in Section 4.3.2, the order quantity in one period under the DB policy is determined by balancing the backlogging penalty and the sum of the marginal holding cost and the outdating cost. In a problem with integer demand and order quantity, randomized dual-balancing policy is used. But because we choose two possible integer order quantities with a particular probability for each one, the average order quantity is still the quantity balancing the two sets of costs. On the other hand, for the LA policy, the feasible integer order quantity can be examined one by one to get the desired order quantity that minimizes the expected total cost in the current period. This order quantity under the LA policy can be either smaller or larger than the average order quantity under the DB policy, which is determined by the shapes of the two curves in the DB policy. For example, consider a system with the Uniform distribution, l=3 and two cost combinations h=10, b=10, o=5 and h=0.1, b=10, o=1, in the first period. Figure 5 and 6 illustrate the backlogging penalty Π^t and the sum of the marginal holding and the outdating costs H^t+O^t with different

cost combinations. The abscissa of the intersection of the two curves is the desired q^1 under the DB policy.

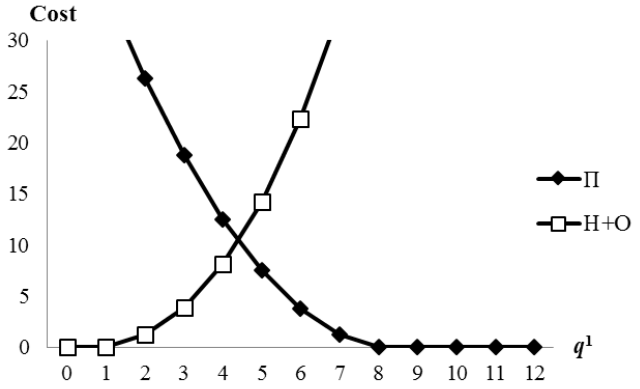

Figure 5. Cost for h=10, o=5

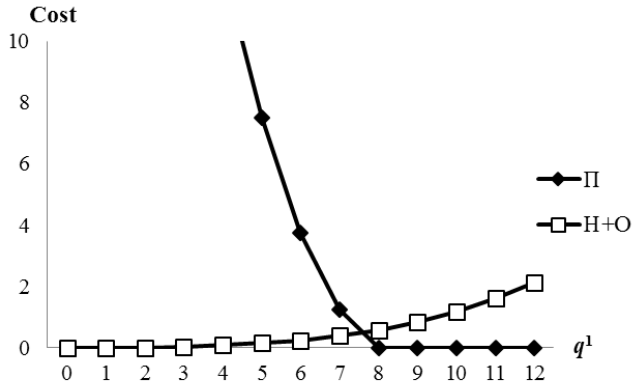

Figure 6. Cost for h=0.1, o=1

Note that the values of b, h and o affect the decreasing or increasing rate of the curves. In Figure 5, h and o are larger than those in Figure 6, so the curve representing H^t+O^t increases much faster than that in Figure 6. The value of q^1 under the DB policy varies simultaneously, which can be observed in Figures 5 and 6 as well. For the LA policy, when the curve representing H^t+O^t increases fast in Figure 5, ordering 4 units (smaller than that under the DB policy) results in a lower total cost, while in the Figure 6, 8 units need to be ordered (larger than that under the DB policy). Therefore, in situations having relatively large h and o, the target inventory level under the DB policy is usually larger than that under the LA policy.

4.2 Systems with Lost-Sales and No Ordering Capacity Constraint

4.2.1 Performance of DB and LA Policies

Table 8 shows ρ^{DB} and ρ^{LA} in a perishable inventory problem with lost-sales (T=6). We observe that the maximum of ρ^{LA} and ρ^{DB} are less than 100%, showing that both LA and DB policies lead to an expected total cost that is less than twice the optimal expected total cost.

Table 8. ρ^{LA} and ρ^{DB} with Lost-Sales and No Capacity Constraint (%, T=6)

Policy	l	Uniform		Binomial		Distribution-3	
		Mean	Max	Mean	Max	Mean	Max
LA	1	2.12	11.75	1.90	7.24	1.64	4.83
	2	3.33	15.39	2.44	14.60	3.10	11.22
	3	3.34	11.86	2.13	10.46	3.25	14.43
DB	1	23.67	85.03	15.04	45.01	28.63	80.06
	2	25.59	87.05	16.67	50.04	31.69	89.86
	3	26.89	87.66	15.87	49.19	32.57	93.80

We also observe that the average of ρ^{LA} is much smaller than the average of ρ^{DB}. Therefore, in a perishable inventory problem with lost-sales and no limited order capacity, the LA policy, on average, outperforms the DB policy significantly.

Considering different truncated marginal holding costs, we observe that, similar to a problem with backlogs, both ρ^{LA} and ρ^{DB} have the smallest average value when l=1. Therefore, LA and DB policies for l=1 is the best one to approximate the optimal policy in a lost-sales problem without an ordering capacity constraint in each period.

Table 9 shows that ρ^{DB} and ρ^{LA} in the 8-period perishable inventory problem with lost-sales, which illustrates a similar result with that in Table 8. We observe that all ρ^{DB} and ρ^{LA} are less than 100% in the columns indicating the maximum relative gap, which demonstrates that the expected total costs under both DB and LA policies are less than twice the optimal expected total cost. Second, the LA policy significantly outperforms the DB policy for all three demand distributions. Third, both ρ^{LA} and ρ^{DB} have the smallest average value when l=1.

Table 9. ρ^{LA} and ρ^{DB} with Lost-Sales and No Capacity Constraint (%, T=8)

Policy	l	Uniform		Binomial		Distribution-3	
		Mean	Max	Mean	Max	Mean	Max
LA	1	2.37	10.42	1.86	7.92	2.63	14.01
	2	3.29	8.47	2.39	5.95	4.51	12.81
	3	4.42	15.74	2.86	11.89	5.15	17.00
DB	1	24.02	79.30	15.83	45.66	31.77	82.71
	2	24.70	87.38	16.63	39.43	34.82	83.86
	3	27.57	81.72	19.59	41.28	33.39	87.91

Next, we examine the impact of the cost parameters on the performance of the DB policy. Figure 7 and 8 show ρ^{DB} for different h, b when the demand in each period follows Uniform and Binomial distributions, respectively (l=3, T=6). Similar to Section 4.1.1, ρ^{DB} tends to be larger as the per-unit backlogging penalty b increases, while tends to be smaller as the per-unit holding cost h increases.

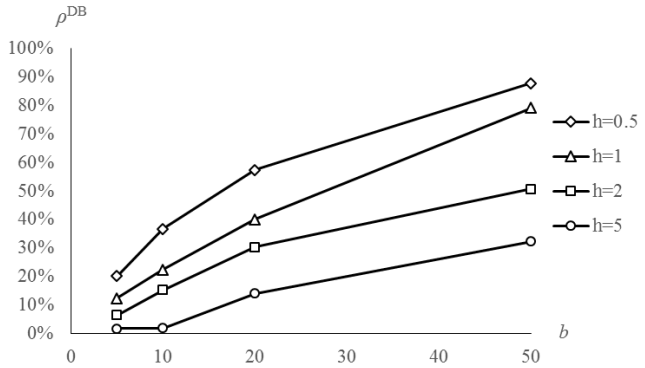

Figure 7. ρ^{DB} for Uniform Demand

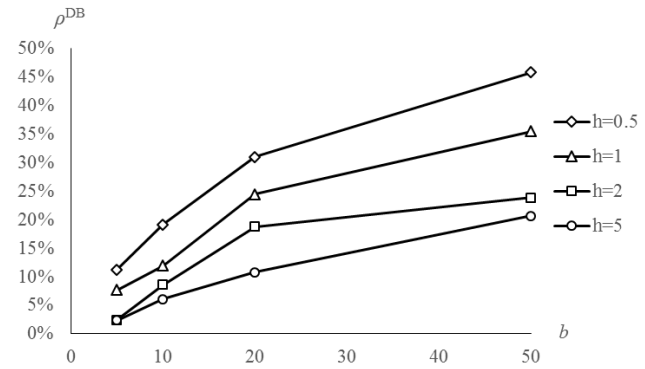

Figure 8. ρ^{DB} for Binomial Demand

Figure 9 and 10 show ρ^{DB} for different h and o.

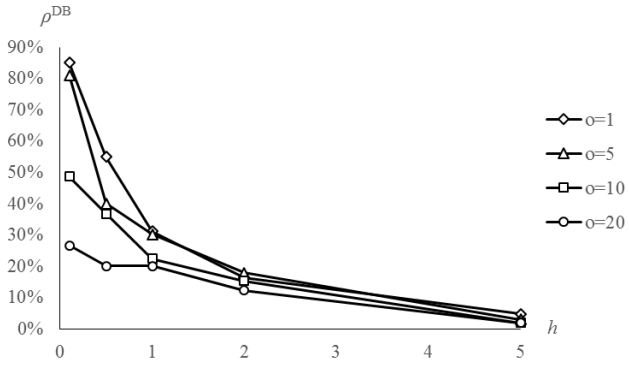

Figure 9. ρ^{DB} for Uniform Demand

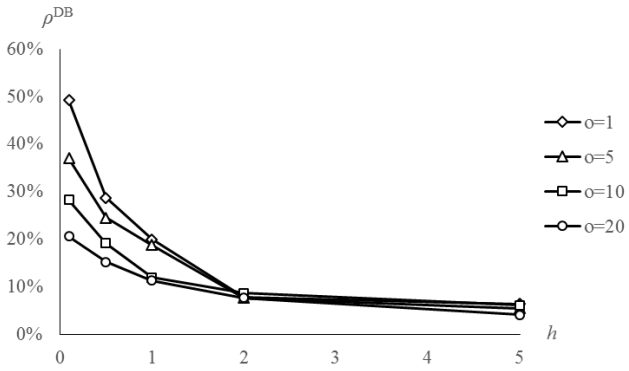

Figure 10. ρ^{DB} for Binomial Demand

We observe that ρ^{DB} tends to be smaller as both the per-unit holding cost h and the per-unit outdating cost o increase. Thus, increasing h or o leads to a better performance of the DB policy.

4.2.2 Optimal and Heuristic Policies

Table 10 shows the average optimal target inventory level over 6 periods for Uniform distribution and b=10 in a perishable inventory system with lost-sales and no capacity constraint. We observe that the average optimal target inventory level decreases in h and o, similar to the result in a problem with backlogs listed in Table 4. Because there is no order capacity in each period, the backlogs do not affect the system satisfying new demand. Thus, the problems with backlogs and lost-sales have a similar optimal target inventory level. The optimal target

inventory levels for Binomial and Distribution-3 distributions are also similar to those in a system with backlogs.

Table 10. Average Optimal Target Inventory Level (T=6)

h	o	Optimal Y^t
0.1	1, 2, 5, 10	8
	20	7.90
0.2	2	8
0.25	5	8
0.4	2	8
0.5	1, 5, 10	8
	20	7.45
1	0.5, 1, 2, 2.5, 5	8
	10	7.44
	20	7.35
2	1, 5, 10, 20	7
2.5	5	7
4	2, 10, 20, 40	6
5	1, 5, 10, 20	6
10	5, 20, 25, 50, 100	4

Table 11 shows the average optimal target inventory level in the 8-period problem for the Uniform distribution. We observe the same trend observed for the average optimal target inventory level for h and o.

Table 11. Average Optimal Target Inventory Level (T=8)

h	o	Optimal Y^t
0.1	1, 5, 10	8
	20	7.92
0.5	1, 5, 10	8
	20	7.41
1	1, 5	8
	10	7.40
	20	7.27
2	1, 5, 10, 20	7
5	1, 5, 10, 20	6

Then, we examine the ordering policy for LA and DB policies in a problem with lost-sales.

In our numerical examples, we observe that Y^t under the LA policy with l=3 is lower than the optimal Y^t. Some examples for Uniform distribution are listed in Table 12. The same result can be observed for the other cost parameters with Uniform, Binomial and Distribution-3 distributions as well. Furthermore, the total target inventory level over 6 periods under the LA policy (l=3) is obviously less than the optimal total target inventory level.

Table 12. Target Inventory Level under LA and Optimal Policies (T=6)

Cost Combinations			Policies	Target Inventory Level in Each Period					
h	*b*	*o*		1	2	3	4	5	6
0.1	10	20	OPT	8	8	8	7.395	8	8
			LA	7	7.236	7.310	7.313	8	8
1	10	20	OPT	7	7	7.075	7.016	8	8
			LA	7	7	7	7	8	8
2.5	10	5	OPT	7	7	7	7	7	7
			LA	6	6	6.047	6.029	7	7
5	10	1	OPT	6	6	6	6	6	6
			LA	5	5	5.014	5.009	5.014	6

In the 8-period problem, the target inventory level under the LA policy in each period is also lower than the optimal target inventory level. Some numerical examples for the Uniform distribution are listed in Table 13, and the same result can be observed for the other cost parameters with Uniform, Binomial and Distribution-3 distributions as well.

Table 13. Target Inventory Level under LA and Optimal Policies (T=8)

Cost Combinations			Policies	Target Inventory Level in Each Period							
h	*b*	*o*		1	2	3	4	5	6	7	8
0.1	10	20	OPT	8	8	8	8	8	7.386	8	8
			LA	7	7.263	7.295	7.276	7.268	7.302	8	8
1	10	20	OPT	7	7	7.046	7.051	7.040	7.023	8	8
			LA	7	7	7	7	7	7	8	8
2	10	20	OPT	7	7	7	7	7	7	7	7
			LA	6	6.133	6.133	6.131	6.140	6.110	7	7
5	10	1	OPT	6	6	6	6	6	6	6	6
			LA	5	5	5.019	5.011	5.015	5.014	5.011	6

The relationship between the target inventory level under LA and DB policies in a problem with lost-sales is the same as that in a problem with backlogs: the LA policy results in a larger Y^t when h and o are small, while the DB policy results in a larger Y^t when h and o are large.

4.3 Systems with Backlogs and an Ordering Capacity Constraint

4.3.1 Performance of DB and LA Policies

Table 14 shows ρ^{DB} and ρ^{LA} in a problem with backlogs and an ordering capacity constraint in each period. The maximum of ρ^{DB} and ρ^{LA} are less than 100%, indicating that both LA and DB policies lead to an expected total cost that is less than twice the optimal expected total cost.

We compare the average values of ρ^{LA} and ρ^{DB}, and observe that ρ^{LA} is smaller than ρ^{DB} for all distributions, illustrating that the LA policy, on average, outperforms the DB policy in a problem with backlogs and an ordering capacity constraint. Furthermore, comparing the ρ^{LA} and ρ^{DB} in a problem with backlogs and no order capacity (Table 2), ρ^{DB} for Binomial distribution and ρ^{LA} become larger, while ρ^{DB} for Uniform and Distribution-3 distributions become smaller. This indicates that in a problem with backlogs, adding an ordering capacity constraint has different effects on the performance of LA and DB policies for different distributions.

Table 14. ρ^{LA} and ρ^{DB} with Backlogs and a Capacity Constraint (%)

Policy	l	Uniform		Binomial		Distribution-3	
		Mean	Max	Mean	Max	Mean	Max
LA	1	7.74	37.33	6.42	20.99	13.19	49.98
	2	10.83	52.84	6.93	34.85	9.83	63.74
	3	12.56	53.63	4.72	21.38	10.14	56.22
DB	1	16.28	62.20	20.08	94.53	23.99	73.17
	2	19.79	58.85	16.47	50.65	30.12	95.89
	3	23.65	68.32	23.40	93.00	23.38	77.09

4.3.2 Optimal and heuristic Policies

When a perishable inventory system has an order capacity constraint in each period, order quantity q^t is limited, which may result in more backlogs. We examine target inventory level Y^t in each period in a perishable inventory problem with backlogs and an ordering capacity constraint **u**. The cost parameters can be divided into three groups based on the optimal Y^t in the first period, which are 8, 7 and 6, respectively. Table 15 shows the maximum and minimum of Y^t in each period for each group and Uniform distribution.

Table 15. Target Inventory Level under the Optimal Policy

Cost Combinations	Period t	1	2	3	4	5	6
	u^t	8	4	6	5	8	5
h<4 and any o	Max Y^t	8	7.643	8.330	7.256	9.827	7.923
h=4 and o=2, 10	Min Y^t	8	6.230	6.263	5.267	6.638	5.446
h=4 and o=20, 40	Max Y^t	7	5.739	6.033	5.089	6.530	5.366
h=5 and any o	Min Y^t	7	5.253	5.283	4.780	5.683	5.025
h=10 and any o	Max Y^t	6	4.327	4.343	3.857	4.710	3.577
	Min Y^t	6	4.190	4.263	3.330	4.602	3.447

We observe that, similar to a problem without a capacity constraint, when h and o are small, the system tends to order more units to avoid the backlogging penalty. We also observe that when the order capacity is relatively large, specifically, in the first, third and fifth periods, the corresponding Y^t are larger than those in the other periods, based on our numerical examples. Therefore, generally, the system tends to order more units when the per-unit backlogging penalty b is larger than the per-unit holding cost h and the order capacity permits. These two characteristics can also be observed for both Binomial and Distribution-3 distributions.

Then, we compare the optimal solutions for problems with and without an ordering capacity constraint in each period. Theoretically, in a dynamic program, adding a constraint may make the optimal solution worse. Based on our numerical examples, we observe that the optimal expected total cost in a problem with a capacity constraint in each period is always

larger than that in a problem with the same cost parameters and without the capacity constraint. Some numerical examples are given in Table 16. The same result can be observed for the other cost parameters.

Table 16. Comparison of Optimal Expected Total Cost

Demand Distributions	Cost Combinations			Optimal Expected Total Cost	
	h	*b*	*o*	No **u**	**u**
Uniform	0.1	10	20	10.209	16.937
	1	10	20	26.492	35.369
	2.5	10	5	48.095	56.242
	5	10	1	78.854	89.470
Binomial	0.1	10	20	2.829	3.325
	1	10	20	14.858	16.195
	2.5	10	5	28.070	31.292
	5	10	1	45.964	48.711
Distribution-3	0.1	10	20	16.749	24.262
	1	10	20	36.275	40.937
	2.5	10	5	60.730	64.400
	5	10	1	84.948	101.370

Table 17 shows the maximum and minimum of Y^t in each period under the LA policy for l=3 and Uniform distribution. We observe that when h and o are small, the system tends to order more units.

Table 17. Target Inventory Level under the LA Policy

Cost Combinations	Period *t*	1	2	3	4	5	6
	u^t	8	4	6	5	8	5
$h \leq 5$ and any o	Max Y^t	8	7.51	8.23	7.25	9.17	7.67
	Min Y^t	8	6.19	6.21	5.09	5.66	5.01
h=10 and o=5, 20, 25, 50	Max Y^t	7	5.42	5.36	4.36	4.95	3.52
	Min Y^t	7	4.97	5.02	4.22	4.79	3.40
h=10 and o=100	Max Y^t	6	4.57	4.50	3.90	4.71	3.58
	Min Y^t	6	4.57	4.50	3.90	4.71	3.58

When there is a capacity constraint in each period, Y^t under the LA policy is larger than the optimal Y^t in most periods when the per-unit holding cost h is relatively large, for example,

h=5 and 10. On the other hand, when h is small, such as $h \leq 2$, based on our numerical examples, Y^t under the LA policy is smaller than the optimal Y^t in most periods. Some examples for Uniform distribution are given in Table 18. A similar relationship between Y^t under the LA policy and the optimal Y^t can be observed for Binomial and Distribution-3 distributions.

Table 18. Target Inventory Level under LA and Optimal Policies

Cost Combinations			Policies	Target Inventory Level in Each Period					
h	b	o		1	2	3	4	5	6
0.1	10	20	OPT	8	6.910	7.273	6.897	9.570	7.670
			LA	8	7.140	7.270	6.740	8.880	7.450
1	10	20	OPT	8	6.792	7.005	6.232	7.561	6.997
			LA	8	6.760	6.810	6.160	7.450	6.710
2.5	10	5	OPT	8	6.607	6.620	5.980	6.740	6.083
			LA	8	6.850	7.080	6.030	6.690	5.920
5	10	5	OPT	7	5.364	5.413	4.838	5.713	5.051
			LA	8	6.370	6.320	5.250	5.820	5.190

Then, we examine the relationship between Y^t under LA and DB policies. We observe that for Uniform distribution, Y^t under the LA policy is larger than that under the DB policy in all periods with only two exceptions. Comparing to a problem with backlogs and without an ordering capacity constraint, the quantity of backlogs may increase, thus, the slope of the curve representing the backlogging penalty in Figure 1 and 2 increases. Therefore, Y^t under the LA policy is larger than that under the DB policy in more periods. For example, Figure 11 shows the backlogging penalty Π^t and the sum of the marginal holding cost and the outdating cost H^t+O^t for Uniform distribution, l=3 and cost combination h=10, b=10, o=5. Comparing to Figure 1, the slope of the curve representing the backlogging penalty Π^t is much higher.

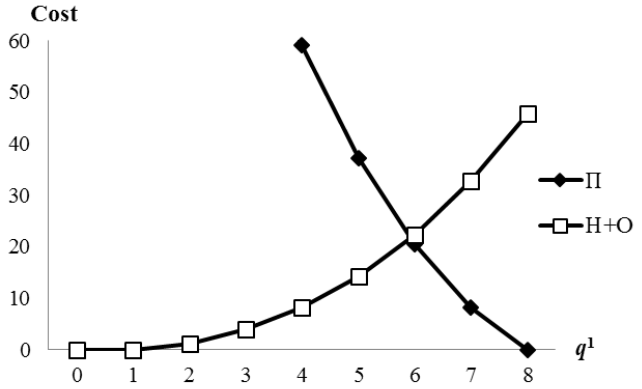

Figure 11. Cost for h=10, o=5 with an Order Capacity Constraint

4.4 Systems with Lost-Sales and an Ordering Capacity Constraint

4.4.1 Performance of DB and LA Policies

Table 19 shows ρ^{LA} and ρ^{DB} for different values of l. The maximum of ρ^{LA} and ρ^{DB} for each distribution and l are less than 100%. This demonstrates that the expected total costs under both LA and DB policies are less than twice the optimal expected total cost in problems with lost-sales and an ordering capacity constraint.

Table 19. ρ^{LA} and ρ^{DB} with Lost-Sales and a Capacity Constraint (%)

Policy	l	Uniform		Binomial		Distribution-3	
		Mean	Max	Mean	Max	Mean	Max
LA	1	7.35	33.08	34.20	55.62	8.61	25.66
	2	4.83	25.30	26.75	54.75	6.30	26.85
	3	6.03	26.22	25.98	57.58	5.82	24.30
DB	1	15.12	43.23	15.64	60.66	19.78	51.67
	2	15.55	42.44	12.78	31.81	21.45	68.73
	3	15.09	48.23	14.15	34.10	21.06	72.53

Comparing the average values of ρ^{LA} and ρ^{DB}, we observe that ρ^{LA} outperforms ρ^{DB} for Uniform and Distribution-3 distributions. But ρ^{LA} is larger than ρ^{DB} for Binomial distribution. Therefore, in problems with lost-sales and an ordering capacity constraint in each period, LA and DB policies perform differently for different distributions. Also, comparing ρ^{LA} and ρ^{DB} in

a problem with lost-sales and no order capacity (Table 8), generally, when there is an order capacity constraint in each period, ρ^{LA} becomes larger, while ρ^{DB} becomes smaller. This indicates that in a problem with lost-sales, adding an ordering capacity constraint improves the performance of the DB policy, and deteriorates the performance of the LA policy.

4.4.2 Optimal and Heuristic Policies

We examine target inventory level Y^t in a perishable inventory problem with lost-sales and a capacity constraint **u**. Similar to the result in Section 4.3.2, the cost parameters can be divided into four groups based on the optimal Y^t in the first period, which are 8, 7, 6 and 4, respectively. Table 20 shows the maximum and minimum of the optimal Y^t for different groups. Recall that in a problem with backlogs and an ordering capacity constraint, the optimal Y^t tends to be smaller as h and o increase. We observe these characteristic in Table 20 as well. Similar result can be observed for Binomial and Distribution-3 distributions.

Table 20. Target Inventory Level under the Optimal Policy

Cost Combinations	Period t	1	2	3	4	5	6
	u^t	8	4	6	5	8	5
h<2 and any o	Max Y^t	8	7.560	8.542	7.588	10.067	8.202
h=2 and o=1, 5	Min Y^t	8	6.172	6.679	6.293	7	6.366
h=2 and o=10, 20	Max Y^t	7	5.866	6.612	6.284	7	6.400
h=2.5 and o=5	Min Y^t	7	5.843	6.590	6.240	7	6.322
h=4 and any o	Max Y^t	6	5.178	6	5.656	6	5.632
h=5 and any o	Min Y^t	6	5.109	6	5.606	6	5.606
h=10 and any o	Max Y^t	4	4	4	4	4	4
	Min Y^t	4	4	4	4	4	4

Then, we compare the optimal solutions in problems with and without an ordering capacity constraint. Some results are given in Table 21. We observe that optimal expected total cost in a problem with a capacity constraint is always larger than that in a problem without the capacity constraint.

Table 21. Comparison of Optimal Expected Total Cost

Demand Distributions	Cost Combinations			Optimal Expected Total Cost	
	h	*b*	*o*	No **u**	**u**
Uniform	0.1	10	20	9.830	13.304
	1	10	20	26.433	31.427
	2.5	10	5	47.383	50.870
	5	10	1	78.855	80.082
Binomial	0.1	10	20	2.925	3.231
	1	10	20	14.816	15.512
	2.5	10	5	28.363	29.923
	5	10	1	46.203	47.028
Distribution-3	0.1	10	20	16.863	18.759
	1	10	20	36.309	38.100
	2.5	10	5	60.888	61.128
	5	10	1	86.063	86.869

Then, we examine Y^t under the LA policy for Uniform distribution, and observe that Y^t tends to be smaller as *h* and *o* increase. Similar to the result in Section 4.3.2, we observe that Y^t under the LA policy is larger than the optimal Y^t in most periods with large *h* and *o*, and smaller than the optimal Y^t in most periods with small *h*. The effect of *o* on the target inventory level under the LA policy is larger than that in a problem with backlogs. Some numerical examples are given in Table 22.

Table 22. Target Inventory Level under LA and Optimal Policies

Cost Combinations			Policies	Target Inventory Level in Each Period					
h	*b*	*o*		1	2	3	4	5	6
0.1	10	20	OPT	8	6.904	7.506	6.916	9.920	7.960
			LA	8	6.765	7.370	6.985	9.380	7.755
1	10	20	OPT	8	6.471	6.913	6.368	8	7.290
			LA	8	6.850	7.280	6.515	8.005	7.310
2.5	10	5	OPT	7	5.866	6.590	6.240	7	6.400
			LA	8	6.275	6.740	6.395	7.010	6.320
5	10	5	OPT	6	5.129	6	5.624	6	5.613
			LA	7	5.245	6.015	5.360	6	5.625

5 Conclusion

We consider a single-item, periodic-review, stochastic perishable inventory problem under both backlogging and lost-sales assumptions, with and without an ordering capacity constraint in each period. To approximate the optimal inventory level at the beginning of each period, we extend the DB and LA policies under different assumptions by using marginal cost accounting scheme, and show numerically that both policies result in an expected total cost less than twice the optimal expected total cost for all the examples considered in this study. We also show that, on average, the LA policy significantly outperforms the DB policy in most situations. For order quantity, we observe that the target inventory level under the LA policy is not larger than the optimal periodic target inventory level in problems with both backlogs and lost-sales and without an ordering capacity constraint. We also analyze the relationship between periodic target inventory levels under LA and DB policies.

There are several possible directions for the future research. First, we study the problem for three independent and identically distributed (i.i.d.) demands, separately. Some other types of demands can be used to examine the policies in the future, such as ADI demands, autoregressive demands, MMFE demands and Markov modulated demands [7]. Furthermore, multiple classes of demands can also be considered. Also, this thesis assumes that the lead time is 0, ignoring the time interval between placing and receiving an order, which may not hold in practice. Thus, problems with a positive lead time can be studied in the future.

References

[1] U. Abdulwahab and M. I. Wahab, "Approximate Dynamic Programming Modeling for a Typical Blood Platelet Bank," *COMPUTER & INDUSTRIAL ENGINEERING,* vol. 78, pp. 259-270, 2014.

[2] J. Ma, T. Lei and G. E. Okudan, "EOQ-based Inventory Control Policies for Perishable Items: The Case of Blood Plasma Inventory Management," in *The Industrial and Systems Engineering Research Conference*, San Juan, 2013.

[3] US Department of Health and Human Services, "The 2011 National Blood Collection and Utilization Survey Report," 2013.

[4] Joint Industry Unsaleables Leadership Team, "2008 Joint Industry Unsaleables Report," 2008.

[5] S. Nahmias, Perishable Inventory Systems, New York: Springer, 2011.

[6] R. Levi, M. Pál, R. O. Roundy and D. B. Shmoys, "Approximation Algorithms for Stochastic Inventory Control Models," *MATHEMATICS OF OPERATIONS RESEARCH,* vol. 32, no. 2, pp. 284-302, 2007.

[7] X. Chao, X. Gong, C. Shi and H. Zhang, "Approximation Algorithms for Perishable Inventory System," *OPERATIONS RESEARCH*, vol. 63, pp. 585-601, 2015.

[8] V. A. Truong, "Approximation Algorithm for the Stochastic Multi-period Inventory Problem via a Look-Ahead Optimization Approach," *MATHEMATICS OF OPERATIONS RESEARCH,* vol. 39, pp. 1039-1056, 2014.

[9] C. Zhang, T. Ayer and C. C. White III, "2-Approximation Algorithms for Perishable Inventory Control When FIFO Is an Optimal Issuing Policy," *Working Paper*, 2016.

[10] R. Levi, G. Janakiraman and M. Nagarajan, "A 2-Approximation Algorithm for Stochastic Inventory Control Models with Lost Sales," *MATHEMATICS OF OPERATIONS RESEARCH*, vol. 33, no. 2, p. 351–374, 2008.

[11] S. Nahmias and W. P. Pierskalla, "Optimal ordering policies for a product that perishes in

two periods subject to stochastic demand," *NAVAL RESEARCH LOGISTICS,* vol. 20, pp. 207-229, 1973.

[12] S. Nahmias, "Optimal ordering policies for perishable inventory-II," *OPERATIONS RESEARCH,* vol. 23, pp. 735-749, 1975a.

[13] B. Fries, "Optimal ordering policy for A perishable commodity with fixed lifetime," *OPERATIONS RESEARCH,* vol. 23, pp. 46-61, 1975.

[14] K. G. Kempf, P. Keskinocak and R. Uzsoy, Planning production and inventories in the extended enterprise: a state of the art handbook, New York: Springer, 2011.

[15] M. A. Cohen, "Analysis of Single Critical Number Ordering Policies for Perishable Inventories," *OPERATIONS RESEARCH,* vol. 24, pp. 726-741, 1976.

[16] E. Brodheim, C. Derman and G. Prastacos, "On the Evaluation of a Class of Inventory Policies for Perishable Products Such as Blood," *MANAGEMENT SCIENCE,* vol. 21, pp. 1320-1325, 1975.

[17] S. Nahmias, "A comparison of alternative approximations for ordering perishable inventory," *INFOR: INFORMATION SYSTEMS AND OPERATIONAL RESEARCH,* vol. 13, pp. 175-184, 1975.

[18] S. Nahmias, "Myopic approximations for the perishable inventory problem," *MANAGEMENT SCIENCE,* vol. 22, pp. 1002-1008, 1976.

[19] S. Nahmias, "Comparison between two dynamic perishable inventory models," *OPERATIONS RESEARCH,* vol. 25, pp. 168-172, 1977.

[20] S. Nahmias, "Higher-Order Approximation for the Perishable-Inventory Problem," *OPERATIONS RESEARCH,* vol. 25, pp. 630-640, 1977.

[21] P. Nandakumar and T. E. Morton, "Near Myopic Heuristics for the Fixed-life Perishability Problem," *MANAGEMENT SCIENCE,* vol. 39, pp. 1490-1498, 1993.

[22] W. L. Cooper, "Pathwise Properties and Performance Bounds for a Perishable Inventory System," *OPERATIONS RESEARCH,* vol. 49, pp. 455-466, 2001.

[23] X. Chen, Z. Pang and L. Pan, "Coordinating inventory control and pricing strategies for perishable," *OPERATIONS RESEARCH,* vol. 62, pp. 284-300, 2014.

[24] Y. Li, A. Lim and B. Rodrigues, "Pricing and inventory control for a perishable product," *MANUFACTURING & SERVICE OPERATIONS MANAGEMENT,* vol. 11, pp. 538-542, 2009.

[25] R. Levi, R. O. Roundy, D. B. Shmoys and V. A. Truong, "Approximation Algorithms for Capacitated Stochastic Inventory Control Models," *OPERATIONS RESEARCH,* vol. 56, no. 5, p. 1184–1199, 2008.