Intervals with few Prime Numbers

by

Dan Wolczuk

A thesis presented to
the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2003

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This thesis is dedicated to the loving memory of George and Alice Wolczuk.

Acknowledgements

First, I would like to thank Dr. Cameron Stewart for all his assistance and understanding with my thesis and degree. I would also like to thank Shonn Martin, Lis D'Alessio and Kim Gingerich for all the support, help and kindness they gave me. I would further like to thank my readers Yu-Ru Liu and Michael Rubinstein for their corrections and Agnieszka Zygmunt for all her encouragement. Finally, I will give a very special thank you to Carol Clemson whose moral support, encouragement and friendship made this thesis possible.

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CHAPTER 1

Introduction

In this thesis we will discuss some of the tools used in the study of the number of primes in short intervals. In particular, we will discuss a large sieve density estimate due to Gallagher and two classical differential delay equations arising in number theory, namely the Buchstab function and the Dickman function. We will also show how these tools have been used by Maier and Stewart to prove a new result in this area.

In chapter 2, after giving a brief introduction to Dirichlet characters and L-functions, we will discuss developments of Linnik's large sieve method and of density theorems for L-functions. Finally, we will give Gallagher's result and show how it can be applied to study the number of primes in arithmetic progressions.

In chapter 3 and 4 we give some of the known results on the Buchstab function and the Dickman function respectively. In chapter 5 we show how values for these functions may be computed and we use these values to compute values for and graph a function introduced by Maier and Stewart in the paper mentioned above.

First, we start by giving a brief summary of some of the results on the distribution of prime numbers.

1.1. The prime number theorem

Over the last century and a half, there has been significant progress in the study of the distribution of the primes. The most important result to date is the prime number theorem which tells us that the number of primes less than or equal to x, denoted by $\pi(x)$, is asymptotic to $x/\log x$ as $x \to \infty$. This was first conjectured in 1792 by Gauss. More precisely, he conjectured that $\pi(x) \sim \mathrm{Li}(x)$ where $\mathrm{Li}(x)$ is the logarithmic integral of x defined by

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

Around 1850, Tschebycheff was able to prove with elementary methods that there exists constants c and c_1 with $0 < c < 1 < c_1$ such that

$$c\frac{x}{\log x} < \pi(x) < c_1\frac{x}{\log x},$$

for all $x \geq 2$. Moreover, he computed values for c and c_1 which are close to 1. The prime number theorem was finally proven in 1896 independently by de la Vallee Poussin and Hadamard. Both proofs relied heavily on the determination of a large zero-free region for the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty}(1/n^s)$ which is defined on the half plane Re $s > 1$, but can be analytically continued to the whole complex plane (see Apostol [1] chapter 12).

As usual we let p_n denote the n-th prime number and $d_n = p_{n+1} - p_n$. The prime number theorem tells us that the average size of d_n is $\log p_n$ and the density of the primes in the interval $(x, x + cx)$ is asymptotic to $1/\log x$ for any positive constant c. This leads us to ask two questions: how much can the size of d_n deviate from the average, and for which functions $\Phi(x)$ do we have

$$\pi(x + \Phi(x)) - \pi(x) \sim \frac{\Phi(x)}{\log x}? \tag{1.1.1}$$

1.2. Small gaps and the twin prime conjecture

A natural question to ask in the study of smaller than average gaps between prime numbers is what is the value of

$$E = \liminf_{n\to\infty} \frac{d_n}{\log p_n}.$$

Trivially by the prime number theorem we have that $E \leq 1$. In 1940, Erdös [28] obtained the first non-trivial unconditional result (previous results by Hardy and Littlewood and later by Rankin [72,74] were conditional to the Extended Riemann Hypothesis) by showing that $E < 1$. Subtle improvements on the upper bound for E were made by Rankin [73] in 1947 who proved $E < 57/59$, by Ricci [79] who showed $E < 15/16$ in 1954, and by Wang Yaun, Xie Sheng-gang, Yu Kun-rui [87] in 1965 who gave $E < 29/32$. A major breakthrough was made by Bombieri and Davenport [7] in 1966. They used the large sieve to get $E \leq (2+\sqrt{3})/8$. Many slight improvements on their result were made by Pilt'jai [69], Huxley [45] and by Fouvry and Grupp [31]. Fouvry and Grupp in fact showed that $d_n \leq 0.4342 \log p_n$ holds for a positive proportion of the primes. The best value of E obtained to date is by Maier [58] in 1988 who showed that $E \leq 0.248$.

We actually expect that $E = 0$ and one method of showing this would be to prove the twin prime conjecture. That is to prove that $d_n = 2$ for infinitely many n. When $d_n = 2$ we say that the pair p_n and p_{n-1} are twin primes. These pairs were characterized in 1949 by Clement [20]. He showed that the positive integers n and $n+2$, $n \geq 2$, are twin primes if and only if

$$4((n-1)!+1)+n \equiv 0 \pmod{n(n+2)}.$$

Let $\pi_2(x)$ denote the number of primes p such that $p \leq x$ and $p+2$ is also prime. In 1919, Brun [10] proved there is an effectively computable integer x_0 such that if $x \geq x_0$ then

$$\pi_2(x) < \frac{100x}{\log^2 x}.$$

Brun [11] also proved in 1919 the convergence of the sum

$$\sum \left(\frac{1}{p} + \frac{1}{p+2}\right),$$

where the sum is taken over all primes p such that $p+2$ is also prime. This shows that if there are infinitely many twin primes then they are extremely sparse among the primes since we know that the $\sum 1/p$ taken over all primes p is divergent.

In 1966, Bombieri and Davenport [7] used sieve methods to prove that

$$\pi_2(x) \leq 4C\frac{x}{\log^2 x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right),$$

where C is the twin prime constant given by

$$C = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right).$$

The value $C = 1.32032\ldots$ was computed by Wrench [90] in 1961.

Improvements to Bombieri and Davenport's result were made by Fouvry and Iwaniec [32] in 1983. They proved that 4 could be replaced by $34/9 + \epsilon$. In 1986, Bombieri, Friedlander and Iwaniec [9] show that this could be further improved to $3.5 + \epsilon$ for any $\epsilon > 0$. The best result to date is by Jie Wu [89], in 1990, who proved that

$$\pi_2(x) \leq 3.418C\frac{x}{\log^2 x}.$$

However, this is far off Hardy and Littlewood's conjecture that

$$\pi_2(x) \sim \frac{Cx}{\log^2 x}.$$

Renyi [76], expanding on work of Brun, was able to attack the twin prime problem in a new way. He showed, in 1947, that there are infinitely many primes p such that $p+2$ has at most k factors. Buchstab [14] proved in 1967 that one could take $k = 3$, and Chen [18,19] announced in 1966 (published in 1973; 1978) that one could take $k = 2$.

For the number $\pi_2^{(2)}(x)$ of primes $p \leq x$ where $p + 2$ has at most 2 prime factors, Chen proved that for large values of x we have

$$\pi_2^{(2)}(x) \geq aC\frac{x}{\log^2 x},$$

where $a = 0.67$. In 1978, Chen [19] improved the constant a to $a = 0.81$ and the best result to date is from 1990 by Wu [89] who showed the result for $a = 1.05$.

1.3. Large Gaps

We will now look at results on when the size of d_n is greater than the average. The first main result in this area was by Backlund [2] in 1929. He proved for any $\epsilon > 0$ there are infinitely many n such that

$$p_{n+1} - p_n > (2 - \epsilon) \log p_n.$$

In 1930, Brauer and Zeitz [4] were able to prove that $(2-\epsilon)$ could be replaced by $(4-\epsilon)$. Then in 1931 Westzynthius [88], was able to prove that, for infinitely many n,

$$p_{n+1} - p_n > \frac{2e^\gamma \log p_n \log_3 p_n}{\log_4 p_n},$$

where $\gamma = 0.5772\cdots$ denotes Euler's constant and where we have adopted the notation that $\log_k x$ indicates k iterations of the logarithm (for example $\log_2 x = \log\log x$). This is a remarkable result as it shows that the size of d_n exceeds the average by a factor which tends to infinity.

Improvements on Westzynthius' result were made in 1934 by Ricci [79] who showed, for a positive number c_1 there are infinitely many n such that

$$p_{n+1} - p_n > c_1 \log p_n \log_3 p_n.$$

Erdös [27] made further improvements in 1935 by proving the following theorem.

THEOREM 1. *For a positive number c_2 there are infinitely many n such that*

$$p_{n+1} - p_n > \frac{c_2 \log p_n \log_2 p_n}{(\log_3 p_n)^2}.$$

We will now give Erdös' proof of this result. We start with the following lemmas.

LEMMA 2. *If N_0 is the number of integers $m \leq p_n \log p_n$ whose greatest prime factor is less than $p_n^{1/(20\log_2 p_n)}$ then $N_0 = o(p_n/\log^2 p_n)$.*

Lemma 3. *We can find a constant c so that the number of primes p less than $cp_n \log p_n/(\log_2 p_n)^2$ and such that $p+1$ is not divisible by any prime between $\log p_n$ and $p_n^{1/(20\log_2 p_n)}$ is less than $p_n/4\log p_n$.*

He proved Lemma 2 on considering the number of different prime factors of the integers m, and Lemma 3 is a direct application of Brun's method. The following lemma follows directly from Lemma 2 and 3.

Lemma 4. *Let T be the set of primes t satisfying $p_n/2 < t \le p_n$ and let R denote the set of primes r such that $\log p_n < r \le p_n^{1/(20\log_2 p_n)}$. Denote by $A = \{a_1, a_2, \ldots, a_k\}$ the union of the set of integers less than or equal to $p_n \log p_n$ whose prime factors are all in R and the set of primes p with $p_n/2 < p < cp_n \log p_n/(\log_2 p_n)^2$ and not congruent to -1 to any modulus $r \in R$, where c is the constant found in Lemma 3.*

Then for p_n sufficiently large, $|T| > |A|$.

Proof of Theorem 1: (Erdös [27])

Let S be the set of primes s satisfying

$$p_n^{1/(20\log_2 p_n)} < s \le p_n/2,$$

and let R, T and A be defined as in Lemma 4. We then find an integer z which satisfies

$$0 < z < p_1 p_2 \cdots p_n,$$

$$z \equiv 0 \pmod q, \quad z \equiv 1 \pmod r, \quad z \equiv 0 \pmod s,$$

$$z + a_i \equiv 0 \pmod t \quad (1 \le i \le k),$$

for all primes q with $1 < q \le \log p_n$, $t \in T$, $r \in R$ and $s \in S$. We see that the last congruence is possible since by Lemma 4 there are more t's than a's.

Consider the integers

$$z, z+1, \ldots, z+l,$$

where l is a positive integer which satisfies $l < c_3 p_n \log p_n / (\log_2 p_n)^2$. We will prove the result by showing that $z+b$ is not relatively prime to $p_1 p_2 \cdots p_n$ for all positive integers $b < l$. To do this we will show that each b falls into one of the following classes:

$(i)\, b \equiv 0 \pmod{q}$, for some q

$(ii)\, b \equiv -1 \pmod{r}$, for some r

$(iii)\, b \equiv 0 \pmod{s}$, for some s

$(iv)\, b$ is an a_i, for some i.

To see this, first observe that b can not be divisible by an $r \in R$ and by a prime greater than $\frac{1}{2} p_n$ since then for sufficiently large n we would have

$$b > \frac{1}{2} p_n r > \frac{1}{2} p_n \log p_n > l.$$

Thus, if b does not satisfy (i) or (iii), it is either a product of primes from R or b is not divisible by any $r \in R$, $s \in S$ or primes q with $q \leq \log p_n$. In the former case, we see that b satisfies (iv), while in the latter case, b must be a prime, since otherwise

$$b > \left(\frac{1}{2} p_n\right)^2 > l,$$

for sufficiently large n. But then, we have

$$\frac{1}{2} p_n < b < \frac{c_3 p_n \log p_n}{(\log_2 p_n)^2},$$

and hence b either satisfies (ii) or (iv).

Thus, $z+b$ is not relatively prime to $p_1 p_2 \cdots p_n$.

Additionally, if $p_1, p_2, \ldots, p_m$ are the primes less than or equal to x then it follows from the above argument and Bertrand's postulate, $p_m \geq x/2$, that $z+b$ is not relatively prime to $p_1 p_2 \cdots p_m$ if $b < c_4 x \log x / (\log_2 x)^2$, where c_4 is a constant independent of x.

Let $x = \frac{1}{2} \log p_n$. By the prime number theorem we see that, for sufficiently large n, the product of primes not exceeding x is less than $\frac{1}{2} p_n$. Thus, since $b < \frac{1}{2} p_n$, we can find

$$k = \frac{c_4 \log p_n \log_2 p_n}{(\log_3 p_n)^2},$$

consecutive integers less than p_n which are each divisible by a prime less than $\frac{1}{2}\log p_n$. Hence there are at least $k - \frac{1}{2}\log p_n > \frac{1}{2}k$ consecutive composite integers and the result follows. □

In 1938, Chang [16] obtained a simpler proof for Erdös' result by eliminating the need to use Brun's method. Shortly after, Rankin [71] was able to slightly improve Erdös' result by sharpening Lemma 2 to be the following.

Lemma 2′. If $N(e^u)$ is the number of positive integers not exceeding e^u which contain no prime factors greater than

$$\exp\left(\frac{u \log_2 u}{a \log u}\right),$$

then

$$N(e^u) < \frac{e^u}{u^{a-1-e_1}}$$

for any fixed $e_1 > 0$ and for $u > u_0(e_1)$.

Taking $e^u = p_n \log p_n$ and $a = 5$ in Lemma 2′ and replacing

$$p_n^{1/(20 \log_2 p_n)} \text{ by } p_n^{\log_3 p_n/(5 \log_2 p_n)}$$

and

$$\frac{p_n \log p_n}{(\log_2 p_n)^2} \text{ by } \frac{p_n \log p_n \log_3 p_n}{(\log_2 p_n)^2},$$

in Erdös' or Chang's proof, Rankin proved that

$$p_{n+1} - p_n > \frac{c_3 \log p_n \log_2 p_n}{(\log_3 p_n)^2} \log_4 p_n. \tag{1.3.1}$$

With some further slight modifications of Chang's proof he also obtained that for any $\epsilon > 0$, we may take $c_3 = (1/3 - \epsilon)$.

Since 1938, the only new results on the lower bound of the size of d_n have concerned improvements of the coefficient c_3 in Rankin's result. In 1963, Rankin [75] proved that we can take $c_3 = (e^\gamma - \epsilon)$ for each $\epsilon > 0$. Using advanced techniques and a combinatorial argument Maier and Pomerance [59] improved Rankin's constant c_3 by a factor of $1.31256\cdots$. In 1996, Pintz [68] refined Maier and Pomerance's combinatorial argument and showed that we may take $c_3 = 2(e^\gamma - \epsilon)$.

1.4. Consecutive large gaps between primes

In 1949, Erdös [29] decided to look for chains of large gaps between primes. He proved that

$$\limsup_{n\to\infty}(\min(d_n, d_{n+1})/\log n) = \infty$$

and asked if for a fixed k does

$$\limsup_{n\to\infty}(\min(d_{n+1}, \ldots, d_{n+k})/\log n) = \infty?$$

Maier [56] proved in 1981 that we in fact have

$$\limsup_{n\to\infty}\frac{\min(d_{n+1},\ldots,d_{n+k})}{\log n\log_2 n\log_4 n/\log_3^2 n} > 0. \tag{1.4.1}$$

In particular, this extends Rankin's result (1.3.1) from single gaps to k consecutive gaps.

He proved this result using the Erdös-Rankin method with the modification that he needs to find such an interval with k gaps instead of one. To obtain this result he first calls an integer $q > 1$ a "good" modulus if the Dirichlet L-function $L(s,\chi) \neq 0$ for all characters $\chi \mod q$ and all $s = \sigma + it$ with

$$\sigma > 1 - c_1/\log(q(|t|+1)),$$

and he notes that if c_1 is sufficiently small then for all $q > 1$ either q is good or, by Page's theorem, there is a unique exceptional real zero of some quadratic character mod q. From this definition Maier proved the following two lemmas, where $P(x)$ denotes the product of all primes $p < x$, and $\pi(x; q, a)$, as usual, denotes the number of primes $p \le x$ such that $p \equiv a \pmod q$.

LEMMA 5. *There exists a constant $c > 0$ such that there exist arbitrarily large values of x for which the modulus $P(x)$ is good in terms of c.*

LEMMA 6. *If q is a good modulus and $x \ge q^D$, where the constant D depends only on the value of c in Lemma 5 then*

$$\pi(x; q, a) \gg x/(\phi(q) \log x),$$

uniformly for $(a, q) = 1$ where $\phi(x)$ denotes the Euler ϕ function.

Lemma 6 is deduced from work of Gallagher, see chapter 2, and gives us that we have exceptionally regular distribution of primes in arithmetic progressions mod $P(x)$. Maier then proved some sieve arguments and an upper bound for prime pairs p, $p + i$ for "small" i to prove the result.

1.5. Density of primes in small intervals

In 1943, Selberg [83] considered the density of primes in "small" intervals of the form $(x, x + \Phi(x))$ for functions $\Phi(x)$ which are positive, increasing and have the following properties:

$$\begin{aligned} &\frac{\Phi(x)}{x} \text{ is decreasing for } x > 0, \\ &\frac{\Phi(x)}{x} \to 0 \text{ as } x \to \infty \qquad (1.5.1) \\ &\Phi(x) \to \infty. \end{aligned}$$

Previous to this, functions of the form $\Phi(x) = x^\theta$ had already been considered. In particular, Hoheisel [42] showed in 1930 that if we take $\theta = 1 - 1/33000 + \epsilon$, for any

$\epsilon > 0$, then for sufficiently large x

$$\pi(x + x^\theta) - \pi(x) = (1 + O(1))\frac{x^\theta}{\log x}.$$

In 1937, Ingham [47] sharpened the result to $\theta = 5/8 + \epsilon$. Further improvements were made by Montgomery [62] in 1969, who showed we may take $\theta = 3/5 + \epsilon$ and Huxley [44] in 1972 proved we could take $\theta = 7/12 + \epsilon$. Using developments on the linear sieve and analytic information on the Riemann zeta function, Iwaniec and Jutila [48] gave in 1979 the value $\theta = 13/25 + \epsilon$ which was quickly improved to $\theta = 11/20 + \epsilon$ by Heath-Brown and Iwaniec [38]. The best result to date is by Baker, Harman and Pintz [3] in 2001 who showed the result for $\theta = 0.525$.

On the other hand, we see that Rankin's result (1.3.1), shows that (1.1.1) is false for $\Phi(x) = \frac{\log x \log_2 x}{(\log_3 x)^2} \log_4 x$, and so Selberg asked what restrictions need to be imposed on $\Phi(x)$ so that (1.1.1) holds. He observed that under the assumption of the Riemann hypothesis that one could prove that (1.1.1) holds if

$$\frac{\Phi(x)}{\sqrt{x}\log x} \to \infty \text{ as } x \to \infty.$$

and he proved, assuming the Riemann hypothesis, that for functions $\Phi(x)$ satisfying (1.5.1) equation (1.1.1) holds for all x, except possibly if x belongs to an exceptional set S where the Lebesgue measure of $S_y = S \cap (0, y)$ is

$$m(S_y) < Ay\left(\frac{\log^2 y}{\Phi(y)}\right)^{1/4} = o(y)$$

for an absolute positive constant A.

In 1985, Maier [57] was able to show that these exceptions in Selberg's result do occur for functions $\Phi(x)$ growing faster than $\log^2 x$. In particular, he proved that for $\Phi(x) = \log^\lambda x$, $\lambda > 1$ that

$$\limsup_{x\to\infty} \frac{\pi(x + \Phi(x)) - \pi(x)}{\Phi(x)/\log x} > 1$$

and

$$\liminf_{x\to\infty} \frac{\pi(x+\Phi(x))-\pi(x)}{\Phi(x)/\log x} < 1. \tag{1.5.2}$$

For the proof, which is quite similar to the proof of (1.4.1), he used Lemma 6 and that

$$\pi(x) = \mathrm{Li}(x)(1+O(e^{-\sqrt{\log x}}))$$

(see Narkiewicz [66] chapter 5) to prove:

LEMMA 7. *Let q be a good modulus, $x/2 \le h \le x$ and $x \ge q^D$ where $\log q \ge D \ge D_0$ for a positive constant D_0 which depends only on the constant c in Lemma 5. Then*

$$\pi(x+h;q,a)-\pi(x;q,a) = \frac{1}{\phi(q)}(Li(x+h)-Li(x))(1+O(e^{-cD}+e^{-\sqrt{\log x}})),$$

for $(a,q)=1$, and where the constant implied by $O(\ )$ also depends only on c.

He also used results on the Buchstab function $\omega(u)$, see chapter 3. In particular, that the function $F(u) = \omega(u) - e^{-\gamma}$ changes sign in every interval of length 1, and Buchstab's result that, for $\lambda > 1$,

$$\lim_{u\to\infty} u^{-\lambda}\phi(u^\lambda, u) = e^\gamma\omega(u)\prod_{p<u}(1-\frac{1}{p}),$$

where

$$\phi(x,y) = |\{n \le x : (n, P(y)) = 1\}|.$$

1.6. Recent Results

Maier and Stewart [60] recently expanded on Maier's result, studying equation (1.1.1) where

$$\Phi(x) = (\log x)^{1+s(x)}$$

for non-increasing functions $s(x)$ defined on the positive real numbers which satisfy

$$s(x)^{-1} = O(\log_2 x/\log_4 x),$$

$$s(x) - s(2x) = o(1/\log_2 x),$$

and

$$s(x) - s(x^{3/2}) = o((s(x))^{3/2}).$$

To state their result we must first define the following functions. Let $\omega(u)$ denote the Buchstab function defined as the unique, positive, continuous function satisfying

$$\omega(u) = \frac{1}{u}, \qquad 1 \le u \le 2,$$
$$(u\omega(u))' = \omega(u-1), \qquad u \ge 2,$$

and let $\rho(u)$ denote the Dickman function defined as the unique, continuous solution of

$$\rho(u) = 1, \text{ if } 0 \le u \le 1$$

$$u\rho'(u) = -\rho(u-1), \text{ if } u > 1$$

(see chapter 3 and chapter 4 respectively). For u, v non-negative real numbers, let

$$f(u,v) = v(\log(1+u) + \rho(v(1+u)).$$

As proved in chapter 5, there exists a unique positive real number θ for which

$$\min_{v\ge 1} f(\theta, v) = e^{\gamma}/2,$$

and it can be computed that $\theta = 0.500462161\ldots$. We then define a function $g(y)$ on the non-negative real numbers by

$$g(y) = \begin{cases} \inf_{v\ge 1} f(y,v) & \text{for } \lambda < \theta \\ \inf_{u \ge y} e^{\gamma}\omega(1+u) & \text{for } y \ge \theta \end{cases}$$

Using these definitions they proved the following theorem.

THEOREM 8. *For any $\epsilon > 0$ there are arbitrarily large integers x such that*

$$\pi(x + (\log x)^{1+s(x)}) - \pi(x) < (1+\epsilon) g(s(x)) (\log x)^{s(x)}.$$

They observed that if we take $s(x) = \lambda$, $\lambda \in \mathbb{R}^+$, we obtain Maier's result (1.5.2). From properties of $g(y)$, see chapter 5, they also deduced that, for $\epsilon > 0$ and functions $s(x)$ as above which also satisfy $\lim_{x\to\infty} s(x) = 0$, there are arbitrarily large integers x with

$$\pi(x + (\log x)^{1+s(x)}) - \pi(x) < (1+\epsilon)\frac{s(x)\log(1/s(x))}{\log\log(1/s(x))}(\log x)^{s(x)}.$$

Hence that if we take $s(x) = \log_3 x/\log_2 x$ we get Rankin's result (1.3.1). In particular, we see that this result interpolates between Rankin's result and the result of Maier.

We now sketch Maier and Stewart's proof starting with the following lemmas.

Lemma 9. *Let ϵ be a positive real number and let x and y be real numbers. If $u = \log x/\log y \le (\log x)^{\frac{3}{8}-\epsilon}$ then*

$$\psi(x,y) = x\rho(u)(1+o(1))$$

as $x \to \infty$, where $\psi(x,y)$ denotes the number of positive integers at most x with all prime factors less than or equal to y, and $\rho(u)$ is the Dickman function.

This result is due to de Bruijn [24], see also Lemma 3.20 in Norton [67]. In Chapter 4 we will discuss further properties of the Dickman function and prove a slightly stronger result than Lemma 9 due to Hildebrand.

Lemma 10. *Let $\phi(x,y)$ denote the number of positive integers $\le x$ with no prime factors less than y and let $\omega(u)$ be the Buchstab function. Further let x and y be real numbers and put $u = \log x/\log y$. If u is fixed and $u > 1$ then, as $x \to \infty$, we have*

$$\phi(x,y) = xe^{\gamma}\omega(u)\prod_{p\le y}\left(1-\frac{1}{p}\right)(1+o(1)).$$

This is a result of Buchstab [12] which we will prove in Chapter 3 along with other properties of the Buchstab function.

Proof of Theorem 7: (Maier and Stewart [60])

Let ϵ be a real number with $0 < \epsilon < 1$ and let $\delta = \delta(\epsilon)$ be a real number which is dependent on ϵ and satisfies $0 < \delta < 1$. Further, let $D = D(\delta)$ be a positive integer which depends on δ. As above, θ is the real number for which $g(\theta) = e^\gamma/2$ and we let $\beta = \lim_{x\to\infty} s(x)$. Pick $v_0 \geq 1.7$ such that $f(s((2\prod_{p\leq x} p)^D), v)$ is minimized at v_0. For each sufficiently large positive integer z put

$$P(z) = \begin{cases} \prod_{z^{1/v_0} \leq p \leq z} p & \text{if } \beta < \theta \\ \prod_{p\leq z} p & \text{if } \beta \geq \theta, \end{cases}$$

and $\Delta(z) = P(z)^D$.

We now choose our interval length, denoted by U, depending of the value of β and a real number λ which is dependent on β. In particular, we choose for a suitable δ

$$U = \begin{cases} (((1+\delta)zD)^{1+\lambda}), & \text{if } \beta \geq \theta \text{ and } \lambda > \beta \\ (((1+\delta)zD)^{1+s(\Delta(z))}), & \text{otherwise.} \end{cases}$$

Let $\mathfrak{R}$ denote the set of positive integers $\leq U$ which are coprime with $P(z)$, and let S denote the number of primes of the form $P(z)k + l$ with $P(z)^{D-1} < k \leq 2P(z)^{D-1}$ and $1 \leq l \leq U$. For each integer l we use Lemma 7 to estimate S.

We find that for z and D sufficiently large, there is some k such that the number of primes in the interval $[P(z)k + 1, P(z)k + U]$ is at most

$$(1+\delta)\frac{|\mathfrak{R}|}{\log(P(z)^D)} \prod_{p \mid P(z)} \left(1 - \frac{1}{p}\right)^{-1}. \tag{1.6.1}$$

To estimate the size of $|\mathfrak{R}|$ we put

$$\mathfrak{R}_1 = \{1 \leq n \leq U: \text{ the greatest prime factor of } n \text{ is less than } z^{1/v_0}\}, \text{ and}$$

$$\mathfrak{R}_2 = \{1 \leq n \leq U : n \text{ is divisible by a prime } p \text{ with } p > z\},$$

so that $\mathfrak{R} \subseteq \mathfrak{R}_1 \cup \mathfrak{R}_2$.

We now use Lemma 9 and the fact that the Dickman function $\rho(u)$ is non-decreasing to prove that for sufficiently large z we have

$$|\mathfrak{R}_1| \leq U\rho(v_0(1+s(\Delta(z))))(1+o(1)).$$

We may estimate the size of $\mathfrak{R}_2$ from above by

$$|\mathfrak{R}_2| \leq U \sum_{z<p\leq U} \frac{1}{p}.$$

Using the fact that there exists a positive constant B such that

$$\sum_{p\leq z} \frac{1}{p} = \log\log z + B + O\left(\frac{1}{\log z}\right),$$

and that for z sufficiently large we have $\Delta(z) < \exp(2zD)$ we get

$$|\mathfrak{R}_2| \leq U(\log(1+s(\Delta(z)))(1+o(1)).$$

This gives us

$$|\mathfrak{R}| \leq U\rho(v_0(1+s(\Delta(z))))(1+o(1)) + U(\log(1+s(\Delta(z)))(1+o(1)),$$

and the result in this case follows from the way we defined U and $s(x)$ and by selecting δ sufficiently small and z sufficiently large.

For $\beta \geq \theta$ we use Lemma 10 to get

$$|\mathfrak{R}| \leq (1+\delta)U \prod_{p \mid P(z)} \left(1-\frac{1}{p}\right) e^{\gamma}\omega(1+\lambda).$$

Thus, by (1.6.1), the number of primes in the interval $[P(z)k+1, P(z)k+U]$ is at most

$$(1+\delta)^2 \frac{U}{\log(P(z)^D)} e^{\gamma}\omega(1+\lambda).$$

Again, the result follows for sufficiently large z from the way we defined U, $s(x)$ and by selecting an appropriate δ. $\square$

CHAPTER 2

Density Estimates

2.1. Dirichlet Characters and L-functions

We start by giving some basic definitions and properties of Dirichlet characters and L-functions.

Definition. A complex-valued function χ defined on $\mathbb{Z}$ is a Dirichlet character modulo q if it satisfies the following three properties.

(i) $\chi(n) = 0$ if and only if $(n, q) > 1$,

(ii) $\chi(n + q) = \chi(n)$,

(iii) $\chi(n_1 n_2) = \chi(n_1)\chi(n_2)$.

The character which takes a value of 1 for all n with $(n, q) = 1$ is called the principal character and is denoted by χ_0.

Definition. Let d be a divisor of q and χ a character modulo d. We say the character χ_1 mod q, defined by

$$\chi_1(n) = \begin{cases} \chi(n) & \text{if } (n, q) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

is induced by χ mod d and that χ_1 mod q is imprimitive if d is a proper divisor of q. A character which is not induced by any character modulo d for any divisor $d > 1$ of q is called a primitive character. The smallest d for which χ mod d induces χ_1 mod q is called the conductor of χ.

An important property of Dirichlet characters is their orthogonality relation which is given by

$$\sum_{\chi} \chi(n) = \begin{cases} \phi(q) & \text{if } n \equiv 1 \bmod q \\ 0 & \text{otherwise,} \end{cases}$$

where the summation runs over all $\phi(q)$ characters modulo q.

We now define, for each character χ to the modulus q, the Gaussian sum $\tau(\chi)$ by

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e(m/q)$$

where $e(x)$ denotes $e^{2\pi i x}$. In the following two lemmas we prove a few properties of $\tau(\chi)$ which we will require below.

LEMMA 11. *We have*

$$\chi(n)\tau(\overline{\chi}) = \sum_{a=1}^{q} \overline{\chi}(a) e(na/q), \qquad \text{for } (n,q) = 1, \tag{2.1.1}$$

and

$$|\tau(\chi)|^2 = q. \tag{2.1.2}$$

Proof: (Davenport [21])

If $(n, q) = 1$ then

$$\chi(n)\tau(\overline{\chi}) = \sum_{a=1}^{q} \overline{\chi}(a)\chi(n)e(a/q).$$

Putting $mn \equiv 1 \pmod q$ we get

$$\chi(n)\tau(\overline{\chi}) = \sum_{a=1}^{q} \overline{\chi}(am)e(a/q),$$

and (2.1.1) follows.

Now observe that from (2.1.1) we have

$$|\chi(n)|^2|\tau(\overline{\chi})|^2 = \sum_{a_1=1}^{q} \sum_{a_2=1}^{q} \overline{\chi}(a_1)\chi(a_2)e(n(a_1 - a_2)/q).$$

If we now sum over a complete set of residues modulo q then the sum of $|\chi(n)|^2$ is $\phi(q)$ by the orthogonality relation and unless $a_1 \equiv a_2 \pmod{q}$ the sum of the exponentials is 0 . Hence

$$\phi(q)|\tau(\chi)|^2 = q \sum_a |\chi(a)|^2 = q\phi(q),$$

as required. □

LEMMA 12. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of complex numbers where $a_n = 0$ if n has any prime factor $\leq Q$. Then*

$$\sum_\chi |\tau(\chi)|^2 \left| \sum_y^{y+z} a_n \chi(n) \right|^2 = \phi(q) \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_y^{y+z} a_n e(a/q) \right|^2 . \tag{2.1.3}$$

Proof:

From (2.1.1) we have

$$\tau(\overline{\chi}) \sum_y^{y+z} a_n \chi(n) = \sum_{a=1}^{q} \left(\overline{\chi}(a) \sum_y^{y+z} a_n e(na/q) \right).$$

The result follows on taking the square of the modulus of both sides and summing over χ. □

Dirichlet, in his study of the distribution of primes in arithmetic progressions, defined an L-function as follows.

Definition. An L-function is a series of the form

$$L = L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

for complex s with Re $s > 1$.

Since $|\chi(n)| \leq 1$ it is easy to see that $L(s,\chi)$ is analytic in the half-plane Re $s > 1$. We in fact can prove (see Apostol [1] Theorem 12.5) for the principal character χ_1 modulo q, $L(s,\chi)$ is analytic everywhere except for a simple pole as $s = 1$ with residue $\phi(q)/q$. For $\chi \neq \chi_1$, $L(s,\chi)$ is an entire function.

It also can be proved (see Karatsuba [52] Chapter 8) that L-functions satisfy the Euler product

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Furthermore, logarithmic differentiation on the Euler product gives us

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha \text{ for some prime } p \text{ and positive integer } \alpha \\ 0 & \text{otherwise} \end{cases}.$$

2.2. The Large Sieve

In 1941, Linnik introduced the large sieve. He considered how many of the residue classes modulo a prime p are represented among an arbitrary, finite set of integers. Let N be a natural number and let $\mathcal{N}$ be a non-empty subset of the set of positive integers $\leq N$. Let Z denote the cardinality of $\mathcal{N}$ and let $Z(a,p)$ denote the number of elements of $\mathcal{N}$ falling into the congruence class a modulo p, so that

$$Z(a,p) = \sum_{\substack{n \in \mathcal{N} \\ n \equiv a \bmod p}} 1.$$

For each prime p, we measure the regularity of the distribution of the elements of $\mathcal{N}$ among the residue classes modulo p by the variance

$$D(p) = \sum_{a=0}^{p-1} \left(Z(a,p) - \frac{Z}{p}\right)^2.$$

Improving on Linnik's method, Renyi [77] proved in 1950, for $X \geq 2$,

$$\sum_{p \leq X} pD(p) \ll Z^{2/3} N^{4/3} X^{1/3},$$

provided $X \leq N^{3/5}$. In addition to this, using a new method, he was able to prove that

$$\sum_{p \leq X} pD(p) \ll Z(N + X^3).$$

Roth [82] further improved on these results by proving that if $N \geq 2$ and $X \geq N^{1/2}(\log N)^{-1/2}$ then

$$\sum_{p \leq X} pD(p) \ll ZX^2 \log X.$$

Bombieri [6], in 1965, not only extended Roth's result but also gave the following generalization of the large sieve. Let a_n be an arbitrary sequence of complex numbers and define the exponential sum $S(\alpha)$ as

$$S(\alpha) = \sum_{y}^{y+z} a_n e(n\alpha).$$

THEOREM 13. *We have*

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |S(a/q)|^2 \ll (Q^2 + z) \sum_{y}^{y+z} |a_n|^2.$$

Proof: (Gallagher [33])

Let F be any complex-valued function with continuous derivative and period 1. We average the inequality

$$|F(a/q)| \leq |F(\alpha)| + \int_{a/q}^{\alpha} |dF(\beta)|$$

over the interval $I(a/q)$ of length $1/Q^2$ centered at a/q to get

$$|F(a/q)| \leq Q^2 \int_{I(a/q)} |F(\alpha)|\, d\alpha + \frac{1}{2} \int_{I(a/q)} |F'(\beta)| d\beta.$$

The intervals $I(a/q)$ with $1 \leq a \leq q$, $(a,q) = 1$ and $q \leq Q$ do not overlap, modulo 1. Hence

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |F(a/q)| \leq Q^2 \int_0^1 |F(\alpha)| d\alpha + \frac{1}{2} \int_0^1 |F'(\beta)| d\beta.$$

If we put $F = S^2$ then the first integral on the right is $\sum_y^{y+z} |a_n|^2$ and the second integral is

$$\int_0^1 |S(\beta)S'(\beta)|d\beta \leq \left(\int_0^1 |S(\beta)|^2 d\beta \cdot \int_0^1 |S'(\beta)|^2 d\beta\right)^{1/2}.$$

Observe that the first integral is again $\sum_y^{y+z} |a_n|^2$. Before estimating the second integral, we may first multiply $S(\beta)$ by $e(-m\beta)$ for a suitable m so that the range for n becomes $|n| \leq \frac{1}{2}z$, since this leaves $|S(\beta)|$ unchanged. The exponential function satisfies

$$\int_0^1 e(nx)dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and hence the second integral is

$$\sum_{|n| \leq z/2} |2\pi i n \cdot a_n|^2 \leq (\pi z)^2 \sum_y^{y+z} |a_n|^2.$$

Thus we get

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |S(a/q)|^2 \leq (Q^2 + \pi z) \sum_y^{y+z} |a_n|^2,$$

as desired. □

2.3. Density Theorems

Let $L(s, \chi)$ be any Dirichlet L-function with a character χ to modulus $q > 2$ and let $N_\chi(\alpha, T)$ denote the number of zeros of $L(s, \chi)$ in the rectangle

$$\alpha \leq \sigma \leq 1, \quad |t| \leq T,$$

where $1/2 \leq \alpha \leq 1$. Linnik [54] proved, using a complicated method, that for $\lambda \in [0, \log q]$ and $T = \min(\lambda^{100}, \log^3 q))$

$$\sum_\chi N_\chi(1 - \lambda/\log q, T) < e^{c_1 \lambda}, \tag{2.3.1}$$

for a positive constant c_1. Linnik used this theorem to prove that the least prime $p(l,k)$ in the arithmetic progression

$$l, l+k, l+2k, \ldots$$

for $0 < l < k$, $(l,k) = 1$, $k \geq 3$, satisfies $p(l,k) < k^{c_2}$ for an absolute constant c_2.

Rodosskii [81] gave a simpler proof for (2.3.1) but only for $T = e^{\lambda}/\log q$. The method was further refined by Knapowski [51] in 1962 and Jutila [50] in 1970.

Expanding on this, Fogels [30] used ideas from Linnik's paper and Turan's power sum method (see Turan [85]) to prove

$$\sum_{\chi} N_{\chi}(\alpha, T) \ll T^{c_3(1-\alpha)}, \tag{2.3.2}$$

for c_3 is a positive constant and $T \geq q$. Fogels applied this theorem to produce an improved result on the number of primes in an arithmetic progression.

The following "large sieve" density theorem was proved by Bombieri [6] in 1965 and was the first of its type.

THEOREM 14. *Let $\mathcal{N}$ be a finite set of positive integers and let*

$$M = \max_{q \in \mathcal{N}} q, \text{ and}$$

$$D = \max_{q \in \mathcal{N}} d(q),$$

where $d(q)$ denotes the number of divisors of q. Then

$$\sum_{q \in \mathcal{N}} \frac{1}{\phi(q)} \sum_{\chi} |\tau(\chi)|^2 N_{\chi}(\alpha, T) \ll DT(M^2 + MT)^{4(1-\alpha)/(3-2\alpha)} \log^{10}(M+T)$$

uniformly with respect to $\mathcal{N}$, for $1/2 \leq \alpha \leq 1$, $T \geq 2$.

Jutila [49] and Montgomery [63] simultaneously generalized the classical density theorems and the large sieve density theorem of Bombieri. They proved "hybrid" density theorems by sieving over χ and q instead of just one or the other. They obtained results of the form

$$\sum_{q \le Q} \sum_{\chi}{}^{*} N_{\chi}(\alpha, T) \ll (Q^2 T)^{c_4(1-\alpha)} \log^b(QT), \tag{2.3.3}$$

for a positive constant c_4 and where $\sum_{\chi}^{*}$ indicates that the sum is taken over the primitive characters modulo q.

Moreover, Montgomery has shown that

$$N_{\chi}(\alpha, T) \ll T^{c_5(1-\alpha)} \log^{13} T,$$

where c_5 can be taken to be $\frac{5}{2}$. This is a remarkable result, as if we could take $c_5 = 2$ in (2.3.3) we would largely have the same result as can be deduced from the Riemann Hypothesis, and so this case has become known as the "density hypothesis".

2.4. A Large Sieve Density Estimate

We will now proceed to prove a large sieve density estimate and application due to Gallagher [34]. In particular, we will prove that for any $b > 0$ there exists positive numbers a and c_6 such that

$$\sum_{q \le T} \sum_{\chi}{}^{*} N_{\chi}(\alpha, T) \ll T^{c_6(1-\alpha)} \quad (T \ge 1),$$

and

$$\sum_{q \le Q} \sum_{\chi}{}^{*} \left| \sum_{x \le p \le x+h} \chi(p) \log p \right| \ll h \exp\left(-a \frac{\log x}{\log Q} \right),$$

for $x/Q \le h \le x$ and $\exp(\log^{1/2} x) \le Q \le x^b$.

Following Gallagher's proof, we will prove a general mean value estimate for exponential sums, a large sieve estimate for character sums with prime argument due to Bombieri and Davenport, and we will give an application of Turan's power sum lemma.

Let

$$S(t) = \sum c(v) \cdot e(vt) \tag{2.4.1}$$

be an absolutely convergent exponential sum with complex coefficients and where the frequencies v run over an arbitrary sequence of real numbers.

LEMMA 15. *Let $\delta = \Theta/T$, with $0 < \Theta < 1$. Then*

$$\int_{-T}^{T} |S(t)|^2 dt \ll_\Theta \int_{-\infty}^{\infty} \left| \delta^{-1} \sum_{|v-x| \le \frac{1}{2}\delta} c(v) \right|^2 dx. \tag{2.4.2}$$

Proof: (Gallagher [34]) Let

$$C_\delta(x) = \delta^{-1} \sum_{|v-x| \le \frac{1}{2}\delta} c(v)$$

so that we may write the integral on the right of (2.4.2) as

$$\int_{-\infty}^{\infty} |C_\delta(x)|^2 dx.$$

We now put $F_\delta = \delta^{-1}$ if $|x| \le \frac{1}{2}\delta$, otherwise we put $F_\delta = 0$ so that we have

$$C_\delta(x) = \sum c(v) F_\delta(x - v).$$

Taking Fourier transforms of C_δ,

$$\begin{aligned}
\hat{C}_\delta(x) &= \int_{-\infty}^{\infty} C_\delta(x) e(xt) dx \\
&= \sum_v c(v) \int_{-\infty}^{\infty} F_\delta(x - v) e(xt) dx \\
&= \sum_v c(v) e(vt) \int_{-\infty}^{\infty} F_\delta(x - v) e((x - v)t) dx \\
&= S(t) \hat{F}_\delta(t).
\end{aligned}$$

Since the series (2.4.1) converges absolutely, C_δ is a bounded integrable function, and hence is square integrable. By Plancherel's theorem,

$$\int_{-\infty}^{\infty} |C_\delta(x)|^2 dx = \int_{-\infty}^{\infty} |S(t)\hat{F}_\delta(t)|^2 dt.$$

The result now follows since

$$\hat{F}_\delta(t) = \frac{\sin \pi\delta t}{\pi\delta t} \gg 1$$

for $|t| \leq T$. □

We now define

$$S(t) = \sum a_n n^{it} \tag{2.4.3}$$

to be an absolutely convergent Dirichlet series, and we apply Lemma 15 to obtain:

Theorem 16. *We have*

$$\int_{-T}^{T} |S(t)|^2 dt \ll T^2 \int_0^\infty \left| \sum_{y}^{y\mathfrak{t}} a_n \right|^2 \frac{dy}{y}, \text{ with } \mathfrak{t} = e^{1/T} \tag{2.4.4}$$

Proof: (Gallagher [34])

Taking $\Theta = 1/(2\pi)$ in (2.4.2) we have

$$\int_{-T}^{T} |S(t)|^2 \ll \int_{-\infty}^{\infty} \left| 2\pi T \sum_{x}^{x+\delta} a_n \right|^2 dx.$$

Making the substitution $\log y = 2\pi x$, we get

$$\int_{-T}^{T} |S(t)|^2 \ll T^2 \int_0^\infty \left| \sum_{y}^{y\mathfrak{t}} a_n \right|^2 \frac{dy}{y},$$

where $\mathfrak{t} = e^{1/T}$. □

We now apply Theorem 16 to sums of the form

$$S(\chi, t) = \sum a_n \chi(n) n^{it}. \tag{2.4.5}$$

Let

$$S(\chi) = \sum_{y}^{y+z} a_n \chi(n), \text{ and } S(\alpha) = \sum_{y}^{y+z} a_n e(n\alpha).$$

We now prove a large sieve estimate due to Bombieri and Davenport [8].

LEMMA 17. *Assume $a_n = 0$ if n has any prime factors $\leq Q$. For $T \geq 1$ we have*

$$\sum_{q \leq Q} \log(Q/q) \sum_{\chi}{}^{*} \left| \sum_{y}^{y+z} a_n \chi(n) \right|^2 \ll (Q^2 + z) \sum_{y}^{y+z} |a_n|^2. \tag{2.4.6}$$

Proof: (Gallagher [34])

From Lemma 11 we have, for $(n, q) = 1$,

$$\chi(n)\tau(\overline{\chi}) = \sum_{a=1}^{q} \overline{\chi}(a) e(na/q).$$

Since $a_n = 0$ if n has a prime factor $\leq Q$, we get for $q \leq Q$

$$\tau(\overline{\chi}) S(\chi) = \sum_{a=1}^{q} \chi(a) S(a/q).$$

Squaring the absolute value of both sides and using the othogonality relation of χ we get

$$\sum_{\chi} |\tau(\overline{\chi}) S(\chi)|^2 = \phi(q) \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |S(a/q)|^2.$$

Hence, by Theorem 13, we get

$$\sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{\chi} |\tau(\overline{\chi}) S(\chi)|^2 \ll (Q^2 + z) \sum_{y}^{y+z} |a_n|^2. \tag{2.4.7}$$

If f is the conductor of χ, then $q = fr$, and $|\tau(\overline{\chi})|^2 = f$ or 0 depending on whether r is square-free and prime to f or not (see Davenport [21] page 148). Also, $S(\chi) = S(\psi)$ where ψ is the primitive character to modulus f which induces χ since $a_n = 0$ if n has any prime factors $\leq Q$. Hence the left side of (2.4.7) is

$$\sum_{f \leq Q} \frac{f}{\phi(f)} \left(\sum_{\substack{r \leq Q/f \\ (r,f)=1}} \frac{\mu^2(r)}{\phi(r)} \right) \left(\sum_{\psi \bmod f}{}^{*} |S(\psi)|^2 \right) \geq \sum_{q \leq Q} \log(Q/q) \sum_{\chi}{}^{*} \left| \sum_{y}^{y+z} a_n \chi(n) \right|^2, \tag{2.4.8}$$

since we have (see van Lint and Richert [55] Chapter 1)

$$\sum_{\substack{r\le x\\(r,f)=1}} \frac{\mu^2(r)}{\phi(r)} \ge \frac{\phi(f)}{f}\log x.$$

□

THEOREM 18. *Using the same assumptions as in Lemma 17 we have*

$$\sum_{q\le Q}\log(Q/q)\sum_{\chi}{}^{*}\int_{-T}^{T}|S(\chi,t)|^2dt \ll \sum (Q^2T+n)|a_n|^2. \tag{2.4.9}$$

Proof: (Gallagher [34])

By Theorem 16, we have that

$$\sum_{q\le Q}\log(Q/q)\sum_{\chi}{}^{*}\int_{-T}^{T}|S(\chi,t)|^2dt \ll T^2\sum_{q\le Q}\log(Q/q)\sum_{\chi}{}^{*}\int_{0}^{\infty}\left|\sum_{y}^{y\mathfrak{t}} a_n\chi(n)\right|^2\frac{dy}{y}.$$

Now applying Lemma 17 we find this is

$$\ll T^2\int_0^\infty (Q^2+y(\mathfrak{t}-1))\sum_{y}^{y\mathfrak{t}}|a_n|^2\frac{dy}{y}.$$

Observe that the coefficient of $|a_n|^2$ is

$$T^2Q^2\int_{n/\mathfrak{t}}^{n}\frac{dy}{y}+T^2(\mathfrak{t}-1)\int_{n/\mathfrak{t}}^{n}dy = TQ^2+T^2(\mathfrak{t}-1)(1-\mathfrak{t}^{-1})n \ll Q^2T+n,$$

provided $T\ge 1$. □

Let $L(s,\chi)$ be an L-function to modulus $\le T$ and let $w=1+iv$ with $|v|\le T$. If $\chi=\chi_0$ we also assume that $|v|\ge 2$. Put $\mathcal{L}=\log T$ and let

$$S_{x,y}(\chi,v)=\sum_{x}^{y}\frac{\chi(p)\log p}{p^w}.$$

We now show by Turan's method that if L has a zero near w then $S_{x,y}$ is large for suitable x,y.

THEOREM 19. *Let r_0, A, and B be positive numbers. If $L(s,\chi)$ has a zero in the disc $|s-w| \le r$, with $\mathcal{L} = \log T$ satisfying $\mathcal{L}^{-1} \le r \le r_0$, then there exists a positive number C such that for each $x \ge T^A$, we have*

$$\int_x^{x^B} |S_{x,y}(\chi, v)| \frac{dy}{y} \gg x^{-Cr} \log^2 x. \tag{2.4.10}$$

Proof: (Gallagher [34]) We have

$$\frac{L'}{L}(s,\chi) = \sum \frac{1}{s-\rho} + O(\mathcal{L}), \quad |s-w| \le \frac{1}{2},$$

where ρ runs over the zeroes in the disc $|s-w| \le 1$. By Cauchy's inequality it follows that

$$\frac{\mathcal{D}^k}{k!} \frac{L'}{L}(s,\chi) = (-1)^k \sum \frac{1}{(s-\rho)^{k+1}} + O(4^k \mathcal{L}), \quad |s-w| \le \frac{1}{4}.$$

We choose $s = w + r$ and estimate the contribution of the terms with $|\rho - w| > \lambda$, where $r \le \lambda \le 1$. These are $\ll 2^j \lambda \mathcal{L}$ terms with $2^j \lambda < |\rho - w| \le 2^{j+1}\lambda$, and each of these terms is $\ll (2^j\lambda)^{-(k+1)}$. Hence for $k \ge 1$, the contribution is $\ll \sum_{j \ge 0} (2^j \lambda)^{-k} \mathcal{L} \ll \lambda^{-k} \mathcal{L}$. Thus for $\mathcal{L}^{-1} \le r \le \lambda \le 1/4$, we have

$$\frac{\mathcal{D}^k}{k!} \frac{L'}{L}(s,\chi) = (-1)^k {\sum}' \frac{1}{(s-\rho)^{k+1}} + O(\lambda^{-k} \mathcal{L}), \tag{2.4.11}$$

where $'$ indicates that ρ now runs over the zeros with $|\rho - w| \le \lambda$. By Linnik's density lemma (Prachar [70] pg 331) there are $\le A_1 \lambda \mathcal{L}$ such zeros, and by the assumption, $\min |s-\rho| \le 2r$. It follows by Turan's second power-sum theorem (Turan [85] ∮9) that

$$\left| {\sum}' \frac{1}{(s-\rho)^{k+1}} \right| \ge (\mathcal{D} r)^{-(k+1)}$$

for some integer $k \in [K, 2K]$, provided $K \ge A_1 \lambda \mathcal{L}$. Choosing $\lambda = A_2 D r$, with A_2 sufficiently large and D a positive constant, the sum in (2.4.11) then dominates the remainder since for any constant C_0, we have $(Dr)^{-(k+1)} \ge C_0 (A_2 D r)^{-k} \mathcal{L}$ provided $A_k^2 \ge C_0 D r \mathcal{L}$; and this holds for $k \ge A_1 A_2 D r \mathcal{L}$ for large A_2, since $r\mathcal{L} \ge 1$. Thus we get

$$\frac{\mathcal{D}^k}{k!} \frac{L'}{L}(s,\chi) \gg (Dr)^{-(k+1)}$$

for some integer $k \in [K, 2K]$, provided $K \geq Er\mathcal{L}$ for a positive constant E. Here we must assume $r \leq r_0$, since we had $\lambda \leq 1/4$. Rewriting this gives

$$\sum \frac{\Lambda(n)\chi(n)}{n^w} p_k(r \log n) \gg D^{-k}/r, \tag{2.4.12}$$

where $p_k(u) = e^{-u}u^k/k!$. There are constants B_1, B_2 such that $p_k(u) \leq (2D)^{-k}$, for $u \leq B_1 k$, and $p_k(u) \leq (2D)^{-k}e^{-u/2}$, for $u \geq B_2 k$. In fact, putting $u = vk$ and using $k! \geq (k/e)^k$, we get $p_k(u) \leq (ve^{1-v})^k$, from which these inequalities follow easily.

Given $x \geq T^A$, with $A = B_1 E$, put $K = B_1^{-1} r \log x$, so that $K \geq Er\mathcal{L}$. Let $k \in [K, 2K]$ satisfy (2.4.12). We have, with $B = 2B_2/B_1$,

$$\sum_{n \leq x} \frac{\Lambda(n)\chi(n)}{n^w} p_k(r \log n) \ll (2D)^{-k} \sum_{n \leq x} \frac{\Lambda(n)}{n} \ll (2D)^{-k} k/r,$$

$$\sum_{n > x^B} \frac{\Lambda(n)\chi(n)}{n^w} p_k(r \log n) \ll (2D)^{-k} \sum_{n \leq x} \frac{\Lambda(n)}{n^{1+r/2}} \ll (2D)^{-k}/r.$$

It follows that

$$\sum_{x}^{x^B} \frac{\Lambda(n)\chi(n)}{n^w} p_k(r \log n) \gg D^{-k}/r \tag{2.4.13}$$

for suitable large k. We note that we can ensure a large enough k by picking a suitably large A. Since $p_k \leq 1$, the contribution to (2.4.13) of the prime powers $n = p^v$ with $v \geq 2$ is $\ll x^{-1/2}$, and therefore may be ignored. Putting $S(y) = S_{x,y}(\chi, v)$, the remaining sum is

$$\int_x^{x^B} p_k(r \log y) dS(y) = p_k(r \log x^B) S(x^B) - \int_x^{x^B} S(y) p_k'(r \log y) r \frac{dy}{y}.$$

Now observe that

$$p_k(r \log x^B) S(x^B) \ll (2D)^{-k} \sum_{n \leq x^B} \frac{\Lambda(n)}{n} \ll (2D)^{-k} k/r,$$

and, since $p_k' = p_{k-1} - p_k \ll 1$,

$$\int_x^{x^B} |S(y)| \frac{dy}{y} \gg D^{-k}/r^2 \gg x^{-Cr} \log^2 x.$$

The result now follows. □

We now have the tools to prove the following large sieve density estimate of Gallagher [34] which is a generalization of equation (2.3.2).

THEOREM 20. *We have*

$$\sum_{q\le T}\sum_{\chi}{}^{*} N_\chi(\alpha,T) \ll T^{c(1-\alpha)}. \tag{2.4.14}$$

Proof: (Gallagher [34])

We first note that since $N_\chi(\alpha,T)\ll T\log T$ for $q\le T$, it suffices to prove our result for $1-\alpha$ sufficiently small. Furthermore, since the left side is a decreasing function of α and the right side is essentially constant for $1-\alpha\ll\mathcal{L}^{-1}$, if suffices to prove our result for $1-\alpha\gg\mathcal{L}^{-1}$. We also note that since the right side is ≥ 1, we may ignore the boundedly many zeros of $\zeta(\sigma+it)$ $(q=1)$ in $0<\sigma<1$, $|t|\le 2$.

Let $1+iv$ with $|v|\le T$ with the additional constraint that $|v|\ge 2$ if $\chi=\chi_0$. Put $r=2(1-\alpha)$. We may assume that $\mathcal{L}_{-1}\le r\le r_0$. By Theorem 19, if $L(s,\chi)$ has a zero in the disc $|s-w|\le r$, then for $x\ge T^A$, (2.4.10) holds. We choose $x=T^{\max(A,5)}$. By the Schwarz inequality, we get, with $c=4C\cdot\max(A,5)$.

$$T^{c(1-\alpha)}\mathcal{L}^{-3}\int_x^{x^B}|S_{x,y}(\chi,v)|^2\frac{dy}{y}\gg 1.$$

Since, there are $\ll r\mathcal{L}$ zeroes in the disc $|s-w|\le r$, and each zero $\beta=\beta+i\gamma$ with $\beta\ge\alpha$ and $|\gamma|\le T$ is detected in this way over a v-interval of length $\gg r$ in $|v|\le T$, we get

$$N_\chi(\alpha,T)\ll T^{c(1-\alpha)}\mathcal{L}^{-2}\int_x^{x^B}\int_{-T}^{T}|S_{x,y}(\chi,v)|^2dv\frac{dy}{y},$$

and hence, for some $y\in[x,x^B]$,

$$\sum_{q\le T}\sum_{\chi}{}^{*} N_\chi(\alpha,T)\ll T^{c(1-\alpha)}\mathcal{L}^{-1}\sum_{q\le T}\sum_{\chi}{}^{*}\int_{-T}^{T}|S_{x,y}(\chi,v)|^2dv. \tag{2.4.15}$$

Using Theorem 18 we get,

$$\sum_{q\le T^2}\log(T^2/q)\sum_{\chi}{}^{*}\int_{-T}^{T}|S_{x,y}(\chi,v)|^2dv \ll \sum_{x}^{y}(T^5+p)\frac{\log^2 p}{p^2}\ll \mathcal{L}^{-2},$$

since $x\ge T^5$. Hence the double sum on the right of (2.4.15) is $\ll \mathcal{L}$, and the left side is $\ll T^{c(1-\alpha)}$ as required. □

We will now present Gallagher's proof of a useful application of Theorem 20.

There is a constant $c_1>0$ such that at most one primitive L-function to modulus $\le T$ has a zero in the region

$$\sigma > 1-\frac{c_1}{\log T},\quad |t|\le T. \tag{2.4.16}$$

If there is an exceptional zero, it is real, simple and unique.

Denoting the exceptional zero by $1-\delta$, Knapowski [51] shows that, for $c_2>0$, as $\delta\log T\to 0$, the zero-free region in (2.4.16) may be widened to

$$\sigma>1-\frac{c_2}{\log T}\log\left(\frac{ec_1}{\delta\log T}\right), \tag{2.4.17}$$

with $1-\delta$ still the only exception.

Theorem 21. *For $x/Q\le h\le x$ and $\exp(\log^{1/2}x)\le Q\le x^b$, we have*

$$\sum_{q\le Q}\sum_{\chi}{}^{*}\left|\sum_{x\le p\le x+h}\chi(p)\log p\right| \ll h\cdot\exp\left(-a\frac{\log x}{\log Q}\right). \tag{2.4.18}$$

The term $q=1$ must be read as

$$\sum_{x\le p\le x+h}\log p - h,$$

and if there is an exceptional zero $1-\delta$ of $L(s,\chi')$, with $\delta\log Q\le d$, then the corresponding term must be read as

$$\sum_{x\le p\le x+h}\chi'(p)\log p + h\zeta^{-\delta},$$

for some $\zeta \in [x, x+h]$. In the latter case, the bound on the right of (2.4.18) may be reduced.

Proof: (Gallagher [34])

For $q \le T \le x^{1/2}$, we have (Davenport [21] Chapters 17,19)

$$\sum_{n\le x} \chi(n)\Lambda(n) = \delta_\chi x - \sum \frac{x^\rho}{\rho} + O\left(\frac{x\log^2 x}{T}\right)$$

where $\delta_\chi = 1$ or 0 according as $\chi = \chi_0$ or not, and the sum on the right is over the zeros of $L(s,\chi)$ in $0 \le \sigma \le 1$, $|t| \le T$. The terms with $n = p^v$, $v \ge 2$ contribute $\ll x^{1/2}$ to the sum, and therefore may be absorbed into the remainder. Since

$$\frac{(x+h)^\rho}{\rho} - \frac{x^\rho}{\rho} = \int_x^{x+h} y^{\rho-1}dy \ll hx^{\beta-1},$$

for $Q \le T$ and $\beta = Re(\rho)$ we get, using $x/h \le Q$ and $\log x \le \log^2 Q$,

$$\sum_{x\le p\le x+h} \chi(p)\log p \ll h\left(\sum x^{\beta-1} + Q^2/T\right),$$

where, if $q = 1$ or $\chi = \chi'$, the left side must be read as indicated in the statement of the theorem, and the sum on the right is over non-exceptional zeros. Hence

$$\sum_{q\le Q}\sideset{}{^*}\sum_{\chi}\left|\sum_{x\le p\le x+h}\chi(p)\log p\right| \ll h\left(\sum_{q\le Q}\sideset{}{^*}\sum_{\chi}\sum x^{\beta-1} + Q^4/T\right). \tag{2.4.19}$$

Using Theorem 20, and assuming $T^c \le x^{1/2}$, we find that the triple sum on the right is

$$\begin{aligned} -\int_0^1 x^{\alpha-1}d_\alpha\left(\sum_{q\le Q}\sideset{}{^*}\sum_{\chi} N_\chi(\alpha,T)\right) &= \int_0^1 x^{\alpha-1}\log x\sum_{q\le Q}\sideset{}{^*}\sum_{\chi} d\alpha + x^{-1}\sum_{q\le Q}\sideset{}{^*}\sum_{\chi} N_\chi(0,T) \\ &\ll \int_0^{1-\eta(T)} x^{(\alpha-1)/2}\log x d\alpha + x^{-1/2} \\ &\ll x^{-\eta(T)/2}, \end{aligned}$$

where $1-\eta(T)$ is either the bound (2.4.17) or the bound (2.4.16) according to as whether there is an exceptional zero or not.

Choose $T = Q^5$. If there is no exceptional zero, then the parenthesis on the right of (2.4.19) is

$$\ll \exp\left(-c_3 \frac{\log x}{\log Q}\right) + Q^{-1} \ll \exp\left(-a \frac{\log x}{\log Q}\right).$$

If there is an exceptional zero, the parenthesis is, using Siegel's estimate, $\delta \ll T^{-\epsilon}$ for each $\epsilon > 0$,

$$\begin{aligned} &\ll \exp\left(-c_4 \frac{\log x}{\log Q} \log\left(\frac{c_5}{\delta \log Q}\right)\right) + Q^{-1}, \\ &\ll (\delta \log Q)^{c_6 \log x/\log Q} + (\delta \log Q)^2 Q^{-1/2}, \\ &\ll (\delta \log Q)^2 \exp\left(-a \frac{\log x}{\log Q}\right). \end{aligned}$$

□

Finally, we will prove the following result of Maier [56], mentioned in Chapter 1, which he deduced from Theorem 21. We first recall from Chapter 1 that an integer $q > 1$ is called a "good" modulus if the Dirichlet L-function $L(s, \chi) \neq 0$ for all characters χ mod q and all s with

$$\sigma > 1 - c_1/\log(q(|t| + 1)),$$

for $c_1 > 0$.

Lemma 22. *Let q be a good modulus then*

$$\pi(x; q, a) \gg \frac{x}{\phi(q) \log x},$$

uniformly for $(a, q) = 1$ and $x \geq q^D$. The constant D depends only on the value c_1 defining q.

Proof: (Maier [56])

Consider

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n).$$

We now put

$$\psi(x,\chi) = \sum_{n\le x} \chi(n)\Lambda(n),$$

so that we have, see Davenport [21] Chapter 20,

$$\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi} \overline{\chi}(a)\psi(x,\chi).$$

We now estimate this sum using Theorem 21, but we must handle the case $n = 1$, corresponding to principal character, separately. In this case, we get $\psi(x,\chi_0) \gg x/\phi(q)$, Davenport [21] Chapter 20. For $n \ge 2$ we take $Q = x^{1/D}$ and $h = x$ in Theorem 21 to estimate the sum as $(x/\phi(q))\cdot\exp(-a\log x/\log Q)$. Since $D = \log x/\log Q$, we can make the last term arbitrarily small by taking D large. Hence we have that $\psi(x;q,a) \gg \frac{x}{\phi(q)}$ and it immediately follows that

$$\pi(x;q,a) \gg \frac{x}{\phi(q)\log x}.$$

□

CHAPTER 3

The Buchstab Function

3.1. Introduction

Buchstab [12], in 1937, considered the number of terms in arithmetic progression $\leq x$ and coprime with all numbers less than $x^{1/\alpha}$ for $\alpha \geq 2$. He was able to prove the following theorem.

Theorem 23. *Let $\Phi_y(x; k, l)$ denote the number of terms $\leq x$ with no prime factors $< y$ in the arithmetical progression*

$$l, l+k, l+2k, \cdots$$

with $l < k$, $(l, k) = 1$. Then for all given $\alpha > 2$,

$$\Phi_{x^{1/\alpha}}(x; k, l) = \frac{1}{\phi(k)} \cdot \frac{x}{\log x} \cdot \Phi(\alpha) + o\left(\frac{x}{\log x}\right),$$

where,

$$\Phi(\alpha) = 1 + \int_1^{\alpha-1} \frac{dz_1}{z_1} + \int_2^{\alpha-1}\int_1^{z_1-1} \frac{dz_1 dz_2}{z_1 z_2} + \cdots + \int_{[\alpha]-1}^{\alpha-1}\int_{[\alpha]-2}^{z_1-1} \cdots \int_1^{z_{[\alpha]-2}-1} \frac{dz_1 dz_2 \cdots dz_{[\alpha]-1}}{z_1 z_2 \cdots z_{[\alpha]-1}},$$

and $\phi(x)$ is Euler's ϕ function.

Restated, Theorem 23 gives us a formula for counting the number of uncancelled terms in the sieve of Eratosthenes after the numbers $2 \leq m \leq x^{1/\alpha}$ have been removed.

In 1950, de Bruijn [23] introduced the Buchstab function $\omega(u)$, which is defined as the unique, positive, continuous function satisfying the differential delay equation

$$\omega(u) = \frac{1}{u}, \qquad 1 \le u \le 2, \tag{3.1.1}$$

$$(u\omega(u))' = \omega(u-1), \qquad u \ge 2, \tag{3.1.2}$$

and reformulated Buchstab's result as

$$\lim_{y\to\infty} \Phi(y^u, y) y^{-u} \log y = \omega(u), \tag{3.1.3}$$

where $\Phi(x, y)$ denotes the number of positive integers $\le x$ which have no prime factors $< y$. Using Merten's formula, we may rewrite (3.1.3) as

$$\lim_{y\to\infty} \Phi(y^u, y) = e^{\gamma}\omega(u)x \prod_{p<y} (1 - 1/p).$$

Hence we see that Buchstab's result shows that $\Phi(x, y)$ oscillates by a factor of $e^{\gamma}\omega(u)$ from its expected value of $x \prod\limits_{p<y} (1 - 1/p)$. Therefore, we should expect that $\omega(u)$ tends to $e^{-\gamma}$ as $u \to \infty$. In fact, de Bruijn proved that $\omega(u)$ converges to $e^{-\gamma}$ faster than exponentially. This result was quickly improved by Hua [43] who showed that

$$|\omega(u) - e^{-\gamma}| \le e^{-u(\log u + \log_2 u + (\log_2 u/\log u) - 1) + O(u \log u)}.$$

As mentioned in Chapter 1, Maier [56] used the Buchstab function to prove a remarkable result on the number of primes in short intervals. In particular, Maier showed that the function $\omega(u) - e^{-\gamma}$ changes sign in any interval of length 1. Expanding on Maier's work, Cheer and Goldston [17] proved some interesting properties of ω which we prove along with Maier's result below. Additionally, Cheer and Goldston computed values for $\omega(u)$ with $1 \le u \le 11$ to provide numerical constants for Maier's result. Values for $\omega(u)$ with $u \le 500$ were extremely accurately computed by Marsaglia, Zaman and Marsaglia [61], and we will discuss their method in Chapter 5.

3.2. Properties of ω

We will start by proving Theorem 23 and showing how ω may be derived from this result. Let

$$\pi(x;k,l) = \sum_{\substack{p\le x \\ p\equiv l \bmod k}} 1$$

and

$$r(n) = \sum_{p\le n} \frac{1}{p},$$

for primes p and observe that

$$\Phi_{x^{1/2}}(x;k,l) = \pi(x;k,l) + O(\sqrt{x}).$$

To prove Theorem 23 we require the following lemma.

LEMMA 24. *Let u and v be values depending on x which satisfy $2 < u < v < A$ for some constant A. Then*

$$\sum_{x^{1/v}\le p<x^{1/u}} \frac{1}{p\log\frac{x}{p}} = \frac{1}{\log x}\log\frac{v-1}{u-1} + O\left(\frac{1}{\log^2 x}\right)$$

Proof: (Buchstab [12]) We know there is a constant B such that $r(n) = \log\log n + B + O(1/\log n)$. Using this, we see that

$$\begin{aligned}
\sum_{x^{1/v}\le p<x^{1/u}} \frac{1}{p\log\frac{x}{p}} &= \sum_{n=[x^{1/v}]+1}^{[x^{1/u}]} \frac{r(n)-r(n-1)}{\log\frac{x}{n}} \\
&= \sum_{n=[x^{1/v}]+1}^{[x^{1/u}]} r(n)\left(\frac{1}{\log\frac{x}{n}} - \frac{1}{\log\frac{x}{n+1}}\right) + \frac{r([x^{1/u}])}{\log\frac{x}{x^{1/u}+1}} - \frac{r([x^{1/v}])}{\log\frac{x}{x^{1/v}+1}} \\
&= -\sum_{n=[x^{1/v}]+1}^{[x^{1/u}]} \log\log n \frac{\log(1+1/n)}{\log\frac{x}{n}\cdot\log\frac{x}{n+1}} + \frac{\log\log[x^{1/u}]}{\log\frac{x}{\log[x^{1/u}]+1}} -
\end{aligned}$$

$$-\frac{\log\log[x^{1/v}]}{\log\frac{x}{\log[x^{1/v}]+1}} + O\left(\sum_{n=[x^{1/v}]+1}^{[x^{1/u}]} \frac{1}{\log n}\frac{\log(1+\frac{1}{n})}{\log\frac{x}{n}\cdot\log\frac{x}{n+1}}\right) + O\left(\frac{1}{\log^2 x}\right).$$

But

$$\sum_{n=[x^{1/v}]+1}^{[x^{1/u}]} \log\log n \frac{\log(1+1/n)}{\log\frac{x}{n}\cdot\log\frac{x}{n+1}} = \sum_{n=[x^{1/v}]+1}^{[x^{1/u}]} \frac{\log\log n}{n\log\frac{x}{n}\cdot\log\frac{x}{n+1}} + O\left(\frac{1}{\log^2 x}\right)$$

$$= \sum_{n=[x^{1/v}]+1}^{[x^{1/u}]} \frac{\log\log n}{n\left(\log\frac{x}{n}\right)^2} + O\left(\frac{1}{\log^2 x}\right)$$

$$= \int_{x^{1/v}}^{x^{1/u}} \frac{\log\log z}{z\left(\log\frac{x}{z}\right)^2}dz + O\left(\frac{1}{\log^2 x}\right)$$

$$= \frac{1}{\log x}\left(\frac{v\log v}{v-1} - \frac{u\log u}{u-1} - \log\frac{v-1}{u-1}\right) +$$

$$+ \frac{(v-u)\log\log x}{(v-1)(u-1)\log x} + O\left(\frac{1}{\log^2 x}\right),$$

$$\frac{\log\log[x^{1/u}]}{\log\frac{x}{x^{1/u}+1}} - \frac{\log\log[x^{1/v}]}{\log\frac{x}{x^{1/v}+1}} = \frac{\log\log x^{1/u}}{\log x^{1-1/u}} - \frac{\log\log x^{1/v}}{\log x^{1-1/v}} + O\left(\frac{1}{\log^2 x}\right)$$

$$= \frac{(v-u)\log\log x}{(v-1)(u-1)\log x} +$$

$$+ \frac{1}{\log x}\left(\frac{v\log u}{v-1} - \frac{u\log u}{u-1}\right) + O\left(\frac{1}{\log^2 x}\right),$$

and

$$O\left(\sum_{n=[x^{\frac{1}{v}}]+1}^{[x^{\frac{1}{u}}]} \frac{1}{\log n}\cdot\frac{\log(1+\frac{1}{n})}{\log\frac{x}{n}\cdot\log\frac{x}{n+1}}\right) = O\left(\frac{1}{\log^3 x}\sum_{n=[x^{\frac{1}{v}}]+1}^{[x^{\frac{1}{u}}]} \log(1+\frac{1}{n})\right) = O\left(\frac{1}{\log^2 x}\right).$$

Finally we get,

$$\sum_{x^{\frac{1}{v}}<p\le x^{\frac{1}{u}}} \frac{1}{p\log\frac{x}{p}} = \frac{1}{\log x}\log\frac{v-1}{u-1} + O\left(\frac{1}{\log^2 x}\right),$$

and the lemma follows. □

Proof of Theorem 23 : (Buchstab [12])

Let, $2 \le \beta < \alpha$ and $p_r, p_{r+1}, \dots, p_{r+m}$ be the primes between $x^{1/\alpha}$ and $x^{1/\beta}$, so that we have

$$x^{1/\alpha} \le p_r < p_{r+1} < \cdots < p_{r+m} < x^{1/\beta}.$$

If $p_i \not| k$ then we have

$$\Phi_{p_i}(x;k,l) = \Phi_{p_{i+1}}(x;k,l) + \Phi_{p_i}(x;kp_i,l'_{p_i}), \tag{3.2.1}$$

where $l'_{p_i} < kp_i$ and $l'_{p_i} \equiv 0 \pmod{p_i}$. If $p_i \mid k$ then

$$\Phi_{p_i}(x;k,l) = \Phi_{p_{i+1}}(x;k,l). \tag{3.2.2}$$

Consecutive applications of (3.2.1) and (3.2.2) gives

$$\Phi_{x^{1/\alpha}}(x;k,l) = \Phi_{x^{1/\beta}}(x;k,l) + \sum_{\substack{x^{1/\alpha} \le p_i \le x^{1/\beta} \\ p_i \not| k}} \Phi_{p_i}(x;kp_i,l'_{p_i}),$$

hence

$$\Phi_{x^{1/\alpha}}(x;k,l) = \Phi_{x^{1/\beta}}(x;k,l) + \sum_{\substack{x^{1/\alpha} \le p_i \le x^{1/\beta} \\ p_i \not| k}} \Phi_{p_i}(x/p_i;k,l_{p_i}), \tag{3.2.3}$$

since $\Phi_{p_i}(x;kp_i,l'_{p_i}) = \Phi_{p_i}(x/p_i;k,l_{p_i})$, where $l_{p_i} = l'_{p_i}/p_i < k$, because every term of the form $l'_{p_i} + kp_i \cdot r = p_i(l_{p_i} + k \cdot r)$.

We will first prove the result for the interval $2 \le \alpha < 3$. We will give x arbitrary values greater than k^3 except if $\alpha = 3$ then we do not let x be the cube of a prime. However, this restriction is not essential for our goal. Then for every α ($2 \le \alpha \le 3$), $\Phi_{x^{1/\alpha}}(x;k,l)$ and $\Phi_{x^{1/2}}(x;k,l)$ distinguish from each other only on the number of terms in arithmetic progression (3.1.1) of form pq where p, q are primes such that,

$$x^{1/\alpha} \le p < x^{1/2}, \quad p \le q,$$

and for every given p all numbers q make progression with difference k since from $pq \equiv l (\text{mod } k)$ follows that $q \equiv \lambda_p (\text{mod } k)$ where $\lambda_p < k$.

$$\begin{aligned}\Phi_{x^{1/\alpha}}(x;k,l) &= \Phi_{x^{1/2}}(x;k,l) + \sum_{x^{1/\alpha}\le p<x^{1/2}} \sum_{\substack{p\le q\le x/p \\ q\equiv\lambda_p \bmod k}} 1 \\ &= \Phi_{x^{1/2}}(x;k,l) + \sum_{x^{1/\alpha}\le p<x^{1/2}} \sum_{\substack{\sqrt{x/p}<q\le x/p \\ q\equiv\lambda_p \bmod k}} 1 - \sum_{x^{1/\alpha}\le p<x^{1/2}} \sum_{\substack{\sqrt{x/p}<q<p \\ q\equiv\lambda_p \bmod k}} 1,\end{aligned}$$

but

$$\begin{aligned}\sum_{x^{1/\alpha}\le p<x^{1/2}} \sum_{\substack{\sqrt{x/p}<q\le x/p \\ q\equiv\lambda_p \bmod k}} 1 &= \sum_{x^{1/\alpha}\le p<x^{1/2}} \{\pi(x/p;k,\lambda_p) - \pi(\sqrt{x/p};k,\lambda_p)\} \\ &= \sum_{x^{1/\alpha}\le p<x^{1/2}} \pi(x/p;k,\lambda_p) + O(x/\log^2 x),\end{aligned}$$

since

$$\sum_{x^{1/\alpha}\le p<x^{1/2}} \pi(\sqrt{x/p};k,\lambda_p) < \sum_{x^{1/\alpha}\le p<x^{1/2}} \sqrt{x/p} < \sqrt{x}\sum_{n=1}^{\sqrt{x}} 1/\sqrt{n} = O(x^{3/4}).$$

We also have

$$\begin{aligned}\sum_{x^{1/\alpha}\le p<x^{1/2}} \sum_{\sqrt{x/p}<q<p} 1 &= O\left(\sum_{x^{1/\alpha}\le p<x^{1/2}} \frac{p}{\log p}\right) \\ &= O\left(\frac{\sqrt{x}}{\log x}\cdot \sum_{x^{1/\alpha}\le p<x^{1/2}} 1\right) \\ &= O\left(\frac{x}{\log^2 x}\right),\end{aligned}$$

so that

$$\begin{aligned}
&\Phi_{x^{1/\alpha}}(x;k,l)\\
&\quad = \pi(x;k,l) + \sum_{x^{1/\alpha}\le p<x^{1/2}} \pi(x/p;k,\lambda_p) + O(x/\log^2 x)\\
&\quad = \frac{1}{\phi(k)}\left(\sum_{x^{1/\alpha}\le p<x^{1/2}} \frac{x}{p\log\frac{x}{p}} + \frac{x}{\log x}\right) + O\left(\sum_{x^{1/\alpha}\le p<x^{1/2}} \frac{x}{p\log^2\frac{x}{p}}\right) + O\left(\frac{x}{\log^2 x}\right)\\
&\quad = \frac{1}{\phi(k)}\frac{x}{\log x}\{1+\log(\alpha-1)\} + O\left(\frac{x}{\log^2 x}\right),
\end{aligned}$$

using known estimates for $\pi(x;k,l)$ and Lemma 24 with $v=\alpha$ and $u=2$. We have for $2\le\alpha<3$

$$\Phi_{x^{1/\alpha}}(x;k,l) = \frac{1}{\phi(k)}\frac{x}{\log x}\Phi(\alpha) + O\left(\frac{x}{\log^2 x}\right),$$

where

$$\Phi(\alpha) = 1 + \int_1^{a-1}\frac{dz}{z}. \tag{3.2.4}$$

We will use induction to build the function $\Phi(\alpha)$ for all $\alpha\ge 2$. Let N be any integer greater than 2 and suppose that for all coprime k and l and for all α, where α depends on x such that $N-1\le\alpha<N$, we have

$$\Phi_{x^{1/\alpha}}(x;k,l) = \frac{1}{\phi(k)}\frac{x}{\log x}\Phi(\alpha) + O\left(\frac{x}{\log^{1+\tau_N}}\right) \tag{3.2.5}$$

where $\tau_N = \frac{1}{2^{N-3}}$, and $\Phi(\alpha)$ a continuous increasing function of α. We will prove that in this case (3.2.5) holds and for all values $N\le\alpha<N+1$, and if in (5) we change τ_N to τ_{N+1}, then in this new interval

$$\Phi(\alpha) = \Phi(N) + \int_{N-1}^{\alpha-1}\frac{\Phi(z)}{z}dz.$$

For proof we consider values

$$u_s = N + \frac{\alpha-N}{n}\cdot s - 1, \quad s = 0,1,2,\ldots,n,$$

where

$$N \le \alpha < N+1$$

and

$$c_1(\log x)^{\frac{1}{2}\tau_N} \le n \le c_2(\log x)^{\frac{1}{2}\tau_N}.$$

For primes p such that $x^{\frac{1}{u_{s+1}+1}} \le p < x^{\frac{1}{u_s+1}}$, with $s = 0, 1, 2, \dots n-1$, we have,

$$N-1 \le u_s < \frac{\log(x/p)}{\log p} \le u_{s+1} \le N.$$

$$\Phi_s = \sum_{x^{\frac{1}{u_{s+1}+1}} \le p < x^{\frac{1}{u_s+1}}} \Phi_p(x/p; k, l_p) = \sum_{x^{\frac{1}{u_{s+1}+1}} \le p < x^{\frac{1}{u_s+1}}} \Phi_{(x/p)^{\frac{\log p}{\log(x/p)}}}(x/p; k, l_p)$$

$$= \frac{1}{\phi(k)} \cdot \sum_{x^{\frac{1}{u_{s+1}+1}} \le p < x^{\frac{1}{u_s+1}}} \phi\left(\frac{\log(x/p)}{\log p}\right) \cdot \frac{x}{p\log(x/p)} +$$

$$+ O\left(\sum_{x^{\frac{1}{u_{s+1}+1}} \le p < x^{\frac{1}{u_s+1}}} \frac{x}{p(\log(x/p))^{1+\tau_N}}\right)$$

$$= \frac{\phi(\eta_s)}{\phi(k)} \sum_{x^{\frac{1}{u_{s+1}+1}} \le p < x^{\frac{1}{u_s+1}}} \frac{x}{p\log(x/p)} + O\left(\frac{x}{(\log x)^{\tau_N}} \sum_{x^{\frac{1}{u_{s+1}+1}} \le p < x^{\frac{1}{u_s+1}}} \frac{1}{p\log(x/p)}\right),$$

where $u_s < \eta_s \le u_{s+1}$, or , using Lemma 24,

$$\Phi_s = \frac{1}{\phi(k)} \cdot \frac{x}{\log x}\phi(\eta_s) \cdot \log\frac{u_{s+1}}{u_s} + O\left(\frac{x}{(\log x)^{1+\tau_N}}\right). \tag{3.2.6}$$

But we have,

$$\begin{aligned}\sum_{s=0}^{n-1}\{\phi(u_{s+1})-\phi(u_s)\}\log\frac{u_{s+1}}{u_s} &< \sum_{s=0}^{n-1}\{\phi(u_{s+1})-\phi(u_s)\}\cdot\frac{u_{s+1}-u_s}{u_s}\\ &< \frac{1}{n(N-1)}\sum_{s=0}^{n-1}\{\phi(u_{s+1}-\phi(u_s)\}\\ &= \frac{\phi(\alpha-1)-\phi(N-1)}{n(N-1)}\\ &= O\left(\frac{1}{(\log x)^{\frac{1}{2}\tau_N}}\right),\end{aligned}$$

and consequently,

$$\left|\sum_{s=0}^{n-1}\phi(\eta_s)\log\frac{u_{s+1}}{u_s}-\int\limits_{N-1}^{\alpha-1}\phi(z)d(\log z)\right| = O\left(\frac{1}{(\log x)^{\frac{1}{2}\tau_N}}\right).$$

Now we get, summing (3.2.6) over s,

$$\begin{aligned}\sum_{x^{1/\alpha}\le p<x^{1/N}}\Phi_p(x/p;k,l_p) &= \sum_{s=0}^{n-1}\Phi_s\\ &= \frac{1}{\phi(k)}\frac{x}{\log x}\sum_{s=0}^{n-1}\phi(\eta_s)\log\frac{u_{s+1}}{u_s}+O\left(n\cdot\frac{x}{(\log x)^{1+\tau_N}}\right)\\ &= \frac{1}{\phi(k)}\frac{x}{\log x}\int\limits_{N-1}^{\alpha-1}\frac{\phi(z)}{z}dz+O\left(\frac{x}{(\log x)^{1+\tau_{N+1}}}\right),\end{aligned}$$

since

$$\frac{x}{\log x}O\left(\frac{1}{(\log x)^{\frac{1}{2}\tau_N}}\right)=O\left(\frac{nx}{(\log x)^{1+\tau_N}}\right)=O\left(\frac{nx}{(\log x)^{1+\tau_{N+1}}}\right)$$

because

$$\frac{1}{2}\tau_N=\tau_{N+1}.$$

Using (3.2.3) with $\beta = N$ and $x > k^\alpha$ and using (3.2.5) with $\alpha = N$, by our inductive hypothesis, we get

$$\begin{aligned}\Phi_{x^{1/\alpha}}(x;k,l) &= \Phi_{x^{1/N}}(x;k,l) + \sum_{x^{1/\alpha}\le p<x^{1/N}} \Phi_p(x/p;k,l_p)\\ &= \frac{1}{\phi(k)}\frac{x}{\log x}\left\{\Phi(N) + \int\limits_{N-1}^{\alpha-1}\frac{\Phi(z)}{z}dz\right\} + O\left(\frac{x}{(\log x)^{1+\tau_{N+1}}}\right),\end{aligned}$$

hence that

$$\Phi_{x^{1/\alpha}}(x;k,l) = \frac{1}{\phi(k)}\frac{x}{\log x}\Phi(\alpha) + O\left(\frac{x}{(\log x)^{1+\tau_{N+1}}}\right)$$
$$(N \le \alpha \le N+1)$$

where

$$\Phi(\alpha) = \Phi(N) + \int\limits_{N-1}^{\alpha-1}\frac{\Phi(z)}{z}dz. \tag{3.2.7}$$

But it is easy to see that (3.2.7) guarantees that function Φ in the new interval is continuous and has positive derivative and the theorem follows.

□

It is now easy to show that $\Phi(x)/x$ satisfies (3.1.1) and (3.1.2) and hence that $\omega(x) = \Phi(x)/x$. First, we see from (3.2.7) that for $N \ge \alpha < N+1$, $N \ge 2$,

$$\alpha\omega(\alpha) = N\omega(N) + \int\limits_{N-1}^{\alpha-1}\frac{z\omega(z)}{z}dz,$$

and differentiating both sides gives us

$$\frac{d}{d\alpha}(\alpha(\omega(\alpha)) = \omega(\alpha-1),$$

as required. From Theorem 23 we see that

$$\alpha\omega(\alpha) = 1 + \int_1^{\alpha-1}\frac{\Phi(z)}{z}dz = 1 + \log(\alpha-1),$$

for $2 \le \alpha \le 3$. Hence, using (3.1.2), we see that $\omega(\alpha) = 1/\alpha$ for $1 \le \alpha \le 2$.

Following the argument of de Bruijn [23], we will show that $\omega(u) \sim e^{-\gamma}$, where γ denotes Euler's constant, by first showing that there exists a constant A such that $\omega(u) = A + O(\Gamma^{-1}(u+1))$ and then that $A = e^{-\gamma}$.

Rewrite (3.1.2) as $u\omega'(u) = -\omega(u) + \omega(u-1)$, and observe that for $u \geq 2$,

$$\omega'(u) \leq u^{-1} \max_{u-1 \leq t \leq u} |\omega'(t)|.$$

Hence $\omega'(u)$ is bounded for $u \geq 2$. Let $M(u)$ denote the upper bound of $|\omega'(t)|$ for $u \leq t < \infty$ and notice that for $u \geq 3$ we have

$$M(u) \leq u^{-1} M(u-1).$$

Hence

$$\begin{aligned} M(u) &\leq u^{-1}(u-1)^{-1} M(u-2) \\ &\leq (u!)^{-1} M(1). \end{aligned}$$

Using $\Gamma(u+1) = u!$ we find that $M(u) \leq c(\Gamma^{-1}(u+1))$ for some constant $c > 0$ and our result follows.

Thus we have $\lim_{u\to\infty} \omega(u) = A$ and it just remains to show that $A = e^{-\gamma}$. To do this de Bruijn defined the following adjoint equation $h(u)$ of $\omega(u)$. Let

$$h(u) = \int_0^\infty \exp\left(-ux - x + \int_0^x \frac{e^{-t}-1}{t} dt\right) dx,$$

which is analytic for $u > -1$. Now observe that $h(u) \sim \frac{1}{u}$ as $u \to \infty$ and $h(u)$ satisfies

$$uh'(u-1) + h(u) = 0. \tag{3.2.8}$$

Consider

$$f(a) = \int_{a-1}^{a} \omega(u)h(u)du + a\omega(a)h(a-1), \tag{3.2.9}$$

Differentiating and using (3.1.2) and (3.2.8) we see that

$$f'(a) = 0 \quad \text{for } a \geq 2,$$

and hence the value of $f(a)$ does not depend on a. So evaluating (3.2.9) as $a \to \infty$ and at $a = 2$ gives us

$$\lim_{a\to\infty} \int_{a-1}^{a} \omega(u)h(u)du + a\omega(a)h(a-1) = \int_{1}^{2} \omega(u)h(u)du + 2\omega(2)h(1).$$

Since as $a \to \infty$ we have $\omega(a) \to A$ and $h(a) \sim a^{-1}$ we get, by 3.2.8, that

$$\begin{aligned} A &= h(1) - \int_1^2 h'(u-1)du \\ &= \lim_{u\to 0^+} uh'(u-1) \\ &= -\lim_{u\to 0^+} u \int_0^\infty \exp\left(-ux - x + \int_0^x \frac{e^{-t}-1}{t}dt + \log x\right) dx \\ &= e^{-\gamma}, \end{aligned}$$

since

$$\lim_{x\to\infty} \left(\int_0^x \frac{e^{-t}-1}{t}dt + \log x\right) = -\gamma.$$

Hence $A = e^{-\gamma}$ and therefore $\lim_{u\to\infty} \omega(u) = e^{-\gamma}$.

To study the behaviour of $\omega(u)$ around $e^{-\gamma}$ we define

$$W(u) = \omega(u) - e^{-\gamma}.$$

Hence we can restate Hua's result mentioned in 3.1 as

$$|W(u)| \leq e^{-u(\log u + \log_2 u + (\log_2 u/\log u) - 1) + O(u \log u)}.$$

In addition to rapidly converging to $e^{-\gamma}$, we see from results by Maier and by Cheer and Goldston that $\omega(u)$ has extraordinarily regular behaviour around $e^{-\gamma}$. We will first prove the following lemma due to Maier.

LEMMA 25. *$W(u)$ changes sign in every interval of length 1.*

Proof: (Maier [56])

Using $h(u)$ and $f(a)$ defined as above, we also define

$$g(a) = \int_{a-1}^{a} h(u)du + ah(a-1).$$

Differentiating $g(a)$, as we did $f(a)$, gives us

$$g'(a) = 0 \quad \text{ for } a > 0.$$

Since $h(u) \sim \frac{1}{u}$ as $u \to \infty$ we have $g(a) \to 1$ as $a \to \infty$. Hence, since $f(a) = e^{-\gamma}$, we find that

$$\int_{a-1}^{a} W(u)h(u)du + aW(a)h(a-1) = 0.$$

Thus in the interval $[a-1, a]$ we must have either $\omega(u) = e^{-\gamma}$ or $W(u)$ changes sign. But $\omega(u) = e^{-\gamma}$ on $[a-1, a]$ contradicts the values of $\omega(u)$ on $1 \leq u \leq 2$. Hence we have our result.

□

We now define

$$M_+(v) = \max_{u \geq v} W(u), \qquad M_-(v) = \min_{u \geq v} W(u).$$

In light of (3.1.2), $W(u)$ clearly satisfies

$$uW'(u) = -W(u) + W(u-1). \tag{3.2.10}$$

We can also infer from Maier's result that $W(u)$ contains a critical point in every interval of length 1 since $W(u)$ is continuous for $u > 2$. Now, let us denote the zeros of $W(u)$ in increasing size by $\lambda_1, \lambda_2, \ldots$. Let $c_1 = 2$ and denote the critical points of $W(u)$ in increasing size by $c_2, c_3, \ldots$. Cheer and Goldston, expanding on Maier's results, proved the following two theorems.

THEOREM 26. *For $v \geq 2$, we have*

$$M_+(v) = \max_{v \leq u \leq v+2} W(u), \qquad M_-(v) = \min_{v \leq u \leq v+2} W(u).$$

Proof: (Cheer and Goldston [17])

$W(u)$ satisfies, (Hua [43]),

$$uh(u-1)W(u) = -\int_{u-1}^{u} W(t)h(t)dt, \tag{3.2.11}$$

where $h(u)$ is defined as above. We will prove the result for $M_+(v)$ and note that the proof for $M_-(v)$ is similar.

To do this we first show that if $c \geq 3$ is a positive relative maximum, then there will be a value d' with $c-2 \leq d' \leq c$ such that $W(d') > W(c)$.

By (3.2.11) and the mean value theorem for integrals there exists a value d with $c-1 \leq d \leq c$ such that

$$W(d)h(d) = -ch(c-1)W(c).$$

Applying (3.2.11) to $W(d)$ we get a number d' with $d-1 \leq d' \leq d$ such that

$$W(d')h(d') = -dh(d-1)W(d),$$

and hence

$$W(d') = \frac{cdh(c-1)h(d-1)W(c)}{h(d)h(d')} \geq cdW(c),$$

since $h(u)$ is positive and decreasing.

Thus the result holds for all $v \geq c-2$ and hence for all $v \geq 2$ since we can verify by computation, see [17], that the first and second positive relative maximums are at $c_2 = 2.76322\ldots$ and $c_4 = 4.21753\ldots$ and $\omega(c_2) > \omega(c_4)$. $\square$

Thus we see that the maxima and minima of $W(u)$ get smaller in intervals of length greater than 2.

THEOREM 27. *In each interval* $[u, u+1]$*,* $u \geq 2$*,* $W(u)$ *has at most two zeros and at most two critical points. Furthermore, we have* $\lambda_1 < c_1 < \lambda_2 < \cdots$*, where the* c_{2k} *are relative maxima with* $\omega(c_{2k}) - e^{-\gamma} > 0$*, and the* c_{2k-1} *are relative minima with* $\omega(c_{2k-1}) - e^{-\gamma} < 0$*.*

Proof: (Cheer and Goldston [17])

We first observe that the theorem is true for $2 \leq u \leq 3$ since $\omega(u) = (\log(u-1)+1)/u$ in this range. We will now proceed to prove the theorem by induction.

To do this, we suppose the theorem is true up to c_k, $k \geq 2$ which is a positive maximum of $W(u)$. We further assume that the only other critical point of $W(u)$ in the interval $[c_k - 1, c_k]$ is the negative minimum c_{k-1}. Hence $W(U)$ has precisely two zeros λ_{k-1}, λ_k in this interval. We will now prove that $W(u)$ duplicates this behaviour in the next interval $[c_k, c_{k+2}]$.

Differentiating (3.2.10) find that $uW''(u) + W'(u) = -W'(u) - W'(u-1)$. So, if c is a critical point which is not a relative maximum or relative minimum, then $W'(c) = W''(c) = 0$ and hence $W'(c-1) = 0$. Thus, a critical points which is also an inflection point can only occur at $u = c$ if $u = c - 1$ is also a critical point. Therefore, since we assumed that the only critical points in $[c_k - 1, c_k]$ are at c_{k-1} and c_k, we see that the only possible critical points which are not extrema in $[c_k, c_k + 1]$ are at $c_{k-1} + 1$ or $c_k + 1$. We will deal with these cases later.

Writing (3.2.10) as

$$(uW(u))' = W(u-1), \tag{3.2.12}$$

we see the sign of $W(u)$ in the interval $[c_k - 1, c_k]$ determines whether $uW(u)$ is increasing or decreasing in the interval $[c_k, c_k + 1]$. We further observe that since $u > 0$, $uW(u)$ and $W(u)$ have the same sign and the same zeros.

Thus we see that

$uW(u)$ has a zero at $u = \lambda_k$,

$uW(u) > 0$ for $\lambda_k < u \le c_k$,

$uW(u)$ increases for $c_k < u < \lambda_{k-1} + 1$, and

$uW(u)$ decreases for $\lambda_{k-1} + 1 < u < \lambda_k + 1$.

Since $W(u)$ has a zero in every interval of length 1, $W(u)$ must have a zero, λ_{k+1} in $(\lambda_k, \lambda_k + 1)$, and hence $W(u)$ and $uW(u)$ have a unique zero at λ_{k+1}. We also see that $\lambda_{k+1} > \lambda_{k-1} + 1$, so that $\lambda_{k+1} - \lambda_{k-1} > 1$ and $W(\lambda_k + 1) < 0$.

Next, $uW(u)$ will increase for $\lambda_k + 1 < u < \lambda_{k+1} + 1$ and there must be a zero λ_{k+2} in this interval since the interval has length 1. Further, $\lambda_{k+2} > \lambda_k + 1$, so $\lambda_{k+2} - \lambda_k > 1$, and $W(\lambda_{k+1} + 1) > 0$.

We now observe that if $uW(u)$ decreases and $W(u) > 0$ in an interval, then $W(u)$ also decreases in that interval. Similarly, if $uW(u)$ increases and $W(u) < 0$ in an interval, then $W(u)$ increases in that interval. Therefore $W(u)$ decreases in $(\lambda_{k-1} + 1, \lambda_{k+1})$ and increases in $(\lambda_k + 1, \lambda_{k+2})$.

We now want to show that $W(u)$ decreases for $c_k < u \le \lambda_{k-1} + 1$. Let α and β be any two numbers in this interval with $\alpha < \beta$. On integrating (3.2.12), we have

$$\begin{aligned} \beta W(\beta) - \alpha W(\alpha) &= \int_{\alpha-1}^{\beta-1} W(t)dt \\ &< (\beta - \alpha) W(\alpha - 1), \end{aligned}$$

since, by our hypothesis, $W(t)$ is positive and decreasing in the interval $(\alpha - 1, \beta - 1) \subset (c_k - 1, \lambda_{k-1})$. Hence we have

$$\begin{aligned} \beta(W(\beta) - W(\alpha)) &= \beta W(\beta) - \alpha W(\alpha) - (\beta - \alpha) W(\alpha) \\ &< (\beta - \alpha)(W(\alpha - 1) - W(\alpha)) \\ &= (\beta - \alpha)\alpha W'(\alpha), \end{aligned}$$

which gives

$$\alpha W'(\alpha) > \frac{\beta}{\beta - \alpha}(W(\beta) - W(\alpha)). \tag{3.2.13}$$

From this we $W(u)$ decreases in $(c_k, \lambda_{k-1} + 1)$, since $W(u)$ initially decreases and if there were a value α where $W'(\alpha) = 0$, then (3.2.13) would imply $W(\beta) < W(\alpha)$ for all $\beta > \alpha$. Hence α would not be a relative minimum, and there are no critical points which are inflections in this interval. Furthermore, $W(u) > 0$ in $(c_k, \lambda_{k-1} + 1)$, because $uW(u) > 0$ in this interval.

Thus, if we define c_{k+1} as the next relative minimum of $W(u)$ for $u \geq \lambda_{k+1}$, we have that $W(u)$ decrease in (λ_{k+1}, c_{k+1}).

Next consider the interval $(c_{k+1}, \lambda_k + 1)$. We note that $c_{k+1} \neq \lambda_{k+1}$, since equality would imply $\lambda_{k+1} - 1 = \lambda_k$ or λ_{k-1}. But, this is impossible because in the first case we have the interval $(\lambda_k, \lambda_{k+1})$ of length one has no zeros, and the second case contradicts $\lambda_{k-1} > \lambda + k - 1 + 1$.

We now prove $c_{k-1} + 1 < c_{k+1}$. For if not, then either $c_{k+1} < c_{k-1} + 1$, or $c_{k+1} = c_{k-1} + 1$. In the first case let $c_{k+1} < \beta < c_{k-1} + 1$. Then, since $W(t)$ is negative and decreasing in $(c_{k+1} - 1, c_{k-1})$,

$$\begin{aligned}
\beta W(\beta) - c_{k+1} W(c_{k+1}) &= \int_{c_{k+1}-1}^{\beta-1} W(t)dt \\
&< (\beta - c_{k+1}) W(c_{k+1} - 1) \\
&= (\beta - c_{k+1}) W(c_{k+1}),
\end{aligned}$$

where we used the fact that, for $c_k > 2$, we have

$$W(c_k - 1) = W(c_k),$$

due to (3.2.10). Hence, $W(\beta) < W(c_{k+1})$ for any $\beta > c_{k+1}$, contradicting the fact that c_{k+1} is a relative minimum. In the case $c_{k+1} = c_{k-1} + 1$, we have $W(c_{k+1}) = W(c_{k-1})$, and for $\lambda_{k+1} < \beta < c_{k+1} = c_{k-1} + 1$,

$$\begin{aligned} -\beta W(\beta) + c_{k+1} w(C_{k+1}) &= \int_{\beta-1}^{c_{k+1}-1} W(t)dt \\ &> (c_{k+1} - \beta) W(c_{k+1} - 1) \\ &= (c_{k+1} - \beta) W(c_{k+1}), \end{aligned}$$

implying $W(\beta) < W(c_{k+1})$, which is impossible if c_{k+1} is a relative minimum. This argument also shows that $c_{k-1} + 1$ is not an inflection point as mentioned earlier.

Next, we prove that $W(u)$ increases in $(c_{k+1}, \lambda_k + 1)$. Let α and β be any numbers satisfying $c_{k+1} < \alpha < \beta < \lambda_k + 1$. Using the same method we used to prove (3.2.13), we have

$$\alpha W'(\alpha) < \frac{\beta}{\beta - \alpha}(W(\beta) - W(\alpha)).$$

If $W(u)$ did not increase through this interval, then there is a point $u = a$ in the interval where $W'(a) = 0$. Letting $\alpha = a$ implies $W(\beta) - W(\alpha) > 0$ for any $\beta > \alpha$, which shows that $W(u)$ increases. Let c_{k+2} be the next relative maximum of $W(u)$ for $u > \lambda_{k+2}$. Then $W(u)$ increases in (λ_{k+2}, c_{k+2}). The proof that $c_{k+2} > c_k + 1$ is the same as the previous argument that $c_{k+1} > c_{k-1} + 1$, which also shows that $c_k + 1$ is not an inflection. The result now follows by induction.

□

CHAPTER 4

The Dickman Function

4.1. Introduction

Let $\varphi(a,x)$ denote the number of positive integers $t \le x$ with a prime factor $p > t^a$ and let

$$f(a) = \lim_{x\to\infty} \frac{\varphi(a,x)}{x}$$

denote the probability that the greatest prime factor of an integer t will be greater than t^a. In 1930, Dickman proved:

THEOREM 28. *For each $n \ge 1$ and for each a with $\frac{1}{n+1} \le a < \frac{1}{n}$, put $f_n(a) = f(a)$. Then we have*

$$f_1(a) = \int_a^1 \frac{dy}{y} = \log \frac{1}{a} \tag{4.1.1}$$

$$f_n(a) = f_{n-1}\left(\frac{1}{n}\right) + \int_a^{\frac{1}{n}} \frac{1}{y}\left(1 - f_{n-1}\left(\frac{y}{y-1}\right)\right) dy \text{ for } n = 2, 3, \ldots \tag{4.1.2}$$

Letting $\psi(x,y)$ denote the number of positive integers $\le x$ with no prime factors $> y$, we can reformulate Dickman's result as

$$\lim_{x\to\infty} \psi(x, x^{1/a}) = \rho(a)x,$$

where $\rho(u)$, called the Dickman function, is defined on the non-negative real numbers as the unique continuous solution of the differential-difference equation,

$$\rho(u) = 1, \text{ if } 0 \le u \le 1 \tag{4.1.3}$$

$$u\rho'(u) = -\rho(u-1), \text{ if } u > 1. \tag{4.1.4}$$

In 1949, Buchstab [13] was the first to use the differential-difference equation form of the Dickman function as he studied the function $B_l(n,x,y)$ defined as the number of positive integers $m \le x$ such that $m \equiv l \pmod{n}$ has no prime factors greater than y. He proved that

$$B_l(n,x,x^{1/u}) = n^{-1}\rho(u)x + O_{n,u}(x(\log x)^{-1/2}),$$

for $(l,n)=1$, $u>1$ and $x>1$. Furthermore, he proved that, for $u \ge 6$,

$$\rho(u) > \exp(-u(\log u + \log\log u + 6\log\log u/\log u)).$$

In 1951, de Bruijn [25] improved this estimate to

$$\rho(u) = \exp\left(-u\left(\log u + \log\log u - 1 + \frac{\log\log u}{\log u} - \frac{1}{\log u} + O\left(\frac{\log^2\log u}{\log^2 u}\right)\right)\right),$$

for $u \ge 3$. A similar result was proved by Hildenbrand and Tenenbaum [41] in 1993. They proved for $u \ge 1$,

$$\rho(u) = \exp\left(-u\left(\log u + \log\log(u+2) - 1 + O\left(\frac{\log\log(u+2)}{\log(u+2)}\right)\right)\right).$$

de Bruijn [25], also in 1951, introduced the function $\Lambda(x,y)$ defined by

$$\Lambda(x,y) = x\int_0^\infty \rho\left(\frac{\log x - \log t}{\log y}\right) d\frac{[t]}{t},$$

and he improved Dickman's result by proving that, for $x>0$ and $y \ge 2$,

$$|\psi(x,y) - \Lambda(x,y)| < cxu^2R(y)$$

where $R(y)$ is approximately the order of $|\pi(y) - \mathrm{Li}(y)|/y$ and c is a positive constant.

In proving this result, de Bruijn also obtains an asymptotic estimate for $\psi(x,x^{1/u})$ where u is allowed to vary with x. In particular, he showed that for $0 \le u \le (\log x)^{3/8-\epsilon}$

$$\psi(x,x^{1/u}) \sim x\rho(u).$$

Maier extended the range to $0 \le u \le (\log x)^{1-\epsilon}$, and this result was further improved to $1 \le u \le \log x/(\log\log x)^{5/3+\epsilon}$ by Hildebrand [39] in 1986. We shall prove Hildebrand's result below.

In 1988, Goldston and McCurley [35], generalized the function $\psi(x,y)$ to the function $\psi(x,y,Q)$ defined as the number of positive integers $\le x$ that have no prime factors from a set of primes Q that exceed y. To study this function they introduce the modified Dickman function $\rho_\delta(u)$ defined by

$$\rho_\delta(u) = 1, \quad 0 \le u \le 1,$$
$$\rho_\delta(u) = 1 - \delta \int_0^{u-1} \frac{\rho_\delta(t)}{t+1} dt, \quad u \ge 1,$$

where $0 \le \delta \le 1$. Observe that $\rho_1(u) = \rho(u)$. Using this function they proved that for any δ, $0 < \delta < 1$ and $u = \log x/\log y$,

$$\psi(x,y,Q) = x\rho_\delta(u)\left(1 + O\left(\frac{1}{\log y}\right)\right),$$

uniformly for $u \ge 1$ and $y \ge 1.5$.

4.2. Properties of ρ

We will start by giving Dickman's proof of Theorem 28.

Proof: (Dickman [26])

Let p denote the greatest prime factor of t and let $q = t/p$. If $p > t^a$ then $q < t^{1-a}$, hence

$$p^{\frac{1}{a}} > t > q^{\frac{1}{1-a}},$$

and therefore

$$p > q^{\frac{a}{1-a}}. \tag{4.2.1}$$

We will first consider the case when $1/2 \le a < 1$. Observe from (4.2.1) that each integer of the form $t = pq$, for positive integers $q < x^{1-a}$ and where p is a prime satisfying $q^{\frac{a}{1-a}} < p \leq \frac{x}{q}$, is an integer less than or equal to x with a prime factor greater than t^a. Since $a \geq 1/2$ the integers t created this way are unique. Hence

$$\phi(a,x) = \sum_{q<x^{1-a}} \pi\left(\frac{x}{q}\right) - \sum_{2\leq q<x^{1-a}} \pi\left(q^{\frac{a}{1-a}}\right) \tag{4.2.2}$$

since $\pi\left(\frac{x}{q}\right) - \pi\left(q^{\frac{a}{1-a}}\right)$ is the number of primes satisfying the given condition.

So, using the prime number theorem, we get

$$\begin{aligned}\sum_{q<x^{1-a}} \pi\left(\frac{x}{q}\right) &\sim \sum_{q<x^{1-a}} \frac{x/q}{\log(x/q)} \\ &= x \sum_{q<x^{1-a}} \frac{1}{q(\log x - \log q)} \\ &\sim x \int_1^{x^{1-a}} \frac{dq}{q\log x\left(1 - \frac{\log q}{\log x}\right)}.\end{aligned}$$

Let $y = 1 - \frac{\log q}{\log x}$ so that $dy = -\frac{dq}{q\log x}$, the lower bound of our integral is

$$y = 1 - \frac{\log 1}{\log x} = 1$$

and the upper bound is

$$y = 1 - \frac{\log(x^{1-a})}{\log x} = a.$$

Thus we get

$$\begin{aligned}\sum_{q<x^{1-a}} \pi\left(\frac{x}{q}\right) &\sim x \int_1^a -\frac{dy}{y} \\ &= x \int_a^1 \frac{dy}{y} \\ &= x \log\frac{1}{a}.\end{aligned}$$

If $a < b < 1$, then for each integer q with $x^{1-b} \leq q < x^{1-a}$ we have

$$\pi\left(q^{\frac{a}{1-a}}\right) \sim \frac{q^{\frac{a}{1-a}}}{\log q^{\frac{a}{1-a}}}.$$

Since the number of integers $q < x^{1-b}$ is of a lower order of magnitude than the number of integers $q < x^{1-a}$ we have that

$$\begin{aligned}\sum_{2\le q<x^{1-a}} \pi\left(q^{\frac{a}{1-a}}\right) &\sim \sum_{2\le q<x^{1-a}} \frac{q^{\frac{a}{1-a}}}{\log q^{\frac{a}{1-a}}}\\ &= \frac{1-a}{a}\sum_{2\le q<x^{1-a}} \frac{q^{\frac{a}{1-a}}}{\log q}\\ &\sim \frac{1-a}{a}\int_2^{x^{1-a}} \frac{q^{\frac{a}{1-a}}}{\log q}\, dq.\end{aligned}$$

Taking $z = q^{\frac{1}{1-a}}$ we get

$$\begin{aligned}\sum_{2\le q<x^{1-a}} \pi\left(q^{\frac{a}{1-a}}\right) &\sim \frac{1-a}{a}\int_{2^{\frac{1}{1-a}}}^{x} \frac{dz}{\log z}\\ &\sim \frac{1-a}{a}\frac{x}{\log x}.\end{aligned}$$

Thus from (4.2.2) we have

$$\phi(a,x) \sim x\log\frac{1}{a} - \frac{1-a}{a}\frac{x}{\log x} \sim x\log\frac{1}{a}.$$

Therefore

$$f_1(a) = \lim_{x\to\infty}\frac{\phi(a,x)}{x} = \log\frac{1}{a}.$$

To prove (4.1.2) we proceed by induction. Assume the result holds for $n-1$. We again consider the integers of the form $t = pq$, for positive integers $q < x^{1-a}$ and where p is a prime satisfying $q^{\frac{a}{1-a}} < p \le \frac{x}{q}$, is an integer less than or equal to x with a prime factor greater than t^a. If all the integers t generated in this way were unique the argument for the case $n = 1$ would hold and we would have

$$f_n(a) = \log\frac{1}{a} = \log n + \int_a^{1/n}\frac{dy}{y}. \tag{4.2.3}$$

However, it is possible that an integer q contains a prime factor $p' > p$. Assuming $p = t^y$ for some y with $a \le y < \frac{1}{n}$, we get $q = t^{1-y}$. Thus $p = q^z$ where $z = \frac{y}{1-y}$. Observe that

$$z < \frac{\frac{1}{n}}{1 - \frac{1}{n}} = \frac{1}{n-1}$$

and

$$z \ge \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{1}{n}$$

so by our inductive hypothesis the probability that q has a prime factor greater than p is

$$\log \frac{1}{z} = f_{n-1}\left(\frac{y}{1-y}\right).$$

Thus we need to multiply the integrand of (4.2.3) by $1 - f_{n-1}\left(\frac{y}{1-y}\right)$. Therefore

$$f_n(a) = \log \frac{1}{n} + \int_a^{\frac{1}{n}} \frac{1}{y}\left(1 - f_{n-1}\left(\frac{y}{1-y}\right)\right) dy$$

as required.

□

Clearly for $a > 1$ the probability that an integer t has a prime factor greater than t^a is 0 and hence we put $f_0 = 0$. Observe that with $f_0 = 0$ that (4.1.3) coincides with (4.1.4).

It is easy to deduce that $\rho(u) = 1 - f(1/u)$. Since $f_0 = 0$ for $0 < u \le 1$, we have that $f(u) = 0$ and hence that $\rho(u) = 1$ in this range. Further from (4.1.4) for $a < 1$, we put $u = \frac{1}{a}$ so that

$$f_n\left(\frac{1}{u}\right) = f_{n-1}\left(\frac{1}{n}\right) + \int_{\frac{1}{u}}^{\frac{1}{n}} \frac{1}{y}\left(1 - f_{n-1}\left(1 - \frac{y}{y-1}\right)\right) dy.$$

Taking $x = \frac{1}{y}$ so that $dx = -\frac{dy}{y^2}$, we get

$$f_n\left(\frac{1}{u}\right) = f_{n-1}\left(\frac{1}{n}\right) + \int_u^n -\frac{1}{x}\left(1 - f_{n-1}\left(\frac{\frac{1}{x}}{1-\frac{1}{x}}\right)\right)dx$$

$$= f_{n-1}\left(\frac{1}{n}\right) + \int_n^u \frac{1}{x}\left(1 - f_{n-1}\left(\frac{1}{x-1}\right)\right)dx$$

Hence

$$1 - \rho(u) = 1 - \rho(n) + \int_n^u \frac{\rho(x-1)}{x}dx.$$

Differentiating and simplifying gives us

$$\rho'(u) = -\frac{\rho(u-1)}{u},$$

for $u > 1$.

Using this derivation it is easy to prove some basic properties of ρ. First we observe that by definition $0 < f(u) \le 1$ for all $u > 0$ and hence $0 < \rho(u) \le 1$. Using this fact and (4.1.4) we see that $\rho(u)$ is strictly decreasing for $u > 1$. Differentiating (4.1.4) gives us

$$\rho''(u) = -\frac{u\rho'(u-1) - \rho(u-1)}{u^2} = \frac{\rho(u-2) + \rho(u-1)}{u^2} > 0$$

for $u > 2$. Therefore $\rho(u)$ is concave up for $u > 2$.

For $u \le 4$, ρ can be written in terms of simple functions. Since $\rho(u) = 1 - f(1/u)$ we see from (4.1.1) that for $1 \le u \le 2$, we have

$$\rho(u) = 1 - \log u. \tag{4.2.4}$$

Let

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}.$$

We then have, due to Chamayou [15],

$$\rho(u) = 1 - \log u + 1/2\log^2 u + \mathrm{Li}_2(1/u) - \mathrm{Li}_2(1/2) - 1/2\log^2 2, \quad \text{for } 2 \le u \le 3,$$

and for $3 \le u \le 4$

$$\begin{aligned}\rho(u) =& 1 - \log u + \left(\frac{1}{2}\log^2 u + \mathrm{Li}_2(1/u) + \mathrm{Li}_2(-1)\right) - \left\{\frac{1}{4}\left[\mathrm{Li}_3\left(\frac{1}{4}\right) - \mathrm{Li}_3\left(\frac{1}{(u-1)^2}\right)\right]\right.\\ &- \frac{1}{3}(\log^3(u-1) - \log^3 2) + \frac{1}{2}(\log^2(u-1)\log u - \log^2 2 \log 3)\\ &+ \mathrm{Li}_2\left(\frac{1}{u-1}\right)\log\frac{u}{u-1} - \mathrm{Li}_2\left(\frac{1}{2}\right)\log\left(\frac{3}{2}\right) - \mathrm{Li}_2\left(-\frac{1}{u-1}\right)\log(u-2)\\ &\left.+ \mathrm{Li}_2(-1)\log(u/3) + V_1 - \frac{1}{2^2}V_2 + \cdots + \frac{(-1)^p}{p^2}V_p + \cdots\right\}\end{aligned}$$

where

$$V_1 = \log\frac{1}{2} - \log\frac{u-2}{u-1} + \frac{1}{2} - \frac{1}{u-1}$$

and

$$V_{p+1} = V_p + \frac{1}{p+1}\left(\frac{1}{2^{p+1}} - \frac{1}{(u-1)^{p+1}}\right).$$

Our next aim is to prove a uniform asymptotic relation for $\psi(x, x^{1/u})$ due to Hildebrand [39]. To do this, we first need to prove several lemmas containing further properties of the function ρ.

LEMMA 29. *The function ρ satisfies*

(i) $u\rho(u) = \int_{u-1}^{u} \rho(t)dt, \quad u \ge 1,$

(ii) $\rho'(u)/\rho(u) \le \log(u\log^2 u), \quad u \ge e^4$

(iii) $\rho(u-t)/\rho(u) \ll (u\log^2(u+1))^t$ *uniformly for $u \ge 1$ and $0 \le t \le u$.*

Proof: (Hildebrand [39])

(i) We see that the result holds for $u = 1$ since $\rho(1) = 1$ for $0 \le u \le 1$. Taking derivatives on both sides of (i) for $u > 1$, we find that

$$(u\rho(u))' = \rho(u) + u\rho'(u) = \rho(u) - \rho(u-1) = \left(\int_{u-1}^{u} \rho(t)dt\right)'.$$

Hence both sides have the same derivative for $u > 1$, so (i) holds for all $u \ge 1$.

(ii) Let $g(u)$ be the logarithmic derivative of $1/\rho(u)$, hence for $u > 1$

$$g(u) = -\frac{\rho'(u)}{\rho(u)} = \frac{\rho(u-1)}{u\rho(u)}.$$

Since $\rho(u) = 1$ for $0 \leq u \leq 1$ we see from (i) that $0 < \rho(u) \leq 1$ for $u \geq 0$ hence $g(u) > 0$. Also by (i) we have for $u > 2$,

$$\begin{aligned} u &= \int_{u-1}^{u} \frac{\rho(t)}{\rho(u)} dt \\ &= \int_{u-1}^{u} \exp(\log \rho(t) - \log \rho(u)) dt \\ &= \int_{u-1}^{u} \exp(\int_{t}^{u} g(s) ds) dt \end{aligned} \tag{4.2.5}$$

We now claim that g is increasing for $u > 1$. To see this we observe that, by equation (4.2.4), for $1 < u \leq 2$ we have

$$g(u) = \frac{\rho(u-1)}{u\rho(u)},$$

which is a strictly increasing function on $1 < u \leq 2$. For $u > 2$ we have by (4.2.5) that

$$\begin{aligned} \frac{1}{g(u)} = \frac{u\rho(u)}{\rho(u-1)} &= u \exp\left(-\int_{u-1}^{u} g(t) dt\right) \\ &= \int_{u-1}^{u} \exp\left(-\int_{u-1}^{t} g(s) ds\right) dt \\ &= \int_{0}^{1} \exp\left(-\int_{u-1}^{u-1+t} g(s) ds\right) dt. \end{aligned}$$

Taking derivatives on both sides, we get

$$\frac{g'(u)}{g^2(u)} = \int_0^1 \exp\left(-\int_{u-1}^{u-1+t} g(s) ds\right) (g(u-1+t) - g(u-1)) dt.$$

Since $g(u)$ is continuous for $u > 1$ and strictly increasing on $(1, 2]$, it follows that $g'(u) > 0$ for $u \in (2, 2+\delta]$ with some $\delta > 0$. Therefore $g(u)$ is strictly increasing in the interval $(1, 2+\delta]$. Further, we can inductively repeat the argument and hence $g(u)$ is increasing for $u > 1$.

From (4.2.5) and the fact that f is non-decreasing, we find that for $u > 2$,

$$u \geq \int_{u-1}^{u} \exp((u-t)g(u-1))dt = \frac{e^{g(u-1)} - 1}{g(u-1)}.$$

Thus

$$g(u-1) \leq \log((u-1)\log^2(u-1)),$$

for $u \geq e^4 + 1$. The result follows since the function $h(x) = (e^x - 1)/x$ is increasing for $x > 0$ and for $u \geq e^4 + 1$ we have

$$\begin{aligned} h(\log(u-1)\log^2(u-1))) &= \frac{(u-1)\log^2(u-1) - 1}{\log((u-1)\log^2(u-1))} \\ &\geq \frac{(u-1)\log^2(u-1) - 1}{3\log(u-1)} \\ &\geq \frac{4(u-1)\log(u-1) - 1}{3\log(u-1)} > u. \end{aligned}$$

(iii) We first observe that for $0 \leq u < e^4$, $\rho(u)$ is bounded from above and below by absolute positive constants. For $u \geq e^4$ the result follows from (*ii*). □

LEMMA 30. *Uniformly for $y \geq 1.5$ and $1 \leq u \leq \sqrt{y}$ we have*

$$\int_0^u \rho(u-t)y^{-1}dt \ll \rho(u)/\log y.$$

Proof: (Hildebrand [39])

By part (*iii*) of Lemma 29 and that $1 \leq u \leq \sqrt{y}$ we have

$$\begin{aligned} \rho(u)^{-1}\int_0^u \rho(u-t)y^{-1}dt &\ll \int_0^u \left(\frac{u\log^2(u+1)}{y}\right)^t dt \\ &\leq \int_0^{\sqrt{y}} \left(\frac{\log^2(\sqrt{y}+1)}{\sqrt{y}}\right)^t dt \\ &\ll (\log y)^{-1}, \end{aligned}$$

as required. □

LEMMA 31. *Uniformly for* $y \geq 1.5$ *and* $1 \leq u \leq y^{1/4}$ *we have*

$$\sum_{\substack{y<p^m\leq y^u \\ p\leq y}} \frac{\log p}{p^m}\rho\left(u - \frac{\log p^m}{\log y}\right) \ll \rho(u).$$

Proof: (Hildebrand [39])

We may suppose $y \geq y_0$ for some fixed constant $y_0 \geq 1.5$ since for $1.5 \leq y \leq y_0$ and $1 \leq u \leq y^{1/4}$ the assertions holds.

By part (iii) of Lemma 29 we have

$$\frac{1}{\rho(u)\log y}\sum_{\substack{y<p^m\leq y^u \\ p\leq y}} \frac{\log p}{p^m}\rho\left(u - \frac{\log p^m}{\log y}\right) \ll \sum_{\substack{p^m>y \\ p\leq y}} \frac{1}{p^{m(1-\alpha)}},$$

where

$$\alpha = \frac{\log(u\log^2(u+1))}{\log y}.$$

If y_0 is sufficiently large, then the hypotheses $y \geq y_0$ and $u \leq y^{1/4}$ imply $\alpha \leq 1/3$ and then we have

$$\begin{aligned}\sum_{\substack{p^m>y \\ p\leq y}} \frac{1}{p^{m(1-\alpha)}} &\leq \sum_{\substack{p^m>y \\ p\leq y}} \frac{1}{p^{2m/3}} \\ &\ll \sum_{p\leq\sqrt{y}} \frac{1}{y^{2/3}} + \sum_{\sqrt{y}<p\leq y}\sum_{m\geq 2} \frac{1}{p^{2m/3}} \\ &\ll \frac{\sqrt{y}}{y^{2/3}} + \sum_{\sqrt{y}<p} \frac{1}{p^{4/3}} \\ &\ll y^{-1/6} \ll \frac{1}{\log y},\end{aligned}$$

as required. □

Lemma 32. *For every $\epsilon > 0$ and uniformly for $y \geq 1.5$, $u \geq 1$ and $0 \leq \theta \leq 1$ we have*

$$\sum_{p^m \leq y^\theta} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right)$$
$$= \log y \int_{u-\theta}^{u} \rho(t)dt + O_\epsilon(\rho(u)\{1 + u\log^2(u+1)\exp(-(\log y)^{3/5-\epsilon})\}).$$

Proof: (Hildebrand [39])

Let S denote the left-hand side of the above equation. Then by partial summation we have

$$S = m(\theta)\rho(u-\theta) - \int_0^\theta m(t)y^t \frac{d}{dt}(y^{-t}\rho(u-t)dt,$$

where

$$m(t) = \frac{1}{y^t} \sum_{p^m \leq y^t} \log p = 1 + O_\epsilon(\exp(-t(\log y)^{3/5-\epsilon}))$$

by the prime number theorem. Put $V = \log y$ and insert the last estimate into the formula for S Separate the main term and error term, we get

$$S = M + R$$

where

$$\begin{aligned} M &= \rho(u-\theta) - \int_0^\theta y^t \frac{d}{dt}(y^{-t}\rho(u-t))dt \\ &= \rho(u-\theta) + \int_0^\theta \{\rho'(u-t) + V\rho(y-t)\}dt \\ &= V\int_{u-\theta}^{u} \rho(t)dt + \rho(u) \end{aligned}$$

and

$$R \ll_e \rho(u-\theta)\exp(-(\theta V)^{3/5-\epsilon}) + \int_0^\theta \{|\rho'(u-t)| + V\rho(y-t)\}\exp(-(tV)^{3/5-\epsilon})dt.$$

Let

$$U = \log(\log^2(u+1))$$

then by (ii) and (iii) of Lemma 29 we get

$$R \ll_e \rho(u)\exp(\theta U - (\theta V)^{3/5-\epsilon}) + \rho(u)(U+V)\int_0^1 \exp(tU - (tV)^{3/5-\epsilon})dt.$$

Clearly the first term on the right-hand side is of the desired order. Further, for

$$U \leq \frac{1}{2}V^{3/5-\epsilon}$$

we have that

$$\begin{aligned}\rho(u)(U+V)\int_0^1 \exp(tU - (tV)^{3/5-\epsilon})dt &\leq \rho(u)(U+V)\int_0^1 \exp\left(-\frac{1}{2}(tV)^{3/5-\epsilon}\right)dt\\ &\leq \rho(u)\frac{U+V}{V}\int_0^\infty \exp\left(-\frac{1}{2}t^{3/5-\epsilon}\right)dt\\ &\ll \rho(u),\end{aligned}$$

where the last estimate holds if, $0 < \epsilon < 1/2$, as we may suppose. Finally, in the case

$$U > \frac{1}{2}V^{3/5-\epsilon},$$

we have

$$\begin{aligned}\int_0^1 \exp(tU - (tV)^{3/5-\epsilon})dt &\leq \int_{1/2}^1 \exp\left(tU - \left(\frac{1}{2}V\right)^{3/5-\epsilon}\right)dt + \int_0^{1/2} e^{tU}dt\\ &\ll \frac{1}{U}\exp\left(U - \left(\frac{1}{2}V\right)^{3/5-\epsilon}\right) + \frac{1}{U}e^{U/2}\\ &\leq \frac{1}{U}\exp(\left(U - \left(\frac{1}{2}V\right)^{3/5-\epsilon}\right) + \frac{1}{U}\exp\left(U - \frac{1}{4}V^{3/5-\epsilon}\right)\\ &\ll_e \frac{1}{U+V}\exp(U - V^{3/5-2e}),\end{aligned}$$

as required. $\square$

We now prove the following theorem due to Hildebrand.

THEOREM 33. *For any* $\epsilon > 0$,

$$\psi(x, x^{1/u}) = x\rho(u)\left(1 + O_e\left(\frac{u}{\log(u+1)}\log x\right)\right)$$

holds uniformly in the range $x \geq 3$, *and* $1 \leq u \leq \log x/(\log\log x)^{5/3+\epsilon}$.

Proof: (Hildebrand [39])

Let, for $y \geq 1$ and $u \geq 0$, $\Delta(y, u)$ be defined by

$$\psi(y^u, y) = y^u\rho(u)(1 + \Delta(y, u)).$$

For $u \geq 1/2$ put

$$\Delta^*(y, u) = \sup_{1/2 \leq u' \leq u} |\Delta(y, u')|,$$

and

$$\Delta^{**}(y, u) = \sup_{0 \leq u' \leq u} |\Delta(y, u')|.$$

We will now show that the estimate

$$\Delta^*(y, u) \ll_\epsilon \log(u+1)/\log y \tag{4.2.6}$$

holds uniformly in the range

$$y \geq 1.5, \quad \frac{1}{2} \leq u \leq \exp(\log y)^{3/5-\epsilon} \tag{4.2.7}$$

for any fixed $\epsilon > 0$.

First, we remark that in the range $0 \leq u \leq 1$ we have $\psi(y^u) = [y^u]$ and $\rho(u) = 1$ and hence

$$|\Delta(y, u)| \leq y^{-u}.$$

This implies

$$\Delta^{**}(y, u) \leq 1 + \Delta^*(y, u) \tag{4.2.8}$$

for every $u \geq 1/2$ and shows that (4.2.6) holds for $1/2 \leq u \leq 1$.

Moreover, if $1 < u \le 2$, the $\rho(u) = 1 - \log u$ and

$$\begin{aligned}\psi(y^u, y) &= [y^u] - \sum_{y<p\le y^u} \left[\frac{y^u}{p}\right] \\ &= y^u\left(1 - \log u + O\left(\frac{1}{\log y}\right)\right) \\ &= y^u\rho(u)\left(1 + O\left(\frac{1}{\log y}\right)\right).\end{aligned}$$

Thus (4.2.6) holds in the range $y \ge 1.5$, $1/2 \le u \le 2$. We will now use this inital condition and the identity

$$\psi(x,y)\log x = \int_1^x \frac{\psi(t,y)}{t}dt + \sum_{\substack{p^m\le x\\ p\le y}} \psi(x/p^m, y)\log p, \quad (x, y \ge 1) \tag{4.2.9}$$

to prove the result for the entire range. Let $D(n)$ denote the largest prime factor of n. To prove (4.2.9) we evaluate the sum

$$\sum_{\substack{n\le x\\ D(n)\le y}} \log n,$$

in two different ways. First we have

$$\begin{aligned}\sum_{\substack{n\le x\\ D(n)\le y}} \log n &= \sum_{\substack{n\le x\\ D(n)\le y}} \sum_{p^m\le x} \log p \\ &= \sum_{p^m\le x} \log p \sum_{\substack{n\le x\\ D(n)\le y\\ p^m \mid n}} 1 \\ &= \sum_{\substack{p^m\le x\\ p\le y}} \psi\left(\frac{x}{p^m}, y\right)\log p.\end{aligned}$$

On the other hand, partial summation yields

$$\sum_{\substack{n\le x\\ D(n)\le y}} \log n = \psi(x,y)\log x - \int_1^x \frac{\psi(t,y)}{t}dt,$$

and (4.2.9) follows easily.

We now fix $y \geq 1.5$ and $u \geq 1.5$ and rewrite (4.2.9) with $x = y^u$ in terms of $\rho(u)$ and $\Delta(y,u)$. After dividing both sides by $\rho(u)y^u \log y^u$, we get

$$\begin{aligned}
1 + \Delta(y,u) = &\frac{1}{\rho(u)y^u \log y^u} \int_1^{y^u} \rho\left(\frac{\log t}{\log y}\right)\left(1 + \Delta\left(y, \frac{\log t}{\log y}\right)\right) dt \\
&+ \frac{1}{\rho(u)\log y^u} \sum_{\substack{p^m \leq y^u \\ p \leq y}} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right)\left(1 + \Delta\left(y, u - \frac{\log p^m}{\log y}\right)\right).
\end{aligned}$$

Noting that, by part (i) of Lemma 29,

$$\begin{aligned}
1 &= \frac{1}{u\rho(u)} \int_{u-1/2}^{u} \rho(t)dt + \frac{1}{u\rho(u)} \int_{u-1}^{u-1/2} \rho(t)dt \\
&= \alpha(u) + (1 - \alpha(u)),
\end{aligned}$$

say, we infer

$$\begin{aligned}
|\Delta(y,u)| \leq &(1 + \Delta^{**}(y,u))(R_1(y,u) + R_2(y,u)) + R_3(y,u) + R_4(y,u) + \\
&+ \Delta^*(y,u)\frac{1}{\rho(u)\log y^u} + \sum_{p^m \leq \sqrt{y}} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right) + \\
&+ \Delta^*(y, u - 1/2)\frac{1}{\rho(u)\log y^u} \sum_{\sqrt{y} < p^m \leq y} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right) \\
&\leq \Delta^*(y,u)\alpha(u) + \Delta^*(y, u - 1/2)(1 - \alpha(u)) + (1 + \Delta^{**}(y,u)) \sum_{i=1}^{4} R_i(y,u),
\end{aligned}$$

where

$$R_1(y,u) = \frac{1}{\rho(u) y^u \log y^u} \int_1^{y^u} \rho(\log t/\log y) dt,$$

$$R_2(y,u) = \frac{1}{\rho(u)\log y^u} \sum_{\substack{y<p^m\le y^u \\ p\le y}} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right),$$

$$R_3(y,u) = \left| \frac{1}{\rho(u)\log y^u} \sum_{p^m \le \sqrt{y}} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right) - \alpha(u) \right|,$$

$$R_4(y,u) = \left| \frac{1}{\rho(u)\log y^u} \sum_{\sqrt{y} < p^m \le y} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right) - (1-\alpha(u)) \right|.$$

By Lemma 30 and Lemma 31, the error terms $R_1(y,u)$ and $R_2(y,u)$ are of order $O(1/u\log y)$ in the range $1 \le u \le y^{1/4}$ and hence also in range (4.2.7), since for $y \ge y_0$, y_0 being a sufficiently large constant, (4.2.7) implies $1 \le u \le y^{1/4}$, and for $y \le y_0$ and u satisfying (4.2.7) the estimate holds trivially. We also can see that $R_3(y,u)$ and $R_4(y,u)$ are of order $O_\epsilon(1/u\log y)$ for y and u satisfying (4.2.7) by applying Lemma 32 with $\theta = 1/2$, $\theta = 1$ and ϵ replaced by $\epsilon/2$.

Therefore, by (4.2.8), for y and u satisfying (4.2.7), we have

$$(1 + \Delta^{**}(y,u)) \sum_{i=1}^{4} R_i(y,u) = O_\epsilon\left(\frac{1+\Delta^*(y,u)}{u\log y}\right).$$

Observe that

$$\alpha(u) = \frac{1}{u\rho(u)} \int_{u-1/2}^{u} \rho(t)dt \le \frac{1}{2u\rho(u)} \int_{u-1}^{u} \rho(t)dt = \frac{1}{2},$$

since $\rho(u)$ is non-increasing for $u \ge 0$ and by part (i) of Lemma 29. From this we see that the quantity

$$\frac{1}{2}\left(\Delta^*(y,u) + \Delta^*\left(y, u - \frac{1}{2}\right)\right) - \left(\alpha(u)\Delta^*(y,u) + (1-\alpha(u))\Delta^*\left(y, u-\frac{1}{2}\right)\right)$$
$$= \left(\frac{1}{2} - \alpha(u)\right)\left(\Delta^*(y,u) - \Delta^*\left(y, u - \frac{1}{2}\right)\right),$$

is nonnegative since Δ^* is a non-decreasing function of u.

We therefore obtain the estimate

$$|\Delta(y,u)| \leq \frac{1}{2}\left(\Delta^*(y,u) + \Delta^*\left(y, u-\frac{1}{2}\right)\right) + O_\epsilon\left(\frac{1+\Delta^*(y,u)}{u\log y}\right)$$

uniformly for every fixed $\epsilon > 0$ and u, y satisfying (4.2.7), $u \geq 1.5$.

If we now suppose, in addition to (4.2.7), $u \geq 2$ and let $u - \frac{1}{2} \leq u' \leq u$, then we can apply the above estimate with u' in place of u and get, using the monotonicity in u of the function $\Delta^*(y,u)$,

$$\begin{aligned} |\Delta(y,u')| &\leq \frac{1}{2}\left(\Delta^*(y,u') + \Delta^*\left(y, u'-\frac{1}{2}\right)\right) + O_\epsilon\left(\frac{1+\Delta^*(y,u')}{u'\log y}\right) \\ &\leq \frac{1}{2}\left(\Delta^*(y,u) + \Delta^*\left(y, u-\frac{1}{2}\right)\right) + O_\epsilon\left(\frac{1+\Delta^*(y,u)}{u\log y}\right). \end{aligned}$$

The last estimate holds trivially for $\frac{1}{2} \leq u' \leq u - \frac{1}{2}$, since then

$$|\Delta^*(y,u) = \sup_{1/2\leq u'\leq u} |\Delta(y,u')| \leq \frac{1}{2}\left(\Delta^*(y,u) + \Delta^*\left(y, u-\frac{1}{2}\right)\right) + O_\epsilon\left(\frac{1+\Delta^*(y,u)}{u\log y}\right)$$

and hence

$$\Delta^*(y,u) \leq \Delta^*\left(y, u-\frac{1}{2}\right) + O_\epsilon\left(\frac{1+\Delta^*(y,u)}{u\log y}\right)$$

uniformly for every fixed $\epsilon > 0$ and u, y satisfying (4.2.7), $u \geq 2$.

Iterating this inequality, we get

$$\Delta^*(y,u) \leq \Delta^*(y,u_0) + O_\epsilon\left(\frac{\log u}{\log y}(1+\Delta^*(y,u))\right)$$

for some u_0 satisfying $1.5 \leq u_0 \leq 2$. Since, we have already established that (4.2.6) holds in the range $\frac{1}{2} \leq u \leq 2$, we deduce

$$\Delta^*(y,u) \ll_\epsilon \frac{\log u + 1}{\log y}(1+\Delta^*(y,u))$$

and hence (4.2.6) holds for the whole range (4.2.7).

□

CHAPTER 5

Computations

5.1. Computation with the Buchstab function

To compute values of the Buchstab function $\omega(u)$ we use the following power series method of Marsaglia, Zaman and Marsaglia [61]. A.Y Cheer and D.A. Goldson [17] independently derived this method with the exception that they choose to take their power series about an end point of the interval. As we show below, taking the power series about the midpoint gives more rapid convergence. We also note that there seems to be some errors in Marsaglia, Zaman and Marsaglia's formula for the coefficients of the next interval. These have been corrected below.

From Theorem 23 we have that

$$(n+1+z)\omega(n+1+z) = (n+1)\omega(n+1) + \int_0^z \omega(n+y)dy$$

and hence

$$(n+\frac{3}{2}+z)\omega(n+\frac{3}{2}+z) = (n+1)\omega(n+1) + \int_{-\frac{1}{2}}^z \omega(n+\frac{1}{2}+y)dy$$

for $n \in \mathbb{Z}^+$, $-\frac{1}{2} \le z \le \frac{1}{2}$. Thus if we have a power series expansion $a_0 + a_1 z + a_2 z^2 + \cdots$ around the midpoint of the interval $n-1 \le z \le n$ we can find a recursive formula for the coefficients of the power series expansion in the next interval. Let $b_0 + b_1 z + \cdots$ be the power series expansion around the midpoint of the interval $n \le z \le n+1$ then we see that

$$\left(n+\frac{3}{2}+\frac{z}{2}\right)(b_0+b_1z+\cdots)=(n+1)(a_0+\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots)+\int_{-\frac{1}{2}}^{z}(a_0+a_1y+\cdots)dy$$
$$=(n+1)(a_0+\frac{a_1}{2}+\cdots)+a_0z+\frac{a_0}{2}+\frac{a_1}{2}z^2-\frac{a_1}{2^3}+\cdots$$

Hence we see that

$$b_0=\left((n+1+\frac{1}{2})a_0+(n+1-\frac{1}{4})\frac{a_1}{2}+\cdots+(n+1+\frac{(-1)^j}{2j+2})\frac{a_j}{2^j}+\cdots\right)\Big/\left(n+\frac{3}{2}\right)$$

and for all $i\geq 1$ we have

$$b_i=\left(\frac{a_{i-1}}{i}-b_{i-1}\right)\Big/\left(n+\frac{3}{2}\right). \tag{5.1.1}$$

Let $a_i(n)$ denote the i-th coefficient of the power series expansion for ω around the midpoint of the interval $[n,n+1]$. Thus we have that $a_i(1)=(-1)^i(2/3)^{i+1}$, $i\geq 0$, since $\omega(u)=1/u$ for $1\leq u\leq 2$. We will now show by induction that $|a_i(n)|\leq(2/3)^{i+1}$ for $i\geq 0$ and $n\geq 1$.

Observe that $a_0(n)=\omega(n+3/2)$ hence $a_0(n)\geq 0$ and thus

$$|a_0(n)|\leq\frac{2}{3},$$

since $\omega(1.5)=3/2$, $\omega(2.5)=0.56218$ and by Theorem 28.

For $i>1$ we find, using our inductive hypothesis,

$$\begin{aligned}a_i(n)&\leq\frac{\frac{(2/3)^i}{i}+(2/3)^i}{n+1/2}\\&\leq\frac{3}{2}\left(\frac{2}{3}\right)^i/(5/2)\\&\leq\left(\frac{2}{3}\right)^{i+1}\end{aligned}$$

as required.

Thus, since $|z|\leq 1/2$ the i-th term of our of power series is less than $(1/3)^i$, and hence our power series converges very rapidly.

The values of $\omega(u)$ for $2 \leq u \leq 10$ were computed by programming this method in C++ using my own multi-precision floating point number class. We initialized the coefficient array with the first fifty coefficients for the power series expansion of $\omega(u) = 1/u$ about 1.5. In each iteration, we computed the first fifty coefficients for the next interval $[n, n+1]$, ensuring an accuracy of greater than twenty decimal places, and used this power series to compute values $n + i/10000$ for $1 \leq i \leq 10000$, $i \in \mathbb{Z}$. Figure 1 shows some of these values.

Figure 5.1: values of $\omega(u)$

u	$\omega(u)$	u	$\omega(u)$	u	$\omega(u)$
2	0.5	3.0	0.56091...	4.0	0.56145...
2.1	0.52157...	3.1	0.56267...	4.1	0.56150...
2.2	0.53741...	3.2	0.56164...	4.2	0.56152...
2.3	0.54885...	3.3	0.56109...	4.3	0.56151...
2.4	0.55686...	3.4	0.56086...	4.4	0.56150...
2.5	0.56218...	3.5	0.56082...	4.5	0.56148...
2.6	0.56538...	3.6	0.56091...	4.6	0.56147...
2.7	0.56689...	3.7	0.56106...	4.7	0.56146...
2.8	0.56706...	3.8	0.56121...	4.8	0.56145...
2.9	0.56615...	3.9	0.56135...	4.9	0.56145...

5.2. Computation with the Dickman function

In 1962, Bellman and Kotkin [5] computed values of the Dickman function $\rho(u)$ for $1 \leq u \leq 20$. In 1969, van de Lune and Wattel computed $\rho(u)$ for values up to $u = 1000$, discovering that Bellman and Kotkin's calculations were inaccurate for $u > 9$. Chamayou [15] demonstrated the probabilistic aspect of $\rho(u)$ by computing values for $1 \leq u \leq 6$ using the Monte-Carlo method. The best result are due to Marsaglia, Zaman, and Marsaglia [61] who used the same power series method as they did with the Buchstab function to compute values of $\rho(u)$ for $2 \leq u \leq 2000$.

For our computations of $\rho(u)$ we again use Marsaglia, Zaman, and Marsaglia's power series method.

Observe that

$$\rho(v+1+z) = \rho(v+1) - \int_0^z \frac{\rho(n+y)}{n+1+y}dy$$

satisfies equation (4.1.4). Hence

$$\rho(v+\frac{3}{2}+z) = \rho(v+1) - \int_{-\frac{1}{2}}^z \frac{\rho(n+\frac{1}{2}+y)}{n+\frac{3}{2}+y}dy. \tag{5.2.1}$$

Thus if we have the power series expansion $a_0 + a_1 z + \cdots$ for $\rho(z)$ around $n + \frac{1}{2}$ then we wish to iteratively obtain a power series $b_0 + b_1 z + \cdots$ for $\rho(z)$ around $n + \frac{3}{2}$. Substituting our power series into (5.2.1) and differentiating we get

$$b_1 + 2b_2 z + \cdots = \frac{a_0 + a_1 z + a_2 z^2 + \cdots}{n + 3/2 + z}.$$

Hence

$$b_1 = -\frac{a_0}{n+\frac{3}{2}} \tag{5.2.2}$$

and, for $i \geq 2$,

$$b_i = -\frac{a_{i-1} + (i-1)b_{i-1}}{i(n+\frac{3}{2})}. \tag{5.2.3}$$

To get a formula for b_0 we first put

$$c_0 = \frac{a_0}{n + \frac{3}{2}}$$

and

$$c_i = \frac{a_i - c_{i-1}}{n + \frac{3}{2}},$$

for $i \geq 1$. Then we have

$$\frac{\rho(n + \frac{1}{2} + y)}{n + \frac{3}{2} + y} = \frac{a_0 + a_1 z + \cdots}{n + \frac{3}{2} + y} = c_0 + c_1 y + c_2 y^2 + \cdots .$$

Substituting this into (5.2.1) we find that

$$b_0 = (a_0 + \frac{a_1}{2} + \frac{a_2}{4} + \cdots) - (\frac{c_0}{2} - \frac{c_1}{2^3} + \frac{c_2}{3 \cdot 2^3} - \cdots). \tag{5.2.4}$$

We have $\rho(u) = 1 - \log u$ for $1 \leq u \leq 2$ and hence if we let $a_i(n)$ denote i-th coefficient of the power series expansion in the midpoint of $[n, n+1)$ then $a_0(1) = 1 - \log(3/2)$ and $a_i(1) = (-1)^i \left(\frac{2}{3}\right)^i$ for $i = 1, 2, 3, \ldots$. We will now show that $|a_i(n+1)| < \left(\frac{2}{3}\right)^i$. Since $a_0(n) = \rho(n + 1/2) < 1$ for $n \geq 1$, we see from (5.2.2) that

$$|a_1(n+1)| < \frac{1}{n + 1/2} < \frac{2}{3},$$

for $n > 1$. Using (5.2.3), for $i > 1$, we have

$$\begin{aligned} |a_i(n+1)| &= \frac{|a_{i-1}(n)| + i|a_{i-1}(n+1)|}{(i+1)(n+1/2)} \\ &< \frac{\left(\frac{2}{3}\right)^{i-1} + i\left(\frac{2}{3}\right)^{i-1}}{(i+1)(n+1/2)} \\ &= \frac{\left(\frac{2}{3}\right)^{i-1}}{n+1/2} < \left(\frac{2}{3}\right)^i \end{aligned}$$

and the result follows by induction.

Thus, we once again have rapid convergence of our power series since $|z| \leq 1/2$ and hence the i-th term of our of power series is less than $(1/3)^i$.

To compute values of $\rho(u)$ for $2 \leq u \leq 10$ this method was programmed in C++ using my own multi-precision floating point number class. We initialized the coefficient array with the first fifty coefficients for the power series expansion of $\rho(u) = 1 - \log u$ about 1.5. In each iteration, we computed the first fifty coefficients for the next interval $[n, n+1]$, ensuring an accuracy of greater than twenty decimal places, and used this power series to compute values $n + i/10000$ for $1 \leq i \leq 10000$, $i \in \mathbb{Z}$. Figure 2 shows some of these values.

Figure 5.2: values of $\rho(u) = c(u) \cdot 10^{-d(u)}$

u	$c(u)$	d(u)	u	c(u)	d(u)
2	0.30685...	0	3.6	0.12875...	1
2.1	0.26045...	0	3.7	0.10172...	1
2.2	0.22035...	0	3.8	0.80068...	2
2.3	0.18579...	0	3.9	0.62803...	2
2.4	0.15599...	0	4.0	0.49109...	2
2.5	0.13031...	0	4.1	0.38285...	2
2.6	0.10827...	0	4.2	0.29754...	2
2.7	0.89418...	1	4.3	0.23050...	2
2.8	0.73391...	1	4.4	0.17799...	2
2.9	0.59878...	1	4.5	0.13701...	2
3.0	0.48608...	1	4.6	0.10514...	2
3.1	0.39322...	1	4.7	0.80455...	3
3.2	0.31703...	1	4.8	0.61395...	3
3.3	0.25464...	1	4.9	0.46727...	3
3.4	0.20371...	1	5.0	0.35472...	3
3.5	0.16229...	1	5.1	0.26857...	3

5.3. Computing $g(y)$

Recall that $g(y)$ is defined by

$$g(y) = \begin{cases} \inf_{v\geq 1} f(y,v) & \text{for } y < \theta \\ \inf_{u\geq y} e^{\gamma}\omega(1+u) & \text{for } y \geq \theta \end{cases}$$

where

$$f(u,v) = v(\log(1+u) + \rho(v(1+u))$$

for non-negative real numbers u, v and θ is the unique positive real number for which $\min_{v\geq 1} f(\theta, v) = e^{\gamma}/2$. Observe that for $v(1+u) > 1$ we have

$$\begin{aligned} \frac{\partial f(u,v)}{\partial v} &= \log(1+u) + \rho(v(1+u)) - v(1+u)\rho'(v(1+u)) \\ &= \log(1+u) + \rho(v(1+u)) + \rho(v(1+u)-1). \end{aligned}$$

Hence for $1 < v(1+u) \leq 2$,

$$\frac{\partial f(u,v)}{\partial v} = \log(1+u) + 1 - \log(v(1+u)) - 1 = -\log v.$$

Further, for $v(1+u) > 2$,

$$\begin{aligned} \frac{\partial^2 f(u,v)}{\partial v^2} &= (1+u)\rho'(v(1+u)) - (1+u)\rho'(v(1+u)-1) \\ &= \left(\frac{1}{v - \frac{1}{1+u}}\right)\rho(v(1+u)-2) - \frac{1}{v}\rho(v(1+u)-1) > 0, \end{aligned}$$

since $\rho(u) > \rho(u+1)$ for all $u > 0$. Hence for each $u > 0$ there is a unique real number $v' = v'(u) \geq 1$ for which

$$\log(1+u) + \rho(v'(1+u)) - \rho(v'(1+u)-1) = 0, \tag{5.3.1}$$

and

$$\inf_{v\geq 1} f(u,v) = \min_{v\geq 1} f(u,v) = f(u,v'). \tag{5.3.2}$$

Observe that for $u > 0$, $v \geq 1$

$$\begin{aligned}\frac{\partial f(u,v)}{\partial u} &= \frac{v}{1+u} + v^2\rho'(v(1+u)),\\ &= \frac{v}{1+u}(1-\rho(v(1+u)-1))\\ &> 0\end{aligned}$$

since $\rho(u) \leq 1$ for all $u > 0$. Thus $f(u, v'(u))$ is an increasing continuous function of u and hence there is a unique real number θ such that

$$f(\theta, v'(\theta)) = e^{\gamma}/2.$$

We now put, for $t \geq 1$,

$$h(t) = \rho(t-1) - \rho(t). \tag{5.3.3}$$

Observe that $h(t)$ is strictly decreasing for $t \geq 2$ since $\rho(u)$ is concave for $u \geq 2$. Hence $h(2) = \log 2$ and for each $t > 2$ there is a unique real number u, with $0 < u < 1$, such that

$$h(t) = \log(1+u). \tag{5.3.4}$$

Thus by (5.3.1)

$$v'(u) = \frac{t}{1+u}.$$

Further

$$\begin{aligned}f(u, v'(u)) &= v'(\log(1+u) + \rho(v'(1+u))\\ &= \frac{t}{1+u}(h(t)+\rho(t))\\ &= \frac{t}{1+u}\rho(t-1). \qquad (5.3.5)\end{aligned}$$

Hence u and v' are determined by t, so we wish to find a real number t' such that $t' = (1+\theta)v'$ and $h(t') = \log(1+\theta)$. Taking $t = 2.6$ we find, using (5.3.3),(5.3.4) and

(5.3.5), that $f(u, v') = 0.90384... < e^\gamma/2$ and taking $t = 2.7$ we find that $f(u, v') = 0.86670... > e^\gamma/2$. Thus we see that $2.6 < t' < 2.7$ and hence we have

$$h(t') = \log\left(\frac{t'}{t'-1}\right) - \left(\frac{1}{2}\log^2 t' + \mathrm{Li}_2\left(\frac{1}{t'}\right) - \frac{\pi^2}{12}\right). \tag{5.3.6}$$

From (5.3.5) we obtain

$$\frac{t'}{1+\theta}(1 - \log(t'-1)) = e^\gamma/2.$$

On taking logarithms and simplifying using (5.3.4) and (5.3.6) we obtain

$$\log(t'-1) + \log(1 - \log(t'-1)) + \frac{1}{2}\log^2 t' + \mathrm{Li}_2\left(\frac{1}{t'}\right) = \log\left(\frac{e^\gamma}{2}\right) + \frac{\pi^2}{12}.$$

Using MAPLE, Maier and Stewart calculated that $t' = 2.637994987\ldots$, which gives us $\theta = 0.500462161\ldots$ and $v'(\theta) = 1.758121634\ldots$.

By equation (5.3.2), we see that $f(y, v') = g(y)$ for $y < \theta$. Hence we only need to compute values for the Dickman function ρ and then we can use equations (5.3.3),(5.3.4) and (5.3.5) to calculate values for $g(y)$. In the other case, we first observe that since $\omega(u)$ is continuous, the $\lim_{u\to\infty} \omega(u) = e^{-\gamma}$ and $\omega(u) - e^{-\gamma}$ changes sign in every interval of length 1, we have

$$\inf_{u \geq y} e^\gamma \omega(1+u) = \min_{u \geq y} e^\gamma \omega(1+u).$$

Further since $M_-(v) = \min_{v \leq u \leq v+2} W(u)$ we see that for $y \geq \theta$

$$\begin{aligned} g(y) = \min_{u \geq y} e^\gamma \omega(1+u) &= e^\gamma M_-(y+1) + 1 \\ &= 1 + e^\gamma \min_{y+1 \leq u \leq y+3} W(u). \end{aligned}$$

Hence we only need to find the minimum value of $\omega(u)$ with $y + 1 \leq u \leq y + 3$. We observe that the minimum must occur at either $u = y + 1$ or at a critical point c_k of ω in the interval. Also observe that if c_k is a local minimum then from (2.2) we have $\omega(c_k) = \omega(c_k - 1)$. Moreover, since there are at most two critical points in each interval of length 1, a local minimum and a local maximum, $\omega(c_k - 1 + \epsilon) > \omega(c_k)$ for $0 < \epsilon < 1$.

Thus, we find that if $y+1 \le c_k \le y+2$ then the minimum is at c_k. Otherwise, the minimum occurs at $u = y+1$.

Thus, to compute $g(y)$ for $y \ge \theta$, we just need to compute all the local minima of ω. To do this assume we know the local minimum c_{k-2} and we wish to compute the local minimum c_k. We will apply Newton's method to $\omega'(u)$ to find $\omega'(u) = 0$. Since $u\omega'(u) = \omega(u-1) - \omega(u)$, we have

$$\omega''(u) = \frac{\omega'(u-1) - 2\omega'(u)}{u},$$

and hence we just need to pick a starting point u_0. We know by Hildebrand's result, see Chapter 3, that as $k \to \infty$ $c_k - c_{k-1} \to 1$. Hence for large k we could take $u_0 = c_{k-2} + 2$, however for smaller k this puts u_0 near an inflection point, and hence we get more rapid convergence with Newton's method by taking $u_0 = c_{k-2} + 1.7$.

In this manner we find that, for $y \ge \theta$,

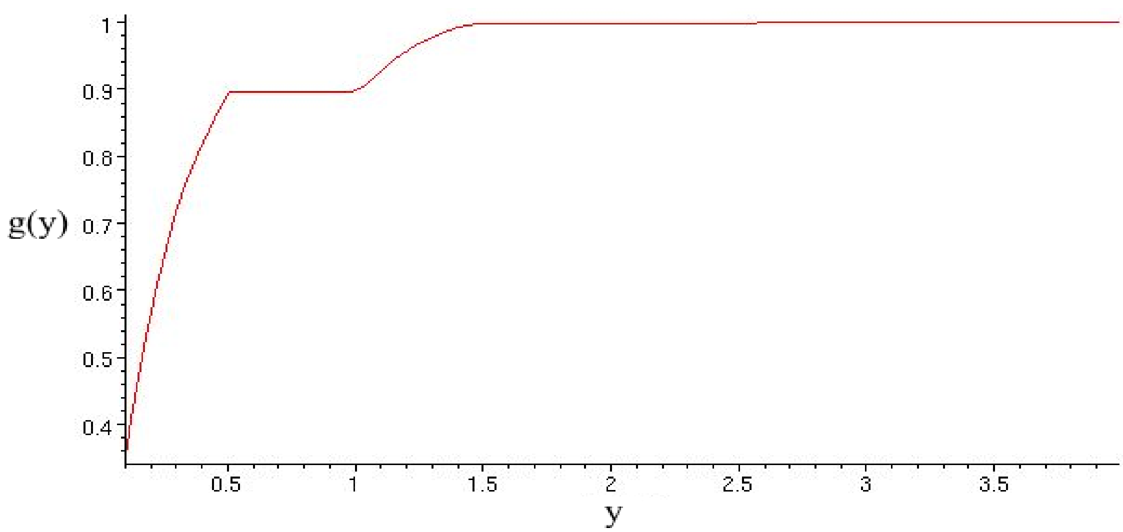
$$g(y) = \begin{cases} e^{\gamma}\omega(2) = 0.89053\ldots & \theta \le y \le 1 \\ e^{\gamma}\frac{\log(y)+1}{y+1} & 1 < y \le 1.46974\ldots \\ e^{\gamma}\omega(3.46974\ldots) = 0.998866\ldots & 1.46974\ldots < y \le 2.46974\ldots \\ e^{\gamma}\omega(y+1) & 2.46974\ldots < y \le 2.99493\ldots \\ e^{\gamma}\omega(4.99493\ldots) = 0.999991\ldots & 2.99493\ldots < y \le 3.99493\ldots \end{cases}$$

In Figure 3 we give some of our values for $g \ge \theta$ and in Figure 4 we give a graph of $g(y)$ for $0 \le y \le 4$ plotted in MAPLE.

Figure 5.3: $g(y)$ for $y < \theta$

t	y	g(y)	t	y	g(y)
2.65	0.49295 . . .	0.88612 . . .	3.65	0.09091 . . .	0.32946 . . .
2.7	0.46221 . . .	0.86670 . . .	3.7	0.08247 . . .	0.30564 . . .
2.75	0.43234 . . .	0.84550 . . .	3.75	0.07471 . . .	0.28288 . . .
2.8	0.40329 . . .	0.82279 . . .	3.8	0.06756 . . .	0.26123 . . .
2.85	0.37503 . . .	0.79759 . . .	3.85	0.06103 . . .	0.24071 . . .
2.9	0.34752 . . .	0.77076 . . .	3.9	0.05506 . . .	0.22133 . . .
2.95	0.32074 . . .	0.74193 . . .	3.95	0.04961 . . .	0.20313 . . .
3.0	0.29465 . . .	0.71104 . . .	4.0	0.04466 . . .	0.18612 . . .
3.05	0.27001 . . .	0.67907 . . .	4.05	0.04018 . . .	0.17029 . . .
3.1	0.24742 . . .	0.64713 . . .	4.1	0.03613 . . .	0.15560 . . .
3.15	0.22668 . . .	0.61537 . . .	4.15	0.03246 . . .	0.14198 . . .
3.2	0.20762 . . .	0.58390 . . .	4.2	0.02914 . . .	0.12938 . . .
3.25	0.19007 . . .	0.55285 . . .	4.25	0.02614 . . .	0.11886 . . .
3.3	0.17390 . . .	0.52230 . . .	4.3	0.02342 . . .	0.10699 . . .
3.35	0.15899 . . .	0.49236 . . .	4.35	0.02097 . . .	0.09709 . . .
3.4	0.14524 . . .	0.46310 . . .	4.4	0.01876 . . .	0.08798 . . .
3.45	0.13255 . . .	0.43460 . . .	4.45	0.01676 . . .	0.07962 . . .
3.5	0.12085 . . .	0.40693 . . .	4.5	0.01497 . . .	0.07195 . . .
3.55	0.11005 . . .	0.38015 . . .	4.55	0.01335 . . .	0.06494 . . .
3.6	0.10009 . . .	0.35431 . . .	4.6	0.01189 . . .	0.05853 . . .

Figure 5.4: Graph of $g(y)$



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