Nevanlinna-Pick Spaces and Dilations

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2016

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

I am the sole author of Chapters 1, 2, 4, 6 and 7. Chapter 3 is joint work with Kenneth R. Davidson and Orr Moshe Shalit. Chapter 5 is joint work with Martino Lupini.

Abstract

The majority of this thesis is devoted to the study of Nevanlinna-Pick spaces and their multiplier algebras. These spaces are Hilbert function spaces in which a version of the Nevanlinna-Pick interpolation theorem from complex analysis holds. Their multiplier algebras occupy an important place at the interface between operator algebras, operator theory and complex analysis.

Over the last few years, the classification problem for these algebras has attracted considerable attention. These investigations were pioneered by Davidson, Ramsey and Shalit, who used a theorem of Agler and McCarthy to identify a given multiplier algebra with the restriction of the multiplier algebra of the universal Nevanlinna-Pick space, namely the Drury-Arveson space, to an analytic variety in a complex ball.

In this thesis, the classification problem is studied from three different angles. In Chapter 3, we investigate multiplier algebras associated to embedded discs in a complex ball. In particular, we exhibit uncountably many embedded discs which are biholomorphic in a strong sense, but whose multiplier algebras are not isomorphic. Motivated by these issues, we use in Chapter 4 a different approach to the classification problem. Thus, we study the spaces and their multiplier algebras directly without making use of the existence of a universal Nevanlinna-Pick space. This allows us to completely classify the multiplier algebras of a special class of spaces on homogeneous varieties. In Chapter 5, we investigate the complexity of this classification problem from the point of view of Borel complexity theory.

In Chapter 6, we show that the Hardy space on the unit disc is essentially the only Nevanlinna-Pick space whose multiplication operators are all hyponormal.

The last part of this thesis is concerned with dilations and von Neumann's inequality. It has been known since the seventies that there are three commuting contractions which do not satisfy von Neumann's inequality. In Chapter 7, we show that every tuple of commuting contractions which forms a multivariable weighted shift dilates to a tuple of commuting unitaries and hence satisfies von Neumann's inequality, thereby providing a positive answer to a question of Shields and Lubin from 1974.

Acknowledgements

First and foremost, I would like to thank my advisor, Ken Davidson. I am deeply grateful for his guidance and support in many aspects of my PhD studies.

I am also indebted to many other mathematicians for valuable discussions. In particular, I am grateful to Raphaël Clouâtre, Jörg Eschmeier, Martino Lupini, John McCarthy, Chris Ramsey and Orr Shalit.

I would also like to thank the whole Pure Mathematics Department at the University of Waterloo for providing a stimulating environment and for making me feel at home.

Finally, the financial support provided by the Ontario Trillium Foundation over the course of my PhD studies is gratefully acknowledged.

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1. Introduction

There are two major themes in this thesis: Nevanlinna-Pick interpolation and dilation theory. Both topics sit at the interface of operator theory, operator algebras and complex analysis. It is the purpose of this introduction to outline some of the history of these topics and thus to put the results obtained in this thesis into perspective. More detailed summaries of the results in this thesis can be found at the beginning of the individual chapters.

A whole century ago, Georg Pick considered the following interpolation problem. Given points $z_1, \ldots, z_n$ in the open unit disc $\mathbb{D}$ in the complex plane and numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, when does there exist an analytic function $f : \mathbb{D} \to \mathbb{C}$ which solves the interpolation problem

$$f(z_i) = \lambda_i \quad (1 \le i \le n)$$

and satisfies the norm constraint

$$||f||_\infty = \sup\{|f(z)| : z \in \mathbb{D}\} \le 1?$$

Observe that without the norm constraint, the interpolation problem can always be solved by a polynomial, but the norm constraint makes the problem non-trivial. Pick's solution [67] is the following theorem.

Theorem (Pick)**.** *Let $z_1, \ldots, z_n \in \mathbb{D}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. There exists an analytic function $f : \mathbb{D} \to \mathbb{C}$ with*

$$f(z_i) = \lambda_i \quad (1 \le i \le n)$$

and $||f||_\infty \le 1$ if and only if the Hermitian $n \times n$ matrix

$$\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \right]$$

is positive semidefinite.

Unaware of Pick's work, Rolf Nevanlinna independently studied this problem and obtained a somewhat different characterization [62, 63].

The original problem of Pick and Nevanlinna is purely function theoretic and does not mention Hilbert spaces in any way. Nevertheless, there is a Hilbert space in the background of Pick's theorem. This space is the Hardy space

$$H^2 = \Big\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(\mathbb{D}) : ||f||^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \Big\}.$$

The Hardy space is a Hilbert space of analytic functions which plays a pivotal role at the intersection of complex analysis and operator theory. It is a reproducing kernel Hilbert space on $\mathbb{D}$, which means that for every $w \in \mathbb{D}$, the linear functional of evaluation at w is bounded on H^2. The function

$$K(z, w) = \frac{1}{1 - z\overline{w}},$$

which is called a Szegő kernel, is the reproducing kernel of H^2. This means that $K(\cdot, w) \in H^2$ for $w \in \mathbb{D}$ and

$$\langle f, K(\cdot, w) \rangle = f(w)$$

for every $f \in H^2$. Its multiplier algebra

$$\operatorname{Mult}(H^2) = \{\varphi : \mathbb{D} \to \mathbb{C} : \varphi \cdot f \in H^2 \text{ for all } f \in H^2\}$$

turns out to be H^∞, the algebra of all bounded analytic functions on the disc. Moreover, the multiplier norm of such a multiplier φ, which is defined to be the norm of the associated multiplication operator on H^2, is simply the supremum norm over $\mathbb{D}$.

Equipped with these notions, Pick's theorem now becomes a theorem about the reproducing kernel Hilbert space H^2: Given points $z_1, \ldots, z_n \in \mathbb{D}$ and values $\lambda_1, \ldots, \lambda_n$, there exists $\varphi \in \operatorname{Mult}(\mathcal{H})$ with

$$\varphi(z_i) = \lambda_i \quad (1 \le i \le n)$$

and multiplier norm at most 1 if and only if the Pick matrix

$$\big[K(z_i, z_j)(1 - \lambda_i \overline{\lambda_j})\big]$$

is positive semidefinite. This operator theoretic approach to Nevanlinna-Pick interpolation was pioneered by Sarason [77], who provided a new proof of Pick's theorem by establishing a precursor of the Sz.-Nagy-Foias commutant lifting theorem [84], see also [85, Section II.2]. A modern account of Sarason's proof can be found in [3, Section 10.6].

Two decades later, Agler [1], Quiggin [71] and McCullough [58] studied the Nevanlinna-Pick interpolation problem for general reproducing kernel Hilbert spaces. It is not hard to see that positivity of the Pick matrix is always a necessary condition for the existence of a solution of the interpolation problem, but it is not always sufficient. If the Hilbert space has the property that this condition is sufficient, then it is called a Nevanlinna-Pick space. Thus, a Nevanlinna-Pick space is a reproducing kernel Hilbert space in which Pick's theorem holds true, and the Hardy space H^2 is the prototypical example of such a space. Another example is the Dirichlet space on the unit disc, thanks to a result due to Agler [1], see also [71].

It turns out that the theory becomes much cleaner if one assumes that Pick's theorem also holds for matrix valued interpolation, which leads to the notion of a complete Nevanlinna-Pick space. We refer the reader to Chapter 2 for precise definitions and a list of examples. A comprehensive treatment of Nevanlinna-Pick spaces can be found in the book [3].

The majority of this thesis is devoted to the study of complete Nevanlinna-Pick spaces and their multiplier algebras. In particular, the classification problem for these multiplier algebras is investigated from several different angles. This line of research was initiated by Davidson, Ramsey and Shalit [24, 25]. A key ingredient in their approach is the Drury-Arveson space H^2_d, which is a natural generalization of the Hardy space to the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$. Specifically, H^2_d is the reproducing kernel Hilbert space on the $\mathbb{B}_d$ with reproducing kernel

$$K(z, w) = \frac{1}{1 - \langle z, w \rangle}.$$

The case $d = \infty$ is allowed, and we understand $\mathbb{C}^\infty$ as ℓ_2 in this case. The Drury-Arveson space, also known as symmetric Fock space, appeared in different guises many times throughout the literature. The reader is referred to the article [8] as well as the survey article [79] for a comprehensive treatment of the different features of the Drury-Arveson space.

The importance of the Drury-Arveson space for the classification problem for multiplier algebras of Nevanlinna-Pick spaces stems from a theorem of Agler and McCarthy [2], according to which every irreducible complete Nevanlinna-Pick space can be identified with the restriction of the Drury-Arveson space to a subset of the unit ball. Davidson, Ramsey and Shalit used this result to identify for every multiplier algebra $\mathcal{M}$ of an irreducible complete Nevanlinna-Pick space an analytic variety V in a complex ball $\mathbb{B}_d$ such that $\mathcal{M}$ is completely isometrically isomorphic and weak-$*$ homeomorphic to $\mathcal{M}_V$. Here $\mathcal{M}_V$ denotes the restriction of the multiplier algebra of the Drury-Arveson space to V, which is naturally identified with the quotient of the multiplier algebra of H^2_d by the ideal of all multipliers which vanish on V. Thus, the general classification problem is reduced to algebras of the form $\mathcal{M}_V$. It is important to note that many spaces, even classical spaces of analytic functions on the unit disc, cannot be realized in a finite dimensional ball in this way.

Another route which leads to the Drury-Arveson space, its multiplier algebra and the algebras $\mathcal{M}_V$ comes from dilation theory of tuples of operators. In single operator theory, the Hardy space H^2 provides an important link between complex analysis and operator theory through dilation theory. The starting point is von Neumann's inequality [88], which states that if T is a contraction on a Hilbert space, then

$$||p(T)|| \leq \sup\{|p(z)| : z \in \mathbb{D}\}$$

holds for all polynomials $p \in \mathbb{C}[z]$. This result can be readily deduced from Sz.-Nagy's dilation theorem [83], see also [85, Chapter I], according to which every contraction T dilates to a unitary operator. A different version, in which the Hardy space is more visible, asserts that every pure contraction T coextends to a direct sum of copies of the unilateral shift M_z on H^2.

In multivariable operator theory, one studies d-tuples $T = (T_1, \ldots, T_d)$ of operators on a Hilbert space $\mathcal{H}$. A natural generalization of the concept of a single contraction to this setting is the notion of a row contraction. Here, the requirement is that the row operator $(T_1, \ldots, T_d) : \mathcal{H}^n \to \mathcal{H}$ be a contraction. If one further assumes that the operators $T_1, \ldots, T_d$ commute, then one can seek a similar link between multivariable complex analysis and multivariable operator theory, and it is here where the Drury-Arveson space enters the picture. The following analogue of von Neumann's inequality is due to Drury [26].

Theorem (Drury). *If $T = (T_1, \ldots, T_d)$ is a commuting row contraction on a Hilbert space, then*

$$||p(T)|| \leq ||p||_{\mathrm{Mult}(H^2_d)}$$

for all polynomials $p \in \mathbb{C}[z_1, \ldots, z_d]$.

The corresponding dilation theorem was established by Müller-Vasilescu [59] and Arveson [8]. However, we remark that Lubin already proved a version of this theorem in [55].

Theorem (Müller-Vasilescu, Arveson). *Every pure commuting row contraction coextends to a direct sum of copies of the d-tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ of multiplication operators on H^2_d.*

In this sense, the d-tuple M_z on the Drury-Arveson space can be regarded as the universal pure commuting row contraction, and the multiplier algebra of H^2_d is the weak-$*$ closed unital operator algebra generated by M_z. If one introduces additional relations between the operators, one is naturally led to quotients of $\mathcal{M}_d$ and hence to the algebras $\mathcal{M}_V$. Indeed, this point of view served as the motivation in [24], see also [69].

We also mention that there exists a corresponding commutant lifting theorem due to Ball, Trent and Vinnikov [9] and Davidson and Le in greater generality [20]. It provides a direct link between dilations and Nevanlinna-Pick interpolation similar to Sarason's work, see [9, Section 5, Example 1].

Generally speaking, the classification scheme of Davidson-Ramsey-Shalit seeks to classify the algebras $\mathcal{M}_V$ in terms of the geometric structure of the underlying varieties V. They first considered the case of homogeneous varieties in a finite dimensional ball [24]. In this case, isomorphism on the level of multiplier algebras translates very nicely to geometric equivalence of the underlying varieties. Specifically, they showed that two multiplier algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are isometrically isomorphic if and only if there is a unitary map on $\mathbb{C}^d$ (equivalently a biholomorphic automorphism of $\mathbb{B}_d$) which maps V onto W. Similarly, they showed that $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic if and only if there exists an invertible linear map on $\mathbb{C}^d$ (equivalently a biholomorphic map) which maps V onto W, provided that the geometry of the underlying varieties is not too complicated. The additional restrictions on the geometry of the varieties were later removed by the author in [40], see also [41].

For general varieties in a finite dimensional ball, Davidson, Ramsey and Shalit showed in [25] that two algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are isometrically isomorphic if and only if the underlying varieties V and W are biholomorphic via an automorphism of $\mathbb{B}_d$. They also showed that if $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, then V and W are biholomorphic, provided that the varieties satisfy some mild geometric conditions. The converse of this statement is not true in general, as Davidson-Ramsey-Shalit exhibited two Blaschke sequences in the unit disc which are biholomorphic, but which give rise to non-isomorphic multiplier algebras. It should be noted, however, that Blaschke sequences are fairly complicated varieties, as they have infinitely many irreducible components.

In the case of one dimensional varieties, the converse is often true if one assumes additional regularity on the boundary. This was proved by Alpay-Putinar-Vinnikov in the case of the unit disc [4], by Arcozzi-Rochberg-Sawyer [6] in the case of finitely connected planar domains, and by Kerr-McCarthy-Shalit [54] in the case of finite Riemann surfaces. Nevertheless, there is still no complete classification of the algebras $\mathcal{M}_V$ even in the one dimensional case, as the regularity assumptions do not hold in general. We refer the reader to the survey article [76] for a comprehensive account on the current state of the art.

In this thesis, the investigations of Davidson, Ramsey and Shalit are continued from three different angles.

Firstly, Chapter 3 investigates algebraic isomorphism for algebras of the type $\mathcal{M}_V$, where V is an embedded disc in a complex ball. In some sense, embedded discs are a particularly simple case of non-homogeneous varieties. In particular, one might expect that they do not exhibit the same pathologies as Blaschke sequences. Nevertheless, among other results, we exhibit an uncountable family of embedded discs which are biholomorphic in a strong sense, but which yield non-isomorphic multiplier algebras. This chapter is based on the paper [19], which was written jointly with Kenneth Davidson and Orr Shalit.

Secondly, Chapter 4 approaches the classification problem from a different angle. Instead of using the universality result of Agler and McCarthy, the multiplier algebras are studied directly. Using this approach, is it possible to classify a broad class of multiplier algebras which were previously inaccessible. The contents of this chapter appeared in [43].

Thirdly, Chapter 5 is concerned with the complexity of the classification problem for multiplier algebras. More precisely, the classification problem is studied from the point of view of Borel complexity theory, a branch of mathematical logic which provides a framework for comparing the complexity of classification problems in mathematics. This chapter is essentially the paper [45], which is joint work with Martino Lupini.

The Hardy space H^2 is a particularly tractable example of a reproducing kernel Hilbert space. It was explained earlier that it is the prototypical example of a Nevanlinna-Pick space. From an operator theoretic point of view, multiplication operators on the Hardy space are fairly well behaved because they are all subnormal, and in particular hyponormal operators. As a consequence, H^∞, the multiplier algebra of H^2, is a uniform algebra, a fact on which much of the classical theory about H^∞ depends, see for instance [33]. In Chapter 6, which is essentially the paper [42], it is shown that this situation is very special: The Hardy space is essentially the only complete Nevanlinna-Pick space whose multiplication operators are all hyponormal.

The first steps into the realm of multivariate operator theory were not undertaken toward a theory for commuting row contractions, but for d-tuples of commuting contractions. This condition is less restrictive. Andô proved that every pair of commuting contractions dilates to a pair of commuting unitaries, thereby establishing an analogue of Sz.-Nagy's dilation theorem in this setting [5]. As a consequence, one sees that the two variable von Neumann inequality holds:

$$||p(T_1, T_2)|| \le \sup\{|p(z_1, z_2)| : (z_1, z_2) \in \overline{\mathbb{D}}^2\}$$

for every polynomial $p \in \mathbb{C}[z_1, z_2]$ and every pair (T_1, T_2) of commuting contractions. Surprisingly, the corresponding result for three commuting contractions is false. First, Parrott [64] gave an example of three commuting contractions which do not dilate to three commuting unitaries (but do satisfy the three variable von Neumann's inequality). A few years later, Kaijser-Varopoulos [87] and Crabb-Davie [14] exhibited three commuting contractions which do not satisfy the three variable version of Neumann's inequality.

Almost immediately after the first counterexamples became known, Lubin and Shields asked if von Neumann's inequality holds for a particularly tractable class of commuting contractions, namely multivariable weighted shifts. Chapter 7, which is essentially the article [44], provides a positive answer to this question.

2. Preliminaries about Nevanlinna-Pick spaces

The purpose of this chapter is to gather basic definitions and results from the theory of reproducing kernel Hilbert spaces and Nevanlinna-Pick interpolation. In particular, we fix the terminology and notation which will be used throughout this thesis. References on the basics of reproducing kernel Hilbert spaces include the classical paper of Aronszajn [7], the book [11] as well as the forthcoming book [66]. The standard reference on Nevanlinna-Pick spaces is the book [3]. The exposition in the first three sections of this chapter follows [41, Appendix I]

2.1. Reproducing kernel Hilbert spaces

Let X be a set and let $\mathcal{H}$ be a Hilbert space of functions on X. We say that $\mathcal{H}$ is a *reproducing kernel Hilbert space* or *Hilbert function space* if for each $x \in X$, the linear functional of point evaluation

$$\mathcal{H} \to \mathbb{C}, \quad f \mapsto f(x),$$

is bounded. By the Riesz representation theorem, there exists for $x \in X$ a function $k_x \in \mathcal{H}$ such that

$$f(x) = \langle f, k_x \rangle \quad \text{for all } f \in \mathcal{H}.$$

The two-variable function

$$K : X \times X \to \mathbb{C}, \quad (x, y) \mapsto k_y(x) = \langle k_y, k_x \rangle,$$

is called the *reproducing kernel of* $\mathcal{H}$. It is easy to see that K is *positive definite* in the sense that for any finite sequence of points $x_1, \ldots, x_n$ in X, the $n \times n$ matrix

$$\big[K(x_i, x_j)\big]_{i,j=1}^n$$

is positive semidefinite.

The following theorem of Moore shows that every positive definite function arises in this way and that a Hilbert function space is uniquely determined by its reproducing kernel.

Theorem 2.1.1 (Moore)**.** *Let X be a set and let $K : X \times X \to \mathbb{C}$ be a positive definite function. Then there exists a unique Hilbert function space $\mathcal{H}$ on X whose reproducing kernel is K.*

Proof. See, for example, [3, Theorem 2.23]. □

It is obvious that the restriction of a positive definite function on a set X to a subset Y of X is again positive definite. The following lemma describes this situation on the level of Hilbert function spaces.

Lemma 2.1.2. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with kernel K. Let $\mathcal{H}\big|_Y$ denote the reproducing kernel Hilbert space on Y with reproducing kernel $K\big|_{Y\times Y}$. Then*

$$\mathcal{H}\big|_Y = \{f\big|_Y : f \in \mathcal{H}\}$$

and the map

$$\mathcal{H} \to \mathcal{H}\big|_Y, \quad f \mapsto f\big|_Y,$$

is a co-isometry.

Proof. See [7, Part I, Section 5]. □

Observe that it is possible to regard ℓ_2 as a Hilbert function space on $\mathbb{N}$. As a Hilbert function space, however, this space is somewhat uninteresting, as it is merely a direct sum of copies of the one dimensional Hilbert function space $\mathbb{C}$, regarded as functions on a singleton. Therefore, most Hilbert function spaces which we consider are *irreducible* in the following sense.

Definition 2.1.3. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X with kernel K. We say that $\mathcal{H}$ (or K) is irreducible if $K(x,y) \neq 0$ for all $x,y \in X$ and $K(\cdot,x)$ and $K(\cdot,y)$ are linearly independent if $x \neq y$.

The following result, which will be used several times, is known as the *Schur product theorem*.

Theorem 2.1.4 (Schur)**.** *Let K, L be positive definite functions on a set X. Then the pointwise product $K \cdot L$ is also positive definite.*

Proof. See [78] or [3, Appendix A]. □

2.2. Multipliers

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X with kernel K. The *multiplier algebra* of $\mathcal{H}$ is

$$\text{Mult}(\mathcal{H}) = \{\varphi : X \to \mathbb{C} : \varphi \cdot f \in \mathcal{H} \text{ for all } f \in \mathcal{H}\}.$$

It should be obvious that $\text{Mult}(\mathcal{H})$ is a unital commutative algebra. An application of the closed graph theorem shows that for every $\varphi \in \text{Mult}(\mathcal{H})$, the associated multiplication operator

$$M_\varphi : \mathcal{H} \to \mathcal{H}, \quad f \mapsto \varphi \cdot f,$$

is bounded. Thus, we may define the *multiplier norm* of φ to be $||\varphi||_{\text{Mult}(\mathcal{H})} = ||M_\varphi||$.

The following characterization of the unit ball of $\text{Mult}(\mathcal{H})$ will be used repeatedly.

Lemma 2.2.1. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X with kernel K. A function $\varphi : X \to \mathbb{C}$ belongs to $\text{Mult}(\mathcal{H})$ and satisfies $||\varphi||_{\text{Mult}(\mathcal{H})} \le 1$ if and only if the function*

$$X \times X \to \mathbb{C}, \quad (x, y) \mapsto K(x, y)(1 - \varphi(x)\overline{\varphi(y)}),$$

is positive definite.

Proof. See, for example, [3, Corollary 2.37]. □

We say that $\mathcal{H}$ has no common zeros if there is no point $x \in X$ such that $f(x) = 0$ for all $f \in \mathcal{H}$. It is not hard to see that this happens if and only if $K(x, x) \neq 0$ for all $x \in X$. In particular, if $\mathcal{H}$ is irreducible, then it has no common zeros.

Lemma 2.2.2. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X without common zeros. Let $\varphi \in \text{Mult}(\mathcal{H})$. Then*

$$M_\varphi^* K(\cdot, x) = \overline{\varphi(x)} K(\cdot, x)$$

for $x \in X$. In particular, $||\varphi||_{\text{Mult}(\mathcal{H})} \ge \sup\{|\varphi(x)| : x \in X\}$.

Proof. The elementary proof can be found, for example, in [3, Section 2.3]. □

In general, the multiplier norm is strictly larger than the supremum norm, and we will see many examples of this phenomenon. The first such example is the Dirichlet space (see Section 2.6 below).

It follows from the last lemma that if $\mathcal{H}$ has no common zeros, then we may recover a multiplier φ from the multiplication operator M_φ via the *Berezin transform*:

$$\varphi(x) = \frac{\langle M_\varphi K(\cdot, x), K(\cdot, x) \rangle}{K(x,x)} \quad (x \in X).$$

In particular, the assignment $\varphi \mapsto M_\varphi$ is injective, which allows us to regard $\operatorname{Mult}(\mathcal{H})$ as a subalgebra of $\mathcal{B}(\mathcal{H})$. It is important to note that $\operatorname{Mult}(\mathcal{H})$ is typically not a selfadjoint algebra. Indeed, it easily follows from the last lemma that if $\mathcal{H}$ is irreducible, then $\operatorname{Mult}(\mathcal{H}) \cap \operatorname{Mult}(\mathcal{H})^* = \mathbb{C}1$.

Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$. We let $\operatorname{Lat}(\mathcal{A})$ denote the lattice of all closed subspaces of $\mathcal{H}$ which are invariant under each $T \in \mathcal{A}$. Moreover, if $\mathcal{N}$ is a collection of closed subspaces of $\mathcal{H}$, let $\operatorname{Alg}(\mathcal{N})$ be the algebra of all bounded operators on $\mathcal{H}$ which leave each $M \in \mathcal{N}$ invariant. Tautologically,

$$\mathcal{A} \subset \operatorname{Alg}(\operatorname{Lat}(\mathcal{A})),$$

and we say that $\mathcal{A}$ is *reflexive* if equality holds. It is straightforward to show that every reflexive algebra is closed in the weak operator topology.

Lemma 2.2.3. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space without common zeros. Then* $\operatorname{Mult}(\mathcal{H})$ *is a reflexive subalgebra of* $\mathcal{B}(\mathcal{H})$*. In particular, it is closed in the weak operator topology.*

Proof (sketch). One verifies that $T \in \mathcal{B}(\mathcal{H})$ is a multiplication operator if and only if T^* leaves $\mathbb{C}K(\cdot, x)$ invariant for every $x \in X$ (the "only if" direction follows from Lemma 2.2.2). Reflexivity of $\operatorname{Mult}(\mathcal{H})$ is immediate from this observation. □

In particular, we see that in the setting of the last lemma, $\operatorname{Mult}(\mathcal{H})$, being a WOT closed subalgebra of $\mathcal{B}(\mathcal{H})$, inherits a weak-$*$ topology from $\mathcal{B}(\mathcal{H})$.

Lemma 2.2.4. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with kernel K without common zeros. Then on bounded subsets of* $\operatorname{Mult}(\mathcal{H})$*, the weak-$*$ topology agrees with the topology of pointwise convergence.*

Proof (sketch). This is a straightforward consequence of the identity

$$\langle \varphi \cdot K(\cdot, x), K(\cdot, y) \rangle = \varphi(y) K(y, x)$$

for all $\varphi \in \operatorname{Mult}(\mathcal{H})$ and $x, y \in X$ and the fact that the linear span of the kernel functions $K(\cdot, x)$ is dense in $\mathcal{H}$. □

Since $\mathrm{Mult}(\mathcal{H})$ is a unital, commutative Banach algebra, it is natural to consider its maximal ideal space (also known as character space)

$$\mathcal{M}(\mathrm{Mult}(\mathcal{H})) = \{\rho : \mathrm{Mult}(\mathcal{H}) \to \mathbb{C} : \rho \text{ is non-zero, linear and multiplicative}\},$$

endowed with the weak-$*$ topology. Clearly, every point in the underlying set X gives rise to the character of evaluation at x. In particular, we see that $\mathrm{Mult}(\mathcal{H})$ semi-simple. However, these point evaluations typically only form a small part of the character space, which is the reason for some of the subtleties in the theory of multiplier algebras. Indeed, even in the case of H^∞, which is the motivating example for many of the investigations in this thesis, the character space is known to be very complicated (see, for example, [36, Chapter V]). For instance, the existence of interpolating sequences implies that $\mathcal{M}(H^\infty)$ contains a homeomorphic copy of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$ (see, for example, [36, Chapter X]). In particular, the compact Hausdorff space $\mathcal{M}(H^\infty)$ is not metrizable and has cardinality $2^{2^{\aleph_0}}$.

2.3. Vector valued reproducing kernel Hilbert spaces

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X and let $\mathcal{E}$ be an auxiliary Hilbert space. Then one may regard $\mathcal{H} \otimes \mathcal{E}$ as a Hilbert space of $\mathcal{E}$-valued functions on X by identifying an elementary tensor $f \otimes v \in \mathcal{H} \otimes \mathcal{E}$ with the function $x \mapsto f(x) \otimes v$.

If $\mathcal{E}'$ is another auxiliary Hilbert space, then a *multiplier from $\mathcal{H} \otimes \mathcal{E}$ into $\mathcal{H} \otimes \mathcal{E}'$* is a mapping $\Phi : X \to \mathcal{B}(\mathcal{E}, \mathcal{E}')$ such that for all $F \in \mathcal{H} \otimes \mathcal{E}$, the function

$$X \to \mathcal{E}', \quad x \mapsto \Phi(x)F(x),$$

belongs to $\mathcal{H} \otimes \mathcal{E}'$. We write $\Phi \in \mathrm{Mult}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{E}')$. If $\mathcal{E} = \mathcal{E}'$, we simply denote this space by $\mathrm{Mult}(\mathcal{H} \otimes \mathcal{E})$. As in the scalar valued case, an application of the closed graph theorem shows that such a multiplier Φ induces a bounded operator $M_\Phi : \mathcal{H} \otimes \mathcal{E} \mapsto \mathcal{H} \otimes \mathcal{E}'$, and the multiplier norm of Φ is defined to be the operator norm $||M_\Phi||$.

We require the following generalization of Lemma 2.2.1, which is essentially proved in the same way.

Lemma 2.3.1. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X and let $\mathcal{E}, \mathcal{E}'$ be auxiliary Hilbert spaces. A function $\Phi : X \to \mathcal{B}(\mathcal{E}, \mathcal{E}')$ belongs to the closed unit ball of* $\mathrm{Mult}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{E}')$ *if and only if*

$$X \times X \to \mathcal{B}(\mathcal{E}'), \quad (x, y) \mapsto K(x, y)(\mathrm{id}_{\mathcal{E}'} - \Phi(x)\Phi(y)^*)$$

is positive definite. □

Here, in analogy with the scalar valued case, a function $L : X \times X \to \mathcal{B}(\mathcal{E}')$ is said to be *positive definite* if for every finite sequences of points $x_1, \ldots, x_n$ in X, the $n \times n$ operator matrix

$$[L(x_i, x_j)]_{i,j=1}^n$$

is positive.

2.4. Nevanlinna-Pick interpolation

As explained in the introduction, Pick's theorem serves as the motivation for the definition of a Nevanlinna-Pick space.

Definition 2.4.1. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X with reproducing kernel K without common zeros. Given a natural number n, we say that $\mathcal{H}$ (or K) satisfies the *n-point Nevanlinna-Pick property* if whenever $z_1, \ldots, z_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that the matrix

$$\left[K(z_i, z_j)(1 - \lambda_i \overline{\lambda_j})\right]_{i,j=1}^n$$

is positive, there exists a multiplier $\varphi \in \operatorname{Mult}(\mathcal{H})$ such that $||\varphi||_{\operatorname{Mult}(\mathcal{H})} \le 1$ and such that

$$\varphi(z_i) = \lambda_i \quad (i = 1, \ldots, n).$$

We say that $\mathcal{H}$ (or K) satisfies the *Nevanlinna-Pick property* if it satisfies the n-point Nevanlinna-Pick property for all $n \in \mathbb{N}$.

It turns out that a much cleaner theory can be obtained for spaces which satisfy the Nevanlinna-Pick property not just for scalars $\lambda_1, \ldots, \lambda_n$, but also for matrices of arbitrary size. This leads to the notion of a complete Nevanlinna-Pick space. While the title of this thesis simply refers to Nevanlinna-Pick spaces for the sake of brevity, we will be almost exclusively concerned with complete Nevanlinna-Pick spaces.

Definition 2.4.2. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X with reproducing kernel K without common zeros. We say that $\mathcal{H}$ (or K) satisfies the *complete Nevanlinna-Pick property* if whenever $n \in \mathbb{N}$ and $r \in \mathbb{N}$ and $\Lambda_1, \ldots, \Lambda_n \in M_r(\mathbb{C})$ such that the $nr \times nr$-matrix

$$\left[K(z_i, z_j)(1 - \Lambda_i \Lambda_j^*)\right]_{i,j=1}^n$$

is positive, there exists a multiplier $\Phi \in \operatorname{Mult}(\mathcal{H} \otimes \mathbb{C}^r)$ such that $||\Phi||_{\operatorname{Mult}(\mathcal{H} \otimes \mathbb{C}^r)} \le 1$ and such that

$$\Phi(z_i) = \Lambda_i \quad (i = 1, \ldots, n).$$

The prototypical example of a complete Nevanlinna-Pick space is the Hardy space H^2. We will see more examples in Section 2.6.

There is a reformulation of the (complete) Nevanlinna-Pick property which is quite useful as well and which will be used repeatedly. In the setting of Definition 2.4.1, let $Y = \{z_1, \ldots, z_n\}$ consist of n distinct points and define a function φ_0 on Y by $\varphi_0(z_i) = \lambda_i$ for $1 \le i \le n$. Thus, we ask if there exists a multiplier $\varphi \in \operatorname{Mult}(\mathcal{H})$ of norm at most 1 such that $\varphi\big|_Y = \varphi_0$. Using Lemma 2.2.1, one can show that the Pick matrix

$$\big[K(z_i, z_j)(1 - \lambda_i \overline{\lambda_j})\big]_{i,j=1}^n$$

is positive if and only if the function φ_0 belongs to the unit ball of $\operatorname{Mult}(\mathcal{H}\big|_Y)$. It is immediate from the definition of $\mathcal{H}\big|_Y$ that the map

$$R_Y : \operatorname{Mult}(\mathcal{H}) \to \operatorname{Mult}(\mathcal{H}\big|_Y), \quad \varphi \mapsto \varphi\big|_Y,$$

is a unital completely contractive homomorphism (which is also true if Y is not necessarily finite). Consequently, $\mathcal{H}$ satisfies the Nevanlinna-Pick property if and only if for every finite set Y, the restriction map R_Y maps the closed unit ball of $\operatorname{Mult}(\mathcal{H})$ onto the closed unit ball of $\operatorname{Mult}(\mathcal{H}\big|_Y)$. Since the closed unit balls in question are weak-$*$ compact and since the restriction map is weak-$*$-weak-$*$ continuous, this happens if and only if the restriction map is a quotient map. Similarly, $\mathcal{H}$ is a complete Nevanlinna-Pick space if and only if for every finite set $Y \subset X$, the restriction map R_Y is a complete quotient map. Background material on maps between operator spaces can be found in [28].

The following result says that the last observation remains true if Y is not necessarily finite. It is originally due to Quiggin [71, Lemma 3.3]. We give a slightly different proof.

Lemma 2.4.3. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X. Then the following are equivalent:*

(i) *$\mathcal{H}$ is a (complete) Nevanlinna-Pick space.*

(ii) *For every subset $Y \subset X$, the restriction map $R_Y : \operatorname{Mult}(\mathcal{H}) \to \operatorname{Mult}(\mathcal{H}\big|_Y)$ is a (complete) quotient map.*

Proof. The implication (ii) $\Rightarrow$ (i) follows from the discussion preceding the lemma.

Conversely, suppose that $\mathcal{H}$ is a complete Nevanlinna-Pick space, let $Y \subset X$ and suppose that Φ_0 belongs to the unit ball of $\operatorname{Mult}(\mathcal{H}\big|_Y \otimes \mathbb{C}^r)$ for some $r \in \mathbb{N}$. The discussion preceding the lemma shows that for every finite set $F \subset Y$, the weak-$*$ compact set

$$\mathcal{I}_F = \{\Phi \in \operatorname{Mult}(\mathcal{H} \otimes \mathbb{C}^r) : ||\Phi|| \le 1 \text{ and } \Phi\big|_F = \Phi_0\big|_F\}$$

is not empty. Therefore, the family $\{\mathcal{I}_F : F \subset Y \text{ finite}\}$ has the finite intersection property. By weak-$*$ compactness of the unit ball of $\operatorname{Mult}(\mathcal{H} \otimes \mathbb{C}^r)$, there exists

$$\Phi \in \bigcap_{F \subset Y \text{ finite}} \mathcal{I}_Y,$$

and it is clear that $\Phi\big|_Y = \Phi_0$. Thus, R_Y is a complete quotient map. Finally, if $\mathcal{H}$ is merely a Nevanlinna-Pick space, then the above argument for $r = 1$ shows that R_Y is a quotient map. □

2.5. The Agler-McCarthy universality theorem

Let $\mathcal{H}$ be an irreducible reproducing kernel Hilbert space on X with kernel K. We say that K is *normalized* at the point $x_0 \in X$ if $K(x, x_0) = 1$ for all $x \in X$. If K is normalized at some point in X, we say that $\mathcal{H}$ (or K) is *normalized*.

Given $x_0 \in X$, it is always possible to normalize K at x_0 by defining

$$\widetilde{K}(x, y) = \delta(x)\overline{\delta(y)}K(x, y) \quad (x, y \in X),$$

where

$$\delta(x) = \frac{\sqrt{K(x, x)}}{K(x_0, x_0)} \quad (x \in X).$$

The resulting Hilbert function space is simply $\delta \cdot \mathcal{H}$, and the multiplier algebra remains unchanged (see [3, Section 2.6]).

Irreducible complete Nevanlinna-Pick spaces are characterized by a theorem of McCullough and Quiggin (see [58] and [71], and also Section 7.1 in [3]). We require the following version of Agler and McCarthy, which is [3, Theorem 7.31].

Theorem 2.5.1 (McCullough-Quiggin, Agler-McCarthy)**.** *Let $\mathcal{H}$ be an irreducible reproducing kernel Hilbert space on a set X with reproducing kernel K which is normalized at a point in X. Then $\mathcal{H}$ is a complete Nevanlinna-Pick space if and only if the Hermitian kernel $F = 1 - 1/K$ is positive definite.* □

This characterization theorem is the essential step in the proof of the universality theorem of Agler-McCarthy, which was mentioned in the introduction. Let

$$\mathbb{B}_d = \{z \in \mathbb{C}^n : ||z|| < 1\}$$

be the open unit ball in $\mathbb{C}^d$. We allow $d = \infty$, in which case $\mathbb{C}^\infty$ is understood as ℓ_2. Let

$$K(z,w) = \frac{1}{1 - \langle z, w \rangle}.$$

Observe that if $d = 1$, this is simply the Szegő kernel. Writing K as a geometric series in $\langle z, w \rangle$, one may use the Schur product theorem (Theorem 2.1.4) to see that K is positive definite. The *Drury-Arveson space* H^2_d is the reproducing kernel Hilbert space on $\mathbb{B}_d$ with reproducing kernel K. If $d = 1$, this space is the Hardy space on the unit disc. It is an immediate consequence of Theorem 2.5.1 that H^2_d is an irreducible complete Nevanlinna-Pick space. The universality theorem of Agler-McCarthy shows that it is in fact a universal such space.

Theorem 2.5.2 (Agler-McCarthy)**.** *If $\mathcal{H}$ is a separable normalized irreducible complete Nevanlinna-Pick space on a set X with kernel K, then there exists $m \in \mathbb{N} \cup \{\infty\}$ and an embedding $j : X \to \mathbb{B}_m$ such that*

$$K(z,w) = k_m(j(z), j(w)) \quad (z, w \in X).$$

In this case, $f \mapsto f \circ j$ defines a unitary operator from $H^2_m\big|_{j(X)}$ onto $\mathcal{H}$.

Proof. See [2], Theorem 3.1 in [9], or Theorem 8.2 and Theorem 7.31 in [3]. □

In the setting of the last theorem, let $U : H^2_d\big|_{j(X)} \to \mathcal{H}$ denote the unitary map. Then conjugation by U defines a unital completely isometric isomorphism and weak-$*$-weak-$*$ homeomorphism

$$\operatorname{Mult}(H^2_d\big|_{j(X)}) \to \operatorname{Mult}(\mathcal{H}), \quad \varphi \mapsto \varphi \circ j.$$

In particular, every multiplier algebra of a separable normalized irreducible complete Nevanlinna-Pick space can be identified with $\operatorname{Mult}(H^2_d\big|_X)$ for some $d \in \mathbb{N} \cup \{\infty\}$ and some $X \subset \mathbb{B}_d$. To simplify notation, let $\mathcal{M}_X = \operatorname{Mult}(H^2_d\big|_X)$. Davidson-Ramsey-Shalit define a *variety* to be the common zero set in $\mathbb{B}_d$ of a family of H^2_d functions [25, Section 2] (these agree with common zero sets of families of functions in $\operatorname{Mult}(H^2_d)$ by [3, Theorem 9.27]).

Example 2.5.3. If $d = 1$ then the varieties in $\mathbb{D}$ are precisely $\mathbb{D}$ itself, all finite subsets of $\mathbb{D}$ as well as all *Blaschke sequences*. Recall that a sequence (z_n) in $\mathbb{D}$ is said to be a Blaschke sequence if

$$\sum_n (1 - |z_n|) < \infty.$$

Background material on Blaschke sequences can be found in [36, Chapter II, Section 2].

The following result is [25, Proposition 2.2], and it shows that we may without loss of generality assume that X is a variety.

Proposition 2.5.4 (Davidson-Ramsey-Shalit). *Let $X \subset \mathbb{B}_d$ and let V be the smallest variety which contains X, that is,*

$$V = \{z \in \mathbb{B}_d : f(x) = 0 \text{ for all } f \in H^2_d \text{ such that } f\big|_X = 0\}.$$

Then the restriction map from $H^2_d\big|_V$ to $H^2_d\big|_X$ is a unitary.

Proof. The restriction map is always a co-isometry by Lemma 2.1.2. Moreover, if $f \in H^2_d\big|_V$ vanishes on X, then it vanishes on V by definition of V, hence it is a unitary. □

Finally, we observe that Lemma 2.4.3 shows that $\mathcal{M}_V$ is the complete quotient of $\operatorname{Mult}(H^2_d)$ by the weak-$*$ closed ideal of all multipliers which vanish on V. Thus, the definition of $\mathcal{M}_V$ given here is consistent with the definition given in the introduction.

2.6. Examples

We finish this chapter by recording some examples of complete Nevanlinna-Pick spaces.

Example 2.6.1. (a) As mentioned earlier, the Hardy space H^2 on the unit disc, and more generally the Drury-Arveson space H^2_d on the unit ball $\mathbb{B}_d$, is a complete Nevanlinna-Pick space.

(b) The Dirichlet space

$$\mathcal{D} = \Big\{ f \in \mathcal{O}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty \Big\}$$

is a complete Nevanlinna-Pick space when endowed with the norm

$$||f||^2_{\mathcal{D}} = ||f||^2_{H^2} + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z),$$

where A denotes the normalized planar Lebesgue measure on $\mathbb{D}$ (see [3, Corollary 7.41]). This space plays an important role in operator theory, complex analysis and harmonic analysis, see for example [29].

The choice of norm is crucial. For instance, $\mathcal{D}$ is not a complete Nevanlinna-Pick space when endowed with the norm

$$|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$$

or with the Sobolev type norm

$$\int_{\mathbb{D}} |f(z)|^2 \, dA(z) + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z).$$

Observe that $||z^n||^2_{\mathcal{D}} = n+1$. In particular, $||z^n||^2_{\mathrm{Mult}(\mathcal{D})} \geq n+1$, so the multiplier norm strictly dominates the supremum norm in this case. In fact, it is possible to show that equality holds (see [29, Exercise 5.1.1] or Lemma 3.7.2 below). Hence, $\mathrm{Mult}(\mathcal{D})$ is not a uniform algebra on any set.

(c) For $s \leq 0$, let $\mathcal{H}_s$ be the reproducing kernel Hilbert space on $\mathbb{D}$ with reproducing kernel

$$K(z,w) = \sum_{n=0}^{\infty} (n+1)^s (z\overline{w})^s.$$

Then $\mathcal{H}_s$ is a complete Nevanlinna-Pick space, see [3, Corollary 7.41]. If $s = 0$, then $\mathcal{H}_0$ is the Hardy space H^2, if $s = -1$, then $\mathcal{H}_s$ is the Dirichlet space $\mathcal{D}$. For $-1 \leq s \leq 0$, the spaces $\mathcal{H}_s$ interpolate between these two spaces in the sense of Riesz-Thorin interpolation (see [3, Appendix C]).

(d) The Sobolev space W_1^2 consists of all absolutely continuous functions f on $[0,1]$ with finite Sobolev norm

$$||f||^2_{W_2^1} = \int_0^1 (|f(x)|^2 + |f'(x)|^2) \, dx.$$

This space is a complete Nevanlinna-Pick space (see [3, Theorem 7.43]).

(e) If $w : \mathbb{D} \to \mathbb{R}$ is a positive superharmonic function, then the weighted Dirichlet space with weight w consists of all analytic functions f on $\mathbb{D}$ with finite norm

$$||f||^2_{\mathcal{D}(w)} = ||f||^2_{H^2} + \int_{\mathbb{D}} |f'(z)|^2 w(z) \, dA(z).$$

These spaces are complete Nevanlinna-Pick spaces, see [82].

We also mention some classical reproducing kernel Hilbert spaces which fail the Nevanlinna-Pick property.

Example 2.6.2. (a) The Bergman space L^2_a on the unit disc, which consists of all analytic functions f on the unit disc with finite norm

$$||f||^2_{L^2_a} = \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty$$

is not a Nevanlinna-Pick space, see [3, Example 5.17]. Its reproducing kernel

$$K_{L^2_a}(z,w) = \frac{1}{(1-z\overline{w})^2}$$

is the square of the Szegő kernel. In particular, powers of Nevanlinna-Pick kernels are not Nevanlinna-Pick kernels in general.

(b) The Hardy space $H^2(\mathbb{B}_d)$ on the ball, which consists of all analytic functions f on $\mathbb{B}_d$ with finite norm

$$||f||^2_{H^2(\mathbb{B}_d)} = \sup_{0<r<1} \int_{\partial\mathbb{B}_d} |f(rz)|^2 \, d\sigma(z),$$

where σ denotes the normalized surface measure on $\partial\mathbb{B}_d$, is not a Nevanlinna-Pick space if $d \geq 2$. Similarly, the Bergman space $L^2_a(\mathbb{B}_d)$, which consists of all analytic functions on $\mathbb{B}_d$ with finite norm

$$||f||^2_{L^2_a(\mathbb{B}_d)} = \int_{\mathbb{B}_d} |f(z)|^2 \, dV(z),$$

where V denotes the normalized Lebesgue measure on $\mathbb{B}_d$, is not a Nevanlinna-Pick space. Indeed, there is no Nevanlinna-Pick space on $\mathbb{B}_d$ whose multiplier algebra is $H^\infty(\mathbb{B}_d)$ isometrically, see [3, Section 8.8].

(c) The Hardy space $H^2(\mathbb{D}^d)$ on the polydisc, which consists of all analytic functions f on $\mathbb{B}_d$ with finite norm

$$||f||^2_{H^2(\mathbb{D}^d)} = \sup_{0<r<1} \int_{\mathbb{T}^d} |f(rz)|^2 \, d\sigma(z),$$

where σ denotes the d-fold product of the normalized Lebesgue measure on $\mathbb{T}$, is not a Nevanlinna-Pick space if $d \geq 2$. Indeed, its reproducing kernel

$$K(z,w) = \prod_{i=1}^{d} \frac{1}{1-z_i\overline{w_i}}$$

restricts to the Bergman kernel if we identify the unit disc $\mathbb{D}$ with its image in $\mathbb{D}^d$ under the embedding $z \mapsto (z, z, 0, \ldots, 0)$, and it is easy to see that restrictions of Nevanlinna-Pick kernels are Nevanlinna-Pick kernels.

Nevertheless, there is a more complicated theorem about interpolation in $H^\infty(\mathbb{D}^2) = \mathrm{Mult}(H^2(\mathbb{D}^2))$, which directly generalizes Pick's theorem. It is due to Agler [1], see also [3, Theorem 11.49].

Theorem 2.6.3 (Agler). *Let* $z^{(1)}, \dots, z^{(n)} \in \mathbb{D}^2$ *and* $\lambda^{(1)}, \dots, \lambda^{(n)} \in \mathbb{C}$. *There exists* $f \in H^\infty(\mathbb{D}^2)$ *with*

$$f(z^{(i)}) = \lambda^{(i)} \quad (1 \leq i \leq n)$$

and $||f||_\infty \leq 1$ *if and only if there are positive definite kernels* K *and* L *on* $\{z^{(1)}, \dots, z^{(n)}\} \times \{z^{(1)}, \dots, z^{(n)}\}$ *such that*

$$1 - \lambda^{(i)}\overline{\lambda^{(j)}} = (1 - z_1^{(i)}\overline{z_1^{(j)}})K(\lambda^{(i)}, \lambda^{(j)}) + (1 - z_2^{(i)}\overline{z_2^{(j)}})L(\lambda^{(i)}, \lambda^{(j)})$$

for $1 \leq i, j \leq n$.

There is a version of this result for $d \geq 3$. The difference is that it does not characterize interpolation using functions in the unit ball of $H^\infty(\mathbb{D}^d) = \mathrm{Mult}(H^2(\mathbb{D}^d))$, but using functions in the *Schur-Agler class* (see [3, Section 11.8]). An analytic function f on $\mathbb{D}^d$ is said to belong to the Schur-Agler class if for every commuting tuple $T = (T_1, \dots, T_d)$ of strict contractions on a Hilbert space, we have

$$||f(T_1, \dots, T_d)|| \leq 1.$$

If $d = 1, 2$, then the Schur-Agler class and the unit ball of $H^\infty(\mathbb{D}^d)$ coincide by Sz.-Nagy's dilation theorem [83] and by Andô's theorem [5], respectively. If $d \geq 3$, the examples of Kaijser-Varopoulos [87] and Crabb-Davie [14] show that the Schur-Agler class is a proper subset of the unit ball of $H^\infty(\mathbb{D}^d)$ if $d \geq 3$.

3. Multipliers of embedded discs

3.1. Introduction

The contents of this chapter are joint work with Ken Davidson and Orr Shalit and appeared in [19]. We study the classification problem for the algebras $\mathcal{M}_V$, where $V \subset \mathbb{B}_d$ is a variety. Concretely, we ask: Given two varieties $V, W \subset \mathbb{B}_d$, when are $\mathcal{M}_V$ and $\mathcal{M}_W$ isomorphic?

As mentioned in Chapter 1, this problem was completely resolved for isometric isomorphisms (if $d < \infty$) in [25]. Here we will be concerned with the question of algebraic isomorphism. Since the algebras $\mathcal{M}_V$ are commutative and semi-simple, every algebraic homomorphism between these algebras is automatically norm continuous (see, for example, [17, Proposition 4.2]). In particular, the questions of algebraic and of topological isomorphism are the same.

The main result in [25] regarding algebraic isomorphism is the following theorem.

Theorem 3.1.1 (Davidson-Ramsey-Shalit)**.** *Let V and W be varieties in $\mathbb{B}_d$, with $d < \infty$, which are the union of finitely many irreducible varieties and a discrete variety. Let Φ be a unital algebra isomorphism of $\mathcal{M}_V$ onto $\mathcal{M}_W$. Then there exist holomorphic maps F and G from $\mathbb{B}_d$ into $\mathbb{C}^d$ with coefficients in $\operatorname{Mult}(H^2_d)$ such that*

(1) $F|_W = \Phi^*|_W \quad$ *and* $\quad G|_V = (\Phi^{-1})^*|_V$

(2) $G \circ F|_W = \mathrm{id}_W \quad$ *and* $\quad F \circ G|_V = \mathrm{id}_V$

(3) $\Phi(f) = f \circ F \quad$ *for* $\quad f \in \mathcal{M}_V$*, and*

(4) $\Phi^{-1}(g) = g \circ G \quad$ *for* $\quad g \in \mathcal{M}_W$*.*

In particular, when the multiplier algebras are isomorphic, the two varieties are biholomorphic. In fact, the function F and its inverse G have the additional feature that the component functions are multipliers. Thus, we say that F is a *multiplier biholomorphism*.

In the case of homogeneous varieties (zero sets of a family of homogeneous polynomials), everything works out in the best possible way. The results of [24, 40] combine to show that the multiplier algebras of two homogeneous varieties are algebraically isomorphic if and only if the varieties are biholomorphic. Moreover the two algebras are similar in this case, and there is a linear map that implements a (possibly different) biholomorphism between the homogeneous varieties W and V.

However, in the non-homogeneous case, a number of examples in [25] showed that a complete converse to the theorem above is not possible. One serious issue is that multiplier biholomorphism is not evidently an equivalence relation. The reason is that the composition of two multipliers defined on varieties may not be a multiplier. In fact, multiplier biholomorphism is not an equivalence relation at least when the varieties have infinitely many components (see Remark 3.6.8).

In [25], two types of counterexamples to the converse of the theorem above are exhibited: Blaschke sequences in the unit disc [25, Examples 6.2,8.2] and discs in $\mathbb{B}_\infty$ [25, Examples 6.11, 6.12, 6.13]. We will examine these examples in more detail here. In particular, we give precise conditions for when the multiplier algebras of two embedded discs in $\mathbb{B}_\infty$ of a special type are isomorphic. Our methods allow us to show that there are uncountably many discs which are multiplier biholomorphic such that their multiplier algebras are not isomorphic. Since these embedded discs live in an infinite dimensional ball, they may appear somewhat pathological. However, from the point of view of Nevanlinna-Pick spaces, they are very natural. Indeed, our most important class of examples arises from the spaces of Example 2.6.1 (c), a family of complete Nevanlinna-Pick spaces on the unit disc which interpolate between the Hardy space and the Dirichlet space, and which have been studied classically.

We will also be concerned with proper embeddings of discs into finite dimensional balls $\mathbb{B}_d$. Here the prototype result is due to Alpay, Putinar and Vinnikov [4]:

Theorem 3.1.2 (Alpay-Putinar-Vinnikov)**.** *Suppose that f is an injective holomorphic function of $\mathbb{D}$ onto $V \subset \mathbb{B}_d$ such that*

(1) *f extends to an injective C^2 function on $\overline{\mathbb{D}}$,*

(2) *$f'(z) \neq 0$ on $\overline{\mathbb{D}}$,*

(3) *$\|f(z)\| = 1$ if and only if $|z| = 1$,*

(4) *$\langle f(z), f'(z)\rangle \neq 0$ when $|z| = 1$.*

Then $\mathcal{M}_V$ is isomorphic to H^∞.

We remark that [4] only asks that f be C^1, but in [6, 2.3.6], where this result is generalized to finitely connected planar domains, the authors point out that the proof requires f to be C^2. This result is further extended in [54] to finite Riemann surfaces.

We will show in Section 3.3 that the transversality condition (4) is a consequence of being C^1. We will also show (in Section 3.5) that for a minor weakening of the hypotheses of Theorem 3.1.2, the conclusion is no longer valid.

It remains open if the converse of Theorem 3.1.1 holds if the varieties have only finitely many irreducible components. The finiteness of d and the finiteness of the number of components eliminates all of the counterexamples that we know about.

The remainder of this chapter is organized as follows. In Section 3.2, we record preliminaries regarding embedded discs which will be used throughout this chapter.

In Section 3.3, we show that the transversality assumption in the theorem of Alpay, Putinar and Vinnikov is a consequence of being C^1.

In Section 3.4, we present a modification of an example due to Josip Globevnik . This example provides a proper analytic embedding of the unit disc in to $\mathbb{B}_2$ which extends to be continuous on $\overline{\mathbb{D}}$, but transversality in an appropriate sense fails.

In Section 3.5, we exhibit a proper rational map f of the disc into $\mathbb{B}_2$ which satisfies all of the hypotheses of the theorem of Alpay-Putinar-Vinnikov, except for the fact that the C^∞ extension to $\overline{\mathbb{D}}$ is not injective. In this case, the multiplier algebra is not isomorphic to H^∞.

In Section 3.6, we show that a biholomorphism between varieties which induces an isomorphism between the multiplier algebras must be a bi-Lipschitz map with respect to the pseudohyperbolic distance. Re-examination of the example of the preceding section shows that indeed f fails to be bi-Lipschitz, hence cannot induce an isomorphism. We also give an example which shows that for Blaschke sequences, being bi-Lipschitz does not imply isomorphism.

In Section 3.7, we consider a special class of embeddings of $\mathbb{D}$ into $\mathbb{B}_\infty$. We give conditions for when the multiplier algebras of two such embedded discs are isomorphic. In particular, we determine when such an algebra is isomorphic to H^∞. We also show that the classical scale of complete Nevanlinna-Pick spaces $\mathcal{H}_s$, where $s \in [-1, 0]$, gives rise to uncountably multiplier biholomorphic varieties whose multiplier algebras are not isomorphic.

In Section 3.8, we show that if we extend the scale $\mathcal{H}_s$ to $s < -1$, then we obtain a family of varieties in $\mathbb{B}_\infty$ which are homeomorphic to the compact unit disc. Again, we determine when two multiplier algebras associated to compact embedded discs of a special type are isomorphic.

In Section 3.9, we use interpolating sequences to show that no multiplier algebra on one of these compact embedded discs can be isomorphic to a multiplier algebra from Section 3.7.

3.2. Multipliers on discs and automorphism invariance

Let $f : \mathbb{D} \to V = f(\mathbb{D}) \subset \mathbb{B}^d$ be a proper holomorphic map. In the case of $d < \infty$, it is well known that if f is injective and $f'(z) \neq 0$ for all $z \in \mathbb{D}$, then the complex structure on V as a subset of $\mathbb{C}^d$ coincides with the complex structure induced from the homeomorphism with $\mathbb{D}$. We require the analogous result for the case $d = \infty$.

A function $f : \Omega_1 \to \Omega_2$ between two open balls of two Hilbert spaces is said to be holomorphic if it is Fréchet differentiable at every point. Equivalently, f is holomorphic if around every point in Ω_1 there is some neighborhood in which f is represented by a convergent (vector valued) power series. Background material on holomorphic functions on infinite dimensional domains can be found in [47, Section III.3].

Suppose that $V, W \subset \ell^2$. A function $h : V \to \ell^2$ will be called holomorphic if for every $v \in V$, there is a ball $b_r(v)$ in ℓ^2 and a holomorphic function g on $b_r(v)$ such that $g|_{V \cap b_r(v)} = h|_{V \cap b_r(v)}$. A bijective map f between V and W will be called a biholomorphism provided that both f and f^{-1} are holomorphic.

The following definition is not standard so it is singled out.

Definition 3.2.1. We say that a map f from the unit disc into the open unit ball of a Hilbert space is proper if $\lim_{|z| \to 1} \|f(z)\| = 1$.

When the target space is finite dimensional this definition agrees (in this setting) with the standard definition of "proper map", which is that f is proper if the preimage of every compact set is compact. We require this definition for dealing with maps into infinite dimensional balls.

The following result is well known when the range is contained in $\mathbb{C}^d$ for $d < \infty$. Since we do not have a convenient reference when $d = \infty$, a proof is provided below. We let $\mathbb{B}_\infty$ denote the open unit ball of ℓ^2.

Proposition 3.2.2. *Let $f : \mathbb{D} \to V = f(\mathbb{D}) \subset \mathbb{B}_\infty$ be a proper injective holomorphic function such that $f'(z) \neq 0$ for $z \in \mathbb{D}$. Then f^{-1} is holomorphic. More generally, a function $h : V \to \mathbb{C}$ is holomorphic if and only if $h \circ f$ is holomorphic.*

Proof. Fix $v_0 = f(z_0) \in V$. As $f'(z_0) \neq 0$, we can define

$$P : \mathbb{B}_\infty \to \mathbb{D}, \quad z \mapsto \left\langle z, \frac{f'(z_0)}{||f'(z_0)||} \right\rangle.$$

Then $P \circ f$ is an analytic function on the disc with non-zero derivative at z_0, hence $P \circ f$ is injective in a neighbourhood of z_0.

We claim that there is an $r > 0$ so that P is injective on $b_r(v_0) \cap V$. Assume toward a contradiction that P is not injective in any neighbourhood of v_0 in V. Then there are sequences w_n and $\tilde{w}_n$ in V which converge to v_0 with $w_n \neq \tilde{w}_n$ and $Pw_n = P\tilde{w}_n$. Write $w_n = f(z_n)$ and $\tilde{w}_n = f(\tilde{z}_n)$, and note that $z_n \neq \tilde{z}_n$. Properness of f implies that z_n and $\tilde{z}_n$ are contained in a disc of radius $r < 1$, so by passing to a subsequence, we may assume that $z_n \to z$ and $\tilde{z}_n \to \tilde{z}$ for points $z, \tilde{z}$ in the disc. Thus $f(z) = f(\tilde{z}) = v_0$. Since f is injective, it follows that $z = \tilde{z} = z_0$. But $(Pf)(z_n) = (Pf)(\tilde{z}_n)$, which contradicts the fact that Pf is injective in a neighbourhood of z_0.

Since $P \circ f$ has non-zero derivative at z_0, there exists $\varepsilon > 0$ such that $P \circ f$ is a biholomorphism of $b_\varepsilon(z_0)$ onto its image. By shrinking ε if necessary, we may further assume that $f(b_\varepsilon(z_0)) \subset b_r(v_0)$. Since $Pf(b_\varepsilon(z_0))$ is open, we can find r_0 with $0 < r_0 \leq r$ such that

$$Pb_{r_0}(v_0) \subset Pf(b_\varepsilon(z_0)).$$

Then $g = (Pf\big|_{b_\varepsilon(z_0)})^{-1}P$ is an analytic function on $b_{r_0}(v_0)$.

We claim that g agrees with f^{-1} on $V \cap b_{r_0}(v_0)$. To this end, let $w \in V \cap b_{r_0}(v_0)$. Then $P(w) \in Pb_{r_0}(v_0) \subset Pf(b_\varepsilon(z_0))$, so there exists $z \in b_\varepsilon(z_0)$ such that $Pf(z) = Pw$. Since $f(b_\varepsilon(z_0)) \subset b_r(v_0)$ and since P is injective on $b_r(v_0)$, it follows that $w = f(z)$. Hence $g(w) = (Pf\big|_{b_\varepsilon(z_0)})^{-1}Pf(z) = z = f^{-1}(w)$, as asserted.

The additional claim now follows from the fact that the composition of holomorphic functions is holomorphic. □

The following consequence is well known if the range is contained in $\mathbb{B}_d$ for $d < \infty$.

Corollary 3.2.3. *If $f : \mathbb{D} \to V = f(\mathbb{D}) \subset \mathbb{B}_\infty$ is a proper injective holomorphic function such that $f'(z) \neq 0$ for all $z \in \mathbb{D}$, then the space $H^\infty(V)$ of bounded analytic functions on V coincides with $\{h \circ f^{-1} : h \in H^\infty\}$.* □

If V is a variety in $\mathbb{B}_d$, then the algebra $\mathcal{M}_V$ is an algebra of functions on the variety V. Thus, every $v \in V$ gives rise to the character δ_v of evaluation at v. These are precisely the weak-$*$ continuous characters on $\mathcal{M}_V$ by [25, Proposition 3.2], and we will identify V with a subset of the maximal ideal space $\mathcal{M}(\mathcal{M}_V)$ in this way. As noted at the end of Section 2.2, the point evaluations typically only form a small subset of the maximal ideal space.

Since the coordinate functions in $\mathcal{M}_V$ form a row contraction and since characters are completely contractive, there is a map

$$\pi : \mathcal{M}(\mathcal{M}_V) \to \overline{\mathbb{B}_d}, \quad \rho \mapsto (\rho(z_i))_{i=1}^d.$$

It follows from [21, Theorem 3.2] that if $d < \infty$, then the characters in $\pi^{-1}(\mathbb{B}_d)$ are precisely the point evaluations. Unfortunately, this theorem is not true for $d = \infty$, as the following example shows.

Example 3.2.4. Let (v_n) be a sequence in $\mathbb{B}_\infty$ with the property that $||v_n|| \to 1$, but (v_n) converges weakly to zero. By passing to a subsequence, we may assume that (v_n) is interpolating for $\operatorname{Mult}(H^2_\infty)$ (see Proposition 3.9.1 below). Thus, the unital homomorphism $\Phi : \operatorname{Mult}(H^2_\infty) \to \ell^\infty$ defined by $\Phi(f)(n) = f(v_n)$ is surjective, so its adjoint Φ^* is an embedding of the Stone-Čech compactification $\beta\mathbb{N}$ into the character space of $\operatorname{Mult}(H^2_\infty)$. We claim that every point in $\beta\mathbb{N}\backslash\mathbb{N}$ lies in the fiber over the origin, i.e., $\pi(\Phi^*(\beta\mathbb{N}\backslash\mathbb{N})) = \{0\}$. Indeed, let $\rho \in \beta\mathbb{N} \setminus \mathbb{N}$. Then for every $k \geq 1$, we have

$$(\Phi^*(\rho))(z_k) = \rho((z_k(v_n))) = \lim_{n\to\infty} z_k(v_n) = 0.$$

In particular, we see that there are points in $\pi^{-1}(\mathbb{B}_\infty)$ which are not point evaluations.

We can also use this construction to show that there are algebras $\mathcal{M}_V$ with characters that are fibered over points in $\mathbb{B}_\infty \setminus V$. Let (v_n) be as above, and assume that $v_0 = 0$. Let $f \in \operatorname{Mult}(H^2_\infty)$ satisfy $f(0) = 1$ and $f(v_n) = 0$ for $n \geq 1$. Then $V = f^{-1}(0)$ is a variety such that $0 \notin V$, but the fiber $\pi^{-1}(0)$ contains a copy of $\beta\mathbb{N} \setminus \mathbb{N}$.

We also need a few variants of results in [25]. Consider two biholomorphisms of discs

$$f_i : \mathbb{D} \to V_i = f_i(\mathbb{D}) \subset \mathbb{B}_{d_i} \quad \text{for} \quad i = 1, 2$$

such that V_i are varieties. We allow the case $d_i = \infty$. Suppose that $\Phi : \mathcal{M}_{V_1} \to \mathcal{M}_{V_2}$ is an algebra homomorphism, and let Φ^* be the induced map from $\mathcal{M}(\mathcal{M}_{V_2})$ to $\mathcal{M}(\mathcal{M}_{V_1})$. Composing Φ^* with the evaluation map π at the row contraction $(z_1, \ldots, z_{d_i})$ yields a map $F_\Phi = \pi \circ \Phi^* : \mathcal{M}(\mathcal{M}_{V_2}) \to \overline{\mathbb{B}_{d_i}}$ given by

$$F_\Phi(\rho) = \big(\rho(\Phi(z_i))\big)_{i=1}^{d_i} \quad \text{for} \quad \rho \in \mathcal{M}(\mathcal{M}_{V_2}).$$

In particular, $F_\Phi|_{V_2}$ maps the variety V_2 into $\overline{\mathbb{B}_{d_i}}$.

Theorem 3.2.5. *Let V_1, V_2 be discs in $\mathbb{B}_{d_i}$ as described above. Furthermore, assume that*

(1) *for every $\lambda \in V_1$, the fiber $\pi^{-1}(\lambda) = \{\delta_\lambda\}$, and*

(2) *$\pi(\mathcal{M}(\mathcal{M}_{V_1})) \cap \mathbb{B}_{d_1} = V_1$.*

Let $\Phi : \mathcal{M}_{V_1} \to \mathcal{M}_{V_2}$ be an algebra homomorphism. Then $F = F_\Phi|_{V_2}$ is a holomorphic map with multiplier coefficients. If F is not constant, then F maps V_2 into V_1. In this case, $\Phi^*|_{V_2} = F$ and Φ is given by composition with F, that is,

$$\Phi(h) = h \circ F \quad \textit{for all} \quad h \in \mathcal{M}_{V_1}.$$

In particular, if Φ is injective, then F is not constant. And if Φ is an isomorphism, F is a biholomorphism of V_2 onto V_1.

Proof. Let $F_i = \Phi(z_i)$ for $1 \leq i \leq d_1$. For $v \in V_2$, let δ_v denote the character of evaluation at v. Then

$$F(v) = \pi(\Phi^*(\delta_v)) = \big(\delta_v(\Phi(z_i))\big)_{i=1}^{d} = \big(F_i(v)\big)_{i=1}^{d_1}.$$

Observe that the coefficients F_i are all multipliers. We remarked above that F maps V_2 into $\overline{\mathbb{B}_{d_1}}$.

In a next step, we show that $F \circ f_2$ is holomorphic. If $d_1 < \infty$, this is clear since the functions $h_i = F_i \circ f_2$ are. If $d_1 = \infty$, let $\alpha = (a_i)_{i=1}^{\infty} \in \ell^2$. Then

$$\langle F \circ f_2(z), \alpha \rangle = \sum_{i=1}^{\infty} \bar{a}_i h_i(z).$$

This converges uniformly on V_2 since by the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{n=N}^{\infty} |\bar{a}_n h_n(z)| &\leq \Big(\sum_{n=N}^{\infty} |a_n|^2 \Big)^{1/2} \Big(\sum_{n=N}^{\infty} |h_n(z)|^2 \Big)^{1/2} \\ &\leq \Big(\sum_{n=N}^{\infty} |a_n|^2 \Big)^{1/2} \xrightarrow{N \to \infty} 0.\end{aligned}$$

Therefore $\langle F \circ f_2(v), \alpha \rangle$ is holomorphic for all α, so $F \circ f_2$ is holomorphic. Since f_2 is a biholomorphism, F is holomorphic.

Now we assume that F is not constant, and show that F maps into $\mathbb{B}_{d_1}$. If $\mu = F(\lambda)$ lies in the boundary $\partial\mathbb{B}_d$ for some $\lambda \in V_2$, then $\langle F \circ f_2(z), \mu \rangle$ is a holomorphic function into $\overline{\mathbb{D}}$ which takes the value 1 at a point in $\mathbb{D}$. By the maximum modulus principle, this function is constant. Since the image of F is contained in the closed unit ball, $F \circ f_2$ itself and thus F must be constant. This contradicts our assumption.

Now for $v \in V_2$, $\Phi^*(\delta_v)$ is fibered over the point $F(v)$, which lies in $\mathbb{B}_{d_1}$. By hypotheses (1) and (2), the characters of $\mathcal{M}_{V_1}$ in $\pi^{-1}(\mathbb{B}_{d_1})$ are precisely the point evaluations at points of V_1. Hence F maps V_2 into V_1. Therefore

$$\Phi(h)(v) = \Phi^*(\delta_v)(h) = \delta_{F(v)}(h) = h(F(v))$$

for all $h \in \mathcal{M}_{V_1}$ and $v \in V_2$.

If Φ is injective, it follows as in [25, Lemma 5.4(2)] that F maps V_2 into V_1. The argument there assumed that Φ is an isomorphism, but only injectivity is required. To recall, suppose that F maps V_2 to a single point $\lambda \in \overline{\mathbb{B}_{d_1}}$. Then for every i, we have

$$\Phi(\lambda_i - z_i) = \lambda_i - F_i = 0,$$

hence $z_i = \lambda_i \in \mathcal{M}_{V_1}$ by injectivity of Φ. This is clearly impossible as V_1 consists of more than one point. Therefore F is not constant.

Now assume that Φ is an isomorphism and recall that Φ is automatically a topological isomorphism in the norm topologies. By an adaptation of [24, Section 11.3], the fact that Φ is implemented by composition implies that Φ is weak-$*$ continuous. Since the closed unit ball B_1 of $\mathcal{M}_{V_1}$ is weak-$*$ compact, and since the weak-$*$ topology on $\mathcal{M}_{V_2}$ is Hausdorff, $\Phi\big|_{B_1} : B_1 \to \Phi(B_1)$ is a homeomorphism in the weak-$*$ topologies. Every bounded set in $\mathcal{M}_{V_2}$ is contained in $r\Phi(B_1)$ for some $r > 0$, hence Φ^{-1} is weak-$*$ continuous on bounded sets. It follows from the Krein-Smulian theorem that Φ^{-1} is weak-$*$ continuous. In particular, $(\Phi^{-1})^*$ takes point evaluations to point evaluations.

We deduce that $\Phi^*(V_2) = V_1$, hence F maps V_2 onto V_1. Since $F^{-1} = \pi \circ (\Phi^{-1})^*$, the map F^{-1} is holomorphic with multiplier coefficients. □

Remark 3.2.6. The special hypotheses on the algebra $\mathcal{M}_{V_1}$ always hold when $d_1 < \infty$ by [25, Proposition 3.2]. Proposition 3.2.8 below shows that even when $d_1 = \infty$, condition (2) holds in many cases of interest.

Remark 3.2.7. Besides the special assumptions on $\mathcal{M}_{V_1}$, another issue which makes this a weaker result than Theorem 3.1.1 in the case $d_1 = \infty$ is that we do not know if the map F can be extended to a map from $\mathbb{B}_{d_2}$ into ℓ_2. Better yet, we would like $(F_1, F_2, \ldots) = (\Phi(z_1), \Phi(z_2), \ldots)$ to be a bounded vector-valued multiplier. In this case, $(F_1, F_2, \ldots)$ would extend to a bounded multiplier from $H^2_{d_2} \otimes \ell_2$ into $H^2_{d_2}$, since the restriction map from $\mathrm{Mult}(H^2_{d_2})$ to $\mathcal{M}_{V_2}$ is a complete quotient map.

Observe that if Φ is assumed to be completely bounded, then F is indeed a bounded vector-valued multiplier, as $(z_1, z_2, \ldots)$ is a row contraction. However, if Φ is merely assumed to be bounded, we only know that each F_i extends to a multiplier of $H^2_{d_2}$, but the resulting F does not obviously extend to a bounded map on $\mathbb{B}_{d_2}$.

At least the second condition in the last result holds in many cases of interest. We let $\mathcal{A}_d$ denote the norm closure of the polynomials in $\mathrm{Mult}(H^2_d)$.

Proposition 3.2.8. *Suppose that a variety V in $\mathbb{B}_d$ is the intersection of zero sets of a family $\mathcal{F} \subset \mathcal{A}_d$. Then $\pi(\mathcal{M}(\mathcal{M}_V)) \cap \mathbb{B}_d = V$.*

Proof. Since every point $v \in V$ gives rise to the character of evaluation at v, one inclusion is obvious. To prove the other inclusion, let ρ be a character on $\mathcal{M}_V$ such that $\lambda = \pi(\rho) \in \mathbb{B}_d$. Then

$$\rho(f\big|_V) = f(\rho(z_1), \dots, \rho(z_d)) = f(\lambda)$$

holds for every polynomial f, and hence for every $f \in \mathcal{A}_d$. In particular, as every $f \in \mathcal{F}$ vanishes on V, we deduce that $f(\lambda) = 0$ for all $f \in \mathcal{F}$. Therefore λ belongs to V. □

Remark 3.2.9. When the functions $\mathcal{F}$ defining V belong to $\mathcal{A}_d$, they extend to be continuous on the closed ball. It follows by the same argument that if $\|\lambda\| = 1$ and a character ρ belongs to $\pi^{-1}(\lambda)$, then $f(\lambda) = 0$ for every $f \in \mathcal{F}$. Hence

$$\lambda \in \{z \in \overline{\mathbb{B}_d} : f(z) = 0 \text{ for all } f \in \mathcal{F}\}.$$

In particular, if the set on the right equals $\overline{V}$, we see that $\pi(\mathcal{M}(\mathcal{M}_V)) = \overline{V}$. This is of interest even when $d < \infty$ (cf. [54, Corollary 5.4]).

It is well known that the conformal automorphisms of the unit disc are the Möbius maps

$$\theta : \mathbb{D} \to \mathbb{D}, \quad z \mapsto \lambda \frac{z-a}{1-\overline{a}z},$$

for $a \in \mathbb{D}$ and $|\lambda| = 1$. Moreover, the automorphisms of H^∞ are precisely the maps $C_\theta h = h \circ \theta$. This familiar result is credited to Kakutani in [50, p.143].

If $f : \mathbb{D} \to V = f(\mathbb{D}) \subset \mathbb{B}_d$ is a biholomorphic map onto a variety V, then we can transfer the Möbius maps to conformal automorphisms of V by sending θ to $f \circ \theta \circ f^{-1}$. Since f is a biholomorphism, these are precisely the conformal automorphisms of V. We say that $\mathcal{M}_V$ is *automorphism invariant* if composition with all of these conformal maps yield automorphisms of $\mathcal{M}_V$. A sufficient criterion for automorphism invariance is given in [12, Theorem 3.5]. For further discussion of this property, the reader is referred to Section 8 in [13].

Corollary 3.2.10. *Let V_1, V_2 be discs in $\mathbb{B}_d$ as described above such that V_1 satisfies conditions* (1) *and* (2) *of Theorem* 3.2.5. *Let $\Phi : \mathcal{M}_{V_1} \to \mathcal{M}_{V_2}$ be an algebra isomorphism. Then there is a Möbius map θ of $\mathbb{D}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}_{V_1} & \xrightarrow{\Phi} & \mathcal{M}_{V_2} \\ \downarrow{\scriptstyle C_{f_1}} & & \downarrow{\scriptstyle C_{f_2}} \\ H^\infty & \xrightarrow{C_\theta} & H^\infty \end{array}$$

Proof. By Theorem 3.2.5, $F = \Phi^*|_{V_2}$ is a biholomorphism of V_2 onto V_1, and Φ is implemented by composition with F. We will make use of the fact that $\mathcal{M}_{V_i}$ can be embedded into H^∞ via

$$C_{f_i} h = h \circ f_i \quad \text{for} \quad h \in \mathcal{M}_{V_i}.$$

This map is contractive since the multiplier norm on $\mathcal{M}_{V_i}$ dominates the supremum norm. Observe that $\theta = f_1^{-1} \circ F \circ f_2$ is a biholomorphism of $\mathbb{D}$ onto itself, and thus is a Möbius map. Clearly this makes the diagram commute. □

Suppose that the automorphism θ can be chosen to be the identity or, equivalently, that C_F, where $F = f_1 \circ f_2^{-1}$, is an isomorphism of $\mathcal{M}_{V_1}$ onto $\mathcal{M}_{V_2}$. Then we will say that $\mathcal{M}_{V_1}$ and $\mathcal{M}_{V_2}$ are *isomorphic via the natural map.*

Corollary 3.2.11. *Let V_1, V_2 be discs in $\mathbb{B}_d$ as described above such that V_1 satisfies conditions* (1) *and* (2) *of Theorem* 3.2.5. *If $\mathcal{M}_{V_1}$ or $\mathcal{M}_{V_2}$ is automorphism invariant, then $\mathcal{M}_{V_1}$ and $\mathcal{M}_{V_2}$ are isomorphic if and only if they are isomorphic via the natural map C_F, where $F = f_1 \circ f_2^{-1}$.*

In particular, if $\mathcal{M}_{V_2}$ is isomorphic to H^∞, then C_{f_2} implements the isomorphism.

Proof. The first paragraph is immediate from the preceding corollary. To deduce the additional statement, we let V_1 be the variety defined by the monomials $\{z_2, z_3, \ldots\}$. Then $\mathcal{M}_{V_1}$ is naturally identified with H^∞ and V_1 satisfies conditions (1) and (2) of Theorem 3.2.5. □

3.3. Transversality

Recall that a map of $\mathbb{D}$ into a ball $\mathbb{B}_d$ is proper if $\lim_{|z|\to 1} \|f(z)\| = 1$. If a proper analytic map $f : \mathbb{D} \to \mathbb{B}_d$ extends to be C^1 on $\overline{\mathbb{D}}$, we will say that the image meets the boundary of $\mathbb{B}_d$ *transversally* at $f(z)$ for $z \in \partial\mathbb{D}$ provided that

$$\langle f(z), f'(z) \rangle \neq 0.$$

As noted in the introduction, transversality at the boundary is a hypothesis needed in the theorem of Alpay, Putinar and Vinnikov. In this section, we show that a proper analytic C^1 embedding of the unit disc automatically meets the boundary transversally.

Proposition 3.3.1. *Let $f : \mathbb{D} \to \mathbb{B}_d$ be an analytic map which extends to be continuous at 1 such that $||f(1)|| = 1$. Then*

$$\frac{\mathrm{Re}\langle f(1) - f(z), f(1)\rangle}{1 - |z|} \geq \frac{1 - |\langle f(z), f(1)\rangle|}{1 - |z|} \geq \frac{1 - |a|}{1 + |a|} > 0$$

for all $z \in \mathbb{D}$, where $a = \langle f(0), f(1)\rangle$. We have

$$L = \liminf_{z \to 1, z \in \mathbb{D}} \frac{1 - |\langle f(z), f(1)\rangle|}{1 - |z|} < \infty$$

if and only if the non-tangential limit of

$$\frac{\langle f(1) - f(z), f(1)\rangle}{1 - z}$$

as $z \to 1$ exists. In this case, this limit equals L. In particular, if f extends to be differentiable at 1, then $\langle f(1), f'(1)\rangle > 0$.

Proof. Consider the holomorphic function

$$g : \mathbb{D} \to \mathbb{D}, \quad z \mapsto \langle f(z), f(1)\rangle.$$

An application of the Schwarz-Pick lemma (compare the discussion following Corollary 2.40 in [12]) shows that

$$\frac{1 - |g(z)|}{1 - |z|} \geq \frac{1 - |g(0)|}{1 + |g(0)|} \quad \text{for all } z \in \mathbb{D},$$

from which the first claim readily follows.

The second claim is a direct consequence of the Julia-Carathédory theorem [12, Theorem 2.44]. It follows from the first part that $L > 0$. In particular, if f extends to be differentiable at 1, then

$$\langle f'(1), f(1)\rangle = \lim_{z \to 1} \frac{\langle f(1) - f(z), f(1)\rangle}{1 - z} = L > 0,$$

so that f meets the boundary transversally at $f(1)$. □

The following consequence is immediate.

Corollary 3.3.2. *If $f : \mathbb{D} \to \mathbb{B}_d$ is a proper analytic map which extends to be C^1 on $\overline{\mathbb{D}}$, then $f(\mathbb{D})$ meets the boundary transversally. Indeed, $\langle f(z), f'(z)z\rangle > 0$ for all $z \in \partial\mathbb{D}$.*

Let us examine the geometric meaning of Corollary 3.3.2. For every $n \in \mathbb{N}$ the space $\mathbb{C}^n$ carries the structure of a $2n$-dimensional real Hilbert space with inner product

$$\langle u, v\rangle_{\mathbb{R}} = \operatorname{Re}\langle u, v\rangle.$$

Let f be as in the corollary, and let us assume for simplicity that f extends analytically to an open neighbourhood U of $\overline{\mathbb{D}}$. The derivative $f'(z)$ is a linear map from the complex tangent space of $\mathbb{C}$ at z (which can be identified with $\mathbb{C}$) into the complex tangent space of $f(U)$ at $f(z)$ (which can be identified with a subspace of $\mathbb{C}^d$ of complex dimension 1). Every $z \in \partial\mathbb{D}$ also serves as the outward pointing normal vector of the real submanifold $\partial\mathbb{D}$ at the point z. The derivative $f'(z)$ maps z to the vector $f'(z)z$.

Intuitively, a curve $f(\overline{\mathbb{D}})$ is transversal to $\partial\mathbb{B}_d$ at $f(z)$ (for $z \in \partial\mathbb{D}$) if the real valued inner product of the tangent vector to the curve at $f(z)$ with the outward pointing normal vector at $f(z)$ is positive. But since the outward pointing normal of $\partial\mathbb{B}_d$ at $f(z)$ is (colinear with) $f(z)$, this boils down to the condition $\langle f(z), f'(z)z\rangle_{\mathbb{R}} = \operatorname{Re}\langle f(z), f'(z)z\rangle > 0$. Corollary 3.3.2 gives slightly more information.

The following proposition and corollary clarify further the geometric meaning of Proposition 3.3.1 and Corollary 3.3.2.

Proposition 3.3.3. *Let φ be a differentiable map from the interval $[0,1]$ into the closed unit ball B of a real Hilbert space such that $\|\varphi(1)\| = 1$ and $\langle \varphi'(1), \varphi(1)\rangle > 0$. Then for x near 1*

$$\|\varphi(1) - \varphi(x)\| \sim 1 - \|\varphi(x)\| \sim 1 - x.$$

Here we use the notation $a(x) \sim b(x)$ to mean $\lim_{x\to 1} \frac{a(x)}{b(x)} = c \in (0, \infty)$.

Proof. By differentiability

$$\|\varphi(1) - \varphi(x)\| = \|\varphi'(1)(x-1) + o(1-x)\| \sim 1 - x,$$

since $\varphi'(1) \neq 0$. Moreover $1 - \|\varphi(x)\| \sim 1 - \|\varphi(x)\|^2$ and

$$1 - \|\varphi(x)\|^2 = 1 - \|\varphi(1) + \varphi'(1)(x-1) + o(x-1)\|^2 = 2\langle \varphi'(1), \varphi(1)\rangle(1-x) + o(1-x),$$

and the latter is $\sim 1 - x$. □

Corollary 3.3.4. *Suppose that f is a proper analytic map of $\mathbb{D}$ into a ball $\mathbb{B}_d$, and that f extends to $\mathbb{D} \cup \{1\}$ and is differentiable at 1. Then there exist $c > 0$ such that for all $x \in (0,1)$,*

$$c \leq \frac{\operatorname{dist}(f(x), \partial\mathbb{B}_n)}{\|f(1) - f(x)\|} = \frac{1 - \|f(x)\|}{\|f(1) - f(x)\|} \leq 1. \qquad \square$$

3.4. Tangential embedding

Following the discussion in the previous section we ask: can a proper biholomorphic embedding of the disc into the ball that extends continuously to the boundary meet the sphere tangentially? Proposition 3.3.1 shows that $\langle f'(1), f(1)\rangle$ is always bounded away from 0, when f extends to be differentiable at 1. And the Julia-Carathéodory Theorem shows that differentiability (at least in the direction of $f(1)$) is equivalent to having a bounded differential quotient along some approach to the boundary point. So a possible reformulation of a tangential condition might be that

$$\lim_{x\to 1, x\in(0,1)} \frac{\operatorname{Re}\langle f(1)-f(x), f(1)\rangle}{1-x} = +\infty.$$

A different formulation is used in [6]. They suggest that the tangential condition should be

$$\liminf_{x\to 1,\, x\in(0,1)} \frac{\operatorname{dist}(f(x), \partial\mathbb{B}_n)}{\|f(1)-f(x)\|} = \liminf_{x\to 1,\, x\in(0,1)} \frac{1-\|f(x)\|}{\|f(1)-f(x)\|} = 0.$$

If this is an actual limit, this intuitively says that as x approaches 1 along the real axis, the curve $f(x)$ approaches the boundary much more quickly than it approaches $f(1)$, and hence must approach $f(1)$ along a curve tangent to the boundary.

Corollary 3.3.4 shows that if f is holomorphic and differentiable at 1, then the curve $f(x)$ cannot approach $\partial\mathbb{B}_d$ tangentially in either of these senses. We have been unable to determine a relationship between these two tangential conditions.

We now construct an example of a continuous proper embedding of a disc into $\mathbb{B}_2$ which meets the boundary tangentially in both of these senses. Unfortunately we have been unable to determine whether the multiplier algebra is isomorphic to H^∞.

Example 3.4.1. The following construction is a modification of an example shown to us by Josip Globevnik. There is a proper embedding F of $\mathbb{D}$ into $\mathbb{B}_2$ which extends to be continuous on $\overline{\mathbb{D}}$ such that

$$\lim_{x\to 1,\, x\in(0,1)} \frac{1-\|F(x)\|}{\|F(1)-F(x)\|} = 0,$$

and

$$\lim_{x\to 1,\, x\in(0,1)} \frac{\operatorname{Re}\langle F(1)-F(x), F(1)\rangle}{1-x} = +\infty.$$

Let A be the region in the upper half plane bounded by two semicircles in the upper half of the unit disc which are tangent at 1, and have radii $r_1 = \frac{1}{2}$ and $r_2 = \frac{3}{4}$ together with the line segment $[-\frac{1}{2}, 0]$. The closure of A is shown in Figure 3.1.

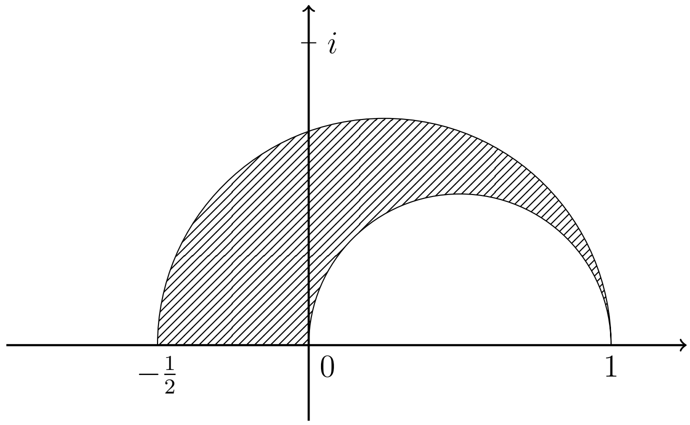

Figure 3.1.: The closure of A

Let f be a conformal map of $\mathbb{D}$ onto A such that $f(1) = 1$. For definiteness, we may assume that $f(-i) = 0$ and $f(i) = -\frac{1}{2}$.

The map f can be achieved by the following sequence of conformal maps. First apply the Möbius map $w \to \frac{w+i}{iw+1}$ which takes $-i$ to 0, i to ∞, carries $\mathbb{D}$ onto the upper half plane, takes 1 to 1, and is analytic in a neighbourhood of 1. Then take the square root map onto the first quadrant, followed by the Möbius map $w \to \frac{w-1}{w+1}$ which carries the quadrant onto the upper half disc. Call the composition of these maps g. Then g maps the disc onto the upper half disc, g takes 1 to 0, and is still analytic in a neighbourhood of 1; and $g(\pm i) = \pm 1$. Now the standard branch of log (with cut along the negative imaginary axis) carries the region onto the half strip bounded by the negative real axis $(-\infty, 0]$, the line segment $[0, \pi i]$ and half line $(-\infty, \pi i]$ parallel to the real line. Then take a final Möbius map $w \to \frac{w-\pi i}{w+2\pi i}$. The composition of all these maps is the desired map f.

Observe that f extends to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{A}$ and satisfies $f(1) = 1$. The map g from $\mathbb{D}$ to the half circle is conformal in a neighbourhood of 1, so $g(e^{it}) \approx at$ where $g'(1) = -ia \neq 0$; in fact, $a = \frac{1}{4}$. Hence $\log g(e^{it}) \approx \log(at)$ for $t > 0$ and $\log g(e^{it}) \approx \log(a|t|) + \pi i$ for $t < 0$. So we obtain that

$$f(e^{it}) \approx \begin{cases} \frac{\log(at)-\pi i}{\log(at)+2\pi i} & \text{for } t > 0 \\ \frac{\log(a|t|)}{\log(a|t|)+3\pi i} & \text{for } t < 0 \end{cases}$$

Hence we may compute that

$$u(e^{it}) := \frac{1}{2} \log\left(1 - |f(e^{it})|^2\right) \approx -\log\log|t|^{-1}.$$

In particular, u is in $L^1(\mathbb{T})$.

Fix $2/3 < r < 1$, and define $\rho(z) = rz + 1 - r$. This maps $\mathbb{D}$ onto a disc of radius r tangent to $\mathbb{D}$ at 1. Therefore $f_1(z) = f(\rho(z))$ maps $\mathbb{D}$ conformally onto a region contained in A which extends to be analytic on a neighbourhood of $\overline{\mathbb{D}} \setminus \{1\}$. It is still true that

$$u_1(e^{it}) := \frac{1}{2} \log \left(1 - |f_1(e^{it})|^2\right)$$

belongs to L^1, but now it is C^∞ except at 1, where it goes to $-\infty$. Hence u_1 extends to a real harmonic function on $\overline{\mathbb{D}}$ which is smooth except at 1, where it goes to $-\infty$. Let $\tilde{u}_1$ be its harmonic conjugate. This is also smooth except at 1. Let $f_2(z) = e^{u_1 + i\tilde{u}_1}$. Then f_2 extends to be continuous on $\overline{\mathbb{D}}$ with $f_2(1) = 0$, and f_2 is smooth except at 1.

Now $|f_1(e^{it})|^2 + |f_2(e^{it})|^2 = 1$ on $\mathbb{T}$. It follows that $F(z) = (f_1(z), f_2(z))$ is a proper map of $\mathbb{D}$ into $\mathbb{B}_2$ that extends to be continuous on $\overline{\mathbb{D}}$, and smooth except at 1. Since f_1 is conformal, F is a biholomorphism of $\mathbb{D}$ onto its image.

It is easy to see that as z approaches 1, $F(z)$ approaches $(1, 0)$ tangentially in the sense that

$$\lim_{x \to 1,\, x \in (0,1)} \frac{1 - \|F(x)\|}{\|F(1) - F(x)\|} = 0.$$

A careful look at the estimates above shows that for $x \in (0, 1)$,

$$f(1-x) \sim \frac{\log(ax) - \frac{\pi}{2} i}{\log(ax) + \frac{3\pi}{2} i} \sim (1 - \frac{c_1}{\log^2 x}) + i \frac{c_2}{\log x}.$$

Hence

$$\mathrm{Re}\langle f(1) - f(x), f(1) \rangle \sim \frac{c_1}{\log^2(1-x)},$$

so that

$$\lim_{x \to 1,\, x \in (0,1)} \frac{\mathrm{Re}\langle F(1) - F(x), F(1) \rangle}{1 - x} = +\infty.$$

3.5. Crossing on the boundary

In this section, we will provide a method for constructing a smooth proper embedding of a disc into a ball such that the multiplier algebra is not isomorphic to H^∞. The idea is to have the boundary cross itself.

Theorem 3.5.1. *Suppose that $f : \mathbb{D} \to \mathbb{B}_d$ is a proper analytic map which satisfies*

(1) *$f|_{\mathbb{D}}$ is injective,*

(2) *f extends to a differentiable map on $\mathbb{D} \cup \{\pm 1\}$, and*

(3) *$f(1) = f(-1)$.*

Suppose that $V = f(\mathbb{D})$ is a variety (in the sense of [25]). Then $f^{-1} \notin \mathcal{M}_V$. In particular, the embedding

$$\mathcal{M}_V \to H^\infty, \quad h \mapsto h \circ f,$$

is not surjective.

Proof. We first make some first order estimates in order to approximate the kernel functions near $f(\pm 1)$. By Proposition 3.3.1, we have $\langle f'(1), f(1) \rangle > 0$. Furthermore, differentiability of f at 1 implies that for small $x > 0$, we have

$$f(1-x) = f(1) - x f'(1) + o(x).$$

Hence

$$\begin{aligned} 1 - \|f(1-x)\|^2 &= \|f(1)\|^2 - \|f(1-x)\|^2 \\ &= \langle f(1), f(1) - f(1-x) \rangle + \langle f(1) - f(1-x), f(1-x) \rangle \\ &= \langle f(1), x f'(1) + o(x) \rangle + \langle x f'(1) + o(x), f(1) + o(1) \rangle \\ &= 2x \langle f'(1), f(1) \rangle + o(x). \end{aligned}$$

Similarly, $\langle f'(-1), f(-1) \rangle < 0$ and for small y with $y > 0$,

$$f(-1+y) = f(-1) + y f'(-1) + o(y)$$

and

$$1 - \|f(-1+y)\|^2 = -2y \langle f'(-1), f(-1) \rangle + o(y).$$

Likewise, for small positive values of x and y, we obtain (using $f(1) = f(-1)$)

$$\begin{aligned} &1 - \langle f(1-x), f(-1+y) \rangle \\ &= 1 - \langle f(1) - x f'(1) + o(x), f(-1) + y f'(-1) + o(y) \rangle \\ &= 1 - \langle f(1), f(-1) \rangle - \langle f(1), y f'(-1) \rangle + \langle x f'(1), f(-1) \rangle + o(x+y) \\ &= x \langle f'(1), f(1) \rangle - y \langle f'(-1), f(-1) \rangle + o(x+y). \end{aligned}$$

Let $s > 0$ so that

$$0 < a := \langle f'(1), f(1) \rangle = -s \langle f'(-1), f(-1) \rangle$$

and set $y = sx$. We have that

$$\frac{(1-\|f(1-x)\|^2)(1-\|f(-1+sx)\|^2)}{\left|1-\langle f(1-x), f(-1+sx)\rangle\right|^2} = \frac{(2ax+o(x))(2ax+o(x))}{(2ax+o(x))^2} = 1+o(1). \tag{3.1}$$

Assume now for a contradiction that f^{-1} is a multiplier and let $C = ||f^{-1}||_{\mathcal{M}_V}$ and $h = f^{-1}/C$, so that $||f||_{\mathcal{M}_V} = 1$.

Applying the Pick condition (see Lemma 2.2.1) to h at the points $\{f(1-x), f(-1+sx)\}$, we see that the matrix

$$\begin{bmatrix} \frac{1-|h(f(1-x))|^2}{1-\|f(1-x)\|^2} & \frac{1-h(f(1-x))\overline{h(f(-1+sx))}}{1-\langle f(1-x),f(-1+sx)\rangle} \\ \frac{1-h(f(-1+sx))\overline{h(f(1-x))}}{1-\langle f(-1+sx),f(1-x)\rangle} & \frac{1-|h(f(-1+sx))|^2}{1-\|f(-1+sx)\|^2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1-C^{-2}(1-x)^2}{1-\|f(1-x)\|^2} & \frac{1+C^{-2}(1-x)(1-sx)}{1-\langle f(1-x),f(-1+sx)\rangle} \\ \frac{1+C^{-2}(1-x)(1-sx)}{1-\langle f(-1+sx),f(1-x)\rangle} & \frac{1-C^{-2}(1-sx)^2}{1-\|f(-1+sx)\|^2} \end{bmatrix}$$

is positive. Taking the determinant and clearing denominators yields

$$\begin{aligned}\big(C^2+(1-x)(1-sx)\big)^2\big(1-\|f(1-x)\|^2\big)\big(1-\|f(-1+sx)\|^2\big)\\ \leq \big(C^2-(1-x)^2\big)\big(C^2-(1-sx)^2\big)\left|1-\langle f(-1+sx), f(1-x)\rangle\right|^2.\end{aligned}$$

Using the estimate from (3.1) and letting x decrease to 0, we obtain

$$(C^2+1)^2 \leq (C^2-1)^2.$$

As this is false, we deduce that $f^{-1} \notin \mathcal{M}_V$. In particular, the coordinate function z is not in the range of the map in the additional statement. □

Now we show that a map with these properties can be obtained.

Theorem 3.5.2. *There is a rational function f with poles outside $\overline{\mathbb{D}}$ and values in $\mathbb{C}^2$ which satisfies the conditions of Theorem 3.5.1, meets $\partial\mathbb{B}_2$ transversally, and is one-to-one except for the fact that $f(-1) = f(1)$, and so that f is a biholomorphism. Then $V = f(\mathbb{D})$ is a variety (in the sense of [25]) such that $\mathcal{M}_V \subsetneqq H^\infty(V)$. In particular, f^{-1} is not a multiplier.*

Proof. Fix $0 < r < 1$, and let

$$b(z) = \frac{z - r}{1 - rz}.$$

Note that $b(\pm 1) = \pm 1$. Define

$$f(z) = \frac{1}{\sqrt{2}}\big(z^2, b(z)^2\big).$$

Then it is clear that f is a rational function with poles outside $\overline{\mathbb{D}}$. Since z and $b(z)$ are automorphisms of the disc, it is easy to see that $||f(z)|| < 1$ if $|z| < 1$ and $||f(z)|| = 1$ if $|z| = 1$.

Since f is analytic on a disc $(1+\varepsilon)\mathbb{D}$ for some $\varepsilon > 0$ and

$$V = f((1+\varepsilon)\mathbb{D}) \cap \mathbb{B}_2,$$

it follows that V is a variety [54]. By Proposition 3.3.1, V meets the boundary transversally at every point.

Note that the first coordinate of $f(z)$ is $z^2/\sqrt{2}$. Hence if $f(w) = f(z)$, we have $w = \pm z$. If $f(z) = f(-z)$, then $b(-z)^2 = b(z)^2$, which is easily seen to have solutions $z \in \{0, \pm 1\}$. Thus $f(-1) = f(1)$ is the only failure to be one-to-one. Moreover,

$$f'(z) = \frac{1}{\sqrt{2}}(2z, 2b(z)b'(z))$$

is never zero since the first coordinate vanishes only at $z = 0$, while

$$2b(0)b'(0) = -2r(1 - r^2) \neq 0.$$

So this map is a biholomorphism. It is now clear that the hypotheses of Theorem 3.5.1 are satisfied. Therefore, $\mathcal{M}_V \subsetneqq H^\infty(V)$ and indeed, f^{-1} is not a multiplier. □

Remark 3.5.3. The fact that f^{-1} is not a multiplier means that this approach will not yield counterexamples to the converse of Theorem 3.1.1.

Corollary 3.2.11 and the automorphism invariance of H^∞ yield the following consequence.

Corollary 3.5.4. *For V given in Theorem 3.5.2, $\mathcal{M}_V$ is not isomorphic to H^∞.*

3.6. Pseudohyperbolic distance

The pseudohyperbolic metric on $\mathbb{B}_d$ is defined by

$$d(z,w) = \|\varphi_w(z)\| = \|\varphi_z(w)\|$$

where φ_w is the conformal automorphism of $\mathbb{B}_d$ onto itself interchanging the points w and 0 given by

$$\varphi_w(z) = \frac{w - P_w z - (1-\|w\|^2)^{1/2}(1-P_w)z}{1-\langle z,w\rangle},$$

where P_w is the orthogonal projection of $\mathbb{C}^d$ onto $\mathbb{C}w$. By [75, Theorem 2.2.2(iv)],

$$d(z,w)^2 = 1 - \frac{(1-\|w\|^2)(1-\|z\|^2)}{|1-\langle w,z\rangle|^2} = 1 - \frac{|K(z,w)|^2}{K(w,w)K(z,z)}, \tag{3.2}$$

where $K(z,w) = (1-\langle z,w\rangle)^{-1}$ is the reproducing kernel of the Drury-Arveson space.

The Schwarz lemma [75, Theorem 8.1.4] in this context states that if F is a holomorphic map of $\mathbb{B}_d$ into $\mathbb{B}_e$, then

$$d(F(z),F(w)) \le d(z,w).$$

Lemma 3.6.1. *Let $V \subset \mathbb{B}_d$ be a variety and let $\lambda,\mu \in V$. Then*

$$d(\lambda,\mu) \le \|\delta_\lambda - \delta_\mu\|_{\mathcal{M}_V^*} \le 2d(\lambda,\mu).$$

Proof. The inequality $\|\delta_\lambda - \delta_\mu\| \le 2d(\lambda,\mu)$ was observed in [25, Lemma 5.3]. For completeness, by the Schwarz lemma

$$\left|\frac{\varphi(\lambda)-\varphi(\mu)}{1-\varphi(\lambda)\overline{\varphi(\mu)}}\right| \le d(\lambda,\mu),$$

for all φ with $||\varphi||_{\mathcal{M}_V} \le 1$ and it follows that

$$\|\delta_\lambda - \delta_\mu\| \le d(\lambda,\mu) \sup_{\|\varphi\|_{\mathcal{M}_V}\le 1} |1-\varphi(\lambda)\overline{\varphi(\mu)}| \le 2d(\lambda,\mu).$$

For the lower bound, let $r = d(\lambda,\mu)$. Equation (3.2) shows that the determinant of the matrix

$$\begin{bmatrix} K(\mu,\mu) & K(\mu,\lambda) \\ K(\lambda,\mu) & K(\lambda,\lambda)(1-r^2) \end{bmatrix}$$

is equal to 0, hence this matrix is positive semidefinite. Since $H^2_d\big|_V$ is a Nevanlinna-Pick space, it follows that there exists a multiplier φ in the unit ball of $\mathcal{M}_V$ such that $\varphi(\mu) = 0$ and $\varphi(\lambda) = r$. Thus, $||\delta_\lambda - \delta_\mu|| \ge r$. ∎

The following result shows that a biholomorphism $F : W \to V$ which induces an isomorphism between $\mathcal{M}_V$ and $\mathcal{M}_W$ is necessarily a bi-Lipschitz map with respect to the pseudohyperbolic distance.

Theorem 3.6.2. *Suppose that $\Phi : \mathcal{M}_V \to \mathcal{M}_W$ is an isomorphism given by composition with a biholomorphism $F : W \to V$. Then there are constants $c, C > 0$ such that*

$$c\, d(\lambda, \mu) \leq d(F(\lambda), F(\mu)) \leq C\, d(\lambda, \mu) \quad \textit{for all} \quad \lambda, \mu \in W.$$

Proof. Put $t = \|\Phi^{-1}\|^{-1}$, and denote by $(\mathcal{M}_V)_1$ and $(\mathcal{M}_W)_1$ the unit balls of $\mathcal{M}_V$ and $\mathcal{M}_W$. Then $t \cdot (\mathcal{M}_W)_1 \subseteq \Phi((\mathcal{M}_V)_1)$, so

$$\begin{aligned} \|\delta_{F(\lambda)} - \delta_{F(\mu)}\| &= \sup_{f \in (\mathcal{M}_V)_1} |\Phi(f)(\lambda) - \Phi(f)(\mu)| \\ &= \sup_{g \in \Phi((\mathcal{M}_V)_1)} |g(\lambda) - g(\mu)| \\ &\geq \sup_{g \in (\mathcal{M}_W)_1} |tg(\lambda) - tg(\mu)| \\ &= t\|\delta_\lambda - \delta_\mu\|. \end{aligned}$$

From the preceding lemma, we deduce that

$$t \cdot d(\lambda, \mu) \leq t \cdot \|\delta_\lambda - \delta_\mu\| \leq \|\delta_{F(\lambda)} - \delta_{F(\mu)}\| \leq 2d(F(\lambda), F(\mu)).$$

This gives one inequality with $c = t/2$. The other inequality follows by symmetry. □

Remark 3.6.3. The proof of Theorem 3.5.1 shows that

$$d(f(1-x), f(-1+sx))^2 = 1 - \frac{(1 - \|f(1-x)\|^2)(1 - \|f(-1+sx)\|^2)}{|1 - \langle f(1-x), f(-1+sx)\rangle|} = o(1).$$

That is, we have

$$\lim_{x \to 0^+} d(f(1-x), f(-1+sx)) = 0.$$

It follows that f does not induce an isomorphism between $\mathcal{M}_V$ and H^∞.

Moreover in the example in Theorem 3.5.2, an easy estimate shows that $\|f'(z)\| \geq \sqrt{2}$ on $\partial\mathbb{D}$. Since f' never vanishes, we have that $\inf_{z \in \mathbb{D}} \|f'(z)\| > 0$. Nevertheless, because of the crossing on the boundary, the previous paragraph shows that the pseudohyperbolic distance is not preserved up to a constant. Thus this property is not just a local condition.

Remark 3.6.4. Using Equation (3.2), it is not hard to see that given points $z, w \in \mathbb{B}_d$ and $\alpha, \beta \in \mathbb{D}$, the Pick matrix

$$\begin{bmatrix} K(z,z)(1-|\alpha|^2) & K(z,w)(1-\alpha\overline{\beta}) \\ K(w,z)(1-\beta\overline{\alpha}) & K(w,w)(1-|\beta|^2) \end{bmatrix}$$

is positive if and only if

$$d(\alpha, \beta) \le d(z, w).$$

This observation shows that the argument in the last remark and the original proof of Theorem 3.5.1 are very closely related.

Example 3.6.5. Consider the sequences

$$v_n = 1 - 1/n^2 \quad \text{and} \quad w_n = 1 - e^{-n^2} \quad \text{for} \quad n \ge 1,$$

and set $V = \{v_n\}_{n=1}^\infty$ and $W = \{w_n\}_{n=1}^\infty$. In [25, Example 6.2] these two varieties were examined, and it was shown that there exist $g, h \in H^\infty$ such that

$$h \circ g|_V = \mathrm{id}_V \quad \text{and} \quad g \circ h|_W = \mathrm{id}_W,$$

while at the same time, since W is interpolating and V is not, $\mathcal{M}_V$ and $\mathcal{M}_W$ are not isomorphic. Theorem 3.6.2 sheds new light on this example. Indeed, we can check that

$$d(v_n, v_{n+1}) = \frac{2n+1}{2n^2+2n} \to 0,$$

while

$$d(w_n, w_{n+1}) = \frac{1 - e^{-2n-1}}{1 + e^{-2n-1} - e^{-n^2-2n-1}} \to 1.$$

Thus the biholomorphisms g and h are not bi-Lipschitz on the varieties, hence they cannot induce an isomorphism.

The following result generalizes this example significantly.

Proposition 3.6.6. *Let $V = \{v_n\}$ be a Blaschke sequence in $\mathbb{D}$. Then there is an interpolating sequence $W = \{w_n\}$ and functions $g, h \in H^\infty$ such that*

$$g(v_n) = w_n \quad \textit{and} \quad h(w_n) = v_n \quad \textit{for all} \quad n \ge 1.$$

Proof. Let $b_a(z) = \frac{\bar{a}}{|a|} \frac{a-z}{1-\bar{a}z}$ for $a \in \mathbb{D}$. Define

$$\delta_n := \prod_{i \ne n} |b_{v_i}(v_n)|.$$

These values are positive because V is a Blaschke sequence. (Carleson's interpolation theorem shows that V is an interpolating sequence if and only if it is strongly separated, i.e. $\inf_{n\geq 1} \delta_n > 0$.) A result of Garnett [35, Theorem 4] (see [36, ch.VII, Exercise 9]) shows that if

$$|a_n| \leq \delta_n (1 + \log \delta_n^{-1})^{-2},$$

there is an $f \in H^\infty$ such that $f(v_n) = a_n$ for $n \geq 1$. Choose (a_n) with $a_n > 0$ satisfying these inequalities, and tending to 0 sufficiently fast that $w_n = 1 - a_n$ is an interpolating sequence. Then $g = 1 - f$ is the desired map. Finally, since W is an interpolating sequence, there is an $h \in H^\infty$ such that $h(w_n) = v_n$ for all $n \geq 1$. □

It is tempting to conjecture that a biholomorphism with multiplier coordinates between two varieties, which is also bi-Lipschitz with respect to the pseudohyperbolic distance d, induces an isomorphism. The following example shows that this fails.

Example 3.6.7. A Blaschke sequence $V = \{v_n\}$ is *separated* if

$$\inf_{m \neq n} d(v_m, v_n) > 0.$$

Interpolating sequences are separated, and are characterized by being strongly separated. However there are Blaschke sequences which are separated but not strongly separated, and thus are not interpolating. For such a sequence V, the maps constructed in Proposition 3.6.6 will be bi-Lipschitz in the pseudohyperbolic metric but the multiplier algebras are not isomorphic.

An explicit example of a separated but not interpolating sequence is given in [27]. Here is a related example which has the additional virtue of having 1 as the only limit point of the sequence. Let

$$v_{n,k} = (1 - 2^{-n}) e^{ik2^{-n}} \quad \text{for} \quad n \geq 1 \text{ and } 0 \leq k < 2^{n/2}.$$

Then set $V = \{v_{n,k} : n \geq 1,\ 0 \leq k < 2^{n/2}\}$. It is routine to verify that this satisfies the Blaschke condition and is separated. In order for the sequence to be interpolating, it is necessary that the measure $\mu = \sum_{n,k} (1 - |v_{n,k}|) \delta_{v_{n,k}}$ be a Carleson measure [36, Theorem VII.1.1]. This means that there is a constant C so that $\mu(S(I)) \leq C|I|$ for every arc $I \subset \mathbb{T}$, where

$$S(I) = \{re^{i\theta} : 1 - |I| \leq r < 1,\ e^{i\theta} \in I\}.$$

But μ is not a Carleson measure: for $p \geq 1$, let

$$S_p = S([0, 2^{-p})) = \{re^{i\theta} : 1 - 2^{-p} \leq r < 1,\ 0 \leq \theta < 2^{-p}\}.$$

Then

$$\frac{1}{2^{-p}} \sum_{v_{n,k} \in S_p} 1 - |v_{n,k}| = 2^p \sum_{n \geq p} 2^{-n} \min\{2^{n-p}, 2^{n/2}\}$$
$$\geq 2^p \sum_{n=p}^{2p} 2^{-n} 2^{n-p} = p + 1.$$

This is not bounded.

Remark 3.6.8. Proposition 3.6.6 raises a fundamental issue in finding a converse to Theorem 3.1.1. The property of having a multiplier biholomorphism between two varieties V and W is not an equivalence relation. The proposition shows that every Blaschke sequence is equivalent to some interpolating sequence. Moreover, examination of the proof shows that if $V = \{v_n\}$ and $X = \{x_n\}$ are Blaschke sequences, there is a common interpolating sequence W which is equivalent to both V and X.

However in general, there is no $h \in H^\infty$ such that $h(V) = X$. To see this, let

$$v_n = 1 - n^{-2} \quad \text{and} \quad x_n = (-1)^n v_n \quad \text{for} \quad n \geq 2.$$

Suppose that h exists. Let $C = \|h\|_\infty$ and $g = C^{-1}h$. Then there is an increasing sequence n_i so that $h(v_{n_i}) > 1/2$ and $h(v_{n_i+1}) < -1/2$. Then $d(v_{n_i}, v_{n_i+1})$ tends to 0, but

$$d(g(v_{n_i}), g(v_{n_i+1})) > d(\tfrac{1}{2C}, \tfrac{-1}{2C}) > 0.$$

This contradicts the Schwarz inequality.

The problem is that if Z is an interpolating sequence and if $g : V \to Z$ and $h : Z \to X$ are multiplier biholomorphisms, then $h \circ g$ may not be a multiplier. We can extend g and h to H^∞ functions on the whole disc, but these extensions do not have norm 1 in general, and thus do not map the disc into the disc. Hence, the extensions cannot be composed.

In this example, the varieties have infinitely many irreducible components. We do not know of any examples with finitely many irreducible components in a finite dimensional ball where multiplier biholomorphism does not imply isomorphism of the multiplier algebras. Obviously, isomorphism is an equivalence relation. Showing that multiplier biholomorphism is not an equivalence relation in this setting therefore requires a counterexample to the hoped-for converse of Theorem 3.1.1.

3.7. A class of discs in $\mathbb{B}_\infty$

We consider a class of embeddings of $\mathbb{D}$ into $\mathbb{B}_\infty$, which were studied in [25, Section 6]. Let $(b_n)_{n=1}^\infty \in \ell^2$ with $||(b_n)||_2 = 1$ and $b_1 \neq 0$. Define $f : \mathbb{D} \to \mathbb{B}_\infty$ by

$$f(z) = (b_1 z, b_2 z^2, b_3 z^3, \dots) \quad \text{for} \quad z \in \mathbb{D}.$$

Then f is a biholomorphism with inverse $g = b_1^{-1} z_1$, and these maps are multipliers. The range $V = f(\mathbb{D})$ is a variety in the sense of [25] because

$$V = \{z \in \mathbb{B}_\infty : b_n z_1^n - b_1^n z_n = 0 \text{ for } n \geq 2\}.$$

Moreover, f extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{V}$. It is easy to see that any two varieties of this type are multiplier biholomorphic.

Define a kernel on $\mathbb{D}$ by

$$K(z,w) = \frac{1}{1 - \langle f(z), f(w) \rangle} \quad \text{for} \quad z, w \in \mathbb{D},$$

and let $\mathcal{H}_f$ be the Hilbert function space on $\mathbb{D}$ with kernel K. It is easy to check that the map

$$U : H^2_\infty\big|_V \to \mathcal{H}_f, \quad h \mapsto h \circ f,$$

is unitary. Moreover, if $\varphi \in \mathcal{M}_V$, then $U M_\varphi U^* = M_{\varphi \circ f}$, hence composition with f also induces a unitarily implemented completely isometric isomorphism $C_f : \mathcal{M}_V \to \operatorname{Mult}(\mathcal{H}_f)$. This observation allows us to work with multiplier algebras of Hilbert function spaces on the disc instead of the algebras $\mathcal{M}_V$.

Thanks to the special form of f, there exists a sequence (a_n) of non-negative real numbers such that

$$K(z,w) = \frac{1}{1 - \sum_{n=1}^\infty |b_n|^2 (z\overline{w})^n} = \sum_{n=0}^\infty a_n (z\overline{w})^n.$$

Hence $\mathcal{H}_f$ is a weighted Hardy space. Background material on these spaces can be found in [13, Section 2.1] and [81, Section 6]. Set $c_n = |b_n|^2$. It was established in [25, Section 6] that the sequence (a_n) satisfies the recursion

$$a_0 = 1 \quad \text{and} \quad a_n = \sum_{k=1}^n c_k a_{n-k} \quad \text{for } n \geq 1. \tag{3.3}$$

Moreover, $a_n \in (0,1]$ for all $n \in \mathbb{N}$.

Remark 3.7.1. The coefficients (a_n) can also be determined in the following way. First, note that as $||(b_n)||_2 = 1$, the function g defined by

$$g(z) = \sum_{n=1}^{\infty} c_n z^n$$

is holomorphic on $\mathbb{D}$ and does not take the value 1 there. Evidently,

$$\frac{1}{1-g(z)} = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all} \quad z \in \mathbb{D}. \tag{3.4}$$

That is, (a_n) is the sequence of Taylor coefficients of $(1-g)^{-1}$ at the origin.

The special form of the kernel K allows us to explicitly compute the multiplier norm of monomials in $\mathcal{H}_f$.

Lemma 3.7.2. *Suppose that $\mathcal{H}$ is a reproducing kernel Hilbert space on $\mathbb{D}$ with kernel*

$$K(z,w) = \sum_{n=0}^{\infty} a_n (z\overline{w})^n,$$

where the sequence (a_n) satisfies a recursion as in (3.3) for some sequence of nonnegative numbers (c_n) with $c_1 \neq 0$. Then

$$||z^n||^2_{\mathrm{Mult}(\mathcal{H})} = ||z^n||^2_{\mathcal{H}} = \frac{1}{a_n} \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. The assumptions imply that $a_n \neq 0$ for all $n \in \mathbb{N}$, so from the general theory of weighted Hardy spaces, we have

$$||z^n||^2_{\mathcal{H}} = \frac{1}{a_n}.$$

Therefore for $n \in \mathbb{N}$,

$$||z^n||^2_{\mathrm{Mult}(\mathcal{H})} = \sup_{k \geq 0} \frac{\|z^{n+k}\|_{\mathcal{H}}}{\|z^k\|_{\mathcal{H}}} = \sup_{k \geq 0} \frac{a_k}{a_{n+k}}.$$

Since $a_0 = 1$, it suffices to show that

$$a_k a_n \leq a_{n+k} \quad \text{for all} \quad k, n \in \mathbb{N}.$$

The proof of this claim proceeds by induction on k. The base case holds since $a_0 = 1$. Assume that $k \geq 1$, and that the assertion has been established for natural numbers smaller than k. Then

$$a_k a_n = \sum_{i=1}^{k} a_{k-i} a_n c_i \leq \sum_{i=1}^{n+k} a_{n+k-i} c_i = a_{n+k}$$

as asserted. ☐

The results of Section 3.2 suggest that we should attempt to verify the properties

1. for every $\lambda \in V$, the fiber $\pi^{-1}(\lambda) = \{\delta_\lambda\}$, and
2. $\pi(\mathcal{M}(\mathcal{M}_V)) \cap \mathbb{B}_d = V$.

We first observe that Proposition 3.2.8 shows that (2) always holds because the functions $\{b_n z_1^n - b_1^n z_n : n \geq 2\}$ are polynomials. In fact, Remark 3.2.9 shows that $\pi(\mathcal{M}(\mathcal{M}_V)) = \overline{V}$.

We do not know if (1) holds in general. It does hold for a large class of examples. In particular, if the ideal of multipliers which vanish at λ coincides with $\overline{(z-\lambda)\operatorname{Mult}(\mathcal{H})}$, then it is clear that any character $\rho \in \pi^{-1}(\lambda)$ must be the character of point evaluation at λ. We do not have a characterization of when this occurs. The following result, without the norm closure, will suffice for our current needs.

Lemma 3.7.3. *Let $f(z) = (b_1 z, b_2 z^2, b_3 z^3, \dots)$ for $z \in \mathbb{D}$, where $\|(b_i)\|_2 \leq 1$. The following assertions are equivalent:*

(i) *For every $g \in \mathcal{M}_V$ with $g(0) = 0$, there is $\widetilde{g} \in \mathcal{M}_V$ such that $g = z_1 \widetilde{g}$.*

(ii) *For every $g \in \operatorname{Mult}(\mathcal{H}_f)$ with $g(0) = 0$, we have $g/z \in \operatorname{Mult}(\mathcal{H}_f)$.*

(iii) *The sequence $\left(\frac{a_n}{a_{n-1}}\right)_{n \geq 1}$ is bounded.*

Proof. The equivalence of (i) and (ii) follows by an application of the isomorphism

$$\mathcal{M}_V \to \operatorname{Mult}(\mathcal{H}_f), \quad g \mapsto g \circ f.$$

Suppose that (iii) holds. Then

$$D : \mathcal{H}_f \to \mathcal{H}_f, \quad h \mapsto \frac{h - h(0)}{z},$$

is a bounded linear map. Indeed, D maps z^n to z^{n-1}, and $||z^n||^2 = \frac{1}{a_n}$. Let $g \in \operatorname{Mult}(\mathcal{H}_f)$ with $g(0) = 0$. Then for every $h \in \mathcal{H}_f$, we have

$$D M_g h = D(gh) = \frac{g}{z} h.$$

This shows that $g/z \in \operatorname{Mult}(\mathcal{H}_f)$ and that $D M_g = M_{g/z}$. Hence, (ii) holds.

Conversely, suppose that (ii) is satisfied. Then

$$\widetilde{D} : \operatorname{Mult}(\mathcal{H}_f) \to \operatorname{Mult}(\mathcal{H}_f), \quad g \mapsto \frac{g - g(0)}{z},$$

is defined and clearly linear. Since convergence in $\mathrm{Mult}(\mathcal{H}_f)$ implies pointwise convergence on $\mathbb{D}$, we conclude with the help of the closed graph theorem that $\widetilde{D}$ is bounded. In particular,

$$\frac{1}{a_{n-1}} = ||z^{n-1}||^2_{\mathrm{Mult}(\mathcal{H}_f)} = ||\widetilde{D}z^n||^2_{\mathrm{Mult}(\mathcal{H}_f)} \leq ||\widetilde{D}||^2 \, ||z^n||^2_{\mathrm{Mult}(\mathcal{H}_f)} = ||\widetilde{D}||^2 \, \frac{1}{a_n},$$

where we have used Lemma 3.7.2. Thus, (iii) holds. □

It is not hard to modify Example 6.12 in [25] to see that the conditions in the preceding lemma are not always satisfied.

Corollary 3.7.4. *Let $f(z) = (b_1 z, b_2 z^2, b_3 z^3, \ldots)$ for $z \in \mathbb{D}$, where $\|(b_i)\|_2 \leq 1$. If $\mathcal{M}_V$ is automorphism invariant and $\sup_{n\geq 1} \frac{a_n}{a_{n-1}} < \infty$, then $\pi^{-1}(\lambda) = \{\delta_\lambda\}$ for every $\lambda \in V$.*

Proof. The result is immediate for $\lambda = 0$ since every $g \in \mathcal{M}_V$ such that $g(0) = 0$ factors as $g = z_1 h$ for some $h \in \mathcal{M}_V$. Thus if $\rho \in \pi^{-1}(0)$, we have $\rho(g) = \rho(z_1)\rho(h) = 0 = \delta_0(g)$. Hence $\rho = \delta_0$. Automorphism invariance readily shows that the same holds for every $\lambda \in V$. □

Suppose now that

$$\tilde{f}(z) = (\tilde{b}_1 z, \tilde{b}_2 z^2, \tilde{b}_3 z^3, \ldots) \quad \text{for} \quad z \in \mathbb{D}$$

is another embedding of the disc into $\mathbb{B}_\infty$ as above, and set $\widetilde{V} = \tilde{f}(\mathbb{D})$. We may define a sequence $(\tilde{a}_n)$ using (3.3) or Remark 3.7.1. We ask: when are $\mathcal{M}_V$ and $\mathcal{M}_{\widetilde{V}}$ isomorphic?

Proposition 3.7.5. *The algebras $\mathcal{M}_V$ and $\mathcal{M}_{\widetilde{V}}$ are isomorphic via the natural map of composition with $f \circ \tilde{f}^{-1}$ if and only if the sequences (a_n) and $(\tilde{a}_n)$ are comparable.*

Suppose that $\mathcal{M}_{\widetilde{V}}$ satisfies (1) $\pi^{-1}(\lambda) = \{\delta_\lambda\}$ for every $\lambda \in \widetilde{V}$ and is automorphism invariant. Then $\mathcal{M}_V$ is isomorphic to $\mathcal{M}_{\widetilde{V}}$ if and only if the sequences (a_n) and $(\tilde{a}_n)$ are comparable. In particular, $\mathcal{M}_V$ is isomorphic to H^∞ if and only if the sequence (a_n) is bounded below.

Proof. Suppose that (a_n) and $(\tilde{a}_n)$ are comparable. The sequence $\{z^n\}$ is an orthogonal basis for $\mathcal{H}_f$ and $\mathcal{H}_{\tilde{f}}$, and Lemma 3.7.2 shows that their norms in $\mathcal{H}_f$ and $\mathcal{H}_{\tilde{f}}$ are comparable. Thus the identity map is an invertible diagonal operator between $\mathcal{H}_f$ and $\mathcal{H}_{\tilde{f}}$. Therefore, $\mathrm{Mult}(\mathcal{H}_f) = \mathrm{Mult}(\mathcal{H}_{\tilde{f}})$, so that $\mathcal{M}_V$ and $\mathcal{M}_{\widetilde{V}}$ are isomorphic via the natural map.

Conversely, if $\mathcal{M}_V$ and $\mathcal{M}_{\widetilde{V}}$ are isomorphic via the natural map, then $\mathrm{Mult}(\mathcal{H}_f) = \mathrm{Mult}(\mathcal{H}_{\tilde{f}})$. Therefore the identity map is an isomorphism between these two semisimple

Banach algebras. Consequently, it is a topological isomorphism. So by Lemma 3.7.2, the sequences (a_n) and $(\tilde{a}_n)$ are comparable.

If $\mathcal{M}_V$ is automorphism invariant and satisfies (1), Corollary 3.2.11 applies. Finally, note that H^2 corresponds to the map $f(z) = (z, 0, 0, \dots)$ and $a_n = 1$ for all $n \geq 1$ because

$$\frac{1}{1-z} = \sum_{n \geq 0} z^n.$$

In general, $0 < \tilde{a}_n \leq 1$, so $(\tilde{a}_n)$ is comparable to (a_n) if and only if it is bounded below. The last claim now follows from the previous paragraph and the automorphism invariance of $H^\infty = \operatorname{Mult}(H^2)$. □

In [25, Example 6.12], an example was given of a variety $V = f(\mathbb{D})$ as above such that $\mathcal{H}_f$ is not isomorphic to H^2 via the identity map (so that $\mathcal{M}_V$ is not similar to H^∞ in the obvious way), and the question was raised whether or not $\mathcal{M}_V$ is isomorphic to H^∞. The above proposition answers this question, showing that those algebras are not isomorphic.

The following result gives a criterion for $\mathcal{M}_V$ being isomorphic to H^∞ in terms of the sequence (b_n) in the definition of the map f.

Corollary 3.7.6. *Let $V = f(\mathbb{D})$ where $f(z) = (b_1 z, b_2 z^2, b_3 z^3, \ldots)$, $\|(b_n)\|_2 = 1$ and $b_1 \neq 0$. Then $\mathcal{M}_V$ is isomorphic to H^∞ if and only if*

$$\sum_{n=1}^\infty n |b_n|^2 < \infty.$$

Proof. We know that $\mathcal{M}_V$ is isomorphic to H^∞ if and only if the sequence (a_n) is bounded below. Define

$$\mu = \sum_{n=1}^\infty n |b_n|^2 \in (0, \infty].$$

By the Erdős-Feller-Pollard theorem (see [31, Chapter XIII, Section 11]),

$$\lim_{n \to \infty} a_n = \frac{1}{\mu},$$

where $\infty^{-1} = 0$. The theorem is applicable since $|b_1|^2 > 0$. Hence, (a_n) is bounded below if and only if this series converges. □

Corollary 3.7.7. *Let V and (b_n) be as in the previous corollary. If $\mathcal{M}_V$ is not isomorphic to H^∞, then the series*

$$g(z) = \sum_{n\geq 1} |b_n|^2 z^n \quad \text{and} \quad (1-g(z))^{-1} = \sum_{n\geq 0} a_n z^n$$

both have radius of convergence 1.

Proof. Since (b_n) is in ℓ^2, the sequence is bounded, and hence the series for g has radius of convergence at least 1. If this radius of convergence is $R > 1$, then the series $\sum_{n=1}^\infty n|b_n|^2$ converges. So Corollary 3.7.6 shows that $\mathcal{M}_V$ is isomorphic to H^∞. Observe that g is bounded on $\mathbb{D}$ by $\|(b_n)\|_2^2 = 1$. In particular, $g(z) \neq 1$ for $z \in \mathbb{D}$, and thus $(1-g(z))^{-1}$ is defined on $\mathbb{D}$. Hence the series for $(1-g(z))^{-1}$ has radius of convergence at least 1. If this radius of convergence were greater than 1, then the only obstruction to

$$g(z) = 1 - \frac{1}{\sum_{n\geq 0} a_n z_n}$$

being defined on a disc of radius $R > 1$ is that $(1-g(z))^{-1}$ has a zero on $\partial\mathbb{D}$. This however implies that g has a pole on the circle, which is impossible because g is bounded on $\mathbb{D}$. Therefore $\sum_{n\geq 0} a_n z^n$ has radius of convergence exactly 1. □

We have seen that not all algebras $\mathcal{M}_V$ are isomorphic to H^∞. In fact, we will now exhibit a whole scale of mutually non-isomorphic algebras of this type. To this end, it is again more convenient to work with the algebras $\text{Mult}(\mathcal{H}_f)$, which are subalgebras of H^∞. The following proposition answer the question of which algebras of functions on $\mathbb{D}$ arise in this way.

Proposition 3.7.8. *An algebra $\mathcal{M}$ of functions on $\mathbb{D}$ arises in the way described above if and only if $\mathcal{M}$ is the multiplier algebra of a Hilbert function space on $\mathbb{D}$ with kernel K of the form*

$$K(z,w) = \sum_{n=0}^\infty a_n (z\overline{w})^n,$$

where $a_0 = 1$ and $a_1 \neq 0$, which satisfies the following two properties:

(1) *K is an irreducible complete Nevanlinna-Pick kernel.*

(2) $\sum_{n=0}^\infty a_n = \infty$.

Proof. Suppose that K satisfies the conditions above. Since K is an irreducible complete Nevanlinna-Pick kernel, the sequence (c_n) defined by

$$\sum_{n=1}^{\infty} c_n z^n = 1 - \frac{1}{\sum_{n=0}^{\infty} a_n z^n}$$

is positive by [3, Theorem 7.33]. The last condition guarantees that

$$\sum_{n=1}^{\infty} c_n = \sup_{0<t<1} \sum_{n=1}^{\infty} c_n t^n = 1.$$

As $a_1 \neq 0$, also $c_1 \neq 0$. Defining $b_n = \sqrt{c_n}$, we see that $\mathcal{M}$ arises as above (compare Remark 3.7.1).

Conversely, suppose that $\mathcal{M}$ arises as above. Then $\mathcal{M}$ is the multiplier algebra of a reproducing kernel Hilbert space on the disc whose kernel is of the desired form. By [3, Theorem 7.33], K is an irreducible complete Nevanlinna-Pick kernel. Finally,

$$\sum_{n=0}^{\infty} a_n = \sup_{0<t<1} \sum_{n=0}^{\infty} a_n t^n = \sup_{0<t<1} \frac{1}{1 - \sum_{n=1}^{\infty} c_n t^n} = \infty$$

because $\sum_{n=1}^{\infty} c_n = 1$. □

Example 3.7.9. For $s \leq 0$, let $\mathcal{H}_s$ be the irreducible complete Nevanlinna-Pick space on $\mathbb{D}$ with kernel

$$K(z, w) = \sum_{n=0}^{\infty} (n+1)^s (z\overline{w})^n,$$

see Example 2.6.1 (c). Recall that $\mathcal{H}_0$ is the Hardy space, and that $\mathcal{H}_{-1}$ is the Dirichlet space. If $-1 \leq s \leq 0$, these spaces satisfy the hypotheses of the last proposition (see also [3, Example 8.8]). Consequently, every multiplier algebra $\mathrm{Mult}(\mathcal{H}_s)$ is isomorphic to an algebra $\mathcal{M}_{V_s}$ where $V_s = f_s(\mathbb{D})$ is a variety and f_s is of the form

$$f_s(z) = (b_{s,1} z, b_{s,2} z^2, \dots) \quad \text{for} \quad z \in \mathbb{D}.$$

Moreover, each $\mathcal{H}_s$ and thus each $\mathrm{Mult}(\mathcal{H}_s)$ is automorphism invariant (see [12, Theorem 3.5]). Condition (iii) of Lemma 3.7.3 holds: $\sup_{n \geq 1} \frac{(n+1)^s}{n^s} = 2^s < \infty$. Thus by Corollary 3.7.4, $\mathcal{M}_{V_s}$ satisfies condition (1). As we observed, condition (2) always holds.

Therefore Proposition 3.7.5 applies. Since the sequences $((n+1)^s)_{n \geq 1}$ are not comparable for distinct values of s, the multiplier algebras $\mathcal{M}_{V_s}$ for $-1 \leq s \leq 0$ are mutually non-isomorphic. In this way, we obtain uncountably many isomorphism classes of algebras $\mathcal{M}_V$.

Consider

$$\langle f_s(z), zf_s'(z)\rangle = \sum_{n=1}^{\infty} n|b_{s,n}|^2|z|^{2n}.$$

This series converges to a finite limit as $|z|$ tends to 1 if and only if $\sum_{n=1}^{\infty} n|b_{s,n}|^2 < \infty$, which by Corollary 3.7.6 holds precisely when $\mathcal{M}_{V_s}$ is isomorphic to H^∞, namely when $s = 0$. Moreover, when $s < 0$, f_s is not C^1 because

$$\lim_{|z|\to 1} \|f_s'(z)\|^2 = \lim_{r\to 1} \sum_{n=1}^{\infty} n^2 r^{2n}|b_{s,n}|^2 = +\infty.$$

A closely related class of examples considered in [6, p.1128–30] are the Besov spaces $B_2^\sigma(\mathbb{D})$ for $0 < \sigma < 1/2$. These spaces coincide as spaces of functions with H_s for $s = -1 + 2\sigma$, although the kernels are somewhat different. Not surprisingly, just as for their embeddings, our embeddings are tangential in the sense that

$$\lim_{x\to 1,\, x\in(0,1)} \frac{1 - \|f_s(xe^{it})\|}{\|f_s(e^{it}) - f_s(xe^{it})\|} = 0$$

as well. Indeed, using that

$$\sum_{n=0}^{\infty} (n+1)^s x^n \approx \Gamma(1+s)(1-x)^{-1-s}$$

as $x \to 1$ from below (see [89, Chap. XIII, p.280, ex. 7]), we see that

$$\begin{aligned} 1 - ||f_s(xe^{it})|| &\sim 1 - ||f_s(xe^{it})||^2 \\ &= 1 - \sum_{n=1}^{\infty} |b_{s,n}|^2 x^{2n} = \Big(\sum_{n=0}^{\infty} (n+1)^s x^{2n}\Big)^{-1} \\ &\sim (1-x^2)^{1+s} \sim (1-x)^{1+s}. \end{aligned}$$

Here, we used the notation $f(x) \sim g(x)$ if $\lim_{x\to 1} f(x)g(x)^{-1} \in (0,\infty)$. On the other hand,

$$\begin{aligned} ||f_s(e^{it}) - f_s(xe^{it})||^2 &= \sum_{n=1}^{\infty} |b_{s,n}|^2(1-x^n)^2 \\ &= 1 - 2\sum_{n=1}^{\infty} |b_{s,n}|^2 x^n + \sum_{n=1}^{\infty} |b_{s,n}|^2 x^{2n} \\ &= 2\Big(\sum_{n=0}^{\infty} (n+1)^s x^n\Big)^{-1} - \Big(\sum_{n=1}^{\infty} (n+1)^s x^{2n}\Big)^{-1}. \end{aligned}$$

Since

$$2\Big(\sum_{n=0}^{\infty}(n+1)^s x^n\Big)^{-1} \approx 2\Gamma(1+s)^{-1}(1-x)^{1+s}$$

and

$$\Big(\sum_{n=1}^{\infty}(n+1)^s x^{2n}\Big)^{-1} \approx \Gamma(1+s)^{-1}(1-x^2)^{1+s} \approx \Gamma(1+s)^{-1}2^{1+s}(1-x)^{1+s},$$

we have

$$||f_s(e^{it}) - f_s(xe^{it})|| \sim (1-x)^{(1+s)/2}.$$

Thus,

$$\lim_{x\to 1,\, x\in(0,1)} \frac{1-\|f_s(xe^{it})\|}{\|f_s(e^{it})-f_s(xe^{it})\|} = 0.$$

Similarly, for $s=-1$, we obtain the same tangential property because

$$1-||f_s(xe^{it})|| \sim \Big(\sum_{n=0}^{\infty}(n+1)^s x^{2n}\Big)^{-1} \sim -\log(1-x)^{-1}$$

and

$$\begin{aligned}||f_s(e^{it}) - f_s(xe^{it})||^2 &= 2\Big(\sum_{n=0}^{\infty}(n+1)^s x^n\Big)^{-1} - \Big(\sum_{n=1}^{\infty}(n+1)^s x^{2n}\Big)^{-1}\\ &\sim -\log(1-x)^{-1}.\end{aligned}$$

It also follows for $-1 \le s < 0$,

$$\begin{aligned}\lim_{x\to 1,\, x\in(0,1)} \frac{\mathrm{Re}\langle f_s(1)-f_s(x), f_s(1)\rangle}{1-x} &= \lim_{x\to 1-}\sum_{n\ge 1}|b_{s,n}|^2\frac{1-x^n}{1-x}\\ &= \sum_{n\ge 1} n|b_{s,n}|^2 = +\infty\end{aligned}$$

by Corollary 3.7.6.

3.8. Embedding closed discs

In this section, we will consider a class of varieties in $\mathbb{B}_\infty$ which includes varieties associated to the spaces $\mathcal{H}_s$ for $s<-1$. Again we define $f:\mathbb{D}\to\mathbb{B}_\infty$ by

$$f(z) = (b_1 z, b_2 z^2, b_3 z^3, \dots) \quad \text{for} \quad z \in \mathbb{D},$$

with $(b_n)_{n=1}^{\infty} \in \ell^2$ and $b_1 \neq 0$. Here, however, we assume that

1. $||(b_n)||_2 = r < 1$, and
2. $\sum_{n\geq 1} |b_n|^2 z^n$ has radius of convergence 1.

Let $V = f(\mathbb{D})$. As observed in the previous section, f extends to a continuous injection of $\overline{\mathbb{D}}$ onto $\overline{V}$. But because $r < 1$, $\overline{V}$ is a compact subset of $r\overline{\mathbb{B}_\infty} \subset \mathbb{B}_\infty$.

As we observed in the previous section, $H^2_\infty\big|_V$ is unitarily equivalent to a reproducing kernel Hilbert space $\mathcal{H}_f$ on $\mathbb{D}$ with kernel

$$K(z,w) = \frac{1}{1 - \sum_{n=1}^\infty |b_n|^2 (z\overline{w})^n} = \sum_{n=0}^\infty a_n (z\overline{w})^n.$$

Setting $c_n = |b_n|^2$, we see as in Remark 3.7.1 that $g(z) = \sum_{n\geq 1} c_n z^n$ determines (a_n) by

$$\frac{1}{1-g(z)} = \sum_{n=0}^\infty a_n z^n \quad \text{for } z \in \mathbb{D}.$$

Now (c_n) is summable and $\|g\|_\infty = r^2 < 1$, so $(1-g)^{-1}$ extends to be continuous on $\overline{\mathbb{D}}$.

Once again, it will be convenient to work with the multiplier algebras $\mathrm{Mult}(\mathcal{H}_f)$. The following result characterizes which algebras of functions on the unit disc arise in this way. It is the analogue of Proposition 3.7.8 in this setting.

Proposition 3.8.1. *An algebra $\mathcal{M}$ of functions on $\mathbb{D}$ arises in the way described above if and only if $\mathcal{M}$ is the multiplier algebra of a Hilbert function space on $\mathbb{D}$ with kernel K of the form*

$$K(z,w) = \sum_{n=0}^\infty a_n (z\overline{w})^n,$$

where $a_0 = 1$ and $a_1 \neq 0$, which satisfies the following two properties:

(1) *K is an irreducible complete Nevanlinna-Pick kernel.*

(2) *$\sum_{n=0}^\infty a_n < \infty$ and the series $\sum_{n=0}^\infty a_n z^n$ has radius of convergence 1.*

Proof. The proof of Proposition 3.7.8 carries over to this setting once we show that $\sum_{n=0}^\infty a_n < \infty$ if and only if $\sum_{n=1}^\infty c_n < 1$, and that in this case, $\sum_{n=0}^\infty a_n z^n$ has radius of convergence 1 if and only if $\sum_{n=1}^\infty c_n z^n$ has radius of convergence 1.

The first claim is immediate from the relation between (a_n) and (c_n). Moreover, since $\sum_{n=0}^\infty a_n$ and $\sum_{n=1}^\infty c_n$ converge, both power series have radius of convergence at least 1. If $\sum_{n=0}^\infty a_n z^n$ extends analytically across $\partial\mathbb{D}$, then so does $\sum_{n=0}^\infty c_n z^n$ by the argument in the proof of Corollary 3.7.7. Conversely, if $g(z) = \sum_{n=1}^\infty c_n z^n$ extends analytically across $\partial\mathbb{D}$, then the only obstruction to $(1-g(z))^{-1} = \sum_{n=0}^\infty a_n z^n$ being defined on a disc of radius $R > 1$ is that $g(z)$ takes the value 1 on $\partial\mathbb{D}$, which is impossible since $\sum_{n=1}^\infty c_n < 1$. $\square$

An example of this are the spaces $\mathcal{H}_s$ of Example 3.7.9 for $s < -1$. The reproducing kernel of the space $\mathcal{H}_s$ is

$$K(z,w) = \sum_{n=0}^{\infty} (n+1)^s (z\overline{w})^n \quad \text{for} \quad z,w \in \mathbb{D}.$$

Since

$$\sum_{n \geq 0} a_n = \sum_{n \geq 0} (n+1)^s < \infty,$$

this space does not fit into the framework of Proposition 3.7.8. However, the series $\sum_{n=0}^{\infty} (n+1)^s z^n$ has radius of convergence 1, so by the previous proposition, the space $\mathcal{H}_s$ fits into the framework of this section for $s < -1$.

The fact that f has radius of convergence 1 means that there is no open neighbourhood U of $\overline{\mathbb{D}}$ such that the functions in $\mathcal{H}_f$ all extend to analytic functions on U. Closely related to this observation is the fact that there is no variety which properly contains $\overline{V}$.

We will now show that while V is not a variety, its compact closure $\overline{V}$ is a variety in $\mathbb{B}_\infty$. This is in stark contrast to the finite dimensional case, since the only compact varieties in $\mathbb{B}_d$ consist of finitely many points if $d < \infty$, see [75, Theorem 14.3.1].

Lemma 3.8.2. *If (b_n) and f are defined as above, then $\overline{V} = f(\overline{\mathbb{D}})$ is the common zero locus of the polynomials $\{b_n z_1^n - b_1^n z_n : n \geq 2\}$; that is,*

$$\overline{V} = V(\{b_n z_1^n - b_1^n z_n : n \geq 2\}).$$

Proof. Note that every point in $\overline{V}$ is a zero of the polynomials $b_n z_1^n - b_1^n z_n$. Conversely, if $\boldsymbol{z} = (z_1, z_2, \ldots)$ satisfies these equations, then setting $z = z_1/b_1$, we find that

$$z_n = b_n z^n \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $(z_1, z_2, \ldots)$ is a point in ℓ^2, we have

$$\infty > \sum_{n=1}^{\infty} |z_n|^2 = \sum_{n=1}^{\infty} |b_n|^2 |z|^{2n}.$$

As the series on the right has radius of convergence 1, it follows that $|z| \leq 1$. Hence $z \in \overline{\mathbb{D}}$ and $\boldsymbol{z} = f(z)$ belongs to $\overline{V}$. □

Remark 3.8.3. $\overline{V}$ is the minimal variety containing V. Hence every function in $H^2_\infty\big|_V$ extends uniquely to a function in $H^2_\infty\big|_{\overline{V}}$, and $\mathcal{M}_V$ can be naturally identified with $\mathcal{M}_{\overline{V}}$ (see [25, Proposition 2.2]). It is $\overline{V}$, not V, which fits into the framework developed in [25].

Another property which distinguishes the algebras $\mathcal{M}_{\overline{V}}$ from the algebras in the preceding section is that the function in $\mathcal{M}_{\overline{V}}$ are continuous on the compact set $\overline{V}$.

Lemma 3.8.4. *If (b_n) and f are defined as above, then $H^2_\infty\big|_{\overline{V}}$ and $\mathcal{M}_{\overline{V}}$ consist of continuous functions on $\overline{V}$.*

Proof. Let $r = ||(b_n)||_2 < 1$. Observe that the Drury-Arveson kernel K is jointly norm continuous on $r\overline{\mathbb{B}_\infty} \times r\overline{\mathbb{B}_\infty}$, hence the map

$$r\overline{\mathbb{B}_\infty} \to H^2_\infty, \quad w \mapsto K(\cdot, w),$$

is continuous, so that all functions in H^2_∞ are norm continuous on $r\overline{\mathbb{B}_\infty}$. Since $\overline{V} \subset r\overline{\mathbb{B}_\infty}$, it follows that all functions in $H^2_\infty\big|_{\overline{V}}$ are continuous on $\overline{V}$. In particular, this is true of $\mathcal{M}_{\overline{V}}$. □

Alternatively, we could have argued with the space $\mathcal{H}_f$ in the last proof. Since $\sum_{n=0}^\infty a_n < \infty$, the reproducing kernel of $\mathcal{H}_f$ extends to a continuous function on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$, so all function in $\mathcal{H}_f$ extend to continuous functions on $\overline{\mathbb{D}}$.

Let $\delta : \overline{V} \to \mathcal{M}(\mathcal{M}_{\overline{V}})$ be the map taking $v \in \overline{V}$ to the character δ_v which evaluates multipliers at v. Since the functions in $\mathcal{M}_{\overline{V}}$ are continuous on $\overline{V}$, the map δ is a homeomorphism onto its image. We do not know if δ is always surjective. Shields [81, Section 9] asks a similar question in the context of spaces of weighted shifts. He answers the question positively when the algebra is strictly cyclic, in which case the multiplier algebra and the Hilbert space coincide as sets. We can use his result here.

Lemma 3.8.5. *Suppose that*

$$\sup_{n\geq 1} \sum_{k=0}^n \left(\frac{a_k a_{n-k}}{a_n} \right) < \infty.$$

Then the natural injection δ of $\overline{V}$ into $\mathcal{M}(\mathcal{M}_{\overline{V}})$ is a homeomorphism. In particular, this is the case if $\overline{V}$ arises from $H_s, s < -1$.

Proof. The results in Section 9 of [81] show that the operator M_z on $\mathcal{H}_f$ is strictly cyclic if the supremum is finite, hence the Gelfand space of $\mathrm{Mult}(\mathcal{H}_f)$ is the closed unit disc. It follows that δ is a homeomorphism. In the case of $\mathcal{H}_s, s < -1$, Example 1 after Proposition 33 in [81] shows that the supremum is finite. □

Now we can establish isomorphism results for this family of compact varieties that parallel the results of the previous section.

Theorem 3.8.6. *Let*

$$f(z) = (b_1 z, b_2 z^2, b_3 z^3, \dots) \quad \textit{and} \quad \tilde{f}(z) = (\tilde{b}_1 z, \tilde{b}_2 z^2, \tilde{b}_3 z^3, \dots)$$

be functions of $\overline{\mathbb{D}}$ *into* $\mathbb{B}_\infty$*, with*

$$\|(b_n)\|_2 = r < 1, \quad \|(\tilde{b}_n)\|_2 = r' < 1 \quad \textit{and} \quad b_1 \tilde{b}_1 \neq 0$$

such that the series

$$\sum_{n \geq 1} |b_n|^2 z^n \qquad \textit{and} \qquad \sum_{n \geq 1} |\tilde{b}_n|^2 z^n$$

both have radius of convergence 1*. Let* $\overline{V} = f(\overline{\mathbb{D}})$ *and* $\overline{\widetilde{V}} = \tilde{f}(\overline{\mathbb{D}})$*.*

(a) $\mathcal{M}_{\overline{V}}$ *and* $\mathcal{M}_{\overline{\widetilde{V}}}$ *are isomorphic via the natural map if and only if the sequences* (a_n) *and* $(\tilde{a}_n)$ *are comparable.*

(b) *Suppose that* $\mathcal{M}_{\overline{V}}$ *satisfies the hypothesis of Lemma 3.8.5:*

$$\sup_{n \geq 1} \sum_{k=0}^{n} \left(\frac{a_k a_{n-k}}{a_n} \right) < \infty \quad \textit{where} \quad \sum_{n \geq 0} a_n z^n = \frac{1}{1 - \sum_{n \geq 1} |b_n|^2 z^n},$$

and is automorphism invariant. Assume that $\mathcal{M}_{\overline{V}}$ *is isomorphic to* $\mathcal{M}_{\overline{\widetilde{V}}}$*. Then the restriction* F *of* Φ^* *to* $\overline{\widetilde{V}}$ *is a homeomorphism of* $\overline{\widetilde{V}}$ *onto* $\overline{V}$ *which is holomorphic on* $\widetilde{V}$ *and* $\Phi(h) = h \circ F$*. There is a Möbius map* θ *so that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}_{\overline{V}} & \xrightarrow{\ \Phi\ } & \mathcal{M}_{\overline{\widetilde{V}}} \\ {\scriptstyle C_f}\big\downarrow & & \big\downarrow{\scriptstyle C_{\tilde{f}}} \\ A(\mathbb{D}) & \xrightarrow{\ C_\theta\ } & A(\mathbb{D}) \end{array}$$

Moreover, they are isomorphic via the natural map of composition with $G = f \circ f'^{-1}$*.*

Proof. (a) This follows as in Proposition 3.7.5.

(b) Since Φ is an isomorphism, Φ^* yields a homeomorphism of the maximal ideal spaces. By Lemma 3.8.5, $\mathcal{M}(\mathcal{M}_{\overline{V}}) = \delta(\overline{V}) \simeq \overline{V}$. So we identify $\mathcal{M}(\mathcal{M}_{\overline{V}})$ with $\overline{V}$. For the other algebra, we have that $\overline{\widetilde{V}}$ is identified with $\delta(\overline{\widetilde{V}})$ as a subset of $\mathcal{M}(\mathcal{M}_{\overline{\widetilde{V}}})$. Let $F : \overline{\widetilde{V}} \to \overline{V}$ be the restriction of Φ^* to this copy of $\overline{\widetilde{V}}$. The argument in the proof of Theorem 3.2.5 again shows that F is holomorphic on $\widetilde{V}$. Now

$$\Phi(h)(\tilde{v}) = \Phi^*(\delta_{\tilde{v}})(h) = \delta_{F(\tilde{v})}(h) = (h \circ F)(\tilde{v}) \quad \text{for} \quad h \in \mathcal{M}_{\overline{V}} \ \text{ and } \ \tilde{v} \in \overline{\widetilde{V}}.$$

Thus $\Phi = C_F$ is a composition operator.

By an adaptation of [24, Section 11.3] as in the proof of Theorem 3.2.5, the fact that Φ is implemented by composition implies that Φ is weak-$*$ continuous. And the argument continues to conclude that Φ^{-1} is also weak-$*$ continuous. In particular, $(\Phi^{-1})^*$ takes point evaluations to point evaluations. As this map is the inverse of F, we deduce that F maps onto $\overline{V}$; and hence $\mathcal{M}(\mathcal{M}_{\overline{\tilde{V}}}) = \overline{\tilde{V}}$.

The commutative diagram is obtained as in the proof of Corollary 3.2.10. The only change is that, since the multipliers are continuous by Lemma 3.8.4, the range is considered as a subalgebra of the disc algebra $A(\mathbb{D})$, rather than in the larger algebra H^∞. Since $\mathcal{M}_{\overline{V}}$ is automorphism invariant, we may apply the automorphism for θ^{-1} to obtain the natural map as in Proposition 3.7.5. □

Example 3.8.7. The spaces $\mathcal{H}_s$ for $s < -1$ yield an uncountable family of varieties in $\mathbb{B}_\infty$ which are homeomorphic to $\overline{\mathbb{D}}$. Their multiplier algebras are automorphism invariant (see [12, Theorem 3.5]) and they satisfy the hypothesis of Lemma 3.8.5. The sequences $((n+1)^s)$ are not comparable for different values of s. Thus by Theorem 3.8.6, they have non-isomorphic multiplier algebras.

3.9. Interpolating sequences

We finish the treatment of the algebras $\mathcal{M}_{\overline{V}}$ of the previous section by showing that under the assumptions of Lemma 3.8.5 these algebras are not isomorphic to an algebra of the type $\mathcal{M}_W$ for any variety W whose closure meets the boundary of the ball. This result should not be surprising, as isomorphism of the algebras yields a homeomorphism of the maximal ideal spaces. In the setting of Lemma 3.8.5 the maximal ideal space is homeomorphic to $\overline{\mathbb{D}}$. The reader may suspect that this is never the case when $\overline{W}$ intersects the boundary.

We will establish this is by showing that any sequence in the ball which converges to the boundary contains an interpolating subsequence. It then follows that $\mathcal{M}_W$ has ℓ^∞ as a quotient, and hence its maximal ideal space contains a copy of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. In particular, it is not metrizable, so it is not homeomorphic to the unit disc. We were not able to show, without imposing any special assumptions, that an algebra $\mathcal{M}_{\overline{V}}$ as in Section 3.8 can never be isomorphic to an algebra of the type occurring in Section 3.7.

A sequence (x_n) in $\mathbb{B}_\infty$ is an *interpolating sequence* for $\mathrm{Mult}(H^2_\infty)$ if the evaluation map

$$\mathrm{Mult}(H^2_\infty) \to \ell^\infty, \quad \varphi \mapsto (\varphi(x_n)),$$

is surjective. The multiplier algebras considered here are all of the form $\mathcal{M}_V$, where V is a variety in $\mathbb{B}_\infty$. These are (complete) quotients of $\mathrm{Mult}(H^2_\infty)$ via the restriction map. So any sequence in V is interpolating for $\mathcal{M}_V$ if and only if it is interpolating for $\mathrm{Mult}(H^2_\infty)$.

Proposition 3.9.1. *Let (z_n) be a sequence in $\mathbb{B}_\infty$ such that $\lim_{n\to\infty} ||z_n|| = 1$. Then (z_n) contains a subsequence which is interpolating for* $\mathrm{Mult}(H^2_\infty)$*.*

Proof. Fix $r \in (0,1)$. We wish to show that there is a subsequence (z_{n_k}) of (z_n) such that for every sequence $(w_k) \in \ell^\infty$ of norm at most r, there is a multiplier $\varphi \in \mathrm{Mult}(\mathcal{H})$ of norm at most 1 such that $\varphi(z_{n_k}) = w_k$. We will recursively construct the subsequence (z_{n_k}) such that for each k and for all $w = (w_i) \in \ell^\infty$ with $||w|| \leq r$, the $k \times k$ matrix

$$A_k(w) = \Big[(1 - w_i \overline{w}_j) K(z_{n_i}, z_{n_j})\Big]_{i,j=1}^{k}$$

is positive and invertible. Once we have achieved this, the Nevanlinna-Pick property yields, for each $w \in \ell^\infty$ with $||w|| \leq r$ and any positive integer k, the existence of a multiplier h_k of norm at most 1 such that $h_k(z_{n_i}) = w_i$ for $1 \leq i \leq k$. Any weak-$*$ cluster point h of the sequence (h_k) will then satisfy $h(z_{n_i}) = w_i$ for all $i \in \mathbb{N}$.

We begin the construction by setting $z_{n_1} = z_1$. Suppose that $k \geq 2$ and that $z_{n_1}, \ldots, z_{n_{k-1}}$ have already been constructed. Given $w = (w_i) \in \ell^\infty$ with $||w|| \leq r$, we set $v_{ij} = 1 - w_i \overline{w_j}$. For $z \in \mathbb{B}_\infty$, we consider the matrix $A(w,z)$ defined by

$$\begin{bmatrix} v_{1,1}K(z_{n_1}, z_{n_1}) & \cdots & v_{1,k-1}K(z_{n_1}, z_{n_{k-1}}) & v_{1,k}K(z_{n_1}, z) \\ \vdots & \ddots & \vdots & \vdots \\ v_{k-1,1}K(z_{n_{k-1}}, z_{n_1}) & \cdots & v_{k-1,k-1}K(z_{n_{k-1}}, z_{n_{k-1}}) & v_{k-1,k}K(z_{n_{k-1}}, z) \\ v_{k,1}K(z, z_{n_1}) & \cdots & v_{k,k-1}K(z, z_{n_{k-1}}) & v_{k,k}K(z,z) \end{bmatrix}.$$

Observe that the first $(k-1)\times(k-1)$ minor equals $A_{k-1}(w)$, which is positive and invertible for all choices of w with $||w|| \leq r$ by our recursive assumption. By Sylvester's criterion, it therefore suffices to show that there exists z_{n_k} with $n_k > n_{k-1}$ such that $\det(A(w, z_{n_k})) > 0$ for all such w. To see that this is possible, note that

$$\lim_{n\to\infty} K(z_n, z_n) = \lim_{n\to\infty} \frac{1}{1 - \|z_n\|^2} = \infty.$$

On the other hand, each $K(z_i, z)$ is bounded. Moreover, by compactness of the unit ball in finite-dimensional spaces, there exists $\delta > 0$ such that

$$\det(A_{k-1}(w)) > \delta$$

for all w with $||w|| \leq r$. Thus, in the expansion of the determinant of $A_n(w,z)$ along the last row, there is one term

$$|v_{kk}K(z,z)\det(A_{k-1}(w))| \geq (1-r^2)\delta K(z,z),$$

which tends to infinity as $z \to 1$ uniformly in w, whereas all other terms are uniformly bounded. Therefore the determinant is eventually strictly positive on the whole r-ball. This establishes the existence of the desired point z_{n_k}, and thus finishes the recursive construction. □

Corollary 3.9.2. *If W is a variety in the ball $\mathbb{B}_d$ for $d \leq \infty$ such that $\overline{W}$ intersects the boundary of the ball, then ℓ^∞ is a quotient of $\mathcal{M}_W$ and hence $\mathcal{M}(\mathcal{M}_W)$ contains a copy of $\beta\mathbb{N}$.*

Proof. Proposition 3.9.1 shows that W contains an interpolating sequence. The restriction map to this sequence is the desired quotient onto ℓ^∞. Hence $\mathcal{M}(\ell^\infty)$, which is homeomorphic to $\beta\mathbb{N}$, embeds as a closed subset of $\mathcal{M}(\mathcal{M}_W)$. □

Thus we obtain the desired consequence.

Proposition 3.9.3. *Let $\overline{V}$ be a compact variety as considered in Theorem 3.8.6 (b), and let $\widetilde{V}$ be a variety as considered in section 3.7. Then there is no unital surjective algebra homomorphism from $\mathcal{M}_{\overline{V}}$ onto $\mathcal{M}_{\widetilde{V}}$. In particular, they are not isomorphic.* □

Combining this observation with Examples 3.7.9 and 3.8.7, we obtain the following consequence.

Corollary 3.9.4. *The Hilbert spaces $\mathcal{H}_s$ have non-isomorphic multiplier algebras for distinct $s \leq 0$.* □

4. A new approach to the classification problem for multiplier algebras

4.1. Introduction

The contents of this chapter appeared in [43]. We continue the study of the isomorphism problem for multiplier algebras of complete Nevanlinna-Pick spaces. In Chapter 3 and in [24, 25, 40, 54], this problem was studied by making use of the universality theorem of Agler and McCarthy (see Section 2.5) to identify a given complete Nevanlinna-Pick space with a restriction of the Drury-Arveson space to an analytic variety. Roughly speaking, the results then typically state that two algebras are isomorphic if and only if the underlying varieties are geometrically equivalent in a suitable sense. For an up-to-date account on these results, the reader is referred to the recent survey article [76].

While this approach has been successful in dealing with the (completely) isometric isomorphism problem (see [25] and [76]), the algebraic (or even completely bounded) isomorphism problem seems to be more difficult. Indeed, we encountered some of these issues in Chapter 3. Essentially the only instance for which the algebraic isomorphism problem has been completely resolved is the case of restrictions of Drury-Arveson space on a finite dimensional ball to homogeneous varieties [24, 40]. The existence of algebraic isomorphisms is also quite well understood for multiplier algebras associated to certain one-dimensional varieties under the assumption of sufficient regularity on the boundary [4, 6, 54]. For more general varieties, however, the situation is far less clear. Moreover, several results in [25] only apply to varieties which are contained in a finite dimensional ball. From the point of view of the study of multiplier algebras of complete Nevanlinna-Pick spaces, this condition is rather restrictive. There are many natural examples of complete Nevanlinna-Pick spaces on the unit disc or, more generally, on a finite dimensional unit ball, which cannot be realized as the restriction of Drury-Arveson space on a finite dimensional ball. Indeed, the classical Dirichlet space, which consists of analytic functions on the unit disc, is such an example (see also Proposition 4.11.8).

In this chapter, we take a different point of view and study the complete Nevanlinna-Pick spaces and their reproducing kernels directly. In particular, we consider a class of spaces on homogeneous varieties in a ball in $\mathbb{C}^d$. This more direct approach has the disadvantage that we can no longer make use of the well developed theory of the Drury-Arveson space. In particular, the tools coming from the non-commutative theory of free semigroup algebras [21, 22, 23] are not available any more.

Nevertheless, the direct approach has certain benefits. Firstly, by studying the spaces directly, we are able to stay within the realm of reproducing kernel Hilbert spaces on subsets of $\mathbb{C}^d$ for finite d. We thus avoid the issues surrounding the Drury-Arveson space H^2_∞ on an infinite dimensional ball, such as the extremely complicated nature of the maximal ideal space of $\operatorname{Mult}(H^2_\infty)$ (cf. Example 3.2.4). Secondly, many spaces of interest are graded in a natural way. Indeed, we consider a class of complete Nevanlinna-Pick spaces of analytic functions on the open unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ which contain the polynomials as a dense subspace and in which homogeneous polynomials of different degree are orthogonal. When identifying such a space with a restriction of the Drury-Arveson space, the grading becomes less visible, since it is usually not compatible with the natural grading on the Drury-Arveson space. By working with the spaces directly, we are able to exploit their graded nature. Finally, when working with two spaces on the same set, one can also ask if their multiplier algebras are equal, rather than just isomorphic.

In addition to this introduction, this chapter has ten sections. In Section 4.2, we gather some preliminaries regarding unitarily invariant spaces.

In Section 4.3, we observe that it is possible to recover the reproducing kernel of a complete Nevanlinna-Pick space from its multiplier algebra. As a consequence, we obtain that two complete Nevanlinna-Pick spaces whose multiplier algebras are equal have the same reproducing kernels, up to normalization.

In Section 4.4, we apply the results of Section 2 to composition operators on multiplier algebras. In particular, we characterize those complete Nevanlinna-Pick spaces of analytic functions on $\mathbb{B}_d$ whose multiplier algebras are isometrically invariant under conformal automorphisms of $\mathbb{B}_d$.

In Section 4.5, we study the notion of algebraic consistency, which, roughly speaking, assures that the functions in a complete Nevanlinna-Pick space are defined on the largest possible domain of definition. It turns out that this notion is closely related to the notion of a variety from [25].

In Section 4.6, we consider a general notion of grading on a complete Nevanlinna-Pick space. The main result in this section asserts that multiplier norm and Hilbert space norm coincide for homogeneous elements.

In Section 4.7, we set the stage for the remainder of this chapter by introducing a family of unitarily invariant complete Nevanlinna-Pick spaces on $\mathbb{B}_d$. The aim is then to investigate the isomorphism problem for multiplier algebras of restrictions of such spaces to homogeneous varieties. This is done by following the route of [24].

In Section 4.8, we study the maximal ideal spaces of the multiplier algebras of spaces introduced in Section 4.7. In particular, we introduce a regularity condition on the maximal ideal space, which we call *tameness*. It is shown that a large collection of spaces, which includes the spaces $\mathcal{H}_s$ of Chapter 3 and their counterparts on $\mathbb{B}_d$, is indeed tame.

In Section 4.9, we recall several results from [24] about holomorphic maps on homogeneous varieties, thereby providing simpler proofs in some instances. We also point out that a crucial argument from [24] can be used to show that the group of unitaries is a maximal subgroup of the group of conformal automorphisms of $\mathbb{B}_d$.

In Section 4.10, we show that the arguments from [24] can be adapted to our setting to show that if two of our multiplier algebras are isomorphic, then they are isomorphic via an isomorphism which preserves the grading.

Finally, Section 4.11 contains the main results about isometric and algebraic isomorphism of the multiplier algebras. We finish by reformulating some of the results in terms of restrictions of Drury-Arveson space, thereby providing a connection to examples in Chapter 3.

4.2. Preliminaries

Let $\mathbb{B}_d$ denote the open unit ball in $\mathbb{C}^d$. Occasionally, we will allow $d = \infty$, in which case $\mathbb{C}^d$ is understood to be ℓ_2. A *unitarily invariant space on* $\mathbb{B}_d$ is a reproducing kernel Hilbert space $\mathcal{H}$ on $\mathbb{B}_d$ with reproducing kernel K which is normalized at 0, analytic in the first component, and satisfies

$$K(Uz, Uw) = K(z, w)$$

for all $z, w \in \mathbb{B}_d$ and all unitary maps U on $\mathbb{C}^d$. Spaces of this type appear throughout the literature, see for example [39, Section 4] or [38, Section 4]. The following characterization of unitarily invariant spaces is well known. Since we do not have a convenient reference for the proof, it is provided below.

Lemma 4.2.1. *Let $d \in \mathbb{N} \cup \{\infty\}$ and let $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ be a function. The following are equivalent:*

(i) *K is a positive definite kernel which is normalized at 0, analytic in the first component, and satisfies $K(z, w) = K(Uz, Uw)$ for all $z, w \in \mathbb{B}_d$ and all unitary maps U on $\mathbb{C}^d$.*

(ii) *There is a sequence $(a_n)_n$ of non-negative real numbers with $a_0 = 1$ such that*

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$$

for all $z, w \in \mathbb{B}_d$.

Proof. (ii) $\Rightarrow$ (i) By the Schur product theorem (Theorem 2.1.4), the map $(z,w) \mapsto \langle z, w \rangle^n$ is positive definite for all $n \in \mathbb{N}$, hence K is positive definite. Clearly, K is normalized at 0 and invariant under unitary maps of $\mathbb{C}^d$. Moreover, for fixed $w \in \mathbb{B}_d$, the series in (ii) converges uniformly in z on $\mathbb{B}_d$, hence K is analytic in the first variable.

(i) $\Rightarrow$ (ii) Let $z_1, w_1, z_2, w_2 \in \mathbb{B}_d$ satisfy $\langle z_1, w_1 \rangle = \langle z_2, w_2 \rangle$. We will show that $K(z_1, w_1) = K(z_2, w_2)$. This will complete the proof, since then, there exists a function $f : \mathbb{D} \to \mathbb{C}$ such that $K(z,w) = f(\langle z, w \rangle)$ for all $z, w \in \mathbb{B}_d$. Since K is analytic in the first component and is normalized at the origin, f is necessarily analytic and satisfies $f(0) = 1$. Positive definiteness of K finally implies that the Taylor coefficients of f at 0 are non-negative, see the proof of [3, Theorem 7.33] and also Corollary 4.6.3 below.

In order to show that $K(z_1, w_1) = K(z_2, w_2)$, first note that for $z, w \in \mathbb{B}_d$, the identity

$$K(\lambda z, w) = K(z, \overline{\lambda} w)$$

holds for all $\lambda \in \mathbb{T}$, as multiplication by a complex scalar of modulus 1 is a unitary map on $\mathbb{C}^d$. Since $K(z, \overline{\lambda} w) = \overline{K(\overline{\lambda} w, z)}$, we see that both sides of the above equation define analytic maps in λ in an open neighbourhood of $\overline{\mathbb{D}}$, hence the above identity holds for all $\lambda \in \overline{\mathbb{D}}$. In particular, we see that

$$K(rz, w) = K(z, rw)$$

for $z, w \in \mathbb{B}_d$ and $r \in [0,1]$. Consequently, we may without loss of generality assume that $||w_1|| = ||w_2||$. Then there exists a unitary map on $\mathbb{C}^d$ which maps w_1 onto w_2. Since K is invariant under unitary maps by assumption, and so is the scalar product $\langle \cdot, \cdot \rangle$, we may in fact suppose that $w_1 = w_2$. Let w denote this vector. Since K is normalized at 0, the claim is obvious if $w = 0$, so assume that $w \neq 0$.

From the assumption $\langle z_1, w \rangle = \langle z_2, w \rangle$, we deduce that there exist vectors $v, r_1, r_2 \in \mathbb{C}^d$ such that $v \in \mathbb{C}w$ and $r_1, r_2 \in (\mathbb{C}w)^\perp$ and such that

$$z_i = v + r_i \quad (i = 1, 2).$$

For $\lambda \in \mathbb{T}$, let U_λ denote the unitary map on $\mathbb{C}^d$ which fixes $\mathbb{C}w$ and acts as multiplication by λ on $(\mathbb{C}w)^\perp$. Then for $i = 1, 2$ and $\lambda \in \mathbb{T}$, we have

$$K(z_i, w) = K(U_\lambda z_i, U_\lambda w) = K(v + \lambda r_i, w).$$

Observe that the right-hand side defines an analytic function in λ in an open neighbourhood of $\overline{\mathbb{D}}$, which is therefore constant. In particular,

$$K(z_1, w) = K(v, w) = K(z_2, w),$$

which completes the proof. □

If $\mathcal{H}$ is a unitarily invariant space on $\mathbb{B}_d$, then it easily follows from the representation of the kernel in part (ii) of the preceding lemma that convergence in $\mathcal{H}$ implies uniform convergence on $r\mathbb{B}_d$ for $0 < r < 1$. Since the kernel functions $K(\cdot, w)$ for $w \in \mathbb{B}_d$ are analytic by assumption, and since finite linear combinations of kernel functions are dense in $\mathcal{H}$, we therefore see that every function in $\mathcal{H}$ is analytic on $\mathbb{B}_d$.

We also require the following straightforward generalization of [3, Theorem 7.33].

Lemma 4.2.2. *Let $d \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{H}$ be a unitarily invariant space on $\mathbb{B}_d$ with reproducing kernel*

$$K(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n,$$

where $a_0 = 1$. Assume that $a_1 > 0$. Then the following are equivalent:

(i) *$\mathcal{H}$ is an irreducible complete Nevanlinna-Pick space.*

(ii) *The sequence $(b_n)_{n=1}^\infty$ defined by*

$$\sum_{n=1}^{\infty} b_n t^n = 1 - \frac{1}{\sum_{n=0}^\infty a_n t^n}$$

for t in a neighbourhood of 0 is a sequence of non-negative real numbers.

In particular, if (ii) holds, then $\mathcal{H}$ is automatically irreducible.

Proof. Observe that

$$1 - \frac{1}{K(z, w)} = \sum_{n=1}^{\infty} b_n \langle z, w \rangle^n.$$

It is known that this kernel is positive if and only if $b_n \geq 0$ for all $n \geq 1$ (see the proof of [3, Theorem 7.33], and also Corollary 4.6.3 below). Consequently, the implication (i) ⇒

(ii) follows from [3, Theorem 7.31], and (ii) $\Rightarrow$ (i) will follow from the same result, once we observe that $\mathcal{H}$ is irreducible in the setting of (ii).

Since $a_0 = 1$ and $a_1 > 0$, the space $\mathcal{H}$ contains the constant function 1 and the coordinate functions (see [39, Proposition 4.1] or [38, Section 4]), from which it readily follows that $K(\cdot, x)$ and $K(\cdot, y)$ are linearly independent if $x \neq y$. We finish the proof by showing that $\sum_{n=0}^{\infty} a_n t^n$ never vanishes on $\mathbb{D}$. Assume toward a contradiction that $t_0 \in \mathbb{D}$ is a zero of $\sum_{n=0}^{\infty} a_n t^n$ of minimal modulus. Then the equality in (ii) holds for all $t \in \mathbb{D}$ with $|t| < |t_0|$, and t_0 is a pole of $\sum_{n=1}^{\infty} b_n t^n$. Since $b_n \geq 0$ for $n \geq 1$, this implies that $|t_0|$ is a pole of $\sum_{n=1}^{\infty} b_n t^n$, and consequently $|t_0|$ is a zero of $\sum_{n=0}^{\infty} a_n t^n$. This is a contradiction, since $a_0 = 1$ and $a_n \geq 0$ for $n \geq 0$, and the proof is complete. □

We observe that the Drury-Arveson space H^2_m is a unitarily invariant complete Nevanlinna-Pick space on $\mathbb{B}_m$. Indeed, it corresponds to the choice $a_n = 1$ for all $n \in \mathbb{N}$ above, since its reproducing kernel is given by

$$k_m(z, w) = \frac{1}{1 - \langle z, w \rangle}.$$

Recall from Section 2.5 the universality theorem of Agler and McCarthy:

Theorem 4.2.3 (Agler-McCarthy)**.** *If $\mathcal{H}$ is a normalized irreducible complete Nevanlinna-Pick space on a set X with kernel K, then there exists $m \in \mathbb{N} \cup \{\infty\}$ and an embedding $j : X \to \mathbb{B}_m$ such that*

$$K(z, w) = k_m(j(z), j(w)) \quad (z, w \in X).$$

In this case, $f \mapsto f \circ j$ defines a unitary operator from $H^2_m\big|_{j(X)}$ onto $\mathcal{H}$.

In this setting, we say that j is *an embedding for $\mathcal{H}$*.

4.3. From multiplier algebras to kernels

We begin by observing that the kernel of a Nevanlinna-Pick space can be recovered from the isometric structure of its multiplier algebra. Results similar to the next proposition are well known, see for example [38] and [3], especially Exercise 8.35. Since we do not have a reference for the exact statement, a complete proof is provided.

Proposition 4.3.1. *Let $\mathcal{H}$ be an irreducible reproducing kernel Hilbert space on a set X with kernel K. Suppose that K is normalized at $x_0 \in X$ and satisfies the two-point Nevanlinna-Pick property. Then*

$$\sup\{\operatorname{Re}\varphi(w) : ||\varphi||_{\mathrm{Mult}(\mathcal{H})} \leq 1 \text{ and } \varphi(x_0) = 0\} = \Big(1 - \frac{1}{K(w,w)}\Big)^{1/2}$$

for every $w \in X$, and this number is strictly positive if $w \neq x_0$. Moreover, there is a unique multiplier φ_w which achieves the supremum if $w \neq x_0$, namely

$$\varphi_w(z) = \frac{1 - \frac{1}{K(z,w)}}{\sqrt{1 - \frac{1}{K(w,w)}}}.$$

Equivalently,

$$K(z,w) = \frac{1}{1 - \varphi_w(z)\varphi_w(w)}.$$

Proof. By the two-point Nevanlinna-Pick property, there exists a contractive multiplier φ with $\varphi(x_0) = 0$ and $\varphi(w) = \lambda$ if and only if the Pick matrix at points (x_0, w),

$$\begin{pmatrix} 1 & 1 \\ 1 & K(w,w)(1 - |\lambda|^2) \end{pmatrix},$$

is positive, which, in turn, happens if and only if

$$K(w,w) \geq \frac{1}{1 - |\lambda|^2}.$$

This proves the formula for the supremum. Moreover, we see that the supremum is actually attained.

Irreducibility of $\mathcal{H}$ implies that $K(w,w) > 1$ if $w \neq x_0$. Indeed, since K is normalized at x_0, we have

$$1 = K(x_0, w) = |\langle K(\cdot, w), K(\cdot, x_0)\rangle| \leq K(w,w)^{1/2}$$

by Cauchy-Schwarz, with equality occurring only if $K(\cdot, w)$ and $K(\cdot, x_0)$ are linearly dependent. Since $\mathcal{H}$ is irreducible, this only happens if $w = x_0$.

Let $\varphi = \varphi_w$ be any multiplier which achieves the supremum. If $z \in X$ is arbitrary, then the Pick matrix at points (x_0, w, z),

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & K(w,z)(1 - \varphi(w)\overline{\varphi(z)}) \\ 1 & K(z,w)(1 - \varphi(z)\overline{\varphi(w)}) & K(z,z)(1 - |\varphi(z)|^2) \end{pmatrix},$$

is positive, since $||\varphi||_{\mathrm{Mult}(\mathcal{H})} \le 1$ (observe that the three-point Nevanlinna-Pick property is not needed for this implication). The determinant of this matrix is

$$-|1 - K(z,w)(1 - \varphi(z)\overline{\varphi(w)})|^2,$$

hence

$$K(z,w)(1 - \varphi_w(z)\overline{\varphi_w(w)}) = 1.$$

Since

$$\varphi_w(w) = \Big(1 - \frac{1}{K(w,w)}\Big)^{1/2},$$

the formula for φ_w follows. In particular, φ_w is unique if $w \neq x_0$. □

The following consequence, which generalizes Section 5.4 in [3], is immediate.

Corollary 4.3.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two irreducible Nevanlinna-Pick spaces on the same set X, with kernels K_1 and K_2, respectively, which are normalized at a point $x_0 \in X$. Then the following are equivalent:*

(i) $\mathrm{Mult}(\mathcal{H}_1) = \mathrm{Mult}(\mathcal{H}_2)$ *isometrically.*

(ii) $\mathrm{Mult}(\mathcal{H}_1) = \mathrm{Mult}(\mathcal{H}_2)$ *completely isometrically.*

(iii) $\mathcal{H}_1 = \mathcal{H}_2$ *isometrically.*

(iv) $K_1 = K_2$.

Proof. The implications (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) are clear. The implication (i) ⇒ (iv) follows from the preceding proposition. □

Observe that the last result is generally false without the assumption that both spaces are Nevanlinna-Pick spaces. Indeed, the Hardy space and the Bergman space on the unit disc both have $H^\infty(\mathbb{D})$ as their multiplier algebra.

We can also use Proposition 4.3.1 to show that certain algebras of functions are not multiplier algebras of complete Nevanlinna-Pick spaces. For $H^\infty(\mathbb{B}_d)$, this is done in Proposition 8.83 in [3].

Corollary 4.3.3. *There is no irreducible reproducing kernel Hilbert space on $\mathbb{D}^d$ for $d \geq 2$ which satisfies the two-point Nevanlinna-Pick property and whose multiplier algebra is $H^\infty(\mathbb{D}^d)$.*

Proof. Let $a = 1/2$ and let $w = (a, a, 0, \dots) \in \mathbb{D}^d$. Let $\varphi \in H^\infty(\mathbb{D}^d)$ be non-constant with $||\varphi||_\infty \le 1$ and $\varphi(0) = 0$. Then $z \mapsto \varphi(z, z, 0, \dots)$ defines an analytic map from $\mathbb{D}$ into $\mathbb{D}$ which fixes the origin, hence $|\varphi(a)| \le 1/2$ by the Schwarz lemma. In particular,

$$\sup\{\mathrm{Re}(\varphi(a)) : ||\varphi||_{H^\infty(\mathbb{D}^d)} \le 1 \text{ and } \varphi(0) = 0\} \le \frac{1}{2}.$$

But there are several functions which realize the value 1/2, for example the coordinate projections z_1 and z_2. In particular, the extremal problem with normalization point 0 in Proposition 4.3.1 does not have a unique solution. Since every non-vanishing kernel can be normalized at an arbitrary point without changing the multiplier algebra (see Section 2.5 or [3, Section 2.6]), it follows that $H^\infty(\mathbb{D}^d)$ is not the multiplier algebra of an irreducible reproducing kernel Hilbert space which satisfies the two-point Nevanlinna-Pick property. □

We now consider a second way of recovering the reproducing kernel of a complete Nevanlinna-Pick space from its multiplier algebra. In contrast to Proposition 4.3.1, this approach uses the operator space structure of the multiplier algebra.

If $\varphi : X \to \mathcal{B}(\mathcal{E}, \mathbb{C})$ is a function with $||\varphi(x)|| < 1$ for all $x \in X$, where $\mathcal{E}$ is an auxiliary Hilbert space, we define a kernel K_φ on X by

$$K_\varphi(z, w) = \frac{1}{1 - \varphi(z)\varphi(w)^*}.$$

Expressing the last identity as a geometric series, we see that K_φ is positive definite.

Proposition 4.3.4. *Let $\mathcal{H}$ be an irreducible reproducing kernel Hilbert space on a set X with kernel K, normalized at $x_0 \in X$. Then K is an upper bound for the set*

$$\Big\{K_\varphi : \varphi \in \mathrm{Mult}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H}) \text{ with } ||M_\varphi|| \le 1 \text{ and } \varphi(x_0) = 0\Big\}$$

with respect to the partial order given by positivity. Moreover, K is the maximum of this set if and only if $\mathcal{H}$ is a complete Nevanlinna-Pick space.

Proof. We first observe that every φ as in the proposition maps X into the open unit ball of $\mathcal{B}(\mathcal{E}, \mathbb{C})$. To this end, let $x \in X$, and consider the Pick matrix associated to $\{x_0, x\}$. Since K is normalized at x_0, and since $\varphi(x_0) = 0$, we obtain

$$\begin{pmatrix} 1 & 1 \\ 1 & K(x,x)(1 - \varphi(x)\varphi(x)^*) \end{pmatrix},$$

so this matrix is positive. In particular, the (2,2)-entry is necessarily bounded above by 1, so that

$$||\varphi(x)||^2 \leq 1 - \frac{1}{K(x,x)} < 1.$$

Now, if $\varphi : X \to \mathcal{B}(\mathcal{E}, \mathbb{C})$ is a multiplier of norm at most 1, then K/K_φ is positive definite by Lemma 2.3.1. If in addition $\varphi(x_0) = 0$, then K_φ is normalized at 0, hence so is K/K_φ. This implies that $K/K_\varphi - 1$ is positive definite (see, for example, the proof of Corollary 4.2 in [37]), and thus also

$$K - K_\varphi = K_\varphi\Big(\frac{K}{K_\varphi} - 1\Big)$$

is positive definite by the Schur product theorem. Consequently, $K_\varphi \leq K$.

If K belongs to the set in the statement of the proposition, then $1 - 1/K$ is positive definite, so $\mathcal{H}$ is a complete Nevanlinna-Pick space by Theorem 2.5.1.

Assume now that $\mathcal{H}$ is a complete Nevanlinna-Pick space, so that we can write

$$K(z,w) = \frac{1}{1 - \langle b(z), b(w)\rangle}$$

for some function $b : X \to \mathbb{B}_\infty$ by Theorem 2.5.2. Consider for $z \in X$ the row operator

$$\varphi(z) = (b_1(z), b_2(z), \dots) \in \mathcal{B}(\ell^2, \mathbb{C}),$$

where the b_i are the coordinate functions of b. Since

$$K(z,w)(1 - \varphi(z)\varphi(w)^*) = 1,$$

we have $\varphi \in \operatorname{Mult}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ with $||\varphi|| \leq 1$, and $K = K_\varphi$. Also, $\varphi(x_0) = 0$ since K is normalized at x_0. □

One advantage of this second approach is that we also obtain information about inclusions of multiplier algebras.

Corollary 4.3.5. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be reproducing kernel Hilbert spaces on the same set X with kernels K_1 and K_2, respectively. Assume that $\mathcal{H}_1$ is an irreducible complete Nevanlinna-Pick space, and suppose that K_1 and K_2 are both normalized at $x_0 \in X$. Then the following are equivalent:*

(i) $\operatorname{Mult}(\mathcal{H}_1) \subset \operatorname{Mult}(\mathcal{H}_2)$, *and the inclusion map is a complete contraction.*

(ii) K_2/K_1 *is positive definite.*

In this case, $\mathcal{H}_1 \subset \mathcal{H}_2$*, and the inclusion map is a contraction.*

Proof. (i) $\Rightarrow$ (ii) Proposition 4.3.4 yields a multiplier $\varphi \in \text{Mult}(\mathcal{H}_1 \otimes \mathcal{E}, \mathcal{H}_1)$ of norm at most 1 such that $K_1 = K_\varphi$. By assumption, φ is a multiplier on $\mathcal{H}_2$ of norm at most 1, hence $K_2/K_1 = K_2/K_\varphi$ is positive. Moreover, another application of the proposition shows that

$$K_1 = K_\varphi \leq K_2,$$

so that $\mathcal{H}_1 \subset \mathcal{H}_2$ and the inclusion map is a contraction.

(ii) $\Rightarrow$ (i) always holds for reproducing kernel Hilbert spaces. Indeed, φ belongs to the unit ball of $\text{Mult}(\mathcal{H}_1 \otimes \ell^2(n), \mathcal{H}_1 \otimes \ell^2(n))$ if and only if

$$K_1(z,w)(I - \varphi(z)\varphi(w)^*)$$

is a positive definite operator valued kernel. By assumption and the Schur product theorem, it follows that

$$K_2(z,w)(I - \varphi(z)\varphi(w)^*)$$

is positive definite, hence $\varphi \in \text{Mult}(\mathcal{H}_2 \otimes \ell^2(n), \mathcal{H}_2 \otimes \ell^2(n))$ with $||\varphi|| \leq 1$. □

We finish this section by observing that the completely bounded version of Corollary 4.3.2 is not true, that is, if the identity map from $\text{Mult}(\mathcal{H}_1)$ to $\text{Mult}(\mathcal{H}_2)$ is merely assumed to be a completely bounded isomorphism, then it does not follow that $\mathcal{H}_1 = \mathcal{H}_2$ as vector spaces.

Example 4.3.6. Let $\mathcal{D}$ be the Dirichlet space on $\mathbb{D}$, whose reproducing kernel is given by

$$K_{\mathcal{D}}(z,w) = -\frac{\log(1 - \overline{w}z)}{\overline{w}z}$$

and let $H^2 = H^2(\mathbb{D})$ be the Hardy space on $\mathbb{D}$ with reproducing kernel

$$K_{H^2}(z,w) = \frac{1}{1 - z\overline{w}}.$$

Then H^2 and $\mathcal{D}$ are complete Nevanlinna-Pick spaces (see, for example, [3, Corollary 7.41]).

Let $(z_n)_{n=0}^\infty$ be a sequence in $(0,1)$ with $z_0 = 0$ and $\lim_{n\to\infty} z_n = 1$ which is interpolating for the multiplier algebra of the Dirichlet space $\mathcal{D}$. Then (z_n) is also interpolating for $H^\infty = \text{Mult}(H^2)$, so if $V = \{z_n : n \in \mathbb{N}\}$, then $\text{Mult}(H^2\big|_V)$ and $\text{Mult}(\mathcal{D}\big|_V)$ are equal as algebras, since they are both equal to ℓ^∞.

In fact, the normalized kernels in $\mathcal{D}\big|_V$ and $H^2\big|_V$ form a Riesz system (see Section 9.3 in [2]), so there is a bounded invertible map

$$A : H^2\big|_V \to \mathcal{D}\big|_V \quad \text{such that} \quad A\Big(\frac{K_{H^2}(\cdot, w)}{||K_{H^2}(\cdot, w)||}\Big) = \frac{K_{\mathcal{D}}(\cdot, w)}{||K_{\mathcal{D}}(\cdot, w)||}$$

for all $w \in V$. A straightforward computation shows that if $\varphi \in \mathrm{Mult}(H^2\big|_V)$, then

$$((A^*)^{-1} M_\varphi A^* f)(w) = \varphi(w) f(w)$$

for $f \in \mathcal{D}\big|_V$ and $w \in V$, so

$$(A^*)^{-1} M_\varphi A^* = M_\varphi.$$

It follows that the identity map between $\mathrm{Mult}(H^2\big|_V)$ and $\mathrm{Mult}(\mathcal{D}\big|_V)$ is given by a similarity.

However, the spaces $H^2\big|_V$ and $\mathcal{D}\big|_V$ are not equal. Indeed, if $f \in \mathcal{D}$, then

$$|f(z)| = |\langle f, K_{\mathcal{D}}(\cdot, z)\rangle| \leq ||f|| \sqrt{K_{\mathcal{D}}(z, z)} \approx ||f|| \sqrt{-\log(1 - z^2)}$$

as $z \to 1$, but there are functions in H^2 which grow faster, such as

$$f(z) = \sum_{n=0}^{\infty} (n+1)^{-3/4} z^n,$$

for which

$$|f(z)| \approx \Gamma\Big(\frac{1}{4}\Big)(1 - z)^{-1/4}$$

as $z \to 1$ from below (see [89, Chap. XIII,p.280,ex. 7]).

4.4. Composition Operators

The methods of the last section also apply to composition operators on multiplier algebras. If K_1 and K_2 are two kernels on a set X, we say that K_1 is a rescaling of K_2 if there exists a nowhere vanishing function $\delta : X \to \mathbb{C}$ such that

$$K_1(z, w) = \delta(z)\overline{\delta(w)} K_2(z, w) \quad (z, w \in X).$$

Rescaling is an equivalence relation on kernels, and two kernels which are equivalent in this sense give rise to the same multiplier algebra (see Section 2.6 in [3]).

Proposition 4.4.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be irreducible complete Nevanlinna-Pick spaces on sets X_1 and X_2 with kernels K_1 and K_2, respectively. Suppose that $F : X_2 \to X_1$ is a bijection. Then the following are equivalent:*

(i) $C_F : \mathrm{Mult}(\mathcal{H}_1) \to \mathrm{Mult}(\mathcal{H}_2), \varphi \mapsto \varphi \circ F$, *is an isometric isomorphism.*

(ii) K_2 *is a rescaling of* $(K_1)_F$, *where* $(K_1)_F = K_1(F(z), F(w))$ *for* $z, w \in X_2$.

In fact, if

$$K_2(z,w) = \delta(z)\overline{\delta(w)}K_1(F(z),F(w)) \quad (z,w \in X_2)$$

for some nowhere vanishing function δ on X_2, then

$$U : \mathcal{H}_1 \to \mathcal{H}_2, \quad f \mapsto \delta(f \circ F),$$

is unitary, and $C_F = \mathrm{Ad}(U)$.

Proof. (i) $\Rightarrow$ (ii). We may assume that K_2 is normalized at a point $x_0 \in X_2$. Define a kernel K on X_2 by

$$K(z,w) = \frac{K_1(F(z),F(w))K_1(F(x_0),F(x_0))}{K_1(F(z),F(x_0))K_1(F(x_0),F(w))}$$

and let $\mathcal{H}$ be the reproducing kernel Hilbert space on X_2 with kernel K. Since K is a rescaling of $(K_1)_F$, the assumption implies that $\mathrm{Mult}(\mathcal{H}) = \mathrm{Mult}(\mathcal{H}_2)$, isometrically. Moreover, K is normalized at x_0, hence $K_2 = K$ by Corollary 4.3.2.

(ii) $\Rightarrow$ (i). This implication holds in general, without the assumption that the kernels are complete Nevanlinna-Pick kernels. To see this, it suffices to show the additional assertion. It is a standard fact from the theory of reproducing kernels that U is unitary. Indeed, the adjoint of U satisfies

$$U^* K_2(\cdot, w) = \overline{\delta(w)} K_1(\cdot, F(w))$$

for all $w \in X_2$, thus the assumption easily implies that U^* is unitary. Moreover, for $f \in \mathcal{H}_2$ and $\varphi \in \mathrm{Mult}(\mathcal{H}_1)$, we have

$$U M_\varphi U^* f = U(\varphi \frac{1}{\delta}(f \circ F^{-1})) = (\varphi \circ F) f,$$

hence $C_F = \mathrm{Ad}(U)$ is a well-defined completely isometric isomorphism. □

The last result applies in particular to automorphisms of multiplier algebras.

Corollary 4.4.2. *Let $\mathcal{H}$ be an irreducible complete Nevanlinna-Pick space on a set X with kernel K, normalized at x_0, and let $F : X \to X$ be a bijection and $a = F^{-1}(x_0)$. Then C_F is an isometric automorphism of* $\operatorname{Mult}(\mathcal{H})$ *if and only if*

$$K(F(z), F(w)) = \frac{K(z,w)K(a,a)}{K(z,a)K(a,w)}$$

for all $z, w \in X$.

Proof. This follows from the preceding proposition as K_F is normalized at a. □

We wish to apply the preceding result to spaces of analytic functions on $\mathbb{B}_d$. The group of conformal automorphisms of $\mathbb{B}_d$ is denoted by $\operatorname{Aut}(\mathbb{B}_d)$. We also allow the case $d = \infty$, see [46] and the references therein.

Proposition 4.4.3. *Let $d \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions on $\mathbb{B}_d$ with kernel K. Assume that K is normalized at 0 and does not vanish anywhere on $\mathbb{B}_d$. Then the identity*

$$K(\varphi(z), \varphi(w)) = \frac{K(z,w)K(a,a)}{K(z,a)K(a,w)} \quad (z, w \in \mathbb{B}_d), \tag{4.1}$$

where $a = \varphi^{-1}(0)$, holds for every $\varphi \in \operatorname{Aut}(\mathbb{B}_d)$ if and only if

$$K(z,w) = \frac{1}{(1 - \langle z, w \rangle)^\alpha}$$

for some $\alpha \in [0, \infty)$.

Proof. It is well-known that (4.1) holds if $K(z,w) = (1 - \langle z, w \rangle)^{-1}$, see [75, Theorem 2.2.5]. When raising this identity to the power of α, care must be taken if α is not an integer. However, (4.1) holds for arbitrary α, and $z, w \in \mathbb{B}_d$ with $||z||$ small, as $K(z,w)$ is close to 1 in this case. Since both sides of (4.1) are analytic in z, it holds for all $z \in \mathbb{B}_d$.

Conversely, suppose that (4.1) holds for all automorphisms φ. Choosing φ to be unitary, it follows that $K(Uz, Uw) = K(z,w)$ for all unitary operators U on $\mathbb{C}^d$. By Lemma 4.2.1, there exists an analytic function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 1$ and non-negative derivatives at 0 such that

$$K(z,w) = f(\langle z, w \rangle).$$

We wish to show that $f(z) = (1 - z)^{-\alpha}$ for some $\alpha \in [0, \infty)$. Since every conformal automorphism of $\mathbb{D}$ extends to a conformal automorphism of $\mathbb{B}_d$ (see [75, Section 2.2.8]), it suffices to prove this for the case $d = 1$.

For $r \in (-1,1)$, consider the conformal automorphism φ_r of $\mathbb{D}$ given by

$$\varphi_r(z) = \frac{r-z}{1-rz}.$$

Then for $z \in \mathbb{D}$ and $w \in (0,1)$, we have

$$f(\varphi_r(z)\varphi_r(w)) = \frac{f(zw)f(r^2)}{f(rz)f(rw)},$$

hence

$$f(\varphi_r(z)\varphi_r(w))f(rz)f(rw) = f(zw)f(r^2).$$

Taking the derivative with respect to r at $r = 0$, and simplifying, we obtain

$$(z+w)(f'(zw)(zw-1) + f(zw)f'(0))) = 0$$

for all $z \in \mathbb{D}$ and $w \in (0,1)$, hence

$$f'(z)(1-z) - f(z)f'(0) = 0.$$

for all $z \in \mathbb{D}$, and $f(0) = 1$. This is a first order linear ODE, whose solutions are given by

$$f(z) = (1-z)^{-\alpha},$$

where $\alpha = f'(0)$. Since $f'(0) \geq 0$, the result follows. □

The desired result about complete Nevanlinna-Pick spaces on $\mathbb{B}_d$ whose multiplier algebras are isometrically automorphism invariant is the following corollary.

Corollary 4.4.4. *Let $d \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions on $\mathbb{B}_d$ with kernel K, normalized at 0. The following are equivalent:*

(i) *$\mathcal{H}$ is an irreducible complete Nevanlinna-Pick space and every $\varphi \in \mathrm{Aut}(\mathbb{B}_d)$ induces an isometric composition operator on $\mathrm{Mult}(\mathcal{H})$.*

(ii) *There exists $\alpha \in (0,1]$ such that*

$$K(z,w) = \frac{1}{(1-\langle z,w\rangle)^\alpha} \quad (z,w \in \mathbb{B}_d).$$

Proof. In light of Corollary 4.4.2 and Proposition 4.4.3, it suffices to show that

$$K(z,w) = \frac{1}{(1-\langle z,w\rangle)^\alpha}$$

is an irreducible complete Nevanlinna-Pick kernel if and only if $\alpha \in (0,1]$.

If $\alpha = 0$, then K is identically 1, and thus not irreducible. If $\alpha > 0$, then Lemma 4.2.2 applies to show that K is an irreducible complete Nevanlinna-Pick kernel if and only if the function $1-(1-x)^\alpha$ has non-negative Taylor coefficients at 0. Observe that

$$1-(1-x)^\alpha = \sum_{k=1}^\infty (-1)^{k+1} \binom{\alpha}{k} x^k,$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}.$$

The coefficient of x^2 in this formula equals

$$-\frac{\alpha(\alpha-1)}{2},$$

which is negative if $\alpha > 1$. Conversely, if $\alpha \le 1$, then all Taylor coefficients are non-negative. □

4.5. Algebraic consistency and varieties

When studying isomorphisms of multiplier algebras, we will usually make an additional assumption, which, roughly speaking, guarantees that the functions in the reproducing kernel Hilbert space are defined on their natural domain of definition.

More precisely, let $\mathcal{H}$ be a Hilbert function space on a set X with $1 \in \mathcal{H}$. A non-zero bounded linear functional ρ on $\mathcal{H}$ is called *partially multiplicative* if $\rho(\varphi f) = \rho(\varphi)\rho(f)$ whenever $\varphi \in \operatorname{Mult}(\mathcal{H})$ and $f \in \mathcal{H}$. We say that $\mathcal{H}$ is *algebraically consistent* if for every partially multiplicative functional ρ on $\mathcal{H}$, there exists $x \in X$ such that $\rho(f) = f(x)$ for all $f \in \mathcal{H}$.

Example 4.5.1. The reproducing kernel Hilbert space $\mathcal{H}$ on $\mathbb{D}$ with kernel

$$K(z,w) = \sum_{n=0}^\infty 2^{-n}(z\overline{w})^n = \frac{1}{1-\frac{1}{2}z\overline{w}}$$

is not algebraically consistent. Indeed, every function in $\mathcal{H}$ extends uniquely to an analytic function on the open disc of radius $\sqrt{2}$ around the origin.

Remark 4.5.2. Our definitions of a partially multiplicative functional and of algebraic consistency are inspired by [12, Definition 1.5] of Cowen-MacCluer, but are slightly different. A non-zero bounded linear functional ρ on $\mathcal{H}$ is partially multiplicative in the sense of Cowen-MacCluer if $\rho(fg) = \rho(f)\rho(g)$ whenever $f, g \in \mathcal{H}$ such that the pointwise product fg belongs to $\mathcal{H}$ as well. The Hilbert function space $\mathcal{H}$ is algebraically consistent in the sense of Cowen-MacCluer if every such functional is given by evaluation at a point in X. We also refer the reader to [57, Section 2]; their *generalized kernel functions* are precisely the elements of $\mathcal{H}$ which give rise to partially multiplicative functionals in the sense of Cowen-MacCluer.

Clearly, every functional which is partially multiplicative in the sense of Cowen-MacCluer is partially multiplicative in our sense. Therefore, every Hilbert function space which is algebraically consistent in our sense is algebraically consistent in the sense of Cowen-MacCluer.

Our definition of algebraic consistency requires that $1 \in \mathcal{H}$ and is only meaningful if $\mathcal{H}$ has "enough" multipliers. This limits its applicability for general Hilbert function spaces. However, it seems to be well-suited for normalized irreducible complete Nevanlinna-Pick spaces. In particular, we will see that in this setting, algebraic consistency in our sense is closely related to the notion of a variety from [25] (see Proposition 4.5.6 below) and behaves well with respect to restrictions of complete Nevanlinna-Pick spaces to subsets (see Lemma 4.5.4 below). Moreover, if $\mathcal{H}$ is a normalized irreducible complete Nevanlinna-Pick space, then $1 \in \mathcal{H}$ and the multiplier algebra contains at least all kernel functions (this known fact can be deduced, for example, from Proposition 4.3.1, as $\psi_w = 1 - 1/K(\cdot, w)$ is a strictly contractive multiplier, so $K(\cdot, w) = \sum_{n=0}^\infty \psi_w^n$ converges absolutely in the Banach algebra $\operatorname{Mult}(\mathcal{H})$). In particular, $\operatorname{Mult}(\mathcal{H})$ is dense in $\mathcal{H}$. It remains open if the two definitions of algebraic consistency agree for normalized irreducible complete Nevanlinna-Pick spaces (see also Remark 4.5.5 below).

The following lemma provides examples of algebraically consistent spaces (compare with [12, Theorem 2.15]). The proof in fact shows that for unitarily invariant spaces, our notion of algebraic consistency and the one of Cowen-MacCluer coincide.

Lemma 4.5.3. *Let $d \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{H}$ be a complete Nevanlinna-Pick space on $\mathbb{B}_d$ with kernel of the form*

$$K(z, w) = \sum_{n=0}^\infty a_n \langle z, w \rangle^n$$

such that $a_0 = 1$ and $a_1 \neq 0$.

(a) *If $\sum_{n=0}^\infty a_n = \infty$, then $\mathcal{H}$ is algebraically consistent on $\mathbb{B}_d$.*

(b) *If* $\sum_{n=0}^{\infty} a_n < \infty$, *but the series* $\sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence* 1, *then the functions in* $\mathcal{H}$ *extend to (norm) continuous functions on* $\overline{\mathbb{B}_d}$, *and* $\mathcal{H}$ *is an algebraically consistent space of functions on* $\overline{\mathbb{B}_d}$.

(c) *If* $\sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence greater than* 1, *then* $\mathcal{H}$ *is not algebraically consistent on* $\mathbb{B}_d$ *or on* $\overline{\mathbb{B}_d}$.

Proof. We begin with some considerations that apply to both (a) and (b). It is known that the condition $a_1 \neq 0$ implies that the function $\langle \cdot, w \rangle$ is a multiplier for $w \in \mathbb{B}_d$ (see, for example, [38, Section 4]). Incidentally, this can also be deduced from Proposition 4.6.4 below. For each i, let

$$\lambda_i = \rho(z_i).$$

We claim that $(\lambda_i) \in \overline{\mathbb{B}_d}$. To this end, let $w \in \mathbb{B}_d$ be finitely supported, say $w_i = 0$ if $i > N$. Then

$$\langle \lambda, w \rangle = \sum_{i=1}^{N} \lambda_i \overline{w_i} = \rho(\langle \cdot, w \rangle).$$

Since ρ is partially multiplicative and non-zero, $\rho(1) = 1$. Thus, we get

$$\rho(K(\cdot, w)) = \sum_{n=0}^{\infty} a_n \rho(\langle \cdot, w \rangle^n) = \sum_{n=0}^{\infty} a_n \rho(\langle \cdot, w \rangle)^n = \sum_{n=0}^{\infty} a_n \langle \lambda, w \rangle^n. \tag{4.2}$$

In either case, the series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1, hence $|\langle \lambda, w \rangle| \leq 1$. Since $w \in \mathbb{B}_d$ was an arbitrary finitely supported sequence, we conclude that $\lambda \in \overline{\mathbb{B}_d}$.

Assume now that $\sum_{n=0}^{\infty} a_n = \infty$. We wish to show that $\lambda \in \mathbb{B}_d$. Suppose for a contradiction that $||\lambda|| = 1$. Observe that (4.2) holds for all $w \in \mathbb{B}_d$, so choosing $w = r\lambda$ for $0 < r < 1$, we see that

$$\sum_{n=0}^{\infty} a_n r^n = \rho(K(\cdot, r\lambda)) \leq ||\rho|| \left(\sum_{n=0}^{\infty} a_n r^{2n} \right)^{1/2} \leq ||\rho|| \left(\sum_{n=0}^{\infty} a_n r^n \right)^{1/2},$$

which is not possible as $\sum_{n=0}^{\infty} a_n = \infty$. Consequently, $\lambda \in \mathbb{B}_d$, and it follows from (4.2) that ρ equals point evaluation at λ. This proves (a).

For the proof of (b), we observe that K extends to a jointly norm continuous function on $\overline{\mathbb{B}_d} \times \overline{\mathbb{B}_d}$, hence all functions in $\mathcal{H}$ extend to norm continuous functions on $\overline{\mathbb{B}_d}$, and $\mathcal{H}$ becomes a reproducing kernel Hilbert space on $\overline{\mathbb{B}_d}$ in this way. Equation (4.2) shows that every partially multiplicative functional is given by point evaluation at a point $\lambda \in \overline{\mathbb{B}_d}$, so that $\mathcal{H}$ is algebraically consistent.

Finally, to show (c), we observe that if $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 1$, then the functions in $\mathcal{H}$ extend uniquely to analytic functions on the ball of radius $\sqrt{R}$. In particular, $\mathcal{H}$ is not algebraically consistent on $\mathbb{B}_d$ or on $\overline{\mathbb{B}_d}$. □

To show that algebraic consistency is closely related to the notion of a variety from [25], we first need a simple lemma.

Lemma 4.5.4. *Let $\mathcal{H}$ be a normalized irreducible complete Nevanlinna-Pick space on a set X which is algebraically consistent. If $Y \subset X$, then $\mathcal{H}\big|_Y$ is an algebraically consistent space of functions on Y if and only if there is a set of functions $S \subset \mathcal{H}$ such that*

$$Y = \{x \in X : f(x) = 0 \text{ for all } f \in S\}.$$

Proof. Suppose that Y is the common vanishing locus of a set $S \subset \mathcal{H}$, and let ρ be a partially multiplicative functional on $\mathcal{H}\big|_Y$. Then $\widetilde{\rho}(f) = \rho(f\big|_Y)$ defines a partially multiplicative functional on $\mathcal{H}$. Since $\mathcal{H}$ is assumed to be algebraically consistent, $\widetilde{\rho}$ is given by point evaluation at a point $y \in X$. We claim that $y \in Y$. To this end, observe that for $f \in S$, we have

$$f(y) = \widetilde{\rho}(f) = \rho(f\big|_Y) = \rho(0) = 0,$$

from which we deduce that $y \in Y$. Since every function in $\mathcal{H}\big|_Y$ is the restriction of a function in $\mathcal{H}$, it follows that ρ is given by evaluation at y. Hence, $\mathcal{H}\big|_Y$ is algebraically consistent.

Conversely, assume that $\mathcal{H}\big|_Y$ is algebraically consistent. Let S be the kernel of the restriction map $\mathcal{H} \to \mathcal{H}\big|_Y$ and let $\widehat{Y}$ denote the vanishing locus of S. Clearly, $Y \subset \widehat{Y}$, and we wish to show that $Y = \widehat{Y}$. To this end, observe that every function $f \in \mathcal{H}\big|_Y$ extends uniquely to a function $\widehat{f} \in \mathcal{H}\big|_{\widehat{Y}}$ of the same norm. Assume for a contradiction that there exists $x \in \widehat{Y} \setminus Y$. Then we obtain a bounded functional ρ on $\mathcal{H}\big|_Y$ which is defined by $\rho(f) = \widehat{f}(x)$. To see that ρ is partially multiplicative, note that if $\varphi \in \operatorname{Mult}(\mathcal{H}\big|_Y)$, then by the Nevanlinna-Pick property (see Lemma 2.4.3), φ extends to a multiplier on $\mathcal{H}\big|_{\widehat{Y}}$, which necessarily equals $\widehat{\varphi}$. Thus, $\widehat{\varphi f} = \widehat{\varphi}\widehat{f}$. Since $\mathcal{H}$ is irreducible, it separates the points of X, so ρ is not equal to point evaluation at a point in Y, a contradiction. Therefore, $Y = \widehat{Y}$. □

Remark 4.5.5. It is the second part of the above proof where the difference between our definition of partially multiplicative functional and the one of Cowen-MacCluer is important. Whereas the functional ρ constructed above is partially multiplicative in our sense, it does not seem to be clear if ρ is partially multiplicative in the sense of Cowen-MacCluer.

Using notation as in the proof, the crucial question is the following: If $f, g \in \mathcal{H}\big|_Y$ such that $fg \in \mathcal{H}\big|_Y$, is $\widehat{fg} = \widehat{f}\widehat{g}$?

It is not hard to see that the following properties are equivalent for a normalized irreducible complete Nevanlinna-Pick space $\mathcal{H}$ on a set X and a subset $Y \subset X$:

(i) Whenever $f, g \in \mathcal{H}\big|_Y$ such that $fg \in \mathcal{H}\big|_Y$, then $\widehat{fg} = \widehat{f}\widehat{g}$.

(ii) Whenever $f, g \in \mathcal{H}\big|_Y$ such that $fg \in \mathcal{H}\big|_Y$, then $\widehat{f}\widehat{g} \in \mathcal{H}\big|_{\widehat{Y}}$.

(iii) Whenever $h_1, h_2, h_3 \in \mathcal{H}$ such that $h_1 = h_2 h_3$ on Y, then $h_1 = h_2 h_3$ on $\widehat{Y}$.

Here, as in the proof, $\widehat{Y}$ denotes the vanishing locus of the kernel of the restriction map $\mathcal{H} \to \mathcal{H}\big|_Y$, which is the smallest common zero set of a family of functions in $\mathcal{H}$ which contains Y. Moreover, for $f \in \mathcal{H}\big|_Y$, the unique extension of f to a function in $\mathcal{H}\big|_{\widehat{Y}}$ is denoted by $\widehat{f}$.

Property (iii) and hence all properties are satisfied if $\mathcal{H} = H^2(\mathbb{D})$, the Hardy space on the unit disc, and $Y \subset \mathbb{D}$ is any subset, since the product of two functions in $H^2(\mathbb{D})$ belongs to $H^1(\mathbb{D})$, and the zero sets of families of functions in $H^2(\mathbb{D})$ and $H^1(\mathbb{D})$ coincide (they are precisely the Blaschke sequences in $\mathbb{D}$, see [36, Section II.2]).

It does not seem to be known if these properties hold if $\mathcal{H} = H^2_d$ for $d \geq 2$ and $Y \subset \mathbb{B}_d$ is an arbitrary subset. If they always hold in this case, then the arguments of this section show that our notion of algebraic consistency and the one of Cowen-MacCluer agree for normalized irreducible complete Nevanlinna-Pick spaces. We also refer the reader to [57, Section 5], where it is shown these properties hold for $\mathcal{H} = H^2_\infty$ and certain special subsets Y of $\mathbb{B}_\infty$.

Let $\mathcal{H}$ be a normalized irreducible complete Nevanlinna-Pick space on X with kernel K. Recall from Section 4.2 that an embedding for $\mathcal{H}$ is an injective function $j : X \to \mathbb{B}_m$ such that

$$K(z, w) = k_m(j(z), j(w)) \quad (z, w \in X),$$

where k_m denotes the kernel of the Drury-Arveson space on $\mathbb{B}_m$. A *variety* in $\mathbb{B}_m$ (see [25, Section 2]) is the common zero set of a family of functions in H^2_m.

Proposition 4.5.6. *Let $\mathcal{H}$ be a normalized irreducible complete Nevanlinna-Pick space on a set X with kernel K. The following assertions are equivalent:*

(i) *$\mathcal{H}$ is algebraically consistent.*

(ii) *There exists an embedding $j : X \to \mathbb{B}_m$ for $\mathcal{H}$ such that $j(X)$ is a variety.*

(iii) *For every embedding $j : X \to \mathbb{B}_m$ for $\mathcal{H}$, the set $j(X)$ is a variety.*

(iv) *Every weak-$*$ continuous character on* $\text{Mult}(\mathcal{H})$ *is given by evaluation at a point in* X.

Proof. Let $j : X \to \mathbb{B}_m$ be an embedding for $\mathcal{H}$, and let $V = j(X)$. Then

$$U : H^2_m\big|_V \to \mathcal{H}, \quad f \mapsto f \circ j,$$

is a unitary operator, and consideration of the map $T \mapsto UTU^*$ shows that U maps $\text{Mult}(H^2_m\big|_V)$ onto $\text{Mult}(\mathcal{H})$. Thus, $\mathcal{H}$ is algebraically consistent if and only if $H^2_m\big|_V$ is. Observe that H^2_m is algebraically consistent by Lemma 4.5.3. Thus, the equivalence of (i), (ii) and (iii) follows from Lemma 4.5.4.

To see that (iii) implies (iv), we note that the identification of $\text{Mult}(H^2_m\big|_V)$ with $\text{Mult}(\mathcal{H})$ from the first part is a weak-$*$-weak-$*$ homeomorphism, since it is implemented by conjugation with a unitary operator. Thus, the result follows from the fact that every weak-$*$ continuous character on $\text{Mult}(H^2_m\big|_V)$ is given by evaluation at a point in V, provided that V is a variety (see [25, Proposition 3.2]).

Conversely, suppose that (iv) holds, and let ρ be a partially multiplicative functional on $\mathcal{H}$. Then the restriction of ρ to $\text{Mult}(\mathcal{H})$ is a character. Since

$$\rho(\varphi) = \rho(M_\varphi 1) \quad \text{for all } \varphi \in \text{Mult}(\mathcal{H}),$$

it is weak-$*$ continuous. By assumption, there is a point $x \in X$ such that $\rho(\varphi) = \varphi(x)$ for all $\varphi \in \text{Mult}(\mathcal{H})$. Since $\text{Mult}(\mathcal{H})$ is dense in $\mathcal{H}$, it follows that ρ is given by evaluation at x. Consequently, $\mathcal{H}$ is algebraically consistent. □

In the setting of the last proposition, we identify X with a subset of the maximal ideal space of $\text{Mult}(\mathcal{H})$ via point evaluations.

Lemma 4.5.7. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be normalized algebraically consistent irreducible complete Nevanlinna-Pick spaces on sets* X_1 *and* X_2*, respectively. Let* $\Phi : \text{Mult}(\mathcal{H}_1) \to \text{Mult}(\mathcal{H}_2)$ *be a unital homomorphism. Then the following assertions are equivalent:*

(i) Φ *is weak-$*$-weak-$*$ continuous.*

(ii) $\Phi^*(X_2) \subset X_1$.

(iii) *There is a map* $F : X_2 \to X_1$ *such that*

$$\Phi(\varphi) = \varphi \circ F$$

for all $\varphi \in \text{Mult}(\mathcal{H}_1)$.

In this case, the map F in (iii) is the restriction of Φ^* to X_2.

Proof. The implication (i) $\Rightarrow$ (ii) follows immediately from the description of the weak-$*$ continuous characters in Proposition 4.5.6. Assume that (ii) holds, and let F denote the restriction of Φ^* to X_2. Then

$$\Phi(\varphi)(\lambda) = \Phi^*(\delta_\lambda)(\varphi) = (\delta_{F(\lambda)})(\varphi) = (\varphi \circ F)(\lambda)$$

for all $\lambda \in X_2$. Hence, Φ is given by composition with F, that is, (iii) holds.

To show that (iii) implies (i), it suffices to show that Φ is weak-$*$-weak-$*$ continuous on bounded sets by the Krein-Smulian theorem. This in turn follows from the general fact that for a bounded net of multipliers, convergence in the weak-$*$ topology is equivalent to pointwise convergence (see Lemma 2.2.4).

Finally, if F is as in (iii), then

$$\varphi(F(x)) = \Phi(\varphi)(x) = \varphi(\Phi^*(x))$$

for all $x \in X_2$ and all $\varphi \in \operatorname{Mult}(\mathcal{H}_1)$, so the assertion follows from the fact that $\operatorname{Mult}(\mathcal{H}_1)$ separates the points of X_1 as $\mathcal{H}_1$ is an irreducible complete Nevanlinna-Pick space (this can be deduced, for example, from Proposition 4.3.1). □

As a consequence, we see that weak-$*$-weak-$*$ homeomorphic isometric isomorphisms between multiplier algebras are always unitarily implemented. In [25], this was shown for spaces which admit an embedding into a finite dimensional ball using different methods. This additional assumption was recently removed in [76] by refining these methods.

Proposition 4.5.8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be normalized algebraically consistent irreducible complete Nevanlinna-Pick spaces on sets X_1 and X_2, respectively. Let $\Phi : \operatorname{Mult}(\mathcal{H}_1) \to \operatorname{Mult}(\mathcal{H}_2)$ be a unital isometric isomorphism. If Φ is a weak-$*$-weak-$*$ homeomorphism, then Φ is given by composition with a bijection $F : Y \to X$ and it is unitarily implemented.*

Proof. Lemma 4.5.7, applied to Φ and Φ^{-1}, shows that Φ is given by composition. Thus, Proposition 4.4.1 implies that Φ is unitarily implemented. □

4.6. Graded complete Nevanlinna-Pick spaces

In this section, we consider reproducing kernel Hilbert spaces which admit a natural grading. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with reproducing kernel K, and let X be equipped with an action of the circle group $\mathbb{T}$. We say that K is $\mathbb{T}$-invariant if

$$\mathbb{T} \to \mathbb{C}, \quad \lambda \mapsto K(\lambda z, w),$$

is continuous for all $z, w \in X$, and

$$K(\lambda z, \lambda w) = K(z, w)$$

for all $\lambda \in \mathbb{T}$ and $z, w \in X$. Then the $\mathbb{T}$-action on X induces a strongly continuous unitary representation

$$\Gamma : \mathbb{T} \to \mathcal{B}(\mathcal{H}), \quad \Gamma(\lambda)(f)(z) = f(\lambda z).$$

Indeed, $\Gamma(\lambda)$ is unitary for $\lambda \in \mathbb{T}$, and for $v, w \in X$, we have

$$\langle \Gamma(\lambda) K(\cdot, w), K(\cdot, v) \rangle = K(\lambda v, w),$$

which is continuous in λ. For $n \in \mathbb{Z}$, let

$$\mathcal{H}_n = \{ f \in \mathcal{H} : \Gamma(\lambda) f = \lambda^n f \text{ for all } \lambda \in \mathbb{T} \}.$$

Then the closed subspaces $\mathcal{H}_n$ are pairwise orthogonal, and it follows from a standard application of the Fejér kernel that

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n.$$

Elements of $\mathcal{H}_n$ are called homogeneous of degree n.

Example 4.6.1. (a) Let $d < \infty$ and $\Omega \subset \mathbb{C}^d$ be open and connected with $0 \in \Omega$ and $\mathbb{T}\Omega \subset \Omega$. Then $\mathbb{T}$ acts on Ω by scalar multiplication. Let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions on Ω with a $\mathbb{T}$-invariant kernel K. It is not hard to see that

$$\mathcal{H}_n = \{ f \in \mathcal{H} : f \text{ is a homogeneous polynomial of degree } n \}$$

for $n \geq 0$, and $\mathcal{H}_n = \{0\}$ for $n < 0$. Concrete examples of this type include many classical spaces on $\mathbb{B}_d$ or $\mathbb{D}^d$, such as the Hardy space and the Dirichlet space.

(b) Let $d \in \mathbb{N} \cup \{\infty\}$, and let $X \subset \mathbb{C}^d$ satisfy $\overline{\mathbb{D}} X \subset X$. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X with a $\mathbb{T}$-invariant kernel K, and assume that for $f \in \mathcal{H}$ and $x \in X$, the function

$$f_x : \overline{\mathbb{D}} \to \mathbb{C}, \quad z \mapsto f(zx),$$

is contained in the disc algebra. Then $\mathcal{H}_n = \{0\}$ for $n < 0$ and for $n \geq 0$, the space $\mathcal{H}_n$ consists of all functions f in $\mathcal{H}$ such that f_x is a multiple of z^n for every $x \in X$.

We require a homogeneous decomposition not only for functions, but also for kernels.

Lemma 4.6.2. *Let K be a $\mathbb{T}$-invariant positive definite kernel on X, possibly with zeroes on the diagonal. Then there are uniquely determined Hermitian kernels K_n on X such that for $z, w \in X$, we have*

(1) $K_n(\lambda z, w) = \lambda^n K_n(z, w)$ *for* $\lambda \in \mathbb{T}$ *and*

(2) $K(z, w) = \sum_{n \in \mathbb{Z}} K_n(z, w)$, *where the series converges absolutely.*

In this case, K_n is the reproducing kernel of the space of homogeneous elements of degree n in $\mathcal{H}$. In particular, each K_n is positive definite.

Proof. Let $\mathcal{H}$ be the reproducing kernel Hilbert space on X with kernel K. For $w \in X$, let

$$K(\cdot, w) = \sum_{n \in \mathbb{Z}} K_n(\cdot, w)$$

be the homogeneous expansion of $K(\cdot, w)$ in $\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n$. Observe that for $f \in \mathcal{H}_n$ and $w \in X$, we have

$$\langle f, K_n(\cdot, w) \rangle = \langle f, K(\cdot, w) \rangle = f(w),$$

hence K_n is the reproducing kernel of $\mathcal{H}_n$, and in particular positive definite. The first property is clear. Since convergence in $\mathcal{H}$ implies pointwise convergence on X, it follows that

$$K(z, w) = \sum_{n \in \mathbb{Z}} K_n(z, w).$$

Positive definiteness of K implies that $|K_n(z, w)|^2 \leq K_n(z, z) K_n(w, w)$, thus $|K_n(z, w)| \leq \max\{K_n(z, z), K_n(w, w)\}$, so the series converges absolutely.

The uniqueness statement follows from the uniqueness of the Fourier expansion of the continuous function

$$\lambda \mapsto K(\lambda z, w) = \sum_{n \in \mathbb{Z}} K_n(\lambda z, w) = \sum_{n \in \mathbb{Z}} \lambda^n K_n(z, w)$$

on $\mathbb{T}$. □

Incidentally, the last lemma provides a simple proof of the following known fact (cf. the proof of Theorem 7.33 in [3]).

Corollary 4.6.3. *Let $(a_n)_n$ be a sequence of complex numbers such that the power series $\sum_{n=0}^\infty a_n t^n$ has a positive radius of convergence R. Let $\mathcal{E}$ be a Hilbert space and let $B_R(0)$ denote the open ball of radius R around 0 in $\mathcal{E}$. Then the function K defined by*

$$K(z,w) = \sum_{n=0}^\infty a_n \langle z, w \rangle^n \quad (z, w \in B_R(0))$$

is a positive definite kernel if and only if $a_n \ge 0$ for all $n \in \mathbb{N}$.

Proof. By the Schur product theorem, $(z,w) \mapsto \langle z, w \rangle^n$ is a positive definite kernel for all $n \in \mathbb{N}$. Thus, the backward direction is clear. Conversely, if K is a positive definite kernel, then an application of Lemma 4.6.2 shows that

$$K_n(z,w) = a_n \langle z, w \rangle^n$$

defines a positive definite kernel for all $n \in \mathbb{N}$. In particular, each K_n is Hermitian, hence $a_n \in \mathbb{R}$. Moreover, if $a_n \le 0$, then $-K_n$ is positive definite as well, hence $K_n = 0$ and thus $a_n = 0$. This observation finishes the proof. □

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with a $\mathbb{T}$-invariant kernel K. Assume that K is normalized at a point in X, so that the constant function 1 is contained in $\mathcal{H}$ and has norm 1. Recall that $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n$. We say that $\mathcal{H}$ is *standard graded* if $\mathcal{H}_0 = \mathbb{C}1$ and $\mathcal{H}_n = \{0\}$ for $n < 0$. All spaces in Example 4.6.1 are standard graded, provided their kernel is normalized at a point. In particular, unitarily invariant spaces on $\mathbb{B}_d$ are standard graded.

In Drury-Arveson space, the multiplier norm of a homogeneous polynomial is equal to its Drury-Arveson norm. This can be shown by embedding H^2_d into the full Fock space (see also [80, Lemma 9.5]). For a special class of complete Nevanlinna-Pick spaces $\mathcal{H}$ on $\mathbb{D}$, it was shown in Lemma 3.7.2 that $||z^n||_{\mathcal{H}} = ||z^n||_{\mathrm{Mult}(\mathcal{H})}$ for all $n \in \mathbb{N}$. The next proposition generalizes these results.

Proposition 4.6.4. *Let $\mathcal{H}$ be an irreducible complete Nevanlinna-Pick space which is standard graded. If $f \in \mathcal{H}$ is homogeneous, then $f \in \mathrm{Mult}(\mathcal{H})$ and $||f||_{\mathrm{Mult}(\mathcal{H})} = ||f||_{\mathcal{H}}$.*

Proof. The proof is an abstract version of the proof of Lemma 3.7.2. Let $K = \sum_{n=0}^\infty K_n$ be the homogeneous decomposition of K from Lemma 4.6.2. In a first step, we will show that for every pair of natural numbers n and k, the kernel

$$K_{n+k} - K_n K_k$$

is positive definite. We proceed by induction on n. The assumption $\mathcal{H}_0 = \mathbb{C}1$ implies that $K_0 = 1$, so this is trivial for $n = 0$. Assume that $n \geq 1$ and that the assertion is true for $0, 1, \dots, n-1$. Since K is a normalized irreducible complete Nevanlinna-Pick kernel, $F = 1 - \frac{1}{K}$ is a positive definite kernel on X by Theorem 2.5.1, and it is clearly $\mathbb{T}$-invariant. Let $F = \sum_{j=0}^{\infty} F_j$ be the homogeneous decomposition of F. Since $K = KF + 1$, we have

$$\sum_{i=0}^{\infty} K_i = \sum_{i=0}^{\infty} \sum_{j=0}^{i} K_{i-j} F_j + 1,$$

where we have used that all series converge absolutely. Since the homogeneous expansion is unique, we may compare homogeneous components in this equation. For $i = 0$, we use that $K_0 = 1$ to obtain $F_0 = 0$. For $i \geq 1$, we therefore get the identity

$$K_i = \sum_{j=1}^{i} K_{i-j} F_j.$$

Using this identity with $i = n + k$ and $i = n$, we deduce that

$$\begin{aligned} K_{n+k} - K_n K_k &= \sum_{j=1}^{n+k} K_{n+k-j} F_j - \sum_{j=1}^{n} K_{n-j} K_k F_j \\ &\geq \sum_{j=1}^{n} (K_{n+k-j} - K_{n-j} K_k) F_j \geq 0 \end{aligned}$$

by induction hypothesis and the Schur product theorem. This finishes the inductive proof.

Now, let $f \in \mathcal{H}$ be homogeneous of degree $n \geq 0$ and suppose that $||f||_{\mathcal{H}} \leq 1$. A well-known characterization of the norm in a reproducing kernel Hilbert space implies that

$$(z, w) \mapsto K(z, w) - f(z)\overline{f(w)}$$

is positive definite. Note that the degree n homogeneous component of this kernel is $K_n(z, w) - f(z)\overline{f(w)}$, which is positive definite by Lemma 4.6.2. Using the Schur product theorem, we deduce that

$$\sum_{k=0}^{\infty} K_k(z, w)(K_n(z, w) - f(z)\overline{f(w)})$$

is positive definite. Since $K_n K_k \leq K_{n+k}$, this implies that

$$0 \leq \sum_{k=0}^{\infty} K_{n+k}(z, w) - \sum_{k=0}^{\infty} (K_k(z, w) f(z)\overline{f(w)}) \leq K(z, w)(1 - f(z)\overline{f(w)}),$$

so that f is a contractive multiplier on $\mathcal{H}$. □

Remark 4.6.5. For Drury-Arveson space, the above proof can be somewhat simplified. In this case, $K_n(z,w) = \langle z, w \rangle^n$, hence $K_{n+k} = K_n K_k$, so that the first step is trivial.

As a consequence, we obtain a simple necessary condition for the complete Nevanlinna-Pick property of a unitarily invariant space.

Corollary 4.6.6. *Let $d \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{H}$ be an irreducible unitarily invariant reproducing kernel Hilbert space on $\mathbb{B}_d$ with reproducing kernel*

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$$

such that $a_0 = 1$. If $\mathcal{H}$ is a complete Nevanlinna-Pick space, then

$$a_n a_k \le a_{n+k}$$

for all $n, k \in \mathbb{N}$.

Proof. The proof of Proposition 4.6.4 shows that if $\mathcal{H}$ is a complete Nevanlinna-Pick space, then

$$K_{n+k} - K_n K_k$$

is positive definite for every $k, n \in \mathbb{N}$. But

$$K_n(z,w) = a_n \langle z, w \rangle^n$$

for $z, w \in \mathbb{B}_d$, hence the result follows. □

Example 4.6.7. In the setting of the last lemma, let $a_n = (n+1)^s$ for $n \in \mathbb{N}$. If $s > 0$, then

$$a_1^2 = 4^s > 3^s = a_2,$$

so $\mathcal{H}$ is not a complete Nevanlinna-Pick space. Observe that if $d = 1$ and $s = 1$, we obtain the well-known fact that the Bergman space on $\mathbb{D}$ is not a complete Nevanlinna-Pick space.

Example 4.6.8. Let us observe that the necessary condition in Corollary 4.6.6 is not sufficient. Let $d = 1$ and define $a_0 = 1, a_1 = \frac{1}{2}, a_n = 1$ for $n \ge 2$, that is, $\mathcal{H}$ is the space on $\mathbb{D}$ with reproducing kernel

$$K(z,w) = \sum_{n=0}^{\infty} a_n (z\overline{w})^n = \frac{1}{1 - z\overline{w}} - \frac{1}{2} z\overline{w}.$$

Then $a_k a_n \le a_{n+k}$ for $n, k \in \mathbb{N}$. However,

$$1 - \frac{1}{K(z,w)} = \frac{1}{2} z\overline{w} + \frac{3}{4}(z\overline{w}) + \frac{1}{8}(z\overline{w})^3 - \frac{5}{16}(z\overline{w})^4 + \text{h.o.t.}.$$

Hence, $\mathcal{H}$ is not a complete Nevanlinna-Pick space by [3, Theorem 7.33].

If $\mathcal{H}$ is a standard graded complete Nevanlinna-Pick space, we let $A(\mathcal{H})$ denote the norm closed linear span of the homogeneous elements in $\mathrm{Mult}(\mathcal{H})$. For example, $A(H^2)$ is the disc algebra.

For standard graded complete Nevanlinna-Pick spaces, there is a bounded version of Corollary 4.3.2.

Proposition 4.6.9. *Let X be a set equipped with an action of $\mathbb{T}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two irreducible complete Nevanlinna-Pick spaces on X with reproducing kernels K_1 and K_2, respectively. Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are standard graded with respect to the action of $\mathbb{T}$ on X. Then the following assertions are equivalent:*

(i) *$\mathcal{H}_1 = \mathcal{H}_2$ as vector spaces.*

(ii) *$\mathrm{Mult}(\mathcal{H}_1) = \mathrm{Mult}(\mathcal{H}_2)$ as algebras.*

(iii) *$A(\mathcal{H}_1) = A(\mathcal{H}_2)$ as algebras.*

(iv) *There exist $c_1, c_2 > 0$ such that $c_1^2 K_2 - K_1$ and $c_2^2 K_1 - K_2$ are positive definite.*

(v) *The identity map $\mathcal{H}_1 \to \mathcal{H}_2$ is a bounded isomorphism which induces similarities $\mathrm{Mult}(\mathcal{H}_1) = \mathrm{Mult}(\mathcal{H}_2)$ and $A(\mathcal{H}_1) = A(\mathcal{H}_2)$.*

Proof. The equivalence of (i) and (iv) is well known. (v) implies (i), (ii) and (iii) is trivial, and (i) implies (v) follows from the closed graph theorem. It remains to see that (ii) or (iii) implies (i). In both cases, $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same homogeneous elements by Proposition 4.6.4. Moreover, since all algebras in question are semi-simple, there are constants $C_1, C_2 > 0$ such that

$$\frac{1}{C_2} ||f||_{\mathrm{Mult}(\mathcal{H}_2)} \le ||f||_{\mathrm{Mult}(\mathcal{H}_1)} \le C_1 ||f||_{\mathrm{Mult}(\mathcal{H}_2)}$$

for every homogeneous element f (see [17, Proposition 4.2]). Since homogeneous elements of different degree are orthogonal in $\mathcal{H}_1$ and $\mathcal{H}_2$, we deduce from Proposition 4.6.4 that there is a bounded isomorphism $\mathcal{H}_1 \to \mathcal{H}_2$ which acts as the identity on homogeneous elements, and hence everywhere. Thus, (i) holds. □

4.7. Restrictions of unitarily invariant spaces

For the remainder of this chapter, we will consider restrictions of unitarily invariant spaces on $\mathbb{B}_d$, and from now on, we will always assume that $d < \infty$.

Suppose that $\mathcal{H}$ is a unitarily invariant space on $\mathbb{B}_d$ with reproducing kernel

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n, \tag{4.3}$$

where $a_0 = 1$ and $a_n \ge 0$ for all $n \in \mathbb{N}$. We will assume that $\mathcal{H}$ has the following properties:

(a) $\mathcal{H}$ contains the coordinate functions.

(b) $\mathcal{H}$ is algebraically consistent on $\mathbb{B}_d$.

(c) $\mathcal{H}$ is an irreducible complete Nevanlinna-Pick space.

For simplicity, we will call a space which satisfies these conditions a *unitarily invariant complete NP-space on* $\mathbb{B}_d$.

The conditions above can also be expressed in terms of the reproducing kernel. If the kernel is given as in (4.3), then (a) is equivalent to demanding that $a_1 > 0$ (see, for example [38, Section 4] or [39, Proposition 4.1]). Lemma 4.5.3 shows that Condition (b) holds if and only if the radius of convergence of the series $\sum_{n=0}^{\infty} a_n t^n$ is 1 (so that $\mathcal{H}$ is defined on $\mathbb{B}_d$) and $\sum_{n=0}^{\infty} a_n = \infty$. In the presence of (a), $\mathcal{H}$ is an irreducible complete Nevanlinna-Pick space if and only if the sequence $(b_n)_{n=1}^{\infty}$ defined by

$$\sum_{n=1}^{\infty} b_n t^n = 1 - \frac{1}{\sum_{n=0}^{\infty} a_n t^n} \tag{4.4}$$

for t in a neighbourhood of 0 is a sequence of non-negative real numbers, see Lemma 4.2.2.

We will also consider spaces on $\overline{\mathbb{B}_d}$. The only difference to the above setting is that here, the functions in $\mathcal{H}$ are assumed to be analytic on $\mathbb{B}_d$ and continuous on $\overline{\mathbb{B}_d}$. Moreover, $\mathcal{H}$ is assumed to be algebraically consistent on $\overline{\mathbb{B}_d}$. In terms of the reproducing kernel K, this means that $\sum_{n=0}^{\infty} a_n < \infty$ but the power series $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence 1 (see Lemma 4.5.3). We call such a space a *unitarily invariant complete NP-space on* $\overline{\mathbb{B}_d}$. We say that $\mathcal{H}$ is a *unitarily invariant complete NP-space* to mean that $\mathcal{H}$ is either a unitarily invariant complete NP-space on $\mathbb{B}_d$ or on $\overline{\mathbb{B}_d}$.

Remark 4.7.1. Let $\mathcal{H}$ be a unitarily invariant complete NP-space as above.

(a) $\mathcal{H}$ is standard graded in the sense of Section 4.6.

(b) The condition that the sequence (b_n) in Equation (4.4) in non-negative is often difficult to check in practice. A sufficient condition for this to hold is that the sequence $(a_n)_n$ is strictly positive and log-convex, i.e.

$$\frac{a_n}{a_{n+1}} \le \frac{a_{n-1}}{a_n} \quad (n \ge 1),$$

see, for example, [3, Lemma 7.38].

(c) Since $\mathcal{H}$ contains the coordinate functions, it follows from Proposition 4.6.4 that the coordinate functions are multipliers. Thus, all polynomials are multipliers. In particular, $\mathcal{H}$ contains all polynomials, so that $a_n > 0$ for all $n \in \mathbb{N}$ (see also [38, Section 4]).

(d) The monomials z^α, where α runs through all multi-indices of non-negative integers of length d, form an orthogonal basis for $\mathcal{H}$. Moreover,

$$||z^\alpha||^2_{\mathcal{H}} = \frac{\alpha!}{|\alpha|! a_{|\alpha|}}$$

for every multi-index α (see, for example, [38, Section 4] or [39, Proposition 4.1]). It follows from unitary invariance that

$$||\langle \cdot, w\rangle^n||^2_{\mathcal{H}_I} = \frac{||w||^{2n}}{a_n}$$

for all $w \in \mathbb{C}^d$ and all $n \in \mathbb{N}$.

Example 4.7.2. For $-1 \leq s \leq 0$, let $\mathcal{H}_s(\mathbb{B}_d)$ be the reproducing kernel Hilbert space on $\mathbb{B}_d$ with kernel

$$K_s(z,w) = \sum_{n=0}^{\infty} (n+1)^{-s} \langle z, w\rangle.$$

Using part (b) of Remark 4.7.1, it is easy to see that $\mathcal{H}_s(\mathbb{B}_d)$ is a unitarily invariant complete NP-space on $\mathbb{B}_d$.

If $s < -1$, the series in the definition of K_s converges on $\overline{\mathbb{B}_d} \times \overline{\mathbb{B}_d}$. Let $\mathcal{H}_s(\overline{\mathbb{B}_d})$ be the reproducing kernel Hilbert space on $\overline{\mathbb{B}_d}$ with this kernel. As above, it is not hard to see that this space is a unitarily invariant complete NP-space on $\overline{\mathbb{B}_d}$.

Closely related to the spaces $\mathcal{H}_s(\mathbb{B}_d)$ for $s \in (-1, 0]$ are the spaces from Corollary 4.4.4. If $\alpha \in (0, 1]$, the space $\mathcal{K}_\alpha$ with reproducing kernel

$$K(z,w) = \frac{1}{(1 - \langle z, w\rangle)^\alpha} \quad (z, w \in \mathbb{B}_d)$$

is a unitarily invariant complete NP-space on $\mathbb{B}_d$. Expressing the reproducing kernel as a binomial series and using part (d) of Remark 4.7.1, it is straightforward to see that $\mathcal{K}_\alpha$ and $\mathcal{H}_{\alpha-1}$ agree as vector spaces, and have equivalent norms.

Remark 4.7.3. While we assume that our spaces on $\mathbb{B}_d$ are invariant under unitary maps, we specifically do not assume that they are invariant under other conformal automorphisms of the unit ball. Such an assumption would simplify some arguments, but the condition of automorphism invariance is often difficult to check in practice. Indeed, even for spaces on $\mathbb{D}$, there does not seem to be a simple criterion for automorphism invariance. We refer the reader to [12, Section 3.1] and [13, Section 8].

We now turn to restrictions of unitarily invariant complete NP-spaces. Suppose that $I \subsetneq \mathbb{C}[z_1, \dots, z_d]$ is a homogeneous ideal. Following [24], we define

$$Z^0(I) = V(I) \cap \mathbb{B}_d$$

and

$$Z(I) = V(I) \cap \overline{\mathbb{B}_d},$$

where

$$V(I) = \{z \in \mathbb{C}^d : f(z) = 0 \text{ for all } f \in I\}$$

denotes the vanishing locus of I. Observe that since I is a proper ideal, $Z^0(I)$ always contains the origin.

If $\mathcal{H}$ is a unitarily invariant complete NP-space on $\mathbb{B}_d$, we define $\mathcal{H}_I = \mathcal{H}\big|_{Z^0(I)}$. If $\mathcal{H}$ is a unitarily invariant complete NP-space on $\overline{\mathbb{B}_d}$, we define $\mathcal{H}_I = \mathcal{H}\big|_{Z(I)}$. Recall from Lemma 2.1.2 that the norm on $\mathcal{H}_I$ is defined in such a way that the restriction map from $\mathcal{H}$ onto $\mathcal{H}_I$ is a co-isometry. Lemma 4.5.4 shows that $\mathcal{H}_I$ is algebraically consistent in both cases. Observe that the circle group acts on $Z^0(I)$ and on $Z(I)$ by scalar multiplication, which gives the spaces $\mathcal{H}_I$ a grading in the sense of Section 4.6. Moreover, the restriction map from $\mathcal{H}$ onto $\mathcal{H}_I$ respects the grading. Thus, a function in $\mathcal{H}_I$ is homogeneous of degree n if and only if it is the restriction of a homogeneous polynomial of degree n.

Since an ideal and its radical have the same vanishing locus, there is no loss of generality in restricting our attention to radical homogeneous ideals. If $I \subsetneq \mathbb{C}[z_1, \dots, z_d]$ is a radical homogeneous ideal, then the ring of polynomial functions on $V(I)$ is canonically isomorphic to the quotient $\mathbb{C}[z_1, \dots, z_d]/I$ by Hilbert's Nullstellensatz. The following lemma, which gives a different description of the space $\mathcal{H}_I$, can be thought of as a Hilbert function space analogue of this fact. Results of this type are certainly well known (cf. [24, Section 6]), but we do not have a convenient reference for the precise statement.

Lemma 4.7.4. *Let $\mathcal{H}$ be a unitarily invariant space on $\mathbb{B}_d$ or on $\overline{\mathbb{B}_d}$ with reproducing kernel K, and let $I \subsetneq \mathbb{C}[z_1, \dots, z_d]$ be a radical homogeneous ideal. Then the closure of I in $\mathcal{H}$ is given by*

$$\overline{I} = \{f \in \mathcal{H} : f\big|_{Z^0(I)} = 0\}.$$

Hence the map

$$\mathcal{H} \ominus I \to \mathcal{H}_I,$$

given by restriction, is a unitary operator. Moreover, $\mathcal{H} \ominus I$ is the closed linear span of the kernel functions $K(\cdot, w)$ for $w \in Z^0(I)$.

Proof. Let
$$R : \mathcal{H} \to \mathcal{H}_I$$
be the restriction map. Then R is a co-isometry by definition of $\mathcal{H}_I$, thus it suffices to show that $\ker R = \overline{I}$. It is clear that $\overline{I} \subset \ker R$. Conversely, let $f \in \ker R$ and let $f = \sum_{n=0}^{\infty} f_n$ be the homogeneous decomposition of f. Then

$$0 = f(tz) = \sum_{n=0}^{\infty} t^n f_n(z)$$

for all $t \in \mathbb{D}$ and all $z \in Z^0(I)$, hence each f_n vanishes on $V(I)$. Consequently, $f_n \in I$ for all $n \in \mathbb{N}$ by Hilbert's Nullstellensatz, thus $f \in \overline{I}$.

Since the restriction map from $\mathcal{H}$ onto $\mathcal{H}_I$ is a co-isometry, it follows that this map is a unitary operator from $\mathcal{H} \ominus I$ onto $\mathcal{H}_I$. Moreover, given $f \in \mathcal{H}$, we see that f is orthogonal to the kernel functions $K(\cdot, w)$ for $w \in Z^0(I)$ if and only if f vanishes on $Z^0(I)$, which happens if and only if $f \in \overline{I}$ by the first part. □

Thus, instead of thinking of $\mathcal{H}_I$ as a space of functions on $Z^0(I)$ or on $Z(I)$, we may also regard it as a subspace of $\mathcal{H}$. The following lemma shows how composition operators act in this second picture of $\mathcal{H}_I$. It is a straightforward generalization of a well-known result about composition operators on reproducing kernel Hilbert spaces (see, for example, [12, Theorem 1.4]).

Lemma 4.7.5. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with reproducing kernel $K_{\mathcal{H}}$, and let $\mathcal{K}$ be a reproducing kernel Hilbert space on a set Y with reproducing kernel $K_{\mathcal{K}}$. Suppose that $Z \subset X$ and $W \subset Y$, and define*

$$I(Z) = \{f \in \mathcal{H} : f\big|_Z = 0\}$$

and

$$I(W) = \{f \in \mathcal{K} : f\big|_W = 0\}.$$

Then for a function $\varphi : W \to Z$, the following are equivalent:

(i) *There exists a bounded composition operator $C_\varphi : \mathcal{H}\big|_Z \to \mathcal{K}\big|_W$ such that $C_\varphi(f) = f \circ \varphi$ for all $f \in \mathcal{H}\big|_Z$.*

(ii) *There exists a bounded operator $T_\varphi : \mathcal{K} \ominus I(W) \to \mathcal{H} \ominus I(Z)$ with $T(K_{\mathcal{K}}(\cdot, w)) = K_{\mathcal{H}}(\cdot, \varphi(w))$ for all $w \in W$.*

In this case,

$$T_\varphi = R_Z^{-1}(C_\varphi)^* R_W,$$

where

$$R_Z : \mathcal{H} \ominus I(Z) \to \mathcal{H}\big|_Z, \quad f \mapsto f\big|_Z,$$

and

$$R_W : \mathcal{K} \ominus I(W) \to \mathcal{K}\big|_W, \quad f \mapsto f\big|_W,$$

denote the unitary restriction maps.

Proof. Suppose that (i) holds. For $w \in W$ and $f \in \mathcal{H} \ominus I(Z)$, we have

$$\begin{aligned}\langle f, R_Z^*(C_\varphi)^* R_W K_{\mathcal{K}}(\cdot, w)\rangle_{\mathcal{H}} &= \langle (f\big|_Z) \circ \varphi, K_{\mathcal{K}}(\cdot, w)\big|_W\rangle_{\mathcal{K}|W} \\ &= f(\varphi(w)) \\ &= \langle f, K_{\mathcal{H}}(\cdot, \varphi(w))\rangle_{\mathcal{H}}.\end{aligned}$$

Since $K_{\mathcal{H}}(\cdot, \varphi(w)) \in \mathcal{H} \ominus I(W)$, we conclude that (ii) holds with $T_\varphi = R_Z^*(C_\varphi)^* R_W$, which also proves the additional assertion.

Conversely, if (ii) holds, let $f \in \mathcal{H} \ominus I(Z)$ and let $w \in W$. Clearly, $R_Z^*(f\big|_Z) = f$ and $R_W^*(K_{\mathcal{K}}(\cdot, w)\big|_W) = K_{\mathcal{K}}(\cdot, w)$, hence

$$\begin{aligned}(R_W T_\varphi^* R_Z^* f\big|_Z)(w) &= \langle R_W T_\varphi^* f, K_{\mathcal{K}}(\cdot, w)\big|_W\rangle_{\mathcal{K}|W} \\ &= \langle f, T_\varphi K_{\mathcal{K}}(\cdot, w)\rangle_{\mathcal{H}} \\ &= \langle f, K_{\mathcal{H}}(\cdot, \varphi(w))\rangle_{\mathcal{H}} \\ &= (f \circ \varphi)(w).\end{aligned}$$

Consequently, (i) holds with $C_\varphi = R_W T_\varphi^* R_Z^*$. □

It may seem restrictive that we only consider restrictions to varieties defined by homogeneous polynomials. Indeed, if $\mathcal{H}$ is a unitarily invariant complete NP-space on $\mathbb{B}_d$ and $X \subset \mathbb{B}_d$ has circular symmetry, i.e. $\mathbb{T}X = X$, then $\mathcal{H}\big|_X$ is standard graded in the sense of Section 4.6. It turns out, however, that algebraic consistency forces X to be a homogeneous variety. More generally, we obtain the following result.

Lemma 4.7.6. *Let $\mathcal{H}$ be a normalized irreducible Hilbert function space of analytic functions on $\mathbb{B}_d$ (respectively of continuous functions on $\overline{\mathbb{B}_d}$ which are analytic on $\mathbb{B}_d$). Let $X \subset \mathbb{B}_d$ (respectively $X \subset \overline{\mathbb{B}_d}$) be a non-empty set which satisfies $\mathbb{T}X \subset X$. If $\mathcal{H}\big|_X$ is algebraically consistent, then $X = Z^0(I)$ (respectively $X = Z(I)$) for some radical homogeneous ideal $I \subsetneq \mathbb{C}[z_1, \ldots, z_d]$.*

Proof. We first consider the case where $\mathcal{H}$ is a space of analytic functions on $\mathbb{B}_d$. Let I be the ideal of all polynomials that vanish on X. Suppose that $f \in \mathcal{O}(\mathbb{B}_d)$ vanishes on X, and let $f = \sum_{n=0}^{\infty} f_n$ be the homogeneous expansion of f. Given $x \in X$, the function

$$\overline{\mathbb{D}} \to \mathbb{C}, \quad \lambda \mapsto f(\lambda x),$$

is contained in the disc algebra and vanishes on $\mathbb{T}$, hence it vanishes identically. Using the homogeneous expansion of f, we see that $f_n(x) = 0$ for all $n \in \mathbb{N}$. Thus, every f_n and hence f vanishes on $Z^0(I)$. This argument also shows that I is a homogeneous ideal. Moreover, every function in $\mathcal{H}\big|_X$ extends uniquely to a function in $\mathcal{O}(\mathbb{B}_d)\big|_{Z^0(I)}$.

Clearly, $X \subset Z^0(I)$. To establish equality, denote for $f \in \mathcal{H}$ the unique extension of f to a function in $\mathcal{O}(\mathbb{B}_d)\big|_{Z^0(I)}$ by $\widehat{f}$. Observe that $\widehat{f}$ in fact belongs to $\mathcal{H}\big|_{Z^0(I)}$ and has the same norm as f. Since $\mathcal{O}(\mathbb{B}_d)\big|_{Z^0(I)}$ is an algebra, we see that $\widehat{\varphi f} = \widehat{\varphi}\widehat{f}$ for all $\varphi \in \operatorname{Mult}(\mathcal{H}\big|_X)$ and $f \in \mathcal{H}\big|_X$. Assume for a contradiction that there exists $x \in Z^0(I) \setminus X$. Then $f \mapsto \widehat{f}(x)$ defines a bounded functional on $\mathcal{H}\big|_X$ which is partially multiplicative. Since $\mathcal{H}$ is irreducible, this functional is not given by evaluation at a point in X. This contradicts algebraic consistency of $\mathcal{H}\big|_X$, hence $X = Z^0(I)$.

Finally, if $\mathcal{H}$ is a space of continuous functions on $\overline{\mathbb{B}_d}$ which are analytic on $\mathbb{B}_d$, then the proof above applies to this setting as well once we replace $Z^0(I)$ with $Z(I)$ and $\mathcal{O}(\mathbb{B}_d)$ with the algebra of all continuous functions on $\overline{\mathbb{B}_d}$ which are analytic on $\mathbb{B}_d$. □

4.8. The maximal ideal space

To classify the multiplier algebras of the spaces $\mathcal{H}_I$ introduced in the last section, we follow the same route as [24]. To this end, we first study the character spaces of these multiplier algebras. We begin with an easier object. Recall that if $\mathcal{H}$ is a standard graded complete Nevanlinna-Pick space, $A(\mathcal{H})$ denotes the norm closure of the span of all homogeneous elements in $\operatorname{Mult}(\mathcal{H})$. If $\mathcal{A}$ is a unital Banach algebra, we let $\mathcal{M}(\mathcal{A})$ denote its maximal ideal space.

Lemma 4.8.1. *Let $\mathcal{H}$ be a unitarily invariant complete NP-space on $\mathbb{B}_d$ or on $\overline{\mathbb{B}_d}$, and let $I \subsetneq \mathbb{C}[z_1, \dots, z_d]$ be a radical homogeneous ideal. Then*

$$\mathcal{M}(A(\mathcal{H}_I)) \to Z(I), \quad \rho \mapsto (\rho(z_1), \dots, \rho(z_d)),$$

is a homeomorphism.

Proof. Let $\rho \in \mathcal{M}(A(\mathcal{H}_I))$. We first show that $\lambda = (\rho(z_1), \dots, \rho(z_d)) \in \overline{\mathbb{B}_d}$. Suppose otherwise, and let $0 < r < 1$ be such that $||r\lambda|| > 1$. If p is a polynomial with homogeneous decomposition $p = \sum_{n=0}^N p_n$, then

$$|p(r\lambda)|^2 \le \Big(\sum_{n=0}^N r^n |\rho(p_n)| \Big)^2 \le \sum_{n=0}^N r^{2n} \sum_{n=0}^N ||p_n||^2_{\mathrm{Mult}(\mathcal{H}_I)}.$$

By Proposition 4.6.4, $||p_n||_{\mathrm{Mult}(\mathcal{H}_I)} = ||p_n||_{\mathcal{H}_I}$, hence this quantity is dominated by

$$\frac{1}{1-r^2} \sum_{n=0}^N ||p_n||^2_{\mathcal{H}_I} = \frac{1}{1-r^2} ||p||^2_{\mathcal{H}_I}.$$

Consequently, $p \mapsto p(r\lambda)$ extends to a well-defined bounded functional $\widetilde{\rho}$ on $\mathcal{H}_I$. It is easy to see that $\widetilde{\rho}$ is partially multiplicative, but

$$(\widetilde{\rho}(z_1), \dots, \widetilde{\rho}(z_d)) = r\lambda \notin \overline{\mathbb{B}_d}.$$

This contradicts the fact that $\mathcal{H}_I$ is algebraically consistent. Clearly, $\lambda \in V(I)$. Thus, if Φ denotes the map from the statement of the lemma, it follows that $\Phi(\rho) \in Z(I)$. It is clear that Φ is continuous, and since the polynomials are dense in $A(\mathcal{H}_I)$ by definition, it is also injective.

Since $\mathcal{M}(A(\mathcal{H}_I))$ is compact, we may finish the proof by showing that Φ is surjective. If $\mathcal{H}$ is a space on $\mathbb{B}_d$, then the elements of $A(\mathcal{H}_I)$ extend to continuous functions on $Z(I)$, as the multiplier norm dominates the supremum norm. If $\mathcal{H}$ is a space on $\overline{\mathbb{B}_d}$, they are already defined on $Z(I)$, so in both cases, every $\lambda \in Z(I)$ gives rise to a character δ_λ given by point evaluation at λ, and this character satisfies $\Phi(\delta_\lambda) = \lambda$. □

The character space of the whole multiplier algebra is often much more complicated. Indeed, if $\mathcal{H}$ is the Hardy space $H^2(\mathbb{D})$, then $\mathrm{Mult}(\mathcal{H}) = H^\infty$, an algebra whose character space is known to be very complicated (see, for example, [36, Chapter V]).

Since every character on $\mathrm{Mult}(\mathcal{H}_I)$ restricts to a character on $A(\mathcal{H}_I)$, we obtain in the setting of the last lemma a continuous map

$$\pi : \mathcal{M}(\mathrm{Mult}(\mathcal{H}_I)) \to Z(I), \quad \rho \mapsto \rho(z_1, \dots, z_d).$$

This map is surjective, as evaluation at a point in $Z^0(I)$ is a character and the character space is compact.

If $\mathcal{H}$ is a space on $\overline{\mathbb{B}_d}$, then $\mathrm{Mult}(\mathcal{H}_I)$ consists of continuous functions on the compact set $Z(I)$. The weak-$*$ continuous characters are precisely the point evaluations at points in

$Z(I)$ by Proposition 4.5.6 and thus form a compact subset of $\mathcal{M}(\mathrm{Mult}(\mathcal{H}))$. The question whether every character is a point evaluation in this setting remains open.

If $\mathcal{H}$ is a space on $\mathbb{B}_d$, then the weak-$*$ continuous characters are point evaluations at points in $Z^0(I)$, again by Proposition 4.5.6, thus they form a proper subset of the maximal ideal space. The next lemma shows that in this case, multipliers can oscillate wildly near the boundary of $\mathbb{B}_d$, and hence the character space of $\mathrm{Mult}(\mathcal{H}_I)$ is rather complicated.

Lemma 4.8.2. *Let $\mathcal{H}$ be a unitarily invariant complete NP-space on $\mathbb{B}_d$ and let $I \subsetneq \mathbb{C}[z_1, \dots, z_d]$ be a radical homogeneous ideal. Let (λ_n) be a sequence in $Z^0(I)$ which satisfies $\lim_{n\to\infty} ||\lambda_n|| = 1$. Then (λ_n) contains a subsequence which is interpolating for* $\mathrm{Mult}(\mathcal{H}_I)$. *In particular, $\pi^{-1}(\lambda)$ contains a copy of $\beta\mathbb{N} \setminus \mathbb{N}$ for every $\lambda \in V(I) \cap \partial\mathbb{B}_d$.*

Proof. The proof of Proposition 3.9.1 shows that it suffices to show that $K_I(\lambda_n, \lambda_n)$ converges to ∞, where K_I denotes the reproducing kernel of $\mathcal{H}_I$. However, if

$$K(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n \quad (z, w \in \mathbb{B}_d)$$

denotes the reproducing kernel of $\mathcal{H}$, then K_I is simply the restriction of K to $Z^0(I)\times Z^0(I)$. Moreover, since $\mathcal{H}$ is algebraically consistent on $\mathbb{B}_d$, we have $\sum_{n=0}^{\infty} a_n = \infty$ by Lemma 4.5.3, thus $K_I(\lambda_n, \lambda_n)$ tends to ∞, as asserted.

For the proof of the additional assertion, we note that for every $\lambda \in V(I) \cap \partial\mathbb{B}_d$, there is an interpolating sequence (λ_n) which converges to λ by the first part. Hence, the algebra homomorphism

$$\mathrm{Mult}(\mathcal{H}_I) \to \ell^\infty, \quad \varphi \mapsto (\varphi(\lambda_n)),$$

is surjective, and its adjoint is a topological embedding of $\beta\mathbb{N} \setminus \mathbb{N}$ into $\pi^{-1}(\lambda)$. □

We now turn to the fibers of π over points in the open ball. Let $\mathcal{H}$ be a unitarily invariant complete NP-space on $\mathbb{B}_d$ or on $\overline{\mathbb{B}_d}$, and let

$$\pi : \mathcal{M}(\mathrm{Mult}(\mathcal{H})) \to \overline{\mathbb{B}_d}, \quad \rho \mapsto (\rho(z_1), \dots, \rho(z_d)),$$

be the map from above. For $\lambda \in \mathbb{B}_d$, the fibers $\pi^{-1}(\lambda)$ always contains the character of evaluation at λ. If one allows the case $d = \infty$, then these fibers can be much larger, see Example 3.2.4. We say that $\mathcal{H}$ is *tame* if the fibers of π over $\mathbb{B}_d$ are singletons. Equivalently, if ρ is a character on $\mathrm{Mult}(\mathcal{H})$ such that $\lambda = \pi(\rho) \in \mathbb{B}_d$, then ρ is the character of evaluation at λ. Note that even if $\mathcal{H}$ is a space on $\overline{\mathbb{B}_d}$, we do not impose a condition on fibers over the boundary. It remains open whether there are non-tame spaces if $d < \infty$. We also mention that for spaces on $\mathbb{D}$, the question of when the fibers of π are singletons already appears in [81] (see Question 3 on page 78).

Example 4.8.3. Perhaps the easiest example of a tame space is the Hardy space $H^2(\mathbb{D})$ on the unit disc. Let us briefly recall the well-known argument, which we will generalize below. Suppose ρ is a character on $H^\infty(\mathbb{D}) = \mathrm{Mult}(H^2(\mathbb{D}))$ such that $\lambda = \rho(z) \in \mathbb{D}$. If $\varphi \in H^\infty(\mathbb{D})$, then

$$\varphi_\lambda = \frac{\varphi - \varphi(\lambda)}{z - \lambda} \in H^\infty(\mathbb{D})$$

by the maximum modulus principle, and

$$\varphi = \varphi(\lambda) + (z - \lambda)\varphi_\lambda.$$

Since ρ is a character and since $\rho(z - \lambda) = 0$, it follows that

$$\rho(\varphi) = \varphi(\lambda),$$

thus ρ is the character of evaluation at λ.

Remark 4.8.4. If $\mathcal{H}$ is tame and if $I \subsetneq \mathbb{C}[z_1, \dots, z_d]$ is a radical homogeneous ideal, then $\mathcal{H}_I$ is similarly well-behaved. More precisely, if ρ is a character on $\mathrm{Mult}(\mathcal{H}_I)$ such that $\pi(\rho) \in \mathbb{B}_d$ (and hence $\pi(\rho) \in Z^0(I)$), then ρ is the character of evaluation at λ. Indeed, this follows from tameness of $\mathcal{H}$ and from the fact that the restriction map from $\mathrm{Mult}(\mathcal{H})$ to $\mathrm{Mult}(\mathcal{H}_I)$ is surjective, since $\mathcal{H}$ is a Nevanlinna-Pick space.

Proposition 3.2 in [25] shows that H^2_d is tame (for $d < \infty$). The argument in [25] uses a result about characters on the non-commutative free semigroup algebra $\mathcal{L}_d$ from [21], and the fact that $\mathrm{Mult}(H^2_d)$ is a quotient of $\mathcal{L}_d$ [22]. Since this does not apply to unitarily invariant complete NP-spaces besides H^2_d, we will use a different argument similar to the one in Example 4.8.3. The underlying principle, however, is always the same, namely a factorization result for elements in the Banach algebra in question. In the following proposition, we record some sufficient conditions for tameness in decreasing order of generality.

Proposition 4.8.5. *Let $\mathcal{H}$ be a unitarily invariant complete NP-space with reproducing kernel $K(z,w) = \sum_{n=0}^\infty a_n \langle z, w\rangle^n$. Consider the following conditions:*

(a) *Gleason's problem can be solved in $\mathrm{Mult}(\mathcal{H})$. That is, given $\lambda \in \mathbb{B}_d$ and $\varphi \in \mathrm{Mult}(\mathcal{H})$, there are $\varphi_1, \dots, \varphi_d \in \mathrm{Mult}(\mathcal{H})$ such that*

$$\varphi - \varphi(\lambda) = \sum_{i=1}^d (z_i - \lambda_i)\varphi_i.$$

(b) *For every* $\lambda \in \mathbb{B}_d$*, the space*

$$\sum_{i=1}^{d} (z_i - \lambda_i)\mathcal{H} \subset \mathcal{H}$$

is closed in $\mathcal{H}$*.*

(c) *The condition*

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1$$

holds.

Then $(c) \Rightarrow (b) \Rightarrow (a)$*, and each of* $(a), (b), (c)$ *implies that* $\mathcal{H}$ *is tame.*

Proof. We first show that (a) implies that $\mathcal{H}$ is tame. Let ρ be a character on $\operatorname{Mult}(\mathcal{H})$ with $\pi(\rho) = \lambda \in \mathbb{B}_d$ and let $\varphi \in \operatorname{Mult}(\mathcal{H})$. By assumption, there are $\varphi_1, \ldots, \varphi_d \in \operatorname{Mult}(\mathcal{H})$ such that

$$\varphi - \varphi(\lambda) = \sum_{i=1}^{d} (z_i - \lambda_i)\varphi_i.$$

Since the right-hand side is contained in the kernel of the multiplicative linear functional ρ, it follows that

$$\rho(\varphi) = \varphi(\lambda),$$

hence ρ is the character of point evaluation at λ.

(b) $\Rightarrow$ (a) We use a factorization theorem for multipliers on complete Nevanlinna-Pick spaces to show that (a) is satisfied (cf. Section 4 of [37]). We first claim that

$$\sum_{i=1}^{d} (z_i - \lambda_i)\mathcal{H} = \{f \in \mathcal{H} : f(\lambda) = 0\}.$$

Indeed, to see the nontrivial inclusion, suppose that $f \in \mathcal{H}$ vanishes at λ. Since the polynomials form a dense subset of $\mathcal{H}$, there is a sequence (p_n) of polynomials which converges to f in $\mathcal{H}$. Then $(p_n - p_n(\lambda))$ is a sequence of polynomials vanishing at λ which converges to f, as evaluation at λ is continuous. Observe that the space on the left-hand side contains all polynomials vanishing at λ and is closed by assumption. Thus, f belongs to the space on the left-hand side, as asserted.

Hence, if $\varphi \in \operatorname{Mult}(\mathcal{H})$ with $\varphi(\lambda) = 0$, then $\operatorname{ran}(M_\varphi)$ is contained in the range of the row multiplication operator

$$(M_{z_1 - \lambda_1}, \ldots, M_{z_d - \lambda_d}).$$

Let $z-\lambda$ denote the $\mathcal{B}(\mathbb{C}^d,\mathbb{C})$-valued multiplier $(z_1-\lambda_1,\dots,z_d-\lambda_d)$. Then by the Douglas lemma, there exists $c>0$ such that

$$M_\varphi M_\varphi^* \le c^2 M_{z-\lambda} M_{z-\lambda}^*.$$

In this situation, a factorization theorem valid for multiplier algebras of complete Nevanlinna-Pick spaces (see, for example, Theorem 8.57 in [3]) implies the existence of a $\mathcal{B}(\mathbb{C},\mathbb{C}^d)$-valued multiplier Ψ such that

$$c(z-\lambda)\Psi = \varphi.$$

Writing

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix},$$

we see that

$$\varphi = \sum_{i=1}^d (z_i-\lambda_i)(c\psi_i).$$

Consequently, Gleason's problem can be solved in $\mathrm{Mult}(\mathcal{H})$, so (a) holds.

(c) $\Rightarrow$ (b) The proof uses the notion of essential Taylor spectrum (see, for example, Section 33 and 34 in [60], Section 2.6 in [30], or [15]). By Theorem 4.5 (2) in [39], the assumption that a_n/a_{n+1} converges to 1 implies that the essential Taylor spectrum of $M_z=(M_{z_1},\dots,M_{z_d})$ equals $\partial\mathbb{B}_d$, hence the d-tuple $(M_{z_1}-\lambda_1,\dots,M_{z_d}-\lambda_d)$ is a Fredholm tuple for all $\lambda=(\lambda_1,\dots,\lambda_d)\in\mathbb{B}_d$. In particular, the last coboundary map in the Koszul complex has closed range, thus the row operator

$$(M_{z_1}-\lambda_1,\dots,M_{z_d}-\lambda_d)$$

has closed range for all $\lambda\in\mathbb{B}_d$. Consequently, (b) holds. □

Example 4.8.6. The spaces $\mathcal{H}_s(\mathbb{B}_d)$, $\mathcal{H}_s(\overline{\mathbb{B}_d})$ and $\mathcal{K}_\alpha$ in Example 4.7.2 all satisfy condition (c) of the preceding proposition and are hence tame.

Remark 4.8.7. (a) The regularity condition $\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=1$ is not uncommon in the study of unitarily invariant kernels, see for example Section 4 in [38]. Proposition 4.5 in [38] shows that this condition automatically holds if $\sum_{n=0}^\infty a_n=\infty$ and (a_n) is eventually decreasing.

(b) If $(a_n)_n$ is log-convex (see part (b) of Remark 4.7.1), then $\lim_{n\to\infty}\frac{a_n}{a_{n+1}}$ always exists in $[0,\infty]$. Since $\mathcal{H}$ is assumed to be algebraically consistent on $\mathbb{B}_d$ or on $\overline{\mathbb{B}_d}$, the power series $\sum_{n=0}^\infty a_n z^n$ has radius of convergence 1 (see Lemma 4.5.3), hence $\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=1$ is automatic in this case.

(c) As mentioned after Lemma 3.7.3, it is possible to modify Example 6.12 in [25] to construct a unitarily invariant complete NP space $\mathcal{H}$ on $\mathbb{D}$ which violates condition (a) at $\lambda = 0$, and hence all conditions of the preceding proposition. It is not known if this space $\mathcal{H}$ is tame.

(d) The idea to use the factorization theorem to solve Gleason's problem in $\text{Mult}(\mathcal{H})$ already appears in [37], where this was done for the multiplier algebra of the Drury-Arveson space. The main difference between the two arguments is that in [37], it was shown that Gleason's problem can be solved in $\text{Mult}(H^2_d)$ for $\lambda = 0$, and automorphism invariance of $\text{Mult}(H^2_d)$ was used to deduce the general case. The argument here does not require automorphism invariance.

We finish this section by observing that tameness is also implied by the presence of a Corona theorem. In practice, this result is of very limited use, since establishing tameness is usually much easier than establishing a Corona theorem. Indeed, it is very easy to see that $H^2(\mathbb{D})$ is tame (see Example 4.8.3), whereas the Corona theorem for $H^\infty(\mathbb{D}) = \text{Mult}(H^2(\mathbb{D}))$ is hard. Nevertheless, since there are no known examples of complete Nevanlinna-Pick spaces on $\mathbb{B}_d$ for which the Corona theorem fails, the next result explains the lack of examples of spaces which are not tame.

Proposition 4.8.8. *Let $\mathcal{H}$ be a unitarily invariant complete NP-space on $\mathbb{B}_d$. If the set of all point evaluations at points in $\mathbb{B}_d$ is weak-$*$ dense in the maximal ideal space of* $\text{Mult}(\mathcal{H})$*, then $\mathcal{H}$ is tame.*

Proof. Let ρ be a character on $\text{Mult}(\mathcal{H})$ such that $\pi(\rho) = \lambda \in \mathbb{B}_d$. By assumption, there is a net of points (λ_α) in $\mathbb{B}_d$ such that δ_{λ_α} converges to ρ in the weak-$*$ topology. Hence, $\lambda_\alpha = \pi(\delta_{\lambda_\alpha})$ converges to $\lambda = \pi(\rho)$. Since the multipliers are continuous on $\mathbb{B}_d$, it follows that δ_{λ_α} converges to δ_λ in the weak-$*$ topology, whence $\rho = \delta_\lambda$. □

4.9. Holomorphic maps on homogeneous varieties

In the last section, we saw that the maximal ideal space of an algebra of the type $A(\mathcal{H}_I)$ or $\text{Mult}(\mathcal{H}_I)$ contains a copy of the homogeneous variety $Z^0(I)$. We will see in the next section that under suitable conditions, algebra homomorphisms between our algebras induce holomorphic maps between the varieties. Thus, we will require some results about holomorphic maps on homogeneous varieties. The arguments presented in the first part of this section (up to Lemma 4.9.6) already appeared in the author's Master's thesis [41, Section 3.3].

Throughout this section, let $I, J \subsetneq \mathbb{C}[z_1, \ldots, z_d]$ be radical homogeneous ideals. We say that a map $F : Z^0(I) \to \mathbb{C}^{d'}$, where $d' \in \mathbb{N}$, is holomorphic if for every $z \in Z^0(I)$, there exists an open neighbourhood U of z and a holomorphic function G on U which agrees with F on $U \cap Z^0(I)$.

We require the following variant of the maximum modulus principle.

Lemma 4.9.1. *Let $F : Z^0(I) \to \overline{\mathbb{B}_d}$ be a holomorphic map. If F is not constant, then $F(Z^0(I)) \subset \mathbb{B}_d$.*

Proof. We may assume that $\{0\} \subsetneq Z^0(I)$. Suppose that there exists $w \in Z^0(I)$ such that $||F(w)|| = 1$ and choose $\widetilde{w} \in Z^0(I)$ satisfying $w \in \mathbb{D}\widetilde{w}$. The ordinary maximum modulus principle shows that the holomorphic function

$$\mathbb{D} \to \overline{\mathbb{D}}, \quad t \mapsto \langle F(t\widetilde{w}), F(w) \rangle,$$

is the constant function 1. Consequently, $F(t\widetilde{w}) = F(w)$ for all $t \in \overline{\mathbb{D}}$, and in particular $F(0) = F(w) \in \partial\mathbb{B}_d$. Now, if $z \in Z^0(I)$ is arbitrary, another application of the maximum modulus principle shows that the function

$$\mathbb{D} \to \overline{\mathbb{D}}, \quad t \mapsto \langle F(tz), F(0) \rangle,$$

is the constant function 1, hence $F(z) = F(0)$. Thus, F is constant. □

The next goal is to show that every biholomorphism between $Z^0(I)$ and $Z^0(J)$ which fixes the origin is the restriction of an invertible linear map. This result is Theorem 7.4 in [24], where it was established by adjusting the proof of Cartan's uniqueness theorem from [75, Theorem 2.1.3]. We provide a simpler proof, which only uses the Schwarz lemma from ordinary complex analysis. We begin with the following variant of the Schwarz lemma.

Lemma 4.9.2. *Let $d' \in \mathbb{N}$ and let $F : Z^0(I) \to \mathbb{B}_{d'}$ be a holomorphic map such that $F(0) = 0$. Then $||F(z)|| \leq ||z||$ for all $z \in Z^0(I)$. If equality holds for some $z \in Z^0(I) \setminus \{0\}$, then there exists $w_0 \in \partial\mathbb{B}_{d'}$ such that*

$$F\Big(t\frac{z}{||z||}\Big) = tw_0 \tag{4.5}$$

for all $t \in \mathbb{D}$. In particular, F maps the disc $\mathbb{C}z \cap \mathbb{B}_d$ biholomorphically onto the disc $\mathbb{C}F(z) \cap \mathbb{B}_{d'}$ in this case.

Proof. We may assume that $\{0\} \subsetneq Z^0(I)$. Let $z \in Z^0(I) \setminus \{0\}$, suppose that $F(z) \neq 0$ and define $w_0 = F(z)/||F(z)||$. By the classical Schwarz lemma, the function

$$f : \mathbb{D} \to \mathbb{D}, \quad t \mapsto \left\langle F\Big(t\frac{z}{||z||}\Big), w_0 \right\rangle,$$

satisfies $|f(t)| \leq |t|$ for all $t \in \mathbb{D}$. The first statement now follows by choosing $t = ||z||$.

If $||F(z)|| = ||z||$, then $f(||z||) = ||z||$, thus f is the identity by the Schwarz lemma. Since $||F(t\frac{z}{||z||})|| \leq |t|$ for all $t \in \mathbb{D}$ by the first part, Equation (4.5) holds. The last assertion is now obvious. □

The desired result about biholomorphisms which fix the origin follows as an application of the last lemma.

Proposition 4.9.3 ([24, Theorem 7.4])**.** *Let $F : Z^0(I) \to Z^0(J)$ be a biholomorphism such that $F(0) = 0$. Then there exists an invertible linear map A on $\mathbb{C}^d$ which maps $V(I)$ isometrically onto $V(J)$ such that $A\big|_{Z^0(I)} = F$.*

Proof. We may again assume that $\{0\} \subsetneq Z^0(I)$. Let G be a holomorphic map which is defined on a neighbourhood U of 0 and which coincides with F on $U \cap Z^0(I)$. Let A_0 be the derivative of G at 0. Lemma 4.9.2, applied to F and its inverse, shows that $||F(z)|| = ||z||$ for all $z \in Z^0(I)$, so the second part of the same lemma applies. Taking the derivative with respect to t in Equation (4.5) for fixed $z \in Z^0(I) \setminus \{0\}$, we see that w_0 necessarily satisfies $w_0||z|| = A_0 z$, hence

$$F(z) = ||z||w_0 = A_0 z.$$

Thus, $A_0\big|_{Z^0(I)} = F$, and A_0 is isometric on $Z^0(I)$ since F is. Linearity of A_0 implies that A_0 maps $V(I)$ isometrically onto $V(J)$.

Finally, the same argument, applied to F^{-1} in place of F, shows that there exists a linear map B_0 on $\mathbb{C}^d$ such that $B_0\big|_{Z^0(J)} = F^{-1}$. From this, we deduce that A_0 restricts to a linear isomorphism from the linear span of $Z^0(I)$ onto the linear span of $Z^0(J)$. Thus, if we let A be an invertible extension of $A_0\big|_{Z^0(I)}$ to $\mathbb{C}^d$, then A satisfies all the requirements of the proposition. □

We also crucially require a result from [24], which, loosely speaking, allows us to repair biholomorphisms which do not fix the origin. This result is contained in the proof of Proposition 4.7 in [24]. The proof in [24] proceeds in two steps. In a first step, tools from algebraic geometry and knowledge about the structure of conformal automorphisms of $\mathbb{B}_d$ are used to reduce the statement about arbitrary homogeneous varieties to the case of discs. The second step, which deals with the case of discs, is an argument from plane conformal geometry.

It turns out that the first step, namely the reduction to discs, also follows immediately from Lemma 4.9.2.

Lemma 4.9.4. *Let $F : Z^0(I) \to Z^0(J)$ be a biholomorphism with $F(0) \neq 0$. Let $b = F(0)$ and let $a = F^{-1}(0)$. Then $||a|| = ||b||$ and F maps the disc $D_1 = \mathbb{C}a \cap \mathbb{B}_d$ biholomorphically onto the disc $D_2 = \mathbb{C}b \cap \mathbb{B}_d$.*

Proof. Let

$$f : \mathbb{D} \to Z^0(J), \quad t \mapsto F\Big(t\frac{a}{||a||}\Big),$$

and let φ be an automorphism of $\mathbb{D}$ which maps 0 to $||a||$ and vice versa. Then $h = f \circ \varphi$ satisfies the assumptions of Lemma 4.9.2, hence

$$||b|| = ||h(||a||)|| \leq ||a||.$$

By symmetry, $||a|| \leq ||b||$, so $||a|| = ||b||$. It now follows from the second part of Lemma 4.9.2 that h maps $\mathbb{D}$ biholomorphically onto the disc D_2. The result follows. □

The second step is essentially the following lemma. For $\lambda \in \mathbb{T}$, let U_λ denote the unitary map on $\mathbb{C}^d$ defined by

$$U_\lambda(z) = \lambda z$$

for $z \in \mathbb{C}^d$.

Lemma 4.9.5 (Davidson-Ramsey-Shalit [24])**.** *Let φ be a conformal automorphism of $\mathbb{D}$. The set*

$$\{(U_\lambda \circ \varphi^{-1} \circ U_\mu \circ \varphi)(0) : \lambda, \mu \in \mathbb{T}\} \subset \mathbb{D}$$

is a closed disc around 0 which contains the point $\varphi^{-1}(0)$.

Proof. We repeat the relevant part of the proof of Theorem 7.4 in [24]. The assertion is trivial if φ fixes the origin, so we may assume that $\varphi(0) \neq 0$. Then

$$C = \{(U_\mu \circ \varphi)(0) : \mu \in \mathbb{T}\}$$

is the circle around 0 with radius $|\varphi(0)|$. Since automorphisms of $\mathbb{D}$ map circles to circles, it follows that the set $\varphi^{-1}(C)$ is a circle which obviously passes through 0. Moreover, $\varphi^{-1}(0)$ is contained in the interior of the circle $\varphi^{-1}(C)$ as 0 is contained in the interior of C. Thus

$$\{(U_\lambda \circ \varphi^{-1} \circ U_\mu \circ \varphi)(0) : \lambda, \mu \in \mathbb{T}\} = \{U_\lambda(\varphi^{-1}(C)) : \lambda \in \mathbb{T}\}$$

is a closed disc around 0 which contains $\varphi^{-1}(0)$. □

Observe that if $I \subsetneq \mathbb{C}[z_1, \ldots, z_d]$ is a radical homogeneous ideal, then U_λ leaves $Z^0(I)$ and $Z(I)$ invariant for each $\lambda \in \mathbb{T}$. Combining Lemmata 4.9.4 and 4.9.5, we obtain the result from [24] which allows us to repair biholomorphisms which do not fix the origin.

Lemma 4.9.6 (Davidson-Ramsey-Shalit [24])**.** *Let $I, J \subsetneq \mathbb{C}[z_1, \ldots, z_d]$ be radical homogeneous ideals and suppose that $F : Z^0(I) \to Z^0(J)$ is a biholomorphism. Then there are $\lambda, \mu \in \mathbb{T}$ such that the biholomorphism*

$$F \circ U_\lambda \circ F^{-1} \circ U_\mu \circ F : Z^0(I) \to Z^0(J)$$

fixes the origin.

Proof. The assertion is trivial if $F(0) = 0$, so we may assume that $F(0) \neq 0$. It follows then from Lemma 4.9.4 that it suffices to consider the case where $d = 1$ and where $Z^0(I) = Z^0(J) = \mathbb{D}$, the unit disc. An application of Lemma 4.9.5 shows that there are $\lambda, \mu \in \mathbb{T}$ such that

$$F^{-1}(0) = (U_\lambda \circ F^{-1} \circ U_\mu \circ F)(0),$$

hence $F \circ U_\lambda \circ F^{-1} \circ U_\mu \circ F$ fixes the origin. □

We finish this section by giving another application of the crucial Lemma 4.9.5 of Davidson, Ramsey and Shalit [24]. We will show that the group of unitaries is a maximal subgroup of $\operatorname{Aut}(\mathbb{B}_d)$, the group of conformal automorphisms of $\mathbb{B}_d$. Since the group $\operatorname{Aut}(\mathbb{B}_d)$ is well studied, it is likely that this has been observed before. Nevertheless, even when $d = 1$, the only result in this direction that seems to be widely known is the fact that the group of unitaries is a maximal compact subgroup of $\operatorname{Aut}(\mathbb{B}_d)$.

Recall that for $a \in \mathbb{B}_d$, there exists an automorphism φ_a of $\mathbb{B}_d$ defined by

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle} \quad (z \in \mathbb{B}_d),$$

where P_a is the orthogonal projection of $\mathbb{C}^d$ onto the subspace spanned by a, $Q_a = I - P_a$ and $s_a = (1 - |a|^2)^{1/2}$. Then φ_a is an involution which interchanges 0 and a (see, for example, [75, Theorem 2.2.2]). Moreover, every $\varphi \in \operatorname{Aut}(\mathbb{B}_d)$ is of the form $\varphi = U \circ \varphi_a$, where U is unitary and $a = \varphi^{-1}(0)$ [75, Theorem 2.2.5]. We begin with a preliminary lemma.

Lemma 4.9.7. *Let $G \subset \operatorname{Aut}(\mathbb{B}_d)$ be a subsemigroup which contains all unitary maps and let $\mathcal{O}$ denote the orbit of 0 under G. Then the following assertions hold:*

(a) *G is a subgroup of* $\operatorname{Aut}(\mathbb{B}_d)$.

(b) *A point* $a \in \mathbb{B}_d$ *belongs to* $\mathcal{O}$ *if and only if* $\varphi_a \in G$.

(c) $G = \operatorname{Aut}(\mathbb{B}_d)$ *if and only if* $\mathcal{O} = \mathbb{B}_d$.

Proof. (a) If $\varphi \in G$, then $\varphi = U\varphi_a$ for some unitary map U and $a \in \mathbb{B}_d$. Then $\varphi_a \in G$. Since φ_a is an involution, it follows that $\varphi^{-1} = \varphi_a U^{-1} \in G$. Hence, G is a group.

(b) For the proof of the non-trivial implication, suppose that $a \in \mathcal{O}$ and let $\varphi \in G$ with $a = \varphi(0)$. Then $\varphi^{-1} \in G$ by part (a) and $(\varphi^{-1})^{-1}(0) = a$, hence

$$\varphi^{-1} = U \circ \varphi_a$$

for some unitary map U. Since $U \in G$, it follows that $\varphi_a \in G$, as asserted.

(c) This follows immediately from (b) and the description of the automorphisms of $\mathbb{B}_d$ in terms of unitary maps and the involutions φ_a. □

We now show that the group of rotations is a maximal subgroup of the group $\operatorname{Aut}(\mathbb{D})$. We will then deduce the higher-dimensional analogue from this result.

Lemma 4.9.8. *The group of rotations is a maximal subgroup of the group* $\operatorname{Aut}(\mathbb{D})$.

Proof. Let G be a subgroup of $\operatorname{Aut}(\mathbb{D})$ which properly contains the group of rotations. Let $\mathcal{O}$ be the orbit of 0 under G. We wish to show that $\mathcal{O} = \mathbb{D}$, which is equivalent to the assertion by part (c) of Lemma 4.9.7.

We first claim that $\overline{\mathbb{D}}\mathcal{O} \subset \mathcal{O}$. To this end, let $a \in \mathcal{O}$. Then $\varphi_a \in G$ by part (b) of Lemma 4.9.7. An application of Lemma 4.9.5 now shows that $\mathcal{O}$ contains the closed disc of radius $|a|$ around 0, which proves the claim.

We finish the proof by showing that $\mathcal{O}$ contains points of modulus arbitrarily close to 1. Since G contains a non-rotation automorphism, $\mathcal{O} \neq \{0\}$. Clearly, $\mathcal{O}$ is rotationally invariant, hence there exists $r > 0$ such that $r \in \mathcal{O}$ and therefore $\varphi_r \in G$ by part (b) of Lemma 4.9.7. Consider the hyperbolic automorphism f defined by

$$f(z) = \varphi_r(-z) = \frac{r+z}{1+rz}$$

for $z \in \mathbb{D}$. Then $f \in G$. Moreover, it is well known and easy to see that

$$\lim_{n \to \infty} f^n(0) = 1,$$

where f^n denotes the n-fold iteration of f. Thus, the proof is complete. □

We are now ready to prove a multivariate analogue of the last lemma.

Proposition 4.9.9. *The group of unitary maps on* $\mathbb{C}^d$ *is a maximal subsemigroup of* $\operatorname{Aut}(\mathbb{B}_d)$.

Proof. Suppose that G is a subsemigroup of $\operatorname{Aut}(\mathbb{B}_d)$ which properly contains the group of unitary maps. Then G is a subgroup by part (a) of Lemma 4.9.7. Let $\mathcal{O}$ denote the orbit of 0 under G. Our goal is to show that $\mathcal{O} = \mathbb{B}_d$ (see part (c) of Lemma 4.9.7). Since G contains all unitaries, it suffices to show that $\mathbb{D}e_1 \subset \mathcal{O}$, where e_1 denotes the first standard basis vector of $\mathbb{C}^d$.

To this end, let

$$H = \{\varphi \in G : \varphi(\mathbb{D}e_1) = \mathbb{D}e_1\}.$$

Identifying $\mathbb{D}e_1$ with $\mathbb{D}$ we obtain a subgroup

$$\widetilde{H} = \{\varphi\big|_{\mathbb{D}} : \varphi \in H\}$$

of $\operatorname{Aut}(\mathbb{D})$. Clearly, $\widetilde{H}$ contains all rotations U_λ for $\lambda \in \mathbb{T}$. Since G contains a non-unitary automorphism, $\{0\} \neq \mathcal{O}$. Moreover, $\mathcal{O}$ is invariant under unitary maps, hence there exists $r > 0$ such that $re_1 \in \mathcal{O}$ and thus $\varphi_{re_1} \in G$ by part (b) of Lemma 4.9.7. Observe that $\varphi_{re_1} \in H$, so $\widetilde{H}$ contains the non-rotation automorphism φ_r. It now follows from Lemma 4.9.8 that $\widetilde{H} = \operatorname{Aut}(\mathbb{D})$. Since $\operatorname{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$, the definition of $\widetilde{H}$ implies that $\mathbb{D}e_1 \subset \mathcal{O}$, which completes the proof. □

There is an immediate consequence for collections of functions on $\mathbb{B}_d$ which are unitarily invariant.

Corollary 4.9.10. *Let* $S \neq \emptyset$ *be a collection of functions on* $\mathbb{B}_d$ *and define*

$$G = \{\varphi \in \operatorname{Aut}(\mathbb{B}_d) : f \circ \varphi \in S \textit{ for all } f \in S\}.$$

Assume that G *contains* $\mathcal{U}$*, the group of unitary maps on* $\mathbb{C}^d$*. Then either* $G = \mathcal{U}$ *or* $G = \operatorname{Aut}(\mathbb{B}_d)$.

Proof. It is clear that G is a subsemigroup of $\operatorname{Aut}(\mathbb{B}_d)$, so the result follows from Proposition 4.9.9. □

The last result applies in particular to reproducing kernel Hilbert spaces $\mathcal{H}$ on $\mathbb{B}_d$ with a kernel of the form

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n \quad (z, w \in \mathbb{B}_d).$$

In this case, by the closed graph theorem, G is also the set of all automorphisms of $\mathbb{B}_d$ which induce a bounded composition operator on $\mathcal{H}$. Moreover, G contains all unitaries. Thus, the last result says that such a space $\mathcal{H}$ is either invariant under all automorphisms of $\mathbb{B}_d$, or under unitaries only.

4.10. Existence of graded isomorphisms

The question of when two algebras of the type $\operatorname{Mult}(\mathcal{H}_I)$ are isomorphic is more difficult than the question about equality of multiplier algebras studied in Section 4.6. The chief reason is that isomorphisms do not necessarily respect the grading. Thus, our goal is to establish the existence of graded isomorphisms. As in [24], this will follow from an application of Lemma 4.9.6.

Throughout this section, let $\mathcal{H}$ and $\mathcal{K}$ be unitarily invariant complete NP-spaces on $\mathbb{B}_d$ or on $\overline{\mathbb{B}_d}$ and let $I, J \subsetneq \mathbb{C}[z_1, \dots, z_d]$ be radical homogeneous ideals. We allow the case where $\mathcal{H}$ is a space on $\mathbb{B}_d$, and $\mathcal{K}$ is a space on $\overline{\mathbb{B}_d}$, or vice versa. We will consider the multiplier algebras $\operatorname{Mult}(\mathcal{H}_I)$ and $\operatorname{Mult}(\mathcal{K}_J)$, as well as their norm closed versions $A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$. To cover both cases, we first study homomorphisms from $A(\mathcal{H}_I)$ into $\operatorname{Mult}(\mathcal{K}_J)$. We identify the maximal ideal space of $A(\mathcal{H}_I)$ with $Z(I)$ by Lemma 4.8.1. Similarly, we identify $Z^0(J)$ with a subset of the maximal ideal space of $\operatorname{Mult}(\mathcal{K}_J)$ via point evaluations. The following lemma should be compared to Proposition 7.1 and Lemma 11.5 in [24].

Lemma 4.10.1. *Let $\mathcal{H}$ and $\mathcal{K}$ as well as $I, J \subsetneq \mathbb{C}[z_1, \dots, z_d]$ be as above.*

(a) *If $\Phi : A(\mathcal{H}_I) \to \operatorname{Mult}(\mathcal{K}_J)$ is an injective unital homomorphism, then Φ^* maps $Z^0(J)$ holomorphically into $Z^0(I)$.*

(b) *If $\Phi : \operatorname{Mult}(\mathcal{H}_I) \to \operatorname{Mult}(\mathcal{K}_J)$ is an injective unital homomorphism and weak-$*$-weak-$*$ continuous, then Φ^* maps $Z^0(J)$ holomorphically into $Z^0(I)$.*

(c) *If $\Phi : \operatorname{Mult}(\mathcal{H}_I) \to \operatorname{Mult}(\mathcal{K}_J)$ is an injective unital homomorphism, and if $\mathcal{H}$ is tame, then Φ^* maps $Z^0(J)$ holomorphically into $Z^0(I)$, and Φ is weak-$*$-weak-$*$ continuous.*

Proof. (a) Clearly, Φ^* maps $Z^0(J)$ into $Z(I)$, and the j-th coordinate of Φ^* is given by $\Phi(z_j) \in A(\mathcal{H}_I)$, hence $F = \Phi^*\big|_{Z^0(J)}$ is holomorphic. Lemma 4.9.1 shows that the range of F contains points in $\partial\mathbb{B}_d$ only if F is constant. In this case, $\Phi(z_j) = \lambda_j$, where $(\lambda_1, \dots, \lambda_d) \in \partial\mathbb{B}_d$. Since Φ is unital and injective, it follows that $\lambda_j = z_j$ on $Z^0(I)$, which is absurd. Thus, the range of F is contained in $Z^0(I)$.

(b) By definition of the map $\pi : \mathcal{M}(\operatorname{Mult}(\mathcal{H}_I)) \to Z(I)$, part (a) implies that $\pi \circ \Phi^*$ is holomorphic and maps $Z^0(J)$ into $Z^0(I)$. Since Φ is weak-$*$-weak-$*$ continuous, $\Phi^*(Z^0(J))$ consists of point evaluations by Lemma 4.5.7, so the assertion follows.

(c) Again by part (a), $\pi \circ \Phi^*$ is holomorphic and maps $Z^0(J)$ into $Z^0(I)$. Since $\mathcal{H}$ is tame, we conclude that Φ^* maps $Z^0(J)$ into the set (of point evaluations at points in) $Z^0(I)$ (see Remark 4.8.4). If $\mathcal{K}$ is a space on $\mathbb{B}_d$, Lemma 4.5.7 therefore implies that Φ is weak-$*$-weak-$*$ continuous. Now, assume that $\mathcal{K}$ is a space on $\overline{\mathbb{B}_d}$. If $\mathcal{H}$ is a space on $\overline{\mathbb{B}_d}$ as well, then $\Phi^*(Z(J)) \subset Z(I)$ by continuity of Φ^*, thus Φ is again weak-$*$-weak-$*$ continuous by Lemma 4.5.7.

It remains to consider the case where $\mathcal{H}$ is a space on $\mathbb{B}_d$ and $\mathcal{K}$ is space on $\overline{\mathbb{B}_d}$. We claim that $\Phi^*(Z^0(J))$ is contained in a ball of radius $r < 1$. This will finish the proof, as $\Phi^*(Z(J)) \subset rZ(I) \subset Z^0(I)$ by continuity, so once again, the assertion follows from Lemma 4.5.7. Suppose that $\Phi^*(Z^0(J))$ contains a sequence $(\Phi^*(\lambda_n))$ with $||\Phi^*(\lambda_n)|| \to 1$. By passing to a subsequence, we may assume that (λ_n) converges to a point $\lambda \in Z(J)$. Lemma 4.8.2 shows that there is a multiplier $\varphi \in \text{Mult}(\mathcal{H}_I)$ such that $(\varphi(\Phi^*(\lambda_n)))$ does not converge. However,

$$\varphi(\Phi^*(\lambda_n)) = (\Phi(\varphi))(\lambda_n),$$

and $\Phi(\varphi) \in \text{Mult}(\mathcal{K}_J)$ is a continuous function on $Z(J)$. This is a contradiction, and the proof is complete. □

For isomorphisms, we obtain the following consequence.

Corollary 4.10.2. *Let $\mathcal{H}$ and $\mathcal{K}$ as well as $I, J \subsetneq \mathbb{C}[z_1, \dots, z_d]$ be as above.*

(a) *If $\Phi : A(\mathcal{H}_I) \to A(\mathcal{H}_J)$ is an isomorphism, then Φ^* maps $Z^0(J)$ biholomorphically onto $Z^0(I)$.*

(b) *Let $\Phi : \text{Mult}(\mathcal{H}_I) \to \text{Mult}(\mathcal{H}_J)$ be an isomorphism, and assume that $\mathcal{H}$ is tame or that Φ is weak-$*$-weak-$*$ continuous. Then Φ^* maps $Z^0(J)$ biholomorphically onto $Z^0(I)$, and Φ is a weak-$*$-weak-$*$ homeomorphism.*

Proof. (a) immediately follows from part (a) of the preceding lemma.

(b) By part (c) of the last lemma, Φ is weak-$*$-weak-$*$ continuous in both cases. Since it is also a homeomorphism in the norm topologies, the Krein-Smulian theorem combined with weak-$*$ compactness of the unit balls shows that Φ^{-1} is weak-$*$-weak-$*$ continuous as well (see, for example, the argument at the end of the proof of Theorem 3.2.5). Thus, part (b) of the last lemma also applies to Φ^{-1}, so that Φ^* is a biholomorphism between $Z^0(J)$ and $Z^0(I)$. □

For $n \in \mathbb{N}$, let $(\mathcal{H}_I)_n$ denote the space of all homogeneous elements of $\mathcal{H}_I$ of degree n. Recall that $(\mathcal{H}_I)_n \subset A(\mathcal{H}_I)$ for all $n \in \mathbb{N}$. We say that a homomorphism $\Phi : A(\mathcal{H}_I) \to \text{Mult}(\mathcal{K}_J)$ is graded if

$$\Phi((\mathcal{H}_I)_n) \subset (\mathcal{K}_J)_n$$

for all $n \in \mathbb{N}$. Graded isomorphisms admit a particularly simple description in terms of their adjoint.

Lemma 4.10.3. *Let $\mathcal{H}$ and $\mathcal{K}$ as well as $I, J \subsetneq \mathbb{C}[z_1, \dots, z_d]$ be as above, and suppose that $\Phi : A(\mathcal{H}_I) \to A(\mathcal{K}_J)$ is an isomorphism (respectively that $\Phi : \operatorname{Mult}(\mathcal{H}_I) \to \operatorname{Mult}(\mathcal{K}_J)$ is a weak-$*$-weak-$*$ continuous isomorphism). Then the following are equivalent:*

(i) *Φ is graded.*

(ii) $\Phi^*(0) = 0$.

(iii) *There exists an invertible linear map A on $\mathbb{C}^d$ which maps $V(J)$ isometrically onto $V(I)$ such that Φ is given by composition with A, that is,*

$$\Phi(\varphi) = \varphi \circ A$$

for all $\varphi \in A(\mathcal{H}_I)$ (respectively $\varphi \in \operatorname{Mult}(\mathcal{H}_I)$).

Proof. (iii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii) Let $\lambda = \Phi^*(0) \in Z(I)$. If $\lambda \neq 0$, then there is a homogeneous element $\varphi \in A(\mathcal{H}_I)$ of degree 1 such that $\varphi(\lambda) \neq 0$. Corollary 4.10.2 implies that $\Phi^*(0) \in Z^0(I)$, hence

$$\Phi(\varphi)(0) = \varphi(\Phi^*(0)) = \varphi(\lambda) \neq 0.$$

In particular, $\Phi(\varphi)$ is not homogeneous of degree 1, hence Φ is not graded.

(ii) $\Rightarrow$ (iii) By Corollary 4.10.2, Φ^* maps $Z^0(J)$ biholomorphically onto $Z^0(I)$. Since $\Phi^*(0) = 0$, Proposition 4.9.3 therefore yields an invertible linear map A which maps $V(J)$ isometrically onto $V(I)$ such that Φ^* coincides with A on $Z^0(J)$. It follows that

$$\Phi(\varphi) = \varphi \circ A$$

on $Z^0(J)$ for all $\varphi \in A(\mathcal{H}_I)$ (respectively $\varphi \in \operatorname{Mult}(\mathcal{H}_I)$). Moreover, if Φ is a map from $A(\mathcal{H}_I)$ onto $A(\mathcal{K}_J)$, then this identity holds on $Z(J)$ by continuity.

Assume now that Φ is a map from $\operatorname{Mult}(\mathcal{H}_I)$ onto $\operatorname{Mult}(\mathcal{K}_J)$. If $\mathcal{K}$ is a space on $\mathbb{B}_d$, we are done. If $\mathcal{K}$ and $\mathcal{H}$ are spaces on $\overline{\mathbb{B}_d}$, then $\Phi(\varphi) = \varphi \circ A$ again holds on $Z(J)$ by continuity. We finish the proof by showing that the remaining case where $\mathcal{K}$ is a space on $\overline{\mathbb{B}_d}$, $\mathcal{H}$ is a space on $\mathbb{B}_d$ and $V(J) \neq \{0\}$ does not occur. Indeed, in this case, $V(I) \neq \{0\}$ and Φ^* would map $Z(J)$ onto a necessarily compact subset of $Z^0(I)$ by Lemma 4.5.7. This contradicts the fact that Φ^* maps $Z^0(J)$ onto $Z^0(I)$. $\square$

We mention that in the case where $\mathcal{H} = \mathcal{K} = H^2_d$, the Drury-Arveson space, isomorphisms as above are called vacuum-preserving in [24].

The desired consequence about the existence of graded isomorphisms is the following result.

Proposition 4.10.4. *Let $\mathcal{H}$ and $\mathcal{K}$ as well as $I, J \subsetneq \mathbb{C}[z_1, \ldots, z_d]$ be as above.*

(a) *If $A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$ are algebraically (respectively isometrically) isomorphic, then there exists a graded algebraic (respectively isometric) isomorphism from $A(\mathcal{H}_I)$ onto $A(\mathcal{K}_J)$.*

(b) *If $\operatorname{Mult}(\mathcal{H}_I)$ and $\operatorname{Mult}(\mathcal{K}_J)$ are algebraically (respectively isometrically) isomorphic via a weak-$*$-weak-$*$ continuous isomorphism, then there exists a graded weak-$*$-weak-$*$ continuous algebraic (respectively isometric) isomorphism from $\operatorname{Mult}(\mathcal{H}_I)$ onto $\operatorname{Mult}(\mathcal{K}_J)$.*

Proof. By Lemma 4.10.3, it suffices to show in each case that there exists an isomorphism whose adjoint fixes the origin. We will achieve this by applying Corollary 4.10.2 and Lemma 4.9.6. To this end, observe that for $\lambda \in \mathbb{T}$, the unitary map U_λ on $\mathbb{C}^d$ given by multiplication with λ induces a unitary composition operator C_{U_λ} on $\mathcal{H}_I$. If $\varphi \in \operatorname{Mult}(\mathcal{H}_I)$, then

$$C_{U_\lambda} M_\varphi C_{U_\lambda}^* = M_{\varphi \circ U_\lambda},$$

hence $\Phi^I_\lambda(M_\varphi) = C_{U_\lambda} M_\varphi C_{U_\lambda}^*$ defines an isometric, weak-$*$-weak-$*$ continuous automorphism of $\operatorname{Mult}(\mathcal{H}_I)$ which maps $A(\mathcal{H}_I)$ onto $A(\mathcal{H}_I)$. Clearly, the adjoint of this automorphism, restricted to $Z^0(I)$, is given by multiplication with U_λ. The same result holds for $\mathcal{K}_J$ in place of $\mathcal{H}_I$.

Suppose now that Φ is an isomorphism between $A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$ (respectively a weak-$*$-weak-$*$ continuous isomorphism between $\operatorname{Mult}(\mathcal{H}_I)$ and $\operatorname{Mult}(\mathcal{K}_J)$). By Corollary 4.10.2, the adjoint Φ^* maps $Z^0(J)$ biholomorphically onto $Z^0(I)$. From Lemma 4.9.6, we infer that there exist $\lambda, \mu \in \mathbb{T}$ such that the map

$$\Phi^* \circ U_\lambda \circ (\Phi^*)^{-1} \circ U_\mu \circ \Phi^*$$

fixes the origin. This map is the adjoint of

$$\Phi \circ \Phi^I_\mu \circ \Phi^{-1} \circ \Phi^J_\lambda \circ \Phi,$$

which is an isomorphism between $A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$ (respectively a weak-$*$-weak-$*$ continuous isomorphism between $\operatorname{Mult}(\mathcal{H}_I)$ and $\operatorname{Mult}(\mathcal{K}_J)$). Moreover, it is isometric if Φ is isometric, which finishes the proof. □

4.11. Isomorphism results

We are now ready to establish the main results about isomorphism of multiplier algebras of spaces of the type $\mathcal{H}_I$. We will usually make an assumption which guarantees that the Hilbert function spaces have dimension at least 2. In projective algebraic geometry, the maximal ideal of $\mathbb{C}[z_1, \ldots, z_d]$ which is generated by the coordinate functions $z_1, \ldots, z_d$ is called the *irrelevant ideal* (see [90, Chapter VII]). This is because the vanishing locus of this ideal in $\mathbb{C}^d$ is just the origin, hence the projective vanishing locus in $\mathbb{P}^{d-1}(\mathbb{C})$ is empty. We will say that a radical homogeneous ideal of $\mathbb{C}[z_1, \ldots, z_d]$ is *relevant* if it is proper and not equal to the irrelevant ideal. By the projective Nullstellensatz, the projective vanishing locus of every such ideal I is not empty, thus $Z^0(I) \subset \mathbb{C}^d$ always contains a disc.

Proposition 4.11.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be unitarily invariant complete NP-spaces, and let I and J be relevant radical homogeneous ideals in $\mathbb{C}[z_1, \ldots, z_d]$. Let $\Phi : A(\mathcal{H}_I) \to A(\mathcal{K}_J)$ be a graded algebraic isomorphism (respectively $\Phi : \mathrm{Mult}(\mathcal{H}_I) \to \mathrm{Mult}(\mathcal{K}_J)$ a graded weak-$*$-weak-$*$ continuous isomorphism).*

Then $\mathcal{H} = \mathcal{K}$ as vector spaces, and there exists an invertible linear map A which maps $V(J)$ isometrically onto $V(I)$ such that Φ is given by composition with A. Moreover, A induces a bounded invertible composition operator

$$C_A : \mathcal{H}_I \to \mathcal{K}_J, \quad f \mapsto f \circ A,$$

such that

$$\Phi(M_\varphi) = C_A M_\varphi (C_A)^{-1}$$

for all $\varphi \in A(\mathcal{H}_I)$ (respectively $\varphi \in \mathrm{Mult}(\mathcal{H}_I)$). In particular, Φ is given by a similarity.

Proof. By Lemma 4.10.3, there exists an invertible linear map A which maps $V(J)$ isometrically onto $V(I)$ and such that Φ is given by composition with A. Since all Banach algebras under consideration are semi-simple, Φ and its inverse are (norm) continuous (see [17, Proposition 4.2]). Thus, if $f \in \mathcal{H}_I$ is homogeneous, then Proposition 4.6.4 shows that

$$||f \circ A||_{\mathcal{K}_J} = ||f \circ A||_{\mathrm{Mult}(\mathcal{K}_J)} \leq ||\Phi|| \, ||f||_{\mathrm{Mult}(\mathcal{H}_I)} = ||\Phi|| \, ||f||_{\mathcal{H}_I},$$

so there exists a bounded operator $C_A : \mathcal{H}_I \to \mathcal{K}_J$ such that

$$C_A f = f \circ A$$

holds for every polynomial f, and hence for all $f \in \mathcal{H}_I$. Consideration of Φ^{-1} shows that C_A is invertible. Moreover, for $\varphi \in \mathrm{Mult}(\mathcal{H}_I)$ and $f \in \mathcal{K}_J$, we have

$$C_A M_\varphi (C_A)^{-1} f = (\varphi \circ A) f,$$

hence Φ is given by conjugation with C_A.

We finish the proof by showing that $\mathcal{H}$ and $\mathcal{K}$ coincide as vector spaces. To this end, let

$$K_{\mathcal{H}}(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n$$

and

$$K_{\mathcal{K}}(z,w) = \sum_{n=0}^{\infty} a'_n \langle z, w\rangle^n$$

denote the reproducing kernels of $\mathcal{H}$ and $\mathcal{K}$, respectively. Since I and J are radical, Lemma 4.7.4 implies that the restriction maps $R_I : \mathcal{H} \ominus I \to \mathcal{H}_I$ and $R_J : \mathcal{K} \ominus J \to \mathcal{K}_J$ are unitary. Let

$$T_A = R_I^{-1}(C_A)^* R_J \in \mathcal{B}(\mathcal{K} \ominus J, \mathcal{H} \ominus I).$$

Then T_A is bounded and invertible, and Lemma 4.7.5 combined with Lemma 4.7.4 implies that

$$T_A K_{\mathcal{K}}(\cdot, w) = K_{\mathcal{H}}(\cdot, Aw)$$

for all $w \in Z^0(J)$. Using the homogeneity of J, it is easy to deduce from $K_{\mathcal{K}}(\cdot,w) \in \mathcal{K} \ominus J$ for $w \in Z^0(J)$ that $\langle \cdot, w\rangle^n \in \mathcal{K} \ominus J$ for all $w \in Z^0(J)$ and all $n \in \mathbb{N}$. Similarly, $\langle \cdot, z\rangle^n \in \mathcal{H} \ominus I$ for all $z \in Z^0(I)$ and all $n \in \mathbb{N}$. Moreover, C_A and hence T_A respects the degree of homogeneous polynomials. Consequently,

$$T_A a'_n \langle \cdot, w\rangle^n = a_n \langle \cdot, Aw\rangle^n \tag{4.6}$$

for all $n \in \mathbb{N}$ and all $w \in V(J)$. Using part (d) of Remark 4.7.1 and the fact that $||Aw|| = ||w||$, we see that

$$||a'_n \langle \cdot, w\rangle^n||^2_{\mathcal{K}_J} = a'_n ||w||^{2n}$$

and that

$$||a_n \langle \cdot, Aw\rangle^n||^2_{\mathcal{H}_I} = a_n ||w||^{2n}.$$

Since J is relevant, $V(J)$ contains a non-zero vector w, hence

$$||(C_A^*)^{-1}||^2 \le \frac{a_n}{a'_n} \le ||C_A^*||^2$$

for all $n \in \mathbb{N}$, from which it immediately follows that $\mathcal{H} = \mathcal{K}$ as vector spaces (see part (d) of Remark 4.7.1). $\square$

Using the same methods as in the last proof, we obtain a version of Proposition 4.11.1 for isometric isomorphisms.

Proposition 4.11.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be unitarily invariant complete NP-spaces, and let I and J be relevant radical homogeneous ideals in $\mathbb{C}[z_1, \dots, z_d]$. Let $\Phi : A(\mathcal{H}_I) \to A(\mathcal{K}_J)$ be a graded isometric isomorphism (respectively $\Phi : \mathrm{Mult}(\mathcal{H}_I) \to \mathrm{Mult}(\mathcal{K}_J)$ a graded weak-$*$-weak-$*$ continuous isometric isomorphism).*

Then $\mathcal{H} = \mathcal{K}$ as Hilbert spaces, and there exists a unitary map U which maps $V(J)$ onto $V(I)$ such that Φ is given by composition with U. Moreover, U induces a unitary composition operator

$$C_U : \mathcal{H}_I \to \mathcal{K}_J, \quad f \mapsto f \circ U,$$

such that

$$\Phi(M_\varphi) = C_U M_\varphi (C_U)^{-1}$$

for all $\varphi \in A(\mathcal{H}_I)$ (respectively $\varphi \in \mathrm{Mult}(\mathcal{H}_I)$). In particular, Φ is unitarily implemented.

Proof. Proposition 4.11.1 and its proof show that there exists an invertible linear map U which maps $V(J)$ isometrically onto $V(I)$ such that U induces a unitary composition operator

$$C_U : \mathcal{K}_J \to \mathcal{H}_I, \quad f \mapsto f \circ U,$$

and such that Φ is given by conjugation with C_U. Since C_U is a unitary operator, the last part of the proof of Proposition 4.11.1 shows that $a_n = a'_n$ for all $n \in \mathbb{N}$ in the notation of the proof, and hence $\mathcal{H} = \mathcal{K}$ as Hilbert spaces.

Finally, setting $n = 1$ in Equation (4.6), we see that

$$T_U \langle \cdot, w \rangle = \langle \cdot, Uw \rangle$$

for all $w \in V(J)$, and hence for all w in the linear span of $V(J)$. Since T_U is a unitary operator, part (d) of Remark 4.7.1 implies that U is isometric on the linear span of $V(J)$. Hence, U is a unitary map from the linear span of $V(J)$ onto the linear span of $V(I)$. Changing U on the orthogonal complement of $\mathrm{span}(V(J))$ if necessary, we can therefore achieve that U is a unitary map on $\mathbb{C}^d$. □

The last result, combined with Proposition 4.10.4, provides a necessary condition for the existence of an isometric isomorphism between two algebras of the form $A(\mathcal{H}_I)$, namely condition (iii) in the next theorem. This condition turns out to be sufficient as well. We thus obtain our main result regarding the isometric isomorphism problem. It generalizes [24, Theorem 8.2]. For a bounded invertible operator S between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let

$$\mathrm{Ad}(S) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}), \quad T \mapsto STS^{-1},$$

be the induced isomorphism between $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$.

Theorem 4.11.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be unitarily invariant complete NP-spaces, and let I and J be relevant radical homogeneous ideals in $\mathbb{C}[z_1, \dots, z_d]$. Then the following are equivalent:*

(i) *$A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$ are isometrically isomorphic.*

(ii) *$\operatorname{Mult}(\mathcal{H}_I)$ and $\operatorname{Mult}(\mathcal{K}_J)$ are isometrically isomorphic via a weak-$*$-weak-$*$ continuous isomorphism.*

(iii) *$\mathcal{H} = \mathcal{K}$ as Hilbert spaces and there is a unitary map U on $\mathbb{C}^d$ which maps $V(J)$ onto $V(I)$.*

If $\mathcal{H}$ or $\mathcal{K}$ is tame, then this is equivalent to

(iv) *$\operatorname{Mult}(\mathcal{H}_I)$ and $\operatorname{Mult}(\mathcal{K}_J)$ are isometrically isomorphic.*

If U is a unitary map on $\mathbb{C}^d$ as in (iii), then U induces a unitary composition operator

$$C_U : \mathcal{H}_I \to \mathcal{K}_J, \quad f \mapsto f \circ U,$$

and $\operatorname{Ad}(C_U)$ maps $A(\mathcal{H}_I)$ onto $A(\mathcal{K}_J)$ and $\operatorname{Mult}(\mathcal{H}_I)$ onto $\operatorname{Mult}(\mathcal{K}_J)$.

Proof. It follows from Proposition 4.10.4 and Proposition 4.11.2 that (i) or (ii) implies (iii). Moreover, if one of the spaces is tame, then Corollary 4.10.2 (b) shows the equivalence of (ii) and (iv).

Conversely, suppose that (iii) holds. Since $\mathcal{H} = \mathcal{K}$ is unitarily invariant, U induces a unitary composition operator $\widehat{C}_U \in \mathcal{B}(\mathcal{H})$. If K denotes the reproducing kernel of $\mathcal{H}$, then

$$(\widehat{C}_U)^* K(\cdot, w) = K(\cdot, Uw)$$

for all $w \in Z^0(J)$ (or $w \in Z(J)$ if $\mathcal{H}$ is a space on $\overline{\mathbb{B}_d}$). Since $\mathcal{H} \ominus I$ and $\mathcal{H} \ominus J$ are spanned by kernel functions (see Lemma 4.7.4), the implication (ii) $\Rightarrow$ (i) in Lemma 4.7.5 shows that U induces a unitary composition operator $C_U : \mathcal{H}_I \to \mathcal{H}_J$.

Then for $\varphi \in \operatorname{Mult}(\mathcal{H}_I)$ and $f \in \mathcal{K}_J$,

$$C_U M_\varphi (C_U)^{-1} f = (\varphi \circ U) \cdot f,$$

hence $\operatorname{Ad}(C_U)$ maps $\operatorname{Mult}(\mathcal{H}_I)$ into $\operatorname{Mult}(\mathcal{H}_J)$ and $A(\mathcal{H}_I)$ into $A(\mathcal{H}_J)$. If we consider $\operatorname{Ad}(C_{U^{-1}})$, we see that $\operatorname{Ad}(C_U)$ is an isomorphism from $\operatorname{Mult}(\mathcal{H}_I)$ onto $\operatorname{Mult}(\mathcal{H}_J)$ and from $A(\mathcal{H}_I)$ onto $A(\mathcal{H}_J)$. Hence, (i) and (ii) hold, and the additional assertion is proven. □

For algebraic isomorphisms, the situation is more difficult. Proposition 4.10.4 and Proposition 4.11.1 show that if $A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$ are algebraically isomorphic, then $\mathcal{H} = \mathcal{K}$ as vector spaces and there exists an invertible linear map A on $\mathbb{C}^d$ which maps $V(J)$ isometrically onto $V(I)$. Note that here, A will in general only be isometric on $V(J)$ and not on all of $\mathbb{C}^d$. In this case, it is no longer obvious that A induces an algebraic isomorphism between $A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$. The reason why the proof of Theorem 4.11.3 does not carry over is that now, A does not induce a composition operator on all of $\mathcal{H}$. In the case of $\mathcal{H} = H^2_d$, this problem already appeared in [24], where it was solved under additional assumptions on the geometry of $V(J)$. The general case was settled in [40]. Fortunately, we can use a crucial reduction from [24] and the main result of [40] in our setting as well.

Lemma 4.11.4. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space on $\mathbb{B}_d$ (or on $\overline{\mathbb{B}_d}$) with a reproducing kernel of the form*

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n,$$

where $a_n > 0$ for all $n \in \mathbb{N}$. Suppose that $I \subsetneq \mathbb{C}[z_1, \ldots, z_d]$ is a radical homogeneous ideal. If A is a linear map on $\mathbb{C}^d$ which is isometric on $V(I)$, then there exists a bounded operator

$$T_A : \mathcal{H} \ominus I \to \mathcal{H} \quad \text{such that} \quad T_A K(\cdot, w) = K(\cdot, Aw)$$

for all $w \in Z^0(I)$ (respectively $w \in Z(I)$ if $\mathcal{H}$ is a space on $\overline{\mathbb{B}_d}$).

Proof. The first part of the proof is a straightforward adaptation of the proof of [40, Proposition 2.5]. Let

$$V(I) = V_1 \cup \ldots \cup V_r$$

be the decomposition of $V(I)$ into irreducible homogeneous varieties and let $\widehat{I}$ be the vanishing ideal of $\operatorname{span} V_1 \cup \ldots \cup \operatorname{span} V_r$. Then $\widehat{I} \subset I$ by Hilbert's Nullstellensatz. By [24, Proposition 7.6], the linear map A is isometric on $V(\widehat{I})$, so we may assume without loss of generality that

$$V(I) = V_1 \cup \ldots \cup V_r$$

is a union of subspaces, so

$$I = I_1 \cap \ldots \cap I_r,$$

where I_j is the vanishing ideal of V_j. Then by a variant of [40, Lemma 2.3],

$$\mathcal{H} \ominus I = \overline{\mathcal{H} \ominus I_1 + \ldots + \mathcal{H} \ominus I_r}.$$

We define T_A on the dense subspace of $\mathcal{H} \ominus I$ consisting of polynomials by

$$T_A p = p \circ A^*.$$

Then $T_A \langle \cdot, w \rangle^n = \langle \cdot, Aw \rangle^n$ for all $w \in V(I)$. Using the fact that $\mathcal{H}$ is unitarily invariant and that A is isometric on each V_j, it is not hard to see that the map T_A is isometric on $\mathcal{H} \ominus I_j$ for every j (cf. [40, Lemma 2.2]). As in the proof of [40, Proposition 2.5], we may therefore finish the proof by showing that the algebraic sum

$$\mathcal{H} \ominus I_1 + \ldots + \mathcal{H} \ominus I_r$$

is closed.

If $\mathcal{H} = H^2_d$, this is the main result of [40]. More generally, in the present setting, there exists a unique unitary operator

$$U : H^2_d \to \mathcal{H} \quad \text{with} \quad U(p) = \sqrt{a_n} p$$

for every homogeneous polynomial p of degree n (see, for example [39, Proposition 4.1]). Since each I_j is a homogeneous ideal, $U(I_j) = I_j$ and hence $U(H^2_d \ominus I_j) = \mathcal{H} \ominus I_j$ for $1 \le j \le r$. Consequently, closedness of the algebraic sum

$$\mathcal{H} \ominus I_1 + \ldots + \mathcal{H} \ominus I_r$$

follows from the special case where $\mathcal{H} = H^2_d$. □

With the help of Lemma 4.11.4, we can now prove the main result regarding the algebraic isomorphism problem. It generalizes [24, Theorem 8.5] and [40, Theorem 5.9]. Observe that since the algebras $A(\mathcal{H}_I)$ and $\mathrm{Mult}(\mathcal{H}_I)$ are semi-simple, algebraic isomorphisms are automatically norm continuous.

Theorem 4.11.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be unitarily invariant complete NP-spaces, and let I and J be relevant radical homogeneous ideals in $\mathbb{C}[z_1, \ldots, z_d]$. Then the following are equivalent:*

- (i) *$A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$ are algebraically isomorphic.*
- (ii) *$\mathrm{Mult}(\mathcal{H}_I)$ and $\mathrm{Mult}(\mathcal{K}_J)$ are isomorphic via a weak-$*$-weak-$*$ continuous isomorphism.*
- (iii) *$\mathcal{H} = \mathcal{K}$ as vector spaces and there is an invertible linear map A on $\mathbb{C}^d$ which maps $V(J)$ isometrically onto $V(I)$.*

If $\mathcal{H}$ or $\mathcal{K}$ is tame, then this is equivalent to

- (iv) *$\mathrm{Mult}(\mathcal{H}_I)$ and $\mathrm{Mult}(\mathcal{K}_J)$ are algebraically isomorphic.*

If A is an invertible linear map on $\mathbb{C}^d$ as in (iii), then A induces a bounded invertible composition operator

$$C_A : \mathcal{H}_I \to \mathcal{K}_J, \quad f \mapsto f \circ A,$$

and $\mathrm{Ad}(C_A)$ maps $A(\mathcal{H}_I)$ onto $A(\mathcal{K}_J)$ and $\mathrm{Mult}(\mathcal{H}_I)$ onto $\mathrm{Mult}(\mathcal{K}_J)$.

Proof. It follows from Proposition 4.10.4 and Proposition 4.11.1 that (i) or (ii) implies (iii). Moreover, if one of the spaces is tame, then Corollary 4.10.2 (b) once again shows the equivalence of (ii) and (iv).

Assume that (iii) holds. Since $\mathcal{H} = \mathcal{K}$ as vector spaces, the formal identity

$$E : \mathcal{H} \to \mathcal{K}, \quad f \mapsto f,$$

is bounded and bounded below by the closed graph theorem. By Lemma 4.11.4, there exists a bounded operator

$$T : \mathcal{K} \ominus J \to \mathcal{K} \quad \text{such that} \quad TK_{\mathcal{K}}(\cdot, w) = K_{\mathcal{K}}(\cdot, Aw)$$

for all $w \in Z^0(J)$ (respectively $w \in Z(J)$). Let $T_A = E^*T$. Then

$$T_A(K_{\mathcal{K}}(\cdot, w)) = E^* K_{\mathcal{K}}(\cdot, Aw) = K_{\mathcal{H}}(\cdot, Aw)$$

for all w, from which we deduce with the help of Lemma 4.7.4 that T_A maps $\mathcal{K} \ominus J$ into $\mathcal{H} \ominus I$. Replacing A with A^{-1}, we see that $T_A \in \mathcal{B}(\mathcal{K} \ominus J, \mathcal{H} \ominus I)$ is invertible. It now follows from Lemma 4.7.5 that A induces a bounded invertible composition operator $C_A : \mathcal{H}_I \to \mathcal{K}_J$. As in the proof of Theorem 4.11.3, we see that $\mathrm{Ad}(C_A)$ is the desired isomorphism. □

Just as in [24], we obtain from the geometric rigidity result [24, Proposition 7.6] a rigidity result for our algebras. It generalizes [24, Theorem 8.7]. The author is grateful to the anonymous referee of [43] for pointing out this corollary.

Corollary 4.11.6. *Let $\mathcal{H}$ be a unitarily invariant complete NP-space and let I and J be relevant radical homogeneous ideals in $\mathbb{C}[z_1, \dots, z_d]$. Suppose that $V(I)$ or $V(J)$ is irreducible.*

(a) *If $A(\mathcal{H}_I)$ and $A(\mathcal{H}_J)$ are algebraically isomorphic, then $A(\mathcal{H}_I)$ and $A(\mathcal{H}_J)$ are unitarily equivalent.*

(b) *If $\mathrm{Mult}(\mathcal{H}_I)$ and $\mathrm{Mult}(\mathcal{H}_J)$ are isomorphic via a weak-∗-weak-∗ continuous isomorphism, then $\mathrm{Mult}(\mathcal{H}_I)$ and $\mathrm{Mult}(\mathcal{H}_J)$ are unitarily equivalent.*

(c) *If $\mathcal{H}$ or $\mathcal{K}$ is tame and $\mathrm{Mult}(\mathcal{H}_I)$ and $\mathrm{Mult}(\mathcal{H}_J)$ are algebraically isomorphic, then $\mathrm{Mult}(\mathcal{H}_I)$ and $\mathrm{Mult}(\mathcal{H}_J)$ are unitarily equivalent.*

Proof. In each case, Theorem 4.11.5 shows that there exists an invertible linear map A on $\mathbb{C}^d$ which maps $V(J)$ isometrically onto $V(I)$. In particular, $V(I)$ and $V(J)$ are both irreducible. Proposition 7.6 in [24] implies that A is isometric on the linear span of $V(J)$, and hence can be chosen to be unitary. All assertions now follow from Theorem 4.11.3. □

Let us apply Theorems 4.11.3 and 4.11.5 in the setting where $\mathcal{H}$ and $\mathcal{K}$ are given by log-convex sequences (see part (b) of Remark 4.7.1). This includes in particular the spaces $\mathcal{H}_s(\mathbb{B}_d)$ and $\mathcal{H}_s(\overline{\mathbb{B}_d})$ of Example 4.7.2. If $\boldsymbol{a} = (a_n)_n$ is a sequence of positive real numbers such that the series

$$\sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence 1, we write $\mathcal{H}(\boldsymbol{a})$ for the reproducing kernel Hilbert space with reproducing kernel

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n.$$

If $\sum_{n=0}^{\infty} a_n = \infty$, this is a reproducing kernel Hilbert space on $\mathbb{B}_d$, and if $\sum_{n=0}^{\infty} a_n < \infty$, this a space on $\overline{\mathbb{B}_d}$.

Corollary 4.11.7. *Let $\boldsymbol{a} = (a_n)$ and $\boldsymbol{a'} = (a'_n)$ be two log-convex sequences of positive real numbers such that*

$$a_0 = 1 = a'_0$$

and

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1 = \lim_{n \to \infty} \frac{a'_n}{a'_{n+1}}.$$

Let $\mathcal{H} = \mathcal{H}(\boldsymbol{a})$ and $\mathcal{K} = \mathcal{H}(\boldsymbol{a'})$. Let $I, J \subset \mathbb{C}[z_1, \dots, z_d]$ be two relevant radical homogeneous ideals of polynomials. Then the following are equivalent:

(i) *$A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$ are isometrically isomorphic.*

(ii) *$\operatorname{Mult}(\mathcal{H}_I)$ and $\operatorname{Mult}(\mathcal{K}_J)$ are isometrically isomorphic.*

(iii) *$a_n = a'_n$ for all $n \in \mathbb{N}$ and there exists a unitary map U on $\mathbb{C}^d$ which maps $V(J)$ onto $V(I)$.*

Moreover, the following assertions are equivalent as well:

(i) *$A(\mathcal{H}_I)$ and $A(\mathcal{K}_J)$ are algebraically isomorphic.*

(ii) *$\operatorname{Mult}(\mathcal{H}_I)$ and $\operatorname{Mult}(\mathcal{K}_J)$ are algebraically isomorphic.*

(iii) *There exist constants $C_1, C_2 > 0$ such that*

$$C_1 \leq \frac{a_n}{a'_n} \leq C_2$$

for all $n \in \mathbb{N}$ and there exists an invertible linear map A on $\mathbb{C}^d$ which maps $V(J)$ isometrically onto $V(I)$.

Proof. The assumptions on $\boldsymbol{a}$ and $\boldsymbol{a'}$ imply that $\mathcal{H}$ and $\mathcal{K}$ are unitarily invariant complete NP spaces on $\mathbb{B}_d$ or on $\overline{\mathbb{B}_d}$ (see the beginning of Section 4.7). Proposition 4.8.5 (c) shows that $\mathcal{H}$ and $\mathcal{K}$ are tame. The first set of equivalences is now an immediate consequence of Theorem 4.11.3. To prove the second set of equivalences, in light of Theorem 4.11.5, it suffices to show that $\mathcal{H} = \mathcal{K}$ as vector spaces if and only if there exist constants $C_1, C_2 > 0$ such that

$$C_1 \leq \frac{a_n}{a_n'} \leq C_2$$

for all $n \in \mathbb{N}$. To this end, observe that if $\mathcal{H} = \mathcal{K}$ as vector spaces, then the formal identity $E : \mathcal{H} \to \mathcal{K}, f \mapsto f$, is bounded and invertible by the closed graph theorem, hence the existence of the constants follows from the description of the norm in part (d) of Remark 4.7.1. The other implication follows from part (d) of Remark 4.7.1 as well. □

We finish this chapter by considering the last result about algebraic isomorphism from the point of view adopted in [25] and in Chapter 3. That is, we will identify a multiplier algebra $\operatorname{Mult}(\mathcal{H}_I)$ with an algebra of the form $\mathcal{M}_V = \operatorname{Mult}(H^2_\infty\big|_V)$ for a suitable variety $V \subset \mathbb{B}_\infty$.

We first show that most of our examples of unitarily invariant complete NP-spaces cannot be embedded into a finite dimensional ball. More generally, let $\mathcal{H}$ be an irreducible complete Nevanlinna-Pick space on $\mathbb{B}_d$ with reproducing kernel of the form

$$K(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n,$$

where $a_0 = 1$. Recall that an embedding for $\mathcal{H}$ is an injective function $j : \mathbb{B}_d \to \mathbb{B}_m$ for some $m \in \mathbb{N} \cup \{\infty\}$ such that

$$\langle j(z), j(w) \rangle = 1 - \frac{1}{\sum_{n=0}^{\infty} a_n \langle z, w \rangle^n}$$

for all $z, w \in \mathbb{B}_d$. By Lemma 4.2.2, there is a sequence (c_n) of non-negative real numbers such that

$$1 - \frac{1}{\sum_{n=0}^{\infty} a_n \langle z, w \rangle^n} = \sum_{n=1}^{\infty} c_n \langle z, w \rangle^n$$

for all $z, w \in \mathbb{B}_d$. Since

$$\langle z, w \rangle^n = \sum_{|\alpha|=n} \binom{n}{\alpha} z^\alpha \overline{w}^\alpha = \langle \psi_n(z), \psi_n(w) \rangle,$$

where

$$\psi_n : \mathbb{C}^d \to \mathbb{C}^{\binom{n+d-1}{n}}, \quad z \mapsto \left(\sqrt{\binom{n}{\alpha}} z^\alpha \right)_{|\alpha|=n},$$

an embedding j for $\mathcal{H}$ can be explicitly constructed by setting

$$j(z) = (\sqrt{c_1}\psi_1(z), \sqrt{c_2}\psi_2(z), \sqrt{c_3}\psi_3(z), \dots).$$

Using the fact that $\sum_{n=1}^\infty c_n \le 1$, it is not hard to see that j is an analytic map from $\mathbb{B}_d$ into $\mathbb{B}_m$ which extends to a norm continuous map from $\overline{\mathbb{B}_d}$ to $\overline{\mathbb{B}_m}$. If $d = 1$, these embeddings are simply the embeddings considered in Sections 3.7 and 3.8.

In particular, we see that if only finitely many of the c_n are non-zero, then $\mathcal{H}$ admits an embedding into a finite dimensional ball, that is, there exists $m < \infty$ and an embedding $j : \mathbb{B}_d \to \mathbb{B}_m$ for $\mathcal{H}$. In fact, this property characterizes unitarily invariant spaces which admit an embedding into a finite dimensional ball.

Proposition 4.11.8. *Let $\mathcal{H}$ be an irreducible complete Nevanlinna-Pick space on $\mathbb{B}_d$ with reproducing kernel of the form*

$$K(z, w) = \sum_{n=0}^\infty a_n \langle z, w \rangle^n,$$

where $a_0 = 1$. Then $\mathcal{H}$ admits an embedding into a finite dimensional ball if and only if the analytic function f on $\mathbb{D}$ defined by

$$f(t) = \frac{1}{\sum_{n=0}^\infty a_n t^n}$$

is a polynomial.

Proof. With notation as in the discussion preceding the proposition, observe that

$$1 - f(t) = \sum_{n=1}^\infty c_n t^n$$

for all $t \in \mathbb{D}$. Hence, f is indeed an analytic function on $\mathbb{D}$, and f is a polynomial if and only if all but finitely many c_n are zero.

For the proof of the remaining implication, suppose that $\mathcal{H}$ admits an embedding j into $\mathbb{B}_m$ for some $m < \infty$. From

$$1 - \frac{1}{K(z, w)} = \langle j(z), j(w) \rangle_{\mathbb{C}^m},$$

we deduce that the rank of the kernel $L = 1 - 1/K$ is at most m in the sense that for any finite collection of points $\{z_1, \dots, z_n\}$, the matrix

$$\Big(L(z_i, z_j)\Big)_{i,j=1}^n$$

has rank at most m. Let $\mathcal{K}$ denote the reproducing kernel Hilbert space on $\mathbb{B}_d$ with reproducing kernel L. Since

$$\langle L(\cdot, w), L(\cdot, z) \rangle_{\mathcal{K}} = L(z, w),$$

and since $\mathcal{K}$ is spanned by the kernel functions $L(\cdot, z)$ for $z \in \mathbb{B}_d$, it follows that the dimension of $\mathcal{K}$ is at most m. However, L also admits the representation

$$L(z, w) = \sum_{n=1}^\infty c_n \langle z, w \rangle^n,$$

hence for every $n \in \mathbb{N}$ with $c_n \neq 0$, the space $\mathcal{K}$ contains the monomial z_1^n, and different monomials are orthogonal. Consequently, $c_n = 0$ for all but finitely many n, so f is a polynomial. □

As a consequence, we see that all spaces in Example 4.7.2 besides the Drury-Arveson space do not admit an embedding into a finite dimensional ball.

Corollary 4.11.9. *Let $\mathcal{H}$ be a unitarily invariant complete NP-space with reproducing kernel of the form*

$$K(z, w) = \sum_{n=0}^\infty a_n \langle z, w \rangle^n.$$

If $\mathcal{H}$ admits an embedding into a finite dimensional ball, then the sequence (a_n) converges to a positive real number, and hence $\mathcal{H} = H^2_d$ as vector spaces. In particular, the space $\mathcal{H}_s(\mathbb{B}_d)$ for $-1 \le s < 0$, the space $\mathcal{H}_s(\overline{\mathbb{B}_d})$ for $s < -1$, and the space $\mathcal{K}_\alpha$ for $0 < \alpha < 1$ do not admit an embedding into a finite dimensional ball.

Proof. Assume that $\mathcal{H}$ admits an embedding into a finite dimensional ball. Proposition 4.11.8 implies that there exists $N \in \mathbb{N}$ and non-negative real numbers $c_1, \dots, c_N$ such that

$$\sum_{n=0}^\infty a_n t^n = \frac{1}{1 - \sum_{n=1}^N c_n t^n}.$$

Observe that $c_1 > 0$ as $a_1 > 0$. Since the power series on the left-hand side has radius of convergence 1, this rational function in t has a pole on $\partial \mathbb{D}$. Because $a_n \ge 0$ for all

$n \in \mathbb{N}$, this is only possible if $\sum_{n=0}^{\infty} a_n = \infty$, from which we deduce that $\sum_{n=1}^{N} c_n = 1$. Let $r = \sum_{n=1}^{N} n c_n$. In this setting, the Erdős-Feller-Pollard theorem (see [31, Chapter XIII, Section 11]) implies that $\lim_{n\to\infty} a_n = 1/r > 0$. The remaining assertions are now obvious. □

Suppose now that $\boldsymbol{a}$ is a sequence as in Corollary 4.11.7, and assume first that we are in the case where $\sum_{n=0}^{\infty} a_n = \infty$. Let $j_{\boldsymbol{a}} : \mathbb{B}_d \to \mathbb{B}_\infty$ denote the embedding for $\mathcal{H}(\boldsymbol{a})$ which was constructed above. Note that $c_1 \neq 0$ as $a_1 \neq 0$, that the coordinates $j_{\boldsymbol{a}}$ are polynomials (in fact monomials), and that the first d coordinates are given by $(\sqrt{c_1} z_1, \dots, \sqrt{c_1} z_d)$. In particular, $j_{\boldsymbol{a}} : \mathbb{B}_d \to V_{\boldsymbol{a}}$ is invertible, where $V_{\boldsymbol{a}}$ denotes the range of $j_{\boldsymbol{a}}$. An inverse of $j_{\boldsymbol{a}}$ is given by

$$j_{\boldsymbol{a}}^{-1}(z) = \frac{1}{\sqrt{c_1}} (z_1, \dots, z_d) \tag{4.7}$$

for $z \in V_{\boldsymbol{a}}$.

Suppose now that $I \subset \mathbb{C}[z_1, \dots, z_d]$ is a relevant radical homogeneous ideal. Then the restriction of $j_{\boldsymbol{a}}$ to $Z^0(I)$ is an embedding for $\mathcal{H}(\boldsymbol{a})_I$. Since $\mathcal{H}(\boldsymbol{a})_I$ is algebraically consistent, the image

$$V_{\boldsymbol{a},I} = j_{\boldsymbol{a}}(Z^0(I)) \subset \mathbb{B}_\infty$$

is a variety by Proposition 4.5.6. Moreover, j_a maps $Z^0(I)$ biholomorphically onto $V_{\boldsymbol{a},I}$. This discussion also applies to the case where $\sum_{n=0}^{\infty} a_n < \infty$ by simply replacing $\mathbb{B}_d$ with $\overline{\mathbb{B}_d}$ and $Z^0(I)$ with $Z(I)$ above. In this case, $\sum_{n=1}^{\infty} c_n < 1$ and $j_{\boldsymbol{a}}$ maps $Z(I)$ homeomorphically onto $V_{\boldsymbol{a},I}$ and $Z^0(I)$ biholomorphically onto its image.

Let $m \in \mathbb{N} \cup \{\infty\}$. For a variety $V \subset \mathbb{B}_m$, let $\mathcal{M}_V = \operatorname{Mult}(H^2_m\big|_V)$. Following Chapter 3, we say that two varieties $V, W \subset \mathbb{B}_m$ are *multiplier biholomorphic* if there exists a homeomorphism $F : V \to W$ such that every coordinate of F is in $\mathcal{M}_V$ and every coordinate of F^{-1} is in $\mathcal{M}_W$. If $m < \infty$, then such a map is automatically a biholomorphism in the usual sense. This definition is motivated by [25, Theorem 5.6], which states that if $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, then V and W are multiplier biholomorphic, provided that $m < \infty$ and V and W satisfy some mild geometric assumptions. Moreover, there are examples of two discs in $\mathbb{B}_2$ which are biholomorphic, but not multiplier biholomorphic (see Section 3.5).

However, already the results in Sections 3.7 and 3.8 show that there are multiplier biholomorphic discs in $\mathbb{B}_\infty$ whose multiplier algebras are not isomorphic. It turns out that for the varieties $V_{\boldsymbol{a},I}$ constructed above, of which the discs from Sections 3.7 and 3.8 are a special case, the multiplier biholomorphism classes only depend on the ideal I and on summability of the sequence $\boldsymbol{a}$. They do not detect any other properties of the sequence

$\boldsymbol{a}$. In light of Corollary 4.11.7, this means that the relation of multiplier biholomorphism fails rather dramatically at distinguishing the isomorphism classes of the algebras $\mathcal{M}_{V_{\boldsymbol{a},I}}$.

Proposition 4.11.10. *Let $\boldsymbol{a} = (a_n)$ and $\boldsymbol{a'} = (a'_n)$ be sequences as in Corollary 4.11.7, and let $I, J \subset \mathbb{C}[z_1, \dots, z_d]$ be relevant radical homogeneous ideals. Let $V_{\boldsymbol{a},I}$ and $V_{\boldsymbol{a'},J}$ be the varieties defined above. Then the following are equivalent:*

(i) *$V_{\boldsymbol{a},I}$ and $V_{\boldsymbol{a'},J}$ are multiplier biholomorphic.*

(ii) *The sequences $\boldsymbol{a}$ and $\boldsymbol{a'}$ are either both summable or both not summable, and there exists an invertible linear map A on $\mathbb{C}^d$ which maps $V(J)$ isometrically onto $V(I)$.*

Proof. (i) $\Rightarrow$ (ii) Observe that $V_{\boldsymbol{a},I}$ is homeomorphic to $Z^0(I)$ if $\boldsymbol{a}$ is not summable and homeomorphic to $Z(I)$ if $\boldsymbol{a}$ is summable. Since $Z(I)$ is compact and $Z^0(I)$ is not, it follows that if (i) holds, then $\boldsymbol{a}$ and $\boldsymbol{a'}$ are either both summable or both not summable. In the non-summable case, $Z^0(I)$ and $Z^0(J)$ are biholomorphic. In the summable case, there is a homeomorphism $F : Z(I) \to Z(J)$ which is analytic on $Z^0(I)$ and whose inverse is analytic on $Z^0(J)$. Then Lemma 4.9.1 implies that $Z^0(I)$ and $Z^0(J)$ are biholomorphic. Finally, an application of Lemma 4.9.6 and Proposition 4.9.3 shows that there exists an invertible linear map A on $\mathbb{C}^d$ which maps $V(J)$ isometrically onto $V(I)$.

(ii) $\Rightarrow$ (i) Let us first assume that $\boldsymbol{a}$ and $\boldsymbol{a'}$ are both not summable, and let $j_{\boldsymbol{a}} : \mathbb{B}_d \to V_{\boldsymbol{a}}$ and $j_{\boldsymbol{a'}} : \mathbb{B}_d \to V_{\boldsymbol{a'}}$ be the embeddings constructed earlier. Then $F = j_{\boldsymbol{a}} \circ A \circ j_{\boldsymbol{a'}}^{-1}$ maps $V_{\boldsymbol{a'},J}$ homeomorphically onto $V_{\boldsymbol{a},I}$. From Equation (4.7) and the fact that the coordinates of $j_{\boldsymbol{a}}$ are polynomials, we deduce that the coordinates of F are polynomials in $z_1, \dots, z_d$. Similarly, the coordinates of F^{-1} are polynomials in $z_1, \dots, z_d$, hence F is a multiplier biholomorphism.

After replacing $\mathbb{B}_d$ with $\overline{\mathbb{B}_d}$, the same argument applies in the situation where $\boldsymbol{a}$ and $\boldsymbol{a'}$ are both summable. Hence, the proof is complete. □

5. The Borel complexity of the classification problem

5.1. Introduction

The contents of this chapter are joint work with Martino Lupini and appeared in [45]. We investigate the classification problem for multiplier algebras $\mathcal{M}_V$ from the perspective of Borel complexity theory. Our main result is that the classification problem for multiplier algebras $\mathcal{M}_V$ up to algebraic isomorphism is intractable in the sense of Borel complexity theory.

Theorem 5.1.1. *The multiplier algebras $\mathcal{M}_V$, where V is a variety in $\mathbb{B}_\infty$, are not classifiable by countable structures up to algebraic isomorphism.*

This means that there is no explicit way to classify the multiplier algebras $\mathcal{M}_V$ using countable structures as complete invariants. A more precise version of the statement of Theorem 5.1.1 will be given in Section 5.2. The proof of Theorem 5.1.1 is presented in Sections 5.3 and 5.5. In order to prove Theorem 5.1.1 we develop in Section 5.4 the theory of turbulence for Polish groupoids. This is a generalization of Hjorth's theory of turbulence for Polish group actions from [49], see also [34, Chapter 10].

We also study the (completely) isometric classification problem for multiplier algebras $\mathcal{M}_V$ associated to varieties in $\mathbb{B}_d$ with d finite. By Theorem 4.4 and Theorem 5.10 in [25], this amounts to classifying varieties in $\mathbb{B}_d$ up to $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence. We are able to exactly determine the complexity of such a task.

Theorem 5.1.2. *For any $d \in \mathbb{N}$, the relation of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence of varieties in $\mathbb{B}_d$ is essentially countable, and has maximum complexity among essentially countable equivalence relations.*

In particular, Theorem 5.1.2 shows that the relation of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence of varieties in $\mathbb{B}_d$ is not smooth. In fact, any class of complete invariants would have to be as complex as conceivable. We will explain in more detail the content of Theorem 5.1.2 in Section 5.2. The proof of Theorem 5.1.2 is provided in Section 5.6.

The remainder of this chapter is organized as follows. In Section 5.2, we recall necessary basic notions and results from Borel complexity theory. In Section 5.3, we reduce Theorem 5.1.1 to a non-classification result for certain sequences in $(0, 1]$. In order to establish this result, we develop the theory of turbulence for Polish groupoids in Section 5.4. In Section 5.5, we establish the desired non-classification result for sequences. Finally, in Section 5.6, the proof of Theorem 5.1.2 is provided.

5.2. Borel complexity theory

Borel complexity theory studies the relative complexity of classification problems in mathematics, and offers tools to detect and prove obstructions to classification. In this framework, a classification problem is regarded as an *equivalence relation* on a *standard Borel space*. Perhaps after a suitable parametrization, this covers most of classification problems in mathematics. For example, varieties in $\mathbb{B}_d$ for $d \in \mathbb{N} \cup \{\infty\}$ are a collection $\mathcal{V}_d$ of nonempty closed subsets of $\mathbb{B}_d$. We will verify in the next section that $\mathcal{V}_d$ is a Borel subset of the space of nonempty closed subsets of $\mathbb{B}_d$ endowed with the Effros Borel structure [53, Section 12.C]. This shows that varieties form a standard Borel space when endowed with the induced Borel structure [53, Proposition 12.1]. The relation of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence of varieties in $\mathbb{B}_d$ can then be regarded as an equivalence relation on this standard Borel space. Similarly, the multiplier algebras $\mathcal{M}_V$ are naturally parametrized by the varieties themselves, and one can regard algebraic isomorphisms of the algebras $\mathcal{M}_V$ as an equivalence relation on the standard Borel space of varieties described above.

Borel complexity theory aims at comparing the complexity of different classification problems. The fundamental notion of comparison is Borel reducibility. Recall that a *Polish space* is a separable topological space which is homeomorphic to a complete metric space. A *standard Borel space* is a measurable space which is isomorphic to the Borel space of a Polish space. If E and F are equivalence relations on standard Borel spaces X and Y respectively, then a *Borel reduction* from E to F is a Borel function $f : X \to Y$ with the property that

$$f(x)Ff(x') \quad \text{if and only if} \quad xEx'$$

for every $x, x' \in X$. The relation E is *Borel reducible* to F—in formulas $E \leq_B F$—if there exists a Borel reduction from E to F. This amounts to saying that one can assign to the elements of X complete invariants up to E that are F-equivalence classes, and moreover such an assignment is *constructive* in the sense that it is given by a Borel map at the level of the spaces. We say that E and F are Borel bireducible, and write $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$. The notion of Borel reducibility was first introduced in [32, Definition 2]. A complete survey on Borel complexity theory can be found in [34].

Some distinguished equivalence relations are used as benchmarks of complexity to draw a hierarchy of classification problems in mathematics. The first natural benchmark is provided by the relation $=_{\mathbb{R}}$ of equality of real numbers. An equivalence relation is *smooth* if it is Borel reducible to $=_{\mathbb{R}}$. (One can replace $\mathbb{R}$ with any other standard Borel space [53, Theorem 15.6].) For example, the relation of isomorphism of finite-splitting trees is smooth [34, Theorem 13.2.3].

Smooth equivalence relations represent the lowest level complexity. A more ample class is given by considering Borel equivalence relations that are countable or essentially countable. An equivalence relation E on a standard Borel space X is *Borel* if it is a Borel subset of the product $X \times X$. A Borel equivalence relation E is *countable* if its classes are countable, and *essentially countable* if it is Borel reducible to a countable one. Clearly, a smooth equivalence relation is, in particular, essentially countable. The relation E_0 of tail equivalence of binary sequences is countable but not smooth [34, Subsection 6.1]. More generally the orbit equivalence relation of a Borel action of a countable group on a standard Borel space is countable. There exists a countable Borel equivalence relation E_∞ that has maximum complexity among (essentially) countable Borel equivalence relations. One can describe E_∞ as the relation of isomorphism of locally finite trees or graphs [34, Theorem 13.2.4]. In the proof of Theorem 5.1.2, we will use the following equivalent description of E_∞. Let F_2 be the free group on two generators and $\{0,1\}^{F_2}$ the space of subsets of F_2 endowed with the product topology. The group F_2 naturally acts on $\{0,1\}^{F_2}$ by translation. The corresponding orbit equivalence relation $E(F_2, 2)$ is Borel bireducible with E_∞ [34, Theorem 7.3.8].

A more generous notion of classifiability for equivalence relations is being *classifiable by countable structures*. An equivalence relation is classifiable by countable structures if it is Borel reducible to the relation of isomorphism within some Borel class of structures in some first order language. Equivalently an equivalence relation is classifiable by countable structures if it is Borel reducible to the orbit equivalence relation of a continuous action of S_∞ on a Polish space [34, Section 3.6]. The Polish group S_∞ is the group of permutations of $\mathbb{N}$ with the topology of pointwise convergence [34, Section 2.4]. Any (essentially) countable equivalence relation is in particular classifiable by countable structures [48, Lemma 2.4, Lemma 2.5]. Again, there exists an equivalence relation of maximum complexity among those that are classifiable by countable structures. Such an equivalence relation can be described, for instance, as the relation of isomorphism of countable trees or graphs [32, Theorem 1].

5.3. Varieties and unitarily invariant kernels

If X is a Polish space, then the space $F(X)$ of nonempty closed subsets of X is a standard Borel space when endowed with the Effros Borel structure [53, Section 12.C]. This is the Borel structure generated by the sets

$$\{K \in F(X) : K \cap U \neq \varnothing\}$$

where U ranges over the open subsets of X.

For $d \in \mathbb{N} \cup \{\infty\}$ let $\mathcal{V}_d \subset F(\mathbb{B}_d)$ be the set of varieties in $\mathbb{B}_d$, where $\mathbb{B}_d$ is endowed with the norm topology. For $d \leq d'$ the canonical inclusion $\mathbb{B}_d \subset \mathbb{B}_{d'}$ induces a Borel injection from $\mathcal{V}_d$ into $\mathcal{V}_{d'}$.

Proposition 5.3.1. *The set $\mathcal{V}_d$ of varieties in $\mathbb{B}_d$ is a Borel subset of $F(\mathbb{B}_d)$.*

Proof. Observe that $\mathcal{V}_d$ is the image of $F(H^2_d)$ under the Borel map which assigns to a closed subset S of H^2_d the variety $V(S)$ of common zeros of elements of S. Therefore $\mathcal{V}_d$ is analytic. By [53, Theorem 14.7] it remains to show that $\mathcal{V}_d$ is co-analytic.

To this end, suppose that $x \in \mathbb{B}_d$, $\varepsilon > 0$ and $F \subset \mathbb{B}_d$ is finite, say $F = \{x_1, \ldots, x_n\}$. By the Nevanlinna-Pick property of H^2_d, there exists a multiplier φ in the unit ball of $\operatorname{Mult}(H^2_d)$ which vanishes on F and satisfies $|\varphi(x)| \geq \varepsilon$ if and only if the matrix

$$A(x, F, \varepsilon) = \begin{bmatrix} K(x,x)(1-|\varepsilon|^2) & K(x,x_1) & \ldots & K(x,x_n) \\ K(x_1,x) & K(x_1,x_1) & \ldots & K(x_1,x_n) \\ \vdots & \ddots & \ddots & \vdots \\ K(x_n,x) & K(x_n,x_1) & \ldots & K(x_n,x_n) \end{bmatrix}$$

is positive semidefinite, which does not depend on the order of the points in F.

We now claim that a closed subset $V \subset \mathbb{B}_d$ is a variety if and only if for every $x \in \mathbb{B}_d$, either $x \in V$ or there exists a rational $\varepsilon > 0$ such that for all finite sets $F \in F(\mathbb{B}_d)$, either $F \cap (\mathbb{B}_d \setminus V) \neq \emptyset$ or $A(x, F, \varepsilon)$ is positive. This formula shows that $\mathcal{V}_d$ is co-analytic by [53, Proposition 37.1], and hence finishes the proof. To show the non-trivial implication of the claim, suppose that the last statement holds. Then for every $x \in \mathbb{B}_d \setminus V$, there exists $\varepsilon > 0$ such that for all finite sets $F \subset \mathbb{B}_d$, the weak-$*$ compact set

$$\mathcal{I}_F = \{\varphi \in \operatorname{Mult}(H^2_d) : ||\varphi||_{\operatorname{Mult}(H^2_d)} \leq 1 \text{ and } |\varphi(x)| \geq \varepsilon \text{ and } \varphi\big|_F 0\}$$

is not empty. Clearly, these sets have the finite intersection property, hence there exists a multiplier φ which belongs to each of the $\mathcal{I}_F$, and this φ vanishes on V and satisfies $|\varphi(x)| \geq \varepsilon$. Consequently, V is a variety. $\square$

We can now state Theorem 5.1.1 more precisely as follows.

Theorem 5.3.2. *The equivalence relation on the space $\mathcal{V}_\infty$ of varieties in $\mathbb{B}_\infty$ defined by $V \sim W$ if and only if $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic is not classifiable by countable structures.*

In fact, we show that this equivalence relation is not even classifiable by countable structures when restricted to set of all varieties of the form $V_{\boldsymbol{a}}$, which were constructed in the discussion preceding Proposition 4.11.10.

Recall from Section 4.7 that a unitarily invariant complete NP-space on $\mathbb{B}_d$ is a reproducing kernel Hilbert space $\mathcal{H}(\boldsymbol{a})$ on $\mathbb{B}_d$ with reproducing kernel of the form

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$$

for $z, w \in \mathbb{B}_d$, where $\boldsymbol{a} = (a_n)$ is a sequence of positive numbers such that $a_0 = 1$, the power series $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence 1, $\sum_{n=0}^{\infty} a_n = \infty$, and there exists a sequence $\boldsymbol{b} = (b_n)$ of non-negative numbers such that

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{1 - \sum_{n=1}^{\infty} b_n t^n}. \tag{5.1}$$

We let $\mathcal{A} \subset (0,\infty)^{\mathbb{N}}$ denote the set of such sequences. It is not difficult to see that $\mathcal{A}$ is a Borel subset of $(0,\infty)^{\mathbb{N}}$ endowed with the product topology. Indeed the first three conditions are clearly Borel. For the last one, one can observe that given a sequence of positive numbers $\boldsymbol{a}$ such that $a_0 = 1$ and such that $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence 1, there is a unique sequence $\boldsymbol{b}$ of real numbers such that Equation 5.1 holds for t in a neighbourhood of the origin. This sequence $\boldsymbol{b}$ can be recursively computed from $\boldsymbol{a}$, which shows that the set of all $\boldsymbol{a}$ such that $\boldsymbol{b}$ is non-negative is Borel.

It follows from the universality theorem of Agler and McCarthy (see Section 2.5) that for every $\boldsymbol{a} \in \mathcal{A}$, there exists a variety $V_{\boldsymbol{a}} \subset \mathbb{B}_\infty$ such that $\mathcal{M}_{V_{\boldsymbol{a}}}$ is completely isometrically isomorphic to $\operatorname{Mult}(\mathcal{H}(\boldsymbol{a}))$. In fact, the variety $V_{\boldsymbol{a}}$ is the image of $\mathbb{B}_d$ under the embedding $j_{\boldsymbol{a}} : \mathbb{B}_d \to \mathbb{B}_\infty$ constructed in the discussion preceding Proposition 4.11.8. One can use the explicit definition of the embedding $j_{\boldsymbol{a}}$ in terms of $\boldsymbol{a}$ to show that the map $\boldsymbol{a} \mapsto V_{\boldsymbol{a}}$ is Borel. Therefore, in order to establish Theorem 5.3.2, it is enough to prove the following result.

Theorem 5.3.3. *Let $d \in \mathbb{N}$. The relation $\sim_d$ on the space $\mathcal{A}$ defined by $\boldsymbol{a} \sim_d \boldsymbol{a}'$ if and only if $\operatorname{Mult}(\mathcal{H}(\boldsymbol{a}))$ and $\operatorname{Mult}(\mathcal{H}(\boldsymbol{a}'))$ are algebraically isomorphic is not classifiable by countable structures.*

To prove this result, we will consider the special class of unitarily invariant complete NP-spaces of Corollary 4.11.7. We say that a (necessarily non-increasing) sequence $\boldsymbol{a} = (a_n)$ in $(0,1]^{\mathbb{N}}$ is *admissible log-convex* if $a_0 = 1$, $(\frac{a_n}{a_{n+1}})_n$ is non-increasing and converges to 1, and $\sum_{n=0}^{\infty} a_n = \infty$. Let $\mathcal{A}_0 \subset (0,1]^{\mathbb{N}}$ be the Borel set of admissible log-convex sequences. Log-convexity of $\boldsymbol{a}$ implies that there exists a sequence (b_n) of non-negative numbers as in Equation (5.1), see [3, Lemma 7.38]. Therefore, $\mathcal{A}_0 \subset \mathcal{A}$.

We consider on $\mathcal{A}_0$ the relation $E_{\mathcal{A}_0}$ defined by $\boldsymbol{a} E_{\mathcal{A}_0} \boldsymbol{a}'$ if and only if $\boldsymbol{a}$ and $\boldsymbol{a}'$ have the same growth. This means that there are constants $c, C > 0$ such that $c \leq a'_n / a_n \leq C$ for every $n \in \mathbb{N}$. The equivalence of (ii) and (iii) in Corollary 4.11.7 shows that the relations $\sim_d$ and $E_{\mathcal{A}_0}$ coincide on $\mathcal{A}_0$. Therefore, it only remains to show that the relation $E_{\mathcal{A}_0}$ is not classifiable by countable structures. This will be proved in Section 5.5.

We mention that the same proof also shows that the algebras $A(\mathcal{H})$ for $\mathcal{H}$ a unitarily invariant complete NP-space on $\mathbb{B}_d$ are not classifiable by countable structures up to algebraic isomorphism. Here $A(\mathcal{H})$ denotes the closure of the polynomials in $\mathrm{Mult}(\mathcal{H})$. One can also observe that, for $d \in \mathbb{N}$, the collection $\mathcal{K}$ of kernels of unitarily invariant complete NP-spaces on $\mathbb{B}_d$ is Borel. It follows from Theorem 5.3.3 that the relation on $\mathcal{K}$ defined by $K \sim K'$ if and only if $\mathrm{Mult}(\mathcal{H}(K))$ and $\mathrm{Mult}(\mathcal{H}(K'))$ are algebraically isomorphic is not classifiable by countable structures. Here, $\mathcal{H}(K)$ denotes the reproducing kernel Hilbert space with kernel K.

5.4. Turbulence for Polish groupoids

The main goal of this section is to introduce the notion of *turbulence* for Polish groupoids, and to generalize to this setting Hjorth's turbulence theorem. A *groupoid* is a small category where every arrow is invertible. If G is a groupoid, then an *object* of G can be identified with the corresponding identity arrow. This allows one to identify the set of objects with a subset G^0 of G. There are *source and range maps* $s, r : G \to G^0$ that map every arrow to the corresponding source and range. The set of *composable arrows* is $G^2 = \{(\gamma, \rho) : s(\gamma) = r(\rho)\}$. Composition of arrows can be seen as a function $G^2 \to G$, and similarly inversion of arrows is a function from G to G.

A *Polish groupoid* is a groupoid endowed with a topology that

1. has a countable basis of Polish open sets,
2. makes composition and inversion of arrows continuous and open,
3. makes for every $x \in G^0$ the set Gx of arrows of G with source x a Polish subspace of G, and

4. makes the set of objects G^0 a Polish subspace of G.

Polish groupoids have been introduced and studied in [72, 73]. In [56] several fundamental results about Polish group actions are generalized to Polish groupoids. We assume in the following that G is a Polish groupoid. The *orbit equivalence relation* of G is the equivalence relation E_G on G^0 defined by xE_Gy if and only if there exits $\gamma \in G$ such that $s(\gamma) = x$ and $r(\gamma) = y$. If $A, B \subset G$ we let AB be the set of all compositions $\gamma\rho$ for $\gamma \in A$ and $\rho \in B$ such that $r(\rho) = s(\gamma)$. We write $A\gamma$ for $A\{\gamma\}$ when $A \subset G$ and $\gamma \in G$. In particular if $x \in G^0$ then Ax is the set of elements of A with source x. If X is a G_δ subset of G^0, denote by $G|_X$ the Polish groupoid $XGX = \{\gamma \in G : s(\gamma), r(\gamma) \in X\}$ endowed with the subspace topology. This is called the *restriction* of G to X. If x is an object of G and V is a neighborhood of x in G, then the *local orbit* $\mathcal{O}(x, V)$ is the set of all points that can be reached from x by applying elements of V. In formulas

$$\mathcal{O}(x, V) = \bigcup_{n \in \mathbb{N}} r(V^n x).$$

Definition 5.4.1. An object x of G is *turbulent* if for every neighborhood V of x the local orbit $\mathcal{O}(x, V)$ is somewhere dense. The groupoid G is *generically preturbulent* if the set of turbulent objects with dense orbit is a comeager subset of G^0. If moreover every orbit is meager, then G is *generically turbulent*.

In the rest of this section we will often tacitly use the following version of the classical Kuratowsky-Ulam theorem, see [56, Lemma 2.9.1].

Theorem 5.4.2. *Suppose that X is a second countable topological space, Y is a Polish space, and $f : X \to Y$ is open and continuous. If $A \subset X$ is analytic, then A is comeager if and only if $f^{-1}\{y\} \cap A$ is comeager in $f^{-1}\{y\}$ for comeager many $y \in Y$.*

For example, it follows from Theorem 5.4.2 that if X is a dense G_δ subspace of G^0 and G is generically (pre)turbulent, then $G|_X$ is generically (pre)turbulent.

Suppose that H is a Polish group and Y is a Polish H-space, i.e. a Polish space endowed with a continuous action of H. Let G be the Polish action groupoid associated with the Polish H-space Y as in [56, Subsection 2.7]. Observe that the orbit equivalence relation E_G coincides with the orbit equivalence relation E_H^Y. Furthermore it is not difficult to verify that G is a generically (pre)turbulent groupoid as in Definition 5.4.1 if and only if Y is a generically (pre)turbulent H-space in the sense of [34, Definition 10.3.3].

Recall the following terminology from Borel complexity of equivalence relations. If E and F are equivalence relations on standard Borel spaces X and Y, then an (E, F)-homomorphism is a function $f : X \to Y$ that maps E-classes into F-classes. A generic

(E, F)-homomorphism is a function $f : X \to Y$ that is an (E, F)-homomorphism when restricted to some comeager subset of X. An equivalence relation E on a standard Borel space X is *generically* S_∞*-ergodic* if for every Polish S_∞-space Y and every Baire-measurable generic $(E, E^Y_{S_\infty})$-homomorphism, there exists a comeager subset of X that is mapped by f into a single S_∞-orbit. It is well known that an equivalence relation is classifiable by countable structures if and only if it is Borel reducible to the orbit equivalence relation of a Polish S_∞-space, see [34, Theorem 11.3.8].

The following is the main consequence of turbulence for Polish groupoids.

Theorem 5.4.3. *Suppose that G is a generically preturbulent Polish groupoid. Then the associated orbit equivalence relation E_G is generically S_∞-ergodic.*

Corollary 5.4.4. *If G is a generically turbulent Polish groupoid, then the orbit equivalence relation E_G is not classifiable by countable structures.*

Theorem 5.4.3 generalizes the original result of Hjorth [49, Section 3] from Polish group actions to Polish groupoids. Polish groupoids provide a natural setting to present the proof of Hjorth's turbulence theorem even in the case of Polish group actions. Indeed in the course of the proof one looks at the action "restricted" to a (not necessarily invariant) G_δ subspace, see for example [34, Theorem 10.4.2]. Such a restriction is not a Polish group action in general, even when one starts with a Polish group action. It is nonetheless a Polish groupoid.

The following lemma is the groupoid analog of [49, Lemma 3.17]. In the following we write $\forall^*\gamma \in X$ to mean "for a comeager set of $\gamma \in X$".

Lemma 5.4.5. *Suppose that G is a Polish groupoid, H is a Polish group, and Y is a Polish H-space. If $f : G^0 \to Y$ is a Baire-measurable generic $\left(E_G, E^Y_H\right)$-homomorphism, then there exists a comeager subset C of G^0 such that for every $x \in C$ and every open neighborhood W of 1_H in H there exists a neighborhood V of x such that for every $x' \in s(V) \cap C$ and for a comeager set of $\gamma \in Vx'$,*

$$f(r(\gamma)) \in Wf(x').$$

Proof. After replacing G with the restriction of G to a dense G_δ subset of G^0, we can assume that f is a continuous $\left(E_G, E^Y_H\right)$-homomorphism [34, Exercise 2.3.2]. Furthermore it is enough to prove that for every open neighborhood W of 1_H there is a comeager subset C of X such that for every $x \in C$ there exists a neighborhood V of x in G such that $\forall x' \in s[V] \cap C$, $\forall^*\gamma \in Vx'$, $f(r(\gamma)) \in Wf(x')$. Fix an open neighborhood W of 1_H and an open neighborhood W_0 of 1_H such that $W_0^{-1} = W_0$ and $W_0^2 \subset W$. Fix a sequence (h_n) in H such that

$$\bigcup_{n \in \mathbb{N}} W_0 h_n = H.$$

For every $n \in \mathbb{N}$, the set

$$B_n = \{(z, y) \in Y \times Y \mid z \in W_0 h_n y\}$$

is analytic. Therefore the set

$$A_n = \{\gamma \in G : f(r(\gamma)) \in W_0 h_n f(s(\gamma))\}$$

is analytic by [53, Proposition 22.1]. By [53, Proposition 8.22] there exists an open subset O_n of G such that $O_n \bigtriangleup A_n$ is meager. Set $D_n = A_n \cap O_n$, and observe that $D_n D_n^{-1}$ is a comeager subset of $O_n O_n^{-1}$. Since G is the union of A_n for $n \in \mathbb{N}$, the union O of O_n for $n \in \mathbb{N}$ is an open dense subset of G. In particular $r(O)$ is an open subset of G^0. Define now, for $n \in \mathbb{N}$, $\widetilde{O}_n$ to be the set of $\gamma \in O_n$ such that $r(\gamma)$ does not belong to the closure of the union of $r(O_i)$ for $i < n$. Let $\widetilde{O}$ be the union of $\widetilde{O}_n$ for $n \in \mathbb{N}$, and observe that $r[\widetilde{O}]$ is an open dense subset of G^0. For every $n \in \mathbb{N}$ set $\widetilde{D}_n = D_n \cap \widetilde{O}_n$ and observe that $\widetilde{D}_n$ is a comeager subset of $\widetilde{O}_n$. Therefore there exists a comeager subset C_n of $r[\widetilde{O}_n] = s[\widetilde{O}_n \widetilde{O}_n^{-1}]$ such that for every $x \in C_n$, $\widetilde{D}_n \widetilde{D}_n^{-1} x$ is a comeager subset of $\widetilde{O}_n \widetilde{O}_n^{-1} x$. Define C to be the union of C_n for $n \in \mathbb{N}$, and observe that C is a comeager subset of G^0. We claim that C satisfies the desired conclusions. Fix $x \in C$ and $n \in \mathbb{N}$ such that $x \in C_n$. We have that $\widetilde{O}_n \widetilde{O}_n^{-1}$ is an open neighborhood of x. Furthermore for every $x' \in C_n = C \cap s[\widetilde{O}_n \widetilde{O}_n^{-1}]$, $\widetilde{D}_n \widetilde{D}_n^{-1} x'$ is comeager in $\widetilde{O}_n \widetilde{O}_n^{-1} x'$. If $\rho, \gamma \in \widetilde{D}_n$, then

$$f(r(\gamma)) \in W_0 h_n f(s(\gamma)) \quad \text{and} \quad f(r(\rho)) \in W_0 h_n f(s(\rho)).$$

Therefore

$$f(r(\rho\gamma^{-1})) = f(r(\rho)) \in W_0 h_n f(s(\rho)) \subset W_0 W_0^{-1} f(s(\rho\gamma^{-1})) \subset W f(s(\rho\gamma^{-1})).$$

This concludes the proof. ☐

We now explain how one can deduce Theorem 5.4.3 from Lemma 5.4.5.

Proof of Theorem 5.4.3. Fix an enumeration $(V_k)_{k \in \mathbb{N}}$ of a basis of Polish open subsets of G, and a compatible complete metric d_Y on Y bounded by 1. Suppose that d is the metric in S_∞ defined by

$$\log_2 d(\sigma, \rho) = -\min \{n \in \mathbb{N} : \sigma(n) \neq \rho(n)\}.$$

for $\sigma, \rho \in S_\infty$. We also consider the complete metric

$$D(\sigma, \rho) = d(\sigma, \rho) + d(\sigma^{-1}, \rho^{-1})$$

on S_∞. Define e to be the identity of S_∞, and

$$N_k = \left\{ \sigma \in S_\infty : d(\sigma, e) < 2^{-k} \right\}$$

for $k \in \mathbb{N}$. As in the proof of Hjorth's turbulence theorem for Polish group actions [34, Theorem 10.4.2], one can deduce from Lemma 5.4.5 that there exists a dense G_δ subset C_0 of G^0 with the following properties:

- $f|_{C_0}$ is a continuous $(E_G, E^Y_{S_\infty})$-homomorphism,
- every element of C_0 has dense orbit,
- for every $m \in \mathbb{N}$ and $x \in V_m \cap C_0$ the local orbit $\mathcal{O}(x, V_m)$ is somewhere dense,
- for every $x \in C_0$ and $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $x \in V_m$ and $\forall x' \in s(V_m)$, $\forall^* \gamma \in V_m x'$, $f(r(\gamma)) \in N_k f(x')$.

Let C be the set of $x \in C_0$ such that $\forall^* \gamma \in Gx$, $r(\gamma) \in C_0$, and observe that C is a dense G_δ subset of G^0 [56, Lemma 2.10.6]. After replacing G with the restriction $G|_C$ of G to C, and V_k with $V_k \cap G|_C$, we can assume that $C = G^0$.

Fix $x_0, y_0 \in G^0$. We claim that $f(x) E^Y_{S_\infty} f(y)$. We will define by recursion on $i \geq 0$ elements x_i, y_i of G^0, g_i, h_i of S_∞, and $n_x(i), n_y(i)$ of $\mathbb{N}$, such that the following conditions hold:

- $g_0 = h_0 = e$,
- $x_0 \in V_{n_x(0)}$ and $y_0 \in V_{n_y(0)}$,
- $g_i f(x) = f(x_i)$ and $h_i f(y) = f(y_i)$,
- $x_{i+1} \in V_{n_x(i)} \cap \mathcal{O}\left(x_i, V_{n_x(i)}\right)$ and $y_{i+1} \in V_{n_y(i)} \cap \mathcal{O}\left(y_i, V_{n_y(i)}\right)$,
- the d_Y-diameter of $f\left(G^0 \cap V_{n_x(i)}\right)$ is at most 2^{-i},
- $\mathcal{O}\left(x_i, V_{n_x(i)}\right)$ is dense in $V_{n_y(i)} \cap G^0$ and $\mathcal{O}\left(y_i, V_{n_y(i)}\right)$ is dense in $V_{n_x(i+1)} \cap G^0$,
- $d(g_i, g_{i+1}) \leq 2^{-i}$ and $d(h_i, h_{i+1}) \leq 2^{-i}$,
- if $i > 0$ and $k_x(i) = \max\left\{ g_i(\lambda), g_i^{-1}(\lambda) \mid \lambda \leq i \right\}$, then $\forall z \in s\left(V_{n_x(i)}\right)$, $\forall^* \gamma \in V_{n_x(i)} z$, $f(r(\gamma)) \in N_{k_x(i)} f(z)$,
- if $i \geq 0$ and $k_y(i) = \max\left\{ h_i(\lambda), h_i^{-1}(\lambda) \mid \lambda \leq i \right\}$, then $\forall z \in s\left(V_{n_y(i)}\right)$, $\forall^* \gamma \in V_{n_y(i)} z$, $f(r(\gamma)) \in N_{k_y(i)} f(z)$.

Granted the construction, the sequences $(g_i), (h_i)$ in S_∞ are D-Cauchy and hence converge to elements $g, h \in S_\infty$. Furthermore $d_Y(g_i f(x), h_i f(y)) \to 0$ and hence $gf(x) = hf(y)$. This concludes the proof that $f(x) E^Y_{S_\infty} f(y)$.

We assume recursively that we have defined $x_i, y_i, g_i, h_i, n_x(i), n_y(i)$ and explain how to define $x_{i+1}, g_{i+1}, n_x(i+1)$. The definition of $y_{i+1}, h_{i+1}, n_y(i+1)$ is similar. We have that the local orbit $\mathcal{O}\left(y_i, V_{n_y(i)}\right)$ is somewhere dense. Pick a nonempty open subset W of $V_{n_y(i)}$ that is contained in the closure of $\mathcal{O}\left(y_i, V_{n_y(i)}\right)$. By recursive hypothesis we have that $\mathcal{O}\left(x_i, V_{n_x(i)}\right)$ is dense in W. Let $\gamma_0, \ldots, \gamma_{\ell-1} \in V_{n_x(i)}$ such that, setting $z_j = s(\gamma_j)$ for $j < \ell$ and $z_\ell = r(\gamma_{\ell-1})$, one has that $z_0 = x_i$, $z_\ell \in W$, and $z_{j+1} = r(\gamma_j)$ for $j < \ell$. Since by inductive assumption we have that $\forall z \in s\left(V_{n_x(i)}\right)$, $\forall^* \gamma \in V_{n_x(i)} z$, $f(r(\gamma)) \in N_{k_x(i)} f(z)$, after modifying the sequence $(\gamma_0, \ldots, \gamma_{\ell-1})$ we can assume that, for every $j < \ell$, $f(z_{j+1}) = p_j f(z_j)$ for some $p_j \in N_{k_x(i)}$. Therefore $f(z_\ell) = pf(z)$ where $p = p_{\ell-1} p_{\ell-2} \cdots p_0 \in N_{k_x(i)}$. We may then let $x_{i+1} = z_\ell$, $g_{i+1} = pg_i$, $k_x(i+1) = \max\left\{g_{i+1}(\lambda), g_{i+1}^{-1}(\lambda) : \lambda \leq i+1\right\}$, and $n_x(i+1) \in \mathbb{N}$ such that $x_{i+1} \in V_{n_x(i+1)}$ and $\forall x' \in s\left(V_{n_x(i+1)}\right)$, $\forall^* \gamma \in V_{n_x(i+1)} x'$, $f(r(\gamma)) \in N_{k_x(i+1)} f(x')$. This concludes the definition of $x_{i+1}, g_{i+1}, n_x(i+1)$. □

5.5. Admissible log-convex sequences

Recall from Section 5.3 that a sequence $\boldsymbol{a}$ in $(0,1]^{\mathbb{N}}$ is *admissible log-convex* if $a_0 = 1$, $(\frac{a_n}{a_{n+1}})_n$ is non-increasing and converges to 1, and $\sum_n a_n = \infty$. The set $\mathcal{A}_0 \subset (0,1]^{\mathbb{N}}$ of admissible log-convex sequences is Borel. We consider the relation $E_{\mathcal{A}_0}$ on A defined by $\boldsymbol{a} E_{\mathcal{A}_0} \boldsymbol{a}'$ if and only if $\boldsymbol{a}$ and $\boldsymbol{a}'$ have the same growth, in formulas $c \leq a'_n / a_n \leq C$ for some constants $c, C > 0$ and for every $n \in \mathbb{N}$. The main goal of this section is to prove the following result:

Proposition 5.5.1. *Admissible log-convex sequences are not classifiable by countable structures up to the relation of having the same growth.*

However, it is not difficult to verify that admissible log-convex sequences are classifiable by the orbits of a Polish group action up the relation of having the same growth. This means that there exists a continuous Polish group action $G \curvearrowright X$ such that $E_{\mathcal{A}_0}$ is Borel reducible to the orbit equivalence relation E^X_G. The crucial point is that if

$$B = \{(-\log(a_n)) : (a_n) \in A\} \subset (0,\infty)^{\mathbb{N}},$$

then

$$H = \{z \in \ell^\infty : \text{ there exist } x, y \in B \text{ with } x - y = z\}$$

is a subgroup of ℓ^∞ which is separable in the ℓ^∞-metric, and two sequences $(a_n), (a'_n)$ in A have the same growth if and only if $(-\log(a_n))$ and $(-\log(a'_n))$ belong to the same H-orbit under translation.

The rest of this section is dedicated to the proof of Proposition 5.5.1. Consider the equivalence relation F on $(0,1)^{\mathbb{N}}$ defined by

$$\boldsymbol{s} F \boldsymbol{s}' \quad \text{if and only if} \quad \sup_n \left| \sum_{k<n} \left(\prod_{i \leq k} s_i - \prod_{i \leq k} s'_i \right) \right| < \infty.$$

Define furthermore the Borel function

$$\begin{array}{rcl} (0,1)^{\mathbb{N}} & \to & (0,1]^{\mathbb{N}} \\ \boldsymbol{s} & \mapsto & f(\boldsymbol{s}) = \exp\left(-\sum_{k<n} \prod_{i \leq k} s_i \right)_{n \in \mathbb{N}} \end{array}$$

where the empty sum is 0. Observe that for $s \in (0,1)^{\mathbb{N}}$, we have that $f(\boldsymbol{s})_0 = 1$, $f(\boldsymbol{s})$ is log-convex and $f(\boldsymbol{s})_n / f(\boldsymbol{s})_{n+1} \geq 1$ for all $n \in \mathbb{N}$. Let $X \subset (0,1)^{\mathbb{N}}$ be the set of $\boldsymbol{s} \in (0,1)^{\mathbb{N}}$ such that $f(\boldsymbol{s}) \in \mathcal{A}_0$. Using the fact that $f(\boldsymbol{s}) \in \mathcal{A}_0$ if and only if $f(\boldsymbol{s})$ is not summable, it is not difficult to verify that X is a dense G_δ subset of $(0,1)^{\mathbb{N}}$. The restriction $f|_X$ of f to X is a Borel reduction from $F|_X$ to $E_{\mathcal{A}_0}$. It is thus enough to show that $F|_X$ is not classifiable by countable structures.

Lemma 5.5.2. *F has meager classes.*

Proof. Fix $\boldsymbol{s} \in (0,1)$. We want to show that the F-class of $\boldsymbol{s}$ is meager. We can assume without loss of generality that $\prod_{i \leq k} s_i \to 0$ for $k \to \infty$, as the set of such $\boldsymbol{s}$ is a comeager subset of $(0,1)^{\mathbb{N}}$. Fix $m \in \mathbb{N}$ and let K_m be the (closed) set of $\boldsymbol{t} \in (0,1)^{\mathbb{N}}$ such that, for every $n \in \mathbb{N}$,

$$\left| \sum_{k<n} \left(\prod_{i \leq k} s_i - \prod_{i \leq k} t_i \right) \right| \leq m.$$

Observe that if $\boldsymbol{t}^0 \in K_m$ and $n_0 \in \mathbb{N}$ then the element $\boldsymbol{t}$ of $(0,1]^{\mathbb{N}}$ defined by

$$t_i = \begin{cases} t_i^0 & \text{for } i \leq n_0, \\ 1 - 2^{-i} & \text{otherwise} \end{cases}$$

does not belong to K_m. Therefore K_m is nowhere dense. Finally observe that the F-class of $\boldsymbol{s}$ is $\bigcup_m K_m$. □

Consider now the relation E on $(0,1)^{\mathbb{N}}$ defined by

$$\boldsymbol{s} E \boldsymbol{s}' \text{ if and only if } \sum_{n\in\mathbb{N}} \left| \prod_{i\leq k} \frac{s_i}{s_i'} - 1 \right| < \infty.$$

We will see below that E is an equivalence relation. Since F has meager classes, $X \subset (0,1)^{\mathbb{N}}$ is comeager, and $E \subset F$, it is not hard to see that, in order to prove that $F|_X$ is not classifiable by countable structures, it is enough to show that E is generically S_∞-ergodic. Indeed if $F|_X$ is classifiable by countable structures, then $F|_X$ admits a Borel reduction f to the orbit equivalence relation of an S_∞-space Y [34, Theorem 11.3.8]. Since $E \subset F$, f is a Borel $\left(E|_X, E^Y_{S_\infty}\right)$-homomorphism. By generic S_∞-ergodicity of E, there exists a comeager subset C of X that is mapped by f into a single S_∞-orbit. Since F has meager classes, C contains at least two F-equivalence classes, contradicting the fact that f is a Borel reduction from F to $E_{S^Y_\infty}$.

Let now Γ be the subgroup of $\mathbb{R}_+^{\mathbb{N}}$ containing those sequences $\boldsymbol{g}$ such that

$$\sum_n \left| \prod_{k\leq n} g_k - 1 \right| < \infty.$$

Observe that Γ is indeed a subgroup of $\mathbb{R}_+^{\mathbb{N}}$. In fact suppose that $\boldsymbol{g}, \boldsymbol{h} \in \Gamma$. Fix $n_0 \in \mathbb{N}$ such that

$$\left| \prod_{k\leq n} g_k - 1 \right| \leq \frac{1}{2}$$

for every $n \geq n_0$. Then

$$\sum_{n\in\mathbb{N}} \left| \prod_{k\leq n} g_k^{-1} - 1 \right| \leq \sum_{n<n_0} \left| \prod_{k\leq n} g_k^{-1} - 1 \right| + 2 \sum_{n\in\mathbb{N}} \left| \prod_{k\leq n} g_k - 1 \right| < \infty$$

and hence $\boldsymbol{g}^{-1} \in \Gamma$. Furthermore

$$\sum_{n\in\mathbb{N}} \left| \prod_{k\leq n} g_k h_k - 1 \right| \leq \sum_{n<n_0} \left| \prod_{k\leq n} g_k h_k - 1 \right| + \frac{3}{2} \sum_{n\in\mathbb{N}} \left| \prod_{k\leq n} h_k - 1 \right| + \sum_n \left| \prod_{k\leq n} g_k - 1 \right| < \infty$$

and hence $\boldsymbol{gh} \in \Gamma$. Since $\boldsymbol{s} E \boldsymbol{s}'$ if and only if $\boldsymbol{s}/\boldsymbol{s}' \in \Gamma$, it follows in particular that E is an equivalence relation.

Define the bi-invariant metric d_Γ on Γ by setting

$$d_\Gamma(\boldsymbol{g}, \boldsymbol{h}) = \sum_{n\in\mathbb{N}} \left| \prod_{k\leq n} g_k - \prod_{k\leq n} h_k \right|.$$

We claim that d_Γ induces a Polish topology on Γ. To this end, consider the injective map $\Phi : \mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$\boldsymbol{a} \mapsto \left(\left(\prod_{k \leq n} a_k\right) - 1\right)_n .$$

Observe that the restriction of Φ to Γ is an isometry from (Γ, d_Γ) to ℓ^1 endowed with the ℓ^1-metric. Furthermore the image of Γ under Φ is a G_δ subset of ℓ^1, since $\boldsymbol{b} \in \Phi(\Gamma)$ if and only if $b_n > -1$ for every $n \in \mathbb{N}$. Since a G_δ subspace of a Polish space is Polish [53, Theorem 3.11], this concludes the proof that d_Γ induces a Polish topology on Γ.

If $\boldsymbol{g} \in \Gamma$ and $\boldsymbol{s} \in (0,1)^{\mathbb{N}}$, define $\boldsymbol{gs} \in \mathbb{R}_+^{\mathbb{N}}$ by

$$(\boldsymbol{gs})_n = g_n s_n.$$

Consider now the groupoid

$$G = \left\{(\boldsymbol{g}, \boldsymbol{s}) \in \Gamma \times (0,1)^{\mathbb{N}} : \boldsymbol{gs} \in (0,1)^{\mathbb{N}}\right\}.$$

Composition and inversion of arrows in G are defined by

$$(\boldsymbol{g}, \boldsymbol{s})(\boldsymbol{h}, \boldsymbol{t}) = (\boldsymbol{gh}, \boldsymbol{t})$$

whenever $\boldsymbol{ht} = \boldsymbol{s}$, and

$$(\boldsymbol{g}, \boldsymbol{s})^{-1} = \left(\boldsymbol{g}^{-1}, \boldsymbol{gs}\right).$$

Being a closed subset of $\Gamma \times (0,1)^{\mathbb{N}}$, G is Polish with the induced topology. Clearly composition and inversion of arrows are continuous. Furthermore the map $(\boldsymbol{1}, \boldsymbol{s}) \mapsto \boldsymbol{s}$ allows one to identify the set of objects of G with $(0,1)^{\mathbb{N}}$. It remains to show that composition of arrows is open. To this purpose it is enough to show that the source map

$$\begin{array}{rcl} G & \to & (0,1)^{\mathbb{N}} \\ (\boldsymbol{g}, \boldsymbol{s}) & \mapsto & \boldsymbol{s} \end{array}$$

is open, see [74, Exercise I.1.8]. Suppose that $(\boldsymbol{g}, \boldsymbol{s}) \in G$, and U is an open neighborhood of $(\boldsymbol{g}, \boldsymbol{s})$. Thus there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that U contains all the pairs $(\boldsymbol{h}, \boldsymbol{t}) \in G$ such that $d_\Gamma(\boldsymbol{g}, \boldsymbol{h}) < \varepsilon$ and $|s_n - t_n| < \varepsilon$ for $n \leq N$. Suppose that $\varepsilon > \eta > 0$ is such that $g_n(s_n + \eta) < 1$ for every $n \leq N$. Consider the neighborhood W of $\boldsymbol{s}$ consisting of those $\boldsymbol{t} \in (0,1)^{\mathbb{N}}$ such that $|s_n - t_n| < \eta$ for every $n \leq N$. We claim that $s(U) \supset W$. In fact if $\boldsymbol{t} \in W$ we have that for $n \leq N$,

$$g_n t_n \leq g_n(s_n + \eta) < 1$$

and therefore $(\boldsymbol{g}, \boldsymbol{t}) \in U$.

In the following lemma we establish that G is a turbulent Polish groupoid. Together with Theorem 5.4.3, this implies that its associated orbit equivalence relation E is generically S_∞-ergodic, concluding the proof of Proposition 5.5.1.

Lemma 5.5.3. *Any element* $\boldsymbol{s}$ *of* $(0,1)^{\mathbb{N}}$ *is a turbulent object with dense orbit for the Polish groupoid* G.

Proof. It is easy to see that the orbit of $\boldsymbol{s}$ is dense. It remains to show that for any neighborhood V of $(\mathbf{1}, \boldsymbol{s})$ in G the local orbit $\mathcal{O}(\boldsymbol{s}, V)$ is somewhere dense. Without loss of generality we can assume that there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that, if

$$U = \left\{ \boldsymbol{t} \in (0,1)^{\mathbb{N}} : \forall n \leq n_0, \left| \frac{t_n}{s_n} - 1 \right| < \varepsilon \right\}$$

and

$$W = \{ \boldsymbol{g} \in \Gamma : d_\Gamma(\boldsymbol{g}, \mathbf{1}) < \varepsilon \},$$

then $V = (W \times U) \cap G$. We claim that the local orbit $\mathcal{O}(\boldsymbol{s}, V)$ dense in U. Fix $\boldsymbol{t} \in U$ and $n_1 \geq n_0$. Let $N \in \mathbb{N}$, to be determined later. Set

$$g_k = \begin{cases} \sqrt[N]{t_k / s_k} & \text{for } k \leq n_1, \\ \prod_{j \leq n_1} \sqrt[N]{s_j / t_j} & \text{for } k = n_1 + 1, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that, for N large enough, we have that $\boldsymbol{g} \in \Gamma$, $d(\boldsymbol{g}, \mathbf{1}) < \varepsilon$, and $\boldsymbol{g}^i \boldsymbol{s} \in U$ for every $i \leq N$. Finally observe that $g_k^N s_k = t_k$ for $k \leq n_1$. This concludes the proof that the local orbit $\mathcal{O}(\boldsymbol{s}, V)$ is dense in U. Since this is true for every neighborhood V of $\boldsymbol{s}$ in G, $\boldsymbol{s}$ is a turbulent point for G. $\square$

5.6. Conformal equivalence of varieties

Fix $d \in \mathbb{N}$ and let $\mathcal{V}_d$ be the space of varieties in $\mathbb{B}_d$. Denote by $\mathrm{Aut}(\mathbb{B}_d)$ be the group of conformal automorphisms of $\mathbb{B}_d$. Recall that the *pseudohyperbolic distance* d on $\mathbb{B}_d$ is defined by

$$d(a, b) = \| \varphi_a(b) \|,$$

where $\|\cdot\|$ is the usual Euclidean norm and φ_a is the conformal automorphism of $\mathbb{B}_d$ which interchanges 0 and a defined in [75, Subsection 2.2.1]. Then d is a proper metric (since its closed balls coincide with Euclidean closed balls) that induces the usual topology on $\mathbb{B}_d$. Furthermore, $\mathrm{Aut}(\mathbb{B}_d)$ is a closed subgroup of the group of isometries of $(\mathbb{B}_d, d)$, and hence a locally compact Polish group when endowed with the compact-open topology. More information about conformal automorphisms of $\mathbb{B}_d$ can be found in [75, Chapter 2]. Consider the Borel action of $\mathrm{Aut}(\mathbb{B}_d)$ on $\mathcal{V}_d$ defined by $(\alpha, V) \mapsto \alpha(V)$. Observe that the relation $E^{\mathcal{V}_d}_{\mathrm{Aut}(\mathbb{B}_d)}$ of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence of varieties in $\mathbb{B}_d$ is the orbit equivalence relation associated with this action. Therefore, it follows from [52, Theorem 1.1] that $E^{\mathcal{V}_d}_{\mathrm{Aut}(\mathbb{B}_d)}$ is essentially countable.

The remainder of this section is devoted to proving Theorem 5.1.2, asserting that $E^{\mathcal{V}_d}_{\mathrm{Aut}(\mathbb{B}_d)}$ has in fact maximum complexity among essentially countable equivalence relations. As explained in the introduction, the same conclusion will then apply to the relation of (completely) isometric isomorphism of multiplier algebras $\mathcal{M}_V$, where $V \in \mathcal{V}_d$.

Observe that the canonical inclusion of $\mathbb{B}_d$ into $\mathbb{B}_{d+1}$ induces an inclusion of $\mathcal{V}_d$ into $\mathcal{V}_{d+1}$. According to the following proposition, this inclusion is a Borel reduction from the relation of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence on $\mathcal{V}_d$ to the relation of $\mathrm{Aut}(\mathbb{B}_{d+1})$-conformal equivalence on $\mathcal{V}_{d+1}$. We mention that this result also follows from [25, Theorem 4.4].

Proposition 5.6.1. *Let $X, Y \subset \mathbb{B}_d$ be subsets. Then X and Y are conformally equivalent via an element of* $\mathrm{Aut}(\mathbb{B}_d)$ *if and only if they are conformally equivalent via an element of* $\mathrm{Aut}(\mathbb{B}_{d+1})$.

Proof. By [75, Section 2.2.8], every conformal automorphism of $\mathbb{B}_d$ extends to a conformal automorphism of $\mathbb{B}_{d+1}$. This establishes one direction.

Conversely, suppose that $F \in \mathrm{Aut}(\mathbb{B}_{d+1})$ maps X onto Y, and let $G \subset \mathrm{Aut}(\mathbb{B}_{d+1})$ denote the subgroup of all automorphisms which fix $\mathbb{B}_d$. We wish to show that X and Y are G-equivalent. Since $\mathrm{Aut}(\mathbb{B}_d)$ acts transitively on $\mathbb{B}_d$ [75, Theorem 2.2.3], and since every element of $\mathrm{Aut}(\mathbb{B}_d)$ extends to an element of G, the subgroup G acts transitively on $\mathbb{B}_d$. We may therefore assume that $0 \in X$ and $0 \in Y$. By Proposition 2.4.2 in [75] and the discussion preceding it, F maps the affine span of X onto the affine span of Y. Hence, F maps $\mathrm{span}(X) \cap \mathbb{B}_d$ onto $\mathrm{span}(Y) \cap \mathbb{B}_d$, where span denotes the linear span. Since $\mathrm{span}(X) \cap \mathbb{B}_d$ and $\mathrm{span}(Y) \cap \mathbb{B}_d$ are themselves unitarily equivalent to complex balls of dimension $e \le d$, and since automorphisms of $\mathbb{B}_e$ extend to automorphisms of higher-dimensional balls, we conclude that there exists a map $\widetilde{F} \in G$ such that $F|_{\mathrm{span}(X)\cap\mathbb{B}_d} = \widetilde{F}|_{\mathrm{span}(X)\cap\mathbb{B}_d}$. This completes the proof. □

Therefore to establish the desired lower bound on the complexity of $E^{\mathcal{V}_d}_{\mathrm{Aut}(\mathbb{B}_d)}$ it suffices to consider the case $d = 1$, hence $\mathbb{B}_d = \mathbb{D}$, the unit disc. The elements of $\mathcal{V}_1 \setminus \{\mathbb{D}\}$ are precisely the (possibly finite) *Blaschke sequences*. Recall from Section 5.2 that the orbit equivalence relation $E(F_2, 2)$ associated with the left translation action of the free group F_2 on its subsets has maximum complexity among essentially countable equivalence relation. We will now show that $E(F_2, 2)$ is Borel reducible to the relation $E^{\mathcal{V}_1}_{\mathrm{Aut}(\mathbb{D})}$ of $\mathrm{Aut}(\mathbb{D})$-conformal equivalence of Blaschke sequences. To this end, we will adapt the proof of [48, Theorem 4.1].

The lower bound in [48, Theorem 4.1] is achieved by encoding the action of F_2 on $\{0,1\}^{F_2}$ by translation. The crucial point in this proof is that $\mathrm{Aut}(\mathbb{D})$ contains a copy of F_2 such that the orbit of every point in $\mathbb{D}$ is discrete. We require something stronger, namely that the orbit of every point is a Blaschke sequence.

Proposition 5.6.2. *There exists a discrete group $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ which is isomorphic to F_2 such that*

$$\sum_{g\in\Gamma} (1 - |g(z)|) < \infty$$

for every $z \in \mathbb{D}$.

Proof. Let g_1 and g_2 be two conformal automorphisms of $\mathbb{D}$ which generate a Schottky group (see Chapter II, Section 1 in [16]), and let Γ be the group generated by g_1 and g_2. Then Γ is isomorphic to F_2 by [16, Chapter II, Proposition 1.6]. By the same proposition, the closure of the Dirichlet domain $\mathcal{D}_0(\Gamma)$ of Γ contains nontrivial arcs in $\partial\mathbb{D}$ (see [16, Chapter I, Section 2.3] for the definition of the Dirichlet domain). In particular, the Lebesgue measure of $\overline{\mathcal{D}_0(\Gamma)} \cap \partial\mathbb{D}$ is strictly positive. In this situation, [86, Theorem XI.4] applies to show that

$$\sum_{g\in\Gamma} (1 - |g(0)|) < \infty.$$

Finally, the argument preceding Theorem XI.3 in [86] shows that this sum is finite if 0 is replaced with an arbitrary point $z \in \mathbb{D}$. □

It seems worthwhile to give a concrete example of two conformal automorphisms of $\mathbb{D}$ which generate a group Γ as in the statement of the proposition. Let $\mathbb{H}$ denote the upper half-plane in $\mathbb{C}$. Recall that $\mathbb{D}$ and $\mathbb{H}$ are conformally equivalent via the Cayley map

$$\begin{aligned} \mathbb{H} &\to \mathbb{D} \\ z &\mapsto \frac{z-i}{z+i}. \end{aligned}$$

This map induces an isomorphism of topological groups between $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$. Moreover, $\mathrm{Aut}(\mathbb{H})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ via the map that assigns to the matrix

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ the corresponding *Möbius transformation*

$$z \mapsto \frac{az+b}{cz+d}.$$

Let $\Phi : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathbb{D})$ denote the isomorphism obtained by composing the two isomorphisms above. The group Λ considered in the proof of [48, Theorem 4.1] is generated by the images of

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

under Φ. The group Λ is isomorphic to F_2, but the orbit of 0 under Λ is not a Blaschke sequence. This follows from the following facts:

- Λ has finite index in $\mathrm{PSL}(2, \mathbb{Z})$, and
- the orbit of 0 under $\Phi\left(\mathrm{PSL}_2(\mathbb{Z})\right)$ is not a Blaschke sequence, as its conical limit set on $\partial\mathbb{D}$ has positive Lebesgue measure, see [16, Chapter II, Section 3.1].

Moreover, Λ is not a Schottky group, but just a generalized Schottky group in the sense of [16, Chapter II, Section 1.1]. However, if we let $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ denote the group generated by the images of

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix},$$

then it is not hard to see that Γ is indeed a Schottky group, and thus satisfies the conclusion of the proposition.

In the proof of the next theorem, we require the following elementary observation.

Lemma 5.6.3. *Let (X, d) be a metric space and let $x^{(0)}, x^{(1)}, x^{(2)}$ and $y^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}$ by points in X such that*

$$d(x^{(i)}, x^{(j)}) = d(y^{(i)}, y^{(j)})$$

for $0 \leq i, j \leq 2$ and such that the distances $d(y^{(i)}, y^{(j)})$ are all distinct for $0 \leq i < j \leq 3$. If $\theta : X \to X$ is an isometry such that

$$\theta(\{x^{(0)}, x^{(1)}, x^{(2)}\}) \subset \{y^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}\},$$

then $\theta(x^{(i)}) = y^{(i)}$ for $0 \leq i \leq 2$.

Proof. The assumptions on the distances and the fact that θ is an isometry imply that

$$\theta(\{x^{(0)}, x^{(1)}\}) = \{y^{(0)}, y^{(1)}\},$$
$$\theta(\{x^{(0)}, x^{(2)}\}) = \{y^{(0)}, y^{(2)}\}, \text{ and}$$
$$\theta(\{x^{(1)}, x^{(2)}\}) = \{y^{(1)}, y^{(2)}\}.$$

This is only possible if $\theta(x^{(i)}) = y^{(i)}$ for $0 \leq i \leq 2$. □

We are now ready to prove the main result of this section.

Theorem 5.6.4. *The relation $E(F_2, 2)$ is Borel reducible to the relation of* Aut($\mathbb{D}$)*-conformal equivalence of Blaschke sequences.*

Proof. The proof is an adaptation of the proof of the lower bound in [48, Theorem 4.1]. The details are as follows.

Let Γ be a group as in Proposition 5.6.2. We will identify F_2 with Γ. Moreover, let d be the pseudohyperbolic metric on $\mathbb{D}$, and for $z \in \mathbb{D}$ and $\varepsilon > 0$, let

$$D_\varepsilon(z) = \{y \in \mathbb{D} : d(y, z) < \varepsilon\}.$$

We will explicitly construct four Blaschke sequences

$$B_i = \{x_g^{(i)} : g \in F_2\}$$

for $0 \leq i \leq 3$ and find $\varepsilon > 0$ with the following properties:

1. $gx_h^{(i)} = x_{gh}^{(i)}$ for $g, h \in F_2$ and $0 \leq i \leq 3$,
2. $x_g^{(i)} \in D_{\varepsilon/5}(x_g^{(0)})$ for $g \in F_2$ and $0 \leq i \leq 3$,
3. $D_{\varepsilon/2}(x_g^{(0)}) \cap (B_0 \cup B_1 \cup B_2 \cup B_3) = \{x_g^{(i)} : 0 \leq i \leq 3\}$,
4. The distances $d(x_g^{(i)}, x_g^{(j)})$ do not depend on $g \in F_2$ and are all distinct and positive for $0 \leq i < j \leq 3$.

The construction proceeds as follows. Let $x_1^{(0)} \in \mathbb{D}$ be arbitrary and set $x_g^{(0)} = g(x_1^{(0)})$ for $g \in F_2$. Let $B_0 = \{x_g^{(0)} : g \in F_2\}$. Then B_0 is a Blaschke sequence. In particular, there exists $\varepsilon > 0$ such that

$$D_\varepsilon(x_1^{(0)}) \cap B_0 = \{x_1^{(0)}\}.$$

Choose distinct points $x_1^{(i)} \in D_{\varepsilon/5}(x_1^{(0)}) \setminus \{x_1^{(0)}\}$ for $i \in \{1, 2, 3\}$ such that the pseudohyperbolic distances $d(x_1^{(i)}, x_1^{(j)})$ for $i < j$ are all different from each other, and define

$x_g^{(i)} = g(x_1^{(i)})$ for $i \in \{1,2,3\}$ and $g \in F_2$. Moreover, set $B_i = \{x_g^{(i)} : g \in F_2\}$. Using the fact that every $g \in F_2$ is an isometry with respect to d, properties (1)–(4) are now easy to verify.

Given $A \subset F_2$, let

$$V_A = B_0 \cup B_1 \cup B_2 \cup \{x_g^{(3)} : g \in A\}.$$

We will show that $A = gB$ for some $g \in F_2$ if and only if V_A and V_B are $\mathrm{Aut}(\mathbb{D})$-conformally equivalent. Clearly, if $g \in F_2$ such that $gA = B$, then $g(V_A) = V_B$, hence V_A and V_B are $\mathrm{Aut}(\mathbb{D})$-conformally equivalent. Conversely, assume that there exists $\theta \in \mathrm{Aut}(\mathbb{D})$ with $\theta(V_A) = V_B$. We will show that there exists $g \in F_2$ such that $\theta = g$. Since $x_1^{(0)} \in V_A$, there exists $g \in F_2$ and $i \in \{0,1,2,3\}$ such that $\theta(x_1^{(0)}) = x_g^{(i)}$. Observe that for $k \in \{1,2\}$, we have

$$d(\theta(x_1^{(k)}), x_g^{(i)}) = d(\theta(x_1^{(k)}), \theta(x_1^{(0)})) = d(x_1^{(k)}, x_1^{(0)}) < \varepsilon/5$$

by Condition (2). By the same condition, $d(x_g^{(i)}, x_g^{(0)}) < \varepsilon/5$, hence

$$\theta(x_1^{(k)}) \in D_{\varepsilon/2}(x_g^{(0)}).$$

Therefore, Condition (3) implies that

$$\theta(\{x_1^{(0)}, x_1^{(1)}, x_1^{(2)}\}) \subset \{x_g^{(i)} : 0 \leq i \leq 3\}.$$

In light of Condition (4), an application of Lemma 5.6.3 shows that $\theta(x_1^{(i)}) = x_g^{(i)}$ for $0 \leq i \leq 2$. This means that θ and g are two Möbius transformations which agree on three points. Consequently, $\theta = g$, see for example [70, Theorem 10.10]. We finish the proof by showing that $gA = B$. Note that if $h \in A$, then $x_h^{(3)} \in V_A$. Therefore, $x_{gh}^{(3)} = g(x_h^{(3)}) = \theta(x_h^{(3)}) \in V_B$, so $gh \in B$. This shows that $gA \subset B$. Similarly, $g^{-1}B \subset A$, so $gA = B$, as desired. □

6. Nevanlinna-Pick spaces with hyponormal multiplication operators

6.1. Introduction

The contents of this chapter appeared in [42]. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with kernel K. In this chapter, we study the relationship between two possible properties of $\mathcal{H}$: the complete Nevanlinna-Pick property and hyponormality of multiplication operators. The definition of the complete Nevanlinna-Pick property can be found in Section 2.4.

The second property we consider is hyponormality of multiplication operators, that is, the property that for every multiplier φ on $\mathcal{H}$, the corresponding multiplication operator $M_\varphi \in \mathcal{B}(\mathcal{H})$ satisfies $M_\varphi M_\varphi^* \le M_\varphi^* M_\varphi$. While multiplication operators are not normal in typical examples, they are subnormal and hence hyponormal for a number of reproducing kernel Hilbert spaces, including Hardy and Bergman spaces on domains in $\mathbb{C}^d$.

Two results concerning weighted Hardy spaces serve as a motivation for the study of the relationship between the two properties. Suppose for a moment that $\mathcal{H}$ is a reproducing kernel Hilbert space on the open unit disc $\mathbb{D}$ with kernel K of the form

$$K(z,w) = \sum_{n=0}^{\infty} a_n (z\overline{w})^n \quad (z,w \in \mathbb{D}),$$

where (a_n) is a sequence of positive numbers with $a_0 = 1$ (i.e. $\mathcal{H}$ is a unitarily invariant space on the unit disc in the sense of Section 4.2). Note that the classical Hardy space H^2 corresponds to the choice $a_n = 1$ for all n, in which case we recover the Szegő kernel $(1 - z\overline{w})^{-1}$. We assume that multiplication by the coordinate function z induces a bounded multiplication operator M_z on $\mathcal{H}$. Equivalently, the sequence (a_n/a_{n+1}) is bounded. Then the operator M_z is hyponormal if and only if

$$\frac{a_n}{a_{n-1}} \ge \frac{a_{n+1}}{a_n} \quad \text{for all } n \ge 1$$

(see Section 7 in [81], and note that the sequence $(\beta(n))$ there is related to (a_n) via $a_n = \beta(n)^{-2}$). On the other hand, a sufficient condition for $\mathcal{H}$ being a complete Nevanlinna-Pick space is that the reverse inequalities

$$\frac{a_n}{a_{n-1}} \le \frac{a_{n+1}}{a_n} \quad \text{for all } n \ge 1$$

hold (see Lemma 7.38 and Theorem 7.33 in [3]). Since this condition is not necessary, the two results do not immediately tell us anything new about weighted Hardy spaces satisfying both the Nevanlinna-Pick property and hyponormality of multiplication operators. Nevertheless, they seem to indicate that the presence of both properties is special.

The aim of this chapter is to show that the Hardy space is essentially the only complete Nevanlinna-Pick space whose multiplication operators are hyponormal. Recall that a reproducing kernel Hilbert space $\mathcal{H}$ with kernel K on a set X is called *irreducible* if $K(x, y)$ is never zero for $x, y \in X$ and if $K(\cdot, x)$ and $K(\cdot, y)$ are linearly independent for different $x, y \in X$. We call a set $A \subset \mathbb{D}$ a *set of uniqueness* for H^2 if the only element of H^2 which vanishes on A is the zero function. The main result now reads as follows.

Theorem 6.1.1. *Let $\mathcal{H}$ be an irreducible complete Nevanlinna-Pick space on a set X with kernel K such that all multiplication operators on $\mathcal{H}$ are hyponormal. Then one of the following possibilities holds:*

(1) *X is a singleton and $\mathcal{H} = \mathbb{C}$.*

(2) *There is a set of uniqueness $A \subset \mathbb{D}$ for H^2, a bijection $j : X \to A$ and a nowhere vanishing function $\delta : X \to \mathbb{C}$ such that*

$$K(\lambda, \mu) = \delta(\lambda)\overline{\delta(\mu)}\, k(j(\lambda), j(\mu)),$$

where $k(z, w) = (1 - z\overline{w})^{-1}$ denotes the Szegő kernel. Hence,

$$H^2 \to \mathcal{H}, \quad f \mapsto \delta(f \circ j),$$

is a unitary operator. If X is endowed with a topology such that K is separately continuous on $X \times X$, then j is continuous. If $X \subset \mathbb{C}^n$ and K is holomorphic in the first variable, then j is holomorphic.

Since the Hardy space H^2 is a complete Nevanlinna-Pick space whose multiplication operators are hyponormal, it is easy to see that the same is true for every space as in part (2). Hence, this result characterizes Hilbert function spaces with these two properties.

Remark 6.1.2. (a) It is well known that sets of uniqueness for H^2 are characterized by the Blaschke condition (see, for example, [36, Section II 2]): A set $A \subset \mathbb{D}$ is a set of uniqueness for H^2 if and only if

$$\sum_{a \in A} (1 - |a|) = \infty.$$

(b) The condition that $K(x, y)$ is never zero is not very restrictive. Indeed, if we drop this condition, then X can be partitioned into sets (X_i) such that the restriction of $\mathcal{H}$ to each X_i (compare the next section) is an irreducible complete Nevanlinna-Pick space (see [3, Lemma 7.2]). This yields a decomposition of $\mathcal{H}$ into an orthogonal direct sum of irreducible complete Nevanlinna-Pick spaces $\mathcal{H}_i$. It is not hard to see that this decomposition is reducing for multiplication operators. Hence, all multiplication operators on $\mathcal{H}$ are hyponormal if and only if this is true for each summand. We omit the details.

Before we come to the proof of the main result of this chapter, let us consider an application to Hilbert function spaces in higher dimensions. In particular, this applies to holomorphic Hilbert function spaces on the open unit ball in $\mathbb{C}^n$ for $n \geq 2$. Standard examples of such spaces either have the property that all multiplication operators are hyponormal (such as Hardy and Bergman space) or have the Nevanlinna-Pick property (such as the Drury-Arveson space, see the next section), but not both. This is not a coincidence.

Corollary 6.1.3. *Let $n \geq 3$ be a natural number, and let $U \subset \mathbb{R}^n$ be an open set. Then there is no irreducible complete Nevanlinna-Pick space on U which consists of continuous functions and whose multiplication operators are all hyponormal.*

Proof. Assume toward a contradiction that $\mathcal{H}$ is such a Hilbert function space, and let K be its kernel. Since the functions in $\mathcal{H}$ are continuous, it follows that K is separately continuous. Hence, Theorem 6.1.1 implies that there is a continuous injection $j : U \to \mathbb{D}$. But this is impossible if $n \geq 3$ due to Brouwer's domain invariance theorem [10]. □

The remainder of this chapter is organized as follows. In Section 6.2, we will use the Agler-McCarhty universality theorem to embed a complete Nevanlinna-Pick space as in Theorem 6.1.1 into the Drury-Arveson space. The proof of Theorem 6.1.1 is then presented in Section 6.3.

6.2. Embedding into Drury-Arveson space

As a first step in the proof of Theorem 6.1.1, we will embed the complete Nevanlinna-Pick space $\mathcal{H}$ into the Drury-Arveson space using the Agler-McCarthy universality theorem.

Using notation as before, we write $\mathbb{B}_d$ for the open unit ball in $\ell^2(d)$, where d is a cardinal. The Drury-Arveson space H_d^2 is the reproducing kernel Hilbert space on $\mathbb{B}_d$ with kernel

$$k_d(z,w) = \frac{1}{1 - \langle z, w \rangle}.$$

If $d = 1$, this is the Hardy space H^2. For $d \geq 2$, Arveson [8] exhibited multipliers on H_d^2 which are not hyponormal by showing that their spectral radius is strictly less than their multiplier norm. Indeed, if z_1 and z_2 denote the coordinate functions on $\mathbb{C}^2$, then $M_{z_1 z_2}$ is not hyponormal on H_2^2, as

$$||M_{z_1 z_2} z_1 z_2||^2 = \frac{1}{6} < \frac{1}{4} = ||M^*_{z_1 z_2} z_1 z_2||^2$$

(see [8, Lemma 3.8]). This observation readily generalizes to $d \geq 2$.

Given a subset $Y \subset \mathbb{B}_d$, we write $H_d^2\big|_Y$ for the reproducing kernel Hilbert space on Y with kernel $k_d\big|_{Y \times Y}$. If

$$I(Y) = \{f \in H_d^2 : f\big|_Y = 0\}$$

denotes the kernel of the restriction map, then Lemma 2.1.2 implies that

$$H_d^2 \ominus I(Y) \to H_d^2\big|_Y, \quad f \mapsto f\big|_Y, \tag{6.1}$$

is a unitary. We will require the universality theorem of Agler and McCarthy (see Section 2.5) in the following form.

Theorem 6.2.1. *Let $\mathcal{H}$ be an irreducible complete Nevanlinna-Pick space on a set X with kernel K. Assume that K is normalized at $\lambda_0 \in X$ in the sense that $K(\lambda_0, \mu) = 1$ for all $\mu \in X$. Then there is a cardinal d and an injection $b : X \to \mathbb{B}_d$ with $b(\lambda_0) = 0$ such that*

$$K(\lambda, \mu) = \frac{1}{1 - \langle b(\lambda), b(\mu) \rangle} \quad (\lambda, \mu \in X).$$

Hence,

$$H_d^2 \ominus I(Y) \to \mathcal{H}, \quad f \mapsto (f\big|_Y) \circ b,$$

is a unitary operator, where $Y = b(X)$. □

In the above setting, let $\mathcal{F}_Y = H_d^2 \ominus I(Y)$. This space is co-invariant under multiplication operators. Clearly, every $\varphi \in \operatorname{Mult}(H_d^2)$ restricts to a multiplier on $H_d^2\big|_Y$, and hence gives rise to the multiplier $(\varphi\big|_Y) \circ b$ on $\mathcal{H}$. If U denotes the unitary operator in Theorem 6.2.1, then

$$U^* M_{(\varphi|_Y \circ b)} U = P_{\mathcal{F}_Y} M_\varphi\big|_{\mathcal{F}_Y}.$$

Thus, if we assume that all multiplication operators on $\mathcal{H}$ are hyponormal, then all operators appearing on the right-hand side of the last identity are hyponormal as well. We will use this fact to show that $\mathcal{F}_Y$ can be identified with H^2.

6.3. Proof of Theorem 6.1.1

The discussion at the end of the last section suggests studying compressions of multiplication operators to co-invariant subspaces such that the compressed operator is hyponormal. We need the following simple observation.

Lemma 6.3.1. *Let $\mathcal{H}$ be a Hilbert space, let $T \in \mathcal{B}(\mathcal{H})$ and let $M \subset \mathcal{H}$ be a co-invariant subspace for T. Suppose that the compression of T to M is hyponormal. If $f \in M$ with $||T^*f|| = ||Tf||$, then $Tf \in M$.*

Proof. Since M is co-invariant under T, and since $P_M T\big|_M$ is hyponormal, we have

$$||T^*f|| \le ||P_M Tf|| \le ||Tf|| = ||T^*f||.$$

Consequently, $||P_M Tf|| = ||Tf||$, and hence $Tf \in M$. □

We will apply this observation to multiplication operators on H^2_d. Since the coordinate functions z_i are multipliers on H^2_d, it follows from unitary invariance of the Drury-Arveson space that all functions of the form $\langle \cdot, w\rangle$ for $w \in \ell^2(d)$ are multipliers on H^2_d.

Lemma 6.3.2. *Suppose that $\mathcal{F} \subset H^2_d$ is a closed subspace which is co-invariant under multiplication operators. Let $z \in \mathbb{B}_d$, and suppose that the compression $P_{\mathcal{F}} M_{\langle \cdot, z\rangle}\big|_{\mathcal{F}}$ is hyponormal. Then the following assertions hold.*

(a) *If $1 \in \mathcal{F}$ and $K(\cdot, z) \in \mathcal{F}$, then $\langle \cdot, z\rangle \in \mathcal{F}$.*

(b) *If $\langle \cdot, z\rangle \in \mathcal{F}$, then $\langle \cdot, z\rangle^n \in \mathcal{F}$ for all $n \ge 1$.*

Proof. (a) Clearly, we may assume that $z \neq 0$, and define $w = z/||z||$. Then

$$\iota : H^2 \to H^2_d, \quad \sum_{n=0}^{\infty} a_n \zeta^n \mapsto \sum_{n=0}^{\infty} a_n \langle \cdot, w\rangle^n,$$

is an isometry, where ζ denotes the identity function on $\mathbb{C}$. Under this embedding, the unilateral shift M_ζ on H^2 corresponds to the restriction of $M_{\langle \cdot, w\rangle}$ to the reducing subspace $\iota(H^2)$. In particular, $M_{\langle \cdot, w\rangle}\big|_{\iota(H^2)}$ is an isometry.

Now, consider

$$f = K(\cdot, z) - 1 = \sum_{n=1}^{\infty} \langle \cdot, z\rangle^n \in \mathcal{F}.$$

Observe that f is contained in the range of the isometry $M_{\langle \cdot, w\rangle}\big|_{\iota(H^2)}$, hence

$$||M_{\langle \cdot, w\rangle} f|| = ||f|| = ||M^*_{\langle \cdot, w\rangle} f||.$$

Lemma 6.3.1 implies that $\mathcal{F}$ contains the element $M_{\langle \cdot, z\rangle} f$, and thus also

$$f - M_{\langle \cdot, z\rangle} f = \langle \cdot, z\rangle \in \mathcal{F}.$$

(b) The proof is by induction on n. The base case $n = 1$ holds by assumption. Suppose that $n \geq 2$ and the assertion is true for $n-1$. The same argument as in the proof of part (a), applied to $\langle \cdot, z\rangle^{n-1}$ in place of f, shows that

$$||M_{\langle \cdot, z\rangle} \langle \cdot, z\rangle^{n-1}|| = ||M^*_{\langle \cdot, z\rangle} \langle \cdot, z\rangle^{n-1}||,$$

so that

$$\langle \cdot, z\rangle^n = M_{\langle \cdot, z\rangle} \langle \cdot, z\rangle^{n-1} \in \mathcal{F}$$

by Lemma 6.3.1. □

Given $Y \subset \mathbb{B}_d$, it can happen that there is a larger set $Z \supset Y$ such that every function in $H^2_d\big|_Y$ extends uniquely to a function in $H^2_d\big|_Z$ (cf. Section 4.5). To account for that, we define

$$\overline{Y} = \{z \in \mathbb{B}_d : f(z) = 0 \text{ for all } f \in I(Y)\}.$$

Then $\overline{Y}$ is the largest set which contains Y and satisfies this extension property. Moreover, it is easy to see that

$$\overline{Y} = \{z \in \mathbb{B}_d : K(\cdot, z) \in H^2_d \ominus I(Y)\}.$$

Lemma 6.3.3. *Let $Y \subset \mathbb{B}_d$ be a set with $0 \in Y$, and set $\mathcal{F}_Y = H^2_d \ominus I(Y)$. If the compression $P_{\mathcal{F}_Y} M_{\langle \cdot, w\rangle}\big|_{\mathcal{F}_Y}$ is hyponormal for every $w \in \mathbb{B}_d$, then $\overline{Y}$ is a complex ball, that is,*

$$\overline{Y} = M \cap \mathbb{B}_d$$

for some closed subspace M of $\ell^2(d)$.

Proof. Let M be the closed linear span of $\overline{Y}$. Observe that for all $w \in \overline{Y}$, we have $K(\cdot, w) \in \mathcal{F}_Y$. Since $1 = K(\cdot, 0) \in \mathcal{F}_Y$, part (a) of Lemma 6.3.2 implies that $\langle \cdot, w\rangle \in \mathcal{F}_Y$ for all $w \in \overline{Y}$. It follows that

$$\langle \cdot, v\rangle \in \mathcal{F}_Y \quad \text{for all } v \in M,$$

as $v \mapsto \langle \cdot, v \rangle$ is a conjugate linear isometry. Using part (b) of Lemma 6.3.2, we deduce that

$$K(\cdot, v) = \sum_{n=0}^{\infty} \langle \cdot, v \rangle^n \in \mathcal{F}_Y$$

for all $v \in M \cap \mathbb{B}_d$. This argument shows that $\overline{Y} \supset M \cap \mathbb{B}_d$, and the reverse inclusion is trivial. □

We can now prove the main result.

Proof of Theorem 6.1.1. If X is a singleton, there is nothing to prove. Otherwise, fix $\lambda_0 \in X$. Since K is an irreducible kernel, it is nowhere zero, so we can consider the normalized kernel defined by

$$\widetilde{K}(\lambda, \mu) = \frac{K(\lambda, \mu)}{\delta(\lambda)\overline{\delta(\mu)}},$$

where

$$\delta(\lambda) = \frac{K(\lambda, \lambda_0)}{\sqrt{K(\lambda_0, \lambda_0)}}.$$

Then $\widetilde{K}(\lambda_0, \mu) = 1$ for all $\mu \in X$. Moreover, if $\widetilde{\mathcal{H}}$ denotes the reproducing kernel Hilbert space with kernel $\widetilde{K}$, then

$$\widetilde{\mathcal{H}} \to \mathcal{H}, \quad f \mapsto \delta f$$

is a unitary operator. It is easy to see that $\widetilde{\mathcal{H}}$ also satisfies the hypotheses of Theorem 6.1.1, so we will work with $\widetilde{\mathcal{H}}$ instead of $\mathcal{H}$.

We will show that $\widetilde{\mathcal{H}}$ can be identified with $H^2_{d'}$ for a suitable cardinal d'. It will then follow that d' is necessarily 1. By Theorem 6.2.1, there is an injection $b : X \to \mathbb{B}_d$ for some cardinal d such that $0 = b(\lambda_0) \in b(X)$ and such that

$$\widetilde{K}(\lambda, \mu) = k_d(b(\lambda), b(\mu))$$

holds for all $\lambda, \mu \in X$. Define $Y = b(X)$ and $\mathcal{F}_Y = H^2_d \ominus I(Y)$, and note that $0 \in Y$. The discussion at the end of Section 6.2 now shows that $\mathcal{F}_Y$ satisfies the hypotheses of Lemma 6.3.3, hence

$$\overline{Y} = M \cap \mathbb{B}_d$$

for some closed subspace M. Let d' be the dimension of the Hilbert space M. As X is not a singleton, $d' \neq 0$. Clearly, $\mathcal{F}_Y = \mathcal{F}_{\overline{Y}}$, so that the restriction map from $\mathcal{F}_Y$ into $H^2_d\big|_{\overline{Y}}$ is unitary. If V is an isometry from $\ell^2(d')$ onto $M \subset \ell^2(d)$, we have

$$k_d(V(z), V(w)) = k_{d'}(z, w) \quad \text{for all } z, w \in \mathbb{B}_{d'}.$$

Therefore,

$$\mathcal{F}_Y \to H^2_{d'}, \quad f \mapsto f \circ V,$$

is a unitary operator as well. Combining this map with the unitary from Theorem 6.2.1, we obtain a unitary

$$H^2_{d'} \to \widetilde{\mathcal{H}}, \quad f \mapsto f \circ j,$$

where $j = V^* \circ b$.

By assumption, all multiplication operators on $\widetilde{\mathcal{H}}$ are hyponormal, hence the same is true for $H^2_{d'}$. This is only possible if $d' = 1$ (see the discussion at the beginning of Section 6.2), so that the last operator is in fact a unitary from H^2 onto $\widetilde{\mathcal{H}}$. Injectivity of this operator implies that $A = j(X)$ is a set of uniqueness for H^2. Combining the identities for the various kernels, we see that

$$K(\lambda, \mu) = \delta(\lambda)\overline{\delta(\mu)}k(j(\lambda), j(\mu)) \quad \text{for all } \lambda, \mu \in X, \tag{6.2}$$

as asserted.

To prove the additional assertion, let $\lambda_0 \neq \mu \in X$. Then $j(\mu) \neq 0$, so rearranging equation (6.2), we obtain for j the formula

$$j(\lambda) = \big(\overline{j(\mu)}\big)^{-1}\Big(1 - \frac{\delta(\lambda)\overline{\delta(\mu)}}{K(\lambda, \mu)}\Big).$$

Taking the definition of δ into account, it follows that j is continuous (respectively holomorphic) whenever $K(\cdot, \mu)$ is. □

Remark 6.3.4. (a) Since $d' = 1$ in the last proof, the isometry V is of the form $\lambda \mapsto \lambda w$ for some unit vector w in the one-dimensional space M. It is easy to see that in this situation, the inverse of the unitary

$$\mathcal{F}_Y = \mathcal{F}_{\overline{Y}} \to H^2, \quad f \mapsto f \circ V,$$

is given by

$$H^2 \to \mathcal{F}_Y \subset H^2_d, \quad \sum_{n=0}^{\infty} a_n \zeta^n \mapsto \sum_{n=0}^{\infty} a_n \langle \cdot, w \rangle^n.$$

An isometric embedding of this type was used in the proof of Lemma 6.3.2.

(b) For the most part of the proof of Theorem 6.1.1, we only used hyponormality of operators of the form $P_{\mathcal{F}_Y} M_{\langle \cdot, w \rangle}\big|_{\mathcal{F}_Y}$ for $w \in \mathbb{B}_d$ (notation as above). If $\mathcal{H}$ is an irreducible complete Nevanlinna-Pick space with kernel K, normalized at some point λ_0, then these operators correspond to multiplication operators on $\mathcal{H}$ with multipliers of the form

$$\varphi(\cdot) = \langle b(\cdot), w\rangle \quad (w \in \mathbb{B}_d), \tag{6.3}$$

where b is the injection from Theorem 6.2.1. These multipliers play the role of coordinate functions for Nevanlinna-Pick spaces (see the discussion preceding Beurling's theorem for Nevanlinna-Pick spaces [3, Theorem 8.67]).

The only argument which requires hyponormality of more general multiplication operators is the proof that $d' = 1$. Thus, if we weaken the hypothesis of Theorem 6.1.1 and only require hyponormality of multiplication operators corresponding to functions as in (6.3), then $\mathcal{H}$ will be equivalent to $H^2_{d'}$ (in the sense of part (2) of Theorem 6.1.1) for some cardinal d'.

7. von Neumann's inequality for commuting weighted shifts

7.1. Introduction

The contents of this chapter appeared in [44]. von Neumann's inequality states that

$$||p(T)|| \leq \sup\{|p(z)| : z \in \overline{\mathbb{D}}\}$$

holds for every contraction T on a Hilbert space and every polynomial $p \in \mathbb{C}[z]$, where $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$ [88]. This inequality can be deduced from Sz.-Nagy's dilation theorem, according to which every contraction T on a Hilbert space admits a unitary (power) dilation [83]. Andô's theorem shows that any pair (T_1, T_2) of commuting contractions dilates to a pair of commuting unitaries [5]. As a consequence, we obtain a two variable von Neumann inequality:

$$||p(T_1, T_2)|| \leq \sup\{|p(z_1, z_2)| : (z_1, z_2) \in \overline{\mathbb{D}}^2\}$$

for every polynomial $p \in \mathbb{C}[z_1, z_2]$. The situation for three or more commuting contractions is quite different. Parrott [64] gave an example of three commuting contractions satisfying von Neumann's inequality which do not dilate to commuting unitaries. Kaijser-Varopoulos [87] and Crabb-Davie [14] exhibited three commuting contractions which do not satisfy the three variable version of von Neumann's inequality. More details about this topic can be found in Chapter 5 of the book [65].

In 1974, Shields [81, Question 36] asked if von Neumann's inequality holds for a particularly tractable class of commuting contractions, namely multivariable weighted shifts. He attributes this question to Lubin. This problem is also explicitly mentioned in the proof of Theorem 22 in [51]. Multivariable weighted shifts can be defined as follows. Let $(\beta_I)_{I \in \mathbb{N}^d}$ be a collection of strictly positive numbers with $\beta_0 = 1$ such that for $j = 1, \ldots, d$, the set

$$\{\beta_{I+\varepsilon_j}/\beta_I : I \in \mathbb{N}^d\}$$

is bounded, where $\varepsilon_j \in \mathbb{N}^d$ is the tuple whose j-coordinate is 1 and whose other coordinates are 0. Define a space of formal power series

$$H^2(\beta) = \left\{ f(z) = \sum_{I \in \mathbb{N}^d} a_I z^I : ||f||^2 = \sum_{I \in \mathbb{N}^d} |a_I|^2 \beta_I^2 < \infty \right\}$$

and for $j = 1, \ldots, d$, let M_{z_j} be the unique bounded linear operator on $H^2(\beta)$ such that

$$M_{z_j} z^I = z^{I+\varepsilon_j} \quad \text{for all } I \in \mathbb{N}^d.$$

Then the tuple $(M_{z_1}, \ldots, M_{z_d})$ is called a d-variable weighted shift. More details about multivariable weighted shifts can be found in Section 7.3.

The purpose of this chapter is to provide a positive answer to the question of Lubin and Shields.

Theorem 7.1.1. *Let $T = (T_1, \ldots, T_d)$ be a d-variable weighted shift and assume that each T_j is a contraction. Then T dilates to a d-tuple of commuting unitaries. In particular, T satisfies von Neumann's inequality, that is,*

$$||p(T)|| \le \sup\{|p(z)| : z \in \overline{\mathbb{D}}^d\}$$

for all $p \in \mathbb{C}[z_1, \ldots, z_d]$.

A proof of Theorem 7.1.1 will be given in Section 7.4. In fact, we will show that every contractive d-variable weighted shift satisfies the matrix version of von Neumann's inequality.

It is important that the tuple T in Theorem 7.1.1 is a multivariable weighted shift in the sense described above. Indeed, the three operators of the Crabb-Davie example [14], which do not satisfy von Neumann's inequality, commute and are weighted shifts individually (with some weights equal to zero), but they do not form a 3-variable weighted shift. Furthermore, it is also possible to define multivariable weighted shifts with possibly zero weights, see Remark 7.3.1 (b). In Section 7.5, we will exhibit such a tuple of operators which does not dilate to a tuple of commuting unitaries. This example is similar to Parrott's example [64].

Abstract considerations show that there exists a d-tuple of commuting contractions $(S_1, \ldots, S_d)$ on a Hilbert space such that

$$||p(T_1, \ldots, T_d)|| \le ||p(S_1, \ldots, S_d)||$$

holds for every d-tuple of commuting contractions $(T_1, \dots, T_d)$ and every polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$, and, in fact, for every matrix of polynomials p. This defines an operator algebra structure on $\mathbb{C}[z_1, \dots, z_d]$, which is called the universal operator algebra for d commuting contractions (see [65, Chapter 5]). It follows from Theorem 7.1.1 and from the failure of von Neumann's inequality for three commuting contractions that S cannot be a d-variable weighted shift for $d \geq 3$.

Corollary 7.1.2. *Let $\rho : \mathbb{C}[z_1, \dots, z_d] \to \mathcal{B}(\mathcal{H})$ be an isometric representation of the universal operator algebra for d commuting contractions. If $d \geq 3$, then the d-tuple of operators $(\rho(z_1), \dots, \rho(z_d))$ is not a d-variable weighted shift.* □

This should be compared with the situation for commuting row contractions, that is, commuting tuples $(T_1, \dots, T_d)$ satisfying $\sum_{j=1}^{d} T_j T_j^* \leq 1$. In this case, the universal norm is the multiplier norm on the Drury-Arveson space, and the corresponding d-tuple $(M_{z_1}, \dots, M_{z_d})$ of row contractions is a d-variable weighted shift [8, 26]. Indeed, the tuple $(M_{z_1}, \dots, M_{z_d})$ was first described as a weighted shift.

The remainder of this chapter is organized as follows. In Section 7.2, we provide a general method for establishing von Neumann's inequality for commuting contractions. In Section 7.3, we recall the definition and some basic properties of multivariable weighted shifts. Section 7.4 contains the proof of Theorem 7.1.1. Finally, in Section 7.5, we exhibit an example which shows that Theorem 7.1.1 does not generalize to multivariable weighted shifts with possibly zero weights.

7.2. A general method for establishing von Neumann's inequality

Let $X \subset \mathbb{C}^N$ be a compact set. We say that a function $f : X \to \mathbb{C}$ is *analytic* if it extends to an analytic function in an open neighbourhood of X. We denote by $\partial_0 X$ the Shilov boundary of the algebra of all analytic functions on X. Thus, $\partial_0 X$ is the smallest compact subset K of X such that

$$\sup\{|f(z)| : z \in X\} = \sup\{|f(z)| : z \in K\}$$

holds for every analytic function f on X. For simplicity, we call $\partial_0 X$ the *Shilov boundary of X*. By the maximum modulus principle, $\partial_0 X$ is contained in the topological boundary ∂X, but it may be smaller. Similar to the scalar valued case, we say that a function $F = (F_1, \dots, F_d) : X \to \mathcal{B}(\mathcal{H})^d$ is analytic if each F_j extends to a $\mathcal{B}(\mathcal{H})$-valued analytic function in an open neighbourhood of X.

The next result is motivated by a proof of von Neumann's inequality for matrices due to Nelson [61], see also [65, Exercise 2.16] and [68, Chapter 1].

Proposition 7.2.1. *Let $X \subset \mathbb{C}^N$ be compact and suppose that $T : X \to \mathcal{B}(\mathcal{H})^d$ is an analytic function such that $T(z)$ is a d-tuple of commuting contractions for all $z \in X$. Then the following assertions are true:*

(a) *If the tuple $T(z)$ satisfies von Neumann's inequality for all $z \in \partial_0 X$, then $T(z)$ satisfies von Neumann's inequality for all $z \in X$.*

(b) *If the tuple $T(z)$ dilates to a tuple of commuting unitaries for all $z \in \partial_0 X$, then $T(z)$ dilates to a tuple of commuting unitaries for all $z \in X$.*

Proof. Let $p = (p_{i,j})_{i,j=1}^n$ be an $n \times n$ matrix of polynomials in $\mathbb{C}[z_1, \ldots, z_d]$ and suppose that the inequality

$$||p(T(z))||_{\mathcal{B}(\mathcal{H}^n)} \le ||p||_\infty$$

holds for all $z \in \partial_0 X$, where

$$||p||_\infty = \sup\{||p(w)||_{M_n} : w \in \overline{\mathbb{D}}^d\}.$$

Given $f, g \in \mathcal{H}^n$ of norm 1, observe that the scalar valued function

$$X \to \mathbb{C}, \quad z \mapsto \langle p(T(z))f, g\rangle,$$

is analytic. By assumption, this function is bounded by $||p||_\infty$ on $\partial_0 X$, and hence on X by definition of $\partial_0 X$. Consequently, the inequality

$$||p(T(z))||_{\mathcal{B}(\mathcal{H}^n)} \le ||p||_\infty$$

holds for all $z \in X$. Part (a) now follows by taking $n = 1$ above. Part (b) is a consequence of the general fact that a tuple of commuting contractions satisfies the matrix version of von Neumann's inequality if and only if it dilates to a tuple of commuting unitaries, which follows from Arveson's dilation theorem (see, for example, [65, Corollary 7.7], or [68, Corollary 4.9] for the explicit statement). □

Remark 7.2.2. (a) Proposition 7.2.1 and its proof remain valid in the following more general setting: Suppose that $\mathcal{A} \subset C(X)$ is a uniform algebra with Shilov boundary $X_0 \subset X$. Let $T : X \to \mathcal{B}(\mathcal{H})^d$ be a function such that $T(z)$ is a d-tuple of commuting contractions for all $z \in X$ and such that for all $p \in \mathbb{C}[z_1, \ldots, z_d]$ and all $f, g \in \mathcal{H}$, the scalar valued function

$$X \to \mathbb{C}, \quad z \mapsto \langle p(T(z))f, g\rangle,$$

belongs to the algebra $\mathcal{A}$. If $T(z)$ satisfies von Neumann's inequality (respectively dilates to a d-tuple of commuting unitaries) for all $z \in X_0$, then $T(z)$ satisfies von Neumann's inequality (respectively dilates to a d-tuple of commuting unitaries) for all $z \in X$.

(b) We can recover Nelson's proof of von Neumann's inequality from Proposition 7.2.1 in the following way. Suppose that $T \in M_n(\mathbb{C})$ is a contraction and let $T = UDV$ be a singular value decomposition of T, where $U, V \in M_n(\mathbb{C})$ are unitary and D is a diagonal matrix with entries in $[0,1]$. For $z \in \overline{\mathbb{D}}^d$, define

$$T(z) = U \operatorname{diag}(z_1, \ldots, z_n) V.$$

This defines an analytic map on $\overline{\mathbb{D}}^d$. Moreover, $\partial_0(\overline{\mathbb{D}}^d) = \mathbb{T}^d$, and $T(z)$ is unitary for $z \in \mathbb{T}^d$. By Proposition 7.2.1, it therefore suffices to establish von Neumann's inequality for unitary matrices, which in turn is an immediate consequence of the spectral theorem.

To motivate the proof of Theorem 7.1.1, we first deduce from Proposition 7.2.1 that single contractive weighted shifts satisfy von Neumann's inequality (of course, this also follows from the usual von Neumann's inequality for Hilbert space contractions).

Proposition 7.2.3. *Let T be a unilateral weighted shift which is a contraction. Then T satisfies von Neumann's inequality.*

Proof. A straightforward approximation argument reduces the statement to the case of truncated weighted shifts (see Lemma 7.4.1 below for the details). Let $n \in \mathbb{N}$ and suppose that $T \in M_n(\mathbb{C})$ is a truncated weighted shift with weight sequence $w_1, \ldots, w_{n-1}$ in $\overline{\mathbb{D}}$, that is,

$$T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ w_1 & 0 & \cdots & 0 & 0 \\ 0 & w_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & w_{n-1} & 0 \end{pmatrix}.$$

For $z = (z_1, \ldots, z_{n-1}) \in \overline{\mathbb{D}}^{n-1}$, let $T(z) \in M_n$ be the truncated weighted shift with weight sequence $z_1, \ldots, z_{n-1}$. This defines an analytic map on $\overline{\mathbb{D}}^{n-1}$. Since $\partial_0(\overline{\mathbb{D}}^{n-1}) = \mathbb{T}^{n-1}$, an application of Proposition 7.2.1 shows that it suffices to establish von Neumann's inequality for $T(z)$ if $z \in \mathbb{T}^{n-1}$. However, for $z \in \mathbb{T}^{n-1}$, the operator $T(z)$ is easily seen to be unitarily equivalent to $T(1, 1, \ldots, 1)$ (cf. Corollary 1 in Section 2 of [81]), which evidently dilates to the bilateral shift, and thus satisfies von Neumann's inequality. □

7.3. Preliminaries about weighted shifts

In this section, we review the definition and some basic properties of multivariable weighted shifts. For a comprehensive treatment, the reader is referred to [51]. Let $d \in \mathbb{N}$. We begin by recalling multi-index notation. A multi-index is an element $I \in \mathbb{N}^d$. For $1 \leq j \leq d$, we write ε_j for the multi-index $I = (i_1, \ldots, i_d)$ with $i_j = 1$ and $i_k = 0$ for $k \neq j$. Given a multi-index $I = (i_1, \ldots, i_d)$, we define

$$|I| = i_1 + \ldots + i_d.$$

Moreover, if $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, we write

$$z^I = z_1^{i_1} \ldots z_d^{i_d}.$$

If $T = (T_1, \ldots, T_d)$ is a commuting tuple of operators, we similarly define T^I. Given two multi-indices $I = (i_1, \ldots, i_d)$ and $J = (j_1, \ldots, j_d)$, we say that $I \leq J$ if $i_k \leq j_k$ for $1 \leq k \leq d$.

Now, let $\mathcal{H}$ be a Hilbert space with an orthonormal basis

$$\{e_I : I \in \mathbb{N}^d\}$$

and let

$$\boldsymbol{w} = (w_{I,j})_{(I,j) \in \mathbb{N}^d \times \{1,\ldots,d\}}$$

be a bounded collection of strictly positive numbers satisfying the commutation relations

$$w_{I,j} w_{I+\varepsilon_j,k} = w_{I,k} w_{I+\varepsilon_k,j} \tag{7.1}$$

for all $I \in \mathbb{N}^d$ and $j \in \{1, \ldots, d\}$. The *(d-variable) weighted shift* with weights $\boldsymbol{w}$ is the unique d-tuple of bounded operators $(T_1, \ldots, T_d)$ on $\mathcal{H}$ satisfying

$$T_j e_I = w_{I,j} e_{I+\varepsilon_j} \quad (I \in \mathbb{N}^d, j \in \{1, \ldots, d\}).$$

Observe that the relations (7.1) guarantee that the operators T_j commute. Evidently, T_j is a contraction if and only if $w_{I,j} \leq 1$ for all $I \in \mathbb{N}^d$.

Remark 7.3.1. (a) The definition of multivariable weighted shifts in the introduction is equivalent to the definition given in this section. To see this, suppose that $M_z = (M_{z_1}, \ldots, M_{z_d})$ is a tuple of multiplication operators on a space $H^2(\beta)$ as in the introduction. Then with respect to the orthonormal basis $(z^I / \beta_I)_{I \in \mathbb{N}^d}$, the tuple M_z is the d-variable weighted shift with weights

$$w_{I,j} = \frac{\beta_{I+\varepsilon_j}}{\beta_I}.$$

Conversely, every d-variable weighted shift in the sense of this section is unitarily equivalent to the tuple $(M_{z_1}, \ldots, M_{z_d})$ on $H^2(\beta)$, where

$$\beta_I = ||T^I e_{(0,\ldots,0)}||$$

for $I \in \mathbb{N}^d$, see [51, Proposition 8]. While the definition in terms of multiplication operators is somewhat cleaner, it is more convenient to work with the weights $\boldsymbol{w}$ and not with the weights β in the proof of Theorem 7.1.1. Nevertheless, part of the proof is motivated by this other point of view.

(b) The assumption that all weights $w_{I,j}$ are strictly positive is standard in the study of multivariable weighted shifts. It is of course possible to define multivariable weighted shifts with complex weights in a similar way. According to [51, Corollary 2], every multivariable weighted shift with complex non-zero weights, equivalently, every injective multivariable weighted shift with complex weights, is unitarily equivalent to one with strictly positive weights. However, if we allow for zero weights, then the situation is quite different. Such a tuple is no longer unitarily equivalent to a tuple of the form M_z on $H^2(\beta)$. We will see in Section 5 that the dilation part of Theorem 7.1.1 does not hold in this more general setting.

Just as in the proof of Proposition 7.2.3, we will work with truncated shifts in the proof of Theorem 7.1.1, and we will also need to consider complex and possibly zero weights. For $N \in \mathbb{N}$, define a finite dimensional subspace $\mathcal{H}_N$ of $\mathcal{H}$ by

$$\mathcal{H}_N = \operatorname{span}\{e_I : |I| \leq N\}. \tag{7.2}$$

Suppose that

$$\boldsymbol{w} = (w_{I,j})_{|I| \leq N, j \in \{1,\dots,d\}}$$

is a collection of complex numbers satisfying the commutation relations (7.1) for $|I| \leq N-1$ and $j \in \{1, \dots, d\}$. We call such a collection a *commuting family.* The *(d-variable) truncated weighted shift* with weights $\boldsymbol{w}$ is the unique d-tuple of operators $(T_1, \dots, T_d)$ on $\mathcal{H}_{N+1}$ satisfying

$$T_j e_I = \begin{cases} w_{I,j} e_{I+\varepsilon_j} & \text{if } |I| \leq N, \\ 0 & \text{if } |I| = N+1. \end{cases}$$

Once again, the commutation relations ensure that the operators T_j commute.

We require the following straightforward adaptation of [51, Corollary 2] to truncated weighted shifts.

Lemma 7.3.2. *Let T be a d-variable truncated weighted shift on $\mathcal{H}_{N+1}$ with non-zero weights $\boldsymbol{w} = (w_{I,j})$. Then T is unitarily equivalent to the d-variable truncated weighted shift with weights $(|w_{I,j}|)$.*

Proof. For $|I| \leq N+1$, we define recursively complex numbers λ_I of modulus 1 by $\lambda_{(0,\dots,0)} = 1$ and $\lambda_{I+\varepsilon_j} = \lambda_I w_{I,j}/|w_{I,j}|$ for $|I| \leq N$. As in the proof of [51, Corollary 2], we deduce from the commutation relations (7.1) that this is well defined. If U is the unitary operator on $\mathcal{H}_{N+1}$ satisfying $Ue_I = \lambda_I e_I$, then $(U^*T_1U, \dots, U^*T_dU)$ is the d-variable truncated weighted shift with weights $(|w_{I,j}|)$. □

7.4. Proof of Theorem 7.1.1

The proof of Theorem 7.1.1 is essentially an adaptation of the proof of Proposition 7.2.3 to the multivariate setting. We begin with a straightforward reduction to truncated shifts.

Lemma 7.4.1. *In order to prove Theorem 7.1.1, it suffices to show that every d-variable truncated weighted shift with weights in $(0,1]$ dilates to a d-tuple of commuting unitaries.*

Proof. Let $T = (T_1, \dots, T_d)$ be a d-variable weighted shift on $\mathcal{H}$ such that $||T_j|| \le 1$ for each j, that is, all weights of T belong to $(0,1]$. Let $\mathcal{H}_N$ be the subspace of $\mathcal{H}$ defined in (7.2). Observe that the compressed tuple

$$P_{\mathcal{H}_N} T\big|_{\mathcal{H}_N} = (P_{\mathcal{H}_N} T_1\big|_{\mathcal{H}_N}, \dots, P_{\mathcal{H}_N} T_d\big|_{\mathcal{H}_N})$$

is a d-variable truncated weighted shift with weights in $(0,1]$. Since $\mathcal{H}_N$ is co-invariant under each T_j, we see that

$$p(P_{\mathcal{H}_N} T\big|_{\mathcal{H}_N}) = P_{\mathcal{H}_N} p(T)\big|_{\mathcal{H}_N}$$

holds for every $p \in \mathbb{C}[z_1, \dots, z_d]$. Therefore, for every $p \in \mathbb{C}[z_1, \dots, z_d]$, the sequence $p(P_{\mathcal{H}_N} T\big|_{\mathcal{H}_N})$ converges to $p(T)$ in the strong operator topology as $N \to \infty$. Consequently, if $P_{\mathcal{H}_N} T\big|_{\mathcal{H}_N}$ dilates to a d-tuple of commuting unitaries and thus satisfies the matrix version of von Neumann's inequality for all $N \in \mathbb{N}$, then T satisfies the matrix version of von Neumann's inequality, and therefore dilates to a d-tuple of commuting unitaries. □

The main obstacle when generalizing the proof of Proposition 7.2.3 to multivariable shifts is that multivariable truncated weighted shifts are not parametrized by the points of a polydisc in an obvious way. This is because of the commutation relations (7.1). Instead, we will use Lemma 7.4.2 below.

For the remainder of this section, let us fix $N \in \mathbb{N}$ and set

$$\mathcal{I} = \{(I,j) \in \mathbb{N}^d \times \{1, \dots, d\} : |I| \le N\}.$$

Let X denote the closure of the set of all commuting families $(w_{I,j})_{(I,j)\in\mathcal{I}}$ with $0 < |w_{I,j}| \le 1$ for all $(I,j) \in \mathcal{I}$. Observe that we may regard X as a compact subset of $\mathbb{C}^{|\mathcal{I}|}$.

Lemma 7.4.2. *The Shilov boundary of X is contained in the set*

$$X_0 = \{(w_{I,j}) \in X : |w_{I,j}| = 1 \text{ for all } (I,j) \in \mathcal{I}\}.$$

Proof. Let $\boldsymbol{w} = (w_{I,j}) \in X \setminus X_0$ with $w_{I,j} \neq 0$ for all $(I,j) \in \mathcal{I}$ and let $f : X \to \mathbb{C}$ be a function which extends to be analytic in a neighbourhood of X. We will show that there exists a point $\widetilde{\boldsymbol{w}} = (\widetilde{w}_{I,j}) \in X$ with $\widetilde{w}_{I,j} \neq 0$ for all $(I,j) \in \mathcal{I}$ such that

$$|f(\boldsymbol{w})| \leq |f(\widetilde{\boldsymbol{w}})|$$

and such that

$$\{(I,j) \in \mathcal{I} : |w_{I,j}| = 1\} \subsetneq \{(I,j) \in \mathcal{I} : |\widetilde{w}_{I,j}| = 1\}.$$

Once this has been accomplished, iterating this process finitely many times yields a point $\boldsymbol{v} \in X_0$ such that $|f(\boldsymbol{w})| \leq |f(\boldsymbol{v})|$. Consequently, X_0 is a boundary for the algebra of all analytic functions on X, so $\partial_0 X \subset X_0$.

Let us begin by establishing some terminology which will be used throughout the proof. Let $T = (T_1, \ldots, T_d)$ be the d-variable truncated weighted shift with weights $\boldsymbol{w}$. If $I \subset \mathbb{N}^d$ is a multi-index with $|I| \leq N+1$, we say that I is *good* if

$$||T^I e_{(0,\ldots,0)}|| = 1.$$

Otherwise, we call I *bad* (cf. Remark 7.3.1 (a)). The following observations are immediate:

(a) If I is good and if $J \leq I$, then J is good.

(b) If $(I,j) \in \mathcal{I}$ with $|w_{I,j}| < 1$, then $I + \varepsilon_j$ is bad.

(c) Suppose that $|I| \leq N$. If I is good and $I + \varepsilon_j$ is bad, then $|w_{I,j}| < 1$.

We say that a pair $(I,j) \in \mathcal{I}$ is *scalable* if I is good, but $I + \varepsilon_j$ is bad. It follows from (b) and the choice of $\boldsymbol{w}$ that there exists at least one bad multi-index. Since $(0,\ldots,0)$ is good, we therefore see that there exists at least one scalable pair. Recall that all $|w_{I,j}|$ are assumed to be non-zero, so we may define

$$r = \max\{|w_{I,j}| : (I,j) \text{ is scalable}\}^{-1}.$$

Then $r > 1$ by (c). Let $\overline{D_r(0)} \subset \mathbb{C}$ denote the closed disc of radius r around 0. For $t \in \overline{D_r(0)}$ and $(I,j) \in \mathcal{I}$, define

$$\widehat{w}_{I,j}(t) = \begin{cases} t w_{I,j} & \text{if } (I,j) \text{ is scalable,} \\ w_{I,j} & \text{if } (I,j) \text{ otherwise} \end{cases}$$

and let $\widehat{\boldsymbol{w}}(t) = (\widehat{w}_{I,j}(t))_{(I,j)\in\mathcal{I}}$. We finish the proof by showing that $\widehat{\boldsymbol{w}}(t) \in X$ for every $t \in \overline{D_r(0)}$. Indeed, it then follows from the maximum modulus principle that there exists $t_0 \in \partial D_r(0)$ with

$$|f(\boldsymbol{w})| = |f(\widehat{\boldsymbol{w}}(1))| \leq |f(\widehat{\boldsymbol{w}}(t_0))|,$$

so setting $\widetilde{\boldsymbol{w}} = \widehat{\boldsymbol{w}}(t_0)$, we obtain a point with the desired properties.

Since X is closed, it suffices to show that $\widehat{\boldsymbol{w}}(t) \in X$ for all $t \in \overline{D_r(0)} \setminus \{0\}$. Clearly, $0 < |\widehat{w}_{I,j}(t)| \leq 1$ for these t, so we need to show that $\widehat{\boldsymbol{w}}(t)$ is a commuting family, that is, we need to show that

$$\widehat{w}_{I,j}(t)\widehat{w}_{I+\varepsilon_j,k}(t) = \widehat{w}_{I,k}(t)\widehat{w}_{I+\varepsilon_k,j}(t)$$

for all $t \in \overline{D_r(0)}$ and all multi-indices I with $|I| \leq N-1$ and $1 \leq j,k \leq d$. Let I be such a multi-index. If I is bad, it follows from (a) that $I+\varepsilon_j$ and $I+\varepsilon_k$ are bad as well, and hence no pairs in $\mathcal{I}$ which appear in the above equation are scalable. If I and $I+\varepsilon_j+\varepsilon_k$ are good, then it follows again from (a) that no pairs in the equation are scalable. Thus, it remains to consider the case where I is good and $I+\varepsilon_j+\varepsilon_k$ is bad. In this case, exactly one of (I,j) and $(I+\varepsilon_j,k)$ is scalable, depending on whether $I+\varepsilon_j$ is good. Similarly, exactly one of (I,k) and $(I+\varepsilon_k,j)$ is scalable. Thus

$$\widehat{w}_{I,j}(t)\widehat{w}_{I+\varepsilon_j,k}(t) = t w_{I,j} w_{I+\varepsilon_j,k} = t w_{I,k} w_{I+\varepsilon_k,j} = \widehat{w}_{I,k}(t)\widehat{w}_{I+\varepsilon_k,j}(t),$$

as asserted. □

We are now ready to prove the main theorem.

Proof of Theorem 7.1.1. According to Lemma 7.4.1, it is enough to establish Theorem 7.1.1 when T is a d-variable truncated weighted shift with weights in $(0,1]$, say T acts on $\mathcal{H}_N$. Given a commuting family $\boldsymbol{w} \in X$, let us write $T(\boldsymbol{w})$ for the d-variable truncated weighted shift on $\mathcal{H}_N$ with weights $\boldsymbol{w}$. Then the range of the analytic map

$$X \to \mathcal{B}(\mathcal{H}_N)^d, \quad \boldsymbol{w} \mapsto T(\boldsymbol{w}),$$

consists of d-tuples of commuting contractions and contains every d-variable truncated weighted shift on $\mathcal{H}_N$ with weights in $(0,1]$. According to Proposition 7.2.1 and Lemma 7.4.2, it therefore suffices to show that $T(\boldsymbol{w})$ dilates to a d-tuple of commuting unitaries if $\boldsymbol{w} \in X_0$. We infer from Lemma 7.3.2 that for $\boldsymbol{w} \in X_0$, the d-tuple $T(\boldsymbol{w})$ is unitarily equivalent to the tuple $T(\mathbf{1})$, where $\mathbf{1}$ denotes the element of X_0 which consists only of 1s. Thus, it remains to show that $T(\mathbf{1})$ dilates to a d-tuple of commuting unitaries. In this case, it is not hard to construct a unitary dilation explicitly.

Indeed, let σ denote the normalized Lebesgue measure on $\mathbb{T}^d$, and for $1 \le k \le d$, let M_{z_k} be the operator on $L^2(\sigma)$ given by multiplication with z_k. Then $M_z = (M_{z_1}, \dots, M_{z_d})$ is a d-tuple of commuting unitaries, and $T(\mathbf{1})$ can be identified with the compression of M_z to the semi-invariant subspace

$$\operatorname{span}\{z^I : I \in \mathbb{N}^d, |I| \le N\},$$

cf. Example 1 in Section 2 of [51]. Therefore, the proof is complete. □

Remark 7.4.3. In the last proof, the tuple of unitaries $(M_{z_1}^*, \dots, M_{z_d}^*)$ on $L^2(\mathbb{T}^d)$ is in fact a regular dilation of the adjoint of $T(\mathbf{1})$ in the sense of [85, Section 9]. Therefore, the adjoint of every tuple $T(\boldsymbol{w})$ for $\boldsymbol{w} \in \partial_0 X$ admits a regular unitary dilation. This is not true for the adjoint of every tuple $T(\boldsymbol{w})$ for $\boldsymbol{w} \in X$.

For example, suppose that $d = 2$ and let

$$w_{I,j} = \begin{cases} 1/2, & \text{if } I = (0,0) \\ 1, & \text{otherwise.} \end{cases}$$

Let $T = (T_1, T_2)$ be the 2-variable weighted shift with weights $(w_{I,j})$. With notation as in Remark 7.3.1 (a), T is unitarily equivalent to (M_{z_1}, M_{z_2}) on $H^2(\beta)$, where $\beta_0 = 1$ and $\beta_I = 1/2$ if $I \ne (0,0)$. A straightforward computation shows that

$$(1 - T_1T_1^* - T_2T_2^* + T_1T_2T_1^*T_2^*)e_{(1,1)} = -3/4 e_{(1,1)},$$

hence T^* does not admit a regular unitary dilation by [85, Theorem 9.1]. Similarly, the truncations $P_{\mathcal{H}_N} T^* \big|_{\mathcal{H}_N}$ do not admit regular unitary dilations if $N \ge 2$.

For the same reason, multivariable truncated weighted shifts do not in general coextend to a (direct sum of) M_z on the Hardy space $H^2(\mathbb{D}^d)$, or, more generally, to a tuple $(V_1, \dots, V_d)$ of doubly commuting isometries (i.e. the V_i commute and $V_i^*V_j = V_jV_i^*$ if $i \ne j$). Indeed, if (V_1, V_2) is a pair of doubly commuting isometries, then

$$(1 - V_1V_1^* - V_2V_2^* + V_1V_2V_1^*V_2^*) = (1 - V_1V_1^*)(1 - V_2V_2^*)$$

is a positive operator, and hence if (T_1, T_2) is a compression of (V_1, V_2) to a co-invariant subspace, then

$$1 - T_1T_1^* - T_2T_2^* + T_1T_2T_1^*T_2^*$$

is a positive operator as well. On the other hand, if $d = 1$, then every contractive weighted shift, being a pure contraction, coextends to a direct sum of copies of the unilateral shift M_z on $H^2(\mathbb{D})$.

7.5. A non-injective counterexample

The definition of multivariable weighted shifts given in Section 7.3 can be generalized to allow complex and possibly zero weights, see Remark 7.3.1 (b). Even though it is customary to assume that all weights are non-zero, we may still ask if Theorem 7.1.1 holds in this greater generality. Observe that the proof of Theorem 7.1.1 breaks down if some of the weights of T are zero. Indeed, the method of scaling certain weights by a complex number t never changes a zero weight into a non-zero one. It is natural to ask, however, if the proof can be modified by introducing non-zero weights in such a way that the commutation relations (7.1) still hold. We will now exhibit an example which shows that this is not possible in general. In fact, the operator tuple in this example does not dilate to a commuting tuple of unitaries.

Let T be a 3-variable weighted shift with not necessarily positive weights $(w_{I,j})$ given by

$$w_{I,j} = \begin{cases} 0, & \text{if } |I| \geq 2 \text{ or } I = \varepsilon_j, \\ a_{i,j}, & \text{if } I = \varepsilon_i \text{ and } i \neq j, \\ \delta_j & \text{if } I = (0,0,0). \end{cases}$$

Here $(a_{i,j})_{i \neq j}$ are six complex numbers of modulus 1 and $(\delta_j)_{1 \leq j \leq 3}$ are three complex numbers of modulus at most 1, all to be determined later. The relevant part of the three dimensional grid $\mathbb{N}^3$ together with the weights $w_{I,j}$ is shown in Figure 7.1.

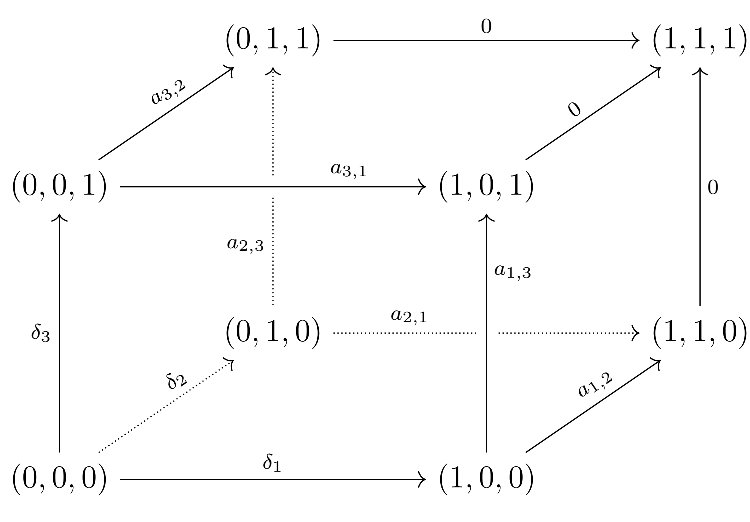

Figure 7.1.: The weights of T

Observe that if $\delta_j = 0$ for all j, then $(w_{I,j})$ satisfies the commutation relations (7.1) regardless of the value of the six weights $a_{i,j}$. On the other hand, the relations

$$\begin{aligned}\delta_1 a_{1,3} &= \delta_3 a_{3,1}\\ \delta_2 a_{2,1} &= \delta_1 a_{1,2}\\ \delta_3 a_{3,2} &= \delta_2 a_{2,3}\end{aligned}$$

show that if one of the δ_j is not zero, then all of them are non-zero. Moreover, multiplying the above equations, we see that in this case, the equation

$$a_{1,3}a_{2,1}a_{3,2} = a_{3,1}a_{1,2}a_{2,3}$$

must hold. For example, let us set $a_{2,1} = -1$ and

$$a_{1,2} = a_{1,3} = a_{2,3} = a_{3,1} = a_{3,2} = 1.$$

If $\delta_j = 0$ for $j = 1, 2, 3$, then we obtain a 3-variable contractive weighted shift T with not necessarily positive weights. However, it is not possible to perturb the first three weights $w_{(0,0,0),j} = \delta_j$ while maintaining commutativity of the operators. Note this also shows that for any $N \geq 2$, the weights $(w_{I,j})$, where $|I| \leq N$, do not belong to the set X of Section 7.4.

We now show that the 3-tuple of commuting contractions T which we just constructed does not dilate to a 3-tuple of commuting unitaries. This is very similar to Parrott's example [64]. Observe that the 6-dimensional space M spanned by the vectors

$$e_{(1,0,0)}, e_{(0,1,0)}, e_{(0,0,1)}, e_{(1,1,0)}, e_{(1,0,1)}, e_{(0,1,1)}$$

contains $\operatorname{ran}(T_j)$ as well as $\ker(T_j)^\perp$ for $j = 1, 2, 3$, so we may restrict our attention to this space. With respect to the orthonormal basis above, the operators T_j are given on M by

$$T_j = \begin{bmatrix} 0 & 0 \\ A_j & 0 \end{bmatrix} \in M_6(\mathbb{C}) \quad (j = 1, 2, 3),$$

where

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

As in the treatment of Parrott's example in [18, Example 20.27], we consider the matrix polynomial

$$p(z_1, z_2, z_3) = \begin{bmatrix} z_1 & z_2 & 0 \\ z_3 & 0 & z_2 \\ 0 & z_3 & -z_1 \end{bmatrix}.$$

It is shown there that

$$\sup\{||p(z_1, z_2, z_3)|| : z_1, z_2, z_3 \in \overline{\mathbb{D}}\} = \sqrt{3}.$$

On the other hand,

$$||p(T_1, T_2, T_3)|| = ||p(A_1, A_2, A_3)||.$$

The submatrix of the 9×9 matrix $p(A_1, A_2, A_3)$ corresponding to the rows $1, 6, 8$ and the columns $2, 4, 9$ is

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

and it is easy to check that this matrix has norm 2. Thus,

$$||p(T_1, T_2, T_3)|| \geq 2.$$

In fact, it is not hard to see that $||p(T_1, T_2, T_3)|| = 2$. Since $2 > \sqrt{3}$, it follows that the commuting contractions T_1, T_2, T_3 do not dilate to commuting unitaries. However, it follows from Section 5 of [64] that T_1, T_2, T_3 do satisfy the scalar version of von Neumann's inequality.

This example also demonstrates that Lemma 7.3.2 and [51, Corollary 6] do not hold in general without the assumption that the weights are non-zero. Indeed, it is not hard to see that if $a_{i,j} = 1$ for all $i \neq j$ in the example above, then T does dilate to a commuting tuple of unitaries.

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