Matchings and Covers in Hypergraphs

by

Michael Szestopalow

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we study three variations of matching and covering problems in hypergraphs. The first is motivated by an old conjecture of Ryser which says that if $\mathcal{H}$ is an r-uniform, r-partite hypergraph which does not have a matching of size at least $\nu + 1$, then $\mathcal{H}$ has a vertex cover of size at most $(r-1)\nu$. In particular, we examine the extremal hypergraphs for the $r = 3$ case of Ryser's conjecture. In 2014, Haxell, Narins, and Szabó characterized these 3-uniform, tripartite hypergraphs. Their work relies heavily on topological arguments and seems difficult to generalize. We reprove their characterization and significantly reduce the topological dependencies. Our proof starts by using topology to show that every 3-uniform, tripartite hypergraph has two matchings which interact with each other in a very restricted way. However, the remainder of the proof uses only elementary methods to show how the extremal hypergraphs are built around these two matchings.

Our second motivational pillar is Tuza's conjecture from 1984. For graphs G and H, let $\nu_H(G)$ denote the size of a maximum collection of pairwise edge-disjoint copies of H in G and let $\tau_H(G)$ denote the minimum size of a set of edges which meets every copy of H in G. The conjecture is relevant to the case where $H = K_3$ and says that $\tau_\triangledown(G) \leq 2\nu_\triangledown(G)$ for every graph G. In 1998, Haxell and Kohayakawa proved that if G is a tripartite graph, then $\tau_\triangledown(G) \leq 1.956\nu_\triangledown(G)$. We use similar techniques plus a topological result to show that $\tau_\triangledown(G) \leq 1.87\nu_\triangledown(G)$ for all tripartite graphs G. We also examine a special subclass of tripartite graphs and use a simple network flow argument to prove that $\tau_\triangledown(H) = \nu_\triangledown(H)$ for all such graphs H.

We then look at the problem of packing and covering edge-disjoint K_4's. Yuster proved that if a graph G does not have a fractional packing of K_4's of size bigger than $\nu^*_\boxtimes(G)$, then $\tau_\boxtimes(G) \leq 4\nu^*_\boxtimes(G)$. We give a complementary result to Yuster's for K_4's: We show that every graph G has a fractional cover of K_4's of size at most $\frac{9}{2}\nu_\boxtimes(G)$. We also provide upper bounds on $\tau_\boxtimes$ for several classes of graphs.

Our final topic is a discussion of fractional stable matchings. Tan proved that every graph has a $\frac{1}{2}$-integral stable matching. We consider hypergraphs. There is a natural notion of fractional stable matching for hypergraphs, and we may ask whether an analogous result exists for this setting. We show this is not the case: Using a construction of Chung, Füredi, Garey, and Graham, we prove that, for all $n \in \mathbb{N}$, there is a 3-uniform hypergraph with preferences with a fractional stable matching that is unique and has denominators of size at least n.

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Dedication

For Janie, Victor, Jamie, and DJ

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Chapter 1

Introduction

A *hypergraph* $\mathcal{H}$ is a pair (V, E) where V is a finite set, called *vertices*, and E is a set of subsets of V, called *edges*. If $\mathcal{H}$ has the property that every edge contains exactly r vertices, then we say that $\mathcal{H}$ is *r-uniform*. In particular, a 2-uniform hypergraph is a *graph*. A *matching* of a hypergraph $\mathcal{H}$ is a set of pairwise disjoint edges of $\mathcal{H}$. As is often the case, we will be interested in large matchings and will use $\nu(\mathcal{H})$ to denote the size of a maximum matching of $\mathcal{H}$. A fundamental problem in graph theory is the following: Given $\mathcal{H}$, can we compute $\nu(\mathcal{H})$? In the case of graphs, we can! Tutte and Berge proved that for all graphs $G = (V, E)$,

$$\nu(G) = \frac{1}{2}\min_{S \subseteq V}(|V| + |S| - \mathrm{odd}(G \backslash S)),$$

where $\mathrm{odd}(G\backslash S)$ is the number of components of $G\backslash S$ with an odd number of vertices [13, 82]. Furthermore, Edmonds developed an algorithm that will find a matching of G with size $\nu(G)$ [24]. On the other hand, the problem of finding maximum matchings becomes significantly more challenging as soon as we leave the world of graphs. Karp proved that, given a 3-uniform, tripartite hypergraph $\mathcal{H}$ and a natural number k, it is NP-complete to decide if $\mathcal{H}$ has a matching of size k [54]. Hence, the problem of finding a maximum matching in a 3-uniform hypergraph is NP-hard.

A related notion is that of a vertex cover: A *vertex cover* or, more simply, a *cover* of a hypergraph $\mathcal{H}$ is a set of vertices which meets every edge of $\mathcal{H}$. In this case, we want small vertex covers. Let $\tau(\mathcal{H})$ denote the size of a minimum vertex cover of $\mathcal{H}$. Once again, it is a fundamental problem to compute $\tau(\mathcal{H})$. However, unlike matchings, computing $\tau(G)$ for an arbitrary graph is also NP-hard [54].

Both of the above problems have fractional versions. Let $\mathcal{H} = (V, E)$ be a hypergraph. A *fractional matching* is a function $\psi : E \to [0,1]$ such that $\sum_{e \in E : v \in e} \psi(e) \leq 1$ for every vertex $v \in V$. A *fractional cover* is a function $\rho : V \to [0,1]$ such that $\sum_{v \in V : v \in e} \rho(v) \geq 1$ for every edge $e \in E$. The relevant parameters become

$$\nu^*(\mathcal{H}) := \max \left\{ \sum_{e \in E} \psi(e) \;\middle|\; \psi \text{ is a fractional matching of } \mathcal{H} \right\}$$

$$\text{and } \tau^*(\mathcal{H}) := \min \left\{ \sum_{v \in V} \rho(v) \;\middle|\; \rho \text{ is a fractional cover of } \mathcal{H} \right\}.$$

While the algorithmics of finding matchings and vertex covers are interesting in their own right, we will focus on comparing the relative sizes of $\nu(\mathcal{H})$, $\nu^*(\mathcal{H})$, $\tau(\mathcal{H})$, and $\tau^*(\mathcal{H})$. For instance, it is straightforward from the definitions to show that

$$\nu(\mathcal{H}) \leq \tau(\mathcal{H}) \leq r\nu(\mathcal{H}) \tag{1.1}$$

for any r-uniform hypergraph. Furthermore, linear programming duality tells us that $\nu(\mathcal{H}) \leq \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq \tau(\mathcal{H})$. It is known that the two inequalites in (1.1) are tight. For an example showing equality in the upper bound, let $\mathcal{P}$ be a finite projective plane of order q. Then $\mathcal{P}$ is a $(q+1)$-uniform hypergraph such that $\nu(\mathcal{P}) = 1$ and $\tau(\mathcal{P}) = q+1$. However, it is natural to ask if we can improve this inequality under some additional assumptions. This question will be the theme for Chapters 3, 4, and 5.

This thesis has three distinct parts: Matchings and covers of 3-uniform, tripartite hypergraphs, packing and covering K_3's and K_4's, and stable matchings. For most of what follows, we will restrict ourselves to r-uniform hypergraphs, where $r \in \{3, 6\}$.

1.1 3-Uniform, Tripartite Hypergraphs

An r-uniform hypergraph is said to be *r-partite* if the vertices of $\mathcal{H}$ can be partitioned into r parts, called *vertex classes*, so that every edge of $\mathcal{H}$ contains exactly one vertex from every vertex class. Our focus is the following famous old conjecture of Ryser.

Conjecture 1.1.1 (Ryser [73]). *If $\mathcal{H}$ is an r-uniform, r-partite hypergraph, then $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$.*

Conjecture 1.1.1 began to appear in the late 1960's. Around the same time, Lovász conjectured a stronger statement: If $\mathcal{H}$ is an r-uniform, r-partite hypergraph, then $\mathcal{H}$ contains a set of vertices S such that $|S| = r-1$ and $\nu(\mathcal{H}\backslash S) \leq \nu(\mathcal{H}) - 1$ [61]. When $r = 2$, Conjecture 1.1.1 is exactly the well-known König-Egerváry Theorem about maximum matchings and minimum covers in bipartite graphs. For $r = 3$, Szemerédi and Tuza showed that $\frac{\tau(\mathcal{H})}{\nu(\mathcal{H})} \leq \frac{25}{9}$ in 1982 [76]. This ratio was subsequently improved to $\frac{8}{3}$ in 1987 by Tuza [87] and to $\frac{5}{2}$ by Haxell in 1995 [40]. Finally in 2001, Aharoni found a very nice topological argument to settle the conjecture in this case.

Theorem 1.1.2 (Aharoni [4]). *If $\mathcal{H}$ is a 3-uniform, tripartite hypergraph, then $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$.*

When $r \geq 4$, much less is known. Nonetheless, Haxell and Scott proved that there is an $\epsilon_r > 0$ such that $\tau(\mathcal{H}) \leq (r - \epsilon_r)\nu(\mathcal{H})$ when $r \in \{4, 5\}$ [47]. In the special case that $\nu(\mathcal{H}) = 1$, Tuza proved Conjecture 1.1.1 for $r \leq 5$ [84, 85]. Very recently, Francetić, Herke, McKay, and Wanless used computational results to verify Conjecture 1.1.1 when $r \leq 9$ and $|e \cap f| = 1$ for all $e, f \in \mathcal{H}$ [29].

In the fractional world, both Lovász's and Ryser's Conjectures are known to be true. In 1975, Lovász proved that $\tau(\mathcal{H}) \leq \frac{r}{2}\nu^*(\mathcal{H})$ [61] and, in 1977, Gyárfás proved that $\tau^*(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$ [37]. Towards Lovász's conjecture, Aharoni, Barat, and Wanless proved the fractional variant of an even stronger statement. Specifically, they showed that in a r-uniform, r-partite hypergraph $\mathcal{H}$, there exists an edge e and a vertex $v \in e$ such that $\nu^*(\mathcal{H}\backslash(e\backslash\{v\})) \leq \nu^*(\mathcal{H}) - 1$ [5].

There has also been research into extremal constructions. Haxell, Narins, and Szabó characterized the 3-uniform, tripartite hypergraphs $\mathcal{H}$ which satisfy $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ [45, 46]. More recently, groups have been focusing on extremal examples when $\nu(\mathcal{H}) = 1$. The classical construction is the *truncated projective plane*: The r-uniform, r-partite hypergraph $\mathcal{P}$ obtained from a projective plane of order $r-1$ by removing any single vertex and the edges that contain it. It is easy to see that $\nu(\mathcal{P}) = 1$ and $\tau(\mathcal{P}) = r-1$. However, truncated projective planes exist only when $r-1$ is a prime power. Furthermore, truncated projective planes are unnecessarily dense; they have a proper subhypergraph which is also extremal. With this in mind, let $f(r)$ denote the minimum integer such that there exists an r-uniform, r-partite hypergraph $\mathcal{H}$ with $f(r)$ edges such that $\nu(\mathcal{H}) = 1$ and $\tau(\mathcal{H}) \geq r-1$. It is not hard to see that $f(2) = 1$ and $f(3) = 3$. In 2009, Mansour, Song, and Yuster showed that $f(4) = 6$ and $f(5) = 9$ [63]. In 2014, Abu-Khazneh and Pokrovskiy showed that $f(6) = 13$ and $f(7) \leq 22$ [3]. In particular, they found the first extremal example for Conjecture 1.1.1 which does not come from a truncated projective plane. Also in 2014, Aharoni, Barat, and

Wanless independently proved that $f(6) = 13$ and found an improved construction to show that $f(7) = 17$ [5].

1.2 Packing and Covering Triangles and K_4's

Let G and H be graphs. We will say that G is *H-free* if G has no subgraph isomorphic to H. An *H-packing* of G is a set of pairwise edge-disjoint subgraphs of G, each of which is isomorphic to H. An *H-cover* of G is a set of edges of G whose deletion creates a H-free graph. Notice that H-packings and H-covers of G correspond to matchings and covers of the hypergraph $\mathcal{H}$ on the edges of G where e is an edge of $\mathcal{H}$ if and only if the vertices of e form a copy of H in G. As an abuse of notation, we will denote the sizes of a maximum H-packing of G and a minimum H-cover of G by $\nu_H(G)$ and $\tau_H(G)$, respectively. A simple consequence of (1.1) is that

$$\nu_H(G) \leq \tau_H(G) \leq |E(H)|\nu_H(G). \tag{1.2}$$

We may also view $\tau_H(G)$ and $\nu_H(G)$ as optimal values of integer programs. Let $\mathcal{L}(G)$ be the set of all copies of H in G. A *fractional H-packing* is a function $\psi : \mathcal{L}(G) \to [0, 1]$ such that $\sum_{K\in\mathcal{L}(G):e\in E(K)} \psi(K) \leq 1$ for every edge e of G. A *fractional H-cover* is a function $\rho : E \to [0, 1]$ such that $\sum_{e\in E(K)} \rho(e) \geq 1$ for every $K \in \mathcal{L}(G)$. As we might expect,

$$\nu_H^*(G) := \max \left\{ \sum_{K\in\mathcal{L}(G)} \psi(K) \;\middle|\; \psi \text{ is a fractional } H\text{-packing of } G \right\}$$

$$\text{and } \tau_H^*(G) := \min \left\{ \sum_{e\in E} \rho(e) \;\middle|\; \rho \text{ is a fractional } H\text{-cover of } G \right\}.$$

Once again, $\nu_H^*(G) = \tau_H^*(G)$ by linear programming duality. Our motivation for studying these parameters comes from a long-standing conjecture of Tuza from 1984 in the case of $H = K_3$.

Conjecture 1.2.1 (Tuza [86])**.** *If G is a graph, then $\tau_\triangledown(G) \leq 2\nu_\triangledown(G)$.*

If true, Conjecture 1.2.1 is the best possible bound; indeed, K_4 satisfies $\tau_\triangledown(K_4) = 2\nu_\triangledown(K_4)$. While still open, Conjecture 1.2.1 is known to be true in many cases. In 1990, Tuza proved several special cases of Conjecture 1.2.1. He showed that the conjecture is true for planar graphs, graphs on n vertices with at least $\frac{7}{16}n^2$ edges, and K_5-free chordal graphs. He also showed that if G is a tripartite graph, then $\tau_\triangledown(G) \leq \frac{7}{3}\nu_\triangledown(G)$ [88]. Later, in 1995, Krivelevich extended the planar case when he proved the conjecture for graphs with no $K_{3,3}$-subdivision. Krivelevich also proved the fractional versions of Conjecture 1.2.1. Specifically, he showed that $\tau^*_\triangledown(G) \leq 2\nu_\triangledown(G)$ and $\tau_\triangledown(G) \leq 2\nu^*_\triangledown(G)$ for every graph G [58].

Haxell and Kohayakawa settled Conjecture 1.2.1 for tripartite graphs in 1998. In particular, for any tripartite graph G, they gave a simple argument to show that $\tau_\triangledown(G) \leq 2\nu_\triangledown(G)$ and a slightly more complicated argument to show that $\tau_\triangledown(G) \leq 1.956\nu_\triangledown(G)$. They also provided two tripartite graphs H_1 and H_2 such that

$$\tau_\triangledown(H_1) = \tau_\triangledown(H_2) = \frac{5}{4}\nu_\triangledown(H_1) = \frac{5}{4}\nu_\triangledown(H_2) \text{ [42]}.$$

The next year, Haxell found the first non-trivial bound for the full conjecture: She showed that $\tau_\triangledown(G) \leq (3 - \frac{3}{23})\nu_\triangledown(G)$ for every graph G [41]. To date, this remains the best known bound for all graphs. Cui, Haxell, and Ma characterized the planar graphs which are extremal for Conjecture 1.2.1. They proved that if G is planar, then $\tau_\triangledown(G) = 2\nu_\triangledown(G)$ if and only if there is a set of pairwise edge-disjoint K_4's S of G such that every triangle of G is contained in a K_4 of S [20].

More recently, Aparna Lakshmanan, Bujtás, and Tuza verified the conjecture for odd-wheel-free graphs, 4-colourable graphs, and triangle-3-colourable graphs [60]. In the same year, Haxell, Kostochka, and Thomassé published results on K_4-free graphs. They showed that if G is a K_4-free planar graph, then $\tau_\triangledown(G) \leq \frac{3}{2}\nu_\triangledown(G)$ [43]. In a second paper, they showed that for all graphs G, if $\tau^*_\triangledown(G) \geq 2\nu_\triangledown(G) - x$, then G contains $\nu_\triangledown(G) - \lfloor 10x \rfloor$ pairwise edge-disjoint copies of K_4 and a further $\lfloor 10x \rfloor$ pairwise edge-disjoint triangles. A consequence of this result is that if G is K_4-free, then $\tau^*_\triangledown(G) \leq 1.8\nu_\triangledown(G)$ [44]. Ghosh and Haxell extended the planar case of Conjecture 1.2.1 to hypergraphs: Let K^d_{d+1} denote the complete d-uniform hypergraph on $d+1$ vertices. They proved that if $\mathcal{H}$ is a d-uniform hypergraph which has a geometric realization in $\mathbb{R}^d$, then $\tau_{K^d_{d+1}}(\mathcal{H}) \leq (\lceil \frac{d}{2} \rceil + 1)\nu_{K^d_{d+1}}(\mathcal{H})$. Based on this evidence, they conjectured that if $\mathcal{H}$ is a 3-uniform hypergraph, then $\tau_{K^3_4}(\mathcal{H}) \leq 3\nu_{K^3_4}(\mathcal{H})$ [35]. In 2015, Puleo used discharging to prove Conjecture 1.2.1 for graphs with no subgraph with average degree at least seven. Some consequences of this work are that Conjecture 1.2.1 is true for toroidal graphs and graphs with no K_5-subdivision [68].

Chapuy, DeVos, McDonald, Mohar, and Scheide considered multigraphs. They extended results of Krivelevich and Haxell to show that if G is a multigraph, then $\tau^*_\triangledown(G) \leq$

$2\nu_{\triangledown}(G)$, $\tau_{\triangledown}(G) \leq 2\nu^*_{\triangledown}(G) - \sqrt{\frac{\nu^*_{\triangledown}(G)}{6}} + 1$, and $\tau_{\triangledown}(G) \leq (3 - \frac{2}{25})\nu_{\triangledown}(G)$. Furthermore, they proved that if G is a multigraph which is embedded in a surface such that every triangle is surface-separating, then $\tau_{\triangledown}(G) \leq 2\nu_{\triangledown}(G)$, which generalizes the planar case for graphs [18].

In terms of K_4's, much less is known. Lovász proved that if $\mathcal{H}$ is an r-uniform, r-partite hypergraph, then $\tau(\mathcal{H}) \leq \frac{r}{2}\nu^*(\mathcal{H})$ [61]. This result implies that if G is a 4-partite graph, then $\tau_{\boxtimes}(G) \leq 3\nu^*_{\boxtimes}(G)$. For the other fractional bound, Yuster proved that $\tau_{\boxtimes}(G) \leq 4\nu^*_{\boxtimes}(G)$ for any G [91].

1.3 Stable Matchings

In 1962, David Gale and Lloyd Shapley introduced the following problem [33]: A community consists of n men and m women. Each person ranks members of the opposite sex in terms of who they would prefer for a spouse. Can we find a set of couples such that if two people are not a couple then at least one of them prefers their partner? The goal of such a set of arrangements is to prevent affairs among unmarried couples; if there are two people who prefer each other to their respective spouses, then, in theory, there is nothing to prevent them from leaving their spouses and marrying each other. We may model this problem as a matching problem in a graph.

Let $\mathcal{H} = (V, E)$ be a hypergraph. For a vertex $v \in V$, a *preference list* L_v of v is a totally ordered list of the edges that contain v. If every vertex of $\mathcal{H}$ has a preference list we will say that $\mathcal{H} = (V, E, \mathcal{L})$ is a *hypergraph with preferences* where $\mathcal{L}$ is the set of vertex preference lists. We will use $h \leq_v e$ to denote the situation where the vertex v prefers the edge e over the edge h and $h <_v e$ to denote the situation where the vertex v strictly prefers the edge e over the edge h. A matching M is a *stable matching* of $\mathcal{H}$ if, for every edge $e \notin M$, at least one vertex of e prefers its matching edge to e. Our motivational problem now becomes the problem of finding a stable matching in a bipartite graph. Figure 1.1 gives an example of a bipartite graph with preferences and a stable matching (bold edges). In their foundational paper, Gale and Shapley proved the following well-known theorem using a very natural and elegant proposal-rejection algorithm.

Theorem 1.3.1 (Gale and Shapley [33]). *Every bipartite graph with preferences has a stable matching.*

The work of Gale and Shapley has numerous practical and theoretical consequences. Arguably the most famous application of their work is the National Resident Matching Program (NRMP). Graduating medical students will apply for acceptance into residency

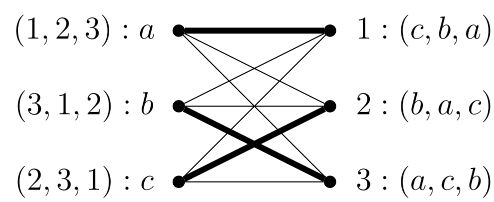

Figure 1.1: Example of a stable matching.

programs at several hospitals. The NRMP then determines where each doctor will do their residency, based on the preferences of the doctors and hospitals. Roth concluded that the success and endurance of the program is due to stable nature of the matchings it produces [71]. In recent years, there has been much research into *kidney exchange* programs (e.g. see [10, 49, 72, 75]). These programs deal with matching kidney donors with patients. From a theoretical perspective, Theorem 1.3.1 played a vital role in Galvin's proof of the Dinitz conjecture [34].

Sadly, Theorem 1.3.1 does not hold for all graphs with preferences. A cycle

$$C = \{v_0v_1, v_1v_2, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_0\}$$

of $G = (V, E, L)$ is a *preference cycle* if $v_{i-1}v_i <_{v_i} v_iv_{i+1}$ for all i modulo n. Notice that if $G = C$ and n is odd, then G does not have a stable matching. However, an odd preference cycle is essentially the only obstruction to the existence of stable matchings in graphs. A *stable partition* of G is a set of edges $S \subseteq E$, with the following properties:

- any component of (V, S) is either a cycle, a single edge, or an isolated vertex;
- each cycle component of (V, S) is a preference cycle; and
- for any $e \in E \backslash S$, there is a vertex v, incident with an edge of S, such that $v \in e$ and $e <_v f$ for any $f \in S$ with $v \in f$.

Notice that if S does not contain any cycles, then S is actually a stable matching. In the case that S contains a preference cycle component of odd length, we say that S is an *odd stable partition*. Using stable partitions, Tan was able to characterize the graphs with preferences which have a stable matching.

Theorem 1.3.2 (Tan [79])**.** *If $G = (V, E, L)$ is a graph with preferences, then G has a stable partition. Furthermore, G has a stable matching if and only if it does not have an odd stable partition.*

The proofs of Theorems 1.3.1 and 1.3.2 have some remarkable consequences:

- Each vertex of V is either matched in every stable matching of G or no stable matching [36],
- all stable matchings of G have the same size [36], and
- there are efficient algorithms to find a stable matching of G or tell us that one does not exist [33, 50].

In a more abstract setting, Knuth showed that if G is a bipartite graph with preferences, then the set of stable matchings forms a finite distributive lattice [55]. Furthermore, Blair proved a converse statement: Every finite distributive lattice is the lattice of stable matchings for some bipartite graph with preferences [16]. It is also completely reasonable to consider vertex preference lists that are not total orders. Indeed, many researchers have studied stable matching problems where the vertices have preference lists that are partially ordered (e.g. see [51, 52, 53, 70]). However, we will not stray from the safety of totally ordered preference lists.

If we venture into the world of hypergraphs with preferences, the situation turns bleak very quickly; none of the desirable properties above hold (e.g. see Section 4.1 in [77]). Furthermore, Hirschberg and Ng proved that the problem of deciding if a hypergraph with preferences has a stable matching is NP-complete [48]. However, all is not lost. As with our other matching problems, we can talk about fractional stable matchings in the hope that the fractional world might reduce some of the difficulties presented by hypergraphs.

Let $\mathcal{H} = (V, E, \mathcal{L})$ be a hypergraph with preferences. A function $\varphi : E \to [0, 1]$ is a *fractional stable matching* if it is a fractional matching and, for each edge $e \in E$, there is a vertex $u \in e$ such that

$$\sum_{e \leq_u h} \varphi(h) = 1.$$

At this point, we note that it is tempting to say that the fractional stable matchings of a fixed hypergraph with preferences are the feasible solutions to the stable matching linear program; indeed, some authors define them this way (e.g. see [1, 2, 80]). However, it is important to note that, while our fractional stable matchings certainly are feasible solutions to the corresponding linear program, the set of fractional stable matchings need not form a convex set (e.g. see Figure 3.2 in [77]).

Using a powerful topological theorem of Scarf [74], Aharoni and Fleiner were able to prove the following result.

Theorem 1.3.3 (Aharoni and Fleiner [7]). *Every hypergraph with preferences has a fractional stable matching.*

Thus, a hypergraph with preferences may not have a stable matching, but it does have a fractional stable matching.

1.4 Outline of Thesis

We begin in Chapter 2 with the necessary prerequisite material from graph theory, combinatorial topology, and linear programming.

In Chapter 3, we examine the extremal hypergraphs for Theorem 1.1.2. In [45] and [46], Haxell, Narins, and Szabó characterized the 3-uniform tripartite hypergraphs $\mathcal{H}$ which satisfy $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. Their work relies heavily on topological arguments and seems difficult to generalize. We reprove their characterization and, with the exception of Theorem 3.2.2 and Lemma 3.2.3 which still rely on topology, use only elementary methods. This represents joint work with P.E. Haxell and T. Szabó.

Our motivational pillar for Chapter 4 is Conjecture 1.2.1. In [42], Haxell and Kohayakawa proved that $\tau_{\triangledown}(G) \leq 1.956\nu_{\triangledown}(G)$ for all tripartite graphs G. We use the techniques from [42] and Theorem 3.2.2 to improve the bound to $\tau_{\triangledown}(G) \leq 1.87\nu_{\triangledown}(G)$ for all tripartite graphs G. This is joint work with P.E. Haxell. We also examine a special subclass of tripartite graphs and use a simple network flow argument to prove that $\tau_{\triangledown}(H) = \nu_{\triangledown}(H)$ for all such graphs H.

In Chapter 5, we replace triangles with copies of K_4. Recently, Yuster proved that $\tau_{K_r}(G) \leq \lfloor \frac{r^2}{4} \rfloor \nu^*_{K_r}(G)$ and, hence, $\tau_{\boxtimes}(G) \leq 4\nu^*_{\boxtimes}(G)$ for any G [91]. In Section 5.1, we give a complementary result to Yuster's for K_4's: We show that $\tau^*_{\boxtimes}(G) \leq \frac{9}{2}\nu_{\boxtimes}(G)$ for any graph G. Unlike Chapters 3 and 4 which ultimately rely on topological methods, the proof of this result yields a polynomial time approximation algorithm for finding fractional K_4-covers in graphs. In Sections 5.2 - 5.5, we give upper bounds on $\tau_{\boxtimes}(G)$ in the cases where G is 4-partite, complete, has low degeneracy, and has no $K_{3,3}$-subdivision.

Chapter 6 is concerned with stable matchings. This chapter will be a slight detour from the previous work: We are interested in the existence of stable matchings in hypergraphs rather than their size. A consequence of Theorem 1.3.2 is that every graph with preferences has a fractional stable matching where the value of every edge is in $\{0, \frac{1}{2}, 1\}$ [7, 79]. Therefore, it is natural to ask if a similar result holds for r-uniform hypergraphs. However, we show that, for all $n \in \mathbb{N}$, there is a 3-uniform hypergraph with preferences with a fractional stable matching that is unique and has denominators of size at least n.

We conclude in Chapter 7 with a short discussion of open problems and a wish list for future work.

Chapter 2

Background Check

Before we begin discussing matchings and covers in hypergraphs, we give a short review of some relevant definitions and theorems from graph theory, combinatorial topology, and linear programming. These sections may be ignored at the reader's own discretion.

Throughout this thesis, we will use the following notation:

- $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$,
- for $k \in \mathbb{N}$, $[k] = \{1, 2, \ldots, k\}$,
- $\mathbb{R}$ is the set of real numbers, and
- $\mathbb{R}_+$ is the set of non-negative real numbers.

2.1 Graph Theory

Recall that a *hypergraph* $\mathcal{H}$ is a pair (V, E) where V is a finite set, called *vertices*, and E is a set of subsets of V, called *edges*. If $\mathcal{H}$ has the property that every edge contains exactly r vertices, then we say that $\mathcal{H}$ is *r-uniform*. In particular, a 2-uniform hypergraph is a *graph*. A *multigraph* is a pair (V, E) where V is a finite set of vertices and E is a multiset of edges. If uv is an edge of a multigraph, then u and v are the *endpoints* of uv. If e and f are distinct edges with the same endpoints in a multigraph, then e and f are *parallel*. A *subhypergraph* $\bar{\mathcal{H}} = (\bar{V}, \bar{E})$ of $\mathcal{H}$ is a hypergraph such that $\bar{V} \subseteq V$ and $\bar{E} \subseteq E$. Usually, we will follow the convention of writing $V(G)$ and $E(G)$ for the set of vertices and edges of a graph G, respectively. However, for a hypergraph $\mathcal{H}$, we will often identify $\mathcal{H}$ with its edge set in order to reduce notation.

A *directed graph*, or *digraph*, is a pair (N, A) where N is a finite set of *nodes* and A is a set of ordered pairs of distinct nodes of N, called *arcs*. For $x, y \in N$, we will use $\vec{xy}$ to distinguish the arc directed from x to y from the corresponding edge, xy, in the underlying graph of D. A *capacitated directed graph*, $D = (N, A, c)$, is a directed graph $G = (N, A)$, together with a function $c : A \to \mathbb{R}_+$, where $c(\vec{e})$ is the *capacity* of the arc $\vec{e}$.

In a graph G, a vertex v is a *neighbour* of vertex u if $uv \in E$. The *neighbourhood* of u, denoted $\Gamma_G(u)$, is the set of neighbours of u. The *degree of* v in G, denoted $deg_G(v)$, is defined to be the number of edges e such that v is an endpoint of e. We will use $\Delta(G)$ to denote the maximum degree of a vertex of G. If $deg_G(u) = 0$ for some vertex u then we say that u is *isolated*. If v is not a vertex of G, then $G + v$ is the graph obtained from G by adding a vertex v and joining v to every vertex of G. If X is a subset of vertices of G, then $G[X]$ is the *subgraph of* G *induced by* X and $G\backslash X$ is the subgraph of G obtained from G by deleting the vertices of X plus any edges that contain a vertex of X. If F is a subset of edges of G, then $G\backslash F$ is the subgraph of G obtained by deleting the edges in F. The *complement graph of* G, denoted G^c, is the graph on $V(G)$ such that e is an edge of G^c if and only if e is not an edge of G.

A *cut set* is a subset $K \subseteq V(G)$ such that $G\backslash K$ has more components G. A *cut vertex* is a cut set of size one. The graph G is l*-connected* if G has at least $l + 1$ vertices and G has no cut set of size at most $l - 1$. In particular, G is *connected* if it is 1-connected. The *connectivity* of G is the maximum k for which G is k-connected. A *block* is a connected graph with no cut vertex. A *component* of G is a maximal connected subgraph of G.

If G and H are graphs, a function $\psi : V(G) \to V(H)$ is an *isomorphism* if it is a bijection and whenever $u, v \in V(G)$, we have $\psi(u)\psi(v) \in E(G)$ if and only if $uv \in E(G)$. We will also say that G and H are *isomorphic*, denoted $G \cong H$, if there is an isomorphism from G to H.

A graph $G = (V, E)$ is *bipartite* if there is a partition (A, B) of V such that every edge of E has exactly one endpoint in A and one endpoint in B. More generally, G is k*-partite* if there exists a partition $(V_1, V_2, \ldots, V_k)$ of V such that every edge of G has at most one endpoint in each part. In particular, if $k = 3$, then G is *tripartite*. Similarly, an r-uniform hypergraph $\mathcal{H}$ is r*-partite* if there is a partition $(V_1, V_2, \ldots, V_r)$ of the vertices of $\mathcal{H}$ such that every edge of $\mathcal{H}$ has exactly one endpoint in each of $V_1, V_2, \ldots, V_r$. Let $\mathcal{Z}$ be a subhypergraph of $\mathcal{H}$. For each $i \in [r]$, we will use $V_i(\mathcal{Z})$ to denote the vertices of $\mathcal{Z}$ that are contained in V_i.

A *path of length* m is a graph P with $m + 1$ distinct vertices $u_0, u_1, \ldots, u_m$ such that

$E(P) = \{u_{i-1}u_i \mid i \in [m]\}$. The edges u_0u_1 and $u_{m-1}u_m$ will be called the *end-edges* of P and the vertices u_0 and u_m will be called the *end-vertices* of P. A *cycle of length n* is a graph, denoted C_n, with n distinct vertices $v_0, v_1, \ldots, v_{n-1}$, where $n \geq 3$ such that $v_iv_{i+1} \in E(C) = \{v_iv_{i+1}$ for all i modulo $n\}$. A *directed cycle of length n* is a directed graph D with n distinct vertices $w_0, w_1, \ldots, w_{n-1}$, where $n \geq 2$ such that w_iw_{i+1} is an arc of D for all i modulo n. A well-known result shows us a very close relationship between a bipartite graph and its set of cycles.

Theorem 2.1.1. *A graph is bipartite if and only if it does not have an odd cycle.*

The graph $G = (V, E)$ is a *complete graph* or *clique* if for each $v \in V$, we have $\Gamma_G(v) = V\backslash\{v\}$. The complete graph on n vertices is denoted K_n. The complete graph on three vertices will be called a *triangle*. The k-partite graph $G = (V_1 \cup V_2 \cup \ldots \cup V_k, E)$ is a *complete k-partite graph* if for all $i \in \{1, 2, \ldots, k\}$ and all $a \in V_i$, we have $\Gamma_G(a) = V\backslash V_i$. If $|V_1| = a$ and $|V_2| = b$, the complete bipartite graph is denoted $K_{a,b}$. The *k-partite Turán graph on n vertices* $\mathcal{T}_k(n)$ is the complete k-partite graph where each vertex class has either $\lceil \frac{n}{k} \rceil$ or $\lfloor \frac{n}{k} \rfloor$ vertices.

Theorem 2.1.2 (Turán [81])**.** *Let G be a graph on n vertices which does not have a subgraph which is isomorphic to K_k for some $k \geq 2$. Then $|E(G)| \leq |E(\mathcal{T}_k(n))|$. Furthermore, $|E(G)| = |E(\mathcal{T}_k(n))|$ if and only if G is isomorphic to $\mathcal{T}_k(n)$.*

Recall that a *matching* of a hypergraph $\mathcal{H}$ is a set of pairwise disjoint edges of $\mathcal{H}$ and a *vertex cover* of $\mathcal{H}$ is subset of $V(\mathcal{H})$ which meets every edge of $\mathcal{H}$. If every vertex of $\mathcal{H}$ is is contained in exactly one edge of M, then M is a *perfect* matching. We will also say that P is a *partial cover* of $\mathcal{H}$ if there is a minimum cover C of $\mathcal{H}$ such that $P \subseteq C$. A fundamental result, due to Egerváry and Kőnig, relates the size of a maximum matching $\nu(G)$ and the size of a minimum cover $\tau(G)$ in a bipartite graph G.

Theorem 2.1.3 (Egerváry [25], Kőnig [57])**.** *If G is a bipartite graph, then $\tau(G) = \nu(G)$.*

Lemma 2.1.4 (Kőnig [56])**.** *Let $m \in \mathbb{N}$ and let G be a bipartite graph. Then, $\Delta(G) \leq m$ if and only if the edges of G can be partitioned into m pairwise disjoint matchings.*

An *independent set* in G is a subset of vertices $X \subseteq V(G)$ such that the graph $G[X]$ has no edges. The *line graph of G*, denoted $L(G)$, is the graph on E where ef is an edge of $L(G)$ if and only if the edges e and f share an endpoint in G. Notice that an independent set in $L(G)$ corresponds to a matching of G. If $\mathcal{H} = (V_1 \cup V_2 \cup V_3, F)$ is a 3-uniform, tripartite hypergraph and $S \subseteq V_1$, then the *link graph $lk_{\mathcal{H}}(S)$* of S is the bipartite multigraph with vertex classes V_2 and V_3 and edge multiset $\{e\backslash z \mid e \in F, z \in e \cap S\}$.

A graph $G = (V, E)$ is *d-degenerate* if there is an ordering $v_1, \ldots, v_n$ of V such that $\deg_{H_i}(v_i) \leq d$ for all $i \in \{1, 2, \ldots, n\}$, where $H_i = G[\{v_1, \ldots, v_i\}]$. A *tree decomposition* of G is a tree $T = (I, F)$ and an assignment of *bags* $X_i \subseteq V$ to vertices $i \in I$ such that

- For each $v \in V$, the bags containing v form a connected subgraph of T and
- if $uv \in E$, then there is a bag that contains both u and v.

The *width* of a tree decomposition is $\max_{i \in I} |X_i| - 1$. The *treewidth* of G is the minimum width of a tree decomposition of G.

A *perfect elimination order* of $G = (V, E)$ is an ordering $v_1, \ldots, v_n$ of V such that for all $i \in \{2, \ldots, n\}$, the subgraph of G induced by $\Gamma(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$ is a clique. Let $k \in \mathbb{N}$. A graph G is called a *k-tree* if G has a perfect elimination order such that $\deg_{H_i}(v_i) = k$ for all $i \in \{k + 1, \ldots, n\}$, where $H_i = G[\{v_1, \ldots, v_i\}]$. A *partial k-tree* is a subgraph of a k-tree. Clearly, partial k-trees are k-degenerate.

Theorem 2.1.5 (van Leeuwen [89])**.** *Let $k \in \mathbb{N}$. A graph has treewidth at most k if and only if it is a partial k-tree.*

A *planar embedding* of a graph is a representation (or drawing) of the graph in $\mathbb{R}^2$ so that edges intersect only at their endpoints. A graph is *planar* if it has a planar embedding. A graph H is a *subdivision* of G if H can be obtained from G by replacing each edge of G by a path of length at least one.

Theorem 2.1.6 (Kuratowski [59])**.** *A graph is planar if and only if it does not have a subgraph which is isomorphic to a subdivision of K_5 or $K_{3,3}$.*

We refer the reader to any standard graph theory textbook for more information (e.g. see [17, 22]).

2.2 Combinatorial Topology

The definitions in this section follow Matoušek [64]. The points $v_1, v_2, \ldots, v_l \in \mathbb{R}^d$ are *affinely dependent* if there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_l$, not all of them 0, such that $\sum_{i=1}^{l} \alpha_i v_i = 0$ and $\sum_{i=1}^{l} \alpha_i = 0$. Otherwise, $v_1, v_2, \ldots, v_l$ are *affinely independent*. The *convex hull* of $v_1, v_2, \ldots, v_l$ is the set of all points of the form $\sum_{i=1}^{l} \lambda_i v_i$ where $\sum_{i=1}^{l} \lambda_i = 1$ and $\lambda_i \geq 0$ for all $i \in [l]$. An *m-simplex* σ is the convex hull of a set A of $m + 1$ affinely independent points. The points in A are called the *vertices* of σ. The convex hull of a set $F \subseteq A$ is a *face* of σ.

A non-empty family $\mathcal{X}$ of simplices is a *geometric simplicial complex* if the following two conditions hold:

- Each face of any simplex $\sigma \in \mathcal{X}$ is also a simplex of $\mathcal{X}$.
- The intersection of any two simplices σ_1 and σ_2 is a face of both σ_1 and σ_2.

The vertex set $V(\mathcal{X})$ of $\mathcal{X}$ is the union of the vertex sets of all simplices of $\mathcal{X}$. A *subdivision* of $\mathcal{X}$ is a geometric simplicial complex $\mathcal{X}^\bullet$ such that

- $\bigcup_{\sigma \in \mathcal{X}} \sigma = \bigcup_{\pi \in \mathcal{X}^\bullet} \pi$, and
- every convex simplex of $\mathcal{X}^\bullet$ is contained in a convex simplex of $\mathcal{X}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be geometric simplicial complexes. A *simplicial map* is a function $g : V(\mathcal{X}) \to V(\mathcal{Y})$ such that the image $\{g(z) \mid z \in \sigma\}$ of every simplex σ of $\mathcal{X}$ is a simplex of $\mathcal{Y}$. Let $k \geq -1$. We say that $\mathcal{X}$ is *k-connected* if for all integers j such that $-1 \leq j \leq k$, for every subdivision Π of the boundary of a $(j+1)$-simplex σ, and for every simplicial map $f : V(\Pi) \to V(\mathcal{X})$, there is a subdivision $\Pi^\bullet$ of σ and a simplicial map $\bar{f} : V(\Pi^\bullet) \to V(\mathcal{X})$ which extends f. The *connectedness* of $\mathcal{X}$, denoted $conn(\mathcal{X})$, is the largest k for which $\mathcal{X}$ is k-connected. In particular, a geometric simplicial complex is -1-connected if and only if it is non-empty. We also define $conn(\emptyset) = -2$. Note that our definition of k-connectedness is slightly non-standard; it is usually defined by extending a continuous function on a j-dimensional sphere to a continuous function on a $(j+1)$-dimensional ball (e.g. see [64]).

Related to geometric simplicial complexes, an *abstract simplicial complex* Σ is a hypergraph with the property that the edge set of Σ is closed under inclusion. An edge of Σ is also called a *simplex*. As an example, the *independence complex* $\mathcal{I}(G)$ of a multigraph G is the abstract simplicial complex whose simplicies are the independent sets of G. If $S \subseteq V(\Sigma)$, the *subcomplex of* Σ *induced by* S, denoted $\Sigma|_S$, is the abstract simplicial complex whose vertex set is S and $\sigma \in \Sigma$ is a simplex of $\Sigma|_S$ if and only if the vertices of σ are contained in S.

The relationship between geometric and abstract simplicial complexes is as follows. The vertex sets of the simplices of a geometric simplicial complex form an abstract simplicial complex. Furthermore, it is known that any abstract simplicial complex can be represented, uniquely up to homeomorphism, as a geometric simplicial complex (e.g. see Munkres [67]). In other words, we may view a geometric simplicial complex and an abstract simplicial complex as different representations of the same object. Therefore, the connectedness of an abstract simplicial complex is the connectedness of its representation as a geometric simplicial complex.

Let G be a multigraph and let xy be an edge of G. The graph G *delete* xy, denoted $G \backslash xy$, is obtained from G by deleting xy. The graph G *explode* xy, denoted $G \divideontimes xy$, is obtained from G by deleting the neighbourhoods of both x and y. The connectedness of $\mathcal{I}(G)$ is related to these two graph operations. Meshulam proved a homological version of Theorem 2.2.1 [65]. The formulation stated here is from Haxell, Narins, and Szabó [45].

Theorem 2.2.1. *If G is a graph and $e \in E(G)$, then either $conn(\mathcal{I}(G)) \geq conn(\mathcal{I}(G \backslash e))$ or $conn(\mathcal{I}(G)) \geq conn(\mathcal{I}(G \divideontimes e)) + 1$.*

A *colouring* of the vertices of an abstract simplicial complex Σ is a function $c : V(\Sigma) \to X$, where X is the set of colours. If a coloured abstract simplicial complex has a simplex σ with the property that each vertex of σ has a distinct colour, then we will say that σ is *fully coloured*. The following theorem with $d = 0$ was implicit in the work of Aharoni and Haxell [8] and stated explicitly, in slightly different language, by Aharoni and Berger [6]. The version we will use here is proven in the work of Haxell, Narins, and Szabó [45].

Theorem 2.2.2. *Let Σ be an abstract simplicial complex whose vertices are coloured from a set X and let $d \geq 0$ be an integer. If, for every $S \subseteq X$, we have that $conn(\Sigma|_S) \geq |S| - d - 2$, then Σ has a fully coloured simplex of size $|X| - d$.*

The following simple lemma will be useful in Chapter 3.

Lemma 2.2.3. *Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph with vertex classes V_1, V_2, and V_3. For any subset $S \subseteq V_1$ we have $\nu(lk_{\mathcal{H}}(S)) \geq |S| - (|V_1| - \tau(\mathcal{H}))$.*

Proof: Let C be a minimum cover of $lk_{\mathcal{H}}(S)$. Then $(V_1 \backslash S) \cup C$ is a cover of $\mathcal{H}$ of size $|V_1| - |S| + \tau(lk_{\mathcal{H}}(S))$. Since $lk_{\mathcal{H}}(S)$ is bipartite, Theorem 2.1.3 tells us that

$$\tau(\mathcal{H}) \leq |V_1| - |S| + \nu(lk_{\mathcal{H}}(S)).$$

A simple rearrangement now yields the lemma. □

For a more comprehensive introduction to topology, we refer the reader to Munkres [67].

2.3 Linear Programming

For us, *linear programming* is the problem of maximizing a linear function of a finite number of real variables subject to a finite number of linear inequalities. Any linear program can be expressed in the following form:

$$\begin{array}{rrcl} \textbf{max } c^T x & & & \text{(P)} \\ \textbf{subject to: } Ax & \leq & b & \\ x & \geq & 0, & \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. This is called the *primal* problem. A *feasible solution* of (P) is a vector $x \in \mathbb{R}^n$ such that $Ax \leq b$ and $x \geq 0$. A feasible solution, x^*, is an *optimal solution* of (P) if $c^T x^* \geq c^T x$ for every feasible solution, x, of (P). Associated with (P) is another linear program:

$$\begin{array}{rrcl} \textbf{min } b^T y & & & \text{(D)} \\ \textbf{subject to: } A^T y & \geq & c & \\ y & \geq & 0. & \end{array}$$

This is the *dual* linear program. The feasible solutions of (P) have a special relationship with the feasible solutions of (D).

Lemma 2.3.1. *If $\bar{x}$ is a feasible solution to* (P) *and $\bar{y}$ is a feasible solution to* (D)*, then $c^T \bar{x} \leq b^T \bar{y}$.*

Proof: We have

$$c^T \bar{x} \leq (A^T \bar{y})^T \bar{x} = \bar{x}^T (A^T \bar{y}) = (A\bar{x})^T \bar{y} \leq b^T \bar{y},$$

where the first inequality follows from (D) and the second inequality follows from (P). □

Corollary 2.3.2. *If $\bar{x}$ is a feasible solution to* (P)*, $\bar{y}$ is a feasible solution to* (D)*, and $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is optimal for* (P) *and $\bar{y}$ is optimal for* (D)*.*

Theorem 2.3.3 (Gale, Kuhn, and Tucker [32]; von Neumann [90])**.** *If* (P) *has an optimal solution x^*, then* (D) *has an optimal solution y^*. Furthermore, $c^T x^* = b^T y^*$.*

Sometimes it is useful, and possibly necessary, to consider solutions of (P) where all the components are integers. If we restrict all the variables of (P) to take integral values, we obtain an *integer linear program*. Although they are notoriously difficult to solve to optimality [54], integer linear programs are very powerful as a modelling tool. Indeed, many combinatorial problems can be formulated as integer linear programs; the problem of finding a maximum matching of a hypergraph can be expressed as:

$$\begin{aligned} \textbf{max } e_E^T x & & & \qquad (\mathrm{P}_{MATCH}) \\ \textbf{subject to: } Mx & \leq & e_V & \\ x \in \{0,1\}^n & & & \end{aligned}$$

where M is the vertex-edge incidence matrix of the hypergraph, and e_E and e_V are the vectors of all 1's in $\mathbb{R}^E$ and $\mathbb{R}^V$, respectively.

A special case of linear programming is network flow theory. Let $D = (N, A, c)$ be a capacitated directed graph and let $s, t \in N$. An (s,t)-*flow* is a function $f : A \to \mathbb{R}_+$ satisfying

- $f(\vec{uv}) \leq c(\vec{uv})$ for all arcs $\vec{uv} \in A$, and
- $\displaystyle\sum_{u:\vec{uv}\in A} f(\vec{uv}) = \sum_{w:\vec{vw}\in A} f(\vec{vw})$ for all $v \in N \backslash \{s,t\}$.

The *value of* f is $\sum_{u:\vec{su}\in A} f(\vec{su})$. An (s,t)-*cut* in D is a set of arcs S such that $D\backslash S$ has no (s,t)-path. The value of S is $\sum_{\vec{su}\in S} f(\vec{su})$. The following fundamental result is a special case of Theorem 2.3.3.

Theorem 2.3.4 (Dantzig and Fulkerson [21], Ford and Fulkerson [28])**.** *Let $D = (N, A, c)$ be a capacitated directed graph and let $s, t \in N$. If f is a maximum valued (s,t)-flow for D and S is a minimum valued (s,t)-cut in D, then*

$$\sum_{r:\vec{rt}\in A} f(\vec{rt}) = \sum_{\vec{xy}\in S} c(\vec{xy}).$$

Finally, we need a technical lemma that will allow us to associate (s,t)-flows in D with subgraphs of the underlying simple graph of D.

Lemma 2.3.5 (Dantzig and Fulkerson [21])**.** *Let $D = (N, A, c)$ be a capacitated directed graph and let $s, t \in N$. If $c(\vec{uv})$ is a non-negative integer for every $\vec{uv} \in A$, then D has a maximum valued (s,t)-flow f such that $f(\vec{uv})$ is a non-negative integer for every $\vec{uv} \in A$.*

For more background reading, see Bertsimas and Tsitsiklis [14].

Chapter 3

3-Uniform, Tripartite Hypergraphs

Recall that an r-uniform hypergraph is *r-partite* if the vertices of $\mathcal{H}$ can be partitioned into r parts, called *vertex classes*, so that every edge of $\mathcal{H}$ contains exactly one vertex from every vertex class. We begin by recalling Ryser's conjecture.

Conjecture 1.1.1 (Ryser [73]). *If $\mathcal{H}$ is an r-uniform, r-partite hypergraph, then $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$.*

In 2001, Aharoni proved the conjecture when $r = 3$ using topological machinery.

Theorem 1.1.2 (Aharoni [4]). *If $\mathcal{H}$ is a 3-uniform, tripartite hypergraph, then $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$.*

In 2014, Haxell, Narins and Szabó characterized the 3-uniform, tripartite hypergraphs $\mathcal{H}$ which satisfy $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ [45, 46]. Our goal in this chapter is to give a new, simpler proof of the characterization of Haxell, Narins and Szabó, as described in the next subsection.

3.1 Home-base Hypergraphs

Let $\mathcal{F}$ denote the truncated projective plane of order two, i.e. the 3-uniform, tripartite hypergraph on six vertices obtained from the projective plane of order two by deleting a single point v and the three edges that contain v. We will also let $\mathcal{R}$ denote the hypergraph obtained from $\mathcal{F}$ by deleting any single edge.

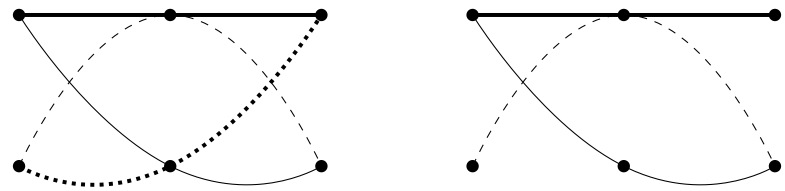

Figure 3.1: The hypergraphs $\mathcal{F}$ and $\mathcal{R}$.

Definition 3.1.1. A 3-uniform, tripartite hypergraph $\mathcal{H}$ is a *home-base hypergraph* if there exist integers $\eta, \mu \geq 0$ such that

(a) $\mathcal{H}$ contains η copies $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\eta$ of $\mathcal{F}$;

(b) $\mathcal{H}$ contains μ copies $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_\mu$ of $\mathcal{R}$;

(c) $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\eta, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_\mu$ are pairwise vertex-disjoint;

(d) $\nu(\mathcal{H}) = \eta + \mu$; and

(e) if e is an edge of $\mathcal{H}$ which is not an edge of $\bigcup_{i=1}^{\eta} \mathcal{F}_i$, then there is a $k \in [\mu]$ such that e contains at least two vertices of degree two in $\mathcal{R}_k$.

Additionally, the set $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\eta, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_\mu\}$ will be called a *spine* of $\mathcal{H}$. See Figure 3.2 for an example of a home-base hypergraph. Haxell, Narins, and Szabó showed that the extremal hypergraphs for Theorem 1.1.2 are precisely the home-base hypergraphs.

Theorem 3.1.2 (Haxell, Narins, Szabó [45, 46])**.** *If $\mathcal{H}$ is a 3-uniform, tripartite hypergraph, then $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ if and only if $\mathcal{H}$ is a home-base hypergraph.*

In [45], Haxell, Narins, and Szabó begin by showing that if $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, then the connectedness of the independence complex of $L(lk_{\mathcal{H}}(V_1))$ is as small as possible, given $\nu(lk_{\mathcal{H}}(V_1))$. They proceed to characterize the bipartite multigraphs G such that the connectedness of the independence complex of $L(G)$ is minimized, with respect to $\nu(G)$. The second paper [46] is dedicated to studying properties of home-base hypergraphs and how $lk_{\mathcal{H}}(V_1)$ lies within $\mathcal{H}$ when $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$.

A disadvantage of the techniques in [45] and [46] is the reliance on topological machinery. These methods seem to present significant challenges if we move away from the case $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. In what follows, we reprove Theorem 3.1.2. In doing so, we significantly reduce the dependence on topology, using it only to find two special matchings of an extremal hypergraph. The remainder of the proof uses only elementary, yet quite intricate and subtle, techniques. Our hope for the future is that these methods will be more easily generalized to cases where $\tau(\mathcal{H}) < 2\nu(\mathcal{H})$.

Our proof starts in Section 3.2 with a brief foray into some topology. We show that for every 3-uniform, tripartite hypergraph $\mathcal{H}$, there is a pair of matchings of $\mathcal{H}$ that interact with each other in a very special way. However, the remaining sections will be noticeably devoid of any topology. In Section 3.3, we show that if $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, then $\mathcal{H}$ contains a special sub-hypergraph $\mathcal{S}$ that is built from the two matchings in Section 3.2. In Sections 3.4 and 3.5, we show that the structure of $\mathcal{S}$ is very restricted. Finally, in Section 3.6, we show that the restricted structure of $\mathcal{S}$ yields a home-base hypergraph, which concludes the characterization.

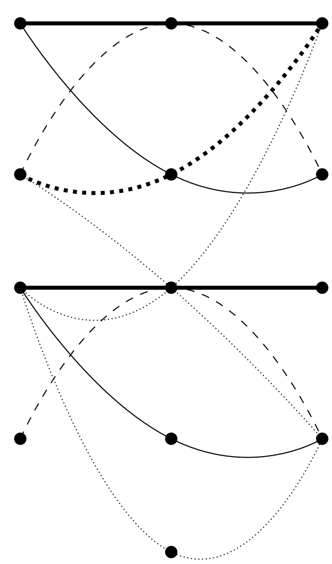

Figure 3.2: A home-base hypergraph.

3.2 Two Matchings of $\mathcal{H}$

Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph. In this section, we show that $\mathcal{H}$ has two disjoint matchings which satisfy Definition 3.2.1. These two matchings will set up the structure we need for Sections 3.3, 3.4, and 3.5.

Definition 3.2.1. Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph with vertex classes V_1, V_2, and V_3 and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be disjoint matchings of $\mathcal{H}$. Then $(\mathcal{M}_1, \mathcal{M}_2)$ is a *good pair of matchings* if the following properties hold:

(a) $|\mathcal{M}_1| + |\mathcal{M}_2| \geq \tau(\mathcal{H})$;

(b) every vertex of V_1 lies in at most one edge of $\mathcal{M}_1 \cup \mathcal{M}_2$; and

(c) every pair of distinct vertices $\{y, z\} \subseteq V_2 \cup V_3$ is contained in at most one edge of $\mathcal{M}_1 \cup \mathcal{M}_2$.

Our first step towards Theorem 3.1.2 is to prove that every 3-uniform, tripartite hypergraph has a good pair of matchings.

Theorem 3.2.2. *If $\mathcal{H}$ is a 3-uniform, tripartite hypergraph, then $\mathcal{H}$ has a good pair of matchings.*

Before we prove Theorem 3.2.2, we recall the following tools from Section 2.2.

Theorem 2.2.1. *If G is a graph and $e \in E(G)$, then either $conn(\mathcal{I}(G)) \geq conn(\mathcal{I}(G \backslash e))$ or $conn(\mathcal{I}(G)) \geq conn(\mathcal{I}(G \divideontimes e)) + 1$.*

Theorem 2.2.2. *Let Σ be an abstract simplicial complex whose vertices are coloured from a set X and let $d \geq 0$ be an integer. If, for every $S \subseteq X$, we have that $conn(\Sigma|_S) \geq |S| - d - 2$, then Σ has a fully coloured simplex of size $|X| - d$.*

Lemma 2.2.3. *Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph with vertex classes V_1, V_2, and V_3. For any subset $S \subseteq V_1$ we have $\nu(lk_{\mathcal{H}}(S)) \geq |S| - (|V_1| - \tau(\mathcal{H}))$.*

Let G be a multigraph and let Y_1 and Y_2 be two copies of a subgraph Y of $L(G)$. A *twin edge* xy is an edge such that $x \in V(Y_1)$, $y \in V(Y_2)$, and x and y are either equal or parallel as edges of G. The Y*-twin graph* is the graph obtained from the disjoint union of Y_1 and Y_2 by adding all of the twin edges. To find a good pair of matchings of $\mathcal{H}$, we show that the independence complexes of $lk_{\mathcal{H}}(S)$-twin graphs are sufficiently connected and apply Theorem 2.2.2.

Lemma 3.2.3. *Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph with vertex classes V_1, V_2, and V_3. Let Y be a subgraph of $L(lk_{\mathcal{H}}(V_1))$ and let M_Y be a matching of $lk_{\mathcal{H}}(V_1)$ such that $M_Y \subseteq V(Y)$. If H is the Y-twin graph, then $conn(\mathcal{I}(H)) \geq |M_Y| - 2$.*

Proof: We construct a sequence of graphs $H_0, H_1, H_2, \ldots, H_t$ in three phases. Set $H_0 = H$. The first phase is as follows. Let M_Y^1 and M_Y^2 be the two copies of M_Y in H. For $i \geq 1$, we choose e_i to be a twin edge xy of H_{i-1} such that either $x \in M_Y^1$ or x is parallel to an edge of M_Y^1. Notice that y is automatically either an edge of M_Y^2 or parallel to an edge of M_Y^2. We then set

$$H_i = \begin{cases} H_{i-1} \backslash e_i \text{ if } conn(\mathcal{I}(H_{i-1})) \geq conn(\mathcal{I}(H_{i-1} \backslash e_i)) \\ H_{i-1} \divideontimes e_i \text{ otherwise.} \end{cases} \tag{3.1}$$

The first phase ends when there are no such twin edges remaining. Let H_α be the graph of the sequence at the end of the first phase. We now proceed to the second phase.

Let K_1 and K_2 be the subgraphs of Y such that H_α is the disjoint union of K_1 and K_2 plus some twin edges. Since we only delete or explode twin edges in the first phase, we see that K_1 and K_2 are isomorphic. Let N_Y^1 and N_Y^2 be the subsets of M_Y^1 and M_Y^2 that remain in K_1 and K_2, respectively. For $i \geq \alpha + 1$, we choose $e_i \in E(K_1) \cap E(H_{i-1})$ such that e_i is incident to a vertex of K_1 which is also an edge of N_X^1 and we define H_i as in (3.1). As before, the second phase ends when there are no such edges to choose. Let H_β be the graph of the sequence at the end of the second phase. Finally, we move on to the third phase.

Let L be the subgraph of K_2 which remains at the end of the second phase. For $i \geq \beta + 1$, we choose $e_i \in E(L) \cap E(H_{i-1})$ such that e_i is incident to a vertex of L which is also an edge of N_Y^2 and define H_i as in (3.1). Once again, the third phase ends when there are no such edges remaining.

We now use our sequence to prove that $conn(\mathcal{I}(H))$ is sufficiently large. Let $j \in [t]$. If we delete e_j, then by the definition of our sequence, we have that

$$conn(\mathcal{I}(H_{j-1})) \geq conn(\mathcal{I}(H_{j-1} \backslash e_j)) = conn(\mathcal{I}(H_j)).$$

If we explode e_j, then $conn(\mathcal{I}(H_{j-1})) < conn(\mathcal{I}(H_{j-1} \backslash e_j))$. However, by Theorem 2.2.1, we have

$$conn(\mathcal{I}(H_{j-1})) \geq conn(\mathcal{I}(H_{j-1} \divideontimes e_j)) + 1 = conn(\mathcal{I}(H_j)) + 1.$$

If k is the number of times we explode an edge in the construction of $H_0, H_1, H_2, \ldots, H_t$, this yields

$$conn(\mathcal{I}(H)) \geq conn(\mathcal{I}(H_t)) + k \geq k - 2 \tag{3.2}$$

since $H = H_0$ and $conn(\mathcal{I}(G)) \geq -2$ for any multigraph G. Notice that if H_t has an isolated vertex, then $conn(\mathcal{I}(H_t)) = \infty$ and the result follows. So, we may assume that H_t has no isolated vertices.

We claim that H_t has no vertex of H which is also an edge of $M_Y^1 \cup M_Y^2$. To see this, suppose that $v \in V(H_t) \cap (M_Y^1 \cup M_Y^2)$. Since v is not isolated, there is an edge f incident to v. Notice that f is not a twin edge, otherwise our algorithm would have chosen it for deletion or explosion in the first phase. But, f is not a non-twin edge either since it would have been chosen for deletion or explosion in the second or third phases. Therefore, every vertex of $M_Y^1 \cup M_Y^2 \subseteq V(H)$ was removed via the explosion of some edge.

In the first phase, since $M_Y \subseteq V(Y)$ is an independent set of Y, any explosion of a twin edge will remove exactly one vertex from $M_Y^1 \subseteq V(H)$ and exactly one vertex from $M_Y^2 \subseteq V(H)$. In the second phase, since Y is a subgraph of $L(lk_{\mathcal{H}}(V_1))$ and M_Y is a matching of $lk_{\mathcal{H}}(V_1)$, every vertex of an H_i is adjacent to at most two vertices of H which are also edges of $M_Y^1 \cup M_Y^2$. Recall that we only explode edges that are incident to a vertex of $M_Y^1 \subseteq V(H)$. This means that every edge explosion removes at most two vertices of $M_Y^1 \subseteq V(H)$. Similarly, every explosion in the third phase removes at most two vertices of $M_Y^2 \subseteq V(H)$. In summary, every edge explosion removes at most two vertices of H which are edges of $M_Y^1 \cup M_Y^2$. Thus, we have $k \geq \frac{|M_Y^1|+|M_Y^2|}{2} = |M_Y|$ and, therefore by (3.2), $conn(\mathcal{I}(H)) \geq |M_Y| - 2$, as required. $\square$

We may now prove Theorem 3.2.2.

Theorem 3.2.2. *If $\mathcal{H}$ is a 3-uniform, tripartite hypergraph, then $\mathcal{H}$ has a good pair of matchings.*

Proof: Let V_1, V_2, and V_3 be the vertex classes of $\mathcal{H}$. For each $S \subseteq V_1$, let H_S be the $L(lk_{\mathcal{H}}(S))$-twin graph. Let $\Sigma = \mathcal{I}(H_{V_1})$, let $X = V_1$, and let $d = |V_1| - \tau(\mathcal{H}) \geq 0$. For each $xyz \in \mathcal{H}$, we colour both copies of vertex yz in H_{V_1} with colour $x \in V_1$. Let $S \subseteq V_1$. By Lemma 2.2.3, there is a matching M_S of $lk_{\mathcal{H}}(S)$ of size at least $|S| - (|V_1| - \tau(\mathcal{H}))$. Furthermore, we see that $M_S \subseteq V(L(lk_{\mathcal{H}}(S)))$. Since $\mathcal{I}(H_{V_1})|_S = \mathcal{I}(H_S)$, Lemma 3.2.3, gives us

$$\begin{aligned} conn(\mathcal{I}(H_{V_1})|_S) &= conn(\mathcal{I}(H_S)) \\ &\geq |M_S| - 2 \\ &\geq |S| - (|V_1| - \tau(\mathcal{H})) - 2. \end{aligned}$$

Therefore, Theorem 2.2.2 yields a fully coloured simplex σ in $\mathcal{I}(H_{V_1})$ of size $|V_1| - (|V_1| - \tau(\mathcal{H})) = \tau(\mathcal{H})$.

Let T be the set of vertices of σ and let L_1 and L_2 be the two copies of $L(lk_{\mathcal{H}}(V_1))$ in H_{V_1}. Notice that the vertices of T together with their colours correspond to edges of $\mathcal{H}$. For each $i \in \{1,2\}$, let $\mathcal{M}_i$ be the set of edges of $\mathcal{H}$ which correspond to the vertices of $T \cap V(L_i)$ together with their colours. Since σ is fully coloured, we see that $\mathcal{M}_1$ and $\mathcal{M}_2$ are disjoint, so that $|\mathcal{M}_1| + |\mathcal{M}_2| \geq \tau(\mathcal{H})$, and that each vertex of V_1 is contained in at most one edge of $\mathcal{M}_1 \cup \mathcal{M}_2$. Furthermore, since σ is a fully coloured simplex of $\mathcal{I}(H_{V_1})$, both $\mathcal{M}_1$ and $\mathcal{M}_2$ are matchings of $\mathcal{H}$. Finally the definition of the $L(lk_{\mathcal{H}}(V_1))$-twin graph ensures that every pair of distinct vertices $\{y,z\} \subseteq V_2 \cup V_3$ is contained in at most one edge of $\mathcal{M}_1 \cup \mathcal{M}_2$, as required. □

3.3 Structure of $\mathcal{H}$

Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph with vertex classes V_1, V_2, and V_3. For the remainder of this chapter, we are interested in the case when $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. In this section, we show that $\mathcal{H}$ has a special subhypergraph called a "standard family". We also establish some helpful properties of standard families.

If $yz \in E(lk_{\mathcal{H}}(V_1))$, the *completion of* yz, denoted by $\pi(yz)$, is the edge of $\mathcal{H}$ which corresponds to yz and we will say yz *completes to* x if xyz is the completion of yz. If $N \subseteq E(lk_{\mathcal{H}}(V_1))$, then $\pi(N)$ will denote the set of edges of $\mathcal{H}$ which are the completions of the edges of N. Alternatively, we will say that $\rho(xyz) = yz$ is the *heart* of xyz. Let $\mathcal{W}$ denote the hypergraph consisting of two edges e and f that intersect in V_1 but not in $V_2 \cup V_3$ along with three distinguished vertices a, b, and c such that $a = e \cap f$, $b = V_2(f)$, and $c = V_3(e)$. Let $b' \in V_2$ and $c' \in V_3$ be the remaining vertices of $\mathcal{W}$. We will say that $\mathcal{W}$ is *crossed* if there is an edge $a'b'c' \in \mathcal{H}$ where $a' \neq a$ and *uncrossed* otherwise.

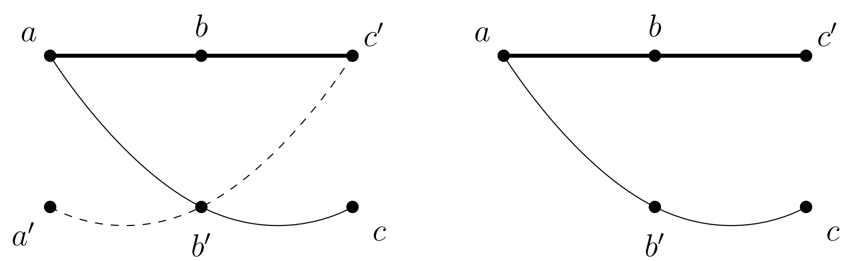

Figure 3.3: Crossed and uncrossed $\mathcal{W}$'s.

A *loose cycle* of $\mathcal{H}$ is a subgraph $\mathcal{B}$ of $\mathcal{H}$ on the vertices $v_1, v_2, \ldots, v_n$ such that $\mathcal{B} = \{v_1v_2v_3, v_3v_4v_5, v_5v_6v_7, \ldots, v_{n-1}v_nv_1\}$. However, we will only be interested in loose cycles of $\mathcal{H}$ that are aligned with the tripartition in a special way. An *aligned loose odd cycle* is a 3-uniform, tripartite hypergraph $\mathcal{U}$ with vertex classes $Y_i \subseteq V_i$ for each $i \in \{1, 2, 3\}$ where $lk_{\mathcal{U}}(Y_1)$ is a path of odd length such that the two end-edges complete to the same vertex of Y_1 and all other edges of $lk_{\mathcal{U}}(Y_1)$ complete to distinct vertices of Y_1. Notice that the completions of the two end-edges of $lk_{\mathcal{U}}(Y_1)$ form a copy of $\mathcal{W}$ (e.g. see Figure 3.4). An *aligned loose even cycle* is a 3-uniform, tripartite hypergraph $\mathcal{V}$ with vertex classes $T_i \subseteq V_i$ for each $i \in \{1, 2, 3\}$ such that $lk_{\mathcal{V}}(T_1)$ is a cycle and every edge of $lk_{\mathcal{V}}(T_1)$ completes to a distinct vertex of T_1. Notice that $lk_{\mathcal{V}}(T_1)$ is an even cycle of $lk_{\mathcal{H}}(V_1)$ (e.g. see Figure 3.4). Since we will not consider non-aligned cycles, we will drop the word "aligned" from now on.

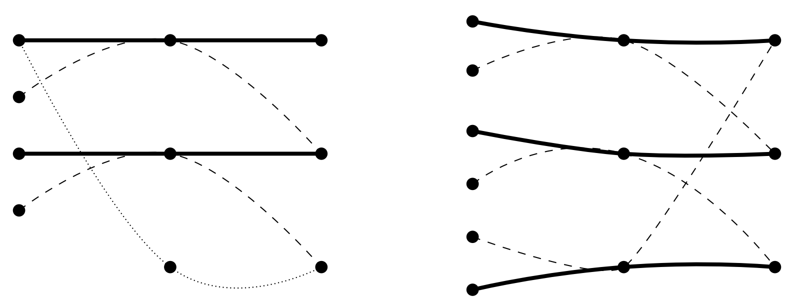

Figure 3.4: Aligned loose 5 and 6-cycles.

Definition 3.3.1. Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph with vertex classes V_1, V_2, and V_3. A *standard family* $\mathcal{S}$ is a subhypergraph of $\mathcal{H}$ such that there exist non-negative integers θ, λ, ω, l_j for each $j \in [\lambda]$, and r_k for each $k \in [\omega]$ with the following properties:

(a) $\mathcal{S}$ has θ distinct copies $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\theta$ of $\mathcal{F}$;

(b) $\mathcal{S}$ has λ distinct loose odd cycles $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_\lambda$, with lengths $2l_j + 1$ for each $j \in [\lambda]$;

(c) for each $j \in [\lambda]$, the copy of $\mathcal{W}$ formed by the two edges of $\mathcal{U}_j$ which meet in V_1 is uncrossed;

(d) $\mathcal{S}$ has ω distinct loose even cycles $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_\omega$, with lengths $2r_k \geq 4$ for each $k \in [\omega]$;

(e) $\mathcal{F}_1, \ldots, \mathcal{F}_\theta, \mathcal{U}_1, \ldots, \mathcal{U}_\lambda, \mathcal{V}_1, \ldots, \mathcal{V}_\omega$ are pairwise vertex-disjoint; and

(f) $\nu(\mathcal{H}) = \theta + \sum_{j=1}^{\lambda} l_j + \sum_{k=1}^{\omega} r_k$.

We will also say that $\mathcal{F}_1, \ldots, \mathcal{F}_\theta, \mathcal{U}_1, \ldots, \mathcal{U}_\lambda, \mathcal{V}_1, \ldots, \mathcal{V}_\omega$ are the *components* of $\mathcal{S}$, $\Phi(\mathcal{S}) = \theta + \lambda$ is the *index* of $\mathcal{S}$, and $(\theta, \lambda, \omega)$ is the *type* of $\mathcal{S}$.

We make special note of the fact that a copy of $\mathcal{R}$ is a loose 3-cycle. Therefore, the spine of a home-base hypergraph is a standard family where $\omega = 0$ and, for each $j \in [\lambda]$, $\mathcal{U}_j$ is a loose 3-cycle. The goal of this section is to prove that if $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, then $\mathcal{H}$ contains a standard family $\mathcal{S}$. Then in Sections 3.4 and 3.5 we show that $\mathcal{S}$ is, in fact, a spine of the home-base hypergraph $\mathcal{H}$.

Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a good pair of matchings of $\mathcal{H}$. Recall that $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings if $(\mathcal{M}_1, \mathcal{M}_2)$ satisfies Definition 3.2.1. For each $i \in \{1, 2\}$, Q_i will be the set of edges of $lk_{\mathcal{H}}(V_1)$ whose completions form $\mathcal{M}_i$ and an $\mathcal{M}_i$*-vertex* is a vertex of V_1 which is contained in an edge of $\mathcal{M}_i$. We will use Q to denote the subgraph of $lk_{\mathcal{H}}(V_1)$ formed by the edges of $Q_1 \cup Q_2$. Notice that since $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, we have $|\mathcal{M}_1| = |\mathcal{M}_2| = \nu(\mathcal{H})$.

3.3.1 Structure of Q

Before we find our standard family $\mathcal{S}$, we examine the graph Q. As we will see, $\mathcal{S}$ will be built around Q.

Lemma 3.3.2. *Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. If $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings of $\mathcal{H}$, then every component of Q is either an even cycle or an even path. Furthermore, every cycle component of Q has length at least four.*

Proof: We first notice that both Q_1 and Q_2 are matchings of $lk_{\mathcal{H}}(V_1)$. Therefore, every component of Q is either a path or a cycle. Furthermore, since $lk_{\mathcal{H}}(V_1)$ is bipartite, any cycle component of Q is even. So suppose, for a contradiction, that Q has a path component J of odd length $2l + 1$. Since both Q_1 and Q_2 are matchings of $lk_{\mathcal{H}}(V_1)$, we may assume without loss of generality that $|Q_1 \cap E(J)| = l + 1$ and $|Q_2 \cap E(J)| = l$. Let

$$\bar{\mathcal{M}}_2 = (\mathcal{M}_2 \backslash \pi(Q_2 \cap E(J))) \cup \pi(Q_1 \cap E(J)).$$

Certainly, $|\bar{\mathcal{M}}_2| = |\nu(\mathcal{H})| + 1$. If $\bar{\mathcal{M}}_2$ is not a matching of $\mathcal{H}$, then there are edges $e, f \in \bar{\mathcal{M}}_2$ such that $e \cap f \neq \emptyset$. Since $\bar{\mathcal{M}}_2 \subseteq \mathcal{M}_1 \cup \mathcal{M}_2$ and $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings, e and f do not meet in V_1. Furthermore, since $\mathcal{M}_1$ and $\mathcal{M}_2$ are matchings of $\mathcal{H}$, we may assume $e \in \mathcal{M}_1$ and $f \in \mathcal{M}_2$ such that $\rho(e) \in Q_1 \cap E(J)$ and $\rho(f) \in Q_2 \cap E(J)$. But now, the definition of $\bar{\mathcal{M}}_2$ says that $f \notin \bar{\mathcal{M}}_2$, which is a contradiction. Therefore, every path component of Q is even.

Now suppose that J is a cycle component of length two and let y and z be the vertices of J. Then there are edges $e \in \mathcal{M}_1$ and $f \in \mathcal{M}_2$ such that $e \cap f = \{y, z\} \subseteq V_2 \cup V_3$. However, since $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings, the vertices y and z contradict Definition 3.2.1 (c). Thus, every cycle component of Q has length at least four. $\square$

Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a good pair of matchings of $\mathcal{H}$ and let M be a matching of $lk_{\mathcal{H}}(V_1)$ of size at least $2\nu(\mathcal{H})$. Notice that such a matching M exists by Lemma 2.2.3. We will say that the triple $(M, \mathcal{M}_1, \mathcal{M}_2)$ is *optimal* if among all matchings of $lk_{\mathcal{H}}(V_1)$ of size at least $2\nu(\mathcal{H})$ and good pairs of matchings of $\mathcal{H}$, the quantity $|M \cap (Q_1 \cup Q_2)|$ is maximized. Let $i \in \{1, 2\}$ and let $e \in M$. Then the edge e is Q_i*-free* if e is disjoint from every edge of Q_i, it is Q_i*-in* if $e \in M \cap Q_i$, and it is Q_i*-touching* otherwise.

Lemma 3.3.3. *Let $(M, \mathcal{M}_1, \mathcal{M}_2)$ be an optimal triple and let $i \in \{1, 2\}$.*

(a) No edge of M is parallel to an edge of $Q_1 \cup Q_2$.

(b) If $yz \in M$ is a Q_i-free edge, then there is a Q_i-in edge $uv \in M$ such that yz and uv complete to the same vertex of V_1. Moreover, every Q_i-in edge is paired in this way with at most one Q_i-free edge of M.

(c) Every edge of Q_i that is not a Q_i-in edge intersects two distinct edges of M.

(d) Every edge of M which is either Q_i-in or Q_i-touching intersects exactly one edge of Q_i.

(e) The number of Q_i-free edges of M is equal to the number of Q_i-in edges of M.

(f) For each Q_i-in edge $uv \in M$ paired with a Q_i-free edge $yz \in M$ as in (b), the component of Q containing uv is a path with one end in $\{u, v\} \cap V_j$ and the other end in $\{y, z\} \cap V_j$ for some $j \in \{2, 3\}$.

(g) No edge $st \in M$ shares one vertex with an edge of Q_1 distinct from st and the other vertex with an edge of Q_2 distinct from st.

Proof: Suppose that $e \in M$ and e is parallel to edge $f \in Q_1 \cup Q_2$. By Lemma 3.3.2, $e \notin Q_1 \cup Q_2$. Therefore, $\bar{M} = (M \backslash \{e\}) \cup \{f\}$ is a matching of $lk_{\mathcal{H}}(V_1)$ such that $|\bar{M}| \geq 2\nu(\mathcal{H})$ and $|\bar{M} \cap (Q_1 \cup Q_2)| > |M \cap (Q_1 \cup Q_2)|$. This contradicts the optimality of $(M, \mathcal{M}_1, \mathcal{M}_2)$ and proves (a).

Let yz be a Q_i-free edge and suppose that yz completes to $a \in V_1$. Notice that a is an $\mathcal{M}_i$-vertex otherwise $\mathcal{M}_i \cup \{ayz\}$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$, which contradicts the maximality of $\mathcal{M}_i$. Let uv be the edge of Q_i which completes to a and let $\bar{\mathcal{M}_i} = \mathcal{M}_i \backslash \{auv\} \cup \{ayz\}$. Since yz is Q_i-free, $\bar{\mathcal{M}_i}$ is a maximum matching of $\mathcal{H}$.

We claim that $(\bar{\mathcal{M}_i}, \mathcal{M}_{3-i})$ is a good pair of matchings. To see this, notice that

$$|\bar{\mathcal{M}_i}| + |\mathcal{M}_{3-i}| = |\mathcal{M}_1| + |\mathcal{M}_2| = 2\nu(\mathcal{H}) = \tau(\mathcal{H})$$

and $V_1(\bar{\mathcal{M}_i} \cup \mathcal{M}_{3-i}) = V_1(\mathcal{M}_1 \cup \mathcal{M}_2)$. Therefore, every vertex of V_1 is in at most one edge of $\bar{\mathcal{M}_i} \cup \mathcal{M}_{3-i}$. Finally, since yz is Q_i-free and not parallel to an edge of Q by part (a), every pair of distinct vertices of $V_2 \cup V_3$ is contained in at most one edge of $\bar{\mathcal{M}_i} \cup \mathcal{M}_{3-i}$. Thus, $(\bar{\mathcal{M}_i}, \mathcal{M}_{3-i})$ is a good pair of matchings. However, if $uv \notin M$, then $|M \cap (\bar{Q}_i \cup Q_{3-i})| > |M \cap (Q_1 \cup Q_2)|$, which contradicts our choice of $\mathcal{M}_1$ and $\mathcal{M}_2$. Thus, uv is a Q_i-in edge which completes to a.

During the above construction of $\bar{\mathcal{M}_i}$, removing auv from $\mathcal{M}_i$ creates an odd component K in $Q \backslash uv$ by Lemma 3.3.2. Since $(\bar{\mathcal{M}_i}, \mathcal{M}_{3-i})$ is a good pair of matchings, Lemma 3.3.2 implies that yz connects to K to form an even cycle or path component of $\bar{Q}_i \cup Q_{3-i}$. Since yz and uv are disjoint edges, this means that the component of Q containing uv is a path and that $\{y, z\} \cap V(K)$ is the unique end-vertex of K which is not $\{u, v\} \cap V(K)$. Thus yz is the only Q_i-free edge in M paired with uv. This proves (b).

Let t be the number of Q_i-in edges. By part (b) we know that the number of Q_i-free edges is at most t. Hence, the number of Q_i-touching edges is at least $2\nu(\mathcal{H}) - 2t$. Since M is a matching of $lk_{\mathcal{H}}(V_1)$, each Q_i-touching edge of M intersects at least one edge of Q_i which is not a Q_i-in edge. Also, each edge of Q_i which is not a Q_i-in edge of M intersects at most two Q_i-touching edges. Since there are $\nu(\mathcal{H}) - t$ edges of $Q_i \backslash M$ and at least $2\nu(\mathcal{H}) - 2t$ Q_i-touching edges, every edge of $Q_i \backslash M$ intersects exactly two Q_i-touching edges, which proves (c). Furthermore, since every Q_i-in edge intersects exactly one edge of Q_i, namely itself, every edge which is either Q_i-in or Q_i-touching intersects exactly one edge of Q_i, which proves (d). We also notice that there are exactly $2\nu(\mathcal{H}) - 2t$ Q_i-touching edges. Since there are t Q_i-in edges, there are also t Q_i-free edges. This verifies (e).

Let $uv \in M$ be a Q_i-in edge which is paired with the Q_i-free edge yz. By the proof of (b), we know that the component J of Q which contains uv is a path with one end in $\{y, z\} \cap V_j$ for some $j \in \{2, 3\}$. If uv is not an end-edge of J, then uv intersects two distinct edges of Q_{3-i}. This contradicts part (d) applied to Q_{3-i}. Thus uv is an end-edge of J and, by Lemma 3.3.2, J has its ends in $\{u, v\} \cap V_j$ and $\{y, z\} \cap V_j$, proving (f).

Finally, suppose $st \in M$ shares one vertex with a Q_1-edge distinct from st and the other vertex with a Q_2-edge distinct from st. Since Q_1 and Q_2 are both matchings of $lk_{\mathcal{H}}(V_1)$, st is both Q_1-touching and Q_2-touching. Therefore, by part (d) and Lemma 3.3.2, there are distinct path components J_s and J_t of Q such that s is an end-vertex of J_s and t is an end-vertex of J_t. Let

$$\tilde{\mathcal{M}}_1 = (\mathcal{M}_1 \backslash \pi(Q_1 \cap E(J_s)) \cup \pi(Q_2 \cap E(J_s))$$

and

$$\tilde{\mathcal{M}}_2 = (\mathcal{M}_2 \backslash \pi(Q_2 \cap E(J_s)) \cup \pi(Q_1 \cap E(J_s)).$$

Since $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings, $\tilde{\mathcal{M}}_1$ and $\tilde{\mathcal{M}}_2$ are both maximum matchings of $\mathcal{H}$. Notice also that $\tilde{Q}_1 \cup \tilde{Q}_2$ has the same components as Q and $V_1(\tilde{\mathcal{M}}_1 \cup \tilde{\mathcal{M}}_2) = V_1(\mathcal{M}_1 \cup \mathcal{M}_2)$. Therefore, $(\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2)$ is a good pair of matchings and $(M, \tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2)$ is an optimal triple. But, now we see that st either intersects two edges of $\tilde{\mathcal{M}}_1$ or two edges of $\tilde{\mathcal{M}}_2$, which contradicts part (d) and proves (g). $\square$

Corollary 3.3.4. *Let $(M, \mathcal{M}_1, \mathcal{M}_2)$ be an optimal triple.*

(a) No cycle component of Q contains an edge of M.

(b) Every path component P of Q has exactly one end-edge in M. Furthermore, $E(P)$ is otherwise disjoint from M.

(c) The quantity $|M \cap (Q_1 \cup Q_2)|$ is equal to the number of path components of Q.

Proof: Parts (b), (e), and (f) of Lemma 3.3.3 tell us that for both $i \in \{1, 2\}$, any Q_i-in edge of M is contained in a path component of Q. Therefore no cycle component of Q contains an edge of M. Let $i \in \{1, 2\}$ and let $yz \in Q_i$ be an end-edge of a path component P of Q with end-vertex y. If yz is a Q_i-in edge of M, then parts (b) and (e) of Lemma 3.3.3 tell us that there is a Q_i-free edge uv of M such that yz and and uv complete to the same vertex of V_1. Furthermore, by Lemma 3.3.3 (f), P has an end-vertex in $\{u, v\}$, but $uv \notin E(P)$. Since $uv \in M$ and M is a matching, this means that P has exactly one end-edge in M.

If yz is not a Q_i-in edge of M, then yz intersects two distinct Q_i-touching edges of M by Lemma 3.3.3 (c). Suppose $xy \in M$ is the Q_i-touching edge incident to y. By parts (d) and (g) of Lemma 3.3.3, x is not incident to an edge of Q. Thus, xy is a Q_{3-i}-free edge of M since $yz \in Q_i$ and y is an end-vertex of P. Therefore, there is a Q_{3-i}-in edge of M, say uv, such that xy and uv complete to the same vertex of V_1, by Lemma 3.3.3 (b). By Lemma 3.3.3 (f), the component P' of Q which contains uv is a path where one of its end-vertices is y since x is not incident to an edge of Q. This means that $P' = P$, $uv \in Q_{3-i}$, and uv is an end-edge of P. Since $uv \in M$ and $yz \notin M$, P has exactly one end-edge in M. To prove the second statement in (b), suppose that P contains an edge $e \in M$ which is not an end-edge of P. Then for some $i \in \{1,2\}$, e intersects two distinct edges of Q_i which contradicts Lemma 3.3.3 (d). Thus, part (b) holds.

Finally, part (b) tells us that the number of path components of Q is at most $|M \cap (Q_1 \cup Q_2)|$. However, for both $i \in \{1,2\}$, any Q_i-in edge of M is contained in a path component of Q, by Lemma 3.3.3 (f). Thus, we have that $|M \cap (Q_1 \cup Q_2)|$ is equal to the number of path components of Q, as required. $\square$

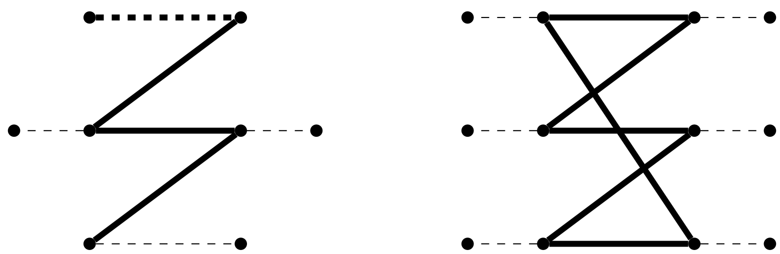

Figure 3.5: Path and cycle components of Q (bold) with M (dashed).

3.3.2 Constructing $\mathcal{S}$

Suppose that $\mathcal{H}$ has a standard family $\mathcal{S}$ of type $(\theta, \lambda, \omega)$. If $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings of $\mathcal{H}$, then $(\mathcal{M}_1, \mathcal{M}_2)$ is *associated to* $\mathcal{S}$ if $\mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathcal{S}$. Specifically, $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$ if $\mathcal{M}_1$ and $\mathcal{M}_2$ can be constructed using the following rules.

- For each $i \in [\theta]$, there is exactly one edge of $\mathcal{M}_1$ and exactly one edge of $\mathcal{M}_2$ in $\mathcal{F}_i$. Furthermore these two edges do not meet in V_1.
- For each $j \in [\lambda]$, $\mathcal{U}_j$ contains l_j edges of $\mathcal{M}_1$ and l_j edges of $\mathcal{M}_2$, where $\mathcal{U}_j$ has length $2l_j + 1$. Furthermore, the edge of $\mathcal{U}_j$ which is not in $\mathcal{M}_1 \cup \mathcal{M}_2$ is one of the two edges which share a vertex of V_1.
- For each $k \in [\omega]$, $\mathcal{V}_k$ contains r_k edges of $\mathcal{M}_1$ and r_k edges of $\mathcal{M}_2$, where $\mathcal{V}_k$ has length $2r_k \geq 4$.

Notice that for each copy of $\mathcal{F}$, there are eight ways to choose the edges in $\mathcal{M}_1$ and $\mathcal{M}_2$. Also, there are four ways to choose the $\mathcal{M}_1$ and $\mathcal{M}_2$ edges in a loose odd cycle and there are two ways to choose the $\mathcal{M}_1$ and $\mathcal{M}_2$ edges in a loose even cycle.

Definition 3.3.5. Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph with vertex classes V_1, V_2, and V_3, let $\mathcal{S}$ be a standard family of type $(\theta, \lambda, \omega)$, and let M be a matching of $lk_{\mathcal{H}}(V_1)$ of size at least $2\nu(\mathcal{H})$. We will say that M is *compatible with* $\mathcal{S}$ if the following hold.

(a) For each $i \in [\theta]$, exactly two edges of $lk_{\mathcal{F}_i}(V_1)$ are in M.

(b) For each $j \in [\lambda]$, exactly two edges of $lk_{\mathcal{U}_j}(V_1)$ are in M. Furthermore, the completions of these two edges meet in V_1.

(c) Every other edge of M intersects $lk_{\mathcal{S}}(V_1)$ in exactly one vertex.

Furthermore, if $\mathcal{P}$ is a component of $\mathcal{S}$, then we will use $M_{\mathcal{P}}$ to denote the set of edges of M which contain a vertex of $\mathcal{P}$.

We now prove the main theorem in this section.

Theorem 3.3.6. *Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ and let $(M, \mathcal{M}_1, \mathcal{M}_2)$ be an optimal triple. Then $\mathcal{H}$ contains a standard family $\mathcal{S}$ such that $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$ and M is compatible with $\mathcal{S}$.*

Proof: For each path component P of Q, Corollary 3.3.4 (b) says that P contains exactly one edge $e \in M$. Parts (b), (e), and (f) of Lemma 3.3.3 tell us there is another edge $f \in M \backslash E(P)$ such that both e and f complete to the same vertex $a \in V_1$ and P has an end-vertex in f. The completions of e and f form a copy of $\mathcal{W}$. Let $P_1, P_2, \ldots, P_t$ be the path components of Q and let $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_t$ be the corresponding copies of $\mathcal{W}$ so that each of $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_\theta$ is crossed and each of $\mathcal{W}_{\theta+1}, \mathcal{W}_{\theta+2}, \ldots, \mathcal{W}_t$ is uncrossed. Recall that $P_j \backslash M$ is a path by Corollary 3.3.4 (b) for each $j \in [t]$. Let $\{a_j, b_j, b'_j, c_j, c'_j\}$ be the vertices of $\mathcal{W}_j$ such that $a_j \in V_1$, $b_j, b'_j \in V_2$, $c_j, c'_j \in V_3$, and b_j and c_j are the end-vertices of $P_j \backslash M$ so that $b_j c'_j, b'_j c_j \in M$. For each $i \in [\theta]$, let $a'_i \in V_1$ be the vertex such that $\pi(b'_i c'_i) = a'_i b'_i c'_i$, which exists by the definition of a crossed $\mathcal{W}$ (e.g. see Figure 3.6).

Claim: For each $i \in [\theta]$, we have $a'_i b_i c_i \in \mathcal{M}_1 \cup \mathcal{M}_2$.

Proof of Claim: Suppose, for a contradiction, that $a'_i b_i c_i \notin \mathcal{M}_1 \cup \mathcal{M}_2$. Since P_i is a path component of Q, there is an $s \in \{1, 2\}$ such that the edge $b'_i c'_i \in E(lk_{\mathcal{H}}(V_1))$ is disjoint from every edge of Q_s. Therefore, a'_i is an $\mathcal{M}_s$-vertex of V_1, otherwise $\mathcal{M}_s \cup \{a'_i b'_i c'_i\}$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$. Now suppose that $a'_i xy \in \mathcal{M}_s$ and $xy \neq b_i c_i$. Without loss of generality, suppose that $c_i \notin xy$. Since $b'_i c'_i$ is disjoint from every edge of Q_s and $b'_i c'_i$ is not, nor parallel to, an edge of Q_{3-s}, we have $\bar{\mathcal{M}}_s = (\mathcal{M}_s \backslash \{a'_i xy\}) \cup \{a'_i b'_i c'_i\}$ is a maximum matching of $\mathcal{H}$ and $(\bar{\mathcal{M}}_s, \mathcal{M}_{3-s})$ is a good pair of matchings of $\mathcal{H}$. Now, if $xy \notin M$, then $(M, \bar{\mathcal{M}}_s, \mathcal{M}_{3-s})$ is an optimal triple. But since $b_i'c_i'$ is disjoint from every edge of Q_s and $c_i \notin xy$, we have $b_i'c_i \in M$ meets $b_i'c_i' \in \bar{Q}_s$ and another edge of $\bar{Q}_s$ at c_i. This contradicts Lemma 3.3.4 (d). Otherwise, if $xy \in M$, then xy is a Q_s-in edge, which implies that $a_i' = a_k$ for some $k \in [t]\backslash\{i\}$ by the construction of $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_t$. In particular, xy does not contain a vertex of P_i. However, $P_i \cup \{b_i'c_i'\}$ is an odd path component of $\bar{Q}_s \cup Q_{3-s}$, which contradicts Lemma 3.3.2 and yields the claim.

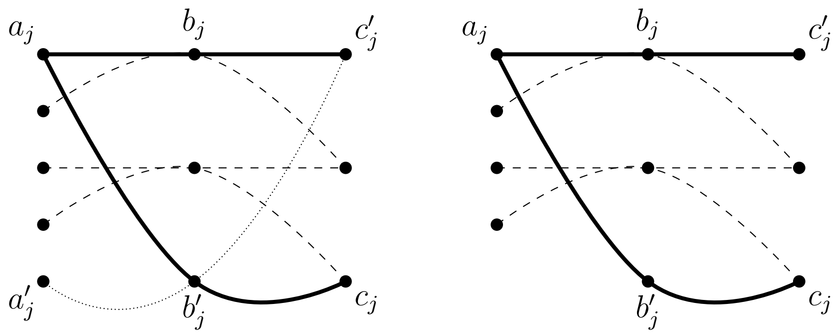

Figure 3.6: Crossed and uncrossed $\mathcal{W}$'s - Building $\mathcal{S}$.

Let $i \in [\theta]$. The claim tells us that P_i is a path of length two. Let $\mathcal{F}_i$ be the copy of $\mathcal{F}$ formed by taking $\mathcal{W}_i$ together with the edges $a_i'b_ic_i$ and $a_i'b_i'c_i'$. Let $\lambda = t - \theta$. For each $j \in [\lambda]$, let $2l_j$ be the length of $P_{\theta+j}$ and let $\mathcal{U}_j$ be the hypergraph formed by $\mathcal{W}_{\theta+j}$ together with the completions of the edges of $P_{\theta+j}$. Since $P_{\theta+j}$ is an even path and $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings, $\mathcal{U}_j$ is a loose odd cycle of length $2l_j + 1$. Let $D_1, D_2, \ldots, D_\omega$ be the cycle components of Q. For each $k \in [\omega]$, we see that the completions of the edges of D_k form a loose even cycle $\mathcal{V}_k$ of length $2r_k \geq 4$ since $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings. Finally, let $\mathcal{S}$ be the hypergraph formed by the union of the following pieces:

$$\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\theta, \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_\lambda, \mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_\omega.$$

Recall that for each $j \in [\lambda]$, $\mathcal{U}_j$ is constructed from a path component of Q whose corresponding copy of $\mathcal{W}$ is uncrossed. Therefore, to show that $\mathcal{S}$ is a standard family, it remains to show that the above pieces are pairwise vertex-disjoint and that $\nu(\mathcal{H}) = \theta + \sum_{j=1}^{\lambda} l_j + \sum_{k=1}^{\omega} r_k$.

By construction, $V_1(\mathcal{S}) = V_1(\mathcal{M}_1 \cup \mathcal{M}_2)$. The claim tells us $a_1', a_2', \ldots, a_\theta' \in V_1(\mathcal{M}_1 \cup \mathcal{M}_2)$ are pairwise distinct. Since $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings and every vertex of $V_1(\mathcal{S})$ corresponds to a unique edge of $\mathcal{M}_1 \cup \mathcal{M}_2$, no two pieces of $\mathcal{S}$ meet in V_1. Furthermore, notice that the link graph of every piece is either a component of Q or a component of Q plus an edge of M. Since every edge of M meets exactly one component by Lemma 3.3.3, no two pieces of $\mathcal{S}$ meet in $V_2 \cup V_3$. Therefore, the subhypergraphs

$$\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\theta, \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_\lambda, \mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_\omega$$

are pairwise vertex-disjoint.

Notice that, by construction, $\mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathcal{S}$. Therefore, since $\mathcal{S}$ is a subhypergraph of $\mathcal{H}$ and $\mathcal{M}_1$ is a maximum matching of $\mathcal{H}$, we have $\nu(\mathcal{H}) = \theta + \sum_{j=1}^{\lambda} l_j + \sum_{k=1}^{\omega} r_k$. Hence, $\mathcal{S}$ is a standard family, as required. We also see that $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$ since $\mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathcal{S}$.

Finally, by construction, we have that $b_j c_j', b_j' c_j \in M$ for all $j \in [t]$. Therefore, by Corollary 3.3.4 (b), each copy of $\mathcal{F}$ and each loose odd cycle contain exactly two edges of M. Furthermore, $b_j c_j'$ and $b_j' c_j$ complete to the same vertex of V_1 for every $j \in [t]$ since $\mathcal{W}_j = \{a_j b_j c_j', a_j b_j' c_j\}$. By Lemma 3.3.3 (d) and (f), every edge of $M \backslash (M \cap E(Q))$ meets exactly one vertex of Q, and hence, exactly one vertex of $lk_{\mathcal{S}}(V_1)$. Thus, M is compatible with $\mathcal{S}$, as required. □

We conclude this section with the following simple observations which will be useful in later sections. Recall from Definition 3.3.1 that the index of $\mathcal{S}$ is $\Phi(\mathcal{S}) = \theta + \lambda$.

Lemma 3.3.7. *Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ and let $\mathcal{S}$ be a standard family of type $(\theta, \lambda, \omega)$. If $(M, \mathcal{M}_1, \mathcal{M}_2)$ is an optimal triple such that $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$ and M is compatible with $\mathcal{S}$, then*

$$\Phi(\mathcal{S}) = \theta + \lambda = |M \cap (Q_1 \cup Q_2)|.$$

Proof: Since $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings, Lemma 3.3.2 tells us that every component of Q is either an even path or even cycle. Since $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$, $\mathcal{V} = \pi(E(P))$ is a loose even cycle of $\mathcal{S}$ if and only if P is an even cycle of Q. Furthermore, Corollary 3.3.4 (c) says that the number of path components of Q is $|M \cap (Q_1 \cup Q_2)|$. Since there are ω loose even cycles of $\mathcal{S}$, we have

$$\Phi(\mathcal{S}) = \theta + \lambda = |M \cap (Q_1 \cup Q_2)|,$$

as required. □

Suppose that $\mathcal{S}$ is a standard family. Recall that there are many good pairs of matchings associated to $\mathcal{S}$. We will often need to specify one which has additional properties. The next lemma ensures that any good pair of matchings we choose yields an optimal triple.

Lemma 3.3.8. *Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, let $(M, \mathcal{M}_1, \mathcal{M}_2)$ be an optimal triple and let $\mathcal{S}$ be a standard family of type $(\theta, \lambda, \omega)$ such that $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$ and M is compatible with $\mathcal{S}$. If $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is any good pair of matchings of $\mathcal{H}$ associated with $\mathcal{S}$, then $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is also an optimal triple.*

Proof: We show that $|M \cap (\bar{Q}_1 \cup \bar{Q}_2)| = |M \cap (Q_1 \cup Q_2)|$. Since $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is associated to $\mathcal{S}$, the graphs Q and $\bar{Q}_1 \cup \bar{Q}_2$ have the same cycle components. Thus, $\bar{Q}_1 \cup \bar{Q}_2$ has exactly $\theta + \lambda$ path components. For each $i \in [\theta]$, let $e_i, f_i \in \mathcal{F}_i$ be edges such that $e_i \in \bar{\mathcal{M}}_1$ and $f_i \in \bar{\mathcal{M}}_2$. Notice that $\rho(e_i)$ and $\rho(f_i)$ form a path of length two in $\bar{Q}_1 \cup \bar{Q}_2$ since $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is a good pair of matchings of $\mathcal{H}$ associated with $\mathcal{S}$. Since $M_{\mathcal{F}_i}$ is a perfect matching of $lk_{\mathcal{F}_i}(V_1)$, exactly one of $\rho(e_i)$ and $\rho(f_i)$ is an edge of M. Also, each path component of $\bar{Q}_1 \cup \bar{Q}_2$ which corresponds to a loose odd cycle of $\mathcal{S}$ contains the heart of exactly one of the two edges forming the copy of $\mathcal{W}$, as $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is a good pair of matchings associated to $\mathcal{S}$. Since the hearts of both edges of the copy of $\mathcal{W}$ are in M, we have that every path component of $\bar{Q}_1 \cup \bar{Q}_2$ contains exactly one edge of M. Thus, by Lemma 3.3.7

$$|M \cap (\bar{Q}_1 \cup \bar{Q}_2)| = \theta + \lambda = |M \cap (Q_1 \cup Q_2)|$$

and, hence, $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is an optimal triple. □

3.3.3 Minimum Covers of $\mathcal{H}$

For the remainder of this chapter, we will assume that $\mathcal{S}$ is a fixed standard family of type $(\theta, \lambda, \omega)$ obtained from an optimal triple as in Theorem 3.3.6 and M is a fixed matching of $lk_{\mathcal{H}}(V_1)$ of size $2\nu(\mathcal{H})$ which is compatible with $\mathcal{S}$.

For each $j \in [\lambda]$, let $\{a_j, b_j, b'_j, c_j, c'_j\}$ be the vertices of $\mathcal{U}_j$ such that $a_j \in V_1$, $b_j, b'_j \in V_2$, $c_j, c'_j \in V_3$, and $a_j b_j c'_j$ and $a_j b'_j c_j$ form the corresponding copy of $\mathcal{W}$. Let $A = \{a_1, a_2, \ldots, a_\lambda\}$, $B = \{b_1, b_2, \ldots, b_\lambda\}$, $B' = \{b'_1, b'_2, \ldots, b'_\lambda\}$, $C = \{c_1, c_2, \ldots, c_\lambda\}$, $C' = \{c'_1, c'_2, \ldots, c'_\lambda\}$, and for each $j \in \{1, 2, 3\}$, let $F^j = V_j(\cup_{i=1}^{\theta} \mathcal{F}_i)$. Finally, let

$$U = (V(\mathcal{S}) \cap (V_2 \cup V_3)) \backslash (B \cup B' \cup C \cup C' \cup F^2 \cup F^3).$$

This section is dedicated to finding minimum covers of $\mathcal{H}$. The following easy lemma will be used extensively throughout the remainder of this chapter.

Lemma 3.3.9. *Let $i \in [\theta]$. If $e \in \mathcal{H}$ such that $e \notin \mathcal{F}_i$, then $\mathcal{F}_i$ contains an edge disjoint from e.*

Lemma 3.3.10. *Let $i \in [\theta]$. If $e \in \mathcal{H}$ such that $e \notin \mathcal{F}_i$, then e does not contain two vertices of $V_2(\mathcal{F}_i) \cup V_3(\mathcal{F}_i)$.*

Proof: Suppose, for a contradiction, that $e \notin \mathcal{F}_i$ but e contains two vertices of $V_2(\mathcal{F}_i) \cup V_3(\mathcal{F}_i)$. Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a good pair of matchings which is associated to $\mathcal{S}$. By Definition 3.2.1 (a), both $\mathcal{M}_1$ and $\mathcal{M}_2$ are maximum matchings of $\mathcal{H}$. Since e is not in a component of $\mathcal{S}$, we see that $e \notin \mathcal{M}_1 \cup \mathcal{M}_2$. Suppose that the V_1-vertex of e is not an $\mathcal{M}_1$-vertex. Let f be the $\mathcal{M}_1$-edge of $\mathcal{F}_i$ and let g be the edge of $\mathcal{F}_i$ which, by Lemma 3.3.9, is disjoint from e. Since the V_1-vertex of e is not an $\mathcal{M}_1$-vertex and since e contains two vertices of $\mathcal{F}_i \cap (V_2 \cup V_3)$, we see that $(\mathcal{M}_1 \backslash \{f\}) \cup \{e, g\}$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$, which contradicts the maximality of $\mathcal{M}_1$. Therefore, the V_1-vertex of e is an M_1-vertex. However, the same argument applied to $\mathcal{M}_2$ tells us that the V_1-vertex of e is also an $\mathcal{M}_2$-vertex. Since $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings, this is not possible. Thus, e does not contain two vertices of $V_2(\mathcal{F}_i) \cup V_3(\mathcal{F}_i)$, as required. $\square$

We now state a helpful result which follows from the definition of loose odd cycles and Figure 3.7.

Lemma 3.3.11. *Let $\mathcal{T}$ be a standard family and let $\mathcal{U}_j$ be a loose odd cycle of $\mathcal{T}$ of length $2l+1$. For each $v \in V_1(\mathcal{U}_j) \cup V_2(\mathcal{U}_j)$ (respectively $w \in V_1(\mathcal{U}_j) \cup V_3(\mathcal{U}_j)$), there is a maximum matching $\mathcal{O}_j$ of $\mathcal{U}_j$ such that no edge of $\mathcal{O}_j$ contains v or c' (respectively w or b').*

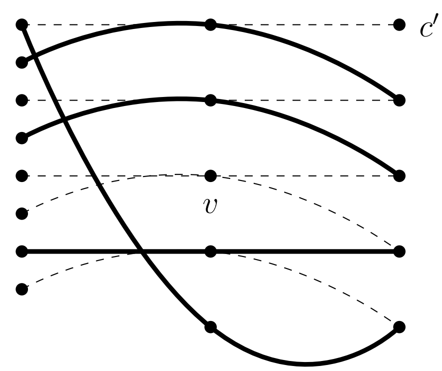

Figure 3.7: A maximum matching (bold edges) that does not contain v or c'.

The next four results tell us about the edges of $\mathcal{H}$ that are not contained in component of $\mathcal{S}$. We will see that such edges interact with $\mathcal{S}$ is a very restricted way.

Lemma 3.3.12. *Let $e \in \mathcal{H}$. If e contains a_s for some $s \in [\lambda]$, then e also contains one of the following:*

- *a vertex of $\{b_s, c_s\}$,*
- *both b_j and c_j for some $j \in [\lambda]$, or*
- *a vertex of U.*

Proof: Suppose that $e = a_s xy$ for some $s \in [\lambda]$ and suppose, for a contradiction, that e contains none of b_s, c_s, both b_j and c_j for some $j \in [\lambda]$, or a vertex of U. We construct a matching $\mathcal{M}$ of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$.

For each $i \in [\theta]$, since $a_s \in e$, and hence $e \notin \mathcal{F}_i$, there is an edge $f_i \in \mathcal{F}_i$ which is disjoint from e by Lemma 3.3.9. For each $k \in [\omega]$, since e does not contain a vertex of U and $a_s \notin V_1(\mathcal{V}_k)$, there is a maximum matching $\mathcal{N}_k$ of $\mathcal{V}_k$ such that e is disjoint from every edge of $\mathcal{N}_k$. For every $j \in [\lambda]$ such that $j \neq s$, there is a maximum matching $\mathcal{O}_j$ of $\mathcal{U}_j$ such that e is disjoint from every edge $\mathcal{O}_j$, by Lemma 3.3.11. Finally, in the loose odd cycle $\mathcal{U}_s$, since $a_s \in e$ but e does not contain b_s, c_s, or a vertex of U, there is a maximum matching $\mathcal{O}_s$ of $\mathcal{U}_s$ such that every edge of $\mathcal{O}_s$ is disjoint from e. Let

$$\mathcal{M} = \bigcup_{i=1}^{\theta} \{f_i\} \cup \bigcup_{j=1}^{\lambda} \mathcal{O}_j \cup \bigcup_{k=1}^{\omega} \mathcal{N}_k.$$

Since $\mathcal{M}$ is a union of maximum matchings of the components of $\mathcal{S}$, $\mathcal{M}$ is a matching of $\mathcal{H}$. Furthermore, since $\mathcal{S}$ is a standard family, $|\mathcal{M}| = \nu(\mathcal{H})$. However, by construction, e is disjoint from every edge of $\mathcal{M}$. Therefore, $\mathcal{M} \cup \{e\}$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$. This contradicts the maximality of $\mathcal{M}$ and yields the lemma. □

Lemma 3.3.13. *Every edge of $\mathcal{H}$ which is not an edge of $\cup_{i=1}^{\theta} \mathcal{F}_i$ and does not contain a_j for any $j \in [\lambda]$ contains a vertex of U or two vertices of $B \cup C$.*

Proof: Let $xyz \in \mathcal{H}$ be an edge which is not an edge of $\cup_{i=1}^{\theta} \mathcal{F}_i$ and does not contain a_j for any $j \in [\lambda]$. Suppose, for a contradiction, that xyz does not contain a vertex of U nor two vertices of $B \cup C$. We build a good pair of matchings $(\mathcal{M}_1, \mathcal{M}_2)$ of $\mathcal{H}$ associated to $\mathcal{S}$ as follows. For each $i \in [\theta]$, xyz does not contain two vertices from $\mathcal{F}_i \cap (V_2 \cup V_3)$ by Lemma 3.3.10. Therefore, we choose the $\mathcal{M}_1$-edge and $\mathcal{M}_2$-edges of $\mathcal{F}_i$ so that yz is disjoint from the corresponding path component of Q. For each $j \in [\lambda]$, since xyz does not contain a vertex of U nor two vertices of $B \cup C$, we choose the $\mathcal{M}_1$-edges and $\mathcal{M}_2$-edges of $\mathcal{U}_j$ so that

yz is disjoint from every edge of Q_1. This choice is possible by Lemma 3.3.11. Finally, since xyz contains no vertex of U, for each $k \in [\omega]$ we choose the $\mathcal{M}_1$-edge and $\mathcal{M}_2$-edges of $\mathcal{V}_k$ so that yz is disjoint from every edge of Q. By construction, $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings that is associated to $\mathcal{S}$. Furthermore, by Lemma 3.3.8, $(M, \mathcal{M}_1, \mathcal{M}_2)$ is an optimal triple. Notice that our choice of $\mathcal{M}_1$ and $\mathcal{M}_2$ ensures that yz is disjoint from every edge of Q_1 and, hence, yz is not equal or parallel to an edge of Q_2.

Since yz is disjoint from every edge of Q_1, we see that x is an $\mathcal{M}_1$-vertex of V_1 otherwise $\mathcal{M}_1 \cup \{xyz\}$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H})+1$. Let xuv be the edge of $\mathcal{M}_1$ which contains x. We now have three cases.

Case 1: Suppose that $xuv \in \mathcal{F}_i$ for some $i \in [\theta]$. By Lemma 3.3.9, there is an edge $e \in \mathcal{F}_i$ which is disjoint from xyz. Consider

$$\mathcal{M}_1' = (\mathcal{M}_1 \backslash \{xuv\}) \cup \{xyz, e\}.$$

Since $\mathcal{S}$ is a standard family, $e \in \mathcal{F}_i$ is disjoint from every edge of $\mathcal{M}_1 \backslash \{xuv\}$. Therefore $\mathcal{M}_1$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$, which is a contradiction.

Case 2: Suppose that $xuv \in \mathcal{U}_j$ for some $j \in [\lambda]$. Since $x \neq a_j$, $uv \notin M$ by Corollary 3.3.4 (b). Let $\bar{\mathcal{M}}_1 = (\mathcal{M}_1 \backslash \{xuv\}) \cup \{xyz\}$. Since yz is disjoint from every edge of Q_1, $\bar{\mathcal{M}}_1$ is a maximum matching of $\mathcal{H}$. Since yz is not equal or parallel to any edge of Q_2, $(\bar{\mathcal{M}}_1, \mathcal{M}_2)$ is a good pair of matchings. Also, since $uv \notin M$, $(M, \bar{\mathcal{M}}_1, \mathcal{M}_2)$ is an optimal triple. Note that this implies that $yz \notin M$.

Let P be the path component of Q which contains uv. Notice that $Q \backslash uv$ contains an odd component $\bar{P}$. Therefore, yz joins $\bar{P}$ to create an even component in $\bar{Q}_1 \cup Q_2$, by Lemma 3.3.2. Since $uv \in Q_1$ and yz are disjoint, yz joins $\bar{P}$ at the end-vertex which is also an end-vertex of P. Since P is a component of Q and $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$, the possible end-vertices of P are b_j, b_j', c_j, or c_j'. Suppose y is a vertex of P. If $y = b_j$ (or c_j), then since $yz \notin M$, we know that $yz \neq b_j c_j'$ ($b_j' c_j$). Since yz is an edge of $\bar{Q}_1$, Lemma 3.3.3 (c) applied to $(M, \bar{\mathcal{M}}_1, \mathcal{M}_2)$ and $i = 1$ says that z is in an edge of M as well. Since xyz contains no vertex of U and no two vertices of $B \cup C$, Lemma 3.3.3 (g) says that $z = c_l'$ (b_l') for some $l \in [\lambda]$. But now, the edge $b_l c_l' \in M$ ($b_l' c_l$) either meets two distinct edges of $\bar{Q}_1$ or b_l is in edge of Q_2 and c_l' is in an edge of $\bar{Q}_1$. This contradicts either Lemma 3.3.3 (d) or Lemma 3.3.3 (g). If $y = b_j'$ (or c_j'), then the edge $b_j' c_j \in M$ ($b_j c_j'$) leads to the same contradiction.

Case 3: Suppose that $xuv \in \mathcal{V}_k$ for some $k \in [\omega]$. This means that uv is in a cycle component P of Q. Since xyz does not contain a vertex of U, the edges yz and uv are disjoint. Furthermore, $uv \notin M$ by Corollary 3.3.4 (a). Let $\bar{\mathcal{M}}_1$ be as in Case 2 and let P be the cycle component of Q that contains uv. Since uv and yz are disjoint edges of $lk_{\mathcal{H}}(V_1)$, P is a cycle, and $(\bar{\mathcal{M}}_1, \mathcal{M}_2)$ is a good pair of matchings, the edge yz does not contain a vertex of P. This means that $P \backslash uv$ is a component of $\bar{Q}_1 \cup Q_2$. But $P \backslash uv$ is a path of odd length which contradicts Lemma 3.3.2.

These three cases yield the result. □

Lemma 3.3.13 implies the following corollary.

Corollary 3.3.14. *Each of $B \cup U$ and $C \cup U$ is a cover of $lk_{\mathcal{H}}(V_1 \backslash (A \cup F^1))$.*

Proof: Let yz be an edge of $lk_{\mathcal{H}}(V_1 \backslash (A \cup F^1))$. By the definition of link graphs, $\pi(yz)$ does not contain a vertex of $A \cup F^1$. In particular, $\pi(yz)$ is not an edge of $\mathcal{F}_i$ for all $i \in [\theta]$ nor does it contain the vertex a_j for any $j \in [\lambda]$. Therefore, by Lemma 3.3.13, $\pi(yz)$ and, hence, yz contain a vertex of U or two vertices of $B \cup C$. Since yz does not contain two vertices of B or two vertices of C, both cases imply that $B \cup U$ and $C \cup U$ are covers of $lk_{\mathcal{H}}(V_1 \backslash (A \cup F^1))$, as required. □

We now prove a refinement of Lemma 3.3.13.

Lemma 3.3.15. *Every edge $e \in \mathcal{H}$ which is not an edge of $\cup_{i=1}^{\theta} \mathcal{F}_i$ and does not contain a_i for any $i \in [\lambda]$ contains a vertex of U or both b_j and c_j for some $j \in [\lambda]$.*

Proof: Let $xyz \in \mathcal{H}$ be an edge which is not an edge of $\cup_{i=1}^{\theta} \mathcal{F}_i$ and does not contain a_i for any $i \in [\lambda]$. Suppose, for a contradiction, that xyz does not contain a vertex of U nor b_j and c_j for any $j \in [\lambda]$. By Lemma 3.3.13, this means that $xyz = xb_kc_l$ for some $k, l \in [\lambda]$. Notice also that $xyz \notin \mathcal{S}$.

Since $yz = b_kc_l$ for some $k, l \in [\lambda]$, there is a good pair of matchings $(\mathcal{M}_1, \mathcal{M}_2)$ associated to $\mathcal{S}$ such that b_kc_l is disjoint from every edge of Q_1. By Lemma 3.3.8, $(M, \mathcal{M}_1, \mathcal{M}_2)$ is an optimal triple. Notice that x is an $\mathcal{M}_1$-vertex of V_1 otherwise $\mathcal{M}_1 \cup \{xb_kc_l\}$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$. Let xuv be the edge of $\mathcal{M}_1$ which meets xb_kc_l where $u \in V_2$ and $v \in V_3$. Since b_kc_l does not intersect an edge of Q_1, $\bar{\mathcal{M}}_1 = (\mathcal{M}_1 \backslash \{xuv\}) \cup \{xb_kc_l\}$ is a maximum matching of $\mathcal{H}$.

Since $xuv \in \mathcal{M}_1$ and $x \neq a_i$ for any $i \in [\lambda]$, we have $uv \notin M$. Furthermore, since b_kc_l is not an edge of Q, $(\bar{\mathcal{M}}_1, \mathcal{M}_2)$ is also a good pair of matchings. Now, since $uv \notin M$, $|M \cap (\bar{Q}_1 \cup Q_2)| = |M \cap (Q_1 \cup Q_2)|$ which implies that $(M, \bar{\mathcal{M}}_1, \mathcal{M}_2)$ is an optimal triple. However, notice that $(\bar{\mathcal{M}}_1, \mathcal{M}_2)$ is not associated to $\mathcal{S}$. By Theorem 3.3.6, there is a standard family $\mathcal{S}'$ such that $(\bar{\mathcal{M}}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}'$. If we apply Corollary 3.3.14 to $\mathcal{S}'$, we see that

$$((C\backslash\{c_l\}) \cup \{v\}) \cup ((U\backslash\{u, v\}) \cup \{b_k, c_l\}) = C \cup (U\backslash\{u\}) \cup \{b_k\}$$

is a cover of $lk_{\mathcal{H}}(V_1\backslash(A \cup F^1))$. Since $C \cup U$ is a cover of $lk_{\mathcal{H}}(V_1\backslash(A \cup F^1))$, every edge which contains b_k also contains a vertex of $C \cup U$. Therefore, since $lk_{\mathcal{H}}(V_1)$ is bipartite and $b_k, u \in V_2$, we see that $C \cup U\backslash\{u\}$ is also a cover of $lk_{\mathcal{H}}(V_1\backslash(A \cup F^1))$. However, this means $A \cup C \cup F^1 \cup (U\backslash\{u\})$ is a cover of $\mathcal{H}$. Furthermore, $|A \cup C \cup F^1 \cup (U\backslash\{u\})| = |M| - 1 = 2\nu(\mathcal{H}) - 1$. This contradicts our assumption that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. □

If we combine Lemmas 3.3.12 and 3.3.15, we obtain the following theorem.

Theorem 3.3.16. *Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, let $(M, \mathcal{M}_1, \mathcal{M}_2)$ be an optimal triple, and let $\mathcal{S}$ be a standard family of type $(\theta, \lambda, \omega)$ such that $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$ and M is compatible with $\mathcal{S}$. Then every vertex subset composed as follows is a minimum cover of $\mathcal{H}$:*

- *for each $i \in [\theta]$, $V(\mathcal{F}_i) \cap V_t$ for some $t \in \{1, 2, 3\}$;*
- *for each $j \in [\lambda]$, two vertices of $\{a_j, b_j, c_j\}$; and*
- *all of U.*

In particular, for each $s \in \{1, 2, 3\}$, the following are minimum covers of $\mathcal{H}$:

- *$A \cup B \cup F^s \cup U$,*
- *$A \cup C \cup F^s \cup U$, and*
- *$B \cup C \cup F^s \cup U$.*

Proof: Let $e \in \mathcal{H}$ and let $\mathcal{C}$ be any set of vertices described above. If $e \in \mathcal{F}_i$ for some $i \in [\theta]$, then e meets $V(\mathcal{F}_i) \cap V_t$ for every $t \in \{1, 2, 3\}$ by definition. So, suppose that $e \notin \mathcal{F}_i$ for any $i \in [\theta]$. If e contains a_s for some $s \in [\lambda]$, then, by Lemma 3.3.12, e also contains one of the following:

- a vertex of $\{b_s, c_s\}$,
- both b_j and c_j for some $j \in [\lambda]$, or

- a vertex of U.

In all three cases, e contains two vertices of $\{a_j, b_j, c_j\}$ for some $j \in [\lambda]$ or a vertex of U. Hence e contains a vertex of $\mathcal{C}$. Thus, we may assume that e does not contain a_s for any $s \in [\lambda]$. By Lemma 3.3.15, e contains both b_j and c_j for some $j \in [\lambda]$ or a vertex of U and, hence, contains a vertex of $\mathcal{C}$. Therefore, $\mathcal{C}$ is indeed a vertex cover of $\mathcal{H}$.

Since $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, to show $\mathcal{C}$ is a minimum vertex cover of $\mathcal{H}$, it suffices to show that $|\mathcal{C}| = 2\nu(\mathcal{H})$. Using the definition of $\mathcal{C}$, we notice the following:

- for each $i \in [\theta]$, $\mathcal{C}$ contains two vertices of $\mathcal{F}_i$;
- for each $j \in [\lambda]$, if $\mathcal{U}_j$ has length $2l_j + 1$, then $\mathcal{C}$ contains $2l_j$ vertices of $\mathcal{U}_j$; and
- for each $k \in [\omega]$, if $\mathcal{V}_k$ has length $2r_k$, then $\mathcal{C}$ contains $2r_k$ vertices of $\mathcal{V}_k$.

Therefore, we have

$$
\begin{aligned}
|\mathcal{C}| &= 2\theta + \sum_{j=1}^{\lambda} 2l_j + \sum_{r=1}^{\omega} 2r_k \\
&= 2\left(\theta + \sum_{j=1}^{\lambda} l_j + \sum_{r=1}^{\omega} r_k\right) \\
&= 2\nu(\mathcal{H}),
\end{aligned}
$$

where the last equality follows from Definition 3.3.1 (f). Thus, $\mathcal{C}$ is a minimum vertex cover of $\mathcal{H}$, as required. □

3.4 Loose Odd Cycles of $\mathcal{S}$

Recall that $\mathcal{S}$ is a fixed standard family of type $(\theta, \lambda, \omega)$ which comes from Theorem 3.3.6 and M is a fixed matching of $lk_{\mathcal{H}}(V_1)$ of size $2\nu(\mathcal{H})$ which is compatible with $\mathcal{S}$. Our next step is to show that for each $j \in [\lambda]$, $\mathcal{U}_j$ is a loose 3-cycle of $\mathcal{S}$ and, hence, a copy of $\mathcal{R}$. We begin with the following observation about certain edges of $lk_{\mathcal{H}}(V_1)$.

Lemma 3.4.1. *If e is an edge of $lk_{\mathcal{H}}(V_1)$ such that e has one end in U and is otherwise disjoint from $B \cup C \cup U$, then e completes to a vertex of $V_1(\mathcal{S})$. Furthermore, if f is an edge of $lk_{\mathcal{S}}(V_1)$ which is incident to e at a vertex of U, then e and f complete to different vertices of $V_1(\mathcal{S})$.*

Proof: First suppose that there are edges $e = yz$ of $lk_{\mathcal{H}}(V_1)$ and $f = yv$ of $lk_{\mathcal{S}}(V_1)$ such that $\pi(yz) = xyz$, $\pi(yv) = uyv$, $y \in U$, and $z \notin B \cup C \cup U$. Suppose, for a contradiction, that either $x \notin V_1(\mathcal{S})$ or $x = u \in V_1(\mathcal{S})$. Notice that since $y \in U$, we see that $uyv \in \mathcal{U}_j$ for some $j \in [\lambda]$ or $uyv \in \mathcal{V}_k$ for some $k \in [\omega]$. We choose a good pair of matchings $(\mathcal{M}_1, \mathcal{M}_2)$ associated to $\mathcal{S}$ as follows: For each $i \in [\theta]$, we choose the $\mathcal{M}_1$-edge and $\mathcal{M}_2$-edge of $\mathcal{F}_i$ so that both edges are disjoint from $\{y, z\}$. Since $y \in U$ and $yz \notin E(lk_{\mathcal{F}_i}(V_1))$, this is possible. For all $j \in [\lambda]$, since $z \notin B \cup C \cup U$, we choose the $\mathcal{M}_1$-edges and $\mathcal{M}_2$-edges of $\mathcal{U}_j$ so that none of them contain z. For each $k \in [\omega]$, we choose the $\mathcal{M}_1$-edges and $\mathcal{M}_2$-edges by choosing a good pair of matchings of $\mathcal{V}_k$. Notice also that z is not a vertex of $\mathcal{V}_k$. By construction, $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings associated to $\mathcal{S}$. Therefore Lemma 3.3.8 says that $(M, \mathcal{M}_1, \mathcal{M}_2)$ is an optimal triple.

Since yz and yv meet in U, we may assume that $\pi(yv) = uyv \in \mathcal{M}_1$. Let $\bar{\mathcal{M}}_1 = (\mathcal{M}_1 \backslash \{uyv\}) \cup \{xyz\}$. By our choice of $(\mathcal{M}_1, \mathcal{M}_2)$, z is not a vertex of Q. Thus, since either $x \notin V_1(\mathcal{S}) = V_1(\mathcal{M}_1 \cup \mathcal{M}_2)$ or $x = u$, we have that $\bar{\mathcal{M}}_1$ is a maximum matching of $\mathcal{H}$ and $(\bar{\mathcal{M}}_1, \mathcal{M}_2)$ is a good pair of matchings. Finally, notice that $yv \notin M$ since $y \in U$, by parts (a) and (b) of Corollary 3.3.4. Therefore, $|M \cap (\bar{Q}_1 \cup Q_2)| = |M \cap (Q_1 \cup Q_2)|$, which implies that $(M, \bar{\mathcal{M}}_1, \mathcal{M}_2)$ is also an optimal triple.

However, since $uyv \in \mathcal{U}_j$ for some $j \in [\lambda]$ or $uyv \in \mathcal{V}_k$ for some $k \in [\omega]$, yv is either an edge of a path component of Q or an edge of a cycle component of Q of length at least four. Therefore, since z is not a vertex of Q, the result of changing $\mathcal{M}_1$ to $\bar{\mathcal{M}}_1$ is that either the cycle containing yv becomes a path component of $\bar{Q}_1 \cup Q_2$ or the path containing yv becomes two paths of $\bar{Q}_1 \cup Q_2$. Since the remaining components of Q remain unchanged in $\bar{Q}_1 \cup Q_2$, this means that $\bar{Q}_1 \cup Q_2$ has more path components than Q. By Corollary 3.3.4 (c), this means that $|M \cap (\bar{Q}_1 \cup Q_2)| > |M \cap (Q_1 \cup Q_2)|$, which contradicts the optimality of $(M, \mathcal{M}_1, \mathcal{M}_2)$. Thus $e = yz$ completes to a vertex of $V_1(\mathcal{S})$ and e and f complete to different vertices of $V_1(\mathcal{S})$, as required. $\square$

Definition 3.4.2. A set of edges X is *bijectively covered* by a set of vertices Y if every edge of X contains exactly one vertex of Y and every vertex of Y is contained in exactly one edge of X.

Notice that if a set of edges X is bijectively covered by a set of vertices Y, then $|X| = |Y|$. The next lemma will be used throughout the remainder of this chapter and follows from the fact that every component of a standard family is either a copy of $\mathcal{F}$ or an aligned loose cycle.

Lemma 3.4.3. *Let $\mathcal{T}$ be a standard family, let $\mathcal{K}$ be a component of $\mathcal{T}$, and let $x \in V_1(\mathcal{K})$. There is a maximum matching $\mathcal{N}$ of $\mathcal{K}$ such that no edge of $\mathcal{N}$ contains x.*

Definition 3.4.4. Let $W \subseteq U$. A *special matching for* W is a matching N of $lk_{\mathcal{H}}(V_1)$ such that N is bijectively covered by W, $(V(N)\backslash W) \cap (B \cup C \cup U) = \emptyset$, and, for each $i \in [\theta]$, $|(V(N)\backslash W) \cap V(\mathcal{F}_i)| \leq 1$.

Special matchings play an important role in our proof of Theorem 3.1.2. If N is a special matching for the U-vertices of a loose odd cycle of $\mathcal{S}$, then the completions of the edges of N behave in a controlled manner.

Lemma 3.4.5. *Let $\alpha \in [\lambda]$ be such that $\mathcal{U}_\alpha$ is a loose odd cycle of $\mathcal{S}$. If N is a special matching for $V(\mathcal{U}_\alpha) \cap U$, then every edge of N completes to a vertex of $V_1(\mathcal{U}_\alpha)$.*

Proof: Suppose, for a contradiction, that there is an edge $yz \in N$ such that $y \in V(\mathcal{U}_\alpha) \cap U$ and $\pi(yz) = xyz$ where $x \notin V_1(\mathcal{U}_\alpha)$. We will find a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$. Notice that $\mathcal{U}_\alpha$ has length at least five since $V(\mathcal{U}_\alpha) \cap U \neq \emptyset$. Since N is a special matching for $V(\mathcal{U}_\alpha) \cap U$, Lemma 3.4.1 says that there is a component $\mathcal{K}$ of $\mathcal{S}$, distinct from $\mathcal{U}_\alpha$, such that $x \in V_1(\mathcal{K})$. We choose a matching $\mathcal{M}$ of $\mathcal{H}$ as follows: For each $i \in [\theta]$, notice that $xyz \notin \mathcal{F}_i$ since $y \in U$. Therefore, there is an edge $f_i \in \mathcal{F}_i$ which is disjoint from xyz, by Lemma 3.3.9. For each $j \in [\lambda]$ such that $j \neq \alpha$, we have $y \notin V(\mathcal{U}_j)$. Furthermore, if $z \in V(\mathcal{U}_j)$, then $z \in \{b'_j, c'_j\}$ by Definition 3.4.4. Therefore Lemma 3.3.11 yields a maximum matching $\mathcal{O}_j$ of $\mathcal{U}_j$ such that no edge of $\mathcal{O}_j$ contains x or z. Thus every edge of $\mathcal{O}_j$ is disjoint from xyz. Also by Lemma 3.3.11, there is a maximum matching $\mathcal{O}_\alpha$ of $\mathcal{U}_\alpha$ such that no edge of $\mathcal{O}_\alpha$ contains $y \in V_2(\mathcal{U})$ or $c'_\alpha \in V_3(\mathcal{U}_\alpha)$ (or $y \in V_3(\mathcal{U})$ or $b'_\alpha \in V_2(\mathcal{U}_\alpha)$). Notice that since $x \notin V_1(\mathcal{U}_\alpha)$, xyz is disjoint from every edge of $\mathcal{O}_\alpha$. Finally, for each $k \in [\omega]$, since y and z are not vertices of $\mathcal{V}_k$, Lemma 3.4.3 tells us there is a maximum matching $\mathcal{N}_k$ of $\mathcal{V}_k$ such that xyz is disjoint from every edge of $\mathcal{N}_k$. Let

$$\mathcal{M} = \bigcup_{i=1}^{\theta} \{f_i\} \cup \bigcup_{j=1}^{\lambda} \mathcal{O}_j \cup \bigcup_{k=1}^{\omega} \mathcal{N}_k \cup \{xyz\}.$$

Since $\mathcal{M}\backslash\{xyz\}$ is a union of maximum matchings of components of $\mathcal{S}$, $\mathcal{M}\backslash\{xyz\}$ is a matching of $\mathcal{H}$. Furthermore, since $\mathcal{S}$ is a standard family, $|\mathcal{M}\backslash\{xyz\}| = \nu(\mathcal{H})$. However, by construction, xyz is disjoint from every edge of $\mathcal{M}\backslash\{xyz\}$. Therefore, $\mathcal{M}$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$, which is a contradiction. Thus, every edge of N completes to a vertex of $V_1(\mathcal{U}_\alpha)$, as required. □

An optimal triple $(M, \mathcal{M}_1, \mathcal{M}_2)$ is *stock* if the end-vertices of every path component of Q are in V_3 and $M \cap (Q_1 \cup Q_2) = M \cap Q_1$. Notice that there is a good pair of matchings associated to $\mathcal{S}$ which, together with M, form a stock optimal triple.

Lemma 3.4.6. *Let $(M, \mathcal{M}_1, \mathcal{M}_2)$ be a stock optimal triple so that $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$ and let $\delta \in [\lambda]$ be such that $\mathcal{U}_\delta$ has length at least five. If N is a special matching for $V(\mathcal{U}_\delta) \cap U$ such that no edge of N contains a vertex of $\mathcal{F}_i$ for any $i \in [\theta]$ and no edge of N completes to a vertex of A, then there is a good pair of matchings $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ such that $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is an optimal triple and $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S}) + 1$ path components.*

Proof: Notice that, since $(M, \mathcal{M}_1, \mathcal{M}_2)$ is a stock optimal triple, we have $a_j b_j c'_j \in \mathcal{M}_1$ for all $j \in [\lambda]$. In particular, no edge of $\mathcal{M}_1 \cup \mathcal{M}_2$ contains b'_j for any $j \in [\lambda]$. Let $\mathcal{N} = \mathcal{U}_\delta \backslash \{a_\delta b_\delta c'_\delta, a_\delta b'_\delta c_\delta\}$ and for each $j \in \{2, 3\}$, let N_j be the edges of N which meet $lk_{\mathcal{U}_\delta}(V_1)$ in V_j. Let $e \in \mathcal{N}$. If $e \in \mathcal{M}_1$ or $b_\delta \in e$ and $e \in \mathcal{M}_2$, then let m_e be the unique edge of N_3 such that $e \cap m_e \neq \emptyset$. Otherwise, let m_e be the unique edge of N_2 such that $e \cap m_e \neq \emptyset$. We define a directed graph Z on $\mathcal{N} \cup N$ as follows: Let $e \in \mathcal{N}$ and $f \in N$. There is an arc from e to f if and only if $f = m_e$ and there is an arc from f to e if and only if f completes to $V_1(e)$. By definition, the underlying graph of Z is bipartite. By Lemma 3.4.1, Z is also simple. Therefore, any directed cycle in Z has length at least four.

Recall that no edge of N completes to a vertex of A. Since N is a special matching for $V(\mathcal{U}_\delta) \cap U$, Lemma 3.4.5 says that every edge of N completes to the V_1-vertex of an edge in $\mathcal{N}$. The definition of Z now ensures that every vertex of Z has out-degree one and, hence, Z has a directed cycle $D = e_1, f_1, e_2, f_2, \ldots, e_t, f_t$ such that $e_s \in \mathcal{N}$ and $f_s \in N$ for all $s \in [t]$. For each $i \in \{1, 2\}$, let $\bar{\mathcal{M}}_i$ be the set of edges of $\mathcal{H}$ obtained from $\mathcal{M}_i$ by replacing each $e \in \mathcal{M}_i \cap V(D)$ by $\pi(m_e)$.

Claim 1: Both $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_2$ are maximum matchings of $\mathcal{H}$.

Proof of Claim 1: Let $l \in \{1, 2\}$. Since N is a special matching for $V(\mathcal{U}_\delta) \cap U$, no two edges of $\rho(\mathcal{N} \cap \mathcal{M}_l)$ are incident to the same edge of N. Therefore, we have $|\bar{\mathcal{M}}_l| = |\mathcal{M}_l| = \nu(\mathcal{H})$. Now, if $\bar{\mathcal{M}}_l$ is not a matching, then there are edges $\alpha, \beta \in \bar{\mathcal{M}}_l$ which are not disjoint. Notice that α and β are not both in $\mathcal{M}_l$ since $\mathcal{M}_l$ is a matching. So, either $\alpha \in \mathcal{M}_l \cap \bar{\mathcal{M}}_l$ and $\beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$ or $\alpha, \beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$. First, suppose that $\alpha \in \mathcal{M}_l \cap \bar{\mathcal{M}}_l$ and $\beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$. If α and β meet in V_1, then by the definitions of Z and D we have $\alpha \in V(D)$. But, if $\alpha \in V(D)$, then by the definition of $\bar{\mathcal{M}}_l$, this means that $\alpha \notin \bar{\mathcal{M}}_l$, which is a contradiction.

Now suppose that α and β meet in $V_2 \cup V_3$. Since N is a special matching for $V(\mathcal{U}_\delta) \cap U$ such that no edge of N contains a vertex of $\mathcal{F}_i$ for all $i \in [\theta]$, α and β meet in $V(\mathcal{U}_\delta) \cap U$ or $\{b'_j, c'_j\}$ for some $j \in [\lambda]$. If α and β meet in $V_2(\mathcal{U}_\delta) \cap U$, then $\beta = \pi(m_\alpha)$ since $\beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$. However, this means that $\alpha \notin \bar{\mathcal{M}}_l$, which is a contradiction. If α and β meet in $\{b'_j, c'_j\}$ for some $j \in [\lambda]$, then since $(M, \mathcal{M}_1, \mathcal{M}_2)$ is a stock optimal triple, $\alpha \cap \beta = \{c'_j\}$. However, since $(M, \mathcal{M}_1, \mathcal{M}_2)$ is a stock optimal triple, the definition of Z tells us that

$\alpha = a_j b_j c'_j \in \bar{\mathcal{M}}_1 \cap \mathcal{M}_1$ and $\rho(\beta) \in N_2$. This means that $\beta \in \bar{\mathcal{M}}_2$. However, since $\alpha \in \bar{\mathcal{M}}_1$, this is a contradiction. Therefore, we have $\alpha, \beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$.

In this case, α and β meet in V_1 since $\rho(\alpha), \rho(\beta) \in N$ and N is a matching of $lk_{\mathcal{H}}(V_1)$. Since $\alpha, \beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$, there are edges $g, h \in \mathcal{M}_l \cap V(D)$ such that $\alpha = \pi(m_g)$ and $\beta = \pi(m_h)$. However, since α and β meet in V_1, this means that m_g and m_h have the same out-neighbour in Z and, hence, are not both vertices of the directed cycle D. Thus $\mathcal{M}_l$ is a maximum matching of $\mathcal{H}$.

Claim 2: The pair $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is a good pair of matchings of $\mathcal{H}$.

Proof of Claim 2: First, we notice that $V_1(\bar{\mathcal{M}}_1 \backslash \mathcal{M}_1 \cup \bar{\mathcal{M}}_2 \backslash \mathcal{M}_2) = V_1(\mathcal{N} \cap V(D))$. Since $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings, Claim 1 and the definitions of $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_2$ tell us that $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_2$ are disjoint matchings of $\mathcal{H}$ such that $|\bar{\mathcal{M}}_1| + |\bar{\mathcal{M}}_2| = 2\nu(\mathcal{H}) = \tau(\mathcal{H})$ and every vertex of V_1 is contained in at most one edge of $\bar{\mathcal{M}}_1 \cup \bar{\mathcal{M}}_2$. Finally, since N is a special matching for $V(\mathcal{U}_\delta) \cap U$, no edge of N is parallel to an edge of Q. This means that every pair of vertices of $V_2 \cup V_3$ is contained in at most one edge of $\bar{\mathcal{M}}_1 \cup \bar{\mathcal{M}}_2$. Therefore, $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is a good pair of matchings.

Now, since $b_j, c'_j \notin U$ for all $j \in [\lambda]$, we have $a_j b_j c'_j \in \bar{\mathcal{M}}_1$ for all $j \in [\lambda]$. Furthermore, no edge of N contains a vertex of $\mathcal{F}_i$ for all $i \in [\theta]$. Therefore, we have $|M \cap (\bar{Q}_1 \cup \bar{Q}_2)| \geq |M \cap (Q_1 \cup Q_2)|$ which implies that $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is an optimal triple. It remains to show that $\bar{Q}_1 \cup \bar{Q}_2$ has more components than Q.

Claim 3: For each $j \in [\lambda]$ such that $j \neq \delta$, if there is an edge of $N \cap V(D)$ which contains a vertex of $\mathcal{U}_j$, then the vertex is b'_j.

Proof of Claim 3: Let $j \in [\lambda]$ such that $j \neq \delta$ and let $e \in N \cap V(D)$. Since N is a special matching for $V(\mathcal{U}_\delta) \cap U$, the only possible vertices of $\mathcal{U}_j$ which are contained in an edge of N are b'_j and c'_j. Suppose, for a contradiction, that $c'_j \in e$ so that $e \in N_2$ and $\pi(e) \in \bar{\mathcal{M}}_2$. By the definitions of $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_2$, $(\bar{\mathcal{M}}_1 \cup \bar{\mathcal{M}}_2) \cap \mathcal{U}_j = (\mathcal{M}_1 \cup \mathcal{M}_2) \cap \mathcal{U}_j$. Since $(M, \mathcal{M}_1, \mathcal{M}_2)$ is a stock optimal triple, $b_j c'_j \in Q_1 \cap \bar{Q}_1$ and b_j is contained in an edge of $Q_2 \cap \bar{Q}_2$. This means that $b_j c'_j$ is a $\bar{Q}_2$-touching edge of M which meets two distinct edges of $\bar{Q}_2$. However, this contradicts Lemma 3.3.3 (d). Thus, if there is an edge of N which contains a vertex of $\mathcal{U}_j$, then the vertex is b'_j.

Claim 3 tells us that, since $(M, \mathcal{M}_1, \mathcal{M}_2)$ is a stock optimal triple, every path component of Q which does not correspond to $\mathcal{U}_\delta$ is also a path component of $\bar{Q}_1 \cup \bar{Q}_2$. Since $(V(N) \backslash V(\mathcal{U}_\delta)) \cap U = \emptyset$, the cycle components of Q are also cycle components of $\bar{Q}_1 \cup \bar{Q}_2$.

Thus, $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S})-1$ path components. We show there are at least two more path components of $\bar{Q}_1 \cup \bar{Q}_2$. Suppose that $uv \in N \cap V(D)$ such that $c'_\delta \notin \{u, v\}$. Since $(V(N)\backslash V(\mathcal{U}_\delta)) \cap (B \cap C \cap U) = \emptyset$ and $(M, \mathcal{M}_1, \mathcal{M}_2)$ is a stock optimal triple, Claim 3 says that the edge uv contains a vertex of degree one in $\bar{Q}_1 \cup \bar{Q}_2$. Furthermore, suppose that $yz \in N \cap V(D)$ such that $c'_\delta \in \{y, z\}$. Then the vertex b_δ has degree one in $\bar{Q}_1 \cup \bar{Q}_2$, otherwise $b_\delta c'_\delta \in M \cap \bar{Q}_1$ is a $\bar{Q}_2$-touching edge which meets two distinct $\bar{Q}_2$-edges of $\bar{Q}_1 \cup \bar{Q}_2$, which contradicts Lemma 3.3.3 (d). Therefore, if D has length at least six or no edge of $N \cap V(D)$ contains c'_δ, then there are at least three additional vertices of degree one in $\bar{Q}_1 \cup \bar{Q}_2$. Since $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is a good pair of matchings, Lemma 3.3.2 says that these three vertices are the end-vertices of at least two path components. This means that $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S})+1$ path components. So, we may assume that the directed cycle D has length exactly four and c'_δ is contained in an edge of $N \cap V(D)$.

Suppose that the vertices of D are e_1, f_1, e_2, and f_2 such that $e_1, e_2 \in \mathcal{N}$, $f_1, f_2 \in N$, and $c'_\delta \in f_1$. From above, we know that b_δ has degree one in $\bar{Q}_1 \cup \bar{Q}_2$. This means that $b_\delta \in e_2$ and, therefore, $f_2 = m_{e_2}$ (e.g. see Figure 3.8). Since no edge of N contains a vertex of $B \cup C$, c_δ has degree at most one in $\bar{Q}_1 \cup \bar{Q}_2$. First suppose that c_δ has degree one in $\bar{Q}_1 \cup \bar{Q}_2$. Since no edge of N contains a vertex of $\mathcal{F}_i$ for all $i \in [\theta]$, Claim 3 implies that f_2 contains a vertex of degree one, say v, in $\bar{Q}_1 \cup \bar{Q}_2$. Now, b_δ, c_δ, and v all have degree one in $\bar{Q}_1 \cup \bar{Q}_2$. As above, this means that $\bar{Q}_1 \cup \bar{Q}_2$ has more path components than Q, as required. So, we may assume that c_δ has degree zero in $\bar{Q}_1 \cup \bar{Q}_2$.

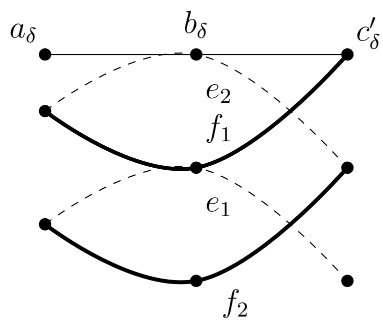

Figure 3.8: A new good pair of matchings: D has length four.

If $b'_\delta \notin f_2$, let $\mathcal{M}^* = \bar{\mathcal{M}}_2 \cup \{a_\delta b'_\delta c_\delta\}$. By Claim 1, $\bar{\mathcal{M}}_2$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H})$. Since $(M, \mathcal{M}_1, \mathcal{M}_2)$ is a stock optimal triple, no edge of $\bar{\mathcal{M}}_2$ contains a_δ. Furthermore, since $b'_\delta \notin f_2$, no edge of $\bar{\mathcal{M}}_2$ contains b'_δ either. Therefore, since the degree of c_δ in $\bar{Q}_1 \cup \bar{Q}_2$ is zero, $\mathcal{M}^*$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H})+1$, which contradicts the maximality of $\bar{\mathcal{M}}_2$. So, we suppose that $b'_\delta \in f_2$ (e.g. see Figure 3.9).

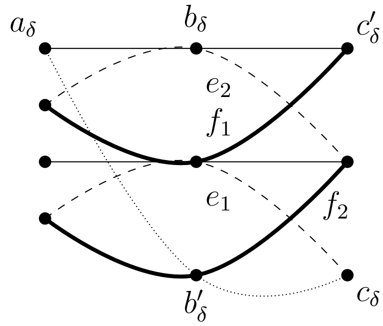

Figure 3.9: A new good pair of matchings: D has length four and $b'_\delta \in f_2$.

Since $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is an optimal triple, Theorem 3.3.6 gives us a standard family $\bar{\mathcal{S}}$ such that $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is associated to $\bar{\mathcal{S}}$ and M is compatible with $\bar{\mathcal{S}}$. Furthermore, the components of $\bar{\mathcal{S}}$ are

$$\mathcal{F}_1, \ldots, \mathcal{F}_\theta, \mathcal{U}_1, \ldots, \mathcal{U}_{\delta-1}, \bar{\mathcal{U}}_\delta, \mathcal{U}_{\delta+1}, \ldots, \mathcal{U}_\lambda, \mathcal{V}_1, \ldots, \mathcal{V}_\omega$$

where $\bar{\mathcal{U}}_\delta$ is the new loose odd cycle corresponding to $\mathcal{U}_\delta$. Specifically,

$$\bar{\mathcal{U}}_\delta = (\mathcal{U}_\delta \backslash \{e_1, e_2\}) \cup \{\pi(f_1), \pi(f_2)\}.$$

Therefore, by Theorem 3.3.16,

$$\mathcal{C} = A \cup (C \backslash \{c_\delta\}) \cup F^2 \cup U \cup \{c'_\delta\}$$

is a minimum cover of $\mathcal{H}$. Consider the partial cover $\mathcal{C} \backslash \{c'_\delta\}$. Since $\mathcal{C}$ is a minimum cover of $\mathcal{H}$, there is an edge $\alpha \in \mathcal{H} \backslash (\mathcal{C} \backslash \{c'_\delta\})$ such that $c'_\delta \in \alpha$. Note that $\alpha \notin \mathcal{F}_i$ for all $i \in [\theta]$. We also note that our choice of $\mathcal{C}$ ensures that $a_j \notin \alpha$ for all $j \in [\lambda]$. Therefore since $U \subseteq \mathcal{C}$, Lemma 3.3.15 says that $\{b'_\delta, c'_\delta\} \subseteq \alpha$. However, this means that in $\mathcal{S}$, the copy of $\mathcal{W}$ which corresponds to $\mathcal{U}_\delta$ is crossed; this contradicts Definition 3.3.1 (c) applied to $\mathcal{S}$. Thus, $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S}) + 1$ path components, as required. $\square$

We are now able to show that all loose odd cycles of $\mathcal{S}$ have length exactly three.

Theorem 3.4.7. *Every loose odd cycle of $\mathcal{S}$ has length exactly three.*

Proof: Suppose, for a contradiction, there is an $r \in [\lambda]$ such that $\mathcal{U}_r$ is a loose $(2l+1)$-cycle where $l \geq 2$. Let $(M, \mathcal{M}_1, \mathcal{M}_2)$ be a stock optimal triple such that $(\mathcal{M}_1, \mathcal{M}_2)$ is associated to $\mathcal{S}$ and M is compatible with $\mathcal{S}$. We first consider the partial cover of $\mathcal{H}$ given by $T_2 = A \cup C \cup F^3 \cup U \backslash (V_2(\mathcal{U}_r) \cap U)$. Since $|V_2(\mathcal{U}_r) \cap U| = l - 1$ and since $A \cup C \cup F^3 \cup U$ is a minimum cover of $\mathcal{H}$, by Theorem 3.3.16, every cover of $\mathcal{H} \backslash T_2$ has size at least $l - 1$. By Lemma 2.2.3, $lk_{\mathcal{H} \backslash T_2}(V_1 \backslash A)$ has a matching N_2 of size $l - 1$. Furthermore, every edge of N_2 has exactly one end in $V_2(\mathcal{U}_r) \cap U$, otherwise, $A \cup C \cup F^3 \cup U$ is not a cover of $\mathcal{H}$. Similarly, we use the partial cover $T_3 = A \cup B \cup F^2 \cup U \backslash (V_3(\mathcal{U}_r) \cap U)$, to find a matching N_3 of $lk_{\mathcal{H} \backslash T_3}(V_1 \backslash A)$ of size $l - 1$ such that every edge of N_3 has exactly one end in $V_3(\mathcal{U}_r) \cap U$.

Let $N = N_2 \cup N_3$ and let $W = (V_2(\mathcal{U}_r) \cup V_3(\mathcal{U}_r)) \cap U$. Since $lk_{\mathcal{H}}(V_1)$ is bipartite, our choices of partial covers imply that N is a matching of $lk_{\mathcal{H}}(V_1)$ and $(V(N) \backslash W) \cap (B \cup C \cup U) = \emptyset$. As we noted above, N is bijectively covered by W. Since no edge of N_2 has an end in F^3 and no edge of N_3 has an end in F^2, $(V(N) \backslash W) \cap V(\mathcal{F}_i) = \emptyset$ for every $i \in [\theta]$. Therefore, N is a special matching for W. Also notice that our choices for T_2 and T_3 ensure that no edge of N completes to a vertex of A. By Lemma 3.4.6, there is an optimal triple $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ such that $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S}) + 1$ path components. But then, by Corollary 3.3.4 (c) and Lemma 3.3.7, we have

$$|M \cap (\bar{Q}_1 \cup \bar{Q}_2)| \geq \Phi(\mathcal{S}) + 1 > |M \cap (Q_1 \cup Q_2)|,$$

which contradicts the optimality of $(M, \mathcal{M}_1, \mathcal{M}_2)$. Hence, $\mathcal{U}_r$ is a loose 3-cycle of $\mathcal{S}$, as required. □

3.5 Loose Even Cycles of $\mathcal{S}$

Recall that $\mathcal{S}$ is a fixed standard family of type $(\theta, \lambda, \omega)$ which comes from Theorem 3.3.6 and M is a fixed matching of $lk_{\mathcal{H}}(V_1)$ which is compatible with $\mathcal{S}$. We also know, by Theorem 3.4.7, that $\mathcal{U}_j$ is a loose 3-cycle for each $j \in [\lambda]$. Our goal in this section is to show that $\mathcal{S}$ has no loose even cycles; that is, we show that $\omega = 0$.

Definition 3.5.1. Suppose that $\mathcal{L}$ is a loose even cycle of $\mathcal{S}$ of length $2l$. A set $L = L_2 \cup L_3$ of edges of $lk_{\mathcal{H}}(V_1)$ of size $2l$ is a *brush for* $\mathcal{L}$ if the following three conditions hold.

(a) For each $i \in \{2, 3\}$, L_i is the set of edges of L which contain a vertex of $V_i(\mathcal{L})$.

(b) The set L is bijectively covered by $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$, as in Definition 3.4.2.

(c) For each $i \in \{2, 3\}$, if $e \in L_i$ and $e' \in L_{5-i}$, then $\nu(\mathcal{S} \backslash (\mathcal{L} \cup \pi(e) \cup e') = \nu(\mathcal{S} \backslash \mathcal{L})$.

Recall from Definition 3.3.5 that if $\mathcal{L}$ is a loose even cycle of $\mathcal{S}$, then $M_{\mathcal{L}}$ is the set of edges of M which contain a vertex of $\mathcal{L}$. As an example, $M_{\mathcal{L}}$ is a brush for $\mathcal{L}$, by Lemma 3.4.3. However, a brush for $\mathcal{L}$ does not necessarily have to be a matching of $lk_{\mathcal{H}}(V_1)$.

Lemma 3.5.2. *Let $\mathcal{L}$ be a loose even cycle of $\mathcal{S}$ and let L be a brush for $\mathcal{L}$. If there are two edges of L which are incident to the same edge of $lk_{\mathcal{L}}(V_1)$ and complete to vertices of $V_1 \backslash V_1(\mathcal{L})$, then every edge of L completes to a vertex of $V_1 \backslash V_1(\mathcal{L})$.*

Proof: Suppose, for a contradiction, there is an edge of L which completes to $V_1(\mathcal{L})$. We show that $\mathcal{H}$ has a matching of size $\nu(\mathcal{H}) + 1$. Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a good pair of matchings associated to $\mathcal{S}$ and let $\alpha_0, \alpha_1, \ldots, \alpha_{2l-1}$ be the edges of $lk_{\mathcal{L}}(V_1)$ in cyclic order such that $\pi(\alpha_0) \in \mathcal{M}_1$. By Definition 3.3.1 (d), we know that $l \geq 2$. By Lemma 3.3.8, $(M, \mathcal{M}_1, \mathcal{M}_2)$ is an optimal triple. Let $L = L_2 \cup L_3$ where $L_2 = \{e_0, e_2, \ldots, e_{2l-2}\}$ and $L_3 = \{e_1, e_3, \ldots, e_{2l-1}\}$ such that for each $k \in [2l-1]$, e_{k-1} and e_k meet α_k, where the subscripts are taken modulo $2l$ (e.g. see Figure 3.10). Furthermore, we assume that e_0 and e_{2l-1} complete to vertices of $V_1 \backslash V_1(\mathcal{L})$ and e_1 completes to a vertex of $V_1(\mathcal{L})$, otherwise the lemma holds.

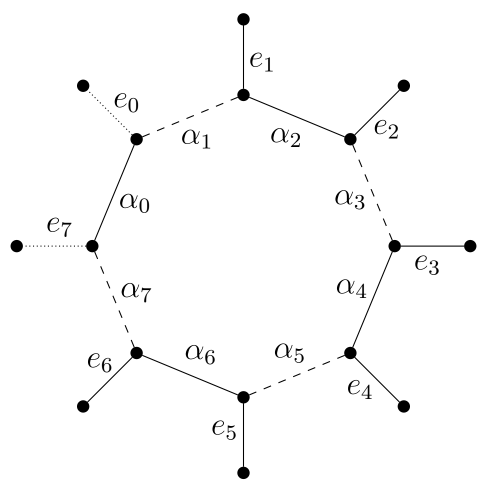

Figure 3.10: A brush for $\mathcal{L}$ when $l = 4$.

Claim 1: Suppose $e_i \in L_t$ and $e_j \in L_{5-t}$ for some $t \in \{2, 3\}$ such that the path $\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_j$ has more Q_n-edges than Q_{3-n}-edges. Let D be the set of $\mathcal{M}_{3-n}$-vertices of $V_1(\mathcal{L})$ which are not the V_1-vertices of any edge in $\mathcal{M}_{3-n} \cap \{\pi(\alpha_{i+1}), \pi(\alpha_{i+2}), \ldots, \pi(\alpha_j)\}$. If one of e_i or e_j completes to a vertex of D and the other completes to either a distinct vertex of D or a vertex of $V_1 \backslash V_1(\mathcal{L})$, then there is a matching of $\mathcal{H}$ of size $|\mathcal{M}_n| + 1$.

Proof of Claim 1: Suppose, without loss of generality, that e_i completes to a vertex of D. Since L is a brush for $\mathcal{L}$, there is a maximum matching $\mathcal{N}$ of $\mathcal{S} \backslash \mathcal{L}$ such that every edge of $\mathcal{N}$ is disjoint from e_i and $\pi(e_j)$. Let $\mathcal{X} = \{\pi(\alpha_{i+1}), \pi(\alpha_{i+3}), \pi(\alpha_{i+5}), \ldots, \pi(\alpha_j)\}$

and $\mathcal{Y} = \{\pi(\alpha_{i+2}), \pi(\alpha_{i+4}), \pi(\alpha_{i+6}), \ldots, \pi(\alpha_{j-1})\}$. Notice that $\mathcal{X} \subseteq \mathcal{M}_n$ and $\mathcal{Y} \subseteq \mathcal{M}_{3-n}$ since $\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_j$ has more Q_n-edges than Q_{3-n}-edges. We also see that no edge of $\mathcal{Y}$ completes to a vertex of D, by definition. Now let $\mathcal{O} = ((\mathcal{M}_n \cap \mathcal{L}) \backslash \mathcal{X}) \cup \mathcal{Y}$ and

$$\mathcal{M} = \mathcal{N} \cup \mathcal{O} \cup \{\pi(e_i), \pi(e_j)\}.$$

Notice that $\mathcal{O}$ is a matching of $\mathcal{L}$ of size $\nu(\mathcal{L}) - 1$ since $\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_j$ has more Q_n-edges than Q_{3-n}-edges. Since $\mathcal{N}$ is a maximum matching of $\mathcal{S} \backslash \mathcal{L}$ and $\mathcal{S}$ is a standard family, $\mathcal{M} \backslash \{\pi(e_i), \pi(e_j)\}$ is a matching of $\mathcal{H}$ of size $|\mathcal{M}_n| - 1$. Now, by construction of $\mathcal{N}$ and since e_i completes to a vertex of $V_1(\mathcal{L})$, both $\pi(e_i)$ and $\pi(e_j)$ are disjoint from every edge of $\mathcal{N}$. Since L is a brush for $\mathcal{L}$, e_i completes to a vertex of D, and e_j completes to either a distinct vertex of D or a vertex of $V_1 \backslash V_1(\mathcal{L})$, both $\pi(e_i)$ and $\pi(e_j)$ are disjoint from every edge in $\mathcal{Y}$. Now, we see that the only edge of $\mathcal{M}_n \cap \mathcal{L}$ which meets $\pi(e_i)$ is $\pi(\alpha_{i+1})$ and the only edge of $\mathcal{M}_n \cap \mathcal{L}$ which meets $\pi(e_j)$ is $\pi(\alpha_j)$. Notice that neither of $\pi(\alpha_i)$ and $\pi(\alpha_j)$ are in $\mathcal{M}$. Finally, since $e_i \in L_t$ and $e_j \in L_{5-t}$ for some $t \in \{2, 3\}$ and e_i and e_j do not complete to the same vertices of V_1, $\pi(e_i)$ and $\pi(e_j)$ are disjoint. Hence, $\mathcal{M}$ is a matching of $\mathcal{H}$ of size $|\mathcal{M}_n| + 1$, as required.

Claim 2: Let $i \in [2l-2]$ and suppose that $\pi(\alpha_i) \in \mathcal{M}_n$ for some $n \in \{1, 2\}$. Then e_i completes to an $\mathcal{M}_n$-vertex of $V_1(\mathcal{L})$.

Proof of Claim 2: Recall that $\pi(\alpha_1) \in \mathcal{M}_2$ and that e_1 completes to a vertex of $V_1(\mathcal{L})$. Suppose, for a contradiction, that e_1 completes to an $\mathcal{M}_1$-vertex of $V_1(\mathcal{L})$. Since $e_0 \in L_2$, $e_1 \in L_3$, $\alpha_1 \in Q_2$, and e_0 completes to a vertex of $V_1 \backslash V_1(\mathcal{L})$, Claim 1 contradicts the maximality of $\mathcal{M}_2$. This proves the $i = 1$ case.

Suppose that $2 \le i \le 2l-2$ and for all $j \in [i-1]$, if $\pi(\alpha_j) \in \mathcal{M}_p$ for some $p \in \{1, 2\}$, then e_j completes to an $\mathcal{M}_p$-vertex of $V_1(\mathcal{L})$. If e_i completes to a vertex of $V_1 \backslash V_1(\mathcal{L})$, then since $e_i \in L_t$ and $e_{i-1} \in L_{5-t}$ for some $t \in \{2, 3\}$, $\pi(\alpha_i) \in \mathcal{M}_n$, and e_{i-1} completes to an $\mathcal{M}_{3-n}$-vertex of $V_1(\mathcal{L})$ by the induction hypothesis, Claim 1 contradicts the maximality of $\mathcal{M}_n$. So suppose that e_i completes to an $\mathcal{M}_{3-n}$-vertex of $V_1(\mathcal{L})$. Notice that by Claim 1, e_{i-1} and e_i complete to the same $\mathcal{M}_{3-n}$-vertex of $V_1(\mathcal{L})$.

Let $\pi(\alpha_s) \in \mathcal{M}_{3-n}$ be such that e_{i-1}, e_i, and α_s complete to the same vertex of V_1. First, suppose that $s \in \{0, 1, 2, \ldots, i-1\}$. If $\pi(\alpha_s) \in \mathcal{M}_1$, then $\alpha_i \in Q_2$, $e_{i-1} \in L_2$, and $e_{2l-1} \in L_3$. Notice also that the path $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{2l-1}$ has more Q_2-edges than Q_1-edges and e_{i-1} completes to an $\mathcal{M}_1$-vertex that is not the V_1-vertex of an edge in $\mathcal{M}_1 \cap \{\pi(\alpha_i), \pi(\alpha_{i+1}), \ldots, \pi(\alpha_{2l-1})\}$. Since e_{2l-1} completes to a vertex of $V_1 \backslash V_1(\mathcal{L})$, Claim 1 contradicts the maximality of $\mathcal{M}_2$. If $\pi(\alpha_s) \in \mathcal{M}_2$, then $s \neq 0$ and we apply the same argument to e_{i-1}, e_0, and the path $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{2l-1}, \alpha_0$ to contradict the maximality of $\mathcal{M}_1$.

Now, suppose that $s \in \{i, i+1, \ldots, 2l-1\}$. Note that $s \neq i$ since $\pi(\alpha_i) \in \mathcal{M}_n$ and $\pi(\alpha_s) \in \mathcal{M}_{3-n}$. Therefore, we have $s \in \{i+1, i+2, \ldots, 2l-1\}$. If $\pi(\alpha_s) \in \mathcal{M}_1$, then $\alpha_i \in Q_2$, $e_0 \in L_2$, and $e_i \in L_3$. Notice also that the path $\alpha_1, \alpha_2, \ldots, \alpha_i$ has more Q_2-edges than Q_1-edges, e_0 completes to a vertex of $V_1 \backslash V_1(\mathcal{L})$, and e_i completes to an $\mathcal{M}_1$-vertex that is not the V_1-vertex of an edge in $\mathcal{M}_1 \cap \{\pi(\alpha_1), \pi(\alpha_2), \ldots, \pi(\alpha_i)\}$. Once again, Claim 1 contradicts the maximality of $\mathcal{M}_2$. Finally, if $\pi(\alpha_s) \in \mathcal{M}_2$, then we apply the same argument to e_i, e_{2l-1}, and the path $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_i$ to contradict the optimality of $\mathcal{M}_1$. Thus, e_i completes to an $\mathcal{M}_n$-vertex of $V_1(\mathcal{L})$, as required.

By Claim 2, we have e_{2l-2} completes to an $\mathcal{M}_1$-vertex of $V_1(\mathcal{L})$, $\alpha_{2l-1} \in Q_2$, and e_{2l-1} completes to a vertex of $V_1 \backslash V_1(\mathcal{L})$. Claim 1 now contradicts the optimality of $\mathcal{M}_2$ and yields the result. □

Definition 3.5.3. Let $L = L_2 \cup L_3$ be a brush of a loose even cycle $\mathcal{L}$ of $\mathcal{S}$. Suppose that L also satisfies the following property:

- Let $e \in L_2$ and let $e' \in L_3$ such that both e and e' complete to vertices of $V_1 \backslash V_1(\mathcal{L})$, but e and e' do not complete to V_1-vertices of the same component of $\mathcal{S}$. Then $\nu(\mathcal{S} \backslash (\mathcal{L} \cup \pi(e) \cup \pi(e'))) = \nu(\mathcal{S} \backslash \mathcal{L})$.

Then we will call L a *strong brush for* $\mathcal{L}$.

Once again, $M_{\mathcal{L}}$ is an example of a strong brush for $\mathcal{L}$, by Lemma 3.4.3. If we have a strong brush for $\mathcal{L}$, then we can improve Lemma 3.5.2.

Lemma 3.5.4. *Let $\mathcal{L}$ be a loose even cycle of $\mathcal{S}$ and let L be a strong brush for $\mathcal{L}$. If there are two edges of L which are incident to the same edge of $lk_{\mathcal{L}}(V_1)$ and complete to vertices of $V_1 \backslash V_1(\mathcal{L})$, then there is a component $\mathcal{L}'$ of $\mathcal{S}$, distinct from $\mathcal{L}$, such that every edge of L completes to a vertex of $V_1(\mathcal{L}')$.*

Proof: Suppose, for a contradiction, that there is no component $\mathcal{L}'$ of $\mathcal{S}$, distinct from $\mathcal{L}$, such that every edge of L completes to $V_1(\mathcal{L}')$. We show that $\mathcal{H}$ has a matching of size $\nu(\mathcal{H}) + 1$. Since L is a brush for $\mathcal{L}$ and there are two edges of L which are incident to the same edge of $lk_{\mathcal{L}}(V_1)$ and complete to vertices of $V_1 \backslash V_1(\mathcal{L})$, Lemma 3.5.2 says that every edge of L completes to a vertex of $V_1 \backslash V_1(\mathcal{L})$. Therefore, there is an edge $xy \in E(lk_{\mathcal{L}}(V_1))$, an edge $m_x \in L_2$ incident to x, and an edge $m_y \in L_3$ incident to y such that m_x and m_y complete to vertices of $V_1 \backslash V_1(\mathcal{L})$, but m_x and m_y do not complete to V_1-vertices of the same component of $\mathcal{S}$.

Since L is a strong brush for $\mathcal{L}$, there is a maximum matching $\mathcal{N}$ of $\mathcal{S} \backslash \mathcal{L}$ such that every edge of $\mathcal{N}$ is disjoint from both $\pi(m_x)$ and $\pi(m_y)$. Let $\mathcal{O}$ be the matching of $\mathcal{L}$ of size $\nu(\mathcal{L}) - 1$ such that no edge of $\mathcal{O}$ contains x or y. Let $\mathcal{M} = \mathcal{N} \cup \mathcal{O} \cup \{\pi(m_x), \pi(m_y)\}$.

Since $\mathcal{N}$ is a maximum matching of $\mathcal{S} \backslash \mathcal{L}$, $\mathcal{O}$ is a matching of $\mathcal{L}$ of size $\nu(\mathcal{L}) - 1$, and $\mathcal{S}$ is a standard family, $\mathcal{M} \backslash \{\pi(m_x), \pi(m_y)\}$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) - 1$. By construction, both $\pi(m_x)$ and $\pi(m_y)$ are disjoint from every edge of $\mathcal{N}$. Furthermore, since L is a strong brush for $\mathcal{L}$ and both m_x and m_y complete to vertices of $V_1 \backslash V_1(\mathcal{L})$, our choice of $\mathcal{O}$ ensures that every edge of $\mathcal{O}$ is disjoint from both $\pi(m_x)$ and $\pi(m_y)$. Finally, $\pi(m_x)$ and $\pi(m_y)$ are disjoint since $m_x \in L_2$, $m_y \in L_3$, and m_x and m_x do not complete to the same vertex of V_1. This means that $\mathcal{M}$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$, which is a contradiction. Thus, there is a component $\mathcal{L}'$ of $\mathcal{S}$, distinct from $\mathcal{L}$, such that every edge of L completes to $V_1(\mathcal{L}')$. $\square$

Recall Definition 3.4.4. In Lemmas 3.5.5 - 3.5.7, we look at properties of special matchings. Ultimately, in Theorem 3.5.11, we will either find a strong brush that is also a special matching or find a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$; both cases will yield contradictions.

Lemma 3.5.5. *Let $W \subseteq \cup_{k=1}^{\omega} V(\mathcal{V}_k) \cap U$ and let $\mathcal{N}$ be the set of edges of $\mathcal{S}$ which contain a vertex of W. If N is a special matching for W such that $V_1(\pi(N)) \subseteq V_1(\mathcal{N})$, then there is a good pair of matchings $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ such that $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is an optimal triple and $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S}) + 1$ path components.*

Proof: Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a good pair of matchings associated to $\mathcal{S}$. By Lemma 3.3.8, $(M, \mathcal{M}_1, \mathcal{M}_2)$ is an optimal triple. For each $e \in \mathcal{N}$, let $m_e \in N$ be an edge such that $e \cap m_e \neq \emptyset$ and $V_1(\pi(m_e)) \in V_1(\mathcal{N})$. Notice that the definitions of N and $\mathcal{N}$ ensure that m_e exists. Similarly to the proof of Lemma 3.4.6, we define a directed graph Z on $\mathcal{N} \cup N$ as follows: Let $e \in \mathcal{N}$ and $f \in N$. There is an arc from e to f if and only if $f = m_e$ and there is an arc from f to e if and only if $V_1(\pi(f)) = V_1(e)$. Since $V_1(\pi(N)) \subseteq V_1(\mathcal{N})$, every vertex of Z has out-degree one and, hence, Z has a directed cycle $D = e_1, f_1, e_2, f_2, \ldots, e_t, f_t$ such that $e_s \in \mathcal{N}$ and $f_s \in N$ for all $s \in [t]$. By definition, the underlying graph of Z is bipartite. By Lemma 3.4.1, it is also simple. Therefore, we have $t \geq 2$. We now build a new good pair of matchings of $\mathcal{H}$.

Let $i \in [\theta]$. Since N is a special matching for W, there are edges $g_i, h_i \in \mathcal{F}_i$ such that (g_i, h_i) is a good pair of matchings of $\mathcal{F}_i$, one of $\rho(g_i)$ and $\rho(h_i)$ is an edge of M, and no edge of N contains a vertex of $g_i \cup h_i$. Let $j \in [\lambda]$. If $c'_j \in m_e$ for some $e \in \mathcal{M}_1 \cap V(D)$, then let (r_j, s_j) be the good pair of matchings of $\mathcal{U}_j$ such that $s_j = a_j b_j c'_j$. Otherwise, let (r_j, s_j) be the good pair of matchings of $\mathcal{U}_j$ such that $r_j = a_j b_j c'_j$. Let $k \in [\omega]$. Let $\mathcal{O}_k$ be the set of edges obtained from $\mathcal{M}_1 \cap \mathcal{V}_k$ by replacing each $e \in \mathcal{M}_1 \cap \mathcal{V}_k \cap V(D)$ by $\pi(m_e)$ and let $\mathcal{T}_k$ be the set of edges obtained from $\mathcal{M}_2 \cap \mathcal{V}_k$ by replacing each $e \in \mathcal{M}_2 \cap \mathcal{V}_k \cap V(D)$ by $\pi(m_e)$. Finally, let

$$\bar{\mathcal{M}}_1 = \bigcup_{i=1}^{\theta} \{g_i\} \cup \bigcup_{j=1}^{\lambda} \{r_j\} \cup \bigcup_{k=1}^{\omega} \mathcal{O}_k$$

and

$$\bar{\mathcal{M}}_2 = \bigcup_{i=1}^{\theta} \{h_i\} \cup \bigcup_{j=1}^{\lambda} \{s_j\} \cup \bigcup_{k=1}^{\omega} \mathcal{T}_k.$$

Notice that b'_j is not contained in an edge of $\bar{\mathcal{M}}_1 \cup \bar{\mathcal{M}}_2$ for any $j \in [\lambda]$.

Claim 1: Both $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_2$ are maximum matchings of $\mathcal{H}$.

Proof of Claim 1: Let $l \in \{1, 2\}$. By construction, $\bar{\mathcal{M}}_l$ contains a maximum matching of $\bigcup_{i=1}^{\theta} \mathcal{F}_i \cup \bigcup_{j=1}^{\lambda} \mathcal{U}_j$. Since N is a special matching for W, no two edges of $\rho(\mathcal{N} \cap \mathcal{M}_l)$ are incident to the same edge of N. Therefore, the definition of $\bar{\mathcal{M}}_l$ ensures that $|\bar{\mathcal{M}}_l| = |\mathcal{M}_l| = \nu(\mathcal{H})$.

Suppose, for a contradiction, that $\bar{\mathcal{M}}_l$ is not a matching. Then there are edges $\alpha, \beta \in \bar{\mathcal{M}}_l$ such that $\alpha \cap \beta \neq \emptyset$. Since $\mathcal{M}_l$ is a matching of $\mathcal{H}$, either $\alpha \in \bar{\mathcal{M}}_l \cap \mathcal{M}_l$ and $\beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$ or $\alpha, \beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$. First, suppose that $\alpha \in \bar{\mathcal{M}}_l \cap \mathcal{M}_l$ and $\beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$. If α and β meet in V_1, then by the definition of Z and D we have $\alpha \in V(D)$. However, this means $\alpha \notin \bar{\mathcal{M}}_l$, which is a contradiction. If α and β meet in V_2, then since N is a special matching for W, the definition of $\bar{\mathcal{M}}_l$ says that we have $\alpha \cap \beta = \{b'_j\}$ and $\alpha = a_j b'_j c_j \in \bar{\mathcal{M}}_l \cap \mathcal{M}_l$ for some $j \in [\lambda]$. However, our construction of $\bar{\mathcal{M}}_l$ ensures that $a_j b'_j c_j \notin \bar{\mathcal{M}}_l$ for all $j \in [\lambda]$. If α and β meet in V_3, then since N is a special matching for W, we have $\alpha \cap \beta = \{c'_j\}$ and $\alpha = a_j b_j c'_j \in \bar{\mathcal{M}}_l \cap \mathcal{M}_l$. Since $c'_j \in \beta \in \bar{\mathcal{M}}_l$, the definition of $\bar{\mathcal{M}}_l$ says that $\alpha \in \bar{\mathcal{M}}_{3-l}$, which is a contradiction. Thus, we have $\alpha, \beta \in \bar{\mathcal{M}}_l \backslash \mathcal{M}_l$.

In this case, since $\rho(\alpha), \rho(\beta) \in N$ and N is a matching of $lk_{\mathcal{H}}(V_1)$, we see that $\alpha \cap \beta \in V_1$ and $\rho(\alpha), \rho(\beta) \in V(D)$. This means that $\rho(\alpha)$ and $\rho(\beta)$ have the same out-neighbour in Z. But then $\rho(\alpha)$ and $\rho(\beta)$ are not both in $V(D)$, which is a contradiction. Thus, $\bar{\mathcal{M}}_l$ is a maximum matching of $\mathcal{H}$.

Claim 2: The pair $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is a good pair of matchings of $\mathcal{H}$.

Proof of Claim 2: Claim 1 tells us that $|\bar{\mathcal{M}}_1| + |\bar{\mathcal{M}}_2| = 2\nu(\mathcal{H}) = \tau(\mathcal{H})$. By the definitions $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_2$, $V_1(\bar{\mathcal{M}}_1 \cup \bar{\mathcal{M}}_2) = V_1(\mathcal{M}_1 \cup \mathcal{M}_2)$. Therefore, since $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings of $\mathcal{H}$, $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_2$ are disjoint matchings of $\mathcal{H}$ and every vertex of V_1 is contained in at most one edge of $\bar{\mathcal{M}}_1 \cup \bar{\mathcal{M}}_2$. Finally, since N is a special matching for W, no edge of N is parallel to an edge of Q. This means that every pair of vertices of $V_2 \cup V_3$ is contained in at most one edge of $\bar{\mathcal{M}}_1 \cup \bar{\mathcal{M}}_2$. Thus, $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is a good pair of matchings of $\mathcal{H}$.

Recall that, for each $i \in [\theta]$, g_i and h_i are edges of $\mathcal{F}_i$ such that one of $\rho(g_i)$ and $\rho(h_i)$ is an edge of M. Also notice that for each $j \in [\lambda]$, our choice of (r_j, s_j) in $\mathcal{U}_j$ ensures that $b_j c'_j$ is an edge of $M \cap (\bar{Q}_1 \cup \bar{Q}_2)$. Therefore, by Lemma 3.3.7, $|M \cap (\bar{Q}_1 \cup \bar{Q}_2)| \geq \Phi(\mathcal{S})$. Since $(M, \mathcal{M}_1, \mathcal{M}_2)$ is an optimal triple and $(\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is a good pair of matchings of $\mathcal{H}$ by Claim 2, $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is also an optimal triple. To conclude, it remains to show that $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S}) + 1$ path components.

Claim 3: For each $j \in [\lambda]$, if there is an edge of $N \cap V(D)$ which contains a vertex of $\mathcal{U}_j$, then the vertex is b'_j.

Proof of Claim 3: Let $j \in [\lambda]$ and let $e \in N \cap V(D)$ such that $\pi(e) \in \bar{\mathcal{M}}_l$ for some $l \in \{1, 2\}$. Since N is a special matching for W, the only possible vertices of $\mathcal{U}_j$ which are contained in e are b'_j and c'_j. Suppose, for a contradiction, that $c'_j \in e$. Since $\pi(e) \in \bar{\mathcal{M}}_l$, Claim 1 and our choice of $\bar{\mathcal{M}}_{3-l}$ tell us that $a_j b_j c'_j \in \bar{\mathcal{M}}_{3-l}$. Therefore, $b_j c'_j \in M$ is a $\bar{Q}_l$-touching edge. Let α be the edge of $\mathcal{U}_j \backslash \{a_j b_j c'_j\}$ which contains b_j. Since $\alpha \in \mathcal{U}_j$ and, hence, $\alpha \notin \mathcal{N}$, we have $\alpha \in \bar{\mathcal{M}}_l$. This means that $b_j c'_j$ is a $\bar{Q}_l$-touching edge of M which meets two distinct edges of $\bar{Q}_l$. However, since $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ is an optimal triple, this contradicts Lemma 3.3.3 (d). Thus, if there is an edge of N which contains a vertex of $\mathcal{U}_j$, then the vertex is b'_j.

By Corollary 3.3.4 (c) and Lemma 3.3.7, Q has $\Phi(\mathcal{S}) = \theta + \lambda$ path components. For each $i \in [\theta]$, our choice of g_i and h_i ensures that $\{g_i, h_i\}$ forms a path component of $\bar{Q}_1 \cup \bar{Q}_2$. Let $j \in [\lambda]$. By Claim 3, the only vertex of $\mathcal{U}_j$ which can be contained in an edge of $(\bar{\mathcal{M}}_1 \backslash \mathcal{M}_1) \cup (\bar{\mathcal{M}}_2 \backslash \mathcal{M}_2)$ is b'_j. Since neither r_j or s_j contains b'_j and $\mathcal{U}_j$ is a loose odd cycle, the edges r_j and s_j correspond to a path component of $\bar{Q}_1 \cup \bar{Q}_2$. In other words, $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S}) = \theta + \lambda$ path components. Now, since N is a special matching for $W \subseteq \cup_{k=1}^{\omega} V(\mathcal{V}_k)$, Claim 3 says that every edge of $N \cap V(D)$ contains a vertex of degree one in $\bar{Q}_1 \cup \bar{Q}_2$. Furthermore, Claim 2 and Lemma 3.3.2 tell us that every such vertex is an end-vertex of an even path component of $\bar{Q}_1 \cup \bar{Q}_2$. However, since $(V(N) \backslash W) \cap (B \cap C \cap U) = \emptyset$, such a path component is distinct from the $\Phi(\mathcal{S}) = \theta + \lambda$ path components above. Thus,

$\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S})+1$ path components. $\square$

Lemma 3.5.6. *Let $\mathcal{L}$ be a loose even cycle of $\mathcal{S}$. If L is a special matching for $V_2(\mathcal{L}) \cup V_3(\mathcal{L}) \subseteq U$, then there are two edges of L which are incident to the same edge of $lk_{\mathcal{L}}(V_1)$ and complete to vertices of $V_1 \backslash V_1(\mathcal{L})$.*

Proof: Suppose, for a contradiction, that every edge of $lk_{\mathcal{L}}(V_1)$ is incident to an edge of L which completes to $V_1(\mathcal{L})$. Let $N \subseteq L$ be the edges which complete to a vertex of $V_1(\mathcal{L})$ and let $W = V(N) \cap V(\mathcal{L})$. Notice that $\mathcal{N} = \mathcal{L}$ is the set of edges of $\mathcal{S}$ which contain a vertex of W. Since L is a special matching for $V_2(\mathcal{L}) \cup V_3(\mathcal{L}) \subseteq U$ and $N \subseteq L$, we have $(V(N)\backslash W) \cap (B \cup C \cup U) = \emptyset$, and $|(V(N)\backslash W) \cap V(\mathcal{F}_i)| \leq 1$ for every $i \in [\theta]$. Finally, since N is a matching of $lk_{\mathcal{H}}(V_1)$ which is bijectively covered by W, N is a special matching for W.

Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a good pair of matchings associated to $\mathcal{S}$. By Lemma 3.3.8, we have that $(M, \mathcal{M}_1, \mathcal{M}_2)$ is an optimal triple. By the definition of N, $V_1(\pi(N)) \subseteq V_1(\mathcal{N})$. Therefore there is an optimal triple $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ such that $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S})+1$ path components, by Lemma 3.5.5. But then, by Corollary 3.3.4 (c) and Lemma 3.3.7, we have

$$|M \cap (\bar{Q}_1 \cup \bar{Q}_2)| \geq \Phi(\mathcal{S}) + 1 > |M \cap (Q_1 \cup Q_2)|,$$

which contradicts the optimality of $(M, \mathcal{M}_1, \mathcal{M}_2)$. Hence, there are two edges of L which are incident to the same edge of $lk_{\mathcal{L}}(V_1)$ and complete to vertices of $V_1 \backslash V_1(\mathcal{L})$. $\square$

Let $\mathcal{L}$ be a loose even cycle component of $\mathcal{S}$. Recall by Definition 3.3.5 that $M_{\mathcal{L}}$ is the set of edges of M that contain a vertex of $\mathcal{L}$ and that $M_{\mathcal{L}}$ is both a strong brush for $\mathcal{L}$ and a special matching for $V(\mathcal{L}) \cap U = V_2(\mathcal{L}) \cup V_3(\mathcal{L})$.

Lemma 3.5.7. *If $\mathcal{S}$ has a loose even cycle component, then there are components $\mathcal{L}$ and $\mathcal{L}'$ of $\mathcal{S}$ such that $\mathcal{L}$ is a loose even cycle, every edge of $M_{\mathcal{L}}$ completes to a vertex of $V_1(\mathcal{L}')$, and $\mathcal{L}'$ is either a loose 3-cycle or a copy of $\mathcal{F}$.*

Proof: By assumption, we have $\omega \geq 1$. Let $k \in [\omega]$. Since $M_{\mathcal{V}_k}$ is a special matching for $V_2(\mathcal{V}_k) \cup V_3(\mathcal{V}_k)$, there are two edges of $M_{\mathcal{V}_k}$ which are incident to the same edge of $lk_{\mathcal{V}_k}(V_1)$ and complete to vertices of $V_1 \backslash V_1(\mathcal{V}_k)$, by Lemma 3.5.6. As $M_{\mathcal{V}_k}$ is also a strong brush for $\mathcal{V}_k$, Lemma 3.5.4 tells us there is a component $\mathcal{P}_k$ of $\mathcal{S}$ such that every edge of $M_{\mathcal{V}_k}$ completes to a vertex of $V_1(\mathcal{P}_k)$.

Suppose, for a contradiction, that for all $k \in [\omega]$, there is a $\bar{k} \in [\omega]$ such that $\mathcal{P}_k = \mathcal{V}_{\bar{k}}$. Let $W = \cup_{k=1}^{\omega}(V_2(\mathcal{V}_k) \cup V_3(\mathcal{V}_k))$, let $N = \cup_{k=1}^{\omega} M_{\mathcal{V}_k}$, and let $\mathcal{N} = \cup_{k=1}^{\omega} \mathcal{V}_k$. Notice that

$N \subseteq M$ is a matching of $lk_{\mathcal{H}}(V_1)$. By our choice of N, $(V(N) \backslash V(\mathcal{L})) \cap (B \cup C \cup U) = \emptyset$ and $|(V(N) \backslash V(\mathcal{L})) \cap V(\mathcal{F}_i)| = 0$ for all $i \in [\theta]$. Thus, N is a special matching for W. Since $V_1(\pi(N)) \subseteq V_1(\mathcal{N})$, Lemma 3.5.5 says that there is an optimal triple $(M, \bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2)$ such that $\bar{Q}_1 \cup \bar{Q}_2$ has at least $\Phi(\mathcal{S}) + 1$ path components. Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a good pair of matchings associated to $\mathcal{S}$. By Lemma 3.3.8, $(M, \mathcal{M}_1, \mathcal{M}_2)$ is an optimal triple. But then, by Corollary 3.3.4 (c) and Lemma 3.3.7, we have

$$|M \cap (\bar{Q}_1 \cup \bar{Q}_2)| \geq \Phi(\mathcal{S}) + 1 > |M \cap (Q_1 \cup Q_2)|,$$

which contradicts the optimality of $(M, \mathcal{M}_1, \mathcal{M}_2)$. Hence, there are components $\mathcal{L}$ and $\mathcal{L}'$ of $\mathcal{S}$ such that $\mathcal{L}$ is a loose even cycle of $\mathcal{S}$, every edge of $M_{\mathcal{L}}$ completes to a vertex of $V_1(\mathcal{L}')$ and $\mathcal{L}'$ is either a loose 3-cycle or a copy of $\mathcal{F}$. $\square$

For the remainder of this section, we assume that $\mathcal{S}$ has a loose even cycle component. Let $\mathcal{L}$ be the loose even cycle of $\mathcal{S}$ of length $2l \geq 4$ and $\mathcal{L}'$ be the loose 3-cycle or copy of $\mathcal{F}$ given by Lemma 3.5.7. Let $\{\alpha, \alpha', \beta, \beta', \gamma, \gamma'\}$ be the vertices of $\mathcal{L}'$ such that $\alpha, \alpha' \in A$, $\beta, \beta' \in B$, and $\gamma, \gamma' \in C$, and let $\{\alpha\beta\gamma', \alpha\beta'\gamma, \alpha'\beta\gamma\}$ be edges of $\mathcal{L}'$ (e.g. see Figure 3.11). Note that $\alpha'\beta'\gamma'$ is an edge of $\mathcal{L}'$ if and only if $\mathcal{L}'$ is a copy of $\mathcal{F}$.

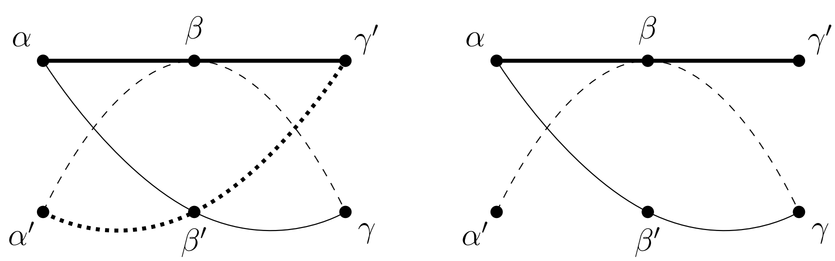

Figure 3.11: The possibilities for $\mathcal{L}'$.

Lemma 3.5.8. *Suppose that $\mathcal{L}'$ is a loose 3-cycle. For some $s \in \{2, 3\}$, there is a set of vertices $W \subseteq V_2(\mathcal{L}) \cup V_3(\mathcal{L})$ and a special matching N for W of size $l+1$ with the following properties:*

(a) there are l edges of N which complete to α' and have exactly one end in $V_s(\mathcal{L})$,

(b) there is one edge of N which completes to α and has exactly one end in $V_{5-s}(\mathcal{L})$, and

(c) either every edge in (a) is in $M_{\mathcal{L}}$ or the edge in (b) is in $M_{\mathcal{L}}$.

Proof: Consider the partial cover of $\mathcal{H}$ given by $T = A \cup B \cup F^2 \cup (U \backslash V_3(\mathcal{L}))$. Since $A \cup B \cup F^2 \cup U$ is a minimum cover of $\mathcal{H}$ by Theorem 3.3.16, every cover of $\mathcal{H} \backslash T$ has size at least $|V_3(\mathcal{L})|$. By Lemma 2.2.3, there is a matching N_3 of $lk_{\mathcal{H} \backslash A}(V_1 \backslash A) \backslash T$ of size $|V_3(\mathcal{L})|$. We also see that every edge of N_3 has exactly one end in $V_3(\mathcal{L})$ and exactly one end in $V_2 \backslash (B \cup C \cup F^2 \cup F^3 \cup U)$, otherwise $A \cup B \cup F^2 \cup U$ is not a cover of $\mathcal{H}$. Let N_2 be the set of edges of $M_{\mathcal{L}}$ with an end in $V_2(\mathcal{L})$. Since $lk_{\mathcal{H}}(V_1)$ is bipartite, $N_2 \cup N_3$ is a matching of $lk_{\mathcal{H}}(V_1)$ of size $2l$. In what follows, suppose that $\mathcal{L} = \mathcal{V}_\omega$.

Claim 1: $N_2 \cup N_3$ is a brush for $\mathcal{L}$.

Proof of Claim 1: By construction, $N_2 \cup N_3$ is bijectively covered by $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$. Let $e \in N_2$ and $e' \in N_3$. Since $e \in M_{\mathcal{L}}$, e does not contain a vertex of $\mathcal{S} \backslash \mathcal{L}$. We also see that since e' has one vertex in $V_3(\mathcal{L}) \subseteq U$ and is otherwise disjoint from $B \cup C \cup F^2 \cup F^3 \cup U$, if e' contains a vertex of a component which is distinct from $\mathcal{L}$, then that component is $\mathcal{U}_j$ for some $j \in [\lambda]$ and $b'_j \in e'$.

We build a maximum matching $\mathcal{M}$ of $\mathcal{S} \backslash \mathcal{L}$ such that every edge of $\mathcal{M}$ is disjoint from $\pi(e)$ and e' as follows: For each $i \in [\theta]$, e' contains no vertex of $\mathcal{F}_i$ and $\pi(e) \notin \mathcal{F}_i$. Therefore, there is an edge $f_i \in \mathcal{F}_i$ which is disjoint from both $\pi(e)$ and e' by Lemma 3.3.9. For each $j \in [\lambda]$, $e \in N_2 \subseteq M_{\mathcal{L}}$ contains no vertex of $V(\mathcal{U}_j)$ by parts (d) and (g) of Lemma 3.3.3. If e completes to a_j, then let $r_j = a'_j b_j c_j$. Otherwise let $r_j = a_j b_j c'_j$. In either case, r_j is disjoint from both $\pi(e)$ and e' since $b'_j \notin r_j$. For each $k \in [\omega - 1]$, $e \in N_2 \subseteq M_{\mathcal{L}}$ does not complete to $V_1(\mathcal{V}_k)$ by Lemma 3.5.7 and neither e nor e' contains a vertex of $\mathcal{V}_k$. Therefore, for each $k \in [\omega - 1]$, let $\mathcal{Y}_k$ be any maximum matching of $\mathcal{V}_k$. Let

$$\mathcal{M} = \bigcup_{i=1}^{\theta} \{f_i\} \cup \bigcup_{j=1}^{\lambda} \{r_j\} \cup \bigcup_{k=1}^{\omega-1} \mathcal{Y}_k.$$

By construction, every edge of $\mathcal{M}$ is disjoint from $\pi(e)$ and e'. Furthermore, $\mathcal{M}$ is a maximum matching of $\mathcal{S} \backslash \mathcal{L}$ since $\mathcal{M}$ is a union of maximum matchings of the components of $\mathcal{S} \backslash \mathcal{L}$ and $\mathcal{S}$ is a standard family.

Now we construct a maximum matching $\bar{\mathcal{M}}$ of $\mathcal{S} \backslash \mathcal{L}$ such that every edge of $\bar{\mathcal{M}}$ is disjoint from e and $\pi(e')$. For each $i \in [\theta]$, $\pi(e') \notin \mathcal{F}_i$. By Lemma 3.3.9, there is an edge $\bar{f}_i \in \mathcal{F}_i$ that is disjoint from $\pi(e')$. For each $j \in [\lambda]$, let $\bar{r}_j = a_j b_j c'_j$. For each $k \in [\omega - 1]$, let $\bar{\mathcal{Y}}_k$ be any maximum matching of $\mathcal{V}_k$ which avoids $\pi(e')$. Note that such a matching of $\mathcal{V}_k$ exists by Lemma 3.4.3 since $\mathcal{V}_k$ is a loose even cycle and e' contains no vertex of $V(\mathcal{V}_k)$. Now let

$$\bar{\mathcal{M}} = \bigcup_{i=1}^{\theta} \{\bar{f}_i\} \cup \bigcup_{j=1}^{\lambda} \{\bar{r}_j\} \cup \bigcup_{k=1}^{\omega-1} \bar{\mathcal{Y}}_k.$$

Since $e \in M_{\mathcal{L}}$, e contains no vertex of $\mathcal{F}_i$ for any $i \in [\theta]$, no vertex of $\mathcal{U}_j$ for any $j \in [\lambda]$, and no vertex of $\mathcal{V}_k$ for any $k \in [\omega - 1]$. Thus, we see that e is disjoint from every edge of $\bar{\mathcal{M}}$. By construction, $\pi(e')$ is disjoint from $\bar{f}_i$ for every $i \in [\theta]$. For each $j \in [\lambda]$, $e' \in N_3$ does not complete to a_j by our choice of T. We also noted earlier that if e' contains a vertex of $\mathcal{U}_j$ for some $j \in [\lambda]$, then e' can contain only b'_j. Since $b'_j \notin \bar{r}_j$, $\pi(e')$ is disjoint from $\bar{r}_j$ for every $j \in [\lambda]$. Finally, $\pi(e')$ is disjoint from every edge of $\bar{\mathcal{Y}}_k$ by construction and, hence, $\pi(e')$ is disjoint from every edge of $\bar{\mathcal{M}}$. Thus, $N_2 \cup N_3$ is a brush, as required.

Claim 2: $N_2 \cup N_3$ is a strong brush for $\mathcal{L}$.

Proof of Claim 2: To show $N_2 \cup N_3$ is a strong brush for $\mathcal{L} = \mathcal{V}_\omega$, suppose that both $e \in N_2$ and $e' \in N_3$ complete to vertices of $V_1 \backslash V_1(\mathcal{L})$, but e and e' do not complete to V_1-vertices of the same component of $\mathcal{S}$. We build a maximum matching $\mathcal{M}^*$ of $\mathcal{S} \backslash \mathcal{L}$ such that every edge of $\mathcal{M}^*$ is disjoint from both $\pi(e)$ and $\pi(e')$. Our choice of T and the fact that $e \in M_{\mathcal{L}}$ tell us that both e and e' have exactly one end in U and are otherwise disjoint from $B \cup C \cup U$. Therefore Lemma 3.4.1 says that both e and e' complete to vertices of $V_1(\mathcal{S})$. Let $i \in [\theta]$. If e completes to a vertex of $V_1(\mathcal{F}_i)$, then let $f_i^* \in \mathcal{F}_i$ be an edge which is disjoint from $\pi(e)$. If e' completes to a vertex of $V_1(\mathcal{F}_i)$, let $f_i^* \in \mathcal{F}_i$ be an edge which is disjoint from $\pi(e')$. Let $j \in [\lambda]$. If e completes to a_j, then let $r_j^* = a'_j b_j c_j$. Otherwise let $r_j^* = a_j b_j c'_j$. Finally, for each $k \in [\omega - 1]$, since e and e' do not complete to the same component of $\mathcal{S}$, there is a maximum matching $\mathcal{Y}_k^*$ of $\mathcal{V}_k$ which is disjoint from both $\pi(e)$ and $\pi(e')$ by Lemma 3.4.3 since $\mathcal{V}_k$ is a loose even cycle and both e and e' contain no vertex of $V(\mathcal{V}_k) \subseteq U$. Now, let

$$\mathcal{M}^* = \bigcup_{i=1}^{\theta} \{f_i^*\} \cup \bigcup_{j=1}^{\lambda} \{r_j^*\} \cup \bigcup_{k=1}^{\omega-1} \mathcal{Y}_k^*.$$

By our choice of T, e' does not contain a vertex of $\mathcal{F}_i$ for any $i \in [\theta]$. Since $e \in M_{\mathcal{L}}$, e does not contain a vertex of any $\mathcal{F}_i$ either. Therefore, our choice of f_i^* ensures that f_i^* is disjoint from both $\pi(e)$ and $\pi(e')$ for every $i \in [\theta]$. Let $j \in [\lambda]$. First suppose that e completes to a vertex of $V_1(\mathcal{U}_j)$. Then since $e \in N_2 \subseteq M_{\mathcal{L}}$ does not contain a vertex of $\mathcal{U}_j$, our choice of r_j^* ensures that r_j^* and $\pi(e)$ are disjoint. We also see that r_j^* is disjoint from $\pi(e')$ since e' does not complete to a vertex of $V_1(\mathcal{U}_j)$ and $b'_j \notin r_j^*$. Now suppose that e' completes to a vertex of $V_1(\mathcal{U}_j)$. Since $e \in M_{\mathcal{L}}$ does not contain a vertex of $\mathcal{U}_j$ and e does not complete to a vertex of $V_1(\mathcal{U}_j)$, r_j^* is disjoint from $\pi(e)$. Also, since $b'_j \notin r_j^*$ and e' does not complete to a_j by our choice of T, r_j^* is disjoint from $\pi(e')$. If neither e nor e' complete to a vertex of $V_1(\mathcal{U}_j)$, then since $e \in M_{\mathcal{L}}$ and $b'_j \notin r_j^*$, we have that both $\pi(e)$ and $\pi(e')$ are disjoint from r_j^*. By construction, $\pi(e)$ and $\pi(e')$ are disjoint from every edge of $\mathcal{Y}_k^*$ for every $k \in [\omega]$. Thus $\pi(e)$ and $\pi(e')$ are disjoint from every edge of $\mathcal{M}^*$ and $N_2 \cup N_3$ is a strong brush for $\mathcal{L}$.

From the definition of $N_2 \cup N_3$, we see that $N_2 \cup N_3$ is also a special matching for $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$. Thus, by Lemmas 3.5.4 and 3.5.6, there is a component $\mathcal{P} \neq \mathcal{L}$ of $\mathcal{S}$ such that every edge of $N_2 \cup N_3$ completes to a vertex of $V_1(\mathcal{P})$. But, by Lemma 3.5.7, every edge of $N_2 \subseteq M_{\mathcal{L}}$ completes to a vertex of $\{\alpha, \alpha'\} = V_1(\mathcal{L}')$. Hence $\mathcal{P} = \mathcal{L}'$ and every edge of $N_2 \cup N_3$ completes to a vertex of $\{\alpha, \alpha'\}$. Furthermore, our choice of T ensures that every edge of N_3 completes to α'. If there is an edge $g \in N_2$ which completes to α, then we set $N = N_3 \cup \{g\}$, $W = V_2(g) \cup V_3(\mathcal{L})$, and $s = 3$ and we are done. So, we may assume that every edge of N_2 completes to α'. Since $N_2 \subseteq M_{\mathcal{L}}$, we may now also assume that every edge of $M_{\mathcal{L}}$ completes to α', otherwise there is a matching $N \subseteq M_{\mathcal{L}}$ and a set $W \subseteq V_2(\mathcal{L}) \cup V_3(\mathcal{L})$ that is the desired special matching for W with $s = 2$.

Let $S = \{\alpha'\} \cup B \cup C \cup F^3 \cup (U \backslash V_2(\mathcal{L}) \cup V_3(\mathcal{L}))$ and consider the bipartite multigraph $H = lk_{\mathcal{H} \backslash S}(V_1 \backslash \{\alpha'\})$. Since $(S \backslash \{\alpha'\}) \cup V_2(\mathcal{L}) \cup V_3(\mathcal{L})$ is a minimum cover of $\mathcal{H}$ by Theorem 3.3.16, every cover of $\mathcal{H} \backslash S$ has size at least $2l - 1$. Therefore, Lemma 2.2.3 tells us there is a matching N^+ of size $2l - 1$ in H. Furthermore, since $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$ is a cover of H, every edge of N^+ contains a vertex of $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$. Since $|V_2(\mathcal{L}) \cup V_3(\mathcal{L})| = 2l$, there are at least $2l - 2$ edges of N^+ that have exactly one end in $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$ and exactly one end in $V_2 \cup V_3 \backslash (B \cup C \cup F^3 \cup U)$.

Since $l \geq 2$ by Definition 3.3.1 (d), there is an edge $h \in N^+$ which has exactly one end in $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$ and exactly one end in $V_2 \cup V_3 \backslash (B \cup C \cup F^3 \cup U)$. Let N^* be the set of edges obtained from $M_{\mathcal{L}}$ by replacing the edge which meets h by h. Note that N^* may not be a matching of $lk_{\mathcal{H}}(V_1)$.

Claim 3: N^* is a brush for $\mathcal{L}$.

Proof of Claim 3: By the definition of N^*, N^* is bijectively covered by $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$. Since $M_{\mathcal{L}}$ is a brush for $\mathcal{L}$, it suffices to check Definition 3.5.1 for $e \in M_{\mathcal{L}} \cap N^*$ and h. Recall that since $e \in M_{\mathcal{L}}$, e does not contain a vertex of $\mathcal{S} \backslash \mathcal{L}$. Let $i \in [\theta]$. Since $\pi(h) \notin \mathcal{F}_i$, there is an edge $\tilde{f}_i \in \mathcal{F}_i$ which is disjoint from $\pi(h)$, by Lemma 3.3.9. Let $j \in [\lambda]$. Since h does not contain a vertex of $B \cup C$ by our choice of S and $\mathcal{U}_j$ is a loose 3-cycle, there is an edge $\tilde{r}_j$ which is disjoint from $\pi(h)$. Let $k \in [\omega - 1]$. Since h contains no vertex of $\mathcal{V}_k$ and $\mathcal{V}_k$ is an even cycle component of $\mathcal{S}$, there is a maximum matching $\tilde{\mathcal{Y}}_k$ of $\mathcal{V}_k$ such that every edge of $\tilde{\mathcal{Y}}_k$ is disjoint from $\pi(h)$, by Lemma 3.4.3. Now we see that

$$\tilde{\mathcal{M}} = \bigcup_{i=1}^{\theta} \{\tilde{f}_i\} \cup \bigcup_{j=1}^{\lambda} \{\tilde{r}_j\} \cup \bigcup_{k=1}^{\omega-1} \tilde{\mathcal{Y}}_k$$

is a maximum matching of $\mathcal{S}\backslash\mathcal{L}$ such that every edge of $\tilde{\mathcal{M}}$ is disjoint from e and $\pi(h)$.

We now construct a maximum matching $\mathcal{M}'$ of $\mathcal{S}\backslash\mathcal{L}$ such that every edge of $\mathcal{M}'$ is disjoint from $\pi(e)$ and h. Let $i \in [\theta]$. Since $e \in M_{\mathcal{L}}$ and h contains at most one vertex of $\mathcal{F}_i$, there is an edge $f_i' \in \mathcal{F}_i$ which is disjoint from both $\pi(e)$ and h, by Lemma 3.3.9. Let $j \in [\lambda]$. Since $e \in M_{\mathcal{L}}$ and h does not contain a vertex of $B \cup C$, there is an edge $r_j' \in \mathcal{U}_j$ which is disjoint from both $\pi(e)$ and h. Let $k \in [\omega - 1]$. Since $e \in M_{\mathcal{L}}$ does not complete to a vertex of $\mathcal{V}_k$ and h does not contain a vertex of $\mathcal{V}_k$, there is a maximum matching $\mathcal{Y}_k'$ of $\mathcal{V}_k$ such that every edge of $\mathcal{Y}_k'$ is disjoint from both $\pi(e)$ and h. Thus,

$$\mathcal{M}' = \bigcup_{i=1}^{\theta}\{f_i'\} \cup \bigcup_{j=1}^{\lambda}\{r_j'\} \cup \bigcup_{k=1}^{\omega-1}\mathcal{Y}_k'$$

is maximum matching of $\mathcal{S}\backslash\mathcal{L}$ such that every edge of $\mathcal{M}'$ is disjoint from $\pi(e)$ and h and, hence, N^* is a brush for $\mathcal{L}$.

Claim 4: N^* is a strong brush for $\mathcal{L}$.

Proof of Claim 4: Suppose that e and h do not complete to V_1-vertices of the same component of $\mathcal{S}$. Let $i \in [\theta]$. Recall that e completes to α'. If $\alpha' \in V_1(\mathcal{F}_i)$, then $\pi(h)$ contains at most one vertex of $V_2(\mathcal{F}_i) \cup V_3(\mathcal{F}_i)$. Thus, since $e \in M_{\mathcal{L}}$, there is an edge $\hat{f}_i \in \mathcal{F}_i$ that is disjoint from both $\pi(e)$ and $\pi(h)$. If $\alpha' \notin V_1(\mathcal{F}_i)$, then $\pi(e)$ is disjoint from every edge of $\mathcal{F}_i$ since $e \in M_{\mathcal{L}}$. Therefore, since $\pi(h) \notin \mathcal{F}_i$, there is an edge $\hat{f}_i \in \mathcal{F}_i$ that is disjoint from $\pi(h)$ by Lemma 3.3.9. Let $j \in [\lambda]$. Since $e \in M_{\mathcal{L}}$, h does not contain a vertex of $B \cup C$, and at most one of e and h completes to a vertex of $V_1(\mathcal{U}_j)$, there is an edge $\hat{r}_j \in \mathcal{U}_j$ which is disjoint from both $\pi(e)$ and $\pi(h)$. Let $k \in [\omega - 1]$. Since e and h do not complete to V_1-vertices of the same component of $\mathcal{S}$, Lemma 3.4.3 tells us that there is a maximum matching $\hat{\mathcal{Y}}_k$ of $\mathcal{V}_k$ such that every edge of $\hat{\mathcal{Y}}_k$ is disjoint from both $\pi(e)$ and $\pi(h)$. Thus

$$\hat{\mathcal{M}} = \bigcup_{i=1}^{\theta}\{\hat{f}_i\} \cup \bigcup_{j=1}^{\lambda}\{\hat{r}_j\} \cup \bigcup_{k=1}^{\omega-1}\hat{\mathcal{Y}}_k$$

is a maximum matching of $\mathcal{S}\backslash\mathcal{L}$ such that every edge of $\hat{\mathcal{M}}$ is disjoint from both $\pi(e)$ and $\pi(h)$. This proves that N^* is a strong brush for $\mathcal{L}$.

Now, since N^* is a strong brush for $\mathcal{L}$ such that at least $2l-1$ edges of N^* complete to vertices of $V_1\backslash V_1(\mathcal{L})$, Lemma 3.5.4 says there is a component $\mathcal{P}$ of $\mathcal{S}$ such that every edge of N^* completes to a vertex of $V_1(\mathcal{P})$. Since N^* contains $2l-1$ edges of $M_{\mathcal{L}}$ and every edge of $M_{\mathcal{L}}$ completes to $\alpha' \in V_1(\mathcal{L}')$, we have $\mathcal{P} = \mathcal{L}'$. By our choice of S, h completes to α. Without loss of generality, suppose h meets $\mathcal{L}$ in V_3. Let $\hat{H} \subset N^*$ be the set of edges which meet $\mathcal{L}$ in V_2 and let $N = \hat{N} \cup \{h\}$. Then for $s = 2$ and $W = V_2(\mathcal{L}) \cup V_3(h)$, N is the desired matching. $\square$

If $\mathcal{L}'$ is a loose 3-cycle of $\mathcal{S}$, we say that α' is *essential* if there are no edges of $\mathcal{H}$ of the form $x\beta\gamma$ where $x \neq \alpha$ or α'.

Lemma 3.5.9. *If $\mathcal{L}'$ is a loose 3-cycle of $\mathcal{S}$, then the vertex α' is essential.*

Proof: Suppose, for a contradiction, that α' is not essential. Our aim is to find a matching of size $\nu(\mathcal{H}) + 1$. Then there is a vertex $p \in V_1$, distinct from α and α' such that $p\beta\gamma \in \mathcal{H}$. Let N be the matching of $lk_{\mathcal{H}}(V_1)$ given by Lemma 3.5.8. Without loss of generality, we may assume that $s = 2$. Recall that $\mathcal{L}$ has length $2l$ and, by Definition 3.3.1 (d), that $l \geq 2$. Thus, by Lemma 3.5.8, there are distinct vertices u, v, w, y, y', and z such that $y, y' \in V_2(\mathcal{L})$, $z \in V_3(\mathcal{L})$, $yz, y'z \in E(lk_{\mathcal{L}}(V_1))$, $yu, y'w, vz \in N$, and $\alpha' yu, \alpha' y' w, \alpha vz \in \mathcal{H}$. We also know that either $yu, y'w \in M_{\mathcal{L}}$ or $vz \in M_{\mathcal{L}}$. Let the vertices $x, x' \in V_1$ be such that $xyz = \pi(yz)$ and $x'y'z = \pi(y'z)$.

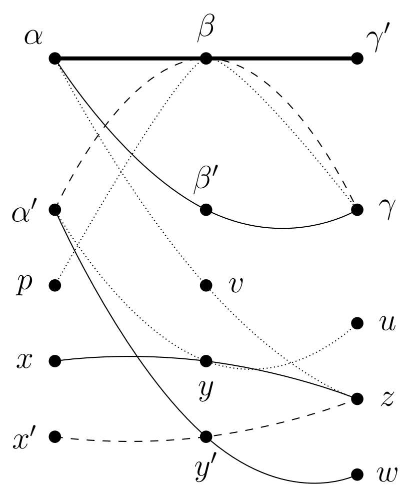

Figure 3.12: Showing that α' is essential.

We choose a good pair of matchings $(\mathcal{M}_1, \mathcal{M}_2)$ associated to $\mathcal{S}$ as follows: Let $i \in [\theta]$. Since N is a special matching for $V_2(\mathcal{L}) \cup \{z\}$, we choose the $\mathcal{M}_1$-edge and $\mathcal{M}_2$-edge of $\mathcal{F}_i$ so that they form a good pair of matchings of $\mathcal{F}_i$ and neither edge contains a vertex of an edge of N. For every $k \in [\omega]$ such that $\mathcal{V}_k \neq \mathcal{L}$, we choose the $\mathcal{M}_1$-edges and $\mathcal{M}_2$-edges of $\mathcal{V}_k$ so that they form a good pair of matchings of $\mathcal{V}_k$. In $\mathcal{L}$, we choose a good pair of matchings of $\mathcal{L}$ so that $xyz \in \mathcal{M}_1$ and $x'y'z \in \mathcal{M}_2$. Now, if $yu, y'w \in M_{\mathcal{L}}$, then for every $j \in [\lambda]$, we choose $a_jb_jc'_j$ to be in $\mathcal{M}_1$ and $a'_jb_jc_j$ to be in $\mathcal{M}_2$. If $vz \in M_{\mathcal{L}}$, then for every $j \in [\lambda]$, we choose $a_jb'_jc_j$ to be in $\mathcal{M}_1$ and $a'_jb_jc_j$ to be in $\mathcal{M}_2$. By construction, $(\mathcal{M}_1, \mathcal{M}_2)$ is a good pair of matchings associated to $\mathcal{S}$.

Claim: No edge of $\mathcal{M}_1 \cup \mathcal{M}_2$ contains u, v, or w.

Proof of Claim: We first suppose that $yu, y'w \in M_{\mathcal{L}}$. In this case neither u nor w is contained in an edge of $\mathcal{M}_1 \cup \mathcal{M}_2$ since every edge of $M_{\mathcal{L}}$ has exactly one end in $\mathcal{S}$, by Definition 3.3.5. Notice that $v \notin B \cup C \cup U$ since N is a special matching for $V_2(\mathcal{L}) \cup \{z\}$. Therefore v is not an edge of $(\mathcal{M}_1 \cup \mathcal{M}_2) \cap \mathcal{V}_k$ for any $k \in [\omega]$. By our choice of $(\mathcal{M}_1, \mathcal{M}_2)$, b'_j is not contained in an edge of $\mathcal{M}_1 \cup \mathcal{M}_2$ for any $j \in [\lambda]$. Furthermore, $v \neq c'_j$ for any $j \in [\lambda]$ since $c'_j \in V_3$. Thus, v is not contained in an edge of $(\mathcal{M}_1 \cup \mathcal{M}_2) \cap \mathcal{U}_j$ for any $j \in [\lambda]$. Finally, for each $i \in [\theta]$, our choice of the $\mathcal{M}_1$-edge and $\mathcal{M}_2$-edge of $\mathcal{F}_i$ ensures that v is not contained in an edge of $(\mathcal{M}_1 \cup \mathcal{M}_2) \cap \mathcal{F}_i$. Therefore, v is not contained in an edge of $\mathcal{M}_1 \cup \mathcal{M}_2$. Similarly, if $vz \in M_{\mathcal{L}}$, no edge of $\mathcal{M}_1 \cup \mathcal{M}_2$ contains u, v, or w.

Now, if p is not an $\mathcal{M}_1$-vertex of V_1, let $\bar{\mathcal{M}}_1$ be the set of edges of $\mathcal{H}$ obtained from $\mathcal{M}_1$ by removing xyz and the $\mathcal{M}_1$-edge of $\mathcal{L}'$ and adding $p\beta\gamma$, $\alpha'yu$, and αvz. Since no edge of $\mathcal{M}_1 \cup \mathcal{M}_2$ contains u or v, we see that $\bar{\mathcal{M}}_1$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$, contradicting the maximality of $\mathcal{M}_1$. Thus, we may assume that p is an $\mathcal{M}_1$-vertex. But, in this case, we let $\bar{\mathcal{M}}_2$ be the set of edges obtained from $\mathcal{M}_2$ by removing $x'y'z$ and the $\mathcal{M}_2$ edge of $\mathcal{L}'$ and adding $p\beta\gamma$, $\alpha'y'w$, and αvz. Similarly $\bar{\mathcal{M}}_2$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$. Thus, the vertex p does not exist and, hence, α' is essential. □

Corollary 3.5.10. *If $\mathcal{L}'$ is loose 3-cycle, then $A \cup \{\alpha'\} \cup (C \backslash \{\gamma\}) \cup F^1 \cup U$ is a minimum cover of $\mathcal{H}$.*

Proof: Suppose, for a contradiction, that $T = A \cup \{\alpha'\} \cup (C \backslash \{\gamma\}) \cup F^1 \cup U$ is not a cover of $\mathcal{H}$. By Theorem 3.3.16, $A \cup C \cup F^1 \cup U$ is a minimum cover of $\mathcal{H}$. So if T is not a cover, there is an edge $e \in \mathcal{H} \backslash T$ such that $\gamma \in e$. Notice that e does not contain a vertex of A or U. Furthermore, it is not an edge of $\mathcal{F}_i$ for any $i \in [\theta]$. Therefore, by Lemma 3.3.15, e also contains the vertex β. Now, Lemma 3.5.9 tells us that α' is essential. This means we have $e \in \{\alpha\beta\gamma, \alpha'\beta\gamma\}$. However, $\{\alpha, \alpha'\} \subseteq T$, which contradicts our assumption that e is not covered by T. Finally, since $A \cup C \cup F^1 \cup U$ is a minimum cover of $\mathcal{H}$ and $|T| = |A \cup C \cup F^1 \cup U|$, T is a minimum cover of $\mathcal{H}$, as required. □

We are finally ready to prove the main result of Section 3.5.

Theorem 3.5.11. *The standard family $\mathcal{S}$ has no loose even cycles.*

Proof: Suppose, for a contradiction, that $\mathcal{S}$ has a loose even cycle component. We aim to find a strong brush for $\mathcal{L}$ that is also a special matching or find a matching of $\mathcal{H}$ of size $\nu(\mathcal{H})+1$ in $\mathcal{H}$. Let $\mathcal{L}$ and $\mathcal{L}'$ be the components of $\mathcal{S}$ given by Lemma 3.5.7 and suppose that $\mathcal{L}$ has length $2l$. We consider a partial cover T of $\mathcal{H}$. If $\mathcal{L}'$ is a loose 3-cycle of $\mathcal{S}$, then we set $T = A \cup \{\alpha'\} \cup (C\backslash\{\gamma\}) \cup F^1 \cup U\backslash V_2(\mathcal{L})$. If $\mathcal{L}'$ is a copy of $\mathcal{F}$, then we set $T = A \cup C \cup F^1 \cup U\backslash V_2(\mathcal{L})$. By Theorem 3.3.16 and Corollary 3.5.10, $T \cup V_2(\mathcal{L})$ is a minimum cover of $\mathcal{H}$ in both cases. Since $|T| = 2\nu(\mathcal{H}) - |V_2(\mathcal{L})|$, every cover of $\mathcal{H}\backslash T$ has size at least $|V_2(\mathcal{L})|$. Therefore, by Lemma 2.2.3, there is a matching N^* of size $|V_2(\mathcal{L})|$ in $lk_{\mathcal{H}\backslash T}(V_1)$. We also see that every edge of N^* has exactly one end in $V_2(\mathcal{L})$, otherwise $T \cup V_2(\mathcal{L})$ is not a minimum cover of $\mathcal{H}$. Furthermore, if $\mathcal{L}'$ is a loose 3-cycle, every edge of N^* has exactly one end in $V_3\backslash((C\backslash\{\gamma\}) \cup U)$ and if $\mathcal{L}'$ is a copy of $\mathcal{F}$, every edge of N^* has exactly one end in $V_3\backslash(C \cup U)$. Recall by Definition 3.3.1 (d), that $l \geq 2$. Therefore, in both cases, there exists an edge $e \in N^*$ such that e does not contain $\gamma \in V_3(\mathcal{L}')$. Note that e does not complete to α or α' by our choices of T.

Case 1: $\gamma' \in e$.

Let $\bar{N}$ be the matching obtained from $M_{\mathcal{L}}$ by removing the edge of $M_{\mathcal{L}}$ which intersects e and then adding e. We claim that $\bar{N}$ is both a special matching for $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$ and a strong brush for $\mathcal{L}$.

Claim 1: $\bar{N}$ is a special matching for $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$.

Proof of Claim 1: Recall that $\bar{N}\backslash\{e\} \subseteq M_{\mathcal{L}}$ and $\gamma' \in e$. By Definition 3.3.5, no edge of $M_{\mathcal{L}}$ contains γ'. Therefore, since $M_{\mathcal{L}}$ is a matching of $lk_{\mathcal{H}}(V_1)$, so is $\bar{N}$. Furthermore, the definition of $\bar{N}$ ensures that $\bar{N}$ is bijectively covered by $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$. Since $M_{\mathcal{L}}$ is a special matching for $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$, $(V(\bar{N}\backslash\{e\})\backslash W) \cap (B \cup C \cup U) = \emptyset$, and, for each $i \in [\theta]$, $|(V(\bar{N}\backslash\{e\})\backslash W) \cap V(\mathcal{F}_i)| = 0$. As $\gamma' \in e$, we have $(V(\bar{N})\backslash W) \cap (B \cup C \cup U) = \emptyset$, and, for each $i \in [\theta]$, $|(V(\bar{N})\backslash W) \cap V(\mathcal{F}_i)| \leq 1$. Thus, $\bar{N}$ is a special matching for $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$, as required.

Claim 2: $\bar{N}$ is a strong brush for $\mathcal{L}$.

Proof of Claim 2: By construction, $\bar{N}$ is bijectively covered by $V_2(\mathcal{L}) \cup V_3(\mathcal{L})$. Let $h \in \bar{N}\backslash\{e\}$ such that h meets $\mathcal{L}$ in $V_3(\mathcal{L})$. Recall that e meets $\mathcal{L}$ in $V_2(\mathcal{L})$ and $\bar{N}\backslash\{e\} \subseteq M_{\mathcal{L}}$.

By Lemma 3.5.7, h completes to a vertex of $\{\alpha, \alpha'\} = V_1(\mathcal{L}')$. By our choices for T, e and h do not complete to V_1-vertices of the same component of $\mathcal{S}$. Therefore, since $M_{\mathcal{L}}$ is a strong brush for $\mathcal{L}$, to show that $\bar{N}$ is a strong brush for $\mathcal{L}$, it suffices to show that the condition from Definition 3.5.3 is satisfied for e and h. In particular, we show that there is a maximum matching $\mathcal{M}$ of $\mathcal{S} \backslash \mathcal{L}$ such that every edge of $\mathcal{M}$ is disjoint from both $\pi(e)$ and $\pi(h)$.

Let $i \in [\theta]$. If $\mathcal{F}_i = \mathcal{L}'$, then since $\pi(h)$ only meets $\mathcal{L}'$ in $V_1(\mathcal{L}')$ and $\pi(e)$ only meets $\mathcal{L}'$ at γ', there is an edge $f_i \in \{\alpha'\beta\gamma, \alpha\beta'\gamma\}$ such that f_i is disjoint from both $\pi(e)$ and $\pi(h)$. Otherwise, since $\pi(e) \notin \mathcal{F}_i$, there is an edge $f_i \in \mathcal{F}_i$ which is disjoint from $\pi(e)$, by Lemma 3.3.9. Let $j \in [\lambda]$. If $\mathcal{U}_j = \mathcal{L}$, then, as above, there is an edge $r_j \in \{\alpha'\beta\gamma, \alpha\beta'\gamma\}$ such that r_j is disjoint from both $\pi(e)$ and $\pi(h)$. Otherwise, since $\pi(e)$ can only meet $\mathcal{U}_j$ in $V_1(\mathcal{U}_j)$, there is an edge $r_j \in \mathcal{U}_j$ which is disjoint from $\pi(e)$. By Theorem 3.4.7, r_j is a maximum matching of $\mathcal{U}_j$ for all $j \in [\lambda]$. Suppose that $\mathcal{L} = \mathcal{V}_\omega$ and let $k \in [\omega - 1]$. Since $\pi(e)$ can only meet $\mathcal{V}_k$ in $V_1(\mathcal{V}_k)$, Lemma 3.4.3 tells us there is a maximum matching $\mathcal{Y}_k$ of $\mathcal{V}_k$ such that every edge of $\mathcal{Y}_k$ is disjoint from $\pi(e)$. Now, let

$$\mathcal{M} = \bigcup_{i=1}^{\theta}\{f_i\} \cup \bigcup_{j=1}^{\lambda}\{r_j\} \cup \bigcup_{k=1}^{\omega-1}\mathcal{Y}_k.$$

Since $\mathcal{M}$ is a union of maximum matchings of components of $\mathcal{S}\backslash\mathcal{L}$, $\mathcal{M}$ is a maximum matching of $\mathcal{S}\backslash\mathcal{L}$. By construction, every edge of $\mathcal{M}$ is disjoint from $\pi(e)$. Since $h \in \bar{N}\backslash\{e\} \subseteq M_{\mathcal{L}}$ and h completes to a vertex of $V_1(\mathcal{L}')$, every edge of $\mathcal{M}$ is also disjoint from $\pi(h)$. Hence, $\bar{N}$ is a strong brush for $\mathcal{L}$.

By Claim 1, Lemma 3.5.6 tells us there are two edges of $\bar{N}$ which meet the same edge of $lk_{\mathcal{L}}(V_1)$ and complete to vertices of $V_1 \backslash V_1(\mathcal{L})$. Since, by Claim 2, $\bar{N}$ is a strong brush for $\mathcal{L}$, Lemma 3.5.4 says that every edge of $\bar{N}$ completes to a vertex of $V_1(\mathcal{L}')$. However, our choices of T ensure that e does not complete to a vertex of $V_1(\mathcal{L}')$. This contradiction yields Case 1.

Case 2: $\gamma, \gamma' \notin e$.

Let $f, g \in lk_{\mathcal{L}}(V_1)$ be the edges incident to e and let m_f and m_g be edges of $M_{\mathcal{L}}$ which meet f and g on the end opposite from e. Since $lk_{\mathcal{H}}(V_1)$ is bipartite, note that e, m_f, and m_g are pairwise disjoint. Since $m_f \in M_{\mathcal{L}}$ and e has exactly one end in $V_2(\mathcal{L})$ and exactly one end in $V_3 \backslash (C \cup U \cup \{\gamma, \gamma'\})$, there is a good pair of matchings $(\mathcal{M}_1, \mathcal{M}_2)$ associated to $\mathcal{S}$ such that every edge of $\mathcal{M}_1 \cup \mathcal{M}_2$ is disjoint from $(m_f \cup e) \backslash V(\mathcal{L})$. Since $\mathcal{L}$ is a loose even cycle, we may assume without loss of generality that $\pi(f) \in \mathcal{M}_1$ and $\pi(g) \in \mathcal{M}_2$. Furthermore, since $M_{\mathcal{L}} \subseteq M$ and M is compatible with $\mathcal{S}$, every edge of $\mathcal{M}_1 \cup \mathcal{M}_2$ is also disjoint from $m_g \backslash V(\mathcal{L})$. By Lemma 3.4.1, e completes to a vertex of $V_1(\mathcal{S}) = V_1(\mathcal{M}_1 \cup \mathcal{M}_2)$. But our choices for T ensure that e does not complete to α or α'. If e completes to an $\mathcal{M}_1$-vertex of V_1, let $\bar{\mathcal{M}}_2$ be the set of edges obtained from $\mathcal{M}_2$ by removing the $\mathcal{M}_2$-edge of $\mathcal{L}'$ and $\pi(g)$, and adding $\pi(e)$, $\pi(m_g)$, and the edge of $\mathcal{L}'$ disjoint from $\pi(m_g)$. Since $\pi(e)$ is disjoint from $V(\mathcal{L}')$, we see that $\bar{\mathcal{M}}_2$ is a matching of $\mathcal{H}$ of size $\nu(\mathcal{H}) + 1$, contradicting the maximality of $\mathcal{M}_2$. However, a similar argument shows that e does not complete to an $\mathcal{M}_2$-vertex of V_1 either. This contradiction yields the theorem. $\square$

3.6 The Characterization

In this section, we complete the characterization of 3-uniform, tripartite hypergraphs satisfying $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. For convenience, we recall the definition of a home-base hypergraph.

Definition 3.1.1. A 3-uniform, tripartite hypergraph $\mathcal{H}$ is a *home-base hypergraph* if there exist integers $\eta, \mu \geq 0$ such that

(a) $\mathcal{H}$ contains η copies $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\eta$ of $\mathcal{F}$;

(b) $\mathcal{H}$ contains μ copies $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_\mu$ of $\mathcal{R}$;

(c) $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\eta, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_\mu$ are pairwise vertex-disjoint;

(d) $\nu(\mathcal{H}) = \eta + \mu$; and

(e) if e is an edge of $\mathcal{H}$ which is not an edge of $\bigcup_{i=1}^{\eta} \mathcal{F}_i$, then there is a $k \in [\mu]$ such that e contains at least two vertices of degree two in $\mathcal{R}_k$.

Proposition 3.6.1. *Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph. If $\mathcal{H}$ has a standard family $\mathcal{S}$ such that every component of $\mathcal{S}$ is either a copy of $\mathcal{F}$ or a loose 3-cycle, then $\mathcal{H}$ is a home-base hypergraph.*

Proof: Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph and let $\mathcal{S}$ be a standard family with θ copies of $\mathcal{F}$, $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\theta$, λ loose odd cycles of length three, $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_\lambda$, and $\omega = 0$ loose even cycles. We verify parts (a) - (e) of Definition 3.1.1. Let $\eta = \theta$ and $\mu = \lambda$. Since a loose 3-cycle is a copy of $\mathcal{R}$, parts (a) and (b) of Definition 3.1.1 are satisfied. Since $\mathcal{S}$ is a standard family, $\mathcal{F}_1, \ldots, \mathcal{F}_\theta, \mathcal{U}_1, \ldots, \mathcal{U}_\lambda$ are pairwise vertex-disjoint and, hence, part (c) is also satisfied. Now, we notice that, for all $j \in [\lambda]$, $\nu(\mathcal{U}_j) = 1$ since $\mathcal{U}_j$ is a loose 3-cycle. Therefore, by Definition 3.3.1 (f),

$$\nu(\mathcal{H}) = \theta + \sum_{j=1}^{\lambda} \nu(\mathcal{U}_j) = \eta + \mu,$$

which confirms part (d).

To show part (e), we first note that, since $\mathcal{S}$ consists of only copies of $\mathcal{F}$ and loose 3-cycles, we have $U = \emptyset$. Now let $e \in \mathcal{H}$ such that $e \notin \bigcup_{i=1}^{\eta} \mathcal{F}_i$. If $a_s \in e$ for some $s \in [\lambda]$, then Lemma 3.3.12 says that e contains b_s, c_s, or both of b_j and c_j for some $j \in [\lambda]$. If $a_s \notin e$ for all $s \in [\lambda]$, then Lemma 3.3.15 tells us that e contains both b_j and c_j for some $j \in [\lambda]$. In both cases, e contains two vertices of degree two in $\mathcal{R}_k$ for some $k \in [\lambda]$. This verifies part (e). Thus, $\mathcal{H}$ is a home-base hypergraph, as required. □

Proposition 3.6.2 (Haxell, Narins, Szabó [46])**.** *If $\mathcal{H}$ is a home-base hypergraph, then $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$.*

Proof: Let $\mathcal{H}$ be a home-base hypergraph. By Theorem 1.1.2, it suffices to show that $\tau(\mathcal{H}) \geq 2\nu(\mathcal{H})$. Let $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\eta, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_\mu\}$ be a spine of $\mathcal{H}$. We notice that $\nu(\mathcal{F}_i) = 1$ and $\tau(\mathcal{F}_i) = 2$ for all $i \in [\eta]$ and $\nu(\mathcal{R}_j) = 1$ and $\tau(\mathcal{R}_j) = 2$ for all $j \in [\mu]$. Since the elements of $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\eta, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_\mu\}$ are pairwise vertex-disjoint, we have

$$\tau(\mathcal{H}) \geq 2(\eta + \mu) = 2\nu(\mathcal{H}),$$

as required. □

We may now complete proof of the characterization.

Theorem 3.1.2 (Haxell, Narins, Szabó [45, 46])**.** *If $\mathcal{H}$ is a 3-uniform, tripartite hypergraph, then $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ if and only if $\mathcal{H}$ is a home-base hypergraph.*

Proof: Let $\mathcal{H}$ be a 3-uniform, tripartite hypergraph. First, we suppose that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. By Theorems 3.3.6, 3.4.7, and 3.5.11, $\mathcal{H}$ has a standard family $\mathcal{S}$ such that each component of $\mathcal{S}$ is either a copy of $\mathcal{F}$ or a loose 3-cycle. Therefore by Proposition 3.6.1, $\mathcal{H}$ is a home-base hypergraph. Conversely, if $\mathcal{H}$ is a home-base hypergraph, then by Proposition 3.6.2, we have $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. Therefore, $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ if and only if $\mathcal{H}$ is a home-base hypergraph, as required. □

Chapter 4

Packing and Covering Triangles

We begin by recalling some definitions. We say that a graph G is *triangle-free* if G has no subgraph isomorphic to a triangle. A *triangle packing* of G is a set of pairwise edge-disjoint triangles of G. A *triangle cover* of G is a set of edges of G whose deletion creates a triangle-free graph. We will denote the sizes of a maximum triangle packing of G and a minimum triangle cover of G by $\nu_\triangledown(G)$ and $\tau_\triangledown(G)$, respectively. Our motivation for this chapter is Tuza's conjecture.

Conjecture 1.2.1 (Tuza [86])**.** *If G is a graph, then $\tau_\triangledown(G) \leq 2\nu_\triangledown(G)$.*

Currently, the best result for all graphs is due to Haxell, who showed that $\tau_\triangledown(G) \leq (3 - \frac{3}{23})\nu_\triangledown(G)$ [41]. However, as described in Section 1.2, many partial results are known (e.g. see [18, 42, 43, 44, 58, 60, 68, 88, 91]). In particular, Tuza's conjecture is true for tripartite graphs. One proof of this fact is as follows. The *triangle hypergraph* of G, denoted $\mathcal{H}_G$, is the 3-uniform hypergraph with vertex set $E(G)$ where efg is an edge of $\mathcal{H}_G$ if and only if e, f, and g are the edges of a triangle of G. Notice that if G is a tripartite graph, then $\mathcal{H}_G$ is a 3-uniform, tripartite hypergraph. We also see that matchings and vertex covers of $\mathcal{H}_G$ correspond exactly to triangle packings and triangle covers of G, respectively. Thus, $\nu(\mathcal{H}_G) = \nu_\triangledown(G)$ and $\tau(\mathcal{H}_G) = \tau_\triangledown(G)$. Now Theorem 1.1.2 implies Conjecture 1.2.1 for tripartite graphs.

However, Haxell and Kohayakawa proved Conjecture 1.2.1 for tripartite graphs in 1998 without topological methods. In fact, they proved that if G is a tripartite graph, then $\tau_\triangledown(G) \leq 1.956\nu_\triangledown(G)$ [42]. Haxell and Kohayakawa also found two tripartite graphs H_1 and H_2 (see Figure 4.1) such that

$$\tau_\triangledown(H_1) = \tau_\triangledown(H_2) = \frac{5}{4}\nu_\triangledown(H_1) = \frac{5}{4}\nu_\triangledown(H_2) \ . \qquad (4.1)$$

For each graph in Figure 4.1, the tripartition is given by the letters A, B, and C; the bold edges form a triangle packing of size four; and the dotted edges form a triangle cover of size five. To see that these are optimal, we consider the *triangle graph* T_G of G; that is, T_G is the graph on the triangles of G such that two vertices of T_G form an edge if and only if the corresponding triangles in G share an edge. Then we see that both T_{H_1} and T_{H_2} are isomorphic to a cycle of length nine. The equalities in (4.1) now follow from the observation that independent sets of T_{H_1} and T_{H_2} correspond to triangle packings of H_1 and H_2 and edge-covers of T_{H_1} and T_{H_2} correspond to triangle covers of H_1 and H_2.

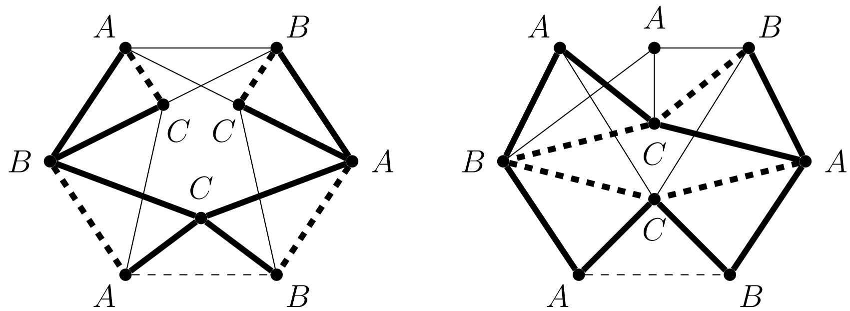

Figure 4.1: Tripartite examples to show that $\tau_\triangledown(G) = \frac{5}{4}\nu_\triangledown(G)$ is possible [42].

In this chapter, we improve the bound of Haxell and Kohayakawa: We show that if G is a tripartite graph, then $\tau_\triangledown(G) \leq 1.87\nu_\triangledown(G)$. While our techniques will be similar to that of [42], we will make use of Theorem 3.2.2 and several additional arguments.

4.1 Tripartite Graphs

Let $G = (V_1 \cup V_2 \cup V_3, E)$ be a tripartite graph and let $\mathcal{P}$ be a triangle packing of G. In this section, we prove that $\tau_\triangledown(G) \leq \frac{28}{15}\nu_\triangledown(G)$.

For each $\lambda \in \{1, 2, 3\}$, E_λ will denote the set of edges of G which do not have an endpoint in V_λ and $E_\lambda(\mathcal{P})$ will denote the subset of $E(\mathcal{P})$ contained in E_λ. An edge e of G is $\mathcal{P}$-*essential* if there exist triangles T and U such that $T \in \mathcal{P}$, T and U share the edge e, and U is otherwise edge-disjoint from $\mathcal{P}$. Finally, we define $W_\lambda(\mathcal{P})$ to be the set of $\mathcal{P}$-essential edges in E_λ and $\eta_\lambda(\mathcal{P}) := |W_\lambda(\mathcal{P})|$. The work in [42] relied on the following two lemmas. We shall do the same.

Lemma 4.1.1 (Haxell and Kohayakawa [42])**.** *Let $G = (V_1 \cup V_2 \cup V_3, E)$ be a tripartite graph and let $\lambda \in \{1, 2, 3\}$ be fixed. If $\mathcal{P}$ is a triangle packing of G, then one of the following statements holds.*

(a) There exists a triangle packing $\mathcal{P}'$ of G such that $|\mathcal{P}'| = |\mathcal{P}|+1$ and $E_{\lambda'}(\mathcal{P}) \subset E_{\lambda'}(\mathcal{P}')$ for both $\lambda' \neq \lambda$.

(b) We have $\tau_\triangledown(G) \leq 2|\mathcal{P}|$.

Lemma 4.1.2 (Haxell and Kohayakawa [42])**.** *Let $G = (V_1 \cup V_2 \cup V_3, E)$ be a tripartite graph and let $\lambda \in \{1, 2, 3\}$ be fixed. If $\mathcal{P}$ is a triangle packing of G, then one of the following statements holds.*

(a) There exists a triangle packing $\mathcal{P}'$ of G such that $|\mathcal{P}'| = |\mathcal{P}| + 1$ and $|E_{\lambda'}(\mathcal{P}) \cap E_{\lambda'}(\mathcal{P}')| \geq |\mathcal{P}| - 1$ for both $\lambda' \neq \lambda$.

(b) We have $\tau_\triangledown(G) \leq 2|\mathcal{P}| - \eta_\lambda(\mathcal{P})$.

The proof of our bound, like that of [42], starts with two triangle packings $\mathcal{P}_1$ and $\mathcal{P}_2$ of G. Using Lemmas 4.1.1 and 4.1.2, we construct two maximal triangle packings $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$. We then choose a suitably small triangle cover of G from $E(\mathcal{P}_1^*) \cup E(\mathcal{P}_2^*)$. One important difference in our method from [42] is the choice of the original triangle packings $\mathcal{P}_1$ and $\mathcal{P}_2$. Let us recall Theorem 3.2.2.

Theorem 3.2.2. *If $\mathcal{H}$ is a 3-uniform, tripartite hypergraph, then $\mathcal{H}$ has a good pair of matchings.*

Recall that the triangle hypergraph $\mathcal{H}_G$ of G is a 3-uniform, tripartite hypergraph. Therefore, we can translate Theorem 3.2.2 into a statement about triangle packings of G.

Theorem 4.1.3. *If $G = (V_1 \cup V_2 \cup V_3, E)$ is a tripartite graph, then there exist two disjoint triangle packings $\mathcal{P}_1$ and $\mathcal{P}_2$ of G with the following properties:*

(a) $|\mathcal{P}_1| + |\mathcal{P}_2| \geq \tau_\triangledown(G)$ and

(b) every edge of E_1 lies in at most one triangle of $\mathcal{P}_1 \cup \mathcal{P}_2$.

Note that in the hypergraph $\mathcal{H}_G$, part (c) of Theorem 3.2.2 is automatically satisfied: If two distinct edges of $E_2 \cup E_3$ are contained in the triangles T_1 and T_2, then $T_1 = T_2$ since G does not have any parallel edges.

For the remainder of this section, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two triangle packings of G given by Theorem 4.1.3 so that $|\mathcal{P}_1| + |\mathcal{P}_2| \geq \tau_\triangledown(G)$ and $|\mathcal{P}_1| \geq \frac{\tau_\triangledown(G)}{2}$. As advertised earlier, we now apply Lemmas 4.1.1 and 4.1.2 to obtain our final two triangle packings, $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$.

Lemma 4.1.4. *If $G = (V_1 \cup V_2 \cup V_3, E)$ is a tripartite graph, then there exist triangle packings $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ of G with the following properties:*

(a) For each $i \in \{1, 2\}$ and $\lambda \in \{2, 3\}$, $\tau_\triangledown(G) \leq 2|\mathcal{P}_i^*| - \eta_\lambda(\mathcal{P}_i^*)$,

(b) $|E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2^*)| \leq 2(|\mathcal{P}_1^*| + |\mathcal{P}_2^*| - \tau_\triangledown(G))$, and

(c) $|E_j(\mathcal{P}_1^*) \cup E_j(\mathcal{P}_2^*)| \leq 2|\mathcal{P}_1^*| + |\mathcal{P}_2^*| + 3\nu_\triangledown(G) - \dfrac{5}{2}\tau_\triangledown(G)$ for some $j \in \{2, 3\}$.

Proof: Define $\nu_\triangledown = \nu_\triangledown(G)$ and $\tau_\triangledown = \tau_\triangledown(G)$. We start with $\mathcal{P}_1$ and repeatedly apply Lemma 4.1.2 with $\lambda \in \{2, 3\}$ to obtain a sequence $\mathcal{P}_1 = \mathcal{X}^1, \mathcal{X}^2, \ldots, \mathcal{X}^l = \mathcal{P}_1^*$ of triangle packings of G such that for all $i \in [l-1]$, $|\mathcal{X}^{i+1}| = |\mathcal{X}^i| + 1$, $|E_1(\mathcal{X}^i) \cap E_1(\mathcal{X}^{i+1})| \geq |\mathcal{X}^i| - 1$, $\tau_\triangledown \leq 2|\mathcal{P}_1^*| - \eta_2(\mathcal{P}_1^*)$, and $\tau_\triangledown \leq 2|\mathcal{P}_1^*| - \eta_3(\mathcal{P}_1^*)$. Recall, by Theorem 4.1.3 (b), that $E_1(\mathcal{P}_1) \cap E_1(\mathcal{P}_2) = \emptyset$. Furthermore, by Lemma 4.1.2 (a), $|E_1(\mathcal{X}^{i+1}) \cap E_1(\mathcal{P}_2)| \leq |E_1(\mathcal{X}^i) \cap E_1(\mathcal{P}_2)| + 2$ for all $i \in [l-1]$. Since there are $|\mathcal{P}_1^*| - |\mathcal{P}_1|$ applications of Lemma 4.1.2, we have $|E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2)| \leq 2(|\mathcal{P}_1^*| - |\mathcal{P}_1|)$. Therefore,

$$\begin{aligned}
|E_1(\mathcal{P}_2) \backslash E_1(\mathcal{P}_1^*)| &= |\mathcal{P}_2| - |E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2)| \\
&\geq |\mathcal{P}_2| - 2(|\mathcal{P}_1^*| - |\mathcal{P}_1|) \\
&= 2|\mathcal{P}_1| + |\mathcal{P}_2| - 2|\mathcal{P}_1^*| \\
&\geq \tau_\triangledown + |\mathcal{P}_1| - 2|\mathcal{P}_1^*|,
\end{aligned}$$

where the last inequality follows from Theorem 4.1.3 (a).

Now, from $\mathcal{P}_2$, we repeatedly apply Lemma 4.1.1 with $\lambda \in \{2, 3\}$ to obtain a sequence $\mathcal{P}_2 = \mathcal{Y}^1, \mathcal{Y}^2, \ldots, \mathcal{Y}^t = \mathcal{P}_2'$ of triangle packings of G such that $|\mathcal{P}_2'| \geq \frac{\tau_\triangledown}{2}$ and, for all $i \in [t-1]$, $E_1(\mathcal{Y}^i) \subset E_1(\mathcal{Y}^{i+1})$. Therefore $E_1(\mathcal{P}_2) \subseteq E_1(\mathcal{P}_2')$ and, hence,

$$|E_1(\mathcal{P}_2') \backslash E_1(\mathcal{P}_1^*)| \geq \tau_\triangledown + |\mathcal{P}_1| - 2|\mathcal{P}_1^*|. \tag{4.2}$$

Finally, we repeatedly apply Lemma 4.1.2 with $\lambda \in \{2, 3\}$ to obtain a sequence $\mathcal{P}_2' = \mathcal{Z}^1, \mathcal{Z}^2, \ldots, \mathcal{Z}^s = \mathcal{P}_2^*$ of triangle packings of G such that for all $i \in [s-1]$, $|\mathcal{Z}^{i+1}| = |\mathcal{Z}^i| + 1$, $|E_1(\mathcal{Z}^i) \cap E_1(\mathcal{Z}^{i+1})| \geq |\mathcal{Z}^i| - 1$, $\tau_\triangledown \leq 2|\mathcal{P}_2^*| - \eta_2(\mathcal{P}_2^*)$, and $\tau_\triangledown \leq 2|\mathcal{P}_2^*| - \eta_3(\mathcal{P}_2^*)$, which now proves (a). Notice that, for all $i \in [s-1]$, $|E_1(\mathcal{Z}^{i+1}) \backslash E_1(\mathcal{P}_1^*)| \geq |E_1(\mathcal{Z}^i) \backslash E_1(\mathcal{P}_1^*)| - 1$ by

Lemma 4.1.2 (a). Since $\mathcal{P}_2^*$ is constructed with at most $|\mathcal{P}_2^*| - \frac{\tau_\triangledown}{2}$ applications of Lemma 4.1.2, we see that (4.2) yields

$$
\begin{aligned}
|E_1(\mathcal{P}_2^*)\backslash E_1(\mathcal{P}_1^*)| &\geq |E_1(\mathcal{P}_2')\backslash E_1(\mathcal{P}_1^*)| - \left(|\mathcal{P}_2^*| - \frac{\tau_\triangledown}{2}\right) \\
&\geq (\tau_\triangledown + |\mathcal{P}_1| - 2|\mathcal{P}_1^*|) - |\mathcal{P}_2^*| + \frac{\tau_\triangledown}{2} \\
&= \frac{3\tau_\triangledown}{2} + |\mathcal{P}_1| - 2|\mathcal{P}_1^*| - |\mathcal{P}_2^*|. \qquad (4.3)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2^*)| &= |\mathcal{P}_2^*| - |E_1(\mathcal{P}_2^*)\backslash E_1(\mathcal{P}_1^*)| \\
&\leq |\mathcal{P}_2^*| - \left(\frac{3\tau_\triangledown}{2} + |\mathcal{P}_1| - 2|\mathcal{P}_1^*| - |\mathcal{P}_2^*|\right) \\
&= 2(|\mathcal{P}_1^*| + |\mathcal{P}_2^*|) - \frac{3\tau_\triangledown}{2} - |\mathcal{P}_1| \\
&\leq 2(|\mathcal{P}_1^*| + |\mathcal{P}_2^*| - \tau_\triangledown), \qquad (4.4)
\end{aligned}
$$

where the last inequality follows from the initial assumption that $|\mathcal{P}_1| \geq \frac{\tau_\triangledown}{2}$. This proves (b).

Let $\mathcal{S}$ be the subset of triangles of $\mathcal{P}_2^*$ which share their E_1-edge with a triangle of $\mathcal{P}_1^*$. Notice that $|\mathcal{S}| = |E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2^*)| \leq 2(|\mathcal{P}_1^*| + |\mathcal{P}_2^*| - \tau_\triangledown)$. For an arbitrary triangle packing $\mathcal{P}$ of G, let $\hat{\nu}_1(\mathcal{P})$ denote the size of a maximum triangle packing $\mathcal{T}$ of G such that every triangle in $\mathcal{T}$ shares only its E_1-edge with a triangle of $\mathcal{P}$. Since $\mathcal{S}$ is a triangle packing of G, at most $\hat{\nu}_1(\mathcal{P}_1^*)$ triangles of $\mathcal{S}$ do not share their E_2 nor E_3-edge with a triangle in $\mathcal{P}_1^*$. Thus, at least $|\mathcal{S}| - \hat{\nu}_1(\mathcal{P}_1^*)$ triangles of $\mathcal{S}$ share either their E_2 or E_3-edge with a triangle in $\mathcal{P}_1^*$. Without loss of generality, we may assume that at least $\frac{1}{2}(|\mathcal{S}| - \hat{\nu}_1(\mathcal{P}_1^*))$ triangles of $\mathcal{S}$ share their E_2-edge with a triangle in $\mathcal{P}_1^*$.

Now consider the triangles in $\mathcal{P}_2^*\backslash\mathcal{S}$. Notice that no triangle of $\mathcal{P}_2^*\backslash\mathcal{S}$ shares an E_1-edge with a triangle of $\mathcal{P}_1^*$. By the definition of essential edges, at most $\eta_3(\mathcal{P}_1^*)$ triangles of $\mathcal{P}_2^*\backslash\mathcal{S}$ do not share their E_2-edge with a triangle in $\mathcal{P}_1^*$. This means that at least $|\mathcal{P}_2^*| - |\mathcal{S}| - \eta_3(\mathcal{P}_1^*)$ triangles of $\mathcal{P}_2^*\backslash\mathcal{S}$ share their E_2-edge with a triangle of $\mathcal{P}_1^*$. Therefore, we have

$$
\begin{aligned}
|E_2(\mathcal{P}_1^*) \cap E_2(\mathcal{P}_2^*)| &\geq \frac{1}{2}(|\mathcal{S}| - \hat{\nu}_1(\mathcal{P}_1^*)) + |\mathcal{P}_2^*| - |\mathcal{S}| - \eta_3(\mathcal{P}_1^*) \\
&= |\mathcal{P}_2^*| - \frac{|\mathcal{S}|}{2} - \eta_3(\mathcal{P}_1^*) - \frac{\hat{\nu}_1(\mathcal{P}_1^*)}{2}
\end{aligned}
$$

and

$$\begin{aligned}
|E_2(\mathcal{P}_1^*) \cup E_2(\mathcal{P}_2^*)| &= |\mathcal{P}_1^*| + |\mathcal{P}_2^*| - |E_2(\mathcal{P}_1^*) \cap E_2(\mathcal{P}_2^*)| \\
&\leq |\mathcal{P}_1^*| + |\mathcal{P}_2^*| - \left(|\mathcal{P}_2^*| - \frac{|\mathcal{S}|}{2} - \eta_3(\mathcal{P}_1^*) - \frac{\hat{\nu}_1(\mathcal{P}_1^*)}{2} \right) \\
&= |\mathcal{P}_1^*| + \frac{|\mathcal{S}|}{2} + \eta_3(\mathcal{P}_1^*) + \frac{\hat{\nu}_1(\mathcal{P}_1^*)}{2}. \qquad (4.5)
\end{aligned}$$

To prove (c), all that remains is to bound $\hat{\nu}_1(\mathcal{P}_1^*)$.

Claim: We have $\hat{\nu}_1(\mathcal{P}_1^*) \leq 6\nu_\triangledown - 4|\mathcal{P}_1^*| - \tau_\triangledown$.

Proof of Claim: By the definition of $\hat{\nu}_1(\mathcal{P}_1^*)$, we have $\hat{\nu}_1(\mathcal{P}_1^*) \leq \eta_1(\mathcal{P}_1^*)$. We repeatedly apply Lemma 4.1.2 with $\lambda = 1$ to obtain a sequence $\mathcal{P}_1^* = \mathcal{Q}^1, \mathcal{Q}^2, \ldots, \mathcal{Q}^k$ of triangle packings of G such that for all $i \in [k-1]$, $|\mathcal{Q}^{i+1}| = |\mathcal{Q}^i| + 1$, $|E_j(\mathcal{Q}^i) \cap E_j(\mathcal{Q}^{i+1})| \geq |\mathcal{Q}^i| - 1$ for each $j \in \{2,3\}$, and $\tau_\triangledown \leq 2|\mathcal{Q}^k| - \eta_1(\mathcal{Q}^k)$.

Let $i \in [k-1]$ and let $\mathcal{A}^i$ be a maximum triangle packing of G such that every triangle of $\mathcal{A}^i$ shares only its E_1-edge with a triangle of $\mathcal{Q}^i$ so that $|\mathcal{A}^i| = \hat{\nu}_1(\mathcal{Q}^i)$. Since $|\mathcal{Q}^{i+1}| = |\mathcal{Q}^i| + 1$ and $|E_j(\mathcal{Q}^i) \cap E_j(\mathcal{Q}^{i+1})| \geq |\mathcal{Q}^i| - 1$ for each $j \in \{2,3\}$, we have

$$|(E_2 \cup E_3) \cap (E(\mathcal{Q}^{i+1}) \backslash E(\mathcal{Q}^i))| \leq 4.$$

Let $\mathcal{X}^i$ be the set of triangles of $\mathcal{A}^i$ which meet an edge of $(E_2 \cup E_3) \cap (E(\mathcal{Q}^{i+1}) \backslash E(\mathcal{Q}^i))$. Since the triangles of $\mathcal{X}^i$ are pairwise edge-disjoint, $|\mathcal{X}^i| \leq 4$ and $|\mathcal{A}^i \backslash \mathcal{X}^i| \geq \hat{\nu}_1(\mathcal{Q}^i) - 4$. Notice that no triangle of $\mathcal{A}^i \backslash \mathcal{X}^i$ contains an edge of $(E_2 \cup E_3) \cap (E(\mathcal{Q}^{i+1}) \backslash E(\mathcal{Q}^i))$. Furthermore, by definition of $\mathcal{A}^i$, no triangle of $\mathcal{A}^i \backslash \mathcal{X}^i$ contains an edge of $(E_2 \cup E_3) \cap (E(\mathcal{Q}^{i+1}) \cap E(\mathcal{Q}^i))$. Thus, $\mathcal{A}^i \backslash \mathcal{X}^i$ is a triangle packing of G such that every triangle of $\mathcal{A}^i \backslash \mathcal{X}^i$ shares only its E_1-edge with a triangle of $\mathcal{Q}^{i+1}$. Hence $\hat{\nu}_1(\mathcal{Q}^{i+1}) \geq \hat{\nu}_1(\mathcal{Q}^i) - 4$ for all $i \in [k-1]$. Since there are $|\mathcal{Q}^k| - |\mathcal{P}_1^*|$ applications of Lemma 4.1.2, this means that $\hat{\nu}_1(\mathcal{Q}^k) \geq \hat{\nu}_1(\mathcal{P}_1^*) - 4(|\mathcal{Q}^k| - |\mathcal{P}_1^*|)$. By definition, $\eta_1(\mathcal{Q}^k) \geq \hat{\nu}_1(\mathcal{Q}^k)$. Thus,

$$\begin{aligned}
\tau_\triangledown &\leq 2|\mathcal{Q}^k| - \eta_1(\mathcal{Q}^k) \\
&\leq 2|\mathcal{Q}^k| - \hat{\nu}_1(\mathcal{Q}^k) \\
&\leq 2|\mathcal{Q}^k| - (\hat{\nu}_1(\mathcal{P}_1^*) - 4(|\mathcal{Q}^k| - |\mathcal{P}_1^*|)) \\
&\leq 6\nu_\triangledown - \hat{\nu}_1(\mathcal{P}_1^*) - 4|\mathcal{P}_1^*|.
\end{aligned}$$

Rearranging the final inequality yields the claim.

Recall, from parts (a) and (b), that $\tau_\triangledown \leq 2|\mathcal{P}_1^*| - \eta_3(\mathcal{P}_1^*)$ and $|E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2^*)| \leq 2(|\mathcal{P}_1^*| + |\mathcal{P}_2^*| - \tau_\triangledown)$. Now, by (4.5), the fact that $|\mathcal{S}| = |E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2^*)|$, and the claim, we have

$$\begin{aligned}
|E_2(\mathcal{P}_1^*) \cup E_2(\mathcal{P}_2^*)| &\leq |\mathcal{P}_1^*| + \frac{|\mathcal{S}|}{2} + \eta_3(\mathcal{P}_1^*) + \frac{\hat{\nu_1}(\mathcal{P}_1^*)}{2} \\
&\leq |\mathcal{P}_1^*| + (|\mathcal{P}_1^*| + |\mathcal{P}_2^*| - \tau_\triangledown) + (2|\mathcal{P}_1^*| - \tau_\triangledown) + \frac{\hat{\nu_1}(\mathcal{P}_1^*)}{2} \\
&\leq 4|\mathcal{P}_1^*| + |\mathcal{P}_2^*| - 2\tau_\triangledown + \frac{1}{2}(6\nu_\triangledown - 4|\mathcal{P}_1^*| - \tau_\triangledown) \\
&\leq 2|\mathcal{P}_1^*| + |\mathcal{P}_2^*| + 3\nu_\triangledown - \frac{5\tau_\triangledown}{2},
\end{aligned}$$

which proves (c), as required. $\square$

We are now ready to prove our main result.

Theorem 4.1.5. *If G is a tripartite graph, then $\tau_\triangledown(G) \leq \frac{28}{15}\nu_\triangledown(G)$.*

Proof: Let G be a tripartite graph. Define $\nu_\triangledown = \nu_\triangledown(G)$ and $\tau_\triangledown = \tau_\triangledown(G)$. We first build a triangle cover of G. Let $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ be the two triangle packings of G given by Lemma 4.1.4. Without loss of generality, we may assume that $|E_2(\mathcal{P}_1^*) \cup E_2(\mathcal{P}_2^*)| \leq 2|\mathcal{P}_1^*| + |\mathcal{P}_2^*| + 3\nu_\triangledown - \frac{5}{2}\tau_\triangledown$. Define

$$\mathcal{C} = (E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2^*)) \cup (E_2(\mathcal{P}_1^*) \cup E_2(\mathcal{P}_2^*)) \cup (W_3(\mathcal{P}_1^*) \cup W_3(\mathcal{P}_2^*)).$$

To see that $\mathcal{C}$ is a triangle cover of G, suppose that T is a triangle of G such that

$$E(T) \cap (E_2(\mathcal{P}_1^*) \cup E_2(\mathcal{P}_2^*) \cup W_3(\mathcal{P}_1^*) \cup W_3(\mathcal{P}_2^*)) = \emptyset.$$

Since $E(T) \cap (W_3(\mathcal{P}_1^*) \cup W_3(\mathcal{P}_2^*)) = \emptyset$ and both $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ are maximal, T intersects a triangle of $\mathcal{P}_1^*$ in E_1 or E_2 and a triangle of $\mathcal{P}_2^*$ in E_1 or E_2. However, $E(T) \cap (E_2(\mathcal{P}_1^*) \cup E_2(\mathcal{P}_2^*)) = \emptyset$ as well. Therefore, $E(T) \cap (E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2^*)) \neq \emptyset$, which implies that $\mathcal{C}$ is a triangle cover of G. Now, since $\mathcal{C}$ is a triangle cover of G, Lemma 4.1.4 tells us that

$$\begin{aligned}
\tau_\triangledown &\leq |E_1(\mathcal{P}_1^*) \cap E_1(\mathcal{P}_2^*)| + |E_2(\mathcal{P}_1^*) \cup E_2(\mathcal{P}_2^*)| + |W_3(\mathcal{P}_1^*) \cup W_3(\mathcal{P}_2^*)| \\
&\leq (2(|\mathcal{P}_1^*| + |\mathcal{P}_2^*| - \tau_\triangledown)) + \left(2|\mathcal{P}_1^*| + |\mathcal{P}_2^*| + 3\nu_\triangledown - \frac{5}{2}\tau_\triangledown\right) + (\eta_3(\mathcal{P}_1^*) + \eta_3(\mathcal{P}_2^*)) \\
&\leq 3\nu_\triangledown + 6|\mathcal{P}_1^*| + 5|\mathcal{P}_2^*| - \frac{13}{2}\tau_\triangledown \\
&\leq 14\nu_\triangledown - \frac{13}{2}\tau_\triangledown,
\end{aligned}$$

where the third inequality uses the fact that $\tau_\triangledown \leq 2|\mathcal{P}_1^*| - \eta_3(\mathcal{P}_1^*)$ and $\tau_\triangledown \leq 2|\mathcal{P}_2^*| - \eta_3(\mathcal{P}_2^*)$. A simple rearrangement yields $\tau_\triangledown \leq \frac{28}{15}\nu_\triangledown$, as required. □

4.1.1 A Special Case

For most of this thesis, we are concerned with the upper bounds in (1.1) and (1.2). In this short section, we consider the lower bound of (1.2). Aparna Lakshmanan, Bujtás, and Tuza showed that $\tau_\triangledown(G) = \nu_\triangledown(G)$ if G is either (K_4, gem)-free or is K_4-free and the triangle graph of G has no induced odd cycles of length at least five [60]. We examine a different class of K_4-free graphs.

Let $\mathfrak{G}_k$ denote the class of tripartite graphs G with the property that there is a bipartite graph H with bipartition (X, Y) such that

$$V(G) = X \cup Y \cup \{u_1, \dots, u_k\}$$

and

$$E(G) = E(H) \cup \{zu_1, \dots, zu_k \mid z \in X \cup Y\}.$$

We use network flow techniques to show that $\tau_\triangledown(G) = \nu_\triangledown(G)$ for all $G \in \mathfrak{G}_k$. To do so, we will rely on the following two observations about triangle packings in tripartite graphs:

- every triangle in G contains exactly one edge of H and exactly one vertex from $\{u_1, \dots, u_k\}$, and
- for each $i \in [k]$, the set of triangles that contain the vertex u_i induces a matching in H.

Proposition 4.1.6. *For any $k \in \mathbb{N}$ and $G \in \mathfrak{G}_k$, we have $\tau_\triangledown(G) = \nu_\triangledown(G)$.*

Proof: Let $k \in \mathbb{N}$ and let $G \in \mathfrak{G}_k$. Let H, X, and Y be as above. We start by constructing a capacitated directed graph $D = (N, A, c)$ where $N := X \cup Y \cup \{s, t\}$,

$$A := \{\vec{xy} \mid x \in X, y \in Y, xy \in E(H)\} \cup \{\vec{sa} \mid a \in X\} \cup \{\vec{bt} \mid b \in Y\},$$

and, for each $\vec{uv} \in A$,

$$c(\vec{uv}) = \begin{cases} 1 & : \ uv \in E(H) \\ k & : \ \text{otherwise} \end{cases}.$$

Notice that an (s,t)-path s, u, v, t in D corresponds to the k triangles of G which contain the edge uv.

Let f be a maximum (s,t)-flow in D. We may assume, via Lemma 2.3.5, that $f(\vec{uv}) \in \mathbb{N} \cup \{0\}$ for every $\vec{uv} \in A$. By the definition of c, f corresponds to a subgraph, H_f, of H with the maximum number of edges subject to $\Delta(H_f) \leq k$. Furthermore, the definition of c tells us that the number of edges of H_f is $\sum_{r:\vec{rt}\in A} f(\vec{rt})$. By Lemma 2.1.4, the edges of H_f correspond to k pairwise disjoint matchings $M_1, M_2, \ldots, M_k$ of H. If we pair up M_i and u_i for all $i \in [k]$, we obtain a triangle packing of G of size $\sum_{r:\vec{rt}\in A} f(\vec{rt})$. To prove the result, it suffices to find a triangle cover of G of size at most $\nu_\triangledown(G)$. Applying Theorem 2.3.4 and the definition of $\nu_\triangledown$, we find that D has an (s,t)-cut S such that

$$\sum_{\vec{uv}\in S} c(\vec{uv}) = \sum_{r:\vec{rt}\in A} f(\vec{rt}) \leq \nu_\triangledown(G). \tag{4.6}$$

The desired triangle cover will be built from S. Let

$$T := \{ab \mid ab \in E(H), \vec{ab} \in S\} \cup \{wu_1, \ldots, wu_k \mid \vec{sw} \in S \text{ or } \vec{wt} \in S\}.$$

Notice that $T \subseteq E(G)$ and $|T| = \sum_{\vec{uv}\in S} c(\vec{uv})$. Suppose, for a contradiction, that T is not a triangle cover of G. Then there exists an edge $ab \in E(H)$ such that the arcs $\vec{sa}, \vec{ab}, \vec{bt} \notin S$. However, $\{\vec{sa}, \vec{ab}, \vec{bt}\}$ is an (s,t)-path in $D \backslash S$, which contradicts the assumption that S is an (s,t)-cut in D. Thus, T is a triangle cover of G. Furthermore, by (4.6) and the definition of $\tau_\triangledown$, we now have

$$\tau_\triangledown(G) \leq |T| = \sum_{\vec{uv}\in S} c(\vec{uv}) \leq \nu_\triangledown(G),$$

which implies that $\tau_\triangledown(G) = \nu_\triangledown(G)$ by (1.2), as required. $\square$

Chapter 5

Packing and Covering K_4's

The goal in this chapter is to prove results about packing and covering edge-disjoint K_4's. In their Ph.D. theses, Lovász and Gyárfás showed that, if $\mathcal{H}$ is an r-uniform, r-partite hypergraph, then $\tau(\mathcal{H}) \leq \frac{r}{2}\nu^*(\mathcal{H})$ [61] and $\tau^*(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$ [37], respectively. These results immediately imply that if G is a 4-partite graph, then $\tau_{\boxtimes}(G) \leq 3\nu^*_{\boxtimes}(G)$ and $\tau^*_{\boxtimes}(G) \leq 5\nu_{\boxtimes}(G)$. More recently, Yuster proved that $\tau_{\boxtimes}(G) \leq 4\nu^*_{\boxtimes}(G)$ for any graph G [91].

We begin in Section 5.1 by discussing fractional K_4-covers. Our main theorem accompanies Yuster's result and says that $\tau^*_{\boxtimes}(G) \leq \frac{9}{2}\nu_{\boxtimes}(G)$ for any graph G. In Section 5.2, we show that $\tau_{\boxtimes}(G) \leq 5\nu_{\boxtimes}(G)$ whenever G is a 4-partite graph. We will make use of Theorem 1.1.2. In Section 5.3, we examine complete graphs. We will see that

$$\lim_{n\to\infty} \frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} = 2.$$

Our proof relies on the existence of certain combinatorial designs. As a consequence of the proof, we show that $\tau_{\boxtimes}(K_n) \leq 3\nu_{\boxtimes}(K_n)$ for all but one value of n. In Section 5.4, we show that if G has degeneracy at most eight, then $\tau_{\boxtimes}(G) \leq 5\nu_{\boxtimes}(G)$. In Section 5.5, we consider planar graphs and, more generally, graphs with no $K_{3,3}$-subdivision. We will show that $\tau_{\boxtimes}(G) \leq 3\nu_{\boxtimes}(G)$ for all such graphs G. Finally, we look at lower bounds in Section 5.6.

It is reasonable for us to assume that every edge of G is in at least one K_4; if e is an edge that is not in a K_4 of G, then e is not in a minimum K_4-cover of G, nor is it contained in a maximum K_4-packing of G. It also safe to assume that G is 2-connected; if G is not 2-connected, the problem of computing $\tau_{\boxtimes}(G)$ and $\nu_{\boxtimes}(G)$ is equivalent to computing $\tau_{\boxtimes}$ and $\nu_{\boxtimes}$ for all of G's blocks.

5.1 Fractional K_4-Covers

Let $G = (V, E)$ be a graph and let $\mathcal{K}(G)$ be the set of all subgraphs of G which are isomorphic to K_4. Recall that a fractional K_4-cover is a function $\rho : E \to [0, 1]$ such that $\sum_{e \in E(K)} \rho(e) \geq 1$ for every $K \in \mathcal{K}(G)$ and that $\tau^*_{\boxtimes}(G)$ denotes the minimum of $\sum_{e \in E} \rho(e)$ over all fractional K_4-covers ρ of G. We aim to show that

$$\tau^*_{\boxtimes}(G) \leq \frac{9}{2}\nu_{\boxtimes}(G).$$

Our technique can be summarized as follows. We start with a packing of G where each subgraph in the packing is one of six special graphs. The value of this packing is based on the number of each type of subgraph in the packing. We then assign values to the edges of the subgraphs so that if we find a K_4 in G that is not covered in the fractional sense, then we can also find a better packing. The terminology used in this section follows that of Haxell, Kostochka, and Thomassé [44]. We will use K_5^- to denote the graph obtained from K_5 by deleting any single edge and L_2 to denote the graph on six vertices consisting of two K_4's sharing exactly one edge. It is routine to check that

$$\nu_{\boxtimes}(K_5) = \nu_{\boxtimes}(K_5^-) = \nu_{\boxtimes}(L_2) = 1.$$

The *central edges* of a K_5^- are the three edges that are shared by both K_4's. The *central edge* of an L_2 is the edge connecting the two vertices of degree five. See Figure 5.1 for the pictures of K_5^- and L_2. The following lemma is immediate, but will be useful in our analysis.

Lemma 5.1.1. *Any graph obtained from K_5, K_5^-, or L_2 by deleting an edge that is not central contains a subgraph which is isomorphic to K_4.*

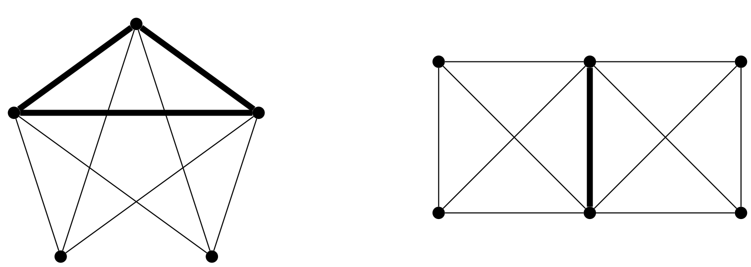

Figure 5.1: The graphs K_5^- and L_2 with bold central edges.

Let $\mathcal{T}$ be a maximum K_4-packing of G and let $F \in \{K_5, K_5^-, L_2, K_4\}$. A $\mathcal{T}$-F is a subgraph of G which is isomorphic to F, contains exactly one element of $\mathcal{T}$, and is otherwise edge-disjoint from $\mathcal{T}$. A *$\mathcal{T}$-pattern* $\mathcal{P}$ of G is a collection of edge-disjoint $\mathcal{T}$-K_5's, $\mathcal{T}$-K_5^-'s, $\mathcal{T}$-L_2's, and $\mathcal{T}$-K_4's that together contain $\mathcal{T}$. An edge of G is *used* if it belongs to some element of $\mathcal{P}$ and *unused* if it does not. If T is a $\mathcal{T}$-K_5^- or $\mathcal{T}$-L_2, we will say that T is *extendible* if there exists a K_4 that contains at least one central edge of T and otherwise only unused edges. A $\mathcal{T}$-K_5^- or $\mathcal{T}$-L_2 that is not extendible is *fixed*.

To each pair $(\mathcal{T}, \mathcal{P})$, where $\mathcal{T}$ is a maximum K_4-packing of G and $\mathcal{P}$ is a $\mathcal{T}$-pattern, we associate the triple $(\alpha_1, \alpha_2, \alpha_3)_{(\mathcal{T},\mathcal{P})}$, where α_1 is the number of $\mathcal{T}$-K_5's in $\mathcal{P}$, α_2 is the number of $\mathcal{T}$-K_5^-'s in $\mathcal{P}$, and α_3 is the number of $\mathcal{T}$-L_2's in $\mathcal{P}$. We will say that the $\mathcal{T}$-pattern $\mathcal{P}$ is *better than* the $\mathcal{T}^*$-pattern $\mathcal{P}^*$ if $(\alpha_1, \alpha_2, \alpha_3)_{(\mathcal{T},\mathcal{P})}$ is bigger than $(\alpha_1^*, \alpha_2^*, \alpha_3^*)_{(\mathcal{T}^*,\mathcal{P}^*)}$ under lexicographical ordering.

We now define a function $\varphi : E \to [0, 1]$ according to the rules below. Let $F \in \mathcal{P}$ and let $e \in E(F)$.

(a) If F is a $\mathcal{T}$-K_5, set $\varphi(e) = \frac{2}{5}$,

(b) If F is an extendible $\mathcal{T}$-K_5^-, set $\varphi(e) = 1$ if e is central and $\varphi(e) = \frac{1}{4}$ otherwise,

(c) If F is a fixed $\mathcal{T}$-K_5^-, set $\varphi(e) = \frac{5}{6}$ if e is central and $\varphi(e) = \frac{1}{3}$ otherwise,

(d) If F is an extendible $\mathcal{T}$-L_2, set $\varphi(e) = 1$ if e is central and $\varphi(e) = \frac{7}{20}$ otherwise, and

(e) If F is a fixed $\mathcal{T}$-L_2, set $\varphi(e) = \frac{3}{4}$ if e is central and $\varphi(e) = \frac{3}{8}$ otherwise, and

(f) If F is a $\mathcal{T}$-K_4, set $\varphi(e) = \frac{3}{4}$.

We also set $\varphi(f) = 0$ for any unused edge f. Notice that for any $e \in E$, $\varphi(e) \in \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{7}{20}, \frac{3}{8}, \frac{2}{5}, \frac{3}{4}, \frac{5}{6}, 1\right\}$. Furthermore, for each $F \in \mathcal{P}$, $\sum_{e\in E(F)} \varphi(e) \le \frac{9}{2}$.

Lemma 5.1.2. *Let $\mathcal{T}$ be a maximum K_4-packing of G and let $\mathcal{P}$ be a $\mathcal{T}$-pattern such that $\mathcal{P}$ is best among all patterns of all maximum K_4-packings. Then φ is a fractional K_4-cover of G.*

Proof: Suppose, for a contradiction, that φ is not a fractional K_4-cover of G. Then there exists a K_4, K, in G such that $\sum_{e\in E(K)} \varphi(e) < 1$. From the definition of φ, we may assume that K is not contained in any element of $\mathcal{P}$, nor does K contain a central edge of any extendible $\mathcal{T}$-K_5^- or $\mathcal{T}$-L_2. Furthermore, K does not contain a central edge of any fixed $\mathcal{T}$-K_5^- or $\mathcal{T}$-L_2. To see this, notice that if K contains an edge e which is central to a fixed

$\mathcal{T}$-K_5^- or $\mathcal{T}$-L_2, then the only edge of K that is used is e. However, this contradicts the assumption that the $\mathcal{T}$-K_5^- or $\mathcal{T}$-L_2 is fixed. We now make the following series of claims.

Claim 1: K contains at most one edge of a $\mathcal{T}$-K_5.

Proof of Claim 1: Let T be a $\mathcal{T}$-K_5. If K contains two edges, say e and f, of T, then the subgraph of T induced by the vertices of e and f contains an edge which is incident to both e and f, say g. Furthermore, since K is isomorphic to K_4, g is also an edge of K. Since $\varphi(e) + \varphi(f) + \varphi(g) = \frac{6}{5} \geq 1$, this is a contradiction.

Claim 2: K contains at most one non-central edge of a $\mathcal{T}$-K_5^-.

Proof of Claim 2: Let T be a $\mathcal{T}$-K_5^- and let $\{a, b, x, y, z\}$ be the vertices of T such that xy, xz, and yz are the central edges. Suppose that K contains two edges, e and f, of T. If $a \in e$ and $a \in f$ or if $\{e, f\}$ is a matching of T, then K also contains a central edge of T. Thus, we may assume that $e = ax$ and $f = bx$. Let $T^+ = G[\{a, b, x, y, z\}]$. In particular, $T^+ \cong K_5$ since $ab \in E(K)$ and there is a vertex p such that $K = G[\{p, a, x, b\}]$. Notice that $p \neq y$ or z since otherwise K contains a central edge of T. See Figure 5.2. We may also assume that ab is used, otherwise $(\mathcal{P} \backslash \{T\}) \cup \{T^+\}$ contains more $\mathcal{T}$-K_5's than $\mathcal{P}$ and, hence, is a better $\mathcal{T}$-pattern than $\mathcal{P}$.

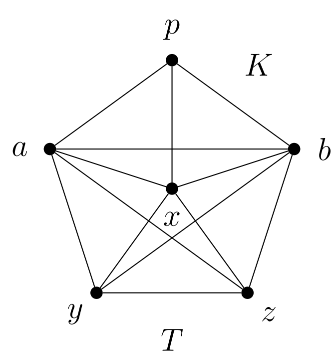

Figure 5.2: Subgraph of G containing T and K.

Since ab, ax, and bx are used and $\varphi(g) \geq \frac{1}{4}$ for every used edge $g \in E(G)$, the edges pa, px, and pb are unused, regardless of whether T is extendible or fixed. If T is fixed, then, since $\varphi(ax) = \varphi(bx) = \frac{1}{3}$ and $\sum_{e \in E(K)} \varphi(e) < 1$, ab is a non-central edge of an extendible $\mathcal{T}$-K_5^-, say A. Let B be the K_4 of A which is edge-disjoint from ab by Lemma 5.1.1. Then $(\mathcal{P} \backslash \{T, A\}) \cup \{T^+, B\}$ is a better $\mathcal{T}$-pattern than $\mathcal{P}$ since it contains more $\mathcal{T}$-K_5's than $\mathcal{P}$.

Thus, T is extendible. If ab is a non-central edge of a $\mathcal{T}$-K_5^- or $\mathcal{T}$-L_2, we may proceed as in the previous paragraph. It remains to check the case where ab belongs to a $\mathcal{T}$-K_5, call it X. Suppose that $X = G[\{a, b, u, v, w\}]$ and let $Z = G[\{b, u, v, w\}]$. We note that p is not a vertex of X since pa and pb are unused edges. Since T is extendible, there is a K_4, Y, consisting only of central edges of T and unused edges. If Y is edge-disjoint from K, then $(\mathcal{P}\backslash\{T, X\}) \cup \{K, Y, Z\}$ contains $\nu_{\boxtimes}(G) + 1$ edge-disjoint K_4's of G as K and Z are also edge-disjoint and neither contain a central edge of T. This contradicts the maximality of $\mathcal{T}$. So, we may assume $E(K) \cap E(Y) = \{px\}$.

Suppose Y contains exactly one central edge of T, say xy. Let $X^- = X\backslash ab$ and $V = G[\{a, b, x, z\}]$. See Figure 5.3. Notice that $V \cong K_4$ and that V, X^-, and Y are pairwise edge-disjoint. Therefore, $(\mathcal{P}\backslash\{T, X\}) \cup \{V, X^-, Y\}$ contains $\nu_{\boxtimes}(G)+1$ edge-disjoint K_4's of G, which contradicts the maximality of $\mathcal{T}$.

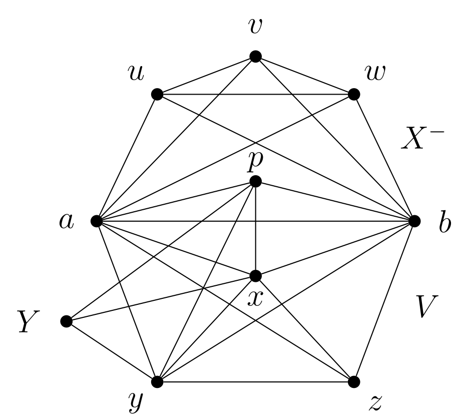

Figure 5.3: Subgraph of G containing V, X^- and Y.

Notice that Y does not contain exactly two central edges of T. Finally, if Y contains all three central edges of T, then $W = G[\{a, p, x, y, z\}] \cong K_5$ and consists only of edges of T and unused edges. See Figure 5.4. Therefore, $(\mathcal{P}\backslash\{T\}) \cup \{W\}$ contains more $\mathcal{T}$-K_5's than $\mathcal{P}$ and, hence, is a better $\mathcal{T}$-pattern than $\mathcal{P}$.

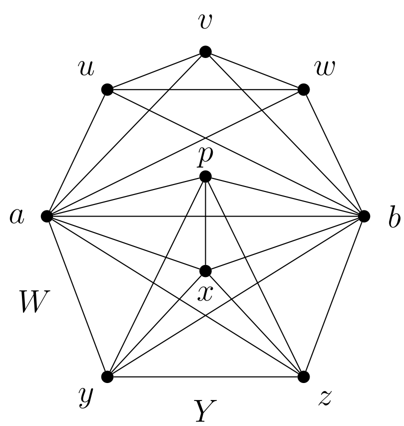

Figure 5.4: Subgraph of G containing W and Y.

Claim 3: K contains at most one non-central edge of a $\mathcal{T}$-L_2.

Proof of Claim 3: Let T be a $\mathcal{T}$-L_2 and let $\{m, n, o, r, s, t\}$ be the vertices of T such that $T[\{m, n, o, r\}]$ and $T[\{o, r, s, t\}]$ are the two copies of K_4. Notice that or is the central edge. If K contains mn and st, then $K = G[\{m, n, s, t\}]$ and $G[\{m, n, o, r, s, t\}] \cong K_6$. Furthermore, since $\varphi(mn) = \varphi(st) \geq \frac{7}{20}$ and $\sum_{e \in E(K)} \varphi(e) < 1$, at least three of the edges in $\{ms, mt, ns, nt\}$ are unused, say ms, mt, and ns. Then, we may replace T by $G[\{m, o, r, s, t\}] \cong K_5$ to obtain a better $\mathcal{T}$-pattern, contradicting our choice of $\mathcal{P}$.

If K contains mn and os, then K also contains the edge mo. Since $\varphi(mn) + \varphi(os) + \varphi(mo) \geq 1$ for both extendible and fixed $\mathcal{T}$-L_2's, this is a contradiction. If K contains mo and rt, then K contains the central edge or, a contradiction. Finally, suppose that K contains mo and os. Then, there is a vertex q such that $K = G[\{q, m, o, s\}]$, as in Figure 5.5. Furthermore, q is not a vertex of T. Indeed, if $q = r$, then K contains the central edge of T and if $q = n$ or t, then we are in the preceding case. Now, we notice that $G[\{q, m, o, r, s\}]$ contains a K_5^-, say S. Since $\varphi(mo) = \varphi(os) \geq \frac{7}{20}$ and $\sum_{e \in E(K)} \varphi(e) < 1$, at least three of $\{ms, qm, qo, qs\}$ are unused edges.

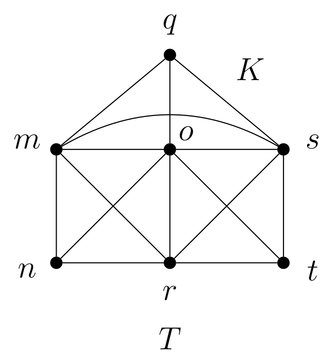

Figure 5.5: Subgraph of G containing T, K, and S.

If T is fixed, then $\varphi(mo) = \varphi(os) = \frac{3}{8}$ and all four of $\{ms, qm, qo, qs\}$ are unused. Notice that S contains only edges of T and unused edges. Thus, $(\mathcal{P}\backslash\{T\}) \cup \{S\}$ is a $\mathcal{T}$-pattern that has the same number of $\mathcal{T}$-K_5's as $\mathcal{P}$, but has more $\mathcal{T}$-K_5^-'s than $\mathcal{P}$. This contradicts our choice of $\mathcal{P}$. If T is extendible, we may assume that only three of $\{ms, qm, qo, qs\}$ are unused, otherwise we may proceed as before. Furthermore, since $\varphi(mo) = \varphi(os) = \frac{7}{20}$, the edge that is used is a non-central edge of an extendible $\mathcal{T}$-K_5^-, say U. Let R be a K_4 that made T extendible and let Q be the copy of K_4 in U that is edge-disjoint from K and T, by Lemma 5.1.1. If R is edge-disjoint from K, then $(\mathcal{P}\backslash\{T, U\}) \cup \{K, Q, R\}$ contains $\nu_{\boxtimes}(G) + 1$ pairwise edge-disjoint K_4's of G. Thus, we assume that R contains the edge qo. See Figure 5.6.

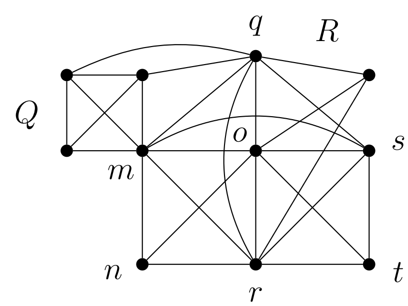

Figure 5.6: Subgraph of G containing K, Q, R, S, and T.

Let $P = G[\{q, m, o, r, s\}]$. Since R contains qo, $P \cong K_5$. Since qo and qr are edges of R, they are unused. Therefore, only one of ms, qm, or qs is used and, by above, it is an edge of U. With the help of Lemma 5.1.1, this implies that $(\mathcal{P}\backslash\{T, U\}) \cup \{P, Q\}$ has more

$\mathcal{T}$-K_5's than $\mathcal{P}$ and, hence, is a better $\mathcal{T}$-pattern than $\mathcal{P}$. These contradictions yield the claim.

Claim 4: K does not contain an edge of a $\mathcal{T}$-K_4.

Proof of Claim 4: Let T be a $\mathcal{T}$-K_4. Since $\varphi(f) = \frac{3}{4}$ for every edge $f \in E(T)$, we suppose that K and T share exactly one edge, say e. Let N be the copy of L_2 formed by $E(K) \cup E(T)$. By the definition of φ, each edge of $E(K)\backslash\{e\}$ is unused. Therefore, $(\mathcal{P}\backslash\{T\}) \cup \{N\}$ is a better $\mathcal{T}$-pattern, which contradicts our choice of $\mathcal{P}$.

Lemma 5.1.1 now tells us that $\nu_{\boxtimes}(G\backslash E(K)) = \nu_{\boxtimes}(G)$. Therefore, a maximum K_4-packing of $G\backslash E(K)$ together with K is a set of $\nu_{\boxtimes}(G) + 1$ edge-disjoint K_4's in G, which contradicts the maximality of $\mathcal{T}$. Thus, φ is a fractional K_4-cover of G, as required. □

We are now ready to prove our main result.

Theorem 5.1.3. *If G is a graph, then $\tau^*_{\boxtimes}(G) \leq \frac{9}{2}\nu_{\boxtimes}(G)$.*

Proof: Let $\mathcal{T}$ be a maximum K_4-packing of G and let $\mathcal{P}$ be a $\mathcal{T}$-pattern such that $\mathcal{P}$ is best among all patterns of all maximum K_4-packings. By Lemma 5.1.2, φ is a fractional K_4-cover of G. For each $F \in \mathcal{P}$, we know that $\nu_{\boxtimes}(F) = 1$. Furthermore, the definition of φ tells us that $\sum_{e \in E(F)} \varphi(e) \leq \frac{9}{2}$. Since the elements of $\mathcal{P}$ are pairwise edge-disjoint and contain a maximum K_4-packing of G, we have

$$\tau^*_{\boxtimes}(G) \leq \sum_{e \in E(G)} \varphi(e) \leq \frac{9}{2}|\mathcal{P}| = \frac{9}{2}\nu_{\boxtimes}(G),$$

as required. □

Unfortunately, we do not have an example to show that the bound in Theorem 5.1.3 is sharp. However, we claim that K_6 satisfies $\tau^*_{\boxtimes}(K_6) = \frac{5}{2}\nu_{\boxtimes}(K_6)$. Since $\nu_{\boxtimes}(K_6) = 1$, it suffices to show that $\tau^*_{\boxtimes}(K_6) = \frac{5}{2}$. To do so, we make the following observation.

Lemma 5.1.4. *Let $n \in \mathbb{N}$. If $\phi : E(K_n) \to [0, 1]$ is the function defined by $\phi(e) = \frac{1}{6}$ for all $e \in E(K_n)$, then ϕ is a minimum fractional K_4-cover of K_n.*

Proof: We begin by noticing that since K_4 has six edges,

$$\sum_{e \in E(K)} \phi(e) = 6\left(\frac{1}{6}\right) = 1,$$

for all $K \in \mathcal{K}(K_n)$. Hence, ϕ is a feasible solution to the K_4-covering linear program for K_n. Let $\mathcal{K}(K_n)$ be the set of all K_4's in K_n and consider the function $\gamma : \mathcal{K}(K_n) \to [0,1]$ defined by $\gamma(K) = \frac{1}{\binom{n-2}{2}}$ for all $K \in \mathcal{K}(K_n)$. Since each edge of K_n is contained in exactly $\binom{n-2}{2}$ copies of K_4, we have

$$\sum_{\substack{K \in \mathcal{K}(G) \\ e \in E(K)}} \gamma(K) = \binom{n-2}{2}\left(\frac{1}{\binom{n-2}{2}}\right) = 1,$$

for each $e \in E(G)$. Therefore, γ is a feasible solution to the K_4-packing linear program for K_n. Finally, since K_n has $\binom{n}{2}$ edges and $\binom{n}{4}$ copies of K_4, a simple calculation yields

$$\sum_{e \in E(K_n)} \phi(e) = \frac{\binom{n}{2}}{6} = \frac{\binom{n}{4}}{\binom{n-2}{2}} = \sum_{K \in \mathcal{K}(K_n)} \gamma(K).$$

Therefore, by Corollary 2.3.2, ϕ is a minimum fractional K_4-cover of K_n, as required. $\square$

Lemma 5.1.4 now tells us that

$$\tau^*_{\boxtimes}(K_6) = \sum_{e \in E(K_6)} \frac{1}{6} = \frac{5}{2},$$

as claimed. Furthermore, let G be a K_7-free graph. If G has the property that every $K \in \mathcal{K}(G)$ lies in exactly one copy of K_6 in G, then G also satisfies $\tau_{\boxtimes}(G) = \frac{5}{2}\nu_{\boxtimes}(G)$.

5.2 4-Partite Graphs

A consequence of the work in [42] is that Conjecture 1.2.1 is true for tripartite graphs. We give an analogous result for packing and covering edge-disjoint K_4's in 4-partite graphs.

Theorem 5.2.1. *If G is a 4-partite graph, then $\tau_{\boxtimes}(G) \leq 5\nu_{\boxtimes}(G)$.*

Proof: We will describe how to build a K_4-packing $\mathcal{P}$ and a K_4-cover $\mathcal{C}$ such that $|\mathcal{C}| \leq 5|\mathcal{P}|$. Let V_0, V_1, V_2, and V_3 be the vertex classes of G and let $w_1, w_2, \ldots, w_t$ be an arbitrary, but fixed ordering of V_0. We construct $\mathcal{P}$ as follows. For each $i \in [t]$, let B_i be the set of edges of a maximum collection of vertex-disjoint triangles in the graph $G[\Gamma(w_i)] \setminus \bigcup_{j=1}^{i-1} B_j$. Let P_i be the set of K_4's obtained by attaching w_i to the triangles formed by the edges of

B_i. Since the triangles of B_i are pairwise vertex-disjoint, P_i is a set of pairwise edge-disjoint K_4's. Furthermore, since no triangle of B_i shares an edge with an triangle in $\bigcup_{j=1}^{i-1} B_j$, we see that $\mathcal{P} := \bigcup_{j=1}^{t} P_j$ is a K_4-packing of G.

We now construct $\mathcal{C}$. For each $i \in [t]$, let $\mathcal{H}_i$ be the 3-uniform, tripartite hypergraph on $\Gamma(w_i)$ where xyz is an edge of $\mathcal{H}_i$ if x, y, and z are the vertices of a triangle in $G[\Gamma(w_i)] \setminus \bigcup_{j=1}^{i-1} B_j$. Notice that P_i corresponds to a maximum matching $\mathcal{M}_i$ of $\mathcal{H}_i$. Let C_i be a minimum vertex cover of $\mathcal{H}_i$. Define

$$\mathcal{C} := \bigcup_{k=1}^{t} \{ab, ac, bc \mid abc \in \mathcal{M}_k\} \cup \{w_k z \mid z \in C_k\}.$$

We claim that $\mathcal{C}$ is a K_4-cover of G. Let K be a K_4 of G with vertices w_s, x, y, and z, where $w_s \in V_0$. If $E(K) \cap \{ab, ac, bc \mid abc \in \mathcal{M}_i\} = \emptyset$ for all $i \in [t]$, then xyz is a non-matching edge of $\mathcal{H}_s$. Since C_s is a vertex cover of $\mathcal{H}_s$, we may assume without loss of generality that $x \in C_s$. However, this means that $w_s x \in \mathcal{C}$, which implies that $\mathcal{C}$ is a K_4-cover of G.

To estimate $|\mathcal{C}|$, notice that $\sum_{j=1}^{t} |\mathcal{M}_j| = \sum_{j=1}^{t} |P_j| = |\mathcal{P}|$ and, by Theorem 1.1.2, $|C_j| \leq 2|\mathcal{M}_j|$ for each $j \in \{1, \ldots, t\}$. Thus,

$$\begin{aligned}
|\mathcal{C}| &= \sum_{j=1}^{s} (3|\mathcal{M}_j| + |C_j|) \\
&\leq 5 \sum_{j=1}^{t} |P_j| \\
&= 5|\mathcal{P}|,
\end{aligned}$$

as required. □

5.3 Complete Graphs

We now turn our attention to complete graphs. Our main result is that

$$\lim_{n\to\infty} \frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} = 2.$$

Along the way, we will show that $\tau_{\boxtimes}(K_n) \leq 3\nu_{\boxtimes}(K_n)$ unless $n = 8$. Let $\mathcal{T}_3(n)$ denote the tripartite Turán graph on n vertices and recall, from Section 2.1, that

$$|E(\mathcal{T}_3(n))| = \begin{cases} \dfrac{n^2}{3} \text{ if } n \equiv 0 \pmod 3 \\ \dfrac{n^2-1}{3} \text{ if } n \equiv 1 \text{ or } 2 \pmod 3. \end{cases} \tag{5.1}$$

By Theorem 2.1.2, any subgraph of K_n with more than $|E(\mathcal{T}_3(n))|$ edges contains a copy of K_4. Therefore we have that $\tau_{\boxtimes}(K_n) \geq \binom{n}{2} - |E(\mathcal{T}_3(n))|$. Alternatively, since $\mathcal{T}_3(n)$ does not contain a copy of K_4, $\tau_{\boxtimes}(K_n) \leq \binom{n}{2} - |E(\mathcal{T}_3(n))|$. Therefore $\tau_{\boxtimes}(K_n) = \binom{n}{2} - |E(\mathcal{T}_3(n))|$. This observation yields the following result.

Lemma 5.3.1. *For all $n \in \mathbb{N}$,*

$$\tau_{\boxtimes}(K_n) = \begin{cases} \dfrac{n(n-3)}{6} \text{ if } n \equiv 0 \pmod 3 \\ \dfrac{(n-1)(n-2)}{6} \text{ if } n \equiv 1 \text{ or } 2 \pmod 3. \end{cases}$$

Before we find a lower bound on $\nu_{\boxtimes}(K_n)$, we first consider the cases when $n \leq 12$.

n	$\nu_{\boxtimes}$	$\tau_{\boxtimes}$	$\frac{\tau_{\boxtimes}}{\nu_{\boxtimes}}$
4	1	1	1
5	1	2	2
6	1	3	3
7	2	5	2.5
8	2	7	3.5
9	3	9	3
10	5	12	2.4
11	6	15	2.5
12	9	18	2

Table 5.1: $\nu_{\boxtimes}$ and $\tau_{\boxtimes}$ for complete graphs on at most twelve vertices

Notice that $n = 8$ is the only case in Table 5.1 where $\frac{\tau_\boxtimes}{\nu_\boxtimes} > 3$. Indeed, we observe that $\tau_\boxtimes(K_8) = 7$ by Lemma 5.3.1. Furthermore, since K_8 contains two vertex-disjoint copies of K_4, we see that $\nu_\boxtimes(K_8) \geq 2$. Now, let T_1, T_2, and T_3 be K_4's of K_8, where T_1 and T_2 are edge-disjoint. Notice that T_3 shares at least two vertices with either T_1 or T_2. This means that T_3 also shares at least one edge with either T_1 or T_2. Thus, $\nu_\boxtimes(K_8) = 2$. Similar arguments can be used to find the values for $\nu_\boxtimes(K_n)$ when $n \leq 7$. For $n \in \{9, 10, 11, 12\}$, it is sufficient for our purposes to know that the values for $\nu_\boxtimes(K_n)$ in Table 5.1 can be attained.

n	$\nu_\boxtimes$	Vertex sets of a maximum K_4-packing
9	3	$\{1,2,3,4\}$, $\{3,5,7,9\}$, $\{4,6,7,8\}$
10	5	$\{1,2,3,4\}$, $\{1,8,9,10\}$, $\{3,5,7,9\}$, $\{2,5,6,10\}$, $\{4,6,7,8\}$
11	6	$\{1,3,4,5\}$, $\{1,6,7,8\}$, $\{1,9,10,11\}$, $\{2,3,6,9\}$, $\{2,4,7,10\}$, $\{2,5,8,11\}$
12	9	$\{1,4,5,6\}$, $\{1,7,8,9\}$, $\{1,10,11,12\}$, $\{2,4,7,10\}$, $\{2,5,8,11\}$, $\{2,6,9,12\}$, $\{3,4,8,12\}$, $\{3,5,9,10\}$, $\{3,6,7,11\}$

Table 5.2: Maximum K_4-packings for K_n when $n \in \{9, 10, 11, 12\}$

Our main tool to estimate $\nu_\boxtimes(K_n)$ when $n \geq 13$ will be a result from combinatorial design theory. A *non-trivial* 2-(n, k, λ)-*design* is a k-uniform hypergraph on n vertices with at least two edges and the property that any pair of distinct vertices is contained in exactly λ edges. We are interested in non-trivial 2-$(n, 4, 1)$-designs. In particular, we want to know when such designs exist.

Theorem 5.3.2 (Hanani [39]). *Let $n \in \mathbb{N}$. There exists a non-trivial 2-$(n, 4, 1)$-design if and only if $n \geq 13$ and $n \equiv 1$ or 4 modulo 12.*

Notice that if a 2-$(n, 4, 1)$-design exists, then the edges of K_n can be partitioned into copies of K_4. In other words, we have the following corollary.

Corollary 5.3.3. *If $n \in \mathbb{N}$ such that $n \geq 13$ and $n \equiv 1$ or 4 modulo 12, then*

$$\nu_\boxtimes(K_n) = \frac{\binom{n}{2}}{6} = \frac{n(n-1)}{12}.$$

When $n \not\equiv 1$ or 4 modulo 12, we will partition the vertices of K_n into sets X and Y so that X has maximum size under the restriction that $|X| \equiv 1$ or 4 modulo 12. We will then build a K_4-packing of K_n from a K_4-packing of $K_{|X|}$ and a triangle packing of $K_{|Y|}$.

Lemma 5.3.4. *Let $n, i \in \mathbb{N}$. If $\nu_{\triangledown}(K_i) \leq n - i$, then*

$$\nu_{\boxtimes}(K_n) \geq \nu_{\boxtimes}(K_{n-i}) + \nu_{\triangledown}(K_i).$$

Proof: We may assume that vertex set of K_n is $[n]$. Let G be the subgraph of K_n induced by $[n-i]$ and let H be the subgraph of K_n induced by $\{n-i+1, n-i+2, \ldots, n\}$. Let $\mathcal{A}$ be a maximum K_4-packing of G and let $\mathcal{B} = \{T_1, T_2, \ldots, T_m\}$ be a maximum triangle packing of H. Let $j \in [m]$ and suppose that the vertex set of T_j is $\{a_j, b_j, c_j\}$. We create a K_4, $\bar{T}_j$, by adding the vertex j and the edges ja_j, jb_j, jc_j to T_j. Since $j \leq m \leq n - i$, this procedure is well-defined. Let $\bar{\mathcal{B}} = \{\bar{T}_1, \ldots, \bar{T}_m\}$. We claim that $\mathcal{P} = \mathcal{A} \cup \bar{\mathcal{B}}$ is a K_4-packing of K_n of size $\nu_{\boxtimes}(K_{n-i}) + \nu_{\triangledown}(K_i)$.

Suppose, for a contradiction, that L and M are K_4's of $\mathcal{P}$ that share an edge. Since $\mathcal{A}$ is a K_4-packing of G, at least one of L and M is in $\bar{\mathcal{B}}$. Furthermore, no K_4 of $\bar{\mathcal{B}}$ contains an edge of G and no K_4 of $\mathcal{A}$ contains an edge of H. Therefore, both L and M are in $\bar{\mathcal{B}}$. However, $\bar{\mathcal{B}}$ was constructed by associating, to each triangle in $\mathcal{B}$, a unique vertex of G. Since the triangles of $\mathcal{B}$ are pairwise edge-disjoint, L and M cannot share any edges, contradicting our assumption and yielding the lemma. $\square$

Before we prove our result, we compile the values of $\nu_{\triangledown}(K_n)$ when $n \leq 8$.

Theorem 5.3.5 (Feder and Subi [27])**.** *If $n \in \mathbb{N}$, then*

$$\nu_{\triangledown}(K_n) = \begin{cases} \dfrac{n(n-2)}{6} & \textit{if } n \equiv 0 \textit{ or } 2 \pmod 6 \\[2ex] \dfrac{n(n-1)}{6} & \textit{if } n \equiv 1 \textit{ or } 3 \pmod 6 \\[2ex] \dfrac{n^2-2n-2}{6} & \textit{if } n \equiv 4 \pmod 6 \\[2ex] \dfrac{n^2-n-8}{6} & \textit{if } n \equiv 5 \pmod 6. \end{cases}$$

Theorem 5.3.5 yields the following table.

n	3	4	5	6	7	8
$\nu_{\triangledown}$	1	1	2	4	7	8

Table 5.3: $\nu_{\triangledown}$ for complete graphs on at most eight vertices

Theorem 5.3.6. *For all $n \in \mathbb{N}$, $\tau_{\boxtimes}(K_n) \leq \frac{7}{2}\nu_{\boxtimes}(K_n)$. Furthermore, if $n \neq 8$, then $\tau_{\boxtimes}(K_n) \leq 3\nu_{\boxtimes}(K_n)$*

Proof: By Table 5.1 above, we may assume that $n \geq 13$. The proof now breaks down into the following cases.

Case 1: $n \equiv 1, 2, 3, 4, 5, 6$, or $7 \pmod{12}$
For each such n, there is an $i \in \{0, 1, 2, 3\}$ such that $n - i \equiv 1$ or 4 modulo 12. Therefore, by Corollary 5.3.3 and Lemma 5.3.4, $\nu_{\boxtimes}(K_n) \geq \frac{(n-3)(n-4)}{12}$. Since Lemma 5.3.1 says that $\tau_{\boxtimes}(K_n) \leq \frac{(n-1)(n-2)}{6}$, we have

$$\frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} \leq 2 + \frac{8n - 20}{n^2 - 7n + 12} \leq 3,$$

since $n \geq 13$.

Case 2: $n \equiv 8 \pmod{12}$
Corollary 5.3.3, Lemma 5.3.4, and Table 5.3 tell us that $\nu_{\boxtimes}(K_n) \geq \frac{(n-4)(n-5)}{12} + 1$ and Lemma 5.3.1 tells us that $\tau_{\boxtimes}(K_n) = \frac{(n-1)(n-2)}{6}$. Therefore,

$$\frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} \leq 2 + \frac{12n - 60}{n^2 - 9n + 32} \leq 3$$

for all $n \geq 20$ satisfying $n \equiv 8 \pmod{12}$.

Case 3: $n \equiv 9 \pmod{12}$
Corollary 5.3.3, Lemma 5.3.4, and Table 5.3 tell us that $\nu_{\boxtimes}(K_n) \geq \frac{(n-5)(n-6)}{12} + 2$ and Lemma 5.3.1 tells us that $\tau_{\boxtimes}(K_n) = \frac{n(n-3)}{6}$. Therefore,

$$\frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} \leq 2 + \frac{16n - 108}{n^2 - 11n + 54} \leq 3$$

for all $n \geq 21$ satisfying $n \equiv 9 \pmod{12}$.

Case 4: $n \equiv 10 \pmod{12}$
Corollary 5.3.3, Lemma 5.3.4, and Table 5.3 tell us that $\nu_{\boxtimes}(K_n) \geq \frac{(n-6)(n-7)}{12} + 4$ and Lemma 5.3.1 tells us that $\tau_{\boxtimes}(K_n) = \frac{(n-1)(n-2)}{6}$. Therefore,

$$\frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} \leq 2 + \frac{20n - 176}{n^2 - 13n + 90} \leq 3$$

for all $n \geq 22$ satisfying $n \equiv 10 \pmod{12}$.

Case 5: $n \equiv 11 \pmod{12}$
Corollary 5.3.3, Lemma 5.3.4, and Table 5.3 tell us that $\nu_{\boxtimes}(K_n) \geq \frac{(n-7)(n-8)}{12} + 7$ and Lemma 5.3.1 tells us that $\tau_{\boxtimes}(K_n) = \frac{(n-1)(n-2)}{6}$. Therefore,

$$\frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} \leq 2 + \frac{24n - 276}{n^2 - 15n + 140} \leq 3$$

for all $n \geq 23$ satisfying $n \equiv 11 \pmod{12}$.

Case 6: $n \equiv 0 \pmod{12}$
Corollary 5.3.3, Lemma 5.3.4, and Table 5.3 tell us that $\nu_{\boxtimes}(K_n) \geq \frac{(n-8)(n-9)}{12} + 8$ and Lemma 5.3.1 tells us that $\tau_{\boxtimes}(K_n) = \frac{n(n-3)}{6}$. Therefore,

$$\frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} \leq 2 + \frac{28n - 336}{n^2 - 17n + 168} \leq 3$$

for all $n \geq 24$ satisfying $n \equiv 0 \pmod{12}$.

These six cases, together with Table 5.1 above, show that the only value of n for which $\tau_{\boxtimes}(K_n) > 3\nu_{\boxtimes}(K_n)$ is $n = 8$, as required. $\square$

Corollary 5.3.7. *The parameters $\tau_{\boxtimes}(K_n)$ and $\nu_{\boxtimes}(K_n)$ satisfy*

$$\lim_{n\to\infty} \frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} = 2.$$

Proof: Lemma 5.3.1 and the proof of Theorem 5.3.6 tell us that for all $n \geq 13$

$$\frac{n(n-3)}{6} \leq \tau_{\boxtimes}(K_n) \leq \frac{(n-1)(n-2)}{6}$$

and

$$\frac{(n-8)(n-9)}{12} \leq \nu_{\boxtimes}(K_n) \leq \frac{n(n-1)}{12}.$$

Therefore,

$$\frac{2(n-3)}{(n-1)} \leq \frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} \leq \frac{2(n-1)(n-2)}{(n-8)(n-9)}$$

for all $n \in \mathbb{N}$. However,

$$\lim_{n\to\infty} \frac{2(n-3)}{(n-1)} = \lim_{n\to\infty} \frac{2(n-1)(n-2)}{(n-8)(n-9)} = 2.$$

Thus, the Squeeze Theorem tells us that

$$\lim_{n\to\infty} \frac{\tau_{\boxtimes}(K_n)}{\nu_{\boxtimes}(K_n)} = 2,$$

as required. $\square$

5.4 Low Degeneracy Graphs

Recall that a graph $G = (V, E)$ is *d-degenerate* if there is an ordering $v_1, \ldots, v_n$ of V such that $\deg_{H_i}(v_i) \leq d$ for all $i \in \{1, 2, \ldots, n\}$, where $H_i = G[\{v_1, \ldots, v_i\}]$. In this section, we show that if a graph G is 8-degenerate, then $\tau_{\boxtimes}(G) \leq 5\nu_{\boxtimes}(G)$. To that end, we define G to be *(d, K_4)-degenerate* if there is an ordering $v_1, \ldots, v_n$ of V such that $\deg_{H_i}(v_i) \leq d$ for all $i \in \{1, 2, \ldots, n\}$, where H_i is the graph obtained from G by deleting the vertices $v_{i+1}, \ldots, v_n$ plus any edges that are not contained in a K_4 of $G \backslash \{v_{i+1}, \ldots, v_n\}$. Notice that if G is d-degenerate, then G is also (d, K_4)-degenerate.

To prove our result, we need several lemmas.

Lemma 5.4.1. *Let $n \in \mathbb{N}$. If (X, Y, Z) is a partition of the vertices of K_n, then*

$$C = E(K_n[X]) \cup E(K_n[Y]) \cup E(K_n[Z])$$

is a K_4-cover of K_n. Furthermore, if X, Y, and Z each contain $\lfloor \frac{n}{3} \rfloor$ or $\lceil \frac{n}{3} \rceil$ vertices, then C is a minimum K_4-cover of K_n.

Proof: Since (X, Y, Z) is a partition of the vertices of K_n, $K_n \backslash C$ is a complete tripartite graph and, hence, C is a K_4-cover of K_n. Furthermore, if X, Y, and Z each contain $\lfloor \frac{n}{3} \rfloor$ or $\lceil \frac{n}{3} \rceil$ vertices, then $K_n \backslash C$ is isomorphic to $\mathcal{T}_3(n)$, the Turán graph on n vertices. Therefore,

$$|C| = \begin{cases} \dfrac{n(n-3)}{6} & \text{if } n \equiv 0 \pmod 3 \\[2ex] \dfrac{(n-1)(n-2)}{6} & \text{if } n \equiv 1 \text{ or } 2 \pmod 3. \end{cases}$$

By Lemma 5.3.1, C is a minimum K_4-cover of K_n, as required. $\square$

Lemma 5.4.2. *If H is a triangle-free graph on at most five vertices, then either*

(a) H is isomorphic to C_5 or

(b) H^c contains two vertex-disjoint cliques, U and W, such that $V(H) = V(U) \cup V(W)$.

Proof: Let H be a triangle-free graph on at most five vertices. If H is not bipartite, then by Theorem 2.1.1, H contains a subgraph which is isomorphic to either C_3 or C_5. However, since H is triangle-free, we see that H is isomorphic to C_5. Otherwise H has a bipartition (X, Y). If we let $U = H^c[X]$ and $W = H^c[Y]$, then U and W are vertex-disjoint cliques of H^c such that $V(H) = V(U) \cup V(W)$, as required. $\square$

Recall that if G is a graph, then $G + v$ is the graph obtained from G by adding a vertex v and joining v to every vertex of G.

Lemma 5.4.3. *Let $d \in \mathbb{N}$ such that $3 \le d \le 8$ and let G be a graph on d vertices. If F is the set of edges of a maximum collection of vertex-disjoint triangles of G, then*

$$\tau_{\boxtimes}((G+v)\backslash F) \le \left(\frac{\lceil \frac{d}{2} \rceil - 2}{3}\right) |F|.$$

Proof: Let the vertices of G be $\{w_1, \ldots, w_d\}$. We may assume that $F \neq \emptyset$, otherwise $G + v$ is K_4-free which implies that $\tau_{\boxtimes}((G + v)\backslash F) = 0$. We first suppose that $F = \{w_1w_2, w_1w_3, w_2w_3, w_4w_5, w_4w_6, w_5w_6\}$, so that $6 \le d \le 8$. Let $X = \{w_1, w_2, w_3\}$, $Y = \{w_4, w_5, w_6\}$ and $Z = (V(G)\backslash(X \cup Y)) \cup \{v\}$. Notice that (X, Y, Z) is a partition of the vertices of the complete graph on $\{w_1, \ldots, w_d, v\}$. If $w_7w_8 \notin E(G)$, then Lemma 5.4.1 tells us that $\{vw_7, vw_8\}$ contains a K_4-cover of $(G + v)\backslash F$. Since $6 \le d \le 8$, we have

$$\tau_{\boxtimes}((G+v)\backslash F) \le 2 = \left(\frac{\lceil \frac{6}{2} \rceil - 2}{3}\right)(6) \le \left(\frac{\lceil \frac{d}{2} \rceil - 2}{3}\right) |F|.$$

If $w_7w_8 \in E(G)$, then $d = 8$ and, by Lemma 5.4.1, $D = \{w_7w_8, vw_7, vw_8\}$ is a K_4-cover of $(G + v)\backslash F$. Thus,

$$\tau_{\boxtimes}((G+v)\backslash F) \le 3 \le \left(\frac{\lceil \frac{8}{2} \rceil - 2}{3}\right)(6) = \left(\frac{\lceil \frac{d}{2} \rceil - 2}{3}\right) |F|.$$

We now suppose that $F = \{w_1w_2, w_1w_3, w_2w_3\}$, so that the graph $H = G\backslash\{w_1, w_2, w_3\}$ is triangle-free. By Lemma 5.4.2, either H is isomorphic to C_5 or H^c contains two vertex-disjoint cliques, U and W, such that $V(H) = V(U) \cup V(W)$. First, suppose that H is isomorphic to C_5 and that w_4, w_5, w_6, w_7, w_8 are the vertices of H in cyclic order. Since $|F| = 3$ and $d = 8$, it suffices to show that $\tau_{\boxtimes}((G+v)\backslash F) \leq 2$. Since H is triangle-free, we see that $G\backslash F$ is K_4-free and that every triangle of $G\backslash F$ contains exactly one of w_1, w_2, or w_3. Without loss of generality, suppose that w_1 has the largest degree among w_1, w_2, and w_3 in $G\backslash F$.

If $\deg_{G\backslash F}(w_1) = 5$, we claim that w_2 and w_3 are not in any triangles of $G\backslash F$. To see this, suppose that $w_2w_4w_5$ is a triangle in $G\backslash F$. Since $\deg_{G\backslash F}(w_1) = 5$, the triangle $w_1w_6w_7$ is vertex-disjoint from $w_2w_4w_5$ which contradicts the maximality of F. Therefore, $\{vw_1\}$ is a K_4-cover of $(G+v)\backslash F$ since every triangle of $G\backslash F$ contains w_1.

Now, suppose that $\deg_{G\backslash F}(w_1) = 4$ and suppose that $\Gamma_{G\backslash F}(w_1) = \{w_4, w_5, w_6, w_7\}$. Recall that every triangle of $G\backslash F$ contains exactly one of w_1, w_2, or w_3. If w_2 (or w_3) is in a triangle of $G\backslash F$, then the triangle is $w_2w_5w_6$ (or $w_3w_5w_6$), otherwise there are two vertex-disjoint triangles in G. Thus, every K_4 of $(G+v)\backslash F$ contains either the edge vw_1 or the edge w_5w_6 and, hence $\{vw_1, w_5w_6\}$ is a K_4-cover of $(G+v)\backslash F$.

Finally, suppose that $\deg_{G\backslash F}(w_1) \leq 3$. We may assume that $\Gamma_{G\backslash F}(w_1)$ induces a path of length at most two in $G\backslash F$ since any edge of $G\backslash F$ where one endpoint is an isolated vertex of $G\backslash F[\Gamma_{G\backslash F}(w_1)]$ and the other endpoint is in $\{w_1, w_2, w_3\}$ is not contained in a triangle of $G\backslash F$. Thus, we may assume that w_1w_4, w_1w_5, and (possibly) w_1w_6 are edges of G. Notice that any triangle of $G\backslash F$ which contains w_2 or w_3 also contains w_4w_5, w_4w_8, or w_5w_6, otherwise there are two vertex-disjoint triangles in G. Next, we see that at least one of $w_2w_4w_8$ and $w_2w_5w_6$ is not a triangle of $G\backslash F$, since $\deg_{G\backslash F}(w_2) \leq \deg_{G\backslash F}(w_1) \leq 3$. Without loss of generality, suppose that $w_2w_5w_6$ is a triangle of $G\backslash F$. Then $w_3w_4w_8$ is not a triangle since $w_2w_5w_6$ and $w_3w_4w_8$ would be two vertex-disjoint triangles in G. Therefore, $\{w_4w_5, w_5w_6\}$ is a K_4-cover of $(G+v)\backslash F$ and $\tau_{\boxtimes}((G+v)\backslash F) \leq 2$, as required.

Otherwise, by Lemma 5.4.2, H^c contains two vertex-disjoint cliques, U and W, such that $V(H) = V(U) \cup V(W)$. Without loss of generality, we may assume that $|V(W)| \leq \lfloor \frac{d-3}{2} \rfloor \leq |V(U)|$. Let $C = \{vw \mid w \in W\}$. By Lemma 5.4.1, $F \cup E(H^c[U]) \cup E(H^c[W])$ is a K_4-cover of K_d. Thus, $C \cup F$ is a K_4-cover of $G+v$ since $E(U) \cup E(W) \subseteq E(H^c)$. Hence C is a K_4-cover of $(G+v)\backslash F$. Now, we see that

$$|C| = |V(W)| \leq \left\lfloor \frac{d-3}{2} \right\rfloor \leq \left(\frac{\lceil \frac{d}{2} \rceil - 2}{3} \right) |F|,$$

since $3 \leq d \leq 8$, as required. $\square$

Lemma 5.4.4. *Let $d \in \mathbb{N}$ such that $d \geq 3$. If G is a K_4-free graph on d vertices, then*

$$\tau_{\boxtimes}(G+v) \leq \begin{cases} 1 & \text{if } d = 3 \\ d-3 & \text{if } d \geq 4. \end{cases}$$

Proof: If $d = 3$, then $G + v$ is a isomorphic to a subgraph of K_4, which implies that $\tau_{\boxtimes}(G+v) \leq 1$. So we assume that $d \geq 4$. We first notice that G contains vertices x, y, and z that do not form a triangle, otherwise G is a complete graph. We now claim that $C = \{vu \mid u \in V(G) \backslash \{x,y,z\}\}$ is a K_4-cover of $G+v$. Let K be a K_4 of $G+v$. Since G is K_4-free, K contains the vertex v. Furthermore, x, y, and z do not form a triangle in G, which implies that K contains an edge vw where $w \in V(G) \backslash \{x,y,z\}$. Since $vw \in E(K) \cap C$, we see that C is a K_4-cover of $G+v$ and $|C| = d-3$, as required. $\square$

We are now ready to prove the main result of this section.

Proposition 5.4.5. *Let $d \in \mathbb{N}$. If G is a (d, K_4)-degenerate graph, then*

$$\tau_{\boxtimes}(G) \leq \left(\left\lceil \frac{d}{2} \right\rceil + 1 \right) \nu_{\boxtimes}(G).$$

Proof: Let $G = (V, E)$ be a (d, K_4)-degenerate graph. By Inequality (1.2), we may assume that $d \leq 8$. We proceed by induction on $n = |V|$. If $n \leq 4$, then we see that $\tau_{\boxtimes}(G) = \nu_{\boxtimes}(G)$. So, we assume that $n \geq 5$ and that $\tau_{\boxtimes}(H) \leq (\lceil \frac{d}{2} \rceil + 1)\nu_{\boxtimes}(H)$ for all (d, K_4)-degenerate graphs H with at most $n-1$ vertices. Let $v_1, \ldots, v_n$ be an ordering of V given by the definition of (d, K_4)-degeneracy. Let $\tilde{G}$ be the graph obtained from G by deleting every edge of G that is not contained in a K_4. Notice that $\tau_{\boxtimes}(G) = \tau_{\boxtimes}(\tilde{G})$ and $\nu_{\boxtimes}(G) = \nu_{\boxtimes}(\tilde{G})$. Define G_1 to be the graph obtained from $\tilde{G}$ by deleting the vertex v_n plus any edges that are not in a K_4 of $\tilde{G} \backslash v_n$ and define G_2 to be the graph $\tilde{G}[\{v_n\} \cup \Gamma_{\tilde{G}}(v_n)]$. Since G_1 is (d, K_4)-degenerate and $|V(G_1)| = n-1$, the inductive hypothesis tells us that $\tau_{\boxtimes}(G_1) \leq (\lceil \frac{d}{2} \rceil + 1)\nu_{\boxtimes}(G_1)$. We also see that $G_2 = (G_2 \backslash v_n) + v_n$ is isomorphic to a subgraph of K_{d+1}.

First, suppose that $\nu_{\boxtimes}(G) = \nu_{\boxtimes}(G_1)$ and let F be the set of edges of a maximum collection of vertex-disjoint triangles of $G_2 \backslash v_n$. Since $|F| \in \{3, 6\}$, notice that $\nu_{\boxtimes}(G_1 \backslash F) \leq \nu_{\boxtimes}(G_1) - \frac{|F|}{3}$, otherwise we can find a K_4-packing of G of size $\nu_{\boxtimes}(G) + 1$ by adding v_n to the triangles formed by the edges in F. Then

$$
\begin{aligned}
\tau_{\boxtimes}(G) &\leq \tau_{\boxtimes}(G_1 \backslash F) + \tau_{\boxtimes}(G_2 \backslash F) + |F| \\
&\leq \left(\left\lceil \frac{d}{2} \right\rceil + 1 \right) \nu_{\boxtimes}(G_1 \backslash F) + \left(\frac{\lceil \frac{d}{2} \rceil - 2}{3} \right) |F| + |F| \qquad (5.2) \\
&\leq \left(\left\lceil \frac{d}{2} \right\rceil + 1 \right) \left(\nu_{\boxtimes}(G_1) - \frac{|F|}{3} \right) + \left(\frac{\lceil \frac{d}{2} \rceil + 1}{3} \right) |F| \\
&= \left(\left\lceil \frac{d}{2} \right\rceil + 1 \right) \nu_{\boxtimes}(G),
\end{aligned}
$$

where (5.2) follows from the inductive hypothesis applied to $G_1 \backslash F$ and Lemma 5.4.3 applied to $G_2 = (G_2 \backslash v_n)) + v_n$. Now suppose that $\nu_{\boxtimes}(G) \geq \nu_{\boxtimes}(G_1) + 1$ and let C be a minimum K_4-cover of G_1. Notice that $(G_2 \backslash v_n) \backslash C$ is K_4-free and has at most d vertices. Then

$$
\begin{aligned}
\tau_{\boxtimes}(G) &\leq |C| + \tau_{\boxtimes}(G_2 \backslash C) \\
&\leq \left(\left\lceil \frac{d}{2} \right\rceil + 1 \right) \nu_{\boxtimes}(G_1) + (d - 3) \qquad (5.3) \\
&\leq \left(\left\lceil \frac{d}{2} \right\rceil + 1 \right) (\nu_{\boxtimes}(G_1) + 1) \qquad (5.4) \\
&\leq \left(\left\lceil \frac{d}{2} \right\rceil + 1 \right) \nu_{\boxtimes}(G),
\end{aligned}
$$

where (5.3) follows from the inductive hypothesis applied to G_1 and Lemma 5.4.4 applied to $G_2 \backslash C$ and (5.4) follows from the assumption that $d \leq 8$. $\square$

The first corollary of Proposition 5.4.5 is immediate from our earlier observation.

Corollary 5.4.6. *If G is an 8-degenerate graph, then $\tau_{\boxtimes}(G) \leq 5\nu_{\boxtimes}(G)$.*

Proposition 5.4.5 also tells us information about graphs with bounded treewidth. Since partial k-trees are k-degenerate, Theorem 2.1.5 yields the following result.

Corollary 5.4.7. *If G is a graph with treewidth at most eight, then $\tau_{\boxtimes}(G) \leq 5\nu_{\boxtimes}(G)$.*

5.5 Planar Graphs

Let $G = (V, E)$ be a planar graph. In [88], Tuza proved that $\tau_{\triangledown}(G) \leq 2\nu_{\triangledown}(G)$. We show that $\tau_{\boxtimes}(G) \leq 3\nu_{\boxtimes}(G)$. It is well known that every planar graph is 5-degenerate. Therefore, Proposition 5.4.5 says that $\tau_{\boxtimes}(G) \leq 4\nu_{\boxtimes}(G)$ whenever G is planar. However, by exploiting the assumption that every edge of G is contained in a K_4 of G, we can improve this bound.

Lemma 5.5.1. *If G is a planar graph with the property that every edge is contained in at least one K_4, then G has a vertex of degree three.*

Proof: Let G be a planar graph such that every edge is contained in at least one K_4. Fix a planar embedding of G and let $\mathcal{K}(G)$ be the set of all K_4's of G. Notice that any planar embedding of K_4 can be obtained from a planar embedding of C_3 by placing a vertex z inside the region bounded by C_3 and adding an edge from z to every vertex of C_3. Therefore, for each $K \in \mathcal{K}(G)$, we define D_K to be the closed region of $\mathbb{R}^2$ which is homeomorphic to the unit disk and whose boundary is the outer copy of C_3 in K. We denote the interior of D_K by $int(D_K)$. Notice that D_K induces a partition of $\mathcal{K}(G)$, namely the K_4's contained in D_K and the K_4's contained in $\mathbb{R}^2 \backslash int(D_K)$. Since G is finite, we may choose a K_4, say K^*, such that D_{K^*} contains the minimum number of K_4's in $\mathcal{K}(G)$. Notice that K^* is the only K_4 contained in D_{K^*}; indeed, if D_{K^*} contained a K_4, say L, such that $L \neq K^*$, then $D_L \subsetneq D_{K^*}$ and K^* is not contained in D_L. This contradicts our choice of K^*.

Now, K^* contains a vertex v such that $v \in int(D_{K^*})$. We claim that v is the desired vertex. Suppose, for a contradiction, that v has a neighbour x that is not a vertex of K^*. Then $x \in int(D_{K^*})$. Furthermore, the edge vx is contained in a K_4 of G. However, such a K_4 is contained in D_{K^*}, which contradicts our choice of K^*. Therefore, v has degree three, as required. $\square$

Lemma 5.5.1 now tells us that planar graphs are $(3, K_4)$-degenerate. This yields the following corollary of Proposition 5.4.5.

Theorem 5.5.2. *If G if a planar graph, then $\tau_{\boxtimes}(G) \leq 3\nu_{\boxtimes}(G)$.*

Recall that Theorem 2.1.6 tells us that a graph G is planar if and only if G does not contain a subgraph that is isomorphic to a subdivision of K_5 or $K_{3,3}$. We conclude this section by examining graphs that contain subdivisions of one of K_5 or $K_{3,3}$. The next observation allows us to extend Theorem 5.5.2 to graphs with no subgraph isomorphic to a subdivision of $K_{3,3}$.

Theorem 5.5.3 (Hall [38], Asano [9])**.** *If G is a 3-connected graph with no subgraph isomorphic to a subdivision of $K_{3,3}$, then G is either planar or isomorphic to K_5.*

The proof of the following result follows in a very similar manner to the proofs of Lemma 5 and Theorem 6 in [58]. We include the details for completeness.

Corollary 5.5.4. *If G is a graph with no subgraph isomorphic to a subdivision of $K_{3,3}$, then $\tau_\boxtimes(G) \leq 3\nu_\boxtimes(G)$.*

Proof: Let $G = (V, E)$ be a graph with no subgraph isomorphic to a subdivision of $K_{3,3}$. We proceed by induction on $n = |V|$. We see that $\tau_\boxtimes(G) = \nu_\boxtimes(G)$ whenever $n \leq 4$. So we assume that $n \geq 5$ and that if H is a graph with at most $n-1$ vertices and no subgraph isomorphic to a subdivision of $K_{3,3}$, then $\tau_\boxtimes(H) \leq 3\nu_\boxtimes(H)$. We first suppose that G is 3-connected. Then by Theorem 5.5.3, G is either planar or isomorphic to K_5. If G is planar, then Theorem 5.5.2 says that $\tau_\boxtimes(G) \leq 3\nu_\boxtimes(G)$. If G is isomorphic to K_5, then we know that $\tau_\boxtimes(G) = 2\nu_\boxtimes(G)$ by Table 5.1 in Section 5.3. Therefore, we may assume that G is not 3-connected. Let u and v be vertices of G such that $G\backslash\{u, v\}$ is not connected and let W be the vertices of a component of $G\backslash\{u, v\}$. Define $G_1 := G[W \cup \{u, v\}]$ and $G_2 := G\backslash W$. Since G does not contain a subdivision of $K_{3,3}$, neither does G_1 nor G_2. Therefore, the inductive hypothesis tells us that $\tau_\boxtimes(G_1) \leq 3\nu_\boxtimes(G_1)$ and $\tau_\boxtimes(G_2) \leq 3\nu_\boxtimes(G_2)$.

Notice that, since there is at most one edge between u and v, a maximum K_4-packing of G_1 and a maximum K_4-packing of G_2 will intersect in at most one edge. Therefore, $\nu_\boxtimes(G)$ will be equal to either $\nu_\boxtimes(G_1)+\nu_\boxtimes(G_2)$ or $\nu_\boxtimes(G_1)+\nu_\boxtimes(G_2)-1$. If $\nu_\boxtimes(G) = \nu_\boxtimes(G_1)+\nu_\boxtimes(G_2)$, then

$$\begin{aligned}\tau_\boxtimes(G) &\leq \tau_\boxtimes(G_1) + \tau_\boxtimes(G_2)\\ &\leq 3\nu_\boxtimes(G_1) + 3\nu_\boxtimes(G_2)\\ &= 3\nu_\boxtimes(G).\end{aligned}$$

If $\nu_\boxtimes(G) = \nu_\boxtimes(G_1)+\nu_\boxtimes(G_2)-1$, every maximum K_4-packing of both G_1 and G_2 contains the edge uv. This means that $\nu_\boxtimes(G_1\backslash uv) = \nu_\boxtimes(G_1) - 1$ and $\nu_\boxtimes(G_2\backslash uv) = \nu_\boxtimes(G_2) - 1$. Furthermore, if C_1 is a K_4-cover of $G_1\backslash uv$ and C_2 is a K_4-cover of $G_2\backslash uv$, then $C_1 \cup C_2 \cup \{uv\}$ is a K_4-cover of G. Thus

$$\begin{aligned}\tau_\boxtimes(G) &\leq \tau_\boxtimes(G_1\backslash uv) + \tau_\boxtimes(G_2\backslash uv) + 1\\ &\leq 3\nu_\boxtimes(G_1\backslash uv) + 3\nu_\boxtimes(G_2\backslash uv) + 1 \qquad (5.5)\\ &= 3(\nu_\boxtimes(G_1) - 1) + 3(\nu_\boxtimes(G_2) - 1) + 1\\ &= 3(\nu_\boxtimes(G_1) + \nu_\boxtimes(G_2) - 1) - 2\\ &\leq 3\nu_\boxtimes(G),\end{aligned}$$

where (5.5) follows from the inductive hypothesis. □

In the case of graphs with no subgraph isomorphic to a subdivision of K_5, we rely on a result of Mader's which bounds the number of edges in such graphs.

Theorem 5.5.5 (Mader [62]). *If $G = (V, E)$ is a graph with $|V| \geq 3$ and no subgraph isomorphic to a subdivision of K_5, then $|E| \leq 3|V| - 6$.*

The relevant consequence of Theorem 5.5.5 is that every graph with no subgraph isomorphic to a subdivision of K_5 is 5-degenerate. Therefore, Proposition 5.4.5 yields the following.

Corollary 5.5.6. *If G is a graph with no subgraph isomorphic to a subdivision of K_5, then $\tau_{\boxtimes}(G) \leq 4\nu_{\boxtimes}(G)$.*

5.6 Additional Remarks

The goal of this chapter was to prove bounds of the form $\tau_{\boxtimes}(G) \leq \alpha\nu_{\boxtimes}(G)$ for several classes of graphs G. Ultimately, we would like to find minimum values for α. Therefore, we now ask the following question: How close are our bounds to being best possible? In Section 5.3 we saw that K_8 satisfies $\tau_{\boxtimes}(G) = \frac{7}{2}\nu_{\boxtimes}(G)$. Hence, α can be at least $\frac{7}{2}$. For planar graphs, α can be at least two. Indeed, the graph G in Figure 5.7 is planar and satisfies $\tau_{\boxtimes}(G) = 2\nu_{\boxtimes}(G)$. In terms of fractional K_4-covers, we saw in Section 5.1 that a minimum fractional K_4-cover of K_6 can be obtained by assigning $\frac{1}{6}$ to every edge. This yields a ratio of $\frac{5}{2}$.

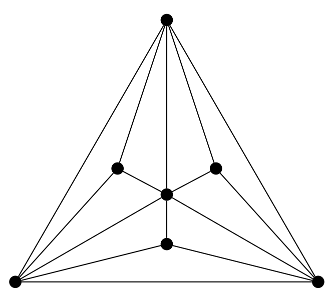

Figure 5.7: Planar graph satisfying $\tau_{\boxtimes}(G) = 2\nu_{\boxtimes}(G)$

Table 5.4 summarizes the known bounds on α. Our goal for the future is to reduce the possible ranges for α. In particular, we suspect that all of our bounds, except for the complete case, are not optimal. It would be interesting to see these optimal bounds.

Case	Range for α
Any graph	$3.5 \leq \alpha \leq 6$
4-partite	$2 \leq \alpha \leq 5$
Complete	$\alpha = 3$, unless K_8
8-degenerate	$3.5 \leq \alpha \leq 5$
No $K_{3,3}$-subdivision	$2 \leq \alpha \leq 3$
$\tau^*_{\boxtimes}(G)$	$2.5 \leq \alpha \leq 4.5$

Table 5.4: Lower bounds for packing and covering K_4's

We conclude this chapter with a few words about Theorem 5.1.3. In Section 5.1, a $\mathcal{T}$-pattern $\mathcal{P}$ is defined for a maximum K_4-packing $\mathcal{T}$. However, it is not necessary for $\mathcal{T}$ to be maximum; we can account for the size of $\mathcal{T}$ by using the 4-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_{(\mathcal{T},\mathcal{P})}$ to measure the quality of $\mathcal{P}$, where α_1 is the size of $\mathcal{T}$, α_2 is the number of $\mathcal{T}$-K_5's in $\mathcal{P}$, α_3 is the number of $\mathcal{T}$-K_5^-'s in $\mathcal{P}$, and α_4 is the number of $\mathcal{T}$-L_2's in $\mathcal{P}$. The proof of Lemma 5.1.2 now yields a procedure for finding a fractional K_4-cover in a graph G: Given $\mathcal{T}$ and $\mathcal{P}$, either φ defines a fractional K_4-cover of size at most $\frac{9}{2}|\mathcal{T}|$ or there is a K_4-packing $\mathcal{T}^+$ and a $\mathcal{T}^+$-pattern $\mathcal{P}^+$ such that $\mathcal{P}^+$ is better than $\mathcal{P}$, as in Claims 1-4. Specifically, the 4-tuple for $\mathcal{P}^+$ is larger than the 4-tuple for $\mathcal{P}$ under lexicographical ordering. Thus, we have a polynomial time algorithm that finds a K_4-packing $\mathcal{T}^*$ and a fractional K_4-cover of size at most $\frac{9}{2}|\mathcal{T}^*|$.

Chapter 6

Stable Matchings

This chapter is concerned with stable matchings. In particular, we examine fractional stable matchings. Recall that, for a hypergraph with preferences $\mathcal{H}$, a function $\varphi : \mathcal{H} \to [0,1]$ is a fractional stable matching if it is a fractional matching and, for each edge $e \in \mathcal{H}$, there is a vertex $u \in e$ such that

$$\sum_{e \leq_u h} \varphi(h) = 1.$$

The vertex u will be called a *witness of* (e, φ). If there exists an $n \in \mathbb{N}$ and, for each $e \in \mathcal{H}$, an $s_e \in [n]$ such that $s_e\varphi(e) \in [n]$, then we will say that φ is a $\frac{1}{n}$*-integral stable matching*.

As we saw in Section 1.3, Theorem 1.3.3 tells us that every hypergraph with preferences has a fractional stable matching. Furthermore, Aharoni and Fleiner noticed in [7] that Tan's main result in [79] implied that every graph with preferences has a $\frac{1}{2}$-integral stable matching. This leads us to wonder if a similar result holds for hypergraphs. In particular, we ask the following question: Given a positive integer r, does there exist a function $f(r)$ such that every r-uniform hypergraph with preferences has a $\frac{1}{n}$-integral stable matching for some $n \leq f(r)$? In this chapter, we provide a negative answer to this question.

6.1 Bounded Denominators

Let us begin with a clarifying example. We use a hypergraph construction due to Chung, Füredi, Garey, and Graham [19]. Let $k \in \mathbb{N}$ and let $\mathcal{H}_k$ be the 3-uniform hypergraph with preferences on the vertex set

$$\{a_1, a_2, \ldots, a_{3k}, a_{3k+1}\} \cup \{b_1, b_2, \ldots, b_k, b_{k+1}\} \cup \{c_1, c_2, \ldots, c_{3k}, c_{3k+1}\}$$

and the edge set

- $X_i = a_{3i-2}a_{3i-1}a_{3i}$, $Y_{2i-1} = a_{3i-2}b_ic_{3i-2}$, and $Y_{2i} = a_{3i-1}b_ic_{3i-1}$ for each $i \in [k]$,
- $Y_{2k+1} = a_{3k}b_{k+1}c_{3k}$,
- $Y_{2k+2} = a_{3k+1}b_{k+1}c_{3k+1}$,
- $Z_0 = a_1a_{3k}a_{3k+1}$, and
- $Z_i = a_{3i}a_{3i+1}c_{3i}$ for each $i \in [k-1]$,

with the vertex preferences given in Table 6.1. Notice that the vertices $c_1, c_2, \ldots, c_{3k}, c_{3k+1}$ are not listed in Table 6.1. We also see that every edge e of $\mathcal{H}_k$ is first in the preference list of some vertex v in Table 6.1. Therefore, since c_i has degree one in $\mathcal{H}_k$ for all $i \in [k+1]$, if ψ is a fractional stable matching of $\mathcal{H}_k$ and $c_i \in e$ is a witness of (e, ψ), then $\psi(e) = 1$ and v is also a witness of (e, ψ). Thus, for the purposes of our analysis, the vertices $c_1, c_2, \ldots, c_{3k}, c_{3k+1}$ can be ignored. However, it is important to note that $c_1, c_2, \ldots, c_{3k}, c_{3k+1}$ still prefer to be contained in an edge of a stable matching than not. As an example, Figure 6.1 shows $\mathcal{H}_2$.

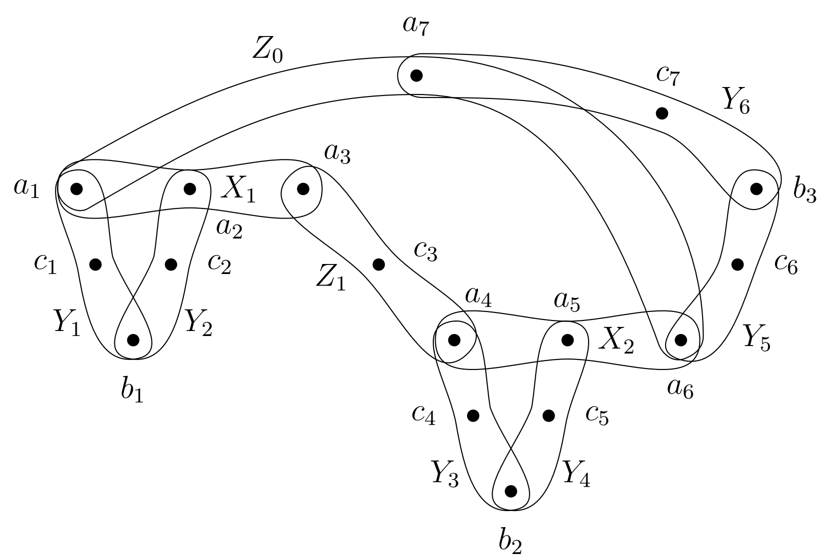

Figure 6.1: The hypergraph $\mathcal{H}_2$.

a_{3i-2} $(i \in [k])$	Y_{2i-1}	X_i	Z_{i-1}
a_{3i-1} $(i \in [k])$	X_i	Y_{2i}	
a_{3i} $(i \in [k-1])$	Z_i	X_i	
a_{3k}	Y_{2k+1}	Z_0	X_k
a_{3k+1}	Z_0	Y_{2k+2}	
b_i $(i \in [k+1])$	Y_{2i}	Y_{2i-1}	

Table 6.1: Vertex preferences for $\mathcal{H}_k$ (most preferred edge on the left).

It is known that for every graph G, $2\nu^*(G)$ is an integer [23, 83]. In [19], Chung, Füredi, Garey, and Graham discussed whether a similar result holds for 3-uniform hypergraphs. They provided a negative answer by showing that, for all rational numbers $q \geq 1$, there is a 3-uniform hypergraph $\mathcal{G}$ such that $\nu^*(\mathcal{G}) = q$. In doing so, they showed that the function $g_k : \mathcal{H}_k \to [0, 1]$ (defined in Table 6.2) is a maximum fractional matching of $\mathcal{H}_k$ for all $k \in \mathbb{N}$.

X_i $(i \in [k])$	$\dfrac{2^{k-i}}{2^{k+1}-1}$
Y_{2i-1} $(i \in [k])$	$\dfrac{2^{k-i}}{2^{k+1}-1}$
Y_{2i} $(i \in [k])$	$1 - \dfrac{2^{k-i}}{2^{k+1}-1}$
Y_{2k+1}	$\dfrac{2^k-1}{2^{k+1}-1}$
Y_{2k+2}	$\dfrac{2^k}{2^{k+1}-1}$
Z_0	$\dfrac{2^k-1}{2^{k+1}-1}$
Z_i $(i \in [k-1])$	$1 - \dfrac{2^{k-i}}{2^{k+1}-1}$

Table 6.2: The fractional matching g_k.

We claim that g_k is also a fractional stable matching of $\mathcal{H}_k$. To see this, note that every edge of $\mathcal{H}_k$ is last in some preference list in Table 6.1. Furthermore, we see that

$$\sum_{e \in \mathcal{H}_k : v \in e} g_k(e) = 1$$

for every vertex v in Table 6.1. In other words, for each $e \in \mathcal{H}_k$, the witness of (e, g_k) is the vertex which ranks e last in its preference list. Thus, g_k is a fractional stable matching of $\mathcal{H}_k$. Notice that $\mathcal{H}_k$ is 3-uniform for every $k \in \mathbb{N}$, yet the denominator for g_k is $2^{k+1} - 1$, which is unbounded as $k \to \infty$. Alternatively, consider the function ψ_k, defined in Table 6.3.

$X_i \ (i \in [k])$	0
$Y_{2i-1} \ (i \in [k])$	0
$Y_{2i} \ (i \in [k])$	1
Y_{2k+1}	$\frac{1}{2}$
Y_{2k+2}	$\frac{1}{2}$
Z_0	$\frac{1}{2}$
$Z_i \ (i \in [k-1])$	1

Table 6.3: The function ψ_k.

Notice that $\mathcal{X} = \{Y_2, Y_4, \ldots, Y_{2k}, Z_1, Z_2, \ldots, Z_{k-1}\}$ is a matching of $\mathcal{H}_k$ and no edge of $\mathcal{X}$ meets Y_{2k+1}, Y_{2k+2}, or Z_0. Furthermore, no vertex of $\mathcal{H}_k$ is contained in all three of Y_{2k+1}, Y_{2k+2}, and Z_0. Hence, ψ_k is a fractional matching of $\mathcal{H}_k$. To show that ψ_k is also a fractional stable matching, we need to find a witness of (e, ψ_k) for every edge $e \in \mathcal{H}_k$. Indeed, we have the following table:

$X_i \ (i \in [k])$	a_{3i}
$Y_{2i-1} \ (i \in [k])$	b_i
$Y_{2i} \ (i \in [k])$	a_{3i-1}
Y_{2k+1}	b_{k+1}
Y_{2k+2}	a_{3k+1}
Z_0	a_{3k}
$Z_i \ (i \in [k-1])$	a_{3i+1}

Table 6.4: Witnesses for the stability of ψ_k.

Thus, for all $k \in \mathbb{N}$, ψ_k is a fractional stable matching of $\mathcal{H}_k$ with denominator 2. This example illustrates the essence of our motivational question: The hypergraph $\mathcal{H}_k$ has a maximum fractional stable matching with large denominators. However, at the expense of the total size of the fractional matching, we can find another fractional stable matching with small denominators. It also shows that, unlike graphs, hypergraphs with preferences may have fractional stable matchings of different sizes. This is potentially helpful in our search for fractional stable matchings with bounded denominators because it allows us to consider a wider range of possible fractional matchings.

However, suppose we modify the preferences for $\mathcal{H}_k$ to obtain the 3-uniform hypergraph with preferences $\mathcal{G}_k$, as shown in Table 6.5 (i.e. the underlying hypergraphs of $\mathcal{H}_k$ and $\mathcal{G}_k$ are the same).

$a_{3i-2} \ (i \in [k])$	Z_{i-1}	X_i	Y_{2i-1}
$a_{3i-1} \ (i \in [k])$	Y_{2i}	X_i	
$a_{3i} \ (i \in [k-1])$	X_i	Z_i	
a_{3k}	X_k	Z_0	Y_{2k+1}
a_{3k+1}	Y_{2k+2}	Z_0	
$b_i \ (i \in [k+1])$	Y_{2i-1}	Y_{2i}	

Table 6.5: Vertex preferences for $\mathcal{G}_k$ (most preferred edge on the left).

Now we have the following result.

Theorem 6.1.1. *For each $k \in \mathbb{N}$, the function g_k is a fractional stable matching of $\mathcal{G}_k$. Furthermore, g_k is the unique fractional stable matching of $\mathcal{G}_k$.*

Proof: By Theorem 1.3.3, $\mathcal{G}_k$ has a fractional stable matching ϕ_k. For each $i \in [k+1]$, let $\alpha_i \in [0,1]$ be such that $\phi_k(Y_{2i}) = \alpha_i$. We show that, for every $e \in \mathcal{G}_k$, $\phi_k(e)$ is determined by exactly one of $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$.

Since ϕ_k is a fractional stable matching of $\mathcal{G}_k$, every edge of $\mathcal{G}_k$ has a witness. In other words, for every edge $e \in \mathcal{G}_k$, there is a vertex $u \in e$ such that

$$\sum_{e \leq_u h} \phi_k(h) = 1.$$

For each $i \in [k]$, a witness of (Y_{2i}, ϕ_k) is either a_{3i-1} or b_i and a witness of (Y_{2k+2}, ϕ_k) is either a_{3k+1} or b_{k+1}. Using Table 6.5, this means that either $\phi_k(Y_{2i}) = \alpha_i = 1$ or $\phi_k(Y_{2i-1}) + \phi_k(Y_{2i}) = \phi_k(Y_{2i-1}) + \alpha_i = 1$ for all $i \in [k+1]$. Therefore, since ϕ_k is also a fractional matching of $\mathcal{G}_k$, both cases yield

$$\phi_k(Y_{2i-1}) = 1 - \alpha_i \tag{6.1}$$

for all $i \in [k+1]$. Now, we have

$$\begin{aligned}\phi_k(Z_0) &\leq \min\{1 - \phi_k(Y_{2k+1}), 1 - \phi_k(Y_{2k+2})\} \\ &= \min\{\alpha_{k+1}, 1 - \alpha_{k+1}\} \\ &\leq \frac{1}{2}\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}\phi_k(X_i) &\leq \min\{1 - \phi_k(Y_{2i-1}), 1 - \phi_k(Y_{2i})\} \\ &= \min\{\alpha_i, 1 - \alpha_i\} \\ &\leq \frac{1}{2}\end{aligned} \tag{6.3}$$

for all $i \in [k]$, since ϕ_k is a fractional matching.

A witness of (Z_0, ϕ_k) is either a_1, a_{3k}, or a_{3k+1}. However, if a_1 is a witness of (Z_0, ϕ_k), then by Table 6.5, $\phi_k(Z_0) = 1$ which contradicts (6.2). Thus a witness of (Z_0, ϕ_k) is either a_{3k} or a_{3k+1} and Table 6.5 tells us that we have

$$\phi_k(Z_0) + \phi_k(X_k) = 1 \text{ or } \phi_k(Z_0) + \phi_k(Y_{2k+2}) = 1. \tag{6.4}$$

Similarly, for each $i \in [k]$, Table 6.5 and (6.3) tell us that the witness of (X_i, ϕ_k) is either a_{3i-2} or a_{3i-1} and this yields

$$\phi_k(X_i) + \phi_k(Z_{i-1}) = 1 \text{ or } \phi_k(X_i) + \phi_k(Y_{2i}) = 1. \tag{6.5}$$

We now make a series of claims.

Claim 1: We have $\phi_k(Z_0) = 1 - \alpha_{k+1}$.

Proof of Claim 1: By (6.4), either $\phi_k(Z_0) + \phi_k(X_k) = 1$ or $\phi_k(Z_0) + \phi_k(Y_{2k+2}) = 1$. If $\phi_k(Z_0) + \phi_k(X_k) = 1$, then by (6.2) and (6.3), we have $\phi_k(Z_0) = \phi_k(X_k) = \frac{1}{2}$. Since ϕ_k is a fractional matching and the vertex a_{3k} is contained in X_k, Y_{2k+1}, and Z_0, (6.1) tells us that $\phi_k(Y_{2k+1}) = 1 - \alpha_{k+1} = 0$ and, by definition, $\phi_k(Y_{2k+2}) = \alpha_{k+1} = 1$. However, we now have $\sum_{e: a_{3k+1} \in e} \phi_k(e) = \phi_k(Z_0) + \phi_k(Y_{2k+2}) = \frac{3}{2}$, which contradicts our assumption that ϕ_k is a fractional matching. Therefore, $\phi_k(Z_0) + \phi_k(Y_{2k+2}) = \phi_k(Z_0) + \alpha_{k+1} = 1$, as required.

Claim 2: For each $i \in [k]$, if $\phi_k(Z_{i-1}) \neq 1$, then $\phi_k(X_i) = 1 - \alpha_i$.

Proof of Claim 2: Let $i \in [k]$ and suppose that $\phi_k(Z_{i-1}) \neq 1$. By (6.5), we have either $\phi_k(X_i) + \phi_k(Z_{i-1}) = 1$ or $\phi_k(X_i) + \phi_k(Y_{2i}) = 1$. If $\phi_k(X_i) + \phi_k(Z_{i-1}) = 1$, then since ϕ_k is a fractional matching and the vertex a_{3i-2} is contained in X_i, Y_{2i-1}, and Z_{i-1}, we have $\phi_k(Y_{2i-1}) = 1 - \alpha_i = 0$ by (6.1) and $\phi_k(Y_{2i}) = \alpha_i = 1$. However, since a_{3i-1} is contained in both X_i and Y_{2i}, this means that $\phi_k(X_i) = 0$ and $\phi_k(Z_{i-1}) = 1$, which is a contradiction. Thus $\phi_k(X_i) + \phi_k(Y_{2i}) = \phi_k(X_i) + \alpha_i = 1$, as required.

Claim 3: For each $i \in [k-1]$, if $\phi_k(Z_{i-1}) \neq 1$, then $\phi_k(Z_i) = \alpha_i$.

Proof of Claim 3: Let $i \in [k-1]$ and suppose that $\phi_k(Z_{i-1}) \neq 1$. Since ϕ_k is a fractional stable matching, a witness of (Z_i, ϕ_k) is either a_{3i} or a_{3i+1}. Thus Table 6.5 tells us that we have either $\phi_k(Z_i) + \phi_k(X_i) = 1$ or $\phi_k(Z_i) = 1$. Since $\phi_k(Z_{i-1}) \neq 1$, Claim 2 implies that $\phi_k(X_i) = 1 - \alpha_i$. So, if $\phi_k(Z_i) + \phi_k(X_i) = 1$, then $\phi_k(Z_i) + \phi_k(X_i) = \phi_k(Z_i) + (1 - \alpha_i) = 1$, which yields $\phi_k(Z_i) = \alpha_i$, as required.

Thus, we assume that $\phi_k(Z_i) = 1$. In this case, since ϕ_k is a fractional matching and a_{3i} is contained in both X_i and Z_i, we have $\phi_k(X_i) = 1 - \alpha_i = 0$ and, by definition, $\phi_k(Y_{2i}) = \alpha_i = 1$. In other words, $\phi_k(Z_i) = \alpha_i$ in this case as well.

Claim 4: For each $i \in [k-1] \cup \{0\}$, we have $\phi_k(Z_i) \neq 1$.

Proof of Claim 4: We proceed by induction on i. If $i = 0$, then by (6.2) we have $\phi_k(Z_0) \leq \frac{1}{2}$. So suppose that $i \geq 1$ and that $\phi_k(Z_{i-1}) \neq 1$. By Claims 2 and 3, $\phi_k(X_i) = 1 - \alpha_i$ and

$\phi_k(Z_i) = \alpha_i$. Now suppose, for a contradiction, that $\phi_k(Z_i) = \alpha_i = 1$. Then $\phi_k(X_i) = \phi_k(Y_{2i-1}) = 1-\alpha_i = 0$ by (6.1) and $\phi_k(Y_{2i}) = \alpha_i = 1$ by definition. A witness of (Y_{2i-1}, ϕ_k) is either a_{3i-2} or b_i. However, since $\phi_k(Y_{2i-1}) = 0$, Table 6.5 tells us that the witness of (Y_{2i-1}, ϕ_k) is a_{3i-2}. Therefore

$$1 = \phi_k(Y_{2i-1}) + \phi_k(X_i) + \phi_k(Z_{i-1}) = \phi_k(Z_{i-1}),$$

which contradicts our assumption. Thus, $\phi_k(Z_i) \neq 1$, as required.

Now, we see that Claims 2, 3, and 4 imply that $\phi_k(X_i) = 1 - \alpha_i$ for all $i \in [k]$ and $\phi_k(Z_j) = \alpha_j$ for all $j \in [k-1]$. To summarize our progress so far, we have the following table of values for ϕ_k:

X_i $(i \in [k])$	$1-\alpha_i$
Y_{2i-1} $(i \in [k])$	$1-\alpha_i$
Y_{2i} $(i \in [k])$	α_i
Y_{2k+1}	$1-\alpha_{k+1}$
Y_{2k+2}	α_{k+1}
Z_0	$1-\alpha_{k+1}$
Z_i $(i \in [k-1])$	α_i

Table 6.6: The possible values for ϕ_k.

Notice that since $\phi_k(X_i) = 1-\alpha_i$ for all $i \in [k]$, (6.2), (6.3), and Claim 1 yield $\alpha_i \geq \frac{1}{2}$ for all $i \in [k+1]$. Therefore, since $\phi_k(Y_{2i-1}) = 1-\alpha_i \leq \frac{1}{2}$ for all $i \in [k+1]$, the stability of ϕ_k tells us that the witness of (Y_{2i-1}, ϕ_k) is a_{3i-2} when $i \in [k]$ and a_{3k} when $i = k+1$. Thus, by Table 6.5, we have $\phi_k(Y_{2i-1}) + \phi_k(X_i) + \phi_k(Z_{i-1}) = 1$ for all $i \in [k]$ and $\phi_k(Y_{2k+1}) + \phi_k(X_k) + \phi_k(Z_0) = 1$. When we substitute in the values from Table 6.6, we are left with the following $k+1$ equations:

$$\begin{aligned} &2\alpha_1 + \alpha_{k+1} = 2 \\ &2\alpha_i - \alpha_{i-1} = 1 \text{ for all } i \in [k]\backslash\{1\} \\ &2\alpha_{k+1} + \alpha_k = 2. \end{aligned} \tag{6.6}$$

This system of equations gives us the matrix equation $A\alpha = b$, where

$$A = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

We see that $\det(A) = 2\det(B) \pm \det(C)$, where B is a lower triangular matrix with only 2's on its diagonal and C is an upper triangular matrix with only ± 1's on its diagonal. Thus, $\det(A) \neq 0$ and ϕ_k is the unique fractional stable matching of $\mathcal{G}_k$.

To conclude, we notice that

$$\alpha_i = 1 - \frac{2^{k-i}}{2^{k+1}-1} \text{ for all } i \in [k] \text{ and}$$

$$\alpha_{k+1} = \frac{2^k}{2^{k+1}-1}$$

is a solution to (6.6) and, hence, the unique solution to (6.6). Furthermore, we see that our solution exactly corresponds to the fractional matching g_k given in Table 6.2. Thus, g_k is the unique fractional stable matching of $\mathcal{G}_k$, as required. □

Chapter 7

Concluding Remarks

In this final chapter, we summarize our earlier work and discuss future directions for research.

Chapter 3 focused on matchings and covers of 3-uniform, tripartite hypergraphs. In [45] and [46], Haxell, Narins, and Szabó characterized the 3-uniform, tripartite hypergraphs $\mathcal{H}$ such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$; they proved Theorem 3.1.2 which says that that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ if and only if $\mathcal{H}$ is a home-base hypergraph. Their work relied heavily on topological arguments which seem to present significant challenges when applied to more general settings. We reproved Theorem 3.1.2 using much less topological machinery. Our hope is that our arguments will lend themselves to situations where $\tau(\mathcal{H}) < 2\nu(\mathcal{H})$.

Ideally, we would like to have a stability version of Theorem 3.1.2. Such a theorem would be along the following lines: If $\mathcal{H}$ is a 3-uniform, tripartite hypergraph such that $\tau(\mathcal{H}) = (2-\epsilon)\nu(\mathcal{H})$, then $\mathcal{H}$ is close to being a homebase hypergraph. Here, "close" would mean that the spine of $\mathcal{H}$ is a disjoint union of $\mathcal{F}$'s, $\mathcal{R}$'s, and a few longer loose cycles instead of only $\mathcal{F}$'s and $\mathcal{R}$'s. This type of theorem would be beneficial in many situations, including several in this thesis.

One such application is the tripartite version of Conjecture 1.2.1. Recall that the triangle hypergraph $\mathcal{H}_G$ of a tripartite graph G is 3-uniform and tripartite. Furthermore, we know that $\mathcal{H}_G$ is not a home-base hypergraph since the presence of an $\mathcal{R}$ or an $\mathcal{F}$ implies that G contains a copy of K_4 as a subgraph. Thus, a stability version of Theorem 3.1.2 has the potential to improve the bound in Theorem 4.1.5.

In Chapter 5 we considered the problem of packing and covering K_4's. We began by proving that $\tau^*_{\boxtimes}(G) \leq \frac{9}{2}\nu_{\boxtimes}(G)$ for all graphs G. We also proved that $\tau_{\boxtimes}(G) \leq 5\nu_{\boxtimes}(G)$ for all 4-partite graphs G. In fact, we proved that the inequality in Theorem 5.2.1 is strict! To see this, we simply have to note that the 3-uniform, tripartite hypergraphs $\mathcal{H}_i$ in the proof of Theorem 5.2.1 are not home-base hypergraphs. Similarly to the above discussion, the presence of an $\mathcal{R}$ or an $\mathcal{F}$ implies that G contains a copy of K_5 as a subgraph. Once again, we see that a stability version of Theorem 3.1.2 could provide an improvement to our work.

We also have some smaller future plans. One idea is to use discharging methods, along the lines of Puleo in [68], to increase the degeneracy assumption in Corollary 5.4.6. We would also like a non-trivial result of the form $\tau_{\boxtimes}(G) \leq (6-\epsilon)\nu_{\boxtimes}(G)$ for all graphs G, and better examples to improve the lower bounds in Table 5.4.

In terms of stable matchings, we are also interested in the following variation which first appeared in [48], where the authors attribute it to Donald Knuth: Let $\mathcal{H}$ be a complete 3-uniform, tripartite hypergraph with vertex classes A, B, and C. Each vertex in A has a totally ordered preference list of the vertices in B, each vertex in B has a totally ordered preference list of the vertices in C, and each vertex in C has a totally ordered preference list of the vertices in A. The problem is to determine if every such instance has a stable matching. Eriksson, Sjöstrand, and Strimling proved that the answer is yes provided that $\max\{|A|, |B|, |C|\} \leq 4$. Furthermore, they conjecture, based on computer evidence, that every instance has at least two stable matchings [26].

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