

Subdividing the cd -index

by

Patrick Dornian

A thesis
presented to the University Of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2016

©Patrick Dornian 2016

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis aims to give the reader an introduction and overview of the cd -index of a poset, as well as establish some new results. We give a combinatorial proof of Ehrenborg and Karu's cd -index subdivision decomposition for Gorenstein* complexes and extend it to a wider class of subdivisions. In doing so, we define a *local cd -index* that behaves analogously to the well studied local h -vector. We examine known cd -index and h -vector bounds, and then use the local cd -index to bound a particular class of polytopes with the cd -index of a stacked polytope. We conclude by investigating the h -vector and local h -vector of posets in full generality, and use an algebra morphism developed by Bayer and Ehrenborg to demonstrate the structural connection between the cd -index subdivision decomposition and the local h -vector subdivision decomposition.

Acknowledgements

I would first like to thank my supervisor Eric Katz for his mathematical guidance, unwavering support, and poor jokes.

Next, I would like to thank my readers Kevin Purbhoo and David Wagner for sacrificing their valuable time in an attempt to parse this tome.

Special thanks to my office mates and friends Cameron, Garnet and Anirudh for creating the best work environment I've ever been a part of (and likely the best I'll ever find).

My parents have spent the last two years having to deal with me ranting about incomprehensible math as a substitute for small talk. Thanks for putting up with me. You supported me through my entire education and raised me with the capacity to see this through. The early math education that you gave me was the start of this long road. The love you provided was invaluable. The money and care packages were pretty good too.

My Waterloo family consists of my loving girlfriend Lilly Zheng and our cats Hodor and Versace. All three provided essential inspiration.

Finally, thanks to all my friends for keeping me sane. In no particular order, the following answered my open facebook invitation for people to acknowledge. I'm almost certainly missing a few names, so I preemptively apologize (Sorry).

Carlo Arcovio	Tavian Barnes
Ryan Hancock	Catherine Maggiori
Aidan Waite	Adam Gomes
Jonathan Dornian	Tevin Straub
Jaqueline Li	Sakib Imtiaz
Alexander Morash	Andrijana Nesic
Kirsten Hattori	Shawn Puthukkeril
Melissa Angyalfi	Mike Daw
Bashar Jabbour	Zach Neshevich
Reagan Elly	Dain Galts
Jamie Waugh	Lily Wang
Andrew Cottle	Francis Williams
Kat Dornian	Ravi Goundalkar
Bill Irons	Steven Ye
Kevin Chapman	Charlie Payne
Jason LeGrow	Nolan Shaw
Steven Sun	Amy Dornian
Stephanie Raphael	Greg Gregory Greggington Lewis
Erin Perri	Benjamin Graf
Darren Stalker	Ella Weatherilt
Prashanth Madhi	Erika Angyalfi
Evan Ferguson	Rachel Dornian
Pavel Shuldiner	Nick Pulos

Dedication

In loving memory of Paul Walker and Rob Ford

*it's been a long day
without you my friend
and i'll tell you all about it
when i see you again.....*

Contents

1	Introduction	1
2	The ab-index and cd-index of a poset	2
3	Shelling the cd-index	7
4	Gorenstein Complexes	18
5	Skeletal Decomposition of a Subdivision	25
6	Subdividing the cd-index	32
7	Upper Bounds, Lower Bounds, and the cd-index of Stacked Polytopes	38
8	Morphisms between the cd-index, the toric h-vector and the local h-vector	52
9	Closing Remarks	62
	References	63

1 Introduction

The f -vector and h -vector of a simplicial complex are classical equivalent tools for encoding its face numbers. Analogously, the flag f -vector and the flag h -vector were developed by Bayer and Billera [1] in 1985 to enumerate flags in posets. If the poset is Eulerian, we may efficiently encode these vectors into a cd -index, a non-commutative generating function developed by Bayer, Fine and Klapper. [3]. The motivating example for an Eulerian poset is typically the face lattice of a convex polytope.

Though the cd -index is efficient, its behavior is difficult to characterize. Given a simple geometric transformation on a polytope, the corresponding action on the cd -index is often hard to express. Its coefficients are non-negative for large classes of posets, but finding a natural combinatorial interpretation of them is an unresolved problem. Stanley uses a shelling argument to show that the cd -index of a regular shellable CW -sphere is non-negative [20]. We paraphrase this result and demonstrate some applications of it as it motivates many of the later techniques we use.

Using commutative algebra, Karu established non-negativity for complete and quasi-convex fans, as well as for Gorenstein* posets [9]. Ehrenborg and Karu extended this result to near-Gorenstein* posets soon after [6]. In the same article, they develop a decomposition theorem for the cd -index of subdivisions of Gorenstein* posets by using the theory of sheaves on fans and posets. We relax their assumptions to a new, slightly broader model of *strongly Eulerian subdivisions* to produce a natural combinatorial model for the decomposition, in the process defining a *local cd -index*. This local cd -index behaves analogous to the local h -vector introduced by Stanley [18].

We then investigate known bounds on the cd -index and the h -vector. Using the local cd -index, we demonstrate a new bound: that the cd -index of a stacked polytope is an upper bound for the cd -index of simplicial spheres that may be triangulated by a shellable complex. Finally, we then examine the connections between the cd -index subdivision decomposition and the general local h -vector decomposition of a subdivision by using techniques developed by Bayer and Ehrenborg in [2].

2 The ab-index and cd-index of a poset

We will first state definitions and notation on partially ordered sets.

Definition 2.1. A *partially ordered set* or *poset* is a set P with a binary relation \leq that satisfies the following three axioms for all elements $s, t, u \in P$:

1. Reflexivity: $t \leq t$.
2. Antisymmetry: If $s \leq t$ and $t \leq s$ then $s = t$.
3. Transitivity: If $s \leq t$ and $t \leq u$ then $s \leq u$.

Given a poset P , we call a subset $C \subseteq P$ a *chain* of P if the elements of C are totally ordered. That is,

$$C = x_0 < x_1 < \cdots < x_k$$

for some $x_i \in P$. Alternatively, these may also be referred to as *flags*. A chain is *maximal* if it is not contained in any other chain of P .

Definition 2.2. Given a poset P with $s \leq t$, we say that t *covers* s and write $s \triangleleft t$ if there exists no $u \in P$ such that $s < u < t$.

Definition 2.3. Given a poset P and $s, t \in P$, we define the *interval* $[s, t]$ by

$$[s, t] = \{x \mid s \leq x \leq t, x \in P\}.$$

Note that $[s, t]$ is also a well defined poset with order relation inherited from P .

We use $\widehat{0}$ and $\widehat{1}$ to denote elements (if they exist) $\widehat{0}, \widehat{1} \in P$ such that for all $s \in P$, $\widehat{0} \leq s$ and $s \leq \widehat{1}$. We say a chain is *degenerate* if it contains $\widehat{0}$ or $\widehat{1}$ (because any chain may be trivially extended by adding a maximal or minimal element).

Definition 2.4. Suppose every maximal chain in P has length n . We then say that P is *graded* with *rank* n .

Definition 2.5. If a poset P is graded, there exists a unique rank function $\rho_P : P \rightarrow \{0, 1, \dots, n\}$ such that:

1. $\rho_P(s) = 0$ if s is a minimal element of P .
2. If $s \triangleleft t$, then $\rho_P(t) = \rho_P(s) + 1$.

When context is clear, we will omit the subscript. If $\rho(s) = k$, we say that s has rank k . Given a nonempty interval $[s, t]$ we define the *length* of $[s, t]$ to be $\rho(s, t) = \rho(t) - \rho(s)$.

Note that if P is graded of rank n and has a $\widehat{0}$ and a $\widehat{1}$ we necessarily have that $\widehat{0}$ is of rank 0 and $\widehat{1}$ is of rank n . We call elements of rank 1 *atoms* of P and elements of rank $n - 1$ to be *coatoms*.

Definition 2.6. We say that a finite graded poset P with $\hat{0}$ and $\hat{1}$ is *Eulerian* if every nonempty interval $[s, t]$ contains an equal number of elements of odd rank and even rank.

Eulerian posets generalize the properties of the Euler characteristic of convex polytopes. We say that a poset is *lower Eulerian* if all its intervals are Eulerian and it has a $\hat{0}$. Lower Eulerian posets are motivated by polyhedral complexes. Note that an Eulerian poset is trivially also lower Eulerian.

For the remainder of this paper, we will let P denote a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$ unless otherwise specified. Note that every maximum length chain in P will have the form $\hat{0} = x_0 < x_1 < \dots < x_n = \hat{1}$, with $\rho(x_i) = i$. Since any maximum length chain must begin with $\hat{0}$ and end with $\hat{1}$, their presence is often taken as a given in our calculations. Unless otherwise specified, all chains are non-degenerate. We use $[n]$ to denote the set $[n] := \{1, 2, \dots, n\}$.

Definition 2.7. For $S \subseteq \{0, 1, \dots, n\}$ we let P_S denote the *S-rank selected subposet* of P ,

$$P_S = \{x \in P : \rho(x) \in S\}$$

Definition 2.8. The *flag f-vector* of P is defined by the function $\alpha_P : 2^{[n]} \rightarrow \mathbb{Z}$ where $\alpha_P(S) =$ the number of maximal chains of P_S .

Where there is no confusion about P , we will often simply denote it by α . We may think of α as a vector indexed by all possible subsets of $[n]$.

Definition 2.9. The *flag h-vector* of P is defined by the function $\beta_P : 2^{[n]} \rightarrow \mathbb{Z}$ where

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{\#(S \setminus T)} \alpha_P(T)$$

Equivalently, by inclusion-exclusion we have the following more elegant expression.

$$\alpha(S) = \sum_{T \subseteq S} \beta(T)$$

Our next goal is to encode the flag f and h -vector into generating functions. To correspond to a given rank selection $S \subseteq [n]$, we define the noncommutative characteristic monomial $u_S = u_1 u_2 \dots u_n$ in the variables a and b by

$$u_i = \begin{cases} a & \text{if } i \notin S \\ b & \text{if } i \in S \end{cases}$$

For example, if $n = 5$ and $S = \{1, 2, 5\}$ we have $u_S = bbaab$. This lets us define the following generating function.

Definition 2.10. The *ab-index* for a graded poset P of rank $n + 1$ with $\hat{0}$ and $\hat{1}$ is the generating function defined by

$$\Psi_P(a, b) = \sum_{S \subseteq [n]} \beta_P(S) u_S.$$

Definition 2.11. The *flag polynomial* for a graded poset P of rank $n + 1$ with $\hat{0}$ and $\hat{1}$ is defined by

$$\Upsilon_P(a, b) = \sum_{S \subseteq [n]} \alpha_P(S) u_S$$

It is easily seen that the flag polynomial and the *ab-index* are equivalent by a linear change of variables. We can transform between them by the following relationship [20].

$$\begin{aligned} \Upsilon_P(a, b) &= \Psi_P(a + b, b), \\ \Psi_P(a, b) &= \Upsilon_P(a - b, b). \end{aligned}$$

When convenient, we may omit the (a, b) argument or use the poset P itself as the argument of the function if context is clear. The following alternative notation is also useful.

Given a poset P of rank $n + 1$, consider some non-degenerate chain $C = x_1 < x_2 < \dots < x_k$. We define the following functions that take chains from P to *ab*-words. First, we set

$$\alpha^C = u_1 u_2 \dots u_n$$

where

$$u_i = \begin{cases} a & \text{if } \nexists \text{ some } j \text{ such that } \rho(x_j) = i \\ b & \text{if } \exists \text{ some } j \text{ such that } \rho(x_j) = i \end{cases}$$

and

$$\beta^C = u_1 u_2 \dots u_n$$

where

$$u_i = \begin{cases} a - b & \text{if } \nexists \text{ some } j \text{ such that } \rho(x_j) = i \\ b & \text{if } \exists \text{ some } j \text{ such that } \rho(x_j) = i \end{cases}.$$

In other words, α^C encodes the ranks of chain C as an *ab*-word where ranks that are present are weighted with a b while missing ranks are weighted with an a .

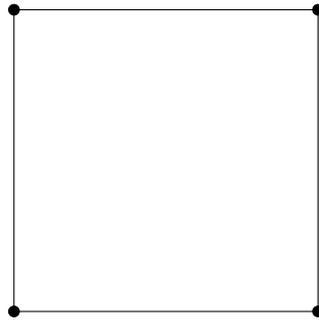
β^C encodes C so that ranks present are weighted with b , and ranks missing are weighted with $(a - b)$.

Proposition 2.12. Given a graded poset P , the following two equations hold (and may serve as alternate definitions of the ab -index and flag polynomial if desired).

$$\Upsilon_P = \sum_{C \text{ a chain of } P} \alpha^C$$

$$\Psi_P = \sum_{C \text{ a chain of } P} \beta^C.$$

Figure 1: The ab -index and flag enumerator of the face poset of a square.



$$\Upsilon_\Gamma = aa + 4ba + 4ab + 8bb$$

$$\Psi_\Gamma = aa + 3ba + 3ab + bb$$

Note that in the following example, our ab -index is symmetric. That is, it is invariant under the involution that switches b 's with a 's and visa versa. This is not a coincidence. Such a statement holds for all Eulerian posets [21].

Lemma 2.13. Let P be an Eulerian poset of rank $n + 1$. Let $S \subseteq [n]$ and denote $\bar{S} = [n] \setminus S$. We then have $\beta_P(S) = \beta_P(\bar{S})$.

The symmetry of the ab -index is a generalization of the symmetry of the h -vector of simplicial polytopes and spheres. That is, it is a generalization of the Dehn-Sommerville equations to flags of a poset [1]. In particular, this allows a particular substitution for the ab -index of an Eulerian poset.

Theorem 2.14. (*Bayer, Fine, Klapper*) [3] Let P be an Eulerian poset. Then there exists a non-commutative polynomial in the variables c and d denoted $\Phi_P(c, d)$ that satisfies $\Phi_P(a + b, ab + ba) = \Psi_P(a, b)$.

We call this polynomial the cd -index. For clarity in particular arguments we may refer to the cd -index of P as any of $\Phi_P(c, d)$, Φ_P , or $\Phi(P)$ depending on context.

Example 2.15. In the above figure of a square with face complex Γ , we have that $\Phi_\Gamma = c^2 + 2d$.

The cd -index lacks the natural symmetry of the ab -index, but represents a vastly more concise encoding of the flag data. The linear span of flag f or h -vectors of graded rank $n + 1$ posets has dimension at most 2^n , corresponding to all possible length n ab -words. However, the linear span of flag f or h -vectors of Eulerian posets of rank $n + 1$ has dimension at most \mathcal{F}_{n+1} where \mathcal{F}_{n+1} is the $(n + 1)^{\text{st}}$ Fibonacci number. This corresponds to the number of cd words of degree n where c has degree 1 and d has degree 2. Stanley demonstrates that this bound is tight [21].

In many cases, the cd -index of a poset has non-negative coefficients. Determining classes of posets for which the cd -index is non-negative is an active area of research. In the next chapter, we discuss Stanley's proof of its non-negativity for a certain class of shellable spheres (including polytopes). In Chapter Four, we discuss Karu's extension of this result to Gorenstein* posets.

3 Shelling the cd -index

The following section develops the theory of shelling and summarizes the relevant results from Chapter Two of *Flag f -vectors and the cd -index* by Stanley [20]. Many of Stanley's techniques motivate our later results so they are included here for reference.

In layman's terms, a shelling is a linear order the facets of a complex such that when 'drawn' sequentially, each facet intersects the previous facets in a continuous segment (with one exception in the one-dimensional case, but this definition suffices for most geometric intuition). It is a useful tool for proving results inductively is the main technique behind early results on the non-negativity of the cd -index.

All polytopes mentioned past this point are convex. No non-convex polytopes are discussed in this thesis. In lieu of writing 'convex' repeatedly, we will refer to 'polytopes' with the implicit understanding that we are referring to the convex case.

Definition 3.1. A *polytopal complex* $\mathcal{C} \subseteq \mathbb{R}^d$ is a finite set of polytopes such that

1. The empty face is present. That is, $\emptyset \in \mathcal{C}$.
2. If $P \in \mathcal{C}$, then all proper faces of P are also in \mathcal{C} .
3. For any $P, Q \in \mathcal{C}$, if $P \cap Q = F$ where $F \neq \emptyset$, then F is a face of both P and Q .

We may analogously define simplicial complexes by replacing all references above to polytopes with simplices. It is easy to see that any complex has a well defined partial order of its faces under inclusion. We will typically be considering the face complex of a polytope P : that is, the set of all its faces under inclusion. Another common construction is the boundary complex ∂P , the set of all faces of P excluding the maximal face of P itself. When context is clear we will use P to denote both the polytope itself, as well as the underlying poset of its face complex. When context is less clear, we will use the notation $F(P)$ for the partial order.

Note that the face poset of a complex is graded with a well defined rank function induced by dimension. Recall that the empty face \emptyset , the minimal face of the complex is typically noted as having dimension -1 . By definition of minimal element, we must have $\rho(\emptyset) = 0$ and it follows from an easy inductive argument that $\rho(F) = \dim F + 1$ for any $F \in P$. That is, vertices are of dimension 0, but rank 1, edges are of dimension 1 but rank 2 and so on. This shift in dimension can lead to endless arithmetic errors if one is careless, so it is important to keep in mind when generalizing theorems on complexes to posets.

The dimension of a complex \mathcal{C} is the maximum dimension of a face in \mathcal{C} . We say that a complex is *pure* if all maximal faces under inclusion have the same dimension.

There are many slight variations on the definition of shelling depending on the authors motives. The following is presented by Ziegler [22] and suffices for most combinatorial purposes.

Definition 3.2. Let Γ be a pure d -dimensional polytopal complex. We say Γ is *shellable* if either Γ is 0-dimensional, or there exists a linear order (called a *shelling*) of its facets F_1, F_2, \dots, F_k that satisfies the following recursive definition:

1. The boundary complex ∂F_1 has a shelling.
2. For $1 < j \leq k$, we have

$$F_j \cap \left(\bigcup_{i=1}^{j-1} F_i \right) = G_1 \cup G_2 \dots G_r$$

for some shelling $G_1, G_2, \dots, G_r, \dots, G_s$ of ∂F_j with $1 \leq r \leq s$. In other words, the intersection between F_j and the union of the previous facets in the shelling is the beginning segment of some shelling of ∂F_j .

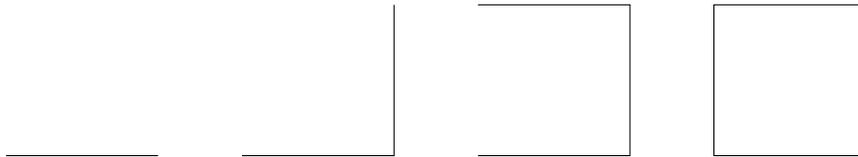


Figure 2: We may shell a polygon by ‘walking’ around its edges.

Shellings have been widely studied and are a useful inductive tool on complexes and polytopes. In particular, any result that can be proven for a shellable complex holds for all polytopes because of the following result of Bruggesser and Mani [5].

Theorem 3.3. Let P be a polytope. Then its boundary complex ∂P is shellable.

In particular, they show that every polytope may be shelled using a construction called a *line shelling* [5].

In proving results on polytopes, it is often of use to generalize to the looser topological framework of *CW*-spheres. The following definitions coincide with Munkres [13].

Definition 3.4. A space is a *cell* of dimension n if it is homeomorphic to the closed ball B^n . It is an *open cell* if it is homeomorphic to $\text{Int}B^n$.

Definition 3.5. A *CW complex* is a topological space X partitioned into open cells e_α such that:

1. X is Hausdorff.

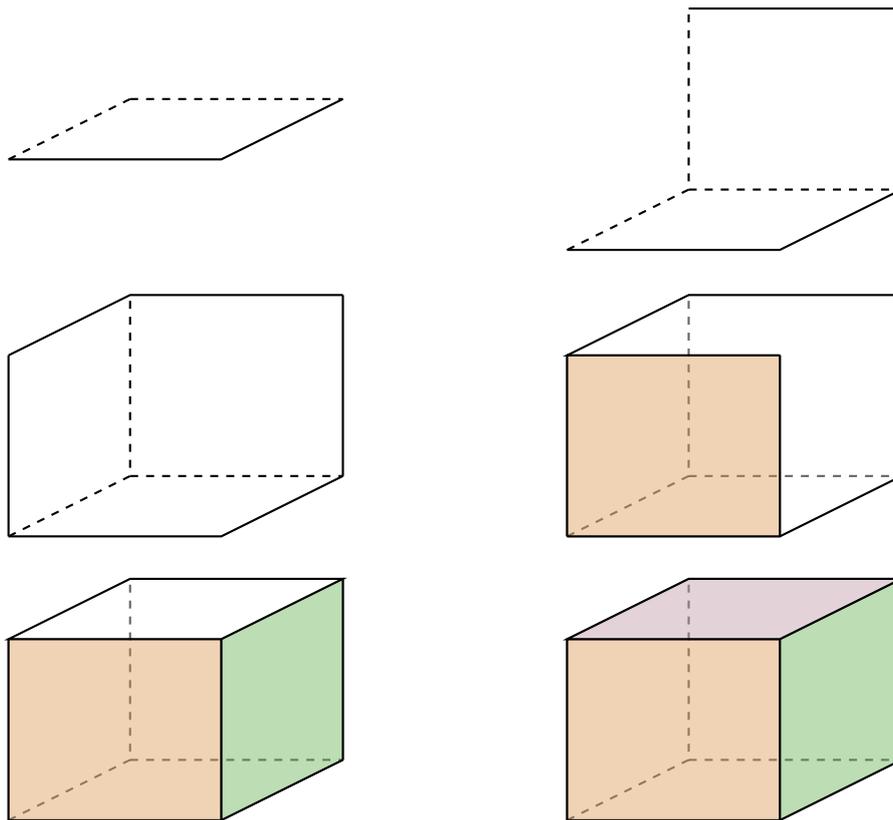


Figure 3: A shelling of a rectangular prism.

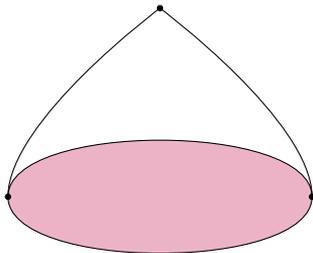


Figure 4: A regular CW -complex with one face, four edges and three vertices.

2. For each open n -cell e_α , there exists a continuous map $f_\alpha : B^n \rightarrow X$ that induces a homeomorphism between $\text{Int}B^n$ and e_α and carries ∂B^n into a finite union of open cells of dimension less than n .
3. X is equipped with the weak topology: A set $A \subseteq X$ is open (or closed) if and only if $A \cap \bar{e}_\alpha$ is open (or closed) in \bar{e}_α for all α .

The final condition is redundant when dealing with complexes of only a finite number of elements (as we are in this paper) but is included for completeness.

If f_α is a homeomorphism for all α , we say that the complex is a *regular CW-complex* (that is, no identifications are made on the boundary of e_α).

Regular CW -complexes are perhaps the most general framework for a topological space that still maintain combinatorial details. Any regular CW -complex Γ of dimension n also has a graded poset structure of rank $n + 1$ under the order relation that given cells $e_x, e_y \in \Gamma$, $e_x \leq e_y$ if and only if $e_x \subseteq \partial e_y$ topologically or $e_x = e_y$. We refer to this as the *face poset* of Γ . We may interpret cells as both topological spaces and as the elements of a poset, and we hope that context ensures clarity. As in the case of polytopes, we refer to the maximal cells of a CW -complex as *facets*. It follows that any polytopal complex Γ is a regular CW -complex with identical poset structure by identifying all d -faces $\sigma \in \Gamma$ with an open d -cell e_σ homeomorphic to its interior and maintaining all inclusion relations. From here on following we will consider faces of any polytopal/simplicial complex Γ to be open cells of a CW -complex when topologically convenient.

If Γ is a regular CW -complex such that its underlying space $|\Gamma|$ is homeomorphic to a sphere, we will call Γ a *CW-sphere*. The face poset of Γ is Eulerian [19]. Additionally, all polytopes are CW -spheres. We may analogously define shelling on regular CW -complexes and it follows that any result that holds for a shellable CW -sphere must hold for a polytope.

The following poset operations are introduced by Stanley in [20] and will be of use to us in both this and further chapters. Though defined on arbitrary posets, they are

geometrically motivated.

Definition 3.6. Given posets P and Q with $\hat{0}$ and $\hat{1}$, we define the *join* $P * Q$ to be the poset on the set

$$P * Q := P \setminus \{\hat{1}\} \cup (Q \setminus \{\hat{0}\})$$

with $x \leq y$ in $P * Q$ if either

1. $x \leq y$ in $P \setminus \{\hat{1}\}$
2. $x \leq y$ in $Q \setminus \{\hat{0}\}$
3. $x \in P \setminus \{\hat{1}\}$ and $y \in Q \setminus \{\hat{0}\}$

We may think of this as placing Q on top of P in the ordering. We remove the maximal element of P , the minimal element of Q and then set every element in Q to be greater than those in P with the pre-existing relations unchanged. It is easy to show that if P and Q are Eulerian, so is $P * Q$.

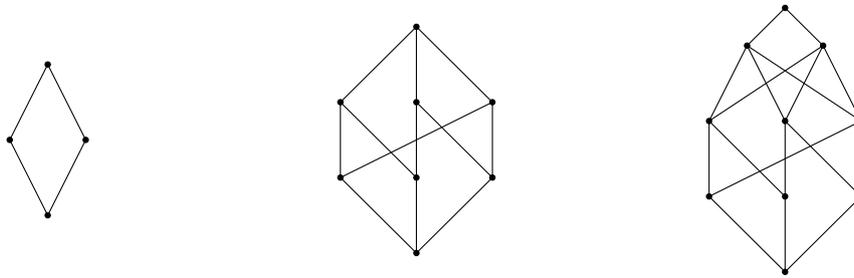


Figure 5: Hasse diagrams of the boolean algebras B_2 , B_3 and the join $B_3 * B_2$ respectively.

Lemma 3.7. [20] Let P and Q be Eulerian posets. Then the cd -index of $P * Q$ is

$$\Phi_{P * Q} = \Phi_P \cdot \Phi_Q.$$

Proof: Chain counting lets us easily see that $\Upsilon_{P * Q}(a, b) = \Upsilon_P(a, b) \cdot \Upsilon_Q(a, b)$. Since we may obtain the cd -index by a series of variable substitutions from the flag polynomial, the result follows. \square

Definition 3.8. Let B_2 denote the boolean algebra of rank two. Let P be a poset. Then we call $P * B_2$ the *suspension* of P and denote it as ΣP .

The geometric intuition for the suspension operator is as follows. Suppose P has a geometric realization as the face poset of a regular $(n - 1)$ -dimension CW -sphere. Consider $\partial P := P \setminus \{\hat{1}\}$. ΣP contains exactly ∂P (as its elements of rank $\leq n$), plus the two facets of rank $n + 1$ introduced from B_2 , along with the maximal element. Since these facets contain every face of ∂P by definition of ΣP , we may view ΣP as the complex induced by embedding ∂P into the equator of an n -sphere with the two new facets corresponding to its northern and southern hemispheres. See Figure 6 for clarification.

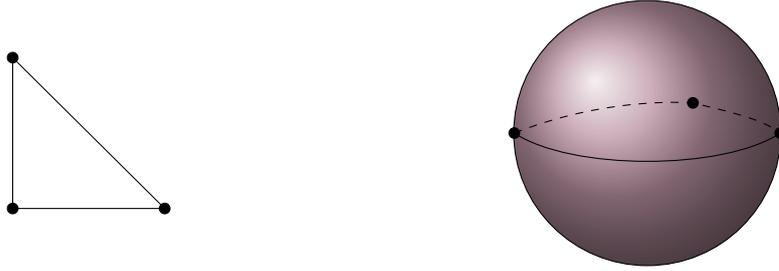


Figure 6: CW -complexes associated to a triangle and its suspension, respectively.

Corollary 3.9. Given an Eulerian poset P , we have that

$$\Phi_{\Sigma P} = \Phi_P \cdot c.$$

Proof: We calculate $\Phi_{B_2} = c$ and apply Lemma 3.7. □

Definition 3.10.

1. We say a poset P is *near-Eulerian* if $P = Q \setminus \{x\}$ where Q is a rank $n + 1$ Eulerian poset and x is a coatom of Q (that is, x has rank n).
2. Given a near-Eulerian poset P , we may uniquely recover the associated Eulerian poset Q by adding a new element x such that $y <_Q x$ for any element $y \in P$ where $[y, \hat{1}]_P$ contains exactly three elements. We call Q the *semisuspension* of P and denote it as $Q := \tilde{\Sigma}P$.

The canonical example of a near-Eulerian poset is the face complex of a polytope with exactly one facet removed. We may think of the semisuspension as ‘capping’ the polytope off with the missing facet. Equivalently, given a regular $n - CW$ -ball Γ , $\tilde{\Sigma}\Gamma$ may be viewed as embedding Γ in a hemisphere of the n -sphere.



Figure 7: A square S and its semisuspension. We adjoin a single new purple 2-face adjacent to all edges and vertices. Think of it as a membrane lying above the pink face.

Since the ab -index and cd -index are only defined for posets with $\hat{1}$, it is of use to have an operation that adjoins $\hat{1}$.

Definition 3.11. Given a complex Ω , we let $P_1(\Omega) = \Omega \cup \{\hat{1}\}$ be the face poset of Ω with a maximal $\hat{1}$ adjoined.

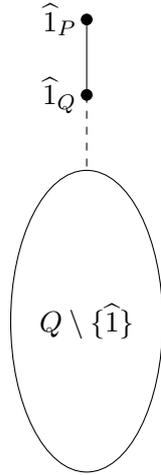


Figure 8: $P_1(Q)$

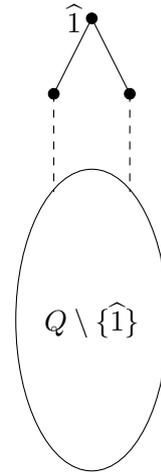


Figure 9: ΣQ

We also may apply this operation to posets that already contain a $\hat{1}$ with the newly adjoined $\hat{1}$ taking precedence as the maximal element.

Definition 3.12. Let Q be an Eulerian poset of rank n . Then $P_1(Q)$ is a near-Eulerian poset of rank $n + 1$.

Proof: Let $\hat{1}_Q$ and $\hat{1}_P$ denote the maximal elements of Q and $P_1(Q)$ respectively. Consider ΣQ and recall that it is Eulerian. Removing either coatom of ΣQ gives a poset isomorphic to $P_1(Q)$, so $P_1(Q)$ is near-Eulerian. This is easy to see via the Hasse diagram above. \square

We now introduce a modified definition of shelling for regular CW -complexes that is required for Stanley's result.

Definition 3.13. [20] Let Ω be an Eulerian n dimensional regular CW -complex. We say that Ω is *spherically shellable* (S -shellable for short) if either Ω is empty, or there exists a linear order of its facets $\sigma_1, \sigma_2 \dots \sigma_k$ that satisfies the following recursive definition:

1. The boundary complex $\partial\sigma_1$ has a shelling.
2. For $1 < i \leq k - 1$, let Γ_i be the closure of the subcomplex of $\partial\sigma_i$ that is not contained in the previously shelled facets. That is,

$$\Gamma_i := cl[\partial\sigma_i \setminus \left(\bigcup_{j=1}^{i-1} \bar{\sigma}_j \cap \bar{\sigma}_i \right)]. \quad (1)$$

We then need that Γ_i is a near-Eulerian $(n - 1)$ - CW -complex and that $\tilde{\Sigma}\Gamma_i$ is S -shellable with initial facet τ (where τ is the new facet introduced by the semi-suspension).

Any line shelling [5] of a polytope satisfies this definition. So any result on S -shellings holds for polytopes. However, not all S -shellings are shellings (via our original definition), and visa versa [20]. We may also inductively find that this definition implies that each Γ_i is a ball and that Ω is a sphere.

We spend the remainder of this chapter giving an overview of Stanley's result that the cd -index of an S -shellable CW -complex is non-negative. This will soon be bogged down in technicalities, so let us state a brief overview here of our ultimate goal. We fix the following notation for the remainder of the chapter.

Let Ω be an n -dimensional regular CW -sphere with an S -shelling $\sigma_1, \sigma_2, \dots, \sigma_k$.

For $1 \leq i \leq k-1$ let $\Omega_i = \widetilde{\Sigma}P_1(\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_i)$. That is, Ω_i is the semisuspension of the complex induced by the first i facets of the shelling. Note that $\Omega_{k-1} = P_1(\Omega)$.

Given Ω_{i-1} , let τ_{i-1} be the new face introduced by the semisuspension of $\sigma_1 \cup \dots \cup \sigma_{i-1}$. We may construct Ω_i from Ω_{i-1} by subdividing τ_{i-1} to 'carve' σ_i from the face. When subdividing Ω_{i-1} to Ω_i , τ_{i-1} is subdivided into exactly two new facets: F_i and τ_i . Furthermore, the subcomplex $\Gamma_i = F_i \cap \tau_i$ coincides exactly with the definition of Γ_i in (1).

Finally, let

$$\check{\Phi}_i = \Phi_{\Omega_i} - \Phi_{\Omega_{i-1}}$$

denote the difference in cd -index between Ω_i and Ω_{i-1} . Note that this is well defined because both complexes are Eulerian posets.

Lemma 3.14. With Γ_i as defined above, we may express $\check{\Phi}_i$ as

$$\check{\Phi}_i = \Phi_{\widetilde{\Sigma}\Gamma_i} \cdot c - \Phi_{\partial\Gamma_i} \cdot (c^2 - d).$$

Proof: This may be established by simple chain counting and clever variable substitution, and is explained in full in [20]. \square

We also may now compute the cd -index of a polytope in terms of complexes of strictly smaller dimension, giving us the following decomposition.

Corollary 3.15. Let Ω be a CW -complex with an S -shelling $\sigma_1, \sigma_2, \dots, \sigma_k$. With notation defined as previous, we may express the cd -index of Ω as

$$\Phi_{\Omega} = \Phi_{\sigma_1} \cdot c + \sum_{i=2}^{k-1} [\Phi_{\widetilde{\Sigma}\Gamma_i} \cdot c - \Phi_{\partial\Gamma_i} \cdot (c^2 - d)].$$

Proof: It is easy to see that

$$\Phi_{\Omega} = \Phi_{\Omega_{k-1}} = \Phi_{\Omega_1} + \sum_{i=2}^{k-1} \check{\Phi}_i. \tag{2}$$

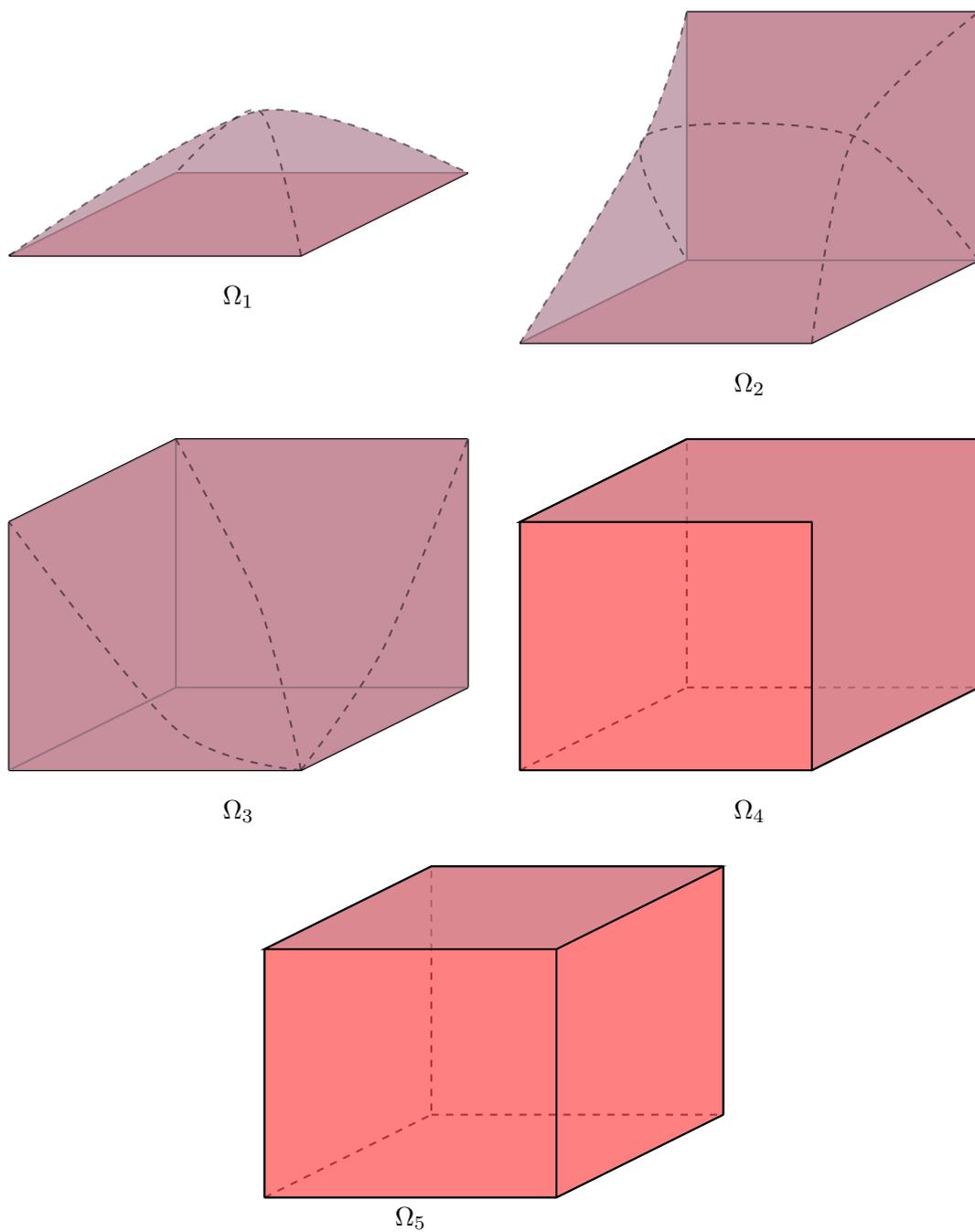


Figure 10: The Ω_i 's of a cube with respect to our previous shelling. The face induced by the semisuspension is drawn in purple while the faces of the shelling are drawn in pink.

Note that given an Eulerian poset Q , $\tilde{\Sigma}P_1(Q) = \Sigma Q$. So $\Omega_1 = \Sigma\sigma_1$. Applying Corollary 3.9 to the first term gives us that $\Phi_{\Omega_1} = \Phi_{\sigma_1} \cdot c$. From Lemma 3.12, we know that $\check{\Phi}_i = \Phi_{\tilde{\Sigma}\Gamma_i} \cdot c - \Phi_{\partial\Gamma_i} \cdot (c^2 - d)$. Substitute each of these values into equation (2) and the result follows. \square

This decomposition is enough to characterize the cd -index for two and three dimensional polytopes.

Lemma 3.16. Consider a 2-polytope P with n vertices. Then $\Phi_P = c^2 + (n - 2)d$.

Proof: There exists some shelling E_1, E_2, \dots, E_n of P . Note that the face poset of any edge E is isomorphic to the boolean lattice B_2 . So we have that $\Phi_{\Omega_1} = \Phi_{E_1}c = c^2$.

Each subsequent edge E_2, \dots, E_{n-1} is adjacent to exactly one other edge in the polygon and intersects it in exactly a single vertex. So for all i , Γ_i is a single vertex, $\Phi_{\tilde{\Sigma}P_1(\Gamma_i)} = c$ and $\Phi_{\partial\Gamma_i} = 1$. This gives us

$$\begin{aligned}\Phi_P &= \Phi_{E_1} \cdot c + \sum_{i=2}^{n-1} [\Phi_{\tilde{\Sigma}\Gamma_i} \cdot c - \Phi_{\partial\Gamma_i} (c^2 - d)] \\ &= c^2 + \sum_{i=2}^{n-1} [c^2 - (c^2 - d)] \\ &= c^2 + (n - 2)d\end{aligned}$$

\square

Lemma 3.17. Let P be a 3-dimensional polytope. Then (with f_i denoting the number of i -faces of P) we have $\Phi_P = c^3 + (f_0 - 2)dc + (f_2 - 2)cd$.

Proof: There exists some shelling F_1, F_2, \dots, F_k of P where $k = f_2$. Let v_i denote the number of vertices of F_i . Applying Lemma 3.16 to F_1 gives us that

$$\Phi_P = (c^2 + (v_1 - 2)d)c + \sum_{i=2}^{k-1} [\Phi_{\tilde{\Sigma}\Gamma_i}c - \Phi_{\partial\Gamma_i}(c^2 - d)]. \quad (3)$$

Note that $\partial\Gamma_i$ for $2 \leq i \leq k - 1$ is 0-dimensional. Furthermore, Γ_i is 1-dimensional and homomorphic to a ball. Specifically, Γ_i is a path of edges. So $\partial\Gamma_i$ is just two points and $\Phi_{\partial\Gamma_i} = c$. Let v_{Γ_i} be the number of vertices of Γ_i . Then $\tilde{\Sigma}\Gamma_i$ is a polygon with v_{Γ_i} vertices and $\Phi_{\tilde{\Sigma}\Gamma_i} = c^2 + (v_{\Gamma_i} - 2)d$. Substituting into equation (2), we may compute the following:

$$\begin{aligned}\Phi_P &= c^3 + (v_1 - 2)dc + \left(\sum_{i=2}^{k-1} [c^2 + (v_{\Gamma_i} - 2)d]c - c(c^2 - d) \right) \\ &= c^3 + (v_1 - 2)dc + \left(\sum_{i=2}^{k-1} (v_{\Gamma_i} - 2)dc + cd \right)\end{aligned}$$

The key observation for our next step is that $v_1 + \sum_{i=2}^{k-1} (v_{\Gamma_i} - 2) = f_0$. This is because we may partition the vertices of P into the sets of those in F_1 , and those in the interior of each Γ_i (since each Γ_i meets the complex $F_1 \cup \dots \cup F_{i-1}$ in exactly its two boundary vertices). Recall that $k = f_2$ and we may then write

$$\Phi_P = c^3 + (f_0 - 2)dc + (f_2 - 2)cd.$$

□

Theorem 3.18. (*Stanley*) [20] Let Ω be an S -shellable regular CW - n -sphere. Then the coefficients of the cd -index of Ω are non-negative. That is, $\Phi_\Omega \geq 0$.

Proof: This requires a finicky double induction on both the number of facets of Ω and the dimension. See [20] for proof.

Corollary 3.19. The cd -index of a polytope has non-negative coefficients.

These results give us a pretty good picture of the non-negativity of the cd -index of geometric complexes. For the further case of non-geometric posets (as well as a wider class of CW -spheres), we turn to the theory of Gorenstein* posets and Gorenstein simplicial complexes.

4 Gorenstein Complexes

We may use the underlying chain structure of a poset to associate to it a simplicial complex. That is, the chains of a poset P can themselves be considered as a poset under inclusion and there exists a simplicial complex whose face poset is isomorphic. This lets us describe arbitrary posets in geometric terms and allows us to apply a toolbox of algebraic topology to the study of posets.

Definition 4.1. Given a poset P , we define a simplicial complex $\Delta(P)$ (called the *order complex* of P) as follows:

- The vertices of $\Delta(P)$ are exactly the elements of P , excluding $\widehat{0}$ and $\widehat{1}$ (if present).
- The faces of $\Delta(P)$ correspond bijectively with the nondegenerate chains of P . That is, for any chain $\widehat{0} < x_0 < x_1 < \dots < x_k < \widehat{1}$ in P , there exists a k -simplex in $\Delta(P)$ with vertices x_0, x_1, \dots, x_k and visa versa.

Note that if P is the face poset of some complex, then $\Delta(P)$ is isomorphic to the face poset of the barycentric subdivision of $P \setminus \{\widehat{1}\}$. We also have that a poset P is Eulerian if and only if $\Delta(P)$ is Eulerian [6]

Also note that this definition of order complex implies a shift between dimension and rank. If P is of rank $n + 1$ and contains $\widehat{1}$ and $\widehat{0}$, then $\Delta(P)$ has dimension $n - 1$.

We also first need to define the operations of the *star* and *link* of a simplicial complex. One must take additional care when describing these operations for infinite complexes, but for the finite case the following definitions will suffice.

Definition 4.2. Let Γ be a pure n -dimensional simplicial complex, and let $\sigma \in \Gamma$ be an i -face.

- The *star* of σ is the n -dimensional simplicial subcomplex induced by the faces that contain σ . That is,

$$\text{Star}_\Gamma(\sigma) = \{x \in \Gamma \mid \exists F \in \Gamma \text{ such that } \sigma \leq F, \text{ and } x \leq F\}.$$

- The *link* of σ is the $(n - i - 1)$ -dimensional simplicial subcomplex given by all the faces of the $\text{Star}_\Gamma(\sigma)$ that do not intersect σ . That is,

$$\text{Link}_\Gamma(\sigma) = \{x \in \text{Star}_\Gamma(\sigma) \mid \sigma \not\leq x\}.$$

This is easier to gather intuition for visually: see figure 9 for clarification. When the underlying complex is clear, we may omit the subscript.

The link is a combinatorially useful structure. The face poset of $\text{Link}_\Gamma(\sigma)$ is isomorphic to the subposet $\{x \mid \sigma \leq x\}$ in Γ . That is, there is a correspondence between

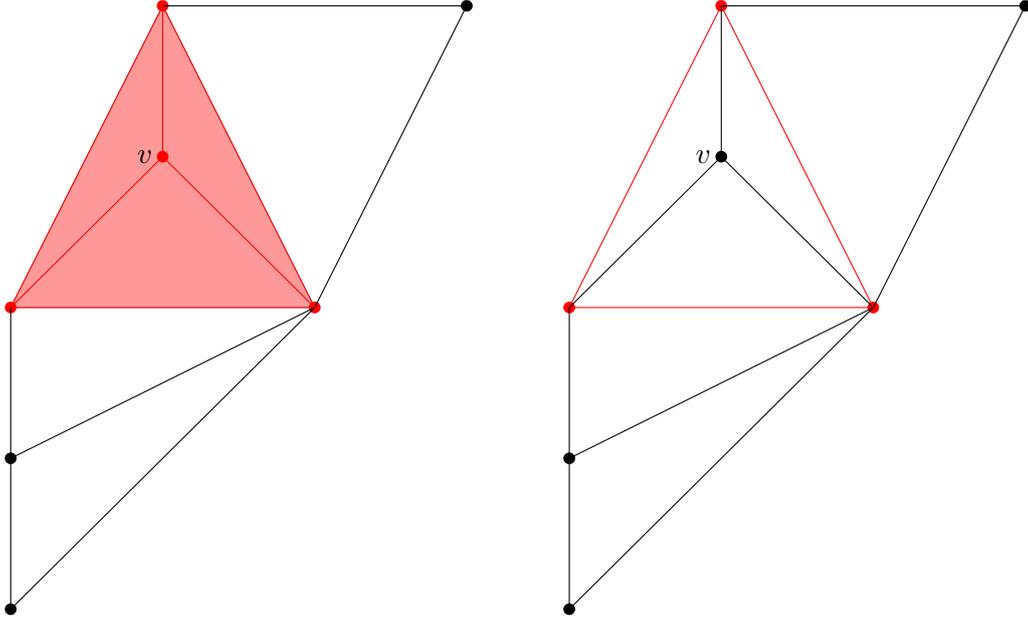


Figure 11: The star and link of v , respectively.

$\widehat{0} \in \text{Link}_\Gamma(\sigma)$ and $\sigma \in \{x \mid \sigma \leq x\}$, the zero dimensional faces of $\text{Link}_\Gamma(\sigma)$ and the $i + 1$ dimensional faces of $\{x \mid \sigma \leq x\}$, and so on. In other words, the link of σ gives us a low dimensional picture of the faces that contain it by shifting those faces down $i + 1$ dimensions.

Gorenstein complexes are a class of well behaved simplicial complexes. They generalize simplicial spheres by being defined to have the same homology groups.

Definition 4.3. We say that an n -dimensional simplicial complex Δ is *Gorenstein* if Δ is a real homology sphere of dimension n . Explicitly, this means that we have that the real reduced homology of Δ is

$$\widetilde{H}_i(\Delta, \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

and for any simplex $x \in \Delta$ with $\dim(x) = m$, we have that the reduced homology of its link is

$$\widetilde{H}_i(\text{Link}_\Delta(x), \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = n - m - 1 \\ 0 & \text{otherwise} \end{cases} .$$

Example 4.4. The boundary of any simplicial polytope or simplicial sphere is a Gorenstein complex.

Example 4.5. Let Δ be an n -dimensional Gorenstein complex with some i -face σ . Then $\text{Link}(\sigma)$ is a Gorenstein complex of dimension $(n - i - 1)$.

We say that a poset P is a *Gorenstein** complex if $\Delta(P)$ is a Gorenstein complex. Note that if P is Eulerian of rank $n + 1$, then $\Delta(P)$ has dimension $n - 1$.

To be near-Gorenstein is to invoke a similar condition on a homology ball [6].

Definition 4.6. We say that a pair of simplicial complexes $(\Delta, \partial\Delta)$ is *near-Gorenstein* if Δ is a real homology ball of dimension n with boundary $\partial\Delta$. That is,

- The complex $\partial\Delta$ is Gorenstein of dimension $n - 1$.
- For every $x \in \Delta$ of dimension m , the reduced homology of its link is

$$\widetilde{H}_i(\text{Link}_\Delta(x), \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = n - m - 1 \text{ and } x \notin \partial\Delta \\ 0 & \text{otherwise} \end{cases} .$$

Note that this implies that the link of any element of $\partial\Delta$ has trivial homology. This includes Δ since by convention the empty face has $\text{Link}(\emptyset) = \Delta$. The motivating example for our purposes is that any full dimensional simplicial subdivision of an n -dimensional polytope P (that is, a subdivision of its maximal face $|P|$ into an n -dimensional simplicial complex) is near-Gorenstein.

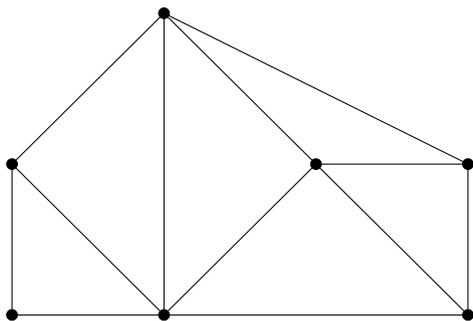


Figure 12: A near-Gorenstein complex.

To define the analogous condition for posets, we must establish a notion of boundary for a poset.

Definition 4.7. Suppose P is a near-Eulerian poset. Recall that we may uniquely obtain its semisuspension $\widetilde{\Sigma}P$ by adding a new coatom τ such that $x \leq_{\widetilde{\Sigma}P} \tau$ for any $x \in P$ such that $[x, \widehat{1}]_P$ is a three element chain. We then define the *boundary* of P to be

$$\partial P := P_1([\widehat{0}, \tau]).$$

This is well defined because $\partial P \subset P$. Also note that ∂P is Eulerian. We refer to any element $x \in P \setminus \partial P$ as an element of (or lying in) the *interior* of P .

Definition 4.8. Let Q be an Eulerian poset. Then we may write $\partial Q := Q \setminus \{\widehat{1}\}$.

This coincides with the polytope definition of a boundary complex, but is not a notation used in literature. We include it here as a formality because it helps some of our summations look more nice.

Example 4.9. Consider the sets $P = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ partially ordered under inclusion (where we have $\widehat{0} = \emptyset$ and $\widehat{1} = \{1, 2, 3\}$). Then $\partial P = \{\{1\}, \{3\}\}$.

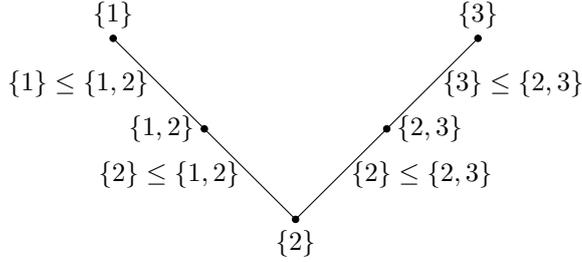


Figure 13: $\Delta(P)$

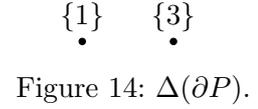


Figure 14: $\Delta(\partial P)$.

Definition 4.10. We say a pair of posets $(P, \partial P)$ is *near-Gorenstein** if $(\Delta(P), \Delta(\partial P))$ is a near-Gorenstein complex.

Note that the above definition implies that ∂P is a Gorenstein* poset. We may see Gorenstein* and near-Gorenstein* posets as a restriction of Eulerian and near-Eulerian posets to those with particularly well behaved order complexes.

We may decompose the ab -index of a near-Eulerian poset P into a term that counts the chains of its boundary, and a remainder term. This is a particularly nice way to approach its chains because ∂P is Eulerian.

Lemma 4.11. Given a near-Eulerian poset P with boundary ∂P , we may write its ab -index as

$$\Psi_P = \ell_P^\Psi + \Psi_{\partial P} \cdot a,$$

where $\ell_P^\Psi := \Psi_{\widetilde{\Sigma}P} - \Psi_{\partial P} \cdot (a+b)$. Furthermore, ℓ_P^Ψ is expressible in the variables $c = a+b$ and $d = ab + ba$. Note that ℓ_P^Ψ is implicitly defined by P .

Proof: Consider the semisuspension $\widetilde{\Sigma}P$. $\widetilde{\Sigma}P$ adds a single new coatom τ . The semisuspension contains chains of exactly two types: those contained in P , and chains in ∂P followed by τ . This gives us the following equations of the flag polynomial:

$$\begin{aligned} \Upsilon_{\widetilde{\Sigma}P} &= \Upsilon_P + \Upsilon_{\partial P} \cdot b. \\ \Upsilon_P &= \Upsilon_{\widetilde{\Sigma}P} - \Upsilon_{\partial P} \cdot b. \end{aligned}$$

Substituting $a - b$ for a gives us the ab -index, which may then be manipulated as follows.

$$\begin{aligned}\Psi_P &= \Psi_{\widetilde{\Sigma}P} - \Psi_{\partial P} \cdot b \\ &= (\Psi_{\widetilde{\Sigma}P} - \Psi_{\partial P} \cdot (a + b)) + \Psi_{\partial P} \cdot a \\ &= \ell_P^\Psi + \Psi_{\partial P} \cdot a.\end{aligned}$$

We omit the a and b arguments of the function ℓ_P^Ψ when context is clear and it is notationally convenient.

To see that ℓ_P^Ψ may be expressed in $c = a + b$ and $d = ab + ba$, note that both $\widetilde{\Sigma}P$ and ∂P are Eulerian and thus there exist cd -indexes $\Phi_{\widetilde{\Sigma}}$ and $\Phi_{\partial P}$. Therefore the polynomial $\ell_P^\Phi(c, d) := \Phi_{\widetilde{\Sigma}P} - \Phi_{\partial P} \cdot c$ is well defined and we have

$$\ell_P^\Phi(a + b, ab + ba) = \ell_P^\Psi(a, b). \quad \square$$

We call $\ell_P^\Psi(a, b)$ and $\ell_P^\Phi(c, d)$ the *local ab -index* and *local cd -index* of P respectively. This result is also described in [6] particularly for Gorenstein* posets, but we deviate from their terminology and notation for reasons that will become clear after chapter 8. Note that the definition implies for that the trivial single element poset $\{\widehat{0}\}$ we have $\ell_0^\Phi = 1$.

We present the following definition analogous to the flag polynomial of a poset (Definition 2.10).

Definition 4.12. We define the *local flag polynomial* of P by

$$\ell_P^\Upsilon(a, b) = \ell_P^\Psi(a + b, b).$$

In the expression $\Psi_P = \ell_P^\Psi + \Psi_{\partial P} \cdot a$, both ℓ_P^Ψ and $\Psi_{\partial P}$ may be rewritten in terms of c and d . So Ψ_P is ‘almost’ a polynomial in c and d . We define the following in accordance with Ehrenborg and Karu [6].

Definition 4.13. Let P be a near-Eulerian poset of rank $n + 1$. Then we define the cd -index of P to be

$$\Phi_P := \ell_P^\Phi + \Phi_{\partial P}.$$

Note that this is a non-homogeneous polynomial. The terms of ℓ_P^Φ are degree n , while the terms of $\Phi_{\partial P}$ are rank $n - 1$ (where c contributes 1 to degree and d contributes 2).

Analyzing the local flag polynomial gives us a more clear combinatorial picture of what exactly the local ab/cd index is enumerating.

Lemma 4.14. Given a near-Eulerian poset P , we have that $\ell_P^\Upsilon = \Upsilon_P - \Upsilon_{P_1(\partial P)}$.

Proof:

$$\begin{aligned}
\ell_P^\Psi &= \Psi_{\tilde{\Sigma}P} - \Psi_{\partial P} \cdot (a + b) \\
\ell_P^\Upsilon &= \Upsilon_{\tilde{\Sigma}P} - \Upsilon_{\partial P} \cdot (a + 2b) \\
&= (\Upsilon_P + \Upsilon_{\partial P} \cdot b) - \Upsilon_{\partial P} \cdot (a + 2b) \\
&= \Upsilon_P - \Upsilon_{\partial P} \cdot (a + b) \\
&= \ell_P^\Upsilon = \Upsilon_P - \Upsilon_{P_1(\partial P)}. \quad \square
\end{aligned}$$

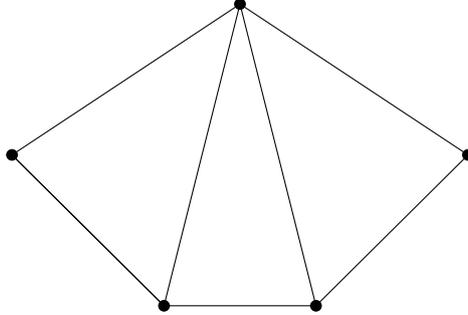
We may think of the local flag polynomial as a difference operator between P and $P_1(\partial P)$. If Π is the face poset of a near-Gorenstein complex, $P_1(\partial\Pi)$ has a natural topological interpretation as the face poset of the CW -complex obtained from $\partial\Pi$ by attaching a maximal face to its boundary. In the next chapter, we examine the concept of the subdivision of a poset that is analogous to a topological subdivision. In general, we find that P is a subdivision of $P_1(\partial P)$, and that ℓ_P^Υ counts the chains introduced by subdividing $P_1(\partial P)$ to P .

Example 4.15. Consider the following two dimensional near-Gorenstein complex Π . Let $\Pi' = P_1(\partial\Pi)$. Then, with Π and Π' as Figures 15 and 16, we have

$$\begin{aligned}
\ell_\Pi^\Upsilon &= \Upsilon_\Pi - \Upsilon_{\Pi'} = 2aba + 2aab + 4bba + 4bab + 4abb + 8bbb. \\
\ell_\Pi^\Psi &= 2aba + 2aab + 2bba + 2bab. \\
\ell_\Pi^\Phi &= 2cd.
\end{aligned}$$

Figure 15: Π

$$\Upsilon_\Pi = aaa + 5baa + 7aba + 3aab + 14bba + 9bab + 9abb + 18bbb$$

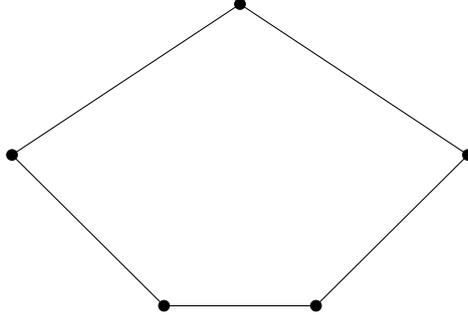


The specification to Gorenstein* and near-Gorenstein* posets is of use because they generalize CW -spheres and CW -balls respectively. Ehrenborg and Karu proved non-negativity results on the cd -index of both Gorenstein* and near-Gorenstein* posets that are natural extensions of Stanley's result on the non-negativity of the cd -index of shellable CW -spheres.

Theorem 4.16. [9] The cd -index of a Gorenstein* poset has non-negative integer coefficients.

Figure 16: Π'

$$\Upsilon_{\Pi'} = aaa + 5baa + 5aba + aab + 10bba + 5bab + 5abb + 10bbb$$



Theorem 4.17. [2] The cd -index of a near-Gorenstein* poset has non-negative integer coefficients.

The proof of these results requires heavy algebraic machinery that is beyond the scope of this thesis. However, non-negativity is a particularly useful characterization for establishing bounds on the cd -index. We will examine some consequences of it in later chapters.

Corollary 4.18. The local cd -index of a near-Gorenstein* poset has non-negative integer coefficients.

Proof: Let P be a near-Gorenstein* poset of rank $n + 1$. Then $\Phi_P = \ell_P^\Phi + \Phi_{\partial P}$. Note that ℓ_P^Φ is homogeneous of degree n (where each c contributes 1 and each d contributes 2 to the degree) and $\Phi_{\partial P}$ is homogenous of degree $n - 1$. Since they have no shared terms and Φ_P has non-negative coefficients, it follows that ℓ_P^Φ must have non-negative coefficients. \square

5 Skeletal Decomposition of a Subdivision

We first present a traditional topological definition of subdivisions.

Definition 5.1. Given face posets of regular CW complexes Π and $\widehat{\Pi}$, we say that $\widehat{\Pi}$ is a *subdivision* of Π if there exists a surjective, order preserving map $\phi : \widehat{\Pi} \rightarrow \Pi$ such that for all $F \in \Pi$ the following conditions hold:

1. $\widehat{F} := \phi^{-1}([\widehat{0}, F])$ is a CW complex of the same dimension as F .
2. $\phi^{-1}(F)$ consists of all the interior cells of \widehat{F} .
3. $|F| = |\widehat{F}|$

If Π and $\widehat{\Pi}$ are Gorenstein, with \widehat{F} near Gorenstein for all $F \in \Pi$, then $\widehat{\Pi}$ is a *Gorenstein subdivision* of Π . As an alternative notation not dependent on $\widehat{\Pi}$, given a subdivision $\phi : \Gamma \rightarrow \Pi$, for any $F \in \Pi$ we let $\Gamma_F := \phi^{-1}([\widehat{0}, F])$ denote the restriction of Γ to F .

Example 5.2. In Figures 14 and 15, Π is a subdivision of Π' .

There are many generalizations of topological subdivisions to arbitrary posets. The following is of particular use to us.

Definition 5.3. Given lower Eulerian posets P and \widehat{P} of rank n , we say \widehat{P} is a *strong Eulerian subdivision* of P if there exists a surjective, order preserving map $\phi : \widehat{P} \rightarrow P$ such that for all $\sigma \in P$,

- $P_1(\phi^{-1}[\widehat{0}, \sigma])$ is a near-Eulerian poset.
- $\phi^{-1}[\widehat{0}, \sigma]$ has rank $\rho_P(\sigma)$.

We may specialize even further by restricting the Eulerian condition to Gorenstein* [6].

Definition 5.4. Let P and \widehat{P} be Gorenstein* posets of rank n . Suppose we have a surjective, order preserving map $\phi : \widehat{P} \rightarrow P$. We say ϕ is a *Gorenstein** subdivision map if for every $\sigma \in P$,

- $(\phi^{-1}[\widehat{0}, \sigma], \phi^{-1}[\widehat{0}, \sigma])$ is a near-Gorenstein* poset.
- $\phi^{-1}[\widehat{0}, \sigma]$ has rank $\rho(\sigma)$.

For shorthand, we let $\widehat{\sigma} := P_1(\phi^{-1}[\widehat{0}, \sigma])$ and $\partial\widehat{\sigma} = P_1(\phi^{-1}[\widehat{0}, \sigma])$. This coincides with our earlier definition of the boundary of a near-Eulerian poset.

Notational aside: This unfortunately leads to the notation $\widehat{1}_\Pi = P_1(\phi^{-1}[\widehat{0}, \widehat{1}_\Pi]) = P_1(\Pi)$. However, in practice such an element is rarely acknowledged because the existence of $\widehat{1}$ in a strongly Eulerian subdivision between Π and $\widehat{\Pi}$ is mostly a formality; note

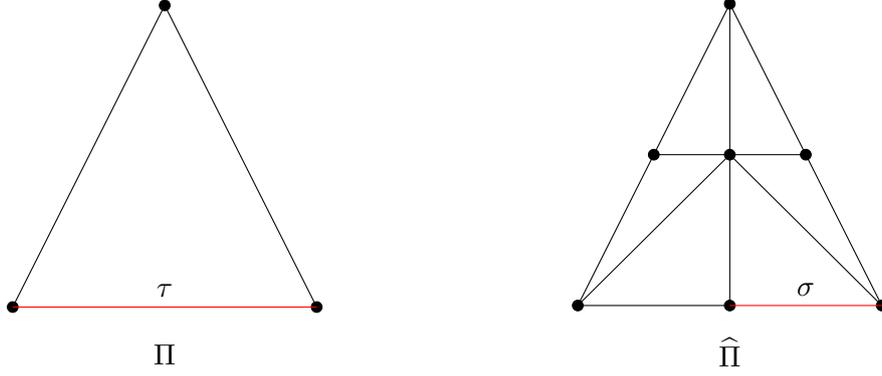


Figure 17: $\widehat{\Pi}$ is a simplicial subdivision of Π . The face $\sigma \in \widehat{\Pi}$ is carried by $\tau \in \Pi$.

that the definition of strongly Eulerian implies that $\phi(\sigma) = \widehat{1}_{\Pi}$ if and only if $\sigma = \widehat{1}_{\widehat{\Pi}}$. Furthermore it is easy to calculate that for any such strongly Eulerian subdivision, $\ell_{\widehat{1}_{\Pi}}^{\phi} = 0$. Since the cd -index and ab -index do not technically count chains including $\widehat{1}$, its presence is often implicit in literature (to the annoyance of this writer). We aim to clarify its presence at the expense of clarity in notation.

It is easy to see that a simplicial subdivision between Gorenstein complexes is both a Gorenstein* subdivision and a strongly Eulerian subdivision.

Definition 5.5. Given a [Gorenstein*/simplicial/Eulerian] subdivision $\phi : \widehat{\Pi} \rightarrow \Pi$ and some face $\sigma \in \widehat{\Pi}$ we define $\phi(\sigma) = \tau \in \Pi$ to be the *carrier* of σ in Π . We say that σ is *carried* by τ .

In the simplicial case, we may geometrically see τ as the minimal face of Π that topologically contains σ . That is, $|\sigma| \subseteq |\tau|$.

Our motivation for the remainder of this chapter, along with most of chapter six, is to find a combinatorial description and elementary proof of the Gorenstein* subdivision cd -index decomposition developed by Ehrenborg and Karu in [6]. Restricting to a special case of Gorenstein simplicial complexes, we find an elegant solution via a process we call ‘skeletal decomposition’ of a subdivision. This gives a clear geometric decomposition when applied to face posets, and generalizes the formula to strongly Eulerian subdivisions.

Definition 5.6. Let ϕ be a strong Eulerian subdivision of rank n Eulerian posets $\phi : \widehat{\Pi} \rightarrow \Pi$. We define the *skeletal posets* Π_i for $0 \leq i \leq n$ as follows:

- $\Pi_n; = \widehat{\Pi}$.
- $\Pi_0; = \Pi$.

- Let $\Pi|_{\geq i+1} := \{\sigma \in \Pi : \rho(\sigma) \geq i+1\}$ and let $\widehat{\Pi}|_{\leq i} := \{\sigma \in \widehat{\Pi} : \rho(\phi(\sigma)) \leq i\}$. For $0 < i < n$, we define

$$\Pi_i := \Pi|_{\geq i+1} \cup \widehat{\Pi}|_{\leq i}$$

to be a poset where for $\sigma, \tau \in \Pi_i$ we have the order relation $\sigma \leq \tau$ if and only if exactly one of the following holds:

1. $\sigma, \tau \in \Pi|_{\geq i+1}$ and $\sigma \leq \tau$ in Π .
2. $\sigma, \tau \in \widehat{\Pi}|_{\leq i}$ and $\sigma \leq \tau$ in $\widehat{\Pi}$.
3. $\sigma \in \widehat{\Pi}|_{\leq i}, \tau \in \Pi|_{\geq i+1}$ and $\phi(\sigma) \leq \tau$ in Π .

In other words, for elements exclusively of rank $\geq i+1$ or exclusively of rank $\leq i$ then their order relation is inherited from Π and $\widehat{\Pi}$ respectively. If $\rho(\sigma) \leq i$ and $\rho(\tau) \geq i+1$ we have that $\sigma \leq \tau$ if and only if the carrier of σ is ordered beneath τ in Π . Note that Π_i is a well defined poset of rank n with a natural rank function. For any $\sigma \in \Pi_i$, we have

$$\rho(\sigma) = \begin{cases} \rho_{\widehat{\Pi}}(\sigma) & \text{if } \sigma \in \widehat{\Pi}|_{\leq i} \\ \rho_{\Pi}(\sigma) & \text{if } \sigma \in \Pi|_{\geq i+1} \end{cases}$$

We then define *skeletal maps* $\phi_i : \Pi_{i+1} \rightarrow \Pi_i$ for $0 \leq i \leq n-1$ by the following.

$$\phi_i(\sigma) = \begin{cases} \phi(\sigma) & \text{if } \sigma \in \widehat{\Pi}|_{\leq i} \text{ and } \rho(\phi(\sigma)) = i+1 \\ \sigma & \text{otherwise} \end{cases}$$

Note that the map only has non-identity outputs for $\sigma \in \widehat{\Pi}|_{\leq i} \subseteq \Pi_{i+1}$, that is, for σ carried by some element of rank $i+1$ in Π .

Lemma 5.7. Let $\Pi, \widehat{\Pi}, \phi, \phi_i$ and Π_i be defined as above. Then

$$\phi_0 \circ \phi_1 \circ \cdots \circ \phi_{n-2} \circ \phi_{n-1} = \phi.$$

Proof: This follows from careful construction of the skeletal posets and maps.

Consider $\sigma \in \widehat{\Pi}$. Fix the rank of its carrier, $\rho_{\Pi}(\phi(\sigma)) = k$.

Then by definition, for all $k \leq i \leq n$, $\sigma \in \Pi_i$. Furthermore, for any $\phi_i : \Pi_{i+1} \rightarrow \Pi_i$ with $k \leq i \leq n-1$, $\phi_i(\sigma) = \sigma$ because σ is carried by an element of rank k . So $\phi_k \circ \phi_{k+1} \circ \cdots \circ \phi_{n-1}(\sigma) = \sigma$.

By definition of ϕ_{k-1} , we have that $\phi_{k-1}(\sigma) = \phi(\sigma)$. Similarly, since $\phi(\sigma)$ has rank k we have that $\phi(\sigma) \in \Pi_i$ for all $0 \leq i \leq k-1$ and that $\phi_i(\phi(\sigma)) = \phi(\sigma)$ for all $0 \leq i \leq k-2$. So $\phi_0 \circ \phi_1 \circ \cdots \circ \phi_{k-2}(\phi(\sigma)) = \phi(\sigma)$. Therefore

$$\begin{aligned} \phi_0 \circ \phi_1 \circ \cdots \circ \phi_{n-2} \circ \phi_{n-1}(\sigma) &= \phi_0 \circ \phi_1 \circ \cdots \circ \phi_{k-2} \circ \phi_{k-1}(\sigma) \\ &= \phi_0 \circ \phi_1 \circ \cdots \circ \phi_{k-2}(\phi(\sigma)) \\ &= \phi(\sigma). \end{aligned}$$

Since this holds for all $\sigma \in \widehat{\Pi}$, it follows that $\phi_0 \circ \phi_1 \circ \cdots \circ \phi_{n-2} \circ \phi_{n-1} = \phi$. \square

It follows that the following commutative diagram holds. We call the process of identifying the skeletal maps and posets of $\phi : \widehat{\Pi} \rightarrow \Pi$ the *skeletal decomposition* of ϕ .

$$\widehat{\Pi} = \Pi_n \xrightarrow{\phi_{n-1}} \Pi_{n-1} \xrightarrow{\phi_{n-2}} \cdots \xrightarrow{\phi_1} \Pi_1 \xrightarrow{\phi_0} \Pi_0 = \Pi$$

ϕ

It is also convenient to define the following maps.

$$\begin{aligned} \phi_i^{\widehat{\Pi}} : \widehat{\Pi} &\rightarrow \Pi_i & \phi_i^{\widehat{\Pi}} &= \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_i \\ \phi_i^{\Pi} : \Pi_i &\rightarrow \Pi & \phi_i^{\Pi} &= \phi_{i-1} \circ \phi_{i-2} \circ \cdots \circ \phi_0 \end{aligned}$$

This gives us

$$\phi = \phi_{n-1} \circ \cdots \circ \phi_0 = \phi_i^{\widehat{\Pi}} \circ \phi_i^{\Pi} \text{ for } 0 \leq i \leq n-1$$

The maps $\phi_i^{\widehat{\Pi}}$ and ϕ_i^{Π} are induced by Π_i . We may think of ϕ_i^{Π} as the map that subdivides Π to Π_i , and $\phi_i^{\widehat{\Pi}}$ as the map that subdivides Π_i to $\widehat{\Pi}$. We have not actually proved that ϕ_i^{Π} and $\phi_i^{\widehat{\Pi}}$ are formal subdivisions, but that is ultimately irrelevant to our use of them and they are convenient to think of as such.

Though seemingly obtuse when applied to general posets, this process is rooted in strong geometric intuition. Suppose that Π and $\widehat{\Pi}$ are simplicial n -dimensional complexes such that $\widehat{\Pi}$ is a subdivision of Π . Π_i is obtained from Π by restricting its subdivision to $\widehat{\Pi}$ to the elements $\sigma \in \Pi$ with $\rho(\sigma) \leq i$. That is, we subdivide the $(i-1)$ -faces of Π and leave the higher dimensional faces untouched (recall that in simplicial/CW complex, elements of rank i correspond to faces of dimension $(i-1)$). Additionally, we can see that Π_{i+1} is obtained from Π_i by subdividing exactly the i -faces of Π_i . In other words, we can see skeletal decomposition of a subdivision as the process of first subdividing the 0-faces of Π , then its 1-faces, and so on until we subdivide its n -faces and achieve $\widehat{\Pi}$. Significantly, though each Π_i is the poset of a geometrically realizable complex, they are not necessarily simplicial complexes (they are best visualized as *CW-complexes*)!

Example 5.8. Let Π be a tetrahedron and let $\widehat{\Pi}$ be achieved by subdividing two of its facets in half, with two new edges meeting at a single new vertex. Note Π_2 is no longer a simplicial complex since the two facets adjacent to the subdivided edge are now both combinatorially equivalent to a quadrilateral.

Figure 18: $\Pi = \Pi_0 = \Pi_1$

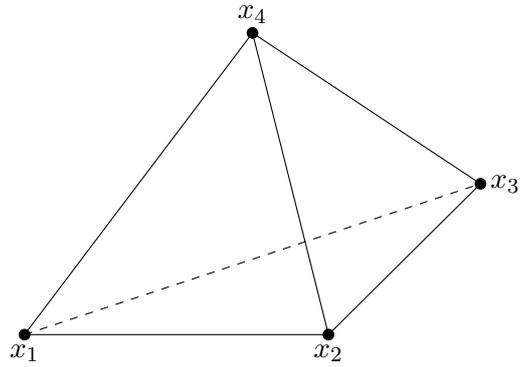


Figure 19: Π_2

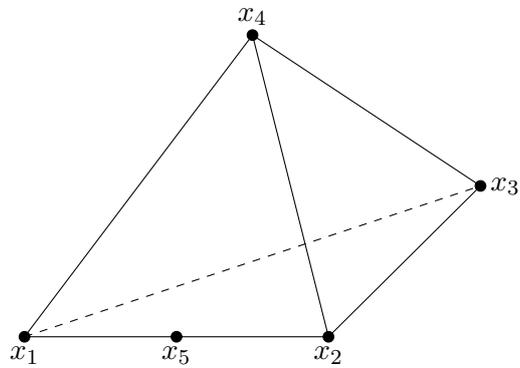
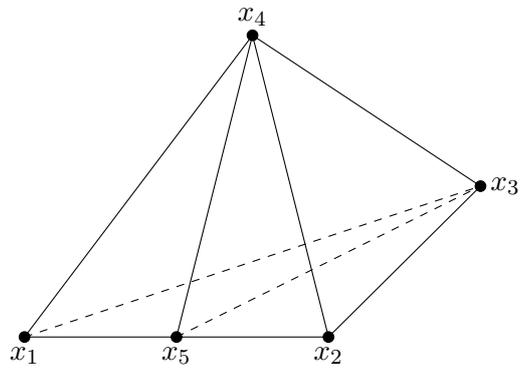


Figure 20: $\Pi_3 = \Pi_4 = \widehat{\Pi}$



We may divide the flags in any Π_i of a skeletal decomposition to be of three distinct combinatorial categories.

Definition 5.9. Let $C = F_0 < F_1 < \dots < F_k$ be a flag of Π_i .

1. If $F_0, \dots, F_k \in \Pi|_{\geq i+1}$, we say that C is an *old* flag.
2. If $F_0, \dots, F_k \in \widehat{\Pi}|_{\leq i}$, we say that C is a *new* flag. If $\rho(\phi_i^\Pi(F_k)) = m$ (that is, F_k is carried by a rank m element of $\widehat{\Pi}$) we say that C *switches* at rank m .
3. If $C = F_0 < \dots < F_j < F_{j+1} < \dots < F_k$ where $F_0, \dots, F_j \in \widehat{\Pi}|_{\leq i}$ and $F_{j+1}, \dots, F_k \in \Pi|_{\geq i+1}$ we say that C is a *mixed* flag. If $\rho(\phi_i^\Pi)(F_j) = m$ we say that C *switches* at rank m .

In other words, a flag of Π_i switches at m if its maximal element in $\widehat{\Pi}|_{\leq i}$ is carried by a rank m element of Π . Additionally, we will also say that a flag switches at σ if $\sigma \in \Pi$ carries its maximal face in $\widehat{\Pi}|_{\leq i}$. Note that by construction, any mixed or new flag of Π_i must switch at some $m \leq i$ because for any $\sigma \in \widehat{\Pi}|_{\leq i}$ we have $\rho(\phi(\sigma)) \leq i$.

Example 5.10. Consider Π_2 of figure 19. We will refer to faces by their vertex sets. $\{x_5\} < \{x_2, x_5\}$ is a new chain. $\{x_5\} < \{x_2, x_5\} < \{x_1, x_2, x_4, x_5\}$ is a mixed chain that switches at rank 2.

Note that even though $\{x_1, x_2, x_4, x_5\}$ contains a vertex not present in Π , we still count it as an ‘old’ face of Π_i because its interior is identical to the interior of the face $\{x_1, x_2, x_4\}$ of Π (recall that in a CW -complex, faces are defined as open cells).

For the remainder of this paper, when not specified we will assume some arbitrary subdivision between n -dimensional Eulerian posets $\phi : \widehat{\Pi} \rightarrow \Pi$.

Note that we may naturally restrict any subdivision map $\phi : \widehat{\Pi} \rightarrow \Pi$ to a face $F \in \Pi$. To make this operation well defined on Π_i for all i , we introduce the following family of restriction maps.

Definition 5.11. For all $0 \leq i \leq n$ and all $F \in \Pi$ we define the restriction of $\widehat{\Pi}_i$ to F by the following map:

$$\chi|_F^i : \Pi_i \rightarrow (\phi_i^\Pi)^{-1}([\widehat{0}, F])$$

$$\chi|_F^i(\sigma) = \begin{cases} \sigma & \text{if } \sigma \in (\phi_i^\Pi)^{-1}([\widehat{0}, F]), \\ \emptyset & \text{otherwise.} \end{cases}$$

$\chi|_F^i$ is just a characteristic function for the faces of Π_i that map to an element of $[\widehat{0}, F]$. In the topological case, given a face F of Π the function identifies all faces of Π_i that are contained in F (or its boundary) and maps all other faces of Π_i to the empty set.

Lemma 5.12. Given a fixed face $F \in \Pi$ and $0 \leq i \leq n - 1$, the following diagrams commute.

$$\begin{array}{ccc}
\widehat{\Pi} = \Pi_n & \xrightarrow{\phi_i^{\widehat{\Pi}}} & \Pi_i \\
\chi|_F^n \downarrow & & \downarrow \chi|_F^i \\
\Pi_n & \xrightarrow{\phi_i^{\widehat{\Pi}}} & \Pi_i \\
\\
\Pi_i & \xrightarrow{\phi_i^{\Pi}} & \Pi_0 \\
\chi|_F^i \downarrow & & \downarrow \chi|_F^0 \\
\Pi_i & \xrightarrow{\phi_i^{\Pi}} & \Pi_0 = \Pi
\end{array}$$

Proof: We may demonstrate the first diagram commutes as follows.

Take $\sigma \in \widehat{\Pi}$. If $\sigma \notin \phi^{-1}([\widehat{0}, F])$ it holds trivially.
So suppose $\sigma \in \phi^{-1}([\widehat{0}, F])$. If $\rho(\phi(\sigma)) \leq i$ we have

$$\phi_i^{\widehat{\Pi}} \circ \chi|_F^n(\sigma) = \phi_i^{\widehat{\Pi}}(\sigma) = \sigma$$

and

$$\chi|_F^i \circ \phi_i^{\widehat{\Pi}}(\sigma) = \chi|_F^i(\sigma) = \sigma$$

and the equation holds.

If $\rho(\phi(\sigma)) \geq i + 1$ then

$$\phi_i^{\widehat{\Pi}} \circ \chi|_F^n(\sigma) = \phi_i^{\widehat{\Pi}}(\sigma) = \phi(\sigma) \in \Pi_{\geq i+1} \subset \Pi_i.$$

$$\chi|_F^i \circ \phi_i^{\widehat{\Pi}}(\sigma) = \chi|_F^i(\phi(\sigma))$$

Since $\phi(\sigma) \in [\widehat{0}, F]$ by definition, we must have $\phi(\sigma) \in (\phi_i^{\Pi})^{-1}([\widehat{0}, F])$ and $\chi|_F^i(\phi(\sigma)) = \phi(\sigma)$. So the equation follows and the diagram commutes.

The proof that

$$\chi|_F^0 \circ \phi_i^{\Pi} = \phi_i^{\Pi} \circ \chi|_F^i$$

is entirely symmetrical. □.

6 Subdividing the cd-index

Given a Gorenstein* subdivision $\phi : \widehat{\Pi} \rightarrow \Pi$, can we express the cd -index of $\widehat{\Pi}$ in terms of the elements of Π ? Ehrenborg and Karu answered this question in the affirmative in Theorem 2.7 of [6]. They achieve the result by using sheaf theory on fans and poset.

By using the theory of skeletal decompositions we may establish this result purely combinatorially by careful chain counting, sidestepping the commutative algebra. Additionally, we extend the subdivision decomposition to hold for any strong Eulerian subdivision. We introduce the theorem below and prove it over the remainder of the chapter.

Theorem 6.1. Let $\phi : \widehat{\Pi} \rightarrow \Pi$ be a strong Eulerian subdivision map between rank n posets. Then

$$\Phi_{\widehat{\Pi}} = \sum_{\sigma \in \Pi} \ell_{\sigma}^{\Phi} \cdot \Phi_{[\sigma, \widehat{1}]}$$

Proof: We claim that for $i \geq 0$, $\sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \ell_{\sigma}^{\mathcal{Y}} \cdot \Upsilon_{[\sigma, \widehat{1}]}$ counts exactly the new chains obtained by subdividing Π_{i-1} to Π_i . That is,

$$\sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \ell_{\sigma}^{\mathcal{Y}} \cdot \Phi_{[\sigma, \widehat{1}]} = \Upsilon_{\Pi_i} - \Upsilon_{\Pi_{i-1}}. \quad (4)$$

Equation (4) will be proven in Lemma 6.2. Assuming its validity (for now), we have

$$\sum_{\sigma \in \Pi} \ell_{\sigma}^{\mathcal{Y}} \cdot \Upsilon_{[\sigma, \widehat{1}]} = \Upsilon_{\Pi_0} + (\Upsilon_{\Pi_1} - \Upsilon_{\Pi_0}) + (\Upsilon_{\Pi_2} - \Upsilon_{\Pi_1}) + \cdots + (\Upsilon_{\Pi_n} - \Upsilon_{\Pi_{n-1}}) = \Upsilon_{\widehat{\Pi}}.$$

Since we may freely convert between the flag polynomial and cd indexes by a variable substitution, it follows that

$$\sum_{\sigma \in \Pi} \ell_{\sigma}^{\Phi} \cdot \Upsilon_{[\sigma, \widehat{1}]} = \Upsilon_{\widehat{\Pi}}$$

if and only if

$$\sum_{\sigma \in \Pi} \ell_{\sigma}^{\Phi} \cdot \Phi_{[\sigma, \widehat{1}]} = \Phi_{\widehat{\Pi}}.$$

The theorem follows. □

We now establish the truth of equation (4).

Lemma 6.2. For some fixed $1 \leq i \leq n$, the following holds:

$$\Upsilon_{\Pi_i} - \Upsilon_{\Pi_{i-1}} = \sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \ell_{\hat{\sigma}}^{\mathcal{X}} \cdot \Upsilon_{[\sigma, \hat{1}]} = \sum_{\substack{C \text{ a chain of } \Pi_i \\ C \text{ switches at rank } i}} \alpha^C - \sum_{\substack{C \text{ a chain of } \Pi_{i-1} \\ C \text{ contains an element of rank } i}} \alpha^C$$

Proof: We demonstrate that each of $\Upsilon_{\Pi_i} - \Upsilon_{\Pi_{i-1}}$ and $\sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \ell_{\hat{\sigma}}^{\mathcal{X}} \cdot \Upsilon_{[\sigma, \hat{1}]}$ equals the right hand side of the equation to prove the lemma.

Claim 1:

$$\Upsilon_{\Pi_i} - \Upsilon_{\Pi_{i-1}} = \sum_{\substack{C \text{ a chain of } \Pi_i \\ C \text{ switches at rank } i}} \alpha^C - \sum_{\substack{C \text{ a chain of } \Pi_{i-1} \\ C \text{ contains an element of rank } i}} \alpha^C$$

That is, $\Upsilon_{\Pi_i} - \Upsilon_{\Pi_{i-1}}$ records exactly the number of flags of Π_i that switch at rank i , minus the flags of Π_{i-1} that contain a rank i -element.

First, recall the definition of Υ_{Π_i} and note that we may write

$$\Upsilon_{\Pi_i} - \Upsilon_{\Pi_{i-1}} = \sum_{C \text{ a chain of } \Pi_i} \alpha^C - \sum_{C \text{ a chain of } \Pi_{i-1}} \alpha^C.$$

Our goal is to cancel chains from this difference until only claim 1 remains.

First, we can see that the old flags of Π_i are in bijective correspondence with the old flags $F_0 < \dots < F_k$ of Π_{i-1} where $\rho_{\Pi_{i-1}}(F_0) \geq i + 1$ because $\phi_{i-1} : \Pi_i \rightarrow \Pi_{i-1}$ acts as the identity for any $\sigma \in \Pi_i$ with $\rho(\phi(\sigma)) > i$. It follows that such flags are eliminated in the difference.

Next, we may note a bijective correspondence between the flags of Π_i that switch at some $j \leq i - 1$ and the flags of Π_{i-1} that switch at j and do not contain an element of rank i . Note that any such flag is present in both Π_i and Π_{i-1} because it decomposes into an initial segment contained in $\widehat{\Pi}_{|\leq i-1}$ followed by a (possibly empty) segment in $\Pi_{|\geq i+1}$. So these flags are eliminated in the difference.

Since all mixed and new flags of Π_i switch at some $j \leq i$, the only flags we have not acknowledged of Π_i are those that switch exactly at rank i . In Π_{i-1} , the only flags that we have not acknowledged are those that contain an element of rank i (including both new, mixed and old chains). The proof of Claim 1 is a result.

Claim 2:

$$\sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \ell_{\hat{\sigma}}^{\mathcal{X}} \cdot \Upsilon_{[\sigma, \hat{1}]} = \sum_{\substack{C \text{ a chain of } \Pi_i \\ C \text{ switches at rank } i}} \alpha^C - \sum_{\substack{C \text{ a chain of } \Pi_{i-1} \\ C \text{ contains an element of rank } i}} \alpha^C.$$

Recall that for any $\sigma \in \Pi$ with σ of rank i and $\hat{\sigma} = P_1(\phi^{-1}[\widehat{0}, \sigma])$ we have that $\ell_{\hat{\sigma}}^{\mathcal{X}} = \Upsilon_{\hat{\sigma}} - \Upsilon_{P_1(\partial\hat{\sigma})}$. By construction, since $[\widehat{0}, \sigma]$ is a rank i subposet of Π , $\phi^{-1}[\widehat{0}, \sigma]$ is a

rank i subposet of Π_i and $\partial\hat{\sigma}$ is a rank i subposet of Π_{i-1} . Note that $\partial\hat{\sigma}$ is well defined for all $\sigma \in \Pi$ because ϕ is a strong Eulerian subdivision (our boundary operator was only defined on near-Eulerian posets). Expand $\ell_{\hat{\sigma}}^{\mathcal{Y}}$ into a difference of flags.

$$\ell_{\hat{\sigma}}^{\Phi} = \Upsilon_{\hat{\sigma}} - \Upsilon_{P_1(\partial\hat{\sigma})} = \sum_{C \text{ a chain of } \hat{\sigma}} \alpha^C - \sum_{C \text{ a chain of } P_1(\partial\hat{\sigma})} \alpha^C \quad (5)$$

Again, we look carefully and see what we can cancel. We may partition the chains of $\hat{\sigma}$ into two sets: Chains contained entirely in $\partial\hat{\sigma}$, and chains $C = C_1 < \tau$ where τ is carried by σ . Similarly, we may partition the chains C of $P_1(\partial\hat{\sigma})$ into two sets: those whose maximal element τ has $\rho(\tau) < i$ (and so C is contained entirely in the boundary of $\hat{\sigma}$), and chains whose maximal element is $\hat{1}_{\partial\hat{\sigma}}$, the single rank i element in $P_1(\partial\hat{\sigma})$.

It follows that chains contained strictly in the boundary of $\hat{\sigma}$ are eliminated in the difference and we have

$$\begin{aligned} \ell_{\hat{\sigma}}^{\mathcal{Y}} &= \sum_{\substack{C=C_1 < \tau \text{ a chain of } \hat{\sigma} \\ \phi(\tau)=\sigma}} \alpha^C - \sum_{C=\dots < \hat{1}_{\partial\hat{\sigma}} \text{ a chain of } P_1(\partial\hat{\sigma})} \alpha^C \\ &= \sum_{\substack{C=C_1 < \tau \text{ a chain of } \hat{\sigma} \\ \phi(\tau)=\sigma}} \alpha^C - \sum_{C \text{ a chain of } \partial\hat{\sigma}} \alpha^C \cdot b. \end{aligned}$$

Summing over all such $\sigma \in \Pi$ of rank i gives the following:

$$\sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \ell_{\hat{\sigma}}^{\mathcal{Y}} \cdot \Upsilon_{[\sigma, \hat{1}]} = \sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \sum_{\substack{C=C_1 < \tau \text{ a chain of } \hat{\sigma} \\ \phi(\tau)=\sigma}} \alpha^C \cdot \Upsilon_{[\sigma, \hat{1}]} - \sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \sum_{C \text{ a chain of } \partial\hat{\sigma}} \alpha^C \cdot b \cdot \Upsilon_{[\sigma, \hat{1}]} \quad (6)$$

Now consider new and mixed chains of Π_i that switch at i . Let C be such a chain. Note that we may uniquely decompose C into $C = C_1 < \tau < C_2$ where $\phi(\tau) = \sigma$ for some $\sigma \in \Pi$ with $\rho(\sigma) = i$, $C_1 < \tau$ is a non-empty chain of $\hat{\sigma}$, and C_2 is a possibly empty chain of $(\sigma, \hat{1}_{\Pi})$ (That is, a nondegenerate chain of $[\sigma, \hat{1}_{\Pi}]$). Since any chain that switches at rank i switches at some unique $\sigma \in \Pi$ of rank i , it follows that

$$\sum_{\substack{C \text{ a chain of } \Pi_i \\ C \text{ switches at rank } i}} \alpha^C = \sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \sum_{\substack{C=C_1 < \tau \text{ a chain of } \hat{\sigma} \\ \phi(\tau)=\sigma}} \alpha^C \cdot \Upsilon_{[\sigma, \hat{1}]} \quad (7)$$

Similarly, consider chains of Π_{i-1} that contain an element of rank i . Let C be such a chain. We may uniquely decompose it into $C = C_1 < \sigma < C_2$ for some $\sigma \in \Pi$ with $\rho(\sigma) = i$ such that C_1 is contained entirely in $\partial\hat{\sigma}$ and C_2 is contained in $(\sigma, \hat{1}_{\Pi})$. Summing this decomposition over all such chains gives us

$$\sum_{\substack{C \text{ a chain of } \Pi_{i-1} \\ C \text{ contains an element of rank } i}} \alpha^C = \sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \sum_{C \text{ a chain of } \partial\hat{\sigma}} \alpha^C \cdot b \cdot \Upsilon_{[\sigma, \hat{1}]} \quad (8)$$

Substitute equations (7) and (8) into (6) to find the result

$$\sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \ell_{\hat{\sigma}}^{\chi} \cdot \Upsilon_{[\sigma, \hat{1}]} = \sum_{\substack{C \text{ a chain of } \Pi_i \\ C \text{ switches at rank } i}} \alpha^C - \sum_{\substack{C \text{ a chain of } \Pi_{i-1} \\ C \text{ contains an element of rank } i}} \alpha^C,$$

thus proving Claim 2.

Therefore,

$$\sum_{\substack{\sigma \in \Pi \\ \rho(\sigma)=i}} \ell_{\hat{\sigma}}^{\chi} \cdot \Upsilon_{[\sigma, \hat{1}]} = \Upsilon_{\Pi_i} - \Upsilon_{\Pi_{i-1}}$$

and the lemma holds. \square

This completes the missing claim of theorem 6.1 and it follows that our cd -index subdivision decomposition holds!

In particular, if we restrict to Gorenstein* subdivisions the following result is an immediate corollary (Stated as a theorem by Ehrenborg and Karu in [6]).

Corollary 6.3. Let Π and $\hat{\Pi}$ be Gorenstein* posets such with a Gorenstein* subdivision $\phi : \hat{\Pi} \rightarrow \Pi$. As a coefficient wise inequality, we then have

$$\Phi_{\hat{\Pi}} \geq \Phi_{\Pi}.$$

Proof: Since ϕ is a Gorenstein* subdivision, it is also strongly Eulerian and we may apply our subdivision theorem.

$$\begin{aligned} \Phi_{\hat{\Pi}} &= \sum_{\sigma \in \Pi} \ell_{\hat{\sigma}}^{\Phi} \cdot \Phi_{[\sigma, \hat{1}]} \\ &= \ell_{\hat{0}}^{\Phi} \cdot \Phi_{[\hat{0}, \hat{1}]} + \sum_{\sigma \in \Pi, \sigma \neq \hat{0}} \ell_{\hat{\sigma}}^{\Phi} \cdot \Phi_{[\sigma, \hat{1}]} \\ &= \Phi_{\Pi} + \sum_{\sigma \in \Pi, \sigma \neq \hat{0}} \ell_{\hat{\sigma}}^{\Phi} \cdot \Phi_{[\sigma, \hat{1}]} \end{aligned}$$

Note that $\hat{\sigma}$ is near-Gorenstein* and $[\sigma, \hat{1}]$ is Gorenstein* for all $\sigma \in \Pi$ by definition of Gorenstein* subdivision and Gorenstein* poset respectively. Since both the cd -index and the local cd -index are non-negative for Gorenstein* and near-Gorenstein* posets, the result follows. \square

Example 6.4. We return to the subdivided tetrahedron of Example 5.8 (pictured in Figures 18, 19 and 20) to demonstrate an application of our subdivision theorem. Let's calculate the cd -index of $\hat{\Pi}$. First consider (as seen in the previous corollary)

$$\Phi_{\widehat{\Pi}} = \Phi_{\Pi} + \sum_{\sigma \in \Pi, \sigma \neq \widehat{0}} \ell_{\sigma}^{\Phi} \cdot \Phi_{[\sigma, \widehat{1}]}$$

By applying our characterization of the cd -index of 3-polytopes (Lemma 3.17), we have $\Phi_{\Pi} = c^3 + 2dc + 2cd$. Also, note that for any $\sigma \in \Pi$ such that $P_1(\sigma) \cong \widehat{\sigma}$ as face posets it easily calculated that $\ell_{\sigma}^{\Phi} = 0$. A similar calculation shows that $\ell_{\widehat{1}_{\Pi}}^{\Phi} = 0$. Topologically, this implies that the only faces $\sigma \in \Pi$ with non-zero local cd -index are those that are non-trivially subdivided. There are exactly three such faces: the edge with a single vertex added to its interior, and the two faces adjacent to that edge that have been divided in two. Call the edge E and the faces F (we need not distinguish between the two of them as long as we count both because they have identical chain structure).



Figure 21: $\ell_E^{\Psi} = ba + ab$ and $\ell_E^{\Phi} = d$.

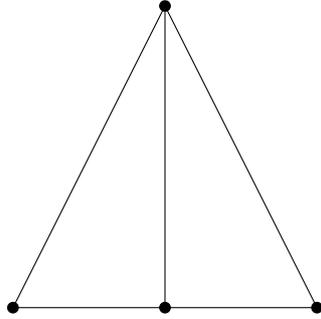


Figure 22: $\ell_F^{\Psi} = aba + aab + bba + bab$ and $\ell_F^{\Phi} = cd$.

We also note that $\Phi_{[E, \widehat{1}]} = c$ and $\Phi_{[F, \widehat{1}]} = 1$. Put it all together and we find

$$\begin{aligned} \Phi_{\widehat{\Pi}} &= \Phi_{\Pi} + \sum_{\sigma \in \Pi, \sigma \neq \widehat{0}} \ell_{\sigma}^{\Phi} \cdot \Phi_{[\sigma, \widehat{1}]} \\ &= \Phi_{\Pi} + \ell_E^{\Phi} \cdot \Phi_{[E, \widehat{1}]} + 2\ell_F^{\Phi} \cdot \Phi_{[F, \widehat{1}]} \\ &= (c^3 + 2dc + 2cd) + dc + 2cd \\ &= c^3 + 3dc + 4cd. \end{aligned}$$

This answer is easy to double check. Our classification of the cd -index of 3-polytopes only depended on shellability. Since $\widehat{\Pi}$ is shellable, we may also apply it to $\widehat{\Pi}$ and note that it gives us the same result for $\Phi_{\widehat{\Pi}}$.

The subdivision decomposition theorem, combined with the described non-negativity conditions on Gorenstein* posets are a powerful tool for establishing upper and lower

bounds on the cd -index via natural combinatorial arguments. In the next chapter, we examine the traditional upper and lower bounds on the face numbers of polytopes and look at the analogous cd -index bounds.

7 Upper Bounds, Lower Bounds, and the cd-index of Stacked Polytopes

Enumerating chains of Eulerian posets is a generalization of the classical theory of enumerating faces of a convex polytope. Though likely familiar to the reader, we present the definitions of f -vector and h -vector for reference.

Definition 7.1. Given a CW $(d-1)$ -complex Γ , let $f_i = |\{\sigma \in \Gamma \mid \dim \sigma = i\}|$ be the number of i -dimensional cells of Γ . With $f_{-1} = 1$ enumerating the single empty cell, we write $f(\Gamma) = [f_{-1}, f_0, \dots, f_{d-1}]$ to be the f -vector of Γ .

In practice, we may assume Γ is a simplicial complex for the remainder of this chapter, but present the above definition for the sake of full generality. The following definition is restricted to simplicial complexes.

Definition 7.2. Given a $(d-1)$ simplicial complex Δ , we define $h(\Delta) = [h_0, h_1, \dots, h_d]$ to be the h -vector of Δ by the relationship

$$\sum_{i=0}^d f_{i-1}(x-1)^{d-i} = \sum_{i=0}^d h_i x^{d-i}. \quad (9)$$

Extracting coefficients, we may explicitly write

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$$

and

$$f_{k-1} = \sum_{i=0}^{d-k} \binom{d-i}{k} h_i.$$

We will often consider the h -polynomial written

$$h(\Delta, x) = h_0 + h_1 x + \dots + h_d x^d.$$

In particular, the h -vector and f -vector of a simplicial complex are the motivation for the flag h -vector β_P and flag f -vector α_P of a poset P as defined in Chapter 2 (Definitions 2.7 and 2.8). In fact, the flag f and h vector of P determine the f and h -vector of its order complex $\Delta(P)$.

Lemma 7.3. [21] Let P be a finite graded poset of rank n with flag f -vector α_P and flag h -vector β_P . We may recover the f -vector and h -vector of its order complex $\Delta(P)$ by the following equations:

$$f_i(\Delta(P)) = \sum_{\substack{|S|=i+1 \\ S \subseteq [n]}} \alpha_P(S)$$

$$h_i(\Delta(P)) = \sum_{\substack{|S|=i \\ S \subseteq [n]}} \beta_P(S)$$

Note that the h -vector and f -vector of a simplicial complex enumerate the same data, since each can be transformed into the other. Though the f -vector is more immediately intuitive, the h -vector possesses many useful properties when applied to convex simplicial polytopes. Many of the results of the field are much more simple to express when using the h -vector rather than the f -vector. The most immediately obvious is the *Dehn-Sommerville equations*.

Theorem 7.4. (*The Dehn-Sommerville Equations*) Let ∂P be the boundary complex of a simplicial d -polytope with $h(\partial P) = [h_0, h_1, \dots, h_d]$. Then the following equations hold:

$$h_i = h_{d-i} \text{ for all } 1 \leq i \leq \lfloor d/2 \rfloor.$$

A proof of these equations (along with a combinatorial interpretation of the h -vector) is given via a shelling argument in [22]. An equivalent formulation of these equations using f -vectors was given in a classical paper by Sommerville [15].

The h -vector of any simplicial polytope is symmetric. This is a concise presentation of the linear relations between the number of faces of different dimension of a polytope and a powerful characterization.

The flag h -vector has analogous symmetry on Eulerian posets. That is, the ab -index of an Eulerian polytope is invariant under the involution that switches a 's and b 's. This relationship, described as the Dehn-Sommerville relations for Eulerian posets, was originally described by Bayer and Billera [1] in different, but equivalent words. It is presented by Stanley in the language of flag f and h -vectors in [21].

This motivates understanding the cd -index as an encoding of the Dehn-Sommerville relations for posets. Given a theorem on the h or f -vector of a polytope or simplicial sphere, we then have the natural question of "is there an analogous result for the cd or ab -index of an Eulerian poset?" We will examine two of the most famous bounding theorems of polytope face enumeration: McMullen's upper bound theorem, and the generalized lower bound theorem.

The first important problem concerns an upper bound. Given a polytope P of dimension d with k vertices, is there a sensible (hopefully tight) upper bound for its f or h vector? We find an answer by looking at the class of *cyclic polytopes*.

Definition 7.5. [8] Consider the function $x : \mathbb{R} \rightarrow \mathbb{R}^d$, $x(t) = (t, t^2, \dots, t^d)$. For any $k \geq d + 1$, we define the cyclic polytope of k vertices in dimension d (denoted $C_d(k)$) as the convex hull of $\{x(t_1), x(t_2), \dots, x(t_k)\}$ where t_1, \dots, t_k are arbitrary real numbers such that $t_1 < t_2 < \dots < t_k$.

This definition may inspire skepticism due to the possible different choices of t_i . However, $C_d(k)$ has a well defined f -vector. That is, no matter the choices of the t_i 's, $f(C_d(k))$ is invariant [8].

Cyclic polytopes are simplicial [22]. They possess many notable combinatorial features. Under consideration of the topic of this thesis, we will ignore most of them ([8] and [22] give a much more thorough explanation). Of particular note is that they have a large amount of faces relative to their number of vertices. We would present a diagram here if not for that their complexity makes all but the most trivial examples of them exceedingly difficult to display intelligently projected on a page.

Cyclic polytopes were conjectured to provide an upper bound for f -vectors by Motzkin in 1957, but a proof for the general case was not established until 1970 by McMullen [22].

Theorem 7.6. (*McMullen's Upper Bound Theorem*) [11] Let P be a d -polytope with k vertices. Then for all integers $0 \leq n \leq d$,

$$\begin{aligned} f_{n-1}(P) &\leq f_{n-1}(C_d(k)) \\ h_n(P) &\leq h_n(C_d(k)) \end{aligned}$$

In other words, P has at most as many n -faces as the corresponding cyclic polytope of the same dimension and vertex number.

We omit McMullen's proof for the sake of brevity. However, it is worth noting that it makes use of both shelling and an h -vector argument.

Perhaps unsurprisingly, the result generalizes to the cd -index, though the correspondence is not trivial. It was not proven until 2000 by Billera and Ehrenborg [4]. Though this result predates Ehrenborg and Karu's decomposition theorem, a key lemma required for the upper bound proof is both generalized (as well as proved more succinctly) by use of a Gorenstein* subdivision decomposition [6]. We present a sketch of their arguments.

First we consider the following lemma presented in [7] and attributed to Richard Stanley.

Lemma 7.7. Let P_1 and P_2 be d -polytopes such that $P_1 \cap P_2 = F$, where F is a single facet of both P_1 and P_2 . Let $P_1 \cup^* P_2 = \partial P_1 \cup \partial P_2 \cup \{\widehat{1}\} \setminus \{F\}$ be the face complex formed by 'joining' P_1 and P_2 at F . That is, $P_1 \cup^* P_2$ contains all faces of P_1 and P_2 except F . We may express the cd -index of $P_1 \cup^* P_2$ as

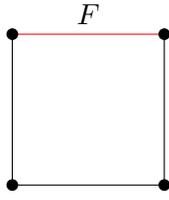


Figure 23: P_1

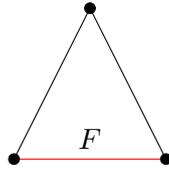


Figure 24: P_2

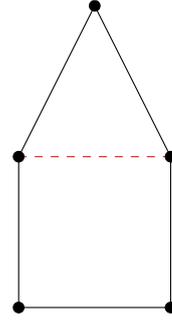


Figure 25: $P_1 \cup^* P_2$

$$\Phi(P_1 \cup^* P_2) = \Phi(P_1) + \Phi(P_2) - \Phi(F) \cdot c.$$

Proof: This follows from a simple chain counting argument. Any nondegenerate chain of $P_1 \cup^* P_2$ is contained in either P_1, P_2 or both of them. The only chains that are contained in both are those that lie in ∂F . The only chains of P_1 and P_2 that are not present in $P_1 \cup^* P_2$ are those whose maximal element is F . So we may count the chains of $P_1 \cup^* P_2$ by summing the chains of P_1 and P_2 , then removing one set of the double counted chains, and both sets of the chains whose maximal element is F (since there is a set contributed by both P_1 and P_2).

Expressed as a flag polynomial, we count chains contained in ∂F by $\Upsilon(F) \cdot a$ and chains with maximal element F by $\Upsilon(F) \cdot b$. Since we remove two sets of the latter such chains, in total we remove $\Upsilon(F) \cdot (a + 2b)$. This gives us

$$\begin{aligned} \Upsilon(P_1 \cup^* P_2) &= \Upsilon(P_1) + \Upsilon(P_2) - \Upsilon(F) \cdot (a + 2b) \\ \Psi(P_1 \cup^* P_2) &= \Psi(P_1) + \Upsilon(P_2) - \Psi(F) \cdot (a + b) \\ \Phi(P_1 \cup^* P_2) &= \Phi(P_1) + \Phi(P_2) - \Phi(F) \cdot c. \end{aligned}$$

□

Joining polytopes together at a face is often a useful construction. The preceding lemma allows us to easily calculate the cd -index of such objects.

Before we continue to the aforementioned ‘key lemma’, we must first define a pyramid operation on posets.

Definition 7.8. Given a poset P , we define the *pyramid* of P by the set

$$\text{Pyr}(P) := P \times \{0, 1\}$$

equipped with the partial order that for $(x, \alpha), (y, \beta) \in \text{Pyr}(P)$ we have

$$(x, \alpha) \leq_{\text{Pyr}(P)} (y, \beta) \iff x \leq_P y \text{ AND } \alpha \leq \beta.$$

Furthermore, if P is a graded poset of rank n , $\text{Pyr}(P)$ is a graded poset of rank $n + 1$ with the rank function

$$\rho_{\text{Pyr}(P)}(x, \alpha) := \rho_P(x) + \alpha.$$

This operation is the natural generalization of taking a pyramid over a polytope. In the case of P being the face poset of a polytope, we can interpret the element $(\widehat{0}, 1)$ as the new vertex induced by the pyramid, and any element $(x, 1)$ as the new face induced by the convex hull of x and the new vertex. Any element of the form $(x, 0)$ corresponds to the old face $x \in P$.

Proposition 7.9. The pyramid operator is non-negative with respect to the cd -index. In particular, for any Eulerian poset P ,

$$\Phi_P \cdot c \leq \Phi_{\text{Pyr}(P)}.$$

We refrain from proving this, but note that it is not too technical. It is best demonstrated by showing that Pyr may be well defined as a derivation on cd -words. See *Coproducts and the cd -index* [7] and *Monotonicity of the cd -index for polytopes* [4] for details.

Finally, we present the key lemma. In [4], an analogous result specifically on polytopes and their pyramids was used to prove the upper bound theorem for cd -indexes. The full result for Gorenstein* lattices was proved by Ehrenborg and Karu in [6]. Recall that a *lattice* is a poset such that any two elements u and v have both a greatest lower bound (denoted $u \wedge v$) and a least upper bound (denoted $u \vee v$). The face poset of any polytope is a Gorenstein* lattice.

Lemma 7.10. [6] Let L be a Gorenstein* lattice and let $x \in L$ be an element such that $x \neq \widehat{0}$ or $\widehat{1}$. We then have the following coefficientwise inequalities:

$$\Phi_L \geq \Phi_{[\widehat{0}, x]} \cdot \Phi_{\text{Pyr}([x, \widehat{1}])} \tag{10}$$

$$\Phi_L \geq \Phi_{\text{Pyr}([\widehat{0}, x])} \cdot \Phi_{[x, \widehat{1}]} \tag{11}$$

Proof Sketch: This may be proved via a direct application of the subdivision decomposition theorem (Theorem 6.1). Recall that the poset join operator $*$ corresponds to cd -index multiplication (Lemma 3.7), so $\Phi([\widehat{0}, x] * \text{Pyr}([x, \widehat{1}])) = \Phi([\widehat{0}, x]) \cdot \Phi(\text{Pyr}([x, \widehat{1}]))$.

To complete the proof, we need to demonstrate two facts.

- Show that $[\widehat{0}, x] * \text{Pyr}([x, \widehat{1}])$ is a Gorenstein* poset.
- Construct a Gorenstein* subdivision $\phi : L \rightarrow [\widehat{0}, x] * \text{Pyr}([x, \widehat{1}])$.

It then follows by the subdivision decomposition theorem (in particular, from Corollary 6.5 we have that a subdivision of a Gorenstein* poset increases the coefficients of the cd -index by a non-negative quantity) that

$$\Phi(L) \geq \Phi([\widehat{0}, x] * \text{Pyr}([x, \widehat{1}])) = \Phi([\widehat{0}, x]) \cdot \Phi(\text{Pyr}([x, \widehat{1}])).$$

We leave the verification of the Gorenstein* condition to Ehrenborg and Karu in [6] and note that the subdivision may be achieved by the map

$$\phi(\sigma) = \begin{cases} \sigma & \text{if } \sigma < x, \\ (\sigma, 1) & \text{if } \sigma \geq x, \\ (\sigma \vee x, 0) & \text{otherwise} \end{cases}$$

giving us inequality (10). We find inequality (11) by considering the dual of L . \square

Corollary 7.11. For any Gorenstein* lattice L and coatom x , we have

$$\Phi_L \geq \Phi_{\text{Pyr}(\widehat{[0,x]})}.$$

Proof: Apply inequality (11) and note that $\Phi_{[x,\widehat{1}]} = 1$ for x a coatom. \square

Though maybe not immediately impressive, pyramids are a powerful poset building block. In particular, given the boolean algebra on n elements B_n , we have $\text{Pyr}(B_n) \cong B_{n+1}$ as posets. Ehrenborg and Karu[6] used these inequalities to easily settle a long standing conjecture of Stanley [20].

Theorem 7.12. Let L be a rank n Gorenstein* lattice. Then

$$\Phi_L \geq \Phi_{B_n}.$$

That is, the cd -index is minimized by boolean algebras over all Gorenstein* lattices.

Proof: Repeatedly apply our previous corollary to get $\Phi(L) \geq \Phi(\text{Pyr}^{n-1}(\widehat{[0,x]}))$ for some atom x of P and note that $\widehat{[0,x]} \cong B_2$. \square

The lemma is also a component of Ehrenborg and Billera's proof of the upper bound theorem for the cd -index [4].

Theorem 7.13. Let P be a d -polytope with k vertices. Then

$$\Phi_P \leq \Phi_{C_d(k)}.$$

Proof Sketch: One can show that by 'pulling' vertices of P one at a time and taking the resulting convex hull, we may deform P into a simplicial polytope Q with the same number of vertices. An argument involving Lemma 7.7 and Lemma 7.10 demonstrates that $\Phi_P \leq \Phi_Q$. Stanley notes in [20] that for a simplicial polytope Q , the cd -index may be entirely determined by its h -vector as a sum of non-negative cd -polynomials determined by the dimension of Q . Since the h -vector is maximized for $C_d(k)$ by McMullen's Upper Bound Theorem, the result follows. \square

So the h -vector and the cd -index of a polytope align on the upper bound theorem.

In general, lower bounds are harder to establish. We first describe a particular polytope important for bounds.

Definition 7.14. Given a simplicial polytope P , we say that a subdivision Γ of P is a *triangulation* if the following holds:

- Every face of Γ is a simplex.
- $\partial\Gamma$ and ∂P are combinatorially isomorphic as simplicial complexes.

That is, Γ subdivides P into simplices such that no new faces are introduced in to the boundary of P .

Definition 7.15. We say that a simplicial d -polytope P is a *stacked polytope* if there exists a triangulation Γ of P such that for any face $\sigma \in \Gamma$ with $\dim \sigma \leq d - 2$, we also have that σ is a face of P . We say that P is $(r - 1)$ -stacked if for any $\sigma \in \Gamma$ with $\dim \sigma \leq d - r$ we have that σ is a face of P . When given an $(r - 1)$ -stacked polytope P , we will implicitly refer to Γ as the *stack subdivision* of P .

Note that a 0-stacked polytope must be a simplex. 1-stacked polytopes are stacked polytopes. Also note that by definition, any r -stacked polytope of dimension d is k -stacked for any $r \leq k \leq d$. For clarity (unless otherwise mentioned) when we refer to an r -stacked polytope P we will assume that r is the smallest such integer for which P is stacked.

A commonly used (and perhaps more intuitive) equivalent definition is that a d -polytope is a stacked polytope if it can be obtained from a d -simplex by repeatedly placing a point beyond a facet and taking the convex hull. That is, we can imagine any stacked polytope (and its underlying triangulation) arising by starting with a simplex, and then ‘stacking’ additional simplexes to it sequentially so that they each intersect with exactly one facet of the previous structure. The only interior (that is, faces not in its boundary) faces of its stack subdivision Γ in such a construction are of dimension $(d - 1)$ or higher.

Stacked polytopes are relevant to the Generalized Lower Bound Theorem. This theorem was original presented as a conjecture by McMullen and Walkup in 1971 [12].

Theorem 7.16. (*The Generalized Lower Bound Theorem*) Given a simplicial d -polytope P , the following hold:

1. $1 = h_0(P) \leq h_1(P) \leq \dots \leq h_{\lfloor \frac{d}{2} \rfloor}$
2. For any integer $1 \leq r \leq \lfloor \frac{d}{2} \rfloor$, $h_{r-1}(P) = h_r(P)$ if and only if P is $(r - 1)$ -stacked.

Part 1 of the theorem is implied by the well known g -theorem for simplicial polytopes that completely characterizes the h -vectors of simplicial polytopes. This particular part of the g -theorem was established by the work of Stanley in [16].

Part 2 was not fully resolved until much more recently. The implication that an $(r - 1)$ -stacked polytope P has $h_{r-1}(P) = h_r(P)$ was resolved by McMullen and Walkup when they posed their conjecture [12]. However, the converse was not proven until 2012 by

Murai and Nevo [14]. Extending the theorem to hold for simplicial spheres remains an active area of research.

This result immediately implies a lower bound for h -vectors analogous to the lower bound given in Theorem 7.12.

Corollary 7.17. The h -vector of the d -simplex is a lower bound for all h -vectors of simplicial d -polytopes.

Proof: [21] It can be easily calculated that the h -vector of the d -simplex is $[1, 1, \dots, 1]$. The result then follows from the lower bound theorem. \square

Stacked polytopes are useful for inducing bounds because of their simple construction. Let Δ_d denote the face poset of the d -simplex. We may simply express the cd -index of stacked polytopes in terms of the cd -index of simplices.

Lemma 7.18. Let P be a stacked d -polytope such that its stack subdivision contains exactly k d -simplices. We then have that

$$\Phi(P) = k \cdot \Phi(\Delta_d) - (k - 1) \cdot \Phi(\Delta_{d-1}) \cdot c.$$

Proof: By definition, we form a stacked polytope by sequentially ‘gluing’ simplices together so that each one intersects the previous complex in exactly one facet (that is, a $(d - 1)$ -simplex). Note that this is exactly the setup of Lemma 7.7. Apply the lemma $(k - 1)$ times and the result follows. \square

In fact, this construction implies that the stack subdivision Γ of a stacked polytope P is shellable. Each d -simplex in the sequence intersects the previous in exactly one facet, whose boundary is a simplex and therefore shellable. It is conjectured that stack subdivision of any $(r - 1)$ -stacked polytope is shellable for $r \leq (d + 1)/2$ [14]. The ‘niceness’ of the resulting cd -index formula can be seen as a consequence of how each simplex intersects the previous ones in exactly one facet.

What each simplex in the shelling order intersects the previous in more than one facet? That is, what can we say about the cd -index of a simplicial sphere that can be triangulated by a shellable complex in general? We spend the remainder of this chapter attempting to find an answer.

Definition 7.19. We say a $(d - 1)$ -dimensional simplicial sphere is *k -shellably triangulizable* if it may be triangulated as a shellable d -ball using exactly k d -simplices.

Note that a stacked d -polytope has exactly $kd + 1$ facets if its stack subdivision contains exactly k d -simplices. So a stacked d -polytope with $kd + 1$ facets is k -shellably triangulizable.

The following facts about shelling a simplex will be useful to us. The first is retrieved from [22].

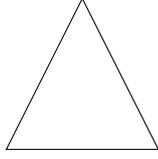


Figure 26: $\text{star}_{\partial\Delta_3}(\Delta_2)$

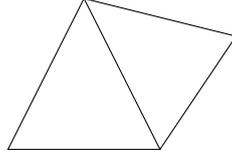


Figure 27: $\text{star}_{\partial\Delta_3}(\Delta_1)$

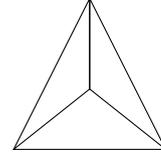


Figure 28: $\text{star}_{\partial\Delta_3}(\Delta_0)$

Proposition 7.20. Consider the boundary complex of a d -simplex $\partial\Delta_d$. Any order $F_1 \dots F_{d+1}$ of its facets is a shelling. Furthermore, any two shellings of $\partial\Delta_d$ are isomorphic.

Proposition 7.21. Let G_1, G_2, \dots, G_{d+1} be a shelling of $\partial\Delta_d$. Then for any $1 \leq r \leq d+1$ we have that $G_1 \cup \dots \cup G_r$ is isomorphic to $\text{star}_{\partial\Delta_d}(\Delta_{d-r})$ as a simplicial complex, where Δ_{d-r} is any $(d-r)$ face of $\partial\Delta_d$.

Proof: Any r facets of a d -simplex intersect in a single $(d-r)$ -simplex [8]. By definition, $\text{star}_{\partial\Delta_d}(\Delta_{d-r})$ is the subcomplex induced by all the facets of $\partial\Delta_d$ that contain Δ_{d-r} and their faces, which is exactly $G_1 \cup \dots \cup G_r$. \square

Corollary 7.22. In any shelling F_1, F_2, \dots, F_k of a simplicial d -dimensional complex, $F_j \cap \left(\bigcup_{i=1}^{j-1} F_i\right) = \text{star}_{\Delta_d}(\Delta_{d-r})$ for some integer $1 \leq r \leq d+1$.

Proof: This follows from the definition of shelling. Since F_j is a d -simplex, $F_j \cap \left(\bigcup_{i=1}^{j-1} F_i\right)$ is the initial segment of a shelling of $\partial\Delta_d$. In our previous proposition we proved this must be some star of a face of $\partial\Delta_d$. \square

Note that for any r , $P_1(\text{star}_{\partial\Delta_d}(\Delta_r))$ is a near-Gorenstein complex of dimension $n-1$.

Consider a shelling F_1, F_2, \dots, F_k of an arbitrary Gorenstein complex. Note that the complex induced by the first j facets of the shelling for $j < k-1$ is near-Gorenstein. We now demonstrate that adding the next facet of the shelling to the complex gives a non-negative increase in the local cd -index.

Lemma 7.23. Let S be a Gorenstein complex with a shelling F_1, F_2, \dots, F_k . Let $\sigma_i = P_1(F_1 \cup F_2 \cup \dots \cup F_i)$ be the subcomplex induced by shelling the first i facets. Then, for any integers $1 \leq i \leq k-2$, we have that

$$\ell_{\sigma_i}^{\Phi} \leq \ell_{\sigma_{i+1}}^{\Phi}.$$

In other words, the local cd -index increases by a non-negative amount at each stage of a shelling.

Proof: Recall that $\ell_{\sigma_i}^{\Gamma} = \Upsilon_{\sigma_i} - \Upsilon_{P_1(\partial\sigma_i)}$. Let $\Gamma := F_{i+1} \cap \sigma_i$. Let $\bar{\Gamma} := F_{i+1} \setminus \text{int}(\Gamma)$ be the complement of Γ relative to F_{i+1} . That is, Γ and $\bar{\Gamma}$ are near-Gorenstein complexes

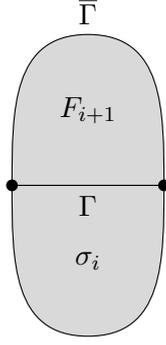


Figure 29: A sketch of σ_{i+1} . Note that the bottom cell represents all of the complex σ_i , while the top cell represents the single face F_{i+1} . $\Gamma = F_{i+1} \cap \sigma_i$ and $\bar{\Gamma} = F_{i+1} \setminus \text{int}(\Gamma)$.

such that $\Gamma \cup \bar{\Gamma} = F_{i+1}$ and $\partial\Gamma = \partial\bar{\Gamma}$. We will count the new chains introduced in $\ell_{\sigma_{i+1}}^{\mathcal{X}}$ relative to $\ell_{\sigma_i}^{\mathcal{X}}$ by considering attaching F_{i+1} to σ_i along Γ .

$$\ell_{\sigma_{i+1}}^{\mathcal{X}} - \ell_{\sigma_i}^{\mathcal{X}} = (\Upsilon_{\sigma_{i+1}} - \Upsilon_{P_1(\partial\sigma_{i+1})}) - (\Upsilon_{\sigma_i} - \Upsilon_{P_1(\partial\sigma_i)}) \quad (12)$$

$$= (\Upsilon_{\sigma_{i+1}} - \Upsilon_{\sigma_i}) - (\Upsilon_{P_1(\partial\sigma_{i+1})} - \Upsilon_{P_1(\partial\sigma_i)}) \quad (13)$$

$$= (\Upsilon_{\Gamma} \cdot b + \Upsilon_{\bar{\Gamma}} \cdot (a+b) - \Upsilon_{\partial\Gamma} \cdot a(a+b)) - (\Upsilon_{\bar{\Gamma}} - \Upsilon_{\Gamma})(a+b) \quad (14)$$

$$= \Upsilon_{\Gamma}(a+2b) - \Upsilon_{\partial\Gamma} \cdot a(a+b). \quad (15)$$

In the above equations, the only non-trivial step is from (13) to (14) by means of a chain counting argument. See Figure 29 for reference. The remainder of the calculations follow from algebraic manipulation. We now outline the counting argument.

Claim 1: $\Upsilon_{\sigma_{i+1}} - \Upsilon_{\sigma_i} = \Upsilon_{\Gamma} \cdot b + \Upsilon_{\bar{\Gamma}} \cdot (a+b) - \Upsilon_{\partial\Gamma} \cdot a(a+b)$.

Since $\sigma_i \subseteq \sigma_{i+1}$ as posets, we may enumerate $\Upsilon_{\sigma_{i+1}} - \Upsilon_{\sigma_i}$ by counting exactly the chains of σ_{i+1} not present in σ_i . Note that any chain that does not contain a face of F_{i+1} must be present in both σ_{i+1} and σ_i and is eliminated in the difference. Any remaining chain must have an initial segment in Γ , $\bar{\Gamma}$, or both (if its initial segment is in $\partial\Gamma$ and contains no max dimensional faces of Γ or $\bar{\Gamma}$). By basic inclusion-exclusion, we count the ab words corresponding to chains with initial segments in Γ and $\bar{\Gamma}$, and then subtract the chains with initial segments in $\partial\Gamma$.

The only chains of σ_{i+1} with an initial segment in Γ that are *not* in σ_i as well are exactly those with F_{i+1} as a maximal element. These may be counted by $\Upsilon_{\Gamma} \cdot b$. The only chains of σ_{i+1} with an initial segment in $\bar{\Gamma}$ that are not in σ_i are of exactly two types. If the initial segment is in the interior of $\bar{\Gamma}$, then the chain is not in σ_i by definition. If the initial segment is in $\partial\Gamma$, it must have F_{i+1} as its maximal element. So we enumerate these chains by $\Upsilon_{\bar{\Gamma}} \cdot (a+b) - \Upsilon_{\partial\Gamma} \cdot aa$. Finally, we may enumerate all chains of σ_{i+1} with

initial segment in $\partial\Gamma$ that are not present in σ_i by $\Upsilon_{\partial\Gamma} \cdot ab$ (because they must have F_{i+1} as a maximal element). So by inclusion-exclusion it follows that

$$\begin{aligned}\Upsilon_{\sigma_{i+1}} - \Upsilon_{\sigma_i} &= \Upsilon_{\Gamma} \cdot b + (\Upsilon_{\bar{\Gamma}} \cdot (a + b) - \Upsilon_{\partial\Gamma} \cdot aa) - \Upsilon_{\partial\Gamma} \cdot ab \\ &= \Upsilon_{\Gamma} \cdot b + \Upsilon_{\bar{\Gamma}} \cdot (a + b) - \Upsilon_{\partial\Gamma} \cdot a(a + b).\end{aligned}$$

Claim 2: $\Upsilon_{P_1(\partial\sigma_{i+1})} - \Upsilon_{P_1(\partial\sigma_i)} = (\Upsilon_{\partial\sigma_{i+1}} - \Upsilon_{\partial\sigma_i})(a + b).$

Any chain of $\partial\sigma_{i+1}$ that is not contained in $\partial\sigma_i$ must lie in the interior $\bar{\Gamma}$. The only chains of $\partial\sigma_i$ not contained in $\partial\sigma_{i+1}$ are those on the interior of Γ . Either of type of chain may be covered by the single maximal element or not. Since the chains contained strictly in the boundary are eliminated in the difference, we may write

$$\Upsilon_{P_1(\partial\sigma_{i+1})} - \Upsilon_{P_1(\partial\sigma_i)} = (\Upsilon_{\bar{\Gamma}} - \Upsilon_{\Gamma})(a + b).$$

Combining these two claims gives that equations (13) and (14) of our argument are equal.

Now, we convert the result of equation (15) into the local cd -index

$$\begin{aligned}\ell_{\sigma_{i+1}}^{\mathcal{R}} - \ell_{\sigma_i}^{\mathcal{R}} &= \Upsilon_{\Gamma}(a + 2b) - \Upsilon_{\partial\Gamma} \cdot ab - \Upsilon_{\partial\Gamma} \cdot aa \\ &= \Upsilon_{\Gamma}(a + 2b) - \Upsilon_{\partial\Gamma} \cdot a(b + a) \\ \ell_{\sigma_{i+1}}^{\Psi} - \ell_{\sigma_i}^{\Psi} &= \Psi_{\Gamma}(a + b) - \Psi_{\partial\Gamma} \cdot (a - b)a \\ &= (\ell_{\Gamma}^{\Psi} + \Psi_{\partial\Gamma} \cdot a)(a + b) - \Psi_{\partial\Gamma} \cdot (a - b)a \\ &= \ell_{\Gamma}^{\Psi}(a + b) + \Psi_{\partial\Gamma}(ab + ba) \\ \ell_{\sigma_{i+1}}^{\Phi} - \ell_{\sigma_i}^{\Phi} &= \ell_{\Gamma}^{\Phi} \cdot c + \Phi_{\partial\Gamma} \cdot d\end{aligned}$$

Since Γ is near-Gorenstein, we have that $\ell_{\Gamma}^{\Phi} \cdot c$ is non-negative. Similarly, since $\partial\Gamma$ is Gorenstein we have that $\Phi_{\partial\Gamma} \cdot d$ is non-negative. It follows that $\ell_{\sigma_i}^{\Phi} \leq \ell_{\sigma_{i+1}}^{\Phi}$. \square

This is a slightly stronger result than we set out to prove. The above equations provide an explicit description of how the local cd -index changes at each stage of a shelling.

Theorem 7.24. Let Π be the face poset of a k -shellably triangulizable $(d - 1)$ -simplicial sphere. Then the cd -index of a stacked d -polytope with $kd + 1$ facets is an upper bound for the cd -index of Π . In other words,

$$\Phi_{\Pi} \leq \Phi_{S_d(k)}$$

where $S_d(k)$ denotes a k -triangulizable stacked d -polytope.

Proof: Let T be the underlying triangulation of Π . Suppose T has shelling order $F_1, F_2 \dots F_k$. For $1 \leq i \leq k$, let $T_i = F_1 \cup \dots \cup F_i$ be the subcomplex of T induced by the first i facets in the shelling order. How can we relate the cd -index of ∂T_i and ∂T_{i+1} ? Note that $\partial T_k = S$.

We get from T_i to T_{i+1} by attaching exactly one k -simplex, F_{i+1} . If F_{i+1} intersects T_i in exactly one facet, then we are simply in the case of lemma 7.7 and have that

$$\Phi_{\partial T_{i+1}} = \Phi_{\partial T_i} + \Phi_{\Delta_d} - \Phi_{\Delta_{d-1}} \cdot c.$$

Otherwise, F_{i+1} intersects T_i in some initial segment of a shelling of ∂F_{i+1} . Since F_{i+1} is a simplex, we know from our previous propositions that $F_{i+1} \cap T_i \cong \text{star}_{\Delta_d}(\Delta_{d-r})$ for some r . Note that $\text{star}_{\Delta_d}(\Delta_{d-r})$ is a near-Gorenstein complex. Also note that the remaining facets of F_{i+1} not in the intersection form a simplicial complex isomorphic to $\text{star}_{\Delta_d}(\Delta_{r-1})$, and that $\partial \text{star}_{\partial \Delta_d}(\Delta_{d-r}) = \partial \text{star}_{\partial \Delta_d}(\Delta_{r-1})$.

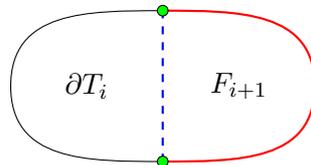


Figure 30: A generalized sketch of ∂T_{i+1} . The red line corresponds to $\text{star}_{\partial \Delta_d}(\Delta_{r-1})$, the green dots to $\partial \text{star}_{\partial \Delta_d}(\Delta_{r-1})$, and the blue dashes to the faces in the interior of $\text{star}_{\partial \Delta_d}(\Delta_{d-r})$ that are in ∂T_i but not ∂T_{i+1} .

We now calculate the ab -index of ∂T_{i+1} relative to ∂T_i . F_{i+1} intersects T_i at $\text{star}_{\partial \Delta_d}(\Delta_{d-r})$. We partition the chains of ∂T_{i+1} into exactly two types: chains that are contained in both ∂T_{i+1} and ∂T_i , and chains that are contained only in ∂T_{i+1} .

All chains of ∂T_i are contained in ∂T_{i+1} with the exception of those whose maximal face lies in the interior of $\text{star}_{\partial \Delta_d}(\Delta_{d-r})$. Subtracting out exactly those chains gives the following quantity.

$$\begin{aligned} & \Upsilon_{\partial T_i} - (\Upsilon_{\text{star}_{\partial \Delta_d}(\Delta_{d-r})} - \Upsilon_{\partial \text{star}_{\partial \Delta_d}(\Delta_{d-r})} \cdot a) \\ &= \Upsilon_{\partial T_i} - (\Upsilon_{\text{star}_{\partial \Delta_d}(\Delta_{d-r})} - \Upsilon_{\partial \text{star}_{\partial \Delta_d}(\Delta_{d-r})} \cdot (a + b) + \Upsilon_{\partial \text{star}_{\partial \Delta_d}(\Delta_{d-r})} \cdot b) \\ &= \Upsilon_{\partial T_i} - (\Upsilon_{\text{star}_{\partial \Delta_d}(\Delta_{d-r})} - \Upsilon_{P_1(\partial \text{star}_{\partial \Delta_d}(\Delta_{d-r}))} + \Upsilon_{\partial \text{star}_{\partial \Delta_d}(\Delta_{d-r})} \cdot b) \\ &= \Upsilon_{\partial T_i} - (\ell_{\text{star}_{\partial \Delta_d}(\Delta_{d-r})}^{\Upsilon} + \Upsilon_{\partial \text{star}_{\partial \Delta_d}(\Delta_{d-r})} \cdot b). \end{aligned} \tag{16}$$

The chains of T_{i+1} that are not contained in T_i are exactly those are contained in the interior of $\text{star}_{\Delta_d}(\Delta_{r-1})$ which may be counted by

$$\Upsilon_{\text{star}_{\partial \Delta_d}(\Delta_{r-1})} - \Upsilon_{\partial \text{star}_{\partial \Delta_d}(\Delta_{r-1})} \cdot a. \tag{17}$$

Now, remembering that $\partial\text{star}_{\partial\Delta_d}(\Delta_{r-1}) \cong \partial\text{star}_{\partial\Delta_d}(\Delta_{d-r})$, we sum (16) and (17) to compute $\Phi_{\partial T_{i+1}}$.

$$\begin{aligned}\Upsilon_{\partial T_{i+1}} &= \Upsilon_{\partial T_i} - (\ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-r})}^{\Upsilon} + \Upsilon_{\partial\text{star}_{\partial\Delta_d}(\Delta_{d-r})} \cdot b) + \Upsilon_{\text{star}_{\partial\Delta_d}(\Delta_{r-1})} - \Upsilon_{\partial\text{star}_{\partial\Delta_d}(\Delta_{r-1})} \cdot a \\ \Psi_{\partial T_{i+1}} &= \Psi_{\partial T_i} - (\ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-r})}^{\Psi} + \Psi_{\partial\text{star}_{\partial\Delta_d}(\Delta_{d-r})} \cdot b) + \Psi_{\text{star}_{\partial\Delta_d}(\Delta_{r-1})} - \Psi_{\partial\text{star}_{\partial\Delta_d}(\Delta_{r-1})} \cdot (a-b) \\ &= \Psi_{\partial T_i} - \ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-r})}^{\Psi} + \Psi_{\text{star}_{\partial\Delta_d}(\Delta_{r-1})} - \Psi_{\partial\text{star}_{\partial\Delta_d}(\Delta_{r-1})} \cdot a \\ &= \Psi_{\partial T_i} + \ell_{\text{star}_{\partial\Delta_d}(\Delta_{r-1})}^{\Psi} - \ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-r})}^{\Psi} \\ \Phi_{T_{i+1}} &= \Phi_{\partial T_i} + \ell_{\text{star}_{\partial\Delta_d}(\Delta_{r-1})}^{\Phi} - \ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-r})}^{\Phi}.\end{aligned}$$

In particular, consider the final terms $\ell_{\text{star}_{\partial\Delta_d}(\Delta_{r-1})}^{\Phi} - \ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-r})}^{\Phi}$. If Π is a stacked polytope, then $r = 1$ since F_{i+1} will intersect exactly one facet of T_i . Given a stacked Π , this holds for all $1 \leq i \leq k-1$.

Claim: $\ell_{\text{star}_{\partial\Delta_d}(\Delta_{r-1})}^{\Phi} - \ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-r})}^{\Phi}$ is maximized when $r = 1$.

Note that if this claim holds, the theorem follows. For all $i < k$, the cd -index of ∂T_{i+1} will increase from ∂T_i by at most the amount it would if we were stacking a simplex onto ∂T_i .

By applying Lemma 7.23 to a shelling of a d -simplex, we have that for all $1 \leq i \leq d-1$

$$\ell_{\text{star}_{\partial\Delta_d}(\Delta_i)}^{\Phi} \leq \ell_{\text{star}_{\partial\Delta_d}(\Delta_{i-1})}^{\Phi}.$$

That is,

$$\ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-1})}^{\Phi} \leq \ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-2})}^{\Phi} \leq \dots \leq \ell_{\text{star}_{\partial\Delta_d}(\Delta_1)}^{\Phi} \leq \ell_{\text{star}_{\partial\Delta_d}(\Delta_0)}^{\Phi}.$$

Since all of the above stars are near-Gorenstein complexes, their local cd -indexes are non-negative and it follows that for any integer $1 \leq r \leq d$

$$\ell_{\text{star}_{\partial\Delta_d}(\Delta_{r-1})}^{\Phi} - \ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-r})}^{\Phi} \leq \ell_{\text{star}_{\partial\Delta_d}(\Delta_{r-1})}^{\Phi} - \ell_{\text{star}_{\partial\Delta_d}(\Delta_{d-r})}^{\Phi}.$$

It follows that the claim holds and our theorem is an immediate consequence. \square

We may easily find a rough upper bound on the cd -index of a simplicial d -polytope with k -facets by the following construction.

Corollary 7.25. Let P be a simplicial polytope with k facets. Then the cd index of P has the cd -index of $S_d(k)$ as an upper bound. That is,

$$\Phi_P \leq \Phi_{S_d(k)}.$$

Proof: Note that we may triangulate P into some near-Gorenstein complex Γ by placing a point x on its interior, and then defining the faces of Γ to be

$$\Gamma := \mathcal{F}(P) \cup \{\text{conv}(F \cup x) \mid F \in \mathcal{F}(P)\}$$

where $\mathcal{F}(P)$ is the face set of P and ‘conv’ denotes the convex hull. The facets of Γ are exactly the simplices $\text{conv}(F \cup x)$ for F a facet of P . Since P is a polytope, it has a shelling F_1, F_2, \dots, F_k . It is simple to see that this induces a shelling $\text{conv}(F_1 \cup x), \text{conv}(F_2 \cup x) \dots \text{conv}(F_k \cup x)$ of Γ . So P is k -shellably triangulizable and we can apply our previous theorem. \square

This is the most trivial possible bound that arises from our theorem and is likely a much worse upper bound than a cyclic polytope. However, if one was studying a class of specific polytopes that triangulates nicely (i.e. into a number of simplices significantly smaller than the number of its facets) into a shellable complex, Theorem 7.24 may yet generate some useful bounds.

8 Morphisms between the cd -index, the toric h -vector and the local h -vector

In the previous chapter, we discussed the classical h -vector for simplicial complexes. We now investigate the *toric* and *local* h -vector. These were introduced by Richard Stanley in [17] and [18] and have been the subject of active research ever since. The toric h -vector is an extension of the h -vector to arbitrary Eulerian posets that maintains most of its ‘nice’ properties. The toric h -vector is a tool that bridges between polyhedral combinatorics and algebraic geometry. The local h -vector is used in decomposing the h -vector relative to a subdivision. This decomposition behaves analogously to our cd -index subdivision decomposition. In [2], Bayer and Ehrenborg define an algebra morphism that maps the cd -index of a poset P to the toric h -vector of P . Our ultimate goal is to demonstrate that this morphism gives a natural map between the cd -index subdivision decomposition and the h -vector subdivision decomposition, and use this to connect the local cd -index to the local h -vector.

We begin by defining the local h -vector with respect to a simplex. We eventually will generalize to Eulerian posets and the toric h -vector. Though doing so immediately would perhaps lead to less exposition, the theory of the toric h -vector and local h -vector for arbitrary Eulerian posets becomes exceedingly technical very quickly. We encourage the reader to study the simplicial case to develop intuition before moving on to the general case.

To align ourselves with the notation of [18], recall that given a vertex set V , we may denote the simplex with vertices labeled by V as 2^V . Furthermore we may identify faces uniquely by their vertex set, giving us a face for any $W \subseteq V$. This definition is consistent with that of an abstract simplicial complex.

Definition 8.1. [18] Let V be some vertex set with $|V| = d$. Given a simplicial subdivision Γ of 2^V , we define the polynomial $\ell_V(\Gamma, x) := \ell_0 + \ell_1 x + \dots + \ell_d x^d$ by the following equation:

$$h(\Gamma, x) = \sum_{W \subseteq V} \ell_W(\Gamma_W, x). \quad (18)$$

We call $\ell_V(\Gamma, x)$ the *local h -polynomial* of Γ . Note that $\ell_V(\Gamma, x)$ is implicitly dependent on the base simplex 2^V of the subdivision. We analogously call $\ell_V(\Gamma) := [\ell_0, \ell_1, \dots, \ell_d]$ the *local h -vector* of Γ .

It may not be immediately obvious that this actually defines the local h -polynomial. However, this is actually its most concise expression. Inverting the previous formula using the method of inclusion-exclusion, we explicitly find that

$$\ell_V(\Gamma, x) = \sum_{W \subseteq V} (-1)^{|V \setminus W|} h(\Gamma_W, x). \quad (19)$$

We present some simple low-dimension examples from Stanley. [18]. These may be

calculated relatively easily by recursive techniques. Things get more complicated in dimensions 3 and larger.

Example 8.2. Let Γ be a subdivision of 2^V .

- $\ell_\emptyset(\emptyset, x) = 1$.
- For $|V| > 0$, consider the trivial subdivision $\Gamma = 2^V$. Then $\ell_V(2^V, x) = 0$.
- For $|V| = 2$, 2^V is a line segment. Suppose Γ adds t interior points to the line. Then $\ell_V(\Gamma, x) = tx$.
- For $|V| = 3$, calculate $h(\Gamma, x) = h_0 + h_1x + h_2x^2 + h_3x^3$. Then $\ell_V(\Gamma, x) = h_2x + h_3x^2$.

We use the above for reference to work through a local h -vector decomposition of the barycentric subdivision of a triangle.

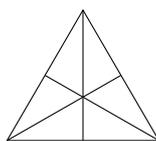


Figure 31: The barycentric subdivision of a triangle.

Example 8.3. Consider a triangle Δ_2 and let Γ be the barycentric subdivision of it pictured in figure 30. Counting faces, we have $f(\Gamma) = [1, 7, 12, 6]$ (the empty face, seven vertices, 12 edges and 6 2-faces). We may compute that

$$h(\Gamma, x) = 1 + 4x + x^2.$$

Any vertex V is not subdivided and has a local h -vector of 0. Note that any edge E of Δ_2 has exactly one vertex introduced into its interior by Γ . So $\ell_E(\Gamma_E, x) = x$. And from the final bullet of our previous example, for the single maximal face F of Δ_2 we have $\ell_F(\Gamma_F, x) = x + x^2$. Plugging these values into equation (18) gives

$$\begin{aligned} h(\Gamma, x) &= \ell_\emptyset(\Gamma_\emptyset, x) + 3\ell_V(\Gamma_V, x) + 3\ell_E(\Gamma_E, x) + \ell_F(\Gamma_V, F) \\ &= 1 + 3 \cdot 0 + 3 \cdot x + (x + x^2) \\ &= 1 + 4x + x^2 \end{aligned}$$

and verifies our earlier computation of $h(\Gamma, x)$.

Note that every local h -vector computed so far has been symmetrical. That is for any local h -vector $\ell(\Gamma) = [\ell_0, \ell_1, \dots, \ell_d]$ we have $\ell_i = \ell_{d-i}$ for all $0 \leq i \leq d$. This is not a coincidence. However, we refrain from stating this as a theorem until we reach the general case.

Restricting ourselves only to subdivisions of a simplex is unsatisfying. It turns out that the local h -vector may be applied to subdivisions simplicial complexes in general. Stanley presents the following theorem in [18].

Theorem 8.4. Let Δ be a pure d -dimensional complex with a simplicial subdivision Γ . Then the following equation holds.

$$h(\Gamma, x) = \sum_{F \in \Delta} \ell_F(\Gamma_F, x) h(\text{link}_\Delta F, x). \quad (20)$$

Note that if Δ is a simplex, $h(\text{link}_\Delta F, x) = 1$ for all $F \in \Delta$ and the above theorem reduces to equation (16), our definition of the local h -vector. Recall that the face poset of $\text{link}_\Delta F$ is exactly the interval $[F, \widehat{1}]_\Delta$ (that is, all the faces of Δ that contain F). If we replace every occurrence of the h -polynomial with the cd -index, this formula appears to be very similar to the cd -index subdivision decomposition. There exists a general version of this theorem (that we will see shortly) that makes the resemblance even more apparent.

We now introduce the toric and local h -vector in full generality. We mostly align ourselves with the conventions and notation of [10].

Definition 8.5. Let P be an Eulerian poset of rank n . We define the g -polynomial $g(P, x)$ of P recursively as follows:

1. If $n = 0$, $g(P, x) = 1$.
2. If $n > 0$, $g(P, x) \in \mathbb{Z}[x]$ is the unique polynomial with degree $< n/2$ that satisfies

$$x^n g(P, 1/x) = \sum_{\sigma \in P} g([\widehat{0}, \sigma], x) (x-1)^{n-\rho(\sigma)}. \quad (21)$$

We define the g -vector of P by $g(P) := [g_0, g_1, \dots, g_m]$ where $m = \lfloor (n-1)/2 \rfloor$.

Definition 8.6. Let P be a lower Eulerian poset of rank n . We then define the h -polynomial $h(P, x)$ of P to be the unique polynomial $h(P, x) \in \mathbb{Z}[x]$ satisfying

$$x^n h(P, 1/x) = \sum_{\sigma \in P} g([\widehat{0}, \sigma], x) (x-1)^{n-\rho(\sigma)}. \quad (22)$$

We define the h -vector of P by $h(P) := [h_0, h_1, \dots, h_n]$.

Note that if P is Eulerian, we have $g(P, x) = h(P, x)$. Though notationally dense, it is easy to see that this h -polynomial reduces to the one previously defined for simplicial complexes (as in Definition 7.2) after applying the following proposition.

Proposition 8.7. [21] For any rank n boolean algebra B_n , we have $g(B_n, x) = 1$.

If P is the face poset of a simplicial complex, then $[\widehat{0}, \sigma] \cong B_k$ for any $\sigma \in P$ and for some $k \leq n$. So all g terms disappear and we have

$$\begin{aligned} x^n h(P, 1/x) &= \sum_{\sigma \in P} (x-1)^{n-\rho(\sigma)} \\ &= \sum_{i=0}^n f_{i-1} (x-1)^{n-i} \end{aligned}$$

which is exactly the definition of the h -polynomial presented in equation (10).

Definition 8.8. Let P be a Eulerian poset of rank $n > 0$. Then we call $h(P \setminus \{\widehat{1}\}, x)$ the *toric h -vector* of P .

Example 8.9. [21] Consider the boolean algebra on $d+1$ elements, B_{d+1} where $d \geq 0$. Then

$$h(B_{d+1} \setminus \{\widehat{1}\}) = 1 + x + \cdots + x^d.$$

Recall that the h -vector of the boundary complex of a polytope holds a number of particularly significant combinatorial properties (we spent much of Chapter 7 discussing them!). The toric h -vector generalizes many of these properties to Eulerian posets. In particular, for any Eulerian P , $h(P \setminus \{\widehat{1}\}, x)$ is symmetric and we may generalize the Dehn-Sommerville relations beyond simplicial polytopes.

Theorem 8.10. (*The Generalized Dehn-Sommerville Equations*) Let P be an Eulerian poset of rank $d+1$ with $h(P \setminus \{\widehat{1}\}, x) = h_0 + h_1x + \cdots + h_dx^d$. Then for all $0 \leq i \leq d$,

$$h_i = h_{d-i}.$$

Proof: This follows from a Möbius inversion argument. See Stanley's textbook [21].

We also mentioned that the h -vector of the boundary complex of a simplicial polytope is both positive and unimodal in the lower bound theorem. Does this result hold for the toric h -vector? In the case of the face posets of rational polytopes, Stanley demonstrated the affirmative [17].

Theorem 8.11. (*Stanley*) Let P be the face poset of a rational d -polytope (that is, a polytope such that all its vertices have rational coordinates). Consider its toric h -vector $h(\partial P) = [h_0, h_1, \dots, h_d]$. Then, for $m = \lfloor d/2 \rfloor$ we have

$$1 \leq h_0 \leq h_1 \leq \cdots \leq h_m.$$

This result comes from interpreting the h_i coefficients as the dimensions of the even cohomology groups of a projective toric variety associated with the normal fan of P [10]. Much of the modern interest in the theory of polytopes and fans is motivated by their

connections to toric varieties and algebraic geometry.

We now define yet another type of subdivision, originally presented by Katz and Stapledon [10].

Definition 8.12. Let $\phi : \Gamma \rightarrow \Pi$ be an order preserving and rank increasing function between lower Eulerian posets Γ and Π . We say that ϕ is *strongly surjective* if

- ϕ is surjective.
- For all $z \in \Gamma$ and $x \in \Pi$ such that $\phi(z) \leq x$, there exists some $y \in \Gamma$ such that $z \leq y$, $\phi(y) \leq x$ and $\rho_\Gamma(y) = \rho_\Pi(x)$.

In other words, if z is carried by x in Π , there must be some y in Γ of equal rank to x that is ordered above z .

Definition 8.13. Let $\phi : \Gamma \rightarrow \Pi$ be an order preserving, rank increasing and strongly surjective function between lower Eulerian posets Γ and Φ . We say ϕ is a *strong formal subdivision* if and only if the following condition holds for all $z \in \Gamma$ and $x \in \Pi$ such that $\phi(z) \leq x$:

$$\sum_{\substack{\phi(y) \leq x \\ z \leq y}} (-1)^{\rho_\Pi(x) - \rho_\Gamma(y)} = \begin{cases} 1 & \text{if } \phi(z) = x \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

This generalizes the idea of a formal subdivision as defined by Stanley in [18].

Lemma 8.14. Any strongly Eulerian subdivision is a strong formal subdivision.

Proof: Recall the definition of a strongly Eulerian subdivision. That is, given a surjective order preserving function $\phi : \Gamma \rightarrow \Pi$ between rank n lower Eulerian posets, for any $x \in \Pi$ we have the following:

- $P_1(\phi^{-1}[\widehat{0}, x])$ is a near-Eulerian poset.
- $\phi^{-1}[\widehat{0}, x]$ has rank equal to $\rho_\Pi(x)$.

The second condition immediately implies strong surjectivity. Now let us verify that equation (23) holds.

Let $x \in \Pi$ be an element of rank n , and let $z, w \in P_1(\phi^{-1}[\widehat{0}, x])$. Because $P_1(\phi^{-1}[\widehat{0}, x])$ is near-Eulerian, the following equation holds.

$$\sum_{z \leq y \leq w} (-1)^{\rho(w) - \rho(y)} = \begin{cases} 1 & \text{if } w = \widehat{1} \text{ and } \phi(z) \neq x \\ 0 & \text{otherwise.} \end{cases}$$

Take $w = \widehat{1}$ and we may calculate

$$\sum_{z \leq y \leq \widehat{1}} (-1)^{n+1-\rho(y)} = \begin{cases} 1 & \text{if } \phi(z) \neq x \\ 0 & \text{otherwise.} \end{cases}$$

$$\sum_{z \leq y < \widehat{1}} (-1)^{n+1-\rho(y)} = \begin{cases} 0 & \text{if } \phi(z) \neq x \\ -1 & \text{otherwise.} \end{cases}$$

However, elements $y \in P_1(\phi^{-1}[\widehat{0}, x])$ such that $z \leq y < \widehat{1}$ are exactly the elements of Γ such that $z \leq y$ and $\phi(y) \leq x$. Noting that $\rho_\Pi(x) = n$ and dividing by -1 gives us the result that

$$\sum_{\substack{\phi(y) \leq x \\ z \leq y}} (-1)^{\rho_\Pi(x) - \rho_\Gamma(y)} = \begin{cases} 0 & \text{if } \phi(z) \neq x \\ 1 & \text{otherwise.} \end{cases}$$

which is equivalent to equation (24) and completes the lemma. \square

It immediately follows that Gorenstein* subdivisions are strong formal subdivisions. Additionally, any CW -subdivision (or polyhedral subdivision) is a strong formal subdivision [17].

We now present the local h -vector in its full generality, as defined by [10].

Definition 8.15. Let $\phi : \Gamma \rightarrow \Pi$ be a strong formal subdivision between lower Eulerian Γ and Eulerian Π . We recursively define the *local h -polynomial* $\ell_\Pi(\Gamma, x) \in \mathbb{Z}[x]$ to be the polynomial satisfying

$$h(\Gamma, x) = \sum_{\sigma \in \Pi} \ell_{[\widehat{0}, \sigma]}(\Gamma_\sigma, x) g([\sigma, \widehat{1}], x). \quad (24)$$

For $\ell_\Pi(\Gamma, x) = \ell_0 + \ell_1 x + \dots + \ell_n$, we write $\ell_\Pi(\Gamma) = [\ell_0 \dots \ell_n]$ and call $\ell_\Pi(\Gamma)$ the local h -vector of Γ (relative to Π).

If Π is a simplex, all g terms are 1 and this reduces to our original definition of the local h -polynomial in equation (19). Much like that definition, this feels ultimately unsatisfying because we would like to describe $\ell_\Pi(\Gamma, x)$ explicitly. However, this presentation of the local h -polynomial is much cleaner than its explicit definition, which we now present for posterity [10].

Proposition 8.16. Given $\phi : \Gamma \rightarrow \Pi$, a strong formal subdivision between lower Eulerian Γ and Eulerian Π , we may define $\ell_\Pi(\Gamma, x)$ by

$$\ell_\Pi(\Gamma, x) = \sum_{\sigma \in \Pi} h(\Gamma_\sigma, x) (-1)^{\rho(\sigma, \widehat{1})} g([\sigma, \widehat{1}]^*, x)$$

where $[\sigma, \widehat{1}]^*$ is the dual poset of $[\sigma, \widehat{1}]$ (that is, the poset on the same underlying set with all order relations reversed).

Demonstrating why these two formulations of the local h -vector are equivalent is highly technical. We have been ignoring much of the underlying theory whenever possible in an effort to be concise. At its most general, the h -vector may be studied from the perspective of kernels and acceptable functions on the incidence algebra of a poset. Kernels and acceptability operators are a broad theory generalizing the study of elements of an algebra that are invariant with respect to some involution. For a detailed treatment, see Katz and Stapledon [10] and/or Stanley [18].

Recall that we earlier discussed the symmetry of the local h -vector. We now formalize that statement [10].

Theorem 8.17. Let $\phi : \Gamma \rightarrow \Pi$ be a strong formal subdivision, where Γ is a rank n lower Eulerian poset and Π is an Eulerian poset. Then

$$\ell_{\Pi}(\Gamma, x) = x^n \ell_{\Pi}(\Gamma, 1/x).$$

So we have a symmetry in the local h -vector analogous to the Dehn-Sommerville equations. Can we say anything about non-negativity or unimodality of the local h -vector? In some cases, yes. Given a polyhedral complex Π , we say that Γ is a *regular subdivision* if it arises from projecting the ‘lower faces’ of some convex polytope P on to Π (we refrain from a formal definition). Stanley demonstrated the following [18].

Theorem 8.18. (*Stanley*) Let P be a rational d -polytope. Suppose Γ is some rational polytopal subdivision of P (that, is all vertices of the subdivision may be realized with rational coordinates). Let $\ell_{\Pi}(\Gamma, x) = \ell_0 + \ell_1 x + \cdots + \ell_d$ Then the following holds:

1. $\ell_i \geq 0$ for all i .
2. If additionally Γ is a regular subdivision, we have $0 \leq \ell_0 \leq \ell_1 \leq \cdots \leq \ell_m$ where $m = \lfloor d/2 \rfloor$.

Like his proof of the unimodality of the toric h -vector for rational polytopes, this is not a very combinatorial proof and instead goes through the intersection cohomology of toric varieties.

We now aim to demonstrate a correspondence between the local cd -index and the local h -vector via an algebra morphism developed by Bayer and Ehrenborg in [2]. The following coproduct structure is described in depth in [7]. Note that we differ from their notation slightly (namely, we hesitate to denote a coproduct as Δ because we have spent most of this document using that to refer to simplices).

Definition 8.19. Let $\mathbb{Z}\langle a, b \rangle$ denote the algebra of non-commutative indeterminates a and b over \mathbb{Z} .

- Define $M : \mathbb{Z}\langle a, b \rangle \otimes \mathbb{Z}\langle a, b \rangle \rightarrow \mathbb{Z}\langle a, b \rangle$ to multiply the tensors together. That is, for $u, v \in \mathbb{Z}\langle a, b \rangle$ we have

$$M(u \otimes v) = uv.$$

- We define a coproduct $C : \mathbb{Z}\langle a, b \rangle \rightarrow \mathbb{Z}\langle a, b \rangle \otimes \mathbb{Z}\langle a, b \rangle$ by the following. For any monomial $w = w_1 w_2 \dots w_n$ we define

$$C(w_1 \dots w_n) = \sum_{i=1}^n w_1 \dots w_{i-1} \otimes w_{i+1} \dots w_n$$

and extend to arbitrary elements of $\mathbb{Z}\langle a, b \rangle$ by linearity.

For our coproduct, we may use Sweedler notation to improve readability at the expense of utility and instead write for any element $w \in \mathbb{Z}\langle a, b \rangle$,

$$C(w) = \sum_w w_{(1)} \otimes w_{(2)}.$$

For any poset P , we have the ab -index $\Psi_P \in \mathbb{Z}\langle a, b \rangle$.

We first need to establish some notation and define a few more maps before we can express Bayer and Ehrenborg's morphism [2].

- Let $U_{\leq m}[\cdot] : \mathbb{Z}[x] \rightarrow \mathbb{Z}[x]$ denote the map that truncates all terms of degree larger than m . That is, for $f(x) = \sum_{i=0}^k a_i x^i$ we have

$$U_{\leq m}[f(x)] = \sum_{i=0}^{\min\{m, k\}} a_i x^i$$

- We define the algebra map $\kappa : \mathbb{Z}\langle a, b \rangle \rightarrow \mathbb{Z}[x]$ by $\kappa(a) = x - 1$, $\kappa(b) = 0$ and extend linearly for all ab -words and elements of $\mathbb{Z}\langle a, b \rangle$. Note that this implies for any poset P of rank n , $\kappa(\Psi_P) = (x - 1)^{n-1}$.

We may now present a morphism between the ab -index and the toric h -vector [2].

Theorem 8.20. (*Bayer and Ehrenborg*) Let f and g be linear maps $f, g : \mathbb{Z}\langle a, b \rangle \rightarrow \mathbb{Z}[x]$ defined by the following intertwined recursive definition:

- For any monomial v ,

$$\begin{aligned} f(v) &= \kappa(v) + M \circ (g \otimes \kappa) \circ C(v) \\ &= \kappa(v) + \sum_v g(v_{(1)}) \kappa(v_{(2)}) \end{aligned}$$

and extend linearly.

- For any monomial v of degree n with $m = \lfloor n/2 \rfloor$ set

$$g(v) = U_{\leq m}[(1-x)f(v)]$$

and extend linearly.

Then, for any graded poset P with $\widehat{0}$ and $\widehat{1}$, $f(\Psi_P) = h(P \setminus \{\widehat{1}\}, x)$ and $g(\Psi_P) = g(P, x)$ where $h(P \setminus \{\widehat{1}\}, x)$ is the toric h -polynomial of P and $g(P, x)$ is the g -polynomial of P .

Proof: This may be proved by a simple inductive argument. See [2] for details.

We claim that these morphisms are well behaved with respect to our ab -index subdivision decomposition.

Lemma 8.21. Let $\phi : \widehat{\Pi} \rightarrow \Pi$ be a strong Eulerian subdivision between Eulerian posets. For any $\sigma \in \Pi$ we have

$$f(\ell_{\widehat{\sigma}}^{\Psi}) = \ell_{\sigma}(\widehat{\sigma}, x).$$

Proof: Recall that $\ell_{\widehat{\sigma}}^{\Psi} = \Psi_{\widehat{\sigma}} - \Psi_{P_1(\partial\widehat{\sigma})}$. So by applying our previous theorem 8.20, we have

$$\begin{aligned} f(\ell_{\widehat{\sigma}}^{\Psi}) &= f(\Psi_{\widehat{\sigma}}) - f(\Psi_{P_1(\partial\widehat{\sigma})}) \\ &= h(\widehat{\sigma} \setminus \{\widehat{1}\}, x) - h(\partial\widehat{\sigma}, x). \end{aligned}$$

Since strong formal subdivisions restricted to order ideals are still strong formal subdivision [10], it follows that by restricting ϕ to σ we have that $\widehat{\sigma} \setminus \{\widehat{1}\}$ and $\partial\widehat{\sigma}$ are both strong formal subdivisions of σ . So we may apply a local h -vector decomposition relative to the face poset of σ , giving us

$$f(\ell_{\widehat{\sigma}}^{\Psi}) = \sum_{F \in \sigma} \ell_F(\widehat{\sigma}_F, x) \cdot h([F, \widehat{1}]_{\sigma}, x) - \sum_{F \in \sigma} \ell_F((\partial\widehat{\sigma})_F, x) \cdot h([F, \widehat{1}]_{\sigma}, x). \quad (25)$$

Note that for any $F \in \sigma$ such that $F \neq \widehat{1}$, we have $F \in \partial\widehat{\sigma}$ and it follows that $\widehat{\sigma}_F \cong \partial\widehat{\sigma}_F$. So for $F \neq \widehat{1}$ we have

$$\ell_F(\widehat{\sigma}_F, x) = \ell_F((\partial\widehat{\sigma})_F, x)$$

So all terms with $F \neq \widehat{1}$ cancel in equation (26). Also, $\widehat{1}_{\sigma}$ is only trivially subdivided with respect to $\partial\widehat{\sigma}$ so it follows that $\ell_{\widehat{1}}((\partial\widehat{\sigma})_{\widehat{1}}, x) = 0$.

The only term left standing is

$$f(\ell_{\widehat{\sigma}}^{\Psi}) = \ell_{\widehat{1}}(\widehat{\sigma}_{\widehat{1}}, x) \cdot h([\widehat{1}, \widehat{1}]_{\sigma}, x) = \ell_{\sigma}(\widehat{\sigma}, x) \cdot 1$$

and the result follows. \square

Since Π is Eulerian, it also follows that for any $\sigma \in \Pi$, $f(\Phi_{[\sigma, \widehat{1}]}) = h([\sigma, \widehat{1}], x)$. We may summarize the result into the following corollary.

Corollary 8.22. Let $\phi : \widehat{\Pi} \rightarrow \Pi$ be a strong Eulerian subdivision, with $f : \mathbb{Z}\langle a, b \rangle \rightarrow \mathbb{Z}[x]$ defined as above. Then the following diagram holds:

$$\begin{array}{ccc}
 \Psi_{\widehat{\Pi}} & = & \sum_{\sigma \in \Pi} \ell_{\widehat{\sigma}}^{\Psi} \cdot \Psi_{[\sigma, \widehat{1}]} \\
 \downarrow f & & \downarrow f \quad \downarrow f \\
 h(\widehat{\Pi} \setminus \{\widehat{1}\}, x) & = & \sum_{\sigma \in \Pi \setminus \{\widehat{1}\}} \ell_{\widehat{\sigma}}(\widehat{\sigma}, x) \cdot h([\sigma, \widehat{1}], x)
 \end{array}$$

Consider restricting $\mathbb{Z}\langle a, b \rangle$ to the subalgebra $\mathbb{Z}\langle c, d \rangle$ where $c = a + b$ and $d = ab + ba$. We then have $\kappa(c) = a + b$ and $\kappa(d) = 0$ and f and g remain well defined. So the above morphisms are also valid for $\Phi_{\widehat{\Pi}}, \ell_{\widehat{\sigma}}^{\Phi}$ and $\Phi_{[\sigma, \widehat{1}]}$ [2]. The morphism f gives a direct correspondence between the cd -index subdivision decomposition and the local h -vector subdivision decomposition.

9 Closing Remarks

We finish with a few open questions. In general, the cd -index contains strictly more combinatorial data than the h -polynomial and we can not recover Φ_P from $h(P, x)$. However, when P is simplicial its flags are entirely determined by its h -vector [18]. Defining some morphism from $\mathbb{Z}[x]$ to $\mathbb{Z}\langle a, b \rangle$ or $\mathbb{Z}\langle c, d \rangle$ that takes the h -polynomial of a simplicial poset to the ab or cd -index while behaving well relative to local decomposition of a subdivision would be a nice construction.

In [10], Katz and Stapledon define the *mixed* h -polynomial of a strong formal subdivision $\phi : \widehat{\Pi} \rightarrow \Pi$. This is a two variable specialization of the h -polynomial that encodes the h -polynomial of $\widehat{\Pi}$ along while tracking the *excess* of each element of $\widehat{\Pi}$ relative to the subdivision Φ . Finding a map from the cd -index to the mixed h -polynomial, or developing a mixed analog of the cd -index that behaves analogously with respect to the local subdivision decomposition would be a natural continuation of the work of this thesis.

References

- [1] Margaret M. Bayer and Louis J. Billera. Generalized Dehn-Sommerville realtions for polytopes, spheres and Eulerian partially ordered sets. *Inventiones mathematicae*, 79:143–157, 1985.
- [2] Margaret M. Bayer and Richard Ehrenborg. The Toric h-vectors of Partially Ordered Sets. *Transactions of the American Mathematical Society*, 352(10):4515–4531, 2000.
- [3] Margaret M. Bayer and Andrew Klapper. A New Index for Polytopes. *Discrete and Computational Geometry*, 6(1):33–47, 1991.
- [4] Louis Billera and Richard Ehrenborg. Monotonicity of the cd-index for polytopes. *Mathematische Zeitschrift*, 233:421–441, 2000.
- [5] Heinz Bruggesser and Peter Mani. Shellable decompositions of cells and spheres. *Math. Scand.*, 29:197–205, 1971.
- [6] Richard Ehrenborg and Kalle Karu. Decomposition theorem for the cd-index of Gorenstein* posets. *Journal of Algebraic Combinatorics*, 26(225-251), 2007.
- [7] Richard Ehrenborg and Margaret Readdy. Coproducts and the cd-index. *Journal of Algebraic Combinatorics*, 8:273–299, 1998.
- [8] Branko Grünbaum. *Convex Polytopes*. Number 221 in Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 2000.
- [9] Kalle Karu. The cd-index of fans and posets. *Compositio Math*, pages 701–718, 2006.
- [10] Eric Katz and Alan Stapledon. Local h-polynomials, Invariants of Subdivisions, and Mixed Ehrhart Theory. Preprint: arXiv 1411.7736v1, Nov 2014.
- [11] Peter McMullen. The Maximum Numbers of Faces of a Convex Polytope. *Mathematika*, 17(2):179–184, 1970.
- [12] Peter McMullen and David Walkup. A generalized lower-bound conjecture for simplicial polytopes. *Mathematika*, 18:264–273, 1971.
- [13] James R. Munkres. *Elements of Algebraic Topology*. Addison-Wesley, 1984.
- [14] Satoshi Murai and Eran Nevo. On the generalized lower bound conjecture for polytopes and spheres. *Acta Mathematica*, 210(1):185–202, 2013.
- [15] Duncan M'Laren Young Sommerville. The relations connecting the angle-sums and volume of a polytope in space of n dimensions. *Proceedings of the Royal Society of London, Series A*, 115:103–119, 1927.

- [16] Richard P. Stanley. The Number of Faces of a Simplicial Convex Polytope. *Advances in Mathematics*, 35:236–238, 1980.
- [17] Richard P. Stanley. Generalized H-vectors, Intersection Cohomology of Toric Varieties, and Related Results. volume 11 of *Advance Studies in Pure Mathematics: Combinatorics and Commutative Algebra*, pages 187–213. North-Holland, 1987.
- [18] Richard P. Stanley. Subdivisions and Local h-Vectors. *Journal of the American Mathematical Society*, 5(4):805–851, 1992.
- [19] Richard P. Stanley. A Survey of Eulerian Posets. In Tibor Bisztriczky et al., editor, *Polytopes: Abstract, Convex and Computational*, Nato Science Series C:, pages 301–333. Kluwer Academic Publishers, Netherlands, 1994.
- [20] Richard P. Stanley. Flag f -vectors and the cd -index. *Mathematische Zeitschrift*, 216:483–499, 1994.
- [21] Richard P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, second edition, 2012.
- [22] Günter M. Ziegler. *Lectures on Polytopes*. Springer, 2007.