Brick Generation and Conformal Subgraphs

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The results in Chapters 2 and 3 are based on the paper [KM16] co-authored with my supervisor U. S. R. Murty.

Abstract

A nontrivial connected graph is *matching covered* if each of its edges lies in a perfect matching. Two types of decompositions of matching covered graphs, namely *ear decompositions* and *tight cut decompositions*, have played key roles in the theory of these graphs. Any tight cut decomposition of a matching covered graph results in an essentially unique list of special matching covered graphs, called *bricks* (which are nonbipartite and 3-connected) and *braces* (which are bipartite).

A fundamental theorem of Lovász (1983) states that every nonbipartite matching covered graph admits an ear decomposition starting with a bi-subdivision of K_4 or of the triangular prism $\overline{C_6}$. This led Carvalho, Lucchesi and Murty (2003) to pose two problems: (i) characterize those nonbipartite matching covered graphs which admit an ear decomposition starting with a bi-subdivision of K_4, and likewise, (ii) characterize those which admit an ear decomposition starting with a bi-subdivision of $\overline{C_6}$.

In the first part of this thesis, we solve these problems for the special case of planar graphs. In Chapter 2, we reduce these problems to the case of bricks, and in Chapter 3, we solve both problems when the graph under consideration is a planar brick.

A nonbipartite matching covered graph G is *near-bipartite* if it has a pair of edges α and β such that $G - \{\alpha, \beta\}$ is bipartite and matching covered; examples are K_4 and $\overline{C_6}$. The first nonbipartite graph in any ear decomposition of a nonbipartite graph is a bi-subdivision of a near-bipartite graph. For this reason, near-bipartite graphs play a central role in the theory of matching covered graphs. In the second part of this thesis, we establish generation theorems which are specific to near-bipartite bricks.

Deleting an edge e from a brick G results in a graph with zero, one or two vertices of degree two, as G is 3-connected. The *bicontraction* of a vertex of degree two consists of contracting the two edges incident with it; and the *retract* of $G - e$ is the graph J obtained from it by bicontracting all its vertices of degree two. The edge e is *thin* if J is also a brick. Carvalho, Lucchesi and Murty (2006) showed that every brick, distinct from K_4, $\overline{C_6}$ and the Petersen graph, has a thin edge.

In general, given a near-bipartite brick G and a thin edge e, the retract J of $G - e$ need not be near-bipartite. In Chapter 5, we show that every near-bipartite brick G, distinct from K_4 and $\overline{C_6}$, has a thin edge e such that the retract J of $G-e$ is also near-bipartite. Our theorem is a refinement of the result of Carvalho, Lucchesi and Murty which is appropriate for the restricted class of near-bipartite bricks.

For a simple brick G and a thin edge e, the retract of $G - e$ may not be simple. It was established by Norine and Thomas (2007) that each simple brick, which is not in any of five well-defined infinite families of graphs, and is not isomorphic to the Petersen graph, has a thin edge such that the retract J of $G - e$ is also simple.

In Chapter 6, using our result from Chapter 5, we show that every simple near-bipartite brick G has a thin edge e such that the retract J of $G - e$ is also simple and near-bipartite, unless G belongs to any of eight infinite families of graphs. This is a refinement of the theorem of Norine and Thomas which is appropriate for the restricted class of near-bipartite bricks.

Acknowledgements

No words can truly express my gratitude to my supervisors Joseph Cheriyan and U. S. R. Murty. Throughout these five years, they have invested tremendous effort in mentoring me, and in doing so, they have played somewhat complementary roles. Professor Murty taught me the beautiful subject of matching theory, and inspired me to seek new results in this area; often when I got stuck, he would rescue me with one of his brilliant ideas. On the other hand, Joseph encouraged me to push the boundaries of what I can achieve, and ensured that I remain focused despite many failures. During difficult times, when I had lost all hopes, they both patiently stood by me and expressed confidence in my abilities. Each of them has spent countless hours listening to me, checking my work as well as helping me write the results. However, most importantly, they have taught me the joy and integrity of mathematics, both in research and in teaching, and they have moulded me into a better person. I am forever indebted to them, and I thank my lucky stars for the opportunity of working under their tutelage.

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These five years at the Department of Combinatorics and Optimization have been the most beautiful years of my life, and my experience has been enriched by the many friends that I made here — to name a few: Ahmad Abdi, Fidel Barrera-Cruz, Gabriel Coutinho, Linda Farczadi, Zhihan Gao, Cheolwon Heo, Rohan Kapadia, Mehdi Karimi, Abbas Mehrabian, Hadi Minooei, Natalie Mullin, Yingkai Ouyang, Devanshu Pandey, Luke Postle, Cristiane Sato, Abhinav Shantanam, Marcel Silva, Leanne Stuive, ... I will cherish their friendships for many years to come.

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Before moving to Waterloo, I was at Georgia Tech. I would like to extend special thanks to Robin Thomas and Santosh Pande. Robin, by virtue of his teaching and enthusiasm, is responsible for making me fall in love with graph theory. He also encouraged me in my pursuit of mathematics, despite knowing that I lacked a rigorous background. While I had no official advisor at Georgia Tech, Santosh was always willing to listen to my future academic plans, and he went out of his way to guide me. Two of my friends from Georgia Tech, namely Akash Kumar and Anand Louis, deserve a special mention; their enthusiasm for mathematics and all things theoretical definitely rubbed off on me.

Finally, I am truly privileged to have an incredibly supportive family. It is primarily due to my parents' teachings and their value systems that both my brother and I have chosen to pursue the academic path in search of truth and beauty. Innumerable discussions with my brother, Ninad, have been instrumental in making many of the more difficult decisions of my life. Both Ninad and his wife, Arushi, flew from Baltimore to attend my defense to make it even more memorable.

It seems as if the universe has conspired in helping me realize my dreams! Thank you.

To my parents, Usha and Bharat.

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Chapter 1

Introduction and summary

This chapter presents a broad survey of the topics relevant to this thesis. Our main results, namely, Theorems 1.10, 1.11, 1.12, 1.22 and 1.24 are presented within the overall context, and we highlight these using a bar on the left side. In addition, Section 1.8 has a list of our results.

1.1 Matching covered graphs

One of the motivations for the study of perfect matchings and edge-colorings was the four-color conjecture. Tait (1880) observed that the four-color conjecture is equivalent to the statement that every 2-connected planar cubic graph is 3-edge-colorable. (The Petersen graph shows that planarity is an essential assumption for this conclusion to hold.)

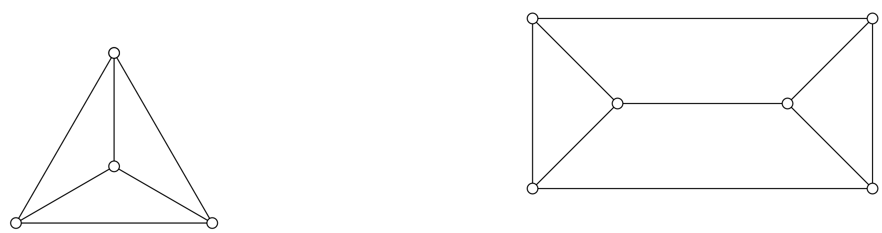
Meanwhile, motivated by a problem about factoring polynomials, Petersen (1891) showed that every 2-connected cubic graph has a perfect matching. Tutte [Tut47] proved his celebrated 1-factor Theorem characterizing graphs which have a perfect matching. (The number of odd components of a graph G is denoted by $\mathsf{odd}(G)$.)

Theorem 1.1 [Tutte's Theorem] *A graph G has a perfect matching if and only if* $\mathsf{odd}(G - S) \leq |S|$ *for each subset S of $V(G)$.*

Tutte deduced as a corollary that, in fact, in a 2-connected cubic graph each edge lies in a perfect matching. Figure 1.1 shows two cubic graphs, namely K_4 and the triangular prism $\overline{C_6}$, which play prominent roles in this thesis.

Let G be a graph that has a perfect matching. A nonempty subset S of its vertices is a *barrier* if it satisfies the equality $\mathsf{odd}(G-S) = |S|$. For distinct vertices u and v of G, it is easily deduced from Tutte's Theorem that the graph $G - \{u, v\}$ has a perfect matching if and only if no barrier of G contains both u and v.

An edge e of G is *admissible* if there is some perfect matching of G that contains e; otherwise it is *inadmissible*. Clearly, an edge is admissible if and only if no barrier of G contains both ends of e.



K_4 $\qquad\qquad$ $\overline{C_6}$

Figure 1.1: The two smallest nonbipartite matching covered graphs

A connected graph with two or more vertices is *matching covered* if each of its edges is admissible. The observation made above implies the following characterization of matching covered graphs. (It can be used to establish, in particular, that every 2-connected cubic graph is matching covered.)

Proposition 1.2 *Let G be a connected graph with a perfect matching. Then G is matching covered if and only if every barrier of G is stable (that is, an independent set).* $\square$

The following fundamental theorem is due to Kotzig (see [LP86, page 150]).

Theorem 1.3 [THE CANONICAL PARTITION THEOREM] *The maximal barriers of a matching covered graph G partition its vertex set.*

For a matching covered graph G, the partition of its vertex set defined by its maximal barriers is called the *canonical partition* of $V(G)$. For instance, for a bipartite matching covered graph $H[A, B]$, the canonical partition of $V(H)$ consists of precisely two parts, namely, its color classes A and B; this is implied by the following proposition which may be derived from the well-known Hall's Theorem. (The neighbourhood of a set of vertices S is denoted by $N(S)$.)

Proposition 1.4 [CHARACTERIZATIONS OF BIPARTITE MATCHING COVERED GRAPHS] *Let $H[A, B]$ denote a bipartite graph on four or more vertices, where $|A| = |B|$. Then the following statements are equivalent:*

(i) *H is matching covered,*

(ii) *$|N(S)| \geq |S| + 1$ for every nonempty proper subset S of A, and*

(iii) *$H - \{a, b\}$ has a perfect matching for each pair of vertices $a \in A$ and $b \in B$.* □

Matching covered graphs are referred to as '1-extendable' graphs in [LP86]. The term 'matching covered' was introduced by Lovász in his seminal work [Lov87] characterizing the matching lattice. For a comprehensive treatment of matching theory and its origins, we refer the reader to Lovász and Plummer [LP86], and to Schrijver [Sch03].

For general graph-theoretical notation and terminology, we essentially follow Bondy and Murty [BM08]. All graphs considered here are loopless; however, we allow multiple (parallel) edges.

It is surprising that matching covered graphs, defined in terms of these seemingly modest axioms, possess a strikingly rich structure. Our investigations, reported in this thesis, are concerned with certain specific questions related to the structure of matching covered graphs, and reinforce the above claim.

This thesis may be viewed as consisting of two main parts. The first part pertains to the problem of characterizing planar matching covered graphs which do not contain specific types of subdivisions of K_4 and $\overline{C_6}$ — a problem that arises from a thirty year old result (Theorem 1.6) of Lovász. In the second part, we explore generation procedures for an important class of matching covered graphs which are referred to as 'near-bipartite bricks'.

Part I - Conformal Subgraphs

1.2 A theorem of Lovász

Several important classes of graphs are characterized by the absence of subdivisions of certain graphs as subgraphs. For example, as was shown by Kuratowski (1930), planar graphs are characterized by the property that they do not contain a subgraph which is a subdivision of either K_5 or of $K_{3,3}$. In the context of matching covered graphs, the notions of subdivision and subgraph need to be employed in a restricted sense, as explained below.

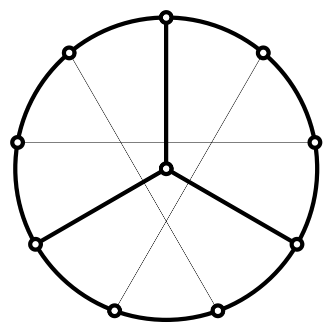


Figure 1.2: The Petersen graph

The length of a path is the number of its edges. A path is *odd* (*even*) if its length is odd (even). To *bi-subdivide an edge* e means to subdivide e by inserting an even number of vertices; or equivalently, to replace e by an odd path. A *bi-subdivision* of a graph J is a graph H obtained from J by means of bi-subdividing a subset of its edges. It is easily verified that any bi-subdivision of a matching covered graph on four or more vertices is also matching covered; however, this is clearly not true for arbitrary subdivisions.

In Figure 1.2, the subgraph whose edges are depicted by the bold lines is a bi-subdivision of K_4. (Bi-subdivisions are also known as totally odd subdivisions. There is an extensive literature dealing with bi-subdivisions of K_4 in the context of chromatic graph theory. See [Zan98] and [Tho01].)

A matching covered subgraph H of a matching covered graph G is *conformal* if the graph $G - V(H)$ has a perfect matching; equivalently, H is conformal if each perfect matching of H extends to a perfect matching of G. In Figure 1.2, the bi-subdivision of

K_4 depicted by the bold lines is a spanning subgraph, whence conformal. In general, a conformal subgraph may not be spanning. (In the literature, conformal subgraphs have been referred to as 'nice' subgraphs by Lovász [Lov83], as 'central' subgraphs by Robertson et al. [RST99], and as 'well-fitted' subgraphs by McCuaig [McC01].)

1.2.1 Ear decompositions

Bipartite graphs

A *single ear* of a graph is an odd path whose internal vertices (if any) have degree two in the graph.

Let H be a bipartite graph and K a subgraph of H. A *bipartite ear decomposition of H starting with K* is a sequence $H_1 \subset H_2 \subset \cdots \subset H_r$ of subgraphs of H such that (i) $H_1 := K$ and $H_r := H$, and (ii) for each i such that $1 \leq i \leq r-1$, the graph H_{i+1} is the union of H_i and exactly one single ear of H_{i+1}.

The following may be deduced from the fact that, for a bipartite matching covered graph $K[A, B]$, the graph $K - \{a, b\}$ has a perfect matching for every pair of vertices $a \in A$ and $b \in B$ (see Proposition 1.4).

Proposition 1.5 *Let H be a bipartite graph and suppose that K is a matching covered subgraph of H. If H admits a bipartite ear decomposition starting with K, then the graph H is also matching covered.* □

It is easily seen that each subgraph in a bipartite ear decomposition of H is a conformal subgraph of H. Conversely, given any conformal matching covered subgraph K of a bipartite matching covered graph H, there exists a bipartite ear decomposition of H starting with K. In particular, since the subgraph $H[e]$ induced by any edge e is conformal, H admits a bipartite ear decomposition starting with $H[e]$. See [LP86, page 124].

Nonbipartite graphs

The 'addition of single ears' is not sufficient to construct nonbipartite matching covered graphs. For instance, it is not possible to obtain K_4 from its conformal subgraph C_4, by means of adding single ears, such that at each step we have a matching covered graph. To fix this, one must allow the 'addition of two single ears simultaneously', as explained below.

A pair of vertex-disjoint single ears is called a *double ear*. Let G be a matching covered graph and H a matching covered subgraph of G. An *ear decomposition of G starting with H* is a sequence $G_1 \subset G_2 \subset \cdots \subset G_r$ of matching covered subgraphs of G such that (i) $G_1 := H$ and $G_r := G$, and (ii) for each i such that $1 \leq i \leq r-1$, the graph G_{i+1} is the union of G_i and exactly one single or double ear of G_{i+1}. We say that G_{i+1} is obtained from G_i by *adding a single ear*, or by *adding a double ear*, as applicable.

A basic result from [LP86, page 182] states that a matching covered subgraph H of a matching covered graph G is conformal if and only if G admits an ear decomposition starting with H. Consequently, every matching covered graph G admits an ear decomposition starting with $G[e]$, where e is any edge of G. Clearly, the second graph in such a sequence is obtained by adding a single ear, and it is a cycle of even length. (It should be noted that a matching covered graph may admit different ear decompositions, possibly of different lengths.)

In other words, every matching covered graph G may be constructed from K_2 by means of adding single or double ears such that, at each step, we have a matching covered graph. A subtle point needs to be made here. A double ear consists of two vertex-disjoint single ears; its addition is justified only if neither of its constituent single ears can be added individually to obtain a matching covered graph. Henceforth, we will implicitly assume this property when considering ear decompositions. (In [LP86], such an ear decomposition is called 'non-refinable'.)

For instance, as noted earlier, every bipartite matching covered graph may be constructed by adding single ears alone. Conversely, any matching covered graph, obtained from a bipartite matching covered graph by adding a single ear, is also bipartite.

Now let G be a nonbipartite matching covered graph, and let $G_1 \subset G_2 \subset \cdots \subset G_r$ be an ear decomposition of G starting with $G_1 \cong K_2$. It follows from the above observation that, at some stage, a double ear is added. Let G_k, where $3 \leq k \leq r$, be the first graph in the sequence obtained by adding a double ear. Then all graphs $G_1, G_2, \ldots, G_{k-1}$ are bipartite, and G_k is nonbipartite. This observation is of significance, especially in the second part of this thesis, and we will return to it in Section 1.6.

A natural question arises from the above observation: given a nonbipartite matching covered graph G, how early can the first double ear be added? Lovász [Lov83] answered this by proving that G admits an ear decomposition $G_1 \subset G_2 \subset \cdots \subset G_r$ starting with $G_1 \cong K_2$ such that either G_3 is a bi-subdivision of K_4, or G_4 is a bi-subdivision of $\overline{C_6}$. This fundamental result of Lovász may be restated as follows.

Theorem 1.6 [LOVÁSZ'S THEOREM] *Each nonbipartite matching covered graph G admits an ear decomposition starting with either a bi-subdivision of K_4 or of $\overline{C_6}$.*

A short proof of the above theorem was given by Carvalho and Lucchesi [CL96]. We remark that, in general, a nonbipartite matching covered graph need not admit an ear decomposition which uses only one double ear addition. For example, every ear decomposition of the Petersen graph requires the addition of at least two double ears (see [LP86, page 178]).

1.2.2 K_4-based and $\overline{C_6}$-based graphs

For a matching covered graph J, we say that G is *J-based* if G contains a conformal subgraph H which is a bi-subdivision of J. Otherwise, we say that G is *J-free*. This notion has played a crucial role in characterizing important classes of matching covered graphs. For example, Little [Lit75] showed that a bipartite matching covered graph is Pfaffian if and only if it is $K_{3,3}$-free.

Lovász's Theorem (1.6) implies that every nonbipartite matching covered graph is either K_4-based, or is $\overline{C_6}$-based, or both. For example, the Petersen graph is K_4-based but $\overline{C_6}$-free; on the other hand, each prism (see page 13) on $4k+2$ vertices, where $k \geq 1$, is $\overline{C_6}$-based but K_4-free; whereas each complete graph on $2k$ vertices, where $k \geq 3$, is K_4-based as well as $\overline{C_6}$-based.

Alternatively, the K_4-based matching covered graphs are precisely those which admit an ear decomposition starting with a bi-subdivision of K_4. An analogous statement holds for $\overline{C_6}$-based graphs.

This led Carvalho, Lucchesi and Murty [CLM03] to pose two problems: (i) determine whether or not a given matching covered graph G is K_4-free, and likewise, (ii) determine whether or not G is $\overline{C_6}$-free. These problems are, in general, unsolved. In the first part of this thesis, we solve these problems for the special case of planar matching covered graphs. In the next two sections, we state the highlights of our solution.

1.3 Bricks

It is well-known that each matching covered graph may be 'decomposed', in an essentially unique manner, into special matching covered graphs called 'bricks' and 'braces'. This procedure is known as the 'tight cut decomposition'.

This decomposition theory was developed by Kotzig, Lovász and Plummer, and its import is due to the fact that the properties of a matching covered graph may often be understood by analysing the properties of its bricks and braces. For instance, a matching covered graph is Pfaffian if and only if each of its bricks and braces is Pfaffian; see [LR91]. Our Theorem 1.10 is another example of this type. We now proceed to describe this decomposition procedure.

1.3.1 Tight cut decomposition

For a nonempty proper subset X of the vertices of a graph G, we denote by $\partial(X)$ the cut associated with X, that is, the set of all edges of G that have one end in X and the other end in $\overline{X} := V(G) - X$. We refer to X and $\overline{X}$ as the *shores* of $\partial(X)$. A cut is *trivial* if any of its shores is a singleton. For a cut $\partial(X)$, we denote the graph obtained by contracting the shore $\overline{X}$ to a single vertex $\overline{x}$ by $G/(\overline{X} \to \overline{x})$. In case the label of the contraction vertex $\overline{x}$ is irrelevant, we simply write $G/\overline{X}$. The two graphs G/X and $G/\overline{X}$ are called the $\partial(X)$-*contractions* of G. In Figure 1.3, the three edges crossing the bold line constitute a nontrivial cut, say $\partial(X)$, and the two $\partial(X)$-contractions are K_4 and $K_{3,3}$.

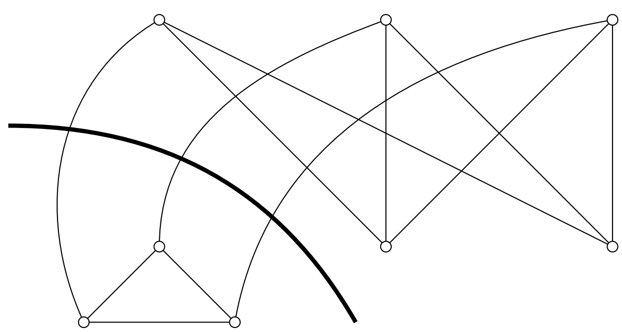


Figure 1.3: A nontrivial tight cut

Let G be a matching covered graph. A cut $\partial(X)$ is a *tight cut* if $|M \cap \partial(X)| = 1$ for every perfect matching M of G. It is easily verified that if $\partial(X)$ is a nontrivial tight cut of G, then each $\partial(X)$-contraction is a matching covered graph that has strictly fewer vertices than G. If either of the $\partial(X)$-contractions has a nontrivial tight cut, then that graph can be further decomposed into even smaller matching covered graphs. We can repeat this procedure until we obtain a list of matching covered graphs, each of which is free of nontrivial tight cuts. This procedure is known as a *tight cut decomposition* of G.

For instance, if S is a barrier of G, and K is an odd component of $G-S$, then $\partial(V(K))$ is a tight cut of G. Such a tight cut is called a *barrier cut*, and such cuts play an important role in the second part of this thesis. The graph in Figure 1.3 has a barrier cut depicted by the bold line, and each of its contractions (that is, K_4 and $K_{3,3}$) is free of nontrivial tight cuts. Note that, if v is a vertex of degree two then $\{v\} \cup N(v)$ is the shore of a barrier cut. A barrier is *trivial* if it has a single vertex. Note that if G is nonbipartite then each nontrivial barrier gives rise to a nontrivial tight cut.

Now suppose that $\{u, v\}$ is a 2-vertex-cut of G such that $G - \{u, v\}$ has an even component, say K. Then each of the sets $V(K) \cup \{u\}$ and $V(K) \cup \{v\}$ is a shore of a nontrivial tight cut of G. Such a tight cut is called a 2-*separation cut*. The graph in Figure 1.4 has a 2-separation cut, and each of its contractions is K_4 with multiple edges. (We remark that, a graph may have a tight cut which is neither a barrier cut nor a 2-separation cut.)

Let G be a matching covered graph free of nontrivial tight cuts. If G is bipartite then it is a *brace*; otherwise it is a *brick*. Thus, a tight cut decomposition of G results in a list of bricks and braces. For example, a tight cut decomposition of the graph shown in Figure 1.3 yields the brick K_4 and the brace $K_{3,3}$.

In general, a matching covered graph may admit several tight cut decompositions. However, Lovász [Lov87] proved the following remarkable result, and demonstrated its significance by using it to compute the dimension of the matching lattice.

Theorem 1.7 [THE UNIQUE DECOMPOSITION THEOREM] *Any two tight cut decompositions of a matching covered graph yield the same list of bricks and braces (except possibly for multiplicities of edges).*

In particular, any two tight cut decompositions of a matching covered graph G yield the same number of bricks; this number is denoted by $b(G)$. We remark that G is bipartite if and only if $b(G) = 0$.

A graph G, with four or more vertices, is *bicritical* if $G - \{u, v\}$ has a perfect matching for every pair of distinct vertices u and v. For instance, the graph shown in Figure 1.4 is bicritical. The following characterization of bicritical graphs follows immediately from Tutte's Theorem.

Proposition 1.8 [CHARACTERIZATION OF BICRITICAL GRAPHS] *Let G be a graph that has a perfect matching. Then G is bicritical if and only if every barrier of G is trivial.* □

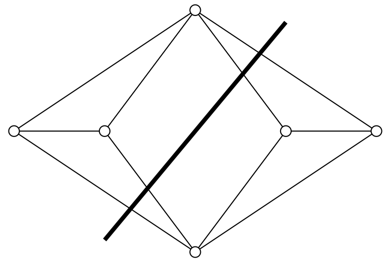


Figure 1.4: A bicritical graph which is not a brick

Equivalently, for a bicritical graph G, the canonical partition of $V(G)$ consists of $|V(G)|$ parts, each of which contains a single vertex. Since a brick is a nonbipartite matching covered graph which is free of nontrivial tight cuts, it follows from the above observations that every brick is 3-connected and bicritical. Edmonds, Lovász and Pulleyblank [ELP82] established the converse; their proof was based on LP-duality.

Theorem 1.9 [Characterization of Bricks] *A graph G is a brick if and only if it is 3-connected and bicritical.*

In particular, a brick is free of nontrivial barriers and of 2-vertex cuts. Szigeti [Szi02] obtained a simple proof of Theorem 1.9 which does not use LP-duality.

Throughout this thesis, we shall mainly be interested in nonbipartite matching covered graphs, and especially in bricks. Three cubic bricks, namely K_4, $\overline{C_6}$ and the Petersen graph, occupy a special position in the theory of matching covered graphs.

1.3.2 Reduction to the case of bricks

As discussed in Section 1.2.2, given a planar matching covered graph G, we would like to determine whether or not G is K_4-free, and likewise, whether or not G is $\overline{C_6}$-free.

As a first step, we reduce the above problems to the case of bricks. Let J denote any cubic brick. The key idea is to show that if G is a matching covered graph and $\partial(X)$ is a nontrivial tight cut of G, then G is J-free if and only if each of its $\partial(X)$-contractions is J-free. (We remark that this statement is not true if J is a cubic brace, as can be inferred from the graph in Figure 1.3 with $K_{3,3}$ playing the role of J; and it is also not true if J is an arbitrary brick.) It now follows from the Unique Decomposition Theorem (1.7) that G is J-free if and only if each of its bricks and braces is J-free. (We point out that braces, being bipartite, are trivially J-free.)

Theorem 1.10 [Reduction to the case of Bricks] *Let J denote any cubic brick. A nonbipartite matching covered graph is J-free if and only if each of its bricks is J-free.*

We emphasize that the above theorem has nothing to do with planarity. We present a proof of Theorem 1.10 in Chapter 2.

It is straightforward to see that, for a planar matching covered graph, each of its bricks and braces is also planar. In view of Theorem 1.10, it suffices to characterize K_4-free planar bricks and $\overline{C_6}$-free planar bricks. We discuss our solutions to these problems in Section 1.4.

1.3.3 Norine-Thomas bricks

Here, we shall describe five infinite families of bricks; namely, odd wheels, prisms, Möbius ladders, truncated biwheels and staircases; we refer to these as the *Norine-Thomas families* for reasons explained in Section 1.5. Furthermore, we say that a brick is *Norine-Thomas* if it belongs to any of these families, or if it is isomorphic to the Petersen graph. We adopt the terminology of Carvalho et al. [CLM08].

Odd Wheels. The *odd wheel* W_{2k+1}, for $k \geq 1$, is defined to be the join of an odd cycle C_{2k+1} and K_1. See Figure 1.6a. The smallest odd wheel is K_4. If $k \geq 2$, then W_{2k+1} has exactly one vertex of degree $2k+1$, called its *hub*, and the edges incident at the hub are called its *spokes*. The remaining $2k+1$ vertices lie on a cycle, called the *rim*, and they are referred to as *rim vertices*.

Each member of the remaining four families contains a bipartite matching covered subgraph which is either a 'ladder' or a 'partial biwheel'. These bipartite graphs are also the main building blocks of additional families of bricks which are of interest in Section 1.7.3. For this reason, we start with a description of these two families of bipartite graphs.

Ladders. Let $x_0x_1 \ldots x_j$ and $y_0y_1 \ldots y_j$ be two vertex-disjoint paths, where $j \geq 2$. The graph K obtained by the union of these two paths, and by adding edges x_iy_i for $0 \leq i \leq j$, is called a *ladder*, and its edges joining x_i and y_i are referred to as its *rungs*. See Figure 1.5. The two rungs x_0y_0 and x_jy_j are *external*, and the remaining rungs are *internal*. We say that K is *odd* (*even*) if it has an odd (even) number of rungs.

Partial Biwheels. Let $x_0x_1 \ldots x_{2j+1}$ be an odd path, where $j \geq 1$. The graph K obtained by adding two new vertices u and w, joining u to vertices in $\{x_0, x_2, \ldots, x_{2j}\}$, and joining w to vertices in $\{x_1, x_3, \ldots, x_{2j+1}\}$, is called a *partial biwheel*; the vertices x_0 and x_{2j+1} are referred to as its *ends*, whereas u and w are referred to as its *hubs*; and an edge incident with a hub is called a *spoke*. See Figure 1.5. The two spokes ux_0 and wx_{2j+1} are *external*, and the remaining spokes are *internal*.

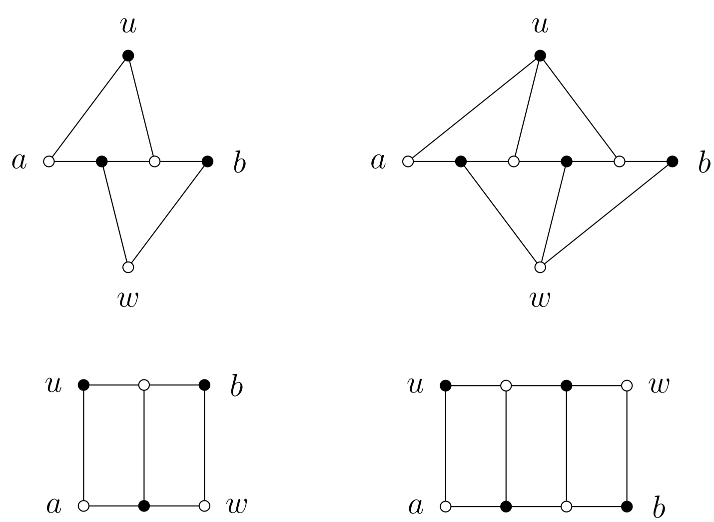


Figure 1.5: Partial biwheels (top) and Ladders (bottom)

When referring to a ladder or to a partial biwheel, say $K[A, B]$, with external rungs/spokes au and bw, we adopt the convention that $a, w \in A$ and $b, u \in B$; furthermore, when K is a partial biwheel, u and w shall denote its hubs; as shown in Figure 1.5. (Sometimes, we may also use subscript notation, such as A_i, B_i, a_iu_i and b_iw_i where i is an integer, and this convention extends naturally.)

It should be noted that a partial biwheel of order six is also a ladder. However, a partial biwheel of order eight or more has only two vertices of degree two, namely, its ends; whereas every ladder has four such vertices. We remark that, a *biwheel*, as defined by McCuaig [McC01], has order at least eight and contains an additional edge joining its ends; and these constitute an important class of braces.

We now proceed to describe the remaining four Norine-Thomas families using ladders and partial biwheels.

Prisms, Möbius Ladders and Truncated Biwheels. Let $H[A, B]$ denote either a ladder or a partial biwheel of order n, with external rungs/spokes au and bw, and let G be the graph obtained from H by adding two edges, namely, aw and bu. If H is an odd ladder then G is a *prism* and it is denoted by P_n, see Figure 1.6b. If H is an even ladder then G is a *Möbius ladder* and it is denoted by M_n, see Figure 1.6f. Finally, if H is a partial biwheel then G is a *truncated biwheel* and it is denoted by T_n, see Figure 1.6c. Note that $\overline{C_6}$ is the smallest prism as well as the smallest truncated biwheel. For convenience, we shall consider K_4 to be the smallest Möbius ladder.

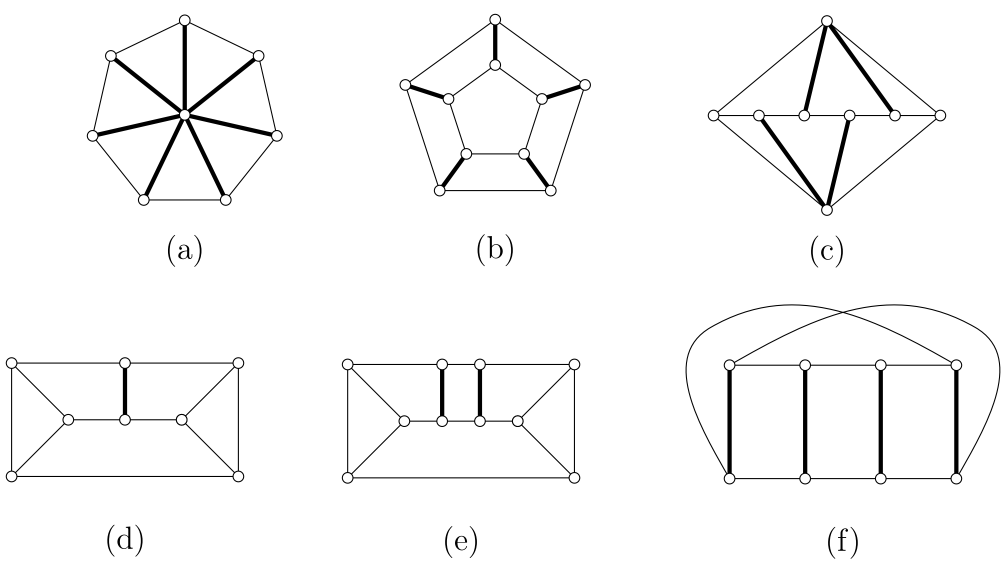


Figure 1.6: (a) Odd wheel W_7, (b) Prism P_{10}, (c) Truncated biwheel T_8, (d) Odd staircase St_8, (e) Even staircase St_{10}, (f) Möbius ladder M_8

Staircases. Let $K[A_1, B_1]$ denote a ladder of order n, with external rungs a_1u_1 and b_1w_1. Then the graph G obtained from K, by adding two new vertices a_2 and b_2, and by adding five new edges $a_1a_2, u_1a_2, b_1b_2, w_1b_2$ and a_2b_2, is called a *staircase*, and it is denoted by St_{n+2}. We say that G is an *odd* (*even*) staircase if K is an odd (even) ladder. See Figures 1.6d and 1.6e.

1.4 Planar bricks

In this section, we state our characterizations of K_4-free planar bricks and of $\overline{C_6}$-free planar bricks. Observe that, for a cubic brick J, a matching covered graph G is J-free if and only if the underlying simple graph of G is J-free. We may thus restrict our attention to simple planar bricks.

A well-known result due to Whitney [Whi33] says that every simple 3-connected planar graph has a unique embedding in the plane. We may thus refer to the faces of simple planar bricks without any ambiguity.

In what follows, the number of odd faces will play a key role. Observe that, for a 3-connected planar graph G, the number of its odd faces is the same as the number of odd faces of its underlying simple graph. Being nonbipartite, each planar brick has at least two odd faces.

Observe that every Norine-Thomas brick, except for the Petersen graph and the Möbius ladders of order eight or more, is planar. Among the planar ones, every prism, truncated biwheel and even staircase has precisely two odd faces. On the other hand, every odd staircase has exactly four odd faces; and since each face of an odd wheel is odd, every odd wheel has at least four odd faces.

1.4.1 K_4-free planar bricks

We begin by noting that K_4 has exactly four odd faces, and so does any bi-subdivision of K_4. This immediately implies that each K_4-based planar graph has four or more odd faces. (In particular, prisms, truncated biwheels and even staircases are K_4-free.)

We establish that the converse is also true when the graph under consideration is a brick; that is, we show that every planar brick with four or more odd faces is K_4-based. (In particular, odd staircases and odd wheels are K_4-based.) This leads us to the following compact characterization of K_4-free planar bricks.

Theorem 1.11 [Characterization of K_4-free Planar Bricks] *A planar brick is K_4-free if and only if it has precisely two odd faces.*

It is important to note that, in general, a planar 3-connected matching covered graph with four or more odd faces may not be K_4-based, as shown by the graph in Figure 3.2. In other words, the bicriticality property of bricks is indispensable; recall Theorem 1.9.

1.4.2 $\overline{C_6}$-free planar bricks

Observe that $\overline{C_6}$ has two vertex-disjoint odd cycles, and thus every $\overline{C_6}$-based graph inherits this property. Consequently, the odd wheels are $\overline{C_6}$-free. By investigating the odd cycles of St_8 (see Figure 1.6d), one may easily verify that it is $\overline{C_6}$-free. More generally, each odd staircase is $\overline{C_6}$-free. We show that apart from these two infinite families, there is one exceptional $\overline{C_6}$-free simple planar brick which we call the *Tricorn* (see Figure 1.8).

Theorem 1.12 [Characterization of $\overline{C_6}$-free Planar Bricks] *A planar brick is $\overline{C_6}$-free if and only if its underlying simple graph is either an odd wheel, or an odd staircase, or the Tricorn.*

We emphasize that Theorems 1.10, 1.11 and 1.12 together provide a complete characterization of K_4-free planar matching covered graphs, and of $\overline{C_6}$-free planar matching covered graphs. These results also appear in [KM16].

We present proofs of Theorems 1.11 and 1.12 in Chapter 3. The principal tool we use for proving these results is the brick generation procedure established by Norine and Thomas [NT07]. We discuss their result, and a related result of Carvalho, Lucchesi and Murty [CLM06], in the next section.

Part II - Brick Generation

1.5 Removable edges

As discussed in Section 1.3, the properties of a matching covered graph can often be deduced by analysing its bricks and braces. This has led researchers to develop inductive tools for studying the properties of bricks and braces; these have found useful applications.

McCuaig [McC01] described a procedure for generating simple braces, and used it in [McC04] to derive a structural characterization of Pfaffian bipartite matching covered graphs. Carvalho, Lucchesi and Murty [CLM06] established a generation procedure for bricks, and applied it to show that the only 'solid' planar bricks are the odd wheels; this may also be deduced from our Theorem 1.12 since solid bricks are also $\overline{C_6}$-free. Norine and Thomas [NT07] established a generation procedure for simple bricks; our proofs of Theorems 1.11 and 1.12 rely heavily on their result.

In this section, we review the aforementioned works of Carvalho et al. [CLM06], and of Norine and Thomas [NT07]. We shall find it convenient to state all of the results using the terminology of Carvalho et al. [CLM06, CLM08].

An edge e of a matching covered graph G is *removable* if $G-e$ is also matching covered; otherwise it is *non-removable*. For example, each edge of the Petersen graph is removable. All bricks in the Norine-Thomas families, except for K_4 and $\overline{C_6}$, have removable edges; these are indicated in Figure 1.6 by the bold lines.

We remark that the notion of a removable edge is intrinsically related to ear decompositions. To see this, note that an edge e of a matching covered graph G is removable if and only if G admits an ear decomposition in which the edge e is the last (single) ear added.

The following was established by Lovász [Lov87].

Theorem 1.13 [Removable Edge Theorem] *Every brick distinct from K_4 and $\overline{C_6}$ has a removable edge.*

We point out that, if e is a removable edge of a brick G, then $G-e$ may not be a brick. For instance, $G-e$ may have vertices of degree two.

In what follows, we will define three types of removable edges: 'b-invariant' edges, 'thin' edges and 'strictly thin' edges, in that order. In the context of bricks, each type is a specialization of the preceding type as depicted in Figure 1.7; for instance, an arrow from 'thin' to 'b-invariant' indicates that a thin edge is a special type of b-invariant edge.

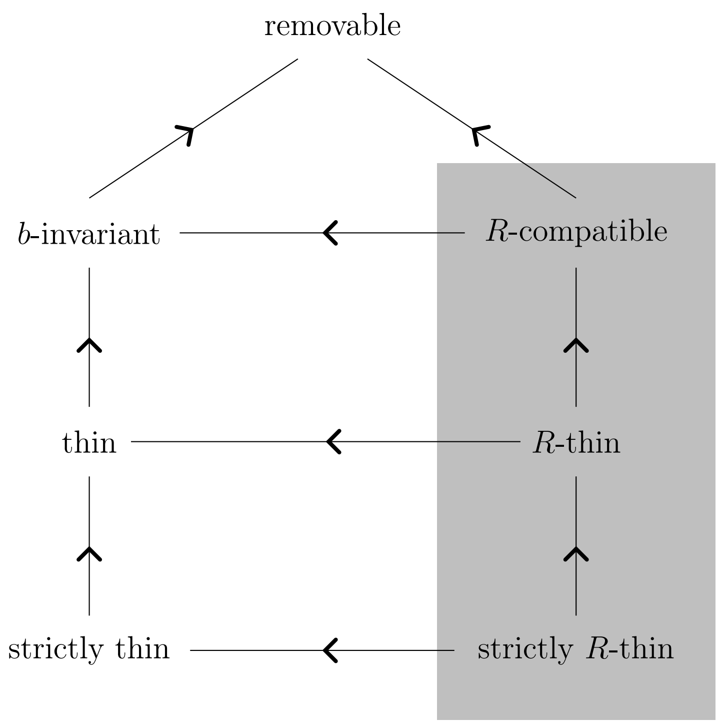


Figure 1.7: Types of removable edges in bricks; the ones in the shaded area are only applicable to near-bipartite bricks and they are introduced in Section 1.7

1.5.1 Near-bricks and b-invariant edges

Recall that $b(G)$ denotes the number of bricks of a matching covered graph G (in any tight cut decomposition), and it is well-defined due to the Unique Decomposition Theorem (1.7). A *near-brick* is a matching covered graph with $b(G) = 1$. Clearly, every brick is a near-brick. However, the converse is not true. For instance, the graph shown in Figure 1.3 is a near-brick but it is not a brick. When proving theorems concerning bricks, one often needs the flexibility of dealing with the wider class of near-bricks, whose properties are akin to those of bricks.

A removable edge e of a matching covered graph G is *b-invariant* if $b(G-e) = b(G)$. In particular, if G is a brick then e is b-invariant if and only if $G-e$ is a near-brick. For example, in every member of the Norine-Thomas families, each removable edge is b-invariant. On the other hand, it is easily verified that if G is the Petersen graph and e is any edge, then $b(G-e) = 2$. Thus each edge of the Petersen graph is removable, but none of them is b-invariant.

Confirming a conjecture of Lovász, the following result was proved by Carvalho, Lucchesi and Murty [CLM02a].

Theorem 1.14 [b-INVARIANT EDGE THEOREM] *Every brick distinct from K_4 and $\overline{C_6}$ and the Petersen graph has a b-invariant edge.*

In [CLM02a], Carvalho et al. established a generalization of the above theorem. For instance, their result shows that the staircase St_8 is the only brick with a unique b-invariant edge, which is depicted in Figure 1.6d by a bold line.

1.5.2 Bicontractions, retracts and bi-splittings

Let G be a matching covered graph and let v be a vertex of degree two, with two distinct neighbours u and w. The *bicontraction* of v is the operation of contracting the two edges vu and vw incident with v. Note that $X := \{u, v, w\}$ is the shore of a tight cut of G, and that the graph resulting from the bicontraction of v is the same as the $\partial(X)$-contraction G/X, whereas the other $\partial(X)$-contraction $G/\overline{X}$ is isomorphic to C_4 (possibly with multiple edges).

The *retract* of G is the graph obtained from G by bicontracting all its degree two vertices. The above observation implies that the retract of a matching covered graph is also matching covered. Carvalho et al. [CLM05] showed that the retract of a matching covered graph is unique up to isomorphism. It is important to note that even if G is simple, the retract of G may have multiple edges.

The operation of bi-splitting is the converse of the operation of bicontraction. Let H be a graph and let v be a vertex of H of degree at least two. Let G be a graph obtained from H by replacing the vertex v by two new vertices v_1 and v_2, distributing the edges in

H incident with v between v_1 and v_2 such that each gets at least one, and then adding a new vertex v_0 and joining it to both v_1 and v_2. Then we say that G is obtained from H by *bi-splitting* v into v_1 and v_2. It is easily seen that if H is matching covered, then G is also matching covered, and that H can be recovered from G by bicontracting the vertex v_0 and denoting the contraction vertex by v.

1.5.3 Thin edges

A b-invariant edge e of a brick G is *thin* if the retract of $G-e$ is a brick. As the graph $G-e$ can have zero, one or two vertices of degree two, the retract of $G-e$ is obtained by performing at most two bicontractions, and it has at least $|V(G)|-4$ vertices.

For example, if G is an odd wheel of order six or more and if e is any spoke, then the retract of $G-e$ is a smaller odd wheel with multiple edges; thus, each spoke of G is a thin edge. More generally, if G belongs to any of the Norine-Thomas families, and if e denotes any removable edge, then the retract of $G-e$ is a smaller Norine-Thomas brick with multiple edges; consequently, e is thin. It should be noted that, in general, a b-invariant edge may not be thin.

The original definition of a thin edge, due to Carvalho et al. [CLM06], was in terms of barriers; 'thin' being a reference to the fact that the barriers of $G-e$ are sparse. This viewpoint will also be useful to us in Chapter 5 where further explanation is provided. Carvalho, Lucchesi and Murty [CLM06] used their b-invariant Edge Theorem (1.14) to derive the following stronger result.

Theorem 1.15 [THIN EDGE THEOREM] *Every brick distinct from K_4 and $\overline{C_6}$ and the Petersen graph has a thin edge.*

The following is an immediate consequence of the above theorem.

Theorem 1.16 [CLM06] *Given any brick G, there exists a sequence $G_1, G_2, \ldots, G_k$ of bricks such that:*

(i) G_1 is either K_4 or $\overline{C_6}$ or the Petersen graph,

(ii) $G_k := G$, and

(iii) for $2 \le i \le k$, there exists a thin edge e_i of G_i such that G_{i-1} is the retract of $G_i - e_i$.

Carvalho et al. [CLM06] also described four elementary 'expansion operations' which may be applied to any brick to obtain a larger brick with at most four more vertices. Each of these operations consists of bi-splitting at most two vertices and then adding a suitable edge. Given a brick J, the application of any of these four operations to J results in a brick G such that G has a thin edge e with the property that J is the retract of $G - e$. Thus, any brick may be generated from one of the three basic bricks (K_4 and $\overline{C_6}$ and the Petersen graph) by means of these four expansion operations.

One of the problems with this brick generation procedure is that, even if $G_k = G$ is a simple brick, there is no guarantee that all the intermediate bricks $G_2, G_3, \ldots G_{k-1}$ are also simple. In fact, certain bricks cannot be generated by staying within the realm of simple bricks.

1.5.4 Strictly thin edges

A thin edge e of a simple brick G is *strictly thin* if the retract of $G - e$ is simple. As an example, consider the Tricorn, shown in Figure 1.8, which has precisely three removable edges indicated by bold lines; deleting one of them, say e, and taking the retract yields the simple odd wheel W_5. Thus each removable edge of the Tricorn is strictly thin. By contrast, in a Norine-Thomas brick, none of the thin edges is strictly thin.

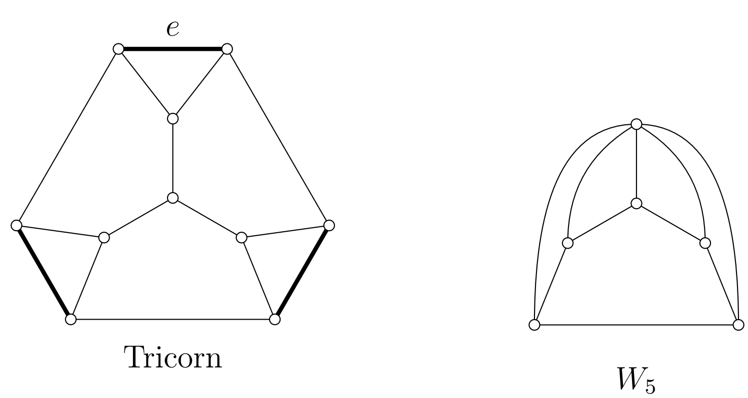


Figure 1.8: Removable edges of the Tricorn

Using this terminology, the theorem of Norine and Thomas [NT07] may be stated as follows.

Theorem 1.17 [STRICTLY THIN EDGE THEOREM] *Let G be a simple brick. If G is free of strictly thin edges then G is either the Petersen graph, or it is an odd wheel, a prism, a Möbius ladder, a truncated biwheel or a staircase.*

Equivalently, the only simple bricks devoid of strictly thin edges are the Norine-Thomas bricks. It should be noted that Norine and Thomas did not state their results in terms of strictly thin edges.

Subsequently, Carvalho et al. [CLM08] used their Thin Edge Theorem (1.15) to deduce the Strictly Thin Edge Theorem (1.17). The following result of Norine and Thomas [NT07] is an immediate consequence of Theorem 1.17.

Theorem 1.18 *Given any simple brick G, there exists a sequence $G_1, G_2, \ldots, G_k$ of simple bricks such that:*

(i) G_1 *is a Norine-Thomas brick,*

(ii) $G_k := G$, *and*

(iii) *for* $2 \leq i \leq k$, *there exists a strictly thin edge* e_i *of* G_i *such that* G_{i-1} *is the retract of* $G_i - e_i$.

The above theorem implies that every simple brick can be generated from one of the Norine-Thomas bricks by repeated application of the four expansion operations such that at each step we have a simple brick.

We remark that Norine and Thomas proved a generalization of Theorem 1.18, which they refer to as the 'splitter theorem for bricks', since it is motivated by the splitter theorem for 3-connected graphs due to Seymour [Sey80]. The notions of thin and strictly thin edges are easily generalized to braces (see [CLM08]). A 'splitter theorem for braces' was established by McCuaig [McC01].

1.6 Near-bipartite graphs

A nonbipartite matching covered graph G is *near-bipartite* if it has a pair $R := \{\alpha, \beta\}$ of edges such that the graph $H := G - R$ is bipartite and matching covered; for instance,

K_4 and $\overline{C_6}$ are the smallest near-bipartite bricks. Observe that the edge α joins two vertices in one color class of H, and that β joins two vertices in the other color class. Consequently, if M is any perfect matching of G then $\alpha \in M$ if and only if $\beta \in M$.

The significance of near-bipartite graphs arises from the theory of ear decompositions (see Section 1.2.1). Observe that if G is any nonbipartite matching covered graph, and if $G_1 \subset G_2 \subset \cdots \subset G_r$ is an ear decomposition of G starting with a conformal bipartite matching covered subgraph G_1, then the first nonbipartite graph in this sequence, say G_k, is a bi-subdivision of a near-bipartite graph (where the double ear added may be viewed as adding two edges, one joining two vertices in one color class of G_{k-1} and another edge joining two vertices in the other color class, and then bi-subdividing those edges). In this sense, near-bipartite graphs constitute the class of nonbipartite matching covered graphs which are closest to being bipartite.

Since the problems of characterizing K_4-free nonplanar bricks and $\overline{C_6}$-free nonplanar bricks do not seem to be tractable with the tools available to us, it may be worthwhile studying these questions for the restricted class of near-bipartite bricks. This approach has been successful in the theory of Pfaffian orientations; although there has been no significant progress in characterizing Pfaffian bricks; Fischer and Little [FL01] were able to characterize Pfaffian near-bipartite graphs.

With this in mind, we undertook to investigate generation procedures which are specific to near-bipartite bricks. We hope that these results can be used to derive characterizations of important classes of near-bipartite bricks.

1.6.1 Removable doubletons

A pair of distinct edges $R := \{\alpha, \beta\}$ of a matching covered graph G is a *removable doubleton* if neither α nor β is removable, but the graph $G - R$ is matching covered. It should be noted that, in general, the graph $G - R$ need not be bipartite. However, Lovász [Lov87] proved that if G is a brick then $G - R$ is indeed bipartite; the following more general result of Carvalho et al. [CLM02b] shows that the conclusion holds even if G is a near-brick.

Theorem 1.19 *Let G be a matching covered graph, and let R be a removable doubleton. Then $b(G - R) = b(G) - 1$.*

The above theorem implies that every near-bipartite graph is a near-brick. In fact, as we will see in Chapter 4, the unique brick of a near-bipartite graph is also near-bipartite.

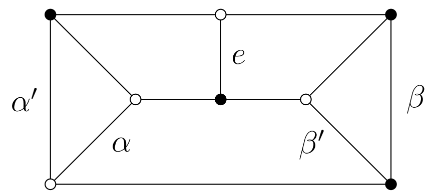


Figure 1.9: The staircase St_8

A graph may have several removable doubletons; for instance, K_4 and $\overline{C_6}$ have three; the staircase St_8 shown in Figure 1.9 has two, namely $R := \{\alpha, \beta\}$ and $R' := \{\alpha', \beta'\}$. It is easily verified that every Norine-Thomas brick (see Section 1.3.3), except for the odd wheels of order six or more and for the Petersen graph, is near-bipartite; among these, only the truncated biwheels, of order eight or more, have a unique removable doubleton.

1.7 Generating near-bipartite bricks

The difficulty in using either Theorem 1.16 or Theorem 1.18 as an induction tool for studying near-bipartite bricks, is that even if $G_k := G$ is a near-bipartite brick, there is no guarantee that all of the intermediate bricks $G_1, G_2, \ldots G_{k-1}$ are also near-bipartite.

For instance, the brick shown in Figure 1.10a is near-bipartite with a (unique) removable doubleton $R := \{\alpha, \beta\}$. Although the edge e is thin; the retract of $G - e$, as shown in Figure 1.10b, is not near-bipartite since it has three edge-disjoint triangles.

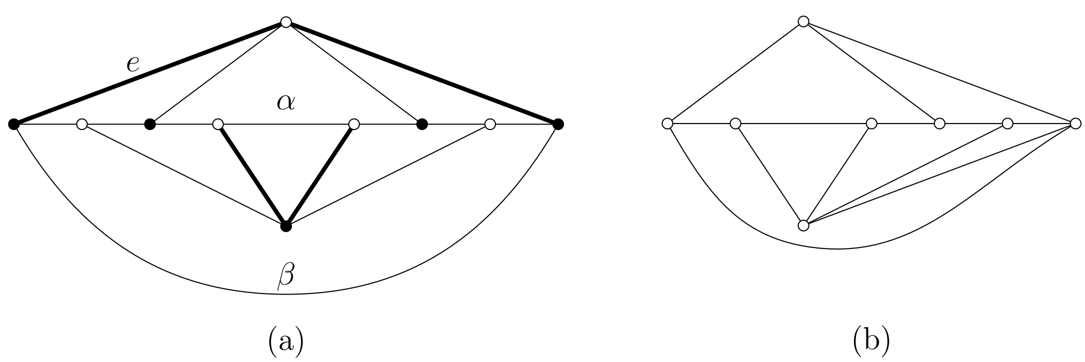


Figure 1.10: (a) A near-bipartite brick G with a thin edge e ; (b) The retract of $G - e$ is not near-bipartite

In other words, deleting an arbitrary thin edge may not preserve the property of being near-bipartite. In this sense, neither the Thin Edge Theorem (1.15) nor the Strictly Thin Edge Theorem (1.17) is adequate for obtaining inductive proofs of results that pertain only to the class of near-bipartite bricks.

To fix this problem, we decided to look for a thin edge whose deletion preserves the property of being near-bipartite. Recall that a graph may have several removable doubletons. We find it convenient to fix a removable doubleton R (of the brick under consideration), and then look for a thin edge whose deletion preserves this removable doubleton. To make this precise, we will first define a special type of removable edge which we call 'R-compatible'.

1.7.1 R-compatible edges

We use the abbreviation *R-graph* for a near-bipartite graph G with (fixed) removable doubleton R, and we shall refer to $H := G - R$ as its *underlying bipartite graph*. In the same spirit, an *R-brick* is a brick with a removable doubleton R.

A removable edge e of an R-graph G is *R-compatible* if it is removable in H as well. Equivalently, an edge e is R-compatible if $G - e$ and $H - e$ are both matching covered. For instance, the staircase St_8, shown in Figure 1.9, has two removable doubletons $R := \{\alpha, \beta\}$ and $R' := \{\alpha', \beta'\}$, and its unique removable edge e is R-compatible as well as R'-compatible.

Now, let G denote the R-brick shown in Figure 1.10a, where $R := \{\alpha, \beta\}$. The thin edge e is incident with an edge of R at a cubic vertex; consequently, $H - e$ has a vertex whose degree is only one, and so it is not matching covered. In particular, e is not R-compatible.

The brick shown in Figure 1.11 has two distinct removable doubletons $R := \{\alpha, \beta\}$ and $R' := \{\alpha', \beta'\}$. Its edges a_1u_1 and b_1w_1 are both R'-compatible, but neither of them is R-compatible. (In Section 1.7.3, we will generalize the graphs in Figures 1.10a and 1.11 to infinite families which play important roles in our work.)

Observe that, if e is an R-compatible edge of an R-graph G, then R is a removable doubleton of $G - e$, whence $G - e$ is also an R-graph; in particular, $G - e$ is near-bipartite. By Theorem 1.19, $G - e$ is a near-brick; and this proves the following. (See Figure 1.7.)

Proposition 1.20 *Every R-compatible edge is b-invariant.* □

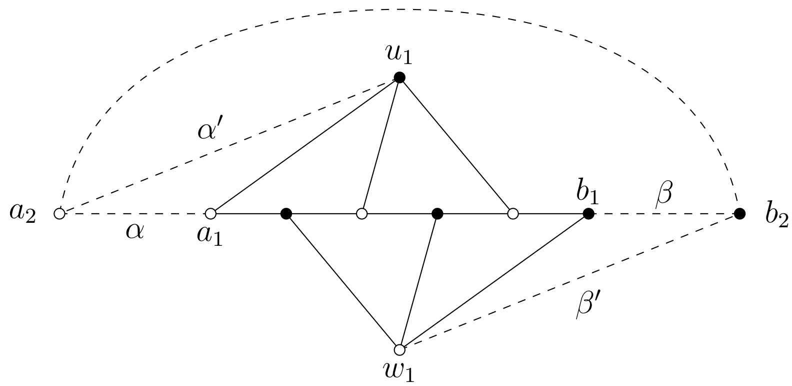


Figure 1.11: The edges a_1u_1 and b_1w_1 are R'-compatible, but they are not R-compatible

Furthermore, as we will see in Chapter 4, if e is an R-compatible edge of an R-brick G then the unique brick J of $G - e$ is also an R-brick; in particular, J is near-bipartite. The following is a special case of a theorem of Carvalho, Lucchesi and Murty [CLM99].

Theorem 1.21 [R-COMPATIBLE EDGE THEOREM] *Every R-brick distinct from K_4 and $\overline{C_6}$ has an R-compatible edge.*

In [CLM99], they proved a stronger result. In particular, they showed the existence of an R-compatible edge in R-graphs with minimum degree at least three, and used it to establish a generalization of Lovász's Theorem (1.6). (They did not use the term 'R-compatible'.) Using the notion of R-compatibility, we now define a thin edge whose deletion preserves the property of being near-bipartite.

1.7.2 R-thin edges

A thin edge e of an R-brick G is *R-thin* if it is R-compatible. Equivalently, an edge e is R-thin if it is R-compatible as well as thin, and in this case, the retract of $G - e$ is also an R-brick. See Figure 1.7.

As noted earlier, the staircase St_8, shown in Figure 1.9, has two removable doubletons R and R'. Its unique removable edge e is R-thin as well as R'-thin; to see this, note that the retract J of $St_8 - e$ is isomorphic to K_4 with multiple edges, and each of R and R' is a removable doubleton of J. It is easily verified that if G is a Norine-Thomas brick which is near-bipartite, then each of its thin edges is R-thin for some removable doubleton R.

It is desirable to characterize R-bricks free of R-thin edges, as this would yield a generation theorem for near-bipartite bricks analogous to Theorem 1.16. Using the R-compatible Edge Theorem (1.21) of Carvalho et al., we proved the following stronger result.

Theorem 1.22 [R-THIN EDGE THEOREM] *Every R-brick distinct from K_4 and $\overline{C_6}$ has an R-thin edge.*

We present a proof of the above theorem in Chapter 5. Our proof uses tools from the work of Carvalho et al. [CLM06], and the overall approach is inspired by their proof of the Thin Edge Theorem (1.15). The following is an immediate consequence of Theorem 1.22.

Theorem 1.23 *Given any R-brick G, there exists a sequence $G_1, G_2, \ldots, G_k$ of R-bricks such that:*

(i) G_1 *is either* K_4 *or* $\overline{C_6}$*,*

(ii) $G_k := G$*, and*

(iii) for $2 \leq i \leq k$*, there exists an R-thin edge* e_i *of* G_i *such that* G_{i-1} *is the retract of* $G_i - e_i$*.*

It follows from the above theorem that every near-bipartite brick can be generated from one of K_4 and $\overline{C_6}$ by means of the expansion operations. However, as in the case of Theorem 1.16, it has the shortcoming that, even if $G_k = G$ is a simple near-bipartite brick, the intermediate bricks $G_1, G_2, \ldots, G_k$ are not guaranteed to be simple. As before, we shall overcome this hurdle using the notion of strictly thin edges.

1.7.3 Strictly R-thin edges

An R-thin edge e of a simple R-brick G is *strictly R-thin* if it is strictly thin. In other words, a strictly R-thin edge e is one which is R-compatible as well as strictly thin; and in this case, the retract of $G - e$ is also a simple R-brick. See Figure 1.7.

For instance, let G denote the R-brick shown in Figure 1.12(a), where $R := \{\alpha, \beta\}$. The retract of $G - e$ is the truncated biwheel T_8 shown in Figure 1.12(b); consequently, e is strictly R-thin.

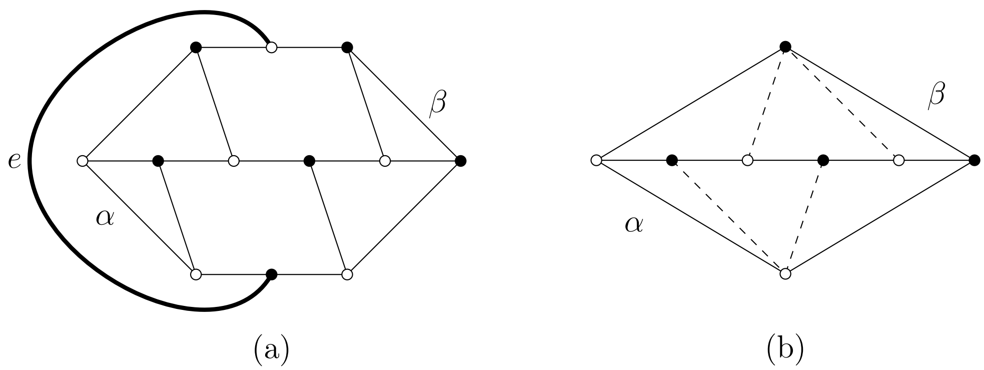


Figure 1.12: Edge e is strictly R-thin

Recall that the Norine-Thomas bricks are precisely those simple bricks which are free of strictly thin edges. In particular, every R-brick, which is a member of the Norine-Thomas families, is free of strictly R-thin edges. A natural question arises as to whether there are any simple R-bricks, different from the Norine-Thomas bricks, which are also free of strictly R-thin edges. It turns out that there indeed are such bricks; we have already encountered two examples in Figures 1.10a and 1.11, as explained below.

Let G denote the R-brick, shown in Figure 1.10a, where $R := \{\alpha, \beta\}$ is its unique removable doubleton. It can be checked that G has precisely four strictly thin edges, depicted by bold lines; these are similar under the automorphisms of the graph. As noted earlier, if e is any of these edges, then e is not R-compatible; furthermore, the retract of $G - e$ is isomorphic to the graph shown in Figure 1.10b, which is not even near-bipartite as it has three edge-disjoint triangles. Thus, the generation of G using the Norine-Thomas procedure cannot be achieved within the class of near-bipartite bricks.

Now, let G denote the brick shown in Figure 1.11; it has two removable doubletons $R := \{\alpha, \beta\}$ and $R' := \{\alpha', \beta'\}$. It may be verified that G has precisely two strictly thin edges, namely a_1u_1 and b_1w_1, each of which is R'-compatible but neither is R-compatible. In particular, G is free of strictly R-thin edges; in this sense it is similar to the graph in Figure 1.10a. On the other hand, G has strictly R'-thin edges; if e is any such edge then the retract of $G - e$ is a simple near-bipartite brick with removable doubleton R'. In this sense, G is different from the graph in Figure 1.10.

We will introduce seven infinite families of simple R-bricks which are free of strictly R-thin edges, and are different from the Norine-Thomas families. The members of these will be described using their specific bipartite subgraphs, each of which is either a ladder or a partial biwheel; see Figure 1.5. The occurrence of these subgraphs may be justified as follows. Let G be a simple R-brick which is free of strictly R-thin edges. If e is any R-thin edge of G, at least one end of e is cubic and the retract of $G - e$ has multiple edges. These strictures can be used to deduce that G contains either a ladder or a partial biwheel, or both, as subgraphs.

In our descriptions of these families, we use α and β to denote the edges of the (fixed) removable doubleton R. Apart from R, a member may have at most one removable doubleton which will be denoted as $R' := \{\alpha', \beta'\}$. We adopt the notational conventions stated in Section 1.3.3. (Recall that a partial biwheel of order six is also a ladder; for this reason, some of our families overlap.)

Pseudo-Biwheels. Let $K[A_1, B_1]$ denote a partial biwheel, of order at least eight, and with external spokes a_1u_1 and b_1w_1. Then the graph G obtained from K, by adding two new vertices a_2 and b_2, and by adding five new edges $\alpha := a_1a_2, \alpha' := u_1a_2, \beta := b_1b_2, \beta' := w_1b_2$ and a_2b_2, is called a *pseudo-biwheel*. Figure 1.11 shows the smallest pseudo-biwheel.

It is worth comparing the above with our desription of staircases in Section 1.3.3. Although a pseudo-biwheel G is free of strictly R-thin edges, the two external spokes of K, namely a_1u_1 and b_1w_1, are both strictly R'-thin.

In order to describe the members of the remaining six families, we need two (sub)graphs. For $i \in \{1, 2\}$, let $K_i[A_i, B_i]$ denote either a ladder or a partial biwheel with external rungs/spokes a_iu_i and b_iw_i, such that K_1 and K_2 are disjoint.

Double Biwheels, Double Ladders and Laddered Biwheels of Type I. Let the graph G be obtained from $K_1 \cup K_2$, by adding edges $\alpha := a_1a_2$ and $\beta := b_1b_2$, by identifying vertices u_1 and u_2, and by identifying vertices w_1 and w_2. There are three possibilities depending on the graphs K_1 and K_2. In the case in which K_1 and K_2 are both partial biwheels, G is a *double biwheel of type I*. Likewise, in the case in which K_1 and K_2 are both ladders, G is a *double ladder of type I*. Finally, when one of K_1 and K_2 is a partial biwheel and the other one is a ladder, G is a *laddered biwheel of type I*.

A member of any of these families has a unique removable doubleton R, and is free of strictly R-thin edges. The graph in Figure 1.10a is the smallest member of each of these families, although its drawing is suggestive of a double biwheel. Figure 1.13a shows a double ladder. A laddered biwheel is obtained from the graph in Figure 1.13b by identifying u_1 with u_2, and likewise, w_1 with w_2.

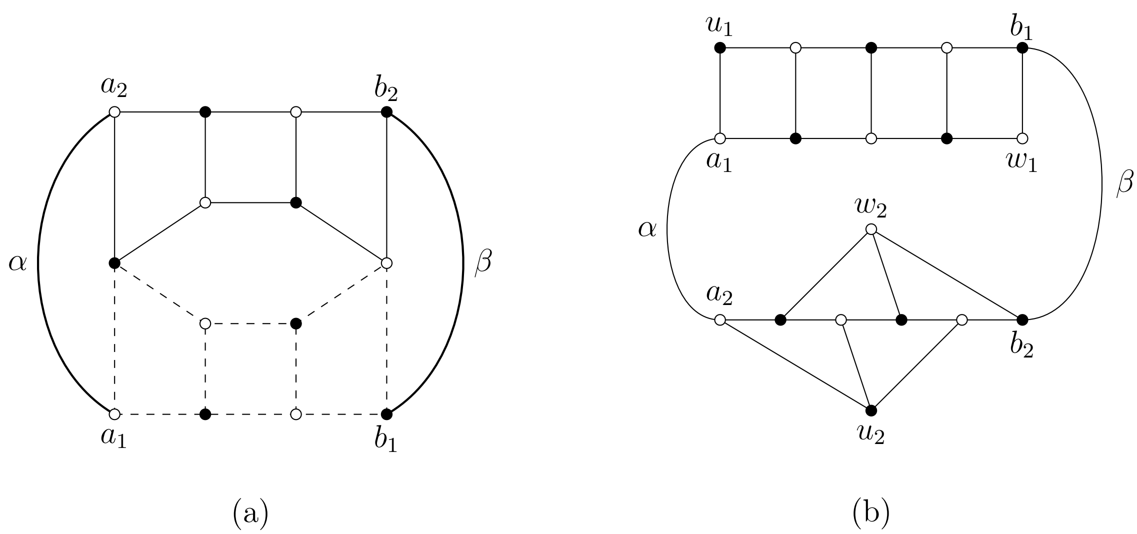


Figure 1.13: (a) A double ladder of type I ; (b) A laddered biwheel of type I is obtained by identifying u_1 with u_2 and likewise w_1 with w_2

Double Biwheels, Double Ladders and Laddered Biwheels of type II. Let the graph G be obtained from $K_1 \cup K_2$, by adding four edges, namely, $\alpha := a_1a_2$, $\beta := b_1b_2$, $\alpha' := u_1w_2$ and $\beta' := w_1u_2$. As before, we have three possibilities. In the case in which K_1 and K_2 are both partial biwheels of order at least eight, G is a *double biwheel of type II*. Likewise, in the case in which K_1 and K_2 are both ladders, G is a *double ladder of type II*. Finally, when one of K_1 and K_2 is a partial biwheel of order at least eight, and the other one is a ladder, G is a *laddered biwheel of type II*.

A member of any of these families has two removable doubletons R and R', and it is free of strictly R-thin edges. However, a double biwheel or a laddered biwheel as shown in Figure 1.14 has strictly R'-thin edges; these are the external spokes of a partial biwheel of order at least eight as depicted by the bold lines in the figure.

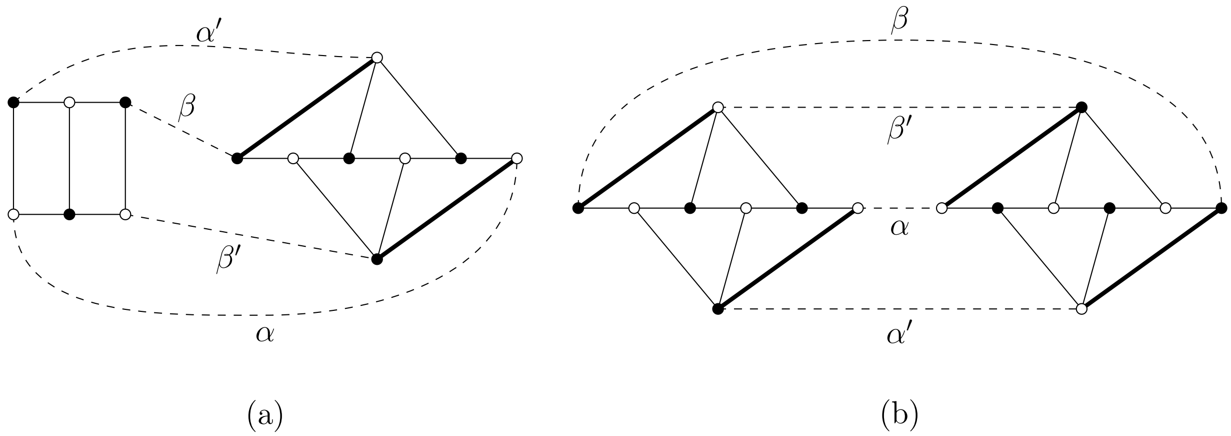


Figure 1.14: (a) A laddered biwheel of type II ; (b) A double biwheel of type II

On the other hand, a double ladder, as shown in Figure 1.15, is free of strictly R'-thin edges as well. This may be explained as follows. Every double ladder is cubic, and it has precisely four strictly thin edges; these are the external rungs of the two ladders, depicted by bold lines in the figure. One end of any such edge, say e, is incident with an edge of R and the other end is incident with an edge of R'; since each end of e is cubic, it is neither R-compatible nor R'-compatible.

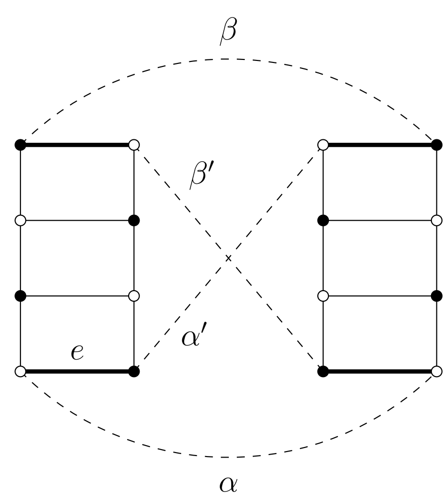


Figure 1.15: A double ladder of type II

Using a strengthening of the R-thin Edge Theorem (1.22), we proved that the seven families described above and four of the Norine-Thomas families are the only simple R-bricks which are free of strictly R-thin edges.

Theorem 1.24 [STRICTLY R-THIN EDGE THEOREM] *Let G be a simple R-brick. If G is free of strictly R-thin edges then G belongs to one of the following infinite families:*

(i) Truncated biwheels

(ii) Prisms

(iii) Möbius ladders

(iv) Staircases

(v) Pseudo-biwheels

(vi) Double biwheels of type I

(vii) Double ladders of type I

(viii) Laddered biwheels of type I

(ix) Double biwheels of type II

(x) Double ladders of type II

(xi) Laddered biwheels of type II

We present a proof of the above theorem in Chapter 6. Our proof is inspired by the proof of the Strictly Thin Edge Theorem (1.17) given by Carvalho et al. [CLM08], and uses several of their results and techniques.

We shall denote by $\mathcal{N}$ the union of all of the eleven families which appear in the statement of Theorem 1.24. The following is an immediate consequence.

Theorem 1.25 *Given any simple R-brick G, there exists a sequence $G_1, G_2, \ldots, G_k$ of simple R-bricks such that:*

(i) $G_1 \in \mathcal{N}$,

(ii) $G_k := G$, *and*

(iii) for $2 \leq i \leq k$, *there exists an R-thin edge e_i of G_i such that G_{i-1} is the retract of $G_i - e_i$.*

In other words, every simple R-brick can be generated from some member of $\mathcal{N}$ by repeated application of the expansion operations such that at each step we have a simple R-brick.

Finally, recall that members of three of the aforementioned families do have strictly R'-thin edges, where $R' := \{\alpha', \beta'\}$ in our description of these families; these are pseudo-biwheels, double biwheels of type II and laddered biwheels of type II. In view of this, we say that a strictly thin edge e of a simple near-bipartite brick G is *compatible* if it is R-compatible for some removable doubleton R. We thus have the following theorem (with eight infinite families) alluded to in the abstract.

Theorem 1.26 *Let G be a simple near-bipartite brick. If G is free of compatible strictly thin edges then G belongs to one of the following infinite families:*

(i) Truncated biwheels

(ii) Prisms

(iii) Möbius ladders

(iv) Staircases

(v) Double biwheels of type I

(vi) Double ladders of type I

(vii) Laddered biwheels of type I

(viii) Double ladders of type II

Four of the families in the above theorem are Norine-Thomas families; these are free of strictly thin edges. As we did in Figure 1.10, it may be verified that if G is a member of any of the remaining four families and e is any strictly thin edge of G then the retract J of $G - e$ is not near-bipartite. (For example, consider the graph G and edge e shown in Figure 1.15, and let J be the retract of $G-e$. It can be checked that J has four odd cycles, C_0, C_1, C_2 and C_3, such that C_1, C_2 and C_3 are edge-disjoint with C_0, and furthermore, there is no single edge which belongs to all three of them.)

1.8 Summary of main contributions

Here, we summarize the main results proved in this thesis.

Part I - Conformal Subgraphs (these results have appeared in [KM16])

- In Chapter 2, for any cubic brick J, we reduce the problem of characterizing J-free graphs to that of characterizing J-free bricks. In particular, we prove Theorem 1.10, which states that a nonbipartite matching covered graph G is J-free if and only if each of its bricks is J-free.

- In Chapter 3, we establish our characterizations of K_4-free planar bricks and $\overline{C_6}$-free planar bricks:
  - We prove Theorem 1.11 which states that a planar brick is K_4-free if and only if it has precisely two odd faces.
  - We prove Theorem 1.12 which states that a planar brick is $\overline{C_6}$-free if and only if its underlying simple graph is either an odd wheel, or an odd staircase, or the Tricorn.

Part II - Brick Generation

- In Chapter 5, we present a proof of the R-thin Edge Theorem (1.22) which states that every R-brick distinct from K_4 and $\overline{C_6}$ has an R-thin edge. This yields a generation procedure for near-bipartite bricks.
- In Chapter 6, we present a proof of the Strictly R-thin Edge Theorem (1.24) which gives a complete characterization of those simple R-bricks which are free of strictly R-thin edges. This yields a generation procedure for simple near-bipartite bricks.

Chapter 2

Conformal subgraphs and tight cuts

We recall Lovász's Theorem (1.6) which says that every nonbipartite matching covered graph is K_4-based, or is $\overline{C_6}$-based, or both. As discussed in Section 1.2.2, in the first part of this thesis, our goal is to characterize those planar matching covered graphs which are K_4-free, and those which are $\overline{C_6}$-free.

In this chapter, we reduce these problems to the case of bricks by proving Theorem 1.10, which is restated below.

Theorem 1.10 [Reduction to the case of Bricks] *Let J denote any cubic brick. A nonbipartite matching covered graph is J-free if and only if each of its bricks is J-free.*

It suffices to prove Theorem 2.8, which states that for any nontrivial tight cut C, the given matching covered graph G is J-free if and only if both C-contractions of G are J-free.

Recall that a matching covered subgraph H of a matching covered graph G is conformal if every perfect matching of H extends to a perfect matching of G. The following proposition is an easy consequence.

Proposition 2.1 *Let H be a conformal subgraph of a matching covered graph G, and let C be a tight cut of G. Then $C \cap E(H)$ is a tight cut of H.* $\square$

2.1 Bi-subdivisions of bricks

Let H be a bi-subdivision of a simple brick J. We refer to the vertices of H of degree three or more as *branch vertices*, and the remaining vertices as *subdivision vertices*. Each branch vertex of H corresponds to a unique vertex of J, and we refer to both of these using the same label. As shown in Figure 2.1, for each edge uv of J, there is a unique odd path in H, denoted P_{uv}, between vertices u and v, such that each internal vertex (possible none) of P_{uv} is a subdivision vertex. If $P_{uv} := w_1w_2 \ldots w_{2k}$ where $w_1 := u$ and $w_{2k} := v$, we say that an edge w_iw_{i+1} is of *odd* parity if i is odd; otherwise of *even* parity. Note that the first and last edges (possibly not distinct) of the path P_{uv} are both of odd parity, regardless of the order in which it is traversed.

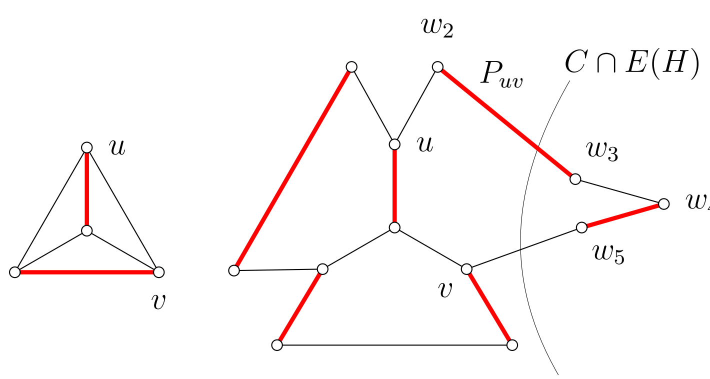


Figure 2.1: Perfect matchings M_J, M_H and M_G

Let G be a J-based simple matching covered graph. By definition, G has a conformal subgraph H which is a bi-subdivision of J. With each perfect matching M_J of J, we associate a perfect matching M_H of H as follows: for each edge uv in M_J, the set M_H contains precisely the odd edges of the path P_{uv}, and for each edge uv in $E(J) - M_J$, the set M_H contains precisely the even edges of P_{uv}. In fact, every perfect matching of H arises this way. Thus $M_J \to M_H$ is a bijective correspondence between the sets of perfect matchings of J and H. Since H is a conformal subgraph of G, the matching M_H can be extended to a perfect matching M_G of G. This extension is, in general, not unique. Figure 2.1 illustrates these observations, where the edges depicted in bold lines indicate the perfect matchings M_J, M_H and M_G.

Now let C be a nontrivial tight cut of G. An edge uv of J is a *C-crossing edge* if the path P_{uv} meets the cut C in at least one edge, that is, $E(P_{uv}) \cap C$ is nonempty; and in this case, we say that P_{uv} is a *C-crossing path.* Furthermore, if $|E(P_{uv}) \cap C| = 1$ we say that P_{uv} crosses C once, and if $|E(P_{uv}) \cap C| = 2$ we say that P_{uv} crosses C twice, and so on. Referring to Figure 2.1, the edge uv of J is the only C-crossing edge, and the corresponding C-crossing path P_{uv} crosses C twice — once in the edge w_2w_3 of even parity, and once in the edge w_5v of odd parity.

In the following result, we establish some simple properties of C-crossing paths with respect to a given tight cut C of G. In its proof, we shall make implicit use of Proposition 2.1.

Proposition 2.2 *Let J be a brick, and G be a J-based matching covered graph. Let H be a conformal subgraph of G which is a bi-subdivision of J, and let C be a nontrivial tight cut of G.*

(i) For a C-crossing path P_{uv}, any two C-crossing edges of P_{uv} must be of opposite parity. (Thus, $|C \cap E(P_{uv})| \leq 2$.)

(ii) If a C-crossing path P_{uv} crosses C twice, then there are no other C-crossing paths.

(iii) If P_{uv} and P_{uw} are two C-crossing paths, then each of them must cross C in an edge of odd parity.

Proof: Suppose first that a path P_{uv} crosses C at least twice, and let e and f be two distinct C-crossing edges. In case both e and f are odd edges, let M_J be any perfect matching of J containing the edge uv, and in case both e and f have even parity, let M_J be a perfect matching of J which does not contain the edge uv. Then the perfect matching M_H of H contains both e and f. Since H is a conformal subgraph of G, there would then be a perfect matching M_G of G containing e and f, implying that $|M_G \cap C| \geq 2$. This is impossible because C is a tight cut. This proves the first part of the assertion.

Now suppose P_{st} and P_{uv} are two distinct C-crossing paths. Note that s need not be distinct from u and v, and the same holds for t. Assume that P_{uv} crosses C twice; let f_1 and f_2 be the two C-crossing edges. Without loss of generality, assume that f_1 is an odd edge and f_2 is an even edge. Let e be a C-crossing edge of P_{st}. In case e is an odd edge, let M_J be a perfect matching of J which contains the edge st, and in case e is an even edge, let M_J be a perfect matching of J which does not contain the edge st. Depending on whether M_J contains the edge uv or not, the perfect matching M_H of H contains either f_1 or f_2, respectively. By choice of M_J, it follows that M_H contains e. Since H is a conformal subgraph of G, we can extend M_H to a perfect matching M_G of G, leading to the same contradiction as before.

Now suppose P_{uv} and P_{uw} are two distinct C-crossing paths. It follows that each of these paths crosses C exactly once, say in edges e and f, respectively. In case e is odd and f is even, let M_J be a perfect matching of J which contains the edge uv, and in case both e and f have even parity, let M_J be a perfect matching of J which contains neither uv nor uw (such a perfect matching exists since u has at least three neighbours in J). Then the perfect matching M_H of H contains both e and f. Since H is a conformal subgraph of G, we can extend M_H to a perfect matching M_G of G, leading to the same contradiction as before. □

2.2 Three lemmas on bricks

Here, we state three useful lemmas; the first two of these are specific to cubic bricks. The following is an immediate consequence of a result of Plesník [LP86, Theorem 3.4.2].

Lemma 2.3 *Let J be a cubic brick, and e_1 and e_2 be two edges of J. Then $J - e_1 - e_2$ has a perfect matching.* □

Lemma 2.4 *For any vertex v of a cubic brick J, the graph $J - v - N_J(v)$ has a perfect matching.*

Proof: Let u_1 and u_2 denote two neighbours of v in J. By Theorem 1.9, J is bicritical, whence $J - u_1 - u_2$ has a perfect matching. The restriction of this matching to the edge set of $J - v - N_J(v)$ is a perfect matching of that graph. □

The following result is an easy consequence of the above lemma.

Corollary 2.5 *Let H be a bi-subdivision of a cubic brick J, and let v be a branch vertex of H. Then the graph $H - v - N_H(v)$ has a perfect matching.* □

Lemma 2.6 *Let J be a brick (not necessarily cubic), let Y be a subset of $V(J)$ with $|Y| \geq |\overline{Y}| \geq 2$, and let uv be an edge of J with $u \in Y$ and $v \in \overline{Y}$. Then there exists a perfect matching M_J of J such that:*

$$|M_J \cap (\partial(Y) - uv)| \geq 2 \tag{2.1}$$

<ins>Proof</ins>: If J has just four vertices, then J is K_4, and the statement is obvious. So, we may assume that $|Y| \geq 3$. If $|Y|$ is odd, take any perfect matching M_J of J with $|M_J \cap \partial(Y)| \geq 3$. (Such a perfect matching must exist because J is a brick, and $\partial(Y)$ is a nontrivial odd cut of J.) If $|Y|$ is even, then take take any perfect matching M_J of J with $|M_J \cap \partial(Y - u)| \geq 3$. It is now easy to see that M_J satisfies the required inequality. □

2.3 Cubic bricks

We shall now proceed to prove Theorem 2.8 which implies the main result of this chapter, namely Theorem 1.10.

Observe that, if H is a bi-subdivision of a cubic brick J, then bicontracting a vertex of degree two of H results in another bi-subdivision of J. Furthermore, any bi-subdivision of H is a bi-subdivision of J as well. This leads us to the following proposition, which we shall use implicitly in the proof of Theorem 2.8.

Proposition 2.7 *Let H be a bi-subdivision of a cubic brick J.*

(i) Let P be an even path in H all of whose internal vertices are subdivision vertices. Then the graph obtained from H by deleting the internal vertices of P and identifying its ends is also a bi-subdivision of J.

(ii) Any graph obtained from H by replacing an edge of H by an odd path is also a bi-subdivision of J. □

Theorem 2.8 *Let J be a cubic brick and G be a matching covered graph. Let $C := \partial(X)$ be a nontrivial tight cut in G. Then G is J-based if and only if at least one of $G/(\overline{X} \to \overline{x})$ and $G/(X \to x)$ is J-based.*

Proof: Suppose that G is J-based, that is, G has a conformal subgraph H such that H is a bi-subdivision of J. Let Y and $\overline{Y}$, respectively, denote the sets of branch vertices in X and $\overline{X}$. Adjust notation so that $|Y| \geq |\overline{Y}|$. We shall first show that $|\overline{Y}| \leq 1$. Assume to the contrary that $|Y| \geq |\overline{Y}| \geq 2$. In this case, clearly, H must have at least three C-crossing paths, and all of these cross C in exactly one edge (by Proposition 2.2). Suppose that two of the C-crossing edges e_1 and e_2 are even edges, then Lemma 2.3 implies that there exists a perfect matching M_H of H which contains both e_1 and e_2. By extending M_H to a perfect matching M_G of G, we have $|M_G \cap C| \geq 2$. This is impossible because C is a tight cut of G. Hence we may assume that at most one C-crossing edge is an even edge. If there is such an edge e, let P_{uv} be the C-crossing path which contains the edge e, where $u \in Y$ and $v \in \overline{Y}$. If there is no such edge, let $u \in Y$ and $v \in \overline{Y}$ be two arbitrary vertices of J which are adjacent. By Lemma 2.6, there exists a perfect matching M_J of J satisfying (2.1). The perfect matching M_H of H that corresponds to M_J meets the tight cut $\partial(X)$ in at least two edges, resulting in a contradiction. Thus, $|\overline{Y}| \leq 1$. We split the rest of the proof into two cases depending on whether $|\overline{Y}|$ is zero or one.

If $|\overline{Y}| = 0$, then all the branch vertices of H lie in X. It follows from Proposition 2.2 that there is at most one C-crossing path. In case there are no C-crossing paths, H itself is a conformal subgraph of $G/\overline{X}$. Otherwise, let P_{uv} be the unique C-crossing path. Proposition 2.2 implies that P_{uv} crosses C in exactly two edges which are of different parities; that is, an odd number of subdivision vertices of P_{uv} lie in $\overline{X}$. Let H_1 denote the subgraph of $G/(\overline{X} \to \overline{x})$, obtained from H, by identifying all the subdivision vertices in $\overline{X}$ with the single vertex $\overline{x}$. Then H_1 is a bi-subdivision of J (Proposition 2.7), and is a conformal subgraph of $G/\overline{X}$.

Finally, if $|\overline{Y}| = 1$, there is precisely one branch vertex, say v, of H which lies in $\overline{X}$. By Proposition 2.2, it follows that the only C-crossing paths of H are those with one end v, and that each of them crosses C exactly once in an edge of odd parity. In other words, each of these C-crossing paths has an even number of subdivision vertices (possibly zero) which lie in $\overline{X}$. Consider the subgraph H_1 of $G/(\overline{X} \to \overline{x})$, obtained from H, by replacing all of these subdivision vertices and the branch vertex v with the single vertex $\overline{x}$. As in the previous case, note that H_1 is a bi-subdivision of J, and is a conformal subgraph of $G/\overline{X}$. This concludes the proof of the 'if' part of the assertion.

Now, to prove the converse, suppose that $G/(\overline{X} \to \overline{x})$ is J-based; that is, $G/(\overline{X} \to \overline{x})$ has a conformal subgraph H_1 such that H_1 is a bi-subdivision of J. In case the vertex $\overline{x} \notin V(H_1)$, then H_1 itself is a subgraph of G. It can be easily checked that H_1 is a conformal subgraph of G.

Now suppose $\overline{x} \in V(H_1)$. It may either be a subdivision vertex of H_1 or a branch vertex.

In the former case, let u_1, u_2, belonging to X, be the neighbours of $\overline{x}$ in H_1, and in the latter case, let u_1, u_2, u_3, belonging to X, be the neighbours of $\overline{x}$ in H_1. Each u_i must clearly have a neighbour in G that belongs to $\overline{X}$. Select one such neighbour v_i of u_i and identify the edge $u_i\overline{x}$ of H_1 with the edge u_iv_i of G. (We admit the possibility that $v_i = v_j$, for $i \neq j$.) Furthermore, let M_i be a perfect matching of G containing the edge u_iv_i, such that the restriction of M_i to $E(H_1)$ is a perfect matching of H_1. Let P denote the (M_1, M_2)-alternating path in G starting with the edge u_1v_1 of M_1, and ending with the edge v_2u_2 of M_2. In addition, when $\overline{x}$ is a branch vertex of H_1, let Q denote the (M_3, M_2)-alternating path in G starting with the edge u_3v_3 of M_3, and ending with the edge v_2u_2 of M_2.

Having established the notation common to both cases, for clarity, let us first deal with the case in which $\overline{x}$ is a subdivision vertex of H_1. In this case, let H be the subgraph of G, obtained from H_1, by replacing the path $u_1\overline{x}u_2$ of length two by the even path P. It is easy to see that H is a bi-subdivision of J and that the restriction of M_1 (or of M_2) to $E(H)$ is a perfect matching of H, implying that H is a conformal subgraph of G.

Now, suppose that $\overline{x}$ is a branch vertex of H_1. In this case, let w be the first vertex of the (M_3, M_2)-alternating u_3u_2-path Q that lies on the (M_1, M_2)-alternating u_1u_2-path P. Let P_1 and P_2 denote the u_1w- and u_2w-segments of P, respectively, and let P_3 denote the u_3w-segment of Q. These three paths have the end vertex w in common, but are otherwise disjoint. Let H denote the subgraph of G obtained from H_1 by replacing, for $i = 1, 2, 3$, the edge $u_i\overline{x}$ by the path P_i. Clearly H is a conformal subgraph of G because the restriction of M_2 to $E(G - V(H))$ is a perfect matching of $G - V(H)$. The graph H would be a bi-subdivision of J as well if all the P_i are odd paths. The path P_3, being an alternating path starting and ending with an edge of M_3, is clearly odd. The path P is an even path as it is an alternating path which starts with an edge of M_1 and ends with an edge of M_2. Let us proceed to show that the two segments P_1 and P_2 of P must both be of odd length. If this is not the case, both P_1 and P_2 are even, and the vertices v_1, v_2 and v_3 would have to be distinct, and the tree $P_1 \cup P_2 \cup P_3$ would have a perfect matching, say N', containing the three edges u_1v_1, u_2v_2 and u_3v_3. By Corollary 2.5, the graph $H_1 - \overline{x} - N_{H_1}(\overline{x})$ has a perfect matching, say N''. Then $N = N' \cup N''$ would be a perfect matching of H containing the three edges u_1v_1, u_2v_2 and u_3v_3. As H is a conformal subgraph of G, there would be a perfect matching of G containing N, and it would have three edges in common with C. This is impossible because C is a tight cut of G.

Hence the three paths P_1, P_2 and P_3 must all be odd paths [1], and H is a bi-subdivision of J, with w as one of its branch vertices. This completes the proof of the assertion. □

[1]Each of P_1 and P_2 is an (M_1, M_2)-alternating path; P_1 starts and ends with an M_1-edge, whereas P_2 starts and ends with an M_2-edge.

As noted earlier, Theorem 1.10 immediately follows from the above theorem. Cláudio Lucchesi communicated to us an alternative proof of Theorem 2.8 which is based on the theory of ear decompositions.

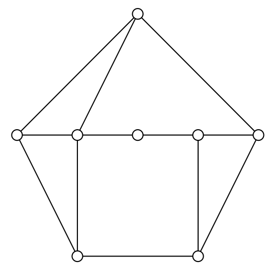


Figure 2.2: A matching covered graph whose unique brick is W_5

We conclude this chapter by noting that the statement of Theorem 2.8 would not be valid if J were an arbitrary brick or if J were a cubic brace, as explained below.

Let G be the matching covered graph shown in Figure 2.2. The unique brick J of G, obtained by bicontracting the degree two vertex, is isomorphic to the odd wheel W_5. In particular, J is W_5-based. However, since the maximum degree in G is four, and W_5 has a vertex of degree five, no bi-subdivision of W_5 can be a conformal subgraph of G.

Now, suppose that G is the matching covered graph shown in Figure 1.3. The unique brace J of G is isomorphic to $K_{3,3}$. In particular, J is $K_{3,3}$-based. However, it can be easily verified that G is $K_{3,3}$-free. (To see this, suppose that G has a conformal subgraph H which is a bi-subdivision of $K_{3,3}$. Since G and $K_{3,3}$ are both cubic, no subgraph of G is isomorphic to $K_{3,3}$. Thus H is a spanning subgraph of G. Being bipartite, H can not use all three edges of the unique triangle T of G. It follows from the degrees of the vertices that $H = G - f$, where f is any edge of T. Observe that H, although a subdivision of $K_{3,3}$, is not a bi-subdivision, contrary to our hypothesis.)

In the next chapter, we shall establish our characterizations of K_4-free planar bricks and $\overline{C_6}$-free planar bricks.

Chapter 3

Planar bricks

In the last chapter, we reduced the problems of characterizing K_4-free and $\overline{C_6}$-free matching covered graphs to the case of bricks.

In Sections 3.4 and 3.5, we shall establish our characterizations of K_4-free planar bricks (Theorem 1.11), and of $\overline{C_6}$-free planar bricks (Theorem 1.12), respectively — thus solving the above problems for planar matching covered graphs. These theorems are restated below.

Theorem 1.11 [Characterization of K_4-free Planar Bricks] *A planar brick is K_4-free if and only if it has precisely two odd faces.*

Theorem 1.12 [Characterization of $\overline{C_6}$-free Planar Bricks] *A planar brick is $\overline{C_6}$-free if and only if its underlying simple graph is either an odd wheel, or an odd staircase, or the Tricorn.*

Our proofs for each of these results rely on the generation procedure for simple bricks established by Norine and Thomas; see Theorem 1.18.

Observe that, for a cubic brick J, a matching covered graph G is J-free if and only if the underlying simple graph of G is J-free. Thus we may restrict ourselves to simple planar graphs and bricks (throughout this chapter). We now proceed to discuss planar graphs and their embeddings.

A planar graph may have different embeddings in the plane. In any embedding of a 2-connected planar graph in the plane, every face is bounded by a cycle. A *facial cycle* in such an embedding is a cycle of the graph that bounds a face. It is easy to see that if C is any facial cycle of a 3-connected planar graph, then C cannot have any chords and that $G - V(C)$ is connected (see [BM08, page 266]). Furthermore, it can be verified that if u and v are two nonadjacent vertices of C, then there is a uv-path in G that is internally disjoint from C.

Two embeddings of a 2-connected planar graph in the plane are regarded as the same if the facial cycles in the two embeddings are the same; otherwise they are different. According to a well-known result due to Whitney [Whi33], every simple 3-connected planar graph has a unique embedding in the plane. In particular, every planar brick has a unique embedding. Thus, we may refer to faces of planar bricks without any ambiguity. Even when we are dealing with planar matching covered graphs which are not bricks, we restrict ourselves to graphs with a given embedding, that is to plane graphs, and thereby avoid any ambiguity as to which cycles are facial.

According to a deep result due to Tutte, every 3-connected planar graph has an embedding in the plane in which all facial cycles are convex polygons. Tutte [Tut84, Theorem XI.63] also proved the following relevant result.

Proposition 3.1 *The boundaries of any two faces of a simple 3-connected planar graph have at most two vertices in common, and if they do have two vertices in common, then those vertices are adjacent.* □

3.1 Number of odd faces

A face F in a 2-connected plane graph G is *even* or *odd* according to the parity of the length of the cycle of G that bounds F. We denote the number of odd faces in G by $\mathsf{f}_{\mathsf{odd}}(G)$. (This function $\mathsf{f}_{\mathsf{odd}}(G)$ will play a crucial role in Section 3.4.) We note that $\mathsf{f}_{\mathsf{odd}}(G)$ is always even, and it is zero if and only if G is bipartite.

Proposition 3.2 *If G is a planar brick, then $\mathsf{f}_{\mathsf{odd}}(G) \geq 2$. Furthermore, if G is not an odd wheel, then it has at least two vertex-disjoint odd faces.*

Proof: The inequality $\mathsf{f}_{\mathsf{odd}}(G) \geq 2$ follows from the fact that G is not bipartite.

Now, suppose that G is not an odd wheel. As per a result of Carvalho et al. [CLM06], G has a nontrivial cut $C := \partial(X)$ such that both C-contractions of G are nonbipartite matching covered graphs; whence $G[X]$ and $G[\overline{X}]$ are both nonbipartite. Thus G has two vertex-disjoint odd cycles; consequently, G has at least two vertex-disjoint odd faces. $\square$

Let G be a plane matching covered graph, let v_0 be a vertex of degree two of G, with v_1 and v_2 as its two neighbours. Suppose that H is the plane graph obtained from G by bicontracting v_0 and denoting the resulting contraction vertex by v. Then there is a natural one-to-one correspondence between the sets of faces of G and H (see Figure 3.1). As shown in the figure, F_2 and F_4 are the only faces of G whose bounding cycles, viewed as a set of edges, are different from the bounding cycles of the corresponding faces of H — in G, these faces have the path $v_1v_0v_2$ in their boundary, whereas in H they just have v instead of $v_1v_0v_2$. It follows that the parity of a face of H is the same as the parity of the corresponding face of G.

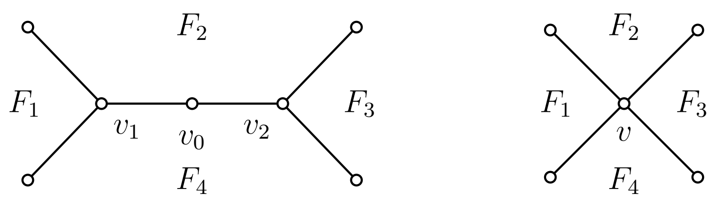


Figure 3.1: Correspondence between faces of G and H

Let G be a plane brick, and let e be an edge of G. If e lies inside an odd face of $G - e$, then $\mathsf{f}_{\mathsf{odd}}(G-e) = \mathsf{f}_{\mathsf{odd}}(G)$; and if e lies inside an even face of $G - e$, then either $\mathsf{f}_{\mathsf{odd}}(G-e) = \mathsf{f}_{\mathsf{odd}}(G)$, or $\mathsf{f}_{\mathsf{odd}}(G-e) = \mathsf{f}_{\mathsf{odd}}(G) - 2$. Now suppose that H is the retract of $G - e$. Since H is obtained from $G - e$ by either zero, one, or two bicontractions, $\mathsf{f}_{\mathsf{odd}}(H) = \mathsf{f}_{\mathsf{odd}}(G-e)$. We thus have the following relationship between the number of odd faces of G and H.

Proposition 3.3 *Let e be a strictly thin edge in a plane brick G, and let H be the retract of $G - e$. Then $\mathsf{f}_{\mathsf{odd}}(H) \leq \mathsf{f}_{\mathsf{odd}}(G) \leq \mathsf{f}_{\mathsf{odd}}(H) + 2$.* $\square$

It is easily verified that if H is a subgraph of a plane 2-connected graph G, then $\mathsf{f}_{\mathsf{odd}}(G) \geq \mathsf{f}_{\mathsf{odd}}(H)$. Note that K_4 has exactly four odd faces, and so does any bi-subdivision of K_4. This immediately implies the following.

Proposition 3.4 *Let G be a K_4-based plane matching covered graph. Then* $\mathsf{f}_{\mathsf{odd}}(G) \geq 4$. $\Box$

The above provides a necessary condition for a plane matching covered graph to be K_4-based. It is not sufficient in general. For instance, let G denote the graph shown in Figure 3.2. Observe that G has precisely four odd faces. However, it is K_4-free, as can be verified using Theorem 1.10 — its tight cut decomposition results in two bricks, each isomorphic to $\overline{C_6}$ (which is K_4-free), and the cube (which is a brace).

However, we show that for planar bricks this condition is indeed sufficient, that is, if a planar brick has four or more odd faces then it is indeed K_4-based. Our proof of this assertion relies on the fact that in a planar brick with precisely two odd faces, each even facial cycle is conformal. In order to prove this by induction we need to use a stronger induction hypothesis concerning all facial cycles. This involves the notion of critical graphs.

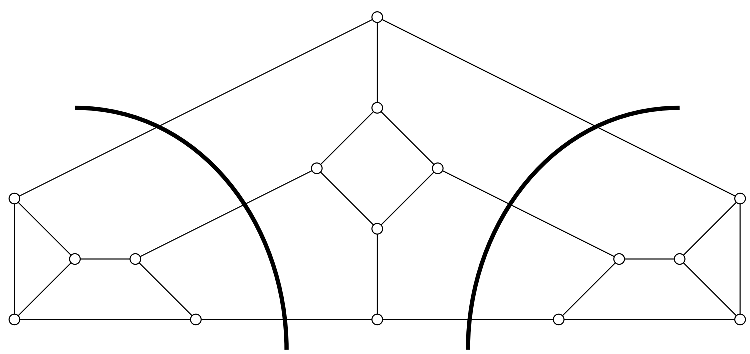


Figure 3.2: K_4-free graph with four odd faces — the bold lines indicate tight cuts

3.2 Critical graphs

A graph G is *critical* if for any vertex v, the graph $G - \{v\}$ has a perfect matching. An *ear decomposition* of a 2-connected critical graph G is a sequence $G_1 \subset G_2 \subset \cdots \subset G_r$ of 2-connected critical subgraphs of G such that (i) G_1 is an odd cycle and $G_r := G$, and (ii) for each i such that $1 \leq i \leq r-1$, G_{i+1} is the union of G_i and exactly one single ear P_{i+1} of G_{i+1}. The following is a well-known result. See [LP86, page 196].

Theorem 3.5 *A 2-connected graph G is critical if and only if it has an ear decomposition $G_1 \subset G_2 \subset \cdots \subset G_r$, starting with an odd cycle G_1. Furthermore, G_1 can be chosen to contain any arbitrary vertex.*

Let H be a graph and v be a vertex of H of degree at least two. We recall the bi-splitting operation defined in Section 1.5.2, where v is replaced by two new vertices v_1 and v_2, and a third vertex v_0 is introduced and joined to both v_1 and v_2. The next result proves that bi-splitting a vertex of a 2-connected critical graph yields a graph which is also 2-connected and critical.

Lemma 3.6 *Let H be a 2-connected critical graph, and $v \in V(H)$. If G is obtained from H by bi-splitting v, then G is also a 2-connected critical graph.*

Proof: We adopt the notation from the preceding paragraph. Since H is 2-connected and critical, Theorem 3.5 implies that H admits an ear decomposition $H_1 \subset H_2 \subset \cdots \subset H_r$, where H_1 can be chosen to be an odd cycle which contains the vertex v. For each i such that $1 \leq i \leq r-1$, we have $H_{i+1} := H_i \cup P_{i+1}$ where P_{i+1} is a single ear of H_{i+1}. To show that G is also 2-connected and critical, we will extend this ear decomposition of H to an ear decomposition of G, say $G_1 \subset G_2 \subset \cdots \subset G_r$ — such that $G_{i+1} := G_i \cup Q_{i+1}$ and Q_{i+1} is a single ear of G_{i+1}.

Let s and t be the unique distinct neighbours of v in the odd cycle H_1. Note that s must be a neighbour of exactly one of v_1 and v_2 in the graph G, and a similar statement holds for t. We divide the rest of the proof into two cases, depending on the neighbourhoods of v_1 and v_2:

Case 1: In G, the vertices s and t are both neighbours of v_1. (The case when s and t are both neighbours of v_2 is analogous.)

Let G_1 be the same as H_1, except that v_1 plays the role of vertex v. Let H_k be the first subgraph in the sequence $H_1 \subset H_2 \subset \cdots \subset H_r$, such that it contains an edge yv for some $y \in N_G(v_2)$. We note that $k > 1$. For all j such that $1 < j < k$, we define the odd path Q_j to be the same as P_j, except that v_1 plays the role of v, if applicable. Suppose $P_k := vy \ldots z$ for some vertex z in H_{k-1}. Now we define Q_k to be the odd path, obtained by stretching P_k, as follows: let v_1 play the role of v, and replace the edge vy by the odd path $v_1v_0v_2y$ — that is, $Q_k := v_1v_0v_2y \ldots z$. For the single ears P_j that follow (that is, $j > k$), we define the odd path Q_j to be the same as P_j, except that the role of

v is played by either v_1 or v_2, as appropriate. More precisely, suppose $P_j := vw\dots x$. If $w \in N_G(v_1)$, then we define Q_j to be $v_1w\dots x$, and if $w \in N_G(v_2)$, then we define Q_j to be $v_2w\dots x$. For each i such that $1 \le i \le r-1$, we let $G_{i+1} := G_i \cup Q_{i+1}$. We observe that the sequence $G_1 \subset G_2 \subset \cdots \subset G_r$ is an ear decomposition of G.

Case 2: In G, the vertex s is a neighbour of v_1, whereas t is a neigbour of v_2.

Suppose the odd cycle $H_1 := svt\dots s$. Now we define G_1 to be the odd cycle, obtained by stretching H_1, as follows: replace the vertex v by the even path $v_1v_0v_2$ — that is, $G_1 := sv_1v_0v_2\dots s$. For each single ear P_j, we define the odd path Q_j to be the same as P_j, except that the role of v is played by either v_1 or v_2, as in the previous case. We let $G_{i+1} := G_i \cup Q_{i+1}$ for each i such that $1 \le i \le r-1$, and observe that the sequence $G_1 \subset G_2 \subset \cdots \subset G_r$ is an ear decomposition of G.

In each case, we obtain an ear decomposition of G. Thus, Theorem 3.5 implies that G is 2-connected and critical. □

3.3 Index of a thin edge

As mentioned earlier, all of our proofs in this chapter use the Strictly Thin Edge Theorem (1.17), which says that every (simple) brick G, except for the Norine-Thomas bricks, has a strictly thin edge e; and in this case, the retract of $G-e$ is also a simple brick. To carry out the case analysis, we shall find it convenient to associate each thin edge e with a number, called its *index*, which is:

- *zero*, if both ends of e have degree four or more in G;
- *one*, if exactly one end of e has degree three in G;
- *two*, if both ends of e have degree three in G and e does not lie in a triangle;
- *three*, if both ends of e have degree three in G and e lies in a triangle.

The following proposition is easily verified.

Proposition 3.7 *Let G be a brick, let e be a thin edge of G, and let H be the retract of $G-e$. If the index of e is zero, then $H = G-e$. If the index of e is one, then $G-e$ has precisely one vertex of degree two; and H has just one contraction vertex, and its degree is at least four. If the index of e is two, then $G-e$ has precisely two vertices of degree two, and they have no common neighbour; and H has two contraction vertices, and their degrees are at least four. If the index of e is three, then $G-e$ has precisely two vertices of degree two, and they have a common neighbour; and H has just one contraction vertex, and its degree is at least five.* □

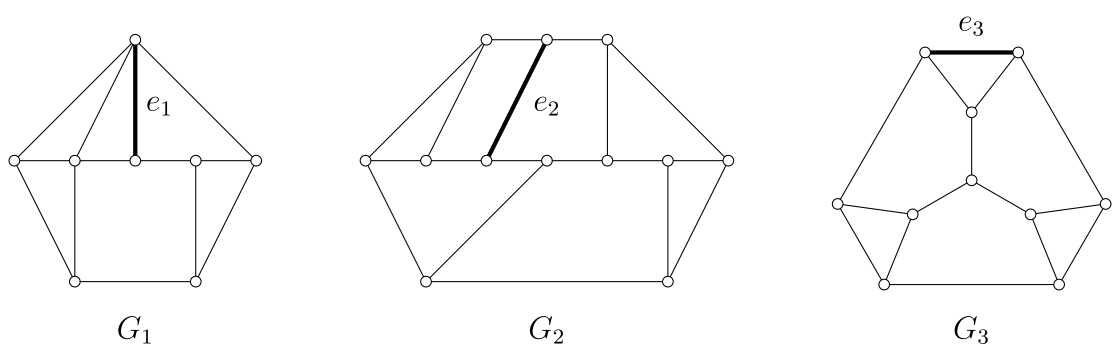


Figure 3.3: Strictly thin edges of indices one, two and three

Figure 3.3 illustrates strictly thin edges of indices one, two and three. The edge e_1 in G_1 has index one, and the retract of $G_1 - e_1$ is the odd wheel on six vertices. The edge e_2 in G_2 has index two, and the retract of $G_2 - e_2$ is G_1. The edge e_3 in G_3 has index three, and the retract of $G_3 - e_3$ is again the odd wheel on six vertices.

Let G be a brick, let e be a thin edge of G, and let J be a cubic brick. If G is J-free, then clearly $G-e$ is also J-free. Since the retract of $G-e$ is the brick of $G-e$, by applying Theorem 2.8, we have the following relevant fact.

Proposition 3.8 *Let G be a brick and e be a thin edge of G. Let H be the retract of $G-e$. For any cubic brick J, if G is J-free then H is also J-free.* □

In particular, if G is a K_4-free ($\overline{C_6}$-free) brick, and $G_1, G_2, \ldots, G_r$ is a sequence of bricks as in Theorem 1.18, then each G_i is also K_4-free ($\overline{C_6}$-free).

3.4 K_4-free planar bricks

In this section we will establish that a planar brick is K_4-free if and only if it has exactly two odd faces (Theorem 1.11). Our proof of this result relies on the fact that, in a planar brick with exactly two odd faces, every even facial cycle is conformal.

It should be noted that an arbitrary planar brick may not satisfy this property. For example, in the brick G_3, shown in Figure 3.3, the outer face is even but not conformal. Furthermore, an arbitrary planar matching covered graph with precisely two odd faces, may not satisfy this property either. For instance, the outer face of the graph shown in Figure 3.4 is not conformal. However, it is easily verified that in a plane bipartite matching covered graph, every facial cycle is conformal; this was also shown by McCuaig [McC04].

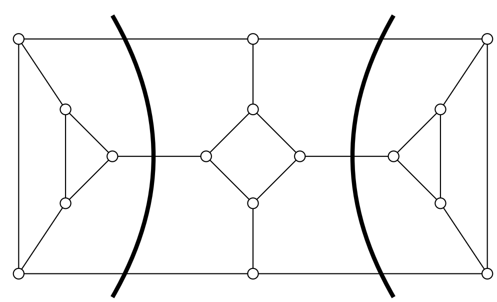


Figure 3.4: Matching covered graph with two odd faces — the bold lines depict tight cuts

As alluded to earlier, proving the above fact concerning even facial cycles by induction requires proving a statement concerning all facial cycles. Before formally stating and proving that assertion concerning facial cycles, we shall set up the required notation and make a few useful observations.

Let G be a planar brick with $\mathsf{f_{odd}}(\mathsf{G}) = 2$. Let $e := u_0v_0$ be a strictly thin edge of G, and let H be the retract of $G - e$. It follows from Proposition 3.3 that $\mathsf{f_{odd}}(\mathsf{H}) = 2$. Furthermore, as H is obtained from $G - e$ by either zero, one or two bicontractions, there is a one-to-one correspondence between the sets of faces of H and $G - e$. For any face Φ of $G - e$, we shall denote by Φ' the corresponding face of H.

We shall denote the two faces of G whose bounding cycles share the edge e by F_1 and F_2, and by F the unique face of $G - e$ which contains both u_0 and v_0. (Thus F' is the face of H that corresponds to F.) Let $u_1 \in V(F_1)\backslash v_0$ and $u_2 \in V(F_2)\backslash v_0$ be neighbours of u_0. Let

$v_1 \in V(F_1)\backslash u_0$ and $v_2 \in V(F_2)\backslash u_0$ be neighbours of v_0. Note that, for $i \in \{1, 2\}$, if $u_i = v_i$ then F_i is an odd face. Since an odd wheel has at least four odd faces, G is not an odd wheel and thus by Proposition 3.2, at most one of F_1 and F_2 is an odd face. We assume that F_2 is an even face and hence that $u_2 \neq v_2$, and admit the possibility that $u_1 = v_1$.

Depending on the index of the edge e, there are four possible scenarios. If the index of e is zero, then $H = G - e$. If the index of e is one, exactly one end of e, say u_0, has degree three; the other end v_0 has degree at least four. In this case, H is obtained from $G - e$ by shrinking $\{u_1, u_0, u_2\}$ to a single vertex; we shall denote the resulting contraction vertex by $\mathbf{u}$ (see Figure 3.5a). If the index of e is two, both ends of e have degree three. In this case, H is obtained from $G - e$ by shrinking $\{u_1, u_0, u_2\}$ to a single vertex, and $\{v_1, v_0, v_2\}$ to a single vertex; we shall denote the two resulting contraction vertices by $\mathbf{u}$ and $\mathbf{v}$, respectively (see Figure 3.5b). If the index of e is three, then both ends of e have degree three and $u_1 = v_1$, and H is obtained from $G - e$ by the bicontractions of u_0 and v_0. This amounts to shrinking $\{u_2, u_0, u_1, v_0, v_2\}$ to a single vertex; we shall denote the resulting contraction vertex by $\mathbf{w}$ (see Figure 3.5c).

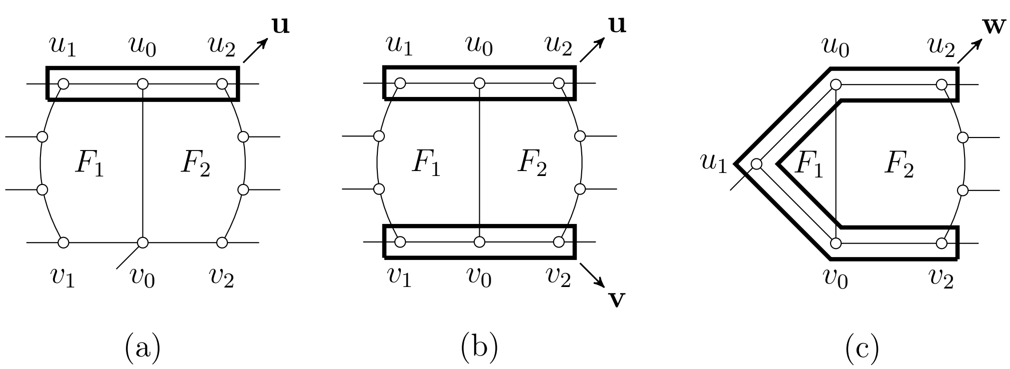


Figure 3.5: (a) Index of e is one, (b) Index of e is two, (c) Index of e is three

Proposition 3.9 *Each vertex on F, other than u_0 and v_0, has a neighbour outside F.*

Proof: If there is an edge of $G - e$ joining two non-consecutive vertices on F, then it would be a chord of F' in H. This is not possible because H is a simple 3-connected graph and F' is a face of H. □

Proposition 3.10 *If the index of e is one, any face Φ of G which contains u_1 and u_2 also contains u_0. If the index of e is two or three, then any face which contains u_1 and u_2 also contains u_0, and any face which contains v_1 and v_2 also contains v_0.*

Proof: Let us consider the case in which e has index one. By the proof of Proposition 3.9, u_1 and u_2 are not adjacent. Thus, if a face Φ contains both u_1 and u_2 but not u_0, the two u_1u_2-segments of the bounding cycle of Φ contain internal vertices, and there would have to exist a path connecting them which is internally disjoint from the bounding cycle. Such a path would have to pass through u_0. This is impossible because the degree of u_0 is three. The case in which the index of e is two or three is similar. □

Proposition 3.11 *Suppose that Φ is an odd face of G, and $x \notin V(\Phi)$ is any vertex. Then x has at least two neighbours in G which do not lie in $V(\Phi)$.*

Proof: Suppose not. Then there exist edges xs and xt which are consecutive in the cyclic order around x, such that $s, t \in V(\Phi)$. Note that s and t must be adjacent since otherwise $\{s, t\}$ is a 2-separation. Now, the triangle $xstx$ bounds an odd face that is not disjoint from the odd face Φ, which contradicts Proposition 3.2. □

Proposition 3.12 *Suppose the face F_1 is odd, and let $u_0x_1x_2 \ldots x_{2k}v_0u_0$ be the cycle bounding F_2, where $x_1 := u_2$ and $x_{2k} := v_2$. Then there exist vertices $y_1, y_2, \ldots, y_{2k}$, which are not on the boundary of F such that x_iy_i is an edge for $1 \leq i \leq 2k$, and $y_i \neq y_{i+1}$ for $i \in \{1, 3, \ldots, 2k-1\}$.*

Proof: By Proposition 3.9, each x_i has a neighbour outside $V(F)$. Suppose that there is only one vertex $y \notin V(F)$ that is adjacent to both x_i and x_{i+1} for some $i \in \{1, 3, \ldots, 2k-1\}$. Then $yx_ix_{i+1}y$ would be the boundary of a triangular face, and the graph H would then have two odd faces which are not vertex-disjoint. □

With this preparation, we are now ready to state and prove the result concerning the conformality of even facial cycles.

Theorem 3.13 *Let G be a planar brick with $\mathsf{f_{odd}}(G) = 2$. Then:*

(i) each even facial cycle is conformal, and,

(ii) for each odd facial cycle of G, the graph obtained from G by deleting the vertices of that cycle is a 2-connected critical graph.

Proof: As noted earlier, the two odd faces of G must be vertex-disjoint. We use induction on the number of edges to prove the theorem.

If G is a Norine-Thomas brick, then G is either a prism, or an even staircase, or a truncated biwheel. In each case, it can be easily checked that the assertion holds.

Hence we may assume that G is not a Norine-Thomas brick. It follows that G has a strictly thin edge e such that the retract of $G - e$, say H, is a planar brick with strictly fewer edges than G. As already noted, $\mathsf{f_{odd}(H)} = 2$, and thus H has two vertex-disjoint odd faces. We shall adopt the notation described earlier in this section. It should be remembered that F is the face of $G - e$ that contains the edge e, and that, for any face Φ of $G - e$, the face of H that corresponds to Φ is denoted by Φ'. (In particular, F' is the face of H that corresponds to the face F of $G - e$.) If e has index zero, or if $V(\Phi')$ does not contain any contraction vertex, then $\Phi' = \Phi$. Otherwise, Φ' and Φ are different. When this is the case, careful analysis is required.

Case 1: Both F_1 and F_2 are even.

Since F_1 is even, e cannot be of index three. We shall divide the analysis into three cases depending on the index. Cases dealing with indices one and two are very similar.

Index of e is zero: In this case, $H = G - e$ and F' is an even face of H. Hence, by induction, F' is conformal in H, implying that $F_1 \cup F_2$ is conformal in G. But each of F_1 and F_2 is conformal in $F_1 \cup F_2$. We deduce that both F_1 and F_2 are conformal in G.

Let Φ be any even face of G distinct from F_1 and F_2. Then Φ is also a face of H. By induction, Φ is conformal in H. Since $G - V(\Phi)$ is either $H - V(\Phi)$ or $H - V(\Phi) + e$, it follows that Φ is conformal in G.

Now let Φ be an odd face of G. Then Φ is also a face of H. By induction, $H - V(\Phi)$ is a 2-connected critical graph. But $G - V(\Phi)$ is either $H - V(\Phi)$ or $H - V(\Phi) + e$. It follows that $G - V(\Phi)$ is a 2-connected critical graph.

Index of e is one: We first note that if Φ is any face of G distinct from F_1 and F_2, it follows from Proposition 3.10 that $\{u_0, u_1, u_2\} \cap V(\Phi)$ is either $\{u_0, u_1, u_2\}$, or $\{u_1\}$ or $\{u_2\}$, or it is empty. The main problem to contend with in analysing this case is that even if a face Φ of G contains just one vertex of $\{u_0, u_1.u_2\}$, the corresponding face Φ' in H contains the contraction vertex $\mathbf{u}$.

Let us note that F' is an even face of H, but it contains the contraction vertex. However, $H - V(F')$ and $G - V(F_1 \cup F_2)$ are still identical, implying that F_1 and F_2 are conformal subgraphs of G (as in the index zero case).

Suppose that $\Phi \neq F_1, F_2$ is an even face of G. Then Φ' is an even face of H and, by induction, $H - V(\Phi')$ has a perfect matching, say M. If $\{u_0, u_1, u_2\} \cap V(\Phi) = \{u_0, u_1, u_2\}$, then $G - V(\Phi) = H - V(\Phi')$ and M itself is a perfect matching of $G - V(\Phi)$. In each of the other cases, either $M + u_0u_2$ or $M + u_0u_1$, as appropriate, is a perfect matching of $G - V(\Phi)$. (Thus, suppose f is the edge of M which is incident with the contraction vertex $\mathbf{u}$ in $H - V(\Phi')$. In $G - V(\Phi)$, this edge might be incident with either u_1 or with u_2. In the former case, $M + u_0u_2$ is a perfect matching of $G - V(\Phi)$, and in the latter case, $M + u_0u_1$ is the desired perfect matching.)

Now suppose that Φ is an odd face of G. Then Φ' is an odd face of H and, by induction, $H' := H - V(\Phi')$ is a 2-connected critical graph. Let us first note that if $u_1 \notin V(\Phi)$ then, by Proposition 3.11, u_1 has at least one neighbour, different from u_0, which does not lie in $V(\Phi)$. (A similar statement holds for u_2.)

If $\{u_0, u_1, u_2\} \cap V(\Phi) = \{u_0, u_1, u_2\}$, then $G - V(\Phi) = H - V(\Phi')$, and thus $G - V(\Phi)$ is a 2-connected critical graph.

If $\{u_0, u_1, u_2\} \cap V(\Phi) = \emptyset$, then the contraction vertex $\mathbf{u}$ is a vertex of $H' := H - V(\Phi')$. Furthermore, by the observations made earlier, among the edges incident with $\mathbf{u}$ in H', there is at least one which is incident with u_1 in G, and at least one incident with u_2. Thus the graph $G - V(\Phi)$ can be obtained from H' by appropriately bi-splitting $\mathbf{u}$. Thus, by Lemma 3.6, $G - V(\Phi)$ is 2-connected and critical.

Now, suppose that $\{u_0, u_1, u_2\} \cap V(\Phi) = \{u_1\}$. In this case, the contraction vertex $\mathbf{u}$ does belong to Φ', and not to $H' := H - V(\Phi')$. However, at least one neighbour of u_2, say $y \neq u_0$, is in H'. The graph $G - V(\Phi)$ can be obtained from H' by adding the ear $yu_2u_0v_0$, and then any remaining edges incident with u_2. It follows that $G - V(\Phi)$ is 2-connected and critical. The case in which $\{u_0, u_1, u_2\} \cap V(\Phi) = \{u_2\}$ is similar.

Index of e is two: The face F' contains both the contraction vertices, and $G - V(F) = H - V(F')$, implying, as before, the conformality of the even faces F_1 and F_2.

For any face $\Phi \neq F_1, F_2$ of G, $\{u_0, u_1, u_2, v_0, v_1, v_2\} \cap V(\Phi)$ is either empty; or is one of the four singletons $\{u_1\}$, $\{u_2\}$, $\{v_1\}$, and $\{v_2\}$; or is one of the two doubletons $\{u_1, v_1\}$ and $\{u_2, v_2\}$; or is one of $\{u_0, u_1, u_2\}$ and $\{v_0, v_1, v_2\}$.

Suppose that $\Phi \neq F_1, F_2$ is an even face of G, and let M be a perfect matching of $H - V(\Phi')$. Then one of $M + u_0u_1 + v_0v_1$, $M + u_0u_1 + v_0v_2$, $M + u_0u_2 + v_0v_1$, $M + u_0u_2 + v_0v_2$, $M + v_0v_1$, $M + v_0v_2$, $M + u_0u_1$, or $M + u_0u_2$, as appropriate, is a perfect matching of $G - V(\Phi)$.

Now suppose that Φ is an odd face of G, and let $H' := H - V(\Phi')$. By induction, H' is a 2-connected critical graph. We need to deduce from it that $G - V(\Phi)$ is 2-connected and critical.

If $\{u_0, u_1, u_2, v_0, v_1, v_2\} \cap V(\Phi)$ is empty, then neither $\mathbf{u}$ nor $\mathbf{v}$ lies in $V(\Phi')$, and $G - V(\Phi)$ can be obtained from H' by bi-splitting $\mathbf{u}$ and $\mathbf{v}$ successively, and is thus 2-connected and critical.

If $\{u_0, u_1, u_2, v_0, v_1, v_2\} \cap V(\Phi) = \{u_0, u_1, u_2\}$, then the contraction vertex $\mathbf{u} \in V(\Phi')$, but $\mathbf{v} \notin V(\Phi')$. In this case, $G - V(\Phi)$ can be obtained from H' by bi-splitting $\mathbf{v}$. (The case in which the intersection is $\{v_0, v_1, v_2\}$ is similar.)

If $\{u_0, u_1, u_2, v_0, v_1, v_2\} \cap V(\Phi)$ is a singleton, then the bi-splitting operation, followed by the addition of an ear of length three and any remaining edges does the trick. If the intersection is a doubleton, say $\{u_1, v_1\}$, then u_1 and v_1 are adjacent. We note that u_2 and v_2 are not adjacent since otherwise H is not simple. Both $\mathbf{u}$ and $\mathbf{v}$ belong to $V(\Phi')$, and in G, each of u_2 and v_2 would have a neighbour in $V(F_2) - \{u_0, v_0\}$, say s and t respectively, which are distinct. In this case adding the ear $su_2u_0v_0v_2t$ to H', and then adding any remaining edges incident with u_2 and v_2 yields $G - V(\Phi)$. This completes the analysis of Case 1.

Case 2: F_1 is odd and F_2 is even.

Index of e is zero: In this case, F' is an odd face of H. By induction, $H' := H - V(F')$ is a 2-connected critical graph. Using this, we now show that F_2 is conformal. There must be a neighbour y of u_1 that is not in F', and hence in $V(H')$. Let M be a perfect matching of $H' - y$. Let G' denote the graph induced by the edge set $E(F_1) + yu_1 - u_1u_0 - u_0v_0 - v_0v_1$. Since G' is an odd path, it has a perfect matching, say M'. Then $M \cup M'$ is a perfect matching of $G - V(F_2)$.

Let Φ be an even face of G distinct from F_2. Then Φ is also an even face of H. Conformality of Φ in H implies the conformality of Φ in G.

Now let us turn to odd faces of G. First let Φ be the odd face distinct from F_1. Then Φ is also an odd face of H. By induction, $H - V(\Phi)$ is a 2-connected critical graph. Since F_1 and Φ are vertex-disjoint, we have $G - V(\Phi) = H - V(\Phi) + e$, and thus $G - V(\Phi)$ is 2-connected and critical.

Finally, consider the odd face F_1. To show that $G - V(F_1)$ is a 2-connected critical graph, we show that the 2-connected critical graph $H' := H - V(F')$ can be augmented using appropriate ear additions to obtain $G - V(F_1)$. Let us label the vertices such that the cycle bounding F_2 is $u_0 x_1 x_2 \dots x_{2k} v_0 u_0$, where $x_1 = u_2$ and $x_{2k} = v_2$. Then, by Proposition 3.12, there exist vertices $y_1, y_2, \dots, y_{2k} \notin V(F)$, which are neighbours of $x_1, x_2, \dots, x_{2k}$, respectively, such that $y_i \neq y_{i+1}$ for $i \in \{1, 3, \dots, 2k-1\}$. Adding the ears $y_1 x_1 x_2 y_2$, $y_3 x_3 x_4 y_4, \dots, y_{2k-1} x_{2k-1} x_{2k} y_{2k}$ to H' successively results in a 2-connected critical graph whose vertex set is $V(G) - V(F_1)$. Now any remaining missing edges can be added as ears of length one to obtain the graph $G - V(F_1)$. It follows that $G - V(F_1)$ is 2-connected and critical.

Index of e is one: Since F' is an odd face of H, by induction, $H - V(F')$ is a 2-connected critical graph. The fact that F_2 is conformal in G follows by an argument analogous to the one used in the index zero case.

Let $\Phi \neq F_2$ be an even face of G. Then Φ' is an even face of H and, by induction, $H' := H - V(\Phi')$ has a perfect matching, say M. If $\{u_0, u_1, u_2\} \cap V(\Phi) = \{u_0, u_1, u_2\}$, then M is a perfect matching of $G - V(\Phi)$ as well. If $\{u_0, u_1, u_2\} \cap V(\Phi)$ is either empty, or is a singleton $\{u_1\}$ or $\{u_2\}$, then either $M + u_0 u_1$ or $M + u_0 u_2$, as appropriate, is a perfect matching of $G - V(\Phi)$, and thus Φ is conformal.

Now let $\Phi \neq F_1$ be an odd face of G. Then Φ' is an odd face of H and is vertex-disjoint from F'. By induction, $H' := H - V(\Phi')$ is 2-connected and critical. Let us first note that since $u_1 \notin V(\Phi)$, by Proposition 3.11, u_1 has at least one neighbour, different from u_0, which does not lie in $V(\Phi)$. (A similar statement holds for u_2.) Thus, the graph $G - V(\Phi)$ can be obtained from H' by appropriately bi-splitting $\mathbf{u}$, which implies that $G - V(\Phi)$ is 2-connected and critical.

Finally, consider the odd face F_1. As in the index zero case, $G - V(F_1)$ can be obtained from the 2-connected critical graph $H - V(F')$ using appropriate ear additions, and is thus 2-connected and critical.

Index of e is two: As before, we note that $H - V(F')$ is a 2-connected critical graph. The conformality of F_2 follows using an argument analogous to the previous cases.

Let $\Phi \neq F_2$ be an even face of G, and let M be a perfect matching of $H - V(\Phi')$. Then one of $M + u_0u_1 + v_0v_1$, $M + u_0u_1 + v_0v_2$, $M + u_0u_2 + v_0v_1$, $M + u_0u_2 + v_0v_2$, $M + v_0v_1$, $M + v_0v_2$, $M + u_0u_1$, or $M + u_0u_2$, as appropriate, is a perfect matching of $G - V(\Phi)$.

Now let $\Phi \neq F_1$ be an odd face of G. By induction, $H' := H - V(\Phi')$ is 2-connected and critical. Since Φ' and F' are vertex-disjoint, $G - V(\Phi)$ can be obtained from H' by bi-splitting $\mathbf{u}$ and $\mathbf{v}$ successively, and is thus 2-connected and critical.

We note that $G - V(F_1)$ can be obtained from the 2-connected critical graph $H - V(F')$ by adding ears appropriately, and is thus itself 2-connected and critical.

Index of e is three: First we note that $u_1 = v_1$. Recall that H is obtained from $G - e$ by shrinking $\{u_2, u_0, u_1, v_0, v_2\}$ to a single vertex, and that the resulting contraction vertex is denoted by $\mathbf{w}$. Observe that u_2 and v_2 are neither adjacent, nor do they have a common neighbour, since otherwise H would not be simple. For any face $\Phi \neq F_1, F_2$ of G, $\{u_0, u_1, u_2, v_0, v_2\} \cap V(\Phi)$ is either empty; or is one of the three singletons $\{u_1\}$, $\{u_2\}$ and $\{v_2\}$; or is one of $\{u_0, u_1, u_2\}$ and $\{v_0, v_1, v_2\}$.

In H, the face F' is an odd face, and by induction, $H' := H - V(F')$ is a 2-connected critical graph. Let y be a neighbour of u_1 that is not in F', and let M be a perfect matching of $H' - y$. Then $M + yu_1$ is a perfect matching of $G - V(F_2)$, and thus F_2 is conformal.

Let $\Phi \neq F_2$ be an even face of G, and let M be a perfect matching of $H - V(\Phi')$. Then one of $M + u_0u_1 + v_0v_2$, $M + u_0u_2 + v_0v_2$, $M + u_0u_2 + v_0v_1$, $M + u_0u_2$, or $M + v_0v_2$, as appropriate, is a perfect matching of $G - V(\Phi)$.

Now let $\Phi \neq F_1$ be an odd face of G. Since Φ' and F' are vertex-disjoint faces of H, neither u_2 nor v_2 lies in $V(\Phi)$. By Proposition 3.11, u_2 has at least one neighbour, different from u_0, which does not lie in $V(\Phi)$. Similarly, v_2 has at least one neighbour, different from v_0, which does not lie in $V(\Phi)$. Now, we observe that $G - V(\Phi)$ can be obtained from the 2-connected critical graph $H - V(\Phi')$ by appropriately bi-splitting the vertex $\mathbf{w}$ into u_1 and u_2, and thereafter appropriately bi-splitting the vertex u_1 into u_1 and v_2, and adding the remaining edge u_0v_0. Thus, $G - V(\Phi)$ is 2-connected and critical.

Finally, consider the odd face F_1. Let s and t be neighbours of u_2 and v_2, respectively, such that $s, t \notin V(F_2)$. As noted earlier, s and t are distinct. We note that s, t lie in the 2-connected critical graph $H' := H - V(F')$. Let $F_2 := u_0x_1x_2 \ldots x_{2k}v_0u_0$, where $x_1 = u_2$ and $x_{2k} = v_2$. Adding the ear $sx_1x_2 \ldots x_{2k}t$ to H' results in a 2-connected critical graph whose vertex set is $V(G) - V(F_1)$. Now any remaining missing edges can be added as ears of length one to obtain the graph $G - V(F_1)$, which is thus 2-connected and critical. This completes the analysis of Case 2.

Thus, all the facial cycles of G possess the desired properties, and this completes the proof of Theorem 3.13. $\square$

We will use Theorem 3.13 to characterize K_4-free planar bricks. However, before that we need another technical result, which proves the existence of a conformal bi-subdivision of K_4 whenever a plane matching covered graph contains a certain configuration.

Lemma 3.14 *Let G be a plane matching covered graph. Let F_1 and F_2 be two odd faces which share exactly one edge e such that $G - V(F_1 \cup F_2)$ has a perfect matching. Then G is K_4-based.*

Proof: We note that the subgraph with vertex set $V(F_1 \cup F_2)$ and edge set $E(F_1 \cup F_2) - e$ is an even cycle; denote this cycle by C, and label its vertices such that $C := w_1 w_2 \ldots w_{2k} w_1$, $w_2 \in V(F_2)$ and $e := w_1 w_{2j+1}$ for some $1 \leq j < k$. See Figure 3.6. Observe that w_1 and w_{2j+1} belong to the same colour class of C. We refer to the vertices of C which belong to the same colour class as w_1 as *black* vertices, and the remaining vertices as *white* vertices.

By hypothesis, $G - V(C)$ has a perfect matching; taking the union of such a matching and the set $\{w_1w_2, w_3w_4, \ldots, w_{2k-1}w_{2k}\}$, we obtain a perfect matching M of G itself. Let M_e denote some perfect matching of G which contains e. Clearly, the edges w_1w_2, $e := w_1w_{2j+1}$, and $w_{2j+1}w_{2j+2}$ lie in an (M_e, M)-alternating cycle; we let P denote the unique (M_e, M)-alternating path which starts at w_2 and ends at w_{2j+2} and does not contain e. See Figure 3.6. The edges in wavy lines depict the matching M, and those in thick lines indicate the matching M_e.

Since the path P has origin in $V(F_2) - \{w_1, w_{2j+1}\}$ and terminus in $V(F_1) - \{w_1, w_{2j+1}\}$, it must clearly have a segment which has its origin in $V(F_2) - \{w_1, w_{2j+1}\}$ and terminus in $V(F_1) - \{w_1, w_{2j+1}\}$ and is otherwise vertex-disjoint with C. Our goal is to show that there is at least one such segment of P, say Q, both of whose ends are white vertices; in general, the path P may have some segments which do not satisfy this property.

The first time that P leaves $V(F_2)$, it must clearly be from a white vertex. Suppose that its next visit to $V(F_2)$, if any, is at a white vertex, say w_{2i}. Let $P[w_2, w_{2i}]$ denote the (w_2, w_{2i})-segment of P, and let C' denote the cycle $w_1w_2P[w_2, w_{2i}]w_{2i}w_{2i+1} \ldots w_{2j+1}w_1$. The black vertex w_{2i-1} and the terminal vertex w_{2j+2} of P would be in different regions determined by the cycle C'. Since P must continue on from w_{2i} to w_{2i-1} and eventually terminate at w_{2j+2}, this is impossible because G is a plane graph. Thus all re-entrances of P in to $V(F_2)$, if any, must be at black vertices; which implies that all exits of P from

$V(F_2)$ must be from white vertices. A similar argument shows that all entries of P into $V(F_1)$ are at white vertices. We conclude that there must be a segment Q of P starting at a white vertex $x \in V(F_2) - \{w_1, w_{2j+1}\}$ and ending at a white vertex $y \in V(F_1) - \{w_1, w_{2j+1}\}$ which is internally-disjoint from $V(F_1) \cup V(F_2)$, as shown in Figure 3.6.

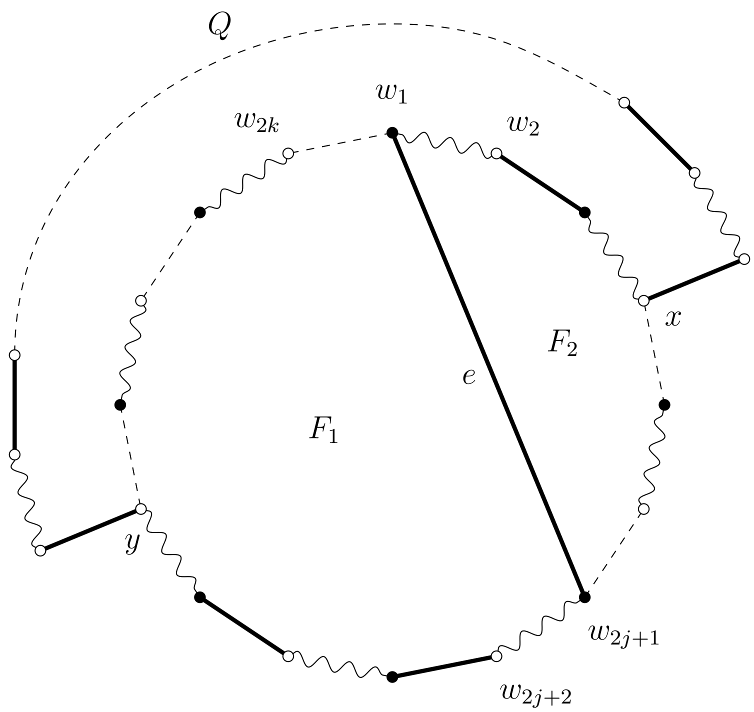


Figure 3.6: A conformal bi-subdivision of K_4

Let H be the subgraph of G with vertex set $V(C) \cup V(Q)$ and edge set $E(C) \cup E(Q) + e$. Clearly, H is a bi-subdivision of K_4. It is a conformal subgraph of G because the restriction of the perfect matching M to $E(H)$ is a perfect matching of H. □

We now proceed to prove the main result of this section.

3.4.1 Proof of Theorem 1.11

Let G be a planar brick. We may assume that G is simple. By Proposition 3.4, if G has exactly two odd faces then G is K_4-free. It remains to prove the converse.

We show that if $\mathsf{f}_{\mathsf{odd}}(G) \geq 4$, then G must be K_4-based. We use induction on the number of edges. If G is a Norine-Thomas brick, then it must be either an odd wheel or an odd staircase. In either case, it can be easily checked that G is K_4-based.

Hence we may assume that G is not a Norine-Thomas brick. It follows that G has a strictly thin edge e such that the retract of $G - e$, say H, is a planar brick with strictly fewer edges than G. Proposition 3.3 implies that $2 \leq \mathsf{f}_{\mathsf{odd}}(H) \leq \mathsf{f}_{\mathsf{odd}}(G)$. If $\mathsf{f}_{\mathsf{odd}}(H) \geq 4$, then it follows from the induction hypothesis and Proposition 3.8 that G is K_4-based. We may thus assume that $\mathsf{f}_{\mathsf{odd}}(H) = 2$. Proposition 3.3 implies that $\mathsf{f}_{\mathsf{odd}}(G) = 4$.

Let us now see how it is possible for the number of odd faces of G to be four, whereas the number of odd faces of H is only two. The edge e is drawn in a face F of $G - e$, giving rise to two faces F_1 and F_2 of G. Any face $\Phi \neq F_1, F_2$ of G is a face of $G - e$ as well, and thus corresponds to a unique face Φ' of H, whose parity is the same as that of Φ. In case of the face F' of H that corresponds to the face F of $G - e$, the parity of F' is the sum (modulo 2) of the parities of the two faces F_1 and F_2. Thus the reduction in the number of odd faces in going from G to H can occur only if both F_1 and F_2 are odd and F' is even.

Since $\mathsf{f}_{\mathsf{odd}}(H) = 2$, the even face F' is conformal in H by Theorem 3.13. This implies that the face F is a conformal even face of $G - e$. In other words, F_1 and F_2 are two odd faces of G which share exactly one edge e such that $G - V(F_1 \cup F_2)$ has a perfect matching. It follows from Lemma 3.14 that G is K_4-based. $\square$

Recall that a matching covered graph is a near-brick if it has precisely one brick. It can be shown that Theorem 1.11 in fact holds for all plane near-bricks.

3.5 $\overline{C_6}$-free planar bricks

Observe that $\overline{C_6}$ has two vertex-disjoint odd cycles, whence each $\overline{C_6}$-based graph inherits this property. In particular, odd wheels are $\overline{C_6}$-free. By investigating the odd cycles of an odd staircase, it may be verified that these are also $\overline{C_6}$-free. (The remaining planar Norine-Thomas bricks have exactly two odd faces, whence they are K_4-free and $\overline{C_6}$-based.)

Apart from the odd wheels and the odd staircases, there is one exceptional $\overline{C_6}$-free simple planar brick. This graph, which we call the *Tricorn*, is the unique planar brick G

with a strictly thin edge e of index three such that the retract of $G-e$ is the odd wheel W_5. See Figure 3.7.

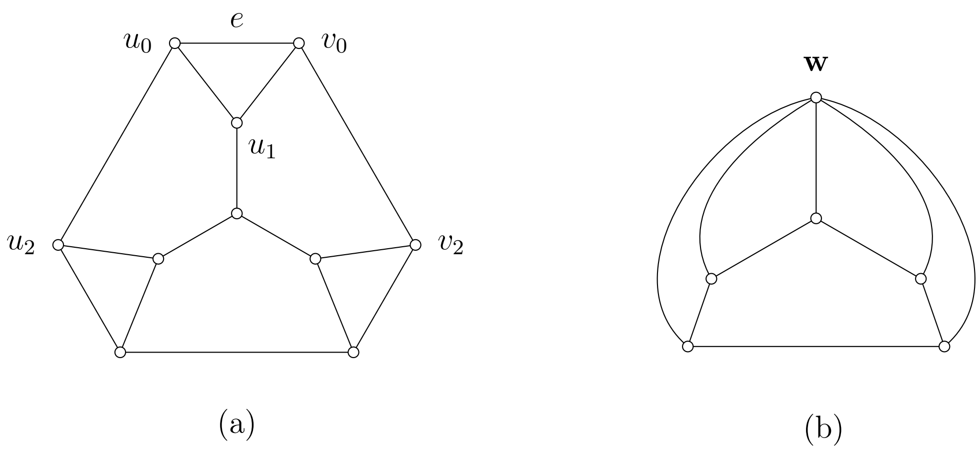


Figure 3.7: (a) Tricorn, (b) W_5

To see that the Tricorn is $\overline{C_6}$-free, consider any subgraph H of the Tricorn which is a bi-subdivision of $\overline{C_6}$. Then H consists of two vertex-disjoint odd cycles C_1 and C_2 together with a 3-linkage linking C_1 and C_2 (three disjoint paths linking three vertices of C_1 with three vertices of C_2). The Tricorn has precisely three 5-cycles and no two of them are disjoint. So, one of C_1 and C_2 has to be a triangle because the Tricorn has ten vertices. Assume without loss of generality that C_1 is a triangle. If C_2 is one of the other two triangles, any 3-linkage linking C_1 and C_2 includes a path of length two, and so the resulting subgraph would not be a bi-subdivision of $\overline{C_6}$. If C_2 is the unique 5-cycle disjoint from C_1, again any 3-linkage linking C_1 and C_2 includes a path of length two. Finally, if C_2 is the unique 7-cycle disjoint from C_1, the unique 3-linkage linking C_1 and C_2 consists of three paths of length one. However, the ends of these three paths on C_2 do not effect a bi-subdivision of $\overline{C_6}$. We, thus have the following.

Proposition 3.15 *Odd wheels, odd staircases and the Tricorn are $\overline{C_6}$-free.* □

We now proceed to show that if G is a planar brick, e is a strictly thin edge of G, and the retract of $G-e$ is an odd wheel W_{2k+1}, then

(i) either G is $\overline{C_6}$-based,

(ii) or the index of e is three, $k = 2$, and G is the Tricorn.

If e were a strictly thin edge of index two, then the retract of $G - e$ would have at least two vertices of degree exceeding three (see Proposition 3.7). Since W_{2k+1} has at most one vertex of degree greater than three, this rules out the case in which e has index two. We examine the other three possibilities in the following propositions.

Proposition 3.16 *Let G be a brick, and let $e := u_0v_0$ be an edge of G such that $G - e$ is an odd wheel W_{2k+1}. Then G is $\overline{C_6}$-based.*

Proof: Clearly, $2k + 1 \geq 5$ and both ends of e are rim vertices of W_{2k+1}. Label the hub of W_{2k+1} as h, and the rim vertices in cyclic order as $r_0, r_1, \ldots, r_{2k}$ such that $r_0 = u_0$ and $r_{2j} = v_0$ for some $1 \leq j \leq k-1$. Then the two vertex-disjoint odd cycles $r_0r_1 \ldots r_{2j}r_0$ and $r_{2k}hr_{2j+1}r_{2j+2} \ldots r_{2k}$, together with the three edges r_0r_{2k}, r_1h and $r_{2j}r_{2j+1}$ linking the two cycles, constitute a spanning subgraph of G, which is a bi-subdivision of $\overline{C_6}$. □

Proposition 3.17 *Let G be a planar brick, and let $e := u_0v_0$ be a strictly thin edge of index one such that the retract of $G - e$ is an odd wheel W_{2k+1}. Then G is $\overline{C_6}$-based.*

Proof: Since e has index one, exactly one end of e has degree three. Suppose that u_0 has degree three and that u_1 and u_2 are its two neighbours in $G - e$. The contraction vertex resulting from the bicontraction of u_0 has degree at least four, and thus $2k+1 \geq 5$. Since the hub h of W_{2k+1} is the only vertex of degree greater than three, it must be the contraction vertex in the retract of $G - e$. Thus the vertices of G, other than u_0, u_1 and u_2, must be rim vertices of W_{2k+1}; the neighbours of u_1 and u_2, other than u_0, together must consist of the set of all the rim vertices.

In $G - e$, the degree of u_0 is two, and v_0 is a vertex on the rim. Thus v_0 must have degree exactly four in G, and exactly one of u_1 and u_2 is a neighbour of v_0. Without loss of generality, assume that u_1 is a neighbour of v_0. Since G is planar, all the neighbours of u_1 and those of u_2 (other than u_0) must appear consecutively on the rim, and exactly one of the neighbours of v_0 on the rim should be adjacent to u_2. Let $r_0 = v_0$, and label the remaining rim vertices in cyclic order as $r_1, r_2, \ldots, r_{2k}$ such that r_{2k} is a neighbour of u_2. Then $r_0u_0u_1r_0$ and $r_{2k}u_2r_{2k-1}r_{2k}$ are two triangles in G. The odd paths r_0r_{2k}, u_0u_2, and $u_1r_1r_2 \ldots r_{2k-1}$ link pairs of vertices of those two triangles. Together, they constitute a spanning subgraph of G, which is a bi-subdivision of $\overline{C_6}$. □

Proposition 3.18 *Let G be a planar brick, and let $e := u_0v_0$ be a strictly thin edge of index three such that the retract of $G - e$ is an odd wheel W_{2k+1}. If $k = 2$, then G is the Tricorn; otherwise, G is $\overline{C_6}$-based.*

Proof: In this case, both u_0 and v_0 have degree two in $G - e$, and have exactly one common neighbour. Let $u_1 = v_1$ be the common neighbour of u_0 and v_0, and let u_2 and v_2 be the other two neighbours of u_0 and v_0 in $G - e$, respectively. (See Figure 3.5c.) The shrinking of $\{u_2, u_0, u_1, v_0, v_2\}$ results in the contraction vertex of the retract of $G - e$, which has degree at least five. Thus we have $k \geq 2$, and that the hub of W_{2k+1} must be the contraction vertex. It follows that in G, the vertices other than u_2, u_0, u_1, v_0, v_2 must be the vertices on the rim of W_{2k+1}. Among the rim vertices, u_1 has at least one neighbour, and u_2 and v_2 have at least two each. Label the rim vertices as $r_0, r_1, \ldots, r_{2k}$ such that $r_0, r_1, \ldots, r_i$ are the neighbours of u_1 on the rim, and r_{i+1} is a neighbour of u_2; in this order, let r_j be the last vertex on the rim that is adjacent to u_2. Thus, according to this convention, r_{j-1} is adjacent to u_2, and r_{j+1} and r_{j+2} are both adjacent to v_2.

Let us first consider the case in which $2k + 1 = 5$. The rim consists of the five vertices r_0, r_1, r_2, r_3 and r_4. It will have to be the case that r_0 is adjacent to u_1, the two vertices r_1 and r_2 are adjacent to u_2, and r_3 and r_4 are adjacent to v_2. Clearly, in this case, G has to be the Tricorn.

Now suppose $2k + 1 \geq 7$. We shall consider three different cases and, in each case indicate a spanning subgraph H of G which is a bi-subdivision of $\overline{C_6}$. (Each of these cases, taking $k = 4$, is illustrated in Figure 3.8.)

Case 1: Vertex u_1 has at least two neighbours on the rim. (In this case, both r_0 and r_1 are neighbours of u_1.) See Figure 3.8a.

The two triangles $r_{j-1}u_2r_jr_{j-1}$ and $r_{j+2}v_2r_{j+1}r_{j+2}$ — together with the odd paths $r_{j-1}r_{j-2}\ldots r_1u_1r_0r_{2k}r_{2k-1}\ldots r_{j+2}$ and $u_2u_0v_0v_2$ and r_jr_{j+1} linking them — constitute the desired spanning subgraph H.

If u_1 has just one neighbour, namely r_0, on the rim, then the parities of the number of neighbours of u_2 and v_2 on the rim are the same. We consider two cases according to their common parity.

Case 2: Vertex r_0 is the only neighbour of u_1 on the rim, and both u_2 and v_2 have an odd number of neighbours on the rim. See Figure 3.8b.

The two triangles $u_1u_0v_0u_1$ and $r_{j-1}u_2r_jr_{j-1}$ — together with the odd paths $u_1r_0r_1 \ldots r_{j-1}$ and u_0u_2 and $v_0v_2r_{2k}r_{2k-1} \ldots r_{j+1}r_j$ linking them — constitute the desired spanning subgraph H.

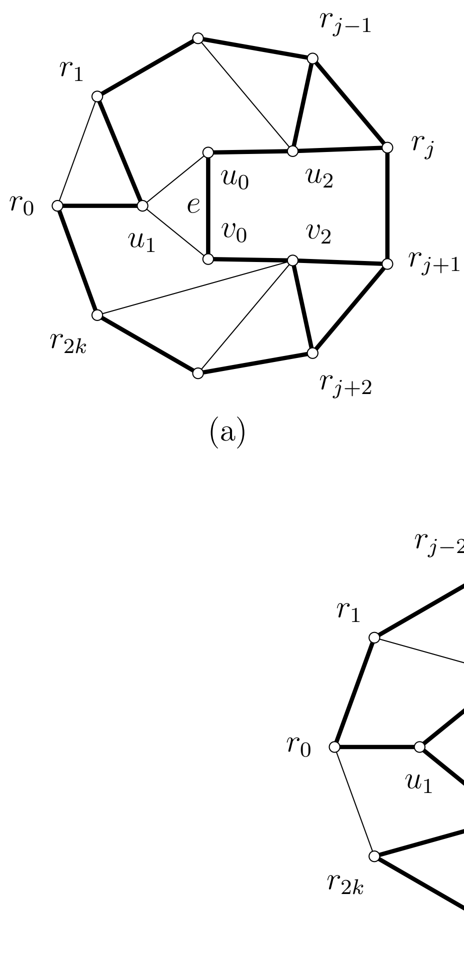


Figure 3.8: Index of e is three

Case 3: Vertex r_0 is the only neighbour of u_1 on the rim, and both u_2 and v_2 have an even number of neighbours on the rim. See Figure 3.8c.

Since $2k+1 \geq 7$, at least one of u_2 and v_2 has at least four neighbours on the rim. Without loss of generality, assume that u_2 has at least four neighbours on the rim. The two triangles $u_1u_0v_0u_1$ and $r_{j-2}u_2r_{j-1}r_{j-2}$ — together with the odd paths $u_1r_0r_1 \dots r_{j-2}$ and u_0u_2 and $v_0v_2r_{2k}r_{2k-1} \dots r_jr_{j-1}$ linking them — constitute the desired spanning subgraph H.

In each of the above cases, we see that G is $\overline{C_6}$-based. □

Next, we shall prove that if G is a planar brick, e is a strictly thin edge of G, and the retract of $G - e$ is an odd staircase St_{2k+4}, then G is $\overline{C_6}$-based. Observe that if e is of index one or more, then the retract of $G - e$ would have a vertex of degree four or more (see Proposition 3.7). Since a staircase is cubic, the only possibility is that e is of index zero, that is, G is obtained from St_{2k+4} by adding the edge e.

Proposition 3.19 *Let G be a brick, and let e be an edge of G such that $H := G - e$ is an odd staircase St_{2k+4}. Then G is $\overline{C_6}$-based.*

Proof: In order to have a convenient labelling of the vertices, we shall redefine odd staircases as follows. Let $r_0r_1 \dots r_k$ and $s_0s_1 \dots s_k$ be two vertex-disjoint paths where $k \geq 2$ and k is even. The odd staircase St_{2k+4} is the graph obtained by the union of these two paths, along with two new vertices x and y, and joining r_i to s_i for $0 \leq i \leq k$, and joining x to r_0 and s_0, y to r_k and s_k, and x and y to each other. Figure 3.9 shows this labelling for the smallest odd staircase St_8.

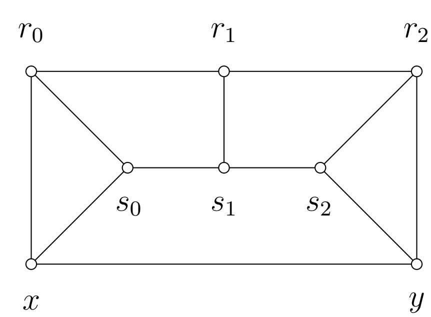


Figure 3.9: The odd staircase St_8

We label the vertices of $H := St_{2k+4}$ as in the preceding paragraph. We shall divide the proof into several cases, depending on the ends of e. In each case, we find a bi-subdivision of $\overline{C_6}$, which is a conformal subgraph of G.

<u>Case 1</u>: $e := xr_{2j+1}$ for some $0 \leq 2j+1 \leq k$.

The two odd cycles $xr_{2j+1}r_{2j}\ldots r_0x$ and yr_ks_ky — together with the three odd paths xy and $r_{2j+1}r_{2j+2}\ldots r_k$ and $r_0s_0s_1\ldots s_k$ linking them — constitute a spanning subgraph of G, which is a bi-subdivision of $\overline{C_6}$.

<u>Case 2</u>: $e := xr_{2j}$ for some $0 < 2j \leq k$.

The two odd cycles xr_0s_0x and $yr_kr_{k-1}\ldots r_{2j}r_{2j-1}s_{2j-1}s_{2j}\ldots s_ky$ — together with the three odd paths xr_{2j} and $r_0r_1\ldots r_{2j-1}$ and $s_0s_1\ldots s_{2j-1}$ linking them — constitute a spanning subgraph of G, which is a bi-subdivision of $\overline{C_6}$.

<u>Case 3</u>: $e := r_ir_j$ such that $i < j$ and $i \equiv j \equiv 0 \pmod 2$.

The two odd cycles $r_ir_{i+1}\ldots r_jr_i$ and $xs_0s_1\ldots s_kyx$ — together with the three odd paths r_is_i and $r_{i+1}s_{i+1}$ and r_js_j linking them — constitute a conformal subgraph of G, which is a bi-subdivision of $\overline{C_6}$.

<u>Case 4</u>: $e := r_ir_j$ such that $i < j$ and $i \equiv j \equiv 1 \pmod 2$.

The two odd cycles $r_ir_{i+1}\ldots r_jr_i$ and $xr_0s_0s_1\ldots s_kr_kyx$ — together with the three odd paths r_is_i and $r_{i+1}s_{i+1}$ and r_js_j linking them — constitute a conformal subgraph of G, which is a bi-subdivision of $\overline{C_6}$.

<u>Case 5</u>: $e := r_ir_j$ such that $i < j$ and $i \not\equiv j \pmod 2$.

Observe that $j - i$ is at least three. The two odd cycles $xr_0r_1\ldots r_ir_{i+1}s_{i+1}s_i\ldots s_0x$ and $yr_kr_{k-1}\ldots r_jr_{j-1}s_{j-1}s_j\ldots s_ky$ — together with the three odd paths r_ir_j and $r_{i+1}r_{i+2}\ldots r_{j-1}$ and $s_{i+1}s_{i+2}\ldots s_{j-1}$ linking them — constitute a spanning subgraph of G, which is a bi-subdivision of $\overline{C_6}$.

<u>Case 6</u>: $e := r_is_j$ such that $i < j$ and $i \equiv j \pmod 2$.

The two triangles xr_0s_0x and ys_kr_ky — together with the three odd paths xy and $r_0r_1\ldots r_is_js_{j+1}\ldots s_k$ and $s_0s_1\ldots s_is_{i+1}r_{i+1}r_{i+2}\ldots r_k$ linking them — constitute a conformal subgraph of G, which is a bi-subdivision of $\overline{C_6}$.

Case 7: $e := r_i s_j$ such that $i < j$ and $i \not\equiv j \pmod 2$.

Without loss of generality, assume that $i \equiv 0 \pmod 2$. The two odd cycles $r_i r_{i+1} \dots r_j s_j r_i$ and $y r_k s_k y$ — together with the three odd paths $r_i r_{i-1} \dots r_0 s_0 x y$ and $r_j r_{j+1} \dots r_k$ and $s_j s_{j+1} \dots s_k$ linking them — constitute a conformal subgraph of G, which is a bi-subdivision of $\overline{C_6}$.

In each of the above cases, we see that G is $\overline{C_6}$-based. □

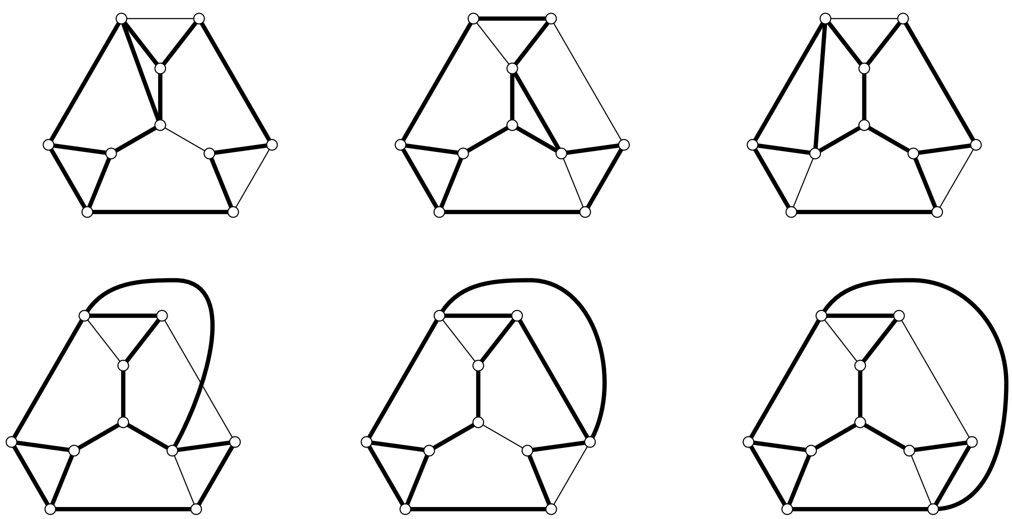


Figure 3.10: Adding an edge to the Tricorn

Finally, we show that adding an edge to the Tricorn results in a $\overline{C_6}$-based brick. Depending on the ends of e, there are six cases to be checked (up to isomorphism). The various possibilities are shown in Figure 3.10, and in each case, the edges depicted by the bold lines constitute a spanning bi-subdivision of $\overline{C_6}$. We thus have the following.

Proposition 3.20 *Let G be a brick, and let e be an edge of G such that $H := G - e$ is the Tricorn. Then G is $\overline{C_6}$-based.* □

We are now ready to prove the main result of this section.

3.5.1 Proof of Theorem 1.12

Let G be a planar brick. We may assume that G is simple. As discussed, if G is an odd wheel or an odd staircase or the Tricorn, then G is $\overline{C_6}$-free. We now establish the converse.

We use induction on the number of edges. If G is a Norine-Thomas brick, then the result holds trivially.

Hence we may assume that G is not a Norine-Thomas brick, and thus, G has a strictly thin edge e such that the retract of $G - e$, say H, is a $\overline{C_6}$-free planar brick with strictly fewer edges than G. By induction, H is either an odd wheel, or an odd staircase, or the Tricorn. We note that odd staircases and the Tricorn are cubic graphs, and that an odd wheel has at most one vertex of degree exceeding three. If the index of e is two, then H would have at least two vertices of degree at least four; thus we rule out this possibility.

If H is an odd staircase or the Tricorn, then e must be a strictly thin edge of index zero, and $H = G - e$. Using Proposition 3.19 or 3.20, as appropriate, we conclude that G must be $\overline{C_6}$-based, which is a contradiction.

If H is an odd wheel, then using Proposition 3.16, 3.17 or 3.18, as appropriate, we conclude that G is either $\overline{C_6}$-based which contradicts the hypothesis, or otherwise G is isomorphic to the Tricorn and we are done. □

3.6 Nonplanar K_4-free and $\overline{C_6}$-free bricks

There is an extensive class of bricks, known as solid bricks [CLM06], which are $\overline{C_6}$-free and are of particular interest in matching theory. For example, each nonbipartite Möbius ladder is a solid brick, and thus $\overline{C_6}$-free. The Petersen graph is an example of a $\overline{C_6}$-free brick which is not solid.

There do exist infinite families of nonplanar K_4-free bricks. The smallest such brick, which we refer to as the *Trellis*, is shown in Figure 3.11.

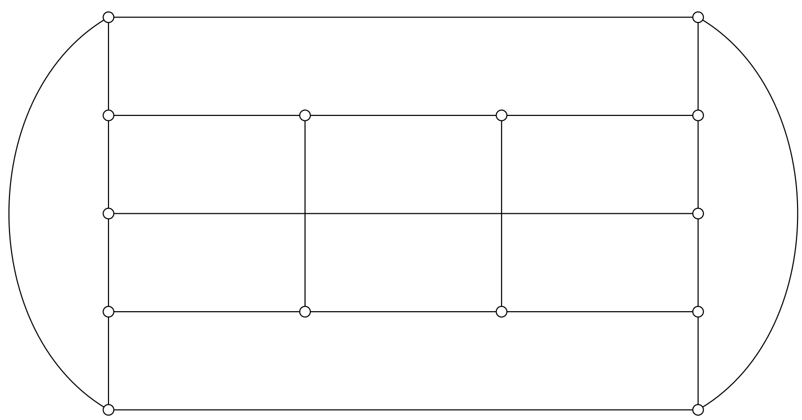


Figure 3.11: The Trellis — a nonplanar K_4-free brick

Chapter 4

Near-bipartite graphs

Here, we will examine properties of near-bipartite graphs that are relevant to us in Chapters 5 and 6. Recall that an R-graph G is a near-bipartite graph with a fixed removable doubleton R. We will adopt the following notational conventions.

Notation 4.1 *For an R-graph G, we shall denote by $H[A, B]$ the underlying bipartite graph $G - R$. We let α and β denote the constituent edges of R, and we adopt the convention that $\alpha := a_1a_2$ has both ends in A, whereas $\beta := b_1b_2$ has both ends in B.*

As we will see, certain pertinent properties of G are closely related to those of H. For this reason, we also review well-known facts concerning bipartite matching covered graphs.

4.1 The exchange property

Recall that an edge of a matching covered graph is removable if its deletion results in another matching covered graph. The removable edges of a bipartite graph satisfy an 'exchange property' and its proof immediately follows from bipartite ear decompositions; see Section 1.2.1.

Proposition 4.2 [Exchange Property of Removable Edges] *Let H denote a bipartite matching covered graph, and let e denote a removable edge of H. If f is a removable edge of $H - e$, then:*

(i) f *is removable in* H*, and*

(ii) e *is removable in* $H - f$*.*

Proof: Observe that the graph $H - f$ may be obtained from the matching covered graph $H - e - f$ by adding a single ear (that is, edge e). Thus, by Proposition 1.5, $H - f$ is matching covered. This proves *(i)*. Statement *(ii)* follows immediately since $H - f - e$ is matching covered. □

The following is a generalization of Proposition 4.2, and it is applicable to certain situations that arise in Chapter 6:

Proposition 4.3 *Let* K *be a conformal matching covered subgraph of a bipartite matching covered graph* H*. Let* e *denote a removable edge of* K*. Then* e *is removable in* H *as well.*

Proof: Since K is a conformal matching covered subgraph, H admits a bipartite ear decomposition starting with K, say $H_1 \subset H_2 \subset \cdots \subset H_r$. Note that $H_1 = K$ and $H_r = H$. Each graph in this sequence includes the edge e. Now, consider the bipartite ear decomposition $H_1 - e \subset H_2 - e \subset \cdots \subset H_r - e$ of the graph $H - e$. Since $H_1 - e = K - e$ is matching covered, Proposition 1.5 implies that $H_r - e = H - e$ is also matching covered, that is, e is removable in H. □

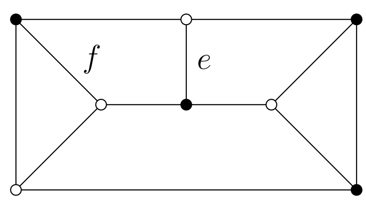


Figure 4.1: f is removable in $St_8 - e$, but it is not removable in St_8

We point out that the conclusion of Proposition 4.2 does not hold, in general, for arbitrary removable edges of nonbipartite graphs. For instance, as shown in Figure 4.1, the edge f is removable in the matching covered graph $St_8 - e$, but it is not removable in St_8. However, as we prove next, the exchange property does hold for R-compatible edges. Recall that an R-compatible edge of an R-graph G is one which is removable in G as well as in the underlying bipartite graph $H := G - R$; see Section 1.7.1.

Proposition 4.4 [EXCHANGE PROPERTY OF R-COMPATIBLE EDGES] *Let G be an R-graph, and let e denote an R-compatible edge of G. If f is an R-compatible edge of $G - e$, then:*

(i) f is R-compatible in G, and

(ii) e is R-compatible in $G - f$.

Proof: Let $H := G - R$. Since f is R-compatible in $G - e$, each of the graphs $G - e - f$ and $H - e - f$ is matching covered. To deduce *(i)*, we need to show that each of $G - f$ and $H - f$ is matching covered. Since f is removable in $H - e$, it follows from Proposition 4.2 that f is removable in H as well. That is, $H - f$ is matching covered.

Next, we note that the edge e is admissible in $H - f$. Thus e is admissible in $G - f$. As $G - e - f$ is matching covered, we conclude that $G - f$ is also matching covered. This proves *(i)*. Statement *(ii)* follows immediately, since each of $G - f - e$ and $H - f - e$ is matching covered. □

4.2 Non-removable edges of bipartite graphs

Let $H[A, B]$ denote a bipartite graph, on four or more vertices, that has a perfect matching. Using the well-known Hall's Theorem, it can be shown that an edge f of H is inadmissible (that is, f is not in any perfect matching of H) if and only if there exists a nonempty proper subset S of A such that $|N(S)| = |S|$ and f has one end in $N(S)$ and its other end is not in S.

Now suppose that H is matching covered, and let e denote a non-removable edge of H. Then some edge f of $H - e$ is inadmissible. This fact, coupled with the above observation, may be used to arrive at the following characterization of non-removable edges in bipartite matching covered graphs; see Figure 4.2.

Proposition 4.5 [CHARACTERIZATION OF NON-REMOVABLE EDGES] *Let $H[A, B]$ denote a bipartite matching covered graph on four or more vertices. An edge e of H is non-removable if and only if there exist partitions (A_0, A_1) of A and (B_0, B_1) of B such that $|A_0| = |B_0|$ and e is the only edge joining a vertex in B_0 to a vertex in A_1.* □

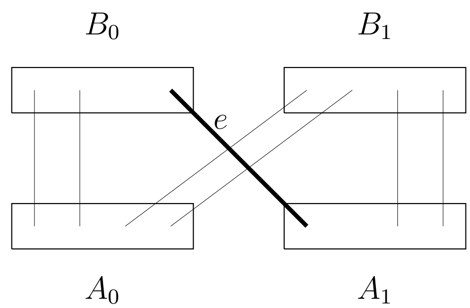


Figure 4.2: Non-removable edge of a bipartite graph

In Figure 4.2, e is the only edge with one end in B_0 and the other in A_1. Consequently, any edge f, with one end in A_0 and the other in B_1, is inadmissible once e is deleted. This fact yields the following corollary. (A 4-cycle is referred to as a *quadrilateral*.)

Corollary 4.6 *Suppose that Q is a quadrilateral of a bipartite matching covered graph H, and let e and f denote two nonadjacent edges of Q. If f is admissible in $H - e$ then e is removable in H.* $\square$

In our work, we will often be interested in finding an R-compatible edge incident at a specified vertex v of an R-brick G. As a first step, we will upper bound the number of edges of $\partial(v)$, which are non-removable in the underlying bipartite graph $H := G - R$. For this purpose, the next lemma of Lovász and Vempala [LV] is especially useful. It is an extension of Proposition 4.5. See Figure 4.3.

Lemma 4.7 [The Lovász-Vempala Lemma] *Let $H[A, B]$ denote a bipartite matching covered graph, and $b \in B$ denote a vertex of degree $d \geq 3$. Let $ba_1, ba_2, \ldots, ba_d$ be the edges of H incident with b. Assume that the edges $ba_1, ba_2, \ldots, ba_r$ where $0 < r \leq d$ are non-removable. Then there exist partitions $(A_0, A_1, \ldots, A_r)$ of A and $(B_0, B_1, \ldots, B_r)$ of B, such that $b \in B_0$, and for $i \in \{1, 2, \ldots, r\}$: (i) $|A_i| = |B_i|$, (ii) $a_i \in A_i$, and (iii) $N(A_i) = B_i \cup \{b\}$; in particular, ba_i is the only edge between B_0 and A_i.* $\square$

Observe that, as per the notation in the above lemma, if ba_1 and ba_2 are non-removable edges, then the vertices a_1 and a_2 have no common neighbour distinct from b. That is, there is no quadrilateral containing edges ba_1 and ba_2 both. This proves the following corollary of Lovász and Vempala [LV].

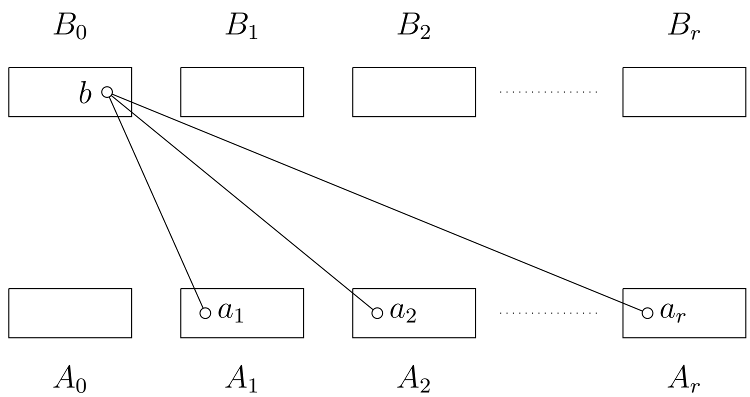


Figure 4.3: Non-removable edges incident at a vertex

Corollary 4.8 *Let H denote a bipartite matching covered graph, and b denote a vertex of degree three or more. If e and f are two edges incident at b which lie in a quadrilateral Q then at least one of e and f is removable.* □

We conclude with an easy application of the Lovász-Vempala Lemma in the context of near-bipartite bricks.

Corollary 4.9 *Let G be an R-brick, and let $H := G - R$. Then for any vertex b, at most two edges of $\partial_H(b)$ are non-removable in H.*

Proof: We adopt Notation 4.1; assume without loss of generality that $b \in B$. If b has only two distinct neighbours in H then the assertion is easily verified. Now suppose that b has at least three distinct neighbours in H, and let d denote the degree of b in H.

Suppose instead that there are $r \geq 3$ non-removable edges incident with b; we denote these as $ba_1, ba_2, \ldots, ba_r$. Then, by the Lovász-Vempala Lemma (4.7), there exist partitions $(A_0, A_1, \ldots, A_r)$ of A and $(B_0, B_1, \ldots, B_r)$ of B, such that $b \in B_0$, and for $i \in \{1, 2, \ldots, r\}$: (i) $|A_i| = |B_i|$, (ii) $a_i \in A_i$, and (iii) $N_H(A_i) = B_i \cup \{b\}$. See Figure 4.3.

Observe that, for $i \in \{1, 2, \ldots, r\}$, every vertex of A_i is isolated in $H - (B_i \cup \{b\})$; consequently, $B_i \cup \{b\}$ is a nontrivial barrier of H. Since G is free of nontrivial barriers (by Theorem 1.9), adding the edges of R must kill each of these barriers. In particular, α must have an end in each A_i for $i \in \{1, 2, \ldots, r\}$. This is not possible, as $r \geq 3$; thus we have a contradiction. This completes the proof of Corollary 4.9. □

4.3 Barriers and tight cuts

We begin with a property of removable edges related to tight cuts which is easily verified; it holds for all matching covered graphs, and will be useful to us in Chapter 5.

Proposition 4.10 *Let G be a matching covered graph, and $\partial(X)$ a tight cut of G, and e an edge of $G[X]$. Then e is removable in $G/\overline{X}$ if and only if e is removable in G.* $\square$

Let us revisit the notion of a barrier cut. If S is a barrier of a matching covered graph G and K is an odd component of $G - S$ then $\partial(V(K))$ is a tight cut of G, and is referred to as a barrier cut. In Sections 4.3.1 and 4.3.2, among other things, we will see that every nontrivial tight cut of a bipartite or of a near-bipartite graph is a barrier cut.

4.3.1 Bipartite graphs

Suppose that X is an odd subset of the vertex set of a bipartite graph $H[A, B]$. Then, clearly one of the two sets $A \cap X$ and $B \cap X$ is larger than the other; the larger of the two sets, denoted X_+, is called the *majority part of* X; and the smaller set, denoted X_-, is called the *minority part of* X.

The following proposition is easily derived, and it provides a convenient way of visualizing tight cuts in bipartite matching covered graphs. See Figure 4.4.

Proposition 4.11 [Tight Cuts in Bipartite Graphs] *A cut $\partial(X)$ of a bipartite matching covered graph H is tight if and only if the following hold:*

(i) $|X|$ is odd and $|X_+| = |X_-| + 1$, consequently $|\overline{X}_+| = |\overline{X}_-| + 1$, and

(ii) there are no edges between X_- and $\overline{X}_-$. $\square$

Observe that, in the above proposition, X_+ and $\overline{X}_+$ are both barriers of H. It follows that every tight cut of a bipartite matching covered graph is a barrier cut.

Recall that, for a bipartite matching covered graph $H[A, B]$, its maximal barriers are precisely its color classes A and B. Now let S denote a nontrivial barrier of H which is not maximal, and adjust notation so that $S \subset B$. It may be inferred from Proposition 4.11

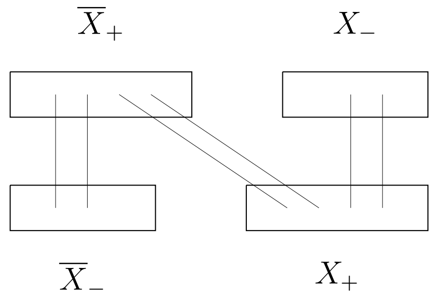


Figure 4.4: Tight cuts in bipartite matching covered graphs

that $H - S$ has precisely $|S| - 1$ isolated vertices each of which is a member of A, and it has precisely one nontrivial odd component K which gives rise to a nontrivial barrier cut of H, namely $\partial(V(K))$.

Since braces are bipartite matching covered graphs which are free of nontrivial tight cuts, Proposition 4.11 may be used to obtain the following characterizations of braces.

Proposition 4.12 [Characterizations of Braces] *Let $H[A, B]$ denote a bipartite graph of order six or more, where $|A| = |B|$. Then the following statements are equivalent:*

(i) *H is a brace,*

(ii) *$|N(S)| \geq |S| + 2$ for every nonempty subset S of A such that $|S| < |A| - 1$, and*

(iii) *$H - \{a_1, a_2, b_1, b_2\}$ has a perfect matching for any four distinct vertices $a_1, a_2 \in A$ and $b_1, b_2 \in B$.* □

4.3.2 Near-bipartite graphs

Let G denote an R-graph. We adopt Notation 4.1. For an odd subset X of $V(G)$, we define its *majority part* X_+ and its *minority part* X_- by regarding it as a subset of $V(H)$.

Observe that, if X is the shore of a tight cut in G then it is the shore of a tight cut in H as well. This observation, coupled with Proposition 4.11, may be used to derive the following characterization of tight cuts in near-bipartite graphs.

Proposition 4.13 [Tight Cuts in Near-bipartite Graphs] *A cut $\partial(X)$ of an R-graph G is tight if and only if the following hold:*

(i) X *is odd and* $|X_+| = |X_-| + 1$, *and consequently,* $|\overline{X}_+| = |\overline{X}_-| + 1$,

(ii) there are no edges between X_- *and* $\overline{X}_-$; *adjust notation so that* $X_- \subset A$,

(iii) one of α *and* β *has both ends in a majority part; adjust notation so that* α *has both ends in* $\overline{X}_+$, *and*

(iv) β *has at least one end in* $\overline{X}_-$.

Consequently, X_+ *is a nontrivial barrier of* G. *Moreover, the* $\partial(X)$*-contraction* G/X *is near-bipartite with removable doubleton* R, *whereas the* $\partial(X)$*-contraction* $G/\overline{X}$ *is bipartite.*

Proof: A simple counting argument shows that if all of the statements *(i)* to *(iv)* hold then $\partial(X)$ is indeed a tight cut of G. See Figure 4.5. Now suppose that $\partial(X)$ is a tight cut; as noted earlier, $\partial(X) - R$ is a tight cut of H. Thus *(i)* and *(ii)* follow immediately from Proposition 4.11. Adjust notation so that $X_- \subset A$.

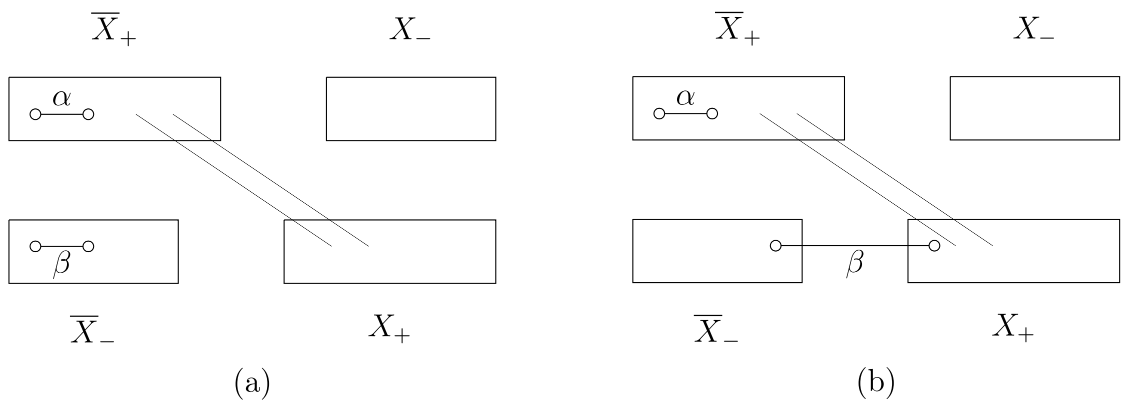


Figure 4.5: Tight cuts in near-bipartite graphs

As each perfect matching of G which contains α must also contain β, we infer that at most one of α and β lies in $\partial(X)$. Furthermore, if α has both ends in X_-, and likewise, if β has both ends in $\overline{X}_-$, then a simple counting argument shows that any perfect matching M of G containing α and β meets $\partial(X)$ in at least three edges; this is a contradiction.

The above observations imply that at least one of α and β has both ends in a majority part; this proves *(iii)*. As in the statement, adjust notation so that α has both ends in $\overline{X}_+$.

Now, if β has both ends in X_+ then it is easily seen that α and β are both inadmissible. This proves *(iv)*. Note that, either β has both ends in $\overline{X}_-$ as shown in Figure 4.5a, or it has one end in $\overline{X}_-$ and the other end in X_+ as shown in Figure 4.5b.

Note that X_+ is a nontrivial barrier of G, and that $G/\overline{X}$ is bipartite. We let $G_1 := G/X$ denote the other $\partial(X)$-contraction. Observe that $H_1 := H/X$ is bipartite and matching covered. Furthermore, in G_1, α has both ends in one color class of H_1, and likewise, β has both ends in the other color class of H_1; this is true for each of the two cases shown in Figure 4.5. Since $H_1 = G_1 - R$, we infer that G_1 is near-bipartite with removable doubleton R. This completes the proof of Proposition 4.13. □

Recall that a near-brick is a matching covered graph whose tight cut decomposition yields exactly one brick. The following is an immediate consequence of Proposition 4.13.

Corollary 4.14 *An R-graph G is a near-brick, and its unique brick is also near-bipartite with removable doubleton R.* □

In other words, a near-bipartite graph G is a near-brick, and its unique brick, say J, inherits its removable doubletons. The *rank* of G, denoted $\mathsf{rank}(G)$, is the order of the unique brick of G. That is, $\mathsf{rank}(G) := |V(J)|$.

Proposition 4.13 shows that every tight cut of a near-bipartite graph is a barrier cut. Now, let S denote a nontrivial barrier of an R-graph G, and adjust notation so that $S \subset B$. It may be inferred from Proposition 4.13 that $G - S$ has precisely $|S| - 1$ isolated vertices each of which is a member of A, and it has precisely one nontrivial odd component K which yields a nontrivial tight cut of G, namely $\partial(V(K))$. Thus there is a bijective correspondence between the nontrivial barriers of G and its nontrivial tight cuts.

4.4 The Three Case Lemma

Recall that a removable edge e of a brick G is b-invariant if $G - e$ is a near-brick. In this section, we will discuss a lemma of Carvalho, Lucchesi and Murty [CLM02b] that pertains to the structure of such near-bricks, that is, those which are obtained from a brick by deleting a single edge. This lemma is used extensively in their works [CLM02a, CLM06, CLM12], and it will play a vital role in Chapter 5.

We will restrict ourselves to the case in which G is an R-brick and e is R-compatible. (By Proposition 1.20, e is b-invariant.) We adopt Notation 4.1. As the name of the lemma suggests, there will be three cases, depending on which we say that the 'index' of e is zero, one or two. In particular, the index of e (defined later) will be zero if $G - e$ is a brick.

Now consider the situation in which $G-e$ is not a brick; that is, $G-e$ has a nontrivial tight cut. By Proposition 4.13, $G-e$ has a nontrivial barrier; let S be such a barrier which is also maximal, and adjust notation so that $S \subset B$. We let I denote the set of isolated vertices of $(G-e)-S$; note that $I \subset A$. Since G itself is free of nontrivial barriers, we infer that one end of e lies in I and its other end lies in $B-S$. This observation, coupled with the Canonical Partition Theorem (1.3) and the fact that e has only two ends, implies that $G-e$ has at most two maximal nontrivial barriers; furthermore, if it is has two such barriers then one is a subset of A and the other is a subset of B.

The *index* of e, denoted $\mathsf{index}(e)$, is the number of maximal nontrivial barriers in $G-e$. (This notion is closely related to the 'index of a thin edge' defined in Section 3.3. In fact, for an R-thin edge, these are equivalent; see Proposition 4.17.) It follows from the preceding paragraph that the index of e is either zero, one or two; and these form the three cases. This is the gist of the lemma; apart from this, it provides further information in the index two case which is especially useful to us. We now state the Three Case Lemma [CLM06], as it is applicable to an R-compatible edge of an R-brick; see Figures 4.6 and 4.7. (The reason for the asymmetry in our notation in Case (2) is discussed in Section 4.4.2.)

Lemma 4.15 [THE THREE CASE LEMMA] *Let G be an R-brick, and e an R-compatible edge. Let $H[A, B] := G - R$. Then one of the following three alternatives holds:*

(0) *$G - e$ is a brick.*

(1) *$G-e$ has only one maximal nontrivial barrier, say S. Adjust notation so that $S \subset B$. Let I denote the set of isolated vertices of $(G-e)-S$. Then $I \subset A$, and e has one end in I and other end in $B-S$.*

(2) *$G-e$ has two maximal nontrivial barriers, say S_1 and S_2^*. Adjust notation so that $S_1 \subset B$ and $S_2^* \subset A$. Let I_1 denote the set of isolated vertices of $(G-e)-S_1$, and I_2^* the set of isolated vertices of $(G-e)-S_2^*$. Then the following hold:*

   (i) $I_1 \subset A$ and $I_2^* \subset B$;

   (ii) e has one end in $I_1 - S_2^*$ and other end in $I_2^* - S_1$;

   (iii) $S_2 := S_2^* - I_1$ is the unique maximal nontrivial barrier of $(G-e)/X_1$, where $X_1 := S_1 \cup I_1$; furthermore, S_2 is a barrier of $G-e$ as well, and $I_2 := I_2^* - S_1$ is the set of isolated vertices of $(G-e)-S_2$. □

Now, let e denote an R-compatible edge of an R-brick G. By the *rank* of e, denoted $\mathsf{rank}(e)$, we mean the rank of the R-graph $G-e$. That is, $\mathsf{rank}(e) := \mathsf{rank}(G-e)$. Recall that e is R-thin if the retract of $G-e$ is a brick. In particular, every R-compatible edge of index zero is R-thin, and these are the only edges whose rank equals $n := |V(G)|$.

In what follows, we will further discuss the cases in which the index of e is either one or two; in each case, we shall relate the rank of e with the information provided by the Three Case Lemma, and we examine the conditions under which e is R-thin. These discussions are especially relevant to an important result in Chapter 5, namely, Lemma 5.17.

We adopt Notation 4.1. Let y and z denote the ends of e such that $y \in A$ and $z \in B$. Note that, if y is cubic, then the two neighbours of y in $G-e$ constitute a barrier of $G-e$; a similar statement holds for z. It follows that if both ends of e are cubic then the index of e is two.

4.4.1 Index one

Suppose that the index of e is one. As in case (1) of the Three Case Lemma, we let S denote the unique maximal nontrivial barrier of $G-e$, and I the set of isolated vertices of $(G-e)-S$. Note that $|I| = |S|-1$. We adjust notation so that $S \subset B$ and $I \subset A$; see Figure 4.6. Observe that $y \in I$ and $z \in B-S$.

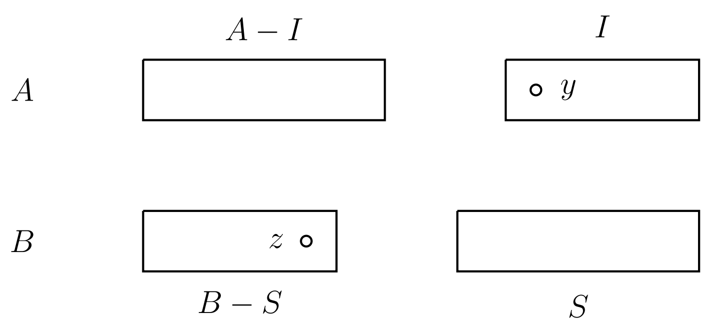


Figure 4.6: An R-compatible edge of index one

In this case, $G-e$ has a unique nontrivial tight cut $\partial(X)$, where $X := S \cup I$. Consequently, $(G-e)/X$ is the brick of $G-e$, and the rank of e is $|V(G)-X|+1$. Furthermore, e is R-thin if and only if $|S| = 2$; and in this case, y is cubic, $N(y) = S \cup \{z\}$, and $\mathsf{rank}(e) = n-2$.

4.4.2 Index two

Suppose that the index of e is two. As in case (2) of the Three Case Lemma, we let S_1 denote one of the two maximal nontrivial barriers of $G-e$, and I_1 the set of isolated vertices of $(G-e)-S_1$, adjusting notation so that $S_1 \subset B$ and $I_1 \subset A$. Note that $|I_1| = |S_1| - 1$ and that $y \in I_1$; see Figure 4.7.

Now, let S_2^* denote the unique maximal nontrivial barrier of $G-e$ which is a subset of A, and I_2^* the set of isolated vertices of $(G-e)-S_2^*$. As in the index one case (see Figure 4.6), we would like to break $V(G)$ into disjoint subsets in order to be able to compute the rank of e. However, this is complicated by the possibility that $S_2^* \cap I_1$ may be nonempty. This explains the asymmetry in our notation in case (2). Fortunately, it turns out that $S_2 := S_2^* - I_1$ is the only maximal nontrivial barrier of $(G-e)/X_1$, where $X_1 := S_1 \cup I_1$. Furthermore, S_2 is a barrier of $G-e$ as well, and $I_2 := I_2^* - S_1$ is the set of isolated vertices of $(G-e)-S_2$. Note that $|I_2| = |S_2| - 1$ and that $z \in I_2$; see Figure 4.7. We let $X_2 := S_2 \cup I_2$.

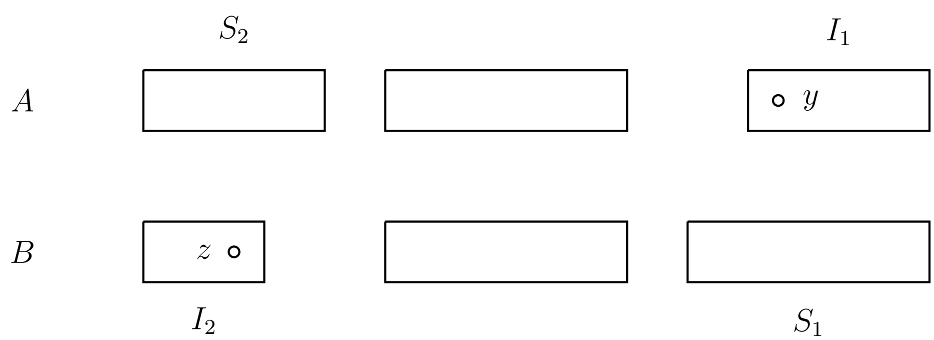


Figure 4.7: An R-compatible edge of index two

In this case, $\partial(X_1)$ and $\partial(X_2)$ are both tight cuts of $G-e$; more importantly, $\partial(X_2)$ is the unique tight cut of $(G-e)/X_1$. Consequently, $((G-e)/X_1)/X_2$ is the brick of $G-e$, and the rank of e is $|V(G) - X_1 - X_2| + 2$.

Furthermore, e is R-thin if and only if $|S_1| = 2 = |S_2|$; and in this case, y and z are both cubic, $N(y) = S_1 \cup \{z\}$ and $N(z) = S_2 \cup \{y\}$, and $\mathsf{rank}(e) = n-4$; also, by switching the roles of S_1 and S_2^*, we infer that $|S_2^*| = 2$.

4.4.3 Index and Rank of an R-thin Edge

The following characterization of R-thin edges is immediate from our discussion in the previous two sections.

Proposition 4.16 [CHARACTERIZATION OF R-THIN EDGES IN TERMS OF BARRIERS] *An R-compatible edge e of an R-brick G is R-thin if and only if every barrier of $G - e$ has at most two vertices.* □

In summary, if the index of e is zero then e is thin and its rank is $n := |V(G)|$. If the index of e is one then $\mathsf{rank}(e) \leq n-2$, and equality holds if and only if e is thin. Likewise, if the index of e is two then $\mathsf{rank}(e) \leq n-4$, and equality holds if and only if e is thin.

We conclude by showing that, for an R-thin edge, the notion of index used here is equivalent to the one defined in Section 3.3.

Proposition 4.17 *Let G be an R-brick, and e an R-thin edge. Then the following statements hold:*

(i) $\mathsf{index}(e) = 0$ *if and only if both ends of e have degree four or more in G;*

(ii) $\mathsf{index}(e) = 1$ *if and only if exactly one end of e has degree three in G; and*

(iii) $\mathsf{index}(e) = 2$ *if and only if both ends of e have degree three in G and e does not lie in a triangle.*

Proof: We note that $\mathsf{index}(e) = 0$ if and only if $G - e$ is free of nontrivial barriers, that is, $G - e$ is a brick; and since e is a thin edge, the latter holds if and only if both ends of e have degree four or more in G. This proves *(i)*.

Let $n := |V(G)|$. We note that $\mathsf{index}(e) = 1$ if and only if $\mathsf{rank}(e) = n - 2$; and since e is a thin edge, the latter holds if and only if exactly one end of e has degree three in G.

Now suppose that $\mathsf{index}(e) = 2$, whence $\mathsf{rank}(e) = n - 4$, and consequently, both ends of e have degree three in G. Conversely, if both ends of e have degree three in G then $G - e$ has two nontrivial barriers which lie in different color classes of $(G - e) - R$, and thus $\mathsf{index}(e) = 2$; furthermore, since e is R-compatible, neither end of e is incident with an edge of R and thus e does not lie in a triangle. □

Chapter 5

Generating near-bipartite bricks

In this chapter, we establish the generation procedure for near-bipartite bricks discussed in Section 1.7.2. Recall that an edge of an R-brick is R-thin if it is R-compatible as well as thin. Our goal is to prove Theorem 1.22, which is restated below.

Theorem 1.22 [R-THIN EDGE THEOREM] *Every R-brick distinct from K_4 and $\overline{C_6}$ has an R-thin edge.*

In fact, we will show something stronger, which is especially useful in the proof of the main result of Chapter 6. Let G be an R-brick distinct from K_4 and $\overline{C_6}$. Then, by Theorem 1.21 of Carvalho et al., G has an R-compatible edge; let e be any such edge. Recall from Chapter 4 that there are two parameters associated with e: the rank of e is the order of the unique brick of $G-e$; and, the index of e is the number of maximal nontrivial barriers of $G-e$, which by the Three Case Lemma (4.15) is either zero, one or two. Using these parameters, we may state our stronger theorem as follows.

Theorem 5.1 *Let G be an R-brick which is distinct from K_4 and $\overline{C_6}$, and let e denote an R-compatible edge of G. Then one of the following alternatives hold:*

- *either e is R-thin,*
- *or there exists another R-compatible edge f such that:*

  (i) f has an end each of whose neighbours in $G-e$ lies in a barrier of $G-e$, and

  (ii) $\mathsf{rank}(f)+\mathsf{index}(f)>\mathsf{rank}(e)+\mathsf{index}(e)$.

Since the rank and index are bounded quantities, the above theorem immediately implies the R-thin Edge Theorem (1.22). Our proof uses tools from the work of Carvalho et al. [CLM06], and the overall approach is inspired by their proof of the Thin Edge Theorem (1.15).

The following proposition shows that condition *(ii)* in Theorem 5.1 is implied by a weaker condition involving only the rank function.

Proposition 5.2 *Suppose that e and f denote two R-compatible edges of an R-brick G. If $\mathsf{rank}(f) > \mathsf{rank}(e)$ then $\mathsf{rank}(f) + \mathsf{index}(f) > \mathsf{rank}(e) + \mathsf{index}(e)$.*

Proof: Note that, since the rank of an edge is even, $\mathsf{rank}(f) > \mathsf{rank}(e)+1$. As the index of an edge is either zero, one or two, we only need to examine the case in which $\mathsf{index}(e) = 2$ and $\mathsf{index}(f) = 0$. However, in this case, $\mathsf{rank}(f) = n$ and $\mathsf{rank}(e) \leq n - 4$ where $n := |V(G)|$, and thus the conclusion holds. □

In the statement of Theorem 5.1, if the given R-compatible edge e is thin, then the assertion is vacuously true. Thus, in its proof, we may assume that e is not thin. It then follows from Proposition 4.16 that $G-e$ has a barrier with three or more vertices; let S be such a barrier. In the next section, we introduce the notion of a candidate edge (relative to e and S) which, as we will see, is an R-compatible edge that satisfies condition *(i)* in the statement of Theorem 5.1, and has rank at least that of e.

5.1 The candidate set $\mathcal{F}(e, S)$

Let G be an R-brick, and let $e := yz$ denote an R-compatible edge which is not thin. We first set up some notation which is used throughout this chapter.

Notation 5.3 *We shall denote by $H[A, B]$ the underlying bipartite graph $G - R$. We let $R := \{\alpha, \beta\}$; and we adopt the convention that $\alpha := a_1a_2$ has both ends in A, whereas $\beta := b_1b_2$ has both ends in B. Adjust notation so that $y \in A$ and $z \in B$.*

The reader is advised to review Section 4.3.2 before proceeding further. Let S be a barrier of $G-e$ such that $|S| \geq 3$, and I the set of isolated vertices of $(G-e)-S$. Adjust notation so that $S \subset B$ and $I \subset A$, as shown in Figure 5.1a. Observe that $X := S \cup I$ is the shore of a tight cut in $G - e$, as well as in $H - e$. By Proposition 4.13, α has both ends in $A - I$; whereas β either has both ends in $B - S$, or it has one end in $B - S$ and another in S. We denote the bipartite matching covered graph

$$(H - e)/\overline{X} \to \overline{x}$$

by $H(e, S)$. Note that its color classes are the sets $I \cup \{\overline{x}\}$ and S; see Figure 5.1b.

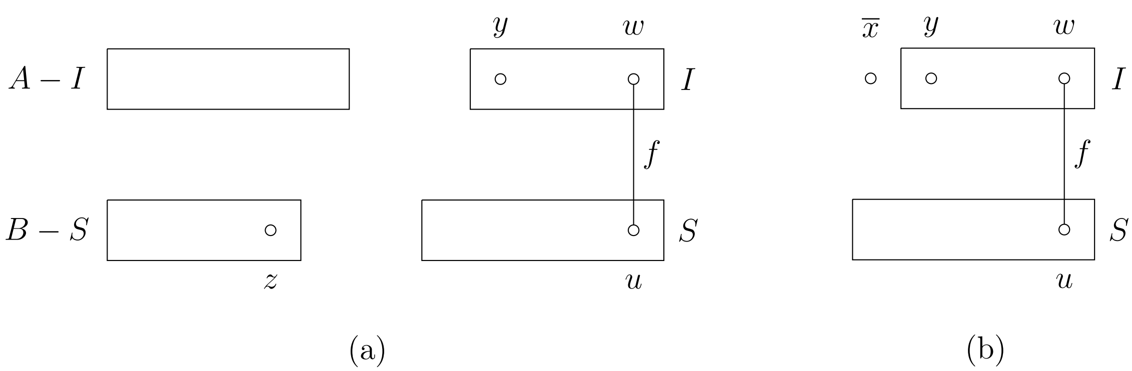


Figure 5.1: (a) S is a barrier of $G - e$ such that $|S| \geq 3$; (b) the bipartite graph $H(e, S)$

Definition 5.4 [The Candidate Set $\mathcal{F}(e, S)$] *We denote by $\mathcal{F}(e, S)$ the set of those removable edges of $H(e, S)$ which are not incident with the contraction vertex $\overline{x}$, and we refer to it as the candidate set (relative to e and the barrier S of $G - e$), and each member of $\mathcal{F}(e, S)$ is called a candidate edge.*

We remark that Carvalho et al. [CLM06] used a similar notion. Since their work concerns general bricks (that is, not just near-bipartite ones), they consider the graph $(G - e)/\overline{X} \to \overline{x}$ and its removable edges which are not incident with the contraction vertex. See Lemma 23 and Theorem 24 in [CLM06].

Now, let $f := uw$ denote a member of the candidate set $\mathcal{F}(e, S)$, as shown in Figure 5.1b. The end w of f lies in I, and all of the neighbours of w, in $G - e$, lie in the barrier S; consequently, f satisfies condition *(i)*, Theorem 5.1. It should be noted that e and f are adjacent if and only if w is the same as y. We now show that f is an R-compatible edge and it has rank at least that of e. The argument pertaining to ranks is the same as that in [CLM06, Lemma 26].

Proposition 5.5 [Properties of Candidate Edges] *Every member of $\mathcal{F}(e,S)$ is an R-compatible edge of $G-e$, and of G, and has rank at least that of e. Conversely, each R-compatible edge of $G-e$, which is incident with a vertex of I, is a member of $\mathcal{F}(e,S)$.*

Proof: Let f be any member of $\mathcal{F}(e,S)$, as shown in Figure 5.1b. We will use Proposition 4.10 to show that f is R-compatible in $G-e$.

Observe that $H(e,S)$ is one of the C-contractions of $H-e$, where $C := \partial(X) - e - R$ is a tight cut. Since f is removable in $H(e,S)$ and $f \notin C$, Proposition 4.10 implies that f is removable in $H-e$ as well. A similar argument shows that f is removable in $G-e$. Thus, f is R-compatible in $G-e$; the exchange property (Proposition 4.4) implies that f is R-compatible in G as well.

Note that since both ends of f are in the bipartite shore X, the brick of $G-e-f$ is the same as the brick of $G-e$. In particular, $\mathsf{rank}(G-e-f) = \mathsf{rank}(G-e)$. On the other hand, note that if D is any tight cut of $G-f$ then $D-e$ is a tight cut of $G-e-f$, whence $\mathsf{rank}(G-f) \geq \mathsf{rank}(G-e-f)$. Thus $\mathsf{rank}(f) \geq \mathsf{rank}(e)$. This proves the first statement.

Now suppose that f is an R-compatible edge of $G-e$ which is incident at some vertex of I. In particular, $H-e-f$ is matching covered; that is, f is removable in $H-e$. By Proposition 4.10, f is removable in $H(e,S)$. This completes the proof of Proposition 5.5. $\square$

In summary, we have shown that every candidate edge is R-compatible; furthermore, it satisfies condition *(i)*, Theorem 5.1; and it has rank at least that of e.

The following property of candidate sets will be useful in dealing with those nontrivial barriers of $G-e$ which are not maximal.

Corollary 5.6 *Let S^* be any barrier of $G-e$. If $S \subset S^*$ then $\mathcal{F}(e,S) \subset \mathcal{F}(e,S^*)$.*

Proof: Let f be a member of $\mathcal{F}(e,S)$. Then f is incident with some vertex of I, say w. Note that w also lies in I^* which denotes the set of isolated vertices of $(G-e)-S^*$.

As f is a member of $\mathcal{F}(e,S)$, Proposition 5.5 implies that f is R-compatible in $G-e$. Consequently, since f is incident at $w \in I^*$, the last assertion of Proposition 5.5, with S^* playing the role of S, implies that f is a member of $\mathcal{F}(e,S^*)$. Thus $\mathcal{F}(e,S) \subset \mathcal{F}(e,S^*)$. $\square$

Now, we will prove two lemmas; each of which gives an upper bound on the number of non-removable edges incident at a vertex of the bipartite graph $H(e, S)$, which is distinct from the contraction vertex $\overline{x}$. Both of them are easy applications of the Lovász-Vempala Lemma (4.7); we will use arguments similar to those in the proof of Corollary 4.9.

Lemma 5.7 *Let u denote a vertex of S which has degree three or more in $H(e, S)$. Then at most two edges of $\partial(u) - \beta$ are non-removable in $H(e, S)$. Furthermore, if precisely two edges of $\partial(u) - \beta$ are non-removable in $H(e, S)$ and if vertices u and $\overline{x}$ are adjacent then the edge $u\overline{x}$ is non-removable in $H(e, S)$.*

Proof: Assume that there are $k \geq 1$ non-removable edges incident with the vertex u, namely, $uw_1, uw_2, \ldots, uw_k$. Then, by Lemma 4.7, there exist partitions $(A_0, A_1, \ldots, A_k)$ of $I \cup \{\overline{x}\}$, and $(B_0, B_1, \ldots, B_k)$ of S, such that $u \in B_0$, and for $j \in \{1, 2, \ldots, k\}$: (i) $|A_j| = |B_j|$, (ii) $w_j \in A_j$ and (iii) $N(A_j) = B_j \cup \{u\}$. See Figure 5.2.

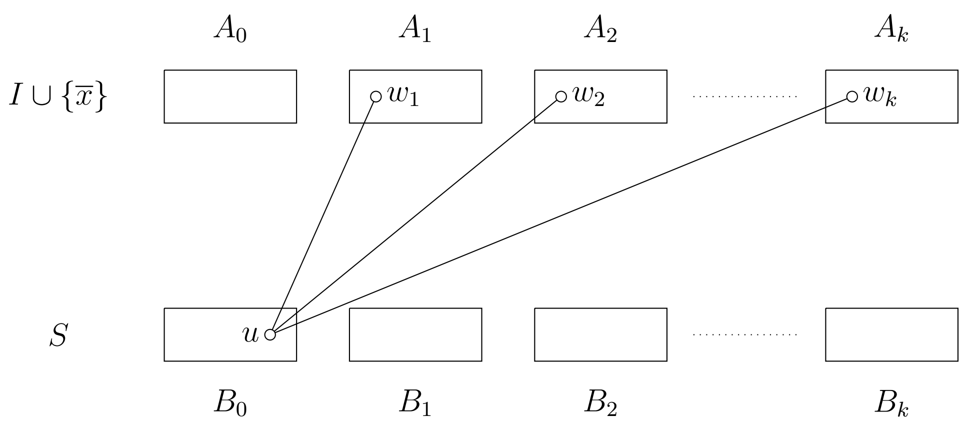


Figure 5.2: Illustration for Lemma 5.7

For $1 \leq j \leq k$, note that $B_j \cup \{u\}$ is a barrier of $H(e, S)$. Moreover, if the set A_j contains neither the contraction vertex $\overline{x}$ nor the end y of e, then $B_j \cup \{u\}$ is a barrier of G itself, which is not possible as G is a brick. We thus arrive at the conclusion that $k \leq 2$, which proves the first part of the assertion.

Now consider the case when $k = 2$. It follows from the above argument that one of the vertices y and $\overline{x}$ lies in the set A_1, whereas the other vertex lies in the set A_2. Adjust notation so that $y \in A_1$ and $\overline{x} \in A_2$. Observe that if u and $\overline{x}$ are adjacent, then $u\overline{x}$ is the unique edge between B_0 and A_2, and it is non-removable in $H(e, S)$ by assumption. This completes the proof of Lemma 5.7. □

Now we turn to the examination of non-removable edges of $H(e, S)$ incident with vertices in I. The proof is similar to that of Lemma 5.7, except that the roles of the color classes S and $I \cup \{\overline{x}\}$ are interchanged.

Lemma 5.8 *Let w denote a vertex of I which has degree three or more in $H(e, S)$. Then at most two edges of $\partial(w) - e$ are non-removable in $H(e, S)$. Furthermore, if precisely two edges of $\partial(w) - e$ are non-removable in $H(e, S)$ then the following hold:*

(i) an end of β lies in S; adjust notation so that $b_1 \in S$,

(ii) in $H(e, S)$, the vertices b_1 and $\overline{x}$ are nonadjacent,

(iii) if b_1 and w are adjacent then the edge $b_1 w$ is non-removable in $H(e, S)$, and

(iv) w is distinct from the end y of e.

Proof: Suppose that there exist $k \geq 1$ non-removable edges incident at the vertex w, namely, $wu_1, wu_2, \ldots, wu_k$. Then, by Lemma 4.7, there exist partitions $(A_0, A_1, \ldots, A_k)$ of the color class $I \cup \{\overline{x}\}$, and $(B_0, B_1, \ldots, B_k)$ of the color class S, such that $w \in A_0$, and for $j \in \{1, 2, \ldots, k\}$: (i) $|A_j| = |B_j|$, (ii) $u_j \in B_j$ and (iii) $N(B_j) = A_j \cup \{w\}$. See Figure 5.3.

For $1 \leq j \leq k$, note that $A_j \cup \{w\}$ is a barrier of $H(e, S)$. Furthermore, if the contraction vertex $\overline{x}$ is not in A_j, or if an end of the edge β is not in B_j, then $A_j \cup \{w\}$ is a barrier of G itself, which is absurd since G is a brick. Clearly, this would be the case for some $j \in \{1, 2, \ldots, k\}$ if $k \geq 3$. We conclude that $k \leq 2$, thus establishing the first part of the assertion.

Now suppose that $k = 2$. It follows from the preceding paragraph that an end of β lies in B_1 or in B_2. This proves *(i)*. Adjust notation so that $b_1 \in B_1$. Furthermore, the contraction vertex $\overline{x}$ lies in A_2. Consequently, vertices b_1 and $\overline{x}$ are nonadjacent; this verifies *(ii)*. Note that if b_1 and w are adjacent, then the edge $b_1 w$ is the unique edge between A_0 and B_1, and it is non-removable in $H(e, S)$ by assumption. This proves *(iii)*. Finally, consider the case in which $w = y$, where y is the end of e in I. Observe that the neighbourhood of $A_0 - y$ lies in the set B_0 in the graph $H(e, S)$ as well as in G, whence B_0 is a barrier of G. We conclude that $|B_0| = 1$, and that y is the only vertex of A_0.

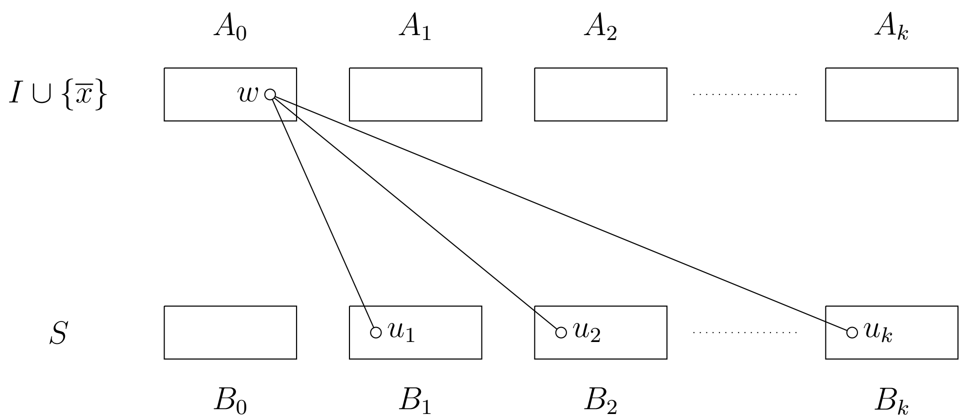


Figure 5.3: Illustration for Lemma 5.8

Furthermore, the neighbourhood of A_1 lies in $B_1 \cup B_0$, and thus $B_1 \cup B_0$ is a nontrivial barrier in $H(e, S)$ as well as in G, which is absurd. We conclude that w is distinct from the end y of e; thus *(iv)* holds. This completes the proof of Lemma 5.8. $\square$

The above lemma implies that each vertex of I, except possibly the end y of e, is incident with at least one candidate. Furthermore, if y has degree three or more in $H(e, S)$ then y is incident with at least two candidates; and likewise, if any other vertex of I, say w, has degree four or more then w is incident with at least two candidates. We thus have the following corollary which is used in the next section.

Corollary 5.9 *The candidate set $\mathcal{F}(e, S)$ has cardinality at least $|S| - 2$. (In particular, the set $\mathcal{F}(e, S)$ is nonempty.) Furthermore, if $\mathcal{F}(e, S)$ is a matching then each vertex of I is cubic in G and $|\mathcal{F}(e, S)| = |S| - 2$.* $\square$

As we will see later, by a result of Carvalho et al. (Corollary 5.19), if the candidate set $\mathcal{F}(e, S)$ is not a matching then it has a member whose rank is strictly greater than that of e. For this reason, in the proof of Theorem 5.1, we will mainly have to deal with the case in which the candidate set is a matching.

5.1.1 When the candidate set is a matching

In this section, we suppose that the candidate set $\mathcal{F}(e,S)$ is a matching. We will make several observations, and these will be useful to us in Section 5.3 where the proof of Theorem 5.1 is presented. For all of the figures in the rest of this chapter, the solid vertices are those which are known to be cubic in the brick G; the hollow vertices may or may not be cubic.

Since $\mathcal{F}(e,S)$ is a matching, Corollary 5.9 implies that every vertex of I is cubic in G, as shown in Figure 5.4. Furthermore, each of these vertices, except for the end y of e, is incident with exactly one candidate edge; in particular, $|\mathcal{F}(e,S)| = |I| - 1 = |S| - 2$.

Notation 5.10 *We let $w_1, w_2, \ldots, w_k$ denote the vertices of $I - y$, where $k := |S| - 2$, and for $1 \le j \le k$, denote the edge of $\mathcal{F}(e,S)$ incident with w_j by f_j and its end in S by u_j.*

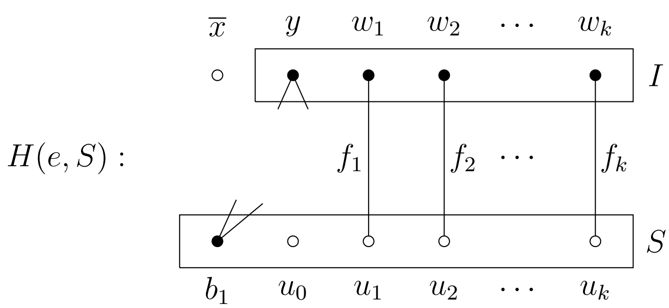


Figure 5.4: When $\mathcal{F}(e,S)$ is a matching

Note that, since $\mathcal{F}(e,S)$ is a matching, the vertices $u_1, u_2, \ldots, u_k$ are distinct, as shown in Figure 5.4. Since every vertex of I is incident with two non-removable edges of $H(e,S)$, we deduce the following by assertions *(i)*, *(ii)* and *(iii)* of Lemma 5.8, respectively:

(1) an end of β lies in S; adjust notation so that $b_1 \in S$,

(2) in $H(e,S)$, vertices b_1 and $\overline{x}$ are nonadjacent; consequently, in G, all neighbours of b_1, except b_2, lie in I, and

(3) b_1 is distinct from each of $u_1, u_2, \ldots, u_k$.

Furthermore, since b_1 is not incident with any member of $\mathcal{F}(e, S)$, Lemma 5.7 implies that it has precisely two neighbours in I; in particular, b_1 is cubic in G.

Notation 5.11 *We let u_0 denote the vertex of S which is distinct from $b_1, u_1, u_2, \ldots, u_k$. That is, $S = \{b_1, u_0, u_1, u_2, \ldots, u_k\}$. (See Figure 5.4.)*

As the vertex u_0 is not incident with any candidate, we conclude using Lemma 5.7 that u_0 has at most one neighbour in I. Observe that if u_0 has no neighbours in I then $(S - u_0) \cup \{z\}$ is a barrier of G (where z is the end of e which is not in I), which is absurd as G is a brick. Thus, u_0 has precisely one neighbour in I.

We note that if y is the unique neighbour of u_0 in the set I, then $S - u_0$ is a barrier of G, which leads us to the same contradiction as before. We thus conclude that u_0 has precisely one neighbour in the set $I - y$, and that its remaining neighbours lie in $\overline{X}$; see Figure 5.5. In particular, in $H(e, S)$, there are are least two edges between u_0 and $\overline{x}$.

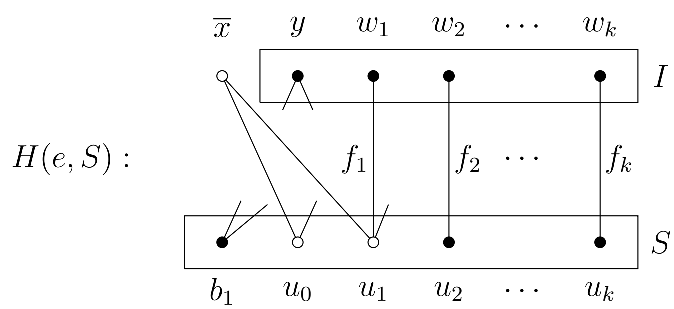


Figure 5.5: u_0 and u_1 are the only vertices adjacent with the contraction vertex $\overline{x}$

Finally, since each vertex u_j in the set $\{u_1, u_2, \ldots, u_k\}$ is incident with exactly one candidate, Lemma 5.7 implies that u_j must satisfy one of the following conditions:

(i) either u_j has some neighbour in the set $\overline{X}$ and it has precisely two neighbours in the set I,

(ii) or otherwise, u_j has no neighbours in the set $\overline{X}$ and it has precisely three neighbours in the set I.

Observe, by counting degrees of the vertices in I, that there are precisely $3k+2$ edges with one end in I and the other end in S. Of these $3k+2$ edges, precisely two are incident with b_1, and only one is incident with u_0. Thus there are $3k-1$ edges with one end in I and the other end in $\{u_1, u_2, \ldots, u_k\}$. It follows immediately that exactly one vertex among $u_1, u_2, \ldots, u_k$ satisfies condition (i); every other vertex satisifes condition (ii).

Notation 5.12 *We adjust notation so that u_1 is the only vertex in $\{u_1, u_2, \ldots, u_k\}$ which has neighbours in $\overline{X}$. (See Figure 5.5.)*

Adopting the notation introduced thus far, the next proposition summarizes our observations in terms of the brick G.

Proposition 5.13 [When the Candidate Set is a Matching] *The following hold:*

(i) each vertex of I is cubic,

(ii) b_1 is cubic and its neighbours lie in $I \cup \{b_2\}$,

(iii) u_0 has precisely one neighbour in $I - y$, and all of its remaining neighbours lie in $\overline{X}$,

(iv) u_1 has precisely two neighbours in I, and all of its remaining neighbours lie in $\overline{X}$,

(v) if $|S| \geq 4$, then each vertex of $S - \{b_1, u_0, u_1\}$ has precisely three neighbours and these neighbours lie in I. □

Observe that, if the barrier S has precisely three vertices, then the candidate set $\mathcal{F}(e, S)$ has only one edge (that is, $f_1 = u_1 w_1$); in this case, all of the edges of $G[X]$ are determined by Proposition 5.13, as listed below, and as shown in Figure 5.6. (Note that the underlying simple graph of $H(e, S)$ is a ladder of order six whose cubic vertices are u_1 and w_1.)

Remark 5.14 *Suppose that $|S| = 3$. Then the following hold:*

(i) the three neighbours of b_1 are y, w_1 and b_2,

(ii) u_0 is adjacent with w_1, and all of its remaining neighbours lie in $\overline{X}$,

(iii) u_1 is adjacent with y and with w_1, and all of its remaining neighbours lie in $\overline{X}$.

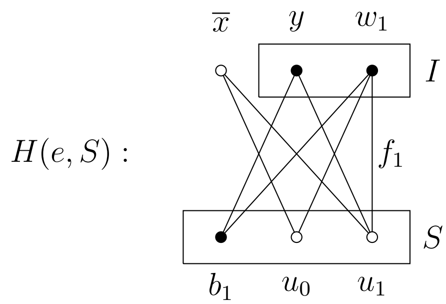


Figure 5.6: When $\mathcal{F}(e,S)$ is a matching, and S has only three vertices

We shall now consider the situation in which $|S| \geq 4$, that is, $k \geq 2$. Note that, as per our notation, $f_1 = u_1w_1$ is the only candidate whose end in S (that is, u_1) has a neighbour in $\overline{X}$. In this sense, f_1 is different from the remaining candidates $f_2, f_3, \ldots, f_k$. In the following proposition, we first show that b_1 is nonadjacent with the end w_1 of f_1. Consequently, b_1 is adjacent with at least one of $w_2, w_3, \ldots, w_k$; we shall assume without loss of generality that b_1 is adjacent with w_2, as shown in Figure 5.7. In its proof, we will apply the Lovász-Vempala Lemma (4.7) to the graph $H(e,S)$, first at w_1, and then at w_2; each of these applications is a refinement of the situation in Lemma 5.8.

Proposition 5.15 *Suppose that $|S| \geq 4$. Then the following hold:*

(i) b_1 and w_1 are nonadjacent; adjust notation so that b_1w_2 is an edge of G,

(ii) y is adjacent with each of b_1 and u_2, and

(iii) u_0 and w_2 are nonadjacent.

Proof: First, we apply Lemma 4.7 to the graph $H(e,S)$ at vertex w_1. Since $f_1 = u_1w_1$ is the only removable edge incident with w_1, there exist partitions (A_0, A_1, A_2) of $I \cup \{\overline{x}\}$, and (B_0, B_1, B_2) of S, such that $w_1 \in A_0$, and $|A_j| = |B_j|$ for $j \in \{0,1,2\}$, vertex u_1 lies in B_0, and the remaining two neighbours of w_1 lie in B_1 and in B_2, respectively. Furthermore, $N(B_1) = A_1 \cup \{w_1\}$ and $N(B_2) = A_2 \cup \{w_1\}$.

Suppose that b_1 is a neighbour of w_1, and adjust notation so that $b_1 \in B_1$. The contraction vertex $\overline{x}$ lies in A_2, since otherwise $A_2 \cup \{w_1\}$ is a nontrivial barrier in G. We will deduce that each of the sets B_0, B_1 and B_2 is a singleton, and thus the barrier S has precisely three vertices, contrary to the hypothesis.

First of all, note that the neighbourhood of $B_1 - b_1$ is contained in A_1, and thus if $|A_1| \geq 2$ then A_1 is a nontrivial barrier in G; we conclude that $|A_1| = 1$ and that $B_1 = \{b_1\}$. Observe that the contraction vertex $\overline{x}$ is only adjacent with u_1, which lies in B_0, and with u_0. Thus the neighbourhood of $B_2 - u_0$ is contained in $(A_2 - \overline{x}) \cup \{w_1\}$, whence the latter is a barrier of G; we infer that $A_2 = \{\overline{x}\}$; consequently, the unique vertex of B_2 has precisely two neighbours, namely w_1 and $\overline{x}$. It follows that $B_2 = \{u_0\}$. Since the vertex w_1 is cubic, the neighbourhood of $B_0 - u_1$ is contained in $(A_0 - w_1) \cup A_1$, whence the latter is a barrier of G; we infer that $A_0 = \{w_1\}$, thus $B_0 = \{u_1\}$. It follows that $|S| = 3$, contrary to our hypothesis. Thus b_1 and w_1 are nonadjacent; this proves *(i)*. As in the statement of the proposition, adjust notation so that b_1 and w_2 are adjacent; see Figure 5.7.

To deduce *(ii)* and *(iii)*, we apply Lemma 4.7 to the graph $H(e, S)$ at vertex w_2. Similar to the earlier situation, there exist partitions (A_0, A_1, A_2) of $I \cup \{\overline{x}\}$, and (B_0, B_1, B_2) of S, such that $w_2 \in A_0$, and $|A_j| = |B_j|$ for $j \in \{1, 2, 3\}$, vertex u_2 lies in B_0, and the remaining two neighbours of w_2 lie in B_1 and in B_2, respectively. Adjust notation so that b_1 lies in B_1. Also, $N(B_1) = A_1 \cup \{w_2\}$ and $N(B_2) = A_2 \cup \{w_2\}$. As before, we conclude that $\overline{x}$ lies in A_2, and that $|A_1| = |B_1| = 1$.

Observe that the unique vertex of A_1 has all of its neighbours in the set $B_0 \cup B_1$. We will show that $B_0 = \{u_2\}$; this implies that the unique vertex of A_1 has precisely two neighbours, and so it must be the end y of e; this immediately implies *(ii)*.

Note that the neighbourhood of $A_0 - w_2$ is contained in B_0. Thus, if $|A_0| \geq 2$ then y lies in A_0 (since otherwise B_0 is a barrier of G). If $|A_0| \geq 3$ then B_0 is a barrier of $G - e$ with three or more vertices. (Note that the barrier B_0 is contained in the barrier S.) Since no end of β lies in B_0, it follows from our earlier observations that the candidate set $\mathcal{F}(e, B_0)$ is not a matching. However, by Corollary 5.6, $\mathcal{F}(e, B_0)$ is a subset of $\mathcal{F}(e, S)$, and the latter is a matching; this is absurd. We conclude that A_0 has at most two vertices, that is, either $A_0 = \{w_2\}$ or $A_0 = \{y, w_2\}$. Now suppose that $A_0 = \{y, w_2\}$. The unique vertex of A_1 is adjacent with b_1, and thus statement *(i)* implies that $w_1 \notin A_1$. Assume without loss of generality that $A_1 = \{w_3\}$. Since w_3 is cubic, we conclude that its neighbourhood is precisely $B_0 \cup B_1$, and thus $B_0 = \{u_2, u_3\}$. Observe that $Q := w_3u_2w_2b_1w_3$ is a quadrilateral in $H(e, S)$ containing the vertex w_3, and thus by Corollary 4.8, one of the edges w_3u_2 and w_3b_1 is removable in $H(e, S)$; however, this contradicts our hypothesis since the only removable edges are the members of $\mathcal{F}(e, S)$. We thus conclude that $A_0 = \{w_2\}$. As explained earlier, $A_1 = \{y\}$, and thus y is adjacent with each of b_1 and u_2; this proves *(ii)*.

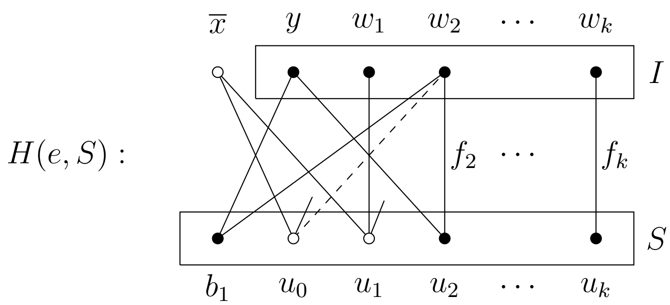


Figure 5.7: When $\mathcal{F}(e, S)$ is a matching, and S has four or more vertices; the vertices u_0 and w_2 are nonadjacent

Now suppose that u_0 and w_2 are adjacent. Observe that $u_1 \in B_2$, and thus all of its neighbours lie in A_2, whence $|A_2| \geq 3$. The neighbourhood of $B_2 - \{u_0, u_1\}$ is contained in $A_2 - \overline{x}$, whence the latter is a nontrivial barrier of G, which is a contradiction. We thus conclude that u_0 and w_2 are nonadjacent; this proves *(iii)*, and completes the proof of Proposition 5.15. □

5.2 The Equal Rank Lemma

Here, we present an important lemma which is used in the proof of Theorem 5.1. This lemma considers the situation in which G is an R-brick and $e := yz$ is an R-compatible edge of index two that is not thin, and f is a candidate relative to a barrier of $G - e$ such that f is also of index two and its rank is equal to that of e. The reader is advised to review the Three Case Lemma (4.15) and Section 4.4.2 before proceeding further.

The Equal Rank Lemma (5.17) relates the barrier structure of $G - f$ to that of $G - e$. More specifically, the lemma establishes subset/superset relationships between eight sets of vertices: the barriers S_1 and S_2 of $G - e$ (as in Case 2 of Lemma 4.15) and their corresponding sets of isolated vertices I_1 and I_2, and likewise, the barriers S_3 and S_4 of $G - f$ and their corresponding sets of isolated vertices I_3 and I_4. Among other things, the lemma shows that $S_1 \cup I_1 \cup S_2 \cup I_2 = S_3 \cup I_3 \cup S_4 \cup I_4$. We now introduce the relevant notation more precisely.

Since e is of index two, by the Three Case Lemma, $G - e$ has precisely two maximal nontrivial barriers, and since e is not thin, at least one of these barriers, say S_1, has three or more vertices (see Proposition 4.16). We adopt Notation 5.3 for the brick G and edge e. Assume without loss of generality that $S_1 \subset B$, and let I_1 denote the set of isolated vertices of $(G-e)-S_1$. We shall denote by S_2 the maximal nontrivial barrier of $(G-e)/X_1$ where $X_1 := S_1 \cup I_1$, and by I_2 the set of isolated vertices of $(G-e)-S_2$. Note that the end z of e lies in I_2 which is a subset of B, whereas the other end y of e lies in I_1 which is a subset of A. See Figure 5.8 (top).

By Corollary 5.9, the candidate set $\mathcal{F}(e, S_1)$ is nonempty, and by Proposition 5.5, each of its members is an R-compatible edge whose rank is at least that of e. Now, let $f := uw$ be a member of $\mathcal{F}(e, S_1)$ such that $u \in S_1$ and $w \in I_1$, and suppose that the index of f is two. The following result of Carvalho et al. [CLM06, Lemma 32] plays a crucial role in our proof of the Equal Rank Lemma (5.17).

Lemma 5.16 *Assume that* $\mathsf{index}(e) = \mathsf{index}(f) = 2$. *If* $\mathsf{rank}(e) = \mathsf{rank}(f)$ *then* S_2 *is a subset of a barrier of* $G - f$. □

We shall let S_3 denote the maximal nontrivial barrier of $G - f$ which is contained in the color class B, and I_3 the set of isolated vertices of $(G - f) - S_3$. Furthermore, let S_4 denote the maximal nontrivial barrier of $(G - f)/(S_3 \cup I_3)$, and I_4 the set of isolated vertices of $(G-f)-S_4$. Note that the end u of f lies in I_4, and its other end w lies in I_3. See Figure 5.8 (bottom). We are now ready to state the Equal Rank Lemma using the notation introduced so far.

Lemma 5.17 [The Equal Rank Lemma] *Assume that* $\mathsf{index}(e) = \mathsf{index}(f) = 2$. *If* $\mathsf{rank}(e) = \mathsf{rank}(f)$ *then the following statements hold:*

(i) e *and* f *are nonadjacent,*

(ii) $S_3 \subseteq S_1 - u$ *and* $I_3 \subseteq I_1 - y$,

(iii) $S_2 \subset S_4$ *and* $I_2 \subset I_4$,

(iv) $S_1 \cup I_2 = S_3 \cup I_4$ *and* $S_2 \cup I_1 = S_4 \cup I_3$,

(v) $N(u) \subseteq S_2 \cup I_1$, *and*

(vi) e *is a member of the candidate set* $\mathcal{F}(f, S_4)$.

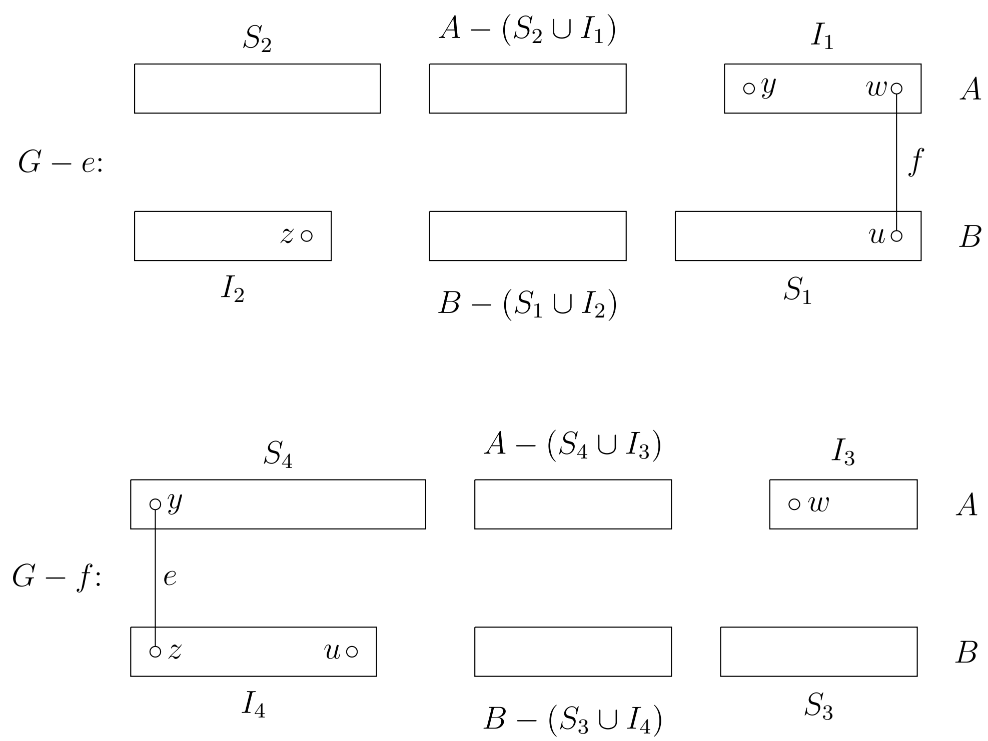


Figure 5.8: The Equal Rank Lemma

Proof: We examine the graph $G - e - f$ in order to prove *(i)* and *(ii)*. Clearly, S_3 is a barrier of $G-e-f$. Observe that, since f has an end in S_1, every barrier of $G-e-f$ which contains S_1 is a barrier of $G-e$ as well. Since S_1 is a maximal barrier of $G-e$, we infer that S_1 is a maximal barrier of $G - e - f$ as well. By the Canonical Partition Theorem (1.3), to prove that S_3 is a subset of S_1, it suffices to show that $S_1 \cap S_3$ is nonempty. To see this, note that $w \in I_1 \cap I_3$, and thus any neighbour of w in $G-e-f$ lies in $S_1 \cap S_3$. Furthermore, since $u \notin S_3$, we conclude that $S_3 \subseteq S_1 - u$; this proves part of *(ii)*. In particular, $z \notin S_3$. Consequently, $y \notin I_3$, and thus y and w are distinct. This proves *(i)*.

Now we prove the remaining part of *(ii)*. Let $v \in I_3$, that is, v is isolated in $(G-f)-S_3$. Consequently, v is isolated in $(G - f) - S_1$. Since f has an end in S_1, we infer that v is isolated in $(G - e) - S_1$, that is, $v \in I_1$. Thus $I_3 \subseteq I_1 - y$. This proves *(ii)*.

We will now prove *(iii)* and *(iv)*. We begin by showing that S_2 is a subset of S_4. By Lemma 5.16, S_2 is a subset of the unique maximal nontrivial barrier of $G - f$ which is contained in the color class A, say S_4^*. By the Three Case Lemma (4.15), $S_4^* = S_4 \cup I'$ for some (possibly empty) subset I' of I_3. That is, S_2 is a subset of $S_4 \cup I'$. Note that S_2 and I_1 are disjoint; by *(ii)*, $S_2 \cap I' = \emptyset$. Thus, $S_2 \subseteq S_4$.

Since the ranks of e and f are equal, it follows that $|A - (S_2 \cup I_1)| = |A - (S_4 \cup I_3)|$ and likewise, $|B - (S_1 \cup I_2)| = |B - (S_3 \cup I_4)|$. In order to prove *(iv)*, it suffices to prove the following claim.

Claim 5.17.1 *$A - (S_2 \cup I_1) \subseteq A - (S_4 \cup I_3)$ and $B - (S_1 \cup I_2) \subseteq B - (S_3 \cup I_4)$.*

Proof: Let $v_1 \in A - (S_2 \cup I_1)$. By *(ii)*, $v_1 \notin I_3$. To prove that v_1 lies in $A - (S_4 \cup I_3)$, it suffices to show that $v_1 \notin S_4$.

Now, let v_2 be any vertex in S_2. We have already shown that $S_2 \subseteq S_4$, and thus $v_2 \in S_4$. Note that, if v_1 also belongs to the barrier S_4, then $(G - f) - \{v_1, v_2\}$ would not have a perfect matching. In the following paragraph, we will show that $(G - e - f) - \{v_1, v_2\}$ has a perfect matching, say M; consequently, $v_1 \notin S_4$.

Let H_1 be the graph $(G-e-f)/\overline{X_1} \to \overline{x_1}$, and let H_2 be the graph $(G-e-f)/\overline{X_2} \to \overline{x_2}$ where $X_2 := S_2 \cup I_2$. Note that H_1 and H_2 are bipartite matching covered graphs. Let $J := ((G-e-f)/X_1 \to x_1)/X_2 \to x_2$. Note that J is the brick of $G-e-f$. Let M_J be a perfect matching of $J - \{x_2, v_1\}$. Let g denote the edge of M_J incident with the contraction vertex x_1. Let M_1 be a perfect matching of H_1 which contains g. Let M_2 be a perfect matching of $H_2 - \{v_2, \overline{x_2}\}$. Observe that $M := M_1 + M_J + M_2$ is the desired matching.

Now, let $v \in B - (S_1 \cup I_2)$. By *(ii)*, $v \notin S_3$. To prove that v lies in $B - (S_3 \cup I_4)$, it suffices to show that $v \notin I_4$. To see this, note that since J is a brick, by Theorem 1.9, $J - \{x_1, x_2\}$ is connected; thus, v is not isolated in $(G - f) - S_4$, that is, $v \notin I_4$. □

It follows from *(ii)* and *(iv)* that the end y of e lies in S_4, and thus S_2 is a proper subset of S_4. Also, we infer from *(ii)* and *(iv)* that I_2 is a subset of I_4. Furthermore, the end u of f lies in I_4, whence I_2 is a proper subset of I_4. This proves *(iii)*.

It remains to prove *(v)* and *(vi)*. As noted above, $u \in I_4$. Thus, all neighbors of u in G lie in $S_4 \cup \{w\} \subseteq S_4 \cup I_3$. It follows from *(iv)* that $N(u) \subseteq S_2 \cup I_1$. This proves *(v)*.

Finally, we prove *(vi)*. Recall that $H(f, S_4)$ denotes the bipartite matching covered graph $(H - f)/\overline{X_4} \to \overline{x_4}$ where $X_4 := S_4 \cup I_4$, and that $\mathcal{F}(f, S_4)$ is the set of those removable edges of $H(f, S_4)$ which are not incident with the contraction vertex $\overline{x_4}$. Since f is R-compatible in $G - e$ (by Proposition 5.5), the exchange property (Proposition 4.4) implies that e is R-compatible in $G - f$. Now, since the end z of e lies in I_4, the last assertion of Proposition 5.5 implies that e is a member of $\mathcal{F}(f, S_4)$. This proves *(vi)*, and finishes the proof of the Equal Rank Lemma. $\square$

5.3 Proof of Theorem 5.1

Before we proceed to prove Theorem 5.1, we state two results of Carvalho et al. [CLM06] which are useful to us. Suppose that G is an R-brick and e is an R-compatible edge which is not thin. We let S_1 denote a maximal nontrivial barrier of $G - e$ such that $|S_1| \geq 3$, and let f denote a member of the candidate set $\mathcal{F}(e, S_1)$.

Note that, since e is not thin, its rank is at most $n - 4$ where $n := |V(G)|$. If the index of f is zero then its rank is n, and in particular, it is greater than that of e. The following result of Carvalho et al. [CLM06, Lemma 31] shows that this conclusion holds even if the index of f is one.

Lemma 5.18 *Suppose that f is a member of the candidate set $\mathcal{F}(e, S_1)$. If the index of f is one then* $\mathsf{rank}(f) > \mathsf{rank}(e)$. $\square$

The following corollary of Lemmas 5.16 and 5.18 was used implicitly by Carvalho et al. [CLM06] in their proof of the Thin Edge Theorem (1.15). We provide its proof for the sake of completeness.

Corollary 5.19 *Assume that the index of e is two. If the candidate set $\mathcal{F}(e, S_1)$ contains two adjacent edges, say f and g, then at least one of them has rank strictly greater than* $\mathsf{rank}(e)$.

Proof: We know by Proposition 5.5 that each of f and g has rank at least $\mathsf{rank}(e)$. If either of them has rank strictly greater than that of e then there is nothing to prove. Now, suppose that $\mathsf{rank}(f) = \mathsf{rank}(g) = \mathsf{rank}(e)$. It follows from Lemma 5.18 that both f and g are of index two. We intend to arrive at a contradiction using Lemma 5.16. We let I_1 denote the set of isolated vertices of $(G - e) - S_1$, and S_2 denote the unique maximal nontrivial barrier of $(G-e)/(S_1 \cup I_1)$. By Lemma 5.16, S_2 is a subset of a barrier of $G-f$, and likewise, S_2 is a subset of a barrier of $G-g$.

Consider two distinct vertices of S_2, say v_1 and v_2. Let M be a perfect matching of the graph $G-\{v_1, v_2\}$. (Such a perfect matching exists as G is a brick.) As noted above, S_2 is a subset of a barrier of $G-f$. In particular, v_1 and v_2 lie in a barrier of $G-f$, whence $(G-f)-\{v_1, v_2\}$ has no perfect matching. Thus f lies in M. Likewise, g also lies in M. This is absurd since f and g are adjacent. We conclude that one of f and g has rank strictly greater than $\mathsf{rank}(e)$. This completes the proof of Corollary 5.19. □

We now proceed to prove Theorem 5.1.

Proof of Theorem 5.1: As in the statement of the theorem, let e denote an R-compatible edge of an R-brick G. If the edge e is thin, then there is nothing to prove. Now consider the case in which e is not thin. By the Three Case Lemma (4.15), $G-e$ has either one or two maximal nontrivial barriers, and by Proposition 4.16, at least one such barrier has three or more vertices. Our goal is to establish the existence of another R-compatible edge f which satisfies conditions *(i)* and *(ii)* in the statement of Theorem 5.1.

Recall that each candidate edge (relative to e and a barrier of $G-e$ with three or more vertices) is an R-compatible edge of G which satisfies condition *(i)* of Theorem 5.1 and has rank at least $\mathsf{rank}(e)$. (See Definition 5.4 and Proposition 5.5.) Furthermore, if a candidate has rank strictly greater than $\mathsf{rank}(e)$, then by Proposition 5.2, it also satisfies condition *(ii)* of Theorem 5.1, and in this case we are done. Keeping these observations in view, we now use Lemma 5.18 to get rid of the case in which index of e is one.

Claim 5.20 *We may assume that the index of e is two.*

Proof: Suppose not. Then the index of e is one, and we let S denote the unique maximal nontrivial barrier of $G-e$. As discussed earlier, $|S| \geq 3$. Let f denote a member of the candidate set $\mathcal{F}(e, S)$, which is nonempty by Corollary 5.9. If the index of f is zero then its rank is clearly greater than $\mathsf{rank}(e)$, and by Lemma 5.18, this conclusion holds even if the index of f is one. Now consider the case in which f is of index two. Since $\mathsf{rank}(f) \geq \mathsf{rank}(e)$, we conclude that f satisfies condition *(ii)*, Theorem 5.1. Thus, irrespective of its index, the edge f satisfies both conditions *(i)* and *(ii)*, and we are done. □

We shall now invoke Corollary 5.19 to dispose of the case in which the candidate set (relative to some barrier of $G-e$) is not a matching.

Claim 5.21 *We may assume that if S is a nontrivial barrier (not necessarily maximal) of $G - e$ with three or more vertices then the corresponding candidate set $\mathcal{F}(e, S)$ is a matching.*

Proof: Suppose that the candidate set $\mathcal{F}(e, S)$ is not a matching, and thus it contains two adjacent edges, say f and g. We let S^* denote the maximal nontrivial barrier of $G - e$ such that $S \subseteq S^*$. By Corollary 5.6, edges f and g are members of $\mathcal{F}(e, S^*)$ as well. Since e is of index two (by Claim 5.20), Corollary 5.19 implies that at least one of f and g, say f, has rank strictly greater than that of e. Thus f satisfies both conditions *(i)* and *(ii)*, Theorem 5.1, and we are done. □

Now, since e is of index two (by Claim 5.20), the graph $G - e$ has precisely two maximal nontrivial barriers. Among these two, we shall denote by S_1 the barrier which is bigger (breaking ties arbitrarily if they are of equal size), and by I_1 the set of isolated vertices of $(G - e) - S_1$. Thus $|S_1| \geq 3$. Let y and z denote the ends of e. We adopt Notation 5.3. Assume without loss of generality that S_1 is a subset of B, and thus by the Three Case Lemma (4.15), the end y of e lies in I_1.

As the candidate set $\mathcal{F}(e, S_1)$ is a matching (by Claim 5.21), we invoke the observations made in Section 5.1.1, with S_1 playing the role of S, and I_1 playing the role of I, and likewise, $X_1 := S_1 \cup I_1$ playing the role of X. In particular, we adopt Notations 5.10, 5.11 and 5.12 and we apply Proposition 5.13. See Figure 5.9.

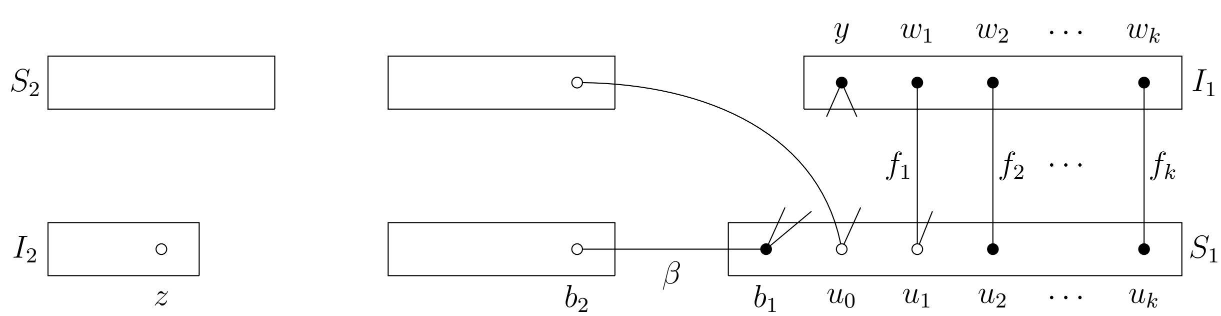


Figure 5.9: Index of e is two, and S_1 is the largest barrier of $G - e$

We let S_2 denote the unique maximal nontrivial barrier of $(G-e)/X_1$, and I_2 the set of isolated vertices of $(G-e)-S_2$. By the Three Case Lemma (4.15), the end z of e lies in I_2, as shown in Figure 5.9. Note that $|S_2| \leq |S_1|$ by the choice of S_1.

Note that, as per statements *(iv)* and *(v)* of Proposition 5.13, the edge $f_1 = u_1w_1$ is the only member of the candidate set $\mathcal{F}(e, S_1)$ whose end in the barrier S_1 (that is, vertex u_1) has some neighbour which lies in $\overline{X_1}$. Also, if $|S_1| = 3$ then f_1 is the unique member of $\mathcal{F}(e, S_1)$. For these reasons, it will play a special role.

Claim 5.22 *We may assume that* $\mathsf{rank}(f_1) = \mathsf{rank}(e)$. *Consequently, the following hold:*

(i) the index of f_1 *is two,*

(ii) all neighbours of u_1 *lie in* $S_2 \cup I_1$, *and*

(iii) the vertex u_0 *has at least one neighbour in the set* $A - (S_2 \cup I_1)$.

Proof: By Proposition 5.5, f_1 is an R-compatible edge which has rank at least that of e, and it satisfies condition *(i)*, Theorem 5.1. If $\mathsf{rank}(f_1) > \mathsf{rank}(e)$, then by Proposition 5.2, f_1 satisfies condition *(ii)* as well, and we are done. We may thus assume that $\mathsf{rank}(f_1) = \mathsf{rank}(e)$. It follows from Lemma 5.18 that the index of f_1 is two; that is, *(i)* holds. Since e and $f_1 = u_1w_1$ are of equal rank and of index two each, the Equal Rank Lemma (5.17)*(v)* implies that each neighbour of u_1 lies in the set $S_2 \cup I_1$, and this proves *(ii)*. We shall now use this fact to deduce *(iii)*.

Since H is bipartite and matching covered, Proposition 1.4*(ii)* implies that the neighbourhood of the set $A-(S_2 \cup I_1)$, in the graph H, has cardinality at least $|A-(S_2 \cup I_1)|+1$, and since $|A-(S_2 \cup I_1)| = |B-(S_1 \cup I_2)|$, we conclude that the set $A-(S_2 \cup I_1)$ has at least one neighbour which is not in $B-(S_1 \cup I_2)$; it follows from Proposition 5.13 and statement *(ii)* proved above that the only such neighbour is the vertex u_0 of barrier S_1. In other words, the vertex u_0 has at least one neighbour in the set $A-(S_2 \cup I_1)$ as shown in Figure 5.9; this proves *(iii)*, and completes the proof of Claim 5.22. □

We shall now consider two cases depending on the cardinality of S_1.

Case 1: $|S_1| \geq 4$.

We invoke Proposition 5.15, with S_1 playing the role of S, and we adjust notation accordingly. See Figure 5.10. Observe that $Q := u_2w_2b_1yu_2$ is a quadrilateral of G which contains the edge $f_2 = u_2w_2$. Since f_2 is a candidate, it is an R-compatible edge whose rank is at least that of e, and it satisfies condition *(i)*, Theorem 5.1. We will use the quadrilateral Q and the Equal Rank Lemma to conclude that f_2 has rank strictly greater than that of e, and thus it satisfies condition *(ii)* as well.

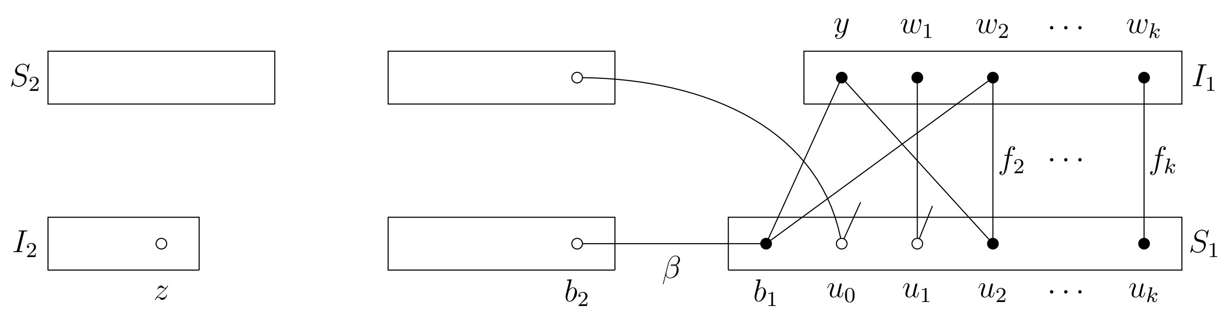


Figure 5.10: When $|S_1| \geq 4$

Now, let v denote the neighbour of w_2 which is distinct from u_2 and b_1. Clearly, $v \in S_1$; by Proposition 5.15*(iii)*, v is distinct from u_0.

Since each end of f_2 is cubic, it is an R-compatible edge of index two. We first set up some notation concerning the barrier structure of $G - f_2$. We denote by S_3 the maximal nontrivial barrier of $G - f_2$ which is a subset of B, and by I_3 the set of isolated vertices of $(G-f_2)-S_3$. We let S_4 denote the unique maximal nontrivial barrier of $(G-f_2)/(S_3 \cup I_3)$, and I_4 the set of isolated vertices of $(G - f_2) - S_4$. By the Three Case Lemma (4.15), the end u_2 of f_2 lies in I_4, and its end w_2 lies in I_3. Also, since $w_2 \in I_3$, $v \in S_3$.

Now, suppose for the sake of contradiction that $\mathsf{rank}(f_2) = \mathsf{rank}(e)$. Then we may apply the Equal Rank Lemma (5.17) to conclude that $S_1 \cup I_2 = S_3 \cup I_4$ and that $S_2 \cup I_1 = S_4 \cup I_3$. Furthermore, by Claim 5.22*(iii)*, the vertex u_0 has a neighbour in $A - (S_4 \cup I_3)$, and thus $u_0 \notin I_4$. We infer that $u_0 \in S_3$. We have thus shown that v and u_0 are distinct vertices of the barrier S_3 of $G - f_2$. Consequently, $(G - f_2) - \{v, u_0\}$ has no perfect matching; we will now use the quadrilateral $Q = u_2w_2b_1yu_2$ to contradict this assertion.

Since G is a brick, $G - \{v, u_0\}$ has a perfect matching, say M. If f_2 is not in M then we have the desired contradiction. Now suppose that $f_2 \in M$. Since v and u_0 both lie in the color class B of H, we conclude that $\alpha \in M$ and that $\beta \notin M$. See Figure 5.10. Note that each of v and u_0 is distinct from b_1, and that the neighbours of b_1 are precisely b_2, w_2 and y. Since $\beta = b_1b_2$ is not in M, and since $f_2 = u_2w_2$ lies in M, it must be the case that yb_1 lies in M. Now observe that the symmetric difference of M and Q is a perfect matching of $(G - f_2) - \{v, u_0\}$, and thus we have the desired contradiction.

We conclude that $\mathsf{rank}(f_2) > \mathsf{rank}(e)$, and thus f_2 is the desired R-compatible edge which satisfies both conditions *(i)* and *(ii)*, Theorem 5.1.

<u>Case 2</u>: $|S_1| = 3$.

We note that since S_1 has precisely three vertices, by Remark 5.14, all of the edges of $G[X_1]$ are determined (where $X_1 = S_1 \cup I_1$). See Figure 5.11. Furthermore, f_1 is the only member of the candidate set $\mathcal{F}(e, S_1)$, and by Claim 5.22, its index is two and its rank is equal to $\mathsf{rank}(e)$. We will examine the barrier structure of $G - f_1$ using the Equal Rank Lemma (5.17), and argue that some edge adjacent with the given edge $e = yz$ (that is, either incident at y, or incident at z) is R-compatible and that its rank is strictly greater than $\mathsf{rank}(e)$. Observe that, since $\mathsf{index}(e) = 2$, each edge adjacent with e satisfies condition *(i)*, Theorem 5.1.

We let S_3 denote the unique maximal nontrivial barrier of $G - f_1$ which is a subset of B, and I_3 the set of isolated vertices of $(G - f_1) - S_3$. We denote by S_4 the unique maximal nontrivial barrier of $(G - f_1)/(S_3 \cup I_3)$, and by I_4 the set of isolated vertices of $(G - f_1) - S_4$. See Figure 5.11. By the Three Case Lemma (4.15), the end u_1 of f_1 lies in I_4, and its end w_1 lies in I_3. Since each of b_1 and u_0 is a neighbour of w_1 in $G - f_1$, they both lie in the barrier S_3. By Lemma 5.17*(ii)*, with f_1 playing the role of f, we conclude that $S_3 = \{b_1, u_0\}$ and that $I_3 = \{w_1\}$, as shown in the figure.

Observe that by the choice of S_1, the barrier S_2 of $G - e$ contains either two or three vertices. However, irrespective of the cardinality of S_2, it follows from the above and from Lemma 5.17*(iv)* that $S_4 = S_2 \cup \{y\}$ and that $I_4 = I_2 \cup \{u_1\}$. In particular, the barrier S_4 of $G - f_1$ contains either three or four vertices. Note that the end z of e lies in I_2 which is a subset of I_4, and its end y lies in S_4. Furthermore, Lemma 5.17*(vi)* implies that e is a member of the candidate set $\mathcal{F}(f_1, S_4)$.

Claim 5.23 *We may assume that e is the only member of $\mathcal{F}(f_1, S_4)$ which is incident with z. Furthermore, we may assume that $|S_2| = 2$.*

<u>Proof</u>: Suppose there exists an edge g incident with z such that g is distinct from e and that $g \in \mathcal{F}(f_1, S_4)$. By Proposition 5.5, g is an R-compatible edge of the brick G. We now

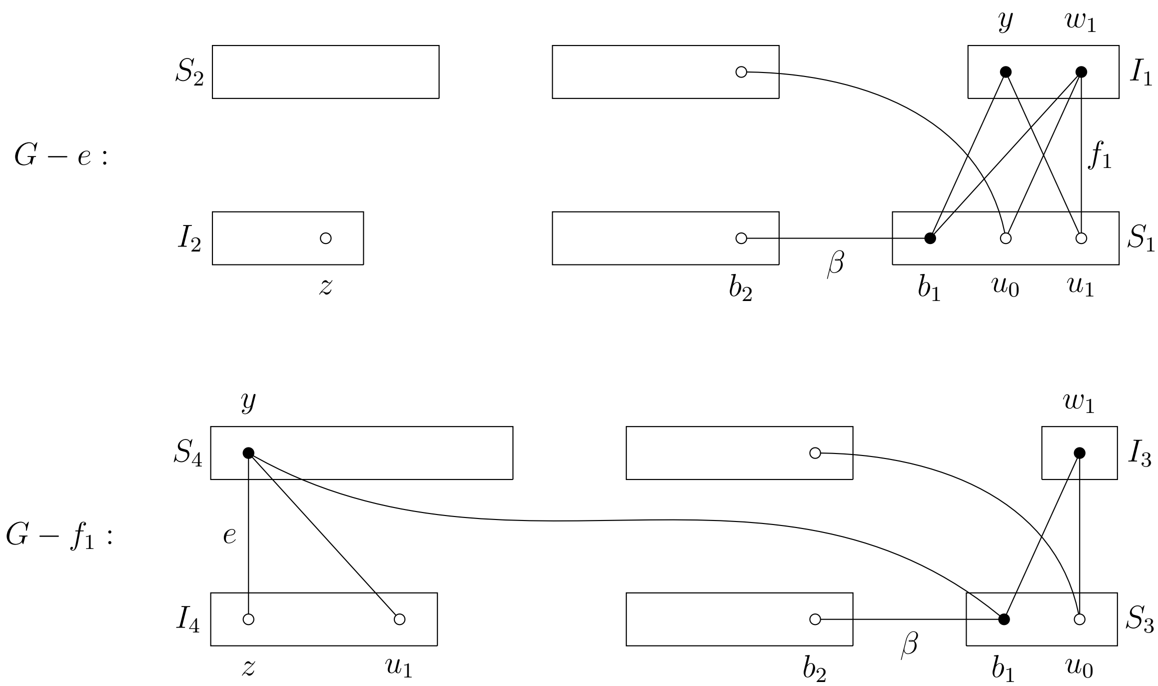


Figure 5.11: When $|S_1| = 3$

apply Corollary 5.19 (with f_1 playing the role of e, and with edges e and g playing the roles of f and g); at least one of e and g has rank strictly greater than $\mathsf{rank}(f_1)$. However, by Claim 5.22, the ranks of e and f_1 are equal; consequently, $\mathsf{rank}(g) > \mathsf{rank}(f_1) = \mathsf{rank}(e)$. By Propostion 5.2, the edge g satisifes condition *(ii)*, Theorem 5.1, and it satisfies condition *(i)* because it is adjacent with the edge e, and thus we are done. So we may assume that e is the only member of $\mathcal{F}(f_1, S_4)$ which is incident with z. Using this, we shall deduce that the barrier S_2 of $G - e$ has only two vertices.

Suppose to the contrary that $|S_2| = 3$. By Claim 5.21, the candidate set $\mathcal{F}(e, S_2)$ is a matching. Consequently, as we did in the case of S_1, we may now invoke the observations made in Section 5.1.1, with S_2 playing the role of S, and I_2 playing the role of I, and likewise, $X_2 := S_2 \cup I_2$ playing the role of X. In particular, by Remark 5.14, all of the edges of $G[X_2]$ are determined. It is worth noting that S_2 is also a maximal barrier of $G - e$ (by the choice of S_1). That is, each of S_1 and S_2 is a maximal barrier of $G - e$ with exactly three vertices. Keeping this symmetry in view, we now choose appropriate notation for those vertices of X_2 which are relevant to our argument. See Figure 5.12.

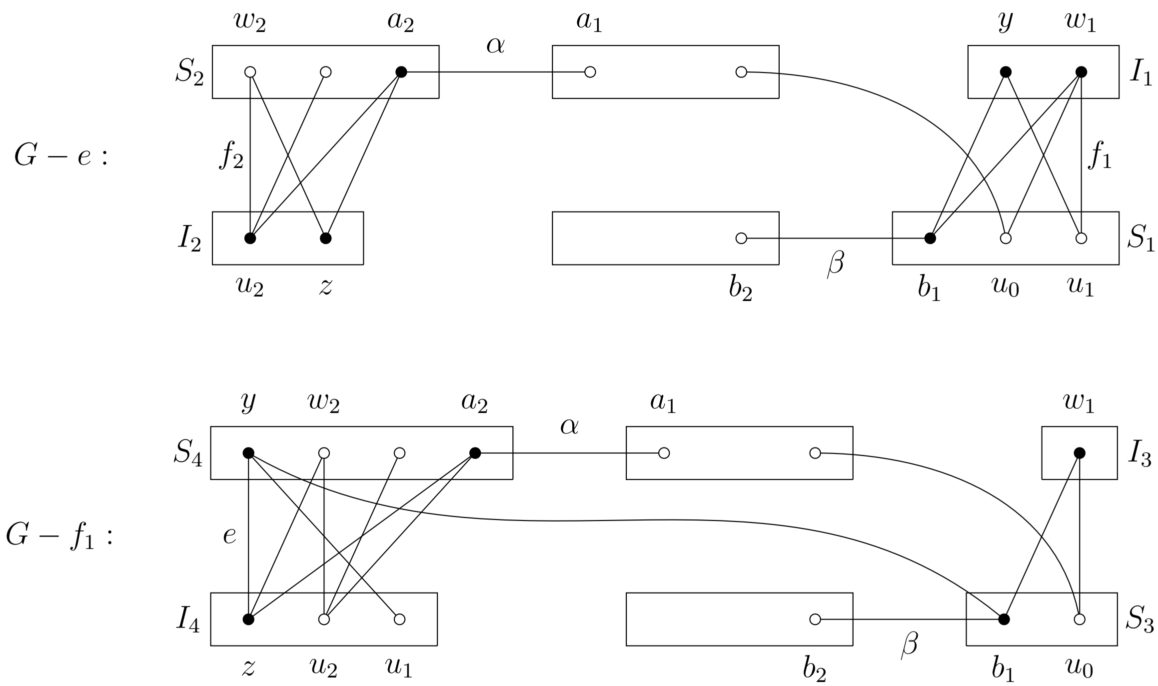


Figure 5.12: When $|S_1| = |S_2| = 3$

We shall let $f_2 := u_2w_2$ denote the unique member of the candidate set $\mathcal{F}(e, S_2)$, where $u_2 \in I_2$ and $w_2 \in S_2$. In particular, $I_2 = \{u_2,z\}$. One of the ends of $\alpha = a_1a_2$ lies in the barrier S_2; we adjust notation so that $a_2 \in S_2$. Consequently, w_2 and a_2 are distinct vertices of S_2. The vertex a_2 is cubic, and its neighbours are z, u_2 and a_1. The vertex w_2 is adjacent with z and u_2, and all of its remaining neighbours lie in $\overline{X_2}$.

Observe that $Q := zw_2u_2a_2z$ is a quadrilateral of the bipartite graph $H(f_1, S_4)$ which contains the vertex z whose degree is three. Consequently, by Corollary 4.8, at least one of zw_2 and za_2 is removable in $H(f_1, S_4)$. However, since a_2 has degree two in $H(f_1, S_4)$, za_2 is non-removable; whence zw_2 is removable. It follows that zw_2 is a member of the candidate set $\mathcal{F}(f_1, S_4)$; this contradicts our first assumption. We conclude that the barrier S_2 has only two vertices, and this completes the proof of Claim 5.23. $\square$

By Proposition 4.16, an R-compatible edge of index two is thin if and only if its rank is $n - 4$; where $n := |V(G)|$. Observe that, since $|S_1| = 3$ and $|S_2| = 2$, the rank of e is $n - 6$, and in this sense, it is very close to being thin; the same holds for the edge f_1. We will establish a symmetry between the barrier structure of $G - e$ and that of $G - f_1$; see Figure 5.13. Thereafter, we will argue that the edge $g := yu_1$ is an R-thin edge of index two; in particular, it is R-compatible and its rank is $n - 4$, and thus it satisfies condition *(ii)*, Theorem 5.1. Since g is adjacent with e, it satisfies condition *(i)* as well.

Since $|S_2| = 2$, the set I_2 contains only the end z of e, and the neighbourhood of z is precisely the set $S_2 \cup \{y\} = S_4$. Also, $I_4 = I_2 \cup \{u_1\} = \{z, u_1\}$, and by Claim 5.23, $e = yz$ is the only member of the candidate set $\mathcal{F}(f_1, S_4)$ which is incident with z. In other words, z is incident with only one removable edge of the bipartite graph $H(f_1, S_4)$, namely, the edge e. We now deduce some consequences of this fact using standard arguments.

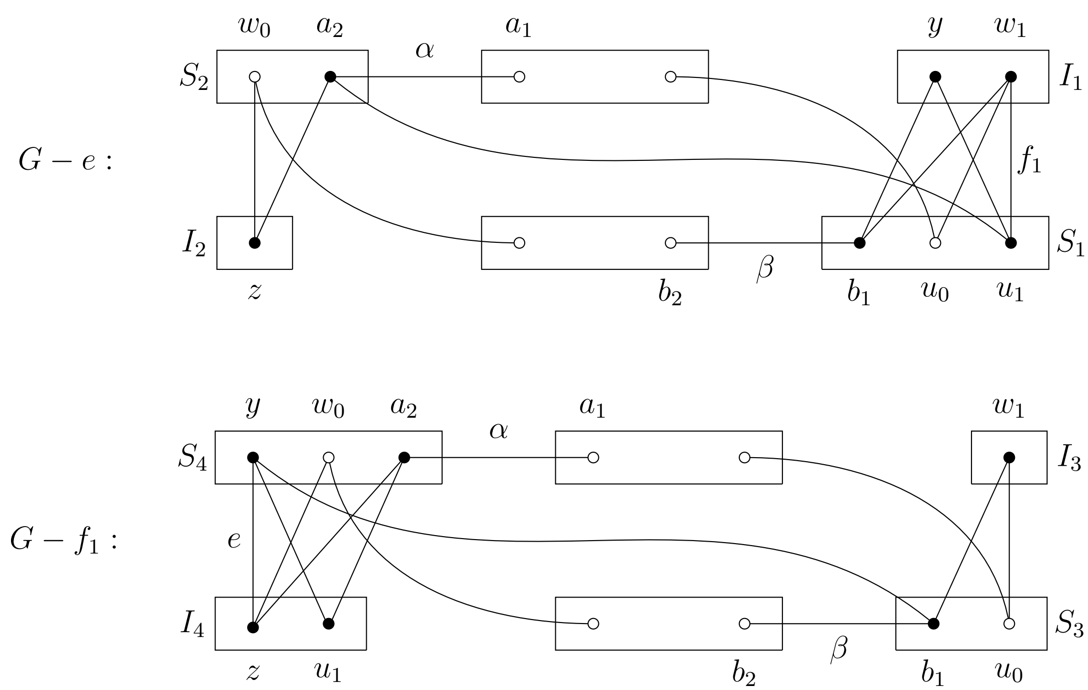


Figure 5.13: When $|S_1| = 3$ and $|S_2| = 2$

First of all, by Lemma 5.8*(i)*, an end of the edge $\alpha = a_1a_2$ lies in the barrier S_4. Adjust notation so that $a_2 \in S_4$. By statement *(ii)* of the same lemma, a_2 has no neighbours in the set $\overline{X_4}$ where $X_4 := S_4 \cup I_4$. Consequently, the neighbourhood of a_2 is precisely $I_4 \cup \{a_1\} = \{z, u_1, a_1\}$. Clearly, y and a_2 are distinct vertices of S_4, and we denote by w_0 the remaining vertex of S_4. Note that $S_2 = \{w_0, a_2\}$.

Next, we observe that if the vertices u_1 and w_0 are adjacent then $Q := zw_0u_1a_2z$ is a quadrilateral of the bipartite graph $H(f_1, S_4)$ and it contains the vertex z which has degree three; by Corollary 4.8, one of the two edges zw_0 and za_2 is removable; however, this contradicts the fact that $e = yz$ is the only removable edge incident with z. Thus, the vertices u_1 and w_0 are nonadjacent. It follows that u_1 is cubic, and its neighbourhood is precisely $\{y, a_2, w_1\}$.

Observe that we have six cubic vertices whose neighbourhoods are fully determined; these are: the ends y and z of e, the ends u_1 and w_1 of f_1, the end b_1 of β, and the end a_2 of α. There is a symmetry between the barrier structure of $G - e$ and that of $G - f_1$; as is self-evident from Figure 5.13. We have not determined the degrees of the two vertices u_0 and w_0; observe that if these vertices are not adjacent with each other then u_0 has at least two neighbours in $A - (S_2 \cup I_1)$ and likewise, w_0 has at least two neighbours in $B - (S_1 \cup I_2)$; whereas if u_0w_0 is an edge of G then u_0 has at least one neighbour in $A - (S_2 \cup I_1)$ and likewise, w_0 has at least one neighbour in $B - (S_1 \cup I_2)$.

As mentioned earlier, we now proceed to prove that $g = yu_1$ is an R-thin edge. We let $J := ((G - e)/X_1 \to x_1)/X_2 \to x_2$ denote the unique brick of $G - e$, where $X_1 = S_1 \cup I_1$ and $X_2 := S_2 \cup I_2$. Note that J is near-bipartite with removable doubleton R.

Claim 5.24 *The edge $g = yu_1$ is R-thin. (That is, g is an R-compatible edge of index two and its rank is $n - 4$.)*

Proof: Observe that $Q := yu_1w_1b_1y$ is a quadrilateral in $H = G - R$ which contains the cubic vertex y. By Corollary 4.8, at least one of the edges $g = yu_1$ and yb_1 is removable in H. Note that yb_1 is not removable, whence g is removable in H. To conclude that g is R-compatible, it suffices to show that edges α and β are admissible in $G - g$. We shall prove something more general, which is useful in establishing the thinness of g as well.

Observe that, in $G - g$, the vertex y has neighbour set $\{z, b_1\}$, and vertex u_1 has neighbour set $\{w_1, a_2\}$. We will show that, if v_1 and v_2 are distinct vertices of the color class B such that $\{v_1, v_2\} \neq \{z, b_1\}$, then $(G - g) - \{v_1, v_2\}$ has a perfect matching, say M. This has two consequences worth noting. First of all, if $\{v_1, v_2\} = \{b_1, b_2\}$ then $M + \beta$ is a perfect matching of $G - g$ which contains α and β both, whence g is an R-compatible edge of G. Secondly, it shows that $\{z, b_1\}$ is a maximal nontrivial barrier of $G - g$. An analogous argument establishes that $\{w_1, a_2\}$ is also a maximal nontrivial barrier of $G - g$, and consequently Proposition 4.16 implies that g is indeed R-thin.

As mentioned above, suppose that v_1 and v_2 are distinct vertices of B such that $\{v_1, v_2\} \neq \{z, b_1\}$. Let N be a perfect matching of $G - \{v_1, v_2\}$. In what follows, we consider different possibilities, and in each of them, we exhibit a perfect matching M of $(G - g) - \{v_1, v_2\}$. If $g \notin N$ then clearly $M := N$. Now suppose that $g \in N$. Note that, since $v_1, v_2 \in B$, the edge α lies in N and β does not lie in N. If $b_1 \notin \{v_1, v_2\}$, then the edge b_1w_1 lies in N, and we let $M := (N - g - b_1w_1) + f_1 + yb_1$.

Now consider the case in which $b_1 \in \{v_1, v_2\}$, and adjust notation so that $b_1 = v_1$. Thus $v_2 \neq z$, whence $zw_0 \in N$. Also, w_1u_0 lies in N. Observe that v_2 lies in the set $B - (S_1 \cup I_2)$. First, we consider the case when u_0w_0 is an edge of G. Observe that the six cycle $C := u_1yzw_0u_0w_1u_1$ is N-alternating and it contains the edge g. In this case, let M denote the symmetric difference of N and C.

Finally, consider the situation in which u_0w_0 is not an edge of G. (In this case, to construct M, we will not use the matching N.) As noted earlier, since u_0 and w_0 are nonadjacent, w_0 has at least two distinct neighbours in the set $B - (S_1 \cup I_2)$. In particular, w_0 has at least one neighbour, say v', which lies in $B - (S_1 \cup I_2)$ and is distinct from v_2. Now, let M_J be a perfect matching of $J - \{v', v_2\}$. Observe that $\alpha \in M_J$ and $\beta \notin M_J$. Note that, in the matching M_J, the contraction vertex x_1 is matched with some vertex in $A - (S_2 \cup I_1)$, which is a neighbour of u_0 in the graph G. Now, we let $M := M_J + w_0v' + f_1 + e$.

In every scenario, M is a perfect matching of $(G - g) - \{v_1, v_2\}$, as desired. Thus, as discussed earlier, g is R-compatible as well as thin. This proves Claim 5.24. □

In summary, we have shown that $g = yu_1$ is an R-compatible edge which satisfies both conditions *(i)* and *(ii)*, Theorem 5.1. This completes the proof. □

Chapter 6

Generating simple near-bipartite bricks

Here, we will use Theorem 5.1 from the last chapter to establish the generation procedure for simple near-bipartite bricks discussed in Section 1.7.3. Recall that, for a simple R-brick G, a strictly R-thin edge e is one which is R-compatible as well as strictly thin, and in this case, the retract of $G - e$ is also a simple R-brick. We will prove Theorem 1.24, which is restated below.

Theorem 1.24 [Strictly R-thin Edge Theorem] *Let G be a simple R-brick. If G is free of strictly R-thin edges then G belongs to one of the following infinite families:*

(i) Truncated biwheels

(ii) Prisms

(iii) Möbius ladders

(iv) Staircases

(v) Pseudo-biwheels

(vi) Double biwheels of type I

(vii) Double ladders of type I

(viii) Laddered biwheels of type I

(ix) Double biwheels of type II

(x) Double ladders of type II

(xi) Laddered biwheels of type II

There are eleven infinite families in the statement of the above theorem. The first four of these (truncated biwheels, prisms, Möbius ladders and staircases) are Norine-Thomas families, that is, they are free of strictly thin edges; these are described in Section 1.3.3. All of the remaining seven families contain strictly thin edges, and these are described in Section 1.7.3. We denote by $\mathcal{N}$ the union of all of these eleven families.

Recall the definitions of ladders and partial biwheels from Section 1.3.3. In our descriptions of the aforementioned eleven families, we constructed their members using either one or two disjoint bipartite matching covered graphs, each of which is either a ladder or a partial biwheel, and thereafter, adding a few vertices and/or edges and possibly identifying two pairs of vertices. As we will see, these constructions are indicative of how these graphs appear in our proof of Theorem 1.24. In the next section, we will define two special types of subgraphs, namely, an 'R-biwheel configuration' and an 'R-ladder configuration'; we will conclude the section with a proof sketch of Theorem 1.24.

Throughout this chapter, we adopt the following notational and figure conventions.

Notation 6.1 *For a simple R-brick G, we shall denote by $H[A, B]$ the underlying bipartite graph $G-R$. We let α and β denote the constituent edges of R, and we adopt the convention that $\alpha := a_1a_2$ has both ends in A, whereas $\beta := b_1b_2$ has both ends in B. We denote by $V(R)$ the set $\{a_1, a_2, b_1, b_2\}$. Furthermore, in all of the figures, the hollow vertices are in A, and the solid vertices are in B.*

6.1 R-configurations

We will also adopt the following notational conventions for a subgraph which is either a ladder or a partial biwheel.

Notation 6.2 *When referring to a subgraph K of H, such that K is either a ladder or a partial biwheel with external rungs/spokes au and bw, we adopt the convention that $a, w \in A$ and $b, u \in B$; furthermore, when K is a partial biwheel, u and w shall denote its hubs; as shown in Figures 6.1 and 6.3. (We may also use subscript notation, such as a_iu_i and b_iw_i where i is an integer, and this convention extends naturally.)*

6.1.1 R-biwheel configurations

Let K be a subgraph of H such that K is a partial biwheel with external spokes au and bw; see Figure 6.1. We say that K is an *R-biwheel configuration* of G if it satisfies the following conditions:

(i) in G, the hubs u and w are both noncubic, and every other vertex of K is cubic,

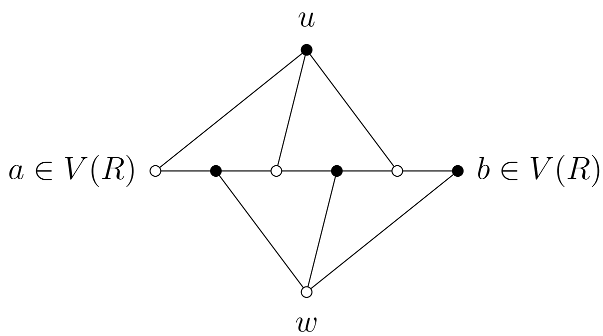


Figure 6.1: An R-biwheel configuration; in G, the free corners (hubs) u and w are noncubic, and every other vertex is cubic.

(ii) the ends of K, namely a and b, both lie in $V(R)$, and,

(iii) in G, every internal spoke of K is an R-thin edge whose index is one.

A pseudo-biwheel, as shown in Figure 6.2, has two removable doubletons $R := \{\alpha, \beta\}$ and $R' := \{\alpha', \beta'\}$. The subgraph K, depicted by solid lines, is an R-biwheel configuration. (To see this, note that every internal spoke of K is an R-thin edge of index one.) However, K is not an R'-biwheel configuration because its ends a and b are not incident with edges of R'.

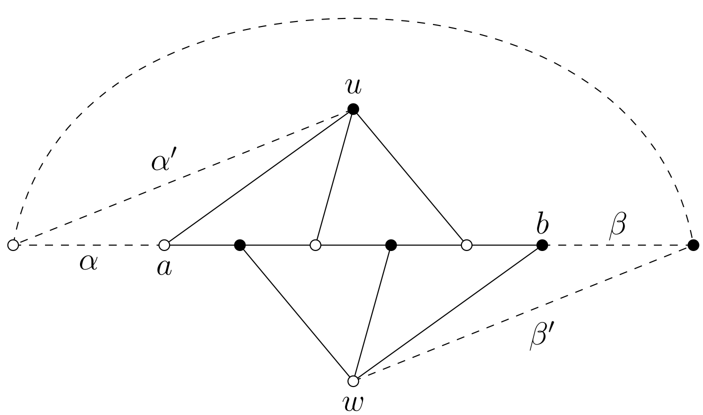


Figure 6.2: A pseudo-biwheel has only one R-biwheel configuration

6.1.2 R-ladder configurations

Let K be a subgraph of H such that K is a ladder with external rungs au and bw; see Figure 6.3. We say that K is an *R-ladder configuration* of G if it satisfies the following conditions:

(i) in G, every vertex of K, except possibly for u and w, is cubic,

(ii) the vertices a and b both lie in $V(R)$, and,

(iii) in G, every internal rung of K is an R-thin edge whose index is two.

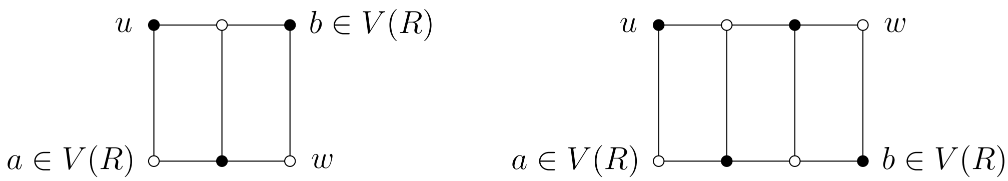


Figure 6.3: Two R-ladder configurations of different parities; each vertex, except possibly for the free corners u and w, is cubic in G

A prism of order n has $\frac{n}{2}$ removable doubletons. If $R := \{\alpha, \beta\}$ is a fixed removable doubleton of a prism G of order ten or more, then the graph $H = G - R$ is itself an R-ladder configuration, as shown in Figure 6.4. (An analogous statement holds for Möbius ladders of order eight or more.)

6.1.3 Corners, rungs and spokes

We shall often need the flexibility of referring to a subgraph K which is either an R-ladder configuration or an R-biwheel configuration, and in this case, we simply write that K is an *R-configuration*. Additionally, we may also state that K has external rungs/spokes au and bw (possibly with subscript notation); in this case, we implicitly adopt the conventions stated in Notation 6.2, and we refer to a, u, b and w as the *corners* of K. Furthermore, as shown in Figures 6.1 and 6.3, we will assume that $a, b \in V(R)$. We refer to u and w as the *free corners* of K; these may lie in $V(R)$ as in Figure 6.4, or they may not lie in $V(R)$ as in Figure 6.2. Observe that any vertex of K, which is not a corner, does not lie in $V(R)$.

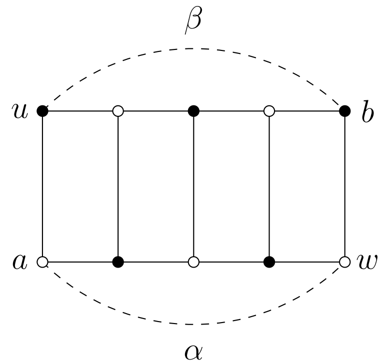


Figure 6.4: A prism has only one R-ladder configuration

For any two distinct rungs/spokes of an R-configuration K, say e and f, we say that e and f are *consecutive*, or equivalently, that e is *consecutive with* f, whenever an end of e which is not a free corner is adjacent with an end of f which is also not a free corner. Clearly, each internal rung (spoke) is consecutive with two rungs (spokes); whereas each external rung (spoke) is consecutive with only one rung (spoke) and the latter is internal. Now, let e denote an internal rung (spoke) of K, and let f and g denote the two rungs (spokes) with which e is consecutive. By definition, e is an R-thin edge of G. Observe that f and g are multiple edges in the retract of $G - e$; consequently, e is not strictly thin.

6.1.4 Two distinct R-configurations

A laddered biwheel of type II, as shown in Figure 6.5, has two removable doubletons $R := \{\alpha, \beta\}$ and $R' := \{\alpha', \beta'\}$. Observe that the graph obtained by removing the edge set $R \cup R'$ has two connected components, of which one is an R-ladder configuration with external rungs a_1u_1 and b_1w_1, and the other is an R-biwheel configuration with external spokes a_2u_2 and b_2w_2. In this case, the two R-configurations are vertex-disjoint.

On the other hand, a double ladder of type I, as shown in Figure 6.6, has only one removable doubleton $R := \{\alpha, \beta\}$ and it has two R-ladder configurations which share their free corners u_1 and w_1, but are otherwise vertex-disjoint. One of these is depicted by dashed lines, and it has external rungs a_1u_1 and b_1w_1, whereas the other one has external rungs a_2u_1 and b_2w_1.

The reader is advised to check that members of all of the eleven families that appear in Theorem 1.24, except for K_4 and $\overline{C_6}$, have either one or two R-configurations for an

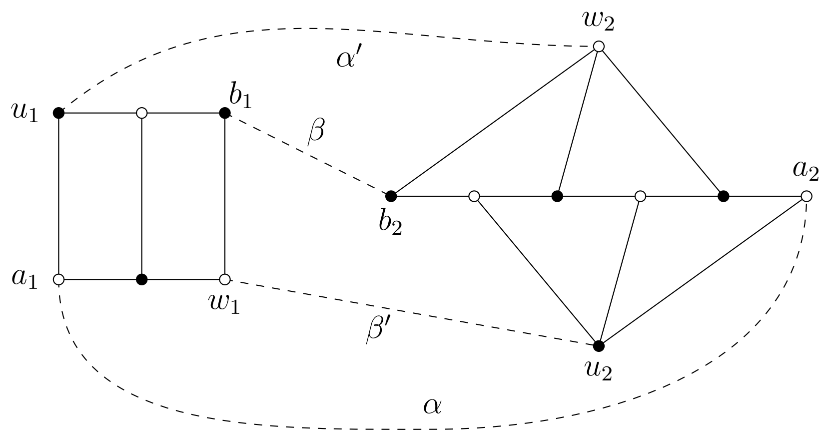


Figure 6.5: A laddered biwheel of type II has two vertex-disjoint R-configurations

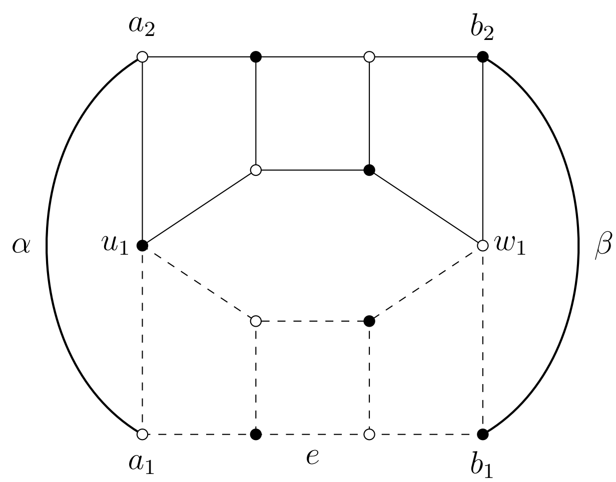


Figure 6.6: A double ladder of type I has two R-configurations which share their free corners but are otherwise vertex-disjoint

appropriately chosen removable doubleton R. (The choice of R matters only in the case of three families, namely, pseudo-biwheels, double biwheels of Type II and laddered biwheels of Type II. Figure 6.2 shows a pseudo-biwheel and its two removable doubletons.)

In order to sketch a proof of Theorem 1.24, we will require a few results which are stated next; their proofs will appear in later sections. In particular, the following proposition states that two distinct R-configurations are either vertex-disjoint, or they have the same free corners but are otherwise vertex-disjoint; its proof appears in Section 6.3.1.

Proposition 6.3 [R-CONFIGURATIONS ARE ALMOST DISJOINT] *Let G be a simple R-brick, and let K_1 denote an R-configuration with free corners u_1 and w_1. If K_2 is any R-configuration distinct from K_1, then precisely one of the following statements holds:*

(i) K_1 and K_2 are vertex-disjoint, or,

(ii) u_1 and w_1 are the free corners of K_2, and K_2 is otherwise vertex-disjoint with K_1.

By the above proposition, the only vertices that can be possibly shared between two distinct R-configurations are their respective free corners. The remaining two corners of each R-configuration lie in $V(R)$. Since $|V(R)| = 4$, we immediately have the following consequence.

Corollary 6.4 *A simple R-brick has at most two distinct R-configurations.* $\square$

For instance, if G is a Norine-Thomas brick or if it is a pseudo-biwheel then it has only one R-configuration. On the other hand, if G is a double biwheel or a double ladder or a laddered biwheel, then it has two R-configurations, say K_1 and K_2. Furthermore, if G is of type II then K_1 and K_2 are vertex-disjoint as in Proposition 6.3*(i)*; whereas, if G is of type I then K_1 and K_2 have the same free corners but they do not have any other vertices in common as in Proposition 6.3*(ii)*.

6.1.5 The R-biwheel and R-ladder Theorems

It is easily verified that if G is any R-brick in $\mathcal{N}$, then every R-thin edge of G lies in an R-configuration. Here, we state two theorems which show that this is not a coincidence.

Now, let G be a simple R-brick which is free of strictly R-thin edges. Given any R-thin edge e of G, we may invoke one of these theorems (depending on the index of e) to find an R-configuration K containing the edge e. In particular, if the index of e is one, we apply Theorem 6.5 and in this case K is an R-biwheel configuration; whereas, if the index of e is two, we apply Theorem 6.6 and in this case K is an R-ladder configuration.

Theorem 6.5 [R-BIWHEEL THEOREM] *Let G be a simple R-brick which is free of strictly R-thin edges, and let e denote an R-thin edge whose index is one. Then G contains an R-biwheel configuration, say K, such that e is an internal spoke of K.*

The proof of the above theorem appears in Section 6.2.2, and it is along the same lines as the proof of [CLM08, Theorem 4.6].

Given the statement of Theorem 6.5, one would expect that, likewise, if e is an R-thin edge whose index is two then G contains an R-ladder configuration, say K, such that e is an internal rung of K. Unfortunately, this is not true, in general. Consider the double ladder of type I, shown in Figure 6.6; e is an R-thin edge of index two, and although it is part of an R-ladder configuration, it is not a rung of that ladder. We instead prove the following slightly weaker statement concerning R-thin edges of index two.

Theorem 6.6 [R-LADDER THEOREM] *Let G be a simple R-brick which is free of strictly R-thin edges, and let e denote an R-thin edge whose index is two. Then G contains an R-ladder configuration, say K, such that $e \in E(K)$.*

The proof of the above theorem appears in Section 6.2.3 and it is significantly longer than that of the R-biwheel Theorem (6.5). These two theorems (6.5 and 6.6) are central to our proof of the Strictly R-thin Edge Theorem (1.24).

6.1.6 Proof Sketch of Theorem 1.24

As in the statement of the theorem, let G be a simple R-brick which is free of strictly R-thin edges. Our goal is to show that G is a member of one of the eleven infinite families which appear in the statement of the theorem, that is, to show that $G \in \mathcal{N}$. We adopt Notation 6.1.

We may assume that G is different from K_4 and $\overline{C_6}$, and thus, by the R-thin Edge Theorem (1.22), G has an R-thin edge, say e_1. Depending on the index of e_1, we invoke either the R-biwheel Theorem (6.5) or the R-ladder Theorem (6.6) to deduce that G has an R-configuration, say K_1, such that $e_1 \in E(K_1)$. We shall let a_1u_1 and b_1w_1 denote the external rungs/spokes of K_1, and adjust notation so that u_1 and w_1 are its free corners.

We will show that either u_1 and w_1 both lie in $V(R)$, or otherwise neither of them lies in $V(R)$. In the former case, we will conclude that G is either a prism or a Möbius ladder or a truncated biwheel, and we are done.

Now suppose that $u_1, w_1 \notin V(R)$. In this case, we will show that either G is a staircase or a pseudo-biwheel, and we are done; or otherwise, G has an R-compatible edge which is not in $E(K_1)$. In the latter case, we will apply Theorem 5.1 to deduce that G has an R-thin edge, say e_2, which is not in $E(K_1)$. Depending on the index of e_2, we may once again use either the R-biwheel Theorem (6.5) or the R-ladder Theorem (6.6) to conclude that G has an R-configuration, say K_2, such that $e_2 \in E(K_2)$.

By Proposition 6.3, either K_1 and K_2 are vertex-disjoint, or otherwise K_2 has the same free corners as K_1 but is otherwise vertex-disjoint with K_1. In the latter case, we will conclude that G is either a double biwheel or a double ladder or a laddered biwheel, each of type I, and we are done.

Now suppose that K_1 and K_2 are vertex-disjoint. We will argue that either G is a double biwheel or a double ladder or a laddered biwheel, each of type II, and we are done; or otherwise, G has an R-compatible edge which is not in $E(K_1 \cup K_2)$. In the latter case, we will once again apply Theorem 5.1 to conclude that G has an R-thin edge, say e_3, which is not in $E(K_1 \cup K_2)$. As usual, depending on the index of e_3, we invoke either the R-biwheel Theorem (6.5) or the R-ladder Theorem (6.6) to deduce that G has an R-configuration, say K_3, such that $e_3 \in E(K_3)$.

We have thus located three distinct R-configurations in the brick G, namely, K_1, K_2 and K_3. However, this contradicts Corollary 6.4, and completes the proof sketch of the Strictly R-thin Edge Theorem (1.24).

6.2 R-thin edges

Here, we will prove the R-biwheel Theorem (6.5) and the R-ladder Theorem (6.6). Our proofs are inspired by the work of Carvalho et al. [CLM08]. In the next section, we will review conditions under which an R-thin edge is not strictly thin, and we will state a few key lemmas (6.10, 6.11 and 6.12) from [CLM08] which are used in our proofs.

6.2.1 Multiple edges in retracts

Throughout this section, G is a simple R-brick, and we adopt Notation 6.1. Furthermore, we shall let e denote an R-thin edge which is not strictly thin, and J the retract of $G - e$. Since e is not strictly thin, J is not simple, and we shall let f and g denote two multiple (parallel) edges of J. It should be noted that since J is also an R-brick, neither edge of R is a multiple edge of J. In particular, f and g do not lie in R.

We denote the ends of e by letters y and z with subscripts 1; that is, $e := y_1z_1$. Adjust notation so that $y_1 \in A$ and $z_1 \in B$. If either end of e is cubic, then we denote its two neighbours in $G - e$ by subscripts 0 and 2. For example, if y_1 is cubic then $N(y_1) = \{z_1, y_0, y_2\}$.

As G is simple, it follows that J has a contraction vertex which is incident with both f and g. We infer that one end of e, say y_1, is cubic, and that f is incident with y_0, and g is incident with y_2. See Figure 6.7. As noted earlier, $f \notin R$; consequently, e and f are nonadjacent. Likewise, e and g are nonadjacent.

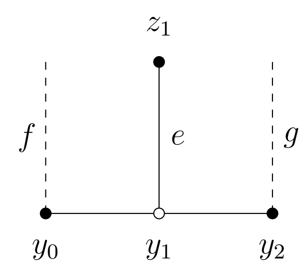


Figure 6.7: f and g are multiple edges in the retract J of $G - e$; the vertex y_1 is cubic.

We will consider two separate cases depending on whether the edges f and g are adjacent (in G) or not. In the case in which they are adjacent, we shall denote their common end by w, as shown in Figure 6.8a. Now suppose that f and g are nonadjacent. Since they are multiple (parallel) edges of J, we infer that both ends of e are cubic, and that f and g join the two contraction vertices of J. This proves the following proposition; see Figure 6.8b.

Proposition 6.7 *Suppose that f and g are nonadjacent in G. Then the following hold:*

(i) each end of e is cubic,

(ii) consequently, the index of e is two, and

(iii) one of f and g is incident with z_0 whereas the other one is incident with z_2. □

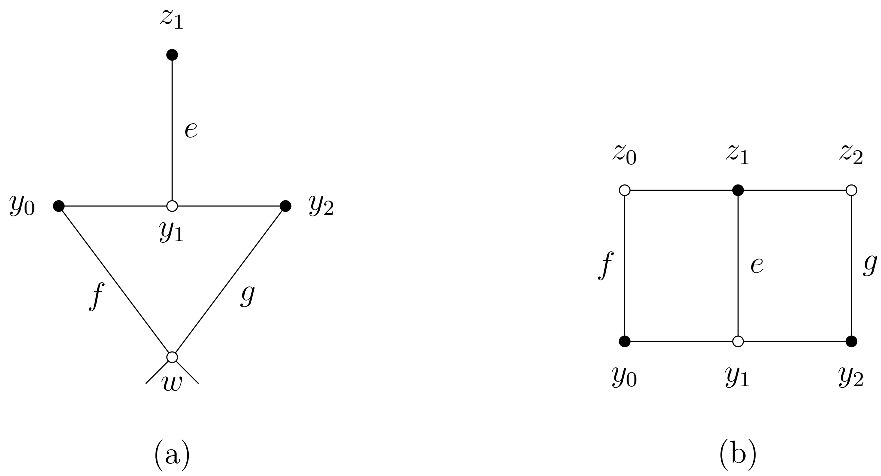


Figure 6.8: (a) when f and g are adjacent; (b) when f and g are nonadjacent

In view of statement *(iii)*, whenever f and g are nonadjacent, we shall assume without loss of generality that $f := y_0z_0$ and $g := y_2z_2$, as shown in Figure 6.8b.

Let us now focus on the case in which f and g are adjacent, as shown in Figure 6.8a. We remark that, in this case, the index of e is not determined; that is, its index could be either one or two depending on the degree of its end z_1. Instead, we are able to say something about the degree of w.

Proposition 6.8 *Suppose that f and g are adjacent in G, and let w be their common end. Then w has degree four or more.*

Proof: First suppose that w is not a neighbour of z_1. In this case, w is not affected by the bicontractions in $G - e$. Consequently, w is a vertex of the brick J, whence it has at least three distinct neighbours. Since f and g are multiple edges, w has degree four or more.

Now suppose that w is a neighbour of z_1. Observe that the neighbours of y_1 are precisely y_0, y_2 and z_1; each of which is adjacent with w. See Figure 6.8a. Note that, if w is cubic, then its neighbourhood is the same as that of y_1; and in this case, $\{y_0, y_2, z_1\}$ is a barrier of the brick G; this is absurd. Thus w has degree four or more. □

Note that f and g, being multiple edges of J, are both R-thin in J. We shall now examine conditions under which one of them, say f, fails to be R-thin in G. This may be the case for three different reasons; firstly, f is non-removable in the bipartite graph $H = G-R$; secondly, f is non-removable in G; and thirdly, f is removable in G but it is not thin.

We begin with the situation in which f is non-removable in H. Note that, if an end of f is cubic (in G) and if it also lies in $V(R)$, then it has degree two in H, rendering f non-removable. We will now argue that the converse also holds.

Lemma 6.9 *The edge f is non-removable in H if and only if it has a cubic end which lies in $V(R)$.*

Proof: Suppose that f has no cubic end which lies in $V(R)$. Consequently, each end of f has degree two or more in $H - f$. Furthermore, since e and f are nonadjacent, each end of f has degree two or more in $H - e - f$ as well. We will argue $H - e - f$ is matching covered, that is, f is removable in $H - e$. The exchange property (Proposition 4.2) then implies that f is also removable in H.

Note that f is a multiple edge of $J - R$, whence $J - R - f$ is matching covered. Recall that any graph obtained from a matching covered graph by means of bi-splitting a vertex is also matching covered. (See Section 1.5.2.) We will argue that $H - e - f$ may be obtained from $J - R - f$ by means of bi-splitting one or two vertices.

Note that J is obtained from $G - e$ by means of bicontracting one or two vertices (of degree two); likewise, $J - R$ may be obtained from $H - e$ by means of bicontractions. Conversely, $H - e$ may be obtained from $J - R$ by means of bi-splitting one or two vertices; these are the contraction vertices of J. As noted earlier, since each end of f has degree two or more in $H - e - f$, we may similarly obtain $H - e - f$ from $J - R - f$ by means of bi-splitting the same vertices. As discussed above, $H - e - f$ is matching covered; consequently, f is removable in H. $\square$

We now turn to the situation in which f is non-removable in G. For convenience, we will state two lemmas (6.10 and 6.11), depending on the index of e. These appear in the work of Carvalho et al. [CLM08, Lemma 4.2] as a single lemma. (In their work, they deal with the more general context in which e is a thin edge of a brick G, which need not be near-bipartite.)

The first lemma (6.10) considers the scenario in which the index of e is one. By Proposition 6.7*(ii)*, f and g are adjacent; and by Proposition 6.8, their common end w is non-cubic.

Lemma 6.10 [CLM08] *Suppose that the index of e is one. If f is non-removable in G then f has a cubic end which is adjacent with both ends of e. (In particular, the cubic end of f lies in $V(R)$.)* $\square$

As w is non-cubic, y_0 is the cubic end of f, and it is adjacent with z_1, as shown in Figure 6.9a. Clearly, the edge joining y_0 and z_1 is none other than β.

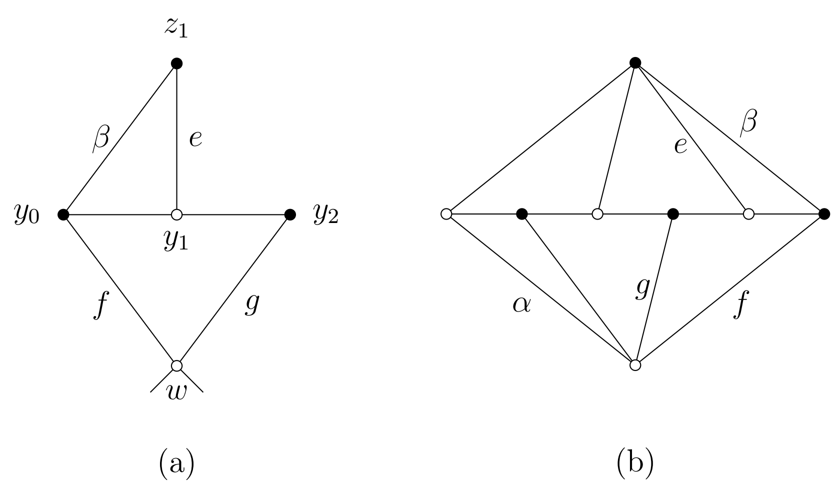


Figure 6.9: Illustration for Lemma 6.10

The situation in Lemma 6.10 arises in truncated biwheels, as shown in Figure 6.9. Note that, every perfect matching which contains e also contains f, rendering f non-removable.

The second lemma (6.11) deals with the scenario in which the index of e is two, that is, each end of e is cubic.

Lemma 6.11 [CLM08] *Suppose that the index of e is two. If f is non-removable in G then the following hold:*

(i) each end of f is cubic,

(ii) consequently, f and g are nonadjacent, and

(iii) the ends of f have a common neighbour.

(In particular, one of the ends of f is cubic and it also lies in $V(R)$.) $\square$

By statement *(i)*, each end of f is cubic; thus f and g are nonadjacent (see Proposition 6.8). By Proposition 6.7, and as per our notation, $f = y_0z_0$ and $g = y_2z_2$, as shown in Figure 6.10a. By statement *(iii)*, y_0 and z_0 have a common neighbour, say x. Clearly, one of xy_0 and xz_0 is an edge of R, depending on whether x lies in A or in B; however, these cases are symmetric. Adjust notation so that $x \in B$; thus xy_0 is the edge β. Using the fact that G is free of nontrivial barriers, it is easily verified that x is not an end of g.

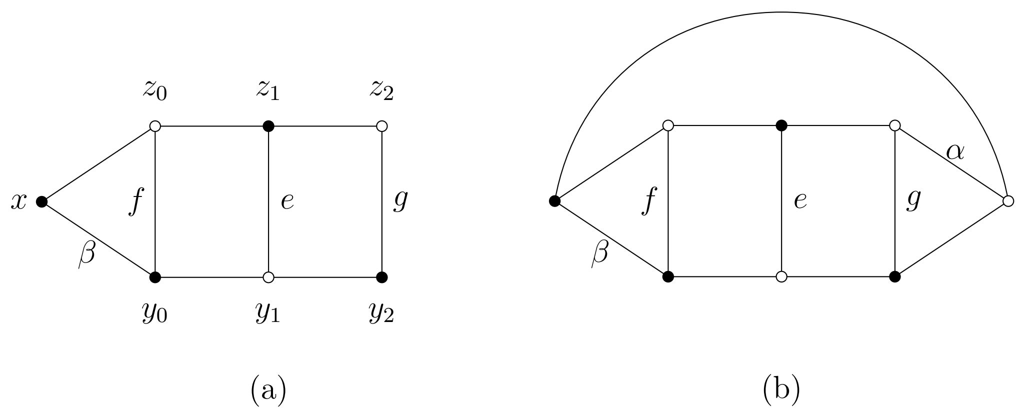


Figure 6.10: Illustration for Lemma 6.11

The situation in Lemma 6.11 is observed in staircases, as shown in Figure 6.10b. The edge f is non-removable since every perfect matching which contains e also contains f.

Finally, we turn to the case in which f is removable in G but it is not thin. This is handled by Lemma 6.12 which appears in the work of Carvalho et al. [CLM08, Lemma 4.3].

Lemma 6.12 [CLM08] *If f is removable in G but it is not thin then the following hold:*

(i) *the index of e is two,*

(ii) *f and g are adjacent and their common end w is not adjacent with any end of e,*

(iii) *g is a thin edge, and*

(iv) *$N(y_0) \subseteq N(z_1) \cup \{w\}$; recall that y_0 is the other end of f, and z_1 is the end of e not adjacent with y_0.* □

The lemma concludes that the index of e is two; that is, its end z_1 is cubic, and as per our notation, the neighbours of z_1 are precisely y_1, z_0 and z_2. Furthermore, it concludes that f and g are adjacent and that their common end w is distinct from each of z_0 and z_2, as shown in Figure 6.11a. Another consequence which may be inferred from their proof is that all of the neighbours of y_0 lie in the set $N(z_1) \cup \{w\} = \{w, y_1, z_0, z_2\}$. (This is not stated explicitly in the statement of [CLM08, Lemma 4.3].) Since y_0 has degree at least three, we may adjust notation so that y_0 is adjacent with z_0, and it may or may not be adjacent with z_2.

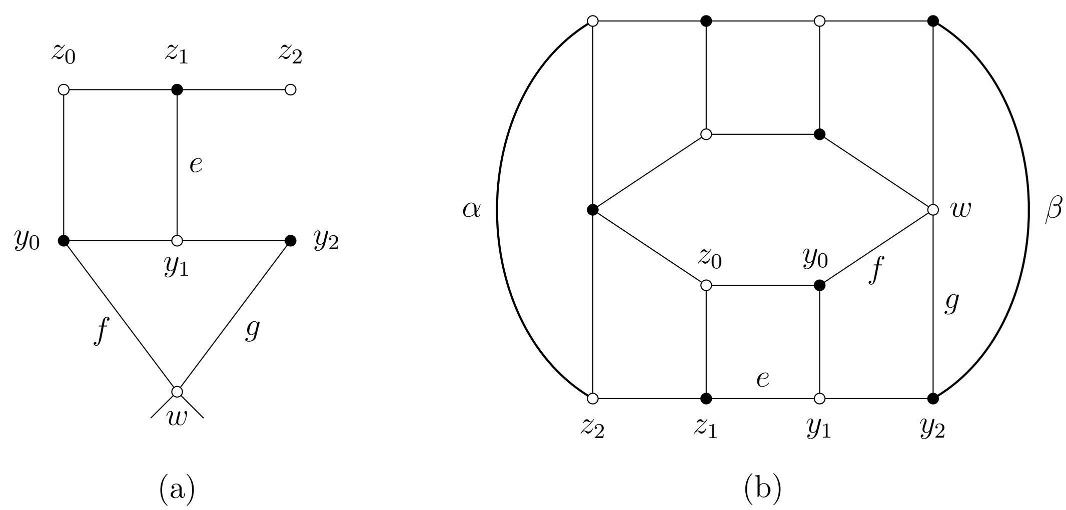


Figure 6.11: Illustration for Lemma 6.12

The situation in Lemma 6.12 is best illustrated by a double ladder of type I in which at least one of the two R-ladder configurations is of order eight, as shown in Figure 6.11b. The edge e is R-thin; deleting it and taking the retract yields the staircase St_{10} with multiple edges, two of which are f and g. It may be verified that both f and g are removable, but of them only g is thin.

6.2.2 Proof of the R-biwheel Theorem

In this section, we prove the R-biwheel Theorem (6.5); our proof is along the same lines as that of [CLM08, Theorem 4.6]. Before that, we need one more lemma pertaining to the structure of R-thin edges of index one (in an R-brick which is free of strictly R-thin edges).

Lemma 6.13 *Let G be a simple R-brick which is free of strictly R-thin edges, e an R-thin edge whose index is one, and y_1 the cubic end of e. Let y_0 and y_2 denote the neighbours of y_1 in $G-e$. Then y_0 and y_2 are both cubic, and they have a common neighbour w which is non-cubic. Let $f := wy_0$ and $g := wy_2$. Furthermore, the following statements hold:*

(i) if f is not R-compatible then $y_0 \in V(R)$, and

(ii) if f is R-compatible then it is R-thin and its index is one.

(Similar statements also apply to g.)

Proof: Let J denote the retract of $G-e$, that is, J is obtained from $G-e$ by bicontracting the vertex y_1. By hypothesis, e is not strictly thin, whence J has multiple edges. This implies that G has a vertex w, distinct from y_1, that is adjacent to both y_0 and y_2, as shown in Figure 6.8a. As in the statement of the lemma, let $f := wy_0$ and $g := wy_2$. By Proposition 6.8, w has degree four or more.

First consider the case in which f is not R-compatible. That is, either f is not removable in H or it is not removable in G, and it follows from Lemma 6.9 or from Lemma 6.10, respectively, that the end y_0 of f is cubic and it lies in $V(R)$.

Now consider the case in which f is R-compatible. Since the index of e is one, Lemma 6.12 implies that f is thin, whence it is R-thin. By hypothesis, f is not strictly R-thin. Consequently, the end y_0 of f is cubic, and the index of f is one. Applying a similar argument to the edge g, we may conclude that y_2 is also cubic. $\square$

Proof of the R-biwheel Theorem (6.5): As in the statement of the theorem, let G be a simple R-brick which is free of strictly R-thin edges, and let e denote an R-thin edge whose index is one. Our goal is to show that G has an R-biwheel configuration of which e is an internal spoke.

As in the statement of Lemma 6.13, we let y_1 denote the cubic end of e, and y_0 and y_2 the neighbours of y_1 in $G-e$. By the lemma, y_0 and y_2 are both cubic, and they have a common neighbour w which is non-cubic. We denote by u the non-cubic end of e, as shown in Figure 6.12. Observe that $y_0y_1y_2$ is a path in $H-\{u,w\}$.

We let $P := v_1v_2\dots v_j$, where $j \geq 3$, be a path of maximum length in the graph $H-\{u,w\}$ that has the following properties (see Figure 6.13):

(i) y_1 is an internal vertex of P,

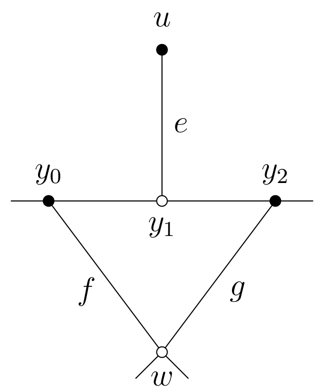


Figure 6.12: e is an R-thin edge of index one; y_0, y_1 and y_2 are cubic; u and w are non-cubic

(ii) every vertex of P is cubic in G; furthermore, if it lies in A then it is adjacent with u, and if it lies in B then it is adjacent with w, and

(iii) for every internal vertex v_i of P, the edge that joins v_i to one of u and w is R-thin of index one.

(Note that the path $y_0y_1y_2$ shown in Figure 6.12 satisfies all of the above properties; thus such a path P exists.)

We adjust notation so that v_1 lies in B as shown in Figure 6.13. It should be noted that the other end of P, namely v_j, may lie in A or in B, depending on whether P is an odd path or even. We shall let K denote the subgraph of H, which has vertex set $V(P) \cup \{u, w\}$ and edge set $E(P) \cup \{v_iw : 1 \leq i \leq j \text{ and } i \text{ odd}\} \cup \{v_iu : 1 \leq i \leq j \text{ and } i \text{ even}\}$.

Our goal is to show that K is an R-biwheel configuration. To this end, we need to establish two additional properties of the path P: first, that it is an odd path; and second, that both its ends v_1 and v_j lie in $V(R)$.

We begin by arguing that the two ends of P are nonadjacent (in G). Suppose not, that is, say v_1v_j is an edge of G. Since each vertex of P is cubic, it follows that $V(G) = V(K)$; since otherwise $\{u, w\}$ is a 2-vertex-cut of G, and we have a contradiction. Since G has an even number of vertices, P is of odd length. Furthermore, either G is the same as K, or otherwise, G has an additional edge joining u and w. In both cases, the graph G is bipartite; this is absurd. Thus v_1 and v_j are nonadjacent.

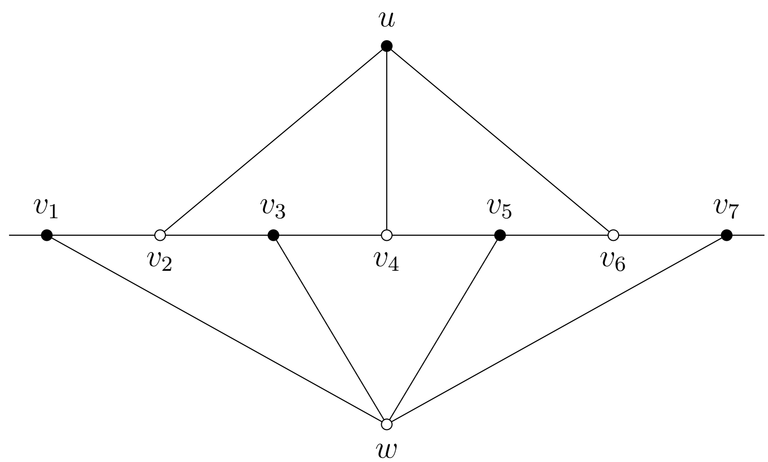


Figure 6.13: Illustration for the R-biwheel Theorem

Now, let f denote the edge v_1w. We will argue that f is not R-compatible, and then use this fact to deduce that $v_1 \in V(R)$. Suppose instead that f is R-compatible. Applying Lemma 6.13*(ii)*, with v_2u playing the role of e, we conclude that f is R-thin and its index is one. Let v_0 denote the neighbour of v_1 which is distinct from v_2 and w; note that $v_0 \in A$. By the preceding paragraph, v_0 is distinct from v_j, and since each vertex of P is cubic, v_0 is not in $V(P)$. Applying Lemma 6.13 again, this time with f playing the role of e, we deduce that v_0 is cubic. Furthermore, v_0 and v_2 have a common neighbour whose degree is four or more; thus v_0 is adjacent with u. Observe that the path v_0v_1P contradicts the maximality of P. We conclude that $f = v_1w$ is not R-compatible. By Lemma 6.13*(i)*, the cubic end v_1 of f lies in $V(R)$.

A similar argument shows that v_j lies in $V(R)$. Since v_1 and v_j are nonadjacent, one of them lies in A and the other one lies in B. (As per our notation, $v_1 \in B$ and $v_j \in A$.) In particular, P is an odd path, and thus K is an R-biwheel configuration. Observe that by property (i) of the path P, the end y_1 of e is an internal vertex of P, whence e is an internal spoke of K, as desired. This completes the proof of Theorem 6.5. $\square$

6.2.3 Proof of the R-ladder Theorem

Here, we prove the R-ladder Theorem (6.6); its proof is significantly longer than that of the R-biwheel Theorem. In its proof, we will need two lemmas (6.14 and 6.15), each of which pertains to the structure of R-thin edges of index two (in an R-brick which is free of strictly R-thin edges); these lemmas correspond to two cases that appear in the proof of Theorem 6.6.

Lemma 6.14 *Let G be a simple R-brick which is free of strictly R-thin edges and $e := y_1z_1$ an R-thin edge whose index is two. Let y_0 and y_2 denote the neighbours of y_1 which are distinct from z_1, and let z_0 and z_2 denote the neighbours of z_1 which are distinct from y_1. Suppose that y_1 is the only common neighbour of y_0 and y_2, and that z_1 is the only common neighbour of z_0 and z_2. Then there are precisely two (nonadjacent) edges, say f and g, between $\{y_0, y_2\}$ and $\{z_0, z_2\}$. Adjust notation so that $f := y_0z_0$ and $g := y_2z_2$. Furthermore, the following statements hold:*

(i) if f is not R-compatible then an end of f is cubic and it lies in $V(R)$, and

(ii) if f is R-compatible then it is R-thin and its index is two.

(Similar statements also apply to g.)

Proof: Let J denote the retract of $G - e$, that is, J is obtained from $G - e$ by bicontracting vertices y_1 and z_1. By hypothesis, e is not strictly thin, whence J has multiple edges. Also, as stated in the assumptions, y_1 is the only common neighbour of y_0 and y_2, and likewise, z_1 is the only common neighbour of z_0 and z_2. It follows that there are precisely two nonadjacent edges between $\{y_0, y_2\}$ and $\{z_0, z_2\}$, as shown in Figure 6.8b. As in the statement, adjust notation so that $f := y_0z_0$ and $g := y_2z_2$.

First consider the case in which f is not R-compatible. That is, either f is not removable in H or it is not removable in G, and it follows from Lemma 6.9 or from Lemma 6.11, respectively, that an end of f is cubic and it lies in $V(R)$.

Now consider the case in which f is R-compatible. Since f and g are nonadjacent, Lemma 6.12 implies that f is thin, whence it is R-thin. It remains to argue that the index of f is two. Suppose to the contrary that an end of f, say z_0, is non-cubic. By hypothesis, f is not strictly R-thin, whence its other end y_0 is cubic. Using the fact that y_1 is the only common neighbour of y_0 and y_2, it is easily verified that the retract of $G - f$ has no multiple edges, that is, f is strictly R-thin; this contradicts the hypothesis. Thus, each end of f is cubic, whence the index of f is two. $\square$

Lemma 6.15 *Let G be a simple R-brick which is free of strictly R-thin edges and $e := y_1z_1$ an R-thin edge whose index is two. Let y_0 and y_2 denote the neighbours of y_1 which are distinct from z_1, and let z_0 and z_2 denote the neighbours of z_1 which are distinct from y_1. Suppose that y_0 and y_2 have a common neighbour w which is distinct from y_1. Let $f := y_0w$ and $g := y_2w$. Then w is non-cubic and is distinct from each of z_0 and z_2. Furthermore, f and g are both removable, y_0 and y_2 are both cubic, and the following statements hold:*

(i) one of f and g is R-compatible; adjust notation so that f is R-compatible;

(ii) f is not thin, and its cubic end y_0 is adjacent with (exactly) one of z_0 and z_2; and,

(iii) g is thin but it is not R-compatible, and its cubic end y_2 lies in $V(R)$.

Proof: Note that f and g are multiple edges in the retract J of $G - e$. Since f and g are adjacent, by Proposition 6.8, their common end w is non-cubic. Consequently, by Lemma 6.11, f and g are both removable. Note that y_0 and y_2 are nonadjacent, since otherwise e is non-removable. In particular, at least one of y_0 and y_2 does not lie in $V(R)$. By Lemma 6.9, at least one of f and g is R-compatible.

We now argue that w is distinct from each of z_0 and z_2. Suppose not, and assume without loss of generality that $w = z_0$. By Lemma 6.12*(ii)*, f and g are both thin; in particular, at least one of them is R-thin. Adjust notation so that f is R-thin. By hypothesis, f is not strictly R-thin, whence the retract of $G - f$ has multiple edges; consequently, the end y_0 of f is cubic. Let v denote the neighbour of y_0 which is distinct from y_1 and z_0. Furthermore, as f is not strictly R-thin, we infer that v and y_1 have a common neighbour which is distinct from y_0; by Proposition 6.8, such a common neighbour is non-cubic. Since z_1 is cubic, we infer that y_2 is non-cubic. By Lemma 6.9, g is R-compatible. As noted earlier, g is thin; whence g is R-thin. Since each end of g is non-cubic, g is strictly R-thin, contrary to the hypothesis. Thus w is distinct from each of z_0 and z_2; see Figure 6.14.

Let us review what we have proved so far. We have shown that y_0 and y_2 are not both adjacent with z_0. An analogous argument shows that y_0 and y_2 are not both adjacent with z_2. By symmetry, z_0 and z_2 are not both adjacent with y_0; likewise, z_0 and z_2 are not both adjacent with y_2. In summary, there are at most two edges between $\{y_0, y_2\}$ and $\{z_0, z_2\}$; and if there are precisely two such edges then they are nonadjacent.

Now we argue that y_0 and y_2 are both cubic. Suppose instead that y_0 is non-cubic; then, by Lemma 6.9, f is R-compatible. Note that since each end of f is non-cubic, if f is thin then it is strictly R-thin, contrary to the hypothesis. So it must be the case that f is not thin. By Lemma 6.12*(iv)*, $N(y_0) \subseteq N(z_1) \cup \{w\} = \{z_0, z_2, y_1, w\}$. As y_0 is non-cubic,

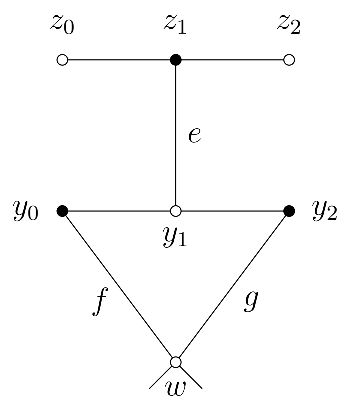


Figure 6.14: Illustration for Lemma 6.15

it must be adjacent with each of z_0 and z_2; however, this contradicts what we have already established in the preceding paragraph. We conclude that y_0 and y_2 are both cubic.

As noted earlier, at least one of f and g is R-compatible. As in statement *(i)* of the lemma, adjust notation so that f is R-compatible. We will now argue that f is not thin.

Suppose instead that f is thin. Let v denote the neighbour of y_0 which is distinct from y_1 and w. By hypothesis, f is not strictly R-thin, whence v and y_1 have a common neighbour which is distinct from y_0; by Proposition 6.8, such a common neighbour is non-cubic. However, this is not possible as each neighbour of y_1 is cubic. Thus, f is not thin. An analogous argument shows that if g is R-compatible then g is not thin.

Since f is removable but it is not thin, by Lemma 6.12*(iv)*, $N(y_0) \subseteq N(z_1) \cup \{w\} = \{z_0, z_2, y_1, w\}$. It follows from our previous observation that y_0 is adjacent with exactly one of z_0 and z_2; adjust notation so that y_0 is adjacent with z_0. This proves statement *(ii)*.

Also, by Lemma 6.12, one of f and g is thin; as per our notation, g is thin. Consequently, g is not R-compatible. By Lemma 6.9, the cubic end y_2 of g lies in $V(R)$. This proves statement *(iii)*, and we are done. □

Proof of the R-ladder Theorem (6.6): As in the statement of the theorem, let G be a simple R-brick which is free of strictly R-thin edges, and let e denote an R-thin edge whose index is two. We shall let y_1 and z_1 denote the ends of e, where $y_1 \in A$ and $z_1 \in B$. Furthermore, we let y_0 and y_2 denote the neighbours of y_1 which are distinct from z_1, and likewise, we let z_0 and z_2 denote the neighbours of z_1 which are distinct from y_1.

Our goal is to show that G has an R-ladder configuration which contains the edge e. As mentioned earlier, we will consider two separate cases which correspond to the situations in Lemmas 6.14 and 6.15, respectively.

Case 1: y_1 is the only common neighbour of y_0 and y_2, and likewise, z_1 is the only common neighbour of z_0 and z_2.

By Lemma 6.14, there are precisely two nonadjacent edges between $\{y_0, y_2\}$ and $\{z_0, z_2\}$. Adjust notation so that y_0z_0 and y_2z_2 are edges of G, as shown in Figure 6.15. Observe that the graph in the figure is a ladder of which e is an internal rung; furthermore, it is a subgraph of H.

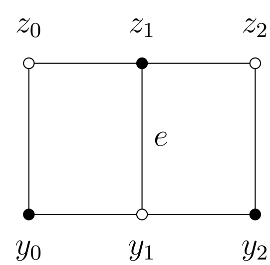


Figure 6.15: The situation in Case 1

We let K be a subgraph of H of maximum order that has the following properties:

(i) K is a ladder and e is an internal rung of K, and

(ii) every internal rung of K is an R-thin edge whose index is two.

Note that the subgraph K is either an odd ladder or an even ladder; see Figure 6.16. We shall denote by au and bw the external rungs of K such that $a, w \in A$ and $b, u \in B$, as shown in the figure. It follows from property (ii) of K that each of its vertices, except possibly a, u, b and w, is cubic in G.

Remark 6.16 *Note that, if $|V(K)| = 6$ then K is the same as the subgraph of H shown in Figure 6.15; in particular, $\{u, b\} = \{y_0, y_2\}$, and likewise, $\{w, a\} = \{z_0, z_2\}$; consequently,* *by our hypothesis, y_1 is the only common neighbour of u and b, and likewise, z_1 is the only common neighbour of w and a.*

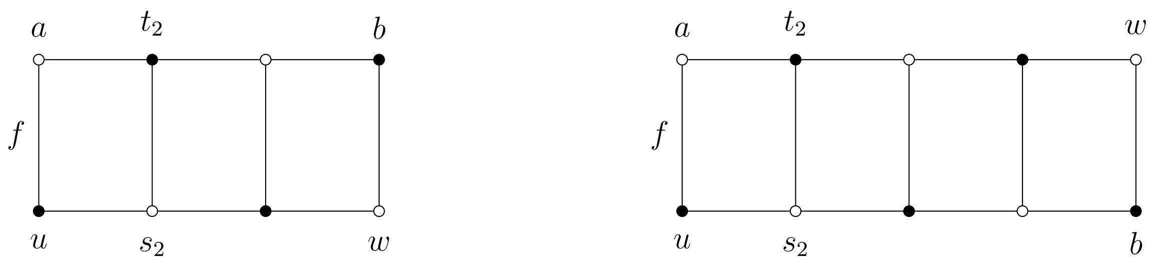


Figure 6.16: Illustration for Case 1 of the R-ladder Theorem

Our goal is to show that K is an R-ladder configuration. To this end, we need to establish that a and b (or likewise, u and w) are both cubic in G and they lie in $V(R)$.

Now, let f denote the edge au. We will argue that f is not R-compatible, and then use this fact to deduce that one of the ends of f is cubic and it lies in $V(R)$. As shown in Figure 6.16, let s_2 denote the neighbour of u in K which is distinct from a, and likewise, let t_2 denote the neighbour of a in K which is distinct from u.

Suppose instead that f is R-compatible. By Lemma 6.14*(ii)*, with s_2t_2 playing the role of e, we conclude that f is R-thin and its index is two. We shall let s_0 denote the neighbour of u which is distinct from s_2 and a, and likewise, let t_0 denote the neighbour of a which is distinct from t_2 and u. Note that $s_0 \in A$ and $t_0 \in B$. It is easily seen that if s_0 is the same as w then $V(K) \cap A$ is a (nontrivial) barrier of G; this is absurd as G is a brick. Thus $s_0 \neq w$, and likewise, $t_0 \neq b$. It follows that $s_0, t_0 \notin V(K)$.

We will use the fact that f is not strictly R-thin to deduce that s_0 and t_0 are adjacent; this will help us contradict the maximality of K. First suppose that s_0 and s_2 have a common neighbour x which is distinct from u. By Proposition 6.8, x is non-cubic. Observe that, if $|V(K)| \geq 8$ then every neighbour of s_2 is cubic; and if $|V(K)| = 6$ then b is the only neighbour of s_2 which is possibly non-cubic. We conclude that $|V(K)| = 6$ and that $x = b$. Now, s_0 is a common neighbour of u and b; this contradicts the hypothesis (see Remark 6.16). We conclude that u is the only common neighbour of s_0 and s_2. An analogous argument shows that a is the only common neighbour of t_0 and t_2. It follows that s_0 and t_0 are adjacent, as f is not strictly thin. Now, let K' denote the subgraph of H

obtained from K by adding the vertices s_0 and t_0, and the edges us_0, s_0t_0 and t_0a; then K' contradicts the maximality of K.

We thus conclude that $f = au$ is not R-compatible. Consequently, by Lemma 6.14*(i)*, with s_2t_2 playing the role of e, at least one of a and u is cubic and it also lies in $V(R)$. Adjust notation so that a is cubic and it lies in $V(R)$.

An analogous argument shows that at least one of b and w is cubic and it lies in $V(R)$; we claim that b must satisfy both of these properties. Suppose not; then w is cubic and it lies in $V(R)$; this means that the edge α of R joins the vertices a and w. Observe that $\{b, u\}$ is a 2-vertex cut of G; this is absurd as G is a brick.

We have shown that a and b both are cubic and they lie in $V(R)$. Thus K is an R-ladder configuration. Observe that, by property (i) of K, the edge e is an internal rung of K. In particular, e is an edge of K, as desired.

Case 2: y_0 and y_2 have a common neighbour which is distinct from y_1, or likewise, z_0 and z_2 have a common neighbour which is distinct from z_1.

As shown in Figure 6.17, assume without loss of generality that y_0 and y_2 have a common neighbour, say w, which is distinct from y_1. We let $f := y_0w$ and $g := y_2w$. We invoke Lemma 6.15 to infer the following: w is non-cubic and it is distinct from each of z_0 and z_2; whereas y_0 and y_2 are both cubic; f and g are both removable edges. Furthermore, adjusting notation as in the lemma, f is R-compatible but it is not thin and its cubic end y_0 is adjacent with one of z_0 and z_2. Assume without loss of generality that y_0 is adjacent with z_0. The edge g is thin but it is not R-compatible and its cubic end y_2 lies in $V(R)$. As per our notation, y_2 is an end of β; we shall let x denote the other end of β.

We will consider two subcases. In the first one, we assume that z_0 is cubic and it lies in $V(R)$; and in the second case, we assume that either z_0 is non-cubic or it is not in $V(R)$.

Case 2.1: z_0 is cubic and it lies in $V(R)$.

In this case, we shall denote by K the subgraph whose vertex set is $\{z_0, z_1, y_0, y_1, w, y_2\}$ and edge set is $\{e, y_1y_2, g, f, y_0z_0, z_0z_1, y_0y_1\}$. Observe that K is a ladder of order six and it is a subgraph of H; furthermore, two of its corners, namely y_2 and z_0, are cubic and they both lie in $V(R)$. To complete the proof in this case, we will show that K is an R-ladder configuration; for this, we only need to prove that the internal rung y_0y_1 is R-thin and its index is two.

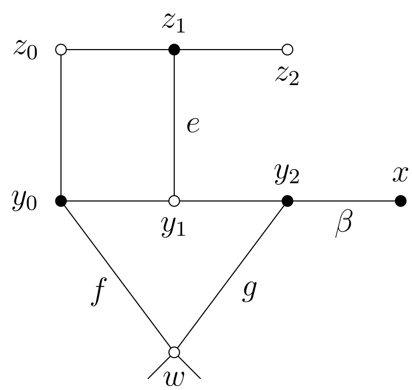


Figure 6.17: The situation in Case 2

We begin by showing that y_0y_1 is R-compatible, that is, y_0y_1 is removable in H as well as in G. Here, we will not require the hypothesis that z_0 is cubic and it lies in $V(R)$.

Claim 6.17 *The edge y_0y_1 is R-compatible.*

Proof: Note that y_0y_1 is removable in the subgraph K. We will argue that K is a conformal subgraph of H, and then use Proposition 4.3 to deduce that y_0y_1 is removable in H.

Let M be any perfect matching of H which contains the edge z_0z_1. Since M does not contain α or β, it is easily verified that $M \cap E(K)$ is a perfect matching of K, whence K is a conformal subgraph of H; consequently, y_0y_1 is removable in H.

To conclude that y_0y_1 is removable in G, we will show that $G - y_0y_1$ has a perfect matching M which contains both α and β. Let N be a perfect matching of $G - \{z_1, x\}$; such a perfect matching exists as G is a brick; note that $\alpha \in N$ and $\beta \notin N$. Clearly, either $y_1y_2 \in N$ or $g \in N$. If $y_1y_2 \in N$, we let $M := (N - y_1y_2) + e + \beta$. On the other hand, if $g \in N$ then $y_0y_1 \in N$, and we let $M := (N - g - y_0y_1) + e + f + \beta$. In either case, M is the desired perfect matching, and this completes the proof. $\square$

We now proceed to show that y_0y_1 is an R-thin edge. To this end, we will use the characterization of R-thin edges in terms of barriers given by Proposition 4.16.

Claim 6.18 *The edge y_0y_1 is R-thin, and its index is two.*

Proof: Observe that, since y_0 and y_1 are both cubic, $G - y_0y_1$ has two maximal nontrivial barriers; one of them, say S_A, is a subset of A and it contains z_0 and w; the other one, say S_B, is a subset of B and it contains z_1 and y_2. In particular, the index of y_0y_1 is two.

We will argue that $S_A = \{z_0, w\}$; our argument does not use the fact that w is non-cubic, and it may be mimicked to show that $S_B = \{z_1, y_2\}$; thereafter, we apply Proposition 4.16 to infer that y_0y_1 is R-thin.

Note that w is in the barrier S_A. Now, let v be any vertex in $A - \{z_0, w\}$. We will show that $(G - y_0y_1) - \{w, v\}$ has a perfect matching M; this would imply that v is not in the barrier S_A. Let N be a perfect matching of $G - \{w, v\}$; note that $\beta \in N$ and $\alpha \notin N$. If $y_0y_1 \notin N$ then let $M := N$, and we are done. Now suppose that $y_0y_1 \in N$. By our hypothesis, z_0 is cubic and it lies in $V(R)$; this means that the three edges incident at z_0 are z_0y_0, z_0z_1 and α. Since, $y_0y_1 \in N$ and $\alpha \notin N$ and $v \neq z_0$, we conclude that $z_0z_1 \in N$. Now, $M := (N - y_0y_1 - z_0z_1) + y_0z_0 + e$ is the desired perfect matching. We conclude that $S_A = \{z_0, w\}$. As discussed in the preceding paragraph, this completes the proof. □

We have shown that the only internal rung of K, namely y_0y_1, is an R-thin edge whose index is two. As discussed earlier, K is indeed an R-ladder configuration, and since it contains e, this completes the proof in this case (2.1).

Case 2.2: Either z_0 is non-cubic or it does not lie in $V(R)$, possibly both.

As per our notation, $z_0 \in A$; it follows from the hypothesis of this case that z_0 has at least one neighbour which lies in $B - \{z_1, y_0\}$; we shall let u denote such a neighbour of z_0, as shown in Figure 6.18. Observe that u is distinct from y_2; however, it is possible that $u = x$.

In this case, we will prove that z_0z_1 is an R-thin edge whose index is two; in particular, z_0 is cubic and $z_0 \notin V(R)$. (If not, we will find a strictly R-thin edge contrary to the hypothesis.) Thereafter, we argue that u is adjacent with z_2; this establishes a certain symmetry between y_0, y_1, y_2, w and z_0, z_1, z_2, u, respectively; see Figure 6.20. We shall exploit this to deduce that y_0y_1 is an R-thin edge (whose index is two), and that z_2 is cubic and it lies in $V(R)$. In the end, we will find an R-ladder configuration of order eight whose internal rungs are y_0y_1 and z_0z_1.

Our first step is to show that z_0z_1 is R-compatible, that is, z_0z_1 is removable in H as well as in G.

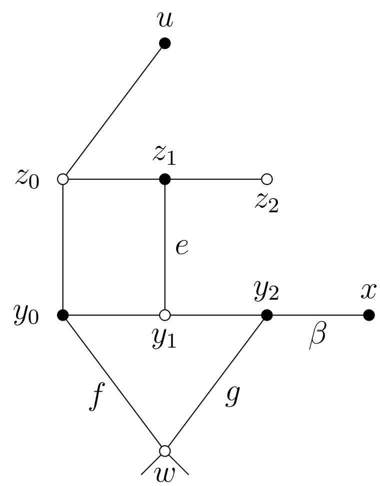


Figure 6.18: The situation in Case 2.2 (all labelled vertices are pairwise distinct, except possibly u and x)

Claim 6.19 *The edge z_0z_1 is R-compatible.*

Proof: Note that $y_0y_1z_1z_0y_0$ is a quadrilateral containing the edges y_0y_1 and z_0z_1. We will show that y_0y_1 is admissible in $H - z_0z_1$, and then invoke Corollary 4.6 to deduce that z_0z_1 is removable in H.

We need to show that $H - z_0z_1$ has a perfect matching M which contains y_0y_1. Let N be any perfect matching of $H - \{u, y_1\}$; such a perfect matching exists by Proposition 1.4. Observe that $g \in N$; consequently, $y_0z_0 \in N$. Now, $M := (N - y_0z_0) + uz_0 + y_0y_1$ is the desired perfect matching. As discussed above, z_0z_1 is removable in H.

To conclude that z_0z_1 is removable in G, we will show that $G - z_0z_1$ has a perfect matching M which contains both α and β. Let N be any perfect matching of G which contains α and β. If $z_0z_1 \notin N$ then let $M := N$, and we are done. Now suppose that $z_0z_1 \in N$. Observe that $y_0y_1 \in N$; furthermore, $M := (N - y_0y_1 - z_0z_1) + e + y_0z_0$ is the desired perfect matching. This completes the proof. $\square$

We proceed to prove that z_0z_1 is an R-thin edge whose index is two. As we did in Claim 6.18, we will use the characterization of R-thin edges given by Proposition 4.16. However, here we need more general arguments since we do not know the degree of z_0.

Claim 6.20 *The edge z_0z_1 is R-thin, and its index is two.*

Proof: Observe that, since z_1 is cubic, $G - z_0z_1$ has a maximal nontrivial barrier, say S_A, which is a subset of A and contains y_1 and z_2. We will first prove that $S_A = \{y_1, z_2\}$.

Let v be any vertex in $A - \{y_1, z_2\}$. We will show that $(G - z_0z_1) - \{z_2, v\}$ has a perfect matching M; this would imply that v is not in the barrier S_A. Let N be a perfect matching of $G - \{z_2, v\}$; note that $\beta \in N$ and $\alpha \notin N$. If $z_0z_1 \notin N$ then let $M := N$, and we are done. Now suppose that $z_0z_1 \in N$, and observe that $y_0y_1 \in N$; consequently, $M := (N - z_0z_1 - y_0y_1) + e + y_0z_0$ is the desired perfect matching. Thus, $S_A = \{y_1, z_2\}$.

Since z_0z_1 is R-compatible, by the Three Case Lemma (4.15), either S_A is the only maximal nontrivial barrier of $G - z_0z_1$, or $G - z_0z_1$ has another maximal nontrivial barrier, say S_B, which is a subset of B. We now argue that, in the former case, z_0z_1 is strictly R-thin, contrary to the hypothesis.

Suppose that S_A is the only maximal nontrivial barrier of $G - z_0z_1$; in this case, the index of z_0z_1 is one. By Proposition 4.16, z_0z_1 is R-thin. Also, z_0 is non-cubic, since otherwise its two neighbours distinct from z_1 would lie in a barrier. Observe that, since z_1 is the only common neighbour of y_1 and z_2, the retract of $G - z_0z_1$ is simple, and thus z_0z_1 is strictly R-thin; this is a contradiction.

It follows that $G - z_0z_1$ has a maximal nontrivial barrier, say S_B, which is a subset of B; in particular, the index of z_0z_1 is two. By the Three Case Lemma (4.15), z_0 is isolated in $(G - z_0z_1) - S_B$; that is, in $G - z_0z_1$, every neighbour of z_0 lies in the barrier S_B. In particular, $u, y_0 \in S_B$. We will prove that $S_B = \{u, y_0\}$.

Let v be any vertex in $B - \{u, y_0\}$. We will show that $(G - z_0z_1) - \{u, v\}$ has a perfect matching M; this would imply that v is not in the barrier S_B. Let N be a perfect matching of $G - \{u, v\}$; note that $\alpha \in N$ and $\beta \notin N$. If $z_0z_1 \notin N$ then let $M := N$, and we are done. Now suppose that $z_0z_1 \in N$. If $y_0y_1 \in N$ then $M := (N - z_0z_1 - y_0y_1) + e + y_0z_0$ is the desired perfect matching. Now suppose that $y_0y_1 \notin N$; then $f, y_1y_2 \in N$, and $M := (N - z_0z_1 - f - y_1y_2) + y_0z_0 + g + e$ is the desired perfect matching. Thus, as discussed above, $v \notin S_B$; consequently, $S_B = \{u, y_0\}$. In particular, z_0 is cubic. Furthermore, by Proposition 4.16, z_0z_1 is R-thin. □

We have shown that z_0z_1 is an R-thin edge and its index is two; in particular, both its ends are cubic. The three neighbours of z_0 are y_0, z_1 and u; see Figure 6.18.

By hypothesis, z_0z_1 is not strictly R-thin; whence the retract of $G - z_0z_1$ has multiple edges. Observe that z_1 is the only common neighbour of y_1 and z_2. Consequently, at least one of the following must hold: either u and y_0 have a common neigbour which is distinct from z_0, or u and z_2 are adjacent. We shall rule out the former case by arriving at a contradiction.

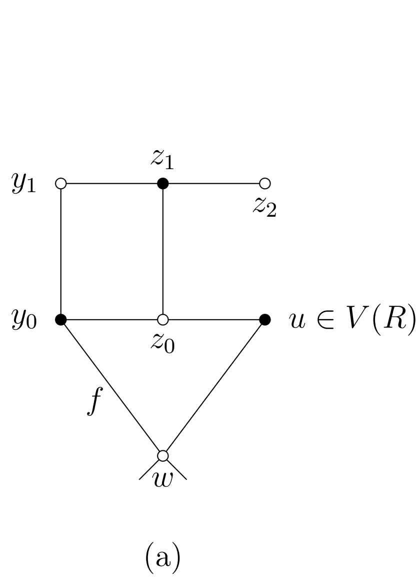


Figure 6.19: When u is adjacent with w

Suppose that u and y_0 have a common neighbour which is distinct from z_0; this is true if and only if u is adjacent with w. We now invoke Lemma 6.15, with z_0z_1 playing the role of e, with u playing the role of y_2, and with uw playing the role of g; see Figure 6.19a, and compare with Figure 6.18. The lemma implies that u is a cubic vertex, and since f is R-compatible, uw is thin but it is not R-compatible; furthermore, $u \in V(R)$. In particular, u is an end of β which implies that $u = x$; see Figures 6.18 and 6.19b. Note that all of the labelled vertices in Figure 6.19b are pairwise distinct; furthermore, each of them except w and possibly z_2, is cubic. Since z_2 has at least one neighbour in B which is distinct from z_1, the graph has more vertices; consequently, $\{w, z_2\}$ is a 2-vertex cut of G; this is a contradiction.

We have shown that z_0 is the only common neighbour of u and y_0; as discussed earlier, this implies that u and z_2 are adjacent; see Figure 6.20. Note that u is now a common neighbour of z_0 and z_2, and it is distinct from z_1; this establishes a symmetry between y_0, y_1, y_2, w, and z_0, z_1, z_2, u, respectively. We invoke Lemma 6.15 to conclude that u is non-cubic, whereas z_2 is cubic and it lies in $V(R)$. Using arguments analogous to those in the proofs of Claims 6.19 and 6.20, we conclude that y_0y_1 is an R-thin edge (whose index is two).

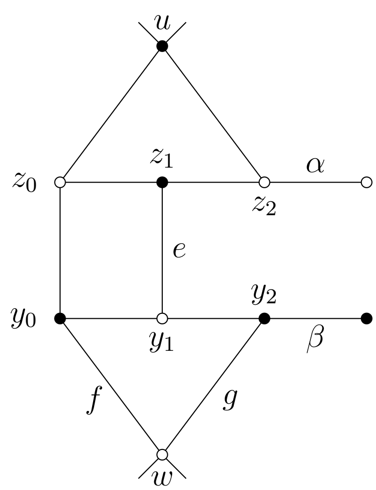


Figure 6.20: Illustration for Case 2.2 of the R-ladder Theorem; u is a common neighbour of z_0 and z_2 which is distinct from z_1

Now, let K denote the subgraph which consists of all of the labelled vertices shown in Figure 6.20, and all of the edges between those vertices which are shown in the figure. Note that K is an R-ladder configuration, and since it contains e, this completes the proof of the R-ladder Theorem (6.6). □

6.3 Properties of R-configurations

In this section, we prove a few results pertaining to R-configurations. These are used in our proof of the Strictly R-thin Edge Theorem (1.24), which appears in the next section.

For the rest of this section, G is a simple R-brick, and we adopt Notation 6.1; furthermore, K_1 is an R-configuration with external rungs/spokes a_1u_1 and b_1w_1. As usual, u_1 and w_1 are the free corners of K_1; see Figure 6.21.

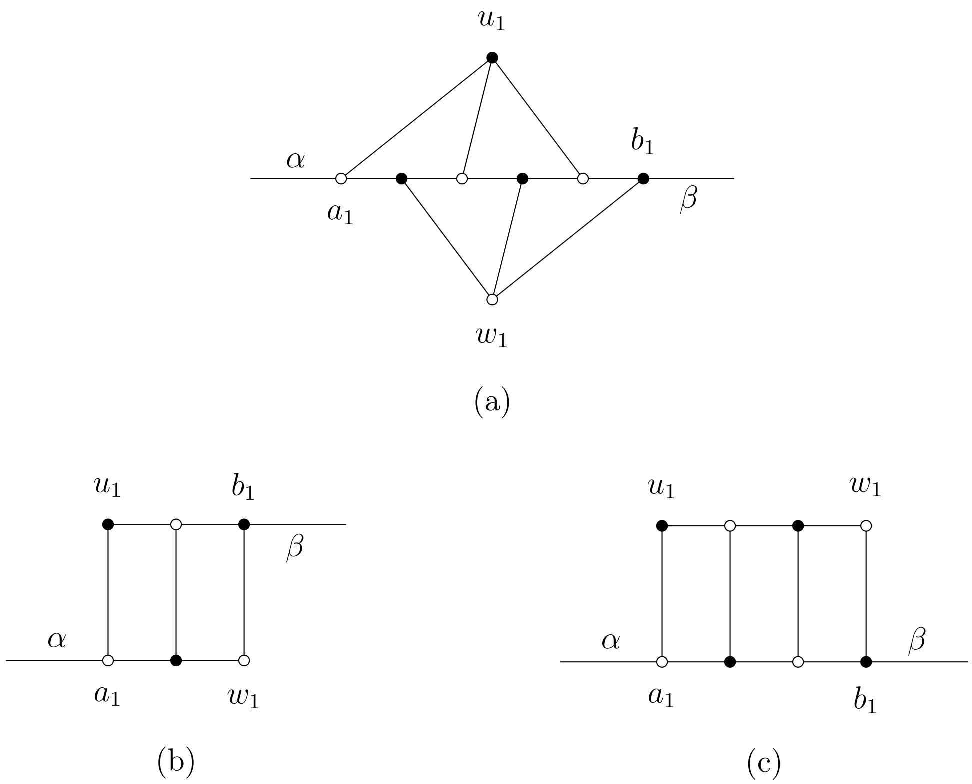


Figure 6.21: The R-configuration K_1

Note that K_1 is either a ladder or a partial biwheel. In either case, it is easily verified that the graph obtained from K_1 by adding two edges, one joining a_1 and b_1, and another joining u_1 and w_1, is a brace. This fact, in conjunction with the characterization of braces provided by Proposition 4.12, yields the following easy observation.

Proposition 6.21 *The following statements hold:*

(i) for every pair of distinct vertices $v_1, v_2 \in A \cap V(K_1)$, the graph $K_1 - \{b_1, u_1, v_1, v_2\}$ has a perfect matching; and likewise,

(ii) for every pair of distinct vertices $v_1, v_2 \in B \cap V(K_1)$, the graph $K_1 - \{a_1, w_1, v_1, v_2\}$ has a perfect matching. □

In the following lemma, we prove some conformality properties of R-configurations; these are useful in subsequent lemmas to show that a certain edge is R-compatible.

Lemma 6.22 *The following statements hold:*

(i) u_1 *lies in* $V(R)$ *if and only if* w_1 *lies in* $V(R)$,

(ii) K_1 *is a conformal matching covered subgraph, and*

(iii) the subgraph induced by $E(K_1) \cup R$ *is conformal.*

Proof: First, we prove *(i)*. Suppose instead that $u_1 \in V(R)$ and $w_1 \notin V(R)$; that is, $u_1 = b_2$, whereas w_1 and a_2 are distinct; see Figure 6.22. For $X := V(K_1) - w_1$, note that every edge in $\partial(X)$, except for α, is either incident with u_1 or with w_1. Recall that if M is any perfect matching, then $\alpha \in M$ if and only if $\beta \in M$. Using these facts, it is easy to see that $\partial(X)$ is a tight cut; this is a contradiction.

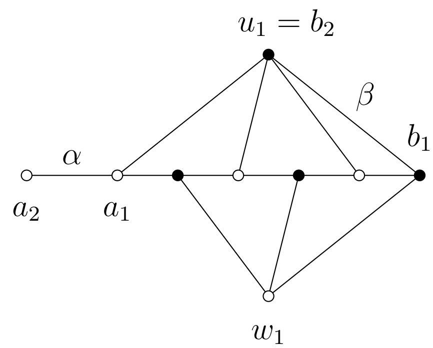


Figure 6.22: $\partial(X)$ is a nontrivial tight cut, where $X := V(K_1) - w_1$

Now, we prove *(ii)*. Since K_1 is either a ladder or a partial biwheel, it is matching covered. To show that K_1 is conformal, we will display a perfect matching M of $G - V(K_1)$. Let N be a perfect matching of H which contains a_1u_1; observe that $M := N - E(K_1)$ is the desired perfect matching.

Note that, if $u_1, w_1 \in V(R)$, then *(iii)* follows immediately from *(ii)*. Now suppose that $u_1, w_1 \notin V(R)$, and let N be a perfect matching of $G - \{a_2, w_1\}$; note that $\beta \in N$. A simple counting argument shows that $M := N - E(K_1) - R$ is a perfect matching of $G - V(K_1) - V(R)$; and this proves *(iii)*. □

In the following two lemmas, apart from other things, we show that under certain circumstances there exists an R-compatible edge which is not in K_1.

Lemma 6.23 *Suppose that $u_1, w_1 \notin V(R)$. Then at most one edge of $\partial(u_1) - E(K_1)$ is not R-compatible. (An analogous statement holds for w_1.)*

Proof: Note that, by Corollary 4.9, at most two edges of $\partial(u_1)$ are non-removable in H; one of these is a_1u_1. Consequently, at most one edge of $\partial(u_1) - E(K_1)$ is non-removable in H. To complete the proof we will show that if e is any removable edge of H such that $e \in \partial(u_1) - E(K_1)$, then e is removable in G as well; for this, it suffices to show a perfect matching M which contains α and β but does not contain e.

Let M_1 be a perfect matching of $G - V(K_1) - V(R)$; such a perfect matching exists by Lemma 6.22*(iii)*. Let M_2 be a perfect matching of $K_1 - \{a_1, b_1\}$; since K_1 is bipartite matching covered, such a perfect matching exists by Proposition 1.4. Now, $M := M_1 \cup M_2 \cup R$ is the desired perfect matching alluded to above, and this completes the proof. □

Lemma 6.24 *Suppose that $u_1, w_1 \notin V(R)$. If $|\partial(u_1) - E(K_1)| \leq 1$ and $|\partial(w_1) - E(K_1)| \leq 1$ then the following statements hold:*

(i) *u_1 and w_1 are nonadjacent,*

(ii) *$\partial(u_1) - E(K_1)$ has exactly one member, say α', and likewise, $\partial(w_1) - E(K_1)$ has exactly one member, say β',*

(iii) *α and α' are adjacent if and only if β and β' are adjacent,*

(iv) *if α and α' are nonadjacent then at most one edge of $\partial(v) - \alpha'$ is not R-compatible, where v denotes the end of α' which is distinct from u_1; an analogous statement holds for β and β'.*

Proof: We first verify *(i)* and *(ii)*. Observe that, if u_1 and w_1 are adjacent, or, if the sets $\partial(u_1) - E(K_1)$ and $\partial(w_1) - E(K_1)$ are both empty, then $\{a_1, b_1\}$ is a 2-vertex cut of G; this is absurd. This proves *(i)*. Note that, if only one of $\partial(u_1) - E(K_1)$ and $\partial(w_1) - E(K_1)$ is nonempty then H has a cut-edge; this is a contradiction. This proves *(ii)*. As in the statement, let α' denote the only member of $\partial(u_1) - E(K_1)$; and likewise, let β' denote the only member of $\partial(w_1) - E(K_1)$. See Figure 6.23.

We now show that *(iii)* holds. Suppose instead that β and β' are adjacent, whereas α and α' are nonadjacent. In particular, β' has ends w_1 and b_2. We let $T := B - V(K_1) - b_2$,

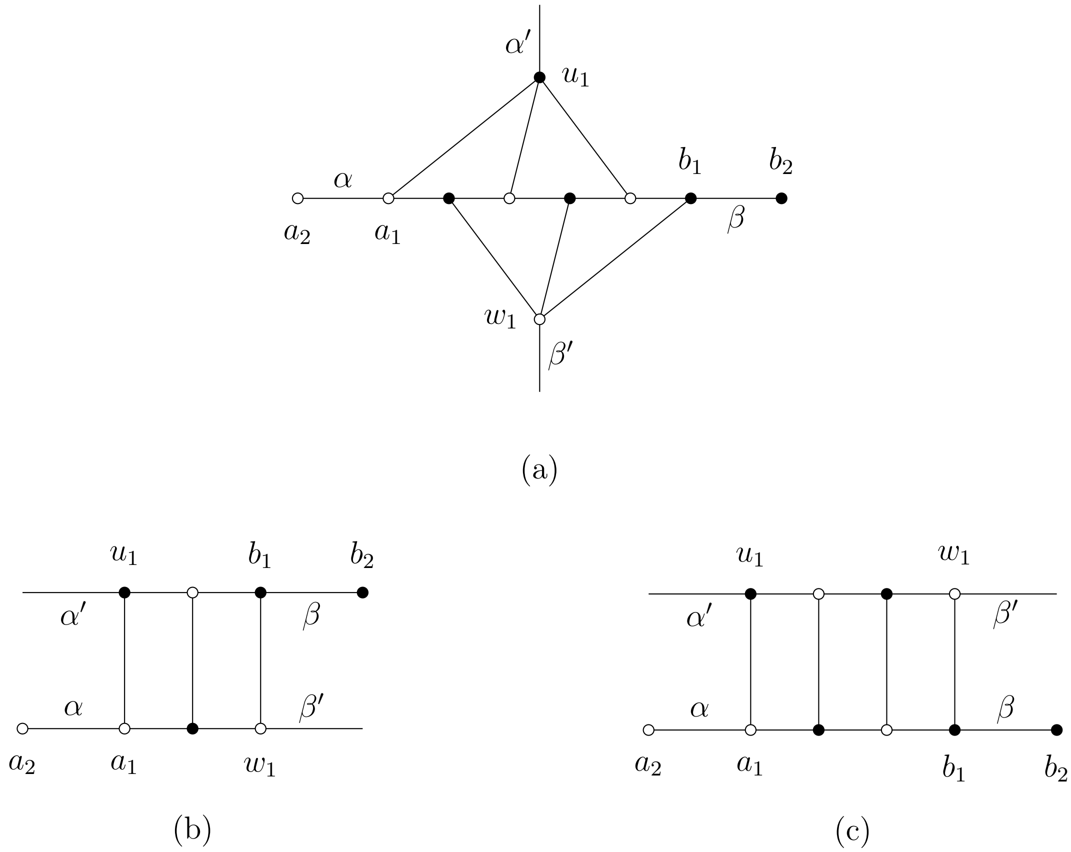


Figure 6.23: When $|\partial(u_1) - E(K_1)| = |\partial(w_1) - E(K_1)| = 1$

and note that T is nonempty. Furthermore, all of the neigbours of T lie in the set $S := A - V(K_1)$; consequently, S is a nontrivial barrier of G; this is absurd.

We now proceed to prove *(iv)*. Suppose that α and α' are nonadjacent; and as in the statement of the lemma, let v denote the end of α' which is distinct from u_1. By *(iii)*, β and β' are also nonadjacent. We will first argue that at most one edge of $\partial(v) - \alpha'$ is non-removable in H.

Observe that $\{\alpha', \beta'\}$ is a 2-cut of H; thus, neither α' nor β' is removable in H. By Corollary 4.9, at most two edges of $\partial(v)$ are non-removable in H; one of these is α'. Consequently, at most one edge of $\partial(v) - \alpha'$ is non-removable in H. To complete the proof we will show that if e is any removable edge of H such that $e \in \partial(v) - \alpha'$, then e is removable in G as well; for this, it suffices to show a perfect matching M which contains α and β but does not contain e.

Let M_1 be any perfect matching of $G - \{a_2, v\}$; note that $\beta \in M_1$. A simple counting argument shows that β' lies in M_1 as well. Now, let M_2 be a perfect matching of $K_1 - \{a_1, u_1, b_1, w_1\}$; such a perfect matching exists due to Proposition 6.21. Observe that $M := (M_1 - E(K_1)) \cup M_2 \cup \{\alpha, \alpha'\}$ is the desired perfect matching alluded to above. As discussed, this completes the proof. □

In the previous two lemmas, we have shown that under certain circumstances there exists an R-compatible edge which is not in K_1. However, in the proof of the Strictly R-thin Edge Theorem (1.24), we will be interested in finding an R-thin edge which is not in K_1. To do so, we will choose an R-compatible edge appropriately, and use Theorem 5.1, in conjunction with the following lemma, to argue that the chosen edge is indeed R-thin.

Lemma 6.25 *Suppose that $u_1, w_1 \notin V(R)$. Let e denote an R-compatible edge which does not lie in $E(K_1)$, let S denote a nontrivial barrier of $G - e$, and I the set of isolated vertices of $(G - e) - S$. Then the following statements hold:*

(i) $S \cap V(K_1)$ contains at most one vertex, and

(ii) $I \cap V(K_1)$ is empty.

Proof: Since e is R-compatible, S is a subset of one of the two color classes of H; assume without loss of generality that $S \subset A$. To establish *(i)*, we will show that if v_1 and v_2 are any two distinct vertices in $V(K_1) \cap A$, then $(G - e) - \{v_1, v_2\}$ has a perfect matching M.

Let M_1 be a perfect matching of $(H - e) - \{v_1, b_2\}$ where b_2 is the end of β which is not in $V(K_1)$; such a perfect matching exists by Proposition 1.4 as $H - e$ is matching covered. A simple counting argument shows that $M \cap \partial(V(K_1))$ contains only one edge, and this edge is incident with the free corner u_1. Let M_2 be a perfect matching of $K_1 - \{b_1, u_1, v_1, v_2\}$; such a perfect matching exists due to Proposition 6.21. Observe that $M := (M_1 - E(K_1)) + M_2 + \beta$ is the desired perfect matching of $(G - e) - \{v_1, v_2\}$, and this proves *(i)*.

We now deduce *(ii)* from *(i)*. Suppose to the contrary that $I \cap V(K_1)$ is nonempty, and let x denote any of its members. Observe that x is adjacent with at least two vertices in $V(K_1)$, and each of these must lie in S; this contradicts *(i)*, and completes the proof. □

6.3.1 Proof of Proposition 6.3

As in the statement of the proposition, let G be a simple R-brick, and let K_1 be an R-configuration with external rungs/spokes a_1u_1 and b_1w_1, where u_1 and w_1 denote the free corners of K_1; see Figure 6.21. Suppose that G has an R-configuration K_2 which is distinct from K_1; that is, K_1 and K_2 are not identical subgraphs of G. We assume that K_1 and K_2 are not vertex-disjoint. Our goal is to deduce that u_1 and w_1 are the free corners of K_2, and that K_2 is otherwise vertex-disjoint with K_1.

We first argue that $u_1, w_1 \notin V(R)$. Note that every vertex of K_1, except possibly u_1 and w_1, is cubic in G. Consequently, if $u_1, w_1 \in V(R)$ then $V(G) = V(K_1)$, since otherwise $\{u_1, w_1\}$ is a 2-vertex cut of G; furthermore, either G is precisely the graph induced by $E(K_1) \cup R$, or otherwise, G has one additional edge joining u_1 and w_1; in either case, it is easily seen that K_1 is the only subgraph with all the properties of an R-configuration; this contradicts the hypothesis. By Lemma 6.22*(i)*, $u_1, w_1 \notin V(R)$.

Claim 6.26 *Let z_1 be any vertex of K_1 which is distinct from u_1 and w_1. If $z_1 \in V(K_2)$ then every edge of K_1 which is incident with z_1 lies in $E(K_2)$.*

Proof: Assume that $z_1 \in V(K_2)$. First consider the case in which $z_1 \in \{a_1, b_1\}$. Note that the degree of z_1 in H is two; consequently, both edges of H incident with z_1 lie in $E(K_2)$.

Now consider the case in which $z_1 \notin \{a_1, b_1\}$. Note that z_1 is cubic. Observe that, for an R-configuration K, any vertex of K, which is not one of its corners, is cubic in K as well as in G. Thus, it suffices to show that z_1 is not a corner of K_2.

Suppose instead that z_1 is a corner of K_2. As $z_1 \notin V(R)$, it is a free corner. Since z_1 is cubic, K_2 is an R-ladder configuration. Also, z_1 must be adjacent with a corner of K_2 which lies in $V(R)$; such a corner is either a_1 or b_1. Adjust notation so that z_1 is adjacent with a_1; thus, both edges of H incident with a_1 lie in $E(K_2)$. Note that a_1z_1 is an external rung of K_2. Also, since u_1 is not a corner of K_2, it is cubic in K_2 and in G. We infer that K_1 is also an R-ladder configuration; see Figure 6.24.

Let y_1 denote the neighbour of u_1 in K_1 which is distinct from a_1, and let v denote the third neighbour of u_1. Note that $y_1, v \in V(K_2)$. Since $|\partial(u_1) - E(K_1)| = 1$, Lemma 6.24*(i)* implies that v is distinct from w_1. Since K_2 is a ladder, a_1z_1 lies in a quadrilateral of K_2; this implies that $y_1z_1 \in E(K_2)$. Note that u_1y_1 is an internal rung of K_2.

Let y_2 denote the neighbour of y_1 which is distinct from u_1 and z_1. Note that $y_2 \in V(K_2)$. Since a_1z_1 and u_1y_1 are rungs of K_2, it must be the case that v and y_2 are adjacent and the

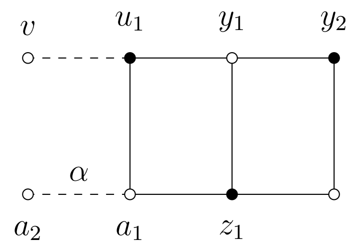


Figure 6.24: Illustration for Claim 6.26; the solid lines show part of the R-configuration K_1

edge joining them is a rung of K_2; however, it is easily seen that v and y_2 are nonadjacent. We thus have a contradiction. This completes the proof of Claim 6.26. □

We will now use Claim 6.26 to deduce that, since K_1 and K_2 are distinct R-configurations, the only vertices of K_1 which may lie in K_2 are its free corners (that is, u_1 and w_1).

Suppose instead that $(V(K_1) - \{u_1, w_1\}) \cap V(K_2)$ is nonempty. Since $K_1 - \{u_1, w_1\}$ is connected, Claim 6.26 implies that $V(K_1) \subseteq V(K_2)$ and $E(K_1) \subseteq E(K_2)$. Furthermore, since $|V(K_1) \cap V(K_2)| \geq 6$, the set $(V(K_2) - \{u_2, w_2\}) \cap V(K_1)$ is also nonempty, where u_2 and w_2 denote the free corners of K_2. By symmetry, $V(K_2) \subseteq V(K_1)$ and $E(K_2) \subseteq E(K_1)$. We conclude that K_1 and K_2 are identical subgraphs of G; contrary to our hypothesis.

Thus, each member of $V(K_1) \cap V(K_2)$ is a free corner of K_1, and by symmetry, it is a free corner of K_2 as well. By our hypothesis, $V(K_1) \cap V(K_2)$ is nonempty; thus, at least one of u_1 and w_1 is a free corner of K_2. Adjust notation so that u_1 is a free corner of K_2. To complete the proof, we will show that w_1 is also a free corner of K_2.

Suppose not, that is, say $V(K_1) \cap V(K_2) = \{u_1\}$, and let w_2 denote the free corner of K_2 distinct from u_1. Observe that the ends a_2 of α and b_2 of β both lie in $V(K_2)$; see Figure 6.25. Furthermore, $|B - V(K_1 \cup K_2)| = |A - V(K_1 \cup K_2)| + 1$. We shall let $T := B - V(K_1 \cup K_2)$. Since every vertex of $K_1 \cup K_2$, except possibly u_1, w_1 and w_2, is cubic, all neighbours of T lie in the set $S := (A - V(K_1 \cup K_2)) \cup \{w_1, w_2\}$. Consequently, S is a nontrivial barrier of G; this is absurd.

Thus, u_1 and w_1 are the free corners of K_2, and K_2 is otherwise vertex-disjoint with K_1. This completes the proof of Proposition 6.3. □

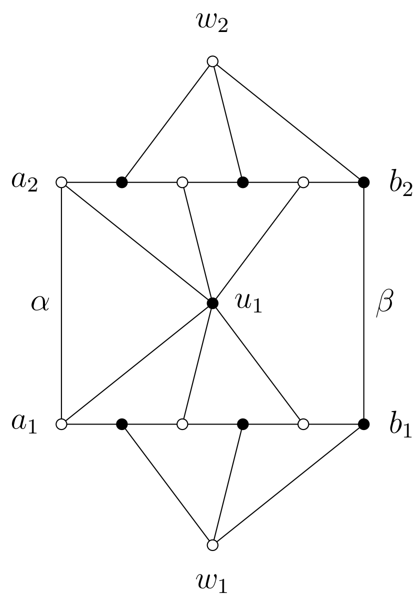


Figure 6.25: When K_1 and K_2 share only one free corner

6.4 Proof of the Strictly R-thin Edge Theorem

As in the statement of the theorem (1.24), let G be a simple R-brick which is free of strictly R-thin edges. Our goal is to show that G is a member of one of the eleven infinite families which appear in the statement of the theorem, that is, to show that $G \in \mathcal{N}$. We adopt Notation 6.1.

We may assume that G is different from K_4 and $\overline{C_6}$, and thus, by the R-thin Edge Theorem (1.22), G has an R-thin edge, say e_1. Depending on the index of e_1, we invoke either the R-biwheel Theorem (6.5) or the R-ladder Theorem (6.6) to deduce that G has an R-configuration, say K_1, such that $e_1 \in E(K_1)$. We shall let a_1u_1 and b_1w_1 denote the external rungs/spokes of K_1, and adjust notation so that u_1 and w_1 are its free corners. See Notation 6.2 and Figure 6.21. Note that a_1 is an end of α and b_1 is an end of β.

By Lemma 6.22, either both free corners u_1 and w_1 lie in $V(R)$, or otherwise, neither of them lies in $V(R)$; let us first deal with the former case.

Claim 6.27 *If $u_1, w_1 \in V(R)$ then G is either a prism, or a Möbius ladder or a truncated biwheel.*

Proof: Suppose that $u_1, w_1 \in V(R)$; that is, $\alpha = a_1w_1$ and $\beta = b_1u_1$. Since every vertex of K_1 is cubic in G, except possibly u_1 and w_1, we conclude that $V(G) = V(K_1)$ as otherwise $\{u_1, w_1\}$ is a 2-vertex cut of G. Furthermore, either G is precisely the graph induced by $E(K_1) \cup R$, or otherwise, G has one additional edge joining u_1 and w_1. In the latter case, u_1w_1 is a strictly R-thin edge, contrary to the hypothesis. In the former case, observe that: if K_1 is an R-biwheel configuration, as shown in Figure 6.21a, then G is a truncated biwheel; if K_1 is an R-ladder configuration of odd parity, as shown in Figure 6.21b, then G is a prism; and if K_1 is an R-ladder configuration of even parity, as shown in Figure 6.21c, then G is a Möbius ladder. □

We may thus assume that neither u_1 nor w_1 lies in $V(R)$. Consequently, the end a_2 of α and the end b_2 of β are both in $V(G) - V(K_1)$.

Claim 6.28 *Either G is a staircase or a pseudo-biwheel, or otherwise, G has an R-compatible edge which is not in $E(K_1)$.*

Proof: We begin by noting that, if $|\partial(u_1) - E(K_1)| \geq 2$, then by Lemma 6.23, some edge of $\partial(u_1) - E(K_1)$ is R-compatible, and we are done; an analogous argument applies when $|\partial(w_1) - E(K_1)| \geq 2$.

Now suppose that $|\partial(u_1) - E(K_1)| \leq 1$ and that $|\partial(w_1) - E(K_1)| \leq 1$. By Lemma 6.24 *(i)* and *(ii)*, u_1 and w_1 are nonadjacent; furthermore, $\partial(u_1) - E(K_1)$ has a single element, say α'; likewise, $\partial(w_1) - E(K_1)$ has a single element, say β'; see Figure 6.23. We let $R' := \{\alpha', \beta'\}$. By *(iii)* of the same lemma, α and α' are adjacent if and only if β and β' are adjacent.

First consider the case in which α and α' are nonadjacent, and as in the statement of Lemma 6.24*(iv)*, let v denote the end of α' which is distinct from u_1; note that $v \notin V(K_1)$. By the lemma, $\partial(v) - \alpha'$ contains an R-compatible edge, and we are done.

Now suppose that α and α' are adjacent; whence β and β' are also adjacent. Note that $\alpha' = u_1a_2$ and $\beta' = w_1b_2$. Every vertex of K_1, except possibly u_1 and w_1, is cubic in G; furthermore, $\partial(u_1) - E(K_1) = \{\alpha'\}$, and likewise, $\partial(w_1) - E(K_1) = \{\beta'\}$. We infer that $V(G) = V(K_1) \cup \{a_2, b_2\}$ as otherwise $\{a_2, b_2\}$ is a 2-vertex cut of G. Furthermore, since each of a_2 and b_2 has degree at least three, there is an edge joining them; and G

is precisely the graph induced by $E(K_1) \cup R \cup R' \cup \{a_2b_2\}$. Observe that if K_1 is an R-biwheel configuration of order at least eight then G is a pseudo-biwheel, and otherwise, G is a staircase. □

We may thus assume that G has an R-compatible edge which is not in $E(K_1)$. We will now use Theorem 5.1 and Lemma 6.25 to deduce that G has an R-thin edge which is not in $E(K_1)$.

Claim 6.29 *G has an R-thin edge, say e_2, which is not in $E(K_1)$.*

Proof: Among all R-compatible edges which are not in $E(K_1)$, we choose one, say e_2, such that $\mathsf{rank}(e_2) + \mathsf{index}(e_2)$ is maximum; we intend to show that e_2 is R-thin. Suppose not; then, by Theorem 5.1, with e_2 playing the role of e, there exists another R-compatible edge f such that (i) f has an end each of whose neighbours in $G - e_2$ lies in a (nontrivial) barrier S of $G - e_2$, and (ii) $\mathsf{rank}(f) + \mathsf{index}(f) > \mathsf{rank}(e_2) + \mathsf{index}(e_2)$.

Let I denote the set of isolated vertices of $(G - e_2) - S$. Condition (i) above implies that f has one end in I and another end in S. By Lemma 6.25, with e_2 playing the role of e, the set $I \cap V(K_1)$ is empty. Since f has one end in I, we infer that f is not in $E(K_1)$; this, combined with condition (ii) above, contradicts our choice of e_2. We thus conclude that e_2 is R-thin. □

Now, depending on the index of e_2, we invoke either the R-biwheel Theorem (6.5) or the R-ladder Theorem (6.6) to deduce that G has an R-configuration, say K_2, such that $e_2 \in E(K_2)$. As e_2 is not in $E(K_1)$ but it is in $E(K_2)$, the R-configurations K_1 and K_2 are clearly distinct. By Proposition 6.3: either K_1 and K_2 are vertex-disjoint; or otherwise, u_1 and w_1 are the free corners of K_2, and K_2 is otherwise vertex-disjoint with K_1. In either case, the end a_2 of α and the end b_2 of β are the two corners of K_2 which are distinct from its free corners. Let us first deal with the case in which K_1 and K_2 are not vertex-disjoint; Figure 6.26 shows an example in which K_1 and K_2 are both R-biwheel configurations.

The proof of the following claim closely resembles that of Claim 6.27.

Claim 6.30 *If K_1 and K_2 are not vertex-disjoint then G is either a double biwheel or a double ladder or a laddered biwheel, each of type I.*

Proof: As noted above, u_1 and w_1 are the free corners of K_2, and K_2 is otherwise vertex-disjoint with K_1. Consequently, the external rungs/spokes of K_2 are a_2u_1 and b_2w_1; see

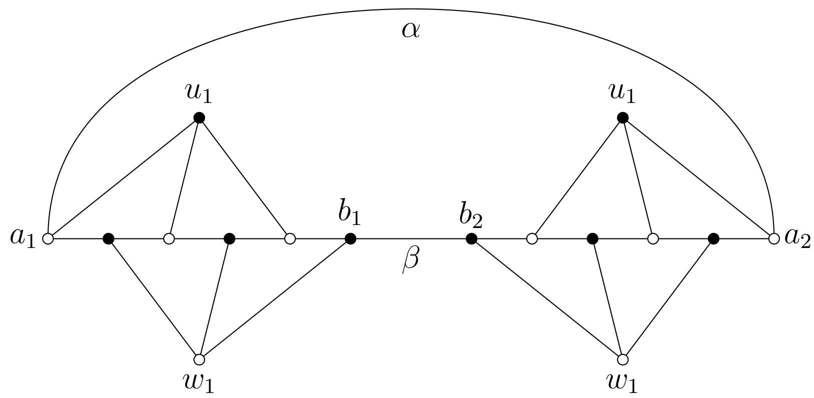


Figure 6.26: When the two R-configurations are not disjoint; the vertices with the same labels are to be identified

Figure 6.26. Since every vertex of $K_1 \cup K_2$ is cubic in G, except u_1 and w_1, we infer that $V(G) = V(K_1) \cup V(K_2)$, as otherwise $\{u_1, w_1\}$ is a 2-vertex cut of G. Furthermore, either G is precisely the graph induced by $E(K_1 \cup K_2) \cup R$, or otherwise, G has one additional edge joining u_1 and w_1. In the latter case, u_1w_1 is a strictly R-thin edge, contrary to the hypothesis. In the former case, observe that if K_1 and K_2 are both R-biwheel configurations then G is a double biwheel of type I; likewise, if K_1 and K_2 are both R-ladder configurations then G is a double ladder of type I; finally, if one of K_1 and K_2 is an R-ladder configuration and the other one is an R-biwheel configuration then G is a laddered biwheel of type I. □

We may thus assume that K_1 and K_2 are vertex-disjoint; and we shall let a_2u_2 and b_2w_2 denote the external rungs/spokes of K_2; in particular, u_2 and w_2 denote the free corners of K_2. Figure 6.27 shows an example in which K_1 is an R-ladder configuration and K_2 is an R-biwheel configuration.

We now find the remaining three families, or show the existence of an R-compatible edge which is not in $E(K_1 \cup K_2)$; the proof is similar to that of Claim 6.28.

Claim 6.31 *Either G is a double biwheel or a double ladder or a laddered biwheel, each of type II, or otherwise, G has an R-compatible edge which is not in $E(K_1 \cup K_2)$.*

<u>Proof</u>: We begin by noting that, if $|\partial(u_1) - E(K_1)| \geq 2$, then by Lemma 6.23, some edge of $\partial(u_1) - E(K_1)$ is R-compatible, and since $u_1 \notin V(K_2)$, such an edge is not in $E(K_2)$,

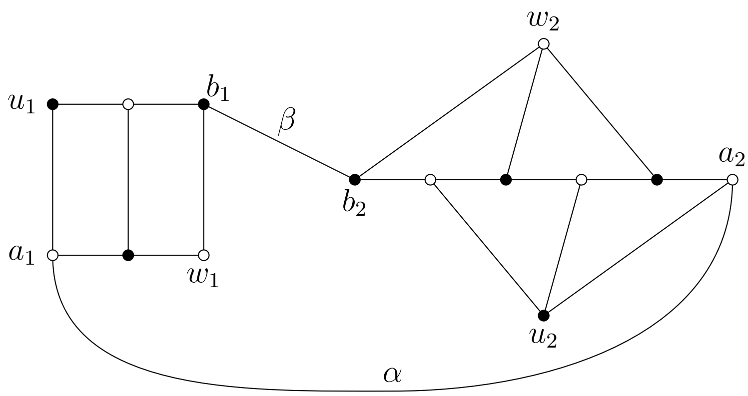


Figure 6.27: When the two R-configurations are disjoint

and we are done; an analogous argument applies when $|\partial(w_1) - E(K_1)| \geq 2$, or when $|\partial(u_2) - E(K_2)| \geq 2$ or when $|\partial(w_2) - E(K_2)| \geq 2$.

Now suppose that, for $i \in \{1, 2\}$, $|\partial(u_i) - E(K_i)| \leq 1$ and $|\partial(w_i) - E(K_i)| \leq 1$; by Lemma 6.24 *(i)* and *(ii)*, u_i and w_i are nonadjacent; furthermore, each of these inequalities holds with equality. Let α' denote the only member of $\partial(u_1) - E(K_1)$, and let β' denote the only member of $\partial(w_1) - E(K_1)$.

First consider the case in which either w_2 is not an end of α' or u_2 is not an end of β'. Assume without loss of generality that w_2 is not an end of α'; thus the end of α' distinct from u_1, say v, is not in $V(K_1 \cup K_2)$. By Lemma 6.24*(iv)*, $\partial(v) - \alpha'$ contains an R-compatible edge; such an edge is not in $E(K_1 \cup K_2)$, and we are done.

Now suppose that w_2 is an end of α' and u_2 is an end of β'. Note that every vertex of $K_1 \cup K_2$, except possibly u_1, w_1, u_2 and w_2, is cubic in G; furthermore, $\partial(u_1) - E(K_1) = \partial(w_2) - E(K_2) = \{\alpha'\}$, and likewise, $\partial(w_1) - E(K_1) = \partial(u_2) - E(K_2) = \{\beta'\}$. We conclude that $V(G) = V(K_1 \cup K_2)$ and $E(G) = E(K_1 \cup K_2) \cup R \cup R'$. Observe that: if K_1 and K_2 are both R-biwheel configurations then G is a double biwheel of type II; likewise, if K_1 and K_2 are both R-ladder configurations then G is a double ladder of type II; finally, if one of K_1 and K_2 is an R-ladder configuration and the other one is an R-biwheel configuration then G is a laddered biwheel of type II. □

We may thus assume that G has an R-compatible edge which is not in $E(K_1 \cup K_2)$. We will now use Theorem 5.1 and Lemma 6.25 to deduce that G has an R-thin edge which is not in $E(K_1 \cup K_2)$. The proof is almost identical to that of Claim 6.29, except that now we have to deal with two R-configurations instead of just one.

Claim 6.32 *G has an R-thin edge, say e_3, which is not in $E(K_1 \cup K_2)$.*

Proof: Among all R-compatible edges which are not in $E(K_1 \cup K_2)$, we choose one, say e_3, such that $\mathsf{rank}(e_3) + \mathsf{index}(e_3)$ is maximum; we intend to show that e_3 is R-thin. Suppose not; then, by Theorem 5.1, with e_3 playing the role of e, there exists another R-compatible edge f such that (i) f has an end each of whose neighbours in $G - e_3$ lies in a (nontrivial) barrier S of $G - e_3$, and (ii) $\mathsf{rank}(f) + \mathsf{index}(f) > \mathsf{rank}(e_3) + \mathsf{index}(e_3)$.

Let I denote the set of isolated vertices of $(G - e_3) - S$. Condition (i) above implies that f has one end in I and another end in S. By Lemma 6.25, with e_3 playing the role of e, the set $I \cap V(K_1)$ is empty; likewise, the set $I \cap V(K_2)$ is empty. Since f has one end in I, we infer that f is not in $E(K_1 \cup K_2)$; this, combined with condition (ii) above, contradicts our choice of e_3. We thus conclude that e_3 is R-thin. □

Now, depending on the index of e_3, we invoke either the R-biwheel Theorem (6.5) or the R-ladder Theorem (6.6) to deduce that G has an R-configuration, say K_3, such that $e_3 \in E(K_3)$. As e_3 is not in $E(K_1 \cup K_2)$ but it is in $E(K_3)$, the R-configuration K_3 is distinct from each of K_1 and K_2. We have thus located three distinct R-configurations in the brick G; namely, K_1, K_2 and K_3. However, this contradicts Corollary 6.4, and completes the proof of the Strictly R-thin Edge Theorem (1.24). □

Chapter 7

Conclusions

Lovász's Theorem (1.6) led Carvalho, Lucchesi and Murty [CLM03] to pose two problems: (i) determine whether or not a given matching covered graph G is K_4-free, and likewise, (ii) determine whether or not G is $\overline{C_6}$-free. In the first part of the thesis, we solved these problems for the special case of planar matching covered graphs.

At a high level, our solution may be viewed as comprising of two steps. In Chapter 2, for any cubic brick J, we reduced the problem of characterizing J-free matching covered graphs to that of characterizing J-free bricks; see Theorem 1.10. In Chapter 3, we characterized K_4-free as well as $\overline{C_6}$-free planar bricks; see Theorems 1.11 and 1.12.

The natural extension of our work is to solve the aforementioned problems (i) and (ii) for general matching covered graphs. In view of our results in Chapters 2 and 3, it suffices to characterize K_4-free nonplanar bricks and $\overline{C_6}$-free nonplanar bricks.

This is reminiscent of an important problem solved in the context of Pfaffian orientations. As per a theorem of Little [Lit75], a bipartite matching covered graph is Pfaffian if and only if it is $K_{3,3}$-free. Several years later, Robertson, Seymour and Thomas [RST99], and independently McCuaig [McC04], gave a structural characterization of $K_{3,3}$-free braces. Recently, in his Ph.D. thesis, Whalen [Wha14] gave a third proof of this theorem. It is worth exploring whether any of these three approaches can be adapted to characterize either K_4-free or $\overline{C_6}$-free nonplanar bricks.

In the second part of the thesis, we established generation theorems which are specific to near-bipartite bricks. In Chapter 5, we proved the R-thin Edge Theorem (1.22), which states that every R-brick distinct from K_4 and $\overline{C_6}$ has an R-thin edge. This is a refinement of the Thin Edge Theorem (1.15) of Carvalho, Lucchesi and Murty, which is appropriate for the restricted class of near-bipartite bricks.

In Chapter 6, we proved the Strictly R-thin Edge Theorem (1.24), which gives a complete characterization of those simple R-bricks which are free of strictly R-thin edges. This is a refinement of the Strictly Thin Edge Theorem (1.17) of Norine and Thomas, which is appropriate for the class of near-bipartite bricks.

It would be interesting to find applications of the R-thin Edge Theorem and the Strictly R-thin Edge Theorem. In particular, since the problems of characterizing K_4-free nonplanar bricks and $\overline{C_6}$-free nonplanar bricks do not seem to be tractable with the inductive tools available to us, it may be worthwhile studying these questions for the restricted class of near-bipartite bricks. This approach has been successful in the theory of Pfaffian orientations; although there has been no significant progress in characterizing Pfaffian bricks; Fischer and Little [FL01] were able to characterize Pfaffian near-bipartite graphs. Related to this, an easy corollary of the Strictly R-thin Edge Theorem is that every nonplanar $\overline{C_6}$-free brick is M_8-based, where M_8 is the Möbius ladder of order eight.

The notions of thin and strictly thin edges are easily generalized to braces; see [CLM08, CLM15]. It was shown by Carvalho, Lucchesi and Murty [CLM08] that every brace of order six or more has a thin edge; their result may also be derived from an earlier theorem of McCuaig [McC01] concerning the existence of strictly thin edges. In their recent work, Carvalho et al. [CLM15] strengthened McCuaig's result to show that every simple brace, which is not in any of four infinite families, has at least two strictly thin edges. They also give examples to show that their result is the best possible.

In view of the above, we would like to show the existence of at least two R-thin edges in every R-brick that is distinct from K_4, $\overline{C_6}$ and the staircase St_8; see Figure 1.9. Such a result is likely to be more useful when trying to solve problems pertaining to near-bipartite bricks using induction; especially, if one can show that there are two R-thin edges which are somewhat far apart.

More generally, it would be interesting to show the existence of at least two thin edges in every brick that is distinct from K_4, $\overline{C_6}$ and St_8. Related to this, we were able to find bricks which have a unique strictly thin edge; one of our examples appears in [CLM15]. We have also found R-bricks which have a unique strictly R-thin edge; in this sense, our Theorem 1.24 is the best possible.

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Glossary of Notation

$(A_0, A_1, \ldots, A_r)$ partition of set, page 73

$\mathcal{F}(e, S)$ candidate set relative to edge and barrier, page 87

$\mathcal{N}$ union of all families which appear in statement of Theorem 1.24, page 31

$\overline{X}$ complement of vertex set (in graph), page 8

$\partial(v)$ cut associated with vertex (of graph), page 74

$\partial(X)$ cut associated with vertex set (of graph), page 8

$\partial_G(v)$ cut associated with vertex of graph, page 74

$\partial_G(X)$ cut associated with vertex set of graph, page 8

$b(G)$ number of bricks in any tight cut decomposition of graph, page 9

$G/(X \to x)$ graph obtained by contracting shore to single vertex, page 8

G/X graph obtained by contracting shore (to single vertex), page 8

$G[e]$ subgraph induced by edge, page 5

$G[X]$ subgraph induced by vertex set, page 94

$H(e, S)$ bipartite matching covered graph defined relative to edge and barrier, page 87

$H[A, B]$ bipartite graph with color classes A and B, page 3

M_n Möbius ladder of order n, page 13

$N(S)$ neighbourhood of set of vertices (in graph), page 3

$N(v)$ neighbourhood of vertex (in graph), page 38

$N_G(S)$ neighbourhood of set of vertices in graph, page 3

$N_G(v)$ neighbourhood of vertex in graph, page 38

P_n prism of order n, page 13

P_{uv} path with ends specified, page 36

$R = \{\alpha, \beta\}$ two edges which constitute a removable doubleton, page 22

St_n staircase of order n, page 14

T_n truncated biwheel of order n, page 13

$V(R)$ set of four vertices which are ends of edges of removable doubleton, page 114

W_n odd wheel of order n, page 11

X_+ majority part of vertex set (of bipartite or near-bipartite graph), page 76

X_- minority part of vertex set (of bipartite or near-bipartite graph), page 76

$\mathsf{f}_{\mathsf{odd}}(G)$ number of odd faces of plane graph, page 44

$\mathsf{odd}(G)$ number of odd components of a graph, page 1

$\mathsf{rank}(e)$ rank of R-compatible edge, page 81

$\mathsf{rank}(G)$ rank of near-bipartite graph, page 79

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