On the Solution of the Hamilton-Jacobi Equation by the Method of Separation of Variables

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The method of separation of variables facilitates the integration of the Hamilton-Jacobi equation by reducing its solution to a series of quadratures in the separable coordinates. The case in which the metric tensor is diagonal in the separable coordinates, that is, orthogonal separability, is fundamental. Recent theory by Benenti has established a concise geometric (coordinate-independent) characterisation of orthogonal separability of the Hamilton-Jacobi equation on a pseudoRiemannian manifold. It generalises an approach initiated by Eisenhart and developed by Kalnins and Miller. Benenti has shown that the orthogonal separability of a system via a point transformation is equivalent to the existence of a Killing tensor with real simple eigenvalues and orthogonally integrable eigenvectors. Applying a moving frame formalism, we develop a method that produces the orthogonal separable coordinates for low dimensional Hamiltonian systems. The method is applied to a two dimensional Riemannian manifold of arbitrary curvature. As an illustration, we investigate Euclidean 2-space, and the two dimensional surfaces of constant curvature, recovering known results. Using our formalism, we also derive the known superseparable potentials for Euclidean 2-space. Some of the original results presented in this thesis were announced in [8, 9, 10].

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Notation and Conventions

Summation notation (index appearing once up and once down):

$$a^i b_i := \sum_{i=1}^{n} a^i b_i$$

Tensor indices in the natural basis: i, j, $\ldots$, z; range: $1, \ldots, n$.

Tensor indices in the moving frame: a, b, $\ldots$, h; range: $1, \ldots, n$.

Set of permutations of r elements: S_r

Symmetrisation of indices, $(i_1 \ldots i_m)$:

$$T_{(i_1 \ldots i_m)} := \frac{1}{m!} \sum_{\sigma \in S_m} T_{i_{\sigma(1)} \ldots i_{\sigma(m)}}$$

Sign of a permutation, $\sigma \in S_r$:

$$\mathrm{sgn}(\sigma) = \begin{cases} 1, & \text{if even} \\ -1, & \text{if odd} \end{cases}$$

Skew-symmetrisation of indices, $[i_1 \ldots i_m]$:

$$T_{[i_1 \ldots i_m]} := \frac{1}{m!} \sum_{\sigma \in S_m} \mathrm{sgn}\ (\sigma) T_{i_{\sigma(1)} \ldots i_{\sigma(m)}}$$

Exclusion of j^{th} index from (skew-)symmetrisation: $|i_j|$

Partial differentiation with respect to position coordinate, q^i: $_{,i}$ or ∂_i

Partial differentiation with respect to momentum coordinate, p_i: ∂^i

(Partial) differentiation with respect to time, t: $\dot{f} := \frac{df}{dt}$ or $\frac{\partial f}{\partial t}$

Metric tensor: $\mathbf{g}(\mathbf{q})$

Metric tensor determinant: $g := \det(g_{ij})_{n\times n}$

Metric: $ds^2 = g_{ij}dq^i dq^j$

Frame vector fields: $E_a = h_a{}^i \frac{\partial}{\partial q^i}$

ω-frame 1-forms: $E^a = h^a{}_i dq^i$

Bracket of two vector fields: $[X, Y] := XY - YX$

Bracket of 1-form, $f = f_i X^i$ and vector field, $Y = Y^i X_i$: $\langle f, Y \rangle := Y^i f_i$

Components of object of anholonomy: $C^k{}_{ij} E_k := [E_i, E_j]$

Components of connection coefficients:

$$\Gamma_{bca} := \frac{1}{2}(E_b g_{ac} - E_a g_{cb} + E_c g_{ba}) - \frac{1}{2}(C_{cba} - C_{bac} + C_{acb})$$

Christoffel symbols of the first kind: $[ij, k] := \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$

Christoffel symbols of the second kind: $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} := g^{il}[jk, l]$

Connection 1-form: $\omega^i{}_j := \Gamma_{kj}{}^i E^k$

Covariant derivative:

$$\begin{aligned} T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s ; c} := \quad & E_c T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} + T^{d_1 \ldots a_r}{}_{b_1 \ldots b_s} \Gamma_{cd_1}{}^{a_1} + \ldots + T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} \Gamma_{cd_r}{}^{a_r} \\ & - T^{a_1 \ldots a_r}{}_{d_{r+1} \ldots b_s} \Gamma_{cb_1}{}^{d_{r+1}} - \ldots - T^{a_1 \ldots a_r}{}_{b_1 \ldots d_{r+s}} \Gamma_{cb_s}{}^{d_{r+s}} \end{aligned}$$

Lie derivative: $\mathcal{L}_X(Y)(x_0) = \left[\frac{d}{dt}(F^*_{-t}Y)\right]_{t=0}$
where F_t is the one parameter group of diffeomorphisms generated by X.

Exterior derivative:

$$d(\omega_{i_1 \ldots i_r} dx^1 \wedge \ldots \wedge dx^r) := \omega_{[i_2 \ldots i_{r+1}, i_1]} dx^1 \wedge \ldots \wedge dx^{r+1}$$

Torsion tensor: $T^i{}_{jk} := \Gamma_{jk}{}^i - \Gamma_{kj}{}^i - C^i{}_{jk}$

Riemann curvature tensor:

$$R^i{}_{jkl} := 2E_{[k}\Gamma_{l]j}{}^i + 2\Gamma_{[l|j|}{}^m \Gamma_{k]m}{}^i - C^m{}_{kl}\Gamma_{mj}{}^i$$

Poisson bracket:

$$\{f, g\} := \sum_{k=1}^{n} \left(\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q^k} - \frac{\partial f}{\partial q^k} \frac{\partial g}{\partial p_k} \right) = \partial^k f \partial_k g - \partial_k f \partial^k g.$$

Schouten bracket:

$$
\begin{aligned}
[P,Q]_{\mathcal{S}}^{i_1\dots i_{p+q-1}} := \;& \Big(\sum_{k=1}^{p} P^{(i_1\dots i_{(k-1)}|\mu|i_k\dots i_{(p-1)}}\Big)\partial_\mu Q^{i_p\dots i_{(p+q-1)})} + \\
& \Big(\sum_{k=1}^{p} (-1)^k P^{[i_1\dots i_{(k-1)}|\mu|i_k\dots i_{(p-1)}}\Big)\partial_\mu Q^{i_p\dots i_{(p+q-1)}]} - \\
& \Big(\sum_{l=1}^{q} Q^{(i_1\dots i_{(l-1)}|\mu|i_l\dots i_{(q-1)}}\Big)\partial_\mu Q^{i_q\dots i_{(p+q-1)})} - \\
& \Big(\sum_{l=1}^{q} (-1)^{(pq+p+q+l)} Q^{[i_1\dots i_{(l-1)}|\mu|i_l\dots i_{(q-1)}}\Big)\partial_\mu Q^{i_q\dots i_{(p+q-1)}]}.
\end{aligned}
$$

Canonical symplectic structure: $\omega_0 := dp_i \wedge dq^i$

Canonical Poisson bi-vector: $P_0 := \dfrac{\partial}{\partial q^i} \wedge \dfrac{\partial}{\partial p_i}$

Chapter 1

Introduction

A problem in mechanics can be mathematically represented in many different forms, each having a variety of solution methods. We are interested in the Hamilton-Jacobi formalism. In this approach, a first order partial differential equation, the Hamilton-Jacobi equation, is the key mathematical object. A powerful solution method of the Hamilton-Jacobi formalism is separation of variables.

In this chapter, we introduce the Hamilton-Jacobi formalism and the method of separation of variables. We briefly discuss the major contributions to separation of variables by Liouville, Stäckel, Levi-Civita, and Eisenhart to extend the basic theory and provide a historical context.

1.1 Hamilton-Jacobi Formalism

In mechanics, the number of *degrees of freedom* for a physical system, say n, is the minimum number of independent quantities that uniquely determines its position at all times. It is often the case that a problem is simplified if we utilise quantities different from the standard Cartesian coordinates. In fact, any set of n *generalised position coordinates*, $\mathbf{q} = (q^1, \dots, q^n)$, that completely describes the motion of the system is adequate. Naturally, the derivatives $\dot{\mathbf{q}}$ are called the *generalised velocities*. Let M be an n dimensional pseudoRiemannian manifold with local coordinates (q^i).

The tangent bundle, TM, has canonical coordinates $(q^i, \dot{q}^i)$, where $i = 1, \ldots, n$.

The basis of the Hamilton-Jacobi formalism in mechanics is the *Hamilton-Jacobi (HJ) equation*,

$$H(p_1, \ldots, p_n, q^1, \ldots, q^n) = E, \tag{1.1}$$

in which H is the *time-independent Hamiltonian (function)*. The variables, (p_i), are the *generalised momenta* defined in terms of the *time-independent Lagrangian*, $L = T - V$ (T and V represent the kinetic and potential energies of the system, respectively), by

$$p_i := \frac{\partial L}{\partial \dot{q}^i}. \tag{1.2}$$

We study only time-independent Hamiltonians in this thesis. A *complete integral* of the HJ equation is a solution of (1.1), $W(\mathbf{q}, \boldsymbol{\alpha})$, depending on the n separation constants, $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$, that satisfies

$$\det \left[\frac{\partial^2 W}{\partial q^i \partial \alpha_j} \right] \neq 0. \tag{1.3}$$

The momenta satisfy the relationship

$$p_i = \frac{\partial W}{\partial q^i}; \tag{1.4}$$

hence, the HJ equation (1.1) may be written as

$$H(q^1, \ldots, q^n, W_{,1}, \ldots, W_{,n}) = E. \tag{1.5}$$

We assume that the Hamiltonian is quadratic in the momenta, (p_i), so that the Hamiltonian of (1.1) and HJ equation (1.5) have the forms

$$H = \frac{1}{2} g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}), \tag{1.6}$$

and

$$H = \frac{1}{2} g^{ij}(\mathbf{q}) W_{,i} W_{,j} + V(\mathbf{q}) = E, \tag{1.7}$$

respectively, where g^{ij} are the contravariant components of the symmetric metric tensor, $\mathbf{g}$, and V is a smooth function.

1.2 Separation of Variables

A Hamiltonian is said to be *(additively) separable* provided the HJ equation (1.7) has a complete integral of the form

$$W(\mathbf{q}, \boldsymbol{\alpha}) = -Et + W_1(q^1, \boldsymbol{\alpha}) + \ldots + W_n(q^n, \boldsymbol{\alpha}). \tag{1.8}$$

The coordinates (q^i) are said to be *separable* with respect to the Hamiltonian and the potential is *compatible with the separable coordinates*, or simply *compatible* or *separable*. If, furthermore, the metric is diagonal, that is,

$$g^{ij} = 0,\ i \neq j, \tag{1.9}$$

then the Hamiltonian system, for which

$$H = \frac{1}{2} g^{ii}(\mathbf{q}) p_i^2 + V(\mathbf{q}), \tag{1.10}$$

is said to be *orthogonally separable.*

In physics, there is also the notion of *product*, or *multiplicative*, separability for the Helmholtz equation with a potential function $U(\mathbf{q})$,

$$\Delta_n \psi + U\psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^i} \left(\sqrt{g} g^{ij} \frac{\partial \psi}{\partial q^j} \right) + U\psi = \lambda \psi,$$

in which the solution has the form $\psi = \prod_i \psi_i(q^i; \boldsymbol{\alpha})$. Product separation of the Helmholtz equation is not investigated in this thesis; however, there is a close relation between it and the additive separation of the HJ equation.

A comprehensive theory of separation of variables must be able to intrinsically characterise (that is, in a coordinate-free manner) separability, and determine the distinct separable coordinate systems. In this thesis, we discuss recent results, by Benenti and others, in the geometric characterisation of separability. We apply a moving frame method for the determination of separable coordinates that extends the work initiated by Eisenhart [14], and developed by others.

First, it is natural to investigate the major results in the field of variable separation.

1.3 Historical Outline

The method of separation of variables has been studied extensively since the middle of the nineteenth century. Other methods have been developed in the second half of this century, some of which we discuss in Chapter 2.

1.3.1 Liouville

Liouville [27] was the first person to study the separability of the Hamiltonian with distinct kinetic and potential energy components,

$$H = T + V = E. \tag{1.11}$$

Given $3n$ functions a_i, c_i, and V_i that are dependent on only the corresponding coordinate, q^i, we define *Liouville systems* (investigated by Liouville in 1849 [28]) as those with a Hamiltonian (1.11) such that

$$T = \frac{c}{2}\sum_{i=1}^{n}\frac{(\dot{q}^i)^2}{a_i} = \frac{1}{2c}\sum_{i=1}^{n}a_i p_i^2,$$

$$V = \frac{1}{c}\sum_{i=1}^{n}V_i,$$

where

$$c = \sum_{i=1}^{n} c_i; \tag{1.12}$$

thus, the HJ equation (1.11) for a Liouville system is

$$H = \frac{1}{2c}\sum_{i=1}^{n}(a_i p_i^2 + 2V_i) = E. \tag{1.13}$$

For a Liouville system, we construct $n-1$ first integrals,

$$I_i = \frac{1}{2} a_i p_i^2 + V_i - c_i H, \tag{1.14}$$

that is, quantities that satisfy $\{I_i, H\} = 0,\ i = 1, \ldots, n$. We derive the fact that only $(n-1)$ of the first integrals are independent by observing that

$$\begin{aligned} \sum_{i=1}^{n} I_i \\ &= cH - \sum_{i=1}^{n} (c_i) H \\ &= 0, \end{aligned}$$

using (1.12) and (1.13). If, in addition, the first integrals satisfy $\{I_i, I_j\} = 0,\ i, j = 1, \ldots, n-1,\ i \neq j$, then $\{I_1, \ldots, I_{n-1}, H\}$ is called an *involutive set*, or is said to be *in involution*. The n first integrals (1.14) including the Hamiltonian, H, (1.13) form an involutive set. Liouville [29] proved a theorem connecting the existence of first integrals with separability.

Theorem 1.1 (Liouville) *A Hamiltonian system with n degrees of freedom that possesses n independent first integrals in involution is integrable by quadratures.*

By the Liouville Theorem, the following theorem is established

Theorem 1.2 *The complete integral for any Liouville system* (1.13) *can be determined (in theory) by the method of separation of variables.*

Using the coordinate transformation $\tilde{q}^i = \int \sqrt{a_i(q^i)}dq^i,\ i = 1, \ldots, n$, we eliminate the a_j dependence from the kinetic energy; therefore, using (1.4), we transform the HJ equation (1.13) into the form

$$H = \frac{1}{2c}\sum_{i=1}^{n}[(W_{,i})^2 + 2V_i] = E.$$

Comparing this form of the HJ equation with the general form (1.7), we conclude that the metric is diagonal (1.9) with non-zero contravariant components $g^{ii} = 1/c$, for $i = 1, \ldots, n$. It follows from the fact that the matrices corresponding to the contravariant and covariant forms of the metric must be inverses that the non-zero covariant components are

$$g_{ii} = \sum_{j=1}^{n} c_j, \ i = 1, \ldots, n.$$

The general form of the line element, also referred to as the *metric*, is

$$ds^2 = g_{ij}dq^i dq^j. \tag{1.15}$$

For Liouville systems, the metric is said to be in *Liouville form*,

$$ds^2 = [c_1(q^1) + \ldots + c_n(q^n)][(dq^1)^2 + \ldots + (dq^n)^2]. \tag{1.16}$$

Morera [33] showed that on a two dimensional Riemannian space of arbitrary curvature, the converse to Theorem 1.2 holds locally.

Theorem 1.3 (Morera) *On a two dimensional Riemannian manifold, any separable metric can be written locally in Liouville form* (1.16).

1.3.2 Stäckel

Stäckel's first major contribution [40] was to find all the separable metrics for an arbitrary two dimensional Riemannian manifold, M_2:

$$\begin{aligned} &I \quad && ds^2 = [c_1(q^1) + c_2(q^2)][(dq^1)^2 + (dq^2)^2], \\ &II \quad && ds^2 = g_{11}(q^1)(dq^1)^2 + 2g_{12}(q^1)dq^1dq^2 + g_{22}(q^1)(dq^2)^2, \\ &III \quad && ds^2 = (dq^1)^2 - 2\cos[c_1(q^1) + c_2(q^2)]dq^1dq^2 + (dq^2)^2. \end{aligned} \tag{1.17}$$

We observe that the type I metrics are in Liouville form. By Theorem 1.3, we know that the other two metrics must be equivalent to some metric in Liouville form. In case II, we define new coordinates $(\tilde{q}^1, \tilde{q}^2)$ by

$$\tilde{q}^1 = \int \frac{\sqrt{g}}{g_{22}} dq^1,$$

$$\tilde{q}^2 = q^2 + \int \frac{g_{12}}{g_{22}} dq^1.$$

In these coordinates, we write type II metrics in the Liouville form

$$ds^2 = g_{22}(q^1(\tilde{q}^1))[(d\tilde{q}^1)^2 + (d\tilde{q}^2)^2].$$

Similarly, by using the transformation to Cartesian coordinates,

$$x = \int \cos(c_1) dq^1 - \int \cos(c_2) dq^2,$$

$$y = \int \sin(c_1) dq^1 + \int \sin(c_2) dq^2,$$

we transform metrics of type III into the Liouville form

$$ds^2 = dx^2 + dy^2.$$

For dimensions greater than two, there are separable systems that are not equivalent to a Liouville system. One such family of systems is that of the Stäckel systems, see [41, 42, 43]. They have Hamiltonians of the form

$$H = \sum_{i=1}^{n} a_i(\mathbf{q}) \Big[\frac{1}{2} p_i^2 + V_i(q^i) \Big]. \tag{1.18}$$

Stäckel proved the following theorem connecting the integrability of a Stäckel system with the existence of a matrix, S, called a Stäckel matrix for the system.

Theorem 1.4 (Stäckel) *A dynamical system with a Hamiltonian of the form* (1.18) *is separable if and only if there exists an $n \times n$ matrix S, with elements $s_{ij} = s_{ij}(q^j)$, such that its determinant does not vanish, and*

$$\sum_{j=1}^{n} s_{ij}(q^j) a_j(\mathbf{q}) = \delta_{i1}. \tag{1.19}$$

If we denote the inverse matrix of S by $A = (a_{ij})$, then the relations

$$a_i = a_{i1},\ i = 1, \dots, n$$

follow immediately from (1.19). The n independent integrals in involution for this system are constructed as

$$I_j = \sum_{i=1}^{n} a_{ij}(\mathbf{q}) \Big[\frac{1}{2} p_i^2 + V_i(q^i) \Big],$$

where $I_1 = H$.

1.3.3 Levi-Civita

The separation of the general HJ equation (1.1) was investigated by Levi-Civita at the beginning of the twentieth century, see [26]. He produced a separability test for a *specific* coordinate system. This is the famous Levi-Civita criterion.

Theorem 1.5 (Levi-Civita) *The HJ equation* (1.1) *separates in a specific set of coordinates,* (q^i)*, if and only if the Hamiltonian satisfies the* $n(n-1)/2$ *equations*

$$\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial q^i \partial q^j} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^j}\frac{\partial^2 H}{\partial q^i \partial p_j} - \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_i \partial q^j} + \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial q^j}\frac{\partial^2 H}{\partial p_i \partial p_j} = 0, \tag{1.20}$$

$$1 \leq i < j \leq n, \; i \neq j,$$

where there is no summation *over the indices.*

Theorem 1.5 provides a straightforward test for separability; however, since it is only a *local* characterisation, it does not, in general, aid in the determination of separable coordinates.

The separability of a Hamiltonian with distinct kinetic and potential energy components (1.11) where V is non-zero requires that the *geodesic* HJ equation,

$$G := \frac{1}{2} g^{ij} p_i p_j = E, \tag{1.21}$$

for which the potential energy *is* zero, separates. Using the notation (1.21), the Hamiltonian (1.6) is represented as $H = G + V$.

To analyse a Riemannian manifold, Levi-Civita introduced a classification system for coordinates. If $\partial_i H$ is divisible by $\partial^i H$, then the coordinate q^i is said to be *first class*. Otherwise, q^i is said to be a *second class* coordinate. We observe that, for a geodesic Hamiltonian (1.21), $\partial_i H$ is quadratic in momenta and $\partial^i H$ is linear in momenta.

Using this taxonomy, Levi-Civita recovered Stäckel's separable metrics (1.17) for two dimensional spaces. He also showed that if all separable coordinates are first class, then the space is necessarily flat, that is, Euclidean.

1.3.4 Eisenhart

Eisenhart was the first mathematician to provide a geometrical characterisation of separation of variables, see [14].

Theorem 1.6 (Eisenhart) *The geodesic Hamiltonian* (1.21) *is orthogonally separable in some coordinate system,* (q^i)*, if and only if the following conditions are satisfied:*

1. *There are* $(n-1)$ *linearly independent quadratic first integrals,*

$$I_a = \mathcal{A}_a^{ij} p_i p_j,$$

that form an involutive set with the Hamiltonian, H*.*

2. *The eigenvalues of* $\mathcal{A}_a^{ij}$*,* λ_a^i *are all distinct, said to be* simple*, and satisfy the determinant equation*

$$\det[\lambda_i^\alpha - \lambda_j^\alpha] \neq 0,$$

for i *fixed and* $\alpha = 2, \ldots, n$*,* $j = 1, \ldots, n$*,* $j \neq i$*.*

3. *The eigenvectors,* $\{E_a\}$*, corresponding to the eigenvalues* $\{\lambda_a^i\}$ *are normal, a concept discussed in Appendix A.*

By taking the hypersurfaces orthogonal to each of the vector fields, E_a, as the coordinate hypersurfaces, we define a separable coordinate system. In these coordinates, the metric component matrix, g^{ij}, and first integral component matrices, $\mathcal{A}^{ij}$, can be simultaneously diagonalised. This follows from the fact that the vector fields are normal.

In Chapter 3, we discuss recent work by Benenti that simplifies and generalises Theorem 1.6. The groundwork of Eisenhart motivates our application of the method of moving frames to the Benenti intrinsic characterisation theorem of orthogonal separability.

Chapter 2

Hamilton-Jacobi Theory

In this chapter, we present the mathematical basis required for the study of mechanics. In agreement with the chronology of their development, we introduce first Lagrange's and then Hamilton's equations. Next we discuss the key tools for the present discussion: first integrals, the Poisson bracket and symplectic structures, and Killing vectors and tensors. This thesis focuses on the Hamilton-Jacobi formalism of mechanics which we proceed to develop including the important results for the solution of the Hamilton-Jacobi equation, the Jacobi and Arnol'd-Liouville theorems. One of the primary solution methods for the Hamilton-Jacobi equation is the method of separation of variables. It is described and applied to the two dimensional harmonic oscillator. Finally, two other algorithms, the Lax and bi-Hamiltonian methods, are introduced, then applied to the non-periodic, finite dimensional Toda lattice.

2.1 Lagrangian and Hamiltonian Formalisms

Hamilton-Jacobi theory is an extension of earlier formalisms of mechanics. We describe the Lagrangian formalism based on the Lagrangian and Lagrange's equations. Then we discuss the development of the Hamiltonian and Hamilton's functions, in the Hamiltonian formalism, from the Lagrangian. Since transformations to separable coordinate systems are key to the study of separation of variables, we discuss the transformations, point and canonical, permitted in mechanics.

2.1.1 Lagrange's Equations

The formulation of Lagrange's equations begins with the concept of a functional, any function of some class of curves. Following [24], we consider a system that occupies positions $\mathbf{q}(t_0)$ and $\mathbf{q}(t_1)$ at times t_0 and t_1, respectively. The fundamental functional in the Lagrangian formalism is the *action*,

$$S = \int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}) dt. \tag{2.1}$$

The function L is called the *Lagrangian* of the system. In this thesis, we investigate only time-independent Lagrangians. An important subset of these systems are *closed* systems, for which the system's particles experience no external forces.

With respect to the Lagrangian, the *generalised momenta* of the system are defined by (1.2) and the *generalised forces* are defined by

$$\dot{p}_i = \partial_i L. \tag{2.2}$$

Hamilton's principal of least action states that the motion of a mechanical system coincides with an extremal of the action and, in the case of a sufficiently short segment of the path, a minimum thereof. It is interesting to note that, because L is a function of only position, and velocity, Hamilton's principal implies that *Newton's principal of determinacy* holds: The motion of a system for all times is completely determined by specifying both the position and velocity vectors at some time, t_0. This is certainly not an intuitive result; however, it follows trivially, from our mathematical formulation of mechanics, that there exists a unique solution to a set of second order ordinary differential equations given the aforementioned initial conditions.

The requirement that S be minimised implies that the first variation of the integral (2.1) vanishes,

$$\delta S = \delta \int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}) dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \Big(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \Big) dt = 0.$$

Integrating the second term by parts, we obtain

$$\Big[\frac{\partial L}{\partial \dot{q}} \delta q \Big] \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \Big(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \Big) \delta q dt = 0. \tag{2.3}$$

At the endpoints the variation of q is zero, that is, $\delta q(t_0) = \delta q(t_1) = 0$; therefore, the integrand of the second term must vanish; thus, for a system with n degrees of freedom, (2.3) implies that the n second order differential equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0 \tag{2.4}$$

are satisfied. The equations of motion (2.4) for the system are called *Lagrange's equations.* Given $2n$ constants, say the positions and velocities at t_0, we may, in theory, determine the trajectories.

2.1.2 Hamilton's Equations

The equations of motion need not be formulated in terms of positions and velocities. Another natural perspective depends on the positions and momenta. We consider an n dimensional manifold, M, in conjunction with its cotangent bundle, T^*M. We represent the local coordinates of M by (q^i), and the corresponding canonical coordinates of T^*M by (q^i, p_i), where $i = 1, \ldots, n$.

The change of independent variables from $(\dot{q}^i)$ to (p_i) may be effected by a Legendre transformation. The Hamilton's function, or *Hamiltonian*, is the Legendre transform of the Lagrangian with respect to the variable $\dot{\mathbf{q}}$ [24],

$$H(\mathbf{p}, \mathbf{q}) = \mathbf{p}\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}). \tag{2.5}$$

The Legendre transform is involutive. That is, the square of the transformation is the identity; thus, by taking the transform of the Hamiltonian, we return to the Lagrangian of the system.

To derive Hamilton's equations, we consider, as in [24], the total differential of the Hamiltonian (2.5),

$$dH = p_i d\dot{q}^i + \dot{q}^i dp_i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i.$$

Using the definitions of the generalised momenta and forces, (1.2) and (2.2), we obtain

$$dH = -\dot{p}_i dq^i + \dot{q}^i dp_i.$$

This leads directly to *Hamilton's equations* for the system,

$$\dot{q}^i = \partial^i H, \quad \dot{p}_i = -\partial_i H. \tag{2.6}$$

We observe that this has transformed the problem from n second order differential equations in n coordinate functions, $\mathbf{q}$, to a system of $2n$ first order differential equations in the $2n$ momenta and coordinate functions, $\mathbf{p}$ and $\mathbf{q}$, respectively. These equations are also called the *canonical equations* because they treat the variables $\mathbf{p}$ and $\mathbf{q}$ symmetrically.

For a mechanical system, we geometrically represent the state of the system using its *phase space*. Consistent with the Hamiltonian formalism, it is a $2n$ dimensional space in which each of the n generalised coordinates, q^i, and n generalised momenta, p_i, plays the role of an independent variable. Each point in the phase space corresponds to a unique system state. As the system evolves in time, a curve in the phase space, called the *phase path* or *phase flow*, is constructed.

2.1.3 Point and Canonical Transformations

As mentioned in Section 1.1, problems may be described using any coordinate system that uniquely determines the state of the system for all times t. The formulation of Lagrange's equations (2.4) is not dependent on any preferred coordinate system; hence, they are invariant under a transformation of the coordinates $(q^1, \ldots, q^n) \to (\tilde{q}^1, \ldots, \tilde{q}^n)$ using a point transformation, $\tilde{q}^i = \tilde{q}^i(\mathbf{q})$. It follows that Hamilton's equations are also invariant.

In the Hamiltonian formalism, because the momenta are treated as variables independent of the coordinates, we may consider transformations of the form

$$\tilde{q}^i = \tilde{q}^i(\mathbf{p}, \mathbf{q}, t), \quad \tilde{p}_i = \tilde{p}_i(\mathbf{p}, \mathbf{q}, t). \tag{2.7}$$

Not all coordinate transformations of this form are desired. We consider only the transformations (2.7) under which Hamilton's equations retain their canonical form,

$$\dot{\tilde{q}}^i = \frac{\partial H}{\partial \tilde{p}_i}, \quad \dot{\tilde{p}}_i = -\frac{\partial H}{\partial \tilde{q}^i}.$$

Such transformations are said to be *canonical*. It has been shown [24] that sufficient conditions for a transformation to be canonical are

$$\{\tilde{q}^i, \tilde{q}^k\} = 0, \quad \{\tilde{p}_i, \tilde{p}_k\} = 0, \quad \{\tilde{p}_i, \tilde{q}^k\} = \delta_i^k. \tag{2.8}$$

The greater diversity in allowable transformations for Hamilton's equations is an advantage because problems may be more readily transformed to a position-momenta coordinate system in which they can be solved; nonetheless, in this thesis, we utilise *only point transformations*,

$$\tilde{q}^i = \tilde{q}^i(\mathbf{q}), \qquad \tilde{p}_i = \frac{\partial q^k}{\partial \tilde{q}^i} p_k. \tag{2.9}$$

2.2 Mathematical Machinery

The study of Hamiltonian mechanics requires special mathematical tools. As mentioned in the Liouville theorem, Theorem 1.1, first integrals are important quantities for separation. The Poisson bracket is key to our mathematical definition of first integrals, and other concepts, such as canonical transformations. The bracket connects naturally with symplectic structures, a fundamental part of the bi-Hamiltonian method. We then introduce Killing tensors and show their significance to mechanics.

2.2.1 First Integrals

An important concept in the solution of Hamilton's (and Lagrange's) equations is that of the first integral of the motion. A *first integral* is a function of the variables fixed by initial conditions, that is, that remains constant along any integral curve.

If the Hamiltonian is independent of a coordinate, say q^i, (and thus also the Lagrangian) then the coordinate is called *ignorable*, or *cyclic*. It is obvious, from (2.6), that if some q^i, say q^1, is ignorable, then the corresponding momentum is constant, $p_1 = c_1$; thus, the momentum is a first integral. Also the coordinate function may be written as $q^1 = c_1 t + d_1$ for constants c_1 and d_1; furthermore, the order of the problem is reduced to $2(n-1)$ because the Hamiltonian may be written [2] as

$$H(c_1, p_2, \dots, p_n, d_1, q^2, \dots, q^n).$$

The most important first integral for a closed system is the energy. We derive this result using the Hamiltonian formalism. By design, it is trivial. Following [24] we begin with the total time derivative of the Hamiltonian,

$$\frac{dH}{dt} = \partial_i H \dot{q}^i + \partial^i H \dot{p}_i.$$

Replacing the partial derivatives of H with respect to coordinates and momenta using Hamilton's equations (2.6), the right hand side vanishes,

$$\frac{dH}{dt} = 0;$$

therefore, if the Hamiltonian is time-independent, then H is constant. The value of this constant is simply the total energy of the system, E; thus, we have established the law of conservation of energy (1.11). In our terminology, the Hamiltonian, H, is a first integral of the system, equal in value to E; therefore, one of the constants of integration, $\boldsymbol{\alpha}$, is E, say $\alpha^n := E$.

Any mechanical system with constant energy is called a *conservative system*; hence, this thesis exclusively examines conservative systems. This set of systems includes, but is not limited to, closed systems.

For a closed mechanical system with n degrees of freedom, there are at most $2n-1$ first integrals. As stated previously, the general solution to a mechanical problem has $2n$ arbitrary constants that must be specified to uniquely determine the solution. Because $\partial H/\partial t = 0$, the time origin may be shifted without changing the problem; thus, one of the constants is simply a translation in time. The remaining $2n-1$ constants can then be represented by functions of the coordinates and momenta (or velocities). These are the first integrals.

2.2.2 Poisson Brackets

We consider the functions $f(\mathbf{p}, \mathbf{q}, t)$ and $g(\mathbf{p}, \mathbf{q}, t)$. The *Poisson bracket* is defined by the sum

$$\{f, g\} := \partial^k f \partial_k g - \partial_k f \partial^k g. \tag{2.10}$$

The Poisson bracket is a bilinear operator. If one of the functions is a position or momentum then the bracket reduces to, respectively,

$$\{f, q^k\} = \partial^k f, \ \{f, p_k\} = -\partial_k f.$$

If both functions are position-momenta coordinates, we obtain the relations

$$\{q^i, q^k\} = 0, \quad \{p_i, p_k\} = 0, \quad \{p_i, q^k\} = \delta_i^k.$$

We recall that these equations are the conditions on new coordinates (2.8), stated in Subsection 2.1.3, sufficient for a canonical transformation.

The most interesting property of the Poisson bracket involves first integrals. We consider the total time derivative of a first integral, $f(\mathbf{p}, \mathbf{q}, t)$, as in [24],

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{k=1}^{n} \left(\frac{\partial f}{\partial q^k} \dot{q}^k + \frac{\partial f}{\partial p_k} \dot{p}_k \right) = 0.$$

Using Hamilton's equations (2.6), this becomes

$$\frac{\partial f}{\partial t} + \{H, f\} = 0.$$

For a first integral without explicit time dependence, we have

$$\{H, f\} \equiv 0.$$

In this case, we say that f is *in involution* with the Hamiltonian. This property holds if and only if f is a first integral.

Another property involves the derivative of the bracket,

$$\frac{\partial}{\partial t}\{f, g\} = \left\{ \frac{\partial f}{\partial t}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} \right\}.$$

Other important properties of the Poisson bracket are the:

Jacobi identity,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{2.11}$$

skew-symmetry,

$$\{f, g\} = -\{g, f\}, \tag{2.12}$$

and Leibniz rule,

$$\{f, gh\} = \{f, g\}h + \{f, h\}g. \tag{2.13}$$

By (2.11), (2.12), and (2.13), a Lie algebra on the space of functions is defined by $\{,\}$.

The Jacobi identity (2.11), with $h = H$, implies that the Poisson bracket of two first integrals, f and g, is also a first integral. This result is known as *Poisson's theorem*. As discussed earlier, there at most $2n - 1$ first integrals for a system; hence, the application of Poisson's theorem does not always produce additional linearly independent first integrals. The Poisson bracket, $\{f, g\}$, may be a constant or functionally dependent on f or g. Similarly, the Leibniz rule (2.13), with $f = H$, implies that the product of two first integrals is a first integral.

2.2.3 Symplectic Structure

Hamilton's equations may be written as a single vector equation by introducing a new notation. Following [36], we define a $2n$ dimensional vector $\mathbf{x}$ by $\mathbf{x} := (\mathbf{p}, \mathbf{q})$; thus, the phase space, which we shall take to be $\mathbb{R}^{2n}$ (in general, some $2n$ dimensional manifold, $\tilde{M}$), contains $\mathbf{x}$. It follows that $\nabla H = (\partial H/\partial \mathbf{p}, \partial H/\partial \mathbf{q})$. By introducing the matrix $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, we may write Hamilton's equations (2.6) as

$$\dot{\mathbf{x}} = J\nabla H(\mathbf{x}). \tag{2.14}$$

This leads to an alternate representation for the Poisson bracket in Euclidean space for functions $\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \in \mathbb{R}^{2n}$,

$$\{\mathbf{f}, \mathbf{g}\} := -(\nabla\mathbf{f}, J\nabla\mathbf{g}), \tag{2.15}$$

where $(,)$ represents the standard Euclidean scalar product.

From (2.14), (2.15) and the skew-symmetry property of the Poisson bracket (2.12), it follows that

$$\{H, x^j\} = (J\nabla H, \nabla x^j) = \sum_{k=1}^{2n} \dot{x}^k \delta_k^j = \dot{x}^j, \; j = 1, \dots, 2n;$$

therefore, the equations of motion may be written in the form

$$X_H := \{H, \mathbf{x}\} = \dot{\mathbf{x}}. \tag{2.16}$$

X_H is called the *Hamiltonian vector field* corresponding to $H(x)$; hence, a Hamiltonian system is characterised by the triple $(\mathbb{R}^{2n}, \{,\}, H(x))$.

The matrix J is nondegenerate; thus, it has an inverse that is clearly $J^{-1} = -J$. This inverse matrix defines a nondegenerate skew-symmetric bilinear form ω_0 on $\mathbb{R}^{2n}$,

$$\omega_0(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, J^{-1}\mathbf{y}).$$

On a general $2n$ dimensional manifold, $\tilde{M}$, a nondegenerate closed 2-form, ω, is called a *symplectic form* or *structure*; thus, the phase space, $\tilde{M}$, equipped with a symplectic structure, ω, is called a *symplectic manifold*, $(\tilde{M}, \omega)$. Using local coordinates, we write ω as,

$$\omega = \omega_{jk}(x) dx^j \wedge dx^k, \ j, k = 1, \dots, 2n.$$

The nondegeneracy of ω is equivalent to the nonvanishing of $\det(\omega_{jk}(x))$ at every point in $\tilde{M}$ so that its inverse, a skew-symmetric matrix $\omega^{jk}(x)$, exists everywhere on $\tilde{M}$. In local coordinates, we write the requirement that ω be closed, that is, $d\omega = 0$, as

$$\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0, \ i, j, k = 1, \dots, 2n.$$

This is equivalent to the Jacobi condition on the Poisson bracket (2.11).

The manifold $\tilde{M} = \mathbb{R}^{2n}$ is equipped with the aforementioned *canonical symplectic form*, ω_0, associated with $-J$,

$$\omega_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} dx^j \wedge dx^k = dp_i \wedge dq^i, \ j, k = 1, \dots, 2n, \ i = 1, \dots, n, \tag{2.17}$$

and its inverse, the *canonical Poisson bi-vector*,

$$P_0 = {\omega_0}^{-1} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}, \ i = 1, \dots, n. \tag{2.18}$$

Theorem 2.1 (Darboux) *At any point* $\mathbf{x}$ *on a symplectic mainfold* $(\tilde{M}, \omega)$*, there exists a local coordinate system in a neighbourhood of* $\mathbf{x}$ *such that* ω *has the standard form*

$$\omega = dp_i \wedge dq^i, \ i = 1, \dots, n.$$

The general Poisson bracket is an extension of the standard Poisson bracket, defined using the Schouten bracket. See Appendix B for a description of the Schouten bracket. With respect to a general Poisson bi-vector, P, the general Poisson bracket is defined by

$$\{f, g\}_P := Pdfdg := [[P, f]_{\mathcal{S}}, g]_{\mathcal{S}}.$$

For the canonical Poisson bi-vector, P_0 (2.18), the general bracket reduces to the standard form (2.10); thus, using Darboux's theorem, Theorem 2.1, we conclude that in the neighbourhood of any point on a manifold, we can find coordinates with respect to which the general Poisson bracket is the standard Poisson bracket.

2.2.4 Killing Tensors

Killing tensors were historically of interest primarily to relativists. More recently, they have become a tool used in classical mechanics to determine first integrals.

We consider a pseudoRiemannian manifold, (M, g), of dimension n. In a local coordinate system, (q^i), a *Killing n-tensor* is defined as a symmetric, covariant tensor field $\mathbf{K}$ on M satisfying the *Killing tensor equation*, originally known as Killing's equations,

$$K_{(i_1 \ldots i_n; i_{n+1})} = 0, \tag{2.19}$$

This is a generalisation of the concept of a *Killing vector* (KV), a one dimensional Killing tensor that satisfies $K_{(i;j)} = (1/2)(K_{i;j} + K_{j;i}) = 0$. A Killing vector may equivalently be defined, see, for example [46], as a contravariant vector satisfying the condition

$$\mathcal{L}_K g = 0,$$

where $\mathcal{L}$ is the Lie derivative.

Another definition of a Killing tensor is provided by the Schouten bracket. The contravariant form of a Killing tensor of any valence must commute with the metric tensor, $\mathbf{g}$, see [46], that is,

$$[\mathbf{K}, \mathbf{g}]_{\mathcal{S}} = 0.$$

We shall consider only Killing vectors and Killing 2-tensors in this thesis. It is to be understood that the term Killing tensor is used to mean Killing 2-tensor.

For a dynamical system, Killing tensors (KTs) correspond to first integrals quadratic in momenta and KVs correspond to first integrals linear in momenta, see [15]. This is shown by calculating the Poisson bracket of the corresponding first integral and the Hamiltonian (1.6).

We consider a contravariant vector, $L^i(\mathbf{q})$, and a symmetric contravariant tensor, $K^{ij}(\mathbf{q})$, with components that are functions of the coordinates. From this, we construct a second first integral, $K^{ij}p_ip_j + L^ip_i + U$, where $U(\mathbf{q})$ is a smooth function of the coordinates. The first integral is necessarily in involution with the Hamiltonian (1.6), that is,

$$\{K^{ij}p_ip_j + L^ip_i + U, \frac{1}{2}g^{ij}p_ip_j + V\} = 0.$$

Expanding, we obtain

$$\begin{aligned}
&\{K^{ij}p_ip_j, \frac{1}{2}g^{ij}p_ip_j\} + \{L^ip_i, \frac{1}{2}g^{ij}p_ip_j\} + \{U, \frac{1}{2}g^{ij}p_ip_j\} \\
&+\{K^{ij}p_ip_j, V\} + \{L^ip_i, V\} + \{U, V\} = 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad &\big(K^{ij}{}_{,k}g^{lk} - K^{kl}g^{ij}{}_{,k}\big)p_ip_jp_l + \Big(L^i{}_{,j}g^{jk} - \frac{1}{2}L^jg^{ik}{}_{,j}\Big)p_ip_k \\
&+\big(-U_{,m}g^{im} + 2K^{im}V_{,m}\big)p_i + L^iV_{,i} = 0.
\end{aligned} \tag{2.20}$$

The terms in (2.20) cubic, quadratic, linear, and constant in momentum must vanish independently.

First we examine the cubic terms. The $p_ip_jp_l$ factor implies that the antisymmetric components with respect to i, j, and l vanish; thus, the symmetric components, with respect to i, j, and l, vanish. Using the contravariant metric tensor components, g^{ij}, to substitute covariant components for the contravariant components, K^{ij}, and expanding the directional derivatives of g^{ij} according to the formula $g^{ij}{}_{,k} = g^{il}g^{mj}g_{ml,k}$ [14], we obtain

$$K_{ij,l} + K_{lj,i} + K_{il,j} - K_{kj}g^{km}(g_{im,l} + g_{lm,i} - g_{il,m})$$

$$-K_{ki}g^{km}(g_{jm,l} + g_{lm,j} - g_{jl,m}) - K_{kl}g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m}) = 0.$$

This is the expansion of the Killing equation (2.19) for a 2-tensor; thus, the K^{ij} are the components of a KT.

Similarly, we conclude that the symmetric (over i and k) components must vanish for the quadratic terms. Since g^{ik} is symmetric, we deduce that

$$L^{(i}{}_{,j}g^{j)k} - \frac{1}{2}L^j g^{ik}{}_{,j} = 0.$$

Rewriting this equation in terms of the covariant form of $\mathbf{L}$ and the directional derivatives of the covariant metric tensor components, we obtain

$$[L_{i,j} + L_{j,i} - L_m g^{mn}(g_{in,j} + g_{jn,i} - g_{ij,n})]g^{il}g^{kj} = 0.$$

This implies that the expression contained in the brackets vanishes. We observe that this expression may be written as $L_{i;j} + L_{j;i}$; therefore, $L_{(i;j)} = 0$ and the L^i are the components of a Killing vector.

By requiring that the linear in momentum terms sum to zero, we derive the equation

$$U_{,i} - 2K_i{}^jV_{,j} = 0.$$

We may write this in the form of a tensor equation,

$$d\mathbf{U} = 2\mathbf{K}dV. \tag{2.21}$$

This equation characterises the potential, U, of a second first integral with respect to the Hamiltonian potential, V; furthermore, it establishes, using the property of the exterior derivative, $d(d\mathbf{U}) = 0$, the compatibility condition between the Hamiltonian potential, V, and associated KT, $\mathbf{K}$,

$$d(\mathbf{K}dV) = 0, \tag{2.22}$$

where $\mathbf{K}$ is the Killing tensor with components $K_i{}^j$.

The final condition, deriving from the term independent of momentum, is

$$L^i V_{,i} = 0.$$

This is equivalent to the vanishing of the Lie derivative of V with respect to $\mathbf{L}$,

$$\mathcal{L}_L V = 0. \tag{2.23}$$

We observe that there exists a coordinate system in which the KV takes the form $L^i = \partial/\partial q^1$, that is, $L^i = \delta^i_1$. Since $\mathbf{L}$ is a KV, it follows that $\partial_1 g_{ij} = 0$. From (2.23), the form of the KV implies that $\partial_1 V = 0$. Because both the metric and potential are independent of q^1, the Hamiltonian (1.6) is independent of q^1; therefore, q^1 is an ignorable coordinate. This links the concept of KVs to ignorable coordinates.

By setting $K^{ij} = 0$ in the second first integral, we obtain a first integral linear in the momentum that corresponds to the KV, $\mathbf{L}$. Similarly, a first integral quadratic in momentum corresponding to the KT, $\mathbf{K}$, is produced by imposing $L^i = 0$.

It is not a trivial task to find Killing n-tensors in an arbitrary manifold; however, Thompson proved that for spaces of constant curvature every Killing tensor is a sum of symmetrised products of KVs [46].

In Euclidean n-space, with the standard metric, $g_{ij} = \delta_{ij}$ in the natural basis, the Lie algebra of KVs has as a basis

$$T_i = \partial_i, \; i = 1, \dots, n,$$

$$R_{ij} = q^j \partial_i - q^i \partial_j, \; 1 \leq i < j \leq n,$$

corresponding to the conservation of linear momentum and angular momentum [46], respectively.

Killing tensors play an integral part in the intrinsic characterisation of additive separation of variables for the HJ equation. This point is elucidated in Chapter 3.

2.3 Hamilton-Jacobi Formalism

We have established the necessary background theory in mechanics and mathematics to discuss the Hamilton-Jacobi formalism. We first derive the key mathematical objects in the theory, the Hamilton-Jacobi equation and its complete integral. The direct approach to solving for the trajectories is provided by the Jacobi theorem, which we describe next. Finally, we include the extension of the Liouville theorem, Theorem 1.1, by Arnol'd.

2.3.1 Hamilton-Jacobi Equation

We initially remove our standard assumption that the system is time-independent; therefore, we treat the Lagrangian and Hamiltonian functions as dependent on time until we explicitly state otherwise later in this subsection. We must derive two preliminary results to develop the Hamilton-Jacobi equation.

Following [24], we view the action (2.1) for the actual path of the system as a function of the coordinates at time t_1. We are interested in comparing the values of S obtained by varying the coordinates at t_1, that is, along paths in a neighbourhood of the actual path. As we derived in Subsection 2.1.1, for the one dimensional case, the first variation of the action is

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q dt.$$

The second term vanishes because the motion satisfies Lagrange's equations. In the first term, we set $\delta q(t_0) = 0$ and replace $\delta q(t_1)$ with δq because t_1 adopts any value of t greater than t_0. Using the relationship (1.2), we arrive at the equivalence $\delta S = p\delta q$. For an n dimensional system, this has the form

$$\delta S = p_i \delta q^i.$$

A relationship between the action and momenta follows directly,

$$S_{,i} = p_i. \tag{2.24}$$

Another necessary result is produced by examining the total time derivative of the action. Directly from the definition of the action (2.1), we observe that

$$\frac{dS}{dt} = L; \tag{2.25}$$

however, by viewing the action as a function of only coordinates and time, it is obvious, using (2.24), that

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + S_{,i}\dot{q}_i = \frac{\partial S}{\partial t} + p_i\dot{q}_i. \tag{2.26}$$

Comparing (2.25) and (2.26), then using (2.5), we obtain

$$\frac{\partial S}{\partial t} + H(\mathbf{p}, \mathbf{q}, t) = 0; \tag{2.27}$$

hence, we have derived a relationship between the action and Hamiltonian. The momenta in (2.27) are replaced using (2.24) to produce the first-order partial differential equation called the *Hamilton-Jacobi equation*,

$$\frac{\partial W}{\partial t} + H\left(q^1, \ldots, q^n, W_{,1}, \ldots, W_{,n}, t\right) = 0, \tag{2.28}$$

where we have relabelled the action, S, as W in agreement with standard notation. Using our new notation, we observe that our derivation of (2.24) also proves the relation (1.4).

There exists a general method for integrating the HJ equation to solve for the system's motion. Before considering it, we introduce two relevant terms.

From the theory of partial differential equations, we have the result that all first order equations have a solution that is unique up to an arbitrary function. This solution is called a *general integral.* A more important concept in mechanics is that of a *complete integral.* For the HJ equation, a complete integral is a solution with $n+1$ independent arbitrary constants that satisfies (1.3). Since the action, W, appears in the HJ equation (2.28) only as a derivative, a complete integral is defined modulo an additive constant, t_0, that may be ignored,

$$W = W(\mathbf{q}, t; \boldsymbol{\alpha}) + t_0, \tag{2.29}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ represents the remaining n independent constants.

2.3.2 Jacobi Theorem

We now establish the connection between the complete integral of the Hamilton-Jacobi equation and solutions of Hamilton's equations. We use the function $W(\mathbf{q}, t; \boldsymbol{\alpha})$ as the generating function for a canonical transformation from the original coordinates $(\mathbf{p}, \mathbf{q})$ to $(\boldsymbol{\alpha}, \boldsymbol{\beta})$; therefore, our new position coordinates are $(\beta^1, \ldots, \beta^n)$ and new momenta coordinates are $(\alpha_1, \ldots, \alpha_n)$. As shown in [24], with this generating function, the new Hamiltonian vanishes everywhere; thus, the transformed Hamilton's equations become

$$\dot{\alpha}_i = 0, \ \dot{\beta}^i = 0,$$

where $\beta^n := t_0$, the constant introduced in the preceding section; therefore, we have $\alpha_i =$ constant, $\beta^i =$ constant. We solve for the position coordinates $\mathbf{q}$ as functions of t, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ using the relationships $\beta^i = \partial W / \partial \alpha_i$ [24]; thus, we have determined the general integral of motion.

The general method for solving the equations of motion is summarised as follows. We derive the Hamilton-Jacobi equation, then find the corresponding complete integral (2.29). We then solve n algebraic equations of the form

$$\frac{\partial W}{\partial \alpha_i} = \beta^i \tag{2.30}$$

to obtain the coordinates $q^i = q^i(t; \boldsymbol{\alpha}, \boldsymbol{\beta})$. The momenta are then calculated using (1.4).

This result is known as Jacobi's Theorem.

Theorem 2.2 (Jacobi) *Let $W = W(\mathbf{q}, t; \boldsymbol{\alpha})$ be a complete integral of the Hamilton-Jacobi equation (2.28) where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ is a set of n arbitrary constants. Let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ be an additional n arbitrary constants. Provided that the determinant condition (1.3) holds, the n relations (2.30) define the n coordinate functions $q^i = q^i(t; \boldsymbol{\alpha}, \boldsymbol{\beta})$ and n momentum functions $p_i = \partial W/\partial q^i$ to produce a general solution to Hamilton's equations (2.6).*

In the case that a complete integral cannot be found, the problem may still be simplified by reduction of the number of degrees of freedom. For example, there may be one or more ignorable coordinates for the system.

Our interest focuses exclusively on conservative systems for which the Hamiltonian has no explicit time dependence. In this case, the action's only time dependence is the term $-Et$. That is, the action may be written in the form

$$W(\mathbf{q}, t; \boldsymbol{\alpha}) = W_0(\mathbf{q}; \boldsymbol{\alpha}) - Et,$$

where $\alpha^n := E$. The Hamilton-Jacobi equation (2.28) simplifies to the standard form (1.5).

2.3.3 Arnol'd-Liouville Theorem

As previously described, the existence of first integrals for a system allows us to reduce the number of degrees of freedom in the associated mathematical problem. It is generally required that we find $2n$ first integrals to integrate a system of $2n$ ordinary differential equations; however, for a set of canonical equations, n integrals is sufficient, if they have the correct form, because each reduces the order of the system of equations by two instead of one. Liouville proved this result in general. Arnol'd formalised the concepts of Liouiville's theorem, Theorem 1.1, and extended it in [2]. Before stating his theorem, we define the concept of a conditionally periodic system. We consider a system that has a motion that is not periodic in any coordinate. That is, the system will not return to any previous state in a finite interval of time. The system is said to be *conditionally periodic* provided that it will pass arbitrarily close to any previous state given a sufficiently large interval of time.

Theorem 2.3 (Arnol'd-Liouville [2]) *Given n functions, $F_1, \ldots, F_n$, in involution on a symplectic manifold, we consider some level set of the functions,*

$$M_f = \{x : F_i(x) = f_i,\ i = 1, \ldots, n\}.$$

Under the assumption that we have independence of the n functions on M_f, that is, that at each point in M_f the 1-forms dF_i are linearly independent, we have the following:

1. *M_f is a smooth manifold that is invariant under phase flow with a Hamiltonian, $H = F_1$.*

2. *If the manifold M_f is compact and connected, then it is diffeomorphic to the n dimensional torus,*

$$T^n = \{(\Phi_1, \ldots, \Phi_n) mod\ 2\pi\}.$$

3. *The phase flow with a Hamiltonian, H, determines a conditionally periodic motion on M_f, that is,*

$$\frac{d\mathbf{\Phi}}{dt} = \boldsymbol{\omega} \ \ where\ \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{f}).$$

4. *The canonical equations with a Hamiltonian, H, can be integrated by quadratures.*

A useful corollary of this theorem exists for two dimensional systems.

Corollary 1 *A mechanical system with two degrees of freedom that possesses a first integral independent of the Hamiltonian is integrable by quadratures; furthermore, a compact connected two dimensional submanifold of the phase space* $H = h$, $F = f$ *is an invariant torus. Motion on it is conditionally periodic.*

2.4 Method of Separation of Variables

An important technique for the determination of a complete integral for the Hamilton-Jacobi equation of a system is the *method of separation of variables.*

We consider the case in which a coordinate, q^j, — without loss of generality q^1 — and the corresponding momentum, $p_1 = W_{,1}$, appear in the HJ equation (1.1) in some combination that is independent of the other coordinates, derivatives and time, $\phi(q^1, W_{,1})$. We may write (1.5) in the form [24]

$$H\left(\mathbf{q}, \frac{\partial W}{\partial \mathbf{q}}, \phi\left(q^1, \frac{\partial W}{\partial q^1}\right)\right) = E. \tag{2.31}$$

In (2.31) and the remainder of this section, $\mathbf{q}$ represents the set of position coordinates excluding q^1.

We proceed to seek a *separated* solution of the form

$$W = \tilde{W}(\mathbf{q}; \boldsymbol{\alpha}) + W_1(q^1) - Et, \tag{2.32}$$

where $\alpha^n := E$. When we substitute this action into (2.31), its form changes to

$$H\left(\mathbf{q}, \frac{\partial \tilde{W}}{\partial \mathbf{q}}, \phi\left(q^1, \frac{\partial W_1}{\partial q^1}\right)\right) = E.$$

This must be an identity for any solution (2.32). Since the only q^1 dependence is in ϕ, ϕ must be constant, say equal to the arbitrary constant α_1. The problem (2.31) has been reduced to

$$\phi\left(q^1, \frac{\partial W_1}{\partial q^1}\right) = \alpha_1, \tag{2.33}$$

$$H\left(\mathbf{q}, \frac{\partial \tilde{W}}{\partial \mathbf{q}}, \alpha_1\right) = E.$$

It is clear that this system is simpler than the original HJ equation (2.31). The first equation is an ordinary differential equation for W_1 that may be solved by quadrature, and the second is a partial differential equation that has two fewer independent variables than the original HJ equation. Ideally, we may separate the remaining $n-1$ coordinates. Then finding the complete integral for the HJ equation is reduced to n quadratures. We say, in this case, that the HJ equation is *completely separable.*

A specific example of separation of variables is the separation of a cyclic coordinate, say q^1. By definition, the Hamiltonian, and hence the HJ equation, are independent of a cyclic coordinate; thus, it is a separable coordinate. In particular, the ordinary differential equation (2.33) becomes $W_{1,1} = \alpha_1$. This implies that $W_1 = \alpha_1 q^1$. The complete integral (2.32) is

$$W = \tilde{W}(\mathbf{q}; \boldsymbol{\alpha}) + \alpha_1 q^1 - Et.$$

We observe that α_1 is the constant value of the corresponding momentum, $p_1 = W_{,1}$, and α_n is the system's energy, E.

For conservative systems, the energy term, E, in (2.31) is produced by the cyclic time variable, t; furthermore, as mentioned in Subsection 2.2.1, E is one of the n first integrals, and hence, constants of motion. If completely separable, the method yields

$$W = \sum_{i=1}^{n} W_i(q^i; \boldsymbol{\alpha}) - Et,$$

where $\alpha^n := E$.

The method of separation of variables encompasses the previous methods of integration using ignorable coordinates. It also includes the more general case in which a coordinate is not cyclic but is separable. In practice, the HJ equation for a system must be represented in appropriate coordinates for it to be separable. A local characterisation of separability, the Levi-Civita criterion, Theorem 1.5, was described in Chapter 1; however, recent work by Benenti, see [4, 5], has produced an intrinsic *coordinate-independent* characterisation of the separability of a system via a point transformation. This is investigated in Chapter 3.

Separation of variables is a powerful method; however, there are other methods used to solve Hamilton's equations. Of the variety of approaches, two, the Lax operator and bi-Hamiltonian methods, are discussed in subsequent sections of this chapter. First, we illustrate the use of separation of variables by applying it to an elementary problem from mechanics, the two dimensional harmonic oscillator.

2.4.1 Example: Two Dimensional Harmonic Oscillator

A standard example for separation of variables is the two dimensional harmonic oscillator, see, for example, [21]. For both independent variables, we have a kinetic energy $T_i = (1/2)p_i\dot{q}^i = (1/2)m(\dot{q}^i)^2$ — using the definition of the momentum in Cartesian coordinates, $p_i = m\dot{q}^i$ — and a potential energy $V_i = 1/2m\omega_i^2(q^i)^2$; thus, the Hamiltonian is

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{m}{2}\left(\omega_1^2(q^1)^2 + \omega_2^2(q^2)^2\right).$$

This system is conservative because there is no explicit time dependence. Using (1.4), the HJ equation (1.7) is

$$\frac{1}{2m}[(W_{,1})^2 + (W_{,2})^2] + \frac{m}{2}\left(\omega_1^2(q^1)^2 + \omega_2^2(q^2)^2\right) = E. \tag{2.34}$$

Neither coordinate is cyclic but both are separable; hence, we seek a solution of the form $W = -Et + W_1(q^1) + W_2(q^2)$. Substituting this form into (2.34), we obtain the separated equations

$$\begin{cases} \dfrac{1}{2m}(W_{1,1})^2 + \dfrac{m}{2}\left(\omega_1^2(q^1)^2\right) = \alpha_1, \\[2ex] \dfrac{1}{2m}(W_{2,2})^2 + \dfrac{m}{2}\left(\omega_2^2(q^2)^2\right) = E - \alpha_1. \end{cases}$$

These ordinary differential equations may be rearranged to obtain equations of the form $W_{i,i} = \sqrt{2mc_i - m^2\omega_i^2(q^i)^2}$ where $c_1 = \alpha_1$ and $c_2 = E - \alpha_1$, respectively. Using inverse trigonometric functions, we integrate to determine the complete integral,

$$\begin{aligned} W = -Et &+ \frac{q^1}{2}\sqrt{2m\alpha_1 - m^2\omega_1^2(q^1)^2} + \frac{\alpha_1}{\omega_1}\arcsin\left(\sqrt{\frac{m}{2\alpha_1}}\omega_1 q^1\right) \\ &+ \frac{q^2}{2}\sqrt{2m(E-\alpha_1) - m^2\omega_2^2(q^2)^2} + \frac{(E-\alpha_1)}{\omega_2}\arcsin\left(\sqrt{\frac{m}{2(E-\alpha_1)}}\omega_2 q^2\right). \end{aligned} \tag{2.35}$$

To solve for the coordinate functions we must solve the n equations (2.30). For a two dimensional system, we have $2 \times 2 - 1 = 3$ separation constants, $\{\alpha_1, E, \beta_1\}$, corresponding to the number of first integrals derived in Subsection 2.2.1, plus the constant representing an allowable shift in the time origin, t_0; therefore, the equations (2.30) take the form

$$\beta_1 = \frac{\partial W}{\partial \alpha_1}, \; t_0 = \frac{\partial W}{\partial E}.$$

Substituting the complete integral (2.35) into these equations, we obtain

$$\begin{cases} \beta_1 = \dfrac{1}{\omega_1}\arcsin\left(\sqrt{\dfrac{m}{2\alpha_1}}\omega_1 q^1\right) - \dfrac{1}{\omega_2}\arcsin\left(\sqrt{\dfrac{m}{2(E-\alpha_1)}}\omega_2 q^2\right), \\ t + t_0 = \dfrac{1}{\omega_2}\arcsin\left(\sqrt{\dfrac{m}{2(E-\alpha_1)}}\omega_2 q^2\right). \end{cases}$$

By inspection, the closed form solution for $\mathbf{q}$ is

$$\begin{cases} q^1 = \dfrac{1}{\omega_1}\sqrt{\dfrac{2\alpha_1}{m}}\sin[\omega_1(\beta_1 + t + t_0)], \\ q^2 = \dfrac{1}{\omega_2}\sqrt{\dfrac{2(E-\alpha_1)}{m}}\sin[\omega_2(t + t_0)]. \end{cases}$$

2.5 Other Complete Integrability Methods

In this section, we outline the Lax and bi-Hamiltonian methods for solving the HJ equation. After describing the non-periodic, finite dimensional Toda lattice, we analyse its integrability using these methods.

2.5.1 Lax Method

A powerful technique used to study dynamical systems is the *Lax method*, also known as the *isospectral deformation method*, formulated by Lax [25]. The essence of this method is to write the equations of motion in a matrix form from which we can derive integrals of the motion. If we generate n functionally independent first integrals in involution, then, by Liouville's theorem, Theorem 1.1, we may conclude that the system is completely integrable.

We follow the development of Toda [49]. It is necessary to construct $n \times n$ real matrices L and B such that the equations of motion for our system are equivalent to the *Lax representation*,

$$\dot{L} = BL - LB = [B, L]. \tag{2.36}$$

The matrices L and B are referred to as a *Lax pair*.

We seek a unitary matrix, that is, U satisfying $UU^* = I$ where $*$ indicates the Hermitian conjugate of a matrix (the transpose of the complex conjugate). Taking the time derivative of this expression, we obtain

$$\frac{dU}{dt}U^* + U\frac{dU^*}{dt} = 0. \tag{2.37}$$

U is defined by the initial value problem

$$\frac{dU}{dt} = BU, \;\; U(0) = \mathrm{I}. \tag{2.38}$$

Using the fact that U is unitary, we rewrite the differential equation of (2.38) in the form $B = (dU/dt)U^*$. With this definition for B, and the properties (2.37) and $(AB)^* = B^*A^*$, it can be shown directly that $B^* = -B$; thus, B is necessarily a *skew-symmetric* matrix. If B were known, then, provided a solution to (2.38) exists, we may, in theory, determine U.

Another implication of (2.37) is that $dU^{-1}/dt = -U^{-1}B$. Using this with (2.36) and (2.38), we conclude that

$$\frac{d}{dt}(U^{-1}LU) = 0;$$

hence, $U^{-1}LU$ is time-independent. We write this condition as

$$L(t) = U(t)L(0)U(t)^{-1}. \tag{2.39}$$

It follows from the fact that U is a unitary matrix, that $L(t)$ and $L(0)$ are unitary equivalent, that is, have the same eigenvalues. Let $\lambda(t)$ and $\phi(t)$ represent an eigenvalue and the corresponding eigenfunction of $L(t)$, respectively. At time $t = 0$, these quantities satisfy the equation $L(0)\phi(0) = \lambda(0)\phi(0)$. This may be rewritten, using (2.39), as

$$L(t)U(t)\phi(0) = \lambda(0)U(t)\phi(0).$$

This equation shows that $U(t)\phi(0)$ is an eigenvector and $\lambda(0)$ is an eigenvalue for every time, t, that is,

$$\lambda(t) = \lambda(0) = \lambda.$$

Since this is true for every λ_i, all the eigenvalues of L are time-independent. Equivalently, we say L evolves in time but conserves its spectrum, $\{\lambda_i\}$, that is, undergoes an isospectral deformation. Although the eigenvalues of L are first integrals, it is easier to generate first integrals by taking the trace of powers of L.

The Lax method has been applied to many known integrable systems; however, the determination of suitable matrices, L and B, may be difficult.

2.5.2 Bi-Hamiltonian Method

Previously, we characterised a Hamiltonian system by the triplet (M, ω, X_H). In the bi-Hamiltonian framework (see, for example, [39]), we must generalise from a symplectic manifold to a Poisson manifold, that is, a manifold equipped with a Poisson bi-vector, P.

A (non-degenerate) *Poisson bi-vector* is a skew-symmetric contravariant tensor of valence two on the manifold, M, that satisfies the condition

$$[P, P]_{\mathcal{S}} = 0 (\Leftrightarrow d\omega = 0,) \tag{2.40}$$

(where $\omega = P^{-1}$). Since ω is nondegenerate by the definition of a symplectic manifold, every symplectic manifold is a *Poisson manifold*, (M, P); hence, every Hamiltonian system may be represented by the triple (M, P, X_H). The Hamiltonian vector field (2.16) is uniquely determined by

$$X_H = PdH = [P, H]_{\mathcal{S}}.$$

A *bi-Hamiltonian system* $(M, P_1, P_2, X_{H1,H2})$ possesses two distinct Hamiltonian representations, that is,

$$X_{H1,H2} = P_1 dH_1 = P_2 dH_2,$$

where the Poisson bi-vectors are *compatible*. Compatibility signifies that the eigenvalues of P_1 and P_2 are in involution which is equivalent to the condition

$$[P_1, P_2]_{\mathcal{S}} = 0. \tag{2.41}$$

Integrability of bi-Hamiltonian systems was investigated for the infinite dimensional case by Magri in [30]. Then, Gelfand and Dorfman [13] produced similar results for finite dimensional Hamiltonian systems. Their work has been extended by Magri and Morosi [31]. The complete integrability of these systems is guaranteed if the following theorem developed by Smirnov [39] is satisfied.

Theorem 2.4 *Consider a bi-Hamiltonian system* $(M, P_1, P_2, X_{H_1,H_2})$ *for a* $2n$ *dimensional manifold,* M*. If the linear operator* $A := P_2 P_1^{-1}$ *has exactly* n *functionally independent eigenvalues, then the dynamical system determined by* X_{H_1,H_2}*, that is,* $\dot{x}(t) = X_{H_1,H_2}(x(t))$*, is completely integrable.*

2.5.3 Example: Non-periodic, Finite Toda Lattice

The Toda lattice was first discussed by Toda in 1967 [47, 48]. The non-periodic, finite dimensional Toda lattice describes the movement of n particles located on a line with an exponential interaction between only adjacent particles. In physical position-momenta coordinates, (q^i), (p_i), its motion is described by the $2n$ equations

$$\begin{aligned} \dot{q}^i &= p_i, \\ \dot{p}_i &= e^{q^{i-1}-q^i} - e^{q^i-q^{i+1}}, \end{aligned} \tag{2.42}$$

where $e^{q^0-q^1} = e^{q^n-q^{n+1}} = 0$. The vector field X_H (2.16) of (2.42) is known to be Hamiltonian and completely integrable, see, for example, [32]. The corresponding Hamiltonian function,

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q^i-q^{i+1}}, \tag{2.43}$$

is constant and equal to the total energy of the system, E.

The periodic Toda lattice represents similarly interacting particles arranged on a circle. Its equations of motion are represented by (2.42) with the proviso that $q^{i+n} := q^i$.

We proceed to establish the complete integrability of the non-periodic finite dimensional Toda lattice using both the Lax and the bi-Hamiltonian methods.

Example of the Lax Method: Non-Periodic, Finite Dimensional Toda Lattice

The Arnol'd-Liouville integrability of the non-periodic, finite dimensional Toda lattice was first established by Flaschka using the Lax method [18].

Our analysis is similar to that of the periodic lattice in [49]. First, we define the Flaschka variables by

$$\begin{aligned} a_i &= \frac{1}{2} e^{(q^i - q^{i+1})/2}, \ i = 1, \dots, n-1, \\ b_i &= \frac{1}{2} p_i, \ i = 1, \dots, n, \end{aligned} \tag{2.44}$$

where we have used different numbering and signs than in Flaschka's paper [18].

In this new coordinate system (2.44), Hamilton's equations for the system (2.42) become

$$\begin{aligned} \dot{a}_i &= a_i(b_i - b_{i+1}), \ i = 1, \dots, n-1, \\ \dot{b}_i &= 2(a_{i-1}^2 - a_i^2), \ i = 1, \dots, n, \end{aligned}$$

where it is understood that $a_0 = 0$.

Using these coordinates, we define the $n \times n$ Jacobi matrices (that is, only the main diagonal and its 2 neighbouring diagonals have non-zero entries) L and B by

$$L = \begin{pmatrix} b_1 & a_1 & & & & & & & \\ a_1 & b_2 & & & & & & & \\ & & \ddots & & & & 0 & & \\ & & & b_{i-1} & a_{i-1} & & & & \\ & & & a_{i-1} & b_i & a_i & & & \\ & & & & a_i & b_{i+1} & & & \\ & & 0 & & & & \ddots & & \\ & & & & & & & b_{n-1} & a_{n-1} \\ & & & & & & & a_{n-1} & b_n \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -a_1 & & & & & & & \\ a_1 & 0 & & & & & & & \\ & & \ddots & & & & 0 & & \\ & & & 0 & -a_{i-1} & & & & \\ & & & a_{i-1} & 0 & -a_i & & & \\ & & & & a_i & 0 & & & \\ & & 0 & & & & \ddots & & \\ & & & & & & & 0 & -a_{n-1} \\ & & & & & & & a_{n-1} & 0 \end{pmatrix}.$$

These matrices satisfy (2.36). They are thus said to form a *Lax pair*; hence, the set of eigenvalues of L is independent of time. Because L is real, its eigenvalues are real; furthermore, since L is an $n \times n$ matrix, it has up to n distinct eigenvalues. It has been proven [17] that because the a_i's are all positive, the eigenvalues are distinct.

The eigenvalues of L are determined by the determinant equation $\det(L - \lambda I) = 0$. This can be expanded into n polynomial equations with respect to the eigenvalues,

$$\lambda_i^n + c_1\lambda_i^{n-1} + \ldots + c_{n-1}\lambda_i + c_n = 0,\ i = 1, \ldots, n,$$

in which the c_i's are functions of a_i and b_i. Simultaneously solving these equations for c_i, we obtain $c_i(a_i, b_i) = c_i(\lambda_1, \ldots, \lambda_n)$ for $i = 1, \ldots, n$. Since the eigenvalues are conserved by the motion, the c_i's are conserved quantities. From (2.36), we observe that n conserved quantities are

$$J_j = \mathrm{Tr}(L^j) = \sum_{i=1}^{n} \lambda_j^i,\ j = 1, \ldots, n,$$

where Tr indicates the diagonal sum.

We can write the n functionally independent first integrals, I_i, $i = 1, \ldots, n$, in terms of the J_j, $j = 1, \ldots, n$; thus, by Liouville's theorem, the non-periodic, finite dimensional Toda lattice is completely integrable.

Example of the Bi-Hamiltonian Method: Non-Periodic, Finite Dimensional Toda Lattice

In considering the non-periodic, finite dimensional Toda lattice, we use the Poisson manifold $(\mathbb{R}^{2n}, P_0)$, where the Poisson bi-vector, P_0 (2.18), is canonical. Following [39] ,we relabel the Hamiltonian, H, (2.43) as H_0.

We define the vector field, Y_P, by

$$Y_P = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} p_j \frac{\partial}{\partial q^i} + \Big(-\sum_{i=1}^{n-1} e^{q^i - q^{i+1}} + \frac{1}{2}\sum_{i=1}^{n} p_i^2\Big)\frac{\partial}{\partial p_i}.$$

On a Poisson manifold, (M, P), the *(contravariant exterior) operator* σ, acts on the space of skew-symmetric contravariant tensors on M, $V^k(M)$. For an arbitrary

$Q \in V^k(M)$,

$$\sigma Q = -[P, Q]_{\mathcal{S}}. \tag{2.45}$$

For a vector $Q \in V^1(M)$, this simplifies to

$$\sigma Q = -\mathcal{L}_Q(P),$$

a property inherited from the Schouten bracket (see Appendix B ($B.2$)). We say that σ is a *coboundary operator* because it satisfies the property

$$\sigma^2 = 0. \tag{2.46}$$

Using σ_0 and Y_P, we define the tensor $P_1 = \mathcal{L}_{Y_P}(P_0) = [P_0, Y_P]_{\mathcal{S}} = -\sigma_0 Y_P$ by

$$P_1 = \sum_{i=1}^{n-1} e^{q^i - q^{i+1}} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i<j}^{n} \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial q^i}. \tag{2.47}$$

Using the second Hamiltonian function for the system, $H_1 = \sum_{i=1}^n p_i$, the Hamiltonian vector field, X_H, may be written in terms of P_1 as $X_H = P_1^{i\alpha} H_{1,\alpha}$. Provided that P_1 is a Poisson bi-vector, we have produced a second Hamiltonian representation. To directly check the condition $[P_1, P_1]_{\mathcal{S}} = 0$ (2.40) is computationally intensive. Instead, we observe that (2.40) is equivalent to $\sigma_1 P_1 = 0$. This holds if P_1 has the form

$$P_1 = P_0 \omega_1 P_0 \tag{2.48}$$

for the second symplectic structure on M,

$$\omega_1 := \sum_{i=1}^{n-1} e^{q^i - q^{i+1}} dq^i \wedge dq^{i+1} + \sum_{i=1}^{n} p_i dq^i \wedge dp_i + \frac{1}{2} \sum_{i<j}^{n} dp_i \wedge dp_j.$$

Das and Okubo derived ω_1 and showed that it is compatible with the canonical symplectic form, ω_0 (2.17), in [12]. In fact, P_1 can be written in the form (2.48); therefore, it is a Poisson bi-vector.

We establish the compatibility of the two Poisson bi-vectors by verifying that the condition (2.41) holds. Using the definitions of σ (2.45) and P_1 (2.47), we derive

$$[P_0, P_1]_{\mathcal{S}} = -\sigma_0 P_1 = 0,$$

using the coboundary property of σ (2.46); hence, we have a bi-Hamiltonian system $(M, P_0, P_1, X_{H_0,H_1})$. The involutive set of first integrals is $\{I_i = (1/i)\mathrm{Tr} P_1 P_0^{-1}\}$ where $P_0^{-1} = \omega_0$; therefore, using Theorem 2.4, we have again established the complete integrability of the non-periodic, finite dimensional Toda lattice.

Chapter 3

Intrinsic Characterisation

In this chapter, we describe the intrinsic characterisation developed by Benenti [4] for the orthogonal separation of variables of the Hamilton-Jacobi equation on a pseudoRiemannian manifold. It provides a definitive criterion for the existence of separable coordinates related to the natural position-momenta coordinates by a point transformation (2.9). This approach uses geometrical objects, Killing tensors, on the manifold rather than local descriptions.

The related mathematical background is presented. Then the theorem is described and proved. We discuss some of the implications of the theory and relate it to the theory of Kalnins and Miller [22]. As an example, we analyse the non-periodic two and three dimensional Toda lattices using the characterisation.

3.1 Intrinsic Characterisation of Orthogonal Separability

We seek a complete integral of the HJ equation (1.7) that is separable. For a separable system, it is clear from (1.4) that $W_{,i}$ is a function of only q^i. In accordance with the Jacobi theorem, Theorem 2.2, by solving these n equations for $(\boldsymbol{\alpha})$ in terms of $(\mathbf{q}, \mathbf{p})$, we produce n independent integrals in involution; hence, the system is completely integrable. Stäckel [44] established that these first integrals are polynomials quadratic in momentum for orthogonal systems (1.10); therefore, as we discussed in Subsection 2.2.4, the coefficients produce Killing two-tensors. This is the key to the intrinsic characterisation of orthogonal separability.

Before investigating the characterisation theorem of Benenti, we show that the eigenvectors of a KT, with distinct eigenvalues, can be used to form a quasi-orthonormal frame that is a rigid moving frame.

3.2 Constructing a Quasi-Orthonormal Frame

Following Eisenhart [14], we establish a quasi-orthonormal frame on a pseudoRiemannian manifold, $(M, \mathbf{g})$, from the eigenvectors of a KT, K_{ij} — a symmetric covariant tensor — with pointwise simple eigenvalues. We begin by considering the determinant equation,

$$|K_{ij} - \lambda g_{ij}| = 0, \tag{3.1}$$

related to the eigenvalue problem,

$$K_{ij} X^j = \lambda g_{ij} X^j. \tag{3.2}$$

The solution set of (3.1) and (3.2) consists of n pairs

$$\{(\lambda_i, X_i) : X_i \neq 0,\ i = 1, \dots, n\}. \tag{3.3}$$

Using the transformation law for a covariant tensor, we observe that, in a different coordinate system, (3.1) may be written

$$|\tilde{K}_{lm} - \lambda \tilde{g}_{lm}| \left| \frac{\partial \tilde{x}^k}{\partial x^i} \right|^2 = 0$$

$$\Leftrightarrow \quad |\tilde{K}_{lm} - \lambda \tilde{g}_{lm}| = 0$$

since the Jacobian matrix for a coordinate transformation is necessarily non-zero. It follows that if λ is an eigenvalue in one coordinate system, it is an eigenvalue in every coordinate system; therefore, the eigenvalues, $\{\lambda_i\}$, are invariant.

We consider a pseudoRiemannian manifold on which the eigenvalues of the solution set (3.3) are *simple*, that is, $\lambda_i \neq \lambda_j$ for $i \neq j$. To establish that the eigenvectors determine an quasi-orthonormal frame, we prove the following propositions:

Proposition 3.1 *The set of eigenvectors, $\{X_i\}$, is linearly independent.*

Proposition 3.2 *Each pair of eigenvectors, $\{X_i, X_j : i \neq j\}$, is orthogonal.*

Proposition 3.3 *If M is a Riemannian manifold, the eigenvalues of K_{ij}, $\{\lambda_i\}$, are real.*

Proposition 3.4 *If the set $\{\lambda_i\}$ is real and simple, then there exists corresponding eigenvectors, $\{X_i\}$, that are real.*

Proposition 3.5 *No eigenvector, X_i, is null.*

Proof of Proposition 3.1

We consider

$$\sum_{k=1}^{n} c_k X_k = 0, \tag{3.4}$$

for constants, c_k, $k = 1, \ldots, n$.

Using the n equations (3.2) with $(\lambda, X) = (\lambda_k, X_k)$, we derive

$$\begin{aligned}\sum_{k=1}^{n} c_k \lambda_k g_{ij} {X_k}^j &= \sum_{k=1}^{n} c_k K_{ij} {X_k}^j \\ &= K_{ij} \Big(\sum_{k=1}^{n} c_k {X_k}^j \Big) \\ &= 0,\end{aligned}$$

using (3.4). It follows that

$$g_{ij} \Big(\sum_{k=1}^{n} c_k \lambda_k {X_k}^j \Big) = 0 \Rightarrow \sum_{k=1}^{n} c_k \lambda_k X_k = 0. \tag{3.5}$$

We observe that (3.4) is equivalent to

$$c_n X_n = \sum_{k=1}^{n-1} (-c_k) X_k. \tag{3.6}$$

Substituting this relation into (3.5), we obtain

$$\sum_{k=1}^{n-1} c_k (\lambda_k - \lambda_n) X_k = 0. \tag{3.7}$$

We now prove, by induction on the dimension n, that the set of eigenvectors associated with a set of simple eigenvalues is linearly independent. In this proof, we use the fact that eigenvectors are necessarily non-zero.

We first consider the case $n = 2$. Setting $n = 2$ in (3.7), we obtain $c_1(\lambda_1 - \lambda_2)X_1 = 0$. It follows, from the fact that the eigenvalues are distinct, that $c_1 = 0$. Substituting $c_1 = 0$ into (3.4), we obtain $c_2 = 0$. Since both coefficients necessarily vanish, the set of eignvectors is linearly independent.

We assume that the hypothesis holds for $n - 1$, then deduce that it holds for n.

The set of eigenvectors $\{X_1, \dots, X_{n-1}\}$ is linearly independent. This implies that the coefficients, $c_k(\lambda_k - \lambda_n)$, of (3.7) vanish. Since the eigenvalues are simple, in particular, $\lambda_k \neq \lambda_n$, $k = 1, \dots, n-1$, we conclude that

$$c_k = 0, \ \ k = 1, \dots, n-1.$$

Substituting these values into (3.6), we deduce that

$$c_n = 0$$

because no eigenvector is zero.

Since the condition that the linear combination of eigenvectors vanishes (3.4) implies that all the coefficients vanish, we conclude that the set of eigenvectors, $\{X_1, \dots, X_n\}$, is linearly independent. This completes the proof by induction. □

Proof of Proposition 3.2

We show directly that any two eigenvectors, X_k and X_l, corresponding to distinct eigenvalues, λ_k and λ_l, $1 \leq k, l \leq n$, $k \neq l$, are mutually orthogonal. From (3.2), we have the equations

$$K_{ij} {X_k}^j = \lambda_k g_{ij} {X_k}^j, \tag{3.8}$$

and

$$K_{ij} {X_l}^j = \lambda_l g_{ij} {X_l}^j. \tag{3.9}$$

We take the difference between ${X_k}^i \times$ (3.9) and ${X_l}^i \times$ (3.8),

$$0 = (\lambda_l - \lambda_k) g_{ij} {X_k}^i {X_l}^j$$

$$\Rightarrow \quad g_{ij} {X_k}^i {X_l}^j = 0$$

since $\lambda_k \neq \lambda_l$; thus, the vectors X_k^i and X_l^j are orthogonal. $\square$

Proof of Proposition 3.3

Any eigenvalue-eigenvector pair, (λ, X), satisfies the equation (3.2). Taking its complex conjugate, we obtain the equation

$$K_{ij} \bar{X}^j = \bar{\lambda} g_{ij} \bar{X}^j.$$

The Killing and metric tensor components are unchanged because they are real. Contracting both sides with X^i, we obtain

$$K_{ij} X^i \bar{X}^j = \bar{\lambda} g_{ij} X^i \bar{X}^j.$$

Using the equation (3.2), we produce the identity

$$(\bar{\lambda} - \lambda) g_{ij} X^i \bar{X}^j = 0. \tag{3.10}$$

We now investigate when $g_{ij} X^i \bar{X}^j$ vanishes. Writing the eigenvector as a sum of its real and imaginary parts, $X^i = U^i + iV^i$, we derive

$$g_{ij} U^i U^j = -g_{ij} V^i V^j. \tag{3.11}$$

If $\mathbf{g}$ is positive or negative definite, then one side of (3.11) is non-zero (since the eigenvector cannot be zero) and the other is zero or of the opposite sign, which is impossible; thus, no eigenvector satisfies the equation $g_{ij} X^i \bar{X}^j = 0$; therefore, (3.10) implies $\bar{\lambda} = \lambda$. $\square$

On a pseudoRiemannian manifold, we may obtain eigenvector solutions to (3.11); hence, we must *assume* that all the eigenvalues are real.

Proof of Proposition 3.4

We consider an eigenvalue-eigenvector pair, (λ, X). We write the eigenvector in the form

$$X^j = U^j + iV^j,$$

where $V^j = 0$ if the eigenvector is real. Substituting this into (3.2), then separating the real and imaginary parts, we obtain

$$K_{ij} U^j = \lambda g_{ij} U^j,$$

$$K_{ij} V^j = \lambda g_{ij} V^j;$$

therefore, we have generated two real eigenvectors corresponding to λ. Since the eigenvalues are distinct, the eigenspaces are one dimensional; hence, $V^j = kU^j$ for some real constant k.

We have shown that for any eigenvalue we can generate an associated real eigenvector. □

Proof of Proposition 3.5

We establish this proposition by contradiction. Without loss of generality, we assume that the eigenvector X_1 is null,

$$g_{ij}{X_1}^i{X_1}^j = 0.$$

Using this relation and Proposition 3.2, we conclude that

$$g_{ij}{X_1}^i{X_k}^j = 0 \tag{3.12}$$

$\forall\ k \in \{1, \ldots, n\}$. Since the eigenspace, V, is spanned by the set $\{X_1, \ldots, X_n\}$, any vector, $Y \in V$, can be expressed as a linear combination of the eigenvectors; hence, it follows from (3.12) that

$$g_{ij}{X_1}^iY^j = 0$$

$\forall\ Y \in V$. This is the degeneracy condition for the metric, $\mathbf{g}$ — a contradiction; thus, X_1 is *not* null. It can be similarly shown that no other eigenvector is null. □

We have proven that on a pseudoRiemannian manifold, a KT, $\mathbf{K}$, with real simple eigenvalues has a set of associated eigenvectors, $\{X_1, \ldots, X_n\}$, that are real, linearly independent, mutually orthogonal and non-null; therefore, the eigenvectors form an orthogonal frame.

For a rigid moving frame, the frame vectors must have constant lengths. Since the vectors, X_k, are not null, we scale the components so that

$$g_{ij}{X_k}^i{X_k}^j = e_k,\ k = 1, \ldots, n,$$

where $e = \pm 1$. That is, we normalise the vectors to produce a quasi-orthonormal frame.

In conclusion, from a KT, $\mathbf{K}$, with real simple eigenvalues, we construct a rigid moving frame on the pseudoRiemannian manifold from the eigenvectors of $\mathbf{K}$.

3.2.1 Introduction to the Moving Frame Formalism

We consider an n dimensional pseudoRiemannian manifold, M. The vectors X_i represent the natural frame corresponding to some system of coordinates, (q^i), that is, $X_i = \partial/\partial q^i$, $i = 1, \ldots, n$. The corresponding co-vectors are $X^i = dq^i$, $i = 1, \ldots, n$. In the natural basis, the metric (1.15) takes the form

$$ds^2 = g_{ij}X^iX^j. \tag{3.13}$$

The *moving frame* vector fields spanning the tangent space, M_p, are defined, with respect to the natural basis, $\{X_1, \ldots, X_n\}$ by

$$E_a = {h_a}^i X_i, \tag{3.14}$$

where ${h_a}^i$ is an $n \times n$ matrix of C^∞ functions. The dual ω-frame 1-forms are defined by

$$E^a = {h^a}_i X^i, \tag{3.15}$$

where ${h^a}_i$ is also an $n \times n$ matrix of C^∞ functions. Since the $\{E_a\}$ and $\{E^a\}$ are dual bases, we have

$$\langle E^a, E_b \rangle = \delta^a_b, \tag{3.16}$$

where the bracket $\langle \omega, X \rangle$ represents the real value of the 1-form, ω, acting on the vector field, X. The property (3.16) follows from the fact that the corresponding natural bases are dual, $\langle dq^i, \partial/\partial q^j \rangle = \delta^i_j$. This leads to the relations $h_a{}^i h^b{}_i = \delta^b_a$ and $h_a{}^i h^a{}_j = \delta^i_j$; hence, $h^a{}_i$ and $h_a{}^i$ may be viewed as inverse matrices.

A *rigid moving frame* is a frame in which the metric tensor components are constant (See Appendix C for a more complete description of the moving frame), that is,

$$g_{ab,c} = 0.$$

Since the eigenvalues of the KT of interest are real and distinct, by selecting the normalised eigenvectors of the KT as the frame vectors, $\{E_a\}$, we obtain, as proven in Section 3.2, a rigid moving frame. In this quasi-orthonormal basis the metric tensor components take the form

$$g_{ab} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1), \tag{3.17}$$

and the metric may be written as

$$ds^2 = (E^1)^2 + \ldots + (E^p)^2 - (E^{p+1})^2 - \ldots - (E^n)^2, \tag{3.18}$$

where the *signature* of the metric, $2p - n$, is independent of the basis by *Sylvester's law of inertia.*

In addition, the KT components,

$$\begin{aligned} K_{ab} &= h_a{}^i h_b{}^j K_{ij}, \\ K^{ab} &= h^a{}_i h^b{}_j K^{ij}, \end{aligned} \tag{3.19}$$

are diagonalised in this frame,

$$K_{ab} = K^{ab} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \tag{3.20}$$

where $\{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of K_{ab} and K^{ab}. The two forms of the tensor, purely contravariant and purely covariant, are identical because the square of the metric is the identity.

3.3 Benenti Theorem

The geometrical approach to separation of variables was initiated by Eisenhart [14, 15], then developed by Benenti [3, 4, 5], Kalnins and Miller [22, 23], Woodhouse [51], and others.

The separability of a system is unaffected by applying a *separated transformation*, $\tilde{q}^i = \tilde{q}^i(q^i)$, $i = 1, \ldots, n$, to a set of separable coordinates. These equivalence classes of coordinates produce (intrinsically) the same complete integral; thus, it is natural to investigate the separability properties of the coordinate hypersurfaces.

On a pseudoRiemannian manifold, M, we define an *orthogonal web* as a family, $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$, of n orthogonal and transversal foliations of hypersurfaces. These submanifolds of dimension $(n-1)$ are defined on $M - \Omega$, where Ω is a closed *singular set*. A set of n real C^∞ functions (q^i) defined on $M - \Omega$ is a *parametrisation* of the orthogonal web provided that dq^i does not identically vanish at any point and the restriction of q^i to its corresponding leaf $\mathcal{S}_i \in \mathcal{S}$ is constant. Locally, orthogonal coordinates adapted to the web are produced by a parametrisation. If the adapted coordinates are separable, then we say the orthogonal web is *separable*; hence, the problem of intrinsically characterising separability is the determination of the geometric properties of a separable web. Benenti has solved this problem by determining necessary and sufficient conditions for separability based on a single KT, see [4].

Theorem 3.1 (Benenti) *A Hamiltonian* (1.6) *on a (pseudo)Riemannian manifold, M, is separable in orthogonal coordinates if and only if there exists a Killing tensor, $\mathbf{K}$, on M, with pointwise (real) simple eigenvalues and orthogonally integrable eigenvectors, that satisfies the potential separability condition* (2.22).

3.3.1 Proof of Benenti Theorem

We prove the Benenti theorem, Theorem 3.1, for a Riemannian manifold, M, then show how the proof may be generalised to a pseudoRiemannian manifold. Although the theory is global, we use local coordinate representations for the proofs. In addition, we assume all objects are C^∞. The essence of the proof is given in [4].

To prove Theorem 3.1, we need two lemmas.

Lemma 3.1 *A Killing tensor,* $\mathbf{K}$*, with (real) simple eigenvalues and orthogonally integrable eigenvectors can be written, in some orthogonal coordinate system,* (u^i)*, in the form*

$$\mathbf{K} = \sum_{i=1}^{n} \lambda_i g^{ii} X_i \otimes X_i, \tag{3.21}$$

where $X_i := \partial/\partial u^i$*. In these coordinates, the Killing tensor equation* (2.19) *may be written as*

$$\lambda_{j,i} = (\lambda_i - \lambda_j)(\ln g^{jj})_{,i}, i \neq j, \tag{3.22}$$

and

$$\lambda_{i,i} = 0. \tag{3.23}$$

Proof of Lemma 3.1

We are interested in the case of distinct eigenvalues; furthermore, the eigenvectors must be orthogonally integrable, that is, the orthogonal distribution, $\Delta^\perp$, must be completely integrable. Because each eigenvector is orthogonal to an $n-1$ dimensional hypersurface, we express this condition as the n equations on the ω-frame 1-forms

$$E^a \wedge dE^a = 0, \;\; a = 1, \ldots, n. \tag{3.24}$$

We rewrite these equations in a useful form. The Froebenius theorem, Theorem A.1, implies that the ω-frame 1-forms satisfy the equations: $E^a = \delta^a_i f^i du^i$, where $f^i = f^i(\mathbf{q})$, and $u^i = u^i(\mathbf{q})$. Since the (u^i) are independent, we choose them as the coordinates of the natural basis; hence, we have

$$E^a = \delta_i^a f^i du^i = f^a du^a, \ a = 1, \dots, n, \tag{3.25}$$

for undetermined functions $f^a(\mathbf{u})$; thus, the frame vectors satisfy

$$E_a = \frac{\delta_a^i}{f^i}\frac{\partial}{\partial u^i} = \frac{1}{f^a}\frac{\partial}{\partial u^a}. \tag{3.26}$$

That is,

$$h_a{}^i = \frac{\delta_a^i}{f^i} = \frac{1}{f^a}, \tag{3.27}$$

and

$$h_i^a = f^i \delta_i^a = f^a. \tag{3.28}$$

Comparing the metric in the moving frame (3.18) with the expressions (3.25), we observe that the metric in the natural basis (3.13) has the form

$$ds^2 = (f^1)^2(X^1)^2 + \ldots + (f^n)^2(X^n)^2, \tag{3.29}$$

where $X^i = du^i$. Of course, the corresponding dual basis is $X_i = \partial/\partial u^i$. From (3.29), we observe that the coordinates, (u^i), are orthogonal (1.9) and

$$\sqrt{g_{ii}} = f^i.$$

Substituting this into (3.27), we obtain

$$h_a{}^i = \sqrt{g^{ii}}\delta_a^i. \tag{3.30}$$

We recall that K^{ab} is symmetric and of the form (3.20). Using this fact and (3.30) in the relations between the components of the KT in the two frames (3.19), we derive the components of (3.21)

$$\begin{aligned} K^{ij} &= h_a{}^i h_b{}^j K^{ab} \\ &= \sqrt{g^{ii}}\delta^i_a\sqrt{g^{jj}}\delta^j_b K^{ab} \\ &= \begin{cases} \lambda_i g^{ii}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \end{aligned}$$

Now we write the Killing tensor equation (2.19) in the local coordinates. In Subsection 2.2.4, we established that **K** is a KT if and only if the Poisson bracket of the related quadratic in momentum polynomial with the geodesic Hamiltonian (1.21) is zero; hence, we calculate the Poisson bracket of $K = \lambda_i g^{ii} p_i^2$ with (1.21) to determine an equivalent system of partial differential equations,

$$\begin{aligned} \{K, H\} &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\{(2\lambda_i g^{ii} p_i \delta_i^j)(g^{aa}{}_{,j}) - (\lambda_{i,j} g^{ii} + \lambda_i g^{ii}{}_{,j})p_i^2(2g^{aa}p_a\delta_a^j)\} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\{\lambda_i g^{ii} g^{jj}{}_{,i} p_i p_j^2 - (\lambda_{i,j} g^{ii} + \lambda_i g^{ii}{}_{,j} g^{jj} p_j p_i^2)\} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ii}[-g^{jj}\lambda_{j,i} + (\lambda_i - \lambda_j)g^{jj}{}_{,i}]p_i p_j^2. \end{aligned}$$

By requiring that the result be equal to zero, we obtain that each term must vanish, that is,

$$g^{jj}\lambda_{j,i} = (\lambda_i - \lambda_j)g^{jj}{}_{,i}.$$

In the case $i \neq j$, this simplifies to (3.22). If $i = j$, then it becomes (3.23). $\square$

Lemma 3.2 *Local orthogonal coordinates (u^i) are separable if and only if the equations*

$$g^{kk}{}_{,ji} = (\ln g^{ii})_{,j} g^{kk}{}_{,i} + (\ln g^{jj})_{,i} g^{kk}{}_{,j}, \;\; i \neq j, \tag{3.31}$$

are satisfied. If the potential, V, is non-zero, then it is separable with respect to the orthogonal separable coordinates (u^i) if and only if the equations

$$V_{,ij} = (\ln g^{ii})_{,j} V_{,i} + (\ln g^{jj})_{,i} V_{,j}, \;\; i \neq j, \tag{3.32}$$

are satisfied.

Proof of Lemma 3.2

The separability of a Hamiltonian system is locally characterised by the Levi-Civita criterion. To determine the characteristic equations for separability in this case, we substitute the Hamiltonian (1.10) into the equations of (1.20) for arbitrary $i \neq j$,

$$\begin{aligned}
&\left(\frac{1}{2}g^{kk}{}_{,i}p_k^2 + V_{,i}\right)\left(\frac{1}{2}g^{kk}{}_{,j}p_k^2\right)(0) - \left(\frac{1}{2}g^{kk}{}_{,i}p_k^2 + V_{,i}\right)(g^{jj}p_j)(g^{ii}{}_{,j}p_i) - \\
&(g^{ii}p_i)\left(\frac{1}{2}g^{kk}{}_{,j}p_k^2 + V_{,j}\right)(g^{jj}{}_{,i}p_j) + (g^{ii}p_i)(g^{jj}p_j)\left(\frac{1}{2}g^{kk}_{,ji}p_k^2 + V_{,ij}\right) = 0 \\
\Leftrightarrow \quad &\frac{1}{2}(-g^{ii}{}_{,j}g^{jj}g^{kk}{}_{,i} - g^{ii}g^{jj}{}_{,i}g^{kk}{}_{,j} + g^{ii}g^{jj}g^{kk}{}_{,ji})p_ip_jp_k^2 + \\
&(-V_{,i}g^{ii}{}_{,j}g^{jj} - V_{,j}g^{ii}g^{jj}{}_{,i} + g^{ii}g^{jj}V_{,ij})p_ip_j = 0.
\end{aligned}$$

The term quartic in momenta must vanish independently of the quadratic term. By requiring that the quartic equals zero for $i \neq j$, we derive

$$g^{ii}g^{jj}g^{kk}{}_{,ji} = g^{ii}{}_{,j}g^{jj}g^{kk}{}_{,i} + g^{ii}g^{jj}{}_{,i}g^{kk}{}_{,j}$$

$$\Leftrightarrow \quad g^{kk}{}_{,ji} = (\ln g^{ii})_{,j}g^{kk}{}_{,i} + (\ln g^{jj}), ig^{kk}{}_{,j};$$

hence, we have established condition (3.31). Similarly, by demanding that the quadratic term equals zero for any $i \neq j$, we obtain (3.32),

$$g^{ii}g^{jj}V_{,ij} = V_{,i}g^{ii}{}_{,j}g^{jj} + V_{,j}g^{ii}g^{jj}{}_{,i}$$

$$\Leftrightarrow \quad V_{,ij} = (\ln g^{ii}), jV_{,i} + (\ln g^{jj}), iV_{,j}.$$

□

Proof of Theorem 3.1

The theorem is first proven for a Riemannian manifold, then generalised to the pseudoRiemannian case. On a Riemannian manifold, the eigenvalues of a symmetric matrix are real. Also the eigenvectors are necessarily non-null by Proposition 3.5.

Initially we assume that the potential, V is zero. The integrability conditions for the system of partial differential equations comprised of (3.22) and (3.23) are

$$\begin{aligned} &[E_i, E_j]\lambda_h = 0 \\ \Leftrightarrow \quad &\lambda_{h,ji} - \lambda_{h,ij} = 0, \ i \neq j. \end{aligned} \tag{3.33}$$

That is, mixed partial derivatives must be equal. We use this fact in the following derivation.

Substituting (3.22) into (3.33), we obtain

$$[(\lambda_j - \lambda_h)(\ln g^{hh})_{,j}]_{,i} - [(\lambda_i - \lambda_h)(\ln g^{hh})_{,i}]_{,j} = 0$$

$$\begin{aligned} \Leftrightarrow \quad & (\lambda_j - \lambda_h)_{,i}(\ln g^{hh})_{,j} + (\lambda_j - \lambda_h)(\ln g^{hh})_{,ji} - (\lambda_i - \lambda_h)_{,j}(\ln g^{hh})_{,i} \\ & -(\lambda_i - \lambda_h)(\ln g^{hh})_{,ij} = 0 \end{aligned}$$

$$\Leftrightarrow \quad (\lambda_j - \lambda_i)(\ln g^{hh})_{,ij} + (\lambda_{j,i} - \lambda_{h,i})(\ln g^{hh})_{,j} - (\lambda_{i,j} - \lambda_{h,j})(\ln g^{hh})_{,i} = 0.$$

We substitute (3.22) into the final equation to obtain

$$\begin{aligned} & (\lambda_j - \lambda_i)[(\ln g^{hh})_{,ij} - (\ln g^{jj})_{,i}(\ln g^{hh})_{,j} - (\ln g^{ii})_{,j}(\ln g^{hh})_{,i} \\ & +(\ln g^{hh})_{,i}(\ln g^{hh})_{,j}] = 0 \end{aligned}$$

$$\Leftrightarrow (\lambda_j - \lambda_i)({g^{hh}}_{,ij} - (\ln g^{jj})_{,i}{g^{hh}}_{,j} - (\ln g^{ii})_{,j}{g^{hh}}_{,i}) = 0, \ i \neq j. \tag{3.34}$$

The condition (2.22) can be written in local coordinates. We observe from (3.21) that ${K^i}_j = \delta^i_j \lambda_i$ for any KT; hence, $\mathbf{K}dV = \lambda_i V_{,i} du^i$. Using this in the condition (2.22), we obtain, for $i \neq j$,

$$(\lambda_i V_{,i})_{,j} - (\lambda_j V_{,j})_{,i} = 0$$

$$\Leftrightarrow \quad \lambda_{i,j} V_{,i} + \lambda_i V_{,ij} - \lambda_{j,i} V_{,j} - \lambda_j V_{,ji} = 0.$$

We simplify these equations using the Killing tensor equations (3.22),

$$(\lambda_i - \lambda_j) V_{,ij} + (\lambda_j - \lambda_i)(\ln g^{ii})_{,j} V_{,i} - (\lambda_i - \lambda_j)(\ln g^{jj})_{,i} V_{,j} = 0$$

$$\Leftrightarrow (\lambda_i - \lambda_j)(V_{,ij} - (\ln g^{ii})_{,j} V_{,i} - (\ln g^{jj})_{,i} V_{,j}) = 0, \ i \neq j. \tag{3.35}$$

We proceed to show that the existence of a suitable KT, $\mathbf{K}$, and potential, V, that satisfy (2.22), implies the existence of separable coordinates (u^i) in the neighbourhood of a point, P, with respect to which V is compatible. Since the KT, $\mathbf{K}$, is assumed to have pointwise simple eigenvalues, $\{\lambda_i\}$, and orthogonally integrable eigenvectors, Lemma 3.1 implies that the Killing tensor equation has the form (3.22), (3.23). It follows that the integrability conditions (3.34) for this system of differential equations are satisfied. Since $\lambda_i \neq \lambda_j$ for $i \neq j$, equations (3.34) imply that the separability conditions (3.31) hold. By Lemma 3.2, we conclude that the coordinates (u^i) are separable.

By assumption, (2.22) holds. In local coordinates, this is equivalent to the equations (3.35). Because the eigenvalues, $\{\lambda_i\}$, are distinct this implies that the equations (3.32) hold. By Lemma 3.2, we conclude that V is compatible with the separable coordinates; therefore, we have orthogonal separable coordinates (u^i) and a compatible separable potential, V, in a neighbourhood of any point.

We now prove the converse: orthogonal separability implies the existence of a KT with simple eigenvalues and a potential, V that satisfies the condition (2.22). We assume orthogonal separable coordinates, (u^i), exist in a neighbourhood of some point, P; therefore, the metric has the form $ds^2 = g_{11}(du^1)^2 + \ldots + g_{nn}(du^n)^2$ (1.9). We further assume that the Killing tensor has the form (3.21). Letting $X^i := du^i$, we define $E^a := \delta^a_i f^i X^i$ and the corresponding dual vectors, E_a, satisfy (3.26). It is clear that the vectors, E_a, $i = i, \ldots, n$, are the eigenvectors of K_{ab} corresponding to the eigenvalues, λ_a, $a = 1, \ldots, n$. By the Frobenius theorem, Theorem A.1, the equations (3.24) hold. Geometrically this means that each eigenvector is orthogonal to a $n-1$ dimensional hypersurface; thus, the eigenvectors are orthogonally integrable.

By Lemma 3.2, the equation (3.31) is satisfied for every pair (i, j), $i \neq j$. Substituting these equations into (3.34), we find that they are trivially satisfied; thus, the integrability conditions for the Killing tensor equations hold. We conclude that the system of linear equations (3.22), (3.23) can be integrated to yield n linearly independent solutions, $\{\lambda_{ai}\}$, $a = 0, \ldots, n-1$. The linear independence of solutions is equivalent to the condition that the determinant of the matrix (λ_{ai}) never vanishes. We view the complete solution (λ_{ai}) as a set of n vectors, $\{\boldsymbol{\lambda}_a\}$, in an n dimensional vector space. Any linear combination, with constant coefficients, of these vectors is a solution of the linear Killing tensor equations (3.22) (3.23). In fact, the general solution is such a combination, for which the coefficients are the n constants of integration. Since $\det(\lambda_{ai}) \neq 0$, we can find a vector $\boldsymbol{\lambda}$ at any point, P, on M such that its components are distinct, that is, $\lambda_i \neq \lambda_j$, $\forall i \neq j$. Since this holds in some neighbourhood of P, we have pointwise simple eigenvalues for the corresponding KT, $\mathbf{K}$, of the form (3.21).

Since the potential, V, is compatible with the separable coordinates, the condition (3.32) is satisfied; therefore, the equations (3.35) hold. This is a local characterisation of the $d(\mathbf{K}dV) = 0$ condition, that is, (2.22) holds.

This establishes Theorem 3.1 on a Riemannian manifold. We now extend the previous arguments to a pseudoRiemannian manifold. To circumvent possible problems produced by the indefinite metric, we require two changes. In the formulae, we replace g^{ii} with $|g^{ii}|$. More importantly, we require that the eigenvalues of the KT be real (which we must assume for a pseudoRiemannian manifold). $\square$

3.3.2 Remarks

Remark 1 *By varying the n constants of integration in the KT, $\mathbf{K}$, we generate an n dimensional space of KTs, $\mathcal{K} = \{\mathbf{K}_a\}$. Every element, $\mathbf{K}_a \in \mathcal{K}$, has common eigenvectors and satisfies the potential condition* (2.22).

Kalnins and Miller [22] have produced a similar result to Theorem 3.1. In the geodesic case, they require the use of n KTs. Using Theorem 3.1, we consider only one KT; however, verifying the orthogonal integrability condition on the eigenvectors can be difficult.

Remark 2 *We observe that* $\lambda_i = 1$ *is a solution of* (3.22) *and* (3.23)*; therefore, the metric tensor,* $\mathbf{g}$*, is necessarily an element of* $\mathcal{K}$*.*

Remark 3 *In a two dimensional manifold, the basis of* $\mathcal{K}$ *is* $(\mathbf{g}, \mathbf{K}_1)$ *where* $\mathbf{K}_1$ *is independent of the metric. The critical set,* Ω*, is the set of points at which* $\mathbf{K}$ *has identical eigenvalues, that is, where* $\mathbf{K}_1$ *and* $\mathbf{g}$ *are proportional. Benenti and Rastelli have shown that distinct and non-constant eigenvalues,* λ_1 *and* λ_2*, form a parametrisation of the separable web [6].*

Remark 4 *As a corollary to the theorem, we observe that* $(n-1)$ *first integrals, in addition to the Hamiltonian, can be computed from* $\mathcal{K}$*. The function* $K^{ab}p_ap_b + U(\mathbf{u})$ *is a first integral provided* $d\mathbf{U} = 2\mathbf{K}dV$ (2.21)*, as derived in Subsection 2.2.4. We consider the* n *KTs,* $\mathbf{K}_a$*, of the form* (3.21)*, each with eigenvalues* $\{\lambda_{ai} g^{ii} : i = 1, \dots, n\}$*. Given smooth functions* $U_a : M \to \mathbb{R}$ *defined locally by* $d\mathbf{U}_a = 2\mathbf{K}_a dV$*, the* n *functions*

$$I_a = \lambda_{ai} g^{ii} p_i^2 + U_a$$

are independent first integrals in involution. Referring to Remark 2, we observe that in the case $\mathbf{K}_0 = \mathbf{g}$*, the corresponding first integral is the Hamiltonian scaled by a factor of* 2*. The remaining* $(n-1)$ *elements of* $\mathcal{K}$ *generate the* $(n-1)$ *additional first integrals.*

3.4 Example: Non-periodic, Finite Toda Lattice

We proceed to apply the Benenti theory to the aforementioned system, the non-periodic, finite dimensional Toda lattice. This system has been studied using most of the known techniques of complete integrability including the Lax and bi-Hamiltonian methods mentioned in Chapter 2. We use the classical approach to Hamilton-Jacobi theory, in which the separable coordinates are related to the generalised physical position-momenta coordinates by a point transformation (2.9).

We show, using Theorem 3.1, that the lattice is separable via a point transformation to separable coordinates in only the two dimensional case. For higher dimensional lattices, there does not exist *any* separable coordinate system related to the physical position-momenta coordinates by a point transformation. In the two dimensional case, the trajectories are explicitly found by separation of variables.

The Levi-Civita criterion, Theorem 1.5, provides a local characterisation of separability. That is, it indicates whether or not a Hamiltonian is written in separable coordinates. We observe that the only mixed second partial derivatives of H that are not identically zero result from

$$\frac{\partial^2 H}{\partial q^i \partial q^{i+1}} = -e^{q^i - q^{i+1}};$$

hence, substituting (2.43) into the $\frac{n}{2}(n-1)$ equations of the Levi-Civita criterion (1.20), we obtain $(n-1)$ equations with single terms,

$$-p_i p_{i+1} e^{q^i - q^{i+1}},$$

that never vanish; therefore, the Toda lattice is not separable in the given set of Cartesian coordinates.

3.4.1 $n = 2$

In this section, we derive an explicit solution for the non-periodic, two dimensional Toda lattice using the Hamilton-Jacobian approach without a priori knowledge of separability. We seek behaviour in agreement with that produced by other methods.

The orthogonal integrability of this system is well established. The existence of orthogonally separable coordinates is verified by showing that the conditions of Theorem 3.1 are satisfied. The eigenvalues are $\{-1, 1\}$ and in a two dimensional space the eigenvectors are necessarily orthogonally integrable. Alternatively, the Bertrand-Darboux-Whittaker theorem [50] guarantees the separability of our system as a Hamiltonian system with two degrees of freedom from the existence of a second first integral quadratic in the momenta, $H_2 = p_1 p_2 - e^{q^1 - q^2}$ [8]. We observe that the required condition on the potential provided by Whittaker [50] — that the expression $(a^2 - b^2)V_{,12} + ab(V_{,22} - V_{,11})$ vanishes for some constants $a, b \in \mathbb{R}$ — is satisfied. The expression $V_{,22} - V_{,11}$ vanishes identically; hence, any $a = \pm b$ suffices. We conclude that orthogonal separable coordinates exist.

The first step is to find separable coordinates. As we showed in the preceding subsection, the HJ equation for this system does not separate in the physical position-momenta coordinates because the Levi-Civita criterion is not satisfied. A translation does not transform the Hamiltonian to a form that satisifies the Levi-Civita criterion. It is natural to proceed by trying a rotation. This choice is reasonable because the system admits a first integral linear in momentum, $H_1 = p_1 + p_2$ (see the bi-Hamiltonian example in Subsection 2.5.2). A suitable point transformation is a rotation of $\pi/4$ radians about the origin,

$$\tilde{q}^1 = \frac{q^1 - q^2}{\sqrt{2}}, \qquad \tilde{q}^2 = \frac{q^1 + q^2}{\sqrt{2}}; \tag{3.36}$$

thus,

$$\tilde{p}_1 = \frac{p_1 - p_2}{\sqrt{2}}, \qquad \tilde{p}_2 = \frac{p_1 + p_2}{\sqrt{2}}.$$

Dropping the tildes, we obtain the transformed Hamiltonian

$$H = 1/2p_1^2 + 1/2p_2^2 + e^{\sqrt{2}q^1}. \tag{3.37}$$

In this form, it is clear that the Levi-Civita criterion is trivially satisfied because the second derivative of the Hamiltonian with repect to any combination of position and momenta variables is zero; hence, we have separability in these coordinates. We note that rotations of $3\pi/4$, $5\pi/4$, or $7\pi/4$ radians also yield separable coordinates; furthermore, any transformation of the form $\tilde{q}^i = f^i(q^i)$ of the new coordinates preserves separability of the system.

We now proceed to find the complete integral W of the system of the form (1.8). Using the separability ansatz (1.8), the Hamilton-Jacobi equation of the system,

$$1/2(W_{,1})^2 + 1/2(W_{,2})^2 + e^{\sqrt{2}q^1} = E,$$

reduces to two ordinary differential equations

$$\begin{cases} 1/2(W_2')^2 = 1/2\alpha^2, \\ 1/2(W_1')^2 + 1/2\alpha^2 + e^{\sqrt{2}q^1} = E. \end{cases} \tag{3.38}$$

We observe that α is the separation constant. Solving (3.38) by quadratures, we obtain a complete integral, W, of the form

$$\begin{aligned} W = {} & -Et + 2\sqrt{E - 1/2\alpha^2 - e^{\sqrt{2}q^1}} \\ & -2\sqrt{E - 1/2\alpha^2}\tanh^{-1}\left(\frac{\sqrt{E - 1/2\alpha^2 - e^{\sqrt{2}q^1}}}{\sqrt{E - 1/2\alpha^2}}\right) + \alpha q^2. \end{aligned} \tag{3.39}$$

According to classical Hamilton-Jacobi theory, the partial derivatives $\partial W/\partial E$ and $\partial W/\partial\alpha$ are first integrals of the system. The orbits of the system may be obtained by solving the equations $\partial W/\partial E = c_1$, $\partial W/\partial\alpha = c_2$ (2.30). Using (3.39), these equations are

$$\begin{cases} -t - \dfrac{1}{\sqrt{E - 1/2\alpha^2}} \tanh^{-1}\left(\dfrac{\sqrt{E - 1/2\alpha^2 - e^{\sqrt{2}q^1}}}{\sqrt{E - 1/2\alpha^2}} \right) = c_1, \\ \\ q^2 + \dfrac{\alpha}{\sqrt{E - 1/2\alpha^2}} \tanh^{-1}\left(\dfrac{\sqrt{E - 1/2\alpha^2 - e^{\sqrt{2}q^1}}}{\sqrt{E - 1/2\alpha^2}} \right) = c_2. \end{cases}$$

Solving these equations for the transformed position coordinates, we obtain a solution for the orbits,

$$\begin{cases} q^1 = 1/\sqrt{2} \ln[(E - 1/2\alpha^2)(1 - \tanh^2((t + c_1)\sqrt{E - 1/2\alpha^2}))], \\ q^2 = \alpha(t + c_1) + c_2. \end{cases}$$

Converting to the original coordinates using the inverse transformation corresponding to (3.36), we produce a *closed form* solution for the orbits,

$$\begin{cases} q^1 = 1/2 \ln[(E - 1/2\alpha^2)(1 - \tanh^2((t + c_1)\sqrt{E - 1/2\alpha^2}))] \\ \qquad +1/\sqrt{2}\alpha(t + c_1) + 1/\sqrt{2}c_2, \\ q^2 = -1/2 \ln[(E - 1/2\alpha^2)(1 - \tanh^2((t + c_1)\sqrt{E - 1/2\alpha^2}))] \\ \qquad +1/\sqrt{2}\alpha(t + c_1) + 1/\sqrt{2}c_2. \end{cases} \tag{3.40}$$

It is obvious from the form of the solution that both c_1 and c_2 are inessential constants. We may eliminate c_1 by a time origin translation and c_2 by a space origin translation.

Substituting $p_2 = W_{,2} = \alpha$, from Hamilton's equation and (3.38), into (3.37), using $H = E$, we see that $E - 1/2\alpha^2$ is strictly positive because the exponential term is strictly positive and the remaining term is non-negative. Since the value of the hyperbolic tangent function varies in the open interval $(-1, 1)$, the second factor

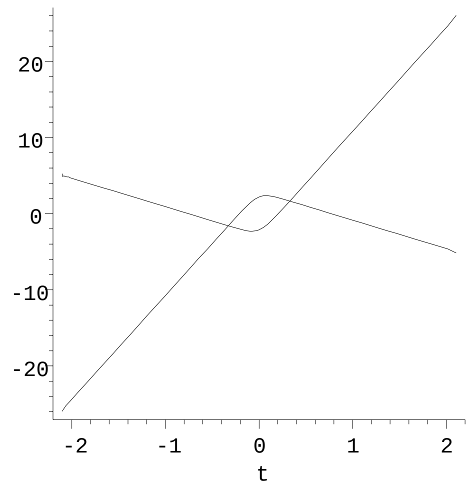

Figure 3.1: Trajectories of two particles in non-periodic Toda lattice where $c_1 = c_2 = 0$ (to eliminate any translation from the origin), $E = 100$, and $\alpha = 7$. This corresponds to some initial value problem.

inside the logarithm term is also strictly positive; therefore, the orbits described by (3.40) are well behaved; however, either of these factors can be arbitrarily small. In fact the second tends to zero as time tends to positive or negative infinity; therefore, the logarithmic term, though bounded above, is *not* bounded below. It follows that the particles exhibit unbounded motion (see Figure 3.1) in agreement with previous results concerning the Toda lattice as a completely integrable Hamiltonian system. It is well known that the non-periodic Toda lattice admits cylinders, rather than tori, as the invariant submanifolds in the theory of Arnol'd-Liouville [2].

Although the Hamilton-Jacobi approach solves the non-periodic Toda lattice with two degrees of freedom, the solution may be obtained more simply using the Hamiltonian formalism. Consider the Hamiltonian for the transformed coordinates. From the corresponding Hamiltonian (3.37), it follows that q^2 is an ignorable coordinate; thus, the system can be reduced to a one dimensional system. We solve Hamilton's equations directly!

Examining the equation

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = -\frac{\partial H}{\partial q^2},$$

we conclude, since H is independent of q^2, that p_2 is constant with respect to time. Setting $p_2 = \alpha$, we obtain

$$q^2 = \alpha t + q_0^2. \tag{3.41}$$

Substituting α for p_2, the Hamiltonian (3.37) becomes

$$H = 1/2p_1^2 + 1/2\alpha^2 + e^{\sqrt{2}q^1}.$$

Using the fact that $H = E$, we solve for p_1. Substituting p_1 into the Hamilton equation,

$$\frac{\mathrm{d}q^1}{\mathrm{d}t} = \frac{\partial H}{\partial p_1},$$

we obtain the first order ordinary differential equation

$$\frac{\mathrm{d}q^1}{\mathrm{d}t} = \sqrt{2}\sqrt{E - 1/2\alpha^2 - e^{\sqrt{2}q^1}},$$

where we have used the fact that $H = E$. This can be separated trivially, then solved by quadratures to obtain

$$t + t_0 = -\frac{1}{\sqrt{E - 1/2\alpha^2}}\tanh^{-1}\left(\frac{\sqrt{E - 1/2\alpha^2 - e^{\sqrt{2}q^1}}}{\sqrt{E - 1/2\alpha^2}}\right).$$

Solving for q^1, we derive

$$q^1 = 1/\sqrt{2}\ln[(E - 1/2\alpha^2)(1 - \tanh^2((t + t_0)\sqrt{E - 1/2\alpha^2}))]. \tag{3.42}$$

Converting (3.41) and (3.42) to the original position-momenta coordinates, we produce a description of the trajectories that agrees with that found previously by the Hamilton-Jacobi method (3.40). The relationships amongst the constants appearing in the solution's different forms may easily be obtained.

3.4.2 $n = 3$

In this section, we study the non-periodic, three dimensional Toda lattice in the framework of Theorem 3.1 since the conclusion is different from the case $n = 2$ and the calculations extend naturally to the general case, $n \geq 3$.

We begin with the condition (2.22) on the characteristic K-tensor $\mathbf{K}$. In this case the potential $V(q^1, q^2) = e^{q^1-q^2} + e^{q^2-q^3}$ and $\mathbf{K}$ is a KT in three dimensional Euclidean space with the standard metric, $\mathbf{g}$, whose components with respect to the Cartesian coordinates $\mathbf{q} = (q^1, q^2, q^3)$ are $g_{ij} = \mathrm{diag}(1, 1, 1)$.

The Killing tensor equation (2.19) with respect to Cartesian coordinates has the form

$$K_{ij,k} + K_{jk,i} + K_{ki,j} = 0. \tag{3.43}$$

Setting $i = j = k$ in (3.43), we immediately observe that

$$K_{ii,i} = 0. \tag{3.44}$$

That is, the components K_{ii} do not depend on the coordinate q^i, for $i = 1,\ 2,\ 3$. If exactly two of the indices are identical, then (3.43) implies a partial differential equation of the form

$$K_{ii,j} = -2K_{ij,i},\ i \neq j. \tag{3.45}$$

In the case that all indices are distinct, (3.43) retains its form

$$K_{ij,k} + K_{jk,i} + K_{ki,j} = 0,\ i \neq j \neq k \neq i. \tag{3.46}$$

The integrability conditions for the Killing tensor equation (3.43) are the $n^2(n+1)^2/4$ equations

$$K_{ij,kl} = K_{ij,lk},\ 1 \leq i \leq j \leq n,\ 1 \leq k \leq l \leq n, \tag{3.47}$$

representing the equality of the mixed second partial derivatives.

Solving the system of differential equations (3.44) (3.45) (3.46) subject to the integrability conditions (3.47), we obtain the following expressions for the covariant components of the tensor $\mathbf{K}$:

$$\begin{cases} K_{11} = a(q^2)^2 + bq^2q^3 + c(q^3)^2 + dq^2 + eq^3 + f, \\ K_{22} = a(q^1)^2 + hq^1q^3 + i(q^3)^2 + jq^1 + kq^3 + l, \\ K_{33} = c(q^1)^2 + nq^1q^2 + i(q^2)^2 + pq^1 + \alpha q^2 + r, \\ K_{12} = -aq^1q^2 - \frac{1}{2}hq^3q^2 - \frac{1}{2}jq^2 - \frac{1}{2}bq^3q^1 - \frac{1}{2}dq^1 + \frac{1}{2}n(q^3)^2 + tq^3 + u, \\ K_{13} = -cq^3q^1 - \frac{1}{2}bq^2q^1 - \frac{1}{2}eq^1 - \frac{1}{2}nq^2q^3 - \frac{1}{2}pq^3 + \frac{1}{2}h(q^2)^2 + wq^2 + \delta, \\ K_{23} = -iq^2q^3 - \frac{1}{2}nq^1q^3 - \frac{1}{2}\alpha q^3 - \frac{1}{2}hq^1q^2 - \frac{1}{2}kq^2 + \frac{1}{2}b(q^1)^2 - (t+w)q^1 + \gamma, \end{cases} \tag{3.48}$$

where the twenty constants indicate the dimension of the related space of KTs, calculated using known formulae, see, for example, [45].

We use the fact that K_{ii} is independent of q^i in (2.22) to derive an equivalent system of partial differential equations. We let $\phi_j = K^i{}_j V_{,i} = K_{ij} V_{,i}$ since the metric is the Kroenecker delta. In this notation, (2.22) becomes

$$\frac{1}{2}(\phi_{i,j} - \phi_{j,i}) dq^j \wedge dq^i = 0, \tag{3.49}$$

for each pair $i \neq j$. Calculating these functions using the Toda potential energy function from (2.43), we obtain

$$\begin{cases} \phi_1 = K_{11}(q^2, q^3) e^{q^1 - q^2} + K_{12}(\mathbf{q})(-e^{q^1-q^2} + e^{q^2-q^3}) + K_{13}(\mathbf{q})(-e^{q^2-q^3}), \\ \phi_2 = K_{12}(\mathbf{q}) e^{q^1 - q^2} + K_{22}(q^1, q^3)(-e^{q^1-q^2} + e^{q^2-q^3}) + K_{23}(\mathbf{q})(-e^{q^2-q^3}), \\ \phi_3 = K_{13}(\mathbf{q}) e^{q^1 - q^2} + K_{23}(\mathbf{q})(-e^{q^1-q^2} + e^{q^2-q^3}) + K_{33}(q^1, q^2)(-e^{q^2-q^3}). \end{cases} \tag{3.50}$$

For $j = 1,\ i = 2$, substituting (3.50) into (3.49), we derive the condition

$$\begin{aligned} &\tfrac{1}{2}[K_{12,1} e^{q^1-q^2} + K_{12} e^{q^1-q^2} + K_{22,1}(-e^{q^1-q^2} + e^{q^2-q^3}) - K_{22} e^{q^1-q^2} \\ &+ K_{23}(-e^{q^2-q^3})] - \tfrac{1}{2}[K_{11,2} e^{q^1-q^2} - K_{11} e^{q^1-q^2} + K_{12,2}(-e^{q^1-q^2} + e^{q^2-q^3}) \\ &+ K_{12}(e^{q^1-q^2} + e^{q^2-q^3}) + K_{13,2}(-e^{q^2-q^3}) + K_{13}(-e^{q^2-q^3})] = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & (K_{12,1} + K_{12} - K_{22,1} - K_{22} - K_{11,2} + K_{11} + K_{12,2} - K_{12}) e^{q^1-q^2} \\ & + (K_{22,1} - K_{23,1} - K_{12,2} - K_{12} + K_{13,2} + K_{13}) e^{q^2-q^3} = 0. \end{aligned} \tag{3.51}$$

Similarly, for $j = 2,\ i = 3$, we obtain

$$\begin{aligned}&[(K_{13,2} - K_{13} - K_{23,2} + K_{23}) - (K_{12,3} - K_{22,3})]e^{q^1-q^2} \\ &+[(K_{23,2} - K_{33,2} - K_{33}) - (K_{22,3} - K_{22} - K_{23,3})]e^{q^2-q^3} = 0,\end{aligned} \tag{3.52}$$

and, for $j = 1,\ i = 3$, we produce

$$\begin{aligned}&[(K_{13,1} + K_{13} - K_{23,1} - K_{23}) - (K_{11,3} - K_{12,3})]e^{q^1-q^2} \\ &+[(K_{23,1} - K_{33,1}) - (K_{12,3} - K_{12} - K_{13,3} + K_{13})]e^{q^2-q^3} = 0.\end{aligned} \tag{3.53}$$

By requiring that the coefficients of the exponential terms of (3.51), (3.52), and (3.53) vanish independently, we obtain the six conditions:

$$\left\{\begin{aligned}&K_{12,1} - K_{22,1} - K_{22} - K_{11,2} + K_{11} + K_{12,2} = 0, \\ &K_{13,2} + K_{22,1} - K_{23,1} - K_{12,2} + K_{13} - K_{12} = 0, \\ &K_{13,2} - K_{13} - K_{23,2} + K_{23} - K_{12,3} + K_{22,3} = 0, \\ &K_{23,2} - K_{33,2} - K_{33} - K_{22,3} + K_{22} + K_{23,3} = 0, \\ &K_{13,1} + K_{13} - K_{23,1} - K_{23} - K_{11,3} + K_{12,3} = 0, \\ &K_{23,1} - K_{33,1} - K_{12,3} + K_{12} + K_{13,3} - K_{13} = 0.\end{aligned}\right. \tag{3.54}$$

We substitute the Killing tensor component equations (3.48) into (3.54) to derive that K_{ij} has the constant form

$$K_{11} = K_{22} = K_{33} = a, \quad K_{12} = K_{13} = K_{23} = b.$$

The characteristic equation of the matrix defining $\mathbf{K}$ is

$$(a - \lambda)^3 - 3b^2(a - \lambda) + 2b^3 = 0$$

$$\Rightarrow \quad (a - \lambda + 2b)(a - \lambda - b)^2 = 0.$$

It follows that if $b \neq 0$, then $\mathbf{K}$ has an eigenvalue with multiplicity two. Otherwise, the only eigenvalue has multiplicity three; thus, there is no characteristic K-tensor $\mathbf{K}$ for the system (2.42) satisfying the conditions of Theorem 3.1; hence, for the Hamiltonian system (2.42), *there is no point transformation* (2.9) *from the original position-momenta coordinates to a separable coordinate system.*

The calculations of this section can be generalised to show that the non-periodic, finite dimensional Toda lattice cannot be separated via a point transformation for any dimension $n \geq 3$. For an arbitrary dimension, $n \geq 3$, the Killing tensor has the constant form $K^{ii} = b =$ constant, $K^{ij} = a =$ constant, $1 \leq i \neq j \leq n$. Our conclusion follows from Theorem 3.1 using the fact that the matrix does *not* have n distinct eigenvalues. Similarly, the n dimensional periodic Toda lattice, for $n \geq 3$, cannot be transformed to a separable coordinate system by a point transformation. The matrix corresponding to the Killing tensor has *all* entries equal, that is, $K^{ij} = a =$constant, $1 \leq i, j \leq n$. For details, see [8].

Chapter 4

Finding Separable Coordinates

Theorem 3.1 is very useful because it provides a criterion to determine whether or not there exist orthogonal separable coordinates, related to the position-momenta coordinates by a point transformation (2.9), for a Hamiltonian (1.6); however, if separable coordinates exist, we need a procedure to find them. In this chapter, we apply the method of moving frames to the intrinsic characterisation theory of Benenti to produce such a formalism for low dimensional Hamiltonian systems.

The method is applied to a two dimensional Riemannian manifold of arbitrary curvature to find the general form of the separable metrics, and their corresponding Killing tensors, separable potentials, and second first integrals. As an example, we investigate Euclidean 2-space, E_2, and the surfaces of constant curvature, recovering known results. Using our formalism, we also recover the superseparable potentials of Euclidean 2-space.

4.1 Using the Moving Frame Formalism

Benenti and Rastelli have developed an algorithm that determines separable coordinates but it is computationally intensive, see, for example, [37]. We propose an alternate method that studies the Killing tensor in a rigid moving frame, the orthonormal frame introduced in subsection 3.2.1.

The dual bases of moving frame vector fields, $\{E_a\}$, and ω-frame 1-forms, $\{E^a\}$, are defined by (3.14), and (3.15), respectively. As discussed in Subsection 3.2.1, in the rigid moving frame, the component matrices of the metric and KT are diagonalised, (3.17) and (3.20), respectively. The orthogonal integrability condition of Theorem 3.1 implies the forms (3.27) and (3.28) of the functions in the definitions of $\{E_a\}$, and $\{E^a\}$ given by (3.14), and (3.15), respectively.

In addition, we must consider Cartan's first and second structure equations. The Levi-Civita connection is torsion-free by definition; therefore, the torsion 2-form vanishes identically. This condition on the first Cartan structure equation, leads to

$$dE^a + \omega^a{}_b \wedge E^b = 0, \tag{4.1}$$

where the connection 1-form, $\omega^a{}_b$, is defined by

$$\omega^a{}_b := \Gamma_{cb}{}^a E^c. \tag{4.2}$$

The curvature of the surface is described by Cartan's second structure equation

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \Theta^a{}_b, \tag{4.3}$$

where

$$\Theta^a{}_b := \frac{1}{2} R^a{}_{bcd} E^c \wedge E^d \tag{4.4}$$

is the curvature 2-form.

By solving the three equations (2.19), (3.24), and (4.1), we obtain all possible separable orthogonal coordinate systems for the pure geodesic Hamiltonian, $H = \frac{1}{2} g^{ij} p_i p_j$, on M. If the potential energy term, V, is non-zero, then the additional restriction (2.22) must be considered. This determines the general form of the separable potential for each coordinate system.

4.2 Two Dimensional Riemannian Manifold

In this section, we investigate a general two dimensional Riemannian manifold, M. We develop the forms of the orthogonal integrability condition on the frame vectors (3.24), Killing tensor equation (2.19) and its integrability conditions, and Cartan's first equation (4.1). Then we solve the cases of the associated equations in the following subsections.

In two dimensions, the orthogonal integrability condition on the Killing tensor eigenvectors, that is, the vectors of the moving frame, (3.24) is automatically satisfied.

With respect to the moving frame, we know that the connection coefficients satisfy the skew-symmetry property,

$$\Gamma_{abc} = -\Gamma_{acb}. \tag{4.5}$$

From the definition of the connection 1-form (4.2), we observe that

$$\omega_{ab} = -\omega_{ba}, \tag{4.6}$$

in the moving frame. Because the manifold is two dimensional, we conclude from (4.5) and (4.6) that the connection coefficients have only four non-zero components, $\Gamma_{112} = -\Gamma_{121}$ and $\Gamma_{212} = -\Gamma_{221}$, and the connection 1-form has only two non-zero components, $\omega_{12} = -\omega_{21}$. To simplify the form of the equations, we introduce the notation

$$\begin{aligned} \alpha(u^1, u^2) &= \Gamma_{112} = -\Gamma_{121}, \\ \beta(u^1, u^2) &= \Gamma_{212} = -\Gamma_{221}. \end{aligned} \tag{4.7}$$

On a two dimensional Riemannian manifold, the metric in the moving frame

(3.17) is

$$g_{ab} = \mathrm{diag}(1, 1). \tag{4.8}$$

This implies that raising and lowering indices using the metric tensor is simplified. The component functions of any quantity in terms of the moving frame are unchanged by these procedures. For example, from (4.7), we have

$$\begin{aligned}
\Gamma_{11}{}^{2} &= \Gamma_{112} = \alpha = -\Gamma_{121} = -\Gamma_{12}{}^{1}, \\
\Gamma_{21}{}^{2} &= \Gamma_{212} = \beta = -\Gamma_{221} = -\Gamma_{22}{}^{1},
\end{aligned} \tag{4.9}$$

and the remaining components are zero.

The Killing tensor equation is a tensor equation; thus, the natural frame form (2.19) is identical to the moving frame form,

$$K_{(ab;c)} = 0. \tag{4.10}$$

In general, the covariant derivative of a 2-covariant tensor may be written using the frame vector fields and connection coefficients as

$$K_{ab;c} = E_c K_{ab} - K_{d_1 b}\Gamma_{ca}{}^{d_1} - K_{a d_2}\Gamma_{cb}{}^{d_2}.$$

Using the facts that the KT is diagonal, the connection coefficients are skew-symmetric on the last two indices (4.5), and the metric is the identity matrix (4.8), we simplify each of the four cases of the Killing tensor equation (4.10):

$$\begin{aligned}
K_{(11;1)} = 0 \quad &\Rightarrow K_{11;1} = 0 \\
&\Rightarrow E_1 K_{11} = 0,
\end{aligned}$$

$$K_{(11;2)} = 0 \quad \Rightarrow \quad K_{11;2} + 2K_{12;1} = 0$$

$$\Rightarrow \quad E_2 K_{11} - 2K_{22}\Gamma_{11}{}^2 - 2K_{11}\Gamma_{12}{}^1 = 0,$$

$$K_{(12;2)} = 0 \quad \Rightarrow \quad K_{22;1} + 2K_{12;2} = 0$$

$$\Rightarrow \quad E_1 K_{22} - 2K_{22}\Gamma_{21}{}^2 - 2K_{11}\Gamma_{22}{}^1 = 0,$$

$$K_{(22;2)} = 0 \quad \Rightarrow K_{22;2} = 0$$

$$\Rightarrow E_2 K_{22} = 0.$$

Using (3.20) and (4.9), we write these equations as

$$\begin{aligned} E_1\lambda_1 &= 0, \\ E_2\lambda_1 &= 2\alpha(\lambda_2 - \lambda_1), \\ E_1\lambda_2 &= 2\beta(\lambda_2 - \lambda_1), \\ E_2\lambda_2 &= 0. \end{aligned} \tag{4.11}$$

We may conclude immediately that λ_1 is independent of u^1 and λ_2 is independent of u^2; furthermore, λ_1 is constant if and only if α is zero and λ_2 is constant if and only if β is zero. The sufficiency of the $\alpha = 0$ condition is trivially true. The necessity follows from the hypothesis of Theorem 3.1 that requires λ_1 and λ_2 be distinct.

We must also investigate the associated integrability conditions for the KT. The first step is to derive the form of the commutator $[E_1, E_2]$. The moving frame is torsion-free; therefore, the structure coefficients satisfy $C^c{}_{ab} = \Gamma_{ab}{}^c - \Gamma_{ba}{}^c$. In general, $[E_a, E_b] = C^c{}_{ab} E_c$; hence,

$$[E_1, E_2] = -\alpha E_1 - \beta E_2, \tag{4.12}$$

using the notation (4.9). We proceed to apply (4.12) to the two non-zero components of the KT, simplifying using the relations (4.11), to produce partial differential equations for α and β with respect to the coordinates. We consider $[E_1, E_2]\lambda_1$,

$$\begin{aligned} & E_1 E_2 \lambda_1 - E_2 E_1 \lambda_1 = -\alpha E_1 \lambda_1 - \beta E_2 \lambda_1 \\ \Rightarrow \quad & 2(\lambda_2 - \lambda_1) E_1 \alpha + 2\alpha(E_1 \lambda_2 - E_1 \lambda_1) = -2\alpha\beta(\lambda_2 - \lambda_1) \\ \Rightarrow \quad & E_1 \alpha (\lambda_2 - \lambda_1) = -3\alpha\beta(\lambda_2 - \lambda_1). \end{aligned}$$

Since Theorem 3.1 requires that the eigenvalues be distinct, we conclude that

$$E_1 \alpha = -3\alpha\beta. \tag{4.13}$$

Investigating the integrability condition $[E_1, E_2]\lambda_2$, we similiarly produce the equation

$$E_2 \beta = 3\alpha\beta. \tag{4.14}$$

Finally, we examine Cartan's equations. From the first equation (4.1), we derive a useful equation for dE^a and also express α and β in terms of f^1, f^2 and their derivatives with respect to the coordinates, u^1 and u^2.

We substitute (4.2) into (4.1) to obtain

$$dE^a = \Gamma_{cb}{}^a E^b \wedge E^c.$$

Using the skew-symmetry on $\Gamma_{cb}{}^{a}$ and the simplifying notation (4.9), we obtain

$$dE^1 = \alpha E^1 \wedge E^2,$$

$$dE^2 = \beta E^1 \wedge E^2.$$

To clarify the process of finding expressions for α and β, we use the original notation (3.15) in (4.1) to derive

$$\begin{aligned} & dE^a + \omega^a{}_b \wedge E^b = 0 \\ \Rightarrow \quad & d(h^a{}_i du^i) + \Gamma_{cb}{}^a E^c \wedge E^b = 0 \\ \Rightarrow \quad & E_f h^a{}_i h^f{}_j du^j \wedge du^i + \Gamma_{cb}{}^a h^c{}_j h^b{}_i du^j \wedge du^i = 0 \\ \Rightarrow \quad & E_f h^a{}_{[i} h^f{}_{j]} + \Gamma_{cb}{}^a h^c{}_{[j} h^b{}_{i]} = 0. \end{aligned} \tag{4.15}$$

We introduce another notation change to avoid ambiguity in the remaining calculations,

$$u := u^1, \ v := u^2.$$

For the coefficient functions of the moving frame vectors and 1-form, we let

$$f(u,v) := f^1(u,v), \ g(u,v) := f^2(u,v).$$

Evaluating the result of (4.15) in the cases $a = 1$ and $a = 2$, we produce

$$\alpha = -\frac{f_{,v}}{fg}, \tag{4.16}$$

where $_{,v} := \partial/\partial v$, and

$$\beta = \frac{g_{,u}}{fg}, \tag{4.17}$$

where $_{,u} := \partial/\partial u$, respectively.

We proceed by examining Cartan's second equation (4.3). In a moving frame on any two dimensional Riemannian manifold, the equations $d\omega^1{}_1 + \omega^1{}_c \wedge \omega^c{}_1 = 0$ and $d\omega^2{}_2 + \omega^2{}_c \wedge \omega^c{}_2 = 0$ are identically satisfied. In the remaining case, we observe that $\omega^1{}_c \wedge \omega^c{}_2 = \omega^1{}_1 \wedge \omega^1{}_2 + \omega^1{}_2 \wedge \omega^2{}_2 = 0$ by the skew-symmetry of $\omega^a{}_b$ inherited from ω_{ab}; hence, (4.3) reduces to $d\omega^1{}_2 = \Theta^1{}_2$. In a two dimensional Riemannian manifold, the curvature 2-tensor can be written, from the definition (4.4), as $\Theta^1{}_2 = \frac{1}{2}R^1{}_{2ed}E^e \wedge E^d$ by using the skew-symmetry on the last pair of indices of the Riemann tensor. Using this fact and the definition of $\omega^a{}_b$, we see that Cartan's second equation is equivalent to

$$d\Gamma_{c2}{}^1 \wedge E^c + \Gamma_{c2}{}^1\ dE^c = \frac{1}{2}R^1{}_{2ed}E^e \wedge E^d.$$

We substitute for dE^c from Cartan's first equation (4.1) to obtain

$$E_d\Gamma_{c2}{}^1 E^d \wedge E^c + \Gamma_{c2}{}^1(-\omega^c{}_d \wedge E^d) = \frac{1}{2}R^1{}_{2ed}E^e \wedge E^d$$

$$\Rightarrow \quad E_{[e}\Gamma_{d]21} - \Gamma_{c21}\Gamma_{[ed]c} = \frac{1}{2}R_{12ed}$$

$$\Rightarrow \quad E_1\Gamma_{221} - E_2\Gamma_{121} - \Gamma_{121}\Gamma_{121} - \Gamma_{221}\Gamma_{122} + \Gamma_{121}\Gamma_{211} + \Gamma_{121}\Gamma_{211} = R_{1212}.$$

Removing the zero terms and writing with the simplified notation (4.7), we obtain the partial differential equation

$$-E_1\beta + E_2\alpha - \alpha^2 - \beta^2 = R_{1212}. \tag{4.18}$$

With this formulation on a general two dimensional Riemannian manifold, we analyse the three cases: α and β both zero, only one of α and β zero, and neither α nor β zero. The general forms of the separable metric

$$ds^2 = (E^1)^2 + (E^2)^2, \tag{4.19}$$

and the associated Killing tensor (3.20), up to some unknown functions or constants, are derived in each case. With the KT determined, we derive the form of the most general separable potential, $V(u,v)$, admitted. We must convert the separability condition described in Chapter 2, $d(\mathbf{K}dV) = 0$ (2.22), into a useful form,

$$\begin{aligned} & d((K^a{}_b E_a V)E^b) = 0 \\ \Rightarrow \quad & d(K^a{}_b E_a V) \wedge E^b + (K^a{}_b E_a V) \wedge dE^b = 0 \\ \Rightarrow \quad & (E_c K^a{}_b)(E_a V)E^c \wedge E^b + K^a{}_b(E_c E_a V)E^c \wedge E^b - (K^a{}_b E_a V)\omega^b{}_d \wedge E^d = 0, \end{aligned}$$

using Cartan's first equation (4.1). It follows that

$$(E_c K^a{}_b)(E_a V)E^c \wedge E^b + K^a{}_b(E_c E_a V)E^c \wedge E^b - (K^a{}_d E_a V)\Gamma_{cb}{}^d E^c \wedge E^b = 0,$$

using the definition of $\omega^a{}_b$ (4.2), then permuting dummy indices in the final term. After eliminating the zero terms and converting to the simplified notation defined by (4.9), (3.20), we derive

$$\begin{aligned} & (E_{[c} K^a{}_{b]})(E_a V) + K^a{}_{[b}(E_{c]} E_a V) - (K^a{}_d E_a V)\Gamma_{[cb]}{}^d = 0 \\ \Rightarrow \quad & (E_1\lambda_2)(E_2 V) + \lambda_2(E_1 E_2 V) + \alpha\lambda_1(E_1 V) - (E_2\lambda_1)(E_1 V) - \lambda_1(E_2 E_1 V) \\ & +\beta\lambda_2(E_2 V) = 0 \end{aligned}$$

$$\Rightarrow \quad 2\beta(\lambda_2 - \lambda_1)(E_2 V) + \lambda_2(E_1 E_2 + \beta E_2)V - 2\alpha(\lambda_2 - \lambda_1)(E_1 V)$$
$$-\lambda_1(E_2 E_1 - \alpha E_1)V = 0,$$

using the Killing tensor equations (4.11). We observe that

$$2\beta(\lambda_2 - \lambda_1)(E_2 V) + (\lambda_2 - \lambda_1)(E_1 E_2 + \beta E_2)V - 2\alpha(\lambda_2 - \lambda_1)(E_1 V) = 0,$$

using the commutator relation (4.12). Since the eigenvalues of the KT must be distinct, we simplify to obtain

$$E_1 E_2 V + 3\beta E_2 V - 2\alpha E_1 V = 0. \tag{4.20}$$

Once V is determined, we derive the second first integral (4.8) as discussed in Remark 4 of Subsection 3.3.2. Since the KT is diagonal, the form of the second first integral in the moving frame is

$$K = \lambda_1 {p_1}^2 + \lambda_2 {p_2}^2 + U(u, v). \tag{4.21}$$

We calculate U by solving the tensor equation (2.21). Writing this condition in the moving frame, $(E_b U)E^b = (2K^a{}_b E_a V)E^b$, we immediately obtain the system

$$\begin{cases} E_1 U = 2\lambda_1 E_1 V, \\ \\ E_2 U = 2\lambda_2 E_2 V. \end{cases} \tag{4.22}$$

Since the momentum is a vector field, its frame components, $p_a = (p_1, p_2)$, are related to its components in the natural basis, $p_i = (p_u, p_v)$, by

$$\begin{aligned} p_a &= {h_a}^i p_i \\ \Rightarrow \quad p_1 &= \frac{1}{f} p_u, \;\; p_2 = \frac{1}{g} p_v, \end{aligned} \tag{4.23}$$

using (3.27); therefore, the Hamiltonian function (1.6) written in the separable coordinates is

$$H = \frac{1}{2}\Big(\frac{p_u^2}{f^2} + \frac{p_v^2}{g^2}\Big) + V(u, v).$$

Using (4.23), we write the second first integral (4.21) in the coordinates,

$$K = \frac{\lambda_1 {p_u}^2}{f^2} + \frac{\lambda_2 {p_v}^2}{g^2} + U(u, v). \tag{4.24}$$

4.2.1 Case I: $\alpha = \beta = 0$

It is an immediate consequence of equations (4.16) and (4.17) with α and β equal to zero, that f is independent of v and g is independent of u, that is, $f = f(u)$ and $g = g(v)$; hence, $E_1 = f(u)du$ and $E_2 = g(v)dv$. This implies that the metric (4.19) has the form $ds^2 = f^2(u)du^2 + g^2(v)dv^2$. Consider $f(u)du$. There exists a coordinate transformation $\tilde{u} = \tilde{u}(u)$ such that $d\tilde{u} = f(u)du$. Simliarly, there exists coordinate transformation $\tilde{v} = \tilde{v}(v)$, such that $d\tilde{v} = g(v)dv$. Removing the tildes to simplify the notation, we have

$$f = g = 1; \tag{4.25}$$

therefore, the metric (4.19) is

$$ds^2 = du^2 + dv^2. \tag{4.26}$$

This is the metric for Cartesian separable coordinates.

We observe that, for this case, Cartan's second equation (4.18) reduces to

$$R_{1212} = 0,$$

that is, the curvature vanishes everywhere, because both α and β are zero. This means that the surface is flat, that is, the manifold is necessarily Euclidean space, E_2; thus, we have proven that Cartesian separable coordinates exist only on Euclidean spaces.

Since $f = g = 1$, we also observe that the frame and natural basis components of vectors and forms are identical; thus, the moving frame is the natural basis.

To determine the KT for Cartesian coordinates, we substitute (4.25), $\alpha = 0$ and $\beta = 0$ into the Killing tensor equations (4.11). This gives the trivial system

$$\lambda_{1,u} = \lambda_{1,v} = \lambda_{2,u} = \lambda_{2,v} = 0,$$

with the solution $\lambda_1 = c_1$ and $\lambda_2 = c_2$, both constant; hence, the KT is

$$\mathbf{K} = \ \mathrm{diag}(c_1, c_2). \tag{4.27}$$

We observe that this may be written as a linear combination of two KTs, $\mathbf{K} = c_1\mathbf{g} + (c_2 - c_1)\mathbf{K_1}$, where $\mathbf{K_1} = \mathrm{diag}(0, 1)$, consistent with the theory of Kalnins and Miller.

We seek the form of the separable potential, V, for this case. Substituting $\alpha = \beta = 0$ into (4.20), we obtain $V_{,vu} = 0$ since $f = g = 1$ (4.25). We immediately conclude that the potential is separable, that is,

$$V = V_1(u) + V_2(v). \tag{4.28}$$

Substituting the form of V into the differential equations for U (4.22), we produce

$$\begin{cases} U_{,u} = 2c_1 V_{1,u}, \\ \\ U_{,v} = 2c_2 V_{2,v}; \end{cases}$$

therefore, U has the form

$$U = 2c_1 V_1(u) + 2c_2 V_2(v). \tag{4.29}$$

Using (4.27), (4.29), we know that the second first integral (4.24) has the form

$$K = c_1 {p_u}^2 + c_2 {p_v}^2 + 2c_1 V_1(u) + 2c_2 V_2(v). \tag{4.30}$$

We want a second first integral that is functionally independent of the Hamiltonian,

$$H = \frac{1}{2}({p_u}^2 + {p_v}^2) + V_1 + V_2;$$

however, clearly (4.30) contains H. Removing a constant multiple times H from (4.30), which is invariant, then dividing by $2(c_1 - c_2)$, we derive a second first integral independent of H,

$$\begin{aligned} & K = 2c_1 H + (c_2 - c_1)p_v^2 + 2(c_2 - c_1)V_2 \\ \Rightarrow \quad & K = \tfrac{1}{2}{p_v}^2 + V_2(v). \end{aligned}$$

Since the separable coordinates are Cartesian, the second first integral in Cartesian coordinates, (x, y), is

$$K = \frac{1}{2}{p_y}^2 + V_2(y). \tag{4.31}$$

4.2.2 Case IIa: $\alpha = 0$, $\beta \neq 0$

We solve for f by substituting $\alpha = 0$ into (4.16) to obtain $f_{,v} = 0$. This implies that f is independent of v, that is, that $f = f(u)$. Performing a coordinate transformation on u, then dropping the tilde, we obtain

$$f = 1. \tag{4.32}$$

Substituting $\alpha = 0$ into (4.14), we obtain $E_2\beta = 0$. This implies that β is independent of v, that is,

$$\beta = \beta(u). \tag{4.33}$$

We proceed to determine the general form of $g(u, v)$. Substituting (4.17) with (4.32) into (4.14), we obtain

$$\frac{1}{g}\left(\frac{g_{,u}}{g}\right)_{,v} = 0 \Rightarrow (\ln g)_{,uv} = 0;$$

hence, $\ln g$ separates as a sum of functions $A(u) + B(v)$, that is, $g = C(u)E(v)$. Again, we find a coordinate system, by transformation on the independent variable v, so that

$$g = g(u); \tag{4.34}$$

hence, the metric has the form

$$ds^2 = du^2 + g^2(u)dv^2. \tag{4.35}$$

We now derive the curvature tensor component in this case. Substituting (4.32) and (4.34) into (4.17), we find that β has the form

$$\beta = \frac{g_{,u}}{g}. \tag{4.36}$$

To calculate R_{1212}, we substitute $\alpha = 0$ and β (4.36) into the simplified Cartan's second structure equation (4.18),

$$R_{1212} = -\left(\frac{g_{,u}}{g}\right)_{,u} - \left(\frac{g_{,u}}{g}\right)^2$$

$$\Rightarrow R_{1212} = -\frac{g_{,uu}}{g}. \tag{4.37}$$

We observe that, in this case, R_{1212} depends on only one coordinate, u. Since the metric (4.35) is independent of v, the coordinate is ignorable; thus, it is natural that the curvature has no v dependence.

Now we derive the KT. Substituting (4.32), (4.34), $\alpha = 0$, and (4.36) into the Killing tensor equations (4.11), we produce a system of differential equations,

$$\begin{cases} \lambda_{1,u} = \lambda_{1,v} = \lambda_{2,v} = 0, \\ \\ \lambda_{2,u} = \dfrac{2g_{,u}}{g}(\lambda_2 - \lambda_1). \end{cases}$$

We conclude immediately that λ_1 is equal to a constant, say c_1. Then the differential equation for λ_2, which is independent of v, is solved as

$$\frac{d\lambda_2}{du} = 2(\ln g)_{,u}(\lambda_2 - c_1)$$

$$\Rightarrow \quad \int \frac{d\lambda_2}{\lambda_2 - c_1} = 2\int (\ln g)_{,u} du$$

$$\Rightarrow \quad \ln(\lambda_2 - c_1) = 2\ln g(u) + d$$

$$\Rightarrow \quad \lambda_2 = c_2 g^2(u) + c_1;$$

hence, the KT is

$$\mathbf{K} = \operatorname{diag}(c_1, c_2 g^2(u) + c_1). \tag{4.38}$$

Again, we observe that the KT can be written as the sum of two KTs: $\mathbf{K} = c_1\mathbf{g} + c_2\mathbf{K_1}$, where $\mathbf{K_1} = \operatorname{diag}(0, g^2(u))$.

We proceed to calculate the separable potential, V. Substituting $\alpha = 0$ and $f = 1$ (4.32) into (4.20), we obtain the partial differential equation

$$3\beta E_2 V + E_1 E_2 V = 0$$

$$\Rightarrow \quad \frac{3\beta}{g} V_{,v} + \left(\frac{1}{g} V_{,v}\right)_{,u} = 0$$

$$\Rightarrow \quad \frac{2g_{,u}}{g^2} V_{,v} + \frac{1}{g} V_{,vu} = 0,$$

where (4.17) has been used. To simplify this problem, we multiply by an integrating factor, g^3,

$$\Rightarrow \quad 2gg_{,u} V_{,v} + g^2 V_{,vu} = 0$$

$$\Rightarrow \quad (g^2 V_{,v})_{,u} = 0$$

$$\Rightarrow \quad V = \frac{h_2(v)}{g^2(u)} + h_3(u),$$

where $h_1(v)$, $h_2(v)$, and $h_3(u)$ are arbitrary functions of one variable; hence, the separable potential has the form

$$V = V_1(u) + \frac{V_2(v)}{g^2(u)}. \tag{4.39}$$

To find U, we substitute this form of V into (4.22),

$$\begin{cases} U_{,u} = 2c_1\Big(V_{1,u} - \dfrac{2g_{,u}}{g^3}V_2\Big), \\ \\ U_{,v} = 2c_2V_{2,v} + \dfrac{2c_1}{g^2}V_{2,v}, \end{cases}$$

$$\Rightarrow \begin{cases} U = 2c_1V_1 + \dfrac{2c_1V_2}{g^2} + h_1(v), \\ \\ U = 2c_2V_2 + \dfrac{2c_1V_2}{g^2} + h_2(u). \end{cases}$$

Equating the two forms of U, we obtain $h_1(v) = 2c_2V_2$ and $h_2(u) = 2c_1V_1$; thus, the potential has the form

$$U = 2c_1V_1(u) + 2c_2V_2(v) + \frac{2c_1V_2(v)}{g^2(u)}. \tag{4.40}$$

Substituting (4.38), (4.40) into (4.24) then removing the Hamiltonian component,

$$H = \frac{1}{2}\Big(p_u^2 + \frac{p_v^2}{g^2}\Big) + V_1 + \frac{V_2}{g^2},$$

we derive the form of the second first integral

$$\begin{aligned} K &= c_1p_u^2 + \frac{(c_2g^2 + c_1)p_v^2}{g^2} + 2c_1v_1 + 2c_2V_2 + \frac{2c_1V_2}{g^2} \\ \Rightarrow \quad K &= 2c_1H + c_2p_v^2 + 2c_2V_2 \\ \Rightarrow \quad K &= \tfrac{1}{2}{p_v}^2 + V_2(v). \end{aligned} \tag{4.41}$$

4.2.3 Case IIb: $\alpha \neq 0$, $\beta = 0$

Despite the sign differences in the associated equations, the solution in this subcase is similar to that of case IIa. The metric can be written in the form

$$ds^2 = f(v)^2 du^2 + dv^2.$$

Modulo an interchange of the coordinates, this is identical to the metric of case IIa. Consistent with this result, the forms of the Killing tensor, potentials V and U and second first integral are identical to those of case IIa after the coordinates are interchanged; hence, we consider only case II: $\alpha = 0$, $\beta \neq 0$ when we specialise to Euclidean 2-space and the surfaces of constant curvature.

4.2.4 Case III: $\alpha\beta \neq 0$

With the moving frame formalism, the investigation of the general case is as simple as either of the previous cases. We begin by proving that the functions $f(u, v)$ and $g(u, v)$ are identical in this case.

In addition to the two equations (4.16), and (4.17), Cartan's first equation, in conjunction with the integrability conditions, implies that the two functions $f(u, v)$ and $g(u, v)$ are identical provided that neither α nor β is zero. Comparing (4.13), and (4.14), we obtain

$$E_1\alpha = -E_2\beta.$$

Substituting (4.16) into the lefthand side and (4.17) into the righthand side, we obtain the sequence of partial differential equations

$$\frac{1}{f}\left(\frac{f_{,v}}{fg}\right)_{,u} = \frac{1}{g}\left(\frac{g_{,u}}{fg}\right)_{,v}$$

$$\Rightarrow \quad \frac{f_{,uv}}{f} - \frac{f_{,u}f_{,v}}{f^2} - \frac{g_{,uv}}{g} + \frac{g_{,u}g_{,v}}{g^2} = 0$$

$$\Rightarrow \quad [(\ln f - lng)_{,u}]_{,v} = 0$$

$$\Rightarrow \quad \left(\ln \frac{f}{g}\right)_{,uv} = 0;$$

therefore, $\ln(f/g)$ is separable. That is, $\ln(f/g) = A(u) + B(v)$ for unknown functions $A(u)$ and $B(v)$. Exponentiating this relationship, we obtain

$$f = gC(u)D(v),$$

where $C(u) = \exp(A(u))$, $D(v) = \exp(B(v))$. We consider $f(u,v)du = g(u,v)C(u)D(v)du$. There exists a coordinate system such that $d\tilde{u} = C(u)du$; hence, we have established that $f(\tilde{u}, v) = g(\tilde{u}, v)D(v)$. Now we consider $g(\tilde{u}, v)dv = g(\tilde{u}, v)D(v)\big(dv/D(v)\big)$. There exists a coordinate system such that $d\tilde{v} = dv/D(v)$. This implies that $f(\tilde{u}, \tilde{v}) = g(\tilde{u}, \tilde{v})$. For brevity of notation, we remove the tildes, then write

$$f(u,v) = g(u,v); \tag{4.42}$$

henceforth, in the calculations for the case III, we use this simplification (4.42). We proceed to determine the general form of the metric.

We substitute (4.16) and (4.17) into (4.13) to obtain

$$\frac{1}{f}\left(-\frac{f_{,v}}{f^2}\right)_{,u} = \frac{3}{f^4} f_{,u} f_{,v}$$

$$\Rightarrow \quad -\left[\frac{f^2{}_{,v}}{2f^3}\right]_{,u} = \frac{3}{f^3}\frac{f^2{}_{,u}}{2f}\frac{f^2{}_{,v}}{2f},$$

$$\Rightarrow \quad -\frac{f^2{}_{,vu}}{2f^3} + \frac{3f^2{}_{,v}}{2f^4}\frac{f^2{}_{,u}}{2f} = \frac{3f^2{}_{,u}f^2{}_{,v}}{4f^5}$$

$$\Rightarrow \quad f^2{}_{,vu} = 0.$$

This implies that f^2 is separable,

$$f^2 = A(u) + B(v). \tag{4.43}$$

We observe that this condition is a direct result of the vanishing of the torsion and the integrability condition on the Killing tensor equations. We conclude that in any two dimensional Riemannian manifold the integrability conditions for the Killing tensor equation imply that the metric may be written as

$$ds^2 = (A(u) + B(v))(du^2 + dv^2). \tag{4.44}$$

We observe that, in agreement with Theorem 1.3, the metric (4.44) is in Liouville form (1.16).

We seek the form of R_{1212} in this case. Substituting $f = g$ from (4.42) into (4.16) and (4.17), we obtain

$$\alpha = \frac{-B_{,v}}{2\sqrt{(A+B)^3}},$$

and

$$\beta = \frac{A_{,u}}{2\sqrt{(A+B)^3}},$$

respectively. Substituting α and β into the simplified form of Cartan's second equation (4.18), we obtain

$$
\begin{aligned}
R_{1212} = & -\frac{1}{\sqrt{A+B}}\left(\frac{A_{,u}}{2\sqrt{(A+B)^3}}\right)_{,u} - \frac{1}{\sqrt{A+B}}\left(\frac{B_{,v}}{2\sqrt{(A+B)^3}}\right)_{,v} \\
& -\frac{(B_{,v})^2}{4(A+B)^3} - \frac{(A_{,u})^2}{4(A+B)^3} \\
\Rightarrow \quad R_{1212} = & \frac{1}{2(A+B)^2}\left[-A_{,uu} - B_{,vv} + \frac{(A_{,u})^2 + (B_{,v})^2}{A+B}\right].
\end{aligned}
$$

Rewriting, we obtain

$$
(A+B)(A_{,uu} + B_{,vv}) + 2(A+B)^3 R_{1212} = (A_{,u})^2 + (B_{,v})^2. \tag{4.45}
$$

We simplify (4.45) in the case that the curvature R_{1212} is constant. After taking the partial derivatives of (4.45) with respect to u, then with respect to v, we produce a separable differential equation,

$$
\begin{aligned}
& A_{,u}B_{,vvv} + B_{,v}A_{,uuu} + 12R_{1212}A_{,u}B_{,v}(A+B) = 0 \\
\Leftrightarrow \quad & \frac{A_{,uuu}}{A_{,u}} + 12R_{1212}A = -\Big(\frac{B_{,vvv}}{B_{,v}} + 12R_{1212}B\Big) = k^2,
\end{aligned} \tag{4.46}
$$

for some constant $k \geq 0$. We observe that the product $A_{,u}B_{,v}$ is non-zero because $\alpha\beta \neq 0$; therefore, the coupled and separated equations in (4.46) are equivalent. In addition, there is no loss of generality in setting the separation constant equal to a non-negative quantity. If it were negative, we would interchange the roles of A and B by performing a coordinate transformation that interchanges the independent variables, u and v, to produce the above form.

We proceed to find the form of the KT. Substituting (4.16) and (4.17) with (4.43) into (4.11), we obtain the following system of differential equations,

$$\lambda_{1,u} = \lambda_{2,v} = 0, \tag{4.47}$$

$$\lambda_{1,v} = \frac{B_{,v}}{A+B}(\lambda_1 - \lambda_2), \tag{4.48}$$

$$\lambda_{2,u} = \frac{A_{,u}}{A+B}(\lambda_2 - \lambda_1). \tag{4.49}$$

From (4.47), we conclude that λ_1 is independent of u; therefore, $\lambda_{1,v}/B_{,v}$ is a function of only v. Using this fact in (4.48), we conclude that $(\lambda_1 - \lambda_2)/(A+B)$ is a function of only v, say

$$\frac{\lambda_1 - \lambda_2}{A+B} = \delta_1(v). \tag{4.50}$$

Similarly, (4.47) implies that λ_2 is independent of v and (4.49) implies that $(\lambda_2 - \lambda_1)/(A+B)$ is a function of only u, say

$$\frac{\lambda_2 - \lambda_1}{A+B} = \delta_2(u). \tag{4.51}$$

Comparing (4.50) and (4.51), we conclude that

$$-\delta_1(v) = \delta_2(u) = \delta,$$

where δ is a separation constant. Using this relationship we substitute (4.50) and (4.51) into (4.48) and (4.49) to obtain the simple relationships

$$\lambda_{1,v} = -\delta B_{,v},$$

$$\lambda_{2,u} = \delta A_{,u},$$

respectively. Solving these equations, we obtain

$$\begin{cases} \lambda_1 = -\delta B(v) + c_1, \\ \\ \lambda_2 = \delta A(u) + c_2. \end{cases} \tag{4.52}$$

Substituting λ_1 and λ_2 into (4.51), we derive

$$\frac{\delta A + c_2 + \delta B - c_1}{A + B} = \delta$$

$$\Rightarrow \quad \delta + \frac{c_2 - c_1}{A + B} = \delta$$

$$\Rightarrow \quad c_1 = c_2 = c.$$

Substituting for c_1 and c_2 in (4.52), we determine that the KT (3.20) is of the form

$$\mathbf{K} = \operatorname{diag}(-\delta B(v) + c, \delta A(u) + c). \tag{4.53}$$

This KT is a linear combination of the metric and $\mathbf{K_1} = \operatorname{diag}(-B(v), A(u))$, $\mathbf{K} = c\mathbf{g} + \delta\mathbf{K_1}$; therefore, we have shown that the KTs produced by the theory of Benenti, Theorem 3.1, using our moving frame method, agree with the theory of Kalnins and Miller [22] in each case.

We use the fact that f and g are identical in this case to simplify the general equation for V (4.20),

$$3\beta E_2 V + E_1 E_2 V - 2\alpha E_1 V = 0$$

$$\Rightarrow \quad \frac{3f_{,u}}{f^3} V_{,v} + \frac{1}{f}\left(\frac{1}{f} V_{,v}\right)_{,u} + \frac{2f_{,v}}{f^3} V_{,u} = 0$$

$$\Rightarrow \quad \frac{2f_{,u}}{f^3} V_{,v} + \frac{1}{f^2} V_{,vu} + \frac{2f_{,v}}{f^3} V_{,u} = 0.$$

To simplify this problem, we multiply by an integrating factor, f^4,

$$\Rightarrow \quad 2ff_{,u}V_{,v} + f^2V_{,vu} + 2ff_{,v}V_{,u} = 0$$

$$\Rightarrow \quad f^2_{,u}V_{,v} + f^2V_{,vu} + f^2_{,v}V_{,u} = 0.$$

The solution is obvious after we substitute for f^2 using (4.43),

$$\Rightarrow \quad A_{,u}V_{,v} + (A+B)V_{,vu} + B_{,v}V_{,u} = 0$$

$$\Rightarrow \quad [(A+B)V]_{,uv} = 0.$$

We observe that the quantity $(A+B)V$ is additively separable; thus, the form of V is

$$V = \frac{V_1(u) + V_2(v)}{A(u) + B(v)}. \tag{4.54}$$

Using the above form of V in (4.22), we integrate to obtain

$$\begin{cases} U = 2(-\delta B + c)\left(\dfrac{V_1 + V_2}{A+B}\right) + h_1(v), \\ \\ U = 2(\delta A + c)\left(\dfrac{V_1 + V_2}{A+B}\right) + h_2(u), \end{cases}$$

$$\Rightarrow \quad \begin{cases} U = \dfrac{[2(-\delta B + c)V_2 + Bh_1] + (-2\delta BV_1 + Ah_1) + 2cV_1}{A+B}, \\ \\ U = \dfrac{[2(\delta A + c)V_1 + Ah_2] + (2\delta AV_2 + Bh_2) + 2cV_2}{A+B}, \end{cases}$$

for arbitrary functions of one variable, $h_1(v)$ and $h_2(u)$. The two representations of U must be equivalent if a solution exists. Comparing the terms dependent on only

v, we observe that

$$h_1 = 2\delta V_2. \tag{4.55}$$

Similarly, if we equate the terms dependent on only u, we derive that

$$h_2 = -2\delta V_1. \tag{4.56}$$

Using (4.55) and (4.56) for h_1 and h_2, respectively, we obtain identical mixed terms for each representation of U; therefore, U is of the form

$$U = \frac{2(-\delta B(v) + c)V_1(u) + 2(\delta A(u) + c)V_2(v)}{A(u) + B(v)}. \tag{4.57}$$

To find the second first integral, we substitute (4.53) and (4.57) into (4.24), and then remove the Hamiltonian,

$$H = \frac{1}{2}\Big(\frac{p_u^2 + p_v^2}{A(u) + B(v)}\Big) + \frac{V_1 + V_2}{A(u) + B(v)},$$

dependence. We obtain

$$K = \frac{(-\delta B + c)p_u^2 + (\delta A + c)p_v^2 + 2(-\delta B + c)V_1 + 2(\delta + c)V_2}{A + B}$$

$$\begin{aligned} \Rightarrow \quad K &= 2cH + \frac{-\delta B p_u^2 + \delta A p_v^2 - 2\delta B V_1 + 2\delta A V_2}{A + B} \\ \Rightarrow \quad K &= \frac{-B(v)p_u^2 + A(u)p_v^2 - 2B(v)V_1(u) + 2A(u)V_2(v)}{2\big[A(u) + B(v)\big]}. \end{aligned} \tag{4.58}$$

We have found the general solution for the metric, KT, potentials, U and V, and second first integral in terms of the arbitrary functions $A(u)$ and $B(v)$ in the Liouville metric (4.44).

On the surfaces of constant curvature and Euclidean space, E_2, we determine the metric for each separable coordinate system. From this, we restrict the form of the admitted KT, potentials and second first integral. We begin by examining Euclidean space, E_2.

4.3 Euclidean 2-Space, E_2

As mentioned in case I, Euclidean space, E_n, is characterised by the vanishing of the curvature tensor component, R_{1212}; therefore, Cartan's second equation (4.18) simplifies to

$$-E_1\beta + E_2\alpha - \alpha^2 - \beta^2 = 0. \tag{4.59}$$

We observe that the vanishing of the curvature is unrelated to the existence of a Killing tensor because (4.59) is independent of (4.13) and (4.14).

The set of differential equations (4.13), (4.14), (4.16), (4.17), and (4.59) are now solved to produce all the separable metrics in Euclidean space, E_2. Since Euclidean space is flat, there exist coordinate transformations from the separable coordinates to Cartesian coordinates in which the metric is Euclidean. At the singular points of the metric, that is, where the eigenvalues of the KT are equal, analytic extensions are used to extend the separable coordinates to the entire space. Once the metric has been derived, we determine the form of the associated Killing tensor, $\mathbf{K}$. This determines the specialised form of the potentials U and V and the second first integral, K, for the corresponding separable coordinates.

4.3.1 Case I: $\alpha = \beta = 0$

This case was completely analysed in the general Riemannian treatment because $\alpha = \beta = 0$ implies that the manifold is Euclidean. The metric (4.26) is Euclidean (since the separable coordinates are Cartesian), the KT is constant (4.27), and the second first integral is (4.31). The forms of the separable potentials, V and U, were determined to be (4.28) and (4.29), respectively. This is the only constant curvature surface in which Cartesian separable coordinates exist.

4.3.2 Case II: $\alpha = 0$, $\beta \neq 0$

In Euclidean 2-space, we have the simplified Cartan's second equation (4.59). Using it, we determine the form of the function $C(u)$ found in the metric (4.35) and KT (4.38) for this case of the general Riemannnian manifold.

Using $\alpha = 0$, (4.32), and (4.33) in (4.59), we obtain a differential equation for β,

$$\frac{d\beta}{du} = -\beta^2.$$

This separates to yield $\beta = (u + c_1)^{-1}$. Translating u to eliminate the constant, we have

$$\beta = \frac{1}{u}. \tag{4.60}$$

We proceed to solve for g by substituting (4.32) and (4.60) into (4.17) to obtain the separable differential equation

$$\frac{g_{,u}}{g} = \frac{1}{u}.$$

The solution is $\ln g = \ln u + h(v)$ for some function of integration $h(v)$. Solving for g, we obtain $g = ue^{h(v)}$. Performing a coordinate transformation on v to eliminate the factor dependent on v, then removing the tilde, we simplify the expression for g,

$$g = u;$$

hence, the metric has the form

$$ds^2 = du^2 + u^2 dv^2. \tag{4.61}$$

We recognise (4.61) as the Euclidean metric in polar coordinates. The transformation to Cartesian coordinates is given by

$$\begin{aligned} x &= u\cos v, \\ y &= u\sin v, \end{aligned} \tag{4.62}$$

where $0 \leq u < \infty,\ 0 \leq v < 2\pi$.

To determine the KT, we recall that $g(u) = u$. Substituting this into the general form of the KT for case II (4.38), we find that

$$\mathbf{K} = \ \mathrm{diag}(c_1, c_2u^2 + c_1), \tag{4.63}$$

a linear combination of the metric, $\mathbf{g}$, and $\mathbf{K_1} = \mathrm{diag}(0, u^2)$.

Substituting $g(u) = u$ into (4.39) and (4.40), we obtain the separable potentials in this case for E_2,

$$V = V_1(u) + \frac{V_2(v)}{u^2}, \tag{4.64}$$

and

$$U = 2c_1V_1(u) + 2c_2V_2(v) + \frac{2c_1V_2(v)}{u^2}.$$

The form of the second first integral was calculated in the associated Riemannian case to be (4.41).

4.3.3 Case III: $\alpha\beta \neq 0$

Using (4.59), we determine the forms of the functions $A(u)$ and $B(v)$ found in the metric (4.44) and KT (4.53). From (4.46), we obtain the simplified equation

$$\frac{A_{,uuu}}{A_{,u}} = -\frac{B_{,vvv}}{B_{,v}} = k^2. \tag{4.65}$$

If $k = 0$, then the system (4.65) has a solution

$$A = c_1 u^2 + c_2 u + c_3,$$
$$B = d_1 v^2 + d_2 v + d_3. \tag{4.66}$$

Substituting these functions into the original partial differential equation for A and B (4.45), then equating coefficients of the u^2, u, v^2, v and constant terms from either side of the resulting equation, we obtain the restrictions

$$c_1 = d_1,$$
$$4c_1(c_3 + d_3) = c_2^2 + d_2^2 \Rightarrow \frac{c_2^2}{4c_1} = c_3 + d_3 - \frac{d_2^2}{4c_1}.$$

Using these relations in (4.66), we obtain

$$\begin{aligned} A + B &= c_1\left(u + \frac{c_2}{2c_1}\right)^2 + c_3 - \frac{c_2^2}{4c_1^2} + c_1\left(v + \frac{d_2}{2c_1}\right)^2 + d_3 - \frac{d_2^2}{4c_1^2} \\ &= c_1\left(u + \frac{c_2}{2c_1}\right)^2 - d_3 + \frac{d_2^2}{4c_1^2} + c_1\left(v + \frac{d_2}{2c_1}\right)^2 + d_3 - \frac{d_2^2}{4c_1^2} \\ &= c_1\left[\left(u + \frac{c_2}{2c_1}\right)^2 + \left(v + \frac{d_2}{2c_1}\right)^2\right]. \end{aligned}$$

By an appropriate coordinate transformation, we translate and scale u and v such that $A + B = u^2 + v^2$. In this coordinate system, the metric (4.44) is

$$ds^2 = (u^2 + v^2)(du^2 + dv^2). \tag{4.67}$$

We recognise (4.67) as the Euclidean metric in parabolic coordinates. The transformation to Cartesian coordinates is given by

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv, \tag{4.68}$$

where $-\infty < u < \infty,\ 0 \le v < \infty$.

The KT (4.53) is

$$\mathbf{K} = \operatorname{diag}(-\delta v^2 + c, \delta u^2 + c). \tag{4.69}$$

In this coordinate system, the KT is a linear combination of the metric and $\mathbf{K_1} = \operatorname{diag}(-v^2, u^2)$.

Using the fact that $A + B = u^2 + v^2$, we conclude from (4.54) and (4.57) that

$$V = \frac{V_1(u) + V_2(v)}{u^2 + v^2}, \tag{4.70}$$

and

$$U = \frac{2(-\delta v^2 + c)V_1(u) + 2(\delta u^2 + c)V_2(v)}{u^2 + v^2}.$$

The second first integral (4.58) takes the form

$$K = \frac{-v^2 p_u^2 + u^2 p_v^2 - 2v^2 V_1(u) + 2u^2 V_2(v)}{2\left(u^2 + v^2\right)}.$$

If $k \neq 0$, the solution to (4.65) is

$$A = c_1 e^{ku} + c_2 e^{-ku} + c_3, \qquad B = d_1 \cos(kv) + d_2 \sin(kv) + d_3. \tag{4.71}$$

Substituting (4.71) into (4.45), then equating coefficients of the e^{2ku}, e^{ku}, e^{-ku},

e^{-2ku}, and constant terms from either side of the resulting equation, we obtain the restrictions

$$c_3 + d_3 = 0,$$
$$4c_1c_2 = d_1^2 + d_2^2. \tag{4.72}$$

We observe that the second restriction implies that c_1 and c_2 have the same sign. We shall assume, for simplicity, that both are positive. Using the first restriction of (4.72) in (4.71), we obtain

$$A + B = c_1e^{ku} + c_2e^{-ku} + d_1\cos(kv) + d_2\sin(kv).$$

Since c_1 and c_2 are both positive, the first two terms may be written as $2\sqrt{c_1c_2}\cosh(ku+\psi)$. Similarly, the final two terms may be written as $b\cos(kv+\phi)$ where $b = -\sqrt{d_1^2+d_2^2} = -2\sqrt{c_1c_2}$ using the second restriction of (4.72); hence, we have shown that

$$A + B = 2\sqrt{c_1c_2}\cosh(ku+\psi) - 2\sqrt{c_1c_2}\cos(kv+\phi).$$

Using standard trigonometric identities, we may write this as

$$\begin{aligned} A + B &= 4\sqrt{c_1c_2}\cosh^2\left(\frac{ku+\psi}{2}\right) - 2\sqrt{c_1c_2} - 4\sqrt{c_1c_2}\cos^2\left(\frac{kv+\phi}{2}\right) \\ &\quad +2\sqrt{c_1c_2} \\ &= 4\sqrt{c_1c_2}\left[\cosh^2\left(\frac{ku+\psi}{2}\right) - \cos^2\left(\frac{kv+\phi}{2}\right)\right]. \end{aligned}$$

By an appropriate coordinate transformation, we translate and scale u and v to convert $A + B$ to the form

$$A + B = a^2(\cosh^2 u - \cos^2 v), \tag{4.73}$$

where $a^2 = 4\sqrt{c_1 c_2}$, with $a > 0$ without loss of generality; therefore, the metric (4.44) is

$$ds^2 = a^2(\cosh^2 u - \cos^2 v)(du^2 + dv^2). \tag{4.74}$$

We recognise (4.74) as the Euclidean metric in elliptic/hyperbolic coordinates. The transformation to Cartesian coordinates is given by

$$\begin{aligned} x &= a\cosh u\cos v, \\ y &= a\sinh u\sin v, \end{aligned} \tag{4.75}$$

where $0 \leq u < \infty,\ 0 \leq v < 2\pi$.

The KT (4.53) is

$$\mathbf{K} = \ \mathrm{diag}(a^2\delta\cos^2(v) + c, a^2\delta\cosh^2(u) + c). \tag{4.76}$$

We observe that the family of KTs related to the elliptic/hyperbolic separable coordinates is $\{\mathbf{g}, \mathbf{K_1}\}$, where $\mathbf{K_1} = \mathrm{diag}(a^2\cos^2 v, a^2\cosh^2 u)$.

Using the fact that $A + B = a^2(\cosh^2 u - \cos^2 v)$ (4.73), we conclude from (4.54) and (4.57) that

$$V = \frac{V_1(u) + V_2(v)}{\cosh^2 u - \cos^2 v}, \tag{4.77}$$

and

$$U = \frac{2(a^2\delta\cos^2(v) + c)V_1(u) + 2(a^2\delta\cosh^2(u) + c)V_2(v)}{\cosh^2 u - \cos^2 v};$$

thus, the independent second first integral (4.58) has the form

$$K = \frac{\cos^2(v)p_u^2 + \cosh^2(u)p_v^2 + 2\cos^2(v)V_1(u) + 2\cosh^2(u)V_2(v)}{2\big(\cosh^2 u - \cos^2 v\big)}.$$

4.3.4 Separable Coordinates in E_2

It is well known [5] that the four separable coordinate systems in Euclidean 2-space (with corresponding metrics) are:

$$\begin{aligned}
&\text{Cartesian, } (x, y): && ds^2 = dx^2 + dy^2, \\
&\text{Polar, } (r, \theta): && ds^2 = dr^2 + r^2 d\theta^2, \\
&\text{Parabolic, } (\xi, \eta): && ds^2 = (\xi^2 + \eta^2)(d\xi^2 + d\eta^2), \\
&\text{Elliptic/Hyperbolic, } (\alpha, \beta): && ds^2 = a^2(\cosh^2\alpha - \cos^2\beta)(d\alpha^2 + d\beta^2),
\end{aligned}$$

where a is a scaling parameter representing half the distance between the focii.

Cartesian coordinates (for which our results are summarised in Table 4.1) are the standard orthogonal coordinates used to describe the natural basis of Euclidean spaces. In fact, any manifold imbedded in Euclidean n-space can be represented locally by a set of Cartesian coordinates $(x^1, \ldots, x^n)$. Using Remark 3 from Subsection 3.3.2, we determine the singular points of the metric. Since the metric in the moving frame is $\mathrm{diag}(1, 1)$ (4.8), $\mathbf{K}$ is proportional to $\mathbf{g}$ if and only if its diagonal components are identical. Since the components are (c_1, c_2) (4.27), there are no singular points.

Polar coordinates (see the summary of our results in Table 4.2) are related to Cartesian coordinates by the formulae (4.62) with $u = r$ and $v = \theta$. The coordinate r represents the distance between the origin and a point, P, and θ is the angle, measured in the counterclockwise direction, between the positive x-axis and the line connecting the origin and P; hence, the coordinate lines are concentric circles for constant r and rays for constant θ. This is shown in Figure 4.1. At any point, P, in the plane — except the origin where $r = 0$ and θ is undefined — there exists a unique representation in polar coordinates; hence, the origin is a singular point. Since the KT (4.63) is proportional to the metric only at this point, where

Cartesian Separable Coordinates	
Metric	$ds^2 = dx^2 + dy^2$
Killing tensor	$\mathrm{diag}(c_1, c_2)$
Separable potential	$V = V_1(x) + V_2(y)$
Second first integral	$\frac{1}{2}{p_y}^2 + V_2(y)$

Table 4.1: Summary for Cartesian coordinates.

$r := u = 0$ and $\theta := v$ is undefined, it is the only singular point.

Parabolic coordinates (see Table 4.3 for a summary of our results) are employed less frequently than polar coordinates but are no less useful. These coordinates are related to Cartesian coordinates by the formulae (4.68) with $u = \xi$, $v = \eta$. The coordinate lines are two families of mutually orthogonal parabolae opening in opposite directions on the x-axis. Figure 4.2 depicts these lines. We observe that the coordinate lines at the origin are parallel; therefore, it is a singular point. Again, the KT (4.69) is proportional to the metric if and only if $\xi := u = 0$ and $\eta := v = 0$, that is, at the origin $(x, y) = (0, 0)$; thus, it is the only singular point. In fact, there are two coincident singular points at the origin, see, for example, [3].

The most complicated separable coordinate system in E_2 are the elliptic/hyperbolic coordinates (see Table 4.4 for our corresponding results). In fact, the other three coordinate systems are degenerate forms of these coordinates. The transformation law to Cartesian coordinates is (4.75) with $u = \alpha$, $v = \beta$. The coordinate lines $\alpha = \alpha_0$ are ellipses and $\beta = \beta_0$ are hyperbolae along the x-axis (see Figure 4.3). The coordinate lines are parallel at the focii of the conics $(x, y) = (\pm a, 0)$. By examining the KT (4.76), we observe it is proportional to the metric provided $\alpha := u = 0$ and

Polar Separable Coordinates	
Metric	$ds^2 = dr^2 + r^2 d\theta^2$
Killing tensor	$\mathrm{diag}(c_1, c_2 r^2 + c_1)$
Separable potential	$V = V_1(r) + \dfrac{V_2(\theta)}{r^2}$
Second first integral	$\frac{1}{2} p_\theta^2 + V_2(\theta)$

Table 4.2: Summary for polar coordinates.

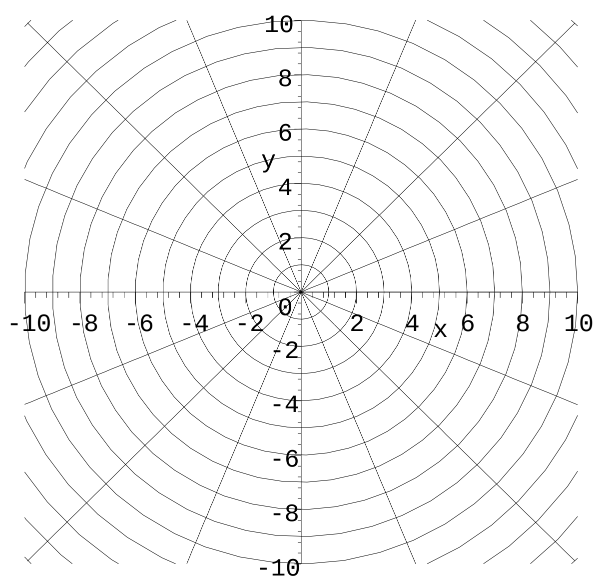

Figure 4.1: Coordinate lines for polar coordinates.

Parabolic Separable Coordinates	
Metric	$ds^2 = (\xi^2 + \eta^2)(d\xi^2 + d\eta^2)$
Killing tensor	$\mathrm{diag}(-\delta\eta^2 + c, \delta\xi^2 + c)$
Separable potential	$\dfrac{V_1(\xi) + V_2(\eta)}{\xi^2 + \eta^2}$
Second first integral	$\dfrac{-\eta^2 p_\xi^2 + \xi^2 p_\eta^2 - 2\eta^2 V_1(\xi) + 2\xi^2 V_2(\eta)}{2\big(\xi^2 + \eta^2\big)}$

Table 4.3: Summary for parabolic coordinates.

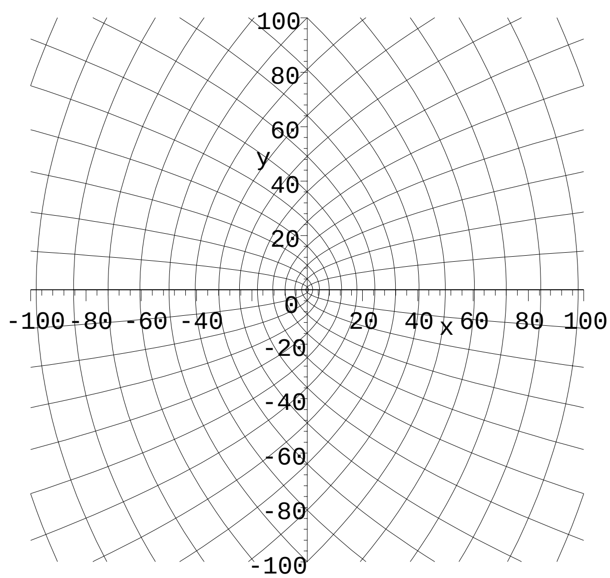

Figure 4.2: Coordinate lines for parabolic coordinates.

Elliptic/Hyperbolic Separable Coordinates	
Metric	$ds^2 = a^2(\cosh^2\alpha - \cos^2\beta)(d\alpha^2 + d\beta^2)$
Killing tensor	$\mathrm{diag}(a^2\delta\cos^2(\beta) + c, a^2\delta\cosh^2(\alpha) + c)$
Separable potential	$\dfrac{V_1(\alpha) + V_2(\beta)}{\cosh^2\alpha - \cos^2\beta}$
Second first integral	$\dfrac{\cos^2(\beta)p_\alpha^2 + \cosh^2(\alpha)p_\beta^2 + 2\cos^2(\beta)V_1(\alpha) + 2\cosh^2(\alpha)V_2(\beta)}{2\big(\cosh^2\alpha - \cos^2\beta\big)}$

Table 4.4: Summary for elliptic/hyperbolic coordinates.

$\beta := v = 0$ or π. Since these points correspond to the focii, $(a, 0)$ and $(-a, 0)$ are the only singular points. This agrees with previous results, see, for example, [3].

By our method, we have produced the four separable coordinate systems in E_2 (4.26), (4.61), (4.67), and (4.74) without a priori knowledge regarding their existence! In addition, we recovered the associated potentials, KTs, and second first integrals, see, for example, [36].

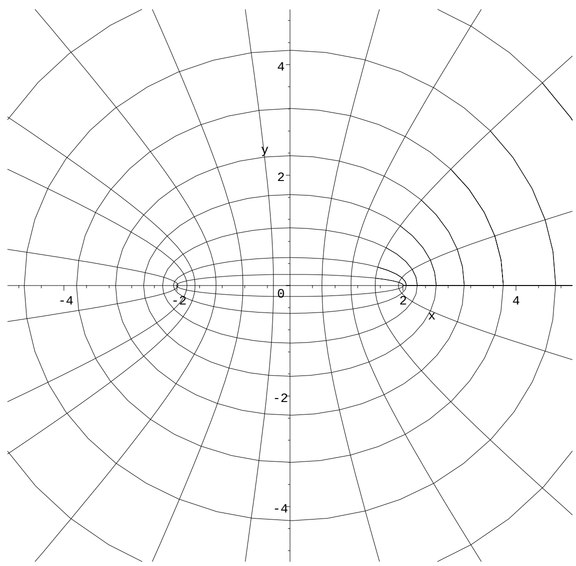

Figure 4.3: Coordinate lines for elliptic/hyperbolic coordinates.

4.4 Two Dimensional Constant Curvature Surfaces

A surface for which the curvature is some non-zero constant, $R_{1212} = R$, is called a *surface of constant curvature.* Geometrically, we locally represent these surfaces as imbeddings in E_3 if $R > 0$, or H_2 if $R < 0$. For $R > 0$, we can represent the surface by the 2-sphere if it is closed. For $R < 0$, the surface can be represented only locally.

As discussed in the treatment of case I for a general Riemannian manifold, the curvature is necessarily zero when both α and β vanish; hence, we begin the analysis with case II.

4.4.1 Case II: $\alpha = 0$, $\beta \neq 0$

In the treatment of a general Riemannian manifold in this case, we found the differential equation for $g(u)$ (4.37). We solve this equation to obtain

$$g_{,uu} = -Rg$$

$$\Rightarrow \quad g = \begin{cases} c_1 \sin(\sqrt{R}u) + c_2 \cos(\sqrt{R}u), \text{ if } R > 0, \\ \\ c_3 e^{\sqrt{-R}u} + c_4 e^{-\sqrt{-R}u}, \text{ if } R < 0. \end{cases}$$

This is equivalent to

$$g = \begin{cases} A \sin(\sqrt{R}u + \delta), \text{ if } R > 0, \\ \\ \left\{ \begin{array}{l} A \sinh(\sqrt{-R}u + \delta), \\ A \cosh(\sqrt{-R}u + \delta), \\ A e^{\sqrt{-R}u}, \end{array} \right. \quad \text{if } R < 0. \end{cases}$$

By translating u and scaling v, we obtain

$$g = \begin{cases} \sin(\sqrt{R}u), \text{ if } R > 0, \\ \\ \begin{cases} \sinh(\sqrt{-R}u), \\ \cosh(\sqrt{-R}u), \quad \text{if } R < 0. \\ e^{-2\sqrt{-R}u}, \end{cases} \end{cases} \tag{4.78}$$

Using $f = 1$ (4.32) and (4.78), for $R > 0$, we write the metric (4.35) as

$$ds^2 = du^2 + \sin^2(\sqrt{R}u)dv^2.$$

To recognise the separable coordinates, we write the metric in the coordinates $\tilde{u} = \sqrt{R}u$, $\tilde{v} = \sqrt{R}v$,

$$ds^2 = \frac{1}{R}\Big[d\tilde{u}^2 + \sin^2(\tilde{u})d\tilde{v}^2\Big].$$

This form of the metric corresponds to regular spherical coordinates that are related to Cartesian coordinates, (x, y, z), by the transformation law

$$\begin{aligned} x &= r\sin\tilde{u}\cos\tilde{v}, \\ y &= r\sin\tilde{u}\sin\tilde{v}, \\ z &= r\cos\tilde{u}, \end{aligned} \tag{4.79}$$

where $r = R^{-1/2} = \text{constant}$, $0 \leq \tilde{u} \leq \pi$, $0 \leq \tilde{v} < 2\pi$; hence, the separable coordinates are spherical coordinates. Spherical coordinates are employed to analyse objects with spherical symmetry. In standard notation, we use the variable names ϕ instead of u and θ in place of v in (4.79). The coordinate ϕ represents the angle between the z-axis and the point, P. The angle, measured in the counterclockwise direction, between the positive x-axis and the line connecting the origin to the pro-

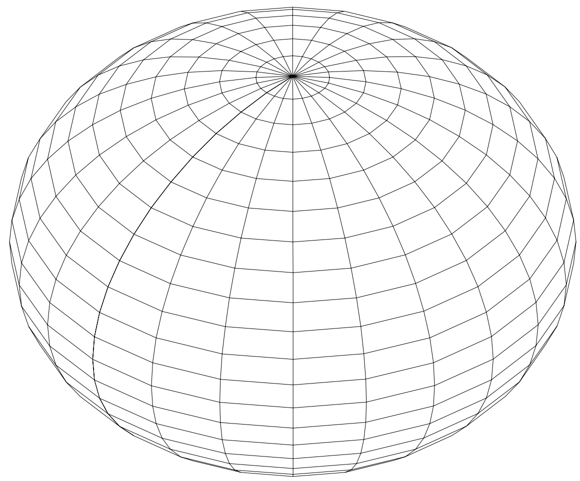

Figure 4.4: Coordinate lines for regular spherical coordinates.

jection of P into the xy-plane is θ; hence, the coordinate lines are circles of latitude for constant ϕ and half great circles for constant θ. This is shown in Figure 4.4. At any point, P, on the 2-sphere — except the poles where $\phi = 0$ or π and θ is undefined — there exists a unique representation in spherical coordinates.

For R negative, there are three cases. The first separable metric from (4.78),

$$ds^2 = du^2 + \sinh^2(\sqrt{-R}u)dv^2, \tag{4.80}$$

is related to the positive definite metric of the coordinates $\tilde{u} = \sqrt{-R}u$, $\tilde{v} = \sqrt{-R}v$,

$$ds^2 = -\frac{1}{R}\Big[d\tilde{u}^2 + \sinh^2(\tilde{u})d\tilde{v}^2\Big].$$

The transformation law (locally) relating Minkowskian coordinates, (t, x, y), to these coordinates is

$$x = \rho \sinh \tilde{u} \cos \tilde{v},$$

$$y = \rho \sinh \tilde{u} \sin \tilde{v},$$

$$t = \rho \cosh \tilde{u},$$

where $\rho = (-R)^{-1/2} = \text{constant}$, $0 \leq \tilde{u} \leq \infty$, $0 \leq \tilde{v} < 2\pi$.

The second separable metric from (4.78),

$$ds^2 = du^2 + \cosh^2(\sqrt{-R}u)dv^2, \tag{4.81}$$

is related to the positive definite metric of the coordinates $\tilde{u} = \sqrt{-R}u$, $\tilde{v} = \sqrt{-R}v$,

$$ds^2 = -\frac{1}{R}\Big[d\tilde{u}^2 + \cosh^2(\tilde{u})d\tilde{v}^2\Big].$$

The transformation law (locally) relating the coordinates (t, x, y) to these coordinates is

$$x = \rho \cosh \tilde{u} \cos \tilde{v},$$

$$y = \rho \cosh \tilde{u} \sin \tilde{v},$$

$$t = \rho \sinh \tilde{u},$$

where $\rho = (-R)^{-1/2} = \text{constant}$, $-\infty < \tilde{u} \leq \infty$, $0 \leq \tilde{v} < 2\pi$.

The coordinates of the final separable metric from (4.78),

$$ds^2 = du^2 + e^{-2\sqrt{-R}u}dv^2, \tag{4.82}$$

can be transformed using $\tilde{u} = \sqrt{-R}u$, $\tilde{v} = \sqrt{-R}v$ to obtain the metric

$$ds^2 = -\frac{1}{R}(d\tilde{u}^2 + e^{-2\tilde{u}}d\tilde{v}^2).$$

These coordinates are (locally) related to the coordinates (t, x, y) by the transformation

$$x = (-R)^{-1/2}e^{-\tilde{u}}\cos\tilde{v},$$

$$y = (-R)^{-1/2}e^{-\tilde{u}}\sin\tilde{v},$$

$$t = (-R)^{-1/2}\left[\sqrt{1-e^{-2\tilde{u}}} - \ln\left(\frac{1+\sqrt{1-e^{-2\tilde{u}}}}{e^{-\tilde{u}}}\right)\right].$$

The KT, and separable potentials, V and U, may be determined in each case by direct substitution into the formulae (4.38), (4.39), and (4.40), respectively. For a discussion of the related singular points, see [37]. The form of the second first integral for all coordinates (4.41) was calculated in the general Riemannian case II calculations. The metric calculations were performed in detail by Olevski while investigating the product separability of the Laplace-Beltrami equation, in [34] where, for $R < 0$, a rigorous geometrical characterisation of the coordinate lines in H_2 is given.

4.4.2 Case III: $\alpha\beta \neq 0$

We proceed from the differential system developed in the general Riemannian case III treatment (4.46) with $R_{1212} = R$,

$$\begin{cases} A_{,uuu} + 12RAA_{,u} = k^2A_{,u}, \\ \\ B_{,vvv} + 12RBB_{,v} = -k^2B_{,v}. \end{cases} \tag{4.83}$$

We solve the differential equation for $A(u)$. It is equivalent to the differential equation for $A_{,u}$,

$$\begin{aligned} (A_{,u})_{,uu} + 6R(A^2)_{,u} &= k^2 A_{,u} \\ \Rightarrow \quad A_{,uu} + 6RA^2 &= k^2 A + l, \end{aligned} \tag{4.84}$$

where l is a constant of integration. The remaining equation is solved by using the following technique. We let

$$\omega(A) = \frac{dA}{du}. \tag{4.85}$$

It follows that

$$A_{,uu} = \omega \frac{d\omega}{dA},$$

using the chain rule. Substituting this relation into the reduced order equation (4.84), we produce a separable differential equation that is integrable

$$\begin{aligned} & \omega \frac{d\omega}{dA} = -6RA^2 + k^2 A + l \\ \Rightarrow \quad & \int \omega d\omega = \int (-6RA^2 + k^2 A + l) dA \\ \Rightarrow \quad & \frac{1}{2}\omega^2 = -2RA^3 + \frac{k^2}{2} A^2 + lA + m \\ \Rightarrow \quad & \omega^2 = -4RA^3 + k^2 A^2 + 2lA + 2m, \end{aligned}$$

where m is a second constant of integration. Using (4.85), we derive the equation for $A_{,u}$

$$(A_{,u})^2 = -4RA^3 + k^2A^2 + 2lA + 2m. \tag{4.86}$$

Similarly, we integrate the differential equation for $B(v)$ (4.83) to obtain

$$(B_{,v})^2 = -4RB^3 - k^2B^2 + 2nB + 2p, \tag{4.87}$$

where n and p are constants of integration.

To obtain the relations amongst the constants, we substitute into the original differential equation (4.45). First, we differentiate (4.86) with respect to u to obtain $A_{,uu}$,

$$A_{,uu} = -6RA^2 + k^2A + l, \tag{4.88}$$

and (4.87) with respect to v to obtain $B_{,vv}$,

$$B_{,vv} = -6RB^2 - k^2B + n, \tag{4.89}$$

using the fact $A_{,u}B_{,v} \neq 0$. Substituting (4.86), (4.87), (4.88), and (4.89) into (4.45), then simplifying, we obtain

$$(n - l)(A - B) - 2(m + p) = 0. \tag{4.90}$$

Differentiating this relation with respect to u, we produce

$$(n - l)A_{,u} = 0$$

$$\Rightarrow \quad n = l.$$

Substituting $n = l$ into (4.90), we obtain the relation $p = -m$; therefore, the solution satisfies the conditions

$$
\begin{aligned}
(A_{,u})^2 &= -4RA^3 + k^2A^2 + 2lA + 2m, \\
(B_{,v})^2 &= -4RB^3 - k^2B^2 + 2lB - 2m.
\end{aligned}
\tag{4.91}
$$

To remove the quadratic terms from the equations of (4.91), we perform a translation by a constant. Shifting A by $a = -k^2/12R$ and B by $-a$, we simplify (4.91),

$$
\begin{aligned}
(A_{,u})^2 &= -4RA^3 + cA + d, \\
(B_{,v})^2 &= -4RB^3 + cB - d,
\end{aligned}
$$

where $c = k^4/12R + 2l$ and $d = (k^2/18R)(k^2/12R + 3l) + 2m$.

It follows that the separation constant, k, can be set to zero without loss of generality. It also follows that

$$
\begin{aligned}
du^2 &= \frac{dA^2}{-4RA^3 + cA + d}, \\
dv^2 &= \frac{dB^2}{-4RB^3 + cB - d}.
\end{aligned}
$$

Since $A_{,u}$ and $B_{,v}$ do not vanish, we adopt (A, B) as coordinates; hence, the metric (4.44) can be written as

$$
ds^2 = (A+B)\left(\frac{dA^2}{-4RA^3 + cA + d} + \frac{dB^2}{-4RB^3 + cB - d}\right). \tag{4.92}
$$

The allowable ranges of the coordinates A and B depend on the roots of the cubic polynomials in the denominators of the metric (4.92). It is necessary that $(A + B)$, and both cubics be positive. To simplify the analysis, we first tranform the coordinates according to $\tilde{A} = A$, $\tilde{B} = -B$. Dropping the tildes, we write the metric in this new coordinate system as

$$ds^2 = (A - B)\left(\frac{dA^2}{-4RA^3 + cA + d} - \frac{dB^2}{-4RB^3 + cB + d}\right).$$

The cubic polynomials in the denominators are now identical. To write the polynomial in *reduced form*, we factor the leading coefficient,

$$ds^2 = -\frac{1}{4R}(A - B)\left(\frac{dA^2}{A^3 + 3hA + j} - \frac{dB^2}{B^3 + 3hB + j}\right), \tag{4.93}$$

where $h = -c/12R$ and $j = -d/4R$.

Since the surface is Riemannian, the metric is positive definite. Without loss of generality, we impose the condition

$$A > B$$

on the coordinate functions A and B.

The analysis depends on the sign of the curvature, R. If $R > 0$, then we require that $A^3 + 3hA + j < 0$ and $B^3 + 3hB + j > 0$; however, if $R < 0$, then it is necessary to impose the conditions $A^3 + 3hA + j > 0$, and $B^3 + 3hB + j < 0$. By varying the zeros of cubic polynomial, we determine all separable coordinate systems. Olevski performed this analysis in [34]. In all cases, the metric is of the form we derived (4.93). We denote the zeros of the cubic polynomial in the denominator of the metric terms by a, b, and c.

For $R > 0$, the analysis produces only one separable coordinate system, the Jacobi elliptic coordinates. The coordinates vary in the intervals between the distinct zeros, that is, $c < B < b < A < a$. The coordinates are related to Cartesian coordinates by

$$\frac{x^2}{C - b} + \frac{y^2}{C - a} + \frac{z^2}{C - c} = 0,$$

where $C = A$ or B.

For $R < 0$, we obtain *six* additional separable coordinate systems. We list the corresponding relations between coordinates, (t, x, y), and the separable coordinates, (A, B), and the intervals of existence for the separable coordinates with respect to the zeros:

1. $$\frac{x^2}{C-b} + \frac{y^2}{C-a} - \frac{t^2}{C-c} = 0,$$

   $$c < b < B < a < A;$$

2. $$\frac{x^2}{C-c} + \frac{y^2}{C-a} - \frac{t^2}{C-b} = 0,$$

   $$B < c < b < a < A;$$

3. $$\frac{y^2}{C-a} - \frac{2dxt + (C-e)(t^2 - x^2)}{(C-e)^2 + d^2} = 0,$$

   $$B < a < A,\ b = e + id,\ c = e - id;$$

4. $$\frac{y^2}{C-a} + \frac{x^2 - t^2}{C-b} + \frac{(x-t)^2}{(C-b)^2} = 0,$$

   $$c = b < B < a < A;$$

5. $$\frac{y^2}{C-a} + \frac{x^2-t^2}{C-b} - \frac{(x-t)^2}{(C-b)^2} = 0,$$

$$B < c = b < a < A;$$

6. $$\left(\frac{t-x}{(C-a)} + y\right)^2 = t^2 - x^2,$$

$$B < c = b = a < A,$$

where $C = A$ or B. Other possible cases exist that are equivalent, by coordinate transformation, to one of these six cases. That is, this is a complete list of the inequivalent cases. For details, including a description of the orthogonal coordinates, see [34]; thus, we have recovered all the known separable coordinate systems for surfaces of constant curvature, see, for example, [21]. The KTs, separable potentials, V and U, and second first integrals may be determined by substitution into the appropriate form from the general two dimensional Riemannian surface analysis. For a discussion of the related singular points, see [37].

4.5 Superseparability in E_2

A Hamiltonian system is said to be *superintegrable*, if it possesses $2n-1$ functionally independent first integrals, that is, two sets of involutive first integrals each with n elements. We are interested in the subset of superintegrable systems for which there are potentials that are separable in multiple coordinate systems, called *superseparable* systems. To complete the study of separation of variables for Euclidean 2-space, E_2, we determine the superseparable potentials. Since the dimension of the surface is two, we seek potentials that separate in two coordinate systems. This provides two first integrals, not necessarily independent, in addition to the Hamiltonian.

The results of Section 4.3 permit the determination of such potentials with relative ease. The results to follow have been obtained previously, by an arguably more complicated approach involving Lie groups, by Winternitz, Smorodinsky et al [19].

4.5.1 Cartesian-Polar

The separable potential for Cartesian coordinates, (x, y), given by (4.28) is $V = V_1(x) + V_2(y)$. We write this in polar coordinates, (r, θ), as

$$V = V_1(r\cos\theta) + V_2(r\sin\theta),$$

where (4.62) has been used. If a potential separates in *both* Cartesian and polar coordinates, by (4.64), we may also write V in the form

$$V = \tilde{V}_1(r) + \frac{\tilde{V}_2(\theta)}{r^2}.$$

The partial derivatives of each form of V must be equal; hence,

$$\begin{aligned} V_{,\theta} &= \frac{\tilde{V}_2{}'}{r^2} = -V_1' r\sin\theta + V_2' r\cos\theta \\ \Rightarrow \quad \tilde{V}_2{}' &= r^3(-V_1'\sin\theta + V_2'\cos\theta), \end{aligned} \tag{4.94}$$

and

$$\begin{aligned} V_{,r} &= \tilde{V_1}' - \frac{2\tilde{V_2}}{r^3} = V_1' \cos\theta + V_2' \sin\theta \\ \Rightarrow\quad \tilde{V_2} &= (\tilde{V_1}' - V_1' \cos\theta - V_2' \sin\theta)\frac{r^3}{2} \qquad (4.95) \\ \Rightarrow\quad \tilde{V_2}' &= \frac{r^3}{2}(V_1'' r \sin\theta\cos\theta + V_1' \sin\theta - V_2'' r\sin\theta\cos\theta - V_2'\cos\theta). \end{aligned}$$

Comparing the final lines of (4.94) and (4.95), we obtain

$$V_1'' r \sin\theta \cos\theta + 3V_1' \sin\theta = V_2'' r \sin\theta\cos\theta + 3V_2'\cos\theta$$

$$\Rightarrow V_1'' + \frac{3V_1'}{r\cos\theta} = V_2'' + \frac{3V_2'}{r\sin\theta}$$

$$\Rightarrow V_1'' + \frac{3V_1'}{x} = V_2'' + \frac{3V_2'}{y} = S, \qquad (4.96)$$

where S is a separation constant. Solving for V_1, we get

$$\begin{aligned} (V_1')' + \frac{3(V_1')}{x} &= S \\ \Rightarrow\quad V_1' &= \frac{S}{4}x + \frac{c}{x^3} \\ \Rightarrow\quad V_1 &= \frac{S}{8}x^2 + \frac{S_1}{x^2}, \end{aligned}$$

where S_1 is a constant of integration. Solving for V_2, we obtain

$$V_2 = \frac{S}{8}y^2 + \frac{S_2}{y^2},$$

where S_2 is a constant of integration; therefore, the superseparable potential in

Cartesian coordinates is

$$V = \frac{S}{8}(x^2 + y^2) + \frac{S_1}{x^2} + \frac{S_2}{y^2}. \tag{4.97}$$

The potential may be written in terms of polar coordinates (4.64) as

$$V = \frac{S}{8}r^2 + \frac{\dfrac{S_1}{\cos^2\theta} + \dfrac{S_2}{\sin^2\theta}}{r^2}. \tag{4.98}$$

4.5.2 Cartesian-Parabolic

The separable potential for Cartesian coordinates, (x, y), given by (4.28) written in parabolic coordinates, (ξ, η), is given by

$$V = V_1\Big(\frac{1}{2}(\xi^2 - \eta^2)\Big) + V_2(\xi\eta), \tag{4.99}$$

where (4.68) has been used. The separability condition for parabolic coordinates (4.70) can be written as

$$\frac{\partial^2[(\xi^2 + \eta^2)V]}{\partial\xi\partial\eta} = 0. \tag{4.100}$$

A superseparable potential, V, that separates in both Cartesian and parabolic coordinates, must satisfy (4.100). Computing the mixed second partial derivative using (4.99), we obtain

$$\begin{aligned}
\frac{\partial^2[(\xi^2 + \eta^2)V]}{\partial\eta\partial\xi} &= 2\xi(-\eta V_1' + \xi V_2') + 2\eta(\xi V_1' + \eta V_2') \\
&\quad + (\xi^2 + \eta^2)(-\xi\eta V_1'' + V_2' + \xi\eta V_2'') \\
&= (\xi^2 + \eta^2)[3V_2' - \xi\eta V_1'' + \xi\eta V_2''],
\end{aligned}$$

from which it follows that

$$3V_2' - \xi\eta V_1'' + \xi\eta V_2'' = 0$$

$$\Rightarrow \quad V_2'' + \frac{3}{y}V_2' = V_1'' = S.$$

The solution for $V_1(x)$ is

$$V_1 = \frac{S}{2}x^2 + S_1 x,$$

where S_1 is an arbitrary constant of integration. The differential equation for V_2 is identical to that encountered while solving the Cartesian-polar case (4.96); hence, the solution is given by,

$$V_2 = \frac{S}{8}y^2 + \frac{S_2}{y^2},$$

where S_2 is a constant of integration; therefore, in Cartesian coordinates, the superseparable potential has the form

$$V = \frac{S}{2}x^2 + \frac{S}{8}y^2 + S_1 x + \frac{S_2}{y^2}.$$

In terms of parabolic coordinates, the potential may be written as

$$V = \frac{\left(\frac{S}{8}\xi^6 + \frac{S_1}{2}\xi^4 + \frac{S_2}{\xi^2}\right) + \left(\frac{S}{8}\eta^6 - \frac{S_1}{2}\eta^4 + \frac{S_2}{\eta^2}\right)}{\xi^2 + \eta^2}.$$

4.5.3 Cartesian-Elliptic/Hyperbolic

The separable potential for Cartesian coordinates, (x, y), given by (4.28) expressed in elliptic/hyperbolic coordinates, (α, β), is given by

$$V = V_1(a\cosh\alpha\cos\beta) + V_2(a\sinh\alpha\sin\beta), \tag{4.101}$$

where (4.75) has been employed. The separability condition for elliptic/hyperbolic coordinates (4.77) can be written as

$$\frac{\partial^2[(\cosh^2\alpha - \cos^2\beta)V]}{\partial\alpha\partial\beta} = 0. \tag{4.102}$$

A superseparable potential, V, that separates in both Cartesian and elliptic/hyperbolic coordinates must satisfy (4.102). Computing the mixed second partial derivative using (4.101), we obtain

$$\begin{aligned}
\frac{\partial^2[(\cosh^2\alpha - \cos^2\beta)V]}{\partial\alpha\partial\beta} = {} & 2\sin\beta\cos\beta(a\sinh\alpha\cos\beta V_1' + a\cosh\alpha\sin\beta V_2') \\
& +2\sinh\alpha\cosh\alpha(-a\cosh\alpha\sin\beta V_1' + \\
& a\sinh\alpha\cos\beta V_2') + (\cosh^2\alpha - \cos^2\beta) \\
& (-a\sinh\alpha\sin\beta V_1' + a\cosh\alpha\cos\beta V_2' \\
& -a^2\sinh\alpha\cosh\alpha\sin\beta\cos\beta V_1'' \\
& +a^2\sinh\alpha\cosh\alpha\sin\beta\cos\beta V_2'').
\end{aligned}$$

It follows from (4.102) and the above that

$$\begin{aligned}
& (\cosh^2\alpha - \cos^2\beta)(-a^2\sinh\alpha\cosh\alpha\sin\beta\cos\beta)V_1'' + (\cosh^2\alpha - \cos^2\beta) \\
& (-3a\sinh\alpha\sin\beta)V_1' + (\cosh^2\alpha - \cos^2\beta)(a^2\sinh\alpha\cosh\alpha\sin\beta\cos\beta)V_2'' \\
& +(\cosh^2\alpha - \cos^2\beta)(3a\cosh\alpha\cos\beta)V_2' = 0
\end{aligned}$$

$$\Rightarrow \quad V_1'' + \frac{3V_1'}{a\cosh\alpha\cos\beta} = V_2'' + \frac{3V_2'}{a\sinh\alpha\sin\beta}$$

$$\Rightarrow \quad V_1'' + \frac{3V_1'}{x} = V_2'' + \frac{3V_2'}{y} = S.$$

We observe that the separable equations for V_1 and V_2 are identical to those encountered while solving the Cartesian-polar case (4.96); therefore, the solution is given by (4.97). Transforming this potential into elliptic/hyperbolic coordinates, we obtain

$$V = \frac{\left(\frac{Sa^2}{8}\sinh^2\alpha\cosh^2\alpha - \frac{S_1}{a^2\cosh^2\alpha} + \frac{S_2}{a^2\sinh^2\alpha}\right)}{\cosh^2\alpha - \cos^2\beta} + \frac{\left(\frac{Sa^2}{8}\sin^2\beta\cos^2\beta + \frac{S_1}{a^2\cos^2\beta} + \frac{S_2}{a^2\sin^2\beta}\right)}{\cosh^2\alpha - \cos^2\beta}. \tag{4.103}$$

4.5.4 Polar-Parabolic

The separable potential for polar coordinates, (r, θ), given by (4.64) written in terms of parabolic coordinates, (ξ, η), is given by

$$V = V_1\Big(\frac{1}{2}(\xi^2+\eta^2)\Big) + \frac{4}{(\xi^2+\eta^2)^2}V_2\Big(\arctan\Big(\frac{2\xi\eta}{\xi^2-\eta^2}\Big)\Big), \tag{4.104}$$

where (4.62) and (4.68) have been used. As previously mentioned, a superseparable potential, V, that separates in parabolic coordinates must satisfy (4.100).

Computing the mixed second partial derivative using (4.104), we obtain

$$\begin{aligned}\frac{\partial^2[(\xi^2+\eta^2)V]}{\partial\eta\partial\xi} &= 2\xi\eta V_1' + 2\xi\eta V_1' + \xi\eta(\xi^2+\eta^2)V_1'' \\ &-8\frac{2\xi^2(\xi^2+\eta^2)V_2' + 2\xi\eta(\xi^2+\eta^2)V_2'' + (\xi^2+\eta^2)^2V_2'}{(\xi^2+\eta^2)^4} \\ &+8\frac{4\xi\eta(\xi^2+\eta^2)V_2 + 4\eta^2(\xi^2+\eta^2)V_2'}{(\xi^2+\eta^2)^4},\end{aligned}$$

from which it follows that

$$\xi\eta(\xi^2+\eta^2)V_1'' + 4\xi\eta V_1' = \frac{16\xi\eta V_2'' + 24(\xi^2-\eta^2)V_2' - 32\xi\eta V_2}{(\xi^2+\eta^2)^3}$$

$$\Rightarrow \quad 2yrV_1'' + 4yV_1' = \frac{2yV_2'' + 6xV_2' - 4yV_2}{r^3}$$

$$\Rightarrow \quad r^4V_1'' + 2r^3V_1' = V_2'' + \frac{3V_2'}{\tan\theta} - 2V_2 = S,$$

where S is a separation constant. To solve for V_1, we integrate twice to obtain

$$V_1 = \frac{S}{2r^2} + \frac{S_1}{r},$$

where S_1 is a constant of integration. The differential equation for V_2 has a particular solution,

$$V_2 = -\frac{S}{2}. \tag{4.105}$$

To find a solution to the related homogeneous equation,

$$\tilde{V}_2'' + \frac{3\tilde{V}_2'}{\tan\theta} - 2\tilde{V}_2 = 0, \tag{4.106}$$

we observe that $\csc^2\theta$ is a solution of (4.106). Since $\csc^2\theta$ factors as $[(1-\cos\theta)(1+\cos\theta)]^{-1}$, we are able to show that

$$W = \frac{S_2}{(1+\cos\theta)} + \frac{S_3}{(1-\cos\theta)} \tag{4.107}$$

satisfies (4.106) where S_2 and S_3 are constants of integration. It follows from (4.105) and (4.107) that the sought solution is

$$V_2 = \frac{S_2}{1+\cos\theta} + \frac{S_3}{1-\cos\theta} - \frac{S}{2};$$

hence, in polar coordinates, the superseparable potential has the form

$$V = \frac{S_1}{r} + \frac{1}{r^2}\left(\frac{S_2}{1+\cos\theta} + \frac{S_3}{1-\cos\theta}\right) \tag{4.108}$$

In terms of parabolic coordinates, the potential may be expressed as

$$V = \frac{\left(2S_1 + \frac{2S_2}{\xi^2}\right) + \left(\frac{2S_3}{\eta^2}\right)}{\xi^2 + \eta^2}.$$

To allow the superseparable potentials to be viewed in the same coordinate system, we seek the form of this potential in Cartesian coordinates. From (4.108), we may immediately rewrite the potential as

$$V = \frac{1}{\sqrt{x^2+y^2}}\left(S_1 + \frac{S_2}{\sqrt{x^2+y^2}+x} + \frac{S_3}{\sqrt{x^2+y^2}-x}\right).$$

4.5.5 Polar-Elliptic/Hyperbolic

We know that the potential (4.97) separates in Cartesian, polar and elliptic/hyperbolic coordinates; therefore, it is a superseparable potential with respect to polar and elliptic/hyperbolic coordinates; however, there may be a more general potential that is superseparable in these two systems. To investigate this possibility, we perform the standard calculation. The separable potential for polar coordinates, (r, θ), (4.64) expressed in elliptic/hyperbolic coordinates, (α, β), is given by

$$V = V_1\left(a\sqrt{\cosh^2\alpha - \sin^2\beta}\right) + \frac{V_2(\arctan(\tanh\alpha\tan\beta))}{a^2(\cosh^2\alpha - \sin^2\beta)}, \tag{4.109}$$

using (4.68) and (4.75).

A superseparable potential, V, that separates in both polar and elliptic/hyperbolic coordinates must satisfy (4.102). Imposing this condition, we obtain, after a lengthy calculation,

$$r^4 V_1'' - r^3 V_1 = V_2'' + 3(\cot\theta - \tan\theta)V_2 - 8V_2. \tag{4.110}$$

Instead of proceeding to solve this differential equation, we re-examine the Cartesian-polar superseparable potential. Transforming the polar separable potential (4.64) into Cartesian coordinates, we obtain

$$V = V_1\left(\sqrt{x^2+y^2}\right) + \frac{V_2\left(\arctan\left(\frac{y}{x}\right)\right)}{x^2+y^2}. \tag{4.111}$$

By imposing the condition for separability in Cartesian coordinates on (4.111) and transforming to polar coordinates, we find that

$$r^4 V_1'' - r^3 V_1' = V_2'' + 3(\cot\theta - \tan\theta)V_2' - 8V_2.$$

Since this equation matches the superseparability condition for polar and elliptic/hyperbolic coordinates (4.110), their solutions must be the same; thus, the sought potential is of the form (4.97). This potential can be written in polar coordinates as (4.98) or in elliptic/hyperbolic coordinates as (4.103).

4.5.6 Parabolic-Parabolic

We observe that there is a second set of distinct parabolic coordinates, (a, b), in E_2 related to the original, (ξ, η), by a rotation of $\pi/2$ in the plane; therefore, the transformation law from (a, b) to Cartesian coordinates, (x, y), is

$$\begin{aligned} x &= -ab, \\ y &= \tfrac{1}{2}(a^2 - b^2), \end{aligned} \tag{4.112}$$

where $-\infty < a < \infty$, $0 \leq b < \infty$. The coordinates lines in this coordinate system are two families of orthogonally intersecting parabolae centred on the origin, opening in opposite directions on the y-axis. The metric associated with these coordinates is (4.67); hence, they are separable. The associated separable potential has the form derived above (4.70), that is,

$$V = \frac{V_1(a) + V_2(b)}{a^2 + b^2};$$

hence, the separability condition is

$$\frac{\partial^2[(a^2+b^2)V]}{\partial b \partial a} = 0. \tag{4.113}$$

We seek a superseparable potential for these two parabolic coordinate systems. The separable potential for the standard parabolic coordinates, (ξ, η), given by (4.70) expressed in the second parabolic coordinates (a, b) is given by

$$V = \frac{V_1\left(\frac{a-b}{\sqrt{2}}\right) + V_2\left(\frac{a+b}{\sqrt{2}}\right)}{a^2+b^2}, \tag{4.114}$$

using (4.68) and (4.112).

Computing the mixed second partial derivative using (4.114), we obtain

$$\frac{\partial^2[(a^2+b^2)V]}{\partial b \partial a} = V_1''\left(-\frac{1}{2}\right) + V_2''\left(\frac{1}{2}\right)$$

Setting this equal to zero, as prescribed by (4.113), we obtain

$$V_1'' = V_2'' = S, \tag{4.115}$$

where S is a separation constant. Integrating, we get

$$\begin{aligned} V_1 &= \frac{S}{2}u^2 + S_1 u + T_1, \\ V_2 &= \frac{S}{2}v^2 + S_2 v + T_2, \end{aligned} \tag{4.116}$$

where S_1, S_2, T_1, and T_2 are constants of integration; therefore, the superseparable potential is

$$V = \frac{S_1 u + S_2 v + S_3}{u^2+v^2},$$

where $S_3 = T_1 + T_2$. In terms of the second parabolic coordinates, the potential may be expressed as

$$V = \frac{\left(\frac{S_1 + S_2}{\sqrt{2}}\right) a + \left(\frac{S_2 - S_1}{\sqrt{2}}\right) b}{a^2 + b^2}.$$

Transforming to Cartesian coordinates, we obtain

$$V = \frac{S_1\sqrt{\sqrt{x^2+y^2}+x} + S_2\sqrt{\sqrt{x^2+y^2}-x} + S_3}{2\sqrt{x^2+y^2}}.$$

4.5.7 Parabolic-Elliptic/Hyperbolic

There may exist potentials that can be separated in both parabolic and elliptic/hyperbolic coordinate systems. We have not, at the time of the writing of this thesis, found a solution to the related differential equations. This is consistent with the paper by Winternitz, et al [19].

4.5.8 Summary of Results

Despite the fact that there are six pairs of coordinate systems in E_2 in which we can construct superseparable potentials, there are only four distinct superseparable potentials. They are listed in Table 4.5. To facilitate their comparison, we write them in Cartesian coordinates.

The utility of superseparable potentials results from our ability to find trajectories as a function of the parameters in the potential, see, for example, [19]. There are several well known physical examples of superseparable systems.

The two dimensional harmonic oscillator discussed in Subsection 2.4.1, with $\omega = \omega_1 = \omega_2$, separates in both Cartesian and polar coordinates. In the physical position-momenta coordinates the system separates. In addition, converting the Hamiltonian of (2.34) to polar coordinates, we observe that V is independent of θ; hence, it is ignorable, and thus separable. As shown in Table 4.5, the potential is

Superseparable Potential	Coordinates
$\frac{S}{8}\left(x^2+y^2\right)+\frac{S_1}{x^2}+\frac{S_2}{y^2}$	Cartesian, Polar, Elliptic/Hyperbolic
$\frac{S}{2}x^2+\frac{S}{8}y^2+S_1x+\frac{S_2}{y^2}$	Cartesian, Parabolic
$\frac{1}{\sqrt{x^2+y^2}}\left(S_1+\frac{S_2}{\sqrt{x^2+y^2}+x}+\frac{S_3}{\sqrt{x^2+y^2}-x}\right)$	Polar, Parabolic
$\frac{S_1\sqrt{\sqrt{x^2+y^2}+x}+S_2\sqrt{\sqrt{x^2+y^2}-x}+S_3}{2\sqrt{x^2+y^2}}$	Parabolic, Parabolic

Table 4.5: Superseparable potentials in Euclidean 2-space.

in the form of a Cartesian-polar-elliptic/hyperbolic superseparable potential with $S = m\omega^2/2$, $S_1 = S_2 = 0$; hence, it also separates in elliptic/hyperbolic coordinates.

Another important example is the central force problem. The Hamiltonian function in polar coordinates is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{m}{r}.$$

The associated trajectories are conics: ellipses, parabolae, and hyperbolae. The potential separates in both polar and parabolic coordinates. It corresponds to $S_1 = -m$, $S_2 = S_3 = 0$.

Chapter 5

Conclusions

The intrinsic characterisation of orthogonal separability of the Hamilton-Jacobi equation developed by Benenti [4] simplifies earlier geometrical theory initiated by Eisenhart [14] and developed recently by Kalnins and Miller [22]. Whereas previous descriptions required a family of n independent Killing tensors, Benenti has developed a theory based on a *single* Killing tensor.

In general, solving the Killing tensor equation to determine Killing tensors on a space of non-constant curvature is non-trivial; furthermore, to apply the theory of Benenti, we require that such a Killing tensor have orthogonally integrable eigenvectors. By adapting a moving frame to the eigenvectors of the Killing tensor, we simultaneously diagonalise the Killing tensor and metric, facilitating the determination of orthogonal separable coordinates on the corresponding pseudoRiemannian manifold. We also circumvent the problem of finding a general Killing tensor on the space.

The moving frame approach, introduced by Darboux and developed by Cartan, permits calculation independent of local coordinates. Moving frames have been used in other areas of mathematics and physics; however, this is the first application of the method of moving frames in the theory of finite dimensional Hamiltonian systems.

From the intrinsic theory of Benenti, we have developed a coordinate-independent method that generates separable coordinates for a Hamiltonian system on the corresponding pseudoRiemannian manifold. Our analysis of two dimensional Riemannian manifolds of *arbitrary* curvature determined an intrinsic characterisation of separability of the Hamiltonian-Jacobi equation. Without a priori assuming separability, we derived the general forms of the separable metrics, and their associated Killing tensors, separable potentials and second first integrals. As an illustration, we used our method based on the moving frame approach to investigate Euclidean 2-space, and the surfaces of constant curvature.

In addition, by applying our method to the separable potentials in Euclidean 2-space, we determined all known superseparable potentials. These calculations are arguably simpler than those of the Lie group method used in [19]. The work initiated in this thesis can be extended to investigate superseparable potentials between separable coordinate systems of E_2 with different axis orientations or origin positions. Additional superseparable potentials can, in principal, be determined by this generalisation. This extension does not yield new results for Cartesian coordinates because both coordinates are ignorable, as the form of the metric indicates. Similarly, for polar coordinates, the orientation is irrelevant since θ is ignorable.

The value of this method derives from its applicability to manifolds of arbitrary curvature and its intrinsic formulation that avoids the complexities related to the use of local coordinates.

Starting with the two dimensional Lorentzian manifolds of arbitrary curvature, we will extend the results of this thesis to Riemannian and Lorentzian manifolds of dimension three and four. The study of four dimensional Lorentzian manifolds is important to the analysis of Hamiltonian systems in General Relativity.

Appendix A

Orthogonal Integrability

An important concept in this thesis is that of orthogonally integrable vector fields. We develop the basic terminology in this appendix as in [7]. We consider a manifold, M, of dimension $m = n + k$. To each $p \in M$, we assign a n dimensional subspace, Δ_p, of the tangent space, M_p. If there exists a neighbourhood of each $p \in M$, say U, for which there are n linearly independent C^∞ vector fields, X_i, $i = 1, \ldots, n$, that form a basis of Δ_q for all $q \in U$, then Δ is a C^∞ *distribution* of dimension n on M. The set $\{X_i\}$ is called a *local basis* of Δ.

A distribution is *involutive* if and only if there exists a local basis, $\{X_i\}$, in a neighbourhood of each point $p \in M$ such that

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ij}^k X_k, \ 1 \leq i, j \leq n,$$

where the C_{ij}^k's are some functions.

If Δ is a C^∞ distribution on M and N is a connected C^∞ submanifold of M such that at each $q \in N$, $N_q \subset \Delta_q$, that is, the tangent space of N coincides with the distribution, we say N is an *integral (sub)manifold* of Δ.

We define n vectors $E_i = \phi^{-1}(\partial/\partial q^i)$, where q^i, $i = 1, \ldots, m$, are local coordinates on M and the chart (U, ϕ) defines a cubic coordinate neighbourhood for each $p \in M$, that is, for each $p \in M$, there exists a chart (U, ϕ) such that $\{\phi(q) : q \in U\}$

defines a cube in $\mathbb{R}^m$. We say that Δ is *completely integrable* if for each $p \in M$, there exists a neighbourhood U such that $\{E_i\}$ is a local basis for Δ on U. This implies that there is an n dimensional integral manifold N through each point $q \in U$ such that $N_q = \Delta_q$.

The concepts of complete integrability and involutivity are related by the Frobenius theorem:

Theorem A.1 (Frobenius) *A distribution, Δ, on a manifold, M, is completely integrable if and only if it is involutive.*

Naturally, an *orthogonal distribution*, $\Delta^\perp$, is the distribution orthogonal to Δ. A vector field is *orthogonally integrable*, or *normal*, if the corresponding orthogonal distribution is completely integrable.

Appendix B

Schouten Bracket

Let $U^q(M)$ represent the space of contravariant q-tensors on a manifold, M. We observe that $U^0(M)$ is the algebra of C^∞ real valued functions on M.

For arbitrary $P \in U^p(M)$, $Q \in U^q(M)$ such that $p+q \geq 1$, the Schouten bracket is a contravariant $(p+q-1)$-tensor defined in [38] by

$$\begin{aligned}
[P,Q]_{\mathcal{S}}^{i_1 \ldots i_{p+q-1}} := \quad & \Big(\sum_{k=1}^{p} P^{(i_1 \ldots i_{(k-1)}|\mu| i_k \ldots i_{(p-1)}}\Big) \partial_\mu Q^{i_p \ldots i_{(p+q-1)})} + \\
& \Big(\sum_{k=1}^{p} (-1)^k P^{[i_1 \ldots i_{(k-1)}|\mu| i_k \ldots i_{(p-1)}}\Big) \partial_\mu Q^{i_p \ldots i_{(p+q-1)}]} - \\
& \Big(\sum_{l=1}^{q} Q^{(i_1 \ldots i_{(l-1)}|\mu| i_l \ldots i_{(q-1)}}\Big) \partial_\mu Q^{i_q \ldots i_{(p+q-1)})} - \\
& \Big(\sum_{l=1}^{q} (-1)^{(pq+p+q+l)} Q^{[i_1 \ldots i_{(l-1)}|\mu| i_l \ldots i_{(q-1)}}\Big) \partial_\mu Q^{i_q \ldots i_{(p+q-1)}]} .
\end{aligned} \tag{B.1}$$

We observe that for P, Q symmetric the second and last terms of $(B.1)$ vanish. Similarly, if P, Q are skew-symmetric, then the first and third terms of $(B.1)$ vanish.

For a contravariant vector $X \in U^1(M)$, the Schouten bracket reduces to the Lie derivative in the direction of the vector field X,

$$[X, Q]_{\mathcal{S}} = \mathcal{L}_X(Q). \tag{B.2}$$

The Schouten bracket also satisfies the properties [11]:

$$[P, Q]_{\mathcal{S}} = (-1)^{pq}[Q, P]_{\mathcal{S}}, \tag{B.3}$$

and, for $R \in U^r(M)$,

$$(-1)^{pq}[[Q, R]_{\mathcal{S}}, P]_{\mathcal{S}} + (-1)^{qr}[[R, P]_{\mathcal{S}}, Q]_{\mathcal{S}} + (-1)^{rp}[[P, Q]_{\mathcal{S}}, R]_{\mathcal{S}} = 0, \tag{B.4}$$

and

$$[P, Q \wedge R]_{\mathcal{S}} = [P, Q]_{\mathcal{S}} \wedge R + (-1)^{pq+q} Q \wedge [P, R]_{\mathcal{S}}. \tag{B.5}$$

The properties $(B.3)$, $(B.4)$, and $(B.5)$, respectively correspond to the skew-symmetry (2.12), Jacobi (2.11), and Leibniz (2.13) properties of the Poisson bracket.

Appendix C

Moving Frame Formalism

We consider an n dimensional pseudoRiemannian manifold, M. In general, a frame in which the directional derivatives of the metric tensor components vanish,

$$g_{ab,c} = 0, \tag{C.1}$$

is called a *(rigid) moving frame.* A moving frame is an invaluable tool for mechanics. It simplifies the mathematical formulation of the intrinsic theory of orthogonal separation of the Hamilton-Jacobi equation described in Chapter 3, and the solution method developed and applied in Chapter 4.

The frame vector fields spanning the tangent space, M_p, are defined, with respect to the natural basis, $\{\partial/\partial q^1, \dots, \partial/\partial q^n\}$, by

$$E_a = h_a{}^i \frac{\partial}{\partial x^i},$$

where $h_a{}^i$ is an $n \times n$ matrix of C^∞ functions. The dual ω-frame 1-forms are defined, with respect to the dual basis, $\{dq^1, \dots, dq^n\}$ by

$$E^a = h^a{}_i dx^i,$$

where $h^a{}_i$ is the $n \times n$ inverse matrix of $h_a{}^i$.

The components of the object of anholonomy, $C^c{}_{ab}$, are defined in any manifold by

$$C^c{}_{ab}E_c := [E_a, E_b],$$

where $[X, Y]$ is the Lie bracket of two vector fields. The bracket is antisymmetric in its arguments. It follows that C_{cab} is antisymmetric in its second and third indices,

$$C_{cab} = -C_{cba}.$$

The connection coefficients are given, in general, by

$$\begin{aligned}\Gamma_{bca} &:= \frac{1}{2}(E_b g_{ac} - E_a g_{cb} + E_c g_{ba}) - \frac{1}{2}(C_{cba} - C_{bac} + C_{acb}) \\ &= \frac{1}{2}(g_{ac,i}h_b{}^i - g_{cb,i}h_a{}^i + g_{ba,i}h_c{}^i) - \frac{1}{2}(C_{cba} - C_{bac} + C_{acb}),\end{aligned}$$

In a rigid moving frame, they take the form

$$\Gamma_{bca} = -\frac{1}{2}(C_{cba} - C_{bac} + C_{acb}), \tag{C.2}$$

in view of $(C.1)$. Using the antisymmetry of C_{cab} in $(C.2)$, we find that the connection coefficients are skew-symmetric in the second and third indices,

$$\Gamma_{bca} = -\Gamma_{bac}. \tag{C.3}$$

Using the definition of the connection 1-form,

$$\omega^a{}_b := \Gamma_{cb}{}^a E^c,$$

we conclude, from $(C.3)$, that it is also skew-symmetric in its indices,

$$\omega_{ab} = -\omega_{ba},$$

The Levi-Civita connection is torsion-free by definition, that is,

$$T_{abc} := \Gamma_{bca} - \Gamma_{cba} - C_{abc} = 0;$$

therefore, the object of anholonomy has the simple form

$$C_{abc} = \Gamma_{bca} - \Gamma_{cba} = 2\Gamma_{[bc]a}.$$

This simplifies the curvature tensor components,

$$R^a{}_{bcd} = 2E_{[c}\Gamma_{d]b}{}^a + 2\Gamma_{[d|b|}{}^e\Gamma_{c]e}{}^a - C^e{}_{cd}\Gamma_{eb}{}^a,$$

which we may write as

$$R^a{}_{bcd} = 2(\Gamma_{[d|b}{}^a{}_{,i|}h_{c]}{}^i + \Gamma_{[d|b|}{}^e\Gamma_{c]e}{}^a - \Gamma_{[cd]}{}^e\Gamma_{eb}{}^a).$$

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