Standard model physics from abstract mathematical objects?

by

Nichol Furey

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0.1 Author's declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

0.2 Abstract

This thesis constitutes a first attempt to derive aspects of standard model particle physics from purely abstract mathematical objects. We begin abstractly with an algebra, and argue that physical concepts such as particles, causality, and irreversible time may emerge from the algebra acting on itself.

We then focus on a special case by considering the algebra $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, the tensor product of the only four normed division algebras over the real numbers. Using nothing more than $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ acting on itself, we set out to find standard model particle representations: a task which occupies the remainder of this text.

From the $\mathbb{C} \otimes \mathbb{H}$ portion of the algebra, we find generalized ideals, and show that they describe concisely all of the Lorentz representations of the standard model.

From the $\mathbb{C} \otimes \mathbb{O}$ portion of the algebra, we find minimal left ideals, and show that they mirror the behaviour of a generation of quarks and leptons under $su(3)_c$ and $u(1)_{em}$. These unbroken symmetries, $su(3)_c$ and $u(1)_{em}$, appear *uniquely* in this model as symmetries of the algebra's ladder operators. Electric charge, here, is seen to be simply a number operator for the system.

We then combine the $\mathbb{C} \otimes \mathbb{H}$ and $\mathbb{C} \otimes \mathbb{O}$ portions of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, and focus on a leptonic subspace, so as to demonstrate a rudimentary electroweak model. Here, the underlying ladder operators are found to have a symmetry generated *uniquely* by $su(2)_L$ and $u(1)_Y$. Furthermore, we find that this model yields a straightforward explanation as to why $SU(2)_L$ acts only on left-handed states.

We then make progress towards a three-generation model. The action of $\mathbb{C} \otimes \mathbb{O}$ on itself can be seen to generate a 64-complex-dimensional algebra, wherein we are able to identify generators of $SU(3)_c$. We apply these generators to the rest of the space, and find that it breaks down into the $SU(3)_c$ representations of exactly three generations of quarks and leptons. Furthermore, we show that these three-generation results can be extended, so as to include $U(1)_{em}$ charges.

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And I want to know the same thing. We all want to know, how's it going to end?

- *Brennan and Waits*

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Chapter 1

Introduction

1.1 Conceptual parsimony

The safest conceivable theory is one which implements the fewest initial assumptions possible. That is, any model based on complicated fundamental objects will by definition specify more detail than would a simple one, and hence would be more likely to be in conflict with reality.

This thesis may be thought of as an experiment of sorts. It is a first attempt to see just how close one can get to standard model predictions, while using as little theoretical input as possible. The original proposal of this idea was put forward in a Part III research essay for a Master's degree in 2006, [33], [32]. Certainly, this project can be seen to be far from complete. However, it does demonstrate how certain carefully-chosen, low-dimensional, mathematical objects can parallel a considerable amount of the standard model's structure.

Over the next 85 pages, we will work towards developing a model which is based on the same principles of unification and simplification, which have driven much of theoretical physics since the 1970s. All objects in our model will be pieced together from the same algebra, although, we will not unify the gauge groups of the standard model in the same sense as do grand unified theories. We will find that both fermions and bosons can arise together from the same algebra. Having said that, we will not implement supersymmetry at this stage. Although the road to unification has been an arduous one, we maintain that it is an idea worthy of staying the course.

1.2 Outline

In Chapter 2 of this thesis, we give a brief sketch of our experiment, which motivates the algebraic model being proposed. In Chapter 3, we introduce the algebra of the complex quaternions, $\mathbb{C}\otimes\mathbb{H}$, and with them, demonstrate all of the Lorentz representations necessary to describe the standard model. Then, in Chapter 4, we review some basic characteristics of Clifford algebras, while in Chapter 5, we review some basic characteristics of the standard model. In Chapter 6 we introduce the complex octonions, $\mathbb{C}\otimes\mathbb{O}$, and show how they can provide a direct route to the unbroken internal symmetries, generated by $su(3)_c$ and $u(1)_{em}$, for one generation of quarks and leptons. These symmetries appear as a special case of *unitary MTIS symmetries,* which are first introduced in this chapter. In Chapter 7, we then combine the $\mathbb{C}\otimes\mathbb{H}$ and $\mathbb{C}\otimes\mathbb{O}$ results to show a rudimentary leptonic model. The unitary MTIS symmetries, found here, happen to be none other than $su(2)_L$ and $u(1)_Y$ on these states. Furthermore, we find that this model can offer an explanation as to why $SU(2)$ acts on states of only one chirality (left). In Chapter 8, we review some of the algebraic structure of $SU(5)$ and $Spin(10)$ grand unified theories, as well as the Pati-Salam model. Finally, in Chapter 9, we go on to demonstrate how the $SU(3)_c$ and $U(1)_{em}$ representations for exactly three generations of standard model fermions can be found, using (paradoxically) nothing more than the eight-complex dimensional algebra, $\mathbb{C}\otimes\mathbb{O}$.

Chapter 2

M-1 Models

We often view mathematics as a tool, a language, as a bookkeeper. Customarily, mathematics is used as a means of keeping tabs on the important physical objects in our theories.

Over the years, though, many have come to notice how strikingly proficient this bookkeeper is in its role. So proficient, in fact, that one might come to suspect that this quiet administrator has secretly been running the whole enterprise.

In the more poignant cases, such as Newtonian mechanics, math is often relegated to perform menial tasks. At best, mathematics plays a secondary role in support of the primary physical picture.

We will open this thesis by suggesting an inversion of the usual hierarchy, via what we call *math-first,* or *M-1 models.* Here, we will argue that rudimentary physical notions, such as particles, events, causality, and irreversible time can arise from purely abstract mathematical objects.

Many earlier authors have expressed ideas, which similarly identify nature as being fundamentally mathematical. Readers are encouraged to see [70], [67], for example.

The ideas outlined in this first section originate from a Cambridge Part III research essay, [33], [32] and have provided the underlying intuition for three papers published during these PhD years, [36], [35], and [34]. With this being said, the results from those three publications may be considered independently from the ideas sketched here in this section, which are still at an early stage of development.

Much of the material in this chapter was submitted in 2014 to FQXi's annual call for essays in fundamental physics, [30].

2.1 Math-second

Consider a satellite orbiting the Earth. Using classical mechanics, we can solve the equations of motion and input initial data in order to arrive at a mathematical description of the satellite's trajectory, $\overrightarrow{x}(t)$. These three functions, $\overrightarrow{x}(t)$, can help to provide important information, like whether or not our satellite is on course to collide with another.

Surely, this $\overrightarrow{x}(t)$ encodes useful information about the system. However, these functions, at best, play an auxiliary role in support of the primary physical picture. Clearly, no one would claim that $\overrightarrow{x}(t)$ itself *is* the satellite.

But what could it mean to say that a physical system *is* its mathematical description? How can a physical idea, like causality, arise from 'just math'?

2.2 Math-first

2.2.1 Causality

Suppose for a moment that nature were an algebra, A. We will start, then, simply with an unevaluated algebraic expression. Consider for example

$$f(e(dc + ba)), \tag{2.1}$$

where a, b, c, d, e, f are elements of A. Taking multiplication to be the propagation along an edge, and addition to be the joining of two edges at a vertex, it can be seen that this unevaluated algebraic expression gives a causal set. Please see Figure (2.1).

A *causal set* is a set S together with the relation, $<$, such that

1. if $x < y$ and $y < z$, then $x < z$ $\quad \forall x, y, z \in S$ $\qquad$ (transitive),

2. if $x < y$ and $y < x$, then $x = y$ $\quad \forall x, y \in S$ $\qquad$ (non-circular),

3. for any given $x, z \in S$, the set of elements $\{y \mid x < y < z\}$ is finite $\qquad$ (locally finite).

In Figure (2.1), S is given by the set of vertices, and the relations $<$ are indicated by the arrows between those vertices.

It is at this early stage only a conjecture that any algebraic expression gives a class of causal sets, a feature which might depend on the details of the particular algebra chosen. For an introduction to causal sets, please see [28], [10] and [56].
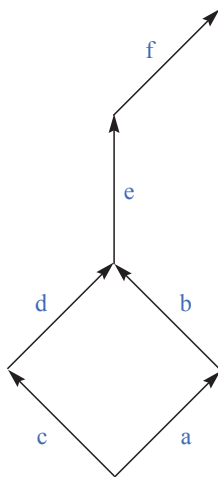
Figure 2.1: The unevaluated algebraic expression, $f(e(dc + ba))$, gives a causal set.

So it seems to be possible that causality, of all things, need not be a fundamental input.

Here we would like to propose an unorthodox interpretation for these causal sets. That is, a causal set is not meant to represent discrete space-time, but is instead meant to represent particle worldlines. In other words, matter exhibits its own causal structure, and there is no such thing as a space-time point. Our picture has particles with no underlying space-time whatsoever, neither continuous, nor discrete.

The question we are then asking here is whether or not space-time can be seen to be as surprisingly unnecessary as was the luminiferous aether from one hundred years ago. Can particles exist *independently,* without the crutch of a fundamental space-time to support them?

Similar in spirit is an earlier spaceless graph model, proposed by Kribs and Markopoulou in [48], where particles emerge at low energies as noiseless subsystems of quantum information processing structures. These models differ from the popular causal set models of Dowker, Sorkin, and Surya, whose vertices represent space-time points. In Dowker, Sorkin, and Surya's models, the vertices specify a position and a time. In the model presented in this thesis, however, the vertices instead specify the internal degrees of freedom of particles, such spin, colour and electric charge, etc.

Over the past ten years, a number of authors have come forward with a variety models, which each call into question the necessity of a fundamental space-time. In 2005, Piazza

did so by proposing to replace localized regions of space with quantum subsystems in [60], [59], an idea which was later developed in 2007 by Piazza and Costa, [61]. In 2010, Van Raamsdonk, [62], proposed building up space-time with quantum entanglement using gauge theory/gravity duality. More recently in 2014, Wieland proposed a model of simplicial gravity, constructed from spinors, [69]. In the same year, W. Edwards described non-embeddable relational configurations, [29]. Also related to these concepts are works by Kempf in 2013, [46], and Saravani, Aslanbeigi, and Kempf in 2015, [53], which discuss how space-time curvature can be encoded in the vacuum entanglement structure of fields.

In 2013, Cortês and Smolin published work on Energetic Causal Sets, [24], [25], which describes a spaceless causal set, constructed out of particles at the fundamental level. This basic concept overlaps significantly with [32]: ideas of L. Smolin's PhD student at the time (the present author). The notes, [32], were written up for L. Smolin in 2011 upon his request, and subsequently emailed to him. (He later innocently forgot about the notes when [24] and [25] were published two years later.) We also point out the relevant earlier work of [48], who proposed a spaceless causal set of quantum information processing systems, with particles emerging in the low energy limit.

2.2.2   Time

Recently, it has been emphasized in the fundamental physics community, [30], that theories like general relativity do a poor job of encapsulating our experience of *events* unfolding. That is, there is nothing in the theory to explain why events happen, but do not 'unhappen'.

In the M-1 model presented here, however, we will suggest a notion of events, which aims to be more in tune with reality. Here, an event is a calculation. Taking our algebra $A = \mathbb{R}$, for example, an event is the evaluation of 6+3 to give 9, or the evaluation of 5*2 to give 10.

Addition and multiplication are examples of uninvertible binary operations. Therefore, an event can be seen to be irreversible, it cannot 'unhappen'. For example, if we are given only the output of 9, it is impossible to tell if 9 came from the inputs of 6+3, or 8+1, or 4+5, etc.

Time, then, is simply a sequence of calculations, and is clearly irreversible. It can be seen to be an entirely local concept within the causal set.

2.2.3   Particles: surviving subspaces

Some carefully chosen algebras, such as the complex Clifford algebra, $A = \mathbb{C}l(2)$, are naturally partitioned into subspaces called *ideals.* Intuitively speaking, an ideal is a special subspace of an algebra because it can survive multiplication by any element in A.

Ideals persisting under multiplication bear a striking resemblance to *particles* persisting under propagation. The proposal, then, is that particles could be singled out in the algebra, thanks to a mathematical incarnation of Darwin's natural selection.

$$\text{Ideals} \sim \text{particles.}$$

Given an algebra, A, a *left ideal, B,* is a subalgebra of A whereby $a\underline{b}$ is in B for all $\underline{b}$ in B, and for any a in A. That is, no matter which a we multiply onto $\underline{b}$, the new product, $\underline{b}' \equiv a\underline{b}$, must be in the subspace B (i.e. the ideal B survives). It is easy to see how $\underline{b}' \equiv a\underline{b}$ could easily describe, for example, a particle $\underline{b}$ undergoing propagation along a.

Theses concepts have a strong connection to well-known physics. In this text, we will first introduce the notion of *generalized ideals.* Taking A to be the complex quaternions, $\mathbb{C} \otimes \mathbb{H} \simeq \mathbb{C}l(2)$, we will then see how generalized ideals lead to left- and right-handed Weyl spinors, [34]. In an analogous construction, starting from the complex octonions, $A = \mathbb{C} \otimes \mathbb{O}$, generalized ideals will be seen to lead to a set of states behaving like a full generation of quarks and leptons, [36].

Ultimately, we intend to merge $\mathbb{C} \otimes \mathbb{H}$ and $\mathbb{C} \otimes \mathbb{O}$ together, via a tensor product over $\mathbb{C}$, resulting in the algebra $A = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. Loosely speaking, we will associate $\mathbb{C} \otimes \mathbb{H}$ with Lorentzian degrees of freedom: spin and chirality, while the octonionic part of the algebra will give rise to the other internal degrees of freedom, such as colour, weak isospin, and charge.

$$\mathbb{C} \otimes \mathbb{H} \sim \text{Lorentz} \qquad \mathbb{C} \otimes \mathbb{O} \sim \text{Other internal: colour, charge, etc.}$$

The Dixon algebra, $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, is the tensor product of the only four normed division algebras over the real numbers: the real numbers, $\mathbb{R}$, the complex numbers, $\mathbb{C}$, the quaternions, $\mathbb{H}$, and the octonions, $\mathbb{O}$. Its connection to particle physics was studied indirectly by Casalbuoni *et al.,* [8], [13], [14], and later much more extensively by Dixon, [26]. Our goal of identifying standard model structure from $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ aligns with that of these earlier authors, however, our implementation of this algebra differs significantly, particularly with respect to chirality and weak isospin, and in our treatment of antiparticles.

2.2.4   Summary and outlook

We would like to entertain the idea that nature could be purely mathematical. In doing so, we have sketched a model, whose only fundamental input is a single algebra, A.

The algebra, A, will be taken to be $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ for reasons that will become clear throughout this thesis. In the future, however, it would be worth investigating whether or not even $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ could be simplified. Explicitly, $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ could be replaced with another mathematical object, which approximates it in some limit, but which lacks any occurrence of uncountable infinities, which are inherent to the real number system.

From the (purely mathematical) algebraic expressions of A, we argue that physical concepts can materialize. This runs backwards to the usual style with which physical theories are constructed. Such resulting physical concepts may include causality, particles, and irreversible time. In particular, particles may be seen to arise as the algebra's most stable subspaces.

With this rough draft of a model, we would like to now develop the idea. But where to begin? Perhaps the most straightforward way to see if the algebra $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ could stand a chance of eventually producing standard model scattering amplitudes, is to see if it can first produce standard model group representations. This task will occupy the rest of the thesis.

Chapter 3

Complex quaternions

3.1 Preamble

A significant challenge within causal set programs has been to explain the existence of 3+1 dimensions. That is, any causal set chosen at random is highly unlikely to have this particular dimension of our choosing.

It is for this reason that we propose here causal sets originating from an algebra, A. The idea then is that a careful choice of algebra will impart on the causal set the desired 3+1 Lorentzian structure. In this section, we will consider $A = \mathbb{C} \otimes \mathbb{H}$, the complex quaternions. Between the left and right action of $\mathbb{C} \otimes \mathbb{H}$ on itself, we will now introduce a surprisingly compact way of describing all of the (3+1) Lorentz representations of the standard model of particle physics.

3.2 Introduction to $\mathbb{C} \otimes \mathbb{H}$

Any element of the complex quaternions can be described as the complex linear combination,

$$c_0 + c_1\, i\epsilon_x + c_2\, i\epsilon_y + c_3\, i\epsilon_z, \tag{3.1}$$

where the $c_n \in \mathbb{C}$. The element i is the usual complex imaginary unit, with $i^2 = -1$, that commutes with all of the elements in the algebra. The quaternionic imaginary units, ϵ_x, ϵ_y, and ϵ_z follow the multiplication rules

$$\epsilon_x{}^2 = \epsilon_y{}^2 = \epsilon_z{}^2 = \epsilon_x \, \epsilon_y \, \epsilon_z = -1, \tag{3.2}$$

which lead to the identities $\epsilon_x\epsilon_y = -\epsilon_y\epsilon_x = \epsilon_z$, $\epsilon_y\epsilon_z = -\epsilon_z\epsilon_y = \epsilon_x$, $\epsilon_z\epsilon_x = -\epsilon_x\epsilon_z = \epsilon_y$. The complex quaternions form an associative algebra, meaning that $(ab)\,c = a\,(bc) \; \forall \; a, \, b, \, c \in \mathbb{C} \otimes \mathbb{H}$. (Note that all tensor products will be assumed to be over $\mathbb{R}$ in this text, unless otherwise stated.)

From their behaviour under multiplication, one may associate $i\epsilon_x$ with the more familiar Pauli matrix $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $i\epsilon_y$ with $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $i\epsilon_z$ with $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Note, however, that the set $\{i\epsilon_x, i\epsilon_y, i\epsilon_z\}$ transforms more symmetrically under complex conjugation than does $\{\sigma_x, \sigma_y, \sigma_z\}$,

$$\sigma_x^* = \sigma_x, \qquad\qquad \sigma_y^* = -\sigma_y, \qquad\qquad \sigma_z^* = \sigma_z,$$

$$\text{versus} \tag{3.3}$$

$$(i\epsilon_x)^* = -i\epsilon_x, \qquad (i\epsilon_y)^* = -i\epsilon_y, \qquad (i\epsilon_z)^* = -i\epsilon_z.$$

Elements of the form $s \equiv r_1\epsilon_x + r_2 i\epsilon_x + r_3\epsilon_y + r_4 i\epsilon_y + r_5\epsilon_z + r_6 i\epsilon_z$ for $r_n \in \mathbb{R}$ can easily be seen to give a representation of the Lie algebra $sl(2, \mathbb{C})$, using the usual commutator, $[s, s'] \equiv ss' - s's$.

Later in Section 3.5, we will find that $L \equiv e^{is} \in \mathbb{C}\otimes\mathbb{H}$ represents an element of $SL(2, \mathbb{C})$, which acts on left-handed Weyl spinors. These Weyl spinors also reside in $\mathbb{C} \otimes \mathbb{H}$, and in fact, all of the results obtained throughout this thesis will follow simply from having an algebra A act on itself.

3.3 Conjugation

In this text, the *complex conjugate* of an element a will be denoted a^*. The conjugate $*$ maps the complex $i \mapsto -i$, in the usual way.

The *quaternion conjugate* of a will be denoted $\widetilde{a}$, and $\sim$ maps the quaternionic $\epsilon_x \mapsto -\epsilon_x$, $\epsilon_y \mapsto -\epsilon_y$, and $\epsilon_z \mapsto -\epsilon_z$.

That which we call the *hermitian conjugate* of a will be denoted $a^\dagger$, and $\dagger$ is the result of performing both the complex and quaternion conjugates simultaneously: $i \mapsto -i$, $\epsilon_x \mapsto -\epsilon_x$, $\epsilon_y \mapsto -\epsilon_y$, and $\epsilon_z \mapsto -\epsilon_z$.

It is important to note that both the quaternion and hermitian conjugates reverse the order of multiplication, as is familiar from matrix multiplication. For example, $(ab)^\dagger = b^\dagger a^\dagger$.

3.4 Generalized ideals

We define a subalgebra B of an algebra A to be a *generalized ideal* if $m(a, \underline{b}) \in B$, $\forall \underline{b} \in B$ and for any $a \in A$, where m is (generalized) multiplication. The notion of a generalized ideal was introduced by this author in [34], and differs from the definition of left ideals by generalizing what is meant by 'multiplication'. Directly from the definition, it can be seen that ideals make up the algebra's most robust subspaces, which persist no matter what a is multiplied onto them.

Starting from the algebra $A = \mathbb{C} \otimes \mathbb{H}$, we will find generalized ideals under three separate notions of generalized multiplication:

$$\text{the complex invariant-action,} \qquad m_c(a, \underline{b}) \equiv a\,\underline{b}\,P + a^*\,\underline{b}\,P^*,$$

$$\text{the hermitian invariant-action,} \qquad m_h(a, \underline{b}) \equiv a\,\underline{b}\,a^\dagger, \qquad (3.4)$$

$$\text{and quaternionic invariant-action,} \qquad m_q(a, \underline{b}) \equiv a\,\underline{b}\,\widetilde{a}.$$

Here, P is a projector in $\mathbb{C} \otimes \mathbb{H}$, to be defined shortly. It should be noted that each of these multiplication rules is constructed so as to preserve conjugation-invariant objects. For example, elements, $\underline{b}$ in $\mathbb{C} \otimes \mathbb{H}$ with the property $\underline{b}^* = \underline{b}$ will maintain this property under m_c, no matter which $a \in \mathbb{C} \otimes \mathbb{H}$ is multiplied onto them.

Taking A to be $\mathbb{C} \otimes \mathbb{H}$, we will first show how the complex invariant-action leads to left- and right-handed Weyl spinors, Majorana spinors, and Dirac spinors. Then we will show how the hermitian invariant-action leads to four-vectors. Finally, we will show how the quaternionic invariant action leads to scalars and the field strength tensor.

$$m_c \qquad \Rightarrow \qquad \psi \qquad \text{(spinors)}$$

$$m_h \qquad \Rightarrow \qquad p_\mu \qquad \text{(four-vectors)}$$

$$m_q \qquad \Rightarrow \qquad \phi \text{ and } F_{\mu\nu} \qquad \text{(scalars and the field strength tensor).}$$

Readers should note that scalars, spinors, four-vectors, and the field strength tensor, mentioned above, account for all of the Lorentz representations of the standard model.

3.5 Complex invariant-action, m_c

3.5.1 Preliminaries

We will now find that left- and right-handed Weyl spinors, Majorana spinors, and Dirac spinors are all generalized ideals under the same complex invariant-action,

$$\underline{b}' = a\,\underline{b}\,P + a^*\,\underline{b}\,P^*. \tag{3.5}$$

For concreteness, we define P to be the projector $\frac{1}{2}\left(1 + i\epsilon_z\right)$, although it is clear that a continuum of other possibilities exist. P and its complex conjugate, P^*, exhibit the properties

$$PP = P, \qquad P^*P^* = P^*, \qquad PP^* = P^*P = 0, \qquad P + P^* = 1. \tag{3.6}$$

Before we begin, though, it will be useful to carry out a change of basis from $\{1,\ i\epsilon_x,\ i\epsilon_y,\ i\epsilon_z\}$ to a new, suggestively named basis, $\{\epsilon_{\uparrow\uparrow},\ \epsilon_{\downarrow\uparrow},\ \epsilon_{\uparrow\downarrow},\ \epsilon_{\downarrow\downarrow}\}$. This new basis will be linked to the operator, $i\epsilon_z \sim \sigma_z$, in that these basis elements will be defined so as to have the properties:

$$
\begin{aligned}
i\epsilon_z\,\epsilon_{\uparrow\uparrow} &= +\,\epsilon_{\uparrow\uparrow} & \epsilon_{\uparrow\uparrow}\,i\epsilon_z &= +\,\epsilon_{\uparrow\uparrow} \\
i\epsilon_z\,\epsilon_{\downarrow\uparrow} &= -\,\epsilon_{\downarrow\uparrow} & \epsilon_{\downarrow\uparrow}\,i\epsilon_z &= +\,\epsilon_{\downarrow\uparrow} \\
i\epsilon_z\,\epsilon_{\uparrow\downarrow} &= +\,\epsilon_{\uparrow\downarrow} & \epsilon_{\uparrow\downarrow}\,i\epsilon_z &= -\,\epsilon_{\uparrow\downarrow} \\
i\epsilon_z\,\epsilon_{\downarrow\downarrow} &= -\,\epsilon_{\downarrow\downarrow} & \epsilon_{\downarrow\downarrow}\,i\epsilon_z &= -\,\epsilon_{\downarrow\downarrow}.
\end{aligned}
\tag{3.7}
$$

In terms of the old basis, we define these new basis vectors to be

$$
\begin{aligned}
\epsilon_{\uparrow\uparrow} &\equiv \tfrac{1}{2}\left(1 + i\epsilon_z\right) & \epsilon_{\downarrow\uparrow} &\equiv \tfrac{1}{2}\left(\epsilon_y + i\epsilon_x\right) \\
\epsilon_{\uparrow\downarrow} &\equiv \tfrac{1}{2}\left(-\epsilon_y + i\epsilon_x\right) & \epsilon_{\downarrow\downarrow} &\equiv \tfrac{1}{2}\left(1 - i\epsilon_z\right).
\end{aligned}
\tag{3.8}
$$

For convenience, we also include here the old basis in terms of the new one,

$$
\begin{aligned}
1 &= \epsilon_{\uparrow\uparrow} + \epsilon_{\downarrow\downarrow} & \epsilon_x &= -i\left(\epsilon_{\downarrow\uparrow} + \epsilon_{\uparrow\downarrow}\right) \\
\epsilon_y &= \epsilon_{\downarrow\uparrow} - \epsilon_{\uparrow\downarrow} & \epsilon_z &= -i\left(\epsilon_{\uparrow\uparrow} - \epsilon_{\downarrow\downarrow}\right).
\end{aligned}
\tag{3.9}
$$

3.5.2   Weyl and Dirac spinors as generalized ideals

Let us now identify two subspaces, which partition the algebra $\mathbb{C} \otimes \mathbb{H}$. These subspaces will be given the suggestive nomenclature Ψ_L and Ψ_R, and be defined as

$$\Psi_\mathrm{L} \equiv \psi_\mathrm{L}^\uparrow \, \epsilon_{\uparrow\uparrow} + \psi_\mathrm{L}^\downarrow \, \epsilon_{\downarrow\uparrow} \qquad\qquad \Psi_\mathrm{R} \equiv \psi_\mathrm{R}^\uparrow \, \epsilon_{\uparrow\downarrow} + \psi_\mathrm{R}^\downarrow \, \epsilon_{\downarrow\downarrow}, \tag{3.10}$$

where $\psi_\mathrm{L}^\uparrow, \, \psi_\mathrm{L}^\downarrow, \, \psi_\mathrm{R}^\uparrow, \, \psi_\mathrm{R}^\downarrow \in \mathbb{C}$. Readers may notice that spin and chirality are analogues of each other in this formalism, as transitions between spin states occur via left multiplication, and transitions between L and R occur via right multiplication.

Straightforward calculation shows that

$$\begin{aligned} \Psi_\mathrm{L} \, P &= \Psi_\mathrm{L} & \Psi_\mathrm{L} \, P^* &= 0, \\ \Psi_\mathrm{R} \, P &= 0 & \Psi_\mathrm{R} \, P^* &= \Psi_\mathrm{R}. \end{aligned} \tag{3.11}$$

Now, the reader may confirm that Ψ_L and Ψ_R are left ideals, as defined in Section 2. That is, no matter which a_1 and a_2 are left multiplied onto them, there exists some $\Psi_\mathrm{L}{}' \equiv \psi_\mathrm{L}^{\uparrow\,'} \, \epsilon_{\uparrow\uparrow} + \psi_\mathrm{L}^{\downarrow\,'} \, \epsilon_{\downarrow\uparrow}$ and $\Psi_\mathrm{R}{}' \equiv \psi_\mathrm{R}^{\uparrow\,'} \, \epsilon_{\uparrow\downarrow} + \psi_\mathrm{R}^{\downarrow\,'} \, \epsilon_{\downarrow\downarrow}$ such that

$$a_1 \Psi_\mathrm{L} = \Psi_\mathrm{L}{}' \qquad\qquad a_2 \Psi_\mathrm{R} = \Psi_\mathrm{R}{}'. \tag{3.12}$$

In other words, the L and R subspaces are each stable under left multiplication.

With this knowledge in hand, it is now almost trivial to see that Ψ_L and Ψ_R are each generalized ideals under the complex invariant-action, m_c. That is, for any $a \in \mathbb{C} \otimes \mathbb{H}$, there exists a Ψ_L' such that

$$\Psi_\mathrm{L}' = a\Psi_\mathrm{L} = a\Psi_\mathrm{L}P = a\Psi_\mathrm{L}P + a^*\Psi_\mathrm{L}P^* = m_c(a, \Psi_\mathrm{L}). \tag{3.13}$$

Likewise, for any $a^* \in \mathbb{C} \otimes \mathbb{H}$, there exists a Ψ_R' such that

$$\Psi_\mathrm{R}' = a^*\Psi_\mathrm{R} = a^*\Psi_\mathrm{R}P^* = a\Psi_\mathrm{R}P + a^*\Psi_\mathrm{R}P^* = m_c(a, \Psi_\mathrm{R}). \tag{3.14}$$

It is furthermore easily seen that the set of elements of the form $\Psi_\mathrm{D} \equiv \Psi_\mathrm{L} + \Psi_\mathrm{R}$ spans all of $\mathbb{C} \otimes \mathbb{H}$, and hence trivially qualifies as a generalized ideal under m_c.

Taking now a to be $L = e^{is}$ from Section 3.2 gives the transformation

$$\Psi_\mathrm{D}' = m_c(L, \Psi_\mathrm{D}) = L\Psi_\mathrm{L} + L^*\Psi_\mathrm{R}. \tag{3.15}$$

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The reader is encouraged to confirm that indeed Ψ_D transforms as a Dirac spinor, Ψ_L transforms as a left-handed Weyl spinor, and Ψ_R transforms as a right-handed Weyl spinor under $SL(2,\mathbb{C})$. In other words, the complex coefficients within Ψ_D transform exactly as would the complex components of a four-dimensional column vector, representing a Dirac spinor, from standard quantum field theory. It is for this reason that the names Ψ_D, Ψ_L, and Ψ_R were given to these generalized ideals early on.

3.5.3   A seamless new way to conjugate Weyl spinors

In quantum field theory textbooks, it is typically explained that a left handed Weyl spinor, Ψ_L, can be conjugated so as to give a right-handed Weyl spinor. The procedure necessary to do so, in the usual matrix-and-column-vector formalism, entails (1) complex conjugating the left-handed spinor's two components, and (2) multiplying this column vector by the matrix $\epsilon = -i\sigma_y = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$,

$$\left[\, \psi_\mathrm{L}^\uparrow,\ \psi_\mathrm{L}^\downarrow \,\right]^\top \ \to (1) \to \ \left[\, \psi_\mathrm{L}^{\uparrow*},\ \psi_\mathrm{L}^{\downarrow*} \,\right]^\top \ \to (2) \to \ \left[\, -\psi_\mathrm{L}^{\downarrow*},\ \psi_\mathrm{L}^{\uparrow*} \,\right]^\top. \tag{3.16}$$

Notice what happens, though, if we write these Weyl spinors instead in terms of the $\mathbb{C}\otimes\mathbb{H}$ algebra:

$$\Psi_\mathrm{L}^* = \psi_\mathrm{L}^{\uparrow*}\,\epsilon_{\uparrow\uparrow}{}^* + \psi_\mathrm{L}^{\downarrow*}\,\epsilon_{\downarrow\uparrow}{}^* = -\psi_\mathrm{L}^{\downarrow*}\,\epsilon_{\uparrow\downarrow} + \psi_\mathrm{L}^{\uparrow*}\,\epsilon_{\downarrow\downarrow}. \tag{3.17}$$

That is, we arrive at the exact same result using nothing more than the complex conjugate: $i \mapsto -i$.

Now, in order to return back to the original left-handed Weyl spinor, in the usual matrix-and-column-vector formalism, we must again (1) complex conjugate the spinor's components, but this time, (2)$'$ multiply by the new matrix $-\epsilon$,

$$\left[\, -\psi_\mathrm{L}^{\downarrow*},\ \psi_\mathrm{L}^{\uparrow*} \,\right]^\top \ \to (1) \to \ \left[\, -\psi_\mathrm{L}^{\downarrow},\ \psi_\mathrm{L}^{\uparrow} \,\right]^\top \ \to (2)' \to \ \left[\, \psi_\mathrm{L}^\uparrow,\ \psi_\mathrm{L}^\downarrow \,\right]^\top. \tag{3.18}$$

In comparison, the same result is achieved in $\mathbb{C}\otimes\mathbb{H}$ more simply by taking the complex conjugate twice,

$$\Psi_\mathrm{L}^{**} = \Psi_\mathrm{L}. \tag{3.19}$$

Readers should note that $* : \Psi_{\mathrm{L}} \mapsto \Psi_{\mathrm{R}}$ is a basis-independent statement in our formalism. It holds, regardless of whether Ψ_{L} is written in the Weyl, Dirac, or Majorana basis of the Dirac algebra. To the best of this author's knowledge, [34] was the first instance where this more streamlined method of conjugating spinors has been proposed.

In considering these two formalisms, one comes to notice that no ϵ-type object was needed in the $\mathbb{C} \otimes \mathbb{H}$ case. Somehow, the complex conjugate $i \mapsto -i$ automatically encoded the information given by ϵ. It is natural to ask, then, why ϵ is necessary in the matrix-and-column-vector formalism.

After comparing equations (3.16) and (3.17), one sees that ϵ was what was needed so as to account for the complex conjugation of basis vectors. In the $\mathbb{C} \otimes \mathbb{H}$ formalism, we naturally take the complex conjugate of *both* the coefficients, and the basis vectors. On the other hand, there is no notion of complex conjugating basis vectors in the matrix-and-column-vector formalism. The matrix ϵ needs to be introduced so as to account for this. The conjugation of Weyl spinors, as explained here, then exposes a naturalness to the purely algebraic formalism.

3.5.4 Majorana spinors as generalized ideals

Given our description above of Dirac spinors, it is now easy to build Majorana spinors. Any Dirac spinor splits into two Majorana spinors, given by

$$\Psi_{\mathrm{M}_1} \equiv \tfrac{1}{2} \left(\Psi_{\mathrm{D}} + \Psi_{\mathrm{D}}^* \right) \qquad \Psi_{\mathrm{M}_2} \equiv \tfrac{1}{2} \left(\Psi_{\mathrm{D}} - \Psi_{\mathrm{D}}^* \right). \tag{3.20}$$

Clearly, Ψ_{M_1} is invariant under complex conjugation, $\Psi_{\mathrm{M}_1}^* = \Psi_{\mathrm{M}_1}$, while Ψ_{M_2} acquires a minus sign, $\Psi_{\mathrm{M}_2}^* = -\Psi_{\mathrm{M}_2}$. The reader is encouraged to confirm that Ψ_{M_1} and Ψ_{M_2} both constitute generalized ideals, and that $m_c(L, \Psi_{\mathrm{M}})$ gives their transformation under $SL(2, \mathbb{C})$.

To summarize, we have just found that left- and right-handed Weyl spinors, Dirac spinors, and Majorana spinors are all simply generalized ideals of the same complex invariant-action, m_c. Furthermore, in every case, their transformation under $SL(2, \mathbb{C})$ can be described succinctly as $\Psi' = m_c(L, \Psi)$.

3.6 Hermitian invariant-action, m_h

We will now find that the hermitian invariant-action, m_h, leads to new generalized ideals, which behave like four-vectors,

$$\underline{b}' = a\,\underline{b}\,a^\dagger. \tag{3.21}$$

Any element of $\mathbb{C}\otimes\mathbb{H}$ can be written as a sum of hermitian $p \equiv p_0 + p_1 i\epsilon_x + p_2 i\epsilon_y + p_3 i\epsilon_z$ and anti-hermitian $\hat{p} \equiv i\hat{p}_0 + \hat{p}_1\epsilon_x + \hat{p}_2\epsilon_y + \hat{p}_3\epsilon_z$ parts, where the p_n and $\hat{p}_n \in \mathbb{R}$. As $ap a^\dagger$ is hermitian and $a\hat{p}a^\dagger$ is antihermitian for any $a \in \mathbb{C}\otimes\mathbb{H}$, it is clear that these two subspaces form generalized ideals under the multiplication $m_h(a, \underline{b}) = a\underline{b}a^\dagger$.

Just as was done in the case for spinors, we may now set $a = L$, where L represents elements of $SL(2,\mathbb{C})$. The hermitian element, p, transforms as $p' = LpL^\dagger$. Matching components, one finds that under this transformation law, p transforms as a contravariant four-vector under the Lorentz group, [41]. Taking the complex conjugate of p describes the transformation a covariant four-vector, $p^{*\prime} = L^*p^*\widetilde{L}$. It can be seen that the antihermitian case for $\hat{p}$ follows analogously.

As an example, let us consider the momentum $p = p_0 + p_1 i\epsilon_x + p_2 i\epsilon_y + p_3 i\epsilon_z$ under a rotation about the z axis by an angle θ. This rotated momentum is given by

$$p' = \exp\left(-\tfrac{\theta\epsilon_z}{2}\right) p \, \exp\left(\tfrac{\theta\epsilon_z}{2}\right) = \left(\cos\tfrac{\theta}{2} - \epsilon_z\sin\tfrac{\theta}{2}\right) p \left(\cos\tfrac{\theta}{2} + \epsilon_z\sin\tfrac{\theta}{2}\right) =$$
$$p_0 + \left(p_1\cos\theta + p_2\sin\theta\right)i\epsilon_x + \left(p_2\cos\theta - p_1\sin\theta\right)i\epsilon_y + p_3 i\epsilon_z, \tag{3.22}$$

as expected.

From this example, readers may note that even though the object $p = p_0 + p_1 i\epsilon_x + p_2 i\epsilon_y + p_3 i\epsilon_z$ looks to have indices which are all tied off, it does not represent a Lorentz scalar. It transforms as a four-vector. Furthermore, even though the coefficients p_μ have space-time indices, they do not transform directly on their own. This is in contrast to objects in the formalism of standard QFT, which would have real numbers, p_μ, transforming under the Lorentz group. Here, we have only the complete $p = p_0 + p_1 i\epsilon_x + p_2 i\epsilon_y + p_3 i\epsilon_z$ transforming under the Lorentz group.

As shown in [41], scalars can be constructed between a covariant vector p and contravariant vector q, as $\frac{1}{2}\left(pq + \widetilde{pq}\right)$, which is simply the real part of pq. Indeed, when $q = p^*$, this gives $p_0^2 - p_1^2 - p_2^2 - p_3^2$.

In brief, we have just seen that four-vectors can be represented simply by generalized ideals under the hermitian multiplicative action, m_h. For the hermitian case, contravariant four vectors transform as $p' = LpL^\dagger$, and covariant four-vectors transform as $p^{*\prime} = L^* p^* \widetilde{L}$, where L represents and element of $SL(2, \mathbb{C})$.

3.7 Quaternionic invariant-action, m_q

Finally, we will now consider the quaternionic invariant-action, m_q. This new action leads to generalized ideals, which are more commonly known as Lorentz scalars, ϕ, and the field strength tensor, F,

$$\underline{b}' = a \, \underline{b} \, \widetilde{a}. \tag{3.23}$$

$\mathbb{C} \otimes \mathbb{H}$ can be partitioned once again, this time into subspaces of the form $\phi \in \mathbb{C}$ and $F = \left(F^{32} + iF^{01}\right)\epsilon_x + \left(F^{13} + iF^{02}\right)\epsilon_y + \left(F^{21} + iF^{03}\right)\epsilon_z$, with $F^{mn} \in \mathbb{R}$. As $\widetilde{\phi} = \phi$, and $\widetilde{F} = -F$, it is clear that each of these two subspaces is closed under the multiplication $\underline{b}' = a\underline{b}\widetilde{a}$ from any element a of the algebra. Hence, they each constitute generalized ideals under m_q.

Lorentz transformations on these two generalized ideals can be found, again, by replacing a with L. For our scalar, ϕ, we have $\phi' = L\phi\widetilde{L}$. Since $\phi \in \mathbb{C}$, it commutes with every element in $\mathbb{C} \otimes \mathbb{H}$, and so $\phi' = L\phi\widetilde{L} = L\widetilde{L}\phi$. It is then easily confirmed that $\widetilde{L} = L^{-1} \ \forall L$, so that ϕ is indeed a Lorentz scalar, $\phi' = \phi$.

In [41] it is shown that the usual field strength tensor is represented by F, which transforms as $F' = LF\widetilde{L}$ under the Lorentz group. $F^* = \left(B_1 + iE_1\right)\epsilon_x + \left(B_2 + iE_2\right)\epsilon_y + \left(B_3 + iE_3\right)\epsilon_z$ gives the field strength $F_{\mu\nu}$, while F gives $F^{\mu\nu}$.

Readers are referred to the upcoming Section 4.9, where the description of $F_{\mu\nu}$ will be extended so as to satisfy parity transformations, defined there.

In summary, we have just seen that the Lorentz scalar, ϕ, and the field strength tensor, F, are simply generalized ideals under the quaternionic invariant-action, m_q. These generalized ideals transform as $\phi' = L\phi\widetilde{L} = \phi$ and $F' = LF\widetilde{L}$ under the Lorentz group, where L represents an element of $SL(2, \mathbb{C})$.

3.8   Bilinears

Bilinears and other scalars can now be built by combining the various ideal representations, whose $SL(2,\mathbb{C})$ factors, L, fit together like lock and key. Noting that $\widetilde{L} = L^{-1}$, let us consider for example the real part of $\Psi_\mathrm{L}^\dagger i\partial\,\Psi_\mathrm{L} + \Psi_\mathrm{R}^\dagger i\partial^*\,\Psi_\mathrm{R}$, where $\partial \equiv \partial_t - i\epsilon_x\,\partial_x - i\epsilon_y\,\partial_y - i\epsilon_z\,\partial_z$. Under a Lorentz transformation,

$$\left\langle \Psi_\mathrm{L}^\dagger i\partial\,\Psi_\mathrm{L} + \Psi_\mathrm{R}^\dagger i\partial^*\,\Psi_\mathrm{R} \right\rangle' = \left\langle \Psi_\mathrm{L}^\dagger L^\dagger\ L^* i\partial\widetilde{L}\ L\Psi_\mathrm{L} + \Psi_\mathrm{R}^\dagger \widetilde{L}\ L i\partial^* L^\dagger\ L^*\Psi_\mathrm{R} \right\rangle$$

$$= \left\langle \Psi_\mathrm{L}^\dagger i\partial\,\Psi_\mathrm{L} + \Psi_\mathrm{R}^\dagger i\partial^*\,\Psi_\mathrm{R} \right\rangle,$$

(3.24)

where $\langle \cdots \rangle$ means to take the real part.

This scalar is the same as the scalar between Dirac spinors of quantum field theory, at a fixed space-time point, $\overline{\Psi}_D i\hat{\partial}\Psi_D = \Psi_D^\dagger \beta i\gamma^\alpha \partial_\alpha \Psi_D$, in the usual matrix-and-column-vector formalism.

3.9   Summary

We have just found a set of generalized ideals, originating from three generalized notions of multiplication, m_c, m_h, and m_q. These generalized ideals led directly to left- and right-handed Weyl spinors, Dirac spinors, and Majorana spinors, four-vectors, scalars, and the field strength tensor. This accounts for *all* of the Lorentz representations of the standard model.

Furthermore, we found that Lorentz transformations can be described concisely by

$$\underline{b}' = m(L, \underline{b}),$$

(3.25)

(or by $\underline{b}' = m(L^*, \underline{b})$). These results were obtained using nothing but the algebra $\mathbb{C} \otimes \mathbb{H}$ acting on itself.

In contrast to the usual matrix-and-column-vector formalism of QFT, we point out a number of advantages offered by the $\mathbb{C} \otimes \mathbb{H}$ formalism:

* A compact way to describe all of the Lorentz representations of the standard model

* A seamless new way of conjugating Weyl spinors, using only the complex conjugate, $i \mapsto -i$

* A minimalistic formalism, making use of just a single algebra acting on itself

* A conceptually new description of Lorentz representations as stable subspaces (generalized ideals) of an algebra.

* A possible explanation as to why higher spin states are not seen experimentally in fundamental particle physics. Higher spin representations of the Lorentz group should be naturally excluded here, if the representation exceeds the number of dimensions describable by $\mathbb{C} \otimes \mathbb{H}$.

3.10   Outlook

It is plain to see that m_c, m_h, and m_q each correspond to an involution of the $\mathbb{C} \otimes \mathbb{H}$ algebra. That is, m_c corresponds to $*$, m_h corresponds to $\dagger$, and m_q corresponds to $\sim$. The complex quaternions, though, have more discrete symmetries available than is shown here, and it would be interesting to see what could come from the corresponding constructions of generalized ideals.

A further proposal is to take the concept of generalized multiplication further by considering *generalized automorphisms.* Just as automorphisms preserve the multiplicative structure of an algebra, generalized automorphisms are defined to preserve the *generalized* multiplicative structure of an algebra. These generalized automorphisms might then qualify as valid candidates for gauge symmetries, in the context of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$.

Another interesting lead to consider is whether or not this work has a connection to Connes' non-commutative geometry, [22], [11], [9]. That is, the complex and/or hermitian invariant-actions, m_c and m_h, introduced here seem to bear some resemblance to the Dirac operators found there. Could the results presented here fit into that formalism? If not, could this work suggest admissible alterations to Connes' axioms?

Yet another line of questioning, currently under investigation, is to see how these $\mathbb{C} \otimes \mathbb{H}$ representations (and later, $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ representations) relate to Jordan algebras.

Chapter 4

Clifford algebras

4.1 Motivation

It is difficult to ignore the presence of Clifford algebras within elementary particle physics. Clifford algebras inevitably appear whenever spinors do, and ultimately underlie the algebra of some well-known grand unified theories, [7]. It was in fact argued in [8], [13], and [14] that Clifford algebras alone are the source of internal structure for quarks and leptons.

But one may ask, where do these Clifford algebras come from? Certainly, there is an infinite number of Clifford algebras available, and nature appears to choose only some of them. How does she make that choice?

The connection between the division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and Clifford algebras is unmistakable. In fact, Clifford algebras were introduced in 1878 by William Kingdon Clifford as an extension of the quaternions, [15].

Later, in Section 6.3, we will show how the action of $\mathbb{C} \otimes \mathbb{O}$ on itself leads to the Clifford algebra $\mathbb{C}l(6)$. From this Clifford algebra we will ultimately extract a one-generation description of quarks and leptons, followed by some further indications of a three-generation model.

4.2 Definition

In [40], [68], a Clifford algebra over $\mathbb{R}$ is defined to be an associative algebra, which is generated by n elements, e_i. These n generators exhibit the properties

$$\{e_i, e_j\} \equiv e_i e_j + e_j e_i = 2\eta_{ij}\mathbb{I}, \tag{4.1}$$

where the entries $\eta_{ij} = 0$ for $i \neq j$, and $\eta_{ii} = \pm 1$ $\forall i$. The symbol $\mathbb{I}$ represents the identity. A Clifford algebra over $\mathbb{R}$ with p generators having the property $e_i^2 = +\mathbb{I}$, and q generators having the property $e_i^2 = -\mathbb{I}$, is referred to as $Cl(p,q)$.

In this text, we will be interested mostly in Clifford algebras over $\mathbb{C}$, referred to as $\mathbb{C}l(n)$. Clearly, taking the algebra's field to be $\mathbb{C}$, instead of $\mathbb{R}$, erases the Clifford algebra's signature, $(p, q) \mapsto (n)$.

Complex Clifford algebras, $\mathbb{C}l(n)$, with even n, each have only one irreducible representation. This irreducible representation has $2^{n/2}$ complex dimensions. For $\mathbb{C}l(n)$ with n odd, there are two inequivalent irreducible representations, each with $2^{(n-1)/2}$ complex dimensions. In this thesis, we will be concerned mostly with $\mathbb{C}l(n)$ for n even.

Below, we include a couple of helpful tables from [52], detailing how Clifford algebras may be faithfully represented by matrices over the rings $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $^2\mathbb{R} \equiv \mathbb{R} \oplus \mathbb{R}$, or $^2\mathbb{H} \equiv \mathbb{H} \oplus \mathbb{H}$. Here, the notation $\mathbb{A}(d)$ refers to $d \times d$ matrices over the ring $\mathbb{A}$.

Real Clifford algebras $Cl(p,q)$ for $p + q < 8$

$\overrightarrow{p-q}$	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
$\downarrow p+q$															
0								$\mathbb{R}$							
1							$\mathbb{C}$		$^2\mathbb{R}$						
2						$\mathbb{H}$		$\mathbb{R}(2)$		$\mathbb{R}(2)$					
3					$^2\mathbb{H}$		$\mathbb{C}(2)$		$^2\mathbb{R}(2)$		$\mathbb{C}(2)$				
4				$\mathbb{H}(2)$		$\mathbb{H}(2)$		$\mathbb{R}(4)$		$\mathbb{R}(4)$		$\mathbb{H}(2)$			
5			$\mathbb{C}(4)$		$^2\mathbb{H}(2)$		$\mathbb{C}(4)$		$^2\mathbb{R}(4)$		$\mathbb{C}(4)$		$^2\mathbb{H}(2)$		
6		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$		$\mathbb{R}(8)$		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$	
7	$^2\mathbb{R}(8)$		$\mathbb{C}(8)$		$^2\mathbb{H}(4)$		$\mathbb{C}(8)$		$^2\mathbb{R}(8)$		$\mathbb{C}(8)$		$^2\mathbb{H}(4)$		$\mathbb{C}(8)$

Furthermore, for Clifford algebras of larger $p + q$, we have the isomorphisms

$$Cl(p, q + 8) \simeq Cl(p, q) \otimes \mathbb{R}(16), \tag{4.2}$$

and similarly,

$$Cl(p + 8, q) \simeq Cl(p, q) \otimes \mathbb{R}(16). \tag{4.3}$$

On the other hand, for the case of complex Clifford algebras, we have $\mathbb{C}l(n) \simeq \mathbb{C}(2^{n/2})$ for n even, and $\mathbb{C}l(n) \simeq {}^2\mathbb{C}(2^{(n-1)/2})$ for n odd.

Complex Clifford algebras $\mathbb{C}l(n)$ for $n < 8$

n $\rightarrow$	0	1	2	3	4	5	6	7
$\mathbb{A}(d) \rightarrow$	$\mathbb{C}$	${}^2\mathbb{C}$	$\mathbb{C}(2)$	${}^2\mathbb{C}(2)$	$\mathbb{C}(4)$	${}^2\mathbb{C}(4)$	$\mathbb{C}(8)$	${}^2\mathbb{C}(8)$

4.3 Examples of Clifford algebras from $\mathbb{C} \otimes \mathbb{H}$

Given the multiplication properties from equation (3.2), it is straightforward to see that $\mathbb{C} \otimes \mathbb{H}$ gives a representation of the real Clifford algebra, $Cl(3,0)$. Please see Figure (4.1).

$$i\epsilon_x\, i\epsilon_y\, i\epsilon_z$$

$$i\epsilon_y i\epsilon_z \qquad\qquad i\epsilon_z i\epsilon_x \qquad\qquad i\epsilon_x i\epsilon_y$$

$$i\epsilon_x \qquad\qquad i\epsilon_y \qquad\qquad i\epsilon_z$$

1

Figure 4.1: The algebra $\mathbb{C} \otimes \mathbb{H}$ written so as to show its $Cl(3,0)$ structure. Here, the Clifford algebra's zero-vector is 1, its generating vectors are $i\epsilon_x$, $i\epsilon_y$, $i\epsilon_z$, its bivectors are $i\epsilon_y i\epsilon_z = -\epsilon_x$, $i\epsilon_z i\epsilon_x = -\epsilon_y$, $i\epsilon_x i\epsilon_y = -\epsilon_z$, and its 3-vector is $i\epsilon_x i\epsilon_y i\epsilon_z = i$. These multivectors are understood to be taken over $\mathbb{R}$.

Alternately, $\mathbb{C} \otimes \mathbb{H}$ also gives a representation of $\mathbb{C}l(2)$, as shown in Figure (4.2).

$$i\epsilon_x \, i\epsilon_y$$

$$i\epsilon_x \qquad\qquad i\epsilon_y$$

$$1$$

Figure 4.2: The algebra $\mathbb{C} \otimes \mathbb{H}$ written so as to show its $\mathbb{C}l(2)$ structure. Here, the Clifford algebra's zero-vector is 1, its generating vectors are $i\epsilon_x$ and $i\epsilon_y$, its bivector is $i\epsilon_x i\epsilon_y = -\epsilon_z$. These multi-vectors are understood to be taken over $\mathbb{C}$.

It was mentioned above that complex Clifford algebras, $\mathbb{C}l(n)$, each have just a single irreducible representation when n is even, and have two inequivalent irreducible representations when n is odd. In the even case, that irreducible representation is $2^{n/2}$-complex-dimensional, whereas, in the odd case, each of the two irreducible representations are $2^{(n-1)/2}$-complex-dimensional. These irreducible representations are commonly known as spinors.

Consider the example of the complex quaternions acting on themselves from the left. This action gives a representation of $\mathbb{C}l(2)$, so that $n = 2$ in this case. We would then expect a single $2^{n/2}$-complex-dimensional irreducible representation for that Clifford algebra. This irreducible representation is none other than the familiar 2-complex-dimensional Weyl spinor.

4.4 Minimal left ideals

For the remainder of this text, we will be interested only in Clifford algebras over $\mathbb{C}$, where n is even. The most common way to model these algebras is to represent them as $2^{n/2} \times 2^{n/2}$ complex matrices, as mentioned above. These complex matrices then act on spinors in the form of $2^{n/2}$-complex-dimensional column vectors.

We would like, however, to avoid resorting to this matrix-and-column-vector formalism,

which posits two separate entities: matrices, and column vectors. Instead, we will opt for a more streamlined formalism, based simply on a single algebra acting on itself. The motivation for such a formalism originates from the M-1 model sketched in Chapter 2.

It should not be surprising that this goal is an obtainable one. That is, there exist $2^{n/2}$-complex-dimensional subalgebras within these Clifford algebras, which serve as irreducible representations. These subalgebras are called *minimal left ideals,* and provide one way of defining spinors, [1].

Given an algebra, A, a *left ideal, B*, is a subalgebra of A whereby $a\underline{b}$ is in B for all $\underline{b}$ in B, and for any a in A. That is, no matter which a we multiply onto $\underline{b}$, the new product, $\underline{b}' \equiv a\underline{b}$, must be in the subspace B (i.e. the ideal B survives).

Now, a *minimal left ideal* is a left ideal which contains no other left ideals other than $\{0\}$ and itself.

4.5 How to identify minimal left ideals

We will now summarize for the reader a procedure for finding minimal left ideals in complex Clifford algebras, where n is even. This procedure is described in [1], which also accommodates other types of Clifford algebra.

In the special case of $\mathbb{C} \otimes \mathbb{H}$, this procedure will allow us to recover the Weyl spinors Ψ_L or Ψ_R, which we found earlier via the complex multiplicative action, m_c. We will then move on to the algebra of $\mathbb{C} \otimes \mathbb{O}$, which can be seen to generate $\mathbb{C}l(6)$. This leads us to minimal left ideals behaving as a full generation of quarks and leptons.

4.5.1 Some definitions

Let V be the vector space over a field $\mathbb{F}$. A *quadratic form, Q,* is a map $Q : V \to \mathbb{F}$ such that $\forall v, w \in V$ and $\lambda \in \mathbb{F}$,

$$Q(\lambda v) = \lambda^2 Q(v) \tag{4.4}$$

and such that the map $B : V \times V \to \mathbb{F}$,

$$B : (v, w) \mapsto Q(v + w) - Q(v) - Q(w) \tag{4.5}$$

is linear in both v and w. This V, together with its quadratic form, is called a *quadratic vector space.*

We will be especially interested in the quadratic vector space given by V and Q, where V is the n-dimensional vector space spanned by the generating elements e_j of definition (4.1). Here, the quadratic form will be given by $Q(v) = \{v, v\}$, and the corresponding bilinear form is then $B(v, w) = 2\{v, w\}$. Note that V here is only the generating subspace of our Clifford algebra, it does not represent the entire Clifford algebra.

A subspace U of V is said to be *totally isotropic* if $B(\alpha_i, \alpha_j) = 0 \; \forall \; \alpha_i, \alpha_j \in U$. In the following construction, we will be interested in *maximal totally isotropic subspaces* of V, which are defined to be totally isotropic subspaces of V with maximal dimension. As explained in [1], for Clifford algebras over $\mathbb{C}$ with n even, the dimension of any maximal totally isotropic subspace will be $n/2$. That is, the maximal totally isotropic subspace makes up exactly half of the Clifford algebra's generating space.

It is interesting to note that our maximal totally isotropic subspace, U of V, generates a Grassmann algebra, seeing as how $B(\alpha_i, \alpha_j) = 2\{\alpha_i, \alpha_j\} = 0 \; \forall \; \alpha_i, \alpha_j \in U$. This fact merits notice, especially in consideration of recent work by László, [50].

4.5.2 Procedure

With these definitions in hand, the procedure for constructing minimal left ideals is straightforward:

1. First identify a quadratic vector space (V, Q). In particular, we will take V to be the generating space of a given Clifford algebra, spanned by the generators $\{e_j\}$ over $\mathbb{C}$, where $j = 1 \dots n$. We will take our quadratic form to be $Q(v) = \{v, v\}$.

2. Identify an MTIS (maximal totally isotropic subspace) U of V as the largest possible subspace of V such that $\{\alpha_i, \alpha_j\} = 0 \; \forall \alpha_i, \alpha_j \in U$.

3. Define the nilpotent object $\Omega \equiv \alpha_1 \alpha_2 \cdots \alpha_{n/2}$, where the α_i are linearly independent basis vectors of U.

4. Construct the projector, $\Omega \Omega^\dagger$, where $\dagger$ takes $i \mapsto -i$, $e_j \mapsto -e_j$, and reverses the order of multiplication.

5. Left multiply the entire Clifford algebra onto $\Omega \Omega^\dagger$ to arrive at a minimal left ideal: $\mathbb{C}l(n) \Omega \Omega^\dagger$.

It should be noted that the above is not the only way in which minimal left ideals may be constructed; for details, please see [1].

The projector, $\Omega\Omega^\dagger$, is an example of a *primitive idempotent.* A primitive idempotent, P, is defined in [52] to be an idempotent ($P^2 = P$), which is not the sum of two annihilating idempotents, $P \neq P_1 + P_2$, where $P_1 P_2 = P_2 P_1 = 0$.

4.6 Minimal left ideals in $\mathbb{C} \otimes \mathbb{H}$

As a specific example, let us now build a minimal left ideal in $\mathbb{C} \otimes \mathbb{H} \simeq \mathbb{C}l(2)$, as per Figure (4.2). Following the procedure from Section (4.5.2),

1. Our vector space, V, is spanned by the elements $i\epsilon_x$ and $i\epsilon_y$ over $\mathbb{C}$. The quadratic form is given by $Q(v) = \{v, v\}$.

2. An MTIS of V is just one-complex-dimensional in this case, and can be spanned by $\alpha_1 = \epsilon_{\downarrow\uparrow} = \frac{1}{2}\left(\epsilon_y + i\epsilon_x\right)$. (Another option would be $\epsilon_{\uparrow\downarrow} = \frac{1}{2}\left(-\epsilon_y + i\epsilon_x\right)$).

3. Since U is only one-dimensional here, the nilpotent object Ω is given simply by $\Omega = \alpha_1 = \epsilon_{\downarrow\uparrow}$.

4. The primitive idempotent, $\Omega\Omega^\dagger$, is then $\Omega\Omega^\dagger = \epsilon_{\downarrow\uparrow}\epsilon_{\downarrow\uparrow}^\dagger = \epsilon_{\downarrow\downarrow}$.

5. Our minimal left ideal is finally given by $\Psi_R = \mathbb{C} \otimes \mathbb{H}\,\Omega\Omega^\dagger = \psi_R^\uparrow \epsilon_{\uparrow\downarrow} + \psi_R^\downarrow \epsilon_{\downarrow\downarrow}$.

Readers will notice that this minimal left ideal matches Ψ_R, found as a generalized ideal in Section (3.5). Furthermore, redoing this procedure by taking the maximal totally isotropic subspace to be instead spanned by $\epsilon_{\uparrow\downarrow}$ yields the familiar $\Psi_L = \psi_L^\uparrow \epsilon_{\uparrow\uparrow} + \psi_L^\downarrow \epsilon_{\downarrow\uparrow}$ from before.

So, we have just shown that we can use this procedure to build a left- or right-handed Weyl spinor from the left action of $\mathbb{C} \otimes \mathbb{H}$ on itself.

4.6.1 Fock space structure

With a little relabelling, it becomes obvious that these minimal left ideals naturally exhibit Fock space structure. Taking $\alpha_1 = \epsilon_{\downarrow\uparrow}$, it then follows that $\alpha_1^\dagger = \epsilon_{\uparrow\downarrow}$, which have the anticommutation relations,

$$\{\alpha_1, \alpha_1\} = \{\alpha_1^\dagger, \alpha_1^\dagger\} = 0 \qquad \{\alpha_1, \alpha_1^\dagger\} = 1. \tag{4.6}$$

Defining the (formal) vacuum states to be $v \equiv \Omega\Omega^\dagger$ and $v^* \equiv \Omega^\dagger\Omega$ for $\Psi_{\rm R}$ and $\Psi_{\rm L}$, respectively, we have

$$\Psi_{\rm R} = \psi_{\rm R}^\uparrow\, \epsilon_{\uparrow\downarrow} + \psi_{\rm R}^\downarrow\, \epsilon_{\downarrow\downarrow} = \psi_{\rm R}^\uparrow\, \alpha_1^\dagger v + \psi_{\rm R}^\downarrow\, v \tag{4.7}$$

$$\Psi_{\rm L} = \psi_{\rm L}^\uparrow\, \epsilon_{\uparrow\uparrow} + \psi_{\rm L}^\downarrow\, \epsilon_{\downarrow\uparrow} = \psi_{\rm L}^\uparrow\, v^* + \psi_{\rm L}^\downarrow\, \alpha_1 v^*. \tag{4.8}$$

That is, $\alpha_1^\dagger$ acts as a raising operator from the vacuum v within $\Psi_{\rm R}$, and $\alpha_1^{*\dagger} = -\alpha_1$ acts as a raising operator from the vacuum v^* within $\Psi_{\rm L}$. It should be clear to the reader that v and v^* represent vacua only in an algebraic sense, and are not meant to represent the zero-particle state.

Such Fock space structure will reappear in other constructions, for example, when we build minimal left ideals from the algebra $\mathbb{C} \otimes \mathbb{O}$. Moreover, it comes up again in the work of László, [50].

4.7   The Dirac algebra

It is not enough for us here to consider minimal left ideals for just $\Psi_{\rm L}$ or for just $\Psi_{\rm R}$, separately. We would like also to combine these two objects together into a single four-complex-dimensional Dirac spinor, as shown in Section 3.5.2.

It had been mentioned earlier that chirality, L and R, is the analogue of spin, $\uparrow$ and $\downarrow$, in this formalism. One can confirm that left multiplying $\Psi_{\rm L}$ and $\Psi_{\rm R}$ by $\mathbb{C} \otimes \mathbb{H}$ causes rotation between their spin states, $\uparrow$ and $\downarrow$, whereas right multiplying $\Psi_{\rm L}$ and $\Psi_{\rm R}$ by $\mathbb{C} \otimes \mathbb{H}$ causes rotation between L and R.

The left action and right action of $\mathbb{C} \otimes \mathbb{H}$ on itself each give a representation of $\mathbb{C}l(2)$. Furthermore, taking ℓ, a, $r \in \mathbb{C} \otimes \mathbb{H}$, one can easily show that $\ell a = ar\ \forall a \Rightarrow \ell, r \in \mathbb{C} \subset \mathbb{C} \otimes \mathbb{H}$. That is, the left action of $\mathbb{C} \otimes \mathbb{H}$ on itself cannot be re-expressed as the right action of $\mathbb{C} \otimes \mathbb{H}$ on itself, and vice versa. It is a property of $\mathbb{C} \otimes \mathbb{H}$ that these two actions are distinct, a feature which does not appear in the case of $\mathbb{C} \otimes \mathbb{O}$. Furthermore, the associativity of $\mathbb{C} \otimes \mathbb{H}$ ensures that the left and right action commute with each other.

Hence, the underlying Clifford algebraic structure when considering both left and right multiplication of $\mathbb{C} \otimes \mathbb{H}$ on itself is $\mathbb{C}l(2) \otimes_{\mathbb{C}} \mathbb{C}l(2)$, where the tensor product is over $\mathbb{C}$.

As $\mathbb{C}l(n) \otimes_{\mathbb{C}} \mathbb{C}l(2) \simeq \mathbb{C}l(n+2)$ for $n \in \mathbb{Z} \geq 0$, we then find that $\mathbb{C}l(2) \otimes_{\mathbb{C}} \mathbb{C}l(2) \simeq \mathbb{C}l(4)$. Furthermore, as the complexification of a real Clifford algebra, $Cl(p,q)$ acts to erase its signature, $\mathbb{C} \otimes Cl(p,q) \simeq \mathbb{C}l(p+q)$, we then find that $\mathbb{C}l(4) \simeq \mathbb{C} \otimes Cl(1,3)$.

In other words, the left and right action of $\mathbb{C} \otimes \mathbb{H}$ on itself gives a representation of the Dirac algebra.

It is straightforward for readers to confirm this fact. Explicitly, the gamma matrices γ^0, γ^1, γ^2, and γ^3, written in the Weyl basis, have the following correspondences in $\mathbb{C} \otimes \mathbb{H}$:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \mapsto 1 \mid i\epsilon_x \qquad\qquad \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} \mapsto i\epsilon_x \mid \epsilon_y$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} \mapsto i\epsilon_y \mid \epsilon_y \qquad\qquad \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \mapsto i\epsilon_z \mid \epsilon_y. \tag{4.9}$$

These findings are likely to appear in future joint work with G. Fiore from INFN.

Here, we make use of the bar notation from earlier authors, [51]. By definition, the operator $a \mid b$ acting on some element c, for $a, b, c \in \mathbb{C} \otimes \mathbb{H}$, is given simply by acb. As an example, consider the object corresponding to γ^0 acting on $\Psi_{\mathrm{D}} \in \mathbb{C} \otimes \mathbb{H}$,

$$\begin{aligned}
1 \, \Psi_{\mathrm{D}} \, i\epsilon_x &= 1 \, (\Psi_{\mathrm{L}} + \Psi_{\mathrm{R}}) \, i\epsilon_x \\
&= \left(\psi_{\mathrm{L}}^{\uparrow} \epsilon_{\uparrow\uparrow} + \psi_{\mathrm{L}}^{\downarrow} \epsilon_{\downarrow\uparrow} + \psi_{\mathrm{R}}^{\uparrow} \epsilon_{\uparrow\downarrow} + \psi_{\mathrm{R}}^{\downarrow} \epsilon_{\downarrow\downarrow} \right) i\epsilon_x \\
&= \left(\psi_{\mathrm{R}}^{\uparrow} \epsilon_{\uparrow\uparrow} + \psi_{\mathrm{R}}^{\downarrow} \epsilon_{\downarrow\uparrow} + \psi_{\mathrm{L}}^{\uparrow} \epsilon_{\uparrow\downarrow} + \psi_{\mathrm{L}}^{\downarrow} \epsilon_{\downarrow\downarrow} \right).
\end{aligned} \tag{4.10}$$

This flips chirality, as one would expect.

From these operators, an object corresponding to $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ can be found to be

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \mapsto -1 \mid i\epsilon_z. \tag{4.11}$$

This thereby confirms the role of P and P^* as $\frac{1}{2}(1 \mp \gamma^5)$, introduced early on in equation (3.5).

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More generally, it can now be seen that $\Psi_\mathrm{D} = \Psi_\mathrm{L} + \Psi_\mathrm{R}$ introduced previously, does in fact behave as a Dirac spinor, acted upon by a complex quaternionic representation of the Dirac algebra, $\mathbb{C} \otimes Cl(1,3)$.

4.8 Right action required

Readers may have noticed that the left action of $\mathbb{C} \otimes \mathbb{H}$ on itself was not enough to describe $\Psi_\mathrm{L} + \Psi_\mathrm{R}$ together as a single irreducible representation. That is, the left action alone has only Ψ_L (or Ψ_R) as an irrep. However, the combined Dirac spinor, $\Psi_\mathrm{L} + \Psi_\mathrm{R}$, does become an irreducible representation of a Clifford algebra when the right action of $\mathbb{C} \otimes \mathbb{H}$ is also included. Readers will find this theme coming up again in future chapters.

4.9 Parity and the field strength tensor

The work in this section is to appear in a joint paper, together with G. Fiore from INFN.

In Section 3.6, we introduced four-vectors, p, as generalized ideals under the hermitian invariant-action, m_h. These can be described as

$$p = p_0 + p_1 i\epsilon_x + p_2 i\epsilon_y + p_3 i\epsilon_z, \tag{4.12}$$

which transform under the Lorentz group as $p' = m_h(L, p) = LpL^\dagger$, for $L \in SL(2, \mathbb{C})$.

Furthermore, in Section 3.7, we introduced the field strength tensor, F, as a generalized ideal under the quaternionic invariant-action, m_q. Explicitly, we had

$$F = (B_1 - iE_1)\,\epsilon_x + (B_2 - iE_2)\,\epsilon_y + (B_3 - iE_3)\,\epsilon_z, \tag{4.13}$$

which transforms under the Lorentz group as $F' = m_q(L, F) = LF\widetilde{L}$. Of course, we also studied spinors and scalars in previous sections, but these will not be relevant for us at the moment.

Given these two Lorentz representations, we might wonder how parity transformations are to be carried out. Under a parity transformation, we would expect

$$p \quad \mapsto \quad p' = p_0 - p_1 i\epsilon_x - p_2 i\epsilon_y - p_3 i\epsilon_z,$$

(4.14)

$$F \quad \mapsto \quad F' = (B_1 + iE_1)\,\epsilon_x + (B_2 + iE_2)\,\epsilon_y + (B_3 + iE_3)\,\epsilon_z.$$

Now, it might be tempting to consider the quaternionic conjugate as the parity conjugate, since $\widetilde{\epsilon_x} = -\epsilon_x$, $\widetilde{\epsilon_y} = -\epsilon_y$, and $\widetilde{\epsilon_z} = -\epsilon_z$. However, the quaternionic conjugate sends $F \mapsto -F$, which does not give the desired result: $E_i \mapsto -E_i$, and $B_j \mapsto B_j$. So how might we define a parity transformation?

In Section 4.7, we demonstrated a representation of the Dirac algebra, $\mathbb{C} \otimes Cl(1,3)$, using both left and right multiplication of $\mathbb{C} \otimes \mathbb{H}$ on itself. From this Clifford algebra, it is then possible to define a parity transformation:

$$\gamma^0 \;\mapsto\; \gamma^0 = 1 \mid i\epsilon_x, \qquad \gamma^j \;\mapsto\; -\gamma^j = -i\epsilon_j \mid \epsilon_y, \tag{4.15}$$

for $j = 1, 2, 3$.

So, the question is now: how do we use this parity transformation on the γ^μ to induce a parity transformation on p and F?

The resolution to this question is not obvious, because p and F are objects which are derived from $\mathbb{C} \otimes \mathbb{H} \simeq \mathbb{C}l(2)$ structure, while the γ^μ belong in $(\mathbb{C} \otimes \mathbb{H}) \otimes_{\mathbb{C}} (\mathbb{C} \otimes \mathbb{H}) \simeq \mathbb{C} \otimes Cl(1,3) \simeq \mathbb{C}l(4)$. One solution, which we will now show, is to generalize the $\mathbb{C}l(2)$ objects, p and F, to two new $\mathbb{C}l(4)$ operators, $\hat{p}$ and $\hat{F}$. We might hope, then, that under the right conditions, $\hat{p}$ and $\hat{F}$ would reduce to p and F. In summary, we would like to see the parity transformations on $\hat{p}$ and $\hat{F}$, given by equation 4.15, automatically induce parity transformations on p and F.

Let us then define $\hat{p}$ to be

$$\hat{p} \equiv p_\mu \gamma^\mu = p_0\, 1 \mid i\epsilon_x + p_j\, i\epsilon_j \mid \epsilon_y, \tag{4.16}$$

for $p_\mu \in \mathbb{R}$, for $\mu = 0, 1, 2, 3$, and $j = 1, 2, 3$. This is none other than the usual $p_\mu \gamma^\mu$ of quantum field theory.

Making use of the Clifford algebraic description of [52], let us define $\hat{F}$ to be

$$\hat{F} \equiv F_{\mu\nu} \gamma^{\mu\nu} = F_{0i} \gamma^{0i} + F_{jk} \gamma^{jk} = E_i\, \epsilon_i \mid \epsilon_z - B_j\, \epsilon_j \mid 1, \tag{4.17}$$

for $\mu, \nu = 0, 1, 2, 3$, $F_{\mu\nu} \in \mathbb{R}$, $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, and $i, j = 1, 2, 3$.

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It is then trivial to see that under a parity transformation given by

$$\gamma^0 \;\mapsto\; \gamma^0 = 1 \,|\, i\epsilon_x, \qquad \gamma^j \;\mapsto\; -\gamma^j = -i\epsilon_j \,|\, \epsilon_y, \tag{4.18}$$

for $j = 1, 2, 3$, the operator, $\hat{p}$ transforms as

$$\hat{p} \;\mapsto\; \hat{p}' = p_0 \, 1 \,|\, i\epsilon_x - p_j \, i\epsilon_j \,|\, \epsilon_y, \tag{4.19}$$

and the operator, $\hat{F}$ transforms as

$$\hat{F} \;\mapsto\; \hat{F}' = -E_i \, \epsilon_i \,|\, \epsilon_z - B_j \, \epsilon_j \,|\, 1, \tag{4.20}$$

as we would expect.

So, we have a new description for four-momenta, $\hat{p}$, and the field strength tensor, $\hat{F}$, which transform as they should under parity. We would now like to know how these relate to the generalized ideals, p and F that we found earlier in Sections 3.6 and 3.7.

We find that the operators $\hat{p}$ and $\hat{F}$ reduce to the generalized ideals p and F, when $\hat{p}$ and $\hat{F}$ are taken to be operators, *acting on spinors* in $\mathbb{C} \otimes \mathbb{H}$. Incorporating the object γ^0, we find that $\hat{p}$ reduces to p, or p^*, on Weyl spinors,

$$\gamma^0 \hat{p} \, \Psi_\mathrm{L} = p^* \, \Psi_\mathrm{L}, \qquad \gamma^0 \hat{p} \, \Psi_\mathrm{R} = p \, \Psi_\mathrm{R}. \tag{4.21}$$

Similarly, we find that $\hat{F}$ reduces to F, or F^*, on Weyl spinors as

$$\hat{F} \, \Psi_\mathrm{L} = -F^* \, \Psi_\mathrm{L}, \qquad \hat{F} \, \Psi_\mathrm{R} = -F \, \Psi_\mathrm{R}. \tag{4.22}$$

Readers may note that a parity transformation on $\hat{p}$ and $\hat{F}$ now automatically induces the correct parity transformation on p and F, when these objects are taken to be operators on spinors.

Finally, we point out that this definition of parity,

$$\gamma^0 \;\mapsto\; \gamma^0 = 1 \,|\, i\epsilon_x, \qquad \gamma^j \;\mapsto\; -\gamma^j = -i\epsilon_j \,|\, \epsilon_y, \tag{4.23}$$

sends $\gamma^5 \mapsto -\gamma^5$, so that left- and right-handed Weyl spinors are swapped under this transformation, as we would expect.

In summary, we have shown that the parity conjugate of the $\mathbb{C} \otimes Cl(1,3) \simeq (\mathbb{C} \otimes \mathbb{H}) \otimes_{\mathbb{C}}$ $(\mathbb{C} \otimes \mathbb{H})$ algebra may be used to induce a parity conjugate on our generalized ideals in $\mathbb{C} \otimes \mathbb{H}$.

Note that we have also demonstrated a way to write down four-vectors, and the field strength tensor, in the Dirac algebra formalism, which makes use of the generalized ideals found in Sections 3.6 and 3.7. Namely,

$$\hat{p} \, \Psi_{\mathrm{D}} = \gamma^0 \, m_c(p^*, \Psi_{\mathrm{D}}), \qquad \hat{F} \, \Psi_{\mathrm{D}} = m_c(-F^*, \Psi_{\mathrm{D}}). \tag{4.24}$$

Chapter 5

Standard model of particle physics

5.1 What it is

The standard model of particle physics is the result of decades of collaboration, which began roughly in the 1930s, and converged finally on its current state in 1979, [63]. It is a mosaic of our best efforts in particle physics over that half century. In the decades since 1979, the standard model has seen little in the way of alterations, and yet has survived rigorous experimental testing, nearly completely unscathed.

A brief history: In 1928, P.A.M. Dirac set up the foundations for quantum electrodynamics (QED), which was later generalized to incorporate the neutrino by E. Fermi, [63]. Feynman, Schwinger, and Tomonaga subsequently developed renormalization theory for QED, [47]. In the mid-1950s, M. Gell-Mann proposed *strangeness* as a new quantum number, while T.D. Lee and C. N. Yang suggested methods for detecting parity violation of the weak force. In that same decade, the W boson was suggested as a mediator of the weak force, following the work of C.N. Yang and R.L. Mills, [63].

In the 1960s, G. Zweig, and independently, M. Gell-Mann, proposed that subatomic particles (now known as quarks) were the constituents of baryonic matter. Electroweak theory was developed by S. Glashow, A. Salam, J.C. Ward, and S. Weinberg, in that same decade, [63]. A mechanism to impart mass on gauge bosons was then proposed by three groups independently: first of all, R. Brout and F. Englert, then, P.W. Higgs, and finally G. Guralnik, C. R. Hagen, and T. Kibble, all in 1964, [47]. In 1971, G. 't Hooft showed that the Glashow-Salam-Ward-Weinberg electroweak model was renormalizable, [66]. M. Gell-Mann, H. Fritzsch, and H. Leutwyler's work in the 1970s resulted in what we know

as quantum chromodynamics (QCD). For a more detailed history of the standard model, readers are encouraged to consult [63].

The particle content of the standard model can be characterized by labelling particles according to how they transform under the standard model's gauge group, $SU(3)_c \times SU(2)_L \times U(1)_Y$. (More accurately, the standard model's gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y / \mathbb{Z}_6$, which will be discussed in Chapter 8.) Here, Y stands for weak hypercharge, and we will be using the weak hypercharge conventions found in [12]. Readers may note the rather arbitrary-looking collection of hypercharges displayed below.

Fermionic (matter) content of the standard model

Symbol	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$(\nu_e,\ e^-)_L$	1	2	$-1/2$
$(\nu_\mu,\ \mu^-)_L$	1	2	$-1/2$
$(\nu_\tau,\ \tau^-)_L$	1	2	$-1/2$
$(u,\ d)_L$	3	2	$1/6$
$(c,\ s)_L$	3	2	$1/6$
$(t,\ b)_L$	3	2	$1/6$
e_R^-	1	1	-1
μ_R^-	1	1	-1
τ_R^-	1	1	-1
u_R	3	1	$2/3$
c_R	3	1	$2/3$
t_R	3	1	$2/3$
d_R	3	1	$-1/3$
s_R	3	1	$-1/3$
b_R	3	1	$-1/3$

Each of the particles above has an anti-particle partner, which transforms as the representation conjugate to that of the original particle. For example, $(\bar{t}, \bar{b})_R$ transforms as $\bar{3}$ under $SU(3)_c$, as a $\bar{2} \simeq 2$ under $SU(2)_L$, and has weak hypercharge $-1/6$.

The standard model also has 12 gauge bosons, whose charges can be summarized by the following table.

Gauge bosonic content of the standard model (prior to electroweak symmetry breaking)

Symbol	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
G_μ^a	8	1	0
W_μ^b	1	3	0
B_μ	1	1	0

Finally, we also have the Higgs boson, which is a scalar particle under the Lorentz group, and transforms under the gauge groups according to the following table.

Higgs field

Symbol	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
ϕ	1	2	1/2

The standard model is believed to be largely valid over a vast range of scales, from the Hubble radius, $10^{30}cm$, down to subatomic scales of $10^{-16}cm$, [63]. It has successfully predicted the existence of a number of particles, including for example, the W and Z bosons, [49], the top quark, tau-neutrino, [63], and most recently, the Higgs boson, [19], [18], [17], [16].

Given the striking complexity of our known universe, it is hard to believe that so much of it can be described by such a short list of elementary particles. Beyond its particle content, the standard model also has a relatively concise list of free parameters, 19 of them, whereby 13 come from the Yukawa sector, 2 from the Higgs sector, 3 from gauge couplings, and 1 from an apparently absent QCD term, [54], [65].

5.2   What it is not

If the standard model is to be faulted for anything, that fault would lie almost exclusively in its incompleteness. The first legitimate fracture in the standard model came with the discovery of neutrino oscillations, which imply that neutrinos do indeed have mass, [54]. Having said that, the standard model is easily amended to accommodate massive neutrinos, albeit, at the expense of having to accept a number of additional free parameters in the theory.

In the absence of any major conflict with experiment, much criticism of the standard model is based in the theory's apparent inability to go beyond. The standard model does not explain dark matter, nor dark energy. Nor, does it describe baryon asymmetry. Furthermore, the standard model has consistently defied unification with gravity.

5.3   What it should be

One might not criticize the standard model too harshly, for simply remaining silent about many of the open problems in physics. However, what is less benign is that the standard model does at times account for various aspects of nature, and yet fails to explain them.

For example, the standard model singles out a particular gauge group, $SU(3)_c \times SU(2)_L \times U(1)_Y / \mathbb{Z}_6$, yet does not explain where this group came from. It further specifies a list of particle representations for this group, Section 5.1, without explaining why these representations were chosen over any other possible set. The standard model does not explain why those representations are organized into three generations. It does not explain why $SU(2)_L$ acts on left-handed spinors, but not on right-handed spinors. It does not explain electric charge quantization. It does not explain the apparently ad hoc arrangement of hypercharges. Furthermore, the standard model does not explain the values of its 19 parameters.

Perhaps the gaps in our understanding could be attributed to the patchwork way in which the standard model came to be. One might argue that what is needed is some form of unifying principles to smooth over the seams. These unifying principles might come in the form of grand unified theories, supersymmetry, M-theory, or perhaps something else.

Chapter 6

One generation of quarks and leptons from $\mathbb{C} \otimes \mathbb{O}$

In Chapter 3, we saw how a four-complex-dimensional algebra, $\mathbb{C} \otimes \mathbb{H}$, supplies an unusually compact description of all the Lorentz representations of the standard model. However, there is more to the standard model than just the Lorentz group. Beyond spin and chirality, standard model fermions also exhibit colour, weak isospin, and charge. Could these physical features also be the result of some algebra acting on itself? If so, which one?

In our work with $\mathbb{C} \otimes \mathbb{H}$, the reader may have noticed the trivial identity,

$$\mathbb{C} \otimes \mathbb{H} = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H}. \tag{6.1}$$

That is, the algebra we have dealt with up until now is the tensor product of three of the four division algebras. It is then only natural to ask, what about $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$? The octonionic algebra, $\mathbb{O}$, possesses eight degrees of freedom, and it is difficult not to notice: so does a generation of quarks and leptons. (Neutrino, charged lepton, three up-type quarks and three down-type quarks).

Despite its counter-culture status, the octonions have long drawn the curiosity of generations of physicists. The algebra is known to appear without warning in apparently disparate areas of mathematics, within algebra, geometry, and topology. However, despite its ubiquity, its practical uses in physics have remained elusive, due to the algebra's non-associativity, which must be handled with care. In the following chapters, we aim to demonstrate to the reader that the octonions' non-associativity is not a impediment, but

instead a gift, and that this misunderstood algebra is really at the heart of the standard model of particle physics.

The findings which we will now describe make a case in support of those who have been long advocating for the existence of a connection between non-associative algebras and particle theory, [42], [43], [64], [57], [26], [27], [34], [35], [36], [11], [3], [2] [5], [6], [44], [45], [55], [8], [13].

One of the earliest breakthroughs along these lines belong to Günaydin and Gürsey, [42], who showed $SU(3)_c$ quark structure in the split octonions. Later, in [43], they showed anti-commuting ladder operators within that model. Our new results stem from the octonionic chromodynamic quark model of [43], and are meant to replace the provisional charges of [34].

In the following pages, we will extend Günaydin and Gürsey's findings of quark structure under $SU(3)_c$ by further demonstrating

1. lepton structure and

2. a natural $U(1)_{em}$ symmetry.

In other words, using only the complex octonions, we will complete the particle content of the model to include a full generation of quarks and leptons, under not only $SU(3)_c$, but under the standard model's two unbroken gauge symmetries.

6.1   A summary of the results to come

Using only the algebra of the complex octonions, which we will introduce, we expose an intrinsic structure to a generation of quarks and leptons. This algebraic structure mimics familiar quantum systems, which have a vacuum state acted upon by raising and lowering operators. In this case, the neutrino poses as the vacuum state, and electrons and quarks pose as the excited states. These results are simply the analogue of the Fock space we found earlier in Section 4.6.1.

With these raising and lowering operators in hand, we are then able to construct a number operator in the usual way,

$$N = \sum_i \alpha_i^\dagger \alpha_i. \tag{6.2}$$

It will be seen that N has eigenvalues given by $\{0, 1, 1, 1, 2, 2, 2, 3\}$. At first sight, these eigenvalues might not look familiar, that is, until they are divided by 3. $N/3$ has eigenvalues

$\{0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1\}$, which can now be recognized as the electric charges of a neutrino (or anti-neutrino), a triplet of anti-down quarks, a triplet of up quarks, and a positron. We will then define our electric charge, Q, as

$$Q \equiv \frac{N}{3}. \tag{6.3}$$

As N must take on integer values, Q must be quantized. Hence, this (incomplete) model offers already at this stage an explanation for the quantization of electric charge, an open problem within the standard model.

As we will show, the remaining states within a generation are related to these particles by complex conjugation, and hence are acted upon by $-Q^*$ in the usual way.

The anti-commuting ladder operators, mentioned above, can be seen to possess a certain symmetry generated uniquely by $su(3)_c$ and $u(1)_{em}$, the two unbroken symmetries of the standard model. These generators of $SU(3)_c$ do indeed match those of [42], and fit in perfectly with Q, the $U(1)_{em}$ generator mentioned above. Under these symmetries, we find that our Fock space, and its complex conjugate, together transform as would a full generation of quarks and leptons.

Ours is certainly not the first instance where Günaydin and Gürsey's model has been adapted. As an extension of their model, [26], [27], Dixon describes electric charge as a mix of quaternionic and octonionic objects. It would be interesting to see if a ladder system could be found, which alternately gives Dixon's Q as a number operator. Readers are encouraged to see [26], [27], or other examples of his extensive work.

Since the time of first writing, more octonionic chromo-electrodynamic models have been found. Most noteworthy of all were three papers written in the late 1970s, [8], [13], and [14], which could also be considered as extensions of Günaydin and Gürsey's model, [43]. In these papers, the authors use two separate ladder systems: system (a) fits with the octonionic ladder operators of [43], and system (b) is introduced as quaternionic. By combining the two systems, they describe the electric charge generator not as a number operator, but as the difference between the number operators of the two systems. References [8], [13], and [14] are important papers, worth careful reconsideration by the community.

Our results differ from earlier versions in that we will be constructing a generation of quarks and leptons explicitly as *minimal left ideals* of a Clifford algebra, generated only by the complex octonions. In doing so, we will use just a single octonionic ladder system, with its complex conjugate. This in turn allows us to (1) define electric charge more simply

as $Q = N/3$, and (2) expose a more direct route to the two unbroken gauge symmetries of the standard model. Furthermore, our formalism naturally relates particles and anti-particles using only the complex conjugate, $i \mapsto -i$, which is not a feature of these earlier models. Finally, as our generation of quarks and leptons will be constructed from Clifford algebra elements, not column vectors, we will then be free to model weak isospin, using *right multiplication* of the original Clifford algebra onto these minimal left ideals.

6.2   Introduction to $\mathbb{C} \otimes \mathbb{O}$

The complex octonions are not a tool commonly used in physics, so we introduce them here.

The generic element of $\mathbb{C} \otimes \mathbb{O}$ is written $\sum_{n=0}^{7} A_n e_n$, where the A_n are complex coefficients. The e_n are octonionic imaginary units $(e_n^2 = -1)$, apart from $e_0 = 1$, which multiply according to Figure 6.1. The complex imaginary unit i commutes the the octonionic e_n.

Figure 6.1:   Multiplication of octonionic imaginary units

Any three imaginary units on a directed line segment in Figure 6.1 act as if they were a triplet of Pauli matrices, σ_m. (More precisely, they behave as $-i\sigma_m$.) For example, $e_6 e_1 = -e_1 e_6 = e_5$, $e_1 e_5 = -e_5 e_1 = e_6$, $e_5 e_6 = -e_6 e_5 = e_1$, $e_4 e_1 = -e_1 e_4 = e_2$, etc.

The multiplication rules for these imaginary units can be defined by setting $e_1 e_2 = e_4$, and then applying the following rules, as shown in [4].

$$e_i e_j = -e_j e_i \qquad i \neq j,$$
$$e_i e_j = e_k \implies e_{i+1} e_{j+1} = e_{k+1}, \tag{6.4}$$
$$e_i e_j = e_k \implies e_{2i} e_{2j} = e_{2k}.$$

It is indeed true that the octonions form a non-associative algebra, meaning that the relation $(ab)c = a(bc)$ does not always hold. The reader can check this by finding three imaginary units, which are not all on the same line segment, and substituting them as in a, b, and c.

The octonionic automorphism group is G_2, which is a 14-dimensional exceptional Lie group. Its Lie algebra may be represented by the generators

$$\Lambda_1 = \tfrac{i}{2}\big(e_1\,(e_5\ \cdot\,) - e_3\,(e_4\ \cdot\,)\big), \qquad \Lambda_8 = \tfrac{i}{2\sqrt{3}}\big(e_1\,(e_3\ \cdot\,) + e_4\,(e_5\ \cdot\,) - 2e_2\,(e_6\ \cdot\,)\big),$$

$$\Lambda_2 = -\tfrac{i}{2}\big(e_1\,(e_4\ \cdot\,) + e_3\,(e_5\ \cdot\,)\big), \qquad g_9 = -\tfrac{i}{2\sqrt{3}}\big(e_1\,(e_5\ \cdot\,) + e_3\,(e_4\ \cdot\,) + 2e_2\,(e_7\ \cdot\,)\big),$$

$$\Lambda_3 = \tfrac{i}{2}\big(e_4\,(e_5\ \cdot\,) - e_1\,(e_3\ \cdot\,)\big), \qquad g_{10} = \tfrac{i}{2\sqrt{3}}\big(e_1\,(e_4\ \cdot\,) - e_3\,(e_5\ \cdot\,) + 2e_6\,(e_7\ \cdot\,)\big),$$

$$\Lambda_4 = \tfrac{i}{2}\big(e_2\,(e_5\ \cdot\,) + e_4\,(e_6\ \cdot\,)\big), \qquad g_{11} = -\tfrac{i}{2\sqrt{3}}\big(e_4\,(e_6\ \cdot\,) - e_2\,(e_5\ \cdot\,) + 2e_1\,(e_7\ \cdot\,)\big),$$

$$\Lambda_5 = \tfrac{i}{2}\big(e_5\,(e_6\ \cdot\,) - e_2\,(e_4\ \cdot\,)\big), \qquad g_{12} = -\tfrac{i}{2\sqrt{3}}\big(e_2\,(e_4\ \cdot\,) + e_5\,(e_6\ \cdot\,) - 2e_3\,(e_7\ \cdot\,)\big),$$

$$\Lambda_6 = \tfrac{i}{2}\big(e_1\,(e_6\ \cdot\,) + e_2\,(e_3\ \cdot\,)\big), \qquad g_{13} = -\tfrac{i}{2\sqrt{3}}\big(-e_1\,(e_6\ \cdot\,) + e_2\,(e_3\ \cdot\,) + 2e_4\,(e_7\ \cdot\,)\big),$$

$$\Lambda_7 = \tfrac{i}{2}\big(e_1\,(e_2\ \cdot\,) + e_3\,(e_6\ \cdot\,)\big), \qquad g_{14} = \tfrac{i}{2\sqrt{3}}\big(-e_1\,(e_2\ \cdot\,) + e_3\,(e_6\ \cdot\,) + 2e_5\,(e_7\ \cdot\,)\big), \tag{6.5}$$

acting on the octonions. Here, the nested brackets indicate that the generators are constructed from chains of octonions, multiplying from right to left.

The eight objects, Λ_i, generate $SU(3) \subset G_2$. This $SU(3)$ may be defined as the subgroup of G_2 which leaves the octonionic unit e_7 invariant. Of course, alternate $SU(3)$ subgroups of G_2 may be found, which correspond to other imaginary units. For a more thorough introduction of $\mathbb{O}$ see [4], [23], [57].

Finally, we define three notions of conjugation on an element a in $\mathbb{C} \otimes \mathbb{O}$. The *complex conjugate* of a, denoted a^*, maps the complex $i \mapsto -i$, as would be expected. The *octonionic conjugate* of a, denoted $\tilde{a}$, takes each of the octonionic imaginary units $e_n \mapsto -e_n$ for $n = 1, \ldots 7$. That which we will call the *hermitian conjugate* of a, denoted $a^\dagger$, performs both of these maps simultaneously, $i \mapsto -i$ and $e_n \mapsto -e_n$ for $n = 1, \ldots 7$. The conjugate

and the hermitian conjugate each reverse the order of multiplication, as is familiar from the hermitian conjugate of a product of matrices.

6.3   Octonionic chain algebra

As a non-associative algebra, the octonions can at times seem temperamental. Equations involving this algebra can quickly become unwieldy, due to the need to repeatedly specify the order of multiplication, by use of brackets. The assumptions we are accustomed to making in associative algebras now do not always apply, and one might be led to wonder how (associative) groups can be described with the (non-associative) octonions.

In this section, however, we explain how this is not the conundrum that it might seem to be. Every multiplication between two octonions can be considered as a linear map of one octonion on the other. As maps are associative by definition, this gives a way of re-describing the action of octonions as an associative algebra.

It is plain to see that left-multiplying one complex octonion, m, onto another, f, provides a map from $f \in \mathbb{C} \otimes \mathbb{O}$ to $f' \equiv mf \in \mathbb{C} \otimes \mathbb{O}$. Subsequently left-multiplying by another complex octonion, n, provides another map: $f \mapsto f'' \equiv n(mf)$. We will call this map $\overleftarrow{nm}$, where the arrow is in place so as to indicate the order in which multiplication occurs. We may extend the chain further by left-multiplying by $p \in \mathbb{C} \otimes \mathbb{O}$, giving $\overleftarrow{pnm} : f \mapsto p(n(mf))$, and so on.

In an associative algebra, A, the exercise of building up chains in order to make new maps would be futile. That is, for m_1, m_2, f in an associative algebra, $m_2(m_1 f)$ can always be summarized as $(m_2 m_1)f = m'f$, where $m' \equiv m_2 m_1 \in A$. However, as the complex octonions form a non-associative algebra, building chains does in fact lead to new maps. For example, consider the map $\overleftarrow{e_3 e_4}$ acting on $f = (e_6 + ie_2)$.

$$\overleftarrow{e_3 e_4} \left(e_6 + ie_2\right) = e_3 \left(e_4 \left(e_6 + ie_2\right)\right) = -1 + ie_7. \tag{6.6}$$

This is not the same as

$$\left(e_3 e_4\right) \left(e_6 + ie_2\right) = \left(e_6\right) \left(e_6 + ie_2\right) = -1 - ie_7, \tag{6.7}$$

and in fact there exists no $a \in \mathbb{C} \otimes \mathbb{O}$ such that $\overleftarrow{e_3 e_4} \left(e_6 + ie_2\right) = a \left(e_6 + ie_2\right)$.

Addition and multiplication are easy to define on this set of chains; we will refer to the resulting algebra as the *complex octonionic chain algebra,* $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$, or simply the *chain algebra* for short. Addition of two maps $N = \overleftarrow{\cdots n_3 n_2 n_1}$ and $P = \overleftarrow{\cdots p_3 p_2 p_1} \in \mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ on f is given by $[N + P] f = Nf + Pf$, where the n_i and $p_j \in \mathbb{C} \otimes \mathbb{O}$. Multiplication, $\circ$, is given simply by the composition of maps,

$$[P \circ N] f = P(N(f)) = \overleftarrow{\cdots p_3 p_2 p_1 \cdots n_3 n_2 n_1} f. \tag{6.8}$$

As the composition of maps is always associative, $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ is an associative algebra. Unconvinced readers are encouraged to check explicitly that $[[A \circ B] \circ C] f = [A \circ [B \circ C]] f$ $\forall\, A, B, C \in \mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ and $\forall\, f \in \mathbb{C} \otimes \mathbb{O}$.

In analogy with the $\mathbb{C} \otimes \mathbb{O}$ case, three notions of conjugation can be defined on $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$. Here, the complex conjugate is the same as before: $i \mapsto -i$. For the octonionic and hermitian conjugates, the definition is also the same as for the $\mathbb{C} \otimes \mathbb{O}$ case, bearing in mind that these conjugates now reverse the order of the chain algebra's multiplication, $\circ$.

Looking more closely at the chains, we notice quickly that

$$\overleftarrow{\cdots e_a e_b \cdots} f = - \overleftarrow{\cdots e_b e_a \cdots} f \quad \forall f \in \mathbb{C} \otimes \mathbb{O}, \tag{6.9}$$

for $a, b = 1, 2, \ldots 7$, when $a \neq b$. Furthermore,

$$\overleftarrow{\cdots e_i e_j e_j e_k \cdots} f = - \overleftarrow{\cdots e_i e_k \cdots} f \quad \forall f \in \mathbb{C} \otimes \mathbb{O}, \tag{6.10}$$

when $i, k = 0, 1, 2, \ldots 7$ and $j = 1, 2, \ldots 7$. With these properties, the chains acting on $\mathbb{C} \otimes \mathbb{O}$ provide a representation of the Clifford algebra $\mathbb{C}l(7)$, where $\{i\overleftarrow{e_1}, i\overleftarrow{e_2}, \ldots i\overleftarrow{e_7}\}$, acting on f, forms the generating set of vectors, [4].

It turns out, though, that $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ does not give a faithful representation of $\mathbb{C}l(7)$. There exists an additional symmetry, which identifies two monomial chains with the same map. For example, $\overleftarrow{e_1 e_2 e_3} f = -\overleftarrow{e_4 e_5 e_6 e_7} f$, $\overleftarrow{e_5 e_7} f = -\overleftarrow{e_1 e_2 e_3 e_4 e_6} f$, $\overleftarrow{e_7} f = \overleftarrow{e_1 e_2 e_3 e_4 e_5 e_6} f$, etc. These 64 equations (duality relations) are readily found by making use of equations (6.9) and (6.10), and also the following form of the identity: $\overleftarrow{e_0} f = -\overleftarrow{e_1 e_2 e_3 e_4 e_5 e_6 e_7} f$. We then see that any element of $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ may be represented as a complex linear combination of chains, of no more than three e_js in length.

The reader is encouraged to check that $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ faithfully represents the 64-complex-dimensional Clifford algebra $\mathbb{C}l(6)$, generated by the set $\{i\overleftarrow{e_1}, i\overleftarrow{e_2}, \ldots i\overleftarrow{e_6}\}$, acting on f.

Figure ([6.2](#)) depicts the complex octonionic chain algebra, organized so as to demonstrate its $\mathbb{C}l(6)$ structure. Starting from the bottom, we have the zero-vector, 1 acting on f, the generating vectors, $\{i\overset{\leftarrow}{e_1}, i\overset{\leftarrow}{e_2}, \ldots i\overset{\leftarrow}{e_6}\}$ acting on f, the bivectors, $\{\overset{\leftarrow}{e_1 e_2}, \ldots \overset{\leftarrow}{e_5 e_6}\}$ acting on f, and so on. Note that we make regular use the identity $\overset{\leftarrow}{e_7} f = \overset{\leftarrow}{e_1 e_2 e_3 e_4 e_5 e_6} f$ so as to avoid writing long chains of multivectors involving only the generators $i\overset{\leftarrow}{e_1}, i\overset{\leftarrow}{e_2}, \ldots i\overset{\leftarrow}{e_6}$.

For readers more comfortable with matrices and column vectors, one may loosely think of $\mathbb{C} \otimes \overset{\leftarrow}{\mathbb{O}}$ as a space of 8×8 complex matrices, whereas the elements, f, would be represented by 8-complex-dimensional column vectors.

$$\overset{\leftarrow}{e_7}$$

$$i\overset{\leftarrow}{e_1 e_7} \quad i\overset{\leftarrow}{e_2 e_7} \quad \cdots \cdots \quad i\overset{\leftarrow}{e_5 e_7} \quad i\overset{\leftarrow}{e_6 e_7}$$

$$\overset{\leftarrow}{e_1 e_2 e_7} \quad \overset{\leftarrow}{e_1 e_3 e_7} \quad \cdots \cdots \quad \overset{\leftarrow}{e_4 e_6 e_7} \quad \overset{\leftarrow}{e_5 e_6 e_7}$$

$$i\overset{\leftarrow}{e_1 e_2 e_3} \quad i\overset{\leftarrow}{e_1 e_2 e_4} \quad \cdots \cdots \quad i\overset{\leftarrow}{e_3 e_5 e_6} \quad i\overset{\leftarrow}{e_4 e_5 e_6}$$

$$\overset{\leftarrow}{e_1 e_2} \quad \overset{\leftarrow}{e_1 e_3} \quad \cdots \cdots \quad \overset{\leftarrow}{e_4 e_6} \quad \overset{\leftarrow}{e_5 e_6}$$

$$i\overset{\leftarrow}{e_1} \quad i\overset{\leftarrow}{e_2} \quad i\overset{\leftarrow}{e_3} \quad i\overset{\leftarrow}{e_4} \quad i\overset{\leftarrow}{e_5} \quad i\overset{\leftarrow}{e_6}$$

$$1$$

Figure 6.2: The 64-complex-dimensional octonionic chain algebra gives a representation of $\mathbb{C}l(6)$. This octonionic chain algebra is a space of maps acting through left multiplication onto any element $f \in \mathbb{C} \otimes \mathbb{O}$.

Now that we have described octonionic left multiplication, by making use of an associative algebra, one might wonder if we have lost something by neglecting right multiplication. It turns out, though, that unlike in the $\mathbb{C} \otimes \mathbb{H}$ case, every complex octonion multiplied from the right may be re-expressed as a sum of chains of octonions, multiplying from the left,

$$fm = c_0 f + \sum_{i=1}^{7} c_i \overleftarrow{e_i} f + \sum_{j=2}^{7} \sum_{i=1}^{j-1} c_{ij} \overleftarrow{e_i e_j} f + \sum_{k=3}^{7} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} c_{ijk} \overleftarrow{e_i e_j e_k} f, \qquad (6.11)$$

for $f, m \in \mathbb{C} \otimes \mathbb{O}$, and for some $c_0, c_i, c_{ij}, c_{ijk} \in \mathbb{C}$. In other words, these right-multiplication maps are already accounted for by $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$.

Readers may also be concerned that in focussing on the associative algebra, $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$, we are losing information from the complex octonions, $\mathbb{C} \otimes \mathbb{O}$, which was tied up in their non-associativity. This does not appear to be the case, however, since, as demonstrated above, it was the non-associativity of the octonions which caused a larger space of maps to be created. It seems, then, that the non-associativity of the octonions re-emerges in $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ as a property necessary to produce an algebra of 64 complex dimensions.

Finally, one might ask, if we are moving to a Clifford algebraic description of octonionic multiplication via $\mathbb{C}l(6)$ anyway, why not just start with $\mathbb{C}l(6)$ in the first place? The answer to this question is two-fold. First of all, in starting only from Clifford algebras, one would be hard-pressed to know which Clifford algebras to choose. That is, an infinite number of Clifford algebras exist, and there appears to be no reason to choose one over any other. Secondly, with an octonionic description of $\mathbb{C}l(6)$, we will be able to map particles into ant-particles, and vice versa, using only the complex conjugate, $i \mapsto -i$. This is typically not the case when Clifford algebras are expressed as matrices with complex components, as was already shown in the $\mathbb{C} \otimes \mathbb{H} \simeq \mathbb{C}l(2)$ case of Section 3.5.3 for left- and right- handed Weyl spinors.

6.4 A system of ladder operators

With the algebra $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ in hand, we can now set out to find a system of ladder operators within the octonionic chain algebra. Consider $\alpha_1 \equiv \frac{1}{2}(-e_5 + ie_4)$, $\alpha_2 \equiv \frac{1}{2}(-e_3 + ie_1)$, and $\alpha_3 \equiv \frac{1}{2}(-e_6 + ie_2)$, similar to that defined in [42]. For all f in $\mathbb{C} \otimes \mathbb{O}$, and assuming right-to-left multiplication, these three lowering operators obey the anticommutation relations

$$\{\alpha_i, \alpha_j\} f = \alpha_i(\alpha_j f) + \alpha_j(\alpha_i f) = 0 \tag{6.12}$$

for all $i, j = 1, 2, 3$. The above can be seen as a generalization of the system in [43]. That is, [43] is recovered by restricting the general f in $\mathbb{C} \otimes \mathbb{O}$ to $f = 1$.

In another slight deviation from [43], we define raising operators as $\alpha_1^\dagger = \frac{1}{2}(e_5 + ie_4)$, $\alpha_2^\dagger = \frac{1}{2}(e_3 + ie_1)$, and $\alpha_3^\dagger = \frac{1}{2}(e_6 + ie_2)$, which obey

$$\{\alpha_i^\dagger, \alpha_j^\dagger\} f = 0 \quad \text{for all } i, j = 1, 2, 3. \tag{6.13}$$

We finally also have

$$\{\alpha_i, \alpha_j^\dagger\} f = \delta_{ij} f \quad \text{for all } i, j = 1, 2, 3. \tag{6.14}$$

With the purpose simplifying notation, we will now implicitly assume all multiplication to be carried out starting at the right, and moving to the left, as was shown in equation (6.12). That is, we will now not write these brackets in explicitly, nor will we include an arrow going from right to left, specifying the direction of multiplication.

Furthermore, we will now be concerned only with operators, such as the α_i, as opposed to the object f. This being the case, it will now be understood that all equations will hold over all f in $\mathbb{C} \otimes \mathbb{O}$, even though f will not be mentioned explicitly. For example, we will now write equation (6.12) simply as

$$\{\alpha_i, \alpha_j\} = \alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad \text{for all } i, j = 1, 2, 3. \tag{6.15}$$

As mentioned earlier, these operators acting on f may be viewed as 8×8 complex matrices acting on f, an eight-complex-dimensional column vector. Taking into account the above paragraph, our equations from here on in can be considered as relations only between the matrices.

Another way of restating the above is to say that the operators are simply elements of $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$, and we will be considering only the elements of $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ from here on in.

6.5 Complex conjugation's analogue

Under complex conjugation, we find an analogous ladder system. Consider $\alpha_1^* = \frac{1}{2}\left(-e_5 - ie_4\right)$, $\alpha_2^* = \frac{1}{2}\left(-e_3 - ie_1\right)$, and $\alpha_3^* = \frac{1}{2}\left(-e_6 - ie_2\right)$. These three lowering operators obey the anticommutation relations

$$\{\alpha_i^*, \alpha_j^*\} = 0 \quad \text{for all } i, j = 1, 2, 3. \tag{6.16}$$

We define raising operators as $\tilde{\alpha}_1 = \frac{1}{2}\left(e_5 - ie_4\right)$, $\tilde{\alpha}_2 = \frac{1}{2}\left(e_3 - ie_1\right)$, and $\tilde{\alpha}_3 = \frac{1}{2}\left(e_6 - ie_2\right)$, which obey

$$\{\tilde{\alpha}_i, \tilde{\alpha}_j\} = 0 \quad \text{for all } i, j = 1, 2, 3. \tag{6.17}$$

Finally, we have also

$$\{\alpha_i^*, \tilde{\alpha}_j\} = \delta_{ij} \quad \text{for all } i, j = 1, 2, 3. \tag{6.18}$$

Using these ladder operators, we will now build *minimal left ideals,* which can be seen to mimic the set of quarks and leptons of the standard model.

6.6 Minimal left ideals

We are now proposing to represent quarks and leptons using minimal left ideals within our space of octonionic operators, $\mathbb{C} \otimes \overleftarrow{\mathbb{O}} \simeq \mathbb{C}l(6)$. That is, within the space of the α_i, $\alpha_j^\dagger$, and their products. A pair of these ideals, S^u and S^d, will be constructed using our procedure from Section 4.5.2:

1. Our vector space, V, is spanned by the elements $ie_1, ie_2, \ldots ie_6$ over $\mathbb{C}$, in keeping with Figure 6.2. The quadratic form is given by $Q(v) = \{v, v\}$.

2. An MTIS of V is three-complex-dimensional in this case, and can be spanned by $\alpha_1 = \frac{1}{2}\left(-e_5 + ie_4\right)$, $\alpha_2 = \frac{1}{2}\left(-e_3 + ie_1\right)$, and $\alpha_3 = \frac{1}{2}\left(-e_6 + ie_2\right)$.

3. Our nilpotent object is then given by $\omega \equiv \alpha_1\alpha_2\alpha_3$.

4. This leads to the primitive idempotent, $\omega\omega^\dagger = \alpha_1\alpha_2\alpha_3\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger$.

5. Our minimal left ideal is finally given by $S^u \equiv \mathbb{C} \otimes \overleftarrow{\mathbb{O}}\; \omega\omega^\dagger$, below.

The eight-complex-dimensional minimal left ideal for the first ladder system is given by

$$S^u \equiv$$

$$
\begin{array}{ccccc}
& & \mathcal{V}\,\omega\omega^\dagger & & \\
+\,\bar{\mathcal{D}}^{\mathrm{r}}\,\alpha_1^\dagger\omega\omega^\dagger & + & \bar{\mathcal{D}}^{\mathrm{g}}\,\alpha_2^\dagger\omega\omega^\dagger & + & \bar{\mathcal{D}}^{\mathrm{b}}\,\alpha_3^\dagger\omega\omega^\dagger \\
+\,\mathcal{U}^{\mathrm{r}}\,\alpha_3^\dagger\alpha_2^\dagger\omega\omega^\dagger & + & \mathcal{U}^{\mathrm{g}}\,\alpha_1^\dagger\alpha_3^\dagger\omega\omega^\dagger & + & \mathcal{U}^{\mathrm{b}}\,\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger \\
& + & \mathcal{E}^{+}\,\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger, & &
\end{array}
\tag{6.19}
$$

where $\mathcal{V}, \bar{\mathcal{D}}^{\mathrm{r}}, \ldots \mathcal{E}^{+}$ are 8 suggestively named complex coefficients.

As

$$\alpha_i\,\omega\omega^\dagger = 0 \qquad \forall i, \tag{6.20}$$

$\omega\omega^\dagger$ plays the role of the vacuum state, where the term *vacuum* is used loosely. Again, it is not to be interpreted as a zero-particle state. However, readers may recognize the similarity between S^u and a Fock space.

The conjugate system of Section 6.5 analogously leads to

$$S^d \equiv$$

$$\bar{\mathcal{V}}\,\omega^\dagger\omega$$

$$\mathcal{D}^{\mathrm{r}}\,\alpha_1\omega^\dagger\omega \quad + \quad \mathcal{D}^{\mathrm{g}}\,\alpha_2\omega^\dagger\omega \quad + \quad \mathcal{D}^{\mathrm{b}}\,\alpha_3\omega^\dagger\omega \tag{6.21}$$

$$+\,\bar{\mathcal{U}}^{\mathrm{r}}\,\alpha_3\alpha_2\omega^\dagger\omega \quad + \quad \bar{\mathcal{U}}^{\mathrm{g}}\,\alpha_1\alpha_3\omega^\dagger\omega \quad + \quad \bar{\mathcal{U}}^{\mathrm{b}}\,\alpha_2\alpha_1\omega^\dagger\omega$$

$$+ \quad \mathcal{E}^-\,\alpha_3\alpha_2\alpha_1\omega^\dagger\omega,$$

where $\bar{\mathcal{V}}, \mathcal{D}^{\mathrm{r}}, \ldots \mathcal{E}^-$ are eight complex coefficients.

This new ideal, (6.21), is linearly independent from the first, (6.19), in the space of octonionic operators. Clearly, the two are related via the complex conjugate, $i \mapsto -i$. In fact, the complex conjugate is *all* that is needed in order to map particles into anti-particles, and vice versa. This was a feature in the models of [43], [35], and also in the context of left- and right-handed Weyl spinors in [34] and Section 3.5.3 of this thesis.

The Clifford algebra, $\mathbb{C}l(6)$, is known to have just a single 8-complex-dimensional irreducible representation, as in S^u, above. In this text, we will none-the-less be including the conjugate ideal, S^d, in analogy to our inclusion of both left- and right-handed Weyl spinors in Section 4.8. Just as in the case with Weyl spinors, rotations between S^u and S^d are enacted via *right* multiplication onto these ideals. S^u and S^d can then be combined into a single irreducible representation under $\mathbb{C}l(6) \otimes_{\mathbb{C}} \mathbb{C}l(2)$, where the factor of $\mathbb{C}l(2)$ accounts for the right action that mixes these two spinors with each other. Unlike in the earlier literature, this additional factor of $\mathbb{C}l(2)$ will originate from right multiplication of our original octonionic operators on these ideals, instead of having to introduce an entirely new Clifford algebra, $\mathbb{C}l(2)$. This topic of doubling the spinor space will come up again in Chapter 9.

As a final note, we point out that another interesting way to obtain anti-particles could be to use the conjugate †, instead of *. In that case, the two minimal left ideals would not be entirely linearly independent from each other. That is, we would find a special Majorana-like property unique to the neutrino: $\left(\omega\omega^\dagger\right)^\dagger = \omega\omega^\dagger$.

6.7 MTIS symmetries: $su(3)_c$ and $u(1)_{em}$

Having obtained these minimal left ideals, we would now like to know how they transform, so as to justify the labels we gave to their coefficients in equations (6.19) and (6.21). Up until now, however, we have not specified under which groups these spinors transform.

A popular choice in the literature for this symmetry group is take the Clifford algebra's spin group. For example, the well-known "SO(10)" grand unified theory is built from the $Spin(10)$ group acting on 16-dimensional spinors. (Here, spinors are defined as irreducible representations of $Spin(10)$.) We point out, though, that $Spin(10)$ is a 45-dimensional group, meaning that 33 generators will need to be explained away in order to arrive finally at $SU(3)_c \times SU(2)_L \times U(1)_Y$. Could there (alternately, or additionally) be another type of symmetry, which leads directly to $SU(3)_c \times SU(2)_L \times U(1)_Y$? Or to the surviving unbroken symmetries, $SU(3)_c \times U(1)_{em}$?

We would now like to point out some symmetries in the construction of our minimal left ideals. These symmetries will be called *unitary MTIS symmetries,* or simply *MTIS symmetries.* (Readers may wish to refer back to Section 4.5.1, where maximal totally isotropic subspaces were first defined.) We propose here to consider MTIS symmetries when building covariant derivatives in gauge theories.

In the case of complex Clifford algebras, $\mathbb{C}l(n)$, with even n, the generating space can always be partitioned into two maximal totally isotropic subspaces, [1], each of dimension $n/2$. For $\mathbb{C}l(6)$ which we have here, the generating space spanned by $\{ie_1, ie_2, \ldots ie_6\}$ is partitioned into an MTIS spanned by $\{\alpha_1, \alpha_2, \alpha_3\}$, and another MTIS spanned by $\{\alpha_1^\dagger, \alpha_2^\dagger, \alpha_3^\dagger\}$. Loosely speaking, MTIS symmetries will preserve this structure.

Unitary MTIS symmetries. We are interested in operator transformations of the form $e^{i\phi_k g_k} \, ie_j \, e^{-i\phi_k g_k}$, where $\phi_k \in \mathbb{R}$ and $g_k \in \mathbb{C} \otimes \overset{\leftarrow}{\mathbb{O}}$. Already, with this constraint, we find that the anti-commutation relations of equations (6.12), (6.13), and (6.14) are preserved. Furthermore, as the name indicates, we will restrict our attention to those transformations under which each MTIS is closed. That is, to first order, the α_i rotate only into themselves, and the $\alpha_i^\dagger$ rotate only into themselves,

$$[\ g_k, \ \sum_i b_i \alpha_i\ ] = \sum_j c_j \alpha_j \qquad [\ g_k, \ \sum_i b_i' \alpha_i^\dagger\ ] = \sum_j c_j' \alpha_j^\dagger, \qquad (6.22)$$

for some complex coefficients, c_j, c_j'. Here, b_i, b_i' are some given complex coefficients, and g_k is a generator of the MTIS symmetry.

Finally, we demand that the group transformation on α_j commute with hermitian conjugation, $\dagger$,

$$e^{i\phi_k g_k} \, \alpha_j^\dagger \, e^{-i\phi_k g_k} = \left(e^{-i\phi_k g_k}\right)^\dagger \, \alpha_j^\dagger \, \left(e^{i\phi_k g_k}\right)^\dagger. \qquad (6.23)$$

Under these conditions, in the case of $\mathbb{C}l(6)$, our unitary MTIS symmetries are then found to be generated uniquely by $su(3)$ and $u(1)$. Explicitly, the $SU(3)$ generators are given by the Λ_i of equation (6.5), and the $U(1)$ generator is found to be given by Q of equation (6.3).

6.8 Ladders to $su(3)_c$ and $u(1)_{em}$

We will now find a compact way of describing the generators of the MTIS symmetries, $su(3)$ and $u(1)$, and apply them to our minimal left ideals.

Consider $\alpha \equiv c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$ and $\alpha' \equiv c_1'\alpha_1 + c_2'\alpha_2 + c_3'\alpha_3$, where the c_i and c_j' are complex coefficients. We can then build hermitian operators, $\mathcal{H}$, of the form

$$\mathcal{H} \equiv \alpha'^\dagger \alpha + \alpha^\dagger \alpha'. \tag{6.24}$$

Taking the most general sum of these objects results in nine hermitian operators:

$$\sum_{\mathcal{H}} \mathcal{H} = r_0 Q + \sum_{i=1}^{8} r_i \Lambda_i, \tag{6.25}$$

where r_0 and r_i are real coefficients. Q is our electromagnetic generator from equation (6.3), and the eight Λ_i can be seen to generate $SU(3)$. Indeed, these Λ_i coincide with those described in equation (6.5), which generate a subgroup of the octonionic automorphism group, G_2. It should be noted that earlier authors, [8], [13], [14], were quite close to these results, but required a second ladder system in order to build Q (and incidentally also all of the isospin down type states).

In terms of ladder operators, the $SU(3)$ generators take the form

$$
\begin{aligned}
&\Lambda_1 = -\alpha_2^\dagger\alpha_1 - \alpha_1^\dagger\alpha_2 \qquad &&\Lambda_2 = i\alpha_2^\dagger\alpha_1 - i\alpha_1^\dagger\alpha_2 \\
&\Lambda_3 = \alpha_2^\dagger\alpha_2 - \alpha_1^\dagger\alpha_1 \qquad &&\Lambda_4 = -\alpha_1^\dagger\alpha_3 - \alpha_3^\dagger\alpha_1 \\
&\Lambda_5 = -i\alpha_1^\dagger\alpha_3 + i\alpha_3^\dagger\alpha_1 \qquad &&\Lambda_6 = -\alpha_3^\dagger\alpha_2 - \alpha_2^\dagger\alpha_3 \\
&\Lambda_7 = i\alpha_3^\dagger\alpha_2 - i\alpha_2^\dagger\alpha_3 \qquad &&\Lambda_8 = -\tfrac{1}{\sqrt{3}}\left[\alpha_1^\dagger\alpha_1 + \alpha_2^\dagger\alpha_2 - 2\alpha_3^\dagger\alpha_3\right],
\end{aligned} \tag{6.26}
$$

all eight of which can be seen to commute with Q, and its conjugate.

We take the operators α and $\alpha^\dagger$ to transform according to

$$e^{i\sum\mathcal{H}}\,\alpha\,e^{-i\sum\mathcal{H}} \quad \text{and} \quad e^{i\sum\mathcal{H}}\,\alpha^\dagger\,e^{-i\sum\mathcal{H}}, \tag{6.27}$$

respectively. The reader may confirm that α transforms as a 3 and $\alpha^\dagger$ transforms as a $\bar{3}$ under $SU(3)$, consistent with the results of [43].

Now, the minimal left ideal, S^u, transforms as

$$e^{i\sum\mathcal{H}}\,S^u\,e^{-i\sum\mathcal{H}} = e^{i\sum\mathcal{H}}\,S^u, \tag{6.28}$$

where the equality holds because $\omega^\dagger\alpha_i^\dagger = 0$ for all i.

We now identify the subspaces of S^u by specifying their electric charges with respect to $U(1)_{em}$, and also which irreducible representation they belong to under $SU(3)_c$. Clearly, i, j and k are meant to be distinct from each other in any given row.

Q	Λ	S^u	ID
0	1	$\omega\omega^\dagger$	ν (or $\bar{\nu}$)
$1/3$	$\bar{3}$	$\alpha_i^\dagger\omega\omega^\dagger$	$\bar{d}_i$
$2/3$	3	$\alpha_i^\dagger\alpha_j^\dagger\omega\omega^\dagger$	u_k
1	1	$\alpha_i^\dagger\alpha_j^\dagger\alpha_k^\dagger\omega\omega^\dagger$	e^+

$$\tag{6.29}$$

So, here we identify a neutrino, ν, (or antineutrino, $\bar{\nu}$), three anti-down type quarks, $\bar{d}_i$, three up-type quarks, u_k, and a positron, e^+.

As the minimal left ideal, S^d, is related to S^u by complex conjugation, we then see that it transforms as

$$e^{-i\sum\mathcal{H}^*}\,S^d\,e^{i\sum\mathcal{H}^*} = e^{-i\sum\mathcal{H}^*}\,S^d, \tag{6.30}$$

where the equality holds because $\omega\alpha_i = 0$ for all i. This leads to the table below.

$-\underline{Q}^*$	$-\underline{\Lambda}^*$	$\underline{S^d}$	$\underline{\text{ID}}$	
0	1	$\omega^\dagger\omega$	$\bar\nu$ (or ν)	
$-1/3$	3	$\alpha_i\omega^\dagger\omega$	d_i	(6.31)
$-2/3$	$\bar{3}$	$\alpha_i\alpha_j\omega^\dagger\omega$	$\bar{u}_k$	
-1	1	$\alpha_i\alpha_j\alpha_k\omega^\dagger\omega$	e^-	

Here, we identify an antineutrino, $\bar\nu$, (or a neutrino, ν), three down-type quarks, d_i, three anti-up type quarks, $\bar{u}_k$, and the electron, e^-.

We have now shown a pair of conjugate ideals, which behave under $SU(3)_c$ and $U(1)_{em}$ as does a full generation of standard model fermions. These are summarized in Figure (6.3).
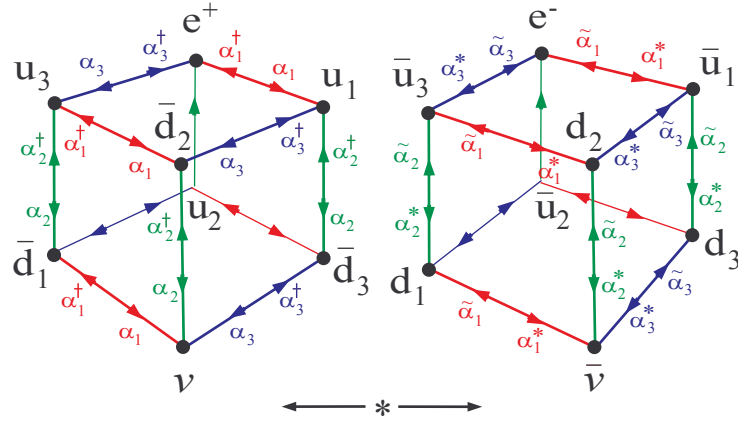
Figure 6.3: States behaving as a full generation of standard model fermions, represented by cubes S^u (left) and S^d (right). Quark and electron states may be viewed as excitations from the neutrino or anti-neutrino. As the "vacuum" represents the neutrino, and not the zero particle state, this model does not constitute a composite model in the usual sense.

6.9   Towards the weak force

Perhaps unexpectedly, it turns out that S^u packages all of the isospin up-type states together, and S^d packages all of the down-type states together. This is of course, if one goes ahead and makes an assumption about the placement of ν into S^u and $\bar{\nu}$ into S^d.

We point out that ω is (automatically) negatively charged, and converts isospin up particles into isospin down, via *right* multiplication on S^u. It thereby exhibits features of the W^- boson. Similarly, $\omega^\dagger$ is positively charged, and converts isospin down particles into isospin up, via *right* multiplication on S^d. In doing so, it exhibits features of the W^+ boson.

Other characteristics of the W bosons do not appear at the level of this chapter. For example, there is nothing to specify that these candidate bosons act only on left-handed particles. We also have no description here for the polarization states of these would-be bosons.

Readers may notice that right multiplication by ω and $\omega^\dagger$ generate a representation of $\mathbb{C}l(2)$, so that their inclusion would mean that we are then interested in a representation of $\mathbb{C}l(6) \otimes_\mathbb{C} \mathbb{C}l(2)$. This is clearly the analogue of the extra $\mathbb{C}l(2)$ which came from right multiplication of $\mathbb{C} \otimes \mathbb{H}$, which related left- and right-handed Weyl spinors with each other. Please see Section 4.8.

6.10   Summary

Using only the complex octonions acting on themselves, we were able to recover a number of aspects of the standard model's structure.

First of all, we introduced unitary MTIS symmetries, which led *uniquely* to the two unbroken gauge symmetries of the standard model, $SU(3)_c$ and $U(1)_{em}$. This new $U(1)_{em}$ generator, Q, happens to be proportional to a number operator, thereby suggesting an unexpected resolution to the question: Why is electric charge quantized?

Then, using octonionic ladder operators, we have built a pair of minimal left ideals, which is found to transform under these unbroken symmetries as does a full generation of quarks and leptons.

If the algebra of the complex octonions is *not* behind the structure of the standard model, it is then a striking coincidence that $SU(3)_c$ and $U(1)_{em}$ both follow readily from its ladder operators.

Chapter 7

Why does $SU(2)_L$ act on only left-handed fermions?

7.1 A spotlight on right multiplication

In Section 4.7, we found a four-complex-dimensional Dirac spinor, $\Psi_D \equiv \Psi_L + \Psi_R$, as an irreducible representation of $\mathbb{C}l(2) \otimes_{\mathbb{C}} \mathbb{C}l(2)$. Here, the second factor of $\mathbb{C}l(2)$ came from right-multiplying $\mathbb{C} \otimes \mathbb{H}$ on itself, and it effected transitions between L and R states.

Likewise, in Section 6.6, we found a 16-complex-dimensional spinor, $S \equiv S^u + S^d$, as an irreducible representation of $\mathbb{C}l(6) \otimes_{\mathbb{C}} \mathbb{C}l(2)$. Again, we see that the factor of $\mathbb{C}l(2)$ came from right multiplication, but this time it effected transitions between isospin up-type and isospin down-type states.

We can now combine these spinors by taking the tensor product of the spaces Ψ_D and S, as in $\Upsilon \equiv \Psi_D \otimes_{\mathbb{C}} S$, resulting in a 64-complex-dimensional spinor space. The Clifford algebra associated with spinors in Υ can easily be seen to be $\mathbb{C}l(6) \otimes_{\mathbb{C}} \mathbb{C}l(2) \otimes_{\mathbb{C}} \mathbb{C}l(2) \otimes_{\mathbb{C}} \mathbb{C}l(2) \simeq \mathbb{C}l(12)$.

In this chapter, we will be interested in only the $\mathbb{C}l(2) \otimes_{\mathbb{C}} \mathbb{C}l(2) \simeq \mathbb{C}l(4)$ sector of this Clifford algebra, which comes exclusively from right multiplication. This sector encodes transitions between L and R states, and also between isospin up-type and down-type states.

In doing so, we will be able to catch a glimpse of how $SU(2)_L$ symmetries are expected to come about in this model. Focussing on the leptonic sector, we will again make use of MTIS symmetries, which were already introduced in Section 6.7.

It turns out that the MTIS symmetries for $\mathbb{C}l(4)$ are generated by $su(2)$ and $u(1)$. This $su(2)$, applied to the minimal right ideal which we will now construct, automatically acts on only one of the fermions' chiralities. In other words, MTIS symmetries seem to be able to offer an explanation for the curious favouritism exhibited by particle physics, for fermions of a particular handedness. Such an explanation is absent in the standard model.

7.2 Leptonic subspace of $\mathbb{C}l(4)$

We will now build a minimal *right* ideal, using the procedure laid out in Section 4.5.2. Readers should note that when multiplying elements constructed from $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, the quaternionic and octonionic imaginary units always commute with each other.

1. Let us consider a Clifford algebra, $\mathbb{C}l(4)$, generated by the vectors, $\{\tau_1 i\epsilon_x, \ \tau_2 i\epsilon_x, \ \tau_3 i\epsilon_x, \ i\epsilon_y\}$, where

$$\tau_1 \equiv \omega + \omega^\dagger, \quad \tau_2 \equiv i\omega - i\omega^\dagger, \quad \tau_3 \equiv \omega\omega^\dagger - \omega^\dagger\omega. \tag{7.1}$$

Notice, that these four generators involve the right-multiplied octonionic object, ω, of Section 6.9, and also the right-multiplied quaternionic objects of Section 4.7.

2. Within this generating space, we may now identify an MTIS spanned by the objects β_1 and β_2 over $\mathbb{C}$, and another MTIS spanned by the objects $\beta_1^\ddagger$ and $\beta_2^\ddagger$ over $\mathbb{C}$. Here, the conjugate, $\ddagger$, maps the complex $i \mapsto -i$, the quaternionic $\epsilon_i \mapsto -\epsilon_i$ and the octonionic $e_j \mapsto -e_j$. It also reverses the order of multiplication. The lowering operators, β_j, will be defined here as

$$\beta_1 \equiv \frac{1}{2}\left(-\epsilon_y + i\epsilon_x\tau_3\right) \quad \text{and} \quad \beta_2 \equiv \omega^\dagger i\epsilon_x. \tag{7.2}$$

Readers may confirm that $\{\beta_i, \beta_j\} = \{\beta_i^\ddagger, \beta_j^\ddagger\} = 0$, and that $\{\beta_i, \beta_j^\ddagger\} \equiv \delta_{ij}$, on the leptonic subspace.

3. The nilpotent object, Ω, may now be constructed as $\Omega = \beta_2\beta_1$.

4. The projector, $\Omega^\ddagger\Omega$, is then found to be $\Omega^\ddagger\Omega = \omega\omega^\dagger\epsilon_{\downarrow\downarrow}$. This idempotent will act as our vacuum state, and can be identified with the spin-down right-handed neutrino, in keeping with the vacua of previous chapters.

5. Finally, right-multiplying $\mathbb{C}l(4)$ onto our projector, we obtain the minimal right ideal,

$$
\begin{aligned}
\Omega^\ddagger\Omega\,\mathbb{C}l(4) &= \mathcal{V}_R^\downarrow\,\omega\omega^\dagger\epsilon_{\downarrow\downarrow} + \mathcal{V}_L^\downarrow\,\omega\omega^\dagger\epsilon_{\downarrow\downarrow}\beta_1^\ddagger + \mathcal{E}_L^{-\downarrow}\,\omega\omega^\dagger\epsilon_{\downarrow\downarrow}\beta_2^\ddagger + \mathcal{E}_R^{-\downarrow}\,\omega\omega^\dagger\epsilon_{\downarrow\downarrow}\beta_1^\ddagger\beta_2^\ddagger \\
&= \mathcal{V}_R^\downarrow\,\omega\omega^\dagger\epsilon_{\downarrow\downarrow} + \mathcal{V}_L^\downarrow\,\omega\omega^\dagger\epsilon_{\downarrow\uparrow} + \mathcal{E}_L^{-\downarrow}\,\omega\epsilon_{\downarrow\uparrow} + \mathcal{E}_R^{-\downarrow}\,\omega\epsilon_{\downarrow\downarrow},
\end{aligned}
\tag{7.3}
$$

where the basis elements have been labelled consistently with previous chapters.

The above minimal right ideal clearly corresponds to spin-down leptons; the spin-up states can easily be found by acting with a $\mathbb{C}\otimes\mathbb{H}$ raising operator, as was shown in Section 4.6.1.

7.3 MTIS symmetries: $su(2)_L$ and $u(1)_Y$

Now that we have constructed a minimal right ideal, we would like to see how it transforms under the MTIS symmetries for this Clifford algebra, as defined in Section 6.7.

Again, MTIS symmetries are those which act on the generating space as $e^{i\phi_k g_k}\mid e^{-i\phi_k g_k}$, which map non-trivially the lowering operators to themselves, and the raising operators to themselves, and which commute with hermitian conjugation, here, $\ddagger$. The generators g_k this time will be elements of $(\mathbb{C}\otimes\mathbb{H})\otimes_{\mathbb{C}}(\mathbb{C}\otimes\overset{\leftarrow}{\mathbb{O}})\equiv\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\overset{\leftarrow}{\mathbb{O}}$.

Readers may confirm that there are four solutions to these constraints, given by

$$
T_1 \equiv \tau_1\,\epsilon_{\uparrow\uparrow}, \quad T_2 \equiv \tau_2\,\epsilon_{\uparrow\uparrow}, \quad T_3 \equiv \tau_3\,\epsilon_{\uparrow\uparrow}, \quad N' \equiv \frac{1}{2}\left(1 + i\epsilon_z\tau_3\right) + \omega^\dagger\omega. \tag{7.4}
$$

The three T_i can be seen to generate $SU(2)$, whereas N' commutes with the T_i, and itself generates $U(1)$. N' may be viewed as a number operator for the $\mathbb{C}l(4)$ system; it is nothing more than the weak isospin operator, Y, on this leptonic subspace. Or, more accurately, $N' = -2Y$. On the other hand, the T_i can be seen to annihilate the right-handed neutrino and right-handed electron states, while transforming the left-handed leptons together as a doublet. Please see Figure 7.1.
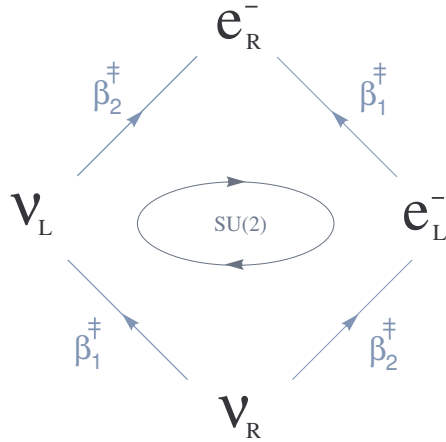
Figure 7.1:   This figure may be thought of as the analogue of Figure 6.3. A minimal right ideal from the Clifford algebra $\mathbb{C}l(4)$, spanned by the states ν_{L}, ν_{R}, e_{L}^-, e_{R}^-. Unitary MTIS symmetries here are given by $su(2)$ and $u(1)$. Under $su(2)$, we find that the minimal right ideal breaks down into two singlets, ν_{R} and e_{R}^-, and a doublet, $\left(\nu_{\mathrm{L}}\; e_{\mathrm{L}}^-\right)$. This model then demonstrates a representation of $su(2)$ which acts automatically on states of only one chirality. On this leptonic subspace, the remaining $u(1)$ turns out to be weak hypercharge.

7.4 Summary and outlook

In this chapter, we have focussed our attention on $\mathbb{C}l(2)\otimes_{\mathbb{C}}\mathbb{C}l(2) \simeq \mathbb{C}l(4)$ structure, coming from the right action on our previously constructed minimal left ideals. From this Clifford algebra, $\mathbb{C}l(4)$, we have then constructed a 4-complex-dimensional minimal right ideal, corresponding to a leptonic subspace of a generation of standard model particles. We then found the unitary MTIS symmetries corresponding to this Clifford algebra. It turns out that the four MTIS symmetry generators correspond to $su(2)_L$ and Y on this leptonic subspace, and furthermore provide an explanation as to why left-handed particles in this model interact via $SU(2)$, whereas right-handed particles do not.

As a final note, we point out that this model produced 4-dimensional spinors, and a 4-dimensional space of symmetries. Likewise, one could pair our 8-dimensional spinors, e.g. S^u of Chapter 6, with the 8-dimensional space of $SU(3)_c$ symmetries. It would then be interesting to investigate whether or not a connection to triality could be made in these models.

Chapter 8

Group representation structure of some grand unified theories

We mentioned in the previous chapter, that combing our results for $\mathbb{C} \otimes \mathbb{H}$ and $\mathbb{C} \otimes \mathbb{O}$ will bring us to consider the Clifford algebra $\mathbb{C}l(12)$. That is, our Dirac spinor, $\Psi_D = \Psi_L + \Psi_R$ was found to be an irreducible representation of $\mathbb{C}l(2) \otimes_{\mathbb{C}} \mathbb{C}l(2)$, and our generation of quarks and leptons of Chapter 6 was found to be an irreducible representation of $\mathbb{C}l(6) \otimes_{\mathbb{C}} \mathbb{C}l(2)$. Taking the tensor product of the two Clifford algebras gives $\mathbb{C}l(2) \otimes_{\mathbb{C}} \mathbb{C}l(2) \otimes_{\mathbb{C}} \mathbb{C}l(6) \otimes_{\mathbb{C}} \mathbb{C}l(2) \simeq \mathbb{C}l(12)$.

In an upcoming paper, [37], we will be analyzing minimal left ideals within $\mathbb{C}l(12)$, whose MTIS symmetries are naturally broken. This breaking occurs, not from implementing an additional Higgs field, but instead due to the fact that rotations between the quarternionic and octonionic sectors are disallowed. One might refer to this as *algebraic symmetry breaking*.

Now, the Clifford algebra, $\mathbb{C}l(12)$, can be rewritten as $\mathbb{C}l(10) \otimes_{\mathbb{C}} \mathbb{C}l(2)$, where we have separated out a factor of $\mathbb{C}l(2)$, corresponding to spin. The remaining $\mathbb{C}l(10)$ algebra is tied in closely with some well-known grand unified theories, which we will now introduce.

8.1 $SU(5)$ unification

8.1.1 Introduction

The $SU(5)$ grand unified theory offers a classic example of what it means to be a grand unified theory. It was proposed in 1974 by H. Georgi and S. Glashow, [39], and can be seen to successfully unify the gauge groups of the standard model. However, the $SU(5)$ model only partially unifies a generation of quarks and leptons.

The group $SU(5)$ has rank 4, as with the standard model's gauge group, G_{SM}; that is, its Lie algebra has a 4-dimensional Cartan subalgebra, [49]. The Lie algebra of $SU(5)$ essentially doubles the gauge symmetry of the standard model by subsuming the standard model's 12 symmetry generators, and then incorporating another 12, for a total of 24.

Irreducible representations for $SU(5)$ are listed as

$$\underline{1},\ \underline{5},\ \underline{10},\ \underline{15},\ \underline{24},\ \underline{35},\ \underline{40},\ \underline{45},\ \underline{50}, \ldots$$

Most descriptions of the $SU(5)$ model give a generation of quarks and leptons, which stretches across the $\underline{5}^*$ and the $\underline{10}$, and optionally, the singlet, $\underline{1}$. This singlet can be included so as to represent the right-handed neutrino. In contrast, gauge bosons lie in the $\underline{24}$, while the Higgs fields typically are represented by the $\underline{5}$ and the $\underline{24}$.

Despite its anticipated candidacy as a suitable fit to nature, the $SU(5)$ grand unified theory has run into trouble with experiment. For example, the theory predicts proton decay, with a lifetime of $10^{31 \pm 2}$ years, depending on the details and source of the calculation, [54], [49]. This is generally seen to be at odds with the findings of the Super-kamiokande experiment, [20], [21], which gives a proton lifetime of $\geq 5.9 \times 10^{33}$ years at the 90% confidence level.

8.1.2 G_{SM} inside $SU(5)$

Here, we detail the embedding of the standard model's gauge group into $SU(5)$, as given in [7].

Elements of $SU(5)$ can be written down as 5×5 complex matrices, wherein we may embed the standard model's gauge group, [7], [49]. Starting with $SU(2)$ and $SU(3)$, we have

$$\begin{pmatrix} h_2 & 0 \\ 0 & h_3 \end{pmatrix} \quad \in \quad SU(5), \tag{8.1}$$

where h_2 is a 2×2 matrix representing elements of $SU(2)$, and h_3 is a 3×3 matrix, representing elements of $SU(3)$. Written in this way, the $SU(2)$ and $SU(3)$ subgroups clearly commute with each other. We might then ask, what groups remain in $SU(5)$, which commute with both of these $SU(2)$ and $SU(3)$ subgroups?

It is straightforward to see, [7], [65], that any 5×5 matrix commuting with

$$\begin{pmatrix} h_2 & 0 \\ 0 & h_3 \end{pmatrix}$$

must be of the form

$$H_1 \quad = \quad \begin{pmatrix} c_2 \, \mathbb{I}_{2\times2} & 0 \\ 0 & c_3 \, \mathbb{I}_{3\times3} \end{pmatrix}, \tag{8.2}$$

for $c_2, c_3 \in \mathbb{C}$. Furthermore, for H_1 to be an element of $SU(5)$, it must have a determinant of one, so that

$$H_1 \quad = \quad \begin{pmatrix} (h_1)^3 \, \mathbb{I}_{2\times2} & 0 \\ 0 & (h_1)^{-2} \, \mathbb{I}_{3\times3} \end{pmatrix}, \tag{8.3}$$

where $h_1 \in \mathbb{C}$.

Putting these all together, we see that it is possible to map any element of $SU(3) \times SU(2) \times U(1)$ into $SU(5)$,

$$\begin{pmatrix} (h_1)^3 \, h_2 & 0 \\ 0 & (h_1)^{-2} \, h_3 \end{pmatrix} \quad \in \quad SU(5), \tag{8.4}$$

where $h_1 \in U(1)$, $h_2 \in SU(2)$, and $h_3 \in SU(3)$.

Now, it turns out that after considering the action of the $SU(2)$ and $SU(3)$ subgroups on the $\underline{5}^*$ and the $\underline{10}$, we see that this $U(1)$ of equation (8.3) is none other than $U(1)_Y$, weak hypercharge. That is, $SU(5)$ theory is able to elegantly explain the strikingly inelegant arrangement of hypercharges asserted by the standard model.

A question now arises, as to whether or not every element of $SU(3) \times SU(2) \times U(1)$ is mapped to a *unique* element of $SU(5)$'s subgroup. That is, we might ask, is this mapping invertible? It turns out that the answer is no, [7], as it is possible to find distinct elements of $SU(3) \times SU(2) \times U(1)$, given by

$$\left(h_3 = z_n^2 \, \mathbb{I}_{3\times3}, \;\; h_2 = z_n^{-3} \, \mathbb{I}_{2\times2}, \;\; h_1 = z_n\right) \quad \in \quad SU(3) \times SU(2) \times U(1), \qquad (8.5)$$

which each map to the same identity element of our $SU(5)$ subgroup,

$$\begin{pmatrix} z_n^3 \, z_n^{-3} \, \mathbb{I}_{2\times2} & 0 \\ 0 & z_n^{-2} \, z_n^2 \, \mathbb{I}_{3\times3} \end{pmatrix} = \mathbb{I}_{5\times5} \quad \in \quad SU(5). \qquad (8.6)$$

The above elements, z_n, are restricted to be sixth roots of unity, $z_n = e^{2\pi i n/6}$, since $z_n^{-3} \, \mathbb{I}_{2\times2} \in SU(2)$ and $z_n^2 \, \mathbb{I}_{3\times3} \in SU(3)$ must each have a determinant of one. This generates the finite group $\mathbb{Z}_6$.

So it turns out that our subgroup of $SU(5)$ is not $SU(3) \times SU(2) \times U(1)$, but rather, $SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$. This could be problematic for $SU(5)$ theory, if the standard model representations were indeed able to distinguish between these six values of z_n.

However, by checking explicitly one can show that the standard model representations are insensitive to these distinct values of z_n. Furthermore, $\mathbb{Z}_6$ is said, [7], to constitute the entire kernel of this representation. This leaves us with the conclusion that the standard model's gauge group is not $SU(3) \times SU(2) \times U(1)$, but rather, $SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$.

8.1.3 Some notes on symmetry breaking

The adjoint Higgs field is responsible for breaking $SU(5)$ symmetry down to $SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$, [65]. In the process, the fermionic $\underline{5}^*$ and $\underline{10}$ break into ($SU(3)_c$, $SU(2)_L$, $U(1)_Y$) irreps as

$$
\begin{aligned}
\underline{5}^* \;\; &\mapsto \;\; \underset{\bar{d}_L}{\left(\underline{3}^*, \, \underline{1}, \, \tfrac{1}{3} \right)} \;\; + \;\; \underset{\ell_L}{\left(\underline{1}, \, \underline{2}^*, \, -\tfrac{1}{2} \right)}, \\[2mm]
\underline{10} \;\; &\mapsto \;\; \underset{\bar{u}_L}{\left(\underline{3}^*, \, \underline{1}, \, -\tfrac{2}{3} \right)} \;\; + \;\; \underset{q_L}{\left(\underline{3}, \, \underline{2}, \, \tfrac{1}{6} \right)} \;\; + \;\; \underset{e_L^+}{\left(\underline{1}, \, \underline{1}, \, 1 \right)}.
\end{aligned}
\qquad (8.7)
$$

On the other hand, gauge generators break down as

$$\underline{24} \;\mapsto\; (\,\underline{8},\,\underline{1},\,0\,) + (\,\underline{1},\,\underline{3},\,0\,) + (\,\underline{1},\,\underline{1},\,0\,) + \Big(\,\underline{3},\,\underline{2}^*,\,-\tfrac{5}{6}\,\Big) + \Big(\,\underline{3}^*,\,\underline{2},\,\tfrac{5}{6}\,\Big), \qquad (8.8)$$

where the $(\,\underline{8},\,\underline{1},\,0\,)$, $(\,\underline{1},\,\underline{3},\,0\,)$, and $(\,\underline{1},\,\underline{1},\,0\,)$ generate $SU(3)_c$, $SU(2)_L$, and $U(1)_Y$, respectively, and the $\Big(\,\underline{3},\,\underline{2}^*,\,-\tfrac{5}{6}\,\Big)$ and $\Big(\,\underline{3}^*,\,\underline{2},\,\tfrac{5}{6}\,\Big)$ generators give rise to 12 new gauge bosons which can be seen to mediate proton decay.

The familiar Higgs doublet, ϕ, can be seen to emerge from the $\underline{5}$ as

$$\underline{5} \;\mapsto\; \underbrace{\Big(\,\underline{3},\,\underline{1},\,-\tfrac{1}{3}\,\Big)}_{\mathcal{H}} \;+\; \underbrace{\Big(\,\underline{1},\,\underline{2},\,\tfrac{1}{2}\,\Big)}_{\phi},$$

$$(8.9)$$

where the $\Big(\,\underline{3},\,\underline{1},\,-\tfrac{1}{3}\,\Big)$ describes a new triplet Higgs field, $\mathcal{H}$.

The Higgs sector, mentioned here, is commonly thought to be the source of two additional outstanding problems for $SU(5)$ theory, [49]. Namely, there is no clear explanation for the large differences between the GUT scale, where $SU(5)$ breaks, and the weak scale, where G_{SM} breaks (GUT hierarchy problem). Secondly, as the Higgs $\underline{5}$ breaks into a doublet under $SU(2)_L$ and a triplet under $SU(3)_c$, both pieces must be accounted for. The Higgs boson, as we now know, has a relatively small mass of about 125 GeV. However, in order to evade proton decay, the remaining triplet must acquire a large mass $\geq 10^{14}$ GeV, and there is no obvious reason for these two mass scales to be so far apart (doublet triplet splitting problem) [49].

8.2   A fermionic binary code

It turns out that there is a very efficient way of describing the standard model's fermions, which is helpful in understanding $SU(5)$, $Spin(10)$, Pati-Salam theories, and incidentally, also upcoming $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ models based on $\mathbb{C}l(12)$, [37]. This is in terms of a binary code, [7], [65]. For a thorough explanation of this topic, the reader is encouraged to consult [7].

This fermionic binary code can be thought of as a sequence of answers to five different yes or no questions:

Is the fermion red?

Is the fermion green?

Is the fermion blue?

Does the fermion have isospin up?

Does the fermion have isospin down?

The set of answers to these five questions can be represented by an exterior algebra, $\Lambda\mathbb{C}^5$, which is $2^5 = 32$-complex dimensional. For example, a fermion which is red, r, and has isospin up, u, would be represented by $r \wedge u$. A fermion which is said to be both red, r, and blue, b, and has both isospin up, u, and isospin down, d, would be represented by $r \wedge b \wedge u \wedge d$, and so on. (It may seem odd for a particle to be both red and blue, or to have both isospin up and isospin down, but this should become clear shortly.) Please see Figure 8.1.
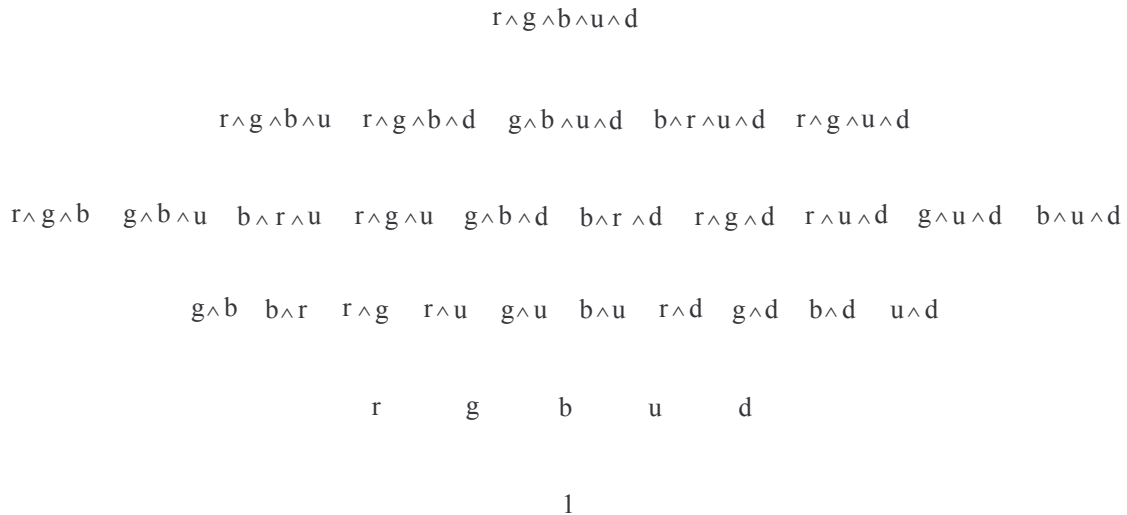
$$r \wedge g \wedge b \wedge u \wedge d$$

$$r \wedge g \wedge b \wedge u \quad r \wedge g \wedge b \wedge d \quad g \wedge b \wedge u \wedge d \quad b \wedge r \wedge u \wedge d \quad r \wedge g \wedge u \wedge d$$

$$r \wedge g \wedge b \quad g \wedge b \wedge u \quad b \wedge r \wedge u \quad r \wedge g \wedge u \quad g \wedge b \wedge d \quad b \wedge r \wedge d \quad r \wedge g \wedge d \quad r \wedge u \wedge d \quad g \wedge u \wedge d \quad b \wedge u \wedge d$$

$$g \wedge b \quad b \wedge r \quad r \wedge g \quad r \wedge u \quad g \wedge u \quad b \wedge u \quad r \wedge d \quad g \wedge d \quad b \wedge d \quad u \wedge d$$

$$r \qquad g \qquad b \qquad u \qquad d$$

$$1$$

Figure 8.1: The exterior algebra $\Lambda\mathbb{C}^5$, representing a fermionic binary code.

Now, complex linear combinations of the basis vectors, r, g, b, u, d, may be acted upon by $SU(5)$, where they form the irreducible representation, $\underline{5}$. By extension, the bivectors of $\Lambda\mathbb{C}^5$ form the $\underline{10}$, the three-vectors form the $\underline{10}^*$, the four-vectors form the $\underline{5}^*$, whereas both the unit, 1, and the five-vector form singlets. We then see that we have exactly

the right $SU(5)$ representations to describe one generation of standard model particles, together with the right-handed neutrino, including all corresponding antiparticles. Please see Figure 8.2.
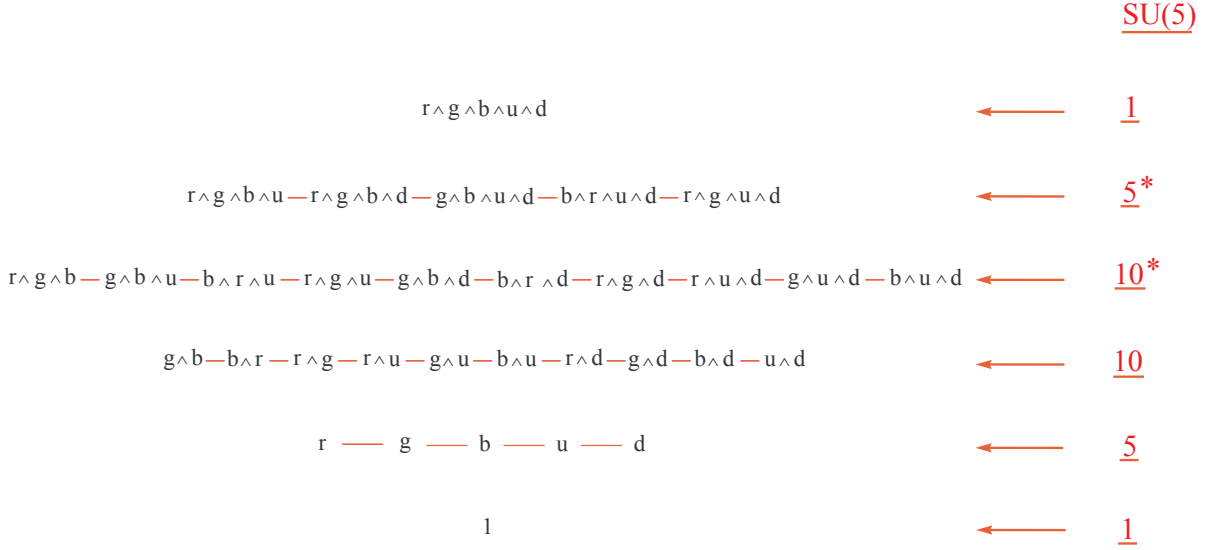
r∧g∧b∧u∧d   ⟵ $\underline{1}$

r∧g∧b∧u — r∧g∧b∧d — g∧b∧u∧d — b∧r∧u∧d — r∧g∧u∧d   ⟵ $\underline{5}^*$

r∧g∧b — g∧b∧u — b∧r∧u — r∧g∧u — g∧b∧d — b∧r∧d — r∧g∧d — r∧u∧d — g∧u∧d — b∧u∧d   ⟵ $\underline{10}^*$

g∧b — b∧r — r∧g — r∧u — g∧u — b∧u — r∧d — g∧d — b∧d — u∧d   ⟵ $\underline{10}$

r —— g —— b —— u —— d   ⟵ $\underline{5}$

1   ⟵ $\underline{1}$

Figure 8.2: The exterior algebra $\Lambda\mathbb{C}^5$, representing a fermionic binary code, broken down into irreps of $SU(5)$. Here, basis elements which are part of the same irreducible representations are connected by red lines.

Upon spontaneous symmetry breaking, the $\underline{5}$ breaks into $\left(\underline{3},\ \underline{1},\ -\frac{1}{3}\right)$ and $\left(\underline{1},\ \underline{2},\ \frac{1}{2}\right)$, and the rest of $\Lambda\mathbb{C}^5$ can then be seen to follow suit. This then allows us to identify each element of $\Lambda\mathbb{C}^5$ with one of the standard model's fermions, as shown in Figure 8.3.

Readers may notice that $SU(5)$ respects the grading of $\Lambda\mathbb{C}^5$, and that all left-handed particles reside in the even grades, while all right-handed particles reside in the odd grades.

8.3 $Spin(10)$ **unification**

Although the $SU(5)$ model was able to draw together the three gauge groups of the standard model into a single group, it only partially unified the standard model's fermions into the $\underline{5}^*$ and the $\underline{10}$. One might wonder if an alternate Lie group might exist, which could fully unify not only the standard model's gauge bosons, but also its fermions.

$$r \wedge g \wedge b \wedge u \wedge d$$
$$\nu_R$$

$$r \wedge g \wedge b \wedge u - r \wedge g \wedge b \wedge d \qquad g \wedge b \wedge u \wedge d - b \wedge r \wedge u \wedge d - r \wedge g \wedge u \wedge d$$
$$\nu_L \qquad\qquad e_L^- \qquad\qquad \bar{d}_L^r \qquad\qquad \bar{d}_L^g \qquad\qquad \bar{d}_L^b$$

$$r \wedge g \wedge b \quad\; g \wedge b \wedge u - b \wedge r \wedge u - r \wedge g \wedge u \quad\; g \wedge b \wedge d - b \wedge r \wedge d - r \wedge g \wedge d \quad\; r \wedge u \wedge d - g \wedge u \wedge d - b \wedge u \wedge d$$
$$e_R^- \qquad \bar{d}_R^r \qquad \bar{d}_R^g \qquad \bar{d}_R^b \qquad\qquad \bar{u}_R^r \qquad \bar{u}_R^g \qquad \bar{u}_R^b \qquad\qquad u_R^r \qquad u_R^g \qquad u_R^b$$

$$g \wedge b - b \wedge r - r \wedge g \qquad r \wedge u - g \wedge u - b \wedge u \qquad r \wedge d - g \wedge d - b \wedge d \qquad u \wedge d$$
$$\bar{u}_L^r \quad \bar{u}_L^g \quad \bar{u}_L^b \qquad u_L^r \quad u_L^g \quad u_L^b \qquad d_L^r \quad d_L^g \quad d_L^b \qquad e_L^+$$

$$r \;\relbar\; g \;\relbar\; b \qquad u \;\relbar\; d$$
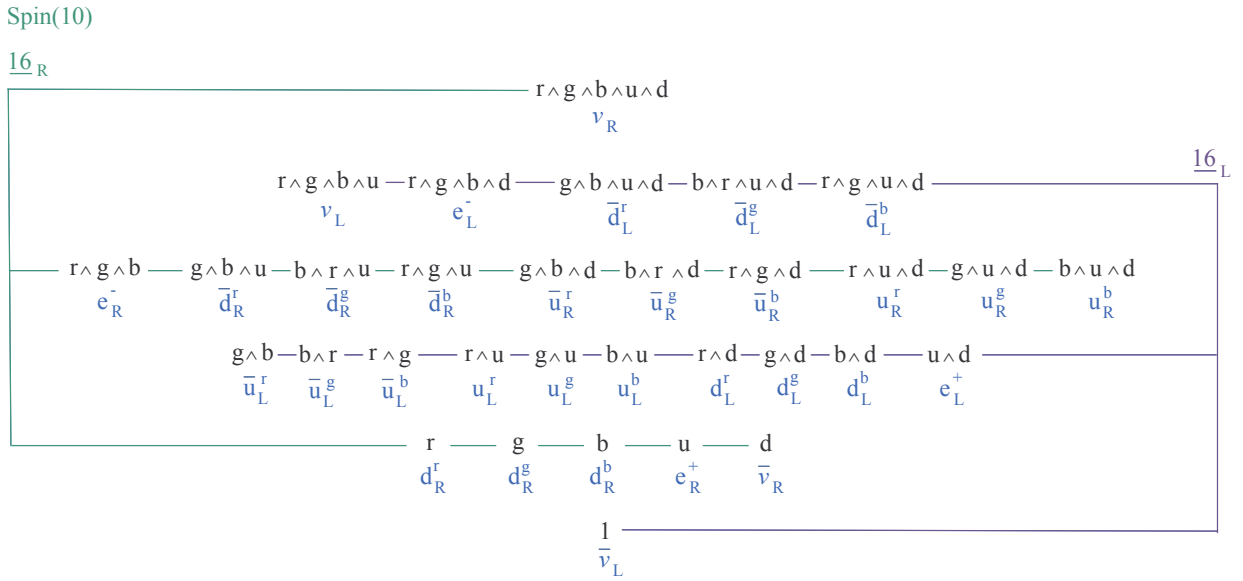$$d_R^r \qquad d_R^g \qquad d_R^b \qquad e_R^+ \qquad \bar{\nu}_R$$

$$1$$
$$\bar{\nu}_L$$

Figure 8.3: The exterior algebra $\Lambda\mathbb{C}^5$, representing a fermionic binary code, with the corresponding standard model particles written underneath. Here, basis elements which are part of the same irreps under G_{SM} are connected by blue lines.

One Lie group which does hit the mark is $Spin(10)$. The $Spin(10)$ grand unified theory is more commonly known in the literature as the "$SO(10)$" model, and was proposed by H. Georgi in 1974/1975, [38], and independently by H. Fritzsch and P. Minkowski in 1975, [31].

$Spin(10)$ is the double cover of the group $SO(10)$, and is of rank 5, [49]. It unifies the standard model's gauge group, and extends it from 12 dimensions to 45. A generation of quarks and leptons in this model is unified into a 16-dimensional spinor, with a conjugate spinor accounting for the required anti-particles. This spinor representation can be seen to naturally account for a right-handed neutrino, [7].

The group, $Spin(10)$, is associated with the Clifford algebra $\mathbb{C}l(10)$, which has an irreducible representation given by a 32-dimensional Dirac spinor. Under $Spin(10)$, this Dirac spinor breaks down into two irreducible representations, known as Weyl spinors, which are each 16-dimensional. Now, it can be shown, [7], that Dirac spinors of the group $Spin(2n)$ can be represented by the exterior algebra, $\Lambda\mathbb{C}^n$. Hence, we see that the exterior algebra, $\Lambda\mathbb{C}^5$, which we just described in the previous section, is capable of acting not only as a representation of $SU(5)$, but also as a representation of $Spin(10)$. Please see

Figure 8.4.

Readers may recognize that this same pattern has appeared more than once in this thesis. That is, minimal left ideals, such as Ψ_L and S^u, have been built using raising operators of the Clifford algebras $\mathbb{C}l(2)$ and $\mathbb{C}l(6)$, respectively. These raising operators may be viewed as generating exterior algebras. For example, by looking at equation (6.19) of Section 6.4, it is easy to see how the Dirac spinor, S^u, is built from the exterior algebra, $\Lambda\mathbb{C}^3$, generated by $\alpha_1^\dagger$, $\alpha_2^\dagger$, and $\alpha_3^\dagger$.

Figure 8.4: The exterior algebra $\Lambda\mathbb{C}^5$, representing a fermionic binary code, with the corresponding standard model particles written underneath. Here, basis elements connected by lines in the even part of the algebra form one irreducible representation under $Spin(10)$, with left-handed chirality. Similarly, those basis elements in the odd part of the algebra form another irreducible representation under $Spin(10)$, with right-handed chirality.

Now, it can be shown that our representation of $SU(5)$ resides inside that of $Spin(10)$, [7]. As with $SU(5)$, the group $Spin(10)$ can be seen to respect the chirality of particles; it does not mix left- and right-handed species. However, it does not respect the grading of $\Lambda\mathbb{C}^5$, as do $SU(5)$ and G_{SM}.

8.4 Pati-Salam model

Before the introduction of $SU(5)$- and $Spin(10)$-theories came an extension to the standard model, now known as the Pati-Salam model. The model is named after its inventors, J. Pati and A. Salam, who published the proposal in 1974, [58].

The Pati-Salam model may be considered to be a more conservative approach to unification. It does not unify the gauge groups of the standard model into a simple Lie group, as with $SU(5)$ and $Spin(10)$. Instead, it is based on the gauge group, $SU(2) \times SU(2) \times SU(4) \simeq Spin(4) \times Spin(6)$, [7], which extends the dimensionality of G_{SM} from 12 to $3 + 3 + 15 = 21$.

One main goal of the Pati-Salam model is to capitalize on the fact that the standard model treats quarks and leptons similarly under weak isospin. Hence, the authors proposed that leptons might be considered as just another type of quark, which happen to be of a fourth colour, beyond the usual red, green and blue of $SU(3)_c$. This fourth colour would then be identified with lepton number, thereby extending $SU(3)_c$ to $SU(4)$, [58].

Perhaps perplexed by the standard model's apparent preference for left-handed particles, Pati and Salam further extended G_{SM} by incorporating an extra factor of $SU(2)$, meant for right-handed particles. With $SU(3)_c$ extended to $SU(4)$ and $SU(2)_L$ extended to $SU(2)_L \times SU(2)_R$, we then arrive at $SU(2) \times SU(2) \times SU(4)$. It turns out that the standard model's $U(1)_Y$ stretches across the right-handed $SU(2)$, and also the $SU(4)$ factor, [7].

The standard model's gauge group may be mapped into $SU(2) \times SU(2) \times SU(4)$ as shown in [7],

$$\left(h_2, \ \begin{pmatrix} (h_1)^3 & 0 \\ 0 & (h_1)^{-3} \end{pmatrix}, \ \begin{pmatrix} h_1 \, h_3 & 0 \\ 0 & (h_1)^{-3} \end{pmatrix} \right) \quad \in \quad SU(2) \times SU(2) \times SU(4), \quad (8.10)$$

for $h_1 \in U(1)$, $h_2 \in SU(2)$, and $h_3 \in SU(3)$.

With the gauge groups in hand, it is now fairly straightforward to build up the fermionic vector spaces proposed in this model. The group $SU(4)$ can be made to act on a four-complex-dimensional vector, $\mathbb{C}^4$, and its conjugate, $\mathbb{C}^{4*}$. For the $SU(2)_L \times SU(2)_R$ sector, we choose a left-handed doublet, $\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}$, and a right-handed doublet, $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}^2$. Putting these all together gives the Pati-Salam representation, [7],

$$
\begin{aligned}
f &\equiv \left((\mathbb{C}^2 \otimes_\mathbb{C} \mathbb{C}) \oplus (\mathbb{C} \otimes_\mathbb{C} \mathbb{C}^2) \right) \otimes_\mathbb{C} \left(\mathbb{C}^4 \oplus \mathbb{C}^{4*} \right) \\
&\simeq (\mathbb{C}^2 \otimes_\mathbb{C} \mathbb{C} \otimes_\mathbb{C} \mathbb{C}^4) \oplus (\mathbb{C} \otimes_\mathbb{C} \mathbb{C}^2 \otimes_\mathbb{C} \mathbb{C}^4) \oplus \\
&\quad \left(\mathbb{C}^2 \otimes_\mathbb{C} \mathbb{C} \otimes_\mathbb{C} \mathbb{C}^{4*} \right) \oplus \left(\mathbb{C} \otimes_\mathbb{C} \mathbb{C}^2 \otimes_\mathbb{C} \mathbb{C}^{4*} \right),
\end{aligned}
\tag{8.11}
$$

which is 32-complex dimensional.

In keeping with the fact that $SU(2) \times SU(2) \times SU(4) \simeq Spin(4) \times Spin(6)$, it turns out that there is another way to describe the fermionic space, f. This is in terms of the Dirac spinors, which form representations of $Spin(4)$ and $Spin(6)$. These Dirac spinors may be expressed as the exterior algebras, $\Lambda\mathbb{C}^2$ and $\Lambda\mathbb{C}^3$, respectively, so that

$$
f \simeq \Lambda\mathbb{C}^2 \otimes_\mathbb{C} \Lambda\mathbb{C}^3.
\tag{8.12}
$$

Let us now consider $\Lambda\mathbb{C}^2$ to be generated by u and d, and $\Lambda\mathbb{C}^3$ to be generated by r, g, and b, which are familiar from Section 8.2. We can now take the tensor product between these two exterior algebras, and arrange them so as to see the similarity between $\Lambda\mathbb{C}^2 \otimes_\mathbb{C} \Lambda\mathbb{C}^3$ and $\Lambda\mathbb{C}^5$ from Section 8.2. Please see Figure 8.5. Clearly, going from the $\Lambda\mathbb{C}^2 \otimes_\mathbb{C} \Lambda\mathbb{C}^3$ fermionic space to the $\Lambda\mathbb{C}^5$ fermionic space, requires simply replacing the tensor product in objects such as $r \otimes u$ with the wedge product, giving $r \wedge u$, [7].

$(r{\wedge}g{\wedge}b){\otimes}(u{\wedge}d)$
ν_R

$(r{\wedge}g{\wedge}b){\otimes}u - (r{\wedge}g{\wedge}b){\otimes}d$ $(g{\wedge}b){\otimes}(u{\wedge}d) - (b{\wedge}r){\otimes}(u{\wedge}d) - (r{\wedge}g){\otimes}(u{\wedge}d)$
ν_L e_L^- $\overline{d}_L^r$ $\overline{d}_L^g$ $\overline{d}_L^b$

$r{\wedge}g{\wedge}b$ $(g{\wedge}b){\otimes}u-(b{\wedge}r){\otimes}u-(r{\wedge}g){\otimes}u$ $(g{\wedge}b){\otimes}d-(b{\wedge}r){\otimes}d-(r{\wedge}g){\otimes}d$ $r{\otimes}(u{\wedge}d)-g{\otimes}(u{\wedge}d)-b{\otimes}(u{\wedge}d)$
e_R^- $\overline{d}_R^r$ $\overline{d}_R^g$ $\overline{d}_R^b$ $\overline{u}_R^r$ $\overline{u}_R^g$ $\overline{u}_R^b$ u_R^r u_R^g u_R^b

$g{\wedge}b-b{\wedge}r-r{\wedge}g$ $r{\otimes}u-g{\otimes}u-b{\otimes}u$ $r{\otimes}d-g{\otimes}d-b{\otimes}d$ $u{\wedge}d$
$\overline{u}_L^r$ $\overline{u}_L^g$ $\overline{u}_L^b$ u_L^r u_L^g u_L^b d_L^r d_L^g d_L^b e_L^+

r —— g —— b u —— d
d_R^r d_R^g d_R^b e_R^+ $\overline{\nu}_R$

1
$\overline{\nu}_L$

Figure 8.5: The tensor product of the exterior algebras $\Lambda\mathbb{C}^2$ and $\Lambda\mathbb{C}^3$, arranged so as to demonstrate its similarity to $\Lambda\mathbb{C}^5$ from Section 8.2. Here, standard model irreducible representations are identified within the diagram.

Chapter 9

Towards a three-generation model

9.1   Introduction

Despite the wide range of proposals to simplify the standard model, most schemes tend to share the same impedances. Few unified models naturally offer more than a single generation of particles, and few are able to evade proton decay without repercussion.

In previous chapters, we were also concerned with finding just a one-generation model of particle physics, based on $\mathbb{C} \otimes \mathbb{O}$. However, this begs the question: could $\mathbb{C} \otimes \mathbb{O}$ provide room for three?

In this chapter, we point out a somewhat mysterious appearance of $SU(3)_c$ representations, which exhibit the behaviour of three full generations of standard model particles. These representations are found in the Clifford algebra $\mathbb{C}l(6)$, arising from the complex octonions. Back in Section 6.3, we explained how this 64-complex-dimensional space came about. With the algebra in place, we will now identify new generators of $SU(3)$ within it. These $SU(3)$ generators then act to partition the remaining part of the 64-dimensional Clifford algebra into six triplets, six singlets, and their antiparticles. That is, the algebra mirrors the chromodynamic structure of exactly three generations of the standard model's quarks and leptons.

Passing from particle to antiparticle, or *vice versa,* requires nothing more than effecting the complex conjugate, $*\colon\ i \mapsto -i$. The entire result is achieved using only the eight-dimensional complex octonions as a single ingredient.

The purpose of this chapter is *not* to offer a completed unified gauge theory, or even a completed description of QCD. Instead, we propose a gateway from which such a theory might be found.

To the best of this author's knowledge, [35] was the first account of these three-generation results found either within the octonions, or $\mathbb{C}l(6)$.

9.2 Lie algebras of $SU(3)_c$

As mentioned earlier, the automorphism group of the octonions is G_2, which is a 14-dimensional exceptional Lie group. Within G_2, we may find a subgroup, $SU(3)$, which is defined as that subgroup of G_2 which keeps the imaginary unit, e_7, invariant. The Lie algebra of this $SU(3)$, acting on $f \in \mathbb{C} \otimes \mathbb{O}$ may be expressed as

$$
\begin{aligned}
\Lambda_1 f &= \tfrac{i}{2}\big(e_1\,(e_5\ f\,) - e_3\,(e_4\ f\,)\big),\\
\Lambda_2 f &= -\tfrac{i}{2}\big(e_1\,(e_4\ f\,) + e_3\,(e_5\ f\,)\big),\\
\Lambda_3 f &= \tfrac{i}{2}\big(e_4\,(e_5\ f\,) - e_1\,(e_3\ f\,)\big),\\
\Lambda_4 f &= \tfrac{i}{2}\big(e_2\,(e_5\ f\,) + e_4\,(e_6\ f\,)\big),\\
\Lambda_5 f &= \tfrac{i}{2}\big(e_5\,(e_6\ f\,) - e_2\,(e_4\ f\,)\big),\\
\Lambda_6 f &= \tfrac{i}{2}\big(e_1\,(e_6\ f\,) + e_2\,(e_3\ f\,)\big),\\
\Lambda_7 f &= \tfrac{i}{2}\big(e_1\,(e_2\ f\,) + e_3\,(e_6\ f\,)\big),\\
\Lambda_8 f &= \tfrac{i}{2\sqrt{3}}\big(e_1\,(e_3\ f\,) + e_4\,(e_5\ f\,) - 2e_2\,(e_6\ f\,)\big).
\end{aligned}
\tag{9.1}
$$

The Lie algebra's commutation relations take the form

$$
\left[\frac{\Lambda_a}{2}, \frac{\Lambda_b}{2}\right]f \equiv \left[\frac{\Lambda_a}{2}\frac{\Lambda_b}{2} - \frac{\Lambda_b}{2}\frac{\Lambda_a}{2}\right]f = ic_{abc}\frac{\Lambda_c}{2}f,
\tag{9.2}
$$

$\forall f \in \mathbb{C} \otimes \mathbb{O}$, with the usual $SU(3)$ structure constants, c_{abc}.

Clearly, the Λ_i, as expressed above, constitute elements of $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$. In earlier references, [42], [43], [26], these Λ_i are shown to act on quark and lepton representations in the eight-dimensional $\mathbb{C} \otimes \mathbb{O}$, or multiple copies thereof. In contrast, here we introduce the Λ_i acting on quark and lepton representations within the 64-dimensional $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$.

Taking a hint from [34], let us now introduce a related representation of $su(3)$, which will draw out structure in $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$, familiar from the behaviour of quarks and leptons.

Consider a resolution of the identity in $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$

$$1f = [\nu + \nu^*]\, f, \tag{9.3}$$

where $\nu \equiv \frac{1}{2}(1 + i\overleftarrow{e_7})$. Both ν and ν^* act as projectors, whereby $\nu\nu = \nu$, $\nu^*\nu^* = \nu^*$, and $\nu\nu^* = \nu^*\nu = 0$.

As

$$\left[\Lambda_n,\, \nu\right] = 0 \qquad \forall n = 1\ldots 8, \tag{9.4}$$

equation (9.2) then leads to

$$\left[\frac{\Lambda_a}{2}\nu,\, \frac{\Lambda_b}{2}\nu\right] = ic_{abc}\frac{\Lambda_c}{2}\nu. \tag{9.5}$$

That is, the eight $\frac{1}{2}\Lambda_n\nu$ form a representation of su(3). Taking the complex conjugate of (9.5) gives

$$\left[-\frac{\Lambda_a^*}{2}\nu^*,\, -\frac{\Lambda_b^*}{2}\nu^*\right] = ic_{abc}\left[-\frac{\Lambda_c^*}{2}\nu^*\right], \tag{9.6}$$

so that the $-\frac{1}{2}\Lambda_n^*\nu^*$ give a further representation.

9.3 Three generations under $SU(3)_c$

Knowing that the $\Lambda_n\nu$ behave as an eight dimensional representation under the action of $[\Lambda_m\nu,\, \cdot\,]$, one might wonder how objects of the more general form $a\nu$ behave under $[\Lambda_m\nu,\, \cdot\,]$.

Obeying $[\Lambda_m\nu,\, \ell_j\nu] = 0 \ \forall m = 1\ldots 8$, we find six $SU(3)$ singlets, whose basis vectors are given by

$$\ell_a \equiv (1 + ie_{13} + ie_{26} + ie_{45})\,\nu,$$
$$\ell_b \equiv (3 - ie_{13} - ie_{26} - ie_{45})\,\nu,$$
$$\ell_c \equiv (-ie_{124} - e_{125} + e_{146} - ie_{156})\,\nu,$$
$$\ell_d \equiv (-ie_1 - e_3 + e_{126} + e_{145})\,\nu, \qquad (9.7)$$
$$\ell_e \equiv (ie_2 + e_6 + e_{123} + ie_{136})\,\nu,$$
$$\ell_f \equiv (ie_4 + e_5 - e_{134} + ie_{135})\,\nu,$$

where the left-pointing arrows were dropped throughout for notational simplicity, and right-to-left multiplication is still meant to occur. The notation e_{ab} is meant here to be shorthand for $e_a(e_b \cdot)$, etc.

The set of basis vectors

$$q_1^R \equiv (-ie_{12} - e_{16} + e_{23} + ie_{36})\,\nu$$
$$q_1^G \equiv (-ie_{24} - e_{25} + e_{46} - ie_{56})\,\nu \qquad (9.8)$$
$$q_1^B \equiv (ie_{14} + e_{15} + e_{34} - ie_{35})\,\nu$$

acts as a triplet under commutation with the $\Lambda_m \nu$. Next, we find five anti-triplets given by

$$\bar{q}_2^R \equiv (ie_{12} - e_{16} + e_{23} - ie_{36})\,\nu$$
$$\bar{q}_2^G \equiv (ie_{24} - e_{25} + e_{46} + ie_{56})\,\nu \qquad (9.9)$$
$$\bar{q}_2^B \equiv (-ie_{14} + e_{15} + e_{34} + ie_{35})\,\nu,$$

$$\bar{q}_3^R \equiv (ie_4 + e_5 + e_{134} - ie_{135})\,\nu$$
$$\bar{q}_3^G \equiv (ie_1 + e_3 + e_{126} + e_{145})\,\nu \qquad (9.10)$$
$$\bar{q}_3^B \equiv (ie_2 + e_6 - e_{123} - ie_{136})\,\nu,$$

$$\bar{q}_4^R \equiv (ie_1 - e_3 + e_{126} - e_{145})\,\nu$$
$$\bar{q}_4^G \equiv (-ie_4 + e_5 + e_{134} + ie_{135})\,\nu \qquad (9.11)$$
$$\bar{q}_4^B \equiv (ie_{124} - e_{125} - e_{146} - ie_{156})\,\nu,$$

$$\bar{q}_5^R \equiv (-ie_2 + e_6 + e_{123} - ie_{136})\, \nu$$
$$\bar{q}_5^G \equiv (ie_{124} - e_{125} + e_{146} + ie_{156})\, \nu \tag{9.12}$$
$$\bar{q}_5^B \equiv (ie_4 - e_5 + e_{134} + ie_{135})\, \nu,$$

$$\bar{q}_6^R \equiv (ie_{124} + e_{125} + e_{146} - ie_{156})\, \nu$$
$$\bar{q}_6^G \equiv (ie_2 - e_6 + e_{123} - ie_{136})\, \nu \tag{9.13}$$
$$\bar{q}_6^B \equiv (-ie_1 + e_3 + e_{126} - e_{145})\, \nu.$$

Taking the complex conjugate, $*\colon i \mapsto -i$, of these 32 basis vectors gives 32 new linearly independent basis vectors. Under commutation with $-\Lambda_m^* \nu^*$,

$$\ell_a^* = (1 - ie_{13} - ie_{26} - ie_{45})\, \nu^*,$$
$$\ell_b^* = (3 + ie_{13} + ie_{26} + ie_{45})\, \nu^*,$$
$$\ell_c^* = (ie_{124} - e_{125} + e_{146} + ie_{156})\, \nu^*,$$
$$\ell_d^* = (ie_1 - e_3 + e_{126} + e_{145})\, \nu^*, \tag{9.14}$$
$$\ell_e^* = (-ie_2 + e_6 + e_{123} - ie_{136})\, \nu^*,$$
$$\ell_f^* = (-ie_4 + e_5 - e_{134} - ie_{135})\, \nu^*$$

transform as $SU(3)$ singlets,

$$q_1^{R*} = (ie_{12} - e_{16} + e_{23} - ie_{36})\, \nu^* \equiv \bar{q}_1^R$$
$$q_1^{G*} = (ie_{24} - e_{25} + e_{46} + ie_{56})\, \nu^* \equiv \bar{q}_1^G \tag{9.15}$$
$$q_1^{B*} = (-ie_{14} + e_{15} + e_{34} + ie_{35})\, \nu^* \equiv \bar{q}_1^B$$

behaves as an anti-triplet,

$$\bar{q}_2^{R*} = (-ie_{12} - e_{16} + e_{23} + ie_{36})\, \nu^* \equiv q_2^R$$
$$\bar{q}_2^{G*} = (-ie_{24} - e_{25} + e_{46} - ie_{56})\, \nu^* \equiv q_2^G \tag{9.16}$$
$$\bar{q}_2^{B*} = (ie_{14} + e_{15} + e_{34} - ie_{35})\, \nu^* \equiv q_2^B$$

behaves as a triplet, and so on.

That is, unlike the standard model, we are able to pass back and forth between particle and anti-particle using *only* the complex conjugate $i \mapsto -i$. This feature appeared early on in the work of [42] for some internal degrees of freedom, and also in [34] and Section 3.5.3 when passing between left- and right-handed Weyl spinors.

9.4   A sample calculation

We introduce to the reader how calculations are carried out in $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ by working through an example. Let us find the action of the first $SU(3)$ generator of the form $\Lambda\nu$, which we will define as $\Lambda_1 \nu \equiv \frac{i}{2} \left(e_{15} - e_{34} \right) \nu$, in accordance with equation (6.26). Let $\Lambda_1 \nu$ act on q_1^R, as defined in equations (9.8):

$$
\begin{aligned}
&\left[\Lambda_1 \nu , \ q_1^R \right] \\
=\ & \left[\tfrac{i}{2} \left(e_{15} - e_{34} \right) \nu , \ \left(-ie_{12} - e_{16} + e_{23} + ie_{36} \right) \nu \right] \\
=\ & \tfrac{i}{2} \Big(\left(e_{15} - e_{34} \right) \left(-ie_{12} - e_{16} + e_{23} + ie_{36} \right) \\
& \quad - \left(-ie_{12} - e_{16} + e_{23} + ie_{36} \right) \left(e_{15} - e_{34} \right) \Big) \nu \\
=\ & \tfrac{i}{2} \Big(-ie_{1512} - e_{1516} + e_{1523} + ie_{1536} \\
& \quad +ie_{3412} + e_{3416} - e_{3423} - ie_{3436} \\
& \quad +ie_{1215} + e_{1615} - e_{2315} - ie_{3615} \\
& \quad -ie_{1234} - e_{1634} + e_{2334} + ie_{3634} \Big) \nu \\
=\ & \tfrac{i}{2} \Big(-ie_{52} - e_{56} + e_{1235} - ie_{1356} \\
& \quad +ie_{1234} + e_{1346} + e_{42} - ie_{46} \\
& \quad +ie_{25} + e_{65} - e_{1235} + ie_{1356} \\
& \quad -ie_{1234} - e_{1346} - e_{24} + ie_{64} \Big) \nu \\
=\ & i \big(ie_{25} - e_{56} - e_{24} - ie_{46} \big) \nu = q_1^G.
\end{aligned}
\tag{9.17}
$$

This is the result we would expect for the first of the $su(3)$ Gell-Mann matrices, Λ_1^{GM}, from the standard model, acting to convert a red basis vector, $\underline{R} \equiv (1,0,0)^{\top}$, into a green basis

vector, $\underline{G} \equiv (0, 1, 0)^\top$.

$$\Lambda_1^{GM} \underline{R} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \underline{G}. \tag{9.18}$$

9.5 Summary for $SU(3)_c$

Finally, we conclude by summarizing the main result of this chapter in Figure 9.1: the breakdown of the 64-dimensional $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ into irreducible representations of $SU(3)$.
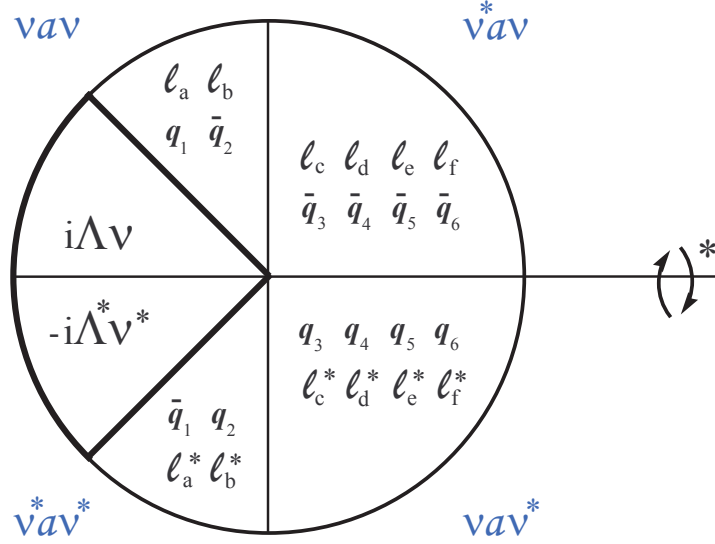

Figure 9.1: The 64-dimensional octonionic chain algebra splits into two sets of $SU(3)$ generators of the form $i\Lambda\nu$ and $-i\Lambda^*\nu^*$, six $SU(3)$ singlets ℓ_j, six triplets q_k, and their complex conjugates. These objects are sectioned off above into four quadrants according to their forms: $\nu a\nu$, $\nu^* a\nu$, $\nu a\nu^*$ and $\nu^* a\nu^*$ for a in the chain algebra. Transforming particles into anti-particles, and *vice versa*, requires only the complex conjugate $*$: $i \mapsto -i$ in our formalism.

Using only the eight-dimensional complex octonions, $\mathbb{C} \otimes \mathbb{O}$, we have explained how to build up a 64-complex-dimensional associative algebra. The $SU(3)$ generators identified

within this algebra then break down the remaining space into six singlets, six triplets, and their antiparticles, with no extra particles beyond these.

These representations are a curious finding. They effortlessly suggest the existence of exactly three generations, they relate particles to antiparticles by using only the complex conjugate $i \mapsto -i$, and finally, they fill these tall orders while working from but a modest eight-complex-dimensional algebra.

9.6 Three generations under $SU(3)_c$ and $U(1)_{em}$

Since the time that these irreducible representations of $SU(3)_c$ were identified within $\mathbb{C} \otimes \overset{\leftarrow}{\mathbb{O}}$, more standard model structure has been uncovered in the algebra. It turns out that we can extend our three-generation results for $SU(3)_c$ to include $U(1)_{em}$ as well (presented for the first time here).

That is, our $SU(3)_c$ action on $\mathbb{C} \otimes \overset{\leftarrow}{\mathbb{O}}$ given by

$$[\Lambda_j \nu, \ a\nu], \qquad [-\Lambda_j^* \nu^*, \ a\nu^*] \tag{9.19}$$

can be generalized, so as to include electric charges. Furthermore, we will be able to incorporate $U(1)_{em}$ by making use of the Q we had already introduced in our one-generation model of Section 6.8,

$$Q = \frac{N}{3} = \sum_{i=1}^{3} \alpha_i^\dagger \alpha_i. \tag{9.20}$$

Consider the following action for the $SU(3)_c$ generators,

$$[\Lambda_j \nu, \ S^* a\nu], \qquad [-\Lambda_j^* \nu^*, \ S a\nu^*],$$
$$[\Lambda_j \nu, \ S a\nu], \qquad [-\Lambda_j^* \nu^*, \ S^* a\nu^*], \tag{9.21}$$

where S is the projector given by

$$S \equiv \frac{1}{2} - \frac{ie_7}{4} + \frac{i}{4} \left(e_{13} + e_{26} + e_{45} \right). \tag{9.22}$$

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Clearly, $S + S^* = 1$, so that we see that this new action (9.21) is no different from the original action (9.19). Although it has not been mentioned before, S is a projector which has come up frequently in complex-octonionic work; it is nothing more than the right-multiplication analogue of $\nu = \frac{1}{2}(1 + ie_7)$. Or in other words,

$$f\, \frac{1}{2}\, (1 + ie_7) = Sf \qquad \forall f \in \mathbb{C} \otimes \mathbb{O}. \tag{9.23}$$

It so happens that $-\Lambda_j^* = \Lambda_j \; \forall j = 1 \ldots 8$, so that the action (9.21) may be rewritten as

$$\begin{aligned}
&[\,\Lambda_j \nu,\; S^* a\nu\,], &&[\,-\Lambda_j^* \nu^*,\; Sa\nu^*\,], \\
&[\,-\Lambda_j^* \nu,\; Sa\nu\,], &&[\,\Lambda_j \nu^*,\; S^* a\nu^*\,].
\end{aligned} \tag{9.24}$$

Now, an action for electric charge may be found, which matches the $SU(3)_c$ action of (9.24). Namely,

$$\begin{aligned}
&[\,Q\nu,\; S^* a\nu\,], &&[\,-Q^* \nu^*,\; Sa\nu^*\,], \\
&[\,-Q^* \nu,\; Sa\nu\,], &&[\,Q\nu^*,\; S^* a\nu^*\,].
\end{aligned} \tag{9.25}$$

Under these actions (9.24) and (9.25), we find $SU(3)_c$ and $U(1)_{em}$ charge assignments, which are consistent with three generations of standard model particles. Below, we relabel the states given earlier in equations (9.7)-(9.16), so as to now specify their electric charges.

At this level, we are not specifying which generation each state belongs to, so for $i = 1, 2, 3$, the three states with electric charge of -1 will be labelled e_i^-; the three states with electric charge of $+1$ will be labelled e_i^+; the three states with electric charge of $2/3$ will be labelled u_i; the three states with electric charge of $-2/3$ will be labelled $\bar{u}_i$; the three states with electric charge $-1/3$ will be labelled d_i, and the three states with electric charge $1/3$ will be labelled $\bar{d}_i$. Since $SU(3)_c$ and $U(1)_{em}$ do not distinguish between neutrinos, ν_i and anti-neutrinos, $\bar{\nu}_i$, we will then label the six states with electric charge of zero as n_i and $\bar{n}_i$, where the symbol n_i could represent either a neutrino or anti-neutrino.

$$n_1 \longleftarrow \ell_a \equiv (1 + ie_{13} + ie_{26} + ie_{45})\, \nu = S\, \ell_a\, \nu,$$

$$n_2 \longleftarrow \ell_b \equiv (3 - ie_{13} - ie_{26} - ie_{45})\, \nu = S^*\, \ell_b\, \nu,$$

$$n_3 \longleftarrow \ell_c \equiv (-ie_{124} - e_{125} + e_{146} - ie_{156})\, \nu = S^*\, \ell_c\, \nu,$$

$$e_1^+ \longleftarrow \ell_d \equiv (-ie_1 - e_3 + e_{126} + e_{145})\, \nu = S\, \ell_d\, \nu,$$

$$e_2^+ \longleftarrow \ell_e \equiv (ie_2 + e_6 + e_{123} + ie_{136})\, \nu = S\, \ell_e\, \nu,$$

$$e_3^+ \longleftarrow \ell_f \equiv (ie_4 + e_5 - e_{134} + ie_{135})\, \nu = S\, \ell_f\, \nu,$$

(9.26)

$$u_1^R \longleftarrow q_1^R \equiv (-ie_{12} - e_{16} + e_{23} + ie_{36})\, \nu = S^*\, q_1^R\, \nu$$

$$u_1^G \longleftarrow q_1^G \equiv (-ie_{24} - e_{25} + e_{46} - ie_{56})\, \nu = S^*\, q_1^G\, \nu$$

$$u_1^B \longleftarrow q_1^B \equiv (ie_{14} + e_{15} + e_{34} - ie_{35})\, \nu = S^*\, q_1^B\, \nu$$

(9.27)

$$\bar{u}_2^R \longleftarrow \bar{q}_2^R \equiv (ie_{12} - e_{16} + e_{23} - ie_{36})\, \nu = S\, \bar{q}_2^R\, \nu$$

$$\bar{u}_2^G \longleftarrow \bar{q}_2^G \equiv (ie_{24} - e_{25} + e_{46} + ie_{56})\, \nu = S\, \bar{q}_2^G\, \nu$$

$$\bar{u}_2^B \longleftarrow \bar{q}_2^B \equiv (-ie_{14} + e_{15} + e_{34} + ie_{35})\, \nu = S\, \bar{q}_2^B\, \nu,$$

(9.28)

$$\bar{u}_3^R \longleftarrow \bar{q}_3^R \equiv (ie_4 + e_5 + e_{134} - ie_{135})\, \nu = S^*\, \bar{q}_3^R\, \nu$$

$$\bar{u}_3^G \longleftarrow \bar{q}_3^G \equiv (ie_1 + e_3 + e_{126} + e_{145})\, \nu = S^*\, \bar{q}_3^G\, \nu$$

$$\bar{u}_3^B \longleftarrow \bar{q}_3^B \equiv (ie_2 + e_6 - e_{123} - ie_{136})\, \nu = S^*\, \bar{q}_3^B\, \nu,$$

(9.29)

$$\bar{d}_1^R \longleftarrow \bar{q}_4^R \equiv (ie_1 - e_3 + e_{126} - e_{145})\, \nu = S\, \bar{q}_4^R\, \nu$$

$$\bar{d}_1^G \longleftarrow \bar{q}_4^G \equiv (-ie_4 + e_5 + e_{134} + ie_{135})\, \nu = S\, \bar{q}_4^G\, \nu$$

$$\bar{d}_1^B \longleftarrow \bar{q}_4^B \equiv (ie_{124} - e_{125} - e_{146} - ie_{156})\, \nu = S\, \bar{q}_4^B\, \nu,$$

(9.30)

$$\bar{d}_2^R \longleftarrow \bar{q}_5^R \equiv (-ie_2 + e_6 + e_{123} - ie_{136})\, \nu = S\, \bar{q}_5^R\, \nu$$

$$\bar{d}_2^G \longleftarrow \bar{q}_5^G \equiv (ie_{124} - e_{125} + e_{146} + ie_{156})\, \nu = S\, \bar{q}_5^G\, \nu$$

$$\bar{d}_2^B \longleftarrow \bar{q}_5^B \equiv (ie_4 - e_5 + e_{134} + ie_{135})\, \nu = S\, \bar{q}_5^B\, \nu,$$

(9.31)

$$\bar{d}_3^R \longleftarrow \bar{q}_6^R \equiv (ie_{124} + e_{125} + e_{146} - ie_{156})\, \nu = S\, \bar{q}_6^R\, \nu$$

$$\bar{d}_3^G \longleftarrow \bar{q}_6^G \equiv (ie_2 - e_6 + e_{123} - ie_{136})\, \nu = S\, \bar{q}_6^G\, \nu$$

$$\bar{d}_3^B \longleftarrow \bar{q}_6^B \equiv (-ie_1 + e_3 + e_{126} - e_{145})\, \nu = S\, \bar{q}_6^B\, \nu.$$

(9.32)

Taking the complex conjugate, $*: i \mapsto -i$, of these 32 basis vectors gives 32 new linearly independent basis vectors:

$$
\begin{aligned}
\bar{n}_1 \longleftarrow \ell_a^* &= (1 - ie_{13} - ie_{26} - ie_{45})\,\nu^* = S^*\,\ell_a^*\,\nu^*, \\
\bar{n}_2 \longleftarrow \ell_b^* &= (3 + ie_{13} + ie_{26} + ie_{45})\,\nu^* = S\,\ell_b^*\,\nu^*, \\
\bar{n}_3 \longleftarrow \ell_c^* &= (ie_{124} - e_{125} + e_{146} + ie_{156})\,\nu^* = S\,\ell_c^*\,\nu^*, \\
e_1^- \longleftarrow \ell_d^* &= (ie_1 - e_3 + e_{126} + e_{145})\,\nu^* = S^*\,\ell_d^*\,\nu^*, \\
e_2^- \longleftarrow \ell_e^* &= (-ie_2 + e_6 + e_{123} - ie_{136})\,\nu^* = S^*\,\ell_e^*\,\nu^*, \\
e_3^- \longleftarrow \ell_f^* &= (-ie_4 + e_5 - e_{134} - ie_{135})\,\nu^* = S^*\,\ell_f^*\,\nu^*,
\end{aligned}
\tag{9.33}
$$

$$
\begin{aligned}
\bar{u}_1^R \longleftarrow q_1^{R*} &= (ie_{12} - e_{16} + e_{23} - ie_{36})\,\nu^* \equiv \bar{q}_1^R = S\,\bar{q}_1^R\,\nu^* \\
\bar{u}_1^G \longleftarrow q_1^{G*} &= (ie_{24} - e_{25} + e_{46} + ie_{56})\,\nu^* \equiv \bar{q}_1^G = S\,\bar{q}_1^G\,\nu^* \\
\bar{u}_1^B \longleftarrow q_1^{B*} &= (-ie_{14} + e_{15} + e_{34} + ie_{35})\,\nu^* \equiv \bar{q}_1^B = S\,\bar{q}_1^B\,\nu^*
\end{aligned}
\tag{9.34}
$$

$$
\begin{aligned}
u_2^R \longleftarrow \bar{q}_2^{R*} &= (-ie_{12} - e_{16} + e_{23} + ie_{36})\,\nu^* \equiv q_2^R = S^*\,q_2^R\,\nu^* \\
u_2^G \longleftarrow \bar{q}_2^{G*} &= (-ie_{24} - e_{25} + e_{46} - ie_{56})\,\nu^* \equiv q_2^G = S^*\,q_2^G\,\nu^* \\
u_2^B \longleftarrow \bar{q}_2^{B*} &= (ie_{14} + e_{15} + e_{34} - ie_{35})\,\nu^* \equiv q_2^B = S^*\,q_2^B\,\nu^*
\end{aligned}
\tag{9.35}
$$

and so on.

9.7   Outlook: From one generation to three

In Section 6.8, we described a one-generation model in $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$, transforming under the symmetry generators Λ_j and Q. In Section 9 we described a three-generation model in $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$, transforming under the symmetry generators $\Lambda_j \nu$ and $Q\nu$. Given that they make use of the same algebras, and very similar symmetry generators, one might wonder if the one-generation and three-generation models could be connected.

Direct verification shows that the one-generation representation and its symmetries do not fit directly into the three-generation model in any obvious way. However, we suspect that the similarity between these two models is no coincidence and there could be a way to go from one to the other[1].

---

[1]For example, one might consider sums of objects of the form $S^u S^{u\dagger}$ and $S^d S^{d\dagger}$ in order to move from

states S^u and S^d in the one-generation model to states in the three-generation model.

Chapter 10

Conclusion

As a non-associative algebra, one might have naturally expected that $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ would not be fit to describe the action of groups. And with no more than 32 complex dimensions, one might have further anticipated that $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ would not have the capacity to describe much of the standard model, whose particle content spans hundreds of degrees of freedom.

Upon closer inspection, though, this algebra can be seen to exhibit a surprising amount of the standard model's structure. Over the years, numerous authors have pointed out various Lorentz representations, within $\mathbb{C} \otimes \mathbb{H}$. In this thesis, we have then gone on to consolidate all of the standard model's Lorentz representations in terms of generalized ideals of $\mathbb{C} \otimes \mathbb{H}$.

In the early seventies, Günaydin and Gürsey showed $SU(3)_c$ quark structure within the octonions, [42]. This thesis then subsequently provided a way to extend this octonionic quark model so as to include leptons and the electromagnetic charge operator. This completes one full generation of quarks and leptons, and describes their behaviour under the unbroken gauge symmetries of the standard model. Our use of minimal left ideals from $\mathbb{C} \otimes \mathbb{O}$ allowed us to provide a straightforward explanation for the quantization of electric charge.

We also demonstrated a rudimentary leptonic model with this algebra, whereby $SU(2)_L$ acts automatically on only left-handed states. We have repeatedly shown the generators of standard model gauge symmetries appearing, uniquely, as symmetries of the algebra's ladder operators.

Finally, within the octonionic sector of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, we found the $SU(3)_c$ and $U(1)_{em}$ representations corresponding to three full generations of quarks and leptons. Given that

most unified theories are based on a single generation, this may be viewed as a rather unusual finding.

Although evidence is accumulating in support of a connection between the standard model and $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, we certainly do not have a complete model, at the moment. However, with every new discovery, it becomes a little more clear that this unlikely algebra is not going away.

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