Hurwitz Trees and Tropical Geometry

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The lifting problem in algebraic geometry asks when a finite group G acting on a curve defined over characteristic $p > 0$ lifts to characteristic 0. One object used in the study of this problem is the Hurwitz tree, which encodes the ramification data of a group action on a disk. In this thesis we explore the connection between Hurwitz trees and tropical geometry. That is, we can view the Hurwitz tree as a tropical curve. After exploring this connection we provide two examples to illustrate the connection, using objects in tropical geometry to demonstrate when a group action fails to lift.

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Dedicated to those whose socioeconomic circumstances prevented them from receiving an education they deserved.

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"Old father, old artificer, stand me now and ever in good stead."

James Joyce, A Portrait of the Artist as a Young Man

Introduction

The goal of this thesis it to study the action of groups on algebraic curves defined over fields of characteristic $p > 0$. Given a group acting on a curve defined over characteristic 0 we may reduce the curve to obtain a group action on a curve over characteristic p. The lifting problem asks the opposite: If we start with a group action on a curve defined over characteristic p, can we lift it to a group action in characteristic 0? In general this is not true. Our approach to this problem is to establish a formal link between tropical geometry and a combinatorial object called a Hurwitz tree. This will allow us to study the lifting problem using tools from tropical geometry, and we demonstrate the usefulness of this approach by providing examples of curves that fail to lift to characteristic 0.

0.0.1 Reduction of curves and lifting of curves

Let R be a ring of characteristic 0 with a maximal ideal m. We assume that $k = R/m$ is a field of characteristic $p > 0$. An algebraic curve $\mathcal{C}$ defined over R can be thought of us the set of zeroes of a polynomial $f(z) \in R[z]$. We assume that we are given a finite group G that acts on $\mathcal{C}$, that is, one that maps $\mathcal{C}$ to itself.

By reducing the coefficients of $f(z)$ modulo m, we end up with a curve C defined over k. The action of G on $\mathcal{C}$ reduces to an action of G on the curve C.

Conversely, one may start with a curve C defined over a field k of characteristic p, along with a group G acting on C. In the lifting problem, we ask if there exists a curve $\mathcal{C}$ defined over a ring R of characteristic 0 along with an action of G on $\mathcal{C}$ such that the action of G on $\mathcal{C}$ reduces to the action of G on C. If such a curve $\mathcal{C}$ exists we say that the G action on C lifts to characteristic 0. In general this it is difficult to find necessary and sufficient conditions on C and G in order for the group action to lift to characteristic 0, and so most results on the lifting problem provide necessary but not sufficient conditions.

0.0.2 Our contributions

In this thesis we shall establish a formal connection between the Hurwitz tree obstruction to lifting defined by Brewis [6] and vanishing functions on tropical curves defined by Katz [17].

In his thesis, Brewis [6] defined a necessary condition that a group action of G on a curve C over characteristic p must satisfy in order to lift to characteristic 0. This condition states that there must exist a combinatorial object called a Hurwitz tree, and thus if a Hurwitz tree does not exist then one may conclude the group action of G on C does not lift. Hurwitz trees have the underlying structure of a graph in the graph-theoretic sense, and have additional data on the leaves and edges of the graph. This data encodes the group action of G on the lift of a curve $\mathcal{C}$ of C.

Katz, in his paper [17], studied the vanishing functions of elements of a ring on a tropical curve. That is, suppose $s \in R$ is an element of a ring R with a maximal ideal m. When we reduce modulo R modulo a maximal ideal m it is possible that s reduces to 0. In this case we say that s vanishes modulo m. Under good circumstances it is possible to quantify the degree of vanishing of s. For example, if $R = \mathbb{Z}$ and $m = (p)$ for a prime number p, we may write an element s of R as $s = p^n q$ where p does not divide q. When $n > 0$ then s vanishes modulo (p), and the degree to which it vanishes is defined to be n.

The vanishing function of an element s as above is a function that measures the extent to which s vanishes modulo the maximal ideals of R. It is defined on a special type of graph called a tropical curve, which is related to the structure of the ring R.

We show that Hurwitz trees are tropical curves, and that the information on the leaves and edges of a Hurwitz tree may be viewed as the vanishing functions of elements. By making this formal connection, we are able to reinterpret the Hurwitz tree obstruction from the perspective of tropical geometry. With this new perspective we then demonstrate two examples where considering vanishing functions on tropical curves allows us to conclude a group action on a curve fails to lift to characteristic 0.

0.1 Overview of thesis

Chapter 1 introduces the lifting problem. The first three sections are intended as background material. The reader is assumed familiar with ring and field theory, Galois theory, and some basic commutative algebra such as modules. Section 1.1 covers representation theory, which is fundamental to the Hurwitz tree obstruction. Section 1.2

covers discrete valuation rings, their completions, and Galois extensions of such rings. Section 1.3 covers the basics of algebraic geometry, including the definition of schemes. These sections are not intended as a complete introduction to the subjects mentioned and many results are stated without proof.

Section 1.4 discusses automorphisms of curves, and Subsection 1.4.1 continues the theme of automorphisms on curves and surfaces by looking at the fixed points of these automorphisms. This is important to us as the fixed points will determine the structure of the Hurwitz tree that will be attached to a group acting on a disk. In section 1.5 we discuss the lifting problem proper. In particular, the global lifting problem of group actions on curves will be shown to be equivalent to the local one of group actions on power series.

Chapter 2 discusses the Hurwitz tree. Following some brief notation in Section 2.1 we will define the Artin and depth characters associated to a finite group of automorphisms acting on an open disk in Section 2.2. These two characters are shown to be closely related in Theorem 2.7, which will be key for demonstrating their "piecewise-linearity". Hurwitz trees are defined in Section 2.3 as an object independent of a group acting on a disk, and then in Section 2.4 we will show how any such group action has an associated Hurwitz tree.

While Section 2.4 provides an elementary way of building a Hurwitz tree, the connection with tropical geometry is best viewed from the point of view of semistable models. As such, Section 2.5 will reiterate the construction of the Hurwitz tree associated with a group action on a disk in terms of semistable models. That this alternate construction is in fact equivalent to the one given before in Section 2.4 is shown in Section 2.6. The way in which Hurwitz trees relate to the lifting problem will be discussed in Section 2.7.

Chapter 3 involves establishing the formal connection between the lifting problem and tropical geometry. The elementary definitions we need from tropical geometry are given in Section 3.1, and Section 3.2 discusses the notion of a function being piecewise linear on a tropical curve. Section 3.3 introduces the vanishing functions of elements. These are piecewise-linear functions that will play the role of both the Artin and depth character on a Hurwitz tree. The connection between Hurwitz trees and tropical curves will be formalized in Section 3.4.

We end the thesis in Chapter 4, which consists of two examples of the lifting problem. The first example in Section 4.1 directly exploits the previous chapters connection between tropical curves and Hurwitz trees to demonstrate the failure of a group action to lift to characteristic 0. The second example in Section 4.3 serves to illustrate a similar purpose.

Chapter 1

The Lifting Problem

This chapter is focused on the lifting problem, which in turn will motivate the definition of the Hurwitz tree in Chapter 2.

The first three sections are intended as background material. Section 1.1 covers representation theory and introduces important characters such as the augmentation character that will appear later in the thesis in Section 2.3. Section 1.2 covers valued fields and discrete valuation rings. The local lifting problem deals with complete discrete valuation rings, and theorems such as the Cohen Structure Theorem (Theorem 1.16) and the Weierstrass Preparation Theorem (Theorem 1.19) are fundamental to studying the lifting problem. Section 1.3 covers the basics of algebraic geometry, such as the Spec construction. Subsection 1.3.3 discusses the technical conditions on schemes that appear in the thesis.

In Section 1.4 we look at automorphisms of schemes. Following this, in Subsection 1.4.1 we look at fixed points of automorphisms of rings $k[[z]]$ and $R[[z]]$ for k a field of characteristic $p > 0$ and R a complete discrete valuation ring of characteristic 0. Lemma 1.37 and Corollary 1.38 in this section provide an important result on how a finite order automorphism acts on the parameter z - this will be important in Section 2.4 when we use group actions on a disk to build Hurwitz trees.

The lifting problem will be introduced in Section 1.5. We will begin by looking at the global lifting problem involving group actions on curves, and then state a result that shows the global problem is equivalent to the local lifting problem where groups act on power series rings. Some elementary results on the structure of groups that can lift to characteristic 0 are proved and and examples provided.

1.1 Representation Theory

This section serves as an introduction to representation theory. Group representations will encode important information on how groups act on curves and surfaces, and are integral to the study of the lifting problem. This section is not meant to serve as a complete introduction, and no proofs will be provided. The primary source for this section is Serre's book *Linear Representations of Finite Groups* [22], especially Chapters 1, 2, and 3 of Serre's book.

Representations are defined in Subsection 1.1.1. Subsection 1.1.2 introduce characters of representations.

Finally, scalar products of characters are defined in Subsection 1.1.3. The augmentation character is defined, which will play a crucial role in Section 2.3 in Chapter 2, where Hurwitz trees are defined.

1.1.1 Group representations

Let V be a vector space over $\mathbb{C}$. We assume V has finite dimension n over $\mathbb{C}$. The *general linear group* $GL(V)$ of V is the set of all invertible linear maps from V to itself. By fixing a basis of V over K we may write any element of $GL(V)$ as an invertible $n \times n$ matrix with coefficients in K.

Definition 1.1.1. A *representation* of a finite group G in V is a group homomorphism

$$\phi : G \to GL(V).$$

Thus for $\sigma \in G$, $\phi(\sigma) \in GL(V)$, and if $v \in V$ then $\phi(\sigma)$ acts on v as $\phi(\sigma)(v)$. When ϕ is understood we will simply write $\sigma(v)$ to denote the action of $\phi(\sigma)$ on $v \in V$.

The *degree* of a representation of G in V is the dimension of V over $\mathbb{C}$.

Definition 1.1.2. Let ϕ and ϕ' be two representations of G in vector spaces V and V', respectively. We say that ϕ is isomorphic to ϕ' if there exists a linear isomorphism $\rho : V \to V'$ such that for all $\sigma \in G$

$$\rho \circ \phi(\sigma) = \phi'(\sigma) \circ \rho.$$

Remark 1.1. In particular, if ϕ and ϕ' are isomorphic representations of G then the degree of ϕ is the same as the degree of ϕ'. One can show isomorphism of representations defines an equivalence class of representations.

Example 1.1. *The identity representation for any finite group G is the homomorphism $\phi : G \to GL(\mathbb{C})$ defined by sending every $\sigma \in G$ to $1 \in \mathbb{C}$. Its degree is 1.*

Example 1.2. *Let G be a finite group of cardinality $|G|$. Let V be the dimension $|G|$ vector space over $\mathbb{C}$ with basis e_σ as σ ranges over all the elements of G.*

Define $\rho : G \to GL(V)$ by sending $\tau \in G$ to the element of $GL(V)$ that sends e_σ to $e_{\tau\sigma}$, then extending it $\mathbb{C}$-linearly. This representation is called the regular representation of G.

Example 1.3. *Recall that given two vector spaces V and V' of dimensions n and m over $\mathbb{C}$, we may define the direct sum vector space $V \oplus V'$ of dimension nm over $\mathbb{C}$. Elements in this vector space are of the form $v \oplus v'$ for $v \in V$ and $v' \in V'$, and $\mathbb{C}$ acts via $x(v \oplus v') = (xv) \oplus (xv')$ for all $x \in \mathbb{C}$.*

Let ϕ and ϕ' be any representations of G in V and V', respectively. Then we may define a new representation of G in $V \oplus V'$ by sending $\sigma \in G$ to $\phi(\sigma) \oplus \phi'(\sigma)$, which acts on an element $v \oplus v'$ as $(\phi(\sigma) \oplus \phi'(\sigma))(v \oplus v') = \phi(g)(v) \oplus \phi'(g)(v')$.

Let $W \subset V$ be a subspace of a vector space V for which we have a representation of G in.

Definition 1.1.3. A subspace $W \subset V$ is said to be *stable under* G if for all $\sigma \in G$ and for all $w \in W$, $\sigma(w) \in W$.

Remark 1.2. Trivially, one finds that W is stable under G if $W = \{0\}$ or $W = V$.

Definition 1.1.4. A representation ρ of G in V is said to be an *irreducible representation* of G if the only subspaces of V stable under G are $\{0\}$ and V.

Remark 1.3. Notably, any representation of G of degree 1 is irreducible, such as the trivial representation.

Example 1.4. *Let ϕ and ϕ' be any two representations of G in V and V', respectively. As in Example 1.3 we may define the representation $\phi \oplus \phi'$ of G in $V \oplus V'$. Then both of $V \oplus \{0\} \subset V \oplus V'$ and $\{0\} \oplus V' \subset V \oplus V'$ are subspaces of $V \oplus V'$, and both are stable under G.*

Example 1.4 shows that direct sum representations of the type in Example 1.3 are never irreducible. In fact, we have the following result:

Theorem 1.4 (Theorem 8 of Serre [22]). *Let ϕ be any representation of G. Then ϕ is isomorphic to the direct sum of irreducible representations of G, and this decomposition of ϕ is unique up to isomorphism.*

Thus the irreducible representations of G determine all the representations of G. As we will see in the following subsections, there are only finitely many irreducible representations of a finite group G.

We conclude this subsection by looking at induced representations, which allow us to find representations of G given a representation of a subgroup of G.

Definition 1.1.5. Let $\phi : G \to GL(V)$ be a representation of G, and for a subgroup H of G let ϕ_H be the restriction of ϕ to H. Let $W \subset V$ be a subspace of V stable under H, $W \neq \{0\}$. Then $\rho : H \to W$ is a representation defined by $\rho = \phi_H$ acting on W.

Note that if $\sigma \in G$ is any element, the subspace $\phi(\sigma)(W) \subset V$ depends only on the left coset σH of σ, as if $\tau \in H$, then $\phi(\tau)(W) = W$ and so $\phi(\sigma\tau)(W) = \phi(\sigma)(W)$.

Let $\sigma \in G$ be a set of representatives of the left cosets of H in G, and let $W_\sigma = \phi(\sigma)(W)$ be as above. As the W_σ are permuted amongst themselves by elements of G, their sum $\sum_{\sigma \in G/H} W_\sigma$ is a stable subspace of V.

We say that ϕ is *induced* by ρ if $V = \bigoplus_{\sigma \in G/H} W_\sigma$.

The following theorem allows us to start with a representation of a subgroup H of G and obtain a unique induced representation.

Theorem 1.5 (Theorem 11 of Serre [22]). *Let H be a subgroup of G. Let ρ be a representation of H in W. Then there exists a representation ϕ of G in a vector space V containing W that is induced by ρ, and it is unique up to isomorphism.*

1.1.2 Characters

Let $M = (a_{ij})$ be an $n \times n$ matrix over a field K. Recall that the *trace* $Tr(M)$ of M is the sum of the entries along its diagonal, i.e. $Tr(M) = \sum_{i=1}^{n} a_{ii}$. If V is a vector space and M is a linear transformation from V to itself, we may then define the trace of M by expressing M in matrix form and taking the usual trace. This definition can be shown to be independent of the basis of V.

Definition 1.1.6. Let $\phi : G \to GL(V)$ be a representation of G in V. We define the *character* of ϕ, $\chi_\phi : G \to \mathbb{C}^*$, via

$$\chi_\phi(\sigma) = Tr(\phi(\sigma)).$$

Example 1.5. *Let ϕ be the trivial character of G. Then the character of ϕ, denoted 1_G, takes value $1_G(\sigma) = 1$ for all $\sigma \in G$. Indeed, $\rho(\sigma)$ is the identity element of $GL(\mathbb{C})$, and in any basis has matrix representation* [1].

Example 1.6. *Let ϕ be the regular representation of G from Example 1.2. Then the character of ϕ, denoted r_G, takes value $|G|$ on 1 and 0 on any nontrivial element of G.*

Indeed, $\phi(1)$ is the identity element of $GL(V)$, and the dimension of $GL(V)$ is $|G|$ whence $r_G(1) = |G|$.

Let $\sigma \neq 1$ be an element of G. Then σ acts on any basis element e_τ of V via $\sigma(e_\tau) = e_{\sigma\tau}$. By assumption on σ, this is not equal to e_τ, and so in the matrix representation of σ all the diagonal entries are 0. Thus $r_G(\sigma) = 0$ for all $\sigma \neq 1$ in G for any representation ϕ of G.

Remark 1.6. The above example illustrates a more general result, namely that $\chi_\phi(1) = n$ where n is the degree of the representation.

The next theorem allows us to find the character of an induced representation (see Definition 1.1.5 for the definition of induced representation).

To state it we first make a definition.

Definition 1.1.7. Let $f : H \to R$ be a map defined on a subgroup H of G to a ring R. We define the *extension by 0 of f from H to G*, denoted by $\hat{f}$, via

$$\hat{f}(\sigma) = \begin{cases} f(\sigma) & \text{if } \sigma \in H \\ 0 & \text{if } \sigma \in G \setminus H. \end{cases}$$

Theorem 1.7 (Theorem 12 of Serre [22])**.** *Let H be a subgroup of G, and let ϕ be a representation of G induced by a representation ρ of H. Fix any system of representatives $\{\tau\}$ of G/H. Then*

$$\chi_\phi(\sigma) = \sum_{\tau \in G/H} \hat{\chi_\rho}(\tau\sigma\tau^{-1}) = \frac{1}{|H|} \sum_{\tau \in G} \hat{\chi_\rho}(\tau\sigma\tau^{-1}).$$

This theorem will be used repeatedly when discussing Hurwitz trees in Chapter 2.

1.1.3 Scalar products of characters

Definition 1.1.8. Given any two representations ϕ and ϕ' of G, we define the *scalar product* between their characters as

$$\langle \chi_\phi, \chi_{\phi'} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_\phi(g) \overline{\chi_{\phi'}}(g),$$

where $\overline{\chi_{\phi'}}(g)$ is the conjugate of $\chi_{\phi'}(g)$ in $\mathbb{C}$.

The following theorem gives us orthonormality relations on the irreducible characters of G.

Theorem 1.8 (Theorem 3 of Serre [22]). *Let ϕ and ϕ' be two irreducible representations of G. Then*

$$\langle \chi_\phi, \chi_{\phi'} \rangle_G = \begin{cases} 1 & \text{if } \phi \cong \phi' \\ 0 & \text{if } \phi \not\cong \phi'. \end{cases}$$

From this theorem we obtain a number of corollaries that will be of use to us.

Corollary 1.9.

- *Two representations of G are isomorphic if and only if their characters are the same.*
- *The regular representation of G is isomorphic to the direct sum of every irreducible character of G, each with multiplicity the degree of the representation.*
- *There are only finitely many irreducible representations of G.*

We now introduced a character for any group G that will prove key in Chapter 2 when defining Hurwitz trees.

Let $u_G : G \to \mathbb{C}$ be the function defined by

$$u_G(\sigma) = r_G(\sigma) - 1_G(\sigma)$$

for all $\sigma \in G$, where r_G and 1_G are the characters of the regular and trivial representation of G, respectively. Then by the Corollary 1.9 we see that u_G is a character of a representation of G, and that it takes value -1 on nontrivial elements of G and $|G| - 1$ on the identity element.

Definition 1.10. The character u_G of G defined above is the *augmentation character* of G.

1.2 Valuation Rings

The local lifting problem looks at group actions on discrete valuation rings. As such, these rings are of vital importance to this thesis, and this section serves as an introduction to them.

Valuations and discrete valuation rings are defined in Subsection 1.2.1, and the associated topology given by the valuation is discussed in Subsection 1.2.1.1. Notably, the ability to view a discrete valuation ring as a unit disk will be used repeatedly in Section 2.5 of Chapter 2.

In Section 1.2.2 we discuss completions of discrete valuation rings. Two key structure theorems are the Cohen Structure Theorem (1.16) and the Weierstrass Preparation Theorem (1.19). Complete discrete valuation rings are the type considered throughout this thesis, and thus the structure theorems will be used throughout

Galois extensions of complete discrete valuation rings are discussed in Subsection 1.2.3. This is important as it will form the basis of our understanding of automorphisms of curves and surfaces in Section 1.4. Ramification (Subsection 1.2.3.1) and Witt vectors (Subsection 1.2.3.2) are important in the local lifting problem. The former will give meaning to both the depth and Artin character of a group acting on an open disk in Section 2.2, and the latter is an important part of the statement of the local lifting problem in Section 1.5.

There are many excellent introductory texts to the subjects in this section. We will draw primarily on Serre's book *Local Fields* [23] as well as Chapters 1 and 2 of Cassels and Fröhlich's book *Algebraic Number Theory* [1].

1.2.1 Valued fields and discrete valuation rings

The local lifting problem, to be discussed in Section 1.5, involves group actions on discrete valuation rings, which we will define in this section. Before that we will discuss valued fields.

Definition 1.2.1. A *totally ordered abelian group* $(\Gamma, \geq)$ is an abelian group Γ along with a binary relation $\geq$ on Γ that satisfies

1. If $x \geq y$ and $x \geq y$ then $x = y$;
2. If $x \geq y$ and $y \geq z$ then $x \geq z$;
3. For any elements x and y of Γ, we have $x \geq y$ or $y \geq x$.

Remark 1.11. Given any set Γ and a binary relation $\geq$ on Γ that satisfies the above three criteria, $\geq$ is said to be a *total order* on Γ. If $\geq$ only satisfies the first two criteria then it is said to be a *partial order*.

Example 1.7. *Consider* $(\mathbb{Z}, \geq)$ *where* $\geq$ *is the usual total order on* $\mathbb{Z}$*. Then* $(\mathbb{Z}, \geq)$ *is a totally ordered abelian group.*

Definition 1.2.2. Let K be a field, K^* the multiplicative group of K and a map $\mathrm{val} : K \to (\Gamma, \geq) \cup \{\infty\}$ from K^* to a totally ordered abelian group $(\Gamma, \geq)$ that satisfies

1. For $x \in K$, $\mathrm{val}(x) = \infty$ if and only if $x = 0$;
2. Given x and y in K, $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$;
3. $\mathrm{val}(x + y) \geq \min\{\mathrm{val}(x), \mathrm{val}(y)\}$ with equality if $\mathrm{val}(x) \neq \mathrm{val}(y)$.

Then (K, val) is said to be a *valued field* with *val* the *valuation*.

Remark 1.12. The type of valuation defined above is a *non-Archimedean valuation*, which is the only type of interest to us. For *Archimedean* valuations the third requirement in the above definition is different.

Definition 1.2.3. Let (K, val) be a valued field. The ring $R = \{x \in K | \mathrm{val}(x) \geq 0\}$ is the *valuation ring* of K with respect to val. The set $m = \{x \in K | val(x) > 0\}$ is the *maximal ideal of val* and it is a prime ideal of R. The field $k = R/m$ is the *residue field of val.*

We are especially interested in the case where Γ is a discrete subgroup of $\mathbb{R}$, such as $\mathbb{Z}$, and $\geq$ is the natural total order on Γ. In this case the valuation is said to be *discrete*, R is a *discrete valuation ring*, and K is a *discrete valuation field.* Equivalently, a discrete valuation ring may be defined as follows.

Definition 1.2.4. A *discrete valuation ring* is a ring R that is a principal ideal domain with a unique non-zero prime ideal (π) and field of fractions K. The element π is referred to as the *uniformizer* of R, and the field $R/(\pi)$ is the *residue field* of R.

Remark 1.13. The following hold true in discrete valuation rings:

- If $u \in R \setminus (\pi)$, then u is a unit of R.
- Any element of R may be written uniquely as $u\pi^n$ for $n \in \mathbb{Z}_{\geq 0}$ and u a unit.
- If K is the fraction field of R, any element of K may be written uniquely as $u\pi^n$ for $n \in \mathbb{Z}$ and u a unit in R.

- Any non-zero ideal I of R is of the form (π^n) for some $n \in \mathbb{Z}_{\geq 1}$

Let x be an element of K, and write $x = u\pi^n$. Define a function $\mathrm{val}_\pi : K^* \to \mathbb{Z}$ by $\mathrm{val}_\pi(u\pi^n) = n$. This is a surjective homomorphism between K^* and $\mathbb{Z}$ that satisfies $\mathrm{val}_\pi(x+y) \geq \min(\mathrm{val}_\pi(x), \mathrm{val}_\pi(y))$ with equality if $\mathrm{val}_\pi(a) \neq \mathrm{val}_\pi(b)$. We set $\mathrm{val}_\pi(0) = \infty$. It is clear from this definition of val_π that R is a discrete valuation ring as initially defined. Conversely, the valuation ring of a valued field where the valuation is discrete can be shown to be a discrete valuation ring as in Definition 1.2.4.

Example 1.8. *Let $R = \mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at a prime number p. This is a principal ideal domain with unique non-zero prime ideal (p), and any element in R may be written uniquely as up^n for $u = \frac{a}{b} \in \mathbb{Q}$ with $p \not| a, b$.*

The residue field of $\mathbb{Z}_{(p)}$ is $k = \mathbb{F}_p$. The discrete valuation val_p on $R = \mathbb{Q}$ is $val_p(x) = n$ where n is the largest power of p dividing x.

Example 1.9. *Let k be any field, and let $k[[z]]$ be the ring of formal power series over k in parameter z. This is a discrete valuation ring whose maximal ideal is (z) and residue field is k. Given any element $f(z) \in k[[z]]$ we may write $f(z) = u(z)z^n$ where $z \not| u(z)$ and $n \in \mathbb{Z}_{\geq 0}$.*

This example of a discrete valuation ring will arise frequently in this thesis. To distinguish the discrete valuation on such a ring from other valuations, we will denote the discrete valuation ord_z. Thus $ord_z(u(z)z^n) = n$.

1.2.1.1 The topology associated to a discrete valuation

Let R be a discrete valuation ring with uniformizer π. There is a natural topology on K that arises from the discrete valuation val_π. Namely, the basis open sets are defined to be sets of the form $Y_{a,\epsilon} = \{x \in K | val_\pi(x-a) > \epsilon\}$ for a fixed element $a \in K$ and ϵ a real number. One can show that open and closed sets coincide under this topology.

This topology is referred to as the π-*adic topology* on K. The homomorphism $R \times R \to R$ defined by $(x, y) \to x - y$ is continuous with respect to this topology. Such an abelian group G (or a ring R in our case) with a topology for which the map sending $(x, y) \to x - y$ is continuous is referred to as a *topological group*.

In particular, the valuation ring R is the disk $\{x \in K | \mathrm{val}_\pi \geq 0\}$.

It is possible to associate to a discrete valuation a *multiplicative valuation* $|-| : K \to \mathbb{R}_{\geq 0}$ by sending $x \in K$ to $|x| = e^{-\mathrm{val}_\pi(x)}$ where e is a fixed real number greater than 1. In this case, R may be viewed as the set $\{x \in K || x| \leq 1\}$. For this reason R is often referred to as the *unit disk*. Beyond this terminology we will not make use of multiplicative valuations and instead work with discrete valuations as previously defined.

1.2.2 Completions of Discrete Valuation Rings

The local lifting problem, stated in Section 1.5, forces us to consider discrete valuation rings that are *complete* with respect to the induced topology. The structure of complete discrete valuation rings of the type we are interested in, as well as other necessary facts on them, will be covered in this subsection.

Let R be a discrete valuation ring with uniformizer π. Let $(x_n)_n$ be a sequence of elements in K. We say that this sequence converges to a limit $x \in K$ if for every $n \in \mathbb{Z}$ there exists some $m_0 \in \mathbb{Z}$ such that $x - x_m \in (\pi)^n$ whenever $m \geq m_0$. Limits are unique if they exist. We now define what it means for a discrete valuation field K to be complete.

Definition 1.2.5. A sequence $(x_n)_n$ of elements in a discrete valuation field K is said to be *Cauchy* if for all $n \in \mathbb{Z}$ there exists some $m_0 \in \mathbb{Z}$ for which

$$\mathrm{val}_\pi x_m - x_{m_0} \in (\pi^n) \text{ whenever } m \geq m_0.$$

Equivalently, $\mathrm{val}_\pi(x_m - x_{m_0}) \geq n$ whenever $m \geq m_0$. If every Cauchy sequence of elements in K has a limit in K, we say that K is *complete* with respect to the valuation val_π.

Remark 1.14. If K is complete, than any power series in the uniformizer π with coefficients units of R is Cauchy, and so converges to a limit. In fact, the Cohen Structure Theorem 1.16 will show any element of K is of this form.

Definition 1.2.6. Let K be a discrete valuation field. The *completion of* K with respect to val_π is a complete discrete valuation field $\bar{K}$ along with a continuous homomorphism ϕ of K into $\bar{K}$ such that any other homomorphism of K into a complete valued field factors uniquely through ϕ.

Proposition 1.15 (Chapter 2.5 of Cassels and Frölich [1]). *Let K be a discrete valuation field with valuation val_π. Then the completion $\bar{K}$ of K exists, extends the valuation on K, and is unique up to unique isomorphism.*

The construction of the completion of a discrete valuation field K is as follows: Let A be the ring of all Cauchy sequences of elements in K, where addition and multiplication are defined entry-wise. A Cauchy sequence that converges to 0 is said to be a *nullsequence*, and one shows that the set all of nullsequences forms a maximal ideal m of A. Let $\bar{K} = A/m$.

The map $\phi : K \to \bar{K}$ sends $x \in K$ to the constant sequence $(x)_n$, where every entry is x. The valuation on $\bar{K}$ is again denoted by val_π and is defined by $\mathrm{val}_\pi((x_n)_n) = \lim_{n\to\infty}\mathrm{val}_\pi(x_n)$, which exists by the assumption $(x_n)_n$ is Cauchy.

Example 1.10. *Recall that $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at (p), is a discrete valuation ring with uniformizer p. To construct the completion of $\mathbb{Q}$ with respect to the valuation val_p, we let A be the set of all Cauchy sequences, i.e. sequences $(a_n)_n$ in $\mathbb{Q}$ such that for any $n \in \mathbb{Z}$ there exists some m_0 with $p^n|(a_m - a_{m_0})$ whenever $m > m_0$.*

After modding out by the nullsequences, we obtain the field $\mathbb{Q}_p$, which we refer to as the p-adic numbers. Any element of $\mathbb{Q}_p$ may be expressed as

$$\sum_{i=n_0}^{\infty} a_i p^i,$$

where $a_i \in \mathbb{F}_p$, the prime field of characteristic p, and $n_0 \in \mathbb{Z}$. The valuation ring of $\mathbb{Q}_p$ is $\mathbb{Z}_p$, the p-adic integers, with maximal ideal (p). Note that $\mathbb{Z}_p/(p) \cong \mathbb{Z}_{(p)}/(p) = \mathbb{F}_p$.

Example 1.11. *Let k be any field, and let $k[t]$ be the ring of polynomials in variable t over k. Given any maximal ideal $(t-a)$ of $k[t]$, we may localize $k[t]$ at this to get $k[t]_{(t-a)}$. This is a discrete valuation ring with field of fractions $k(t)$ and valuation ord_z, where $ord_z(f(t))$ is the largest power of $(t-a)$ dividing $f(t)$.*

The completion of $k(t)$ with respect to this valuation yields the field $k((z))$ with valuation ring $k[[z]]$, $z = (t-a)$. Thus any element of the completion of $k[t]$ may be expressed as a power series in z over k.

1.2.2.1 The Cohen Structure Theorem

We will state the Cohen Structure Theorem, which will allow us to express any element of a complete discrete valuation field as a power series in the uniformizer π. The version we state assumes that R, the valuation ring, is a discrete valuation ring, but more general versions exist. See, for example, Singh's notes [26].

Theorem 1.16 (The Cohen Structure Theorem)**.** *Let R be a complete discrete valuation ring with uniformizer π and residue field k. Let A be a system of representatives of k in R. Then we may write any element x of R as*

$$x = \sum_{i=0}^{\infty} a_i \pi^i,$$

where $a_i \in A$ for all i. In the case that the characteristic of R and k are the same, then we have $R \cong k[[z]]$.

1.2.2.2 Hensel's Lemma

Hensel's lemma will be important when discussing the construction of Hurwitz trees from semistable models in Chapter 2, especially when discussing the reduction map of a model to its special fiber in Section 2.5.

Theorem 1.17 (Hensel's Lemma, Chapter 2 Theorem 4.6 of Neukirch [20])**.** *Let R be a complete discrete valuation ring with uniformizer π and residue field k. Let $f(z)$ be a polynomial in $R[z]$, and let $\bar{f}(z)$ be the image of $f(z)$ in $k[z] = R[z]/(\pi)$. Assume $\bar{f}(z) \neq 0$. If $\bar{a} \in k$ is a root of $\bar{f}(z)$ of multiplicity 1, then there is a unique root $a \in R$ of $f(z)$ of multiplicity 1 such that $a \pmod{\pi} = \bar{a}$.*

That is, we are able to lift roots of multiplicity 1 of a polynomial in $k[z]$ to roots of a polynomial in $R[z]$.

Remark 1.18. The assumption that R is complete can be relaxed to the condition that R is *Henselian*, which is weaker than complete. Notably, a discrete valuation ring is Henselian if and only if it satisfies Hensel's lemma.

Example 1.12. *Suppose we would like to know whether or not the polynomial z^3+3z-7 is irreducible in $\mathbb{Z}_7$, the 7-adic integers. The image of this polynomial in $\mathbb{F}_7[z]$ is then $z^3+3z = z(z^2+3)$, and by inspection we find that 0, 2, and 5 are roots of this polynomial in $\mathbb{F}_7$ with multiplicity 1. By Hensel's lemma we conclude that z^3+3z-7 splits in $\mathbb{Z}_7[z]$.*

1.2.2.3 The Weierstrass Preparation Theorem

We conclude this subsection by looking at the Weierstrass Preparation Theorem. This theorem will prove key when looking at the lifting problem, as we will very often be working with rings of the form $R[[z]]$ where R is a characteristic 0 complete discrete valuation ring whose residue field is a field k of characteristic $p > 0$. We state it here from Elliot [11].

Theorem 1.19. *Let $f(z) = \sum a_n z^n \in R[[z]]$ be a power series over a complete discrete valuation ring R with uniformizer π. Let N be the number defined by $N = min\{n \geq 0 : val_\pi(a_l) \geq val_\pi(a_n) \forall l\}$. Then there exists a polynomial*

$$g(z) = b_0 + \cdots + b_n z^N \in R[z]$$

and a unit

$$u(z) = 1 + c_1 z + \cdots \in R[[z]]$$

such that $f(z) = a_N g(z) u(z)$. *The zeros of* $f(z)$ *are exactly those of* $g(z)$.

The polynomial $g(z)$ in the Weierstrass Preparation Theorem is the *Weierstrass polynomial* of $f(z)$.

Example 1.13. *Let* R *be any complete discrete valuation ring of characteristic* 0. *Let* σ *be a finite order* R*-linear automorphism of* $R[[z]]$ *- that is,* σ *is an invertible* R*-linear homomorphism from* $R[[z]]$ *to itself. Then* $\sigma z \in R[[z]]$, *and we may write*

$$\sigma z - z = a_{m_\sigma} f_{m_\sigma}(z) u(z),$$

where $m_\sigma = min\{n \geq 0 : val_\pi(a_l) \geq val_\pi(a_n) \forall l\}$ *and* f_{m_σ} *is the Weierstrass polynomial of degree* m_σ.

1.2.3 Galois extensions of discrete valuation fields

The goal of this subsection is to discuss ramification in extensions of discrete valuation fields. First we recall the definition of a Dedekind domain.

Definition 1.2.7. Let R be a Noetherian integral domain. Then R is said to be a *Dedekind domain* if for every non-zero prime ideal p of R, R_p is a discrete valuation ring.

Example 1.14. *We claim that* $\mathbb{Z}$ *is a Dedekind domain. Indeed, the non-zero prime ideals are of the form* (p) *for a prime number* p *by the fact* $\mathbb{Z}$ *is a unique factorization domain. From the previous subsections we know that* $\mathbb{Z}_{(p)}$ *is a discrete valuation ring, and so it follows that* $\mathbb{Z}$ *is a Dedekind domain.*

Dedekind domains may not have unique factorization of their elements, however, the ideals of a Dedekind domain do factor uniquely into prime ideals.

Proposition 1.20 (Chapter 1 Proposition 2.2 of Cassels and Frölich [1])**.** *Let* R *be a Dedekind domain. Then if* I *is any ideal of* R, I *factors uniquely as* $\beta_1^{n_1} \ldots \beta_r^{n_r}$ *where the* β_k*'s are prime ideals and* $n_i \geq 1$.

When looking at finite field extensions, we will be interested in the integral closure of rings. We define this here.

Definition 1.2.8. Let $R \subset K$ be a subring of a field K. An element $a \in K$ is said to be *integral over* R if a is the root of a monic polynomial $f(z) \in R[z]$, where a monic polynomial is one of the form $x^n + x_{n-1}x^{n_1} + \ldots + a_0$. If $R \subset A$ is a subring of a ring A, then A is said to be *an integral extension of* R is every element of A is integral over R.

Remark 1.21. Elements being integral over a ring R is a property preserved under addition and multiplication. Thus if $R \subset K$ and L/K is a field extension, than the *integral closure* of R in L is the largest subring of L integral over R.

Example 1.15. *Let $K = \mathbb{Q}$ with subring $\mathbb{Z}$, and let $L = \mathbb{Q}(i)$ be the Gaussian numbers. Let $\mathbb{Z}[i]$ be the Gaussian integers, which is the set $\{x \in \mathbb{C} | x = a + bi, a, b \in \mathbb{Z}\}$. Note that i is integral over $\mathbb{Z}$ with minimal polynomial $f(z) = z^2 - 1$. Thus, by the above remark, $\mathbb{Z}[i]$ is integral over $\mathbb{Z}$, and in fact one may show that it is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$. A subring R in a field K is said to be integrally closed in K if whenever $x \in K$ is integral over R, we have $x \in R$.*

Let R be a discrete valuation ring with fraction field K and uniformizer π. Let L/K be a finite Galois field extension. Then the integral closure A of R in L is a finitely generated R module, and furthermore A is a Dedekind domain.

Definition 1.2.9. Let $\beta \subset A$ be a non-zero prime ideal such that $\beta \cap R = (\pi)$. Then we will say that β *divides* π. Equivalently, $(\pi)A \subset \beta$.

Because A is a Dedekind domain, $(\pi)A$ factors uniquely into a product of ideals. Let e_β be the exponent of β in this decomposition. It is a positive integer referred to as the *ramification exponent* of β in L/K. Thus,

$$(\pi)A = \Pi_{\beta|(\pi)} \beta^{e_\beta}.$$

Finally, note that $R/(\pi) \subset A/\beta$ is a finite field extension. It's degree f_β is the *residue field degree* of β over (π).

Example 1.16. *We return to looking at $\mathbb{Q}(i)/\mathbb{Q}$. In $\mathbb{Q}$ we will consider the subring $\mathbb{Q}_{(2)}$. Note that $2 = (1+i)(1-i)$ in $\mathbb{Z}[i]$. In fact, $(1+i)$ and $(1-i)$ are the same prime ideals in $\mathbb{Z}[i]$. The integral closure of $\mathbb{Z}_{(2)}$ in $\mathbb{Q}(i)$ can be shown to be $\mathbb{Z}[i]_{(1+i)}$, and by definition of the ramification index we find $e_{(1+i)} = 2$. Similarly, the residue fields are found to be the same, and so $f_{(1+i)} = 1$.*

The following result places a strong constraint on the residue field degrees and ramification exponents.

Proposition 1.22 (Chapter 2 Proposition 10.1 of Cassels and Frölich [1])**.** *Let L/K be a Galois field extension of degree n, where K is a discrete valuation field with valuation ring R and uniformizer π. Then*

$$n = \sum_{\beta|(\pi)} e_\beta f_\beta.$$

In particular there are a finite number of ideals that divide (π).

An extension as in the proposition is said to be *totally ramified* if there is a prime ideal $\beta|(\pi)$ of A such that $e_\beta = n$. Thus, the above proposition ensures us that β is the only prime dividing (π) and that the residue fields are the same.

An extension as in the proposition is said to be *unramified* if $e_\beta = 1$ for all $\beta|(\pi)$ and all the residue field extensions are separable.

1.2.3.1 Ramification over complete discrete valuation rings

We restrict our attention to *complete* discrete valuation rings R. If L/K is a finite Galois field extension as before, the following theorem ensures us there is only one prime $\beta|(\pi)$.

Theorem 1.23. *[Chapter 2 Theorem 4.8 of Neukirch [20]] Let K be a field complete with respect to a discrete valuation val_π with valuation ring R. Let L/K be a finite Galois field extension. Then the integral closure of R in L, denoted by A, is again a discrete valuation ring, and L is complete with respect to the topology defined by A.*

Definition 1.2.10. Let R be a discrete valuation ring with maximal ideal (π), and suppose R is contained in a discrete valuation ring A with maximal ideal m. We say that A *dominates* R if $m \cap R = (\pi)$.

Remark 1.24. Thus in the situation of Theorem 1.23 we see that A dominates R.

Let L/K be a finite Galois extension. We will assume henceforth that the residue field extension is separable. There exists an element $x \in A$ such that x generates A as an R-algebra. Let val_π be the valuation on K, and val_L the valuation on L. As we only have one prime above (π), let e and f denote the ramification index and residue field degree, respectively. We will define the ramification groups as in Chapter IV of Serre [23].

Definition 1.2.11. Let $i_G : \mathrm{Gal}(L/K) \to A$ be the function defined by

$$i_G(\sigma) = \mathrm{val}_L(\sigma(x) - x).$$

We define the *ramification groups* G_i of $\mathrm{Gal}(L/K)$ by

$$i_G(\sigma) \geq i+1 \iff \sigma \in G_i.$$

The first ramification group is called the *inertia group.* It is the largest subgroup of $\mathrm{Gal}(L/K)$ that acts trivially on the residue field of L.

One may show that i_G satisfies $i_G(\tau\sigma\tau^{-1}) = i_G(\sigma)$, so that it is constant on conjugacy classes of G. From this we may define a character on G.

Definition 1.2.12. The *Artin character* of G is the character a_G of G defined by

$$a_G(\sigma) = -f i_G(\sigma)$$

whenever $\sigma \neq 1$, and $a_G(1) = -\sum_{\sigma\neq 1} a_G(\sigma)$ otherwise.

That this is a character is shown in Serre [23]. This character will be of importance in discussing the lifting problem.

Consider the following case of field extensions: R will be equal to $k[[z]]$, where k is a field of characteristic $p > 0$. The valuation on R is then ord_z. Let G be a finite group of k-linear automorphisms. The fixed field of $k((z))$ under the action of G is itself a discrete valuation field, and as G is k-linear the residue fields of both discrete valuation rings are k. Thus $k((z))/k((z))^G$ is a totally ramified extension, and we have the following theorem.

Theorem 1.25. *G is isomorphic to the semidirect product of a p-group P and a cyclic group C of order prime to p, i.e. $G \cong P \rtimes C$.*

Groups as in the above theorem are referred to as *cyclic-by-p* groups.

Proof. Our proof will follow that in Chapter IV of Serre [23]. Denote by U^0 the group of invertible elements of $k[[z]]$, and for $i \geq 1$, $U^i = 1 + (z^i) = \{a \in k[[z]] | a = 1 + z^i b, b \in k[[z]]\}$. It is straightforward to show that

$$U^0/U^1 \cong k^* \text{ and } U^i/U^{i+1} \cong k,$$

where $i \geq 1$, and where k^* and k refer to the multiplicative and additive groups of the residue field, respectively.

Define a map from G_i/G_{i+1} to U^i/U^{i+1} that takes $\sigma \in G_i$ to $\sigma(z)/z$. This map defines an isomorphism of G_i/G_{i+1} onto a subgroup of U^i/U^{i+1}, and thus G_0/G_1 is a finite subgroup of the roots of unity of k and so is a cyclic group of order prime to p.

When $i \geq 1$, G_i/G_{i+1} is a subgroup of k and so is a vector space over $\mathbb{F}_p$, whence it is a direct sum of cyclic groups of order p. It follows that G_1 is a p-group.

To finish the proof we will show that there is a subgroup H of G_0 that maps isomorphically onto G_0/G_1.

Let $\sigma \in G$ map to a generator of G_0/G_1, and let N be an integer so that $p^N \equiv 1 \pmod{|G_0/G_1|}$, which is possible as $(|G_0/G_1|, p) = 1$. We may assume $p^N > |G_1|$. Let $h = \sigma^{p^N}$. Then h has order dividing $|G_0/G_1|$ by construction, but also maps to a generator of $|G_0/G_1|$ and so $(h) = G_0/G_1$. This completes the proof. □

1.2.3.2 Witt vectors

Let k be a perfect field of characteristic $p > 0$. We would like to find a complete discrete valuation ring R of characteristic 0 with uniformizer π whose residue field is k.

Suppose such an R exists. Let val_π be the valuation on R. Let $e = \mathrm{val}_\pi(p)$. As $p \pmod{\pi} = 0$, e is a positive integer, and we call it the *absolute ramification index* of R. R is said to be *absolutely unramified* if $e = 1$, whence p may be taken as its parameter. The following theorem tells us such an absolutely unramified ring exists for the field k.

Theorem 1.26 (Chapter III Theorem 3 of Serre [23])**.** *If k is a perfect field of characteristic $p > 0$, there exists an absolutely unramified complete discrete valuation ring $W(k)$ of characteristic 0 with residue field k that is unique up to unique isomorphism.*

This ring is called the *Witt vectors of k*. Suppose R is some other complete discrete valuation ring with residue field k and absolute ramification index e. Then there exists a unique homomorphism of $W(k)$ into R that commutes with the quotient maps to k, and such that R is a free $W(k)$-moduloe of rank e.

Example 1.17. *Let $k = \mathbb{F}_p$. Then $W(k) = \mathbb{Z}_p$, the ring of p-adic integers. Indeed, $\mathbb{Z}_p$ is defined as the completion of the $\mathbb{Z}$ with respect to $val_{(p)}$, and so it is absolutely unramified with residue field equal to $\mathbb{F}_p$.*

1.3 The rudiments of algebraic geometry

This section covers elementary facts from algebraic geometry. We will define *schemes* and in particular *algebraic curves*, which are the focus of the global lifting problem. The definitions and constructions in this section closely follow those in Chapter 2 of Liu's book *Algebraic Geometry and Arithmetic Curves* [19].

Throughout this section, let R be a commutative ring with unity. Though the constructions in this section do not require R to be Noetherian unless stated explicitly, all of the rings considered in this thesis are Noetherian.

Subsection 1.3.1 defines the spectrum of a ring, as well as the underlying topology of this set. The local lifting problem deals with the spectra of affine schemes, and so the definitions here will be seen frequently throughout the thesis.

Subsection 1.3.2 aims to provide an accessible definition of schemes. This first involves defining sheaves and locally ringed spaces in 1.3.2.1, and the definition of schemes follows in 1.3.2.2.

Finally, in Subsection 1.3.3, we look at various conditions on schemes that arise throughout the thesis. Many of these conditions, such as regular, are defined. These are important as they allow us to provide explicit descriptions of the completions of the local rings of smooth curves, which arise in the Local to Global Principle (Theorem 1.39) in Section 1.5. Other, more technical conditions such as mentioned, with the key point being that all the schemes considered in the local lifting problem will satisfy these conditions.

1.3.1 Spec of a ring

We begin by defining the underlying set of an affine scheme.

Definition 1.3.1. Let R be a ring. We define the spectrum of R, $\mathrm{Spec}(R)$, to be the set of all prime ideals.

This is made into a topological space as follows: For any ideal I of R, let $V(I) = \{p \in \mathrm{Spec}(R) | I \subset p\}$. For any element $f \in R$, let $D(f) = \mathrm{Spec}(R) \setminus V((f))$. Then the *Zariski topology* on $\mathrm{Spec}(R)$ is the topology whose closed sets are sets of the form $V(I)$ for an ideal I, and the sets $D(f)$ constitute a basis of open sets.

Example 1.18. *Let K be a field. Then $\mathrm{Spec}(K)$ consists of one point corresponding to the ideal (0), which is both open and closed in the Zariski topology.*

Example 1.19. *Let $R = \mathbb{Z}$. Then Spec(R) consists of all prime ideals (p) for p a prime number, as well as the zero ideal. The closed points are of the form (p), and basic open sets are of the form Spec$(R) \setminus \{(p_1), \ldots, (p_k)\}$ for a finite number of primes $p_1, \ldots, p_k$.*

Let $\phi^{\#} : A \to R$ be a ring homomorphism. Let $p \in$ *Spec*(R) be a prime ideal, and note that $(\phi^{\#})^{-1}(p) = \{x \in A | \phi(x) \in p\}$ is a prime ideal of A. Let $\phi : \text{Spec}(R) \to \text{Spec}(A)$ be the map defined by $\phi(p) = (\phi^{\#})^{-1}(p)$. This will be important for Lemma 1.30.

1.3.2 The definition of schemes

1.3.2.1 Sheaves

Sheaves are a fundamental notion in algebraic geometry, and are key to understanding the local structure of algebraic curves. We start by defining *presheaves.*

Definition 1.3.2. Let Y be a topological space. A *presheaf of abelian groups on Y*, denoted by $\mathcal{O}_Y$, consists of the following data:

- For every open set $U \subset Y$, an abelian group $\mathcal{O}_Y(U)$;
- For any pair of open sets $U \subset V$, a restriction homomorphism of abelian groups $\phi_{VU} : \mathcal{O}_Y(V) \to \mathcal{O}_Y(U)$.

Furthermore, this data must satisfy

- $\mathcal{O}_Y(\emptyset) = 0$;
- $\phi_{UU} = id_U$ for any open set $U \subset Y$;
- For open sets $U \subset V \subset W$, we have $\phi_{WU} = \phi_{VU} \circ \phi_{WV}$.

Remark 1.27. More generally, one may define presheafs of rings or other algebraic objects.

Definition 1.3.3. Let $\mathcal{O}_Y$ be a presheaf on Y. An element $s \in \mathcal{O}_Y(U)$ is *a section of* $\mathcal{O}_Y$ over U.

If $U \subset V$ and $s \in \mathcal{O}_Y(V)$, the image of s under ϕ_{VU} is the *restriction of s to U*, denoted $s|_U$

Definition 1.3.4. Let O_Y be a presheaf on Y. We say that O_Y is a *sheaf* if it satisfies

- If $U \subset Y$ is open and has a covering by open sets $\{U_i\}_i$, and if $s \in \mathcal{O}_Y(U)$ satisfies $s|_{U_i} = 0$ for all i, then $s = 0$.
- Suppose for every U_i in an open cover of U, we have a section $s_i \in \mathcal{O}_Y(U_i)$, and these satisfy $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there exists a section $s \in \mathcal{O}_Y(U)$ such that $s|_{U_i} = s_i$ for all i.

Example 1.20. *Let $Y = \mathbb{R}^2$ be the real plane with the standard metric topology. Define $\mathcal{O}_Y$ by setting $\mathcal{O}_Y(U)$ to be the set of all continuous functions on U from $\mathbb{R}^2$ to $\mathbb{R}$. The restriction maps are the usual restriction of functions on the plane. One verifies that this is a presheaf. If f is a function on an open set U of the plane that is the 0 function on an open cover, then clearly f is itself 0.*

Let U_i be an open cover of U, and let f_i be continuous functions on the U_i that agree on the overlaps. Define f as a function on U by setting $f(x) = f_i(x)$ if $x \in U_i$. This is well-defined by assumption, and as f is locally continuous it is continuous. Thus $\mathcal{O}_Y$ is a sheaf on Y.

Example 1.21. *Let Y be any topological space, and let $p \in Y$. Define the tower sheaf at p by setting $\mathcal{O}_Y(U) = \mathbb{R}$ is $p \in \mathbb{R}$ and 0 if $p \notin U$. The restriction map from V to U is the identity on $\mathbb{R}$ if both sets contain p, and is the 0 map otherwise. This defines a presheaf on Y.*

By definition of the restriction maps, if a section $s \in \mathcal{O}_Y$ is locally 0 then it must be 0. Suppose we have sections s_i on open sets U_i that agree on overlaps. If $U = \cup_i U_i$ doesn't contain p then set $s = 0$. Otherwise there is some i for which $p \in U_i$. Set $s = s_i$. By assumption $s|_{U_j} = s_j$ for all j, and we conclude that this defines a sheaf.

Definition 1.3.5. Let $\mathcal{O}_Y$ be a sheaf on Y, and let $p \in Y$. We define the *stalk* of $\mathcal{O}_Y$ at p, $\mathcal{O}_{Y,p}$, to be the set of all pairs (U, s), where $p \in U$ for U an open set, $s \in \mathcal{O}_Y(U)$ is a section of U, modulo the equivalence relation defined by $(U, s) = (V, t)$ if there exists some open set $W \subset U \cap V$ such that $s|_W = t|_W$.

Example 1.22. *Let $\mathcal{O}_{Y,p}$ be the tower sheaf at p on a Hausdorff space Y. Then the stalk at p, again denoted $\mathcal{O}_{Y,p}$, is just $\mathbb{R}$. If $x \neq p$ is any point, the stalk $\mathcal{O}_{Y,x}$ is 0.*

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two sheaves on Y. A *morphism of sheaves* $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a group homomorphism $f_U : \mathcal{O}_1(U) \to \mathcal{O}_2(U)$ for every open set U that is compatible with the restriction maps. It is an *isomorphism* if it has an inverse morphism.

Let $f : X \to Y$ be a continuous map of topological spaces, and suppose that we have a sheaf $\mathcal{O}_X$ on X. We may define a sheaf $f_*\mathcal{O}_X$ on Y, called the *direct image of* $\mathcal{O}_X$, by setting $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$ for any open set $U \subset Y$.

We may now define locally ringed spaces.

Definition 1.3.6. A *locally ringed space* $\mathcal{O}_Y$ on a topological space Y is a sheaf of rings on Y such that for every $p \in Y$, the stalk $\mathcal{O}_{Y,p}$ is a local ring. The *residue field of* $\mathcal{O}_Y$ at p is the field $\mathcal{O}_{Y,p}/m_p$, where m_p is the maximal ideal of $\mathcal{O}_{Y,p}$.

Definition 1.3.7. Let $(Y, \mathcal{O}_Y)$ be a locally ringed space. A *sheaf of modules* over Y is a sheaf $\mathcal{M}$ such that for every open set $U \subset Y$, $\mathcal{M}(U)$ is a module over $\mathcal{O}_Y(U)$.

Example 1.23. *Let $Y = Spec(\mathbb{Z})$. Let $\mathcal{O}_Y$ be the sheaf defined on the basis sets $D(f)$ by setting $\mathcal{O}_Y(D(f)) = \mathbb{Z}_f$, i.e. the localization of $\mathbb{Z}$ at f. One extends this to a sheaf on Y as follows. For any open set U, let $(U_i)_i$ be an open cover of U by basic open sets. Then $\mathcal{O}_Y(U) = \{(s_i)_i | s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}\}$.*

The stalk at a point $(p) \in Spec(\mathbb{Z})$ is $\mathbb{Z}_{(p)}$ whenever $p \neq 0$, and is $\mathbb{Q}$ otherwise. Thus the stalk is always a local ring, and in fact is a discrete valuation ring for any closed point (p). We conclude that $(Spec(\mathbb{Z}), \mathcal{O}_{Spec(\mathbb{Z})})$ is a locally ringed space.

Remark 1.28. The extension of a sheaf defined on a basis of open sets to all open sets in the above example works more generally for any topological space Y.

Definition 1.3.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two locally ringed spaces. A morphism of ringed spaces is a pair $(f, f^\#)$ such that $f : X \to Y$ is a continuous map, and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves that satisfies for every $x \in X$, the induced homomorphism $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is such that $(f_x^\#)^{-1}(m_x) = m_{f(x)}$.

1.3.2.2 Schemes

We now define *affine schemes.*

Let R be any ring, and let $\mathrm{Spec}(R)$ be the set of prime ideals of R with the Zariski topology. For any basic open set $D(f)$, $f \in R$, set $\mathcal{O}_{\mathrm{Spec}(R)} = R_f$, the localization of R at f. This defines a sheaf on the basic open sets of $\mathrm{Spec}(R)$, and one extends this to a sheaf on $\mathrm{Spec}(R)$ as in Example 1.23 by setting $\mathcal{O}_{\mathrm{Spec}(R)} = \{(s_i)_i | s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}\}$ for an open cover $(U_i)_i$ of U. This is referred to as the *structure sheaf* on $\mathrm{Spec}(R)$, and the topological space with its structure sheaf is in fact a ringed topological space.

Definition 1.3.9. An *affine scheme* is a ringed topological space of the form $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$. A *scheme* is a ringed topological space $(Y, \mathcal{O}_Y)$ with a cover by open sets U_i such that each $(U_i, \mathcal{O}_Y|_{U_i})$ is an affine scheme. Each U_i is an *affine open set.*

For use in Chapter 3 we also define line bundles over schemes.

Definition 1.3.10. Let Y be a scheme. A line bundle over Y is a sheaf of modules $\mathcal{L}$ over Y such that there exists a cover of open sets $(U_i)_i$ of Y satisfying $\mathcal{L}(U_i)$ is a rank 1 module over $\mathcal{O}_Y(U_i)$ for all i.

Proposition 1.29 (Chapter 2 Proposition 3.1 of Liu [19])**.** *Let Spec(R) be any affine scheme, and let p be a prime ideal of R. The stalk $\mathcal{O}_{Spec(R),p}$ of the structure sheaf at p is R_p, the localization of R at p.*

Example 1.24. *Let $R = \mathbb{C}[x]$ be the ring of polynomials with coefficients in $\mathbb{C}$. The non-zero prime ideals of this ring are of the form $(x - a)$ for some $a \in \mathbb{C}$. Let $\mathcal{O}$ denote the structure sheaf on Spec(R). For any non-zero prime $(x - a)$, note that the stalk at the corresponding point in Spec(R) is $\mathbb{C}[x]_{(x-a)}$, which is a discrete valuation ring. Then (Spec$(\mathbb{C}[x]), \mathcal{O})$ is an affine scheme, and we will see that it is an example of an affine curve.*

As $\mathbb{C}[x]_{(x-a)}$ is a discrete valuation ring, the completion of this ring with respect to the discrete valuation is a power series ring of the form $\mathbb{C}[[z]]$ as in Example 1.11.

More generally, Spec$(k[x])$ for any field k is referred to as the affine line over k.

Definition 1.3.11. A *morphism* of schemes is a morphism of the underlying ringed spaces.

Recall from the discussion at the end of Subsection 1.3.1 that to a ring homomorphism we may associate a continuous map between the spectra of the rings. In fact, the map between spectra is a morphism of affine schemes.

Lemma 1.30 (Chapter 2 Proposition 3.25 of Liu [19])**.** *Let X = Spec(A) and Y = Spec(R) be two affine schemes. Then there exists a one-to-one correspondence between morphisms $\phi : X \to Y$ and ring homomorphisms $\phi^{\#} : R \to A$.*

Let Y be a scheme defined over a ring R, and let X be some other scheme defined over R. Then there exists a scheme $Y \times_R X$ over both X and Y called the *base-change* of Y by X.

Let $\phi : Y \to \text{Spec}(R)$ be the morphism from Y to Spec(R). Given any point $p \in \text{Spec}(R)$, let k_p denote the residue field of the local ring at p. The scheme $Y \times_R k_p$ is called the *fiber of ϕ over p*. This is a scheme over k_p, and the points lying in this space are in one-to-one correspondance with the set $\phi^{-1}(p) \subset Y$.

1.3.3 Some technical conditions on schemes

There are numerous topological or algebraic properties that schemes may satisfy. In this subsection we will look at the ones that are encountered in the local lifting problem. These definitions may be found in Liu [19] in their full generality.

Definition 1.3.12. A scheme Y is *connected* if its underlying topological space is connected. That is, Y cannot be written as the union of two nonempty disjoint open sets. It is *irreducible* if it is nonempty and cannot be written as the union of two proper closed subsets. The maximal irreducible subsets of Y are the *irreducible components* (or simply *components*).

Definition 1.3.13. A scheme Y is *Noetherian* if there is a finite cover of Y by affine open subsets that are the spectra of Noetherian rings. A Noetherian scheme Y is said to be *integral* if it is nonempty, and every affine open subset is the spectrum of an integral domain.

The point corresponding to the zero-ideal in an integral scheme is referred to as the *generic point.*

Remark 1.31. Every scheme considered in this thesis is Noetherian. Integrality can also be defined for non-Noetherian schemes, but we won't consider this.

Example 1.25. *The affine line over a field k, Spec$(k[x])$, is connected, irreducible, and integral.*

Example 1.26. *Let $Y = Spec(\mathbb{R}[x,y]/((x+y)(x-y)))$. This is an affine scheme with two irreducible components corresponding to the minimal prime ideals $(x+y)$ and $(x-y)$. Geometrically, these correspond to the lines $y=-x$ and $y=x$ in $\mathbb{R}^2$. Note that Y is connected, but not integral.*

Definition 1.3.14. A scheme Y is said to *lie over* a ring R if there is a morphism of schemes from Y to Spec(R) (integral closure was defined in Subsection 1.2.3. It is *of finite type over* R if there is a cover of affine open sets $U_i = \text{Spec}(X_i)$ of Y such that each X_i is a finitely generated R-algebra.

Recall that the *Krull dimension* of a ring R is the supremum of the lengths of chains of prime ideals in R, where a chain $p_0 \subsetneq p_1 \subsetneq \ldots \subsetneq p_n$ has length n. A local ring R is said to be *regular* if the number of generators of its maximal ideal equals the Krull dimension of R.

Definition 1.3.15. A scheme Y is said to be *normal* if the stalk at every point is an integrally closed ring. A scheme Y is *regular* or *non-singular* at a point p if the local ring at that point is a regular ring. Y is *regular* if it is regular at every point.

Remark 1.32. Regularity implies normality.

Definition 1.3.16. Let Y be a scheme. The *dimension* of Y is the supremum of the lengths of chains of irreducible closed subsets. Y has pure dimension n if all of its components have the same dimension n. A *curve* is a scheme of pure dimension 1, and a surface is a scheme of pure dimension 2. The *codimension* of an irreducible subset $X \subset Y$ is the largest chain of increasing irreducible closed subsets in Y beginning with $\bar{X}$.

Remark 1.33. Let R be a ring. Then one can show that the Krull dimension of R is equal to the dimension of $\mathrm{Spec}(R)$.

Definition 1.3.17. Let Y be a Noetherian scheme of pure dimension n. The free abelian group on the irreducible components of Y of codimension 1 is the *group of Weil divisors of* Y, denoted by $\mathrm{Div}(Y)$.

Example 1.27. *Let k be any field, and let $k[x]$ be the polynomial ring in variable x over k. Then Spec($k[x]$) is a regular curve over k. That its dimension is* 1 *follows from the previous remark. It is clearly regular at the point corresponding to the zero ideal, and we see it also regular at the other prime ideals. Indeed, we know from Example 1.24 that the stalk at these points are discrete valuation rings, which are regular.*

Remark 1.34. Let Y be an irreducible regular curve over a field k. Let p be a point on the curve other than the generic point. Then we know that the stalk $\mathcal{O}_{Y,p}$ is a discrete valuation ring. By the Cohen Structure Theorem (1.16), the completion of this ring $\hat{\mathcal{O}}_{Y,p}$ is a ring of the form $k[[z]]$, a power series ring in the parameter z over k.

There are five other conditions on schemes that arise in this thesis that are more technical. Namely, a scheme may be proper, projective, smooth, flat, and excellent.

A scheme being *proper* over a ring R is analogous to the scheme being compact. Projective schemes over a ring R are isomorphic to closed subsets of certain schemes over R called *projective n-space over R* for positive integers $n \in \mathbb{Z}$. When R is Noetherian as in this thesis, projective implies proper.

We care about smoothness as if a scheme over a field k is smooth, then it is regular and hence normal.

Flat is only of relevance when defining arithmetic surfaces in Chapter 2. The surfaces considered in this thesis are of the form $\mathrm{Spec}(R[[z]])$ for R a complete discrete valuation ring, and it is known that these rings are flat over R.

Excellent is a technical condition that says every affine open set of a scheme is the spectrum of an excellent ring. We will not define excellent rings here, however, we note that the following rings are excellent: Complete Noetherian local rings; Dedekind domains of characteristic 0; all localizations of excellent rings; and all finitely generated algebras over excellent rings are excellent. In particular, as the local lifting problem deals with complete Noetherian local rings and Dedekind domains of characteristic 0, the assumption that rings are excellent is quite reasonable. Virtually all rings considered algebraic geometry and algebraic number theory are excellent.

1.4 Automorphisms of schemes

Having defined schemes in Section 1.3.2, we now turn out attention to *automorphisms* of these schemes.

Definition 1.4.1. Let Y be a scheme defined over a ring R with morphism $\rho : Y \to \operatorname{Spec}(R)$. An *$R$-automorphism* of a scheme is an isomorphism of schemes σ from Y to itself such $\rho = \rho \circ \sigma$.

Remark 1.35. The set of R-automorphisms of a scheme Y form a group under composition, called the R-automorphism group of Y and denoted by $\operatorname{Aut}_R(Y)$.

Example 1.28. *We look at a concrete example of automorphisms, viz. the Möbius transformations of $\mathbb{CP}^1$. Recall that we may identify $\mathbb{CP}^1$ with $\mathbb{C} \cup \{\infty\}$. Let a, b, c, d be complex numbers such that $ad - bc \neq 0$. Then we define the function*

$$f : \mathbb{CP}^1 \to \mathbb{CP}^1 \text{ by } f(z) = \frac{az + b}{cz + d}.$$

By identifying this function with the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

one can readily verify that the inverse matrix, which exists by the assumption $ad-bc \neq 0$, gives the inverse morphism. Thus we have an entire family of automorphisms, where composition is given by multiplication of the respective matrices.

This example shows that there is no limit on the size of the automorphism group of a general scheme, however, we will restrict our attention to finite subgroups of the automorphism group.

The *local* lifting problem deals with the stalks at closed points of a projective scheme.

Definition 1.4.2. Let Y be a scheme over a ring R and G a finite subgroup of $\operatorname{Aut}_R(Y)$. A point $x \in Y$ is a *fixed point* of an automorphism $\sigma \in \operatorname{Aut}_R(Y)$ if $\sigma(x) = x$. The *stabilizer subgroup* of x is the group $G_x \subset G$ such that $\sigma(x) = x$ for all $\sigma \in G_x$.

By Remark 1.34, the completion of the local ring at a closed point x of a regular curve Y over a field k is of the form $k[[z]]$ for some parameter z. If G_x is the stabilizer of x, then G_x also acts on $k[[z]]$. By Lemma 1.30, k-linear automorphisms of $k[[z]]$ are in one-to-one correspondance with R-automorphisms of $\operatorname{Spec}(k[[z]])$.

Let R be a discrete valuation ring of characteristic 0 with uniformizer π and residue field k of characteristic $p > 0$. We will assume R is complete with respect to the associated valuation. Let $G \subset \mathrm{Aut}_k k[[z]]$ be a finite group of k-linear automorphisms.

Note that, by passage to the quotient, we have a map

$$\Psi : G \to \mathrm{Aut}_k k[[z]]$$

that may not be injective. In the local lifting problem we are interested in the case where the map Ψ is injective, and so we will restrict our attention to cyclic-by-p automorphism groups of $R[[z]]$ by Theorem 1.25.

Example 1.29. *Throughout this thesis we will consider the automorphism of* $\mathbb{F}_3[[z]]$ *defined by*

$$\sigma(z) = \frac{z}{1+z} = z(\sum_{i=0}^{\infty}(-1)^i z^i).$$

Notably, this automorphism has order 3*, as one can check that* $\sigma^n(z) = \frac{z}{1+nz}$ *for any nonnegative integer* n*. Thus* $G = \langle\sigma\rangle$ *is a finite automorphism group of* $\mathbb{F}_3[[z]]$*, and it is of the form given in Theorem 1.25.*

1.4.1 Fixed points of automorphisms

Let k be a perfect field of characteristic $p > 0$.

Definition 1.4.2 introduced the notion of a point x on a regular curve Y over k being fixed by a k-automorphism of Y, and we saw that the stabilizer subgroup of x acts on the completion $k[[z]]$ of the local ring $\mathcal{O}_{Y,x}$. Here we will look at fixed points of automorphisms of $\mathrm{Spec}(R[[z]])$, where R is a complete discrete valuation ring dominating the Witt vectors $W(k)$ of k. Witt vectors were defined in Subsection 1.2.3.2.

let σ be an R-linear automorphism of $R[[z]]$. Then by Lemma 1.30, σ corresponds uniquely to an R-automorphism of $\mathrm{Spec}(R[[z]])$, which we also denote by σ. We write the action of σ on z as $\sigma(z) = \sum_{i=0}^{\infty} a_i z^i$, where $a_i \in R$.

Definition 1.4.3. An element $\alpha \in R$ is *fixed by* σ if the prime $(z - \alpha)$ is sent to itself under the action of the automorphism determined by σ.

Remark 1.36. This set of fixed points of R under σ will be denoted by Δ_σ.

Let us make this more explicit. By the Weierstrass Preparation Theorem (Theorem 1.19), write $\sigma(z) - z = a_{m_\sigma} f_{m_\sigma}(z)u(z)$ where $m_\sigma = \min\{n \geq 0 : \text{val}_\pi(a_l) \geq \text{val}_\pi(a_n) \forall l\}$, f_{m_σ} is a Weierstrass polynomial of degree m_σ, and $u(z)$ is a unit. The zeros of this polynomial correspond to the fixed points of σ, and in particular we have that $|\Delta_\sigma| = m_\sigma$.

Fix σ and suppose that it is of finite order n. To obtain a somewhat more explicit form of σ we note the following lemma:

Lemma 1.37. *Suppose* $\sigma(z) \equiv z \pmod{z^2}$*. Then* $\sigma = id$*.*

Proof. We follow Coleman as in [9]. Indeed, suppose otherwise, so that

$$\sigma(z) \equiv z + cz^m \pmod{z^{m+1}},$$

where $c \neq 0$ and $m \geq 2$. Then we have

$$\sigma^n(z) \equiv z + ncz^m \pmod{z^{m+1}}.$$

As n was the order of σ we must have that $nc = 0$, however, as the characteristic of R is 0, n is not a zero divisor and so we conclude that $c = 0$, a contradiction. Thus $\sigma = id$.

□

As a corollary of this lemma we have the following result:

Corollary 1.38. *Let* $\sigma \in Aut_R R[[z]]$ *have finite order* n *and suppose that its image in* $Aut_k k[[z]]$ *is non-trivial. Then we may express its action as*

$$\sigma(z) = \zeta z \left(1 + \sum_{i=1}^{\infty} a_i z^i\right),$$

where ζ *is a primitive* n*-th root of unity.*

This result will be crucial when discussing the leaves of Hurwitz trees in Chapter 2.

1.5 The Lifting Problem

We now introduce the lifting problem.

Let k be an algebraically closed field of characteristic $p > 0$ and let $W(k)$ be the ring of Witt-vectors over k. Let R be a finite, totally ramified extension of $W(k)$ and $K = \text{Frac}(R)$. Suppose we are given a smooth, proper curve C defined over k and a finite automorphism group G of C.

Definition 1.5.1. The pair (C, G) is said to lift to characteristic 0 if there exists a discrete valuation ring R exists that dominates $W(k)$, a smooth projective R-curve $\mathcal{C}$ and an R-linear action of G on $\mathcal{C}$ satisfying

1. $\mathcal{C}\otimes_R$k$\cong C$

2. The G action on $\mathcal{C}$ restricts to the G action on C.

The base change $\mathcal{C} \otimes_R k$ of $\mathcal{C}$ by k was defined in Subsection 1.3.2.2.

The *global lifting problem* in algebraic geometry asks under what circumstances such a pair (C, G) lifts to characteristic 0. A comprehensive introduction to the problem may be found in the notes of Bouw and Wewers [5].

Closely related to the global lifting problem is the local lifting problem. The set-up for the local lifting problem is as follows:

Definition 1.5.2. Let k be a field of characteristic $p > 0$ and let G be a finite group. A local G action is a subgroup of $\text{Aut}_k k[[z]]$ as in Section 1.4. This action is said to *lift to characteristic zero* if there exists a ring R dominating the Witt vectors $W(k)$ and an R-linear G action on $R[[z]]$ that descends to the given action on $k[[z]]$. The *local lifting problem* asks such a pair $(k[[z]], G)$ lifts to characteristic 0.

The global lifting problem can be solved locally via the local-global principle, the proof of which is found in Green's and Matignon's paper [13].

Theorem 1.39. *Let (C, G) be as above. This action lifts to characteristic zero if and only if for all closed points $y \in C$ the induced local action $(\hat{\mathcal{O}}_{C,y}, G_y)$ lifts. Here, $\hat{\mathcal{O}}_{C,y}$ is the completion of the local ring $O_{C,y}$ with respect to its uniformizer, and G_y the stabilizer of the point y.*

From Remark 1.34, the assumption that C is smooth ensures that $\hat{\mathcal{O}}_{C,y}$ is a formal power series ring $k[[z]]$ over k.

We will frequently use the following lemma when discussing rings of the form $R[[z]]$.

Lemma 1.40. *There is a one-to-one correspondence between the non-generic points of the space $Y = Spec(R[[z]])$ and the set $\{x \in \bar{K} | val_\pi(x) \geq 0\}$ modulo the action of the absolute Galois group $Gal(\bar{K}/K)$. In particular, Y may be regarded as the open unit disk in the topology defining R as the valuation ring of K.*

Proof. Let $p \in Y$ be a non-zero prime ideal of $R[[z]]$. By the Weierstrass preparation theorem we may represent it as $(f(z))$ for some irreducible polynomial $f(z) \in R[z]$, with roots in the integral closure of R in $\bar{K}$. All of these roots are $\mathrm{Gal}(\bar{K}/K)$ conjugates. We associate to $(f(z))$ these Galois conjugates.

Conversely, let $x \in \bar{K}$ with $\mathrm{val}_\pi(x) \geq 0$. Then x is integral over R, and so has minimal polynomial $f(z) \in R[z]$, which is the same minimal polynomial of its $\mathrm{Gal}(\bar{K}/K)$ conjugates. We associate to x the point $(f(z)) \in Y$, and we see that this is the inverse of the above map from Y to $\{x \in \bar{K} | \mathrm{val}_\pi(x) \geq 0\}/\mathrm{Gal}(\bar{K}/K)$. □

Remark 1.41. Lemma 1.40 lets us speak of rings of the form $R[[z]]$ as disks being defined by relations of the form $\{z \in \bar{K} || z - a| < \epsilon\}$. As the valuation is discrete, all such disks are both open and closed in the π-adic topology.

In general the local lifting problem remains difficult to solve. Our main approach to the local lifting problem will be to look at the conditions that the group action of G on $R[[z]]$ must satisfy supposing that the action lifts. Especially important is viewing how G acts on the parameter z as well as any fixed points that arise under this action. By Theorem 1.25 we know that G will be isomorphic to a cyclic-by-p group, viz. $G \cong P \rtimes C$ where C is a cyclic group of order prime to p and P is a p-group.

If G is cyclic, a recently proven result in Pop's paper [21] known as the Oort Conjecture ensures us that the pair $(k[[z]], G)$ lifts to characteristic 0.

Theorem 1.42 (The Oort Conjecture)**.** *Let G be a finite cyclic group of k-linear automorphisms of a ring $k[[z]]$, where k is a field of characteristic p. Then the pair $(k[[z]], G)$ lift to characteristic* 0.

When G is abelian we have a particularly simple structure of it. For the proof the reader is referred to Proposition 3.3 in Green's chapter in *Valuation theory and its applications volume II* [12].

Proposition 1.43. *Let $G \subset Aut_k k[[z]]$ be a finite abelian group that admits a lift to characteristic 0. Then G is either cyclic or a p-group.*

Remark 1.44. The Oort Conjecture provides a converse to this proposition when G is cyclic.

It is not difficult to find examples of group actions that fail to lift. The Hurwitz bound, for example, states that given a curve of genus g defined over a field K of characteristic 0, the order of any automorphism group of the curve satisfies $|G| \leq 84(g-1)$. This bound fails to apply in general in characteristic p - it follows that there are curves with group actions that cannot be lifted. The following example is taken from Bouw's notes [5].

Example 1.30. *Consider the curve defined over $\bar{\mathbb{F}}_p$ given by $y^p - y = x^{p+1}$. One checks that this curve has genus $(p-1)p/2$ (see Section 4.3). Consider the group of automorphisms acting on this curve defined as follows:*

Let ζ be a primitive (p^2-1)-st root of unity in $\bar{\mathbb{F}}_p$ and consider automorphisms σ, τ defined by

$$\sigma(x) = x \text{ and } \sigma(y) = y + 1;$$

$$\tau(x) = \zeta x \text{ and } \tau(y) = \zeta^{p+1} y.$$

Then one computes the order of σ to be p and of τ to be $p^2 - 1$. It follows that the order of the group they generate has order $\geq p^3 - p$ and so for large values of p will exceed $84(g-1)$, whence the curve with the G-action cannot lift.

We will show in Chapter 4 that, except for the case $p = 2$, the above group action will never lift even when Hurwitz bound is not violated.

Example 1.31. *We will continue looking at the group action of G on $\mathbb{F}_3[[z]]$ from Example 1.29. Note that by the Oort Conjecture we know that this local G-action does lift to characteristic 0.*

Let R be the totally ramified extension of $W(\mathbb{F}_3)$ given by adding the cubic roots of unity to $W(\mathbb{F}_3$. We will let ζ_3 denote a primitive cubic root of unity, and note that the image of ζ_3 in the residue field $\mathbb{F}_3$ of R is 1. Let $\tilde{\sigma}$ be an R-linear automorphism of $R[[z]]$ given by

$$\tilde{\sigma}(z) = \frac{\zeta_3 z}{1+z}.$$

Then $\tilde{\sigma}$ restricts to the action of σ on $\mathbb{F}_3$ under the reduction map, and furthermore $\tilde{\sigma}^3(z) = \frac{z}{1+z(\zeta_3^2+\zeta_3+1)} = z$, whence it has order 3. This confirms that the local G-action lifts to characteristic 0.

Chapter 2

Hurwitz Trees

As mentioned in Chapter 1, the local lifting problem is in general quite difficult. As such, *sufficient* conditions for a group action to lift are not the focus of much of the research surrounding the problem, but rather *necessary* conditions. That is, if we have a finite group G acting on a ring $k[[z]]$ where k is a characteristic $p > 0$ field, there are certain conditions that must be satisfied in order for the group to lift. Ideally, we would like these *obstructions to lifting* to be as strong as possible to further restrict the groups we need to look at.

This chapter focuses heavily on the Hurwitz tree obstruction, with the first four sections following much of the constructions in Chapter 3 of Brewis' thesis [6]. Before delving into the obstruction, we set up some notation in Section 2.1. Section 2.2 introduces two characters that encode the ramification data of a group acting on a disk - the Artin and depth characters. Following their definitions we will show in Theorem 2.7 that they are closely related.

Hurwitz trees are actually defined in Section 2.3 once we introduce the multiplicative character. The Hurwitz tree object, while complicated to define, is a metric tree with characters on every vertex and edge that satisfy certain relations. The following section, Section 2.4, relates the Hurwitz tree object to a group acting on a disk. We will use results from the first chapter on fixed points of automorphisms. Each leaf of the Hurwitz tree will correspond to a fixed point of the action, and the depth and Artin characters of the tree encode the ramification of the group acting on increasingly small disks about the fixed points.

An alternative construction of the Hurwitz tree is given in Section 2.5. After defining semistable models, we will show that the underlying tree of a Hurwitz tree is just the dual graph of a model of a curve.

The main result of this chapter is given in Section 2.6, where we prove in Theorem 2.21 that the construction offered in Section 2.5 is equivalent to that of Section 2.4. The main motivation for this theorem is that it will allow us to look at Hurwitz trees as tropical curves of the type looked at in Katz's paper [17].

We end the chapter in Section 2.7 by relating Hurwitz trees to the lifting problem. Namely, if a group action lift to characteristic 0 then there must be a Hurwitz tree related to its action on the disk it lifts to.

2.1 Notation

We begin this section with some notation. Let K be a complete discrete valuation field with uniformizer π, valuation val_π and ring of integers R. Furthermore, the residue field k is assumed to be algebraically closed with positive characteristic p. The valuation on K is assumed to be normalized so that $\mathrm{val}_\pi(\pi) = 1$.

Given a finite group G we denote by 1_G the unit character, r_G the regular representation, and u_G the augmentation character, viz. $u_G = r_G - 1_G$. These were defined in Section 1.1.

Finally, let Y denote the R-scheme $\mathrm{Spec}(R[[z]])$.

As before, we will denote by G a finite subgroup of automorphisms of the ring $R[[z]]$ that induces an R-automorphism of Y. Recall from Lemma 1.40 that Y can be identified with the unit disk $\{x \in \bar{K} | \mathrm{val}_\pi(x) \geq 0\}$ modulo $\mathrm{Gal}(\bar{K}/K)$.

2.2 Artin and depth characters

We associate to a G-action on the disk two characters that describe the action in terms of ramification, both with respect to the uniformizer for the ring R (the depth character) and with respect to the parameter z (the Artin character). The relation of the Artin character of G acting on the disk to the Artin character of G acting on $k[[z]]$ (defined in Subsection 1.2.3.1) will be revealed in Section 2.7.

2.2.1 The depth character

We begin with some preliminary definitions.

Definition 2.2.1. Let val_Y be the *Gauss valuation* on $R[[z]]$ defined by

$$\mathrm{val}_Y\left(\sum_i a_i z^i\right) = \min_i\{\mathrm{val}_\pi(a_i)\}.$$

Definition 2.2.2. The depth character δ_Y^G associated to the pair (Y, G) is the class function

$$\delta_Y^G(\sigma) = \begin{cases} -|G|\mathrm{val}_Y(\sigma(z) - z) & \text{if } \sigma \neq e \\ -\sum_{\sigma\neq e}\delta_Y^G(\sigma) & \text{otherwise.} \end{cases}$$

Thus, the depth character measures the ramification with respect to π of G acting on the disk.

Recall from Definition 1.2.11 that the inertia group of G is the largest subgroup of G that acts trivially on the residue field k of R.

Remark 2.1. Note that it is not immediate from the definition that δ_Y^G is a true character rather than a virtual character. It is clear that it is orthogonal to the identity character 1_G. Furthermore, the depth character is everywhere zero if and only if the inertia subgroup is trivial. Equivalently, the G-action on $R[[z]]$ is unramified with respect to π. That this is indeed a character is shown by Brewis and Wewers [7], as well as that this definition is independent of the choice of parameter z.

Example 2.1. *We will continue looking at the G-action on $R[[z]]$ in Example 1.31 by calculating the depth character associated to this action. Note that*

$$\tilde{\sigma}z - z = \frac{z(-1+\zeta_3) - z^2}{1+z} = (\zeta_3 - 1)z + \zeta_3 z^2 + z^3(g(z)),$$

for some $g(z) \in R[[z]]$. *In particular, because* ζ_3 *is a unit, we see that* $val_Y(\tilde{\sigma}z - z) = val_\pi(\zeta_3) = 0$. *Thus* $\delta_Y^G(\tilde{\sigma}) = 0$.

A similar calculation shows that $\delta_Y^G(\tilde{\sigma}^2) = 0$, *whence* $\delta_Y^G \equiv 0$.

The depth and Artin characters both involve elements of $R[[z]]$ of the form $\sigma(z) - z$. By the Weierstrass Preparation Theorem (Theorem 1.19), we may write

$$\sigma(z) - z = a_{m_\sigma} f_{m_\sigma}(z) u(z),$$

where $m_\sigma = \min\{n \geq 0 : \mathrm{val}_\pi(a_l) \geq \mathrm{val}_\pi(a_n) \forall l\}$ and f_{m_σ} is the Weierstrass polynomial of degree m_σ.

Example 2.2. *We continue Example 2.1 by finding the Weierstrass polynomial of* $\tilde{\sigma}z - z$. *Namely, we may take* $m_{\tilde{\sigma}} = 2$, $a_{m_{\tilde{\sigma}}} = -1$, *and* $u(z) = \frac{1}{1+z}$. *Hence we may take* $f_{\tilde{\sigma}} = z^2 + z(1 - \zeta_3)$, *whence the fixed points of* $\tilde{\sigma}$ *are* $z = 0$ *and* $z = 1 - \zeta_3$.

One finds the same Weierstrass polynomial for $\tilde{\sigma}^2$, *whence there are two fixed points of the* G*-action, both of which are fixed by all of* G.

2.2.2 The Artin character

The next character that will be important in the definition of the Hurwitz tree is the Artin character. For an element f of $R[[z]]$ we set

$$\#_Y(f) = \mathrm{ord}_z\left(\overline{\frac{f}{\pi^{\mathrm{val}_Y(f)}}}\right),$$

where the overhead bar denotes the residue class of the term $\frac{f}{\pi^{\mathrm{val}_Y(f)}}$ modulo π. Here ord_z is the valuation function on $k[[z]]$ with respect to z.

Definition 2.2.3. The Artin character a_Y^G associated to the pair (Y, G) is the class function

$$a_Y^G(\sigma) = \begin{cases} -\#_Y(\sigma z - z) & \text{if } \sigma \neq e \\ -\sum_{\sigma \neq e} a_Y^G(\sigma) & \text{otherwise.} \end{cases}$$

Remark 2.2. Note that if the inertia group of G is trivial, then G maps isomorphically onto a subgroup G' of $\mathrm{Aut}_k k[[z]]$. The Artin character as defined above coincides with the Artin character defined in Subsection 1.2.3.1 associated to the G-action on the local ring $k[[z]]$ when this is the case.

Example 2.3. *We continue Example 2.2. Namely, we will calculate the Artin character of the action. Recall that $\tilde{\sigma}z - z = -(z^2 - (\zeta - 1))u(z)$ where the image of $u(z)$ is a unit in $\mathbb{F}_3[[z]]$. Thus*

$$a_Y^G(\tilde{\sigma} = -ord_z(-z^2\overline{u(z)}) = -2.$$

Similarly, one may show that $a_Y^G(\tilde{\sigma}^2) = -2$, and finally we find $a_Y^G(1) = 4$.

This concludes the definition of the depth and Artin characters. We will now state an important result about the Artin character that will provide us a nice interpretation of it in terms of fixed points of the G-action.

Recall from Subsection 1.4.1 that for $\sigma \in G \setminus \{e\}$, $\Delta_\sigma \subset Y$ denotes the set of fixed points of σ. Define $\Delta = \cup_{\sigma \neq e} \Delta_\sigma$, and let $B = \Delta/G$ be the orbit space. For each $b \in B$, let $y \in \Delta$ be an element belonging to b, and G_b the stabilizer of y.

Theorem 2.3. *With the notation as above,*

$$a_Y^G = \sum_{b \in B} u_{G_b}^*.$$

Here $u_{G_b}^*$ denotes the induced character $Ind_{G_b}^G u_{G_b}$. Thus a_Y^G is a true character.

See Subsection 1.1.2 for the definition of induced characters.

Proof. Write

$$\sigma(z) - z = a_{m_\sigma} f_{m_\sigma}(z) u(z)$$

where $m_\sigma = \min\{n \geq 0 : \mathrm{val}_\pi(a_l) \geq \mathrm{val}_\pi(a_n) \forall l\}$ and f_{m_σ} is the Weierstrass polynomial of degree m_σ. The zeros of this polynomial correspond to the fixed points of σ, and in particular we have that $|\Delta_\sigma| = m_\sigma$. We may centre the disk about any such root and thus assume that it is given by (z) and so that $\sigma(z) = \zeta z(1 + \sum_{i=1}^{\infty} a_i z^i)$ (see corollary 1.38 in Chapter 1), whence we see that $\sigma(z) - z$ has $z = 0$ as a simple root. It follows that all such roots are simple. Thus, by definition of the Artin character,

$$a_Y^G(\sigma) = -\#_Y(\sigma z - z) = -m_\sigma = -|\Delta_\sigma|.$$

As $\sum_{b\in B} u^*_{G_b}$ evaluated at σ counts the number of elements in Δ_σ, the result follows. □

Remark 2.4. In the course of the proof it was shown that $a_Y^G = -|\Delta_\sigma|$. In particular, the Artin character associated to the action of G on Y is determined by the set of fixed points of the action of σ for all nontrivial $\sigma \in G$.

Example 2.4. *We will continue Example 2.3. Note from Example 2.2 that the two fixed points of the G-action, $z = 0$ and $z = \zeta_3 - 1$, are distinct modulo the G-action. Hence the set $B = \Delta/G$ has cardinality 2.*

Furthermore, each of the fixed points is fixed by all of G, so $G_b = G$ for all $b \in B$. Thus by the above theorem,

$$a_Y^G = \sum_{b\in B} u^*_{G_b} = u_G + u_G = 2u_G.$$

By the calculations in Example 2.3 we are able to confirm that this is the case, as by definition $u_G(\tilde\sigma) = u_G(\tilde\sigma^2) = -2$ and $u_G(1) = 4$.

2.2.3 Relation between the Artin and depth characters

We end this section by relating the depth and Artin characters through a theorem that will also be important when discussing vanishing functions on tropical curves. First we make a definition.

Definition 2.2.4. Let $a \in R[[z]]$ be a fixed but arbitrary element. We assume that $a \neq 0$. By the *residue class of a point y on the disk Y*, we refer to the set of points $\{x \in R[[z]] | x \equiv y \pmod{a}\}$.

Remark 2.5. Typically a will be chosen to be a power of π, so that the residue class of a point y will be the set of points $x \in Y$ such that $x \equiv y \pmod{\pi^k}$ for some non-negative integer k.

Definition 2.2.5. Let $Y = \mathrm{Spec}(R[[z]])$ be a unit disk. Let $D \subset Y$ be a closed disk of the form $\{x \in \bar K | \mathrm{val}_\pi(x-a) \geq \epsilon\}$ modulo the $\mathrm{Gal}(\bar K/K)$ action, where $a \in R$ is fixed and where $\epsilon > 0$ is an integer. Then ϵ is the *thickness of the annulus $Y \setminus D$*.

Remark 2.6. In the topology defined by a discrete valuation, it can be shown that any point within a disk is at the centre of the disk. Thus in the above definition the disk D is centred within Y, and the thickness of the annulus $Y \setminus D$ is well-defined.

The example to keep in mind for the following theorem is the case where D is the residue class of a fixed point of the G-action. This will be relevent in Section 2.5.

Theorem 2.7. *Let $D \subset Y$ be a closed disk which contains the set Δ and is fixed by the action of G. Then*

$$\delta_D^G = \delta_Y^G + |G|\epsilon s_Y^G,$$

where $s_Y^G = a_Y^G - u_G$ and where $\epsilon \in \mathbb{Z}_{\geq 0}$ is the thickness of the annulus $Y \setminus D$.

Proof. Without loss of generality we may assume that the point y corresponds to $z = 0$. In this case we find that we may view D as elements in Y with valuation $\geq \epsilon$. Choose an element $a \in R$ with $\mathrm{val}_Y(a) = \epsilon$, and consider the new term $z_1 = z/a$. From this is it clear that z_1 may be chosen as a parameter for the disks $D \cong \mathrm{Spec}(R[[z_1]])$.

Write $\sigma z - z = z^k \pi^n u$ where u is a unit in $R[[z]]/\pi$. By definition of z_1 this may be rewritten as

$$z^k \pi^n u = (z_1 a)^k \pi^n u,$$

whence the valuation of $f_\sigma = \sigma z - z$ on Y and D are related by

$$\mathrm{val}_D(f_\sigma) = \mathrm{val}_Y(f_\sigma) + k\mathrm{val}_Y(a) = \mathrm{val}_Y(f_\sigma) + \#_Y(f_\sigma)\epsilon,$$

by definition of both ϵ and the $\#_Y$ function. Thus,

$$\delta_D(\sigma) = -|G|\mathrm{val}_D(\sigma z_1 - z_1) = -|G|\mathrm{val}_D(\sigma z - z) + |G|\epsilon \tag{2.1}$$

$$= -|G|\mathrm{val}_Y(f_\sigma) - |G|\epsilon(\#_Y(f_\sigma) - 1) = \delta_Y(\sigma) + |G|\epsilon s_Y(\sigma) \tag{2.2}$$

□

2.3 Hurwitz Tree definition

In this section the notions of metric trees and multiplicative characters will be introduced. These will then be used to properly define the Hurwitz tree. Some basic familiarity with the notion of a graph is assumed. For reference, see the first chapter in Diestel's book on graph theory [10].

We begin with the definition of a rooted tree.

Definition 2.3.1. Let T be a tree with edge set E and vertex set V. Let $v_0 \in V$ be a distinguished leaf attached to edge e_0. Then (T, v_0) is a *rooted tree* with root v_0 and trunk e_0. We will often denote such a rooted tree by T when v_0 is understood.

Remark 2.8. Note that there is a natural partial order on the vertices of any such rooted tree, where the root vertex is the initial vertex. Given any edge e, let the source vertex $s(e)$ be the unique vertex attached to e in the same component of v_0 in the graph $T \setminus \{e\}$. Let the target vertex $t(e)$ be the other vertex attached to the edge. This determines a natural partial order on the vertex set, where $v \leq w$ if and only if there is an oriented path starting from v and going to w.

Definition 2.3.2. The maximal vertices are referred to as the *leaves*, and the set of leaves is denoted by B. If v is any vertex, we let $B_v = \{w \in B : v \leq w\}$.

Remark 2.9. Note our definition of leaf is different from the standard definition because the root vertex will have distinct properties from the set B.

Definition 2.3.3. A *metric* on a rooted tree is a map $\epsilon : E(T) \to \mathbb{Z}_{\geq 0}$ such that $\epsilon(e) = 0$ if and only if $t(e)$ is a leaf. The pair (T, v_0, ϵ) is a *metric tree* and it will often be denote by T when the root and metric are clear.

Before we define a Hurwitz tree properly, we introduce the notion of a multiplicative character.

Definition 2.3.4. Suppose that $G = \mathbb{Z}/p^m\mathbb{Z} = \langle \sigma \rangle$, the cyclic group of order p^m with $m \geq 0$. The multiplicative character, δ_G^{mult}, is the class function defined as follows: if $a \not\equiv 0 \pmod{p^m}$,

$$\delta_G^{\text{mult}}(\sigma^a) = -\frac{p^{i+1}}{p-1}\text{val}_\pi(p),$$

where $i = \text{ord}_p(a) < m$.

Remark 2.10. One may show as in Brewis' thesis [6] that δ_G^{mult} is a character of G.

As we will see later in Theorem 2.12, the multiplicative character as defined measures the ramification of an automorphism group acting on a disk with one fixed point. We also set

$$\delta_G^{\mathrm{mult}}(1) = -\sum_{a=1}^{p^m-1} \delta_G^{\mathrm{mult}}(\sigma^a) = mp^m \mathrm{val}_\pi(p).$$

The following result concerning the multiplicative character will be important when looking at examples and follows by a direct computation.

Lemma 2.11. *Let χ be an irreducible character of $\mathbb{Z}/p^m\mathbb{Z}$ of order p^n (i.e. the image of G under χ is a cyclic group of order p^n). Then*

$$\langle \delta^{mult}, \chi \rangle = \begin{cases} \frac{np-n+1}{p-1} val_\pi(p) & \text{if } n > 0 \\ 0 & \text{if } n = 0. \end{cases}$$

The inner product $\langle \delta^{\mathrm{mult}}, \chi \rangle$ was defined in Subsection 1.1.3.

We now define a G-Hurwitz tree.

Definition 2.3.5. A *G-Hurwitz tree over K* is a datum $\mathcal{T} = (T, [G_v], a_e, \delta_v)$ where

- $T = (T, \epsilon)$ is a metric tree with root v_0, trunk e_0 and set of leaves B,
- for every vertex v of T, $[G_v]$ is the conjugacy class of a subgroup $G_v \subset G$,
- for every edge e of T, a_e is a character of G defined over K,
- for all vertices v, δ_v is a character of G defined over K.

We refer to G_v as the *monodromy group* and δ_v the *depth of vertex v*. Analogously, a_e is the *Artin character of edge $e \in E$*.

We impose five conditions on the above datum:

1. Let $v \in V$. Then (up to conjugation)

   $$G_{v'} \subset G_v$$

   for all $v' \geq v$ such that v and v' are adjacent. Also,

$$\sum_{v \to v'} [G_v : G_{v'}] > 1,$$

where $v \to v'$ denotes v and v' are adjacent vertices with $v' \geq v$, except in the case $v = v_0$. In this case $G_v = G_{v_0} = G$. Thus away from v_0 we require that anytime a vertex v has a unique vertex v' directly following it on the tree, the monodromy group of v' is strictly contained in that of v.

2. The group G_b associated to any leaf is non-trivial and cyclic.

3. For all $e \in E$,

$$a_e = \begin{cases} \sum_{t(e)=s(e')} a_{e'} & \text{if } t(e) \notin B \\ u^*_{G_b} & \text{otherwise.} \end{cases}$$

4. For all $e \in E$,

$$\delta_{t(e)} = \delta_{s(e)} + \epsilon_e s_e$$

where $s_e = a_e - u^*_{G_{t(e)}}$.

5. For $b \in B$,

$$\delta_b = (\delta^{\text{mult}}_{G_b})^*$$

The *depth and Artin character of the tree* $\mathcal{T}$ are defined to be $\delta_{\mathcal{T}} = \delta_{v_0}$ and $a_{\mathcal{T}} = a_{e_0}$, respectively.

This completes the definition of the Hurwitz tree. One immediate consequence of the definition is an alternate way of expressing the Artin character of an edge. Namely, for $e \in E$ let B_e be the set of leaves that are greater than the terminal vertex $t(e)$ of e with respect to the partial order. Then we have

$$a_e = \sum_{b \in B_e} u^*_{G_b}.$$

2.4 Hurwitz trees and group actions on the disk

We have now defined all we need to attach a Hurwitz tree to a group action on the disk. Recall from Subsection 2.2.2 that $\Delta = \cup_{\sigma \in G} \Delta_\sigma$ is the set of all points on the disk Y fixed by a nontrivial element of G.

Theorem 2.12. *Let $Y = Spec(R[[z]])$ be the unit disk and $G \subset Aut_K(Y)$ be a finite p-group of automorphisms. Suppose that the set of fixed points $\Delta \subset Y$ is nonempty and finite. Then, after a possible finite extension of K there exists a G-Hurwitz tree $\mathcal{T}$ over K with $\delta_{\mathcal{T}} = \delta_Y^G$ and $a_{\mathcal{T}} = a_Y^G$.*

2.4.1 Construction of the Hurwitz tree

We construct the Hurwitz tree associated to such a group action inductively. First we will consider a small open set around each fixed point containing no other fixed points and construct a Hurwitz tree in the case $|\Delta| = 1$, which turns out to be especially simple. This allows us to break our unit disk into smaller regions and construct a Hurwitz tree for each one. After modding out by the G action we may then patch together these smaller Hurwitz trees, whereby all that will remain is to check the five conditions imposed in the definition hold.

To that end, consider the case $|\Delta| = 1$. Without loss of generality we may assume the fixed point y is $z = 0$. Every element of G must fix this point, for if $gy = y'$ for some $g \in G$, $y \neq y'$, then it follows that some conjugate of the stabilizer of y fixes y', and so y' is also a fixed point.

From this it follows that $a_Y^G = u_G$ by Theorem 2.3.

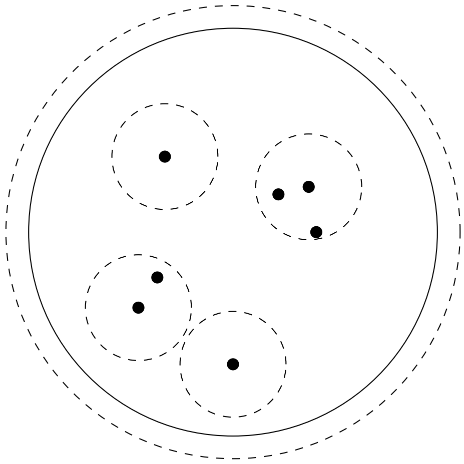
For an element $g \in G$, we can write

$$gz = \chi(g)z(1 + a_1 z + \cdots).$$

where χ is an injective character of G into $K^\times$. We note that the action of G must send a generator of $K((z))$ to another generator, G being the Galois group of $K((z))$ over $K((z))/G$, hence why $z|gz$ but $z^2 \nmid gz$. That χ is injective is a result of Green and Matignon [13]. The injectivity of χ implies that G is cyclic. Finally, suppose that g has order np^m with $(n, p) = 1$. Define $f_g = gz - z$ and note that $z = 0$ is the only root of f_g. Thus $\#(f_g) = 1$ and thus $\mathrm{val}_Y(f_g)$ is exactly the multiplicative character for G.

We define the Hurwitz tree in this case by giving T two vertices, v_0 and v_1, with one edge between them. The metric for the tree is trivial, and we set the depth character

FIGURE 2.1: Residue classes E_i (dashed circles) contained within closed disk D (solid circle). The unit disk Y is the outer dashed circle. Note the residue classes are centred at the fixed points (solid circles), and contain all of the fixed points.



of all vertices as δ_Y^G, the monodromy group to be G for both vertices, and the Artin character of the single edge to be a_Y^G. It follows from the above that this is a Hurwitz tree.

Suppose now that $|\Delta| \geq 2$. Let $D \subset Y$ be the minimal disk containing all the fixed points Δ, and note that D is fixed by all of G.

We may choose finitely many residue classes $(E_j)_{j \in J}$ of points in Δ such that

1. $\Delta \subset \cup_{j \in J} E_j$;
2. $E_j \cap E_i = \emptyset$ whenever $i \neq j$;
3. $E_j \subset D$ for all $j \in J$.

This can be achieved by letting n be the minimal power of π dividing all of the fixed points such that at least two of the fixed points have distinct images modulo π^{n+1}. Let $z_1, \ldots, z_n$ denote the fixed points with distinct images modulo π^{n+1}, and let $E_j = \{x \in K\bar{K} | \mathrm{val}_\pi(x - z_j) \geq n+1\}$. Then by how the z_j were chosen we see that these E_i satisfy the above three conditions. This is illustrated in Figure 2.1.

We obtain the final Hurwitz tree by choosing a set of representatives $J' \subset J$ of J/G and patching together the $\mathcal{T}_j$ at their roots. We label this patched vertex v_1 and add another v_0 to the tree, connected to v_1 by a unique edge e_0. The monodromy group of both v_0 and v_1 are redefined to be the group G.

All vertices other than v_0 and v_1 retain their previous monodromy groups, and for these vertices we redefine their depth characters as $\mathrm{Ind}_{G_v}^G \delta_v$.

If an edge e is not equal to e_0, it previously corresponded to an edge of some $\mathcal{T}_j$ and so had an associated Artin character a_e. We redefine the Artin character for edge e by setting it equal to $\mathrm{Ind}^G a_e$, induced from the subgroup that the Hurwitz tree $\mathcal{T}_j$ was defined for.

All metrics on edges other than e_0 remain as they were before.

Finally, set the metric of e_0 to be $|G|$ multiplied by the thickness of the annulus $Y \setminus D$, which was defined in Subsection 2.2.3. We set $a_{e_0} = a_Y^G$, $\delta_{v_0} = \delta_Y^G$, and $\delta_{v_1} = \delta_D^G$.

2.4.2 Showing the construction yields a Hurwitz tree

It remains to show that the five axioms of the Hurwitz tree hold for this new tree.

Lemma 2.13. *The object $\mathcal{T} = (T, [G_v], a_e, \delta_v)$ defined above is a Hurwitz tree.*

Proof. Observe that the first axiom holds by the inductive hypothesis: for any vertices v, w not equal to v_0 or v_1 this is immediate. For v_1 we have $G_{v_1} = G$, and as $|\Delta| > 1$ we have in particular that there is more than one Hurwitz tree $\mathcal{T}_j$ being patched at the root vertices. Thus,

$$\sum_{v_1 \to v} [G : G_v] > 1.$$

The second and fifth axioms hold immediately by the inductive hypothesis, as if $b \in B$ is any leaf it belongs to one of the $\mathcal{T}_j$ and thus has a cyclic, non-trivial monodromy group, with $\delta_b = (\delta_{G_b}^{mult})^*$, where the $(\delta_{G_b}^{mult})^*$ denotes the induced character of $\delta_{G_b}^{mult}$.

For the third axiom, we note that as $\mathrm{Ind}^G(a_1 + a_2) = \mathrm{Ind}^G a_1 + \mathrm{Ind}^G a_2$ for any characters a_1, a_2, this axiom holds away from e_0 by induction. For the edge e_0 the third axiom holds as a consequence of Theorem 2.3. Namely,

$$a_{e_0} = a_Y^G = \sum_{b \in B} u_{G_b}^*.$$

However, as the E_j were chosen to cover all the fixed points of our action and $J' \subset J$ was a complete set of representatives of J/G, we have

$$\sum_{b \in B} u_{G_b}^* = \sum_{j' \in J'} a_{\mathcal{T}_{j'}}^* = \sum_{s(e) = v_1} a_e,$$

whence the third axiom holds.

Similarly, the fourth axiom holds away from v_0, so we need only check that $\delta_{v_0} = \delta_{v_1} + \epsilon_{e_0} s_{e_0}$. This follows at once by Theorem 2.7. □

This completes the proof of Theorem 2.12.

As a corollary to the above theorem, we may express $a_{\mathcal{T}}$ and $\delta_{\mathcal{T}}$ in a particularly simple form through the use of the Weierstrass Preparation Theorem.

Corollary 2.14. *Keep the notation of Theorem 2.12. Let $\sigma \in G$, $\sigma \neq 1$ and suppose that we may write*

$$\sigma z - z = a_{m_\sigma} f_{m_\sigma}(z) u(z),$$

where $a_{m_\sigma} \in R$ with $val_\pi(a_{m_\sigma}) = n \in \mathbb{Z}_{\geq 0}$, and $f_{m_\sigma}(z)$ the Weierstrass polynomial as in Theorem 1.19 a polynomial in z of degree $m > 0$. Then

$$a_{\mathcal{T}}(\sigma) = -m \text{ and } \delta_{\mathcal{T}}(\sigma) = -|G|n,$$

where $|G|$ is the cardinality of the group G.

Proof. By Theorem 2.12 we may write the Artin character for the Hurwitz tree $\mathcal{T}$ as $a_{\mathcal{T}} = a_Y^G$, and the depth character as $\delta_{\mathcal{T}} = \delta_Y^G$.

The result follows immediately by the definitions of both a_G^Y and δ_G^Y. □

2.5 Hurwitz trees in terms of semistable models

In this section we will present an alternative construction of the Hurwitz tree associated to a group action on a disk in terms of semistable models of curves. Our motivation for this is to better relate Hurwitz trees to tropical curves in the Chapter 3, however, such a construction is also needed if one wishes to construct a G-action with a given Hurwitz tree [6]. We will prove in the Section 2.6 that this definition is equivalent with the one given in Section 2.4.

2.5.1 Arithmetic surfaces

Many of the definitions and results in this subsection are found in Chapter IV of Silverman's book [25]. To begin, we will define an arithmetic surface. Hurwitz trees associated to group actions on disks will be seen as the *special fibers* of such surfaces. Fibers of morphisms were defined in Subsection 1.3.2.2.

Definition 2.5.1. Let R be a Dedekind domain with field of fractions K. An *arithmetic surface* over R is an integral, normal, excellent scheme $\mathcal{C}$, flat and of finite type over R, whose generic fiber $C = \mathcal{C} \times_R \text{Spec}(K)$ is a nonsingular connected projective curve over K, and such that the special fibers (the fibers over closed points) are unions of curves over their respective residue fields.

Remark 2.15. We refer the reader to Subsection 1.3.3 for the properties of the schemes in the above definition. In particular, excellent is a technical condition that is satisfied whenever R is a Dedekind domain of characteristic 0, as is the case in the local lifting problem.

Definition 2.5.2. Let R be a Dedekind domain with field of fractions K. A *model* for a nonsingular connected projective curve C/K is a pair $(\mathcal{C}, \phi)$ of an arithmetic surface $\mathcal{C}$ over R and an isomorphism ϕ between C and the generic fiber $\mathcal{C}_K$ of $\mathcal{C}$.

Example 2.5. *For any Dedekind domain R of characteristic 0 the projective line $\mathbb{P}^1_R$ is a proper and smooth arithmetic surface over R. The generic fiber is $\mathbb{P}^1_K$ and if k_p is the residue field over a prime $p \in Spec(R)$ for a prime $p \neq (0)$, the special fiber over this point is given by $\mathbb{P}^1_{k_p}$. Note that as any Dedekind domain R of characteristic 0 is excellent.*

Definition 2.5.3. With the notation as above, fix a closed point $p \in \text{Spec}(R)$. The fiber $\mathcal{C}_p$ of a model $\mathcal{C}$ of C is said to be the reduction of C at p, or the reduction of C modulo p.

We may define a map from the closed points C^0 of C to the special fiber $\mathcal{C}_p$.

Definition 2.5.4. Keep the notation in the previous definitions. We define the *reduction map* $r : C^0 \to \mathcal{C}_p$ by sending the closed point $z \in C^0$ to $\bar{\{z\}} \cap \mathcal{C}_p$.

Remark 2.16. We will only work in the case where R is a complete discrete valuation ring and $\mathcal{C}/R$ is smooth, i.e. the case of the lifting problem. In this case we may conclude by Hensel's lemma (Theorem 1.17)that the reduction map defined above is surjective.

It is not immediately evident that given a smooth projective curve C/K an arithmetic surface $\mathcal{C}$ exists as above whose generic fiber over R is isomorphic to C. In fact it does as shown in Silverman [25]. Such a model can be chosen to be minimal in the following sense [24] [18]:

Theorem 2.17. *Assume that the genus g of C is ≥ 1. Then there exists a proper regular model $\mathcal{C}^{min}/R$ for C/K such that given any other proper regular model $\mathcal{C}/R$ of C/K and a fixed isomorphism from the generic fiber of $\mathcal{C}$ to the generic fiber of $\mathcal{C}^{min}$, the induced birational map $\mathcal{C} \to \mathcal{C}^{min}$ is an R-morphism. $\mathcal{C}^{min}$ is unique up to unique R-isomorphism.*

As suggested by the Hurwitz tree set-up, we will be primarily interested in the case when R is a discrete valuation ring with uniformizer π.

It is possible that distinct points on our minimal model may have the same reduction modulo (π). To separate these points in the central fiber we may *blow-up* our model. The theory behind blowing-up may be found in Hartshorne [14]. Beginning with a minimal model we may take successive blow-ups to separate points in the central fiber, as in Lemma 2.18.

Definition 2.5.5 (Hartshorne, Chapter 2.7 [14])**.** Let Y be a Noetherian scheme, $\mathcal{I}$ a coherent sheaf of ideals on Y. Let $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{I}^d$, where $\mathcal{I}^0 = \mathcal{O}_Y$. Define $\tilde{Y} = \text{Proj}\mathcal{J}$ as the blowing-up of Y with respect to $\mathcal{J}$.

Lemma 2.18. *Let Δ be a finite set of closed points of the unit disk Spec$(R[[z]])$. Then after a finite number of blow-ups of Spec$(R[[z]])$ we may assume that the points in Δ have pairwise distinct images in the reduction of Spec$(R[[z]])$ at (π).*

Proof. Recall from the discussion in Section 2.1 that we may identify Spec$(R[[z]])$ with the set of points $\{x \in \text{R} | \text{val}_\pi(x) \geq 0\}$ modulo the action of the absolute Galois group of K. Let $\Delta = \{z_1, \ldots, z_m\}$.

Fix some i, and let $n = \min_{j \neq i} \text{val}_Y(z_j - z_i)$, and consider the blow-up of Spec(R) with respect to the ideal $(\pi^n, z - z_i)$. Denote by $\bar{z_j}$ the image of z_j under the reduction map, where z_j is viewed as a point on the blown-up scheme.

We may write $z_i = a_1\pi + \ldots + a_n\pi^n + \ldots$ where $a_i \in k$ by the completeness of R.

The blow-up of Spec($R[[z]]$) at $(z - z_i, \pi^n) = I$ is by definition Proj($\mathcal{J}$), where $\mathcal{J} = \bigoplus_{m\geq 0} I^m$ is a graded module over $R[[z]]$. We may define a map $R[[z]][x_0, x_1] \to \mathcal{J}$ by sending $x_0 \to z - z_i$ and $x_1 \to \pi^n$. From this we find that we may view the blow-up at Spec($R[[z]]$) at I to be the closed subset of $\mathbb{P}^1_{R[[z]]}$ defined by the homogeneous polynomial $(x_0(z - z_i) - x_1\pi^n)$.

When $x_0 \neq 0$, the defining equation is $z - z_i = x_1\pi^n$. In particular, the closed point in Spec($R[[z]]$) corresponding to the prime $(z-z_j)$ for any $z_j \in \Delta$ is then $((z-z_i)+(z_i-z_j))$.

By definition of n, we may factor out π^n from this and write

$$(z - z_i) + (z_i - z_j) = \pi^n(x_1 - \frac{z_i - z_j}{\pi^n}).$$

Notably, the point image of z_i under the reduction map to the special fiber is defined by (x_1), and we see that if $z_j \equiv z_i \pmod{\pi^{n+1}}$, then z_j lies in the fiber over z_i.

Thus the fiber above $\bar{z}_i$ is the set $\{x \in \pi R | x \equiv z_i \pmod{\pi^{n+1}}$. Because the z_i's are pairwise distinct, and by how n was chosen, this process ensures that there is at least one pair (z_i, z_j) with distinct images in the reduction. As there are only finitely many elements in Δ, this process may be repeated finitely many times to separate all the points in the reduction. □

Definition 2.5.6. When the special fiber is reduced and has only ordinary double-points as singularities, we refer to the model as being *semistable*.

When the irreducible components of the special fiber are smooth we say the model is *strongly semistable* [3].

2.5.2 Dual graphs

Fix a discrete valuation ring R and a smooth curve C/K, and suppose that we have a strongly semistable model $\mathcal{C}/R$ for C. Let $\{C_1, ..., C_n\}$ be the irreducible components of $\mathcal{C}_k$, the special fiber of the model. We define the dual graph of the special fiber as follows.

Definition 2.5.7. With the notation as above, the dual graph T of $\mathcal{C}_k$ is the graph on n vertices where vertex v_i corresponds to component C_i. An edge exists between each v_i and v_j for each point of intersection between components C_i and C_j.

The assumption that the model is strongly semistable implies that the dual graph is well-defined with no loop edges, as seen in Baker's paper [3].

Definition 2.5.8. In addition to the above definitions we will also say that a curve C and a finite set of points $\Delta \subset C$, which we refer to as the *marked points* of the curve, is a *stably-marked curve* if C is complete, semistable, connected, and that for every marked point $y \in C$ the stabilizer subgroup of y of the automorphism group of the curve is finite.

2.5.3 Alternate construction of the Hurwitz tree

We will now construct an object $\mathcal{T}$ associated to the action of a finite group G on $R[[z]]$.

Fix a finite group G acting on $R[[z]]$. Assume that for each $\sigma \in G$ we have finitely many points on the disk $\mathrm{Spec}(R[[z]])$ fixed by σ, denoted by Δ_σ. Let $\Delta = \cup_{g \in G} \Delta_g$.

Note that we have an induced action of G on the unit disk $\mathrm{Spec}(R[[z]])$. Define Y to be the minimal model of $\mathrm{Spec}(K[[z]])$ embedded in projective space such that:

- The image of the points in Δ under the reduction map from $Y^0 \to Y \otimes k = Y_k$ are pairwise distinct, where Y^0 denotes the closed points of Y;
- Y_k is a semistable curve over k;
- If b_0 is the image of the strict transform of the generic point of $\mathrm{Spec}(R[[z]]) \otimes k$ in Y, then $(Y, \Delta \cup b_0)$ is stably marked.

The existence of such a Y satisfying these properties follows from Lemma 2.18 above, and the fact that as K is a one-dimensional function field and $Y \otimes K$ is defined over K, we may ensure Y has semistable reduction at the closed point of $\mathrm{Spec}(R)$ by Theorem A.9.3.2. in Hindry and Silverman [16].

Remark 2.19. The proof of Lemma 2.18 allows us an explicit description of the components of the special fiber of the blown-up model Y. Namely, we begin by labeling the elements of Δ as $\{z_1, \ldots, z_m\}$. The first blow-up on $R[[z]]$ is done with respect to the ideal $(z - z_1, \pi^n)$ where n is the minimal valuation needed to define the closed disk D containing Δ.

This gives us an exceptional divisor attached to the generic point of the special fiber Y_k, and we obtain a new vertex v_1 in the special fiber of the blow-up corresponding to this divisor.

For any z_i, we then look at the fiber of the reduction of z_i modulo π, which is the residue class $E_i = \{x | x \equiv z_i \pmod{\pi^n}\}$. We may cover D by these residue classes E_i such that $\Delta \cap D \subset \cup E_i$, then repeat the construction for each residue class.

At this point, the dual graph has a new vertex v_1 corresponding to our initial blow-up that is adjacent to v_0, where v_0 corresponds to the disk Y. Leading away from v_1 we have an edge w_i for each E_i that we have repeated the construction with.

Let $Z_K = (\mathrm{Spec}(R[[z]]) \otimes K)/G$ denote the quotient scheme of Y_K under the action of G. We have a set of closed points $\Delta/G \subset Z_K$, and we define Z to be the minimal model of Z_K with the same criteria as above. Let $Z_k = Z \otimes k$ be the special fiber.

We define the following data from this construction:

- We let T be the dual graph of Z_k.
- To a vertex v, let Z_v be the residue class in Z about a fixed points in Δ/G defining the blow-up from which we obtained v. That is, v was obtained after choosing a residue class $\{x \in R | x \equiv z_i \pmod{\pi^k}\}$ for some fixed point z_i and blowing up with respect to $(z - z_i, \pi^k)$. Thus $Z_v = \{x \in R | x \equiv z_i \pmod{\pi^k}\}$. Let Y_v be any residue class in Y whose image in Z is D_v, so that Y_v is an open disk. We set G_v to be the stabilizer of Y_v.
- With Y_v as above, define δ_v to be $\mathrm{Ind}_{G_v}^G \delta_{Y_v}^{G_v}$. Here, $\delta_{Y_v}^{G_v}$ is the depth character of the action of G_v on the disk Y_v.
- Let e be an edge oriented from v to w, so that w corresponds to a residue class Y_w contained within Y_v with Y_v as above. Set ϵ_e to be the thickness of the annulus $Y_v \setminus Y_w$ multiplied by $|G_v|$.
- Set a_e for e as above to be $\mathrm{Ind}_{G_v}^G a_{D_v}^{G_v}$. Here, $a_{D_v}^{G_v}$ is the Artin character of the action of G_v on Y_v.

Remark 2.20. With the definitions as above, the root vertex v_0 have $Y_{v_0} = Y$ and $v_1 = D$, where D is the smallest closed disk containing the fixed points. Any vertex v other than v_0 adjacent to v_1 has $Y_v = E_j$ for some residue class E_j of a fixed point in Δ.

2.6 Equivalence of the definitions of the Hurwitz tree

In this section we will prove that the data $(T, [G_v], \delta_v, a_e, \epsilon_e)$ defined in the previous section is actually the data of a Hurwitz tree associated to the action of G on the disk Y.

Theorem 2.21. *Let R be a complete DVR with characteristic 0, uniformizer π, field of fractions K and residue field k, where k has characteristic $p > 0$.*

Let $(T, [G_v], \delta_v, a_e, \epsilon_e)$ denote the data defined in the previous section associated to a finite group G acting on $R[[z]]$.

Then $(T, [G_v], \delta_v, a_e, \epsilon_e)$ is the data of a Hurwitz tree associated to the action of G on the disk Spec$(R[[z]])$, as in Section 2.3.

We need to check the five axioms of the Hurwitz tree hold.

First we prove a lemma showing equivalence in the case where Δ, the set of fixed points of the G-action defined in Subsection 2.2.2, one element.

Lemma 2.22. *Keep the notation of the previous theorem. If Δ consists of one element then the theorem holds true.*

Proof. By assumption there is a unique fixed point x of G, which we may assume without loss of generality is $z = 0$.

As in the construction of the previous section, we proceed by taking a closed disk in Y containing Δ. We may take this disk to be Y itself, and as $|\Delta| = 1$ it suffices to take no further blow-ups beyond the first.

This yields two vertices, v_0 and v_1, both corresponding to Y. By definition, $\epsilon_e = 0$, and the stabilizer of Y is G. Every element of G fixes x, and so we conclude as in the proof of Theorem 2.12 that $a_e = u_G$, $\delta_{v_0} = \delta_{v_1} = \delta_G^{mult}$.

This data agrees entirely with the case $|\Delta| = 1$ of Theorem 2.12, and so the result follows. □

To conclude the proof of Theorem 2.21 we will show that the construction of Section 2.5 can be finished inductively in the same way as Theorem 2.12. As the two cases agree in the case Δ is a singleton the result will follow.

Proof of Theorem 2.21. Suppose that $|\Delta| > 1$.

From Remarks 2.19 and 2.20, we begin by covering Δ with a closed disk D and then choosing at least two residue classes E_j, possible by our assumption on $|\Delta|$, such that $D \subset \cup E_j$. Each E_j is an open disk, and we repeat this construction on each E_j until the fixed points have distinct reductions on the special fiber.

Thus each E_j yields, by induction, a Hurwitz tree associated to the action of G_j on E_j. These Hurwitz trees may all be viewed as having v_1 as their root vertex corresponding to the closed disk D, and v_0 corresponds to Y.

The dual graph of Z that we obtain at the end of the construction in Section 2.5 is then represented by choosing a system of representatives of the residue classes E_j modulo G.

Thus the inductive process is entirely the same as that of Theorem 2.12, and as the base case $|\Delta| = 1$ agree the result follows. □

Remark 2.23. This justifies viewing a Hurwitz tree as the dual graph modulo G of the special fiber of a minimal model separating the fixed points of the G-action, as was mentioned but not shown in Brewis' thesis [6]. This will be key to viewing a Hurwitz tree as a tropical curve in Chapter 3.

Example 2.6. *We continue the work done in Example 2.4 by constructing a Hurwitz tree for the action of G on $R[[z]]$. Here, $G = \langle\sigma\rangle$ and R is $W(\mathbb{F}_3$ adjoined the cubic roots of unity, with σ acting on $R[[z]]$ via*

$$\sigma z = \frac{\zeta z}{1+z}.$$

We may regard $Y = Spec(R[[z]])$ as the unit disk on which G acts. Note that the two fixed points of the G-action on $R[[z]]$, namely $z = 0$ and $z = \zeta - 1$, are both contained in the closed disk $D = \{x | val_Y(x) \geq \frac{val_Y(3)}{2}\}$, where $\frac{val_Y(3)}{2} = val_Y(\zeta - 1)$. In particular, the thickness of the annulus $Y \setminus D$ is $\frac{3}{2} val_Y(3)$.

We have the following information:

- *The first vertex v_0 of the tree corresponds to the disk Y, and it is adjacent to vertex v_1 corresponding to D.*
- *The weight of e_0 is the thickness of the annulus $Y \setminus D$ multiplied by $|G|$, whence $\epsilon_{e_0} = \frac{3}{2} val_Y(3)$.*
- *The two fixed points, distinct modulo G, correspond to the existence of two leaves b_1 and b_2 with monodromy group G.*

Because the monodromy groups at the leaves are equal to G*, the first axiom of the Hurwitz tree implies that, in addition to* v_1 *and* v_0*, the leaves* b_1 *and* b_2 *are the only other vertices of the tree. Notably, this corresponds to the fact that the two fixed points are distinct modulo* π^2*, and as choosing* D *corresponds to blowing up with respect to the ideal* $(z, 1-\zeta)$ *we do not need to blow-up the disk beyond this. We set the depth character on* v_0 *to be* 0*, and the depth on the other vertices to be* δ_G^{mult}*, the multiplicative character of* G*.*

Explicitly,

$$\delta_G^{mult}(\sigma) = \delta_G^{mult}(\sigma^2) = \frac{-3}{2} val_Y(3),$$

and

$$\delta_G^{mult}(1) = 3 val_Y(3).$$

The Artin character on e_0 *is the Artin character on the disk* Y*, given explicitly in Example 2.3, and the Artin characters on the edges incident with the leaves are both* u_G*, the augmentation character of* G*.*

The first, second, and fifth axioms of the Hurwitz tree follow by definition, and the third axiom follows by Example 2.4. The fourth axiom requires that $\delta_G^{mult} = \epsilon_{e_0}(a_{e_0} - u_G)$*, and this follows by a direct calculation.*

2.7 Hurwitz trees as an obstruction to lifting

The goal of this subsection is to explain in what capacity the Hurwitz tree functions as an obstruction to the lifting problem. We will return to using the notation where k is an algebraically closed field of positive characteristic and $\phi : G \to \mathrm{Aut}_k(k[[z]])$ is a local G-action.

Recall from Subsection 1.2.3.1 that to such a G-action we may define a character a_ϕ of G, called the Artin character.

The Hurwitz tree obstruction is then

Theorem 2.24. *Let ϕ be a local G-action with the notation as above. If ϕ lifts to characteristic 0 then there exists a G Hurwitz tree $\mathcal{T}$ defined over a field K of characteristic 0 that satisfies*

$$a_{\mathcal{T}} = a_\phi \text{ and } \delta_{\mathcal{T}} = 0.$$

Proof. Suppose the action lifts. By definition of a lift of the local action, the G-action on the ring $R[[z]]$ for some ring R of characteristic 0 descends injectively to the action of G on $k[[z]]$.

Thus if $H \triangleleft G$ is the kernel of this quotient map from G acting on $R[[z]]$ to G acting on $k[[z]]$, then H is trivial. In terms of the ramification groups as defined in Subsection 1.2.3.1, we have $G_i = 0$ for all $i > 0$, whence for any $g \neq 1$ in G we have

$$\mathrm{val}_Y(gz - z) = 0 \text{ and } \overline{gz - z} = \bar{g}z - z,$$

where $\bar{g}$ is the image of g under the quotient map and the expression $\bar{g}z - z$ is in the ring $k[[z]]$.

In particular the depth character $\delta_{\mathcal{T}}$ is 0 by definition, and the Artin character of the root vertex $a_{\mathcal{T}}$ coincides with the Artin character a_ϕ of the G-action on $k[[z]]$.

The existence of the Hurwitz tree with the above properties then follows from Theorem 2.12.

□

Example 2.7. *From Example 2.6 it is clear that the Hurwitz tree constructed for the action of G on $\mathbb{F}_3[[z]]$ is of the type described in Theorem 2.24.*

Indeed, the depth character of this action was calculated in Example 2.1 to be 0*, and the Artin character of the tree was calculated in Example 2.3 to take value* -2 *on* σ *and* σ^2 *and* 4 *on* 1*. We may calculate the Artin character* a_ϕ *of our* G*-action on* $k[[z]]$ *defined in Example 1.29.*

$$a_\phi(\sigma) = -ord_z(\sigma(z) - z) = -ord_z(\frac{z}{1+z} - z) = -2,$$

and similarly we find that $a_\phi(\sigma^2) = -2$*. Thus the Artin character of the tree is the same as that of the Artin character of* G *acting on* $k[[z]]$*.*

Chapter 3

Tropical Curves and Hurwitz Trees

This chapter seeks to establish a formal link between tropical geometry and the lifting problem in algebraic geometry.

Tropical geometry itself is a vast field that we will not attempt to provide a rigorous introduction to. Rather, in Section 3.1, we provide only the necessary definitions that pertain directly to the lifting problem.

An important part of tropical geometry is to view aspects of algebraic geometry in a "Piecewise-linear" sense. In Section 3.2 we discuss rational functions on tropical curves as well as what it means to be harmonic. This will be important for viewing the depth and Artin character of a Hurwitz tree as harmonic functions away from the leaves.

Section 3.3 introduces the vanishing function, a term used in Katz' paper [17]. This function, defined on vertices of the tropical curve, describes the order of vanishing of a section of a line bundle. It is a continuous piecewise linear function that satisfies additional properties.

In Section 3.4 we draw together the previous sections and show that a Hurwitz tree associated to a G-action on a disk is a tropical curve. The underlying tree is the tropical curve, and the depth and Artin characters are analogous to the vanishing functions of elements of the form $\sigma z - z$ for $\sigma \in G$ and z a parameter on the disk.

3.1 Tropical Geometry

In this section the fundamentals of tropical geometry will be highlighted so as to be able to discuss the relation between tropical geometry and Hurwitz trees. This should not be viewed as a complete introduction to tropical geometry - many definitions are omitted in their generality for the sake of brevity. The reader interested in more is referred to *Introduction to Tropical Geometry* by Maclagan and Sturmfels [28].

3.1.1 Definitions

Throughout this chapter, unless stated otherwise, R will refer to a complete discrete valuation ring with field of fractions K, uniformizer π and residue field k.

Given a smooth, connected, projective curve C/K, we will let $\mathcal{C}$ denote a model for C as defined in Section 2.5. G will only be used to denote a group. Graphs will be denoted by other symbols such as T.

We will begin by generalizing our notion of a metric graph.

Definition 3.1.1. A weighted graph is one in which each edge is assigned a positive number, which we refer to as the weight of the edge.

Definition 3.1.2. Given two weighted graphs T and T^* we say that T and T^* are equivalent if they have a common subdivision of the edges, or a refinement, in a weight-preserving manner. That is, the sum of the weights of any new edges refined from an edge should be the same as the original weight. This is an equivalence relation on weighted graphs.

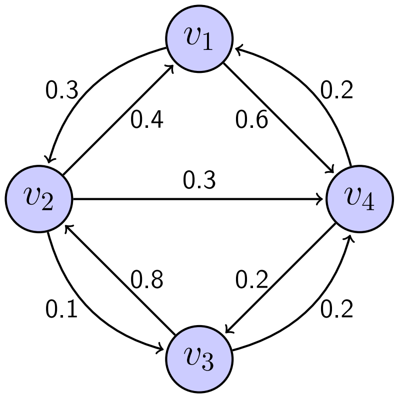
Definition 3.1.3. An equivalent class of weighted graphs is a *metric graph*, and an element of the equivalence class is a *model* for the metric graph.

Remark 3.1. Note that the use of the word *model* in the above definition bears no relation to the models of curves discussed in Section 2.5. This will cause no ambiguity and will be clear from the context.

Remark 3.2. Suppose we are given a weighted graph T. We may regard each edge as a line segment $[0, \epsilon] \subset \mathbb{R}$ where ϵ is the weight of the edge. This allows us to view a metric graph as a compact, connected metric space T [3]. Note that a metric graph is oriented, where an edge is oriented from 0 to ϵ.

Definition 3.1.4. A tropical curve T is a metric graph with a finite number of edges of weight 0.

FIGURE 3.1: An example of a model for a metric graph. Edges are oriented between vertices with their weights displayed. Graph template from texexample.net [15].



Thus, a tropical curve is a metric graph with some edges of length 0. This differs from the definition found in Baker's paper [3], where edges of length 0 are replaced by edges of infinite length. As we will see in Section 3.3, this will not affect our results, but it will allow us to regard Hurwitz trees directly as tropical curves without replacing the leaves by unbounded edges.

3.2 Piecewise linear functions and harmonicity

In the alternative construction of Hurwitz trees via semistable models in Section 2.5 we associated to an arithmetic surface the dual graph of its special fiber. The theory of divisors on schemes finds an analogue on graphs (see Subsection 1.3.3 for the definition of divisors on a scheme), and we will see that divisors on a curve may be *specialized* to give a divisor on the dual graph.

Suppose we have an arithmetic surface $\mathcal{C}/R$ for C/K. Let T denote the dual graph of $\mathcal{C}_k$ as defined in Subsection 2.5.2.

Definition 3.2.1. Given a graph T with vertex set V and edge set E we define the free abelian group on V to be $\mathrm{Div}(T)$, the *divisor group* of T. An element in this group is of the form

$$\Theta = \sum_{v \in V} n_v v, \text{ where } n_v \in \mathbb{Z}.$$

Remark 3.3. There exists a rich theory on divisor groups of graphs and how many classical results in algebraic geometry such as the Riemann-Roch Theorem have analogues on graphs. The interested reader is recommended to read Baker and Norine's paper [4] for a robust look at the topic.

Definition 3.2.2. If Θ and γ are two divisors in $\mathrm{Div}(T)$ we say that $\Theta \geq \gamma$ if $\Theta(v) - \gamma(v) \geq 0$ for all vertices $v \in V(T)$.

Definition 3.2.3. Rational functions $f : T \to \mathbb{R}$ are piecewise affine functions on the metric graph whose slopes between vertices are rational numbers. Given a model T for a metric graph this is equivalent to assigning rational numbers to the edges and vertices so as to be consistent with the underlying metric.

Remark 3.4. When T is a tropical curve we may still define rational functions. We add the stipulation that if e is an edge of length 0 between v_0 and v_1, then $f(v_0) = f(v_1)$.

Example 3.1. *To illustrate this with an example, suppose that our model T for a metric graph consists of two vertices v_0 and v_1 with a single edge e between them, oriented from v_0 to v_1. Assign weight ϵ to edge e. Define $f : T \to \mathbb{R}$ to take the value 0 on v_0 and has slope n on e where $n \in \mathbb{Z}$. Then the value of f at v_1 is uniquely determined to be ϵn.*

We will now introduce the notion of a rational function being harmonic, which is closely tied to the definition used in a paper by Baker and Norine [2].

Definition 3.2.4. Let f be a rational function on an oriented metric graph, so that each edge is directed from one vertex to another. Let $f(e)$ be the slope of f on edge e.

We say that f is harmonic if for every vertex v we have

$$\sum_{e,t(e)=v} f(e) = \sum_{e,s(e)=v} f(e)$$

where the first sum is over all edges with target vertex v, and the second sum is over all edges with source vertex v.

Remark 3.5. By the third axiom of the Hurwitz tree, we may view the Artin characters of the Hurwitz tree to be the slopes of a harmonic function away from the leaves and root vertex for fixed $\sigma \in G$. That is, suppose f is a rational function defined in such a way that the slope of f on edge e is $a_e(\sigma)$. As $a_e(\sigma) = \sum_{e',t(e)=s(e')} a_{e'}(\sigma)$, we see that f is harmonic away from the leaves and root vertex.

Definition 3.2.5. To a rational function on a graph T we associate a divisor via the Laplacian operator Ω, defined as

$$\Omega(f) = -\sum_{v\in V} \sigma_v v,$$

where σ_v is the sum of all the slopes emanating away from vertex v. The group of principal divisors on the graph is the set $\text{Prin}(T) = \{\Omega(f) : \text{f is a rational function}\}$.

In our above example we find that $\Omega(f) = -nv_0 + nv_1$. Note that the orientation of the edges are crucial.

Definition 3.2.6. We define the linear system $L(\Theta)$ of a divisor Θ on T to be the set of all rational functions f that satisfy

$$\Omega(f) + \Theta \geq 0.$$

We will be mostly interested in knowing how to specialize a divisor from a curve to its dual graph T.

Let $\text{Div}(C)$ and $\text{Div}(\mathcal{C})$ denote the group of divisors on C and $\mathcal{C}$, respectively. The assumptions on C and $\mathcal{C}$ ensures that Cartier and Weil divisors are the same; see Baker's paper [3]. We may regard each component C_i of the special fiber of $\mathcal{C}$ as being in $\text{Div}(\mathcal{C})$.

Given any divisor Θ on $\mathcal{C}$ we have an intersection pairing with the C_i defined as

$$(C_i, \Theta) = (C_i \cdot \Theta) = \deg(\mathcal{O}_{\mathcal{C}}(\Theta)|_{C_i}) \in \mathbb{Z}.$$

Definition 3.2.7. The specialization map, as defined in Baker's paper [3], is a map $\rho : \mathrm{Div}(\mathcal{C}) \to \mathrm{Div}(T)$ given by:

$$\rho(\Theta) = \sum_{v_i \in T} (C_i, \Theta) v_i,$$

and we may also define the action of ρ on $\mathrm{Div}(C)$ by composing ρ with the map taking $\Theta \in \mathrm{Div}(C)$ to its Zariski closure.

Remark 3.6. As mentioned in Baker's paper [3], the specialization map sends vertical divisors (those supported entirely on the special fiber) to principal divisors of the dual graph. This will be relevant for the proof of Lemma 3.12.

3.3 Vanishing functions

We will now define the vanishing function of a section of a line bundle of an arithmetic surface. This section closely follows Chapter 7 of Katz [17].

Suppose we are given a line bundle $\mathcal{L}$ over $\mathcal{C}$ and a rational section s of $\mathcal{L}$. Line bundles were defined in Subsection 1.3.2.2.

Remark 3.7. Here, $\mathcal{C}$ is *marked* as in Section 2.5 with a finite number of points, and we assume that it is the minimal model such that these marked points have distinct images in the special fiber. That such a minimal model exists is shown in Lemma 2.18.

Let $\mathcal{C}_k$ denote the special fiber of $\mathcal{C}$, and $\tilde{\mathcal{C}_k}$ the normalization of the special fiber, where the components of $\tilde{\mathcal{C}_k}$ are $\{C_1, ..., C_n\}$. We will denote the normalization map by $\phi : \tilde{\mathcal{C}_k} \to \mathcal{C}_k$.

If we have a point of intersection between C_i and C_j corresponding to an edge e of the dual graph, let p_e denote this point lying on the two components.

Remark 3.8. There is an important distinction between the dual graph T as given in Section 2.5 versus that used by Katz in his paper [17]. Namely, the set of leaves B we defined gives a one-to-one correspondence between the leaves and the fixed points Δ of the G-action on the disk, modulo the G-action. In Katz's paper, the leaves are replaced by unbounded edges, and so there is a one-to-one correspondence between unbounded edges and marked points of the model. We will keep the leaves and prove the results of Katz in this case.

Remark 3.9. The example most relevant to us is where $\mathcal{L}$ the trivial bundle and s is of the form $gz - z$ for $g \neq 1$.

Recall that each component of the special fiber has a valuation associated to it (or more specifically to its generic point). It is the valuation of a section at each of these components that defines the value of the divisor (s) at the C_v's.

To simplify the situation we will always assume that the divisor (s) is supported on $C(K)$. After a possible extension of R this may always be assumed to hold in the case $s = gz - z$. This is important as the image of K-rational points on the special fiber are smooth. See, for example, Section IV.4 of Silverman [25].

Definition 3.3.1. We define the vanishing function $\varpi_s : T \to \mathbb{R} \cup \infty$ via $\varpi_s(v) = (s)(v)$, where $(s)(v)$ is the coefficient of vertex v in the divisor (s). Furthermore, we extend ϖ_s linearly on edges. For a vertex $v \in V$, set $s_v = \phi^* \left(\frac{s}{\pi^{\varpi_s(v)}} \right) |_{C_v}$.

When an edge e is adjacent to a vertex v, let $\mathrm{ord}_{p_e}(s_v)$ be the valuation of s_v at the point $p_e \in C_v$. If the length of the edge e is 0, set the slope of ϖ_s along that edge to be $\mathrm{ord}_{p_e}(s_v)$. When $s = 0$ set $\varpi_s = \infty$.

Remark 3.10. When an edge e between v_0 and v_1 has length 0, we have that $\varpi_s(v_0) = \varpi_s(v_1)$. This is implicit in the above definition, where ϖ_s is extended linearly on edges. Notably, the slope of such an edge is assigned a definite value. As the length of the edge is 0 the piecewise linearity of the vanishing function will not be impacted as long as the vanishing function does not change between the vertices incident with this edge.

In general the support of (s) might involve more than just the C_v's, and so we write

$$(s) = \bar{\Theta} + \sum \varpi_s(v) C_v,$$

where $\bar{\Theta}$ is a horizontal divisor, by which we mean a prime divisor whose image under the structure morphism $\mathcal{C} \to \mathrm{Spec}(R)$ is $\mathrm{Spec}(R)$.

We now state two crucial results from Katz [17].

Lemma 3.11. *Suppose s has K-rational zeros and poles. Let $e \in E(T)$ be an edge adjacent to $v \in V(t)$. Then $\mathrm{ord}_{p_e}(s_v)$ is equal to the slope of ϖ_s along e away from v.*

Proof. When the weight of the edge is 0, this follows directly from the definition of the vanishing function.

For the case of an edge having positive weight, the reader is referred to [17]. □

For the second lemma, we define the divisor γ on T via

$$\gamma = \sum_v \deg(\phi^* \mathcal{L}|_{C_v})(v).$$

Lemma 3.12. *If s is a section of $\mathcal{L}$ that is regular on the generic fiber C and has $\mathbb{K}$-rational zeroes, then $\Omega(\varpi_s) + \gamma \geq 0$. Thus $\varpi_s \in L(\gamma)$.*

Of relevance to us is the case where $\mathcal{L}$ is the trivial bundle and $s = \sigma z - z$ for $\sigma \in G$. As the proof of Lemma 3.12 helps to illustrate how a section of $\mathcal{L}$ provides us with a piecewise-linear function on the dual graph of the special fiber, we include it here.

Proof. By definition of the specialization map and the fact that s is a rational section of the trivial bundle in this case, we have that

$$\gamma(w) = \rho((s))(w) \sim \rho\left(\left(\frac{s}{\pi^{\varpi_s(w)}}\right)\right)(w),$$

where $\sim$ denotes linear equivalence. This follows from the fact that the specialization of a vertical divisor lies in the principal divisors of T; see Chapter 2 of Baker [3].

Now,

$$\left(\frac{s}{\pi^{\varpi_s(w)}}\right) = \bar{\Theta} + \sum_v (\varpi_s(v) - \varpi_s(w))C_v,$$

whence

$$\rho((\frac{s}{\pi^{\varpi_s(w)}}))(w) = \deg((\bar{\Theta} + \sum_v (\varpi_s(v) - \varpi_s(w))C_v) \cdot C_w) \quad (3.1)$$

$$\geq \sum_{e=vw} (\varpi_s(v) - \varpi_s(w)) \quad (3.2)$$

$$= -\Omega(\varpi_s)(w). \quad (3.3)$$

The inequality follows from the fact $\bar{\Theta}$ is effective, and so intersects a component C_w of the special fiber with non-negative multiplicity on smooth points of C_w.

This completes the proof. □

Note that in the case $s = \sigma z - z$ we may conclude a stronger result. Namely, as the divisor $(\sigma z - z)$ is a vertical divisor we may take $\bar{\Theta} = 0$. This gives us

Corollary 3.13. *Suppose s in the above lemma is replaced by $\sigma z - z$. Then with the notation as above, we have that $\Omega(\varpi_s) + \gamma = 0$.*

3.4 Hurwitz Trees as Tropical Curves

In this section we will explore the link between Hurwitz trees and tropical curves. Our goals are two-fold. Suppose we have a Hurwitz tree $\mathcal{T}$ associated to the action of a finite group G on $R[[z]]$, as in Section 2.5 of Chapter 2.

- We first wish to view the underlying metric tree of our Hurwitz tree as a tropical curve. This is more or less immediate from the definitions. There are some subtleties, however, in how the Hurwitz tree as constructed by Brewis will differ from that constructed by Katz, and this will be seen in the metric on the tree.
- Second, we will view the depth and Artin characters of a Hurwitz tree evaluated at $\sigma \in G$ as being closely related to the vanishing function for the section $s = \sigma z - z$. We will find that the difference between the former functions and the latter is dependent upon the metric of the tree and can be explained in terms of rescaling our parameter z on the disk $\text{Spec}(R[[z]])$ to be the uniformizer on progressively smaller disks.

Once these two steps are complete we will be able to view Hurwitz trees associated to group actions on disks as tropical curves with special vanishing functions that satisfy certain axioms.

3.4.1 The underlying tree as a tropical curve

Let G be a finite group acting on $R[[z]]$. From Section 2.5 we may associate to this group action a Hurwitz tree $\mathcal{T}$ with underlying tree T.

The metric ϵ on T on an edge e has already been defined in Section 2.5 as the thickness of the annulus obtained when going from an open disk containing fixed points of the G-action to the residue class of a fixed point on the disk.

Because the Hurwitz tree constructed in 2.5 has only finitely many leaves (corresponding to finitely many fixed points modulo the G-action), the tree obtained in Section 2.5 satisfies the definition of a metric tree, as only finitely many edges have weight 0 and the rest have positive weight.

Remark 3.14. One difference that appears between our Hurwitz trees and those used in Katz's paper [17] are that the metrics on our trees may differ from 1 on bounded edges, whereas in Katz's paper they are all equal to 1.

Two observations yield the explanation for this difference:

- First, the weights on the Hurwitz trees are always rational numbers. Recall from the proof of Theorem 2.7 that the annuli defining our metric are obtained after possibly extending the field K by a totally ramified extension, and so the thickness of the annuli may be in $\mathbb{Q} \setminus \mathbb{Z}$. One may choose a minimal totally ramified field extension L/K and, after replacing K by L, assume that the weights are all integers.

- Second, after assuming the edges of the Hurwitz tree all have integer weights, we may subdivide any edge e of weight $n > 1$ to ensure that all edges have weight 1, thus reducing to the situation of Katz's paper.

We now have a tropical curve (T, ϵ) that serves as the underlying tree of the Hurwitz tree.

3.4.2 Vanishing functions as depth / Artin characters

In this subsection we will explore the connection between the depth and Artin character of a Hurwitz tree as defined in Brewis [6] and the vanishing function defined by Katz on a tropical curve [17].

Definition 3.4.1. For every $\sigma \in G$ with $\sigma \neq 1$, define f_σ by

$$f_\sigma = \sigma z - z.$$

Remark 3.15. Consider the divisor $(f_\sigma) = \sum_v (f_\sigma)(v) C_v$, where $(f_\sigma)(v)$ is the coefficient of (f_σ) on the component C_v of the special fiber corresponding to vertex v. Thus we see that $(f_\sigma)(v) = \varpi_{f_\sigma}(v)$ by definition of the vanishing function for the section f_σ.

Definition 3.4.2. Fix a vertex $v \in T$ corresponding to a component C_v. Let $f \in R[[z]]$ be some element. Then

$$\mathrm{val}_v(f)$$

will denote the order of vanishing of f on component C_v.

Remark 3.16. Note that (f_σ) is a vertical divisor on $\mathrm{Spec}(R[[z]])$; that is, one supported entirely on the special fiber, where by definition

$$(f_\sigma)(v) = \mathrm{val}_v f_\sigma.$$

By definition of the depth character (see Section 2.2) and by Theorem 2.12, we note that on the component v_0 corresponding to Spec($R[[z]]$) this is closely related to the depth character $\delta_{\mathcal{T}} = \delta_G^Y$. That is, $-|G|\text{val}_{v_0} f_\sigma = -|G|\text{val}_Y(\sigma z - z) = \delta_G^Y(\sigma)$.

We may extend the relation between $\delta_v(\sigma)$ and $\varpi_{f_\sigma}(v)$ for any vertex v of the Hurwitz tree via a lemma.

From section 2.5 we know that the depth character δ_v of a vertex v corresponds to induced character of the depth character δ'_v on a residue class E_k of a fixed point contained in Spec($R[[z]]$). That is, $\delta_v = \text{Ind}_{G_k}^G \delta'_v$, where G_k is the stabilizer of the residue class E_j.

Lemma 3.17. *Suppose the unique path from v_0 to v_k on the Hurwitz tree goes through edges $e_0, \ldots, e_k$, with the weights of these edges $\epsilon_0, \ldots, \epsilon_k$, respectively. Then for $\sigma \neq 1 \in G_k$, G_k the monodromy group of v_k, we have*

$$\delta'_{v_k}(\sigma) = -|G_k|\varpi_{f_\sigma}(v_k) + |G_k|(\epsilon_0 + \ldots + \epsilon_k),$$

where $f_\sigma = \sigma z_0 - z_0$ for $z_0 = z$ the parameter defining $\mathcal{C} =$ Spec($R[[z]]$). Here δ'_{v_k} is the depth character on the residue class E_k to which v_k corresponds.

Proof. As in the proof of Theorem 2.7, we may suppose that the parameter on E_k is given by z_k. Thus the depth character of vertex v_k is given by

$$\delta_{v_k}(\sigma) = -|G_k|\text{val}_{v_k}(\sigma z_k - z_k).$$

Suppose in general that v_i precedes v_{i+1} for some vertex v_{i+1}. Then, as in the proof of Theorem 2.7,

$$-|G_k|\text{val}_{v_{i+1}}(\sigma z_{i+1} - z_{i+1}) = -|G_k|\text{val}_{v_{i+1}}(\sigma z_i - z_i) + |G_k|\epsilon,$$

where ϵ is the valuation of the element a defining the equivalence class to which v_{i+1} corresponds.

Iterating this, we may replace z_i with z_{i-1} and so forth, where the next vertex is uniquely determine by the partial order defined by the underlying tree. Thus, for a vertex v_k as in the statement of the lemma,

$$\delta'_{v_k}(\sigma) = -|G_k|\mathrm{val}_{v_k}(\sigma z_k - z_k) \tag{3.4}$$
$$= -|G_k|\mathrm{val}_{v_k}(\sigma z_{k-1} - z_{k-1}) + |G_k|\epsilon_{e_k} \tag{3.5}$$
$$= -|G_k|\mathrm{val}_{v_k}(f_\sigma) + |G_k|(\epsilon_{e_0} + ... + \epsilon_{e_k}) \tag{3.6}$$
$$= -|G_k|\varpi_{f_\sigma}(v_k) + |G_k|(\epsilon_{e_0} + ... + \epsilon_{e_k}), \tag{3.7}$$

where $f_\sigma = \sigma z_0 - z_0$. □

Remark 3.18. Because $\delta_v = \mathrm{Ind}^G \delta'_v$, this provides a direct link between the depth character on a vertex and the vanishing function of f_σ for $\sigma \neq 1$.

Recall from Lemma 3.11 that on an edge e, ϖ_s has slope $\mathrm{ord}_{p_e}(s_v)$, where (from Section 3.3) p_e is the point of intersection between the components of the special fiber corresponding to $t(e)$ and $s(e)$. In the case where $s = f_\sigma$, we see that by definition of ord_{p_e}

$$\mathrm{ord}_{p_e}(f_\sigma)_v = \mathrm{ord}_{z_{s(e)}}\left(\frac{f_\sigma}{\pi^{\mathrm{val}_{v_{s(e)}}(f_\sigma)}}\right) = -\#_{s(e)} f_\sigma, \tag{3.8}$$

where $\mathrm{ord}_{z_{s(e)}}$ is the $z_{s(e)}$-adic valuation on the disk corresponding to vertex $s(e)$.

Thus, for a fixed edge, the function $\mathrm{ord}_{p_e} : G\setminus\{1\} \to \mathbb{Z}$ defined by $\mathrm{ord}_{p_e}(\sigma) = \mathrm{ord}_{p_e}(f_\sigma)_v$ closely resembles the Artin character on that edge. In fact, we claim this is equal to the Artin character of the residue class corresponding to a vertex v which an edge leads from.

Lemma 3.19. *For an edge e of T with weight ϵ and $\sigma \in G_{s(e)}$ not equal to 1, we have*

$$a'_e(\sigma) = \mathit{ord}_{p_e}(f_\sigma)_v.$$

Here, a'_e is the Artin character of the residue class $E_{s(e)}$, where $E_{s(e)}$ corresponds to the vertex $s(e)$ preceding edge e, and $G_{s(e)}$ the monodromy group of $s(e)$.

Proof. In the case where the weight of the edge is 0 (equivalently when $t(e)$ is a leaf), this follows directly from Equation 3.8. Namely, there is a unique fixed point of all the elements in G_b for a leaf b, whence $-\#_{s(e)} f_\sigma = -1 = u_{G_b} = a'_e(\sigma)$ for e the edge incident with b.

To see this in the general case where $\epsilon \neq 0$ we return to Theorem 2.7. Recall that in the course of its proof we showed

$$\mathrm{val}_D(f_\sigma) = \mathrm{val}_Y(f_\sigma) + \#_Y(f_\sigma)\epsilon,$$

where D is a disk inside of Y such that the thickness of the resulting annulus is ϵ. Thus, for $\sigma \neq 1$,

$$\begin{aligned}
\mathrm{ord}_{p_e}(f_\sigma)_v &= \frac{1}{\epsilon}\varpi_{f_\sigma}(t(e)) - \varpi_{f_\sigma}(s(e)) \text{ by lemma 3.11} && (3.9)\\
&= \frac{1}{\epsilon}\mathrm{val}_{t(e)}(f_\sigma) - \mathrm{val}_{s(e)}(f_\sigma) && (3.10)\\
&= -\#_{s(e)}(f_\sigma) = a'_e(\sigma) && (3.11)
\end{aligned}$$

□

Remark 3.20. Thus the depth and Artin characters of the residue classes to which the vertices correspond are closely related to the vanishing functions, with the Artin character being equal to the slope of the vanishing function along edges and the depth character being directly related to the vanishing function at vertices with a difference of the edge lengths. In particular, for an edge e, we have that

$$a_e(\sigma) = \mathrm{Ind}^G a'_e(\sigma) = \sum_{g \in G/G_{s(e)}} a'_e\widehat{(g\sigma g^{-1})} = \sum_{g \in G/G_{s(e)}} \widehat{\mathrm{ord}}_{p_e}(f_{g\sigma g^{-1}}),$$

where $\hat{f}(g\sigma g^{-1})$ equals $f(g\sigma g^{-1})$ if $g\sigma g^{-1} \in G_{s(e)}$ and is 0 otherwise (see Subsection 1.1.2 for definition of induced characters).

Example 3.2. *We calculate the vanishing function of $\sigma z - z$ for $\sigma \in G$ acting on $R[[z]]$ as in Example 2.7. This example showed that $\varpi_{\sigma z - z}(v_0) = 0$. We will calculate $\varpi_{\sigma z - z}(v_1)$, which by Lemma 3.17 is also the value of the vanishing function of $\sigma z - z$ at the leaves.*

By Theorem 2.7 we have

$$\varpi_{\sigma z - z}(v_1) = val_D(\sigma z - z) = val_Y(\sigma z - z) + \#_Y(\sigma z - z)\frac{val_Y(3)}{2} = \#_Y(\sigma z - z)\frac{val_Y(3)}{2}.$$

Note that D is the disk corresponding to v_1 and Y the unit disk corresponding to v_0. By the definition of the Artin character and the calculations in Example 2.3 we conclude that

$$\varpi_{\sigma z-z}(v_1) = val_Y(3).$$

We may then confirm Lemma 3.17 for $\varpi_{\sigma z-z}(v_1)$. Namely,

$$\varpi_{\sigma z-z}(v_1) = \epsilon_{e_0} - \frac{\delta_{v_1}(\sigma)}{|G|} = \frac{val_Y(3)}{2} + \frac{val_Y(3)}{2} = val_Y(3),$$

as desired.

Chapter 4

Examples

In this chapter we will explore examples that illustrate how the Hurwitz tree obstruction can be approached from the point of view of tropical curves, namely in viewing the Artin and depth characters as being closely related to to the vanishing function of sections of the form $\sigma z - z$, where $\sigma \in G$ and z is a parameter on the disk. The examples are:

- the general quaternion group acting on a local power series ring in characteristic 2
- the lifting problem for certain metacyclic groups over fields of characteristic $p > 2$

The first example, in Section 4.1, is found in Chapter 3 of Brewis' thesis [6], where it is shown that the action does not lift due to the Hurwitz tree obstruction. In this section we will make heavy use of the piecewise linearity of the vanishing function of sections and thus the example will serve to illustrate the connection to tropical curves described in Chapter 3.

The second example, in Section 4.3, is inspired by Bouw's notes [5]. For general groups of the type described in the notes, it is shown that the Hurwitz bound (discussed in Section 1.5) prevents the group action from lifting once its cardinality is large enough. Here we will show that even when the Hurwitz bound is not violated the action will not lift due to a violation of the Hurwitz tree obstruction.

4.1 Quaternionic actions

4.1.1 Goal of the quaternion example

The goal of the first example is to show that *simple* quaternionic actions on $k[[z]]$ do not lift to characteristic 0, where $k = \overline{\mathbb{F}}_2$.

We will accomplish this by constructing piecewise linear functions Λ_{χ_0}, Λ_{χ_1}, and Λ_ψ on the Hurwitz tree associated to the G-action assuming it does lift to characteristic 0. Here, χ_0, χ_1, and ψ are characters of G.

At a specific leaf b_0 of the tree we will show in Lemma 4.15 that $\Lambda_{\chi_0}(b_0) + \Lambda_{\chi_1}(b_0) \geq \frac{1}{2}\Lambda_\psi(b_0)$, and in Lemma 4.17 that $\Lambda_{\chi_0}(b_0) + \Lambda_{\chi_1}(b_0) < \frac{1}{2}\Lambda_\psi(b_0)$, thus obtaining a contradiction to the action lifting.

Simple quaternionic actions are defined in the following subsection, along with the characters χ_0 and χ_1. Following this, we will show the existence of the leaf b_0 by demonstrating an equivalent notion of simple actions. The character ψ is defined in Section 4.2, along with the piecewise linear functions Λ_ϕ for a character ϕ of G. Our contradiction will follow.

4.1.2 Set-up for the quaternionic action

Our set-up for this example is as follows. Let

$$G = \mathbb{Q}_{2^{n+1}} = \langle \sigma, \tau | \tau^{2^n} = 1, \tau^{2^{n-1}} = \sigma^2, \sigma\tau\sigma^{-1} = \tau^{-1} \rangle$$

act on a local ring $k[[z]]$, where $k = \overline{\mathbb{F}_2}$, where the bar denotes the algebraic closure. That is, we have an injection

$$\phi : G \hookrightarrow \mathrm{Aut}_k k[[z]].$$

We will make heavy use of the following cyclic subgroups of G, which will be needed to define the characters χ_0 and χ_1 that are used to show simple actions fail to lift:

$$H_0 = \langle \tau \rangle, H_1 = \langle \sigma \rangle, \text{ and } H_2 = \langle \sigma\tau \rangle.$$

Note that H_0 has order 2^n whereas the other two cyclic subgroups have order 4.

Remark 4.1. It is easy to verify that up to conjugation by elements of G the only other cyclic subgroups of G are of the form $\langle \tau^{2r} \rangle$ for r some integer.

We now define the characters χ_0, χ_1, and χ_2. These are essential to the definition of a simple quaternionic action.

Definition 4.1.1. We define 3 one-dimensional characters of G, $\chi_i : G \to \{\pm 1\}$ for $i \in \{0, 1, 2\}$ such that χ_i, when restricted to H_i, is trivial, i.e.

$$\chi_i(g) = 1$$

whenever $g \in H_i$.

Explicitly, and keeping in mind that the χ_i are homomorphisms as they are characters of one-dimensional representations,

$$\chi_0(\tau) = 1, \chi_0(\sigma) = -1,$$

$$\chi_1(\tau) = -1, \chi_1(\sigma) = 1,$$

$$\chi_2(\tau) = -1, \chi_2(\sigma) = -1.$$

Remark 4.2. Note that χ_i defines a *nontrivial* character of the quotient group $G/\langle \tau^2 \rangle \cong \mathbb{Z}_2^2$. Indeed, the χ_i are precisely the pullbacks to G of the three nontrivial characters of $G/\langle \tau^2 \rangle$.

With these characters and subgroups in mind, we define what it is for a G-action on $k[[z]]$ to be *simple*. This definition makes use of the Artin character of a G-action on $k[[z]]$ for k a field, which was defined in section 2.7.

So as to simplify the notation, we will here define, for any two characters ψ and χ of G,

$$\psi(\chi) = \langle \psi, \chi \rangle,$$

where the inner product is defined as in Serre's book [23] to be

$$\langle \psi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g)\chi(g).$$

Definition 4.1.2. Keep the notation as above. We say the G-action is *simple* if $a_\phi(\chi_0) = 2$ and $a_\phi(\chi_1) = a_\phi(\chi_2) \geq 2$.

Remark 4.3.
- *Simple actions* were introduced by Chinburg, Guralnick, and Harbater in [8], where it was shown that such actions satisfy the Bertin obstruction, an obstruction strictly weaker than the Hurwitz tree obstruction.
- As we will see in the next subsection, this may be reinterpreted in terms of the existence of certain fixed points of the G-action in the case of a lift to characteristic 0.
- It is shown in Chapter 3 of Brewis [6] that simple actions can always be defined for G and $k[[z]]$ as above.

Our aim is to show that simple actions do not lift by constructing piecewise linear functions associated to the characters of G that are defined on the Hurwitz tree in the case of a lift. We will do so in two broad steps:

- First, in the following subsection, we will show that the definition of the action being simple is equivalent to the existence of certain fixed points under the G-action in characteristic 0, assuming it lifts.
- Second, in section 4.2 we will construct piecewise linear functions corresponding to the vanishing functions of the sections $gz - z$ for $g \in G$. By the results of Chapter 3 these will be closely related to characters of G, and we can exploit this relation to arrive at a contradiction to simple actions lifting.

4.1.3 Equivalent notion of simple actions

In this subsection we will provide an equivalent definition of the G action being simple, and in doing so prove the existence of a leaf b_0 of the Hurwitz tree on which the piecewise linearity of certain functions will be violated.

Suppose the G-action on $k[[z]]$ lifts to characteristic 0.

Definition 4.1.3. Denote by B the set of leaves of the Hurwitz tree, and B_i the set of leaves with monodromy group H_i (up to equivalence by conjugation). Also, let $B' = B_0 \cup B_1 \cup B_2$, and $B^i = B' \setminus B_i$.

Remark 4.4. Using the harmonicity condition on the Artin character of an edge (the third axiom in the definition of the Hurwitz tree), we know that

$$a_\phi = a_{e_0} = \sum_{b \in B} a_b.$$

The first equality follows from Theorem 2.24, which says the Artin character of G acting on $k[[z]]$ is the Artin character of the Hurwitz tree.

Here, a_b is the Artin character on the edge incident with leaf b. The third axiom tells us that $a_b = u^*_{G_b}$, where G_b is the monodromy group of vertex b and u is the augmentation character, which takes the value -1 for all $g \neq 1$ and $|G_b| - 1$ otherwise. In particular the inner product of the augmentation character with the trivial character is 0. Thus, we may rewrite the above as

$$a_{e_0} = \sum_{b \in B} u^*_{G_b}.$$

We are now able to demonstrate an equivalent notion of the action being simple in the case of a lift.

Lemma 4.5. *Suppose a $\mathbb{Q}_{2^{n+1}}$-action on $k[[z]]$ lifts. Then the action is simple if and only if $|B_0| \geq 2$ and $|B_1| = |B_2| = 1$.*

Proof. Write

$$a_{e_0}(\chi_i) = \sum_{b \in B} u^*_{G_b}(\chi_i) = \sum_{b \in B} \langle u^*_{G_b}, \chi_i \rangle.$$

Using Frobenius reciprocity, this is equivalent to

$$\sum_{b \in B} \langle u_{G_b}, \chi_i|_{G_b} \rangle_{G_b},$$

where we have restricted χ_i to G_b and the inner product is with respect to G_b.

By construction, χ_i is trivial when restricted to H_i or any cyclic subgroup of the form $\langle \tau^{2r} \rangle$, and is a nontrivial irreducible character otherwise when restricted to H_j for $j \neq i$. Thus, as u_G may be written as the sum of all nontrivial characters of G for any group G,

$$\langle u^*_{G_b}, \chi_i \rangle = \begin{cases} 0 & \text{if } [G_b] = H_i \text{ or } [G_b] = \langle \tau^{2r} \rangle \\ 1 & \text{if } [G_b] = H_j \text{ for } j \neq i. \end{cases}$$

Consequentially, we find that

$$a_{e_0}(\chi_i) = |B^i| = |B_j| + |B_k|,$$

where k, i, j are pairwise inequivalent. The latter equality follows from the fact all the B_j's are disjoint.

Returning to the original definition of our action being simple, we end up with four inequalities:

$$|B_1| + |B_2| = 2,$$
$$|B_0| + |B_2| \geq 2,$$
$$|B_1| + |B_0| \geq 2,$$
$$|B_0| + |B_2| = |B_1| + |B_0|.$$

Solving these, we find that the condition of our action being simple implies

$$|B_0| \geq 2, |B_1| = |B_2| = 1.$$

Conversely, if we assume the above equations, we end up with the original definition by working backwards. Thus, our action being simple is equivalent to the existence of leaves of the Hurwitz tree with monodromy groups H_i. □

Remark 4.6. We will denote the unique leaf in B_2 by b_0.

4.2 Piecewise linear functions from characters

Recall from section 3.4 that in the case of a lift to characteristic 0 of a local G-action we get a Hurwitz tree with piecewise linear functions defined for all $g \in G$ with $g \neq 1$. Namely, we obtain functions ϖ_{f_g}, the vanishing functions for sections of the form $f_g = gz - z$, which measure the order of vanishing of f_g on components of the special fiber (corresponding to vertices), and which have slopes $\text{ord}_{p_e}(f_g)$ along edge e.

Furthermore, $\text{ord}_{p_e}(f_g) = -a_e(g)$, and we take this as a definition of $\text{ord}_{p_e}(f_1)$, i.e. $\text{ord}_{p_e}(f_1) = -a_e(1)$.

Similarly, for $g = 1$ we define

$$\varpi_{f_1} = -\sum_{g \neq 1} \varpi_{f_g}.$$

Definition 4.2.1. We define for all vertices $v \in T$ a function $\varpi_v^* : G \to \mathbb{Q}$ via $\varpi_v^*(h) = \sum_{g \in G_v/G} \widehat{\varpi}_{f_{ghg^{-1}}}(v)$.

Similarly, for all edges $e \in T$, define $\text{ord}_{p_e}^* : G \to \mathbb{Q}$ by $\text{ord}_{p_e}^*(h) = \sum_{g \in G_{t(e)}/G} \widehat{\text{ord}}_{p_e}^*(f_{ghg^{-1}})$.

Here $\hat{f}$ denotes a function that is 0 if $g\sigma g^{-1} \notin G_v$ or $G_{t(e)}$, respectively, and is otherwise equal to f.

Remark 4.7. By what was shown in Section 3.4, we see that $\text{ord}_{p_e}^*(h) = -a_e(h)$.

Let ϕ be a character of G.

Definition 4.2.2. We define Λ_ϕ as a function on the vertices of the tree by setting

$$\Lambda_\phi(v) = \langle \phi, \varpi_G^*(v) \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \varpi_{f_g}^*(v).$$

As well, the slope of this function on an edge e is given by

$$\Lambda_\phi(e) = \langle \phi, \text{ord}_{p_e}^* \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \text{ord}_{p_e}^*(f_g).$$

Remark 4.8. We note that Λ_ϕ is a continuous, piecewise-linear function. That is, given any vertex v on the Hurwitz tree and a unique path from v_0 to v along edges e_0 to e_k, $\Lambda_\phi(v) = \sum_e \epsilon_e \Lambda_\phi(e_i)$ where the sum is over the unique edge path. This follows from the continuity of the vanishing function. That is,

$$\Lambda_\phi(v) = \langle \phi, \varpi_G^*(v) \rangle = \langle \phi, \sum_e \epsilon_e \mathrm{ord}_{p_e}^* \rangle = \sum_e \epsilon_e \Lambda_\phi(e),$$

where the sum $\sum_e$ is over the unique path from edge e_0 to the edge whose terminal vertex is v.

Definition 4.2.3. To simplify the notation, for any subset $A \subset B$ of the leaves we will define $\#A_e = |\{b \in B | b > t(e_i) \text{ and } b \in A\}|$.

By the results of the previous section, we have the following lemma.

Lemma 4.9. *Keep the notation as above. We have that*

$$\Lambda_{\chi_i}(e) = -\langle \chi_i, a_e \rangle = -|\{b \in B | b > t(e_i) \text{ and } b \in B^i\}| = -\#B_e^i.$$

The contradiction to the G-action lifting to characteristic 0 will be as follows:

By the piecewise linearity of the Λ_ϕ's, we may calculate $\Lambda_\phi(b_0)$ both directly and by summing over all the slopes along edges leading up to b_0. Here, b_0 is the unique leaf with monodromy group H_2.

Though we will not be able to directly compare the two values due to terms that will depend on the valuation of our ring R, we can construct another piecewise linear function from a character ψ such that Λ_ψ has a greater slope for every edge than the Λ_{χ_i}'s, but is less than the value of $\Lambda_{\chi_0} + \Lambda_{\chi_1}$ at b_0. This will contradict the piecewise linearity of Λ_{χ_i}.

To that end, we define a character of G as follows:

Definition 4.2.4. Let $\chi : H_0 = \langle \tau \rangle \to \mathbb{C}$ be any injective homomorphism. In particular χ is a nontrivial character of H_0. Define

$$\psi = \mathrm{Ind}_{H_0}^G \chi$$

Lemma 4.10. *For any nontrivial cyclic subgroup H of G, $\psi|_h$ is the sum of two non-trivial characters.*

Proof. By the normality of H_0,

$$\psi(g) = \begin{cases} 0 & \text{if } g \notin H_0 \\ 2\chi(g) & \text{if } g \in H_0. \end{cases}$$

Thus $\psi|_{\langle\tau^r\rangle}$ is $2\chi|_{\langle\tau^r\rangle}$ and so is nontrivial for all r such that $\tau^r \neq 1$.

The only other cyclic subgroups of G up to conjugation are H_1 and H_2, both of which are isomorphic to $\mathbb{Z}_4$. Note that it is sufficient to choose H_1 and H_2 as the monodromy groups of the Hurwitz tree are only defined up to conjugation, and none of the results are impacted by choosing a given representative of the conjugacy class.

By definition of ψ,

$$\psi(1) = 2, \psi(\tau^{2^{n-1}}) = -2, \text{ and } \psi(\sigma\tau^k) = 0.$$

In particular we see that $\psi|_H = \lambda_1 + \lambda_2$, where H is any subgroup of G that is isomorphic to $\mathbb{Z}_4$ the λ_i's are irreducible characters of $\mathbb{Z}_4$ satisfying $\lambda_1(1) = i$ and $\lambda_2(1) = -i$. □

Lemma 4.11. *For any edge e of T,*

$$\Lambda_{\chi_0}(e) + \Lambda_{\chi_1}(e) = -\#B_e^0 - \#B_e^1$$

and

$$\Lambda_\psi(e) = -2\#B_e.$$

Proof. Because G_b is a nontrivial cyclic subgroup of G for all $b \in B$, by Lemma 4.10 we have that

$$\langle\psi, u^*_{G_b}\rangle = 2 \text{ for all } b \in B,$$

whence

$$\Lambda_\psi(e) = -2\#B_e.$$

By Lemma 4.9

$$\Lambda_{\chi_0}(e) + \Lambda_{\chi_1}(e) = -\#B_e^0 - \#B_e^1,$$

whence the result.

□

Remark 4.12. We would like to conclude that, as $B^0 \cup B^1 \subset B$,

$$\Lambda_{\chi_0}(e) + \Lambda_{\chi_1}(e) \geq \frac{1}{2}\Lambda_{\psi}(e),$$

however, $B^0 \cap B^1 = b_0 \neq \emptyset$, so we cannot immediately conclude this. To get around this, we make a new definition.

Definition 4.2.5. Let $\Lambda^*_{\chi_i}(e) = \Lambda_{\chi_i}(e) + 1$, and $\Lambda^*_{\psi}(e) = \Lambda_{\psi}(e) + 2$.

Remark 4.13. This eliminates any contribution from the vertex b_0, as now we have that $\Lambda^*_{\chi_i}(e) = -\#A^i_e$, where $A^i_e = B^i_e \setminus \{b_0\}$. Similarly, $\Lambda^*_{\psi}(e) = -\#A_e$, where $A_e = B_e \setminus \{b_0\}$.

Remark 4.14. The continuity of the vanishing function $\Lambda_{\phi}(v)$ for any character ϕ of G is such that

$$\Lambda_{\phi}(v) = \sum_e \epsilon_e \Lambda_{\phi}(e),$$

where the sum is over the unique edge path from e_0 to vertex v. If we replace the slopes $\Lambda_{\chi_i}(e)$ by $\Lambda^*_{\chi_i}(e)$, we obtain a new value $\Lambda^*_{\chi_i}(v)$ at vertex v, namely

$$\Lambda^*_{\chi_i}(v) = \Lambda_{\chi_i}(v) + \sum_e \epsilon_e.$$

We similarly have

$$\Lambda^*_{\psi}(v) = \Lambda_{\psi}(v) + 2\sum_e \epsilon_e.$$

Lemma 4.15. *With the notation as above, we have*

$$\Lambda^*_{\chi_0}(b_0) + \Lambda^*_{\chi_1}(b_0) \geq \frac{1}{2}\Lambda^*_{\psi}(b_0).$$

Proof. As $A^0_e \cup A^1_e \subset A_e$ and $A^0_e \cap A^1_e = \emptyset$ for all edges e leading up to b_0, we have that $\sum_e \epsilon_e(\Lambda^*_{\chi_0}(e) + \Lambda^*_{\chi_1}(e)) \geq \sum_e \epsilon_e(\frac{1}{2}\Lambda^*_{\psi}(e))$ holds true, whence the result. □

Remark 4.16. The functions Λ^*_{ϕ} are closely related to the *density* of a Hurwitz tree in Chapter 3 of Brewis's thesis [6]. Our functions emphasize the continuity of the vanishing functions of elements of the form $\sigma z - z$.

Lemma 4.15 compared the values of $\Lambda^*_{\chi_0}(b_0) + \Lambda^*_{\chi_1}(b_0)$ to $\frac{1}{2}\Lambda^*_{\psi}(b_0)$ by using the continuous piecewise-linearity of these functions to sum their values along the preceding edges. We will now calculate directly the values of these functions at b_0.

Lemma 4.17. *We have that*

$$\Lambda^*_{\chi_0}(b_0) + \Lambda^*_{\chi_1}(b_0) < \frac{1}{2}\Lambda^*_{\psi}(b_0).$$

Proof. We will directly compute both of $\Lambda^*_{\chi_0}(b_0) + \Lambda^*_{\chi_1}(b_0)$ and $\frac{1}{2}\Lambda^*_{\psi}(b_0)$ and compare them.

Recall from section 3.4.2. that we can explicitly relate the depth character to the vanishing function at a vertex v once we know the metric on the tree. Namely, for $g \in G_v$, $g \neq 1$,

$$\delta'_v(g) = -|G_v|\varpi_{f_g}(v) + \alpha_v,$$

where

$$\alpha_v = |G_v|(\epsilon_{e_0} + \cdots + \epsilon_{e_k}).$$

Here we assume that the unique path from v_0 to v is given along edges $e_0, \ldots, e_k$.

At the vertex b_0 we have that $\alpha_{b_0} = |G_v|(\epsilon_{e_0} + \ldots + \epsilon_{e_l})$, $t(e_l) = b_0$, whence $\delta'_{G_{b_0}}(g) = -|G_{b_0}|\varpi_{f_g}(b_0) + \alpha_{b_0}$ by definition of the vanishing function.

As $|G_{b_0}| = 4$ the vanishing function is related to the depth character via

$$\varpi_{f_g}(b_0) = \frac{-1}{4}\delta'_{b_0} + \frac{1}{4}\alpha_{b_0} \tag{4.1}$$

Thus,

$$\Lambda_{\chi}(b_0) = \frac{-1}{4}\langle \chi, \delta_{b_0} + \alpha_{b_0}u^*_{G_{b_0}}\rangle = \frac{-1}{4}\langle \chi, \delta_{b_0}\rangle - \sum_e \epsilon_e,$$

and similarly

$$\Lambda_{\psi}(b_0) = \frac{-1}{4}\langle \psi, \delta_{b_0} - \alpha_{b_0}u^*_{G_{b_0}}\rangle = \frac{-1}{4}\langle \psi, \delta_{b_0}\rangle - 2\sum_e \epsilon_e.$$

Thus, from Remark 4.14 we have $\Lambda^*_{\chi_i}(b_0) = \frac{-1}{4}\langle \chi, \delta_{b_0}\rangle$ and similarly $\Lambda^*_{\psi}(b_0) = \frac{-1}{4}\langle \psi, \delta_{b_0}\rangle$

Recall that at the leaves of a Hurwitz tree the depth character is given by $(\delta^{\text{mult}})^*$.

By Lemma 2.1 we have that

$$\frac{1}{2}\Lambda_{\psi}^*(b_0) = \frac{-1}{4}3\text{val}_{\pi}(2),$$

and

$$\Lambda_{\chi_0}^*(b_0) + \Lambda_{\chi_1}^*(b_0) = \frac{-1}{4}4\text{val}_{\pi}(2),$$

where $\text{val}_{\pi}(2)$ is the valuation of 2 with respect to the uniformizer π of R.

Thus,

$$\Lambda_{\chi_0}^*(b_0) + \Lambda_{\chi_1}^*(b_0) < \frac{1}{2}\Lambda_{\chi}^*(b_0),$$

as desired.

□

Our main theorem follows from the above two lemmas:

Theorem 4.18. *Simple quaternionic G-actions on $k[[z]]$ fail to lift to characteristic* 0.

Proof. By Lemma 4.15,

$$\Lambda_{\chi_0}^*(b_0) + \Lambda_{\chi_1}^*(b_0) \geq \frac{1}{2}\Lambda_{\psi}^*(b_0)$$

and by Lemma 4.17,

$$\Lambda_{\chi_0}^*(b_0) + \Lambda_{\chi_1}^*(b_0) < \frac{1}{2}\Lambda_{\psi}^*(b_0).$$

This is a contradiction, and so the action fails to lift. □

4.3 Metacyclic groups

4.3.1 Goal of the metacyclic example

A group is referred to as metacyclic if it is isomorphic to an extension of a cyclic group by a cyclic group.

This example was inspired by Bouw's and Wewer's notes [5].

The goal of this example is to show that particular instances of metacyclic groups acting on curves in positive characteristic fail to lift to characteristic 0. This will be done through the Hurwitz tree obstruction. Namely, we will show that the harmonicity of the Artin character of the Hurwitz tree (the third axiom of the Hurwitz tree) will be violated if the action is assumed to lift.

In the following subsection we will define the group G and its action on a curve. For every $p > 2$ we will have a different group and curve. The genus of the curve will be calculated in Lemma 4.20, which will imply that the Hurwitz bound will be violated for all $p > 41$ (see Section 1.5 for a discussion on the Hurwitz bound). This will imply in particular that the action fails to lift for $p > 41$. Because the Hurwitz tree obstruction will hold for all odd primes, this will serve to illustrate that the Hurwitz tree obstruction is more informative than the Hurwitz bound for this example.

Following this we will use the theory in Serre's book [23] on local field extensions to find out the form of the Artin character of the G-action. This will allow us to show that the Artin character of the Hurwitz tree - assuming it exists - will differ from the Artin character of the G-action on the curve, thus contradicting Theorem 2.24.

4.3.2 Set-up for metacyclic group example

Let $p > 2$ be a prime.

G will be defined by its action on a curve in $\mathbb{P}^2_{\bar{k}}$, where $\bar{k}$ is the algebraic closure of $k = \mathbb{F}_p$.

Definition 4.3.1. Let the curve $C \subset \mathbb{P}^2$ be defined by $y^p - y = x^{p+1}$.

G is generated by the elements σ and τ, where

$$\sigma(x) = x, \sigma(y) = y + 1$$

and

$$\tau(x) = \zeta x, \tau(y) = \zeta^{p+1} y$$

for $\zeta \in \bar{k}$ a fixed primitive $(p^2 - 1)$ root of unity.

Remark 4.19. Thus $\tau\sigma\tau^{-1}(y) = y + \zeta^{-p-1} \neq \sigma(y)$, whence G is a non-abelian group. In fact, we have that $G \cong \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/(p^2 - 1)\mathbb{Z}$.

We may view G as the Galois group for a tower of fields

$$k(x) \subset k(x)[x_1]/(x_1^{p^2-1} - x) \subset L = k(x_1)[y]/(y^p - y - x_1^{p+1}).$$

This is a Kummer extension of $k(x_1)/k(x)$ followed by an Artin-Schreier extension of $L/k(x_1)$ of degrees $p^2 - 1$ and p, respectively.

We conclude this subsection by showing the Hurwitz bound is violated for $p > 41$. To do so we must calculate the genus of C.

Lemma 4.20. *The genus of the curve $C = k[x, y]/(y^p - y - x^{p+1})$ is $\frac{p(p-1)}{2}$.*

Proof. By Kummer's Theorem [27] both field extensions $k(x_1)/k(x)$ and $L/k(x_1)$ are totally ramified over ∞. Denote the unique point at ∞ in C by P, and in $k(x_1)$ by P'. We will calculate the ramification filtration above ∞ for these two extensions. We focus on the Artin-Schreier extension with Galois group $\langle\sigma\rangle \cong \mathbb{Z}/p\mathbb{Z}$.

By proposition 3.7.8c in Stichtenoth [27] we find that the degree of the different $d(P/P')$ is $(p+2)(p-1)$. Combined with Hilbert's different formula (Proposition 4.9 of Chapter 1 in Cassels and Frohlich [1]), we conclude that the ramification filtration for P'/P is $\mathbb{Z}/p\mathbb{Z} = G_0 = \ldots = G_{p+1} \supset G_{p+2} = \{1\}$.

This allows us to calculate the genus of $k[x][y]/(y^p - y - x^{p+1})$ by looking at the morphism of it to $\mathbb{P}^1_{\bar{k}}$ defined by the action of the Galois group $\langle\sigma\rangle$. Namely, by the Riemann-Hurwitz equation and the fact that ∞ is the only place where the cover is ramified we find that the genus of $k[x][y]/(y^p - y - x^{p+1})$ is $p(p-1)/2$. □

Corollary 4.21. *For G and C as above, the action of G on C does not lift to characteristic 0 for $p > 41$. In particular the Hurwitz bound is violated for such primes.*

Proof. Recall that the Hurwitz bound says that for any curve defined over characteristic 0, any automorphism group G of the curve satisfies $|G| < 84(g-1)$ where g is the genus of the curve.

Suppose the G-action lifts to characteristic 0. The arithmetic surface for which C is the special fiber is such that its generic fiber having the same genus as the special fiber. This follows by the assumptions of the curves to which the lifting problem applies - see Chapter 1 for the details.

Thus G acts on a curve in characteristic 0 of the same genus as C, and by inspection we see that for $p > 41$ the Hurwitz bound is violated, whence the result. □

4.3.3 Showing the metacyclic group action fails to lift

If the G-action define in the previous subsection lifts to characteristic 0, it must lift locally about each closed point by the local-to-global principle (Theorem 1.39), and in particular it must lift locally at the point at ∞ on C. Our aim is to show this fails.

Recall that we may view G as the Galois group for a tower of function fields

$$k(x) \subset k(x)[x_1]/(x_1^{p^2-1} - x) \subset L = k(x_1)[y]/(y^p - y - x_1^{p+1})$$

of the curves $\bar{k}[x]$, $\bar{k}[x_1]$, and C, respectively.

Localizing these curves at ∞ and taking the completion gives us a tower of fields

$$\bar{k}((x)) \subset \bar{k}((x_1)) \subset \bar{k}((z)),$$

for which x, x_1 and z are local parameters at ∞ for the respective curves.

To proceed further, we need a definition.

Definition 4.3.2. Let $k((z))/k((x))$ be a Galois extension with Galois group G. We assume that the integral closure of $k[[x]]$ in $k((z))$ is $k[[z]]$. Define $i_G : G \to \mathbb{Z}$ by

$$i_G(g) = \mathrm{val}_z(gz - z),$$

where val_z is the valuation defining the DVR $k[[z]]$ whose fraction field is $k((z))$.

Remark 4.22.
- As shown in Chapter IV of Serre [23] this definition is independent of a choice of uniformizer z of $k[[z]]$.

- Note that $i_G(g) = -a_\phi(g)$, where a_ϕ is the Artin character of the G-action $\phi : G \to \mathrm{Aut}_k k[[z]]$.

To ease notation we will make a definition.

Definition 4.3.3. Keep the notation of the previous definition, and let H be a normal subgroup of G. For $g \in G$ and $h \in G/H$, we will write $g \to h$ to mean $h = g \pmod{H}$. That is, the image of g in the quotient group G/H is h.

The following lemma from Serre [23] is crucial to our main result:

Lemma 4.23 (Proposition IV.1.3 of [23])**.** *Let H be a normal subgroup of G, where G acts k-linearly on $k[[z]]$. For every $h \in G/H$,*

$$i_{G/H}(h) = \frac{1}{e'} \sum_{g \to h} i_G(g),$$

where $e' = e_{k((z))/K'}$ is the ramification index of $k((z))/K'$ for K' the fixed field of H.

We will use the above lemma to determine the value of the Artin character of G acting on $k[[z]]$ for $g \notin \langle \sigma \rangle$.

Lemma 4.24. *For all $g \in G \setminus \langle \sigma \rangle$, $i_G(g) = 1$.*

Proof. Let $H = \langle \sigma \rangle$.

Note that $k((x_1))/k((x))$ is a Galois Kummer extension, with Galois group $G/H \cong \mathbb{Z}/(p^2-1)\mathbb{Z}$. By considering the cardinalities of the groups, for any $h \in G/H$ we have that

$$|\{g \in G : g \to h\}| = p.$$

By the definition of G, for all $h \in G/H$, $h(x_1) = cx_1$ for some constant c, and so $i_{G/H}(h) = 1$ for all $h \neq 1$.

It follows that $i_G(g) = 1$ for all $g \notin \langle \sigma \rangle$. Certainly each $g \in G$ satisfies $i_G(g) \geq 1$ by the fact the extensions of the fields are totally ramified above ∞ (see the proof of Lemma 4.20), but it is at most one as by Lemma 4.23 and for $g \notin H$

$$i_{G/H}(h) = 1 = \frac{1}{e'} \sum_{g \to h} i_G(g) = \frac{1}{p} \sum_{g \to h} i_G(g).$$

Thus, as $|\{g \in G : g \to h\}| = p$, we must have $i_G(g) = 1$ for all $g \notin \langle \sigma \rangle$. $\square$

Remark 4.25. Thus the Artin character of a Hurwitz tree associated to the lifted action of G takes value -1 for all $g \notin \langle\sigma\rangle$. We will use this to arrive at a contradiction. First we will show $\sigma \in G_b$ for some leaf b of the Hurwitz tree.

Lemma 4.26. *Assume that the action of G on C lifts locally at the point at ∞, so that we have an associated Hurwitz tree $\mathcal{T}$.*

There exists a leaf $b_0 \in B$ of the Hurwitz tree such that $\sigma \in G_{b_0}$.

Proof. By the normality of $\langle\sigma\rangle$ we have that

$$g\sigma g^{-1} \in \langle\sigma\rangle \forall g \in G.$$

Thus if $\sigma \notin G_b$ for all $b \in B$ we would have

$$a_{\mathcal{T}}(\sigma) = 0$$

contradicting the fact that

$$a_{\phi}(\sigma) = -p - 1$$

from the ramification filtration for $\langle\sigma\rangle$ in the Artin-Schreier extension (i.e. $G_i = H_i$ for all subgroups of G). □

Thus, without loss of generality σ is contained in G_{b_0} for some leaf b_0 of the tree.

We may now prove our main result.

Theorem 4.27. *The local G-action at ∞ does not lift to characteristic 0.*

Proof. We consider two possibilities:

- $G_{b_0} = \langle\sigma\rangle$;
- $G_{b_0} \neq \langle\sigma\rangle$.

In the first case the normality of $\langle\sigma\rangle$, combined with the definition of the induced representation, implies that

$$u^*_{G_{b_0}}(\sigma) = -(p^2-1) < -p-1 = a_{\mathcal{T}}(\sigma).$$

As $a_{\mathcal{T}}(\sigma) = \sum_{b\in B} u^*_{G_b}(\sigma)$, this is a contradiction.

In the second case there exists some $g \notin \langle\sigma\rangle$ that generates G_{b_0} of the form $\sigma^m\tau^n$. By left multiplying by σ we have that $\tau^n \neq 1$ is in G_{b_0}, and as this must commute with σ we have that n is some multiple of $p-1$. In particular, we have 1 and τ are in separate cosets of G_{b_0} as $\tau \notin G_{b_0}$. Letting 1 and τ be among the coset representatives of G_{b_0} in G, we find that, by definition of the induced character and as 1 and τ commute with τ^n,

$$u^*_{G_{b_0}}(\tau^n) < -1,$$

contradicting Lemma 4.24.

In either case we arrive at a contradiction, and so the action cannot lift to characteristic 0. □

Remark 4.28. Note that when $p = 2$, G is cyclic. In this case the Oort Conjecture (Theorem 1.42)informs us that the action must lift.

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