Exponentially Dense Matroids

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis deals with questions relating to the maximum density of rank-n matroids in a minor-closed class.

Consider a minor-closed class $\mathcal{M}$ of matroids that does not contain a given rank-2 uniform matroid. The growth rate function is defined by

$$h_{\mathcal{M}}(n) = \max\left(|M| \,:\, M \in \mathcal{M} \text{ simple}, r(M) \le n\right).$$

The Growth Rate Theorem, due to Geelen, Kabell, Kung, and Whittle, shows that the growth rate function is either linear, quadratic, or exponential in n. In the case of exponentially dense classes, we conjecture that, for sufficiently large n,

$$h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - c,$$

where q is a prime power, and k and c are non-negative integers depending only on $\mathcal{M}$. We show that this holds for several interesting classes, including the class of all matroids with no $U_{2,t}$-minor.

We also consider more general minor-closed classes that exclude an arbitrary uniform matroid. Here the growth rate, as defined above, can be infinite. We define a more suitable notion of density, and prove a growth rate theorem for this more general notion, dividing minor-closed classes into those that are at most polynomially dense, and those that are exponentially dense.

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Dedication

To Mum, for teaching me place value with old pennies.

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Chapter 1

Introduction

This thesis deals with questions concerning the density of matroids in a minor-closed class $\mathcal{M}$ that does not contain some uniform matroid $U_{a,b}$, where $2 \le a \le b$. The material is divided into two parts. Part I focuses on the case where $a = 2$; in this case, Kung [26] showed that the number of points in a matroid in $\mathcal{M}$ is bounded by a function of its rank. This function is the 'growth rate function' of $\mathcal{M}$. We will almost exactly determine the growth rate functions for many interesting classes of matroids; we state here two such theorems that are particular highlights.

Theorem 1.0.1. *Let $\ell \ge 2$ be an integer, and q be the largest prime power such that $q \le \ell$. If M is a simple matroid with no $U_{2,\ell+2}$-minor and of sufficiently large rank, then $|M| \le \frac{q^{r(M)}-1}{q-1}$.*

We will also show that this bound is best-possible, and that equality holds if and only if M is a projective geometry over $\mathrm{GF}(q)$.

Theorem 1.0.2. *Let q be a prime power, and $k \ge 0$ be an integer. If M is a simple $\mathrm{GF}(q^2)$-representable matroid with no $\mathrm{PG}(k+1, q^2)$-minor and of sufficiently large rank, then $|M| \le \frac{q^{r(M)+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1}$.*

We show that this bound, too, is best-possible, and for each $r \ge k$ we give an example of a rank-r matroid for which equality holds. This theorem is corollary to a striking result, Theorem 1.5.7, concerning growth rates of minor-closed classes of $\mathrm{GF}(q^2)$-representable matroids in general.

Part II of this thesis addresses the case where $a > 2$. For each $r \ge 2$, it is easy to construct examples of rank-r matroids with no $U_{a,b}$-minor, and arbitrarily many points. However, Geelen and Kabell [13] introduced a coherent notion of density by showing that a rank-r matroid with no $U_{a,b}$ minor can be covered with a bounded number of sets of rank less than a. We do not state the main result of Part II (Theorem 5.4.3) here, but it has the following as a corollary:

Theorem 1.0.3. *For all integers $2 \le a \le b$, and $k \ge 1$, there is an integer p such that, if M is a matroid with no $U_{a,b}$-minor, and no rank-k projective geometry minor, then $E(M)$ admits a partition into at most $r(M)^p$ sets of rank less than a.*

The remainder of this introduction is mostly concerned with Part I of this thesis. We will cover some basic concepts and terminology in matroid theory, discuss important examples of minor-closed classes, mention some related literature, and give proofs of preliminary results that will shed light on later techniques, before stating our main results on growth rates and discussing further potential work. We introduce Part II in Chapter 5.

1.1 Minor-Closed Classes of Matroids

1.1.1 Basics

A matroid is a mathematical object formalising the intuitive notion of dependence. An excellent reference on matroid theory is found in [35]; here, we introduce the basic concepts needed in this thesis. For a reader already familiar with matroid theory, the material in Section 1.1 is standard, with the exception of the definitions of $\varepsilon(M)$, $M /\!\!/ C$ and $h_{\mathcal{M}}(n)$ that soon appear, and the discussion in 1.1.5.

Matroids have many equivalent definitions, and the one we give is in terms of its rank function. A *matroid* M is a pair (E, r), where E, the *ground set* of M, is a finite set, and $r : 2^E \to \mathbb{Z}$, the *rank function* of M, is a function satisfying three axioms:

1. $0 \leq r(X) \leq |X|$ for all $X \subseteq E$, and
2. $r(X) \leq r(Y)$ for all $X \subseteq Y \subseteq E$, and
3. $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$ for all $X, Y \subseteq E$.

We write $|M|$ for $|E|$ and $r(M)$ for $r(E)$; this is the *rank of* M. We view a matroid geometrically, and much of the remainder of our terminology reflects this. If $X \subseteq E$, then the *closure* of X in M, written $\text{cl}(X)$, is the set $\{e \in E : r(X \cup \{e\}) = r(X)\}$; in particular, $X \subseteq \text{cl}(X) = \text{cl}(\text{cl}(X))$. A set $F \subseteq E$ is a *flat* of M if $F = \text{cl}(F)$; a *point*, *line*, or *plane* is a flat of rank 1, 2 or 3 respectively, and a *hyperplane* is a flat of rank $r(M) - 1$. A line L of M is *long* if it contains at least three points of M; in general, we write $\varepsilon(M)$ for the number of points of M, and $\varepsilon_M(X)$ for the number of points of $M|X$, for any $X \subseteq E(M)$.

A set $X \subseteq E$ is *independent* in M if $r(X) = |X|$; otherwise it is *dependent*. A maximal independent set of M is a *basis*, and a minimal dependent set of M is a *circuit*; a matroid is alternatively determined by its independent sets, by its circuits, by its flats, and by its bases.

A *loop* of M is an element $e \in E$ such that $r(\{e\}) = 0$; any other element of E is a *nonloop*. Two nonloops e, f are *parallel* in M if they are contained in the same point of M. The nonloops of M can thus be partitioned into *parallel classes*, each of which is a set of rank 1 in M. If M is loopless, then its parallel classes are exactly its points; a matroid is *simple* if it is loopless, and each parallel class has size 1. The *simplification* of M, for which we write $\text{si}(M)$, is the (simple) matroid whose ground set is the set of points of M, where the rank in $\mathrm{si}(M)$ of a subset of points is given by the rank of their union in M. Thus, $\varepsilon(M) = |\,\mathrm{si}(M)|$. Two matroids M and N are equal *up to simplification* if $\mathrm{si}(M) \cong \mathrm{si}(N)$.

To distinguish between M and other matroids, we usually write $E(M)$ and r_M for the ground set and rank function of M. If $Y \in E(M)$, then the *deletion* and *contraction* of Y in M, denoted respectively by $M\backslash Y$ and M/Y, are matroids with ground set $E(M) - Y$, and rank functions defined by $r_{M\backslash Y}(X) = r_M(X)$, and $r_{M/Y}(X) = r_M(X \cup Y) - r_M(Y)$. A matroid N obtained from M by contracting a set C, and deleting a set D, where $C, D \subseteq E(M)$ are disjoint, is written $M/C\backslash D$, and is referred to as a *minor* of M; the 'minor' relation is a quasi-order on the set of matroids. A minor is a *restriction* or *contraction-minor* of M if $C = \varnothing$ or $D = \varnothing$ respectively; we write $M|X$ for the restriction $M\backslash(E(M) - X)$. For a matroid N, an *N-minor* of a matroid M is a minor of M, isomorphic to N. We define an *N-restriction*, and *N-contraction-minor* similarly. (A matroid isomorphic to $\mathrm{si}(M)$ is obtained from M by deleting all loops and all but one element from each parallel class, so M has a $\mathrm{si}(M)$-restriction.) A class $\mathcal{M}$ of matroids is *minor-closed* if it is closed under taking minors and under isomorphism.

We now define our density measure. If $\mathcal{M}$ is a class of matroids, then the *growth rate function* of $\mathcal{M}$, written $h_{\mathcal{M}}$, is the function defined for all integers $n \geq 0$ by:

$$h_{\mathcal{M}}(n) = \max(\varepsilon(M) : M \in \mathcal{M}, r(M) \leq n).$$

This maximum may fail to exist, so we allow $h_{\mathcal{M}}(n) = \infty$, and if this is the case for some n, then it is clearly the case for all larger n; this in particular holds when $\mathcal{M}$ contains all simple rank-2 matroids. We also remark that, if $\mathcal{M}$ contains matroids of arbitrarily large rank, then $h_{\mathcal{M}}(n) \geq n$ for all $n \geq 0$. If $h_{\mathcal{M}}(n)$ is finite, then we call a rank-n matroid $M \in \mathcal{M}$ *extremal in* $\mathcal{M}$ if $\varepsilon(M) = h_{\mathcal{M}}(n)$; in other words, if M is a matroid of maximum density in $\mathcal{M}$. This term is used differently in [26].

A variation of contraction that we use extensively in this thesis is *projection from a set* C. If $C \subseteq E(M)$, then $M /\!\!/ C$ denotes the matroid with ground set $E(M)$, and rank function defined by $r_{M/\!\!/C}(X) = r_M(X \cup C) - r_M(C)$. We say this matroid is obtained from M by *projecting M from C*. Note that $M/\!\!/C\backslash C = M/C$, and $r_{M/\!\!/C}(C) = 0$, so C contains only loops in $M/\!\!/C$, and $M/\!\!/C$ is obtained from M/C just by adding loops to its ground set. Contraction and projections from sets thus behave very similarly; if N is a loopless matroid obtained from M by deletions and projections from subsets of $E(M)$, then N is a minor of M, and if $\mathcal{M}$ is a minor-closed class that is closed under adding loops, then $\mathcal{M}$ is also closed under projections from sets.

Very seldom are we concerned with non-simple minors of a matroid, minor-closed classes that are not closed under adding loops, or matroid properties that are altered by adding

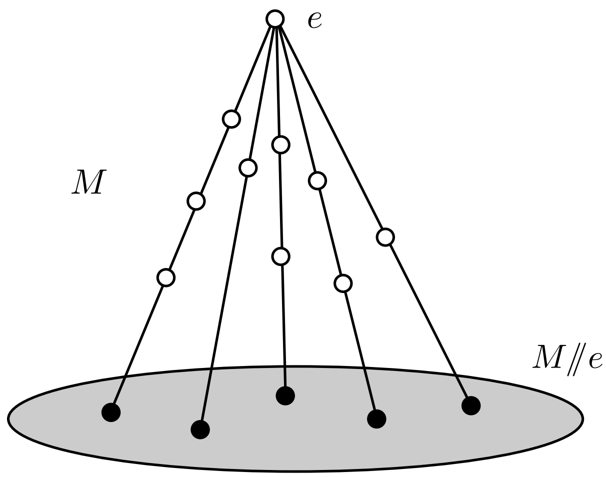

Figure 1.1: A geometric illustration of how points of $M /\!\!/ e$ correspond to lines of M through e. Here, $\varepsilon(M) = 10$ and $\varepsilon(M /\!\!/ e) = 5$.

loops, and we will usually without mention use projection from sets in lieu of contraction for the sake of convenience. For example, lemmas such as the following easily proved one can be stated more concisely in terms of projection:

Lemma 1.1.1. *If e is a nonloop of a matroid M, then a set $P \subseteq E(M)$ is a point of $M /\!\!/ e$ if and only if P is a line of M containing e.*

A geometric view of this lemma can be seen in Figure 1.1.

We conclude by defining an important special example of a matroid. If a and b are integers with $0 \le a \le b$, and M is a matroid such that $|M| = b$, and $r_M(X) = \min(a, |X|)$ for all $X \subseteq E(M)$, then X is a rank-a *uniform matroid* on b elements; we write $M \cong U_{a,b}$. It is easy to check that if M is a rank-2 matroid, then $\mathrm{si}(M) \cong U_{2,\varepsilon(M)}$; in particular, any simple rank-2 matroid is uniform.

1.1.2 Representable Matroids and Projective Geometries

Let $\mathbb{F}$ be a field, and A be an $S \times E$ matrix with entries in $\mathbb{F}$. The matrix A determines a rank function $\mathrm{rank}_A : 2^E \to \mathbb{Z}$ on its subsets of columns. The matroid *represented by* A, for which we write $M(A)$, is the matroid on ground set E, with rank function defined by

$r_{M(A)}(X) = \mathrm{rank}_A(X)$. One can verify that a column of A is a loop of $M(A)$ if and only if it is a zero vector, and that two nonzero columns of A are parallel in $M(A)$ if and only if they are parallel vectors. Moreover, applying elementary row operations and nonzero column scalings to A does not change the function rank_A, and therefore does not change $M(A)$.

A matroid M is $\mathbb{F}$-*representable* if $M = M(A)$ for some matrix A with entries in $\mathbb{F}$. In this case, A is an $\mathbb{F}$-*representation* of M. We write $\mathcal{L}(\mathbb{F})$ for the class of $\mathbb{F}$-representable matroids, and $\mathcal{L}(q)$ for $\mathcal{L}(\mathrm{GF}(q))$ for each prime power q. By considering operations on the relevant matrix, one can show that the class $\mathcal{L}(\mathbb{F})$ is closed under minors and adding loops.

For each integer $r > 0$, any simple, rank-r matroid $M \in \mathcal{L}(q)$ is represented by a matrix over $\mathrm{GF}(q)$ with r rows, in which the columns are nonzero, pairwise nonparallel vectors in $\mathrm{GF}(q)^r$. Each of the $q^r - 1$ nonzero vectors in $\mathrm{GF}(q)^r$ lies in a parallel class of $q-1$ vectors, so we have $|M| \leq \frac{q^r-1}{q-1}$. If equality holds, then M contains exactly one vector from each parallel class, and therefore is determined uniquely up to isomorphism. A simple, rank-r $\mathrm{GF}(q)$-representable matroid M with $|M| = \frac{q^r-1}{q-1}$ is a *projective geometry* over $\mathrm{GF}(q)$, and we write $M \cong \mathrm{PG}(r-1, q)$. The matroid $\mathrm{PG}(r-1, q)$ has every simple rank-r matroid in $\mathcal{L}(q)$ as a restriction. The growth rate function for $\mathcal{L}(q)$ follows from this fact:

Proposition 1.1.2. *$h_{\mathcal{L}(q)}(n) = \frac{q^n-1}{q-1}$ for all n.*

1.1.3 Graphic Matroids

Let $G = (V, E)$ be a graph, possibly with multiple edges and loops. The *cycle matroid of* G, for which we write $M(G)$, is the matroid with ground set E, and rank function given by $r_{M(G)}(X) = |V| - n(X)$, where $n(X)$ denotes the number of components of the graph (V, X). Note that, if G is connected, then $r(M(G)) = |V| - 1$. The matroid $M(G)$ is perhaps more naturally characterised by its set of circuits, which are exactly the edge-sets of cycles of G, or by its collection of independent sets, which are exactly the edge sets of the acyclic subgraphs of G. It is straightforward to see that loops and parallel edges of G correspond exactly to loops and parallel elements of M, and therefore that $M(G)$ is a simple matroid if and only if G is a simple graph.

A matroid M is *graphic* if $M = M(G)$ for some graph G. The class of graphic matroids, for which we write $\mathcal{G}$, is closed under minors and adding loops, and in fact, contracting and deleting elements of $M(G)$ correspond exactly to contracting and deleting edges of G. Moreover, graphic matroids are $\mathbb{F}$-representable for any field $\mathbb{F}$. One can show that any rank-r graphic matroid is equal to $M(G)$ for some graph G on $r+1$ vertices, and therefore that any simple, rank-r graphic matroid is a restriction of the cycle matroid $M(K_{r+1})$ of the $(r+1)$-vertex clique. This gives the growth rate function for this class:

Proposition 1.1.3. *$h_{\mathcal{G}}(n) = \binom{n+1}{2}$ for all n.*

1.1.4 Duality

If M is a matroid, then the *dual* of M, written M^*, is the matroid with ground set $E(M)$, and rank function defined by $r_{M^*}(X) = |X| - r(M) + r_M(E(M) - X)$ for all $X \subseteq E(M)$. Duality, while not directly relevant to the theorems of this thesis, will occur in some examples we consider, and the structural conjectures we discuss later, so we briefly survey it here.

Duality has many attractive properties. The dual of the uniform matroid $U_{a,b}$ is $U_{b-a,b}$. The bases of M^* are exactly the complements of the bases of M, so any matroid M satisfies $M^{**} = M$. Contraction and deletion are dual to each other in the sense that $(M/e)^* = M^*\backslash e$, and $(M\backslash e)^* = M^*/e$ for any $e \in E(M)$. It follows from this that, if $\mathcal{M}$ is a minor-closed class of matroids, then the set of duals of matroids in $\mathcal{M}$ is also a minor-closed class. One example is the class $\mathcal{G}^*$ of duals of graphic matroids; we call these the *cographic matroids*. The dual of an $\mathbb{F}$-representable matroids is also $\mathbb{F}$-representable, and therefore the matroids in $\mathcal{G}^*$ are representable over every field, and the class $\mathcal{L}(\mathbb{F})$ is closed under duality.

A final important property is that matroid duality coincides with planar graphic duality; if G is a planar graph with a planar dual G^*, then $M(G)^* = M(G^*)$. Therefore, all planar-graphic matroids are both graphic and cographic; in fact, one can show that this exactly characterises the class $\mathcal{P}$ of planar-graphic matroids; we have $\mathcal{P} = \mathcal{G} \cap \mathcal{G}^*$. A classical result of Euler states that any simple planar graph on $v \geq 3$ vertices has at most $3v - 6$ edges, with equality holding for planar triangulations; from this, together with some basic calculations, the growth rate function for both the planar-graphic and cographic matroids can be deduced:

Proposition 1.1.4. *$h_{\mathcal{P}}(n) = h_{\mathcal{G}^*}(n) = 3n - 3$ for all $n \geq 2$.*

1.1.5 Projections

We now define a more general quasi-order on matroids than minors. If M is a matroid, then a matroid M^+ is an *extension* of M (by e) if $E(M^+) = E(M) \cup \{e\}$, and $M = M^+\backslash e$.

A *projection* of M is a matroid of the form M^+/e, where M^+ is an extension of M by e. (In [35], projections are referred to as *quotients*.) Note that any projection of M has the same ground set as M. If M^+ is an extension of M by e, and e is parallel in M^+ to some $f \in E(M)$, then M^+ is a *parallel extension of* M (by e). In this case, $M^+/e = M /\!\!/ f$, meaning that projection of M from f is indeed a special case of projection in general.

If all circuits of M^+ containing e have the form $B \cup \{e\}$, where B is a basis of M, then the point e has been added 'as freely as possible' to M, and we say M^+ is the *free extension* of M by e. In this case, we call the matroid M^+/e the *truncation* of M, which we denote $T(M)$. This is another special case of projection. If $r(M) > 2$, then e is contained in no long line of M^+, and $\varepsilon(T(M)) = \varepsilon(M)$.

For an integer $k \geq 0$, a *k-element projection* of M is a matroid obtained by projecting M k times; equivalently, this is a matroid of the form M^+/K, where $M^+ \backslash K = M$, and $|K| = k$. If K is independent and co-independent in M^+, then this projection has rank $r(M) - k$. Generalising a single truncation, we write $T^k(M)$ for the rank-$(r(M) - k)$ matroid obtained by truncating M k times; if $r(M) > k + 1$, then $\varepsilon(T^k(M)) = \varepsilon(M)$.

Unlike projection from a set, projection in general does not preserve properties such as representability and graphicness. However, projections do not behave too wildly, as the following easily proved fact shows:

Proposition 1.1.5. *If $k \geq 0$ is an integer, $\mathcal{M}$ is a minor-closed class of matroids, and $\mathcal{M}'$ is the set of all k-element projections of matroids in $\mathcal{M}$, then $\mathcal{M}'$ is minor-closed.*

In fact, the class $\mathcal{M}'$ contains all k-element truncations of matroids in $\mathcal{M}$, and the growth rate function of $\mathcal{M}'$ can thus be easily computed in terms of that of $\mathcal{M}$:

Proposition 1.1.6. *If $k \geq 0$ is an integer, $\mathcal{M}$ is a minor-closed class of matroids, and $\mathcal{M}'$ is the set of all k-element projections of matroids in $\mathcal{M}$, then $h_{\mathcal{M}'}(n) = h_{\mathcal{M}}(n + k)$ for all $n \geq 2$.*

1.2 Minor-Closed Classes of Graphs

Before discussing density results in matroid theory, we explore what is known for graphs, phrasing the results we state matroidally. We saw earlier that the class $\mathcal{G}$ of graphic matroids has quadratic growth rate function $h_{\mathcal{G}}(n) = \binom{n+1}{2}$, whereas the growth rate function $h_{\mathcal{P}}(n) = 3n - 3$ of the planar-graphic matroids is linear. With regards to more general minor-closed classes of graphs, Mader [30] showed the following:

Theorem 1.2.1. *If $t \geq 1$ is an integer, and $\mathcal{G}(t)$ is the class of graphic matroids with no $M(K_t)$-minor, then $h_{\mathcal{G}(t)}(n) \leq c_t n$.*

Here, c_t denotes a real constant depending only on t. This theorem tells us that excluding *any* clique as a minor gives us a class that, like the planar-graphic matroids, is linearly dense. Any proper minor-closed subclass of the graphic matroids fails to contain some clique $M(K_t)$, so a corollary is the following:

Corollary 1.2.2. *If $\mathcal{H}$ is a proper minor-closed subclass of the graphic matroids, then $h_{\mathcal{H}}(n) \leq c_{\mathcal{H}} n$.*

In this corollary, $c_{\mathcal{H}}$ denotes a real constant depending only on $\mathcal{H}$. Compared with Proposition 1.1.3, this result is striking; although the graphic matroids themselves are quadratically dense, any proper minor-closed subclass is at most linearly dense. This fact hints at the deep structure of such classes that was famously found by Robertson and Seymour in the Graph Minors Structure Theorem [39].

We now return to Theorem 1.2.1. Although we did not mention its explicit value, the upper bound in Mader's proof for c_t in the theorem is exponential in t, and far from best possible, so we explore how it can be improved. To this end, for each integer $t \geq 1$ let c_t denote the infimum of all c such $h_{\mathcal{G}(t)}(n) \leq cn$ for all n. Mader himself showed in [30] that $c_t \geq t - 2$ for all $t \geq 3$, and that equality holds when $t \leq 7$. Jørgensen showed in [20] that $c_8 = 6$, and Song and Thomas in [48] that $c_9 = 7$. Although Bollobás conjectured in [4] that $c_t = t - 2$ for all $t \geq 3$, this is false. Various estimates for c_t were obtained by Kostochka [22, 23] and Fernandez de la Vega [8]; Kostochka [22, 23] and Thomason [49] independently showed that $t\sqrt{\log t}$ is the correct order of magnitude for c_t. Finally, in [50], Thomason determined c_t almost exactly:

Theorem 1.2.3.

$$c_t = (\alpha + o(1))\ t\sqrt{\log t},$$

where $\alpha = 0.319...$ is an explicit constant.

Thomason also showed that random graphs G with fewer than $c_t|V(G)|$ edges have no K_t-minor, meaning that random graphs are extremal in his theorem.

More generally than simply excluding a clique, let $\mathcal{G}(H)$ denote the class of graphic matroids with no $M(H)$-minor. Analogously to c_t, one may define c_H as the minimum c such that $h_{\mathcal{G}(H)}(n) \leq cn$ for all n. So $c_{K_t} = c_t$, and clearly, if H is simple with t vertices, then $c_H \leq c_t$. In [32], Myers and Thomason generalised Theorem 1.2.3 by determining c_H asymptotically for all simple H:

Theorem 1.2.4. *If H is a simple graph on t vertices, then*

$$c_H = (\alpha\gamma(H) + o(1))\ t\sqrt{\log t}.$$

Here, $\alpha = 0.319...$ is the same constant as that in Theorem 1.2.3, and $\gamma(H)$ denotes the solution to a particular optimization problem over vertex-weightings of H. Again, random graphs are extremal.

The most general problem of this sort would be to determine, for an arbitrary proper minor-closed class $\mathcal{H}$, the minimum value of $c_{\mathcal{H}}$ for which Corollary 1.2.2 holds. This problem appears very difficult, and Theorem 1.2.4 is just a starting point; by the Graph Minors Theorem of Robertson and Seymour [41], one would need an analogue of Theorem 1.2.4 that finds one of the graphs in some finite set of graphs as a minor, rather than just one graph H.

1.3 A Weak Growth Rate Theorem

Part I of this thesis concerns the growth rate functions of matroids in a minor-closed class $\mathcal{M}$. In this section, we prove two theorems about these functions, and combine them into a single result that serves as a prelude to the coming Growth Rate Theorem.

We first consider classes excluding a simple rank-2 matroid; If $\ell \geq 0$ is an integer, then we write $\mathcal{U}(\ell)$ for the (minor-closed) class of matroids with no $U_{2,\ell+2}$-minor. The following beautiful theorem of Kung [26] bounds the density of all matroids in this class, comparing $\varepsilon(M)$ to $r(M)$:

Theorem 1.3.1. *Let $\ell \geq 2$ be an integer. If $M \in \mathcal{U}(\ell)$ is a rank-r matroid, then*

$$\varepsilon(M) \leq \frac{\ell^r - 1}{\ell - 1}.$$

Proof. The theorem evidently holds when $r = 0$; suppose inductively that it holds for a given $r \geq 0$. Let M be a rank-$(r+1)$ matroid and e be a nonloop of M. By Lemma 1.1.1, the points of $M/\!\!/e$ are exactly the lines of M through e, and we have $r(M/\!\!/e) = r$, so there are at most $\frac{\ell^r-1}{\ell-1}$ lines of M through e by the inductive hypothesis. Since $M \in \mathcal{U}(\ell)$, each such line contains at most ℓ points other than e, giving $\varepsilon(M) \leq \ell\frac{\ell^r-1}{\ell-1} + 1 = \frac{\ell^{r+1}-1}{\ell-1}$, as required. $\square$

This theorem tells us that any minor-closed class $\mathcal{M}$ either contains all rank-2 uniform matroids, or satisfies $\mathcal{M} \subseteq \mathcal{U}(\ell)$ for some ℓ, and has bounded growth rate function, giving a bound depending on ℓ. As we have observed, if $\mathcal{M}$ contains all rank-2 uniform matroids, then $h_{\mathcal{M}}(2) = \infty$, so $h_{\mathcal{M}}(n) = \infty$ for all $n \geq 2$. Therefore, Theorem 1.3.1 has the following corollary, giving a clear dichotomy between finitely and infinitely dense classes:

Corollary 1.3.2. *If $\mathcal{M}$ is a minor-closed class of matroids, then either*

1. *there is an integer $\ell \geq 2$ such that $\mathcal{M} \subseteq \mathcal{U}(\ell)$, and $h_{\mathcal{M}}(n) \leq \frac{\ell^n - 1}{\ell - 1}$ for all n, or*
2. *$\mathcal{M}$ contains all rank-2 uniform matroids, and $h_{\mathcal{M}}(n) = \infty$ for all $n \geq 2$.*

We remark that, for any prime power q, the matroid $U_{2,q+2}$ is not GF(q)-representable, and therefore $\mathcal{L}(q) \subseteq \mathcal{U}(q)$, and any subclass of $\mathcal{L}(q)$ satisfies 1.

We next prove a theorem that surprisingly divides growth rate functions into the polynomial and the exponential. The proof uses many techniques we will see later, and serves as a simple introduction to the technicalities of this thesis:

Theorem 1.3.3. *If $\mathcal{M}$ is a minor-closed class of matroids, then either*

1. *$h_{\mathcal{M}}(n) \leq n^{c_{\mathcal{M}}}$ for all n, or*
2. *$h_{\mathcal{M}}(n) \geq 2^n - 1$ for all n.*

We obtain the theorem as a corollary of a result due to Kung [26], which finds an exponentially dense minor of a polynomially dense matroid in $\mathcal{U}(\ell)$:

Theorem 1.3.4. *Let $\ell \geq 2$ and $n \geq 1$ be integers. If $M \in \mathcal{U}(\ell)$ is a matroid such that*

$$\varepsilon(M) > (\ell - 1)^{n-1}\binom{r(M)+1}{n-1},$$

then M has a simple rank-n minor N such that $|N| = 2^n - 1$.

Proof. When $n = 1$, all that is required is that M has a rank-1 minor; this is clear, as $\varepsilon(M) > 0$. Fix $n \geq 1$, and suppose inductively that the result holds for n. Let $M \in \mathcal{U}(\ell)$ be a minor-minimal matroid such that $\varepsilon(M) > (\ell - 1)^n\binom{r(M)+1}{n}$. M is clearly simple; let $e \in E(M)$, and $r = r(M)$. By minimality of M, we have $\varepsilon(M /\!\!/ e) \leq (\ell - 1)^n\binom{r}{n}$, so $\varepsilon(M) - \varepsilon(M /\!\!/ e) > (\ell - 1)^n\binom{r-1}{n-1}$.

Consider the set of lines of M containing e. By Lemma 1.1.1, each such line L is a point of $M/\!\!/e$, and thus contributes $|L|-2$ to the difference $\varepsilon(M)-\varepsilon(M/\!\!/e)$. Since $|L|-2 \le \ell-1$ for all such L, it follows that there are at least $\frac{1}{\ell-1}(\varepsilon(M)-\varepsilon(M/\!\!/e)) > (\ell-1)^{n-1}\binom{r-1}{n-1}$ long lines of M through e. Let $\{L_1, \dots, L_t\}$ be the set of long lines through e, and for each $1 \le i \le t$, let x_i and y_i be distinct points in $L_i - e$.

Let $M' = (M/\!\!/e)|\{x_1, \dots, x_t\}$. Now, M' is a simple matroid with $r(M') \le r-1$, and $|M'| = t > (\ell-1)^{n-1}\binom{r-1}{n-1}$, so by the inductive hypothesis, M' has a simple, rank-n minor N' with $|N'| = 2^n - 1$. Let $X' = E(N')$. So $N' = (M'/\!\!/C)|X'$ for some independent set C of $M/\!\!/e$, and by relabelling the x_i if necessary, we may assume that $X' = \{x_1, \dots, x_{2^n-1}\}$. Let $Y' = \{y_1, \dots, y_{2^n-1}\}$, and $N = (M/\!\!/C)|(X' \cup Y' \cup \{e\})$.

We claim that N satisfies the theorem. The set C is independent in $M/\!\!/e$, and x_i is a nonloop of $M/\!\!/e$ for each i, so we have $N|\{e, x_i, y_i\} = M|\{e, x_i, y_i\}$ for each $1 \le i \le 2^n - 1$, meaning that $\{e, x_i, y_i\}$ is a 3-point line of N. Since $N' = (N/\!\!/e)|X'$ is simple, it now follows that N is simple, as any pair of elements of $N \backslash e$ are either distinct points on a line through e, or on distinct lines through e. Moreover, it follows that $r(N) = r(N') + 1 = n + 1$, since N is spanned by $E(N') \cup \{e\}$. Therefore, N is a simple, rank-$(n+1)$ minor of M with $|N| = |X'| + |Y'| + 1 = 2^{n+1} - 1$, giving the theorem. □

Although the techniques in this proof seem coarse, the bound $2^n - 1$ is best-possible, since M could be a binary projective geometry of large rank. Theorem 1.3.3 now follows:

Proof of Theorem 1.3.3. If $\mathcal{M}$ contains all simple rank-2 matroids, then the second outcome is trivially satisfied, so we may assume that $\mathcal{M} \subseteq \mathcal{U}(\ell)$ for some $\ell \ge 2$. Assume that the second outcome fails; therefore, there exists an integer $k \ge 2$ so that $\mathcal{M}$ contains no simple rank-k matroid with $2^k - 1$ points. Theorem 1.3.4 gives $\varepsilon(M) \le (\ell-1)^{k-1}\binom{r(M)+1}{k-1}$ for all $M \in \mathcal{M}$, giving the first outcome for an appropriately large $c_{\mathcal{M}}$. □

In fact, Theorem 1.3.4 implies two other nontrivial results related to growth rates. Since $\mathcal{U}(2) = \mathcal{L}(2)$ [51], and any simple rank-k binary matroid on $2^k - 1$ points is isomorphic to $\mathrm{PG}(k-1, 2)$, specialising Theorem 1.3.4 to the binary matroids yields the following corollary, which was first proved independently by Sauer [42] and Shelah [47]:

Corollary 1.3.5. *If $k \ge 2$ is an integer, and M is a binary matroid with no* $\mathrm{PG}(k-1, 2)$*-minor, then*

$$\varepsilon(M) \le \binom{r(M)+1}{k-1}.$$

Specialising to the case $k = 3$ gives us a further corollary. A matroid representable over every field is *regular*; we denote this minor-closed class by $\mathcal{R}$. One can check that the Fano Matroid $\mathrm{PG}(2,2)$ is not regular, so $\mathcal{R}$ is contained in the class of binary matroids with no $\mathrm{PG}(2,2)$-minor. Since $\mathcal{G} \subseteq \mathcal{R}$, setting $k = 3$ in the above corollary gives the growth rate function for $\mathcal{R}$, which was first determined by Heller [18]:

Corollary 1.3.6. $h_{\mathcal{R}}(n) = \binom{n+1}{2}$.

We now combine Corollary 1.3.2, Theorem 1.3.3, and elementary lower bounds for growth rate functions into a result that divides growth rate functions into four types:

Theorem 1.3.7. *If $\mathcal{M}$ is a minor-closed class of matroids, then one of the following holds:*

1. *$h_{\mathcal{M}}(n)$ is eventually constant, and up to isomorphism, $\mathcal{M}$ contains only finitely many simple matroids, or*
2. *$n \leq h_{\mathcal{M}}(n) \leq n^{c_{\mathcal{M}}}$ for all n, and some integer $c_{\mathcal{M}}$, or*
3. *$2^n - 1 \leq h_{\mathcal{M}}(n) \leq \frac{\ell^n - 1}{\ell - 1}$ for all n, and some integer $\ell \geq 2$, or*
4. *$h_{\mathcal{M}}(n) = \infty$ for all $n \geq 2$, and $\mathcal{M}$ contains all simple rank-2 matroids.*

While this theorem goes some way towards the classification of growth rate functions, we will see in the next section how much it can be improved.

1.4 The Growth Rate Theorem

Results on growth rates are stunningly encapsulated by the following theorem, which motivates this thesis. The theorem was conjectured by Kung in [26], and is a culmination of work of Geelen, Kabell, Kung and Whittle in [17], [14] and [15], which contains its formal statement.

Theorem 1.4.1 (Growth Rate Theorem)**.** *If $\mathcal{M}$ is a minor-closed class of matroids, then one of the following holds:*

1. *$h_{\mathcal{M}}(n) \leq c_{\mathcal{M}} n$ for all n, or*
2. *$\binom{n+1}{2} \leq h_{\mathcal{M}}(n) \leq c_{\mathcal{M}} n^2$ for all n, and $\mathcal{M}$ contains all graphic matroids, or*

3. *there is a prime power q such that $\frac{q^n-1}{q-1} \leq h_{\mathcal{M}}(n) \leq c_{\mathcal{M}} q^n$ for all n, and $\mathcal{M}$ contains all* $\mathrm{GF}(q)$*-representable matroids, or*

4. $h_{\mathcal{M}}(n) = \infty$ *for all $n \geq 2$, and $\mathcal{M}$ contains all simple rank-2 matroids.*

In all cases, $c_{\mathcal{M}}$ denotes a positive real constant depending only on $\mathcal{M}$. Note that the lower bound on $h_{\mathcal{M}}$ in 2 follows from the fact that $\mathcal{M}$ contains all cliques, and the bound in 3 follows from the fact that $\mathcal{M}$ contains all projective geometries over $\mathrm{GF}(q)$.

This theorem is striking, telling us that any growth rate function that is neither infinite nor eventually constant must have a very special type: linear, quadratic or exponential with base equal to a prime power. We call a class satisfying 3 *(base-q) exponentially dense*.

Perhaps more striking is the natural emergence in the theorem of the well-known classes of graphic and representable matroids. This phenomenon does not seem to be isolated, and is seen in a number of other intriguing results, which we will explore in Chapter 5.

1.4.1 Applications

The demarcation between growth rate functions of different classes provided by the Growth Rate Theorem gives a number of interesting applications, which we discuss below.

The first is motivated by Corollary 1.3.5. If p is a prime and $s \geq 2$ is an integer, then the class of $\mathrm{GF}(p)$-representable matroids with no $\mathrm{PG}(s, p)$-minor does not contain $\mathcal{L}(q)$ for any q, but does contain all graphic matroids, and therefore must be quadratically dense. In particular, this tells us that the correct bound for Corollary 1.3.5 is quadratic. We can generalise this construction by observing that, for any $k \geq 1$ and $s \geq 2$, the class of $\mathrm{GF}(p^k)$-representable matroids with no $\mathrm{PG}(s, p)$-minor is also quadratically dense.

Another class that is forced by the theorem to be quadratically dense is the class of $\mathbb{R}$-representable matroids in $\mathcal{U}(\ell)$ for some $\ell \geq 2$; since finite projective planes are not real-representable, this class does not contain $\mathcal{L}(q)$ for any q, but does contain all graphic matroids, and is therefore quadratically dense. This was originally shown with an explicit bound by Kung [25], who proved the following:

Theorem 1.4.2. *If $\ell \geq 2$ is an integer, and M is an $\mathbb{R}$-representable matroid in $\mathcal{U}(\ell)$, then*

$$\varepsilon(M) \leq \left(\ell^{2^\ell - 1} - \ell^{2^\ell - 2}\right) \binom{r(M)+1}{2} - r(M).$$

More quadratically dense classes come from considering fields of different characteristics. Let $\mathcal{F}$ be a set of fields, not all infinite, and not all of the same characteristic, and let $\mathcal{M}$ be the class of matroids representable over all fields in $\mathcal{F}$. It is not hard to see that $\mathcal{M}$ contains no finite projective plane, but does contain all graphic matroids. Moreover, since $\mathcal{F}$ contains a finite field, $\mathcal{M}$ does not contain all simple rank-2 matroids. Therefore, $\mathcal{M}$ must be quadratically dense. This construction, even when $|\mathcal{F}| = 2$, gives rise to many well-known classes whose growth rate functions have been determined or conjectured, including the regular matroids [18], golden-mean matroids [1, 43, 52], dyadic matroids [24, 27], sixth-root-of-unity matroids [36], and near-regular matroids [36]. Not all choices of $\mathcal{F}$ yield different $\mathcal{M}$. In fact, if $\mathrm{GF}(2) \in \mathcal{F}$, then $\mathcal{M}$ is always the class of regular matroids [6]; many of the above classes were defined by Whittle, who showed [53, 54] that if $\mathrm{GF}(3) \in \mathcal{F}$, then there are just five possible $\mathcal{M}$. An important tool when considering classes constructed in this way is the theory of partial fields, which was used to great effect in the case where $\mathrm{GF}(5) \in \mathcal{F}$ by Pendavingh and Van Zwam [37, 38].

We can also construct exponentially dense classes whose growth rate functions are restricted by the theorem. A simple example is, for a prime power q, the class $\mathcal{L}(q^2) \cap \mathcal{L}(q^3)$. This class contains $\mathcal{L}(q)$, but does not contain $\mathcal{L}(q')$ for any $q' > q$, so is base-q exponentially dense. Classes of this sort are treated in detail in Chapter 4.

1.5 This Thesis

Part I of this thesis is concerned with exponential growth rate functions. Our goal is to refine the growth rate theorem, determining growth rate functions not just to within a constant factor, but as precisely as we can. Chapter 2 contains a collection of preliminary results regarding projective geometries, while the theorems in Chapters 3 and 4 deal with two related problems:

Problem 1.5.1. *Classify the functions $h(n)$ that are exponential in n, and occur as the growth rate function of a minor-closed class of matroids.*

Problem 1.5.2. *Let $\mathcal{M}$ be an 'interesting' exponentially dense minor-closed class of matroids. What is the growth rate function $h_{\mathcal{M}}(n)$?*

At first glance, it is not clear that such problems should even be tractable. In fact, there is reason to believe otherwise; the setting of graphic matroids paints a gloomy picture. A case of Problem 1.5.2 in this setting is solved by Theorem 1.2.3, which gives an asymptotic expression for the growth rate function of $\mathcal{G}(t)$, the (linearly dense) class of graphic matroids with no K_t-minor. However, this is far from determining the function itself, and the extremal graphs in this class are random, meaning that any exact expression for $h_{\mathcal{G}(t)}$ is likely to be highly elusive. This is a very basic example, and general linearly dense minor-closed classes of matroids are bound to be even less amenable to analysis of this sort.

As we will see, however, the exponential setting looks promising. We will provide partial answers to Problems 1.5.1 and 1.5.2 in many cases, actually obtaining exact expressions for growth rate functions. We also make a specific conjecture that would resolve Problem 1.5.1 almost completely. The next two sections detail the results we prove in Part I, and the last section concerns the conjecture.

1.5.1 Excluding a Line

Chapter 3 deals with exponentially dense classes that exclude some simple rank-2 matroid. The main result of the chapter is a theorem that finds one of two infinite sequences of 'unavoidable minors' in a sufficiently dense minor-closed class. This theorem is technical in its statement, so we first state a corollary relating more explicitly to growth rate functions.

The corollary answers (for large rank) a very natural instance of Problem 1.5.2. The question is 'what is the growth rate function of the class $\mathcal{U}(\ell)$?' Our answer was conjectured by Kung in [26], and is a restatement of Theorem 1.0.1, together with an equality characterisation.

Theorem 1.5.3. *If $\ell \geq 2$ is an integer, and q is the largest prime power such that $q \leq \ell$, then*

$$h_{\mathcal{U}(\ell)}(n) = \frac{q^n - 1}{q - 1}$$

for all sufficiently large n. Moreover, if n is sufficiently large, and $M \in \mathcal{U}(\ell)$ is a simple rank-n matroid that is extremal in $\mathcal{U}(\ell)$, then M is a projective geometry over $\mathrm{GF}(q)$.

The definition of q is just the largest q such that $\mathrm{PG}(n, q) \in \mathcal{U}(\ell)$. Thus, this theorem states that, for large rank, the densest matroids in $\mathcal{U}(\ell)$ are simply the densest projective geometries in $\mathcal{U}(\ell)$. For example, the theorem implies that the three classes $\mathcal{U}(13)$, $\mathcal{U}(14)$ and $\mathcal{U}(15)$ all eventually have the exact same growth rate function, as the largest projective geometries in all three of these classes are those over $\mathrm{GF}(13)$. The class $\mathcal{U}(16)$, on the other hand, has a larger growth rate function, as it contains all projective geometries over $\mathrm{GF}(16)$.

Since $\mathcal{U}(\ell)$ contains all projective geometries over $\mathrm{GF}(q)$, the lower bound $h_{\mathcal{U}(\ell)}(n) \geq \frac{q^n-1}{q-1}$ is immediate. Two results we have already mentioned provided upper bounds. The first is Theorem 1.3.1, which gives $h_{\mathcal{U}(\ell)}(n) \leq \frac{\ell^n-1}{\ell-1}$. When ℓ is itself a prime power, then $q = \ell$, and this upper bound is tight. When ℓ is not a prime power, then $q < \ell$, so Theorem 1.5.3 shows that the bound provided by Theorem 1.3.1 is an overestimate by an exponential factor. The second, asymptotically tighter, upper bound for $h_{\mathcal{U}(\ell)}(n)$ is given by the Growth Rate Theorem, which implies that $h_{\mathcal{U}(\ell)}(n) \leq cq^n$ for some constant c.

The first case where $q \neq \ell$ is $\ell = 6$; this was addressed for all $n \geq 4$ by Bonin and Kung in [5]. This range of values for n starkly contrasts that of our result, which applies for 'sufficiently large' n. In fact, 'sufficiently large' feels like an understatement; the numbers concerned easily dwarf the number of particles in the universe. A starting source of astronomical numbers is to be found in the values $c_{\mathcal{M}}$ in the Growth Rate Theorem itself, which is essentially invoked twice in our proof. This is inherent in the techniques we use; finding the exact value for $h_{\mathcal{M}}(n)$ for the 'small' values of n not implied by this theorem would require genuinely different methods. However, we conjecture that Theorem 1.5.3 holds for all $n \geq 4$; the theorem is clearly false for $n = 2$, and non-Desarguesian projective planes provide counterexamples to the equality characterisation (as well as the bound, if they exist with order other than a prime power) for $n = 3$.

To state the larger theorem from which Theorem 1.5.3 follows, we need to briefly discuss a special type of single-element extension. If M is a matroid, F is a flat of M, and $e \notin E(M)$, then we write $M +_F e$ for the matroid with ground set $E(M) \cup \{e\}$, so that $(M +_F e)\backslash e = M$, and e is *freely placed* in F (thus, the flats of $M +_F e$ are the flats of M that do not contain F, along with the flats of M that contain F, with e added to them). The matroid $M +_F e$ is the *principal extension of M by F*. In the special case where $F = E(M)$, then $M +_F e$ is the free extension of M, which we saw in Section 1.1.5.

The principal extensions we encounter are of projective geometries; we give some notation for matroids of this sort. Let q be a prime power, and n and k be integers with $1 \leq k \leq n$. If $N \cong \mathrm{PG}(n-1, q)$ is a matroid, $e \notin E(N)$, and F is a rank-k flat of N, then we write $\mathrm{PG}^+(n-1, q, k)$ for any matroid isomorphic to $N +_F e$. Thus, this matroid is the one obtained by freely adding a point to a rank-k flat of a rank-n projective geometry over $\mathrm{GF}(q)$. Since the automorphism group of $\mathrm{PG}(n-1, q)$ acts transitively on its set of flats of rank k, the choice of F is unimportant. The case where $k = 1$ is a parallel extension of N; when $k = 2$, we are principally extending a line of N. The largest possible k gives the matroid $\mathrm{PG}(n-1, q, n)$, obtained by freely extending N itself. For all $k > 1$, the matroid $\mathrm{PG}(n-1, q, k)$ is simple, and has $\mathrm{PG}(n-1, q)$ as a proper spanning restriction, so is not $\mathrm{GF}(q)$-representable.

The main result of Chapter 3 is the following, which finds an infinite sequence of unavoidable minors in any base-q exponentially dense minor closed class that is eventually denser than $\mathcal{L}(q)$:

Theorem 1.5.4. *Let $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids. If there exist infinitely many n such that $h_{\mathcal{M}}(n) > \frac{q^n-1}{q-1}$, then one of the following holds:*

1. $\mathrm{PG}^+(n-1, q, 2) \in \mathcal{M}$ *for all* $n \geq 2$, *or*
2. $\mathrm{PG}^+(n-1, q, n) \in \mathcal{M}$ *for all* $n \geq 2$.

Both these sequences comprise specific matroids that are denser than $\mathrm{PG}(n-1, q)$, thus explicitly certifying the fact that $h_{\mathcal{M}}(n)$ exceeds $h_{\mathcal{L}(q)}(n)$ for arbitrarily large n.

A matroid M is *weakly round* if there is no pair of sets (A, B) such that $A \cup B = E(M)$, and $r_M(A) \geq r(M) - 1$, and $r_M(B) \geq r(M) - 2$. Weak roundness is a notion of matroid connectivity that we will discuss in coming chapters; we mention it here in order to state another corollary of the machinery developed in Chapter 3, which produces a surprisingly long line minor for its modest hypotheses:

Theorem 1.5.5. *For every prime power q, there exists an integer n such that, if M is a weakly round matroid with a $U_{2,q+2}$-restriction and a $\mathrm{PG}(n-1, q)$-minor, then M has a U_{2,q^2+1}-minor.*

A final result of Chapter 3 is the following (recall that a rank-r matroid $M \in \mathcal{M}$ is extremal in $\mathcal{M}$ if $\varepsilon(M) = h_{\mathcal{M}}(r)$):

Theorem 1.5.6. *If $\mathcal{M}$ is an exponentially dense minor-closed class of matroids, then there exist extremal matroids in $\mathcal{M}$, of arbitrarily large rank, that are weakly round.*

1.5.2 Square Fields

We now turn our attention to Chapter 4, whose aim is to further progress towards Problem 1.5.1, in a less general setting than Chapter 3. The chapter takes a close look at growth rate functions of a special type of minor-closed class: we fix a prime power q, and restrict our consideration to the subclasses of $\mathcal{L}(q^2)$ that contain $\mathcal{L}(q)$.

For each prime power q and integer $k \geq 0$, let $\mathcal{P}_{q,k}$ denote the closure under minors and isomorphism of the set of matroids of the form M/K, where M is $\mathrm{GF}(q^2)$-representable, K is a k-subset of $E(M)$, and $M \backslash K \cong \mathrm{PG}(r(M)-1, q)$. The matroids in $\mathcal{P}_{q,k}$ are all $\mathrm{GF}(q^2)$-representable, k-element projections of matroids in $\mathcal{L}(q)$, and the extremal matroids in $\mathcal{P}_{q,k}$ are, up to simplification, k-element projections of projective geometries over $\mathrm{GF}(q)$. In Chapter 4, we prove the following:

Theorem 1.5.7. *Let q be a prime power. If $\mathcal{M}$ is a minor-closed class of matroids such that $\mathcal{L}(q) \subseteq \mathcal{M} \subsetneq \mathcal{L}(q^2)$, then there is an integer $k \geq 0$ such that $\mathcal{P}_{q,k} \subseteq \mathcal{M}$, and $h_{\mathcal{M}}(n) = h_{\mathcal{P}_{q,k}}(n)$ for all sufficiently large n.*

In other words, any proper minor-closed subclass of $\mathcal{L}(q^2)$ that contains $\mathcal{L}(q)$ has its growth rate function eventually determined by that of $\mathcal{P}_{q,k}$ for some k. We also compute the growth rate function of $\mathcal{P}_{q,k}$, giving us an explicit expression for $h_{\mathcal{M}}(n)$ for any such class:

Theorem 1.5.8. *Let q be a prime power. If $\mathcal{M}$ is a minor-closed class of matroids such that $\mathcal{L}(q) \subseteq \mathcal{M} \subsetneq \mathcal{L}(q^2)$, then there is an integer $k \geq 0$ so that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - q\left(\frac{q^{2k}-1}{q^2-1}\right)$$

for all sufficiently large n.

This theorem effectively resolves Problem 1.5.1 in the setting of Chapter 4, in a striking way: there is just a single-parameter family of functions that together encompass (for large n), all possible functions $h_{\mathcal{M}}(n)$ for a class $\mathcal{M}$ with $\mathcal{L}(q) \subseteq \mathcal{M} \subsetneq \mathcal{L}(q^2)$. The statement that n is sufficiently large is necessary here; if $\mathcal{M}$ is a class satisfying the theorem for a particular q and k, and $t \geq 0$ is an arbitrary integer, then the class $\mathcal{M}' = \{M \in \mathcal{L}(q^2) : M \in \mathcal{M} \text{ or } r(M) \leq t\}$, constructing by adding to $\mathcal{M}$ all matroids in $\mathcal{L}(q^2)$ of rank at most t, is a minor-closed class whose growth rate function disagrees with the formula in the theorem for all $n \leq t$.

Theorems 1.5.7 and 1.5.8, along with the tools used in their proofs, impose severe limitations on the behaviour of base-q exponentially dense subclasses of $\mathcal{L}(q^2)$. It makes sense to apply these theorems to answer Problem 1.5.2 for various interesting classes of matroids in this setting.

Perhaps the most natural way to define such a class is just to exclude some fixed projective geometry over $\mathrm{GF}(q^2)$. A consequence of Theorem 1.5.7, that will follow from investigation of $\mathcal{P}_{q,k}$, is the growth rate function for such a class:

Theorem 1.5.9. *Let q be a prime power, and $k \geq 0$ be an integer. Let $\mathcal{M}(k)$ denote the class of $\mathrm{GF}(q^2)$-representable matroids with no $\mathrm{PG}(k+1, q^2)$-minor. Then*

$$h_{\mathcal{M}(k)}(n) = \frac{q^{n+k}-1}{q-1} - q\left(\frac{q^{2k}-1}{q^2-1}\right)$$

for all sufficiently large n.

The 'large n' caveat here, as given by our proof, is just as huge as it is in Chapter 3. Although we do not formally conjecture this, it is likely that in reality, n need not be particularly large for this to apply. A reasonable guess would be that this theorem holds for all $n \geq 2k$.

Another way to construct classes to which the results of this chapter apply is to consider the matroids representable over some set of fields, all with a $\mathrm{GF}(q)$-subfield. They give some interesting applications of Theorems 1.5.7 and 1.5.8. For example, our next result resolves Problem 1.5.2 for the class of matroids representable over both $\mathrm{GF}(q^2)$, and some other extension field of $\mathrm{GF}(q)$ of odd degree. We avoid the phrase 'sufficiently large' in the statement to emphasise that the largeness required depends only on q:

Theorem 1.5.10. *Let q be a prime power. There exists an integer n_q such that, for each odd number $j \geq 3$, if $\mathcal{M} = \mathcal{L}(q^2) \cap \mathcal{L}(q^j)$, then*

$$h_{\mathcal{M}}(n) = \frac{q^{n+1} - 1}{q - 1} - q$$

for all $n \geq n_q$.

The second result concerning other fields is surprising, giving an apparently uncountably large collection of minor-closed classes of matroids whose growth rate functions (not just for large rank) together form a finite set:

Theorem 1.5.11. *Let q be a prime power. There is a finite set $\mathfrak{H}_q$ of integer-valued functions so that, if $\mathcal{F}$ is a set of fields with $\mathrm{GF}(q^2) \in \mathcal{F}$, and all fields in $\mathcal{F}$ have a $\mathrm{GF}(q)$-subfield, but not all fields in $\mathcal{F}$ have a $\mathrm{GF}(q^2)$-subfield, and $\mathcal{M} = \bigcap_{\mathbb{F} \in \mathcal{F}} \mathcal{L}(\mathbb{F})$, then $h_{\mathcal{M}} \in \mathfrak{H}_q$.*

The number of possible sets $\mathcal{F}$ is huge - we are not even restricted to finite fields. This result seems truly strange. One possible (equally strange) explanation is that, despite the huge number of choices for $\mathcal{F}$, the number of classes that they give is finite:

Conjecture 1.5.12. *Let q be a prime power. There is a finite set $\mathfrak{M}_q$ of minor-closed classes of matroids so that, if $\mathcal{F}$ is a set of fields with $\mathrm{GF}(q^2) \in \mathcal{F}$, and all fields in $\mathcal{F}$ have a $\mathrm{GF}(q)$-subfield, but not all fields in $\mathcal{F}$ have a $\mathrm{GF}(q^2)$-subfield, and $\mathcal{M} = \bigcap_{\mathbb{F} \in \mathcal{F}} \mathcal{L}(\mathbb{F})$, then $\mathcal{M} \in \mathfrak{M}_q$.*

1.5.3 The Conjecture

The Growth Rate Theorem and the theorems of Chapters 3 and 4, while not answering Problem 1.5.1 completely, do give us some insight into the general situation. The recurrent theme seems to be that the spectrum of possible exponential growth rate functions is 'discrete', with only a few functions that actually occur, and many 'gaps' containing huge numbers of exponential functions that are not growth rate functions. We will explain this reasoning, eventually arriving at a conjectured answer to Problem 1.5.1.

The first example of these 'gaps' comes from the Growth Rate Theorem. To occur as a growth rate function, an exponentially large function h must satisfy $\frac{q^n-1}{q-1} \leq h(n) \leq cq^n$ for all n, some prime power q, and some constant c. This fact already restricts the set of possible $h_{\mathcal{M}}$.

A next, telling occurrence of gaps comes from a theorem of Chapter 3, which provides a concrete example of a gap between the growth rate functions of two known classes, both exponential with the same base, in which no other growth rate functions lie:

Theorem 1.5.13. *If $\mathcal{M}$ is a minor-closed class of matroids, and q is a prime power, then either $h_{\mathcal{M}}(n) \leq \frac{q^n-1}{q-1}$ for all sufficiently large n, or $h_{\mathcal{M}}(n) \geq \frac{q^{n+1}-1}{q-1} - q$ for all sufficiently large n.*

A class at the lower end of the gap is $\mathcal{L}(q)$, in which the simple extremal matroids are the projective geometries over $\mathrm{GF}(q)$. As we will see in Chapter 4, an example of a class at the other end of this gap is the class of $\mathrm{GF}(q^2)$-representable matroids with no $\mathrm{PG}(2, q^2)$-minor, in which a class of extremal matroids is provided by certain single-element projections of projective geometries over $\mathrm{GF}(q)$. It seems natural to continue in this direction, considering classes whose extremal members are k-element projections of projective geometries over $\mathrm{GF}(q)$ for general k, and perhaps to conjecture the following:

Conjecture 1.5.14. *Let $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids. There exists an integer $k \geq 0$ such that every extremal matroid M in $\mathcal{M}$ of sufficiently large rank is, up to simplification, a k-element projection of a projective geometry over $\mathrm{GF}(q)$.*

This conjecture seems bold, but one's conviction is substantially strengthened by Theorem 1.5.7, which suggests that in the restricted setting of $\mathcal{L}(q^2)$, the conjecture holds; the growth rate function of any base-q exponentially dense minor-closed subclass of $\mathcal{L}(q^2)$ eventually agrees precisely with that of $\mathcal{P}_{q,k}$ for some k, and the simple extremal matroids in $\mathcal{P}_{q,k}$ are k-element projections of projective geometries. (The conjecture for $\mathcal{M} \subseteq \mathcal{L}(q^2)$

is not quite implied by this, as we do not show that the extremal matroids in $\mathcal{P}_{q,k}$ are the *only* simple high-rank extremal matroids in $\mathcal{M}$, although this is very likely to be true.)

We can adapt Conjecture 1.5.14 into a conjecture about growth rates via the following theorem, proved in Chapter 2:

Theorem 1.5.15. *Let q be a prime power, and $k \geq 0$ be an integer. If M is a matroid of rank at least $k+1$ that is a k-element projection of $\mathrm{PG}(r(M)-1+k, q)$, and k is minimal such that M has this property, then there exists an integer d with $0 \leq d \leq \frac{q^{2k}-1}{q^2-1}$ such that*

$$\varepsilon(M) = \frac{q^{r(M)+k}-1}{q-1} - qd.$$

Note that, for every k, the matroid M is approximately q^k times denser than $\mathrm{PG}(r(M)-1, q)$. Extremal matroids in the class $\mathcal{P}_{q,k}$ of Theorem 1.5.7 give examples where d is maximised. In fact, as is shown by Theorem 1.5.8, the setting of Chapter 4 is a very special case of the general problem, as the parameter d of Theorem 1.5.15 is forced to take its largest possible value.

To construct matroids for which $d = 0$, we use truncation; recall that the truncation $T^k(M)$ is the k-element projection of M formed by freely adding a point to M and then contracting the point, k times. If $r(M) > k+1$, then $\varepsilon(T^k(M)) = \varepsilon(M)$, so a matroid $M \cong T^k(\mathrm{PG}(n+k-1, q))$ is a rank-n, k-element projection of $\mathrm{PG}(n+k-1, q)$ with $\varepsilon(M) = \varepsilon(\mathrm{PG}(n+k-1, q)) = \frac{q^{n+k}-1}{q-1}$, giving $d = 0$.

Assuming Conjecture 1.5.14 and applying Theorem 1.5.15 gives our main conjecture on exponential growth rate functions, which would effectively resolve Problem 1.5.1 completely, and would go some way towards Problem 1.5.2:

Conjecture 1.5.16 (Exponential Growth Rate Conjecture)**.** *Let q be a prime power, and $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids. There exists an integer $k \geq 0$, and an integer d with $0 \leq d \leq \frac{q^{2k}-1}{q^2-1}$ so that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$$

for all sufficiently large n.

Theorem 1.5.8 solves this conjecture for the case where $\mathcal{M} \subseteq \mathcal{L}(q^2)$; some more intermediate cases are where $\mathcal{M} \subseteq \mathcal{U}(q^2)$, $\mathcal{M} \subseteq \mathcal{L}(q^k)$ for some k, or where $\mathcal{M} \subseteq \mathcal{L}(\mathbb{K})$ for some arbitrary field $\mathbb{K}$ with a GF(q)-subfield. Many of the techniques in Chapters 3 and 4 should be helpful in a proof of this conjecture; in fact, in Chapter 3 we prove the following, which verifies the conjecture for all classes that exclude a simple rank-2 matroid on not too many points:

Theorem 1.5.17. *Let q be a prime power. If $\mathcal{M}$ is a base-q exponentially dense minor-closed class of matroids such that $\mathcal{M} \subseteq \mathcal{U}(q^2-1)$, then*

$$h_{\mathcal{M}}(n) = \frac{q^n - 1}{q - 1}$$

for all sufficiently large n.

Here, the values of k and d in Conjecture 1.5.16 are both zero, and consequently $d = \frac{q^{2k}-1}{q^2-1}$ takes its maximum possible value. We conjecture that d takes its maximum value for a larger collection of classes:

Conjecture 1.5.18. *Let q be a prime power. If $\mathcal{M}$ is a base-q exponentially dense minor-closed class of matroids with $\mathcal{M} \subseteq \mathcal{U}(q^2+q-1)$, then there is an integer $k \geq 0$ such that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+k} - 1}{q - 1} - q\left(\frac{q^{2k}-1}{q^2-1}\right)$$

for all sufficiently large n.

Theorem 1.5.8 also resolves this conjecture in the case where $\mathcal{M} \subseteq \mathcal{L}(q^2)$. The condition $\mathcal{M} \subseteq \mathcal{U}(q^2+q-1)$ cannot be improved; the class $\{T(M) : M \in \mathcal{L}(q)\}$ of truncations of GF(q)-representable matroids, which we will investigate further in the next chapter, is contained in $\mathcal{U}(q^2+q)$ and has a growth rate function not of the above form.

Chapter 2

Projective Geometries

Recall the following ambitious conjecture:

Conjecture 1.5.14. *Let $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids. There exists an integer $k \geq 0$ such that every extremal matroid M in $\mathcal{M}$ of sufficiently large rank is, up to simplification, a k-element projection of a projective geometry over* $\mathrm{GF}(q)$.

Working with this assumption, it is plain that projective geometries and their projections will play a large role in our investigations of extremal matroids and growth rate functions in this thesis; this chapter unifies various results relating to these matroids.

As encouraged by Conjecture 1.5.14, the last section of this chapter will deal with k-element projections of projective geometries, and the penultimate section with single-element projections in more detail. The other sections treat the various alternative contexts in which projective geometries will arise in the next two chapters; for example, our extremal characterisation in Theorem 1.5.3 requires a way to prove a given matroid is a projective geometry, which is provided in Section 2.2, and Chapter 4's investigation of matroids in $\mathcal{L}(q^2)$ with spanning $\mathrm{PG}(n-1,q)$-restrictions will require us to consider representations of these matroids in more detail; this is done in Section 2.3.

2.1 Basics

Recall that the projective geometry $\mathrm{PG}(n-1,q)$ is the unique simple extremal rank-n matroid in the class $\mathcal{L}(q)$. If A is a matrix with n rows, containing exactly one column from each parallel class of nonzero vectors in $\mathrm{GF}(q^n)$, then $M(A) \cong \mathrm{PG}(n-1,q)$. Projective geometries are highly symmetric, and have many pleasing properties, such as the following two, which are readily verifiable by considering the relevant matrices:

Lemma 2.1.1. *Let $n \geq 1$ be an integer, q be a prime power, and $M \cong \mathrm{PG}(n-1,q)$ be a matroid. If F is a flat of M, then $M|F \cong \mathrm{PG}(r_M(F)-1,q)$.*

Lemma 2.1.2. *Let $n \geq 1$ be an integer, q be a prime power, and $M \cong \mathrm{PG}(n-1,q)$ be a matroid. If $C \subseteq E(M)$, then $\mathrm{si}(M/C) \cong \mathrm{PG}(r(M/C)-1,q)$.*

A third important property regards the rank of intersections of flats, and follows from the familiar formula for the dimension of the intersection of two subspaces of a vector space. If X and Y are sets in a matroid M, then we write $\sqcap_M(X,Y)$ as shorthand for $r_M(X)+r_M(Y)-r_M(X \cup Y)$; this is sometimes called the *local connectivity* between X and Y in M. If $\sqcap_M(X,Y)=0$ (or equivalently if $r_M(X \cup Y) = r_M(X)+r_M(Y)$), then we say that X and Y are *skew* in M.

Lemma 2.1.3. *Let q be a prime power, and $n \geq 1$ be an integer. If F and F' are flats of $M \cong \mathrm{PG}(n-1,q)$, then $r_M(F \cap F') = \sqcap_M(F, F')$.*

A consequence is that any two flats of M are either skew, or have nonempty intersection. We refer to this property as *modularity*, and use it without explicit reference to this lemma.

In particular, if $M \cong \mathrm{PG}(n-1,q)$, and L and H are respectively a line and hyperplane of M, then, then $\sqcap_M(L, H) = 1$, so $L \cap H \neq \varnothing$ by modularity. In fact, this intersection property essentially characterises projective geometries; the following recognition theorem was proved in the context of geometric lattices in [3] (pp. 90-93), and its reformulation to matroids follows from [35], Proposition 6.9.1, Corollary 6.9.3 and 6.9 ex. 2.

Theorem 2.1.4. *If M is a simple matroid such that $r(M) \geq 4$, and every line of M contains at least 3 points and intersects every hyperplane of M, then $M \cong \mathrm{PG}(r(M)-1,q)$ for some prime power q.*

2.2 Recognition

Theorem 2.1.4 gives us a way to show that a given matroid is a projective geometry, using the fact that the lines and hyperplanes intersect. In this section, we derive another recognition method, whose hypothesis is a more 'local' property stating that any two coplanar lines intersect.

A *projective space* P is a pair $(X, \mathcal{L})$, where X is a set, and $\mathcal{L}$ is a set of subsets of X, called the *lines* of P, satisfying the following conditions:

1. $|\mathcal{L}| \geq 3$ for all $L \in \mathcal{L}$, and
2. For each pair x, y of distinct elements of X, there is exactly one line $L_{xy} \in \mathcal{L}$ containing x and y, and
3. If x, y, u, v are distinct elements of X, such that no three lie in a common line, and L_{xy} intersects L_{uv}, then L_{xu} intersects L_{yv}.

If $P = (X, \mathcal{L})$ is a projective space, and $I \subset X$, then the *projective closure* of I in P, for which we write $\mathrm{pcl}_P(I)$, is the minimum set C such that $I \subseteq C$, and $L_{xy} \subseteq C$ for all $x, y \in I$. A set $Y \subseteq X$ is a *projective subspace* of P if $Y = \mathrm{pcl}_P(I)$ for some I; for such a Y, it is clear that $(Y, \bigcup_{x,y \in Y} L_{xy})$ is also a projective space. The *dimension* in a finite projective space P of a set $Z \subseteq X$, written $\dim_P(Z)$, is equal to $\min\{|I| : Z \subseteq \mathrm{pcl}_P(I)\} - 1$.

If M is a simple matroid in which every line contains at least three points, and any two coplanar lines of M intersect, then it is straightforward to check that $E(M)$, together with the set of lines of M, forms a projective space. (In general, M is not determined by this projective space; truncating a matroid of rank at least 4 does not change its set of points and lines, while certainly changing the matroid itself.) In this case, we write $P(M)$ for the projective space given by M, and for each $X \subseteq E(M)$, we write $\mathrm{pcl}_M(X)$ and $\dim_M(X)$ for the projective closure and projective dimensions of X in $P(M)$. Any two coplanar lines of the projective geometry $\mathrm{PG}(n,q)$ intersect, so $P(\mathrm{PG}(n,q))$ is an example of such a projective space. In $P(\mathrm{PG}(n,q))$, the dimension-k projective subspaces are precisely the rank-$(k+1)$ flats of $\mathrm{PG}(n,q)$.

For such an M, the functions pcl_M and $\dim_M$ bear some relation to cl_M and r_M. In general, the following hold, and are not hard to check:

- $X \subseteq \mathrm{pcl}_M(X) \subseteq \mathrm{cl}_M(X) = \mathrm{pcl}_M(\mathrm{cl}_M(X))$ for all $X \subseteq E(M)$.
- $\dim_M(X) \geq r_M(X) - 1$ for all $X \subseteq E(M)$.
- If I is an independent set of M, then $\dim_M(I) = |I| - 1$.

Two projective spaces $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ are *isomorphic* if there is a bijection $\varphi : X \to X'$ that preserves the property of being a line; in other words, $L \in \mathcal{L}$ if and only if $\varphi(L) \in \mathcal{L}'$. If this is the case, then we write $(X, \mathcal{L}) \cong (X', \mathcal{L}')$. Note that $P(M) \cong P(M')$ does not imply $M \cong M'$. Our recognition theorem rests on the the following characterisation of finite projective spaces, which states that any projective space of dimension at least 3 is isomorphic to a projective geometry:

Theorem 2.2.1 (Desargues' Theorem)**.** *If P is a finite projective space such that $\dim(P) \geq 3$, then $P \cong P(\mathrm{PG}(\dim(P), q))$ for some prime power q.*

We now give our recognition method:

Theorem 2.2.2. *Let $r \geq 4$ be an integer. If M is a simple, rank-r matroid such that every line of M contains at least three points, and any two coplanar lines of M intersect, then M has a $\mathrm{PG}(r-1,q)$-restriction for some prime power q, and every line of M contains exactly $q+1$ points.*

Proof. $E(M)$, together with the set of lines of M, is a projective space. Let B be a basis for M, and let $N = M|\mathrm{pcl}_M(B)$; clearly $r(N) = r$. Since B is independent, we have $\dim_M(E(N)) = \dim_M(B) = r - 1$. Each hyperplane H of N satisfies $H = \mathrm{cl}_M(H) =$

$\mathrm{pcl}_M(\mathrm{cl}_M(H)) = \mathrm{pcl}_M(H)$, so is a projective subspace of $E(N)$. Moreover, $H \neq E(N)$, so H is a *proper* projective subspace of $E(N)$, and $\dim_M(H) < \dim_M(E(N)) = r - 1$. On the other hand, $\dim_M(H) \geq r_M(H) - 1 = r - 2$, so $\dim_M(H) = r - 2$ for all hyperplanes H of N.

By Theorem 2.2.1, $P(N) \cong P(\mathrm{PG}(r-1, q))$ for some prime power q, so in particular, every dimension-$(r-2)$ projective subspace of $P(N)$ intersects every dimension-1 projective subspace of $P(N)$. The dimension-1 projective subspaces of $P(N)$ are just the lines of N, so if H is a hyperplane of N, and L is a line of N, then $H \cap L \neq \varnothing$. By Theorem 2.1.4, we have $N \cong \mathrm{PG}(r-1, q)$ for some prime power q.

To see that every line of M contains exactly $q+1$ points, observe that, by Theorem 2.2.1, $P(M) \cong P(PG(n, q'))$ for some n and q', so every line of M has the same number of points, and the lines of M contained in $E(N)$ have $q + 1$ points. □

We remark that this theorem remedies a subtle error in [14], in which it is wrongly assumed ([14], Theorem 5.2) that any simple rank-n matroid M whose set of lines and points gives a projective space is isomorphic to $\mathrm{PG}(n-1, q)$ for some q; the matroid $T(\mathrm{PG}(n, q))$ provides a counterexample for all $n \geq 3$. For their purposes ([14], Lemma 5.3), it suffices to show that M has a $\mathrm{PG}(n-1, q)$-restriction, as implied by Theorem 2.2.2.

2.3 Representations

In this section, we prove a result useful in Chapter 4, establishing that if A is a matrix with entries in a finite field $\mathbb{F}$, then a submatrix of A representing a projective geometry over a subfield of $\mathbb{F}$ can be assumed to only have entries in this subfield. Theorem 2.3.4 is likely equivalent to statements already well-known by projective geometers.

If q is a prime power, we will write $\mathrm{GF}(q)$ for some canonical field with q elements. If $\mathbb{F}$ has $\mathrm{GF}(q)$ as a subfield, M is an $\mathbb{F}$-representable matroid, and R is a restriction of M, then R is a $\mathrm{GF}(q)$*-represented restriction* of M if there is an $\mathbb{F}$-representation A of M such that $A[E(R)]$ has entries only in $\mathrm{GF}(q)$.

Two matrices A and B with entries in a field $\mathbb{F}$ are *projectively equivalent* if there is a sequence of elementary row operations and column scalings of A that gives B. We say that such a B is obtained by applying a *projective transformation* to A. If this is the case, then $M(A) = M(B)$.

Theorem 2.3.4 is closely related to the following:

Theorem 2.3.1 (Fundamental Theorem of Projective Geometry)**.** *Let q be a prime power, and $n \geq 2$ be an integer. The matroid* $\mathrm{PG}(n, q)$ *is uniquely* $\mathrm{GF}(q)$*-representable, up to projective equivalence and field automorphisms.*

We use two well-known results, one from matroid theory [7] and one from algebra [28]:

Theorem 2.3.2. *If M is a binary matroid, and $\mathbb{F}$ is a field, then M has at most one $\mathbb{F}$-representation, up to projective equivalence.*

Theorem 2.3.3 (Subfield Criterion)**.** *Let q be a prime power, and $k \geq 1$ be an integer. The field* $\mathrm{GF}(q^k)$ *has a unique subfield of order q.*

Theorem 2.3.4. *If q is a prime power, $n \geq 3$ is an integer, and $\mathbb{F}$ is a finite extension field of* $\mathrm{GF}(q)$*, then each representation of* $\mathrm{PG}(n-1, q)$ *over $\mathbb{F}$ is projectively equivalent to a representation over* $\mathrm{GF}(q)$*.*

Proof. Let $M \cong \mathrm{PG}(n-1, q)$, and A be an $\mathbb{F}$-representation of M; we may assume that A has an I_n-submatrix. We will show that there is a $\mathrm{GF}(q)$-subfield $\mathbb{F}'$ of $\mathbb{F}$, so that for any pair of distinct columns u and v of A, and $\omega \in \mathbb{F}'$, the vector $u + \omega v$ is parallel to a column of A. This property is preserved by row operations and column scalings, so we will freely apply projective transformations to A.

Let $\{x_1, x_2, x_3\}$ be an independent set of size 3 in M, and e_1, e_2, e_3 be the first three vectors in the standard basis of $\mathbb{F}^n$. The matrix B with column set $\{e_1, e_2, e_3, e_1 - e_2, e_2 - e_3, e_3 - e_1\}$ is a $\mathbb{F}$-representation of the cycle matroid of K_4, and M has an $M(K_4)$-restriction with basis $\{x_1, x_2, x_3\}$, so we may assume by Theorem 2.3.2 that $A_{x_i} = e_i$ for each $i \in \{1, 2, 3\}$, and moreover that all columns of B are columns of A.

Let Z be the set of vectors in $\mathbb{F}^n$ that are parallel to a column of A. Since $M \cong \mathrm{PG}(n-1, q)$ is modular, if L_1 and L_2 are rank-2 subspaces of $\mathbb{F}^n$, each spanned by a pair of vectors in Z, and $w \in L_1 \cap L_2$, then $w \in Z$. For simplicity, we will refer to such subspaces as *lines*, and write $\mathrm{cl}(v_1, v_2)$ for the subspace spanned by vectors $v_1, v_2 \in \mathbb{F}^n$.

For $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$, let $L_{ij} = \mathrm{cl}(e_i, e_j)$, and $F_{ij} = \{\omega \in \mathbb{F} : e_i + \omega e_j \in Z\}$.

Since all lines in $\mathrm{PG}(n-1, q)$ have $q+1$ points, and the elements of F_{ij} are in one-to-one correspondence with the points other than u_j on the line L_{ij}, we have $|F_{ij}| = q$, and since the columns of B are columns of A, the sets F_{ij} contain 0 and -1.

2.3.4.1. *$F_{12} = F_{23} = F_{31}$, and this set is closed under $\mathbb{F}$-inverses.*

Proof of claim: Let $\alpha \in F_{12}$. The lines $\mathrm{cl}(e_1+\alpha e_2, e_3-e_1)$ and L_{23} meet at a point parallel to $e_2+\alpha^{-1}e_3$, so $\alpha^{-1} \in F_{23}$. The lines $\mathrm{cl}(e_2+\alpha^{-1}e_3, e_1-e_2)$ and L_{31} meet at a point parallel to $e_3+\alpha e_1$, so $\alpha \in F_{31}$. Finally, the lines L_{12} and $\mathrm{cl}(e_3+\alpha e_1, e_2-e_3)$ meet at a point parallel to $e_1+\alpha^{-1}e_2$, so $\alpha^{-1} \in F_{12}$. Now, $F_{12} = \{\alpha^{-1} : \alpha \in F_{12}\}$, and the inclusions established give $F_{12} \supseteq F_{23} \supseteq F_{31} \supseteq F_{12}$, giving the claim. □

Let $F = F_{12} = F_{23} = F_{31}$. This second claim, together with the first claim and the fact that F contains -1 and 0, implies that F is a subfield of $\mathbb{F}$.

2.3.4.2. *F is closed under subtraction and multiplication in $\mathbb{F}$.*

Proof of claim: Let $\alpha, \beta \in F$. To see closure under multiplication, observe that $\alpha \in F_{12}$, $\beta \in F_{23}$, so $e_1+\alpha e_2$ and $e_2+\beta e_3$ are both in Z. The lines $\mathrm{cl}(e_1, e_2+\beta e_3)$ and $\mathrm{cl}(e_1+\alpha e_2, e_3)$ meet at a point parallel to $e_1+\alpha e_2+\alpha\beta e_3$, so this vector is in Z. The line $\mathrm{cl}(e_1+\alpha e_2+\alpha\beta e_3, e_2)$ meets L_{31} at $e_3+(\alpha\beta)^{-1}e_1$, so $(\alpha\beta)^{-1} \in F_{31}$, giving $\alpha\beta \in F$ by the first claim.

We have $\alpha, \beta \in F_{12}$, so $e_1+\alpha e_2$ and $e_1+\beta e_2$ are both in Z. The lines $\mathrm{cl}(e_1+\alpha e_2, e_2-e_3)$ and $\mathrm{cl}(e_1+\beta e_2, e_3-e_1)$ meet at a point parallel to $e_1+\beta e_2+(\alpha-\beta)e_3$, and $\mathrm{cl}(e_2, e_1+\beta e_2+(\alpha-\beta)e_3)$ meets L_{31} at a point parallel to $(\beta-\alpha)^{-1}e_1+e_3$, so $(\alpha-\beta)^{-1} \in F_{31}$, giving $\alpha-\beta \in F$ by the first claim. □

By these two claims, F is a subfield of $\mathbb{F}$. We know $|F| = q$, so Theorem 2.3.3 implies that $F = \mathrm{GF}(q)$. We have therefore shown that for all $\omega \in \mathrm{GF}(q)$ and distinct elements $x_1, x_2 \in E(M)$, the vector $A_{x_1}+\omega A_{x_2}$ is parallel to a column of A. We may assume that all columns of I_n are columns of A, so by repeated applications of this fact, it follows that all nonzero vectors in $\mathbb{F}^n$ are parallel to a column of A, which implies the theorem. □

As projective transformations on a submatrix can be extended to ones on a matrix, this theorem has an immediate corollary:

Corollary 2.3.5. *If q is a prime power, $k \geq 2$ is an integer, M is a GF(q^k)-representable matroid, and R is a $\mathrm{PG}(r(M)-1, q)$-restriction of M, then R is $\mathrm{GF}(q)$-represented in M.*

Finally, this theorem implies an appealing lemma when the extension field in question is $\mathrm{GF}(q^2)$. It allows us to describe an arbitrary point of a $\mathrm{GF}(q^2)$-representable matroid M in terms of its position relative to some $\mathrm{PG}(r(M)-1, q)$-restriction:

Lemma 2.3.6. *Let q be a prime power, M be a GF(q^2)-representable matroid, and let R be a $\mathrm{PG}(r(M)-1, q)$-restriction of M. If $e \in E(M)$ is a nonloop, and e is not parallel to a nonloop of R, then there is a unique line L of R so that $e \in \mathrm{cl}_M(L)$.*

Proof. By Corollary 2.3.5, there is a GF(q^2)-representation A of M so that $A[E(R)]$ has entries only in GF(q). Let $e \in E(M \backslash R)$ be a nonloop, and $\omega \in \mathrm{GF}(q^2) - \mathrm{GF}(q)$. Since $\{1, \omega\}$ is a basis for GF(q^2) over GF(q), there are vectors $v, v' \in \mathrm{GF}(q)^n$ so that $A_e = v + \omega v'$. Since $R \cong \mathrm{PG}(r(M) - 1, q)$, the vectors u and v are parallel to columns A_f and $A_{f'}$ of $A[E(R)]$, so $e \in \mathrm{cl}_M(\{f, f'\})$, which is a line of R. By modularity of the lines of R, and the fact that e is not a point of R, this line is unique. □

2.4 Matchings

Extending the definition of two skew sets, we say that a collection $\mathcal{X}$ of sets is *mutually skew* in a matroid M if $r_M\left(\bigcup_{X \in \mathcal{X}} X\right) = \sum_{X \in \mathcal{X}} r_M(X)$: equivalently, if each set in $\mathcal{X}$ is skew to the union of the other sets in $\mathcal{X}$.

A *matching* of a matroid M is a mutually skew set of lines of M. To construct an important class of matroids in Chapter 4, we need to consider matchings in projective geometries. The theorem of this section follows easily from the linear matroid matching theorem of Lovász ([29], Theorem 2), but is significantly weaker, and has a relatively short self-contained proof, so we include it. It gives a partly qualitative necessary condition for the nonexistence of a large matching.

Theorem 2.4.1. *There is an integer-valued function $f_{2.4.1}(q, k)$ so that, for any prime power q and integers $n \geq 1$ and $k \geq 0$, if $M \cong \mathrm{PG}(n-1, q)$ is a matroid, then for every set $\mathcal{L}$ of lines of M, either*

- *$\mathcal{L}$ contains a $(k+1)$-matching of M, or*
- *There is a flat F of M with $r_M(F) \leq k$, and a set $\mathcal{L}_0 \subseteq \mathcal{L}$ with $|\mathcal{L}_0| \leq f_{2.4.1}(q, k)$, such that every line $L \in \mathcal{L}$ either intersects F, or is in $\mathcal{L}_0$. Moreover, if $r_M(F) = k$, then $\mathcal{L}_0 = \varnothing$.*

Proof. Let q be a prime power and $k \geq 1$ be an integer, and set

$$f_{2.4.1}(q, k) = \frac{(q^{2k} - 1)(q^{2k+3} - 1)}{(q-1)^2}.$$

Let $n \geq 1$ be an integer, let $M \cong \mathrm{PG}(n-1, q)$, and $\mathcal{L}$ be a set of lines of M. For every $e \in E(M)$, we write $\deg_{\mathcal{L}}(e) = |\{L \in \mathcal{L} : e \in L\}|$. Let $C \subseteq E(M)$ be a maximal independent set so that

$$\deg_{\mathcal{L}}(e) > \frac{q^{2k+3} - 1}{q - 1}$$

for every $e \in C$. let $C' = C$ if $|C| \leq k$, and C' be a $(k+1)$-subset of C otherwise.

2.4.1.1. *$\mathcal{L}$ contains a $|C'|$-matching. Moreover, if there is a line L in $\mathcal{L}$ skew to C', then $\mathcal{L}$ contains a $(|C'|+1)$-matching.*

Proof of claim: We prove the second part of the claim; the proof of the first part is similar but simpler. Let $|C'| = \{e_1, \ldots, e_{|C'|}\}$. Let j be maximal so that $0 \leq j \leq |C'|$, and so that there is a $(j+1)$-matching $\mathcal{L}_j = \{L, L_1, \ldots, L_j\}$ so that $\mathcal{L}_j \subseteq \mathcal{L}$, and for each $1 \leq i \leq j$, we have $L_i \cap \mathrm{cl}_M(C') = \{e_i\}$. If $j = |C'|$, then $\mathcal{L}_j$ satisfies the claim; we may therefore assume that $j < |C'|$. Since $\mathcal{L}_j$ is a matching, and every line in $\mathcal{L}_j - \{L\}$ meets C' in a point, we have $r_M\left(C' \cup \bigcup_{L' \in \mathcal{L}_j}(L')\right) = |C'| + 2 + j \leq 2|C'| + 1$.

Now, $\deg_{\mathcal{L}}(e_{j+1}) > \frac{q^{2k+3}-1}{q-1} \geq \frac{q^{2|C'|+1}-1}{q-1}$, each line in $\mathcal{L}$ is a point of $M /\!\!/ e_{j+1}$, and M is GF(q)-representable, so there is a set X so that $\mathrm{cl}_M(\{x, e_{j+1}\}) \in \mathcal{L}$ for all $x \in X$, and $r_M(X) > 2|C'| + 1$. There is therefore some $x \in X$ not in $\mathrm{cl}_M(C' \cup \bigcup_{L' \in L_j}(L'))$. Now, $\mathcal{L}_j \cup \{\mathrm{cl}_M(\{x, e_{j+1}\})\}$ is a matching of M, contradicting the maximality of j. □

Suppose that the first outcome of the theorem does not hold; by 2.4.1.1, we may assume that $|C| \leq k$. Let $\mathcal{L}_0$ be the set of lines in $\mathcal{L}$ that are skew to C.

2.4.1.2. *$|\mathcal{L}_0| \leq f_{2.4.1}(q, k)$.*

Proof of claim: By maximality of C, for each $e \notin \mathrm{cl}_M(C)$, we have $\deg_{\mathcal{L}}(e) \leq \frac{q^{2k+3}-1}{q-1}$. Let $\mathcal{L}_0'$ be a maximal matching contained in $\mathcal{L}_0$, and let F' be the flat spanned in M by the lines in $\mathcal{L}_0'$. We may assume that $|\mathcal{L}_0'| \leq k$, so $|F'| \leq \frac{q^{2k}-1}{q-1}$. By maximality of $\mathcal{L}_0'$ and modularity of M, each $L \in \mathcal{L}_0$ contains a point in F', so the claim follows by this bound on $|F'|$, and our degree bound. □

We now set $F = \mathrm{cl}_M(C)$. Since M is modular, every line in $\mathcal{L} - \mathcal{L}_0$ meets F. If $r_M(F) = k$, and $L \in \mathcal{L}_0$, then by 2.4.1.1, $\mathcal{L}$ contains a $(k+1)$-matching. So if $r_M(F) = k$, we must have $\mathcal{L}_0 = \varnothing$. Now, F and $\mathcal{L}_0$ satisfy the second outcome of the lemma. □

2.5 Single-Element Extensions

To find the unavoidable minors $\mathrm{PG}^+(n-1, q, n)$ and $\mathrm{PG}^+(n-1, q, 2)$ of Theorem 1.5.4, we need to consider general single-element extensions of projective geometries. The theory of single-element extensions for arbitrary matroids is not easy, but projective geometries are a very special case; the attractive fact is that any single-element extension of a projective geometry is a principal extension (can be obtained by freely adding a point to a flat):

Lemma 2.5.1. *Let $n \geq 1$ be an integer, and q be a prime power. If M is a simple rank-n matroid, and $e \in E(M)$ satisfies $M \backslash e \cong \mathrm{PG}(n-1,q)$, then $M \cong \mathrm{PG}^+(n-1,q,k)$ for some $1 < k \leq n$.*

Proof. Clearly e is spanned by the flat $E(M \backslash e)$ of $M \backslash e$; suppose for a contradiction that there are two distinct minimal flats F_1 and F_2 of $M \backslash e$ that both span e in M. By minimality, $e \notin \mathrm{cl}_M(F_1 \cap F_2)$. Let $M' = M /\!\!/ (F_1 \cap F_2)$. We know e is a nonloop of M', and since $r_M(F_1 \cap F_2) = \sqcap_M(F_1, F_2)$, the sets F_1 and F_2 are skew in M'. But $e \in \mathrm{cl}_{M'}(F_1)$ and $e \in \mathrm{cl}_{M'}(F_2)$, so $\sqcap_{M'}(F_1, F_2) > 0$, a contradiction. Therefore, there is a unique minimal flat F of $M \backslash e$ that spans e in M. It now follows from the definition that $M = (M \backslash e) +_F e$, and therefore that $M \cong \mathrm{PG}^+(r(M)-1, q, r_M(F))$. □

Given a matroid of the form $\mathrm{PG}^+(n-1,q,k)$, we can contract points, inside or outside of the rank-k flat that has been principally extended, to obtain minors of a similar form. In this way, we can arrive at one of our two unavoidable minors from *any* single-element extension of a large enough projective geometry:

Lemma 2.5.2. *Let q be a prime power, and n, k, n' be integers with $1 < k \leq n$ and $2 \leq 2n' \leq n$, and let $M \cong \mathrm{PG}^+(n-1,q,k)$. If $k < n'$, then M has a $\mathrm{PG}^+(n'-1,q,2)$-minor, and if $k \geq n'$, then M has a $\mathrm{PG}^+(n'-1,q,n')$-minor.*

Proof. We show that $\mathrm{PG}^+(n-1,q,k)$ has a $\mathrm{PG}^+(n-2,q,k-1)$-minor for all $k \geq 2$, and that, if $k < n$, then $\mathrm{PG}^+(n-1,q,k)$ has a $\mathrm{PG}^+(n-2,q,k)$-minor. The lemma will follow from repeated applications of these two facts.

Let $N \cong \mathrm{PG}(n-1,q)$, and F be a rank-k flat of N. Let $N^+ = N +_F e \cong \mathrm{PG}^+(n-1,q,k)$. We have $r(N^+) = r(N) = n$. Let B be a basis for N, containing a basis B_F for $N|F$.

If $k \geq 2$, then let $x \in B_F$. We have $e \in \mathrm{cl}_{N^+}(B_F)$, and moreover B_F is a minimal subset of B that spans e in N^+. Therefore, we have $e \in \mathrm{cl}_{N^+ /\!\!/ x}(B_F - \{x\})$, and $B_F - \{x\}$ is a minimal subset of $B - \{x\}$ that spans e in $N^+ /\!\!/ x$. By Lemma 2.1.2 and Lemma 2.1.1, the matroid $N /\!\!/ x$ has a $\mathrm{PG}(n-2,q)$-restriction $(N /\!\!/ x)|X$ with basis $B - \{x\}$. In this restriction, the rank-$(k-1)$ flat spanned by $B_F - \{x\}$ is a minimal flat that spans e in $(N^+ /\!\!/ x)|(X \cup \{e\})$. Therefore, by Lemma 2.5.1, we have $(N^+ /\!\!/ x)|(X \cup \{e\}) \cong \mathrm{PG}^+(n-2,q,k-1)$, giving the first half of the lemma.

If $k < n$, then $B_F \neq B$; let $y \in B - B_F$. Clearly, B_F is a minimal subset of $B - \{y\}$ that spans e in $N /\!\!/ y$. By a similar argument to the first part, $N^+ /\!\!/ y$ has a $\mathrm{PG}^+(n-2,q,k)$-restriction, giving the lemma. □

In particular, this lemma implies that $\mathrm{PG}^+(n-1,q,2)$ has a $\mathrm{PG}^+(n'-1,q,2)$-minor for all $n' < n$, and a similar statement for $\mathrm{PG}^+(n-1,q,n)$. This motivates the definition of minor-closed classes associated with these matroids: for each prime power q, let $\mathcal{L}^2(q)$ denote the closure of the set $\{\mathrm{PG}^+(n-1,q,2) : n \geq 2\}$ under minors and isomorphism, and $\mathcal{L}^T(q)$ denote the closure of the set $\{\mathrm{PG}^+(n-1,q,n) : n \geq 2\}$ under minors and isomorphism. Clearly $\mathcal{L}^T(q)$ and $\mathcal{L}^2(q)$ contain all projective geometries over $\mathrm{GF}(q)$, so they both contain $\mathcal{L}(q)$. Our first lemma regarding these classes is immediate from Lemma 2.5.2:

Lemma 2.5.3. *If q is a prime power, and $\mathcal{M}$ is a minor-closed class of matroids, then $\mathcal{L}^2(q) \subseteq \mathcal{M}$ if and only if there exist infinitely many n such that $\mathrm{PG}^+(n-1,q,2) \in \mathcal{M}$, and $\mathcal{L}^T(q) \subseteq \mathcal{M}$ if and only if there exist infinitely many n such that $\mathrm{PG}^+(n-1,q,n) \in \mathcal{M}$.*

We now calculate the growth rate functions for these two classes:

Lemma 2.5.4. *If q is a prime power, then $h_{\mathcal{L}^2(q)}(n) = \frac{q^{n+1}-1}{q-1} - q$ for all $n \geq 2$, and $h_{\mathcal{L}^T(q)}(n) = \frac{q^{n+1}-1}{q-1}$ for all $n \geq 2$.*

Proof. We first identify the extremal matroids. Every matroid in $\mathcal{L}^2(q)$ has the form $(N +_L e)/C \backslash D$, where $N \cong \mathrm{PG}(n-1,q)$ for some $n \geq 2$, and L is a line of N, and D is coindependent. One can check by Lemma 2.1.2 that contracting points of $(N +_L e)$ other than e yields a matroid M with simplification isomorphic to either $\mathrm{PG}^+(m-1,q,2)$ or $\mathrm{PG}(m-1,q)$, depending on whether any points on L were contracted. Furthermore, the matroids in $\mathcal{L}^T(q)$ have the form $(N +_{E(N)} e)/C \backslash D$, and any matroid obtained by contracting points other than e from $N +_{E(N)} e$ has simplification isomorphic to $\mathrm{PG}^+(m-1,q,m)$ for some m. Deleting points in a coindependent set cannot increase density, so it follows that, up to simplification, the extremal matroids in $\mathcal{L}^2(q)$ take the form $(N +_L e)/C$, where N is a projective geometry over $\mathrm{GF}(q)$, and $C = \varnothing$ or $C = \{e\}$, and the extremal matroids in $\mathcal{L}^T(q)$ have the form $(N +_{E(N)} e)/C$, where N is a projective geometry over $\mathrm{GF}(q)$, and $C = \varnothing$ or $C = \{e\}$.

If $N \cong \mathrm{PG}(n-1,q)$, and L is a line of N, then $\varepsilon(N +_L e) = \varepsilon(N +_{E(N)} e) = \frac{q^n-1}{q-1} + 1 = \frac{q^{r(N)}-1}{q-1} + 1$. On the other hand, e lies on exactly one long line of $N +_L e$, and this line contains $q+1$ other points, so $\varepsilon((N +_L e)/e) = \varepsilon(N) - q = \frac{q^n-1}{q-1} - q = \frac{q^{r((N+_L e)/e)+1}-1}{q-1} - q$. Since $\frac{q^{m+1}-1}{q-1} - q > \frac{q^m-1}{q-1}$, the matroid $(N +_L e)/e$ is denser than $N +_L e$, and is therefore extremal in $\mathcal{L}^2(q)$. Therefore $h_{\mathcal{L}^2(q)}(n) = \frac{q^{n+1}-1}{q-1} - q$ for all $n \geq 2$. Similarly, the point e lies on no long line of $(N +_{E(N)} e)/e$, so $\varepsilon((N +_{E(N)} e)/e) = \frac{q^{r((N+_{E(N)}e)/e)+1}-1}{q-1}$, and we get $h_{\mathcal{L}^T(q)}(n) = \frac{q^{n+1}-1}{q-1}$ for all $n \geq 2$. $\square$

Note that the extremal matroids in $\mathcal{L}^T(q)$ are truncations of projective geometries over $\mathrm{GF}(q)$, explaining the notation we use for this class. Both these classes satisfy Conjectures 1.5.14 and 1.5.16 with $k = 1$, being denser than $\mathcal{L}(q)$ by roughly a factor of q.

While the class $\mathcal{L}(q)$ contains no simple rank-2 matroid with more than $q+1$ elements, both $\mathcal{L}^2(q)$ and $\mathcal{L}^T(q)$ contain dramatically larger rank-2 matroids. The following lemma, a consequence of the computations in the proof above, shows this.

Lemma 2.5.5. *If q is a prime power, then $\mathrm{PG}^+(2, q, 2)$ has a U_{2,q^2+1}-minor, and $\mathrm{PG}^+(2, q, 3)$ has a U_{2,q^2+q+1}-minor.*

2.6 k-Element Projections

We now prove two theorems relating to k-element projections of a projective geometry, without loss of generality considering just those which are not projections for any smaller k. The first is Theorem 2.6.2, which gives an alternative characterisation of these projections, and provides the bounds for their number of points that led us in Chapter 1 to Conjecture 1.5.16. The second, Theorem 2.6.3, gives the precise number of points in a k-element projection in the special case of Chapter 4, serving to highlight the difference between this specialisation and the general case.

Recall that a collection $\mathcal{X}$ of sets in a matroid M is mutually skew in M if $r_M\left(\bigcup_{X\in\mathcal{X}} X\right) = \sum_{X\in\mathcal{X}} r_M(X)$, and that this is equivalent to the statement that each set in $\mathcal{X}$ is skew to the union of the other sets in $\mathcal{X}$. Our first lemma shows that certain projections of projective geometries do not destroy too many points; the matroid N/K in the following lemma differs from a general k-element projection of $N\backslash K$ only in that K is a *flat* of N:

Lemma 2.6.1. *Let q be a prime power, and $k \geq 0$ be an integer. If N is a matroid, and K is a rank-k flat of N such that $N\backslash K \cong \mathrm{PG}(r(N)-1, q)$, then there is an integer d with $0 \leq d \leq \frac{q^{2k}-1}{q-1}$ such that $\varepsilon(N/K) = \varepsilon(N\backslash K) - qd$.*

Proof. Let $\mathcal{P}$ denote the set of points of N/K. Each $P \in \mathcal{P}$ is a flat of the projective geometry $N\backslash K$, so satisfies $|P| = \frac{q^{r_N(P)}-1}{q-1}$, and $N\backslash K$ is simple, so we have $\varepsilon(N\backslash K) - \varepsilon(N/K) = \sum_{P\in\mathcal{P}}(|P|-1)$. The quantity $|P|-1 = q\left(\frac{q^{r_N(P)-1}-1}{q-1}\right)$ is a multiple of q for all $P \in \mathcal{P}$, so this proves the following:

2.6.1.1. *$\varepsilon(N\backslash K) - \varepsilon(N/K)$ is a multiple of q.*

Let $\mathcal{P}' = \{P \in \mathcal{P} : |P| > 1\}$, so $\varepsilon(N\backslash K) - \varepsilon(N/K) = \sum_{P \in \mathcal{P}'}(|P| - 1)$. Let B be a basis for $\bigcup_{P \in \mathcal{P}'} P$ in N/K, and for each $e \in B$, let $P_e \in \mathcal{P}'$ be the point of N/K containing e. Let $\mathcal{P}_0 = \{P_e : e \in B\}$. The set $\mathcal{P}_0$ is a mutually skew collection of $|B|$ points of N/K. By choosing B appropriately, we may assume that if there is some $Q \in \mathcal{P}'$ of rank at least 3 in N, then $Q \in \mathcal{P}_0$. We show that $\mathcal{P}_0$ is also mutually skew in $N\backslash K$:

2.6.1.2. *$\mathcal{P}_0$ is a mutually skew collection of flats of $N\backslash K$.*

Proof of claim: Suppose not; then there is some $Q \in \mathcal{P}_0$ such that $\sqcap_{N\backslash K}(Q, \bigcup_{P \in \mathcal{P}_0 - \{Q\}} P) > 0$, and by modularity of flats in a projective geometry, there is therefore a point f of $N\backslash K$, spanned by both Q and $\bigcup_{P \in \mathcal{P}_0 - \{Q\}} P$. As $\mathcal{P}_0$ is mutually skew in N/K, there is no such point in N/K, so f is a loop of N/K and is therefore spanned by K in N, contradicting the fact that K is a flat of N. $\square$

Let $X = \bigcup_{P \in \mathcal{P}_0} P$. Each $P \in \mathcal{P}_0$ is in $\mathcal{P}'$, so satisfies $|P| > 1$ and therefore $r_N(P) \geq 2$. We therefore have $r_N(X) = r_N(\bigcup_{P \in \mathcal{P}_0} P) = \sum_{P \in \mathcal{P}_0} r_N(P) \geq 2|\mathcal{P}_0| = 2|B|$, and if equality holds, then $r_N(P) = 2$ for all $P \in \mathcal{P}$. On the other hand, $|B| = r_{N/K}(X) \geq r_N(X) - k \geq 2|B| - k$, so $|B| \leq k$. Since B is a basis of X in N/K, we have $X \subseteq \mathrm{cl}_N(X \cup K)$, so $r_N(X) \leq 2k$. By our choice of $\mathcal{P}_0$ and equality characterisation, we have $r_N(X) = 2k$ if and only if every $P \in \mathcal{P}'$ is a line of $N\backslash K$. Our third claim, in conjunction with the first, gives the lemma.

2.6.1.3. $0 \leq \varepsilon(N\backslash K) - \varepsilon(N/K) \leq q\left(\frac{q^{2k}-1}{q-1}\right)$

Proof of claim: Let $F = \mathrm{cl}_N(X)$; we have $r_N(F) \leq 2k$. By construction of $\mathcal{P}'$ and $\mathcal{P}_0$, all points of $N\backslash K$ that are identified with another point in N/K are points in F, so $\varepsilon(N\backslash K) - \varepsilon(N/K) = \varepsilon((N\backslash K)|F) - \varepsilon((N/K)|F)$.

If $r_N(F) = 2k$, then as we have observed, every $P \in \mathcal{P}'$ is a line of $N\backslash K$. Each such line contains exactly $q+1$ points, q of which are lost when K is contracted. Every point of $M|F$ not in $\mathcal{P}'$ has size 1, so $\varepsilon((N/K)|F) \geq \frac{1}{q+1}\varepsilon((N\backslash K)|F) = \frac{1}{q+1}\left(\frac{q^{2k}-1}{q-1}\right)$. A calculation now gives $\varepsilon((N\backslash K)|F) - \varepsilon((N/K)|F) \leq q\left(\frac{q^{2k}-1}{q^2-1}\right)$, as required.

If $r_N(F) \leq 2k-1$, then $\varepsilon((N\backslash K)|F) - \varepsilon((N/K)|F) \leq \varepsilon((N\backslash K)|F) \leq \frac{q^{2k-1}-1}{q-1} < q\left(\frac{q^{2k}-1}{q^2-1}\right)$, also giving the claim. $\square$

$\square$

Now we prove the main theorem of this section, which implies Theorem 1.5.15.

Theorem 2.6.2. *Let q be a prime power, $k \geq 0$ be an integer, and M be a matroid of rank at least $k+1$. If M is, up to simplification, a k-element projection of* $\mathrm{PG}(r(M)+k-1, q)$*, then the following are equivalent:*

1. *k is the minimal integer such that M is, up to simplification, a k-element projection of* $\mathrm{PG}(r(M)+k-1, q)$*.*
2. *There exists a matroid N, and a rank-k independent flat K of N such that $N \backslash K \cong$* $\mathrm{PG}(r(M)+k-1, q)$*, and* $\mathrm{si}(M) = \mathrm{si}(N/K)$*.*
3. *There is an integer $d \in \{0, 1, \ldots \frac{q^{2k}-1}{q^2-1}\}$ such that $\varepsilon(M) = \frac{q^{r(M)+k}-1}{q-1} - qd$.*

Proof. The previous lemma gives $2 \Rightarrow 3$; we show that $1 \Rightarrow 2$ and $3 \Rightarrow 1$.

Suppose that 1 holds: by definition there is some matroid N, and a rank-k independent set K of N such that $\mathrm{si}(M) = \mathrm{si}(N/K)$, and $N \backslash K \cong \mathrm{PG}(r(M)+k-1, q)$; we just need to show that K is a flat of N. If not, then there is some $e \in E(N \backslash K)$ spanned by K. Therefore, $\mathrm{si}(M) = \mathrm{si}((N/e)/K')$, where K' is a $(k-1)$-subset of K that is independent in N/e. By Lemma 2.1.2, we have $\mathrm{si}(N/e) \cong \mathrm{PG}(r(M)+k-2, q)$, so, up to simplification, M is a $(k-1)$-element projection of $\mathrm{PG}(r(M)+k-2, q)$, contradicting minimality of k.

If 3 holds, then $\varepsilon(M) \geq \frac{q^{r(M)+k}-1}{q-1} - q\left(\frac{q^{2k}-1}{q^2-1}\right)$. If 1 fails, then M is a k'-element projection of $\mathrm{PG}(r(M)+k'-1, q)$ for some $k' < k$, so $\varepsilon(M) \leq \frac{q^{r+k'}-1}{q-1} \leq \frac{q^{r+k-1}-1}{q-1}$. A straightforward computation shows that $\frac{q^{r+k}-1}{q-1} - q\left(\frac{q^{2k}-1}{q^2-1}\right) > \frac{q^{r+k-1}-1}{q-1}$ for all $r \geq k+1$, giving a contradiction. □

We now show that, in the setting of Chapter 4, the value d of the preceding theorem is completely determined by k. Recall that $\mathcal{P}_{q,k}$ denotes the class of minors of matroids of the form N/K, where $N \in \mathcal{L}(q^2)$ and $N \backslash K \cong \mathrm{PG}(r(N)-1, q)$; this is the special type of $\mathrm{GF}(q^2)$-representable k-element projection where the 'intermediate' matroid N is also $\mathrm{GF}(q^2)$-representable.

Theorem 2.6.3. *Let q be a prime power, and $k \geq 0$ be an integer. If $M \in \mathcal{P}_{q,k}$, and k is minimal such that M is a k-element projection of* $\mathrm{PG}(r(M)-1+k, q)$ *up to simplification, then*

$$\varepsilon(M) = \frac{q^{r(M)+k}-1}{q-1} - q\left(\frac{q^{2k}-1}{q^2-1}\right).$$

Proof. Let N and K be given by property 2 of Theorem 2.6.2. Since K is an independent flat of N and $N \backslash K \cong \mathrm{PG}(r(N)-1, q)$, we know N is simple. Let $\mathcal{L}$ denote the set of lines of M that are not skew to K. (Since K does not contain any line in $\mathcal{L}$, each $L \in \mathcal{L}$ satisfies $\sqcap_N(L, K) = 1$).

As $M \in \mathcal{P}_{q,k}$, we may assume that $N \in \mathcal{L}(q^2)$; let $N^+ \cong \mathrm{PG}(r(M)-1, q^2)$ be such that $N = N^+|(E(M) \cup K)$. By Lemma 2.3.6, each point $e \in E(N^+ \backslash M)$ lies in the closure of exactly one line of M, and since K spans no points of $N \backslash K$ in N, every point in $\mathrm{cl}_{N^+}(K)$ has this property. Furthermore, no line of $N \backslash K$ spans two different points of $\mathrm{cl}_{N^+}(K)$, as such a line would itself be spanned by K. By modularity of N^+, the lines of $N \backslash K$ whose closure contains a point of $\mathrm{cl}_{N^+}(K)$ are precisely the lines in $\mathcal{L}$, so we have $|\mathcal{L}| = |\mathrm{cl}_{N^+}(K)| = \frac{q^{2k}-1}{q^2-1}$.

If two lines $L_1, L_2 \in \mathcal{L}$ are not disjoint, then let P be the plane of $N \backslash K$ containing L_1 and L_2, and $e_i \in \mathrm{cl}_{N^+}(K)$ be a point spanned by L_i for each $i = 1, 2$. By considering the plane $\mathrm{PG}(2, q)$ as a restriction of $\mathrm{PG}(2, q^2)$, one can verify that every line of $\mathrm{cl}_{N^+}(P)$ contains a point of P, and hence $\mathrm{cl}_{N^+}(\{e_1, e_2\})$ contains a point of P. This contradicts $\mathrm{cl}_{N^+}(K) \cap E(N \backslash K) = \varnothing$.

Hence $\mathcal{L}$ comprises $\frac{q^{2k}-1}{q^2-1}$ pairwise disjoint lines of $N \backslash K$, with each $L \in \mathcal{L}$ satisfying $\sqcap_N(L, K) = 1$. Each such line contains $q+1$ points of $N \backslash K$, and is contained in a single point of N/K. Thus, q points of N are lost for each line in $\mathcal{L}$, so $\varepsilon(N) - \varepsilon(N/K) \geq q|\mathcal{L}| = q\left(\frac{q^{2k}-1}{q^2-1}\right)$. This gives $\varepsilon(M) = \varepsilon(N/K) \leq \varepsilon(N) - q\left(\frac{q^{2k}-1}{q^2-1}\right) = \frac{q^{r(M)+k}-1}{q-1} - q\left(\frac{q^{2k}-1}{q^2-1}\right)$. By property 3 of Theorem 2.6.2, this must hold with equality. $\square$

Chapter 3

Excluding a Line

In this chapter, we will prove Theorem 1.0.1, which we now restate for convenience:

Theorem 3.0.4. *Let $\ell \geq 2$ be an integer, and q be the largest prime power not exceeding ℓ. If M is a simple rank-r matroid with no $U_{2,\ell+2}$-minor, and with sufficiently large rank, then $|M| \leq \frac{q^{r(M)}-1}{q-1}$.*

3.1 Proof Sketch

To motivate the preliminary material on connectivity and density in this chapter, we start with a sketch of the proof of Theorem 3.0.4. The formal proof of the theorem given later uses more abstract techniques, but contains the same essential line of reasoning; the theorem appears in [16], in which the argument is more transparent, as the main result.

We will define positive integers r_1, r_2, r_3, r_4, where we set $r_1 = 3$, and each r_i is sufficiently large in comparison to its predecessor. (Leaving r_1 arbitrary will imply a stronger result, namely Theorem 1.5.4). Now, we consider a potential counterexample to the theorem; let $M_4 \in \mathcal{U}(\ell)$ be a simple matroid with $r(M) \geq r_4$ and $|M_4| > \frac{q^{r(M_4)}-1}{q-1}$. We wish to derive a contradiction by showing that M_4 has a $U_{2,\ell+2}$-minor; we sometimes use the crude estimate $q^2 + 1 \geq \ell + 2$.

Recall that weak roundness is a connectivity property; M is weakly round if there are no two sets $A, B \subseteq E(M)$ with union $E(M)$, such that $r_M(A) = r(M) - 2$ and $r_M(B) = r(M) - 1$. Material in Section 3.3 implies our first claim:

3.1.1. *M_4 has a simple weakly round restriction M_3, with $r(M_3) \geq r_3$, and $|M_3| > \frac{q^{r(M_3)}-1}{q-1}$.*

The next claim is a corollary of the growth rate theorem, following from the density of M_3; the details are discussed in Section 3.2:

3.1.2. *M_3 has a minor $N_2 \cong \mathrm{PG}(r_2 - 1, q)$.*

Now, let M_2 be a minimal contraction-minor of M_3 such that N_2 is a minor of M_2, and $\varepsilon(M_2) > \frac{q^{r(M_2)}-1}{q-1}$. We say a line L of M is *q-long* if it contains at least $q + 2$ points, or equivalently if $M|L \notin \mathcal{L}(q)$. A simple argument exploiting the minor-minimality of M_2 implies the following:

3.1.3. *Either*

1. *N_2 is a spanning restriction of M_2, or*

2. M_2 *has a* q*-long line* L*.*

Lemma 2.5.5 gives the next claim, that will certify the existence of a $U_{2,\ell+2}$-minor of M_2:

3.1.4. *If* $r \geq 3$*, and* M *is a simple, rank-*r *single-element extension of* $\mathrm{PG}(r-1,q)$*, then* M *has a* U_{2,q^2+1}*-minor.*

This claim allows us to assume that the first outcome of 3.1.3 does not hold, so we may assume that M_2 has a q-long line L, as well as the $\mathrm{PG}(r_2-1,q)$-minor N_2. Using a lemma proved in Section 3.3 that exploits weak roundness of M_2 and density of N_2, we obtain a final claim:

3.1.5. M_2 *has a minor* M_1 *with a spanning* $\mathrm{PG}(r_1-1,q)$*-restriction, such that* $M_1|L = M_2|L$*.*

Now the contradiction we need follows from this claim and 3.1.4.

3.2 Density Theorems

In this section, we formulate a corollary of the Growth Rate Theorem that will allow us to find a large projective geometry minor in a sufficiently dense matroid in $\mathcal{U}(\ell)$. We will use the 'polynomial-exponential' part of the theorem, which was resolved by Geelen and Kabell [14]:

Theorem 3.2.1. *Let* $\mathcal{M}$ *be a minor-closed class of matroids, not containing all simple rank-*2 *matroids. Either*

1. $h_{\mathcal{M}}(n) \leq n^{c_{\mathcal{M}}}$ *for all* n*, or*
2. *There is a prime power* q *so that* $\frac{q^n-1}{q-1} \leq h_{\mathcal{M}}(n) \leq c_{\mathcal{M}} q^n$ *for all* n*, and* $\mathcal{M}$ *contains all* $\mathrm{GF}(q)$*-representable matroids.*

The corollary we use is one that applies to specific dense matroids, rather than minor-closed classes. It is one of many results in this thesis stated 'qualitatively' in terms of a large function whose value is not explicitly given in the statement.

Theorem 3.2.2. *There is a real-valued function $\alpha_{3.2.2}(n, \beta, \ell)$ so that, for any integers $n \geq 1$ and $\ell \geq 2$, and real number $\beta > 1$, if $M \in \mathcal{U}(\ell)$ is a matroid such that $\varepsilon(M) \geq \alpha_{3.2.2}(n, \beta, \ell)\beta^{r(M)}$, then M has a* $\mathrm{PG}(n-1, q)$*-minor for some $q > \beta$.*

Proof. Let $n \geq 1$ and $\ell \geq 2$ be integers, and $\beta > 1$ be a real number. Let

$$\mathcal{M} = \{M \in \mathcal{U}(\ell) : \ M \text{ has no } \mathrm{PG}(n-1, q)\text{-minor for all } q > \beta\}.$$

This class is clearly minor closed and does not contain all simple rank-2 matroids. It suffices to set $\alpha(n, \beta, \ell)$ as some α such that $h_{\mathcal{M}}(r) < \alpha\beta^r$ for all $r \geq 0$.

By Theorem 3.2.1, the class $\mathcal{M}$ is either exponentially or polynomially dense. If $\mathcal{M}$ is exponentially dense, then since $\mathcal{M}$ does not contain arbitrarily large projective geometries over any field with more than β elements, we must have $h_{\mathcal{M}}(r) \leq c_{\mathcal{M}} q^{r(M)}$ for some real number $c_{\mathcal{M}}$, and prime power $q \leq \beta$, and for all $r \geq 0$. Any $\alpha > c_{\mathcal{M}}$ will now satisfy $h_{\mathcal{M}}(r) < \alpha q^r \leq \alpha\beta^r$ for all r, so the required α exists.

If $\mathcal{M}$ is polynomially dense, then $h_{\mathcal{M}}(r) \leq r^{c_{\mathcal{M}}}$ for all r, and, since $\beta > 1$, some sufficiently large α will satisfy $r^{c_{\mathcal{M}}} < \alpha\beta^r$ for all $r \geq 0$, again giving what we want. □

3.3 Connectivity

This section deals with notions of matroid connectivity that will be useful in this thesis. We prove three results. The first is Theorem 1.5.6, which concerns highly connected extremal matroids in minor-closed classes. The next is a lemma that allows us to reduce theorems to the highly connected case, and finally we prove a lemma showing how high connectivity is useful. These last two lemmas were both invoked in the proof sketch introducing this chapter. Instead of traditional notions such as 3-connectivity and internal 4-connectivity, we will use connectivity properties that are more appropriate to the exponentially dense matroids we are considering.

The first such notion is roundness. A matroid M is *round* if $E(M)$ is not the union of two hyperplanes of M. Equivalently, a matroid is round if it is infinitely vertically connected, or if it has no pair of disjoint cocircuits. Compared to usual notions of connectivity, roundness is very strong, and is seldom found in 'sparse' matroids. A graphic matroid M is round if and only if $\mathrm{si}(M)$ is the cycle matroid of a clique, whereas the class of 3-connected (or internally 4-connected) graphic matroids is rich.

For technical reasons, we also consider a slight relaxation of roundness that has already been mentioned: A matroid M is *weakly round* if there is no pair of sets A, B with union

$E(M)$, such that $r_M(A) = r(M) - 2$ and $r_M(B) = r(M) - 1$. Any round matroid, or matroid of rank at most 2, is clearly weakly round.

Roundness is indeed a common property in the world of exponentially dense matroids. If $M \cong \mathrm{PG}(n-1, q)$, then $|M| = \frac{q^n-1}{q-1}$, and any hyperplane H of M satisfies $|H| = \frac{q^{n-1}-1}{q-1} < \frac{1}{2}|M|$, so $E(M)$ is not the union of two hyperplanes, and M is round. Many more round matroids can be formed by deleting a small number of points from M, giving rise to a large class of round matroids.

A traditional difficulty with connectivity notions is determining which matroid operations preserve them. Even for 3-connectivity, this is nontrivial and has been extensively studied. However, roundness and weak roundness give us no such difficulties, and the relevant result feels almost too good to be true - these properties are both preserved by contraction and simplification!

Lemma 3.3.1. *If M is a (weakly) round matroid, then $\mathrm{si}(M)$ is (weakly) round, and if $C \subseteq E(M)$, then $M /\!\!/ C$ is (weakly) round.*

The proof of these facts is straightforward, and we will invoke them freely throughout the thesis.

3.3.1 Extremal Matroids

To prove Theorem 1.5.6, we first require a small lemma, which finds a dense highly connected restriction of an arbitrary matroid:

Lemma 3.3.2. *If M is a matroid, then M has a weakly round restriction N such that $\varepsilon(N) \ge \varphi^{r(N)-r(M)}\varepsilon(M)$, where $\varphi = \frac{1}{2}(1+\sqrt{5})$.*

Proof. we may assume that M is not weakly round, so $r(M) > 2$, and there are sets A, B of M such that $r_M(A) = r(M) - 2$, $r_M(B) = r(M) - 1$, and $E(M) = A \cup B$. Now, since $\varphi^{-1} + \varphi^{-2} = 1$, either $\varepsilon(M|A) \ge \varphi^{-2}\varepsilon(M)$ or $\varepsilon(M|B) \ge \varphi^{-1}\varepsilon(M)$; in the first case, by induction $M|A$ has a weakly round restriction N with $\varepsilon(N) \ge \varphi^{r(N)-r(M|A)}\varepsilon(M|A) \ge \varphi^{r(N)-r(M)+2}\varphi^{-2}\varepsilon(M) = \varphi^{r(N)-r(M)}\varepsilon(M)$, giving the result. The second case is similar. □

We can now prove Theorem 1.5.6, which states that exponentially dense minor-closed classes have many weakly round extremal matroids. In fact, we prove something stronger. We say a class of matroids is *restriction-closed* if it is closed under restriction and isomorphism; all minor-closed classes are restriction-closed.

Theorem 3.3.3. *If $\mathcal{M}$ is a restriction-closed class of matroids such that $h_{\mathcal{M}}(n) \geq 2^{n-1}$ for all $n \geq 1$, then there exist weakly round extremal matroids in $\mathcal{M}$ of arbitrarily large rank.*

Proof. For each integer $n \geq 0$, let $\alpha(n) = \frac{h_{\mathcal{M}}(n)}{\varphi^n}$, where $\varphi = \frac{1}{2}(\sqrt{5}+1)$. Since $2 > \varphi$, and $h_{\mathcal{M}}(n) \geq 2^{n-1}$, we have $\lim_{n\to\infty}(\alpha(n)) = \infty$. Therefore, there exist arbitrarily large m such that $\alpha(n) < \alpha(m)$ for all $0 \leq n < m$. We show that for every such m, the rank-m extremal matroids in $\mathcal{M}$ are all weakly round.

Let m be an integer with this property, and let M be a rank-m extremal matroid in $\mathcal{M}$. If M is not weakly round, then by Lemma 3.3.2, M has a weakly round restriction $N \in \mathcal{M}$ with $\varepsilon(N) \geq \varphi^{r(N)-r(M)}\varepsilon(M) = \varphi^{r(N)-m}h_{\mathcal{M}}(m)$, so $\frac{\varepsilon(N)}{\varphi^{r(N)}} \geq \frac{h_{\mathcal{M}}(m)}{\varphi^m} = \alpha(m)$. But $\frac{\varepsilon(N)}{\varphi^{r(N)}} \leq \alpha(r(N)) < \alpha(m)$, a contradiction. Therefore, M is weakly round, giving the theorem. □

By the Growth Rate Theorem, the hypotheses hold where $\mathcal{M}$ is any exponentially dense minor-closed class, giving Theorem 1.5.6.

A generalisation of roundness is *thickness*; if $d \geq 3$ is an integer, then a matroid M is *d-thick* if $E(M)$ is not the union of fewer than d hyperplanes of M; thus, roundness is equivalent to 3-thickness, and d-thickness implies d'-thickness for all $d' \leq d$.

Theorem 3.3.3 used only base-2 exponential density. For classes with exponential density with a larger base, one can use the the same techniques, along with a 'thickness' version of Lemma 3.3.2, to prove the following stronger result:

Theorem 3.3.4. *Let $q > 2$ be a prime power. If $\mathcal{M}$ is a restriction-closed class of matroids, and $h_{\mathcal{M}}(n) \geq q^{n-1}$ for all $n \geq 1$, then there exist q-thick extremal matroids in $\mathcal{M}$ of arbitrarily large rank.*

When $\mathcal{M}$ is the closure under restriction of the set of affine geometries over $\mathrm{GF}(q)$ (matroids formed from projective geometries over $\mathrm{GF}(q)$ by deleting a hyperplane), the extremal matroids are the affine geometries themselves, which are q-thick and not $(q+1)$-thick, so this theorem is, in a sense, best possible. However, when $\mathcal{M}$ is minor-closed, it seems that we can do slightly better; a simple counting argument shows that the extremal matroids $\mathrm{PG}(n,q)$ in the class $\mathcal{L}(q)$ are $(q+1)$-thick as well as q-thick. Moreover, it seems likely that *all* extremal matroids in $\mathcal{M}$ of sufficiently large rank ought to be similarly highly connected. We conjecture the following:

Conjecture 3.3.5. *If $\mathcal{M}$ is a base-q exponentially dense minor-closed class of matroids, then the extremal matroids in $\mathcal{M}$ of sufficiently large rank are $(q+1)$-thick.*

It is easy to show that Conjecture 1.5.14 would imply this conjecture.

3.3.2 Finding Connectivity

Many theorems in this thesis start with a connectivity reduction. The idea of such a reduction is that, to prove a theorem about matroids in general, we first show that a counterexample to the theorem implies a highly connected counterexample, and then use this additional connectivity assumption to show that such a counterexample cannot exist. Unfortunately, Theorem 3.3.3 is not strong enough to supply this reduction for classes with unknown growth rate function. The following lemma uses similar techniques to find a dense weakly round restriction of a dense matroid in $\mathcal{U}(\ell)$, with respect to a precise exponential density function $g(n)$; in this chapter, we invoke the lemma with $g(n) = \frac{q^n-1}{q-1}$.

Lemma 3.3.6. *There is an integer-valued function $f_{3.3.6}(r,d,\ell)$ so that, for any integers $d \geq 0$, $\ell \geq 2$ and $r \geq d$, and real-valued function $g(n)$ satisfying $g(d) \geq 1$ and $g(n) \geq 2g(n-1)$ for all $n > d$, if $M \in \mathcal{U}(\ell)$ is a matroid such that $r(M) \geq f_{3.3.6}(r,d,\ell)$ and $\varepsilon(M) > g(r(M))$, then M has a weakly round restriction N such that $r(N) \geq r$ and $\varepsilon(N) > g(r(N))$.*

Proof. Let $\varphi = \frac{1}{2}(1+\sqrt{5})$, let $\ell \geq 2$, $r \geq 0$ and $d \geq 0$ be integers, and $g(n)$ be a real-valued function with $g(d) \geq 1$ and $g(n) \geq 2g(n-1)$ for all $n > d$. Observe that $g(n) \geq 2^{n-m}g(m)$ for all m, n with $d \leq m \leq n$. Set $f_{3.3.6}(\ell,d,r)$ to be an integer s so that $s \geq d$ and $2^{-d}(\sqrt{5}-1)^s \geq \frac{\ell^r-1}{\ell-1}$.

Let M be a matroid with $r(M) \geq s$, and $\varepsilon(M) \geq g(r(M))$. By Lemma 3.3.2, there is a weakly round restriction N of M satisfying $\varepsilon(N) > \varphi^{r(N)-r(M)}g(r(M)) \geq \varphi^{-r(M)}2^{r(M)-d} = 2^{-d}(\sqrt{5}-1)^{r(M)} \geq \frac{\ell^r-1}{\ell-1}$, since $r(M) \geq s$. Therefore, by Theorem 1.3.1, we have $r(N) \geq r \geq d$. Now, $\varepsilon(N) > \varphi^{r(N)-r(M)}g(r(M)) \geq \varphi^{r(N)-r(M)}2^{r(M)-r(N)}g(r(N)) \geq g(r(N))$, so N is the required restriction. □

3.3.3 Exploiting Connectivity

We now show how the weak roundness property is useful, deriving a technical lemma that will be employed in both this chapter and the next. In order to accommodate the needs of both chapters, this lemma will be stated in more generality than is necessary for the current chapter.

Recall that two sets $X, Y \subseteq E(M)$ are *skew* in M if $r_M(X \cup Y) = r_M(X) + r_M(Y)$. The first result we need allows us to find, in a dense set X, a dense subset that is skew to a given set of low rank. The base case of this induction was essentially proved in [14]:

Lemma 3.3.7. *Let λ, μ be real numbers with $\lambda > 0$ and $\mu > 1$, let $t \geq 0$ and $\ell \geq 2$ be integers, and let A and B be disjoint sets of elements in a matroid $M \in \mathcal{U}(\ell)$ with $r_M(B) \leq t < r(M)$ and $\varepsilon_M(A) > \lambda\mu^{r_M(A)}$. Then there is a set $A' \subseteq A$ that is skew to B and satisfies $\varepsilon_M(A') > \lambda\left(\frac{\mu-1}{\ell}\right)^t \mu^{r_M(A')}$.*

Proof. We will prove the result by induction on t; the result is trivial if $t = 0$, so our base case is when $r_M(B) = 1$. Let $e \in B$ be a nonloop. We may assume that $r(M) \geq 2$, that A is minimal satisfying $\varepsilon(M|A) > \lambda\mu^{r_M(A)}$, and that $E(M) = A \cup \{e\}$. Let W be a flat of M with $e \notin W$, so that $r_M(W) = r(M) - 2$. Let $H_0, \dots, H_m$ be the hyperplanes of M containing W, where $e \in H_0$. The sets $\{H_i - W : 1 \leq i \leq m\}$ form a partition of $E(M) - W$. Also, $\mathrm{si}(M /\!\!/ W) \cong U_{2,m+1}$, so $m \leq \ell$.

Minimality of A gives $\varepsilon_M(H_0 \cap A) \leq \lambda\mu^{r(M)-1}$, so

$$\varepsilon(M|(A - H_0)) > \lambda(\mu - 1)\mu^{r(M)-1}.$$

The union of the hyperplanes $H_1, \dots, H_m$ contains $E(M) - H_0$, so by a majority argument, there is some $1 \leq i \leq m$ such that

$$\varepsilon_M(A \cap H_i) \geq \frac{1}{m}\varepsilon(M|(A - H_0)) > \lambda\left(\frac{\mu - 1}{\ell}\right)\mu^{r(M)-1}.$$

Set $A' = A \cap H_i$. Now A' is skew to e and therefore to B, and $\varepsilon_M(A')$ is large enough, completing the base case.

Now, suppose that the result holds for some $t \geq 1$. Let A and B be disjoint sets of elements in a matroid M with $r_M(B) \leq t + 1$, and $\varepsilon(M|A) > \lambda\mu^{r_M(A)}$. Let $e \in B$ be a nonloop. By the base case, there is a set $A' \subseteq A$, skew to $\{e\}$, and satisfying $\varepsilon(M|A') > \lambda\left(\frac{\ell}{\mu-1}\right)\mu^{r_M(A')}$. Now $r_{M/\!\!/e}(B) \leq t < r(M /\!\!/ e)$, and the result follows by applying the inductive hypothesis to A'. □

The next lemma is the main technical result of this section. It states that, given a matroid M^+ with a very large projective geometry minor, it is possible to find a set X so that $M^+ /\!\!/ X$ is spanned by a large projective geometry, and contraction of X doesn't reduce the rank of some fixed set B. Moreover, this can be done if the points in X are required to belong to some particular highly connected, spanning restriction M of M^+. In this chapter, we will use this lemma with $M = M^+$, and $M^+|B \cong U_{2,q+2}$; it may be instructive to read the statement with this in mind.

Lemma 3.3.8. *There is an integer-valued function $f_{3.3.8}(n,q,t,\ell)$ so that, for any prime power q, and integers $t \geq 0$, $n \geq 1$, and $\ell \geq 2$, if $M^+ \in \mathcal{U}(\ell)$ and M are matroids, and $B \subseteq E(M^+)$ is a set so that M, M^+ and B satisfy:*

- *$r_{M^+}(B) \leq t$, and*
- *M is a weakly round, spanning restriction of M^+, and*
- *M has a PG($f_{3.3.8}(n,q,t,\ell)-1,q$)-minor N,*

then there is a set $X \subseteq E(M)$ so that $r(M /\!\!/ X) \geq n$, and $M /\!\!/ X$ has a PG$(r(M /\!\!/ X)-1,q)$*-restriction, and $(M^+ /\!\!/ X)|B = M^+|B$.*

Proof. Let q be a prime power, and $t \geq 0$, $n \geq 1$ and $\ell \geq 2$ be integers. Let $n' = \max(n, t+1)$, and $\alpha = \alpha_{3.2.2}(n', q-\frac{1}{2}, \ell)$. Let m be a positive integer large enough so that $m \geq 2t$, and so that

$$\left(\frac{q^m-1}{q-1}\right) \geq \alpha\left(\frac{\ell(q-\frac{1}{2})}{q-\frac{3}{2}}\right)^t (q-\tfrac{1}{2})^m,$$

and set $f_{3.3.8}(n,q,t,\ell) = m$.

Let $M^+ \in \mathcal{U}(\ell)$ be a matroid with a weakly round, spanning restriction M, such that M has a PG$(m-1,q)$-minor $N = M /\!\!/ C \backslash D$, where C is independent in M. Let $B \subseteq E(M^+)$ be a set of rank at most t. We show that the required set $X \subseteq E(M)$ exists.

3.3.8.1. *There is a set $C' \subseteq E(M)$, so that $M /\!\!/ C'$ has a PG($n'-1,q$)-restriction N', and $(M^+ /\!\!/ C')|B = M^+|B$.*

Proof of claim: Let $C_0 \subseteq C$ be maximal so that $(M^+ /\!\!/ C_0)|B = M^+|B$, and let $M_0 = M /\!\!/ C_0$, and $M_0^+ = M^+ /\!\!/ C_0$. By maximality of C_0, we have $C - C_0 \subseteq \mathrm{cl}_{M_0^+}(B)$, and therefore $r_{M_0}(C - C_0) \leq t$.

We have $r_{M'}(E(N)) = r_{M' /\!\!/ (C-C_0)}(E(N)) + r_{M'}(C - C_0) \leq m + t$. Now

$$\begin{aligned}\varepsilon_{M_0}(E(N)) &= \frac{q^m-1}{q-1} \\ &\geq \alpha\ell^t(q-\tfrac{3}{2})^{-t}(q-\tfrac{1}{2})^{m+t} \\ &\geq \alpha(\ell(q-\tfrac{3}{2})^{-1})^t(q-\tfrac{1}{2})^{r_{M_0}(E(N))}.\end{aligned}$$

Applying Lemma 3.3.7 to $E(N)$ and B, with $\mu = q - \frac{1}{2}$, gives a set $A \subseteq E(N)$, skew to B in M_0^+, satisfying $\varepsilon(M_0^+|A) > \alpha(q - \frac{1}{2})^{r(M_0^+|A)}$. By Theorem 3.2.2, the matroid $M_0^+|A = M_0|A$ has a $\mathrm{PG}(n'-1, q')$-minor $N_1 = (M_0|A) /\!\!/ C_1 \backslash D_1$ for some $q' > q - \frac{1}{2}$.

Since A is skew to B in M_0^+, it is also skew to $C - C_0$, so $M_0|A = (M_0 /\!\!/ (C - C_0))|A = N|A$, and therefore $M|A$ is $\mathrm{GF}(q)$-representable, and so is N_1. So $q' = q$, and N_1 is a $\mathrm{PG}(n'-1, q)$-restriction of $M_0 /\!\!/ C_1$. Moreover, $C_1 \subseteq A$, so C_1 is skew to B in M_0^+, so $(M_0^+ /\!\!/ C_1)|B = M_0^+|B = M^+|B$. Therefore, $C' = C_0 \cup C_1$ satisfies the claim. □

Let X be a maximal set satisfying the following:

- $C' \subseteq X \subseteq E(M)$, and
- $(M^+ /\!\!/ X)|B = M^+|B$, and
- N' is a restriction of $M /\!\!/ X$.

If N' is spanning in $M /\!\!/ X$, then X satisfies the lemma. Otherwise, we have $r_{M^+}(B) \le t < n' = r(N') < r(M /\!\!/ X)$. Weak roundness of $M /\!\!/ X$ thus gives some $f \in E(M /\!\!/ X)$ not in $\mathrm{cl}_{M /\!\!/ X}(E(N'))$ or $\mathrm{cl}_{M^+ /\!\!/ X}(B)$. This contradicts maximality of X. □

3.4 The Main Theorem

Before proving Theorem 1.5.4, we state the specialisation of Lemma 3.3.8 we will be using.

Lemma 3.4.1. *There is an integer-valued function $f_{3.4.1}(n, q, \ell)$ so that, for any prime power q, and integers $n \ge 2$ and $\ell \ge 2$, if $M \in \mathcal{U}(\ell)$ is a weakly round matroid with a $\mathrm{PG}(f_{3.4.1}(n, q, \ell)-1, q)$-minor and a $U_{2,q+2}$-restriction, then M has either a $\mathrm{PG}^+(n-1, q, n)$-minor or a $\mathrm{PG}^+(n-1, q, 2)$-minor.*

Proof. Let $n \ge 2$ and $\ell \ge 2$ be integers, and q be a prime power. Set $f_{3.4.1}(n, q, \ell) = f_{3.3.8}(2n, q, 2, \ell)$. Let M be a matroid with a $U_{2,q+2}$-restriction $M|L$, and a $\mathrm{PG}(f_{3.4.1}(n, q, \ell) - 1, q)$-minor. By Lemma 3.3.8, applied with $M = M^+$ and $B = L$, there is a set $X \subseteq E(M)$ so that $r(M /\!\!/ X) \ge 2n$, and $(M /\!\!/ X)|L = M|L$, and $M /\!\!/ X$ has a $\mathrm{PG}(r(M /\!\!/ X) - 1, q)$-restriction R.

Since R has no lines containing more than $q + 1$ points, there must be a nonloop e of $(M /\!\!/ X)|L$ that is not parallel in $M /\!\!/ X$ to any nonloop of R. The matroid $(M /\!\!/ X)|(E(R) \cup \{e\})$ satisfies the hypotheses of Lemma 2.5.1, so by Lemma 2.5.2 has either a $\mathrm{PG}^+(n - 1, q, n)$-minor or a $\mathrm{PG}^+(n - 1, q, 2)$-minor. □

Setting $n = 3$ gives a rephrasing of Theorem 1.5.5, which we restate here, as a nice corollary.

Corollary 3.4.2. *There is an integer-valued function $f_{3.4.2}(q)$ so that, for any prime power q, if M is a weakly round matroid with a $U_{2,q+2}$-restriction and a $\mathrm{PG}(f_{3.4.2}(q)-1, q)$-minor, then M has a U_{2,q^2+1}-minor.*

Proof. For each prime power q, set $f_{3.4.2}(q) = f_{3.4.1}(2, q, q^2 - 1)$. Suppose that M is a weakly round matroid with a $U_{2,q+2}$-restriction and a $\mathrm{PG}(f_{3.4.2}(q) - 1, q)$-minor, and that M has no U_{2,q^2+1}-minor (in other words, $M \in \mathcal{U}(q^2 - 1)$). By Lemma 3.4.1, M has either a $\mathrm{PG}^+(2, q, 2)$ or a $\mathrm{PG}^+(2, q, 3)$-minor. By Lemma 2.5.5, both these matroids have a U_{2,q^2+1}-minor. $\square$

We can now restate and prove our main result, Theorem 1.5.4. This time, we state the theorem in terms of unavoidable subclasses rather than unavoidable minors, using the classes $\mathcal{L}^2(q)$ and $\mathcal{L}^T(q)$ that were defined in Section 2.5. (Recall that these classes are the closure under minors of the sequences $(\mathrm{PG}^+(n-1, q, 2) : n \geq 2)$ and $(\mathrm{PG}^+(n-1, q, n) : n \geq 2)$ respectively; the following statement, and the original statement of Theorem 1.5.4, are equivalent by Lemma 2.5.3.)

Theorem 3.4.3. *Let q be a prime power, and $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids. If there exist infinitely many n such that $h_{\mathcal{M}}(n) > \frac{q^n-1}{q-1}$, then either $\mathcal{L}^2(q) \subseteq \mathcal{M}$, or $\mathcal{L}^T(q) \subseteq \mathcal{M}$.*

Proof. Let $\mathcal{M}$ be a minor-closed class satisfying the hypotheses. By the Growth Rate Theorem, there is some $c_{\mathcal{M}}$ such that $h_{\mathcal{M}}(n) \leq c_{\mathcal{M}} q^n$ for all n, and since $\mathcal{M}$ does not contain all simple rank-2 matroids, there is some $\ell \geq 2$ such that $\mathcal{M} \subseteq \mathcal{U}(\ell)$. Let n_0 be an integer such that $(q+1)^{n-1} > c_{\mathcal{M}} q^n$ for all $n \geq n_0$. The majority of the proof is contained in the following claim:

3.4.3.1. *If $n \geq n_0$, then either $\mathrm{PG}^+(n-1, q, 2) \in \mathcal{M}$, or $\mathrm{PG}^+(n-1, q, n) \in \mathcal{M}$.*

Proof of claim: Let $m_1 = \max(2n, f_{3.4.1}(n, q, \ell))$. Let m_2 be an integer large enough so that $\frac{q^s-1}{q-1} \geq \alpha_{3.2.2}(m_1, q - \frac{1}{2}, \ell)(q - \frac{1}{2})^s$ for all $s \geq m_2$. Let $m_3 = f_{3.3.6}(\ell, 1, m_2)$.

Let $M_3 \in \mathcal{M}$ be a matroid such that $r(M_3) \geq m_3$ and $\varepsilon(M_3) > \frac{q^{r(M_3)}-1}{q-1}$. The function $g(m) = \frac{q^m-1}{q-1}$ satisfies $g(1) = 1$, and $g(m) \geq 2g(m-1)$ for all $m \geq 2$, so by Lemma 3.3.6, M_3 has a weakly round restriction M_2 such that $r(M_2) \geq m_2$, and $\varepsilon(M_2) \geq \frac{q^{r(M_2)}-1}{q-1}$. By definition of m_2, the matroid M_2 satisfies $\varepsilon(M_2) > \alpha_{3.2.2}(m_1, q - \frac{1}{2}, \ell)(q - \frac{1}{2})^{r(M_2)}$, so M_2 has a $\mathrm{PG}(m_1, q')$-minor N for some $q' > q - \frac{1}{2}$. If $q' > q$, then since $m_1 \ge n \ge n_0$, we have $\varepsilon(N) = \frac{(q')^{m_1}-1}{q'-1} \ge (q+1)^{m_1-1} > c_{\mathcal{M}} q^{m_1} \ge h_{\mathcal{M}}(m_1)$, a contradiction; we may thus assume that $q' = q$. Let $N = M_2/\!\!/C \backslash D$, where C is independent in M_2, and N is spanning in $M_2/\!\!/C$. Let C' be a maximal subset of C such that $\varepsilon(M_2/\!\!/C') > \frac{q^{r(M_2/\!\!/C')}-1}{q-1}$.

If $C = C'$, then $\varepsilon(M_2/\!\!/C) > \frac{q^{r(M_2/\!\!/C)-1}}{q-1} = \frac{q^{m_1}-1}{q-1} = \varepsilon(N)$. There is therefore a nonloop e of $M'/\!\!/C$, not parallel to any nonloop of N. But N is spanning in $M_2/\!\!/C$, and $(M_2/\!\!/C)|E(N) = N \cong \mathrm{PG}(m_1-1, q)$ so Lemma 2.5.1 implies that $(M_2/\!\!/C)|(E(N) \cup \{e\}) \cong \mathrm{PG}^+(m_1, q, k)$ for some $1 < k \le m_1$. Lemma 2.5.2 now gives the claim.

We may therefore assume that $C' \neq C$. Let $f \in C - C'$, and let $M = M/\!\!/C'$. By maximality of C, we have $\varepsilon(M) > \frac{q^{r(M)}-1}{q-1}$, and $\varepsilon(M/\!\!/f) \le \frac{q^{r(M/\!\!/f)}-1}{q-1}$; this gives $\varepsilon(M) > q\varepsilon(M/\!\!/f) + 1$. By Lemma 1.1.1 and a majority argument, some line of M must contain at least $q+2$ points, so M has a $U_{2,q+2}$-restriction L. Since $C' \cup \{f\} \subseteq C$, the matroid $M/\!\!/f$ has N has a minor. Moreover, M is weakly round, so the claim follows from Lemma 3.4.1. □

Therefore, either $\mathrm{PG}^+(n-1, q, 2) \in \mathcal{M}$ for arbitrarily large n, or $\mathrm{PG}^+(n-1, q, n) \in \mathcal{M}$ for arbitrarily large n. In either case, the theorem follows from Lemma 2.5.3.

□

This theorem easily implies Theorem 1.5.13, establishing a gap between two exponential growth rate functions.

Theorem 1.5.13. *If $\mathcal{M}$ is a minor-closed class of matroids, and q is a prime power, then either $h_{\mathcal{M}}(n) \le \frac{q^n-1}{q-1}$ for all sufficiently large n, or $h_{\mathcal{M}}(n) \ge \frac{q^{n+1}-1}{q-1} - q$ for all sufficiently large n.*

Proof. We may assume that $\mathcal{M}$ is base-q exponentially dense (otherwise the result follows easily from the Growth Rate Theorem), and that the first outcome does not hold, so $h_{\mathcal{M}}(n) > \frac{q^n-1}{q-1}$ for infinitely many n. The theorem now follows from Theorem 3.4.3 and Lemma 2.5.4.

□

3.5 A Growth Rate Function

In this section, we derive Theorems 1.5.17 and 1.5.3 as corollaries of Theorem 1.5.4.

Theorem 1.5.17. *Let q be a prime power. If $\mathcal{M}$ is a base-q exponentially dense minor-closed class of matroids such that $\mathcal{M} \subseteq \mathcal{U}(q^2-1)$, then $h_{\mathcal{M}}(n) = \frac{q^n-1}{q-1}$ for all sufficiently large n.*

Proof. By the Growth Rate Theorem, we have $h_{\mathcal{M}}(n) \geq \frac{q^n-1}{q-1}$ for all n. We may assume that there exist n arbitrarily large such that equality does not hold, so Theorem 3.4.3 gives either $\mathcal{L}^2(q) \subseteq \mathcal{M}$ or $\mathcal{L}^T(q) \subseteq \mathcal{M}$. Therefore, one of $\mathrm{PG}^+(2,q,2)$ or $\mathrm{PG}^+(2,q,3)$ is in $\mathcal{M}$, so by Lemma 2.5.5, $U_{2,q^2+1} \in \mathcal{M}$, a contradiction. □

We split Theorem 1.5.3 into two parts, one computing the required growth rate function, and the other verifying that the simple extremal matroids are projective geometries.

Theorem 3.5.1. *If $\ell \geq 2$ is an integer, and q is the largest prime power such that $q \leq \ell$, then*

$$h_{\mathcal{U}(\ell)}(n) = \frac{q^n-1}{q-1}$$

for all sufficiently large n.

Proof. Since $\mathrm{PG}(n-1,q) \in \mathcal{U}(\ell)$ for all $n \geq 1$, we have $h_{\mathcal{U}(\ell)}(n) \geq \frac{q^n-1}{q-1}$ for all n. By the Growth Rate Theorem, the class $\mathcal{U}(\ell)$ is therefore base-q' exponentially dense for some $q' \geq q$. If $q' > q$, then $q' > \ell$ by definition of q, and $\mathcal{U}(\ell)$ contains all $\mathrm{GF}(q')$-representable matroids, including $U_{2,q'+1}$, which has a $U_{2,\ell+2}$-restriction, giving a contradiction. Therefore, we may assume that $q' = q$. Note that $q^2+1 \geq \ell+2$, so $\mathcal{M} \subseteq \mathcal{U}(q^2-1)$; the result now follows from Theorem 1.5.17.

□

We remark here that, since we were able to find a U_{2,q^2+1}-minor, the only numerical comparison between q and ℓ that we used is that $q^2 > \ell$. In fact, there is a power of 2 between $q+1$ and $2q$, so we have $2q > \ell$; finding a $U_{2,2q+1}$-minor would have been enough to prove Theorem 3.5.1. In light of this, the fact that we found a U_{2,q^2+1}-minor is surprising.

We now characterise the extremal matroids of large rank in $\mathcal{U}(\ell)$:

Theorem 3.5.2. *Let $\ell \geq 2$ be an integer, and q be the largest prime power not exceeding ℓ. If M is extremal in $\mathcal{U}(\ell)$ and has sufficiently large rank, then $\mathrm{si}(M)$ is a projective geometry over $\mathrm{GF}(q)$.*

Proof. By Theorem 3.5.1, there is an integer r_2 such that $h_{\mathcal{U}(\ell)}(r) = \frac{q^r-1}{q-1}$ for all $r \geq r_2$. Let $m = f_{3.4.2}(q)$, and let r_1 be an integer so that $\frac{q^r-1}{q-1} \geq \alpha_{3.2.2}(m+2, q-\frac{1}{2}, \ell)(q-\frac{1}{2})^r$ for all $r \geq r_1$. Let $r_0 = f_{3.3.6}(\ell, 1, \max(r_1, r_2))$.

Let $M \in \mathcal{U}(\ell)$ be a matroid such that $r(M) \geq \max(4, r_0, r_2+1)$, and $\varepsilon(M) = \frac{q^{r(M)}-1}{q-1}$. We may assume that M is simple; it suffices to show that $M \cong \mathrm{PG}(r(M)-1, q)$.

3.5.2.1. *M is weakly round.*

Proof of claim: Define a function g by $g(n) = \frac{q^n-1}{q-1}$ for all integers $n \geq 0$. Clearly $g(1) = 1$, and $g(n) \geq 2g(n-1)$ for all $n > 1$, so if M is not weakly round, then by Lemma 3.3.6, M has a weakly round restriction M' such that $r(M') \geq r_2$ and $\varepsilon(M') > \frac{q^{r(M')}-1}{q-1} = h_{\mathcal{U}(\ell)}(r(M'))$. This contradicts $M' \in \mathcal{U}(\ell)$. $\square$

By Theorem 3.2.2, M has a $\mathrm{PG}(m+1, q')$-minor for some $q' > q - \frac{1}{2}$; since $q' > \ell$ for any prime power $q' > q$, we must have $q' = q$, otherwise this minor has an ℓ-long line.

3.5.2.2. *Every line of M contains exactly $q+1$ points.*

Proof of claim: If M has a q-long line, then M has a $U_{2,q+2}$-restriction; M also has a $\mathrm{PG}(m-1, q)$-minor, so by Corollary 3.4.2, M has a U_{2,q^2+1}-minor, contradicting $M \in \mathcal{U}(\ell)$. Therefore, M has no q-long lines. Let $e \in E(M)$. Since $r(M /\!\!/ e) \geq r_2$, Theorem 3.5.1 gives $\varepsilon(M /\!\!/ e) \leq \frac{q^{r(M /\!\!/ e)}-1}{q-1}$, giving $\varepsilon(M) \geq q\varepsilon(M /\!\!/ e) + 1$; by Corollary 1.1.1, equality must hold, and every line of M through e contains exactly $q+1$ points. This holds for all e, giving the claim. $\square$

3.5.2.3. *Any two coplanar lines of M intersect.*

Proof of claim: Suppose otherwise; let L_1 and L_2 be disjoint lines of M such that $r_M(L_1 \cup L_2) = 3$. If $e \in L_2$, then $\mathrm{cl}_{M /\!\!/ e}(L_1)$ contains $q+2$ points, so $M /\!\!/ e$ has a $U_{2,q+2}$-restriction. Moreover, M has a $\mathrm{PG}(m+1, q)$-minor, so $M /\!\!/ e$ has a $\mathrm{PG}(m-1, q)$-minor (see [10], Lemma 5.2). Therefore $M /\!\!/ e$ has a U_{2,q^2+1}-minor by Corollary 3.4.2, contradicting $M \in \mathcal{U}(\ell)$. $\square$

By Theorem 2.2.2, M has a $\mathrm{PG}(r(M)-1, q')$-restriction N for some prime power q', and every line of M contains exactly $q'+1$ points. Therefore $q' = q$, and M is a simple matroid with the same number of points as its restriction N, giving $M = N \cong \mathrm{PG}(r(M)-1, q)$. $\square$

Chapter 4

Square Fields

This chapter contains a proof of the following theorem, which precisely determines the eventual behaviour of the growth rate function for all base-q exponentially dense minor-closed subclasses of $\mathcal{L}(q^2)$, verifying Conjecture 1.5.16 for these classes:

Theorem 1.5.8. *Let q be a prime power. If $\mathcal{M}$ is a minor-closed class of matroids such that $\mathcal{L}(q) \subseteq \mathcal{M} \subsetneq \mathcal{L}(q^2)$, then there is an integer $k \geq 0$ so that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - q\left(\frac{q^{2k}-1}{q^2-1}\right)$$

for all sufficiently large n.

We prove the theorem by means of the following result; recall that $\mathcal{P}_{q,k}$ denotes the closure under minors and isomorphism of the set of matroids of the form N/K, where $N \in \mathcal{L}(q^2)$, $N \backslash K \cong \mathrm{PG}(r(N)-1, q)$, and K is a rank-k independent set of N:

Theorem 1.5.7. *Let q be a prime power. If $\mathcal{M}$ is a minor-closed class of matroids such that $\mathcal{L}(q) \subseteq \mathcal{M} \subsetneq \mathcal{L}(q^2)$, then there is an integer $k \geq 0$ such that $\mathcal{P}_{q,k} \subseteq \mathcal{M}$, and $h_{\mathcal{M}}(n) = h_{\mathcal{P}_{q,k}}(n)$ for all sufficiently large n.*

Our argument revolves around identifying the extremal matroids in $\mathcal{P}_{q,k}$, and then finding these matroids as unavoidable minors of arbitrary dense matroids in $\mathcal{L}(q^2)$. The contents of this chapter are submitted for publication by the author [33].

4.1 The Extremal Matroids

Let q be a prime power, and k and n be integers with $0 \leq k \leq n$. Let $N^+ \cong \mathrm{PG}(n-1, q^2)$, and N be a $\mathrm{PG}(n-1, q)$-restriction of N^+. Let F be a rank-k flat of N, and $X = \bigcup_{e \in E(N)} \mathrm{cl}_{N^+}(F \cup \{e\})$. We write $\mathrm{PG}^{(k)}(n-1, q)$ for any matroid isomorphic to $N^+|X$; note that this matroid is uniquely determined. We will show that the matroids $\mathrm{PG}^{(k)}(n-1, q)$ are extremal in $\mathcal{P}_{q,k}$; this section and the next are devoted to their properties and construction.

Since $E(N) \subseteq X \subseteq E(N^+)$, the matroid $\mathrm{PG}^{(k)}(n-1, q)$ is simple, has rank n, and has a $\mathrm{PG}(n-1, q)$-restriction. The size of this matroid is easy to compute; we use the following lemma freely:

Lemma 4.1.1. *If q is a prime power, and $k \geq 0$ and $n \geq k$ are integers, then*

$$|\mathrm{PG}^{(k)}(n-1, q)| = \frac{q^{n+k}-1}{q-1} - q\left(\frac{q^{2k}-1}{q^2-1}\right).$$

Proof. Let N^+, N, F and X be the objects in the definition. Let $F^+ = \mathrm{cl}_{N^+}(F)$. Note that each $f \in X - F^+$ lies in the closure of a unique rank-$(k+1)$ flat of N containing F. There are $\frac{q^{n-k}-1}{q-1}$ such flats (as they are exactly the points of $N/\!\!/F$), and each such flat contains exactly $\frac{q^{2(k+1)}-1}{q^2-1} - \frac{q^{2k}-1}{q^2-1} = q^{2k}$ points of $X - F^+$. The flat F^+ itself contains $\frac{q^{2k}-1}{q^2-1}$ points of X, so we have $|\operatorname{PG}^{(k)}(n-1,q)| = |X| = \frac{q^{2k}-1}{q^2-1} + q^{2k}\left(\frac{q^{n-k}-1}{q-1}\right)$, and the result follows by a calculation. $\square$

This next lemma follows easily from the definition:

Lemma 4.1.2. *Let q be a prime power, and k, k', n, n' be integers with $0 \le k' \le k \le n' \le n$. The matroid $\operatorname{PG}^{(k)}(n-1,q)$ has a $\operatorname{PG}^{(k')}(n-1,q)$ and a $\operatorname{PG}^{(k)}(n'-1,q)$-restriction.*

It is also routine to determine which projective geometries over $\operatorname{GF}(q^2)$ occur as minors of these matroids:

Lemma 4.1.3. *Let q be a prime power, and $k \ge 0$ and $n > k$ be integers. The matroid $\operatorname{PG}^{(k)}(n-1,q)$ has a $\operatorname{PG}(k,q^2)$-restriction, and no $\operatorname{PG}(k+1,q^2)$-minor.*

Proof. Note that $\operatorname{PG}^{(k)}(k,q) \cong \operatorname{PG}(k,q^2)$; applying the previous lemma where $n' = k+1$ gives a $\operatorname{PG}(k,q^2)$-restriction. To see that there is no $\operatorname{PG}(k+1,q^2)$-minor, observe that, if N^+, N, F and X are as in the definition of $\operatorname{PG}^{(k)}$, then every $f \in X$ is either a loop of $N^+/\!\!/F$, or is parallel in $N^+/\!\!/F$ to some $e \in E(N)$; therefore $(N^+|X)/\!\!/F \in \mathcal{L}(q)$. Thus, there is a set of rank at most k (namely, F) in $\operatorname{PG}^{(k)}(n-1,q)$ whose contraction yields a $\operatorname{GF}(q)$-representable matroid. Any contraction-minor of $\operatorname{PG}^{(k)}(n-1,q)$ therefore also has this property, but any matroid with a $\operatorname{PG}(k+1,q^2)$-restriction does not, so $\operatorname{PG}^{(k)}(n-1,q)$ has no such minor. $\square$

We now show the important fact asserted earlier: that the matroids $\operatorname{PG}^{(k)}(n,q)$ are extremal in $\mathcal{P}_{q,k}$. Recall that we defined $\mathcal{P}_{q,k}$ as the closure under minors of the set of matroids of the form N/K, where $N \in \mathcal{L}(q^2)$ and $N \backslash K \cong \operatorname{PG}(r(N)-1,q)$. In fact, one can easily show with Lemma 2.1.2 that every simple matroid of rank at least k in $\mathcal{P}_{q,k}$ is isomorphic to a spanning restriction of such an N/K. We use this fact in the following proof:

Lemma 4.1.4. *Let q be a prime power, and $k \ge 0$ be an integer. If $M \in \mathcal{P}_{q,k}$ is a simple matroid of rank at least $k+1$, then M is isomorphic to a restriction of $\operatorname{PG}^{(k)}(r(M)-1,q)$.*

Proof. Let $M \in \mathcal{P}_{q,k}$; by the above, M is a spanning restriction of N/K, where $N\backslash K \cong \mathrm{PG}(r(N)-1,q)$ for some k-subset K of $E(N)$, and $N \in \mathcal{L}(q^2)$. By the fact we just observed, it is enough to show that $\mathrm{si}(N/K)$ is a restriction of $\mathrm{PG}^{(k)}(r(N/K)-1,q)$.

By Lemma 2.3.6, each $e \in K$ is either a loop of N, parallel to an element of $N\backslash K$, or lies on a line spanned by two elements of $N\backslash K$, so there is a flat F' of $N\backslash K$ such that $r_N(F') \leq 2r_N(K)$, and $K \subseteq \mathrm{cl}_N(F')$. Let B_K be a basis for K, $B_{F'}$ be a basis for F' containing B_K, and B be a basis for N containing $B_{F'}$. Let $F = \mathrm{cl}_N(B_{F'} - B_K)$; we have $r_N(F) = r_N(F') - r_N(K) \leq r_N(K) \leq k$.

By Lemma 2.1.2, we have $\mathrm{si}(N /\!\!/ B_{F'}) = \mathrm{si}(N /\!\!/ F') \cong \mathrm{PG}(r(N /\!\!/ F')-1,q)$; in particular, every nonloop of $N /\!\!/ B_{F'}$ is parallel to a nonloop of the projective geometry $\mathrm{cl}_N(B - B_{F'})$. Therefore, every $f \in E(N/K)$ lies in $\mathrm{cl}_{N/K}(F \cup \{e\})$ for some $e \in \mathrm{cl}_N(B - B_{F'})$. Now, $M = N/K$ is a matroid in $\mathcal{L}(q^2)$, having a spanning $\mathrm{PG}(r(N/K)-1,q)$ restriction $\mathrm{cl}_N(B - B_K)$, in which F is a flat of rank at most K, such that every point of N/K is spanned by F and a point in $\mathrm{cl}_N(B - B_K)$. By definition, $\mathrm{si}(N/K)$ is a restriction of $\mathrm{PG}^{(r_N(F))}(r(N/K)-1,q)$, which is itself a restriction of $\mathrm{PG}^{(k)}(r(N/K)-1,q)$, giving the lemma. $\square$

The growth rate function for $\mathcal{P}_{q,k}$ follows from this effortlessly:

Corollary 4.1.5. *$h_{\mathcal{P}_{q,k}}(n) = |\mathrm{PG}^{(k)}(n-1,q)|$ for all $n > k$.*

4.2 Finding Extremal Matroids

We give in this section a means to construct the matroids $\mathrm{PG}^{(k)}(n,q)$ of the previous section.

Recall that a *matching* of M is a mutually skew set of lines of M (in other words, $\mathcal{L}$ is a matching if $r_M(\bigcup_{L \in \mathcal{L}} L) = 2|\mathcal{L}|$). We define a new property in terms of a matching in a spanning $\mathrm{PG}(n,q)$-restriction:

Let q be a prime power, M be a $\mathrm{GF}(q^2)$-representable matroid, and R be a $\mathrm{PG}(r(M)-1,q)$-restriction of M. By Lemma 2.3.6, each nonloop of e of M is either parallel to a nonloop of R, or there is a unique line L_e of R such that $e \in \mathrm{cl}_M(L_e)$. If $X \subseteq E(M)$ is an independent set of M containing no element parallel to a nonloop of R, and $\{L_e : e \in X\}$ is an $|X|$-matching in R, then we say that X is *R-unstable*.

We construct the extremal matroids by contracting an unstable set:

Lemma 4.2.1. *Let q be a prime power, and let $k \geq 0$, $n > k$, and $n' \geq n + k$ be integers. If a rank-n', $GF(q^2)$-representable matroid M has a $\mathrm{PG}(n'-1, q)$-restriction R, and an R-unstable set of size k, then M has a $\mathrm{PG}^{(k)}(n-1, q)$-minor.*

Proof. By Lemma 4.1.2, it is enough to show that $(M/X)|E(R) \cong \mathrm{PG}^{(k)}(n'-k+1, q)$, as $n' - k \geq n$.

4.2.1.1. *X is a flat of $M|(E(R) \cup X)$.*

Proof. We may assume that $E(M) = E(R) \cup X$. If $|X| = 0$, then the result is trivial, as R is loopless. Suppose inductively that $|X| > 0$, and that the result holds for smaller $|X|$. Let $x_0 \in X$, let $\mathcal{L}$ be the matching associated with X, and L_0 be the line of $\mathcal{L}$ spanning x_0. Let F be the flat of R spanned by the union of the lines in $\mathcal{L} - \{L_0\}$; clearly $X - \{x_0\} \subseteq \mathrm{cl}_M(F)$, and since $\mathcal{L}$ is a matching, we have $M|F = (M/L_0)|F$, and moreover every $e \in \mathrm{cl}_R(F \cup L_0)$ is either a loop of R/L_0, or parallel in R/L_0 to some $f \in F$.

The set $X - x_0$ is R-unstable in M, so inductively $X - \{x_0\}$ is a flat of $M|(F \cup (X - \{x_0\}))$. Since $\sqcap_M(X, L_0) = 1$, the point x_0 is the unique point of $\mathrm{cl}_M(L_0)$ spanned by X in M. If X is not a flat of M, then there is thus some $e \in E(R) - L_0$ such that $e \in \mathrm{cl}_M(X)$. Therefore, $e \in \mathrm{cl}_{M/L_0}(X - \{x_0\})$. But, as we have observed, e is parallel in M/L_0 to some $f \in F$, so $X - \{x_0\}$ spans a point of F in M/L_0, and therefore spans a point of F in M. This contradicts our inductive assumption that $X - \{x_0\}$ is a flat of M. $\square$

Now, $(M \backslash X)|E(R) \cong \mathrm{PG}(n'-1, q)$, so the rank-$(n'-k)$ matroid $N = (M/X)|E(R)$ is a k-element projection of $\mathrm{PG}(r(N)+k-1, q)$, and moreover $N \in \mathcal{P}_{q,k}$. By the claim, X is a rank-k independent flat of M, so Theorem 2.6.2 implies that k is minimal with the property that N is a k-element projection of $\mathrm{PG}(r(N)+k-1, q)$. Therefore, by Theorem 2.6.3, we have $\varepsilon(N) = \frac{q^{r(M)+k}-1}{q-1} - q\left(\frac{q^{2k}-1}{q^2-1}\right) = |\,\mathrm{PG}^{(k)}(r(N)-1, q)|$. By Theorem 4.1.4, $\mathrm{si}(N)$ is a restriction of $\mathrm{PG}^{(k)}(r(N)-1, q))$, so it follows that $\mathrm{si}(N) \cong \mathrm{PG}^{(k)}(r(N)-1, q) = \mathrm{PG}^{(k)}(n'-k+1, q)$, giving the lemma. $\square$

We can make some straightforward observations determining the fields over which these matroids are representable:

Lemma 4.2.2. *Let q be a prime power, and $n \geq 3$ be an integer. If $\mathbb{F}$ is a field with a proper $\mathrm{GF}(q)$-subfield, then $\mathrm{PG}^{(1)}(n-1, q)$ is $\mathbb{F}$-representable, and if $\mathbb{F}$ has no $\mathrm{GF}(q^2)$-subfield, then $\mathrm{PG}^{(2)}(n-1, q)$ is not $\mathbb{F}$-representable.*

Proof. Consider the matroid $M \cong \mathrm{PG}^{+}(n, q, 2)$ (recall that this is the matroid obtained by extending a single line of $R \cong \mathrm{PG}(n, q)$ by a point e). We construct, for any $\mathbb{F}$ with a proper $\mathrm{GF}(q)$-subfield, an $\mathbb{F}$-representable of M; let $\mathbb{F}$ be such such a field, with $\omega \in \mathbb{F} - \mathrm{GF}(q)$. If A is a matrix on $S \cup \{e\}$ such that $A[S]$ is a $\mathrm{GF}(q)$-representation of $\mathrm{PG}(n, q)$, and A_e is a vector with just two nonzero entries, of the form $(\omega, 1, 0, \ldots, 0)$, then it is easy to check that $M(A) \cong M$. Therefore, M is $\mathbb{F}$-representable. In particular, $M \in \mathcal{L}(q^2)$, and clearly $\{e\}$ is an R-unstable set of size 1 in M, so M has a $\mathrm{PG}^{(1)}(n-1, q)$-minor by Lemma 4.2.1. Therefore, $\mathrm{PG}^{(1)}(n-1, q) \in \mathcal{L}(\mathbb{F})$.

Lemma 4.1.3 implies that the matroid $\mathrm{PG}^{(2)}(n-1, q)$ has a $\mathrm{PG}(2, q^2)$-restriction. This matroid admits no representation over a field without a $\mathrm{GF}(q^2)$-subfield. Therefore, if $\mathbb{F}$ has no such subfield, $\mathrm{PG}^{(2)}(n-1, q)$ is not $\mathbb{F}$-representable. $\square$

Using the results established so far, we will prove Theorem 1.5.7 by reducing it to the following unavoidable minor theorem. We devote the remainder of our efforts to its proof.

Theorem 4.2.3. *There is an integer-valued function $f_{4.2.3}(n, q, k)$ so that, for any prime power q, and integers n and k with $0 \leq k < n$, if M is a GF(q^2)-representable matroid such that $r(M) \geq f_{4.2.3}(n, q, k)$ and*

$$\varepsilon(M) > |\,\mathrm{PG}^{(k)}(r(M) - 1, q)|,$$

then M has an $\mathrm{PG}^{(k+1)}(n-1, q)$-minor.

4.3 The Spanning Case

In this section, we show how to construct a $\mathrm{PG}^{(k+1)}$-minor directly from density in the case that we have a dense $\mathrm{GF}(q)$-represented restriction that is spanning and weakly round.

Lemma 4.3.1. *There is an integer-valued function $f_{4.3.1}(n, q, k)$ so that if q is a prime power, n and k are integers with $0 \leq k < n$, and M is a GF(q^2)-representable matroid such that*

- *M has a weakly round, spanning $\mathrm{GF}(q)$-represented restriction R, and*
- *R has a PG($f_{4.3.1}(n, q, k) - 1, q$)-minor, and*
- *$\varepsilon(M) > |\,\mathrm{PG}^{(k)}(r(M) - 1, q)|$,*

then M has an $\mathrm{PG}^{(k+1)}(n-1,q)$*-minor.*

Proof. Let s be an integer so that

$$|\mathrm{PG}^{(k)}(s'-1,q)| > |\mathrm{PG}^{(j)}(s'-1,q)| + f_{2.4.1}(q,k)$$

for all $j<k$ and $s' \geq s$. Set

$$f_{4.3.1}(n,q,k) = \max(s, f_{3.3.8}(n+k,q,2k+2,q^2)).$$

Let $M \in \mathcal{L}(q^2)$ be a matroid with a weakly round, spanning $\mathrm{GF}(q)$-represented restriction R, such that R has a $\mathrm{PG}(f_{4.3.1}(n,q,k)-1,q)$-minor, and $\varepsilon(M) > |\mathrm{PG}^{(k)}(r(M)-1,q)|$. We may assume that M is simple. Let A be a $\mathrm{GF}(q^2)$-representation of M with $r(M)$ rows, so that $A[E(R)]$ has all entries in $\mathrm{GF}(q)$. Let A' be a matrix formed by extending R to a $\mathrm{PG}(r(M)-1,q)$-restriction R' by appending columns with entries in $\mathrm{GF}(q)$ to A. Let $M' = M(A')$; by construction, M' is simple, and M is a spanning restriction of M'.

Let $\mathcal{L}$ be the set of lines of R', and let $\mathcal{L}^+ = \{L \in \mathcal{L} : \mathrm{cl}_{M'}(L) - E(R') \neq \varnothing\}$. Note that $|\mathrm{cl}_{M'}(L)| > q+1$ for all $L \in \mathcal{L}^+$. Our goal is to use $\mathcal{L}^+$ to find an unstable set in a minor.

4.3.1.1. $\mathcal{L}^+$ *contains a* $(k+1)$*-matching of* R'.

Proof of claim: Suppose not. Let $F \subseteq E(R')$ and $\mathcal{L}_0 \subseteq \mathcal{L}^+$ be the sets defined in Theorem 2.4.1. Let $j = r_M(F)$; we know that $0 \leq j \leq k$, and if $j=k$, then $\mathcal{L}_0 = \varnothing$. By Lemma 2.3.6, we have $E(M') = \left(\bigcup_{L \in \mathcal{L}^+} L\right) \cup E(R')$. Let $\mathcal{L}_F = \{L \in \mathcal{L} : |L \cap F| = 1\}$. So each point in $M' \backslash R'$ is either in $\mathrm{cl}_{M'}(F)$, in a line in $\mathcal{L}_F$, or in a line in $\mathcal{L}_0$.

Since F is modular in R', each point of $E(R') - F$ lies on $|F|$ distinct lines in $\mathcal{L}_F$, and each line in $\mathcal{L}_F$ contains exactly q points in $E(R') - F$, so

$$|\mathcal{L}_F| = \frac{|F|(|E(R')| - |F|)}{q} = \frac{(q^j-1)(q^{r(M)}-q^j)}{q(q-1)^2}.$$

Each line of R' contains $q+1$ points of R', and its closure in M' contains at most q^2-q

points of $M'\backslash R'$. We can now estimate $\varepsilon(M')$.

$$\begin{aligned}\varepsilon(M') &= |R'| + |M'\backslash R'| \\ &\le |R'| + \sum_{L\in\mathcal{L}_F\cup\mathcal{L}_0} |L - E(R')| + |\operatorname{cl}_{M'}(F) - E(R')| \\ &\le \frac{q^{r(M)}-1}{q-1} + (q^2-q)(|\mathcal{L}_F| + |\mathcal{L}_0|) + \left(\frac{q^{2j}-1}{q^2-1} - \frac{q^j-1}{q-1}\right) \\ &\le \frac{(q^2-q)(q^j-1)(q^{r(M)}-q^j)}{q(q-1)^2} + \frac{q^{r(M)}-q^j}{q-1} + \frac{q^{2j}-1}{q^2-1} \\ &+ (q^2-q)|\mathcal{L}_0| \\ &= \frac{q^{r(M)+j}-1}{q-1} - q\left(\frac{q^{2j}-1}{q^2-1}\right) + (q^2-q)|\mathcal{L}_0|. \\ &= |\operatorname{PG}^{(j)}(r(M)-1,q)| + (q^2-q)|\mathcal{L}_0|\end{aligned}$$

If $j < k$, then by the fact that $r(M') = r(M) \ge f_{4.3.1}(n,q,k) \ge s$, we have $\varepsilon(M') \le |\operatorname{PG}^{(k)}(r(M)-1,q)|$. If $j = k$, then $\mathcal{L}_0 = \varnothing$, so $\varepsilon(M') \le |\operatorname{PG}^{(k)}(r(M)-1,q)|$. In either case,

$$\varepsilon(M') \le |\operatorname{PG}^{(k)}(r(M)-1,q)| < \varepsilon(M),$$

contradicting the fact that M is a restriction of M'. □

Let $\{L_1,\dots,L_{k+1}\} \subseteq \mathcal{L}^+$ be a $(k+1)$-matching, and let $B = \bigcup_{i=1}^{k+1} L_i$. We have $r_{M'}(B) = 2k+2$. The matroid R is a weakly round, spanning restriction of M', and R has a $\operatorname{PG}(f_{3.3.8}(n+k,q,2k+2,q^2)-1,q)$-minor, so by Lemma 3.3.8, there is a set $X \subseteq E(R)$ so that $r(R/\!\!/X) \ge n+k$, and $R/\!\!/X$ has a $\operatorname{PG}(r(M/\!\!/X)-1,q)$-restriction R_0, and $(M'/\!\!/X)|B = M'|B$.

4.3.1.2. $\operatorname{si}(M'/\!\!/X) \cong \operatorname{si}(M/\!\!/X)$.

Proof of claim: All entries of $A'[E(R')]$ are in $\operatorname{GF}(q)$. In particular, the entries of $A'[X]$ are in $\operatorname{GF}(q)$, so there is a $GF(q^2)$-representation A_0 of $M'/\!\!/X$ such that $A_0[E(R')-X]$ only has entries in $\operatorname{GF}(q)$.

But $E(R_0) \subseteq E(R')-X$, and R_0 is a $\operatorname{GF}(q)$-represented $\operatorname{PG}(r(R/\!\!/X)-1,q)$-restriction of $R/\!\!/X$, so every column of A_0 with entries only in $\operatorname{GF}(q)$ is parallel in A_0 to some element of R_0. All elements of $E(R')$ have this property, and $E(M') = E(M) \cup E(R')$, so the claim follows. □

4.3.1.3. *There is an R_0-unstable set of size $k+1$ in $M'/\!\!/X$.*

Proof of claim: For each $1 \le i \le k+1$, let $L'_i = \mathrm{cl}_{M'/\!\!/X}(L_i)$. Since $(M'/\!\!/X)|B = M'|B$, the set $\{L'_1, \dots, L'_{k+1}\}$ is a $(k+1)$-matching of $M'/\!\!/X$. Moreover, each L_i is spanned by a pair of points of R', and each such point is parallel in $M'/\!\!/X$ to a point of R_0, so for each i, the set $L'_i \cap E(R_0)$ is a line of R_0. Finally, $\varepsilon(M'/\!\!/X|L'_i) \ge \varepsilon(M'|\,\mathrm{cl}_{M'}(L_i)) > q+1$ for each i, so each L'_i contains a point e_i not parallel to any points of R_0. The set $\{e_1, \dots, e_{k+1}\}$ is R_0-unstable in $M'/\!\!/X$. $\square$

By Lemma 4.2.1, the matroid $M'/\!\!/X$ has a $\mathrm{PG}^{(k+1)}(n-1,q)$-minor; by the second claim, so does $M/\!\!/X$. $\square$

4.4 Constellations

If the hypotheses in the previous section fail, then we use a different method to find a $\mathrm{PG}^{(k)}$-minor, defining an auxiliary low-rank matroid that contains a large number of q-long lines:

Let s, q and j be positive integers. A matroid K is an (s,q,j)-*constellation* if

- $r(K) \le s(j+1)$, and
- K has an independent set S of size s such that, for all $e \in S$, there exists a $(K/\!\!/e)$-independent set X_e of size j, such that, for all $f \in X_e$, the line $\mathrm{cl}_K(\{e,f\})$ is q-long.

A constellation is an independent set of points, each of which is the centre of a 'star' of an independent collection of q-long lines. If K is any matroid satisfying the second part of the definition, then $K|\,\mathrm{cl}_K\left(S \cup \bigcup_{e\in S} X_e\right)$ is an (s,q,j)-constellation. Moreover, for any $s' \le s$, an (s,q,j)-constellation has an (s',q,j)-constellation restriction, found by considering an s'-subset of S.

To prove this section's main result, which is a means of constructing $\mathrm{PG}^{(k)}(n,q)$ from a constellation, we require two auxiliary results. The first is our means of finding an unstable set through Theorem 2.4.1:

Lemma 4.4.1. *There is an integer-valued function $f_{4.4.1}(q,k)$ so that, for any prime power q and integer $k \ge 0$, if M is a GF(q^2)-representable matroid, and R is a* GF(q)*-represented* PG($r(M)-1,q$)*-restriction of M, then either*

- *There is an R-unstable set of size $k+1$ in M, or*

- *There is some $C \subseteq E(R)$ so that $r_M(C) \le k$, and $\epsilon(M /\!\!/ C) \le \epsilon(R /\!\!/ C) + f_{4.4.1}(q,k)$.*

Proof. Let q be a prime power, and $k \ge 0$ be an integer. Set $f_{4.4.1}(q,k) = (q^2+1)f_{2.4.1}(q,k)$. Let $M \in \mathcal{L}(q^2)$ be a matroid with a GF(q)-represented PG$(r(M)-1,q)$-restriction R. We may assume that M is simple; let $\mathcal{L}$ be the set of lines L of R such that $|\operatorname{cl}_M(L)| > |\operatorname{cl}_R(L)|$. If $\mathcal{L}$ contains a $(k+1)$-matching of R, then choosing a point from $\operatorname{cl}_M(L) - \operatorname{cl}_R(L)$ for each line L in the matching gives an R-unstable set of size $k+1$. We may therefore assume that $\mathcal{L}$ contains no such matching. Thus, let F and $\mathcal{L}_0$ be the sets defined in the second outcome of Theorem 2.4.1. Let $C = F$, and $D = \cup_{L \in \mathcal{L}_0} L$. We have $|D| \le (q^2+1)|\mathcal{L}_0| \le f_{4.4.1}(q,k)$. By Lemma 2.3.6, each point of $M \backslash R$ lies in the closure of a line in $\mathcal{L}$, so $\varepsilon((M /\!\!/ C) \backslash E(R)) \le \varepsilon((M /\!\!/ C)|D)$; the result now follows. $\square$

The next lemma shows that a large collection of q-long lines guarantees a large collection of points outside a given PG$(r(M)-1,q)$-restriction:

Lemma 4.4.2. *Let q be a prime power, $d \ge 0$ be an integer, M be a GF(q^2)-representable matroid, and R be a* PG$(r(M)-1,q)$*-restriction of M. If $\mathcal{L}$ is a set of q-long lines of M such that $|\mathcal{L}| > \binom{d+1}{2}$, then $\epsilon(M) > \epsilon(R) + d$.*

Proof. We may assume that M is simple; it therefore suffices to show that $|M \backslash R| > d$. By Corollary 2.3.5, R is GF(q)-represented, and clearly, $L - E(R)$ is nonempty for every $L \in \mathcal{L}$. For each $L \in \mathcal{L}$, let $e_L \in L - E(R)$. Let $\mathcal{L}_0 = \{L \in \mathcal{L} : |L \cap E(R)| > 1\}$. Since $L \cap E(R)$ is a line of R for each $L \in \mathcal{L}_0$, Lemma 2.3.6 implies that the points $e_L : L \in \mathcal{L}_0$ are distinct, so $\varepsilon(M) \ge \varepsilon(R) + |\mathcal{L}_0|$. We may thus assume that $|\mathcal{L}_0| \le d$, and therefore that $|\mathcal{L} - \mathcal{L}_0| > \binom{d+1}{2} - d = \binom{d}{2}$.

Now, each $L \in \mathcal{L} - \mathcal{L}_0$ contains at least two points of $M \backslash R$, and no two lines in $\mathcal{L} - \mathcal{L}_0$ contain two common points of $M \backslash R$, so it follows that $|\mathcal{L} - \mathcal{L}_0| \le \binom{|M \backslash R|}{2}$, and therefore that $|M \backslash R| > d$. $\square$

Now we show how to construct PG$^{(k)}(n,q)$ from a constellation. This lemma is loosely analogous to Lemma 3.4.1 of Chapter 3; the constellation in the statement takes the place of the $U_{2,q+2}$-restriction.

Lemma 4.4.3. *There is an integer-valued function $f_{4.4.3}(n,q,k)$ so that, for any prime power q, and integers n and k with $0 \le k < n$, if M is a weakly round, GF(q^2)-representable matroid with an $(f_{4.4.3}(n,q,k),q,k{+}1)$-constellation restriction K, and a PG$(f_{4.4.3}(n,q,k)-1,q)$-minor, then M has a* PG$^{(k+1)}(n-1,q)$*-minor.*

Proof. Let q be a prime power, and n and k be integers with $0 \leq k < n$. Let $d = f_{4.4.1}(q, k)$, and let $s = d(d+1) + k + 1$. Set

$$f_{4.4.3}(n, q, k) = \max(s, f_{3.3.8}(n + k, q, s(k + 2), q^2)).$$

Let $M \in \mathcal{L}(q^2)$ be a weakly round matroid with an $(f_{4.4.3}(n, q, k), q, k)$-constellation restriction K, and a $\mathrm{PG}(f_{4.4.3}(n, q, k) - 1, q)$-minor. We have $M \in \mathcal{L}(q^2) \subseteq \mathcal{U}(q^2)$. By Lemma 3.3.8, applied with $M^+ = M$, and $B = E(K)$, there is some set $X \subseteq E(M)$ so that $r(M /\!\!/ X) \geq n+k$, and $M /\!\!/ X$ has a $\mathrm{PG}(r(M /\!\!/ X)-1, q)$-restriction R, and $(M /\!\!/ X)|E(K) = M|E(K) = K$. Let $M' = M /\!\!/ X$.

4.4.3.1. *M' has an R-unstable set of size $k + 1$.*

Proof of claim: By Lemma 4.4.1, we may assume that there is a set $C \subseteq E(R)$ so that $r_{M'}(C) \leq k$, and $\varepsilon(M' /\!\!/ C) \leq \varepsilon(R /\!\!/ C) + d$. The set S in the constellation K is a rank-$(d(d+1)+k+1)$ set in M'; let $S' \subseteq S$ be an independent set of size $d(d+1)+1$ in $M' /\!\!/ C$. Let $e \in S'$. Since $r_{(M' /\!\!/ e)}(X_e) > k$, there is some $f \in X_e$ so that $\{e, f\}$ is independent in $M' /\!\!/ C$; let $L_e = \mathrm{cl}_{M' /\!\!/ C}(\{e, f\})$. The line L_e contains at least $q + 2$ points in K, and therefore in $M' /\!\!/ C$.

S' is independent in $M' /\!\!/ C$, so no line L_e can contain more than two points of S', giving $|\{L_e : e \in S'\}| \geq \frac{1}{2}|S'| > \binom{d+1}{2}$. The matroid $R /\!\!/ C$ is a spanning restriction of $M' /\!\!/ C$, and $\mathrm{si}(R /\!\!/ C) \cong \mathrm{PG}(n-1-r_{M'}(C), q)$, so Lemma 4.4.2 now implies that $\varepsilon(M' /\!\!/ C) > \varepsilon(R /\!\!/ C)+d$, a contradiction. $\square$

The lemma now follows from Lemma 4.2.1.

$\square$

4.5 The Reductions

We are now in a position to prove Theorem 4.2.3 by showing that it can be reduced to either Lemma 4.3.1 or Lemma 4.4.3. The following technical lemma contains this reduction.

Lemma 4.5.1. *There is an integer-valued function $f_{4.5.1}(m, q, k)$ so that, for any prime power q, and integers $m \geq 1$ and $k \geq 0$, if M is a weakly round, GF(q^2)-representable matroid such that*

- *M has a PG($f_{4.5.1}(m, q, k) - 1, q$)-minor, and*

- $\varepsilon(M) > |\operatorname{PG}^{(k)}(r(M)-1,q)|$,

then one of the following holds:

(i) M has a minor M' such that

- *M' has a weakly round, spanning $\mathrm{GF}(q)$-represented restriction R, and*
- *R has a $PG(m-1,q)$-minor, and*
- *$\varepsilon(M') > |\operatorname{PG}^{(k)}(r(M')-1,q)|$,*

or

(ii) M has a weakly round minor M' with an $(m,q,k+1)$-constellation restriction, and a $PG(m-1,q)$-minor.

Proof. Let q be a prime power, and let $m \geq 1$ and $k \geq 0$ be integers. Let r be an integer large enough so that

$$q^{r'-3m} \geq \alpha_{3.2.2}(q-\tfrac{1}{2},q,m)(q-\tfrac{1}{2})^{r'}$$

for all $r' \geq r$. Let $n = f_{3.3.6}(q^2,3m,r)+2m$. Set $f_{4.5.1}(m,q,k) = n$.

Let $M \in \mathcal{L}(q^2)$ be a weakly round matroid with a $\mathrm{PG}(n-1,q)$-minor, satisfying $\varepsilon(M) > |\operatorname{PG}^{(k)}(r(M)-1,q)|$. We may assume that M is simple, and minor-minimal satisfying the hypotheses. Let $N = M/\!\!/C\backslash D \cong \mathrm{PG}(n-1,q)$, where C is independent, and D is co-independent.

4.5.1.1. *M has a $(|C|,q,k+1)$-constellation restriction.*

Proof of claim: Let $e \in C$. The matroid $M/\!\!/e$ is weakly round and $\mathrm{GF}(q^2)$-representable, and has an N-minor, so

$$\varepsilon(M/\!\!/e) \leq |\operatorname{PG}^{(k)}(r(M/\!\!/e)-1,q)|$$

by minor-minimality of M. Let $\mathcal{L}^+$ be the set of q-long lines of M containing e, and $\mathcal{L}^-$ be the set of all other lines of M containing e. Each line in $\mathcal{L}^-$ contains at most q points other than e, and each line in $\mathcal{L}^+$ contains at most q^2 points other than e. Lemma 1.1.1 gives $\varepsilon(M/\!\!/e) = |\mathcal{L}^+|+|\mathcal{L}^-|$, and $\varepsilon(M) \leq q^2|\mathcal{L}^+|+q|\mathcal{L}^-|+1 = q\varepsilon(M/\!\!/e)+(q^2-q)|\mathcal{L}^+|+1$. Now

$$\begin{aligned}
|\operatorname{PG}^{(k)}(r(M)-1,q)| &< \varepsilon(M)\\
&\leq q\varepsilon(M/\!\!/e)+(q^2-q)|\mathcal{L}^+|+1\\
&\leq q|\operatorname{PG}^{(k)}(r(M/\!\!/e)-1,q)|+(q^2-q)|\mathcal{L}^+|+1.
\end{aligned}$$

This implies that

$$|\mathcal{L}^+| > \frac{1}{q^2-q}\left(|\operatorname{PG}^{(k)}(r(M)-1,q)| - q|\operatorname{PG}^{(k)}(r(M)-2,q)| - 1\right),$$

and a computation gives $|\mathcal{L}^+| > \frac{q^{2k}-1}{q^2-1}$. Let X'_e be a set formed by choosing a point other than e from each line in $\mathcal{L}^+$. Since M is $\mathrm{GF}(q^2)$-representable, it now follows that $r_{M/\!\!/e}(X'_e) > k$; let $X_e \subseteq X'_e$ be an independent set of size $k+1$ in $M/\!\!/e$. The set C, along with $X_e : e \in C$, gives the required constellation. □

Since $n \geq m$, the matroid M also has a $\mathrm{PG}(m-1,q)$-minor, so if $|C| \geq m$, we have outcome (ii) for M by 4.5.1.1. We may therefore assume that $|C| < m$.

4.5.1.2. *There is a weakly round,* $\mathrm{GF}(q)$*-represented restriction* R *of* M *so that* R *has a* $\mathrm{PG}(m-1,q)$*-minor.*

Proof of claim: Since $E(N)$ is a spanning restriction of $M/\!\!/C$, there is a matrix A' representing M over $\mathrm{GF}(q^2)$ of the following form:

$$A' = \begin{array}{c c} & \begin{array}{ccc} C & E(N) & D \end{array} \\ \begin{array}{c} C \\ {[n]} \end{array} & \begin{pmatrix} I_C & Q_1 & Q_2 \\ 0 & B & Q_3 \end{pmatrix} \end{array}$$

where $M(B) \cong \mathrm{PG}(n-1,q)$. By applying Theorem 2.3.4 to the submatrix $A'[[n],E(M)]$, we may assume that all entries of B are in $\mathrm{GF}(q)$. Since $|C| < m$, there are at most $q^{2(m-1)}$ distinct column vectors in Q_1, so there is some $Y \subseteq E(N)$ so that $|Y| \geq q^{-2(m-1)}|E(N)|$, and all columns of the matrix $Q_1[Y]$ are the same. Now,

$$A'[Y] = \begin{pmatrix} Q_1[Y] \\ B[Y] \end{pmatrix},$$

where $Q_1[Y]$ is a matrix of rank at most 1, so by scaling the first $|C|$ rows of $A'[Y]$, we can obtain a matrix of the following form:

$$\begin{pmatrix} P \\ B[Y] \end{pmatrix},$$

where all entries of P are 0 or 1. Applying these same row scalings to A' gives a matrix A representing M over $\mathrm{GF}(q^2)$, in which all entries of $A[Y]$ are in P or $B[Y]$, and therefore in $\mathrm{GF}(q)$.

We have $|Y| \geq q^{-2(m-1)}|E(N)| > q^{n-2m+1}$. Also, $r_M(Y) \leq r(M) \leq n+m-1$, so $|Y| > q^{-3m}q^{r(M|Y)}$. Finally, $M|Y$ is GF(q)-representable, so $r_M(Y) \geq n-2m+2 \geq f_{3.3.6}(q^2, 3m, r)$ by our first lower bound on $|Y|$. The function $g(i)$ defined by $g(i) = q^{i-3m}$ satisfies $g(3m) = 1$ and $g(i) \geq 2g(i-1)$ for all $i > 3m$, so by Lemma 3.3.6, $M|Y$ has a weakly round restriction R with $r(R) \geq r$, and $\varepsilon(R) > q^{r(R)-3m}$.

$A[E(R)]$ is a submatrix of $A[Y]$, so R is a $GF(q)$-represented restriction of M. We have

$$\varepsilon(R) > q^{-3m}q^{r(R)} \geq \alpha_{3.2.2}(q - \tfrac{1}{2}, q, m)(q - \tfrac{1}{2})^{r(R)},$$

so R has a PG$(q', m-1)$-minor for some prime power $q' > q - \frac{1}{2}$. Since R is GF(q)-representable, we must have $q' = q$, so R satisfies the claim. $\square$

Let M' be minor-minimal subject to the following conditions:

- M' is a weakly round minor of M, and
- $\varepsilon(M') > |\,\mathrm{PG}^{(k)}(r(M')-1, q)|$, and
- R is a GF(q)-represented restriction of M'.

If R is spanning in M', then M' and R satisfy outcome (i). We may therefore assume that $r(R) < r(M')$. Since R has a PG$(m-1, q)$-minor, the following claim will give outcome (ii) for M'.

4.5.1.3. *M' has an $(m, q, k+1)$-constellation restriction.*

Proof of claim: We have $m \leq r(R') \leq r(M) - 1$, so by weak roundness of M', the set $E(M') - \mathrm{cl}_{M'}(E(R))$ has rank at least $r(M) - 1 \geq m$ in M; let S be an independent set of size m in M, disjoint from $\mathrm{cl}_{M'}(E(R))$.

For each $e \in S$, the matroid $M' /\!\!/ e$ is weakly round, and we have $R = (M' /\!\!/ e)|(E(R))$, so R is a GF(q)-represented restriction of $M' /\!\!/ e$. By minimality of M', it follows that

$$\varepsilon(M' /\!\!/ e) \leq |\,\mathrm{PG}^{(k)}(r(M' /\!\!/ e) - 1, q)|.$$

The remainder of the proof is very similar to that of 4.5.1.1. $\square$

$\square$

We can now prove Theorem 4.2.3, which we restate here for convenience:

Theorem 4.2.3. *There is an integer-valued function $f_{4.2.3}(n, q, k)$ so that, for any prime power q, and integers n and k with $0 \leq k < n$, if M is a GF(q^2)-representable matroid with $r(M) \geq f_{4.2.3}(n, q, k)$ and*

$$\varepsilon(M) > |\,\mathrm{PG}^{(k)}(r(M) - 1, q)|,$$

then M has a $\mathrm{PG}^{(k+1)}(n - 1, q)$-minor.

Proof. Let q be a prime power, and n and k be integers with $0 \leq k < n$. We define the function $f_{4.2.3}$ as follows. Let

$$m = \max(f_{4.3.1}(n, q, k), f_{4.4.3}(n, q, k)).$$

Let $\alpha = \alpha_{3.2.2}(q - \frac{1}{2}, q^2, m)$. Let r be an integer large enough so that

$$|\,\mathrm{PG}^{(k)}(r' - 1, q, k)| \geq \alpha(q - \tfrac{1}{2})^{r'}$$

for all $r' \geq r$, and let $s = f_{3.3.6}(q^2, 1, r)$. Set $f_{4.2.3}(n, q, k) = s$.

Let $M \in \mathcal{L}(q^2)$ be a matroid with $r(M) \geq s$, and $\varepsilon(M) > |\,\mathrm{PG}^{(k)}(r(M) - 1, q)|$. The function $g(i) = |\,\mathrm{PG}^{(k)}(i - 1, q)|$ can easily be seen to satisfy $g(1) \geq 1$ and $g(i) \geq 2g(i - 1)$ for all $i \geq 2$, so by Lemma 3.3.6, M has a weakly round restriction N with $r(N) \geq r$, and $\varepsilon(N) > |\,\mathrm{PG}^{(k)}(r(N) - 1, q)|$.

By Theorem 3.2.2 and definition of r, N has a $\mathrm{PG}(m - 1, q')$-minor for some $q' > q - \frac{1}{2}$. Since N is GF(q^2)-representable, we have $q' = q^2$ or $q' = q$, so in either case, N has a $\mathrm{PG}(m - 1, q)$-minor. The lemma now follows by applying Lemma 4.5.1 to N, and then either Lemma 4.3.1 or Lemma 4.4.3 to the minor M' of N given by Lemma 4.5.1. $\square$

4.6 The Main Theorems

We now restate and prove our main results, starting with Theorem 1.5.7:

Theorem 1.5.7. *Let q be a prime power. If $\mathcal{M}$ is a minor-closed class of matroids so that $\mathcal{L}(q) \subseteq \mathcal{M} \subsetneq \mathcal{L}(q^2)$, then there is an integer $k \geq 0$ such that $\mathcal{P}_{q,k} \subseteq \mathcal{M}$, and $h_{\mathcal{M}}(n) = h_{\mathcal{P}_{q,k}}(n)$ for all sufficiently large n.*

Proof. Since $\mathcal{M}$ does not contain all GF(q^2)-representable matroids, there is an integer s so that PG$(s, q^2) \notin \mathcal{M}$. We have $\mathcal{P}_{q,0} = \mathcal{L}(q) \subseteq \mathcal{M}$, and by Lemma 4.1.3 and Lemma 4.1.4, we have $\mathcal{P}_{q,s'} \nsubseteq \mathcal{M}$ for all $s' \geq s$; let $k \geq 0$ be maximal so that $\mathcal{P}_{q,k} \subseteq \mathcal{M}$.

Clearly $h_{\mathcal{M}}(n) \geq h_{\mathcal{P}_{q,k}}(n)$ for all n; we need to show that this holds with equality for all large n; suppose that this is not the case. For all integers $m > k$, there is therefore some $M \in \mathcal{M}$ such that $r(M) \geq f_{4.2.3}(m, q, k)$ and $\varepsilon(M) > h_{\mathcal{P}_{q,k}}(r(M)) = |\,\mathrm{PG}^{(k)}(r(M) - 1, q)|$. By Theorem 4.2.3, this M has an PG$^{(k+1)}(m - 1, q)$-minor. Thus, PG$^{(k+1)}(m - 1, q) \in \mathcal{M}$ for all $m > k$, so by Lemma 4.1.4, $\mathcal{P}_{q,k+1} \subseteq \mathcal{M}$, contradicting maximality of k. □

Theorems 1.5.8 and 1.5.9 are now immediate, the first from Corollary 4.1.5, and the second from Corollary 4.1.5 and Lemma 4.1.3:

Theorem 1.5.8. *Let q be a prime power. If $\mathcal{M}$ is a minor-closed class of matroids such that $\mathcal{L}(q) \subseteq \mathcal{M} \subsetneq \mathcal{L}(q^2)$, then there is an integer $k \geq 0$ so that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+k} - 1}{q - 1} - q\left(\frac{q^{2k} - 1}{q^2 - 1}\right)$$

for all sufficiently large n.

Theorem 1.5.9. *Let q be a prime power, and $k \geq 0$ be an integer. Let $\mathcal{M}(k)$ denote the class of* GF(q^2)*-representable matroids with no* PG$(k+1, q^2)$*-minor. Then*

$$h_{\mathcal{M}(k)}(n) = \frac{q^{n+k} - 1}{q - 1} - q\left(\frac{q^{2k} - 1}{q^2 - 1}\right)$$

for all sufficiently large n.

Theorems 1.5.10 and 1.5.11 also have easy proofs:

Theorem 1.5.10. *Let q be a prime power, and $j \geq 3$ be an odd number. If $\mathcal{M} = \mathcal{L}(q^2) \cap \mathcal{L}(q^j)$, then there exists an integer n_q so that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+1} - 1}{q - 1} - q$$

for all $n \geq n_q$.

Proof. Set n_q to be an integer large enough so that Theorem 1.5.9 for q and $k = 1$ applies for all $n \geq n_q$. By Lemma 4.2.2, $\mathcal{M}$ contains $\mathrm{PG}^{(1)}(n-1, q)$ for all $n \geq 0$, but not $\mathrm{PG}(2, q^2)$; Theorem 1.5.9 gives

$$\varepsilon(M) \leq \frac{q^{r(M)+1} - 1}{q - 1} - q = |\,\mathrm{PG}^{(1)}(r(M) - 1, q)|$$

for all M satisfying $r(M) \geq n_q$. But $h_{\mathcal{M}}(n) \geq |\,\mathrm{PG}^{(1)}(n - 1, q)|$ for all n, so the theorem follows. $\square$

Theorem 1.5.11. *Let q be a prime power. There is a finite set $\mathfrak{H}_q$ of integer-valued functions so that, for any set of fields $\mathcal{F}$ such that $\mathrm{GF}(q^2) \in \mathcal{F}$, and all fields in $\mathcal{F}$ have a $\mathrm{GF}(q)$-subfield, but not all fields in $\mathcal{F}$ have a $\mathrm{GF}(q^2)$-subfield, if $\mathcal{M} = \bigcap_{\mathbb{F} \in \mathcal{F}} \mathcal{L}(\mathbb{F})$, then $h_{\mathcal{M}} \in \mathfrak{H}_q$.*

Proof. Let n_0 be an integer large enough so that Theorem 1.5.9 for q and $k = 1$ applies for all $n \geq n_0$. Let $\mathfrak{H}_q$ be the set of integer-valued functions f so that $0 \leq f(n) \leq \frac{q^{2n}-1}{q^2-1}$ for all $0 \leq n < n_0$, and

$$f(n) = \frac{q^{n+1} - 1}{q - 1} - q$$

for all $n \geq n_0$. The set $\mathfrak{H}_q$ is clearly finite. Let $\mathcal{F}$ be a set of fields satisfying the listed conditions, and $\mathcal{M}$ be the class of matroids representable over all fields in $\mathcal{F}$. There is some $\mathbb{F} \in \mathcal{F}$ with no $\mathrm{GF}(q^2)$-subfield, so by Lemma 4.2.2, we know that $\mathrm{PG}^{(1)}(n - 1, q) \in \mathcal{M}$ for all n, and $\mathrm{PG}(2, q^2) \notin \mathcal{M}$. It now follows from $\mathrm{GF}(q^2)$-representability of matroids in $\mathcal{M}$, and a similar argument to the proof of Theorem 1.5.10, that $h_{\mathcal{M}} \in \mathfrak{H}_q$, giving the theorem. $\square$

Part II

Covering Number

Chapter 5

Minors and Structure Theory

5.1 Introduction

The theorems in this thesis exclusively deal with the density of matroids in minor-closed classes not containing some uniform matroid $U_{a,b}$. However, this work is partly motivated by deeper questions about the structure of these classes. In this chapter, we will discuss what is known and conjectured in matroid structure theory for these classes, thereby motivating our work on their extremal behaviour in Part II.

We start by drawing a parallel with graph theory, restating a result of Mader, Theorem 1.2.2.

Theorem 5.1.1. *If $\mathcal{G}$ is a proper minor-closed class of graphs, then $|E(G)| \leq c_{\mathcal{G}}|V(G)|$ for all simple graphs G in $\mathcal{G}$.*

As we have already seen, this tells us that any proper minor-closed class of graphs is linearly dense, in contrast to the quadratically dense class of all graphs. However, the theorem also foreshadows a much deeper result, the Graph Minors Structure Theorem [39], which provides a constructive description of all proper minor-closed classes of graphs. The theorem, the crux of the Graph Minors Project of Robertson and Seymour, states that the graphs in any such class are essentially obtained from graphs 'almost' embeddable on a surface of bounded genus. Our understanding of surface embeddings can thus be extended to arbitrary minor-closed classes, giving the theorem far-reaching consequences, both algorithmic and theoretical. Theorems such as Theorem 5.1.1 can be seen as reflections of this underlying structure.

Analogous results should be attainable in matroid theory; indeed, Geelen, Gerards and Whittle have extended the Graph Minors Project to binary matroids (a discussion written prior to its completion can be found in [12]), obtaining profound structural results such as the following recently announced theorem, which states that the highly connected binary matroids in any proper minor-closed class are essentially graphic or cographic:

Theorem 5.1.2. *For each proper minor-closed subclass $\mathcal{M}$ of the binary matroids, there exists an integer k such that every k-connected matroid in $\mathcal{M}$ has the form $M(A + P)$, where $M(A)$ is either graphic or cographic, and P has rank at most k.*

5.2 Natural Classes

Important in both the Graph Minors Structure Theorem, and Theorem 5.1.2 are 'natural' minor-closed classes: in the former case, graphs that embed on some fixed surface, and in the latter, the graphic and cographic matroids. Emergence of these classes in major theorems is a recurrent phenomenon; the Growth Rate Theorem is another striking example, and in this section we discuss some more results in which natural classes of this sort surprisingly appear.

The first is Seymour's Regular Matroid Decomposition Theorem [45], which we do not phrase in full generality, but has the following consequence, which can be seen as an exact, specialised version of Theorem 5.1.2:

Theorem 5.2.1. *If M is an internally 4-connected, regular matroid such that $|M| > 10$, then M is graphic or cographic.*

The second is the extension of Robertson and Seymour's grid theorem for graphs [40] to matroids, due to Geelen, Gerards and Whittle [11]. In this theorem, the planar-graphic matroids play a fundamental role. The *branch-width* of a matroid is, roughly, a measure of how 'tree-like' it is; see [35] for a definition.

Theorem 5.2.2 (Grid Theorem for Matroids)**.** *Let $\ell \geq 2$ be an integer, and $\mathcal{M}$ be a minor-closed class of matroids such that $U_{2,\ell+2} \notin \mathcal{M}$ and $U_{\ell,\ell+2} \notin \mathcal{M}$. Either*

1. *there is an integer k such that all matroids in $\mathcal{M}$ have branch-width at most k, or*
2. *$\mathcal{M}$ contains all planar-graphic matroids.*

Another result in which graphic matroids appear surprisingly is a beautiful theorem of Seymour [46]:

Theorem 5.2.3. *Let M be a simple, vertically 4-connected binary matroid, and e, f, g be distinct elements of $E(M)$. Either*

1. *e, f and g lie in a common circuit, or*
2. *$M = M(G)$ is graphic, and e, f and g are edges incident with a common vertex of G.*

When M is not binary, this theorem fails; a class of counterexamples is provided by *frame matroids*. A matroid M with basis B is a *framed matroid with frame B* if every $e \in E(M)$ lies in a flat spanned by at most two elements of B. A *frame matroid* is a restriction of a framed matroid; we write $\mathcal{D}$ for the class of frame matroids. The class $\mathcal{D}$ is minor-closed, and a matroid having a representation in which each column has at most two nonzero entries (in particular, any graphic matroid) is clearly in $\mathcal{D}$. One can show that, if M is a framed matroid with frame B, and e, f, g are distinct elements of B, then e, f and g lie in no common circuit of M; this also holds for any restriction of M containing e, f and g. It is straightforward to construct highly vertically connected, non-binary frame matroids; thus, any generalisation of Theorem 5.2.3 must take these matroids into account.

Before stating this section's last theorem, we mention two special subclasses of the frame matroids. If M is a framed matroid with frame B, such that every nonloop e of M is either parallel to an element of B, or is placed freely on the line spanned by two elements of B, then any restriction of M is a *bicircular matroid*; we denote the class of such matroids by $\mathcal{B}$. This class is natural when viewed geometrically - the points of the frame are 'vertices', and all other points lie freely on the rank-2 'edge' between a pair of these vertices. A definition of $\mathcal{B}$ in terms of graphs is found in [35]. One can show that $\mathcal{B}$ is minor-closed, and contains all simple rank-2 matroids, but not all graphic matroids.

The second subclass arises more algebraically. If Γ is a group, then a construction involving group-labelled digraphs gives rise to the Γ-*Dowling matroids*, a class we denote by $\mathcal{D}(\Gamma)$. The full definition appears in [21], in which these matroids are referred to as *voltage-graphic matroids with voltages in* Γ; here, we just mention some key facts. The class $\mathcal{D}(\Gamma)$ is minor-closed, and $\mathcal{D}(\Gamma)$ contains all simple rank-2 matroids and all bicircular matroids if and only if Γ is infinite. Finally, if $\mathbb{F}^*$ is the multiplicative group of a field $\mathbb{F}$, then $\mathcal{D}(\mathbb{F}^*)$ is precisely the class of matroids having an $\mathbb{F}$-representation in which each column contains at most two nonzero entries; in particular, if $\Gamma = \mathrm{GF}(2)^*$ is the trivial group, then $\mathcal{D}(\Gamma) = \mathcal{G}$, the class of graphic matroids.

The Γ-Dowling matroids appear as one of two natural classes in this next theorem of Kahn and Kung [21] concerning varieties of matroids. A minor-closed class $\mathcal{V}$ of matroids is a *variety* if it is closed under taking direct sums, and if for each integer $n \geq 0$, there is a rank-n matroid in $\mathcal{V}$ having every simple rank-n matroid in $\mathcal{V}$ as a restriction; for example, the classes $\mathcal{G}$ and $\mathcal{L}(q)$ are varieties, with cliques and projective geometries satisfying the definition. The theorem, having the following consequence, classifies varieties of matroids completely:

Theorem 5.2.4. *If $\mathcal{V}$ is a variety of matroids containing a 3-connected matroid of rank 3, then either $\mathcal{V} = \mathcal{D}(\Gamma)$ for some finite group Γ, or $\mathcal{V} = \mathcal{L}(q)$ for some prime power q.*

5.3 Structure

In this section, we give a qualitative structure conjecture, due to Geelen, Gerards and Whittle, for the highly connected matroids in a minor-closed class $\mathcal{M}$ not containing some uniform matroid $U_{a,b}$; the statement of the conjecture will be analogous to that of Theorem 5.1.2, asserting that any highly connected matroid in $\mathcal{M}$ is 'close' to a matroid in some 'natural' minor-closed class. The material in this section is covered in much more detail in [9].

We first require a notion of 'close' that is not peculiar to representable matroids. If M and N are matroids on the same ground set E, then we say that M is *adjacent* to N if M is a single-element projection of N or vice versa (if M is a single-element projection of N, then N is a *single-element lift* of M); this defines a graph G_E on the set of matroids on E. We write $d(M, N)$ for the length of the shortest path from M to N in G_E. The rank-0 matroid on E can clearly be obtained any matroid M on E by a sequence of $r(M)$ projections; it follows that G_E is connected, so $d(\cdot, \cdot)$ is a metric on the set of all matroids on E. One can show that $d(M, N) = d(M^*, N^*)$ for any M and N on the same ground set. A crucial fact about this notion of distance is the following, essentially stated in [9] as a consequence of a theorem of Geelen and Kabell [13], which tells us that travelling a bounded distance in this graph does not introduce arbitrarily large new uniform minors. Note that the statement is self-dual.

Lemma 5.3.1. *For all integers $t \geq 1$ and $k \geq 1$, there exists an integer n such that, if M is a matroid with no $U_{n,2n}$-minor, and N is a matroid such that $E(N) = E(M)$, and $d(N, M) \leq k$, then N has no $U_{t,2t}$-minor.*

Now, we state the structure conjecture. In order that the statement is self-dual, we exclude a uniform matroid $U_{n,2n}$ in place of $U_{a,b}$ without loss of generality, as any minor-closed class of matroids not containing all uniform matroids omits some uniform matroid of this special form. The theorem states that all matroids in a minor-closed class $\mathcal{M}$ not containing $U_{n,2n}$ are a bounded distance from a matroid in one of three natural classes:

Conjecture 5.3.2. *Let $n \geq 1$ be an integer, and $\mathcal{M}$ be a minor-closed class of matroids not containing $U_{n,2n}$. There is an integer k such that, for every k-connected matroid $M \in \mathcal{M}$, there is a matroid N so that $d(M, N) \leq k$, and either*

1. *$N \in \mathcal{L}(q)$ for some prime power q such that $U_{n,2n} \notin \mathcal{L}(q)$, or*
2. *One of N or N^* is in $\mathcal{D}$.*

Here, the natural binary classes of graphic and cographic matroids of Theorem 5.1.2 are replaced by the more general classes of frame and co-frame matroids, as well as representable matroids over a small enough finite field; as none of these classes contain arbitrarily large uniform matroids, Lemma 5.3.1 implies that matroids M as described by the conjecture cannot have arbitrarily large uniform minors.

Conjecture 5.3.2, which is likely extremely difficult to prove, would imply that frame matroids and representable matroids play a huge role in the general structure of minor-closed classes of matroids; this would explain the emergence of these classes (and their special subclasses such as the graphic and planar-graphic matroids) in theorems such as those we saw in the last section.

5.4 The Growth Rate Conjecture

Just as Theorem 5.1.1 reflects the structure of minor-closed classes of graphs given by the Graph Minors Structure Theorem, the structure that Conjecture 5.3.2 would give can be seen as a deep explanation of the restricted extremal behaviour implied by the Growth Rate Theorem; growth rate functions are linear, quadratic or exponential because these are the densities of the natural classes. However, the Growth Rate Theorem, inasmuch as its conclusion is helpful, applies only to classes excluding some fixed rank-2 uniform matroid $U_{2,\ell+2}$. It is thus natural to ask what Conjecture 5.3.2 could tell us about the extremal behaviour in a class $\mathcal{M}$ excluding a more general uniform matroid $U_{a,b}$. In this section, we present a conjecture to this effect.

An immediate hurdle in formulation is that, if $\mathcal{M}$ contains all simple rank-2 matroids, then $\varepsilon(M)$ is unbounded as a function of $r(M)$ over the matroids $M \in \mathcal{M}$, and therefore comparing $\varepsilon(M)$ and $r(M)$ no longer gives a sensible notion of density; thus, we introduce a new one, due originally to Geelen and Kabell [13]. If $a \geq 1$ is an integer, then we write $\tau_a(M)$ for the *a-covering number* of M, defined to be the minimum number of sets of rank at most a in M required to cover $E(M)$. Thus, $\tau_1(M) = \varepsilon(M)$. In [13], Geelen and Kabell proved the following, suggesting that τ_a is a reasonable measure of density:

Lemma 5.4.1. *If a and b are integers with $1 \leq a < b$, and M is a matroid with no $U_{a+1,b}$-minor such that $r(M) > a$, then $\tau_a(M) \leq \binom{b-1}{a}^{r(M)-a}$.*

A crude counting argument gives, for any $n \geq a$ and prime power q, that $\frac{q^n-1}{q^a-1} \leq \tau_a(\mathrm{PG}(n-1,q)) \leq \frac{q^n-1}{q-1}$, so the class $\mathcal{L}(q)$ is exponentially dense with respect to τ_a for any $a \geq 1$. Similarly, we have $\frac{n(n+1)}{a(a+1)} \leq \tau_a(M(K_n)) \leq \binom{n+1}{2}$ for any $n \geq 2$, so $\mathcal{G}$ is quadratically dense.

The class $\mathcal{D}$ of frame matroids is a larger quadratically dense class. By the Growth Rate Theorem, any minor-closed subclass of $\mathcal{D}$ with quadratic growth rate function, and not containing all simple rank-2 matroids, contains $\mathcal{G}$. However, when not excluding a line, the bicircular matroids $\mathcal{B}$ provide a quadratically dense subclass not containing $\mathcal{G}$. Fortunately, $\mathcal{B}$ and $\mathcal{G}$ are the only unavoidable subclasses of this sort; one can show that, with respect to τ_a for any $a \geq 2$, any subclass of $\mathcal{D}$ with quadratic density contains either $\mathcal{B}$ or $\mathcal{G}$.

We now give a conjecture, due to Geelen [9], for the extremal behaviour of matroids in a minor-closed class, not containing all uniform matroids of some rank. This conjecture strongly resembles the Growth Rate Theorem.

Conjecture 5.4.2 (Growth Rate Conjecture)**.** *Let $a \geq 1$ be an integer, and $\mathcal{M}$ be a minor-closed class of matroids. Either:*

1. *$\tau_a(M) \leq c_{\mathcal{M}} r(M)$ for all $M \in \mathcal{M}$, or*
2. *$\tau_a(M) \leq c_{\mathcal{M}} r(M)^2$ for all $M \in \mathcal{M}$, and $\mathcal{M}$ contains either all graphic matroids, or all bicircular matroids, or*
3. *there is a prime power q such that $\tau_a(M) \leq c_{\mathcal{M}} q^{r(M)}$ for all $M \in \mathcal{M}$, and $\mathcal{M}$ contains all* GF(q)*-representable matroids, or*
4. *$\mathcal{M}$ contains all rank-$(a+1)$ uniform matroids.*

As ever, $c_{\mathcal{M}}$ denotes a real constant depending only on $\mathcal{M}$. For $a = 1$, this conjecture specialises to the Growth Rate Theorem, as $\tau_1(M) = \varepsilon(M)$, and the bicircular matroids contain all rank-2 uniform matroids. In the general case, both $\mathcal{B}$ and $\mathcal{G}$ appear as the two unavoidable dense subclasses of the class $\mathcal{D}$ of frame matroids. The co-frame matroids are linearly dense, so the first three outcomes of the conjecture reflect the natural classes of Conjecture 5.3.2.

The main result of Part II is the following, which resolves the 'polynomial-exponential' component of the conjecture, implying Theorem 1.0.3 and generalising Theorem 3.2.1:

Theorem 5.4.3. *Let $a \geq 1$ be an integer, and $\mathcal{M}$ be a minor-closed class of matroids, not containing all rank-$(a+1)$ uniform matroids. Either*

1. *$\tau_a(M) \leq r(M)^{c_{\mathcal{M}}}$ for all $M \in \mathcal{M}$, or*
2. *there is a prime power q such that $\tau_a(M) \leq c_{\mathcal{M}} q^{r(M)}$ for all $M \in \mathcal{M}$, and $\mathcal{M}$ contains all* GF(q)*-representable matroids.*

5.5 Small Uniform Matroids

Although Part II of this thesis revolves around excluding a uniform minor of rank at least 2, we will briefly consider excluding the uniform matroids of rank 0 and 1, to which Theorem 5.4.3 does not apply.

A matroid with no $U_{0,b}$-minor has a dual with no $U_{b,b}$-minor, and therefore has corank at most $b-1$, giving the following:

Proposition 5.5.1. *Let $b \geq 0$ be an integer. If M is a matroid with no $U_{0,b}$-minor, then $|M| \leq r(M) + b - 1$.*

In fact, this is an exact characterisation of matroids with no $U_{0,b}$ minor; the converse also holds. Any matroid of corank $b-1$ satisfies this bound, so it is best possible. The result for excluding a rank-1 uniform matroid is essentially proved in [35], without a claim of originality:

Proposition 5.5.2. *Let $b \geq 1$ be an integer. If M is a loopless matroid with no $U_{1,b}$-minor, then $|M| \leq (b-1)r(M)$.*

Proof. The result is plain for $r(M) = 0$; suppose that $r(M) > 0$, and that the proposition holds for matroids of smaller rank. Let C be a cocircuit of M. We have $M/(E(M)-C) \cong U_{1,|C|}$, so $|C| \leq b-1$. By the inductive hypothesis, $|M \backslash C| \leq (b-1)(r(M)-1)$, so $|M| \leq (b-1)(r(M)-1) + b - 1 = (b-1)r(M)$. □

This bound is attained when M is the direct sum of matroids isomorphic to $U_{1,b-1}$, so is also best possible.

The key observation in the above is that M has no cocircuits of size b or greater. Seymour is credited in [34] with the following structure theorem for matroids of this sort, which implies that a matroid with no $U_{1,b}$-minor is sparse:

Theorem 5.5.3. *If M is a connected matroid, and C is a largest cocircuit of M, then the every cocircuit of $M \backslash C$ has size less than $|C|$.*

5.6 Covering Number

We now return to the parameter τ_a in more detail, proving Theorem 5.4.1, and thus showing that τ_a is bounded as a function of rank in a class $\mathcal{M}$, if and only if $\mathcal{M}$ does not contain all rank-$(a+1)$ uniform matroids. This verifies that classes satisfying outcome 4 of the Growth Rate Conjecture comprise exactly the infinitely dense classes. To this end, we write $\mathcal{U}(a,b)$ for the class of matroids with no $U_{a+1,b}$-minor. (This does not extend the notation $\mathcal{U}(\ell)$ of Part I elegantly, as $\mathcal{U}(1,b) = \mathcal{U}(b-2)$, but it helps to keep things concise in what follows). We first deal with the base case separately:

Lemma 5.6.1. *If a and b are integers with $1 \leq a < b$, and $M \in \mathcal{U}(a,b)$ is a rank-$(a+1)$ matroid, then $\tau_a(M) \leq \binom{b-1}{a}$.*

Proof. If B is a basis of M, then $M|B \cong U_{a+1,a+1}$; let $S \subseteq E(M)$ be maximal such that $M|S \cong U_{a+1,|S|}$; we may assume that $|S| < b$. By maximality of S, every $e \in E(M) - S$ is in a flat of M spanned by at most a elements of S. Taking all such flats gives a cover of M of size at most $\binom{|S|}{a} \leq \binom{b-1}{a}$. $\square$

We can now restate and prove the theorem:

Theorem 5.6.2. *Let a and b be integers with $1 \leq a < b$. If $M \in \mathcal{U}(a,b)$ is a matroid of rank at least $a+1$, then $\tau_a(M) \leq \binom{b-1}{a}^{r(M)-a}$.*

Proof. If $r(M) = a+1$, then the theorem follows from Lemma 5.6.1; we may suppose that $r(M) > a+1$, and inductively assume that the result holds for matroids of smaller rank. Let $e \in E(M)$. We have $\tau_{a+1}(M) \leq \tau_a(M/\!\!/e) \leq \binom{b-1}{a}^{r(M)-a-1}$ by induction, and by Lemma 5.6.1, each rank-$(a+1)$ set in M admits a cover with at most $\binom{b-1}{a}$ sets of rank at most a. Therefore $\tau_a(M) \leq \binom{b-1}{a}\tau_{a+1}(M) \leq \binom{b-1}{a}^{r(M)-a}$, as required. $\square$

When $a = 1$, the class $\mathcal{U}(a,b)$ just excludes some simple rank-2 matroid; this theorem can thus be compared to Theorem 1.3.1 - since $\mathcal{U}(\ell) = \mathcal{U}(1, \ell+2)$ the statement is slightly weaker than Kung's. For $a \geq 2$, the huge exponential upper bound provided by this result is much larger than necessary, as is shown by Theorem 5.4.3. However, it does give us a finite/infinite dichotomy:

Corollary 5.6.3. *Let $\mathcal{M}$ be a minor-closed class of matroids, and $a \geq 1$ be an integer. Either:*

1. *There is some $b > a$ such that $\mathcal{M} \subseteq \mathcal{U}(a,b)$, and $\tau_a(M) \leq \binom{b-1}{a}^{r(M)-a}$ for all $M \in \mathcal{M}$, or*
2. *$\mathcal{M}$ contains all rank-$(a+1)$ uniform matroids.*

Finally, we remark that we have not defined an analogue of the growth rate function $h_{\mathcal{M}}$ for the density measure τ_a. We could certainly do this, and rephrase the preceding theorems and conjectures accordingly. However, the 'precision' of the growth rate function, which allowed us to often determine its exact value, appears to be lost for this more general measure (even computing $\tau_a(\mathrm{PG}(n-1,q))$ is nontrivial), and we do not see how to obtain generalisations to τ_a of exact theorems such as those of Chapters 3 and 4.

Chapter 6

Uniform Minors I

In order to prove Theorem 3.2.1, which divided minor-closed classes excluding a line into the exponentially dense and the polynomially dense, Geelen and Kabell proved the following two similar theorems:

Theorem 6.0.4. *There is an integer-valued function $f_{6.0.4}(\ell, n)$ so that, for any integers $\ell \geq 2$ and $n \geq 2$, if $M \in \mathcal{U}(\ell)$ is a matroid such that $\varepsilon(M) \geq r(M)^{f_{6.0.4}(\ell,n)}$, then M has a* $\mathrm{PG}(n-1, q)$*-minor for some prime power q.*

Theorem 6.0.5. *There is a real-valued function $\alpha_{6.0.5}(\ell, n, q)$ so that, for any integers $q \geq 2$, $\ell \geq 2$ and $n \geq 1$, if $M \in \mathcal{U}(\ell)$ is a matroid such that $\varepsilon(M) \geq \alpha_{6.0.5}(\ell, n, q)q^{r(M)}$, then M has a* $\mathrm{PG}(n-1, q')$*-minor for some prime power $q' > q$.*

Analogously, we will prove the following two results about the rank-a covering number τ_a, which together will imply Theorem 5.4.3, in this chapter and the next:

Theorem 6.0.6. *There is an integer-valued function $f_{6.0.6}(a, b, n)$ so that, for any integers $1 \leq a < b$ and $n \geq 2$, if $M \in \mathcal{U}(a, b)$ is a matroid such that $\tau_a(M) \geq r(M)^{f_{6.0.6}(a,b,n)}$, then M has a* $\mathrm{PG}(n-1, q)$*-minor for some prime power q.*

Theorem 6.0.7. *There is a real-valued function $\alpha_{6.0.7}(a, b, n, q)$ so that, for any integers $1 \leq a < b$ and $n \geq 1$ and $q \geq 1$, if $M \in \mathcal{U}(a, b)$ is a matroid satisfying $\tau_a(M) \geq \alpha_{6.0.7}(a, b, n, q)q^{r(M)}$, then M has a* $\mathrm{PG}(n-1, q')$*-minor for some prime power $q' > q$.*

The goal of this chapter is to prove the following partial result:

Theorem 6.0.8. *There is an integer-valued function $f_{6.0.8}(a, b, n, q)$ so that, for any integers $1 \leq a < b$, $q \geq 1$ and $n \geq 1$, if $M \in \mathcal{U}(a, b)$ is a matroid such that $r(M) > 1$ and $\tau_a(M) \geq r(M)^{f_{6.0.8}(a,b,n,q)}q^{r(M)}$, then M has a* $\mathrm{PG}(n-1, q')$*-minor for some prime power $q' > q$.*

While this theorem is slightly weaker than Theorem 6.0.7, letting $q = 1$ gives Theorem 6.0.6. This chapter's first five sections are used to define the terminology and intermediate structures we need, and the bulk of the argument rests on the lemmas in Sections 6.6, 6.7 and 6.8.

6.1 Preliminaries

For a matroid M, in our proof we will often consider collections of subsets of $E(M)$ rather than just the subsets themselves. To this end, we define some new notation.

A common object is a collection of sets of the same rank. If M is a matroid, and $a \geq 1$ is an integer, then $\mathcal{R}_a(M)$ denotes the set $\{X \subseteq E(M) : r_M(X) = a\}$.

Generalising the notion of parallel elements, if $X, X' \subseteq E(M)$, then we write $X \equiv_M X'$ if $\mathrm{cl}_M(X) = \mathrm{cl}_M(X')$; we say that X and X' are *similar* in M. We write $[X]_M = \{X' \subseteq E(M) : X \equiv_M X'\}$ for the 'similarity class' of X in M.

We also extend existing notation in straightforward ways. If $\mathcal{X} \subseteq 2^{E(M)}$, then we write $\mathrm{cl}_M(\mathcal{X})$ for $\mathrm{cl}_M(\bigcup_{X \in \mathcal{X}} X)$, and $r_M(\mathcal{X})$ for $r_M(\mathrm{cl}_M(\mathcal{X}))$. Two sets $\mathcal{X}, \mathcal{X}' \subseteq 2^{E(M)}$ are *similar* in M if $\mathrm{cl}_M(\mathcal{X}) = \mathrm{cl}_M(\mathcal{X}')$.

Analogously to the notion of a simple matroid, we say that $\mathcal{X} \subseteq 2^{E(M)}$ is *simple* in M if the sets in $\mathcal{X}$ are pairwise dissimilar in M. Note that any collection of flats of M is simple. We write $\varepsilon_M(\mathcal{X})$ for the maximum size of a subset of $\mathcal{X}$ that is simple in M, or equivalently the number of different similarity classes of $2^{E(M)}$ containing a set in $\mathcal{X}$. If $\mathcal{X}$ just contains singletons, then $\varepsilon_M(\mathcal{X})$ agrees (up to punctuation) with the parameter ε of Part I.

We also restate Theorem 5.6.2, and two simple corollaries concerning the density of matroids in $\mathcal{U}(a,b)$ relative to that of their minors:

Theorem 5.6.2. *Let a and b be integers with $1 \leq a < b$. If $M \in \mathcal{U}(a,b)$ is a matroid of rank at least $a+1$, then $\tau_a(M) \leq \binom{b-1}{a}^{r(M)-a}$.*

Lemma 6.1.1. *Let a and b be integers with $1 \leq a < b$. If $M \in \mathcal{U}(\ell)$ is a matroid, and $C \subseteq E(M)$, then $\tau_a(M) \leq \binom{b-1}{a}^{r_M(C)} \tau_a(M /\!\!/ C)$.*

Lemma 6.1.2. *Let a and b be integers with $1 \leq a < b$. If $M \in \mathcal{U}(\ell)$ is a matroid, and N is a minor of N, then $\tau_a(M) \leq \binom{b-1}{a}^{r(M)-r(N)} \tau_a(N)$.*

6.2 Thickness and Firmness

Two density-related notions that will occur frequently in our proof are those of *thickness* and *firmness*, which we define and explain in this section.

Thickness is a notion we saw briefly in Chapter 3, and we rephrase it here in terms of covering number; if $d \geq 1$ is an integer, and M is a matroid, then M is *d-thick* if $\tau_{r(M)-1}(M) \geq d$. A set $X \subseteq E(M)$ is *d-thick in M* if $M|X$ is d-thick.

Note that every matroid is 2-thick, and that thickness is monotone in the sense that if $d' \geq d$ and M is d'-thick, then M is d-thick. The following lemma, which generalises Lemma 3.3.1, is fundamental, and we use it freely and frequently in our proof.

Lemma 6.2.1. *Let $d \geq 1$ be an integer. If M is a matroid, N is a minor of M, and $X \subseteq E(N)$ is d-thick in M, then X is d-thick in N.*

Proof. Deleting an element of M outside X, or contracting an element outside $\mathrm{cl}_M(X)$ does not change $M|X$, so it suffices to show that contracting a nonloop $e \in \mathrm{cl}_M(X)$ does not destroy d-thickness of X. This follows from the fact that $\tau_{r(M)-2}(M /\!\!/ e) \geq \tau_{r(M)-1}(M)$. $\square$

Any rank-1 or rank-0 matroid is clearly arbitrarily thick. Convenient examples of thick matroids are uniform matroids - no rank-a set in the matroid $U_{a+1,b}$ contains more than a elements, so $U_{a+1,b}$ is $\lceil \frac{b}{a} \rceil$-thick. Indeed, in sufficiently thick matroids, uniform minors abound:

Lemma 6.2.2. *Let a and b be integers with $1 \leq a < b$. If M is $\binom{b}{a}$-thick and $r(M) > a$, then M has a $U_{a+1,b}$-minor.*

Proof. By Lemma 6.2.1, d-thickness of M is preserved by contraction, so by contracting points if needed, we may assume that $r(M) = a + 1$. Now, $\binom{b-1}{a} < \binom{b}{a} \leq \tau_a(M)$, so the result follows from Theorem 5.6.1. $\square$

This lemma tells us that, qualitatively, searching for a $U_{a+1,b}$-minor is equivalent to searching for an appropriately thick minor of rank greater than a. We take this approach hereon; in fact, nearly all the uniform minors we find will be constructed by implicit use of this lemma.

We now turn to a definition of firmness. If $d \geq 1$ is an integer and M is a matroid, then a set $\mathcal{X} \subseteq 2^{E(M)}$ is *d-firm* in M if all $\mathcal{X}' \subseteq \mathcal{X}$ with $|\mathcal{X}'| > d^{-1}|\mathcal{X}|$ satisfy $r_M(\mathcal{X}') = r_M(\mathcal{X})$.

Firmness is a measure of how 'evenly spread' a collection of sets is. The set of points in a d-point line is d-firm; more generally, the set of a-subsets of $E(U_{a+1,b})$ is $\binom{b}{a}$-firm. Firmness is clearly monotone in the sense that d-firmness implies $(d-1)$-firmness.

Our first lemma relates firmness to thickness in an exact way:

Lemma 6.2.3. *Let $a \geq 1$ and $d \geq 1$ be integers, and M be a matroid. If $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is d-firm in M, and each $X \in \mathcal{X}$ is d-thick in M, then $\mathrm{cl}_M(\mathcal{X})$ is d-thick in M.*

Proof. Let $\mathcal{F}$ be a cover of $M|\,\mathrm{cl}_M(\mathcal{X})$ with flats of smaller rank; we wish to show that $|\mathcal{F}| \geq d$. If a set $X \in \mathcal{X}$ is not contained in any flats in $\mathcal{F}$, then $\{X \cap F : F \in \mathcal{F}\}$ is a cover of $M|X$ with sets of smaller rank, of size at most $|\mathcal{F}|$, so $|\mathcal{F}| \geq d$ by d-thickness of X. We may therefore assume every $X \in \mathcal{X}$ is contained in some $F \in \mathcal{F}$. Now, since $\mathcal{X}$ is d-firm in M and no flat in $\mathcal{F}$ is spanning in $M|\,\mathrm{cl}_M(\mathcal{X})$, each flat in $\mathcal{F}$ contains at most $d^{-1}|\mathcal{X}|$ different sets in $\mathcal{X}$. We thus have $|\mathcal{F}| \geq \frac{|\mathcal{X}|}{d^{-1}|\mathcal{X}|} = d$, as required. $\square$

We will use this lemma to construct the thick sets of rank greater than a that we are frequently seeking. Thus, we often consider a set $\mathcal{X} \subseteq \mathcal{R}_a(M)$ that has no firm subset of rank $> a$ in a minor of M; we are 'excluding' a minor with this structure from $\mathcal{X}$ and M in lieu of excluding $U_{a+1,b}$.

This exclusion allows us to control the number of sets in $\mathcal{X}$ in useful ways; the first of the next two lemmas tells us about the 'absolute' density of $\mathcal{X}$ in M, and the second about the 'relative' density of $\mathcal{X}$ in M as compared to in a minor of M. These results can be compared to Theorems 5.6.2 and 6.1.2.

Lemma 6.2.4. *Let $a \geq 1$ and $d \geq 2$ be integers, M be a matroid with $r(M) > a$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$. If $\varepsilon_M(\mathcal{X}) \geq d^{r(M)-a}$, then there is a set $\mathcal{Y} \subseteq \mathcal{X}$ such that $r_M(\mathcal{Y}) > a$, and $\mathcal{Y}$ is d-firm in M.*

Proof. We may assume that $\mathcal{X}$ is simple. If $r(M) = a + 1$, then the union of any two sets in $\mathcal{X}$ is spanning in M, and $|\mathcal{X}| \geq d$, so $\mathcal{X}$ is d-firm; we assume that $r(M) > a + 1$, and proceed by induction on $r(M)$. If $\mathcal{X}$ is not d-firm, then there is some $\mathcal{X}' \subseteq \mathcal{X}$ with $r_M(\mathcal{X}') < r_M(\mathcal{X})$, and $|\mathcal{X}'| \geq d^{-1}|\mathcal{X}| \geq d^{r(M)-1-a} \geq d^{r_M(\mathcal{X}')-a}$. Moreover, $|\mathcal{X}'| \geq d^{r(M)-1-a} \geq d \geq 2$, so $r_M(\mathcal{X}') > a$. The result follows by applying the inductive hypothesis to $\mathcal{X}'$ in $M|\operatorname{cl}_M(\mathcal{X}')$. $\square$

Lemma 6.2.5. *Let $a \geq 1$ and $d \geq 2$ be integers. If M is a matroid, N is a minor of M, and $\mathcal{X} \subseteq \mathcal{R}_a(M) \cap \mathcal{R}_a(N)$. If $\varepsilon_M(\mathcal{X}) > d^{r(M)-r(N)}\varepsilon_N(\mathcal{X})$, then there is a set $\mathcal{Y} \subseteq \mathcal{X}$ such that $r_M(\mathcal{Y}) > a$, and $\mathcal{Y}$ is d-firm in M.*

Proof. Let $N = M /\!\!/ C \backslash D$, where $r_M(C) = r(M)-r(N)$. Suppose that $\varepsilon_M(\mathcal{X}) > d^{r_M(C)}\varepsilon_N(\mathcal{X})$. By a majority argument applied to the similarity classes of $\mathcal{X}$ in N, there is some $X \in \mathcal{X}$ such that $\varepsilon_M([X]_N \cap \mathcal{X}) \geq d^{r_M(C)} = d^{r_M(X \cup C)-a}$. Now, every set in $[X]_N \cap \mathcal{X}$ is contained in $\operatorname{cl}_M(X \cup C)$, so applying Lemma 6.2.4 to $M|(\operatorname{cl}_M(X \cup C))$ gives the result. $\square$

6.3 Arrangements

We prove two lemmas related to how collections of sets in a matroid 'fit together'.

This first lemma shows that, given $\mathcal{X} \subseteq \mathcal{R}_a(M)$, we can contract a point of M so that the rank of most sets in $\mathcal{X}$ is unchanged:

Lemma 6.3.1. *Let M be a matroid, $a \geq 1$ be an integer, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$. There exists a nonloop $e \in E(M)$ so that*

$$\varepsilon_M(\mathcal{X} \cap \mathcal{R}_a(M /\!\!/ e)) \geq \left(1 - \tfrac{a}{r(M)}\right) \varepsilon_M(\mathcal{X}).$$

Proof. Let $\mathcal{X}'$ be a maximal simple subset of $\mathcal{X}$, and B be a basis of M. Each set in $\mathcal{X}'$ has at most a elements of B in its closure, so $\sum_{f \in B} |\{X \in \mathcal{X}' : f \in \mathrm{cl}_M(X)\}| \leq a|\mathcal{X}'|$. There is therefore some $e \in B$ such that $|\{X \in \mathcal{X}' : e \in \mathrm{cl}_M(X)\}| \leq \frac{a}{|B|}|\mathcal{X}'|$. Every set in $\mathcal{X}'$ that does not span e is in $\mathcal{R}_a(M /\!\!/ e)$, so

$$\begin{aligned}\varepsilon_M(\mathcal{X} \cap \mathcal{R}_a(M /\!\!/ e)) &\geq |\mathcal{X}' \cap \mathcal{R}_a(M /\!\!/ e)| \\ &\geq |\mathcal{X}'| - \tfrac{a}{|B|}|\mathcal{X}'| \\ &= \varepsilon_M(\mathcal{X}) - \tfrac{a}{r(M)}\varepsilon_M(\mathcal{X}),\end{aligned}$$

and the result follows. □

This second lemma is a generalisation of the fact that a graph with many edges contains either a vertex of large degree or a large matching. Recall that a set $\mathcal{W} \subseteq 2^{E(M)}$ is *mutually skew* in a matroid M if $r_M(\bigcup_{W \in \mathcal{W}} W) = \sum_{W \in \mathcal{W}} r_M(W)$.

Lemma 6.3.2. *Let M be a matroid, $a \geq 1$ and $t \geq 1$ be integers, and let $\mathcal{X} \subseteq \mathcal{R}_a(M)$. Either*

(i) There exists $\mathcal{W} \subseteq \mathcal{X}$ so that $|\mathcal{W}| = t$, and $\mathcal{W}$ is mutually skew in M, or

(ii) There is a minor N of M with $r(N) \geq r(M) - at$, a set $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{R}_a(N)$ such that $|\mathcal{Y}| \geq (at)^{-1}|\mathcal{X}|$, and some nonloop $e \in E(N)$ such that $e \in \mathrm{cl}_N(Y)$ for all $Y \in \mathcal{Y}$.

Proof. Let $\mathcal{W}$ be a maximal mutually skew subset of $\mathcal{X}$; we may assume that $k < t$. Let $e_1, \ldots, e_{a|\mathcal{W}|}$ be a basis for $\bigcup_{W \in \mathcal{W}} W$. For each $1 \leq i \leq a|\mathcal{W}|$, let $M_i = M /\!\!/ \{e_1, \ldots, e_i\}$. By maximality of $\mathcal{W}$, each $X \in \mathcal{X} - \mathcal{W}$ satisfies $r_{M_{a|\mathcal{W}|}}(X) < r_M(X) = a$, and this inequality clearly also holds for all $X \in \mathcal{W}$, so for each $X \in \mathcal{X}$ there is some i_X such that $r_{M_{i_X}-1}(X) = a$ and $r_{M_{i_X}}(X) = a-1$. By a majority argument, there is some $1 \leq i_0 \leq a|\mathcal{W}|$ and $\mathcal{Y} \subseteq \mathcal{X}$ such that $|\mathcal{Y}| \geq (a|\mathcal{W}|)^{-1}|\mathcal{X}|$, and $i_Y = i_0$ for all $Y \in \mathcal{Y}$. Since $|\mathcal{W}| < t$, the minor $N = M/\{e_1, \ldots, e_{i_0-1}\}$, along with $\mathcal{Y}$ and e_{i_0}, will satisfy the second outcome. □

6.4 Weighted Covers and Scatteredness

Our main theorems concern upper bounds on the parameter τ_a. It is therefore natural to consider minimum-sized covers of a matroid with sets of rank at most a. However, such a cover has few useful properties, and it seems difficult to make use of one in a proof. We will therefore change the parameter we are considering to one that considers minimal 'weighted' covers. This tweak will force a minimal cover to have many properties that we exploit at length.

If M is a matroid, and $\mathcal{X}, \mathcal{F} \subseteq 2^{E(M)}$, then $\mathcal{F}$ is a *cover of* $\mathcal{X}$ in M if every set in $\mathcal{X}$ is contained in a set in $\mathcal{F}$. A *cover of* M is a cover of $\{\{e\} : e \in E(M)\}$.

If $d \geq 1$ is an integer, and $\mathcal{F} \subseteq 2^{E(M)}$, then we write $\mathrm{wt}_M^d(\mathcal{F})$ for the sum $\sum_{F \in \mathcal{F}} d^{r_M(F)}$. Thus, the 'cost' of a point in $\mathcal{F}$ is d, the 'cost' of a line is d^2, etc. $\mathcal{F}$ is a *d-minimal cover of* $\mathcal{X}$ if $\mathcal{F}$ minimizes $\mathrm{wt}_M^d(\mathcal{F})$ subject to being a cover of $\mathcal{X}$. We write $\tau^d(M)$ for the weight of a d-minimal cover of M. The parameter τ^d will not change too dramatically in a minor:

Lemma 6.4.1. *Let d be a positive integer. If M is a matroid and N is a minor of M, then $\tau^d(N) \geq d^{r(N)-r(M)}\tau^d(M)$.*

Proof. It suffices to show that, for a nonloop $e \in E(M)$, we have $\tau^d(M /\!\!/ e) \geq d^{-1}\tau^d(M)$. If $\mathcal{F}$ is a d-minimal cover of $M /\!\!/ e$, then $\mathcal{F}' = \{\mathrm{cl}_M(F \cup \{e\}) : F \in \mathcal{F}\}$ is a cover of M, so $\tau^d(M) \leq \mathrm{wt}_M^d(\mathcal{F}') = \sum_{F \in \mathcal{F}} d^{r_M(F \cup \{e\})} = \sum_{F \in \mathcal{F}} d^{r_{M /\!\!/ e}(F)+1} = d\,\mathrm{wt}_{M /\!\!/ e}^d(\mathcal{F}) = d\tau^d(M /\!\!/ e)$, giving the result. □

A concept that we will soon use to build a highly structured minor is that of *scatteredness*, another measure of how 'spread out' a collection of sets is. A set $\mathcal{X} \subseteq 2^{E(M)}$ is *d-scattered* in a matroid M if all sets in $\mathcal{X}$ are d-thick in M, and $\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}$ is a d-minimal cover of $\mathcal{X}$ in M.

A scattered set 'refuses' to be efficiently covered with sets of larger rank. Again, we use the symbol d; this same parameter will be passed around the proof of Theorem 6.0.8 in measures of thickness, firmness and scatteredness.

Our first lemma establishes some nice properties in the case where a minimal cover of a set $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is just the ground set of M:

Lemma 6.4.2. *Let $a \geq 1$ and $d \geq 1$ be integers, M be a matroid with $r(M) > a$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$. If all sets in $\mathcal{X}$ are d-thick in M, and $\{E(M)\}$ is a d-minimal cover of $\mathcal{X}$ in M, then $\varepsilon_M(\mathcal{X}) \geq d^{r(M)-a}$, and M is d-thick.*

Proof. $\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}$ is a cover of $\mathcal{X}$ in M; since $\{E(M)\}$ is a d-minimal cover of $\mathcal{X}$, we have $\mathrm{wt}^d_M(\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}) \geq \mathrm{wt}^d_M(\{E(M)\})$, so $d^a \varepsilon_M(\mathcal{X}) \geq d^{r(M)}$, giving the first part of the lemma.

We will now show that M is d-thick. Let $\mathcal{F}$ be a cover of M with flats of smaller rank. If some $X \in \mathcal{X}$ is not contained in F for any set F in $\mathcal{F}$, then $\{X \cap F : F \in \mathcal{F}\}$ is a cover of $M|X$ of size at most $|\mathcal{F}|$ with sets of smaller rank than X, so $|\mathcal{F}| \geq d$ by d-thickness of X. Otherwise, $\mathcal{F}$ is a $\mathcal{X}$-cover, so $\mathrm{wt}^d_M(\mathcal{F}) \geq \mathrm{wt}^d_M(\{E(M)\})$. Therefore $|\mathcal{F}| d^{r(M)-1} \geq d^{r(M)}$, so $|\mathcal{F}| \geq d$. $\square$

Our means of constructing scattered sets is the following lemma:.

Lemma 6.4.3. *Let $d \geq 1$ and $a \geq 1$ be integers, M be a matroid, and $\mathcal{X} \subseteq 2^{E(M)}$. If all sets in $\mathcal{X}$ are d-thick in M, and $\mathcal{F}$ is a d-minimal cover of $\mathcal{X}$ in M, then every subset of $\mathcal{F}$ is d-scattered in M.*

Proof. Let $\mathcal{F}' \subseteq \mathcal{F}$. It is clear from d-minimality of $\mathcal{F}$ that $\mathcal{F}'$ is simple, and that $\mathcal{F}'$ is a d-minimal cover of $\mathcal{F}'$. For each $F \in \mathcal{F}'$, the set $\{F\}$ is a d-minimal cover of $\{X \in \mathcal{X} : X \subseteq F\}$ by d-minimality of $\mathcal{F}$, so by applying Lemma 6.4.2 to $M|F$, we see that F is d-thick in M. Therefore $\mathcal{F}'$ is d-scattered in M. $\square$

In particular, if $\mathcal{F}$ is a d-minimal cover of M itself, then every subset of $\mathcal{F}$ is d-scattered in M, as the singleton $\{e\}$ is d-thick in M for any $e \in E(M)$.

Lemma 6.4.4. *Let $a \geq 1$ and $d \geq 1$ be integers. If M is a matroid, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is d-scattered in M, then $\varepsilon_M(\mathcal{X}) \leq d^{r(M)-a}$.*

Proof. $\{E(M)\}$ is a cover of $\mathcal{X}$ in M, so d-scatteredness of $\mathcal{X}$ gives $d^a \varepsilon_M(\mathcal{X}) = \mathrm{wt}^d_M(\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}) \leq \mathrm{wt}^d_M(\{E(M)\}) = d^{r(M)}$, giving the result. $\square$

The parameter τ^d, for an appropriate d, is what we use to gain traction towards Theorem 6.0.8. Considering this parameter instead of τ_a is not a major change in the setting of excluding $U_{a+1,b}$; indeed, these two parameters differ by at most a constant factor.

Lemma 6.4.5. *If a, b, d are integers with $1 \leq a < b$ and $d \geq \binom{b}{a}$, and $M \in \mathcal{U}(a,b)$ is a matroid, then no d-minimal cover of M contains a set of rank greater than a, and $\tau_a(M) \leq \tau^d(M) \leq d^a \tau_a(M)$.*

Proof. Let $\mathcal{F}$ be a d-minimal cover of M. By Lemma 6.4.3, every set in $\mathcal{F}$ is d-thick, so by Lemma 6.2.2 and definition of d, there is no set of rank greater than a in $\mathcal{F}$. Therefore $\tau_a(M) \leq |\mathcal{F}| \leq \mathrm{wt}^d_M(\mathcal{F}) = \tau^d(M)$. Moreover, if $\mathcal{H}$ is a minimum-sized cover of M with sets of rank at most a, then $\tau^d(M) \leq d^a |\mathcal{H}| = d^a \tau_a(M)$. $\square$

6.5 Pyramids

We now define the intermediate structure that is vital to our proof.

Let $a \geq 1$ and $h \geq 0$ be integers, M be a matroid, $\mathcal{S} \subseteq \mathcal{R}_a(M)$, and $\{e_1, \ldots, e_h\}$ be an independent set of size h in M. For each $0 \leq i \leq h$, let $M_i = M /\!\!/ \{e_1, \ldots, e_i\}$.

We say $(M, \mathcal{S}; e_1, \ldots, e_h)$ is an (a, q, h, d)-*pyramid* if

- $\mathcal{S} \neq \varnothing$, and S is skew to $\{e_1, \ldots, e_h\}$ for all $S \in \mathcal{S}$, and
- for each $0 \leq i < h$, and $S \in \mathcal{S}$, there are sets $S_1, \ldots, S_q \in \mathcal{S}$, pairwise dissimilar in M_i, and each similar to S in M_{i+1}. amd
- S is d-thick in M for all $S \in \mathcal{S}$.

A pyramid is a highly structured, exponential-sized collection of thick sets of rank a. For each $0 \leq i < h$, and each $S \in \mathcal{S}$, contracting e_{i+1} in M_i turn 'collapses' the dissimilar d-thick sets $S_1, \ldots, S_q$ onto the single d-thick set S in M_{i+1}. A geometric view of M_i, and a generic set in $\mathcal{S}$, is shown in Figure 6.1.

When $a = 1$, the set $\mathcal{S}$ simply contains points; in this case, the value of d is irrelevant, and the structure described in the second condition is a set of q other points on a line through e_{i+1}. In fact, the minor N constructed in the proof of Theorem 1.3.4 is a $(1, 2, n, 1)$-pyramid. Pyramids are based on objects of the same name introduced by Geelen and Kabell in [14]; a pyramid in their sense is a special sort of pyramid in our sense, with $a = 1$.

The structure of a pyramid is highly self-similar, and the next two easily proved lemmas concern smaller pyramids inside a pyramid:

Lemma 6.5.1. *Let a, q, h, d be positive integers. If $(M, \mathcal{S}; e_1, \ldots, e_h)$ is an (a, q, h, d)-pyramid, and i and j are integers with $0 \leq i \leq j \leq h$, then*

$$(M /\!\!/ \{e_{i+1}, \ldots, e_j\}, \mathcal{S}; e_1, \ldots, e_i, e_{j+1}, \ldots, e_h)$$

is an $(a, q, h - (j - i), d)$-pyramid.

Lemma 6.5.2. *Let $(M, \mathcal{S}; e_1, \ldots, e_h)$ be an (a, q, h, d)-pyramid, and let N be a minor of $M /\!\!/ \{e_1, \ldots, e_h\}$. If $\mathcal{Y} \subseteq \mathcal{S} \cap \mathcal{R}_a(N)$, then there is a minor M' of M, and an (a, q, h, d)-pyramid $(M', \mathcal{S}'; e_1, \ldots, e_h)$ so that $\mathcal{Y} \subseteq \mathcal{S}' \subseteq \mathcal{S}$, and $N|\mathcal{Y} = (M' /\!\!/ \{e_1, \ldots, e_h\})|\mathcal{Y}$.*

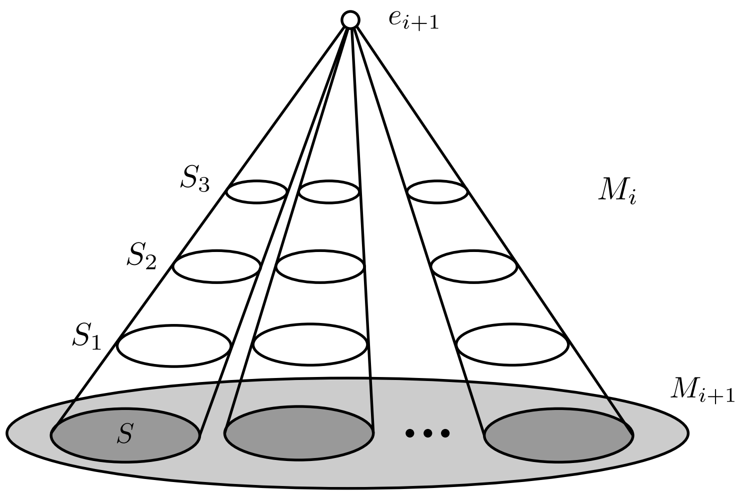

Figure 6.1: A geometric illustration of M_i for each i, in a pyramid with $q = 3$. The sets S_1, S_2 and S_3 are pairwise dissimilar in M_i, but each is similar to S in $M_{i+1} = M_i /\!\!/ e_{i+1}$.

The next lemma is our means of adding a 'level' to a pyramid. In accordance with the definition, it requires a point e and a smaller pyramid on $M/\!\!/e$ such that e 'lifts' each set in the pyramid into $q+1$ distinct sets. The proof, which we omit, is cumbersome but routine.

Lemma 6.5.3. *Let M be a matroid, $e \in E(M)$ be a nonloop, a, d, q, h be integers with $q, a, d \geq 1$ and $h \geq 0$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ be simple in M. Let $\mathcal{X}_{>q} = \{X \in \mathcal{X} : |[X]_{M/\!\!/e} \cap \mathcal{X}| > q\}$. If $M/\!\!/e$ has an $(a, q+1, h, d)$-pyramid minor P such that $\mathcal{S}_P \subseteq \mathcal{X}_{>q}$, then M has an $(a, q+1, h+1, d)$-pyramid minor P' such that $\mathcal{S}_{P'} \subseteq \mathcal{X}$.*

The following lemma shows that a pyramid can be restricted to have bounded rank:

Lemma 6.5.4. *Let $(M, \mathcal{S}; e_1, \ldots, e_h)$ be an (a, q, h, d)-pyramid, and $M_h = M/\!\!/\{e_1, \ldots, e_h\}$, and $S \in \mathcal{S}$. Then there is a restriction M' of M such that*

$$(M', \{S' \in \mathcal{S} : S' \equiv_{M_h} S\}; e_1, \ldots, e_h)$$

is an (a, q, h, d)-pyramid, and $r(M') = a + h$.

Proof. Let $M' = M|\operatorname{cl}_M(S \cup \{e_1, \ldots, e_h\})$, and $\mathcal{S}' = \{S' \in \mathcal{S} : S' \equiv_{M_h} S\}$. Since $r_{M_h}(S) = a$, we have $r(M') = a + h$. Let $0 \leq i < h$. Let $S' \in \mathcal{S}'$, and $S'_1, \ldots, S'_q$ be the sets for i and

S' as given by the definition of a pyramid. Each S'_j is similar to S' in M_i, and therefore also in M_h, so $\{S'_1, \dots, S'_q\} \subseteq \mathcal{S}'$, and $(S'_1 \cup \dots \cup S'_q) \subseteq E(M')$. Therefore, $(M', \mathcal{S}'; e_1, \dots, e_h)$ is an (a, q, h, d)-pyramid. $\square$

Our penultimate lemma verifies the set $\mathcal{S}$ in a pyramid has exponentially many elements:

Lemma 6.5.5. *If $(M, \mathcal{S}; e_1, \dots, e_h)$ is an (a, q, h, d)-pyramid, then $\varepsilon_M(\mathcal{S}) \geq q^h \varepsilon_{M_h}(\mathcal{S})$.*

Proof. When $h = 0$, there is nothing to show. Otherwise, suppose that the result holds for a fixed h, and let $(M, \mathcal{S}; e_1, \dots, e_{h+1})$ be an $(a, q, h+1, d)$-pyramid. We know that $(M /\!\!/ e_1, \mathcal{S}; e_2, \dots, e_{h+1})$ is an (a, q, h, d)-pyramid; now, we have $\varepsilon_{M /\!\!/ e_1}(\mathcal{S}) \geq q^h \varepsilon_{M_{h+1}}(\mathcal{S})$ by the inductive hypothesis. Moreover, for each $S \in \mathcal{S}$, there are pairwise dissimilar sets $S_1, \dots, S_q \in \mathcal{S}$, each similar to S in $M /\!\!/ e_1$. Therefore $\varepsilon_M(\mathcal{S}) \geq q \varepsilon_{M /\!\!/ e_1}(\mathcal{S}) \geq q^{h+1} \varepsilon_{M_{h+1}}(\mathcal{S})$, so the lemma holds. $\square$

Finally, we observe that a pyramid has a restriction with bounded rank, containing an exponential-size subset of $\mathcal{S}$. This lemma follows routinely from Lemmas 6.5.1, 6.5.4 and 6.5.5.

Lemma 6.5.6. *If $(M, \mathcal{S}; e_1, \dots, e_h)$ is an (a, q, h, d)-pyramid, and $0 \leq h' \leq h$, then there is a rank-$(a + h')$ restriction M' of M, and a set $\mathcal{S}' \subseteq \mathcal{S}$, so that $(M', \mathcal{S}'; e_1, \dots, e_{h'})$ is an (a, q, h', d)-pyramid, and $\varepsilon_{M'}(\mathcal{S}') \geq q^{h'}$.*

6.6 Building a Pyramid

In this section, we show that a large d-scattered set allows us to either find a d-firm subset of large rank in a minor, or a large pyramid. The majority of this argument lies in an ugly technical lemma, which we will adapt into two useful corollaries. To understand this lemma, it may be helpful to read it where $a_0 = 1$ and $a = 2$; in this case, $\mathcal{X}$ is a dense d-scattered set of points; the first outcome corresponds to a d-point line minor whose points are in $\mathcal{X}$, the second to a $(1, q+1, h, d)$-pyramid minor, and the third to a minor containing a d-scattered collection of lines built from $\mathcal{X}$.

Lemma 6.6.1. *There is an integer-valued function $f_{6.6.1}(a, d, h, m)$ so that, for all integers a_0, a, d, h, q with $a \geq a_0 \geq 1$, $d \geq 1$, $q \geq 1$, $h \geq 0$, and $m \geq 0$, if M is a matroid with $r(M) \geq f_{6.6.1}(a, d, h, m)$, and a set $\mathcal{X} \subseteq \mathcal{R}_{a_0}(M)$ is d-scattered in M and satisfies $\varepsilon_M(\mathcal{X}) \geq r(M)^{f_{6.6.1}(a,d,h,m)} q^{r(M)}$, then either:*

(i) There is a minor N of M and a set $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{R}_a(N)$ so that $r_N(\mathcal{Y}) > a$, and $\varepsilon_N(\mathcal{Y}) \geq d^{r(N)-a}$, and $\mathrm{cl}_N(\mathcal{Y})$ is d-thick in N, or

(ii) M has an $(a_0, q+1, h, d)$-pyramid minor P with $\mathcal{S}_P \subseteq \mathcal{X}$, or

(iii) There exists an integer a_1 with $a_0 < a_1 \leq a$, a minor M' of M with $r(M') \geq m$, and a set $\mathcal{X}' \subseteq \mathcal{R}_{a_1}(M')$ so that $\mathcal{X}'$ is d-scattered in M', and $\varepsilon_{M'}(\mathcal{X}') \geq r(M')^m q^{r(M')}$.

Proof. Let a_0, a, d, h and q be positive integers such that $a \geq a_0$, and let $m \geq 0$ be an integer. Let $p_0 = 0$, and for each $h > 0$, recursively define p_h to be an integer so that

$$d^{-1}q^r(r-1)^{p_h-1}\left(p_h - 3a(1+d^a)\right) \geq (r-1)^{p_{h-1}}q^{r-1},$$

for all integers $r \geq 2$, and so that $p_h \geq \max(2, d, m+1)$.

We will show for all h that if M is a matroid with $r(M) \geq p_h$, and a set $\mathcal{X} \subseteq \mathcal{R}_{a_0}(M)$ is d-scattered in M and satisfies $\varepsilon_M(\mathcal{X}) \geq r(M)^{p_h} q^{r(M)}$, then one of the three outcomes holds for M; thus, setting $f_{6.6.1}(a, d, h, m) = p_h$ will satisfy the lemma. Our proof is by induction on h. If $h = 0$, then, since $(M, \{X\};)$ is an $(a_0, q+1, 0, d)$-pyramid for any $X \in \mathcal{X}$, the outcome (ii) holds. Now, fix $h > 0$, and suppose that the result holds for smaller h. Let $p = p_h$, and M be minor-minimal so that $r(M) \geq p$, and there exists a d-scattered $\mathcal{X} \subseteq \mathcal{R}_{a_0}(M)$ such that $\varepsilon_{M'}(\mathcal{X}) \geq r(M)^p q^{r(M)}$. Let $r = r(M)$. If $r = p$, then $\varepsilon_M(\mathcal{X}) \geq p^p q^p > d^{p-a_0}$; this contradicts d-scatteredness of $\mathcal{X}$ by Lemma 6.4.4, so we may assume that $r > p$.

By Lemma 6.3.1, there is some $e \in E(M)$ so that $\varepsilon_M(\mathcal{X} \cap \mathcal{R}_{a_0}(M /\!\!/ e)) \geq \left(1 - \frac{a_0}{r}\right)\varepsilon_M(\mathcal{X})$. Let $\mathcal{X}' = \mathcal{X} \cap \mathcal{R}_{a_0}(M /\!\!/ e)$, and $\mathcal{F}$ be a d-minimal cover of $\mathcal{X}'$ in $M /\!\!/ e$ such that $|\mathcal{F}|$ is maximized. We may assume that all sets in $\mathcal{F}$ are flats of $M /\!\!/ e$. The set $\mathcal{F}$ is simple in $M /\!\!/ e$; for each $i \geq 1$, let $\mathcal{F}_i = \mathcal{F} \cap \mathcal{R}_i(M /\!\!/ e)$. We will henceforth assume that (i) and (iii) do not hold.

6.6.1.1. $\mathcal{F} = \bigcup_{a_0 \leq i \leq a} \mathcal{F}_i$.

Proof of claim: Every $F \in \mathcal{F}$ must contain a set in $\mathcal{X}'$, so $\mathcal{F}$ contains no set of rank less than a_0. If $\mathcal{F}$ contains a set F of rank greater than a, then $\{F\}$ is a d-minimal cover of $\{X \in \mathcal{X}' : X \subseteq F\}$ in $M /\!\!/ e$, so by Lemma 6.4.2, the matroid $(M /\!\!/ e)|F$ and the set $\{X \in \mathcal{X}' : X \subseteq F\}$ satisfy (i), a contradiction. $\square$

6.6.1.2. *There is a set $\mathcal{X}'' \subseteq \mathcal{X}'$ that is d-scattered in $M /\!\!/ e$, and satisfies $\varepsilon_M(\mathcal{X}'') \geq q^r\left(r^p - a(1+d^a)r^{p-1}\right)$*

Proof of claim: Each $X \in \mathcal{X}'$ is contained in some set in $\mathcal{F}$; for each $F \in \mathcal{F}$, let $\mathcal{X}_F = \{X \in \mathcal{X}' : X \subseteq F\}$. By Lemma 6.4.4, each $F \in \mathcal{F}$ satisfies $\varepsilon_M(\mathcal{X}_F) = \varepsilon_{M|F}(\mathcal{X}_F) \leq d^{r(M|F)-a_0} \leq d^{a+1-a_0} \leq d^a$. Moreover, each $\mathcal{F}_i$ is simple and d-scattered in $M /\!\!/ e$, so we may assume that $|\mathcal{F}_i| \leq r^m q^r$, as (iii) does not hold. Since $\mathcal{X}'$ is the union of the $\mathcal{X}_F$, we have

$$\begin{aligned}\sum_{F \in \mathcal{F}_{a_0}} (\varepsilon_M(\mathcal{X}_F)) &\geq \varepsilon_M(\mathcal{X}') - \sum_{\substack{a_0 < i \leq a \\ F \in \mathcal{F}_i}} \varepsilon_M(\mathcal{X}_F) \\ &\geq (1 - \tfrac{a_0}{r}) r^p q^r - d^a \sum_{a_0 < i \leq a} |\mathcal{F}_i| \\ &\geq (1 - \tfrac{a}{r}) r^p q^r - a d^a r^m q^r \\ &\geq q^r (r^p - a(1 + d^a) r^{p-1}),\end{aligned}$$

as $p - 1 \geq m$. Let $\mathcal{X}'' = \bigcup_{F \in \mathcal{F}_{a_0}} \mathcal{X}_F$. Now, since $\mathcal{F}_{a_0}$ is simple in $M /\!\!/ e$, and every set in $\mathcal{F}_{a_0}$ and every set in $\mathcal{X}'$ has rank a_0 in $M /\!\!/ e$, no set in $\mathcal{X}''$ is contained in two different sets in F_{a_0}. Therefore $\varepsilon_M(\mathcal{X}'') = \sum_{F \in \mathcal{F}_{a_0}} (\varepsilon_M(\mathcal{X}_F))$. Moreover, d-minimality of $\mathcal{F}$ implies that $\mathcal{F}_{a_0} = \{\mathrm{cl}_{M /\!\!/ e}(X) : X \in \mathcal{X}''\}$ is a d-minimal cover of $\mathcal{X}''$ in $M /\!\!/ e$. Therefore, $\mathcal{X}''$ is d-scattered in $M /\!\!/ e$, giving the claim. $\square$

This next claim completes the proof:

6.6.1.3. *M has an $(a_0, q+1, h, d)$-pyramid minor P, with $\mathcal{S}_P \subseteq \mathcal{X}$.*

Proof of claim: Let $\mathcal{Y}$ be a maximal subset of $\mathcal{X}''$ that is simple in M. Since $\mathcal{X}''$ is d-scattered in both M and $M /\!\!/ e$, so is $\mathcal{Y}$. We have $r(M /\!\!/ e) = r - 1 \geq p$, so minor-minimality of M gives $|\mathcal{Y}| = \varepsilon_{M /\!\!/ e}(\mathcal{Y}) < (r(M /\!\!/ e))^p q^{r(M /\!\!/ e)} = (r-1)^p q^{r-1}$. Let $\mathcal{Y}_{>q} = \{Y \in \mathcal{Y} : |[Y]_{M /\!\!/ e} \cap \mathcal{Y}| > q\}$, and $\mathcal{Y}_{\leq q} = \mathcal{Y} - \mathcal{Y}_{>q}$. Since $\mathcal{Y}$ is d-scattered and simple in M, Lemma 6.4.4 gives $|[Y]_{M /\!\!/ e} \cap \mathcal{Y}| = |\{Y' \in \mathcal{Y} : Y' \subseteq \mathrm{cl}_M(Y \cup \{e\})\}| \leq d^{(a_0+1)-a_0} = d$ for all $Y \in \mathcal{Y}$. Now,

$$\begin{aligned}q^r (r^p - a(1 + d^a) r^{p-1}) &\leq |\mathcal{Y}| \\ &= |\mathcal{Y}_{>q}| + |\mathcal{Y}_{\leq q}| \\ &\leq d\varepsilon_{M /\!\!/ e}(\mathcal{Y}_{>q}) + q\varepsilon_{M /\!\!/ e}(\mathcal{Y}_{\leq q}) \\ &\leq d\varepsilon_{M /\!\!/ e}(\mathcal{Y}_{>q}) + q\varepsilon_{M /\!\!/ e}(\mathcal{Y}) \\ &< d\varepsilon_{M /\!\!/ e}(\mathcal{Y}_{>q}) + q(r-1)^p q^{r-1}.\end{aligned}$$

Rearranging this inequality yields

$$\begin{aligned}\varepsilon_{M/\!\!/e}(\mathcal{Y}_{>q}) &\geq d^{-1}q^r(r^p - (r-1)^p - a(1+d^a)r^{p-1}) \\ &\geq d^{-1}q^r(p(r-1)^{p-1} - a(1+d^a)r^{p-1}) \\ &= d^{-1}q^r(r-1)^{p-1}\left(p - a(1+d^a)\left(\tfrac{r}{r-1}\right)^{p-1}\right).\end{aligned}$$

By hypothesis, $r \geq p$, so $\left(\frac{r}{r-1}\right)^{p-1} \leq \left(\frac{p}{p-1}\right)^{p-1} \leq 2.718\ldots < 3$. This gives

$$\begin{aligned}\varepsilon_{M/\!\!/e}(\mathcal{Y}_{>q}) &> d^{-1}q^r(r-1)^{p-1}(p - 3a(1+d^a)) \\ &\geq r(M/\!\!/e)^{p_{h-1}}q^{r(M/\!\!/e)}\end{aligned}$$

by definition of $p = p_h$. We may assume that outcomes (i) and (iii) both fail for $M/\!\!/e$ and $\mathcal{Y}_{>q}$; therefore, by induction on h, the matroid $M/\!\!/e$ has an $(a_0, q+1, h-1, d)$-pyramid minor P' with $\mathcal{S}_{P'} \subseteq \mathcal{Y}_{>q}$. By Lemma 6.5.3, M has an $(a_0, q+1, h, d)$-pyramid minor P with $\mathcal{S}_P \subseteq \mathcal{Y}_{>q} \subseteq \mathcal{X}$. $\square$

$\square$

Our first corollary, which will be used in the next section, finds a pyramid or a firm set of rank greater than a, starting with a collection of thick rank-a sets. The corollary is obtained by specialising to the case where $a = a_0$, thus rendering the third outcome impossible.

Corollary 6.6.2. *There is an integer-valued function $f_{6.6.2}(a, d, h)$ so that, for any integers a, d, h, q with $h \geq 0$, $a \geq 1$, $d \geq 2$ and $q \geq 1$, if M is a matroid such that $r(M) \geq f_{6.6.2}(a, d, h)$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is a set such that every $X \in \mathcal{X}$ is d-thick in M, and $\varepsilon_M(\mathcal{X}) \geq r(M)^{f_{6.6.2}(a,d,h)}q^{r(M)}$, then either*

(i) there is a minor N of M, and a set $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{R}_a(N)$ so that $r_N(\mathcal{Y}) > a$, and $\mathcal{Y}$ is d-firm in N, or

(ii) M has an $(a, q+1, h, d)$-pyramid minor P, with $\mathcal{S}_P \subseteq \mathcal{X}$.

Proof. Let a, d, h, q be integers with $h \geq 0$, $a \geq 1$, $d \geq 2$ and $q \geq 1$. Set $f_{6.6.2}(a, d, h) = f_{6.6.1}(a, d, h, 0)$. Let M be a matroid such that $r(M) \geq f_{6.6.2}(a, d, h)$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ be a set such that every $x \in \mathcal{X}$ is d-thick in M, and $\varepsilon_M(\mathcal{X}) \geq r(M)^{f_{6.6.2}(a,d,h)}q^{r(M)}$. We consider two cases:

Case 1: $\mathcal{X}$ is d-scattered in M.

By definition of $f_{6.6.2}$, we can apply Lemma 6.6.1 to $\mathcal{X}$. Since there is no integer a_1 with $a < a_1 \leq a$, we know that 6.6.1(iii) cannot hold. If 6.6.1(ii) holds, then we have our result. We may thus assume that 6.6.1(i) holds; now, outcome (i) follows from Lemma 6.2.4.

Case 2: $\mathcal{X}$ is not d-scattered in M.

By definition, $\{\mathrm{cl}_M(X) : X \in \mathcal{X}\}$ is not a d-minimal cover of $\mathcal{X}$ in M, so any d-minimal cover of $\mathcal{X}$ contains a set F of rank greater than a. Let $\mathcal{X}_F = \{X \in \mathcal{X} : X \subseteq F\}$. The cover $\{F\}$ must be a d-minimal cover of $\mathcal{X}_F$, so by Lemma 6.4.2 applied to $M|F$ and $\mathcal{X}_F$, we have $\varepsilon_M(\mathcal{X}_F) \geq d^{r(M|F)-a}$. Again, outcome (i) follows from Lemma 6.2.4. $\square$

The second corollary essentially reduces Theorem 6.0.8 to the case where M is a pyramid:

Corollary 6.6.3. *There is an integer-valued function $f_{6.6.3}(a, b, d, h)$ so that, for any integers a, b, d, h, q with $q \geq 1$, $d \geq 2$, $h \geq 0$, and $1 \leq a < b$, if $M \in \mathcal{U}(a, b)$ is a matroid such that $r(M) > 1$ and $\tau_a(M) \geq r(M)^{f_{6.6.3}(a,b,d,h)} q^{r(M)}$, then there is some $a_0 \in \{1, \ldots, a\}$ such that M has an $(a_0, q + 1, h, d)$-pyramid minor.*

Proof. Let a, b, d, h, q be integers with $q \geq 1$, $d \geq 2$, $h \geq 0$ and $1 \leq a < b$. Let $d' = \max(d, \binom{b}{a})$. We define a sequence of integers $p_{a+1}, \ldots, p_1$; let $p_{a+1} = 0$, and for each $1 \leq i \leq a$, recursively set $p_i = \max(p_{i+1}, f_{6.6.1}(a, d', h, p_{i+1}))$. Note that $p_1 \geq p_2 \geq \ldots \geq p_{a+1}$. Set $f_{6.6.3}(a, b, h, d)$ to be an integer $p \geq p_1$ so that

$$a^{-1}(d')^{-a} r^p \geq r^{p_1}$$

for all integers $r \geq p_1$. Let M be a matroid with $r(M) \geq p$, and $\tau_a(M) \geq r(M)^p q^{r(M)}$.

6.6.3.1. *Let $1 \leq i \leq a$. If $r(M) \geq p_i$, and $\mathcal{X} \subseteq \mathcal{R}_i(M)$ is d'-scattered in M and satisfies $\varepsilon_M(\mathcal{X}) \geq r(M)^{p_i} q^{r(M)}$, then M has an $(a_0, q+1, h, d)$-pyramid minor for some $i \leq a_0 \leq a$.*

Proof of claim: By definition of p_i, we can apply Lemma 6.6.1 to $\mathcal{X}$ in M. If 6.6.1(i) holds, then M has a d'-thick minor of rank greater than a. Since $d' \geq \binom{b}{a}$, this contradicts $M \in \mathcal{U}(a, b)$ by Lemma 6.2.2. Since $d' \geq d$, 6.6.1(ii) gives the claim, so we may assume that 6.6.1(iii) holds. If $i = a$, this is impossible, so the claim is proven. Otherwise, we have the hypotheses for a minor of M and some larger $i \leq a$, so the claim holds by induction. $\square$

Let $\mathcal{F}$ be a d'-minimal cover of M. Clearly $\mathcal{F}$ is simple. By Lemma 6.4.5, we have $\mathrm{wt}_M^{d'}(\mathcal{F}) \geq \tau_a(M)$, and every set in $\mathcal{F}$ has rank at most a, so $\varepsilon_M(\mathcal{F}) = |\mathcal{F}| \geq$

$(d')^{-a}\,\mathrm{wt}_M^{d'}(\mathcal{F}) \geq (d')^{-a} r(M)^p q^{r(M)}$. For each $1 \leq i \leq a$, let $\mathcal{F}_i = \mathcal{F} \cap \mathcal{R}_i(M)$. By a majority argument, some $1 \leq i \leq a$ satisfies $\varepsilon_M(\mathcal{F}_i) = |\mathcal{F}_i| \geq a^{-1}|\mathcal{F}| \geq a^{-1}(d')^{-a} r(M)^p q^{r(M)} \geq r(M)^{p_1} q^{r(M)} \geq r(M)^{p_i} q^{r(M)}$. The set $\mathcal{F}_i$ is d'-scattered in M by Lemma 6.4.3, and $r(M) \geq p \geq p_i$, so the result follows from the claim. $\square$

6.7 Finding Firmness

This section explores what can be done with a large collection $\mathcal{X}$ of thick rank-a sets in a matroid M with no large projective geometry as a minor. We prove a single lemma, which finds a large subcollection of $\mathcal{X}$ that is firm in a minor of M. When $a = 1$, this is equivalent to finding a large rank-2 uniform minor, and thus Theorems 6.0.4 and 6.0.5 appear in the base case of this lemma.

Lemma 6.7.1. *There is an integer-valued function $f_{6.7.1}(a, d, n, q)$ so that, for any positive integers a, d, n, q, if M is a matroid with $r(M) \geq f_{6.7.1}(a, d, n, q)$, and $\mathcal{X} \subseteq \mathcal{R}_a(M)$ is such that every $X \in \mathcal{X}$ is $f_{6.7.1}(n, q, a, d)$-thick in M, and $\varepsilon_M(\mathcal{X}) \geq r(M)^{f_{6.7.1}(a,d,n,q)} q^{r(M)}$, then either*

- *(i) M has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$, or*
- *(ii) There is a minor N of M, and a set $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{R}_a(N)$ so that $r_N(\mathcal{Y}) > a$, and $\mathcal{Y}$ is d-firm in N.*

Proof. Let n, q, d be integers at least 1. Set

$$f_{6.7.1}(1, d, n, q) = \max(f_{6.0.4}(d-2, n), \lceil \alpha_{6.0.5}(d-2, n, q) \rceil).$$

We now define $f_{6.7.1}(a, d, n, q)$ for general a recursively; for each $a > 1$, suppose that $f_{6.7.1}(a-1, d, n, q)$ has been defined. Let h be an integer so that

$$(3a)^{-1} d^{-3a} (q+1)^h \geq (h+a)^{f_{6.7.1}(a-1,d,n,q)} q^{h+a}.$$

Let $s = d^{h-a}$, and let h' be an integer so that

$$(as)^{-1} d^{-as} (q+1)^{h'} \geq (h'+a)^{f_{6.7.1}(a-1,d,n,q)} q^{h'+a};$$

Set $f_{6.7.1}(a, d, n, q) = \max(s+1, f_{6.6.2}(a, d', h+h'))$.

Now, let $a \geq 1$ be an integer, and let M be a matroid with $r(M) \geq f_{6.7.1}(a, d, n, q)$, and let $\mathcal{X} \subseteq \mathcal{R}_a(M)$ be a set whose elements are all $f_{6.7.1}(a, d, n, q)$-thick in M, satisfying $\varepsilon_M(\mathcal{X}) \geq r(M)^{f_{6.7.1}(a,d,n,q)}$. We show that M satisfies (i) or (ii), first resolving the case where $a = 1$, and proceeding by induction on a.

6.7.1.1. *If $a = 1$, then M satisfies (i) or (ii).*

Proof of claim: Every $X \in \mathcal{X}$ is a rank-1 set, so $M' = M|\bigcup_{X \in \mathcal{X}}(X)$ is a matroid with $\varepsilon(M') = \varepsilon_M(\mathcal{X}) \geq r(M')^{f_{6.7.1}(1,d,n,q)} q^{r(M')}$. Clearly $\varepsilon(M') > 1$, so $r(M') > 1$.

If $q = 1$, then $r(M')^{f_{6.7.1}(1,d,n,q)} \geq r(M')^{f_{6.0.4}(d-2,n)}$, so if the first outcome does not hold, then M' has a $U_{2,d}$-minor by Theorem 6.0.4. By construction of M', this minor corresponds to a simple subset of $\mathcal{X}$ in a rank-2 minor of M', containing d pairwise dissimilar rank-1 sets. This is a rank-2, d-firm subset of $\mathcal{X}$ in a minor of M, giving (ii).

If $q > 1$, then since $r(M') > 1$, we have

$$\varepsilon(M') \geq r(M')^{f_{6.7.1}(1,d,n,q)} q^{r(M')} \geq f_{6.7.1}(1,d,n,q) q^{r(M')} \geq \alpha_{6.0.5}(d-2,n,q) q^{r(M')},$$

so the theorem follows from Theorem 6.0.7 in a similar way to the $q = 1$ case. □

Now, assume inductively that $a > 1$, and that $f_{6.7.1}(a', d', n', q')$ as defined satisfies the lemma for all $a' < a$, and all integers $d', q', n' \geq 1$. Suppose further that (i) does not hold for M.

6.7.1.2. *M' has an $(a, q+1, h+h', d')$-pyramid minor $P = (M, \mathcal{S}; e_1, \ldots, e_{h+h'})$, with $\mathcal{S} \subseteq \mathcal{X}$.*

Proof of claim: By definition of $f_{6.7.1}(a,d,n,q)$, we know that $r(M') \geq f_{6.6.2}(a,d,h+h')$, $\varepsilon_{M'}(\mathcal{X}) \geq r(M')^{f_{6.6.2}(a,d,h+h')} q^{r(M')}$, and all sets in $\mathcal{X}$ are d'-thick in M'; we can therefore apply Corollary 6.6.2 to M'. Since $d' \geq d$, outcome 6.6.2(i) does not hold, giving 6.6.2(ii) and hence the claim. □

By Lemma 6.5.6, we may assume that $r(M) = h' + h + a$. Let $J = \{e_1, \ldots, e_h\}$. By Lemma 6.5.1, $(M /\!\!/ J, \mathcal{S}; e_{h+1}, \ldots, e_{h+h'})$ is an $(a, q+1, h', d)$-pyramid, so by Lemma 6.5.5, there is a set $\mathcal{S}' \subseteq \mathcal{S}$ such that $|\mathcal{S}'| \geq (q+1)^{h'}$, and $\mathcal{S}'$ is simple in $M /\!\!/ J$.

6.7.1.3. *There is a set $\mathcal{W} \subseteq \mathcal{S}'$ so that $|\mathcal{W}| = s$, and $\mathcal{W}$ is mutually skew in $M /\!\!/ J$.*

Proof of claim: Suppose there is no such $\mathcal{W}$. By Lemma 6.3.2, there a minor N of $M /\!\!/ J$ such that $r(N) \geq r(M /\!\!/ J) - as$, a set $\mathcal{Y} \subseteq \mathcal{S}' \cap \mathcal{R}_a(N)$ such that $|\mathcal{Y}| \geq (as)^{-1}|\mathcal{S}'|$, and some nonloop e of N so that $e \in \mathrm{cl}_N(Y)$ for all $Y \in \mathcal{Y}$. We will apply the inductive hypothesis on a to $N /\!\!/ e$.

The set $\mathcal{Y} \subseteq \mathcal{S}'$ is simple in $M /\!\!/ J$, so by Lemma 6.2.5, we have

$$\begin{aligned}\varepsilon_N(Y) \geq d^{r(N)-r(M/\!\!/J)}\varepsilon_{M/\!\!/J}(\mathcal{Y}) &\geq d^{-as}|\mathcal{Y}| \geq (as)^{-1}d^{-as}|\mathcal{S}'| \\ &\geq (as)^{-1}d^{-as}(q+1)^{h'} \geq (h'+a)^{f_{6.7.1}(n,q,a-1,d)}q^{h'+a}.\end{aligned}$$

Since $r(N/\!\!/e) < r(N) \leq r(M/\!\!/J) = a + h'$, this gives $\varepsilon_N(\mathcal{Y}) \geq r(N/\!\!/e)^{f_{6.7.1}(n,q,a-1,d)}q^{r(N/\!\!/e)}$. Since $e \in \mathrm{cl}_N(Y)$ for all $Y \in \mathcal{Y}$, we also have $\mathcal{Y} \subseteq \mathcal{R}_{a-1}(N/\!\!/e)$, and $\varepsilon_{N/\!\!/e}(\mathcal{Y}) = \varepsilon_N(\mathcal{Y})$. Moreover, $r(N/\!\!/e) \geq r(M/\!\!/J) - as - 1 \geq a + h' - as - 1 \geq f_{6.7.1}(n, q, a-1, d)$, so by the inductive hypothesis, there is a minor N' of $N/\!\!/e$, and a set $\mathcal{Y}' \subseteq \mathcal{Y} \cap \mathcal{R}_{a-1}(N')$ such that $r_{N'}(\mathcal{Y}') \geq a$, and $\mathcal{Y}'$ is d-firm in N'. If $N' = N/\!\!/(C \cup \{e\})\backslash D$, where $C \cup \{e\}$ is independent in N, then it is straightforward to check that $\mathcal{Y}' \subseteq \mathcal{R}_a(N/\!\!/C)$, and $r_{N/\!\!/C}(\mathcal{Y}') > a$, and $\mathcal{Y}'$ is d-firm in $N/\!\!/C$. This gives (ii).

□

Let $\mathcal{W} = \{W_1, \ldots, W_s\}$. For each $1 \leq i \leq s$, let $\mathcal{S}_i = \{S \in \mathcal{S} : S \equiv_{M/\!\!/J} W_i\}$. By Lemma 6.5.4, there is, for each $1 \leq i \leq s$, a rank-$(a+h)$ restriction M_i of M such that $(M_i, \mathcal{S}_i; e_1, \ldots, e_h)$ is an $(a, q+1, h, d')$-pyramid.

6.7.1.4. *For each $1 \leq i \leq s$, there are distinct sets $V_i, Z_i, Z_i' \in \mathcal{S}_i$ such that $\{V_i, Z_i, Z_i'\}$ is mutually skew in M_i.*

Proof of claim: By Lemma 6.5.5, $\mathcal{S}_i$ has a subset $\mathcal{S}'$ of size $(q+1)^h$ that is simple in M_i. If there is a subset of $\mathcal{S}_i'$ of size 3 that is skew in M_i, then the claim follows. Otherwise, by Lemma 6.3.2, there is a minor N_i of M_i, with $r(N_i) \geq r(M_i) - 3a$, a set $\mathcal{Y} \subseteq \mathcal{S}_i' \cap \mathcal{R}_a(N_i)$ such that $|\mathcal{Y}| \geq (3a)^{-1}d^{-3a}|\mathcal{S}_i'|$, and a nonloop e of N_i so that $e \in \mathrm{cl}_{N_i}(Y)$ for all $Y \in \mathcal{Y}$. The proof is now very similar to that of the previous claim, following from the definition of h. □

Let $\mathcal{V} = \{V_1, \ldots, V_s\}$. Since $V_i \equiv_{M/\!\!/J} W_i$ for each i, the set $\mathcal{V}$ is skew in $M/\!\!/J$. This last claim uses Z_i and Z_i' to contract the elements of $\mathcal{V}$, one by one, into the span of J without reducing their rank, while maintaining the 'skewness' and structure of the elements of $\mathcal{V}$ not yet contracted:

6.7.1.5. *For each $0 \leq i \leq s$, there is a minor N_i of M such that*

(a) $\{V_{i+1}, \ldots, V_s\}$ is skew in $N_i/\!\!/J$, and

(b) for each $i < j \leq s$, $N_i|E(M_j) = M_j$, and

(c) $\{V_1, \ldots, V_i\} \subseteq \mathcal{R}_a(N_i|\operatorname{cl}_{N_i}(J))$*, and* $\{V_1, \ldots, V_i\}$ *is simple in* N_i*.*

Proof of claim: When $i = 0$, the claim is clear, with $N_0 = M'$. Suppose inductively that $1 \le i \le s$, and that the claim holds for smaller i. We will construct N_i by contracting a rank-a set of $N_{i-1}|E(M_i)$. By definition, Z_i' and V_i are similar to W_i in $M_i /\!\!/ J$, so $r_{M_i /\!\!/ J}(Z_i') = r_{M_i /\!\!/ J}(V_i) = a$; Let $I \subseteq Z_i'$ be an independent set of size $(a-1)$ in $M_i /\!\!/ J$. So $\{V_i, Z_i\}$ is a skew pair of rank-a sets in $M_i /\!\!/ I$, and $r(M_i /\!\!/ I) = h + a - (a-1) = h+1$. Since I is independent in $M_i /\!\!/ J$, it is skew to J in M_i, so $r_{M_i /\!\!/ I}(J) = h$. Moreover, $r_{M_i /\!\!/ (J \cup I)}(V_i) = r_{M_i /\!\!/ (J \cup I)}(Z_i) = r_{M_i /\!\!/ (J \cup I)}(Z_i') = 1$, so neither Z_i nor V_i is contained in $\operatorname{cl}_{M_i /\!\!/ I}(J)$.

By the inductive hypothesis, $(N_{i-1} /\!\!/ I)|E(M_i) = M_i /\!\!/ I$, so we can extend the observations just made about $M_i /\!\!/ I$ to apply in $N_{i-1} /\!\!/ I$. Therefore, in the matroid $N_{i-1} /\!\!/ I$, $\{V_i, Z_i\}$ is a skew pair of rank-a sets, each contained in the rank-$(h+1)$ set $E(M_i)$, which itself contains the rank-h set J, and $\operatorname{cl}_{N_{i-1} /\!\!/ I}(J)$ does not contain Z_i or V_i.

For each $1 \le k < i$, let $F_k = \varnothing$ if $r_{N_{i-1} /\!\!/ I}(V_k \cup V_i) > a+1$, and $F_k = \operatorname{cl}_{N_{i-1} /\!\!/ I}(V_k \cup V_i)$ otherwise. Since V_i and Z_i are skew sets of rank $a > 1$ in $N_{i-1} /\!\!/ I$, and F_k is a flat of rank at most $a+1$ containing V_i, it follows that $Z_i \not\subseteq F_k$, so $r_{N_{i-1} /\!\!/ I}(F_k \cap Z_i) < a$. Also, the set $\operatorname{cl}_{N_{i-1} /\!\!/ I}(J)$ does not contain Z_i. The set Z_i is $(d' \ge s+1)$-thick in $N_{i-1} /\!\!/ I$, and there are at most $s-1$ possible k, so there is some $f \in Z_i$ that is not in any of the sets F_k, and not in $\operatorname{cl}_{N_{i-1} /\!\!/ I}(J)$. Set $N_i = N_{i-1} /\!\!/ (I \cup \{f\})$. By choice of f, we have $r_{N_i}(J) = h = r_{N_{i-1}}(J)$, so $I \cup \{f\}$ is skew to J in N_{i-1}; we now show that N_i satisfies (a), (b) and (c).

(a) We have $I \cup \{f\} \subseteq Z_i \cup Z_i'$. The sets Z_i and Z_i' are both similar to V_i in $M_i /\!\!/ J = (N_{i-1} /\!\!/ J)|E(M_i)$, so $I \cup \{f\} \subseteq \operatorname{cl}_{N_{i-1} /\!\!/ J}(V_i)$. $\{V_i, \ldots, V_s\}$ is skew in $N_{i-1} /\!\!/ J$ by the inductive hypothesis, so $\{V_{i+1}, \ldots, V_s\}$ is skew in $N_{i-1} /\!\!/ (J \cup I \cup \{f\}) = N_i /\!\!/ J$.

(b) Let $i < j \le s$. Since $(M_j, \mathcal{S}_j; e_1, \ldots, e_h)$ is an $(a, q+1, h, d)$-pyramid and $V_j \in \mathcal{S}_j$, the set $J \cup V_j$ is spanning in M_j, and J is skew to V_j in M_j. As we saw in (a), the set $I \cup \{f\}$ is skew to J in N_{i-1}, and is skew to V_j in $N_{i-1} /\!\!/ J$. Now, $M_j = N_{i-1}|E(M_j)$ and $M_i = N_{i-1}|E(M_i)$, so

$$\begin{aligned} r_{N_{i-1}}((I \cup \{f\}) \cup (J \cup V_j)) &= r_{N_{i-1} /\!\!/ J}(I \cup \{f\} \cup V_j) + r_{N_{i-1}}(J) \\ &= r_{N_{i-1} /\!\!/ J}(I \cup \{f\}) + r_{N_{i-1} /\!\!/ J}(V_j) + r_{N_{i-1}}(J) \\ &= r_{N_{i-1}}(I \cup \{f\}) + r_{N_{i-1}}(V_j) + r_{N_{i-1}}(J) \\ &= r_{N_{i-1}}(I \cup \{f\}) + r_{N_{i-1}}(V_j \cup J). \end{aligned}$$

Therefore, $I \cup \{f\}$ and $V_j \cup J$ are skew in N_{i-1}. Since $V_j \cup J$ is spanning in M_j, this gives $N_i|E(M_j) = N_{i-1}|E(M_j) = M_j$.

(c) Since $I \cup \{f\}$ is skew to J in N_{i-1}, it is clear that $\{V_1, \ldots, V_{i-1}\} \subseteq \mathcal{R}_a(N_i | \operatorname{cl}_{N_i}(J))$ and that $\{V_1, \ldots, V_{i-1}\}$ is simple in N_i. Moreover, V_i is a rank-a set that is skew to $Z_i \cup Z_i'$ in N_{i-1}, and therefore is skew to $I \cup \{f\}$, so $r_{N_i}(V_i) = a$. It therefore remains to show that V_i is not similar in N_i to any of $V_1, \ldots, V_{i-1}$.

Suppose for a contradiction that $V_i \equiv_{N_i} V_k$ for some $1 \leq k < i$. Either V_i and V_k are similar in $N_{i-1} /\!\!/ I$, or V_i and V_k lie in a common rank-$(a+1)$ flat F of $N_{i-1} /\!\!/ I$, and contracting $f \in F$ makes the two sets similar in N_i. In the first case, this gives $0 = r_{N_{i-1} /\!\!/ (I \cup V_k)}(V_i) \geq r_{N_{i-1} /\!\!/ (I \cup J)}(V_i) = r_{N_{i-1} /\!\!/ J}(V_i) - r_{N_{i-1}}(I) = a - (a-1) = 1$, a contradiction. In the second case, we have $f \in \operatorname{cl}_{N_{i-1} /\!\!/ I}(V_i \cup V_k)$, which does not occur by choice of f.

$\square$

Now, let $N = N_s | \operatorname{cl}_{N_s}(J)$. We have $r(N) \leq h$, and $\mathcal{V}$ is a simple subset of $\mathcal{R}_a(N)$ by construction, so $\varepsilon_N(\mathcal{V}) = |\mathcal{V}| = s = d^{h-a}$. Also, $\mathcal{V} \subseteq \mathcal{X}$ and $d' \geq d$, so every $V \in \mathcal{V}$ is d-thick in N. (ii) now follows by applying Lemma 6.2.4 to $\mathcal{V}$ in N. $\square$

6.8 Upgrading a Pyramid

The goal of this section is to prove that a sufficiently large pyramid minor will be enough to prove Theorem 6.0.8. We show that for very large h and d, an $(a_0, q+1, h, d)$-pyramid will either contain a thick set of rank greater than a, or a large projective geometry over $\mathrm{GF}(q')$ for some $q' > q$. We first prove this when $a_0 = a$, and then show that, for $a_0 < a$, we can find a large pyramid as a minor with a larger a_0, thereby 'upgrading' our pyramid.

An important intermediate object is an $(a_0, q+1, \cdot, \cdot)$-pyramid P 'on top of' a very firm set $\mathcal{X} \subseteq \mathcal{S}_P$ with rank greater than a_0. We construct such objects using the results in the previous section; this is the reason that we need to exclude a projective geometry.

We upgrade a pyramid of height h on top of a firm set by 'lifting' the firm set one step up the pyramid h times, sacrificing a large amount of firmness at each step. Our next two lemmas establish the machinery required for this; the first simply lifts a firm set up a pyramid of height 1:

Lemma 6.8.1. *Let a_0, a, q, d, d' be integers with $1 \leq a_0 \leq a$, $d, d' \geq 2$, and $q \geq 2$. If $(M, \mathcal{S}; e)$ is an $(a_0, q, 1, d')$-pyramid, and $\mathcal{X} \subseteq \mathcal{S}$ is d^{q+2}-firm in $M /\!\!/ e$ and satisfies $r_{M /\!\!/ e}(\mathcal{X}) = a$, then either* *(i) There exists $\mathcal{Y} \subseteq \mathcal{S}$ so that $r_M(\mathcal{Y}) = a + 1$ and $\mathcal{Y}$ is d-firm in M, or*

(ii) There exist sets $\mathcal{X}_1, \ldots, \mathcal{X}_q \subseteq \mathcal{S}$ such that

- *Each $\mathcal{X}_i$ is d-firm in M, and $r_M(\mathcal{X}_i) = a$, and*
- *The $\mathcal{X}_i$ are pairwise dissimilar in M, and each is skew to $\{e\}$ in M, and similar to $\mathcal{X}$ in $M /\!\!/ e$.*

Proof. We may assume that $\mathcal{X}$ is spanning in $M /\!\!/ e$, so $r(M) = a + 1$. Suppose that the first outcome does not hold. Let I be an indexing set for X (i.e. let $\mathcal{X} = \{X^i : i \in I\}$, with $|I| = |\mathcal{X}|$). For each $i \in I$, let $X^i_1, \ldots, X^i_q$ be pairwise dissimilar sets in $\mathcal{S}$, each similar to X^i in M, as given by the definition of a pyramid.

6.8.1.1. *There are sets $\mathcal{X}_1, \ldots, \mathcal{X}_q \subseteq \mathcal{S}$ and $I_1, \ldots, I_q \subseteq I$ such that the following conditions hold:*

- *For each $1 \le j \le q$, the set I_j is the indexing set for $\mathcal{X}_j$ in I (i.e. $\mathcal{X}_j = \{X^i_j : i \in I_j\}$), and*
- *$I \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq I_q$, and*
- *for each $1 \le j \le q$, we have $|\mathcal{X}_j| \ge d^{-j}|\mathcal{X}|$, and $r_M(\mathcal{X}_j) \le a$.*

Proof of claim: We construct the sets in question by induction on j. Suppose that $1 \le j < q$, and that the sets $\mathcal{X}_1, \ldots, \mathcal{X}_{j-1}$ and $I_1, \ldots, I_{j-1}$ have been defined to satisfy the conditions. Let $I_0 = I$, and $\mathcal{X}_0 = \mathcal{X}$; note that $|\mathcal{X}_0| \ge d^0|\mathcal{X}|$. As (i) does not hold, the set $\{\mathcal{X}^i_j : i \in I_{j-1}\}$ is not a rank-$(a + 1)$, d-firm set in M, so we may assume that there is some $\mathcal{X}_j \subseteq \{X^j_i : I \in I_{j-1}\}$ such that $|\mathcal{X}_j| \ge d^{-1}|\{\mathcal{X}^i_j : i \in I_{j-1}\}|$ and $r_M(\mathcal{X}_j) \le a$. Now, $|\mathcal{X}_j| \ge d^{-1}|\{\mathcal{X}^i_j : i \in I_{j-1}\}| = d^{-1}|I_{j-1}| = d^{-1}|\mathcal{X}_{j-1}| \ge d^{-j}|\mathcal{X}|$. The set $\mathcal{X}_j$, along with $I_j = \{i \in I_{j-1} : X^i_j \in \mathcal{X}_j\}$, satisfies the required conditions. □

6.8.1.2. *For each $1 \le j \le q$, the set $\mathcal{X}_j$ is d-firm in M, and $r_M(\mathcal{X}_j) = r_{M /\!\!/ e}(\mathcal{X}_j) = a$.*

Proof of claim. We know that $r_M(\mathcal{X}_j) \le a$; let $\mathcal{X}'_j \subseteq \mathcal{X}_j$ satisfy $|\mathcal{X}'_j| \ge d^{-1}|\mathcal{X}_j|$, and let $I'_j = \{i \in I_j : X^i_j \in \mathcal{X}'_j\}$. Let $\mathcal{X}' = \{X^i : i \in I'_j\}$. By definition of $\mathcal{X}$ and $\mathcal{X}_i$, each set in $\mathcal{X}'$ is similar in $M /\!\!/ e$ to a set in $\mathcal{X}'_j$, and vice versa. We therefore have $|\mathcal{X}'| = |\mathcal{X}'_j|$ and $r_{M /\!\!/ e}(\mathcal{X}') = r_{M /\!\!/ e}(\mathcal{X}'_j)$. Now, $|\mathcal{X}'| = |\mathcal{X}'_j| \ge d^{-1}|\mathcal{X}_j| > d^{-(q+2)}|\mathcal{X}|$, and $\mathcal{X}' \subseteq \mathcal{X}$, so d^{q+2}-firmness of $\mathcal{X}$ gives $r_{M /\!\!/ e}(\mathcal{X}') = r_{M /\!\!/ e}(\mathcal{X}) = a$. Therefore,

$$a \ge r_M(\mathcal{X}_j) \ge r_M(\mathcal{X}'_j) \ge r_{M /\!\!/ e}(\mathcal{X}'_j) = r_{M /\!\!/ e}(\mathcal{X}') = r_{M /\!\!/ e}(\mathcal{X}) = a,$$

and the lemma follows from definition of firmness, and the fact that $r_{M/\!/e}(\mathcal{X}_j) \geq r_{M/\!/e}(\mathcal{X}'_j)$. $\square$

By assumption, the set $\mathcal{X}$ is spanning in the rank-a matroid $M/\!/e$, and by the second part of 6.8.1.2, the set $\mathcal{X}_j$ is also spanning in $M/\!/e$, so $\mathcal{X}_j \cong_{M/\!/e} \mathcal{X}$. Also, the fact that $r_{M/\!/e}(\mathcal{X}_j) = r_M(\mathcal{X}_j)$ implies that $\mathrm{cl}_M(\mathcal{X}_j)$ is skew to $\{e\}$ in M, so the lemma follows from a final claim:

6.8.1.3. *The sets $\mathcal{X}_j : 1 \leq j \leq q$ are pairwise dissimilar in M.*

Proof of claim: Suppose not; let $\mathcal{X}_j$ and $\mathcal{X}_{j'}$ be similar in M, where $1 \leq j < j' \leq q$. By 6.8.1.2, $r_M(\mathcal{X}_j \cup \mathcal{X}_{j'}) = r_M(\mathcal{X}_j) = a$. Let $i \in I_{j'}$. We have $X^i_{j'} \in \mathcal{X}_{j'}$ by definition, and $I_{j'} \subseteq I_j$, so $i \in I_j$ and $X^i_j \in \mathcal{X}_j$. But X^i_j and $X^i_{j'}$ are dissimilar rank-a_0 sets in M, each similar to the rank-a_0 set X^i in $M/\!/e$. Therefore, $e \in \mathrm{cl}_M(X^i_j \cup X^i_{j'})$, and so $e \in \mathrm{cl}_M(\mathcal{X}_j \cup \mathcal{X}'_j) = \mathrm{cl}_M(\mathcal{X}_j)$. This contradicts the fact that $\mathrm{cl}_M(\mathcal{X}_j)$ is skew to $\{e\}$ in M. $\square$

$\square$

The next lemma iterates the previous one h times to upgrade a pyramid completely - here, a_0 is upgraded to a_1 in the second outcome:

Lemma 6.8.2. *Let a_0, a_1, q and d be integers with $1 \leq a_0 \leq a_1$ and $d, q \geq 2$, and let $(M, \mathcal{S}; e_1, \ldots, e_h)$ be an (a_0, q, h, d)-pyramid. For each $0 \leq i \leq h$, let $M_i = M/\!/\{e_1, \ldots, e_i\}$. If $\mathcal{X} \subseteq \mathcal{S}$ is a set so that $r_{M_h}(\mathcal{X}) = a_1$ and $\mathcal{X}$ is $d^{(q+2)^h}$-firm in M_h, then either*

- *(i) There is an integer $1 \leq i \leq h$, and a set $\mathcal{Y} \subseteq \mathcal{S}$ so that $\mathcal{Y}$ is d-firm in M_i, and $r_{M_i}(\mathcal{Y}) > a_1$, or*
- *(ii) There is a set $\mathcal{T}$ so that $(M, \mathcal{T}; e_1, \ldots, e_h)$ is an (a_1, q, h, d)-pyramid.*

Proof. Assume that (i) does not hold; we will build a pyramid-like structure inductively.

6.8.2.1. *For each $0 \leq i \leq h$, there exists a nonempty set $\mathfrak{X}^i$ of subsets of $\mathcal{S}$ satisfying the following:*

- $\mathrm{cl}_M(\mathcal{X})$ *is skew to $\{e_{i+1}, \ldots, e_h\}$ in M_i for all $\mathcal{X} \in \mathfrak{X}^i$, and*
- *For all $\mathcal{X} \in \mathfrak{X}^i$ and i' such that $i \leq i' < h$, there exist sets $\mathcal{X}_1, \ldots, \mathcal{X}_q \in \mathfrak{X}^i$, pairwise dissimilar in $M_{i'}$, and each similar to $\mathcal{X}$ in $M_{i'+1}$.*

- *For all $\mathcal{X} \in \mathfrak{X}^i$, we have $r_{M_i}(\mathcal{X}) = a$, and $\mathcal{X}$ is $d^{(q+1)^i}$-firm in M_i.*

Proof of claim: Let $\mathfrak{X}^h = \{\mathcal{X}\}$. It is clear that $\mathfrak{X}^h$ satisfies all three conditions. Fix $0 \leq i < h$, and suppose that $\mathfrak{X}^{i+1}$ has been defined to satisfy the conditions. Let $\mathcal{X} \in \mathfrak{X}^{i+1}$. We know that $(M_i, \mathcal{S}; e_{i+1})$ is an (a_0, q)-pyramid; by the inductive hypothesis, the set $\mathcal{X}$ satisfies the hypotheses of Lemma 6.8.1 for this pyramid, and $d^{(q+2)^i}$. If 6.8.1(i) holds, then so does outcome (i) of the current lemma, as $d^{(q+2)^i} \geq d$. Otherwise, let $P(\mathcal{X}) = \{\mathcal{X}_1, \ldots, \mathcal{X}_q\}$, where $\mathcal{X}_1, \ldots, \mathcal{X}_q$ are the sets given by 6.8.1(ii). Now, $\mathfrak{X}^i = \bigcup_{\mathcal{X} \in \mathfrak{X}^{i+1}} P(\mathcal{X})$ will satisfy the claim, which now follows inductively. $\square$

Let $\mathcal{T} = \{\mathrm{cl}_M(\mathcal{X}) : \mathcal{X} \in \mathfrak{X}^0\}$. Each set in $\mathfrak{X}^0$ is d-firm in M, so all sets in $\mathcal{T}$ are d-thick by Lemma 6.2.3. It is now clear from the claim that $(M, \mathcal{T}; e_1, \ldots, e_h)$ is an (a, q, h, d)-pyramid. $\square$

Having seen that a pyramid on top of a firm set is a useful object, we now show that such an object can be constructed by Lemma 6.7.1 by excluding a projective geometry.

Lemma 6.8.3. *There is an integer-valued function $f_{6.8.3}(a_0, d, n, q, h)$ so that, for any integers a_0, d, n, q, d', h' with $a_0, d, n, q \geq 1$ and $\min(d', h') \geq f_{6.8.3}(a_0, d, n, q, h)$, if P is an $(a_0, q+1, h', d')$-pyramid on a matroid M, then either*

(i) M has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$, or

(ii) There is a minor M' of M, an $(a_0, q+1, h, d)$-pyramid $(M', \mathcal{S}'; e_1, \ldots, e_h)$ with $\mathcal{S}' \subseteq \mathcal{S}_P$, and a set $\mathcal{Y} \subseteq \mathcal{S}'$ so that $\mathcal{Y}$ is d-firm in $M' /\!\!/ \{e_1, \ldots, e_h\}$ and $r_{M' /\!\!/ \{e_1, \ldots, e_h\}}(\mathcal{Y}) > a_0$.

Proof. Let a_0, d, n, q be integers at least 1. Let h^* be an integer large enough so that $(q+1)^{h^*} \geq (a_0 + h^*)^{f_{6.7.1}(a_0,d,n,q)} q^{a_0 + h^*}$, and $h^* \geq f_{6.7.1}(a_0, d, n, q)$. Set $f_{6.8.3}(a_0, d, n, q, h) = h + h^*$. Now, let h' be d' are integers with $\min(h', d') \geq h + h^*$, and $P = (M, \mathcal{S}; e_1, \ldots, e_{h'})$ be an $(a_0, q+1, h', d')$-pyramid on a matroid M. We show that M satisfies one of the two outcomes; by Lemma 6.5.6, we may assume that $h' = h + h^*$, and that $r(M) = h + h^* + a_0$. Let $M_h = M /\!\!/ \{e_1, \ldots, e_h\}$.

Now, $r(M_h) = h^* + a_0$, and $Q = (M_h, \mathcal{S}; e_{h+1}, \ldots, e_{h+h^*})$ is an $(a_0, q+1, h^*, d')$-pyramid, and by Lemma 6.5.5, $\varepsilon_{M_h}(\mathcal{S}) = (q+1)^{h^*} \geq (h^* + a_0)^{f_{6.7.1}(a_0,d,n,q)} q^{h^* + a_0} = r(M_h)^{f_{6.7.1}(a_0,d,n,q)} q^{r(M_h)}$. Since $d' \geq h \geq f_{6.7.1}(a_0, d, n, q)$, we can apply Lemma 6.7.1 to $\mathcal{S}$ in M_h. We may assume that 6.7.1(i) does not hold, so 6.7.1(ii) does; therefore, there is a minor N of M_h and a set $\mathcal{Y} \subseteq \mathcal{S} \cap \mathcal{R}_a(N)$ such that $r_N(\mathcal{Y}) > a_0$, and $\mathcal{Y}$ is d-firm in N. By Lemma 6.5.2, there is an $(a_0, q+1, h, d')$-pyramid $(M', \mathcal{S}'; e_1, \ldots, e_h)$ so that $\mathcal{Y} \subseteq \mathcal{S}'$, and $N|\mathcal{Y} = (M' /\!\!/ \{e_1, \ldots, e_h\})|\mathcal{Y}$. Since $d' \geq d$, this gives (ii). $\square$

Finally, we combine the lemmas in this section to prove what we want: that any $(\cdot, q+1, h, d)$-pyramid for large h and d contains either a thick minor of rank greater than a, or a large projective geometry over a field larger than $\mathrm{GF}(q)$. This tells us that finding such a pyramid is enough to prove Theorem 6.0.8.

Lemma 6.8.4. *There is an integer-valued function $f_{6.8.4}(a, d, n, q)$ so that, for any integers $n, q, a_0, a, d, d^*, h^*$ with $n, q \geq 1$, $d \geq 2$, $1 \leq a_0 \leq a$, and $\min(h^*, d^*) \geq f_{6.8.4}(a, d, n, q)$, if P is an $(a_0, q+1, h^*, d^*)$-pyramid on a matroid M, then either*

(i) M has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$, or

(ii) M has a d-thick minor N, with $r(N) > a$.

Proof. Let n, q, a_0, a, d be integers with $n, q \geq 1$, $d \geq 2$, and $1 \leq a_0 \leq a$. For each pair of integers i, j with $1 \leq i \leq j \leq a$, recursively define integers h^i_j and d^i_j as follows: (h^i_j and d^i_j are well-defined for all i, j in the range, as h^a_a and d^a_a are defined, and the definitions of h^i_j and d^i_j depend only on pairs (i', j') exceeding (i, j) lexicographically)

$$h^i_j = \begin{cases} f_{6.8.3}(a, d, n, q, 0) & \text{if } j = a \\ f_{6.8.3}(a, d^i_{i+1}, n, q, h^i_{i+1}) & \text{if } j < a \text{ and } i = j \\ h^{i+1}_{i+1} + h^i_{j+1} & \text{if } 1 \leq i < j < a \end{cases}$$

$$d^i_j = \begin{cases} f_{6.8.3}(a, d, n, q, 0) & \text{if } j = a \\ f_{6.8.3}(a, d^i_{i+1}, n, q, h^i_{i+1}) & \text{if } j < a \text{ and } i = j \\ (\max(d^{i+1}_{i+1}, d^i_{j+1}))^{h^{i+1}_{i+1}} & \text{if } 1 \leq i < j < a \end{cases}$$

Note that if (i, j) exceeds (i', j') lexicographically, then $h^i_j \leq h^{i'}_{j'}$ and $d^i_j \leq d^{i'}_{j'}$. We set $f_{6.8.4}(a, d, n, q) = \max(h^1_1, d^1_1)$. The lemma will follow from a technical claim:

6.8.4.1. *Let $1 \leq i \leq j \leq a$, and d and h be integers so that $d \geq d^i_j$ and $h \geq h^i_j$. If $P = (M, \mathcal{S}; e_1, \ldots, e_h)$ is an $(i, q+1, h, d)$-pyramid, and $\mathcal{X} \subseteq \mathcal{S}$ is d-firm in $M /\!\!/ \{e_1, \ldots, e_h\}$ and satisfies $r_{M /\!\!/ \{e_1, \ldots, e_h\}}(\mathcal{X}) = j$, then (i) or (ii) holds for M.*

Proof of claim: By Lemma 6.5.6, we may assume that $h = h^i_j$. If $j = a$, then $h^i_j = d^i_j = f_{6.8.3}(a, d, n, q, 0)$; we can therefore apply Lemma 6.8.3 to P. Outcome 6.8.3(i) gives (i), and applying Lemma 6.2.3 to the $\mathcal{X}$ and M' given by 6.8.3(ii) gives (ii). Suppose inductively that $1 \leq i \leq j < a$, and that the claim holds for all (i', j') lexicographically greater than (i, j).

If $j = i$, then by Lemma 6.8.3, there is a minor M' of M, an $(i, q+1, h^i_{i+1}, d^i_{i+1})$-pyramid $(M', \mathcal{S}'; e_1, \dots, h^i_{i+1})$ on M', and a set $\mathcal{X}' \subseteq \mathcal{S}'$ so that $\mathcal{X}'$ is d^i_{i+1}-firm in $M' /\!\!/ \{e_1, \dots, e^i_{i+1}\}$, and $r_{M'}(\mathcal{X}') \geq i+1$. Let $i' = r_{M'}(\mathcal{X})$. If $i' > a$, then by Lemma 6.2.3, outcome (ii) holds. Otherwise, since $h^i_{i+1} \geq h^i_{i'}$ and $d^i_{i+1} \geq d^i_{i'}$, the lemma follows from the inductive hypothesis.

We may now assume that $1 \leq i < j < a$. For each $0 \leq k \leq h$, write M_k for $M /\!\!/ \{e_1, \dots, e_k\}$. We have $h = h^i_j = h^i_{j+1} + h^{i+1}_{i+1}$. Let $h' = h^i_{j+1}$, and $h'' = h^{i+1}_{i+1}$. By Lemma 6.5.1, $P' = (M_{h'}, \mathcal{S}; e_{h'+1}, \dots, e_{h'+h''})$ is an $(i, q+1, h'', d^i_j)$-pyramid, and $\mathcal{X}$ is d^i_j-firm in $M_h = M_{h'} /\!\!/ \{e_{h'+1}, \dots, e_h\}$. By definition, $d \geq (\max(d^{i+1}_{i+1}, d^i_{j+1}))^{h''}$, so we can apply Lemma 6.8.2 to P'.

If 6.8.2(i) holds for P', then there is some $1 \leq \ell \leq h''$, a set $\mathcal{Y} \subseteq \mathcal{S}$ that is d^i_{j+1}-firm in $M_{h'} /\!\!/ \{e_{h'+1}, \dots, e_{h'+\ell}\} = M_{h'+\ell}$, and satisfies $r_{M_{h'+\ell}}(\mathcal{Y}) > j$; let $j' = r_{M_{h'+\ell}}(\mathcal{Y})$. If $j' > a$, then (ii) follows from Lemma 6.2.3. Otherwise, by Lemma 6.5.1, $P'' = (M /\!\!/ \{e_{h'+1}, \dots, e_{h'+\ell}\}, \mathcal{S}; e_1, \dots, e_{h'})$ is an $(i, q+1, h', d)$-pyramid, and since $d \geq d^i_{j+1} \geq d^i_{j'}$ and $h' = h^i_{j+1} \geq h^i_{j'}$, the pyramid P'' and the set $\mathcal{Y}$ satisfy the hypotheses of the claim for (i, j'). The claim follows by induction.

If 6.8.2(ii) holds for P, then there is a $(j, q+1, h'', d^{i+1}_{i+1})$-pyramid Q on $M_{h'}$. We have $h'' = h^{i+1}_{i+1} \geq h^j_j$, and for any $X \in \mathcal{S}_Q$, the set $\{X\}$ is trivially d^j_j-firm in M_h, so Q and $\{X\}$ satisfy the hypotheses of the claim for $(j+1, j+1)$. Again, the claim follows inductively.

□

Let h^* and d^* be integers with $\min(h^*, d^*) \geq f_{6.8.4}(a, d, n, q)$, and $P = (M, \mathcal{S}; e_1, \dots, e_{h^*})$ be an $(a_0, q+1, h^*, d^*)$-pyramid. For any $X \in \mathcal{S}_P$, the set $\{X\}$ is d^*-firm in $M /\!\!/ \{e_1, \dots, e_{h^*}\}$, and $d^* \geq f_{6.8.4}(a, d, n, q) \geq d^1_1 \geq d^{a_0}_{a_0}$. Moreover, $h^* \geq f_{6.8.4}(a, d, n, q) \geq h^1_1 \geq h^{a_0}_{a_0}$, so the lemma follows by applying the claim to P and $\{X\}$. □

6.9 A Halfway Point

We are now able to prove Theorem 6.0.8, which we restate here for convenience:

Theorem 6.0.8. *There is an integer-valued function $f_{6.0.8}(a, b, n, q)$ so that, for any integers $1 \leq a < b$, $q \geq 1$ and $n \geq 1$, if $M \in \mathcal{U}(a, b)$ is a matroid such that $r(M) > 1$ and $\tau_a(M) \geq r(M)^{f_{6.0.8}(a,b,n,q)} q^{r(M)}$, then M has a* PG$(n-1, q')$*-minor for some prime power $q' > q$.*

Proof. Let a, b, n, q be integers with $n, q \geq 1$ and $1 \leq a < b$. Let $d = \binom{b}{a}$, and $h = f_{6.8.4}(a, d, n, q)$. Set $f_{6.0.8}(a, b, n, q)$ to be an integer p such that $p \geq f_{6.6.3}(a, b, h, h)$, and so that $r^p \geq d^r$ for all r such that $2 \leq r < p$.

Let $M \in \mathcal{U}(a, b)$ be a matroid with $r(M) > 1$, and $\tau_a(M) \geq r(M)^p q^{r(M)}$; we show that M has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$. If $r(M) < p$, then by Theorem 5.6.2, $\tau_a(M) \leq \binom{b-1}{a}^{r(M)} < d^{r(M)} \leq r(M)^p$, a contradiction. So we may assume that $r(M) \geq p$. By Lemma 6.6.3, M has an $(a_0, q+1, h, h)$-pyramid minor for some $1 \leq a_0 \leq a$. By Lemma 6.8.4, M either has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$, giving the theorem, or a d-thick minor of rank greater than a, in which case a contradiction follows from Lemma 6.2.2. $\square$

Theorem 6.0.6, which we also restate, follows by setting $f_{6.0.6}(a, b, n) = f_{6.0.8}(a, b, n, 1)$.

Theorem 6.0.6. *There is an integer-valued function $f_{6.0.6}(a, b, n)$ so that, for any integers $1 \leq a < b$ and $n \geq 2$, if $M \in \mathcal{U}(a, b)$ is a matroid such that $r(M) > 1$ and $\tau_a(M) \geq r(M)^{f_{6.0.6}(a,b,n)}$, then M has a* $\mathrm{PG}(n-1, q)$*-minor for some prime power q.*

Chapter 7

Uniform Minors II

The goal of this chapter is to consider the a-covering number parameter τ_a further. We prove Theorem 6.0.7, using a corollary of Theorem 6.0.8 as a foundation in our proof. We restate both theorems here:

Theorem 6.0.7. *There is a real-valued function $\alpha_{6.0.7}(a,b,n,q)$ so that, for any integers a,b,n,q with $n \ge 1$, $q \ge 2$ and $1 \le a < b$, if $M \in \mathcal{U}(a,b)$ is a matroid such that $\tau_a(M) \ge \alpha_{6.0.7}(a,b,n,q)q^{r(M)}$, then M has a $\mathrm{PG}(n-1,q')$-minor for some $q' > q$.*

Theorem 6.0.8. *There is an integer-valued function $f_{6.0.8}(a,b,n,q)$ so that, for any integers $1 \le a < b$, $q \ge 1$ and $n \ge 1$, if $M \in \mathcal{U}(a,b)$ is a matroid such that $r(M) > 1$ and $\tau_a(M) \ge r(M)^{f_{6.0.8}(a,b,n,q)}q^{r(M)}$, then M has a $\mathrm{PG}(n-1,q')$-minor for some prime power $q' > q$.*

The corollary of Theorem 6.0.8 we will use is the following, which is weaker than Theorem 6.0.7 only in that the outcome $q' > q$ is relaxed to $q' \ge q$:

Corollary 7.0.1. *There is a real-valued function $\alpha_{7.0.1}(a,b,n,q)$ so that, for any integers $1 \le a < b$, $n \ge 1$ and $q \ge 2$, if $M \in \mathcal{U}(a,b)$ is a matroid such that $\tau_a(M) \ge \alpha_{7.0.1}(a,b,n,q)q^{r(M)}$, then M has a $\mathrm{PG}(n-1,q')$-minor for some $q' \ge q$.*

Proof. Let a,b,n,q be integers with $1 < a \le b$, $n \ge 1$ and $q \ge 2$. Let α be a real number such that $\alpha q^r \ge r^{f_{6.0.8}(a,b,n,q-1)}(q-1)^r$ for all $r \ge 0$. Set $\alpha_{7.0.1}(a,b,n,q) = \alpha$. If $M \in \mathcal{U}(a,b)$ satisfies $\tau_a(M) \ge \alpha q^{r(M)}$, then Theorem 6.0.8 and definition of α imply that M has a $\mathrm{PG}(n-1,q')$-minor for some $q' > q-1$, giving the result. $\square$

7.1 Roundness

The first step in our proof of Theorem 6.0.7 is a connectivity reduction, along the same lines as the one found in Chapter 3. In contrast to the care we had to take in Part I when dealing with specific growth rate functions, the huge, unspecified constant in the statement of Theorem 6.0.7 makes the reduction nearly effortless. The more general size parameter τ_a poses no difficulty, and we are not required to weaken our notion of connectivity; roundness in its usual form will suffice.

The reduction is just one easy lemma:

Lemma 7.1.1. *Let $a \ge 1$ and $q \ge 2$ be integers, and $\alpha \ge 0$ be a real number. If M is a matroid with $\tau_a(M) \ge \alpha q^{r(M)}$, then M has a round restriction N such that $\tau_a(N) \ge \alpha q^{r(N)}$.*

Proof. If $r(M) \leq 1$, then M is round; assume that $r(M) > 1$, and M is not round. There are sets $A, B \subseteq E(M)$ such that $r(M|A) < r(M)$, $r(M|B) < r(M)$ and $A \cup B = E(M)$. Now, $\tau_a(M|A) + \tau_a(M|B) \geq \tau_a(M) \geq \alpha q^{r(M)}$, so at least one of $M|A$ or $M|B$ satisfies $\tau_a \geq \frac{1}{2}\alpha q^{r(M)} \geq \alpha q^{r(M)-1}$. The lemma follows by induction. $\square$

Again along the lines of Part I, the way we exploit roundness of M is to contract one restriction of M into the span of another restriction of larger rank. The following lemma encapsulates this:

Lemma 7.1.2. *Let M be a round matroid, and $X, Y \subseteq E(M)$ be sets with $r_M(X) \leq r_M(Y)$. There is a minor N of M so that $N|X = M|X$, $N|Y = M|Y$, and Y is spanning in N.*

Proof. Let $C \subseteq E(M)$ be maximal such that $(M /\!\!/ C)|X = M|X$ and $(M /\!\!/ C)|Y = M|Y$. Since roundness is preserved by contraction, $M /\!\!/ C$ is round. By maximality of C, every $e \in E(M /\!\!/ C)$ is spanned by either X or Y. We have $r_{M /\!\!/ C}(X) = r_M(X) \leq r_M(Y) = r_{M /\!\!/ C}(Y)$. If $r_{M /\!\!/ C}(Y) < r(M /\!\!/ C)$, then $(\mathrm{cl}_{M /\!\!/ C}(X), \mathrm{cl}_{M /\!\!/ C}(Y))$ is a pair of sets of rank less than $r(M /\!\!/ C)$ whose union is $E(M /\!\!/ C)$, contradicting roundness of $M /\!\!/ C$. Therefore, Y is spanning in $M /\!\!/ C$, and $N = M /\!\!/ C$ satisfies the lemma. $\square$

7.2 Stacks

We now define an 'obstruction' to GF(q)-representability. Like a constellation in Chapter 4, this structure takes the form of a matroid of small rank that is 'far' from being GF(q)-representable, due to many of its minors having q-long lines.

If $m \geq 0$ and $q \geq 1$ are integers, then a matroid S is an (m, q)*-stack* if $r(S) = 2m$, and S has a basis $B(S) = \{e_1, e'_1 \ldots, e_m, e'_m\}$ so that, for each $1 \leq i \leq m$, the line spanned by e_i and e'_i in $S /\!\!/ (\{e_1, e'_1 \ldots, e_{i-1}, e'_{i-1}\})$ is q-long.

Note that an empty matroid is a $(0, q)$-stack. We can obtain smaller stacks from a stack in two different ways. The following fact, which we use freely, is clear.

Lemma 7.2.1. *If S is an (m, q)-stack, and $i \in \{0, 1, \ldots, m\}$ is an integer, then*

- *$S|\,\mathrm{cl}_S(\{e_1, e'_1, \ldots, e_i, e'_i\})$ is an (i, q)-stack, and*
- *$S /\!\!/ \{e_1, e'_1, \ldots, e_i, e'_i\}$ is an $(m - i, q)$-stack.*

As a consequence of these hereditary properties, given a set with low connectivity to a huge stack, we can make the set skew to a large stack in a minor by contracting some of the original stack:

Lemma 7.2.2. *Let a, m, q be integers with $m, q \geq 1$ and $a \geq 0$. If M is a matroid with an $((a+1)m, q)$-stack restriction S, and $X \subseteq E(M)$ is a set satisfying $\sqcap_M(X, E(S)) \leq a$, then there exists $C \subseteq E(S)$ so that $(M /\!\!/ C)|E(S)$ has an (m, q)-stack restriction S', and X and $E(S')$ are skew in $M /\!\!/ C$.*

Proof. If $\sqcap_M(X, E(S)) = 0$, then $C = \varnothing$ gives the result. Otherwise, let $m' = (a+1)m$, and let k be minimal so that $\{e_1, e'_1, \ldots, e_k, e'_k\}$ is not skew to X in M. If $k > m$, then the points $e_1, e'_1, \ldots, e_m, e'_m$ span an (m, q)-stack, skew to X in M, so the lemma holds with $C = \varnothing$. If $k \leq m$, then let $M' = M /\!\!/ \{e_1, e'_1, \ldots, e_m, e'_m\}$. The points $\{e_{m+1}, e'_{m+1}, \ldots, e_{m'}, e'_{m'}\}$ span an $(m' - m, q)$-stack S' in M'. We have $m' - m = am$, and by definition of k, we have $\sqcap_{M'}(X, E(S')) \leq \sqcap_M(X, E(S)) - 1 \leq a - 1$, so the result follows by induction on a. $\square$

This low connectivity is obtained via the following lemma, which applies more generally. We will just use the case when $M|Y$ is a stack.

Lemma 7.2.3. *If $M \in \mathcal{U}(a, b)$ is a matroid, and $Y \subseteq E(M)$, then there is a set $X \subseteq E(M)$ so that $\tau_a(M|X) \geq \binom{b-1}{a}^{r_M(Y)-a} \tau_a(M)$, and $\sqcap_M(X, Y) \leq a$.*

Proof. Let B be a basis for M, containing a basis B_Y for $M|Y$. We have $r(M /\!\!/ (B - B_Y)) = r_M(Y)$, so $\tau_a(M /\!\!/ (B - B_Y)) \leq \binom{b-1}{a}^{r_M(Y)-a}$ by Theorem 5.6.2. Applying a majority argument to a minimum a-cover of $M /\!\!/ (B - B_Y)$ gives a set $X \subseteq E(M)$ so that $r_{M /\!\!/ (B-B_Y)}(X) \leq a$, and $\tau_a(M|X) \geq \binom{b-1}{a}^{a-r_M(Y)} \tau_a(M)$. Moreover, $B - B_Y$ is skew to Y in M, so $\sqcap_M(X, Y) \leq \sqcap_{M /\!\!/ (B-B_Y)}(X, Y) \leq a$. $\square$

7.3 Stacking Up

In this section, we show that a sufficiently dense matroid has a dense contraction-minor with a large stack restriction. Our first lemma simply finds this minor:

Lemma 7.3.1. *There is a real-valued function $\alpha_{7.3.1}(a, b, m, q, \lambda)$ so that, for any real number $\lambda > 1$ and integers a, b, m, q with $1 \leq a < b$, $m \geq 0$ and $q \geq 2$, if $M \in \mathcal{U}(a, b)$ is a matroid such that $\tau_a(M) \geq \alpha_{7.3.1}(a, b, m, q) q^{r(M)}$, then there is a contraction-minor M' of M such that M' has an (m, q)-stack restriction, and $\tau_a(M') \geq \lambda q^{r(M')}$.*

Proof. Let $\lambda > 1$ be a real number, and a, b, m, q be integers with $1 \leq a < b$, $m \geq 0$ and $q \geq 2$. Let $d = \max(q+2, \binom{b}{a})$, let $\mu = d^a q^{2m+1}\lambda$, and let $\beta = q^2 d^{a+1}$. Set $\alpha_{7.3.1}(a, b, m, q, \lambda) = \mu\beta^m$. We modify the problem to work with τ^d instead of τ_a:

7.3.1.1. *If $0 \leq i \leq m$, and $M \in \mathcal{U}(a, b)$ is a matroid such that $\tau^d(M) > \beta^i\mu(q^{r(M)} - 1)$, then there is a contraction-minor M' of M such that $r(M') > 2i$, $\tau^d(M') > \mu(q^{r(M')-2i} - 1)$, and M' has an (i, q)-stack restriction.*

Proof of claim: Clearly $r(M) > 0$, so when $i = 0$, then $C = \varnothing$ will satisfy the claim. Fix $i > 0$, and suppose that the claim holds for smaller i. Let M be contraction-minimal such that $\tau^d(M) > \beta^i q^{r(M)}$; we will show that there is a set $C \subseteq E(M)$ such that $r_M(C) < a$, and $M /\!\!/ C$ has a $U_{2,q+2}$-restriction.

Let $e \in E(M)$ be a nonloop. Let $\mathcal{F}$ be a d-minimal cover of M, and $\mathcal{F}'$ be a d-minimal cover of $M /\!\!/ e$. We consider two cases:

Case 1: Every set in $\mathcal{F}$ and $\mathcal{F}'$ has rank 1.

We have $|\mathcal{F}| = \varepsilon(M)$ and $|\mathcal{F}'| = \varepsilon(M /\!\!/ e)$, and moreover $\mathrm{wt}^d_M(\mathcal{F}) = d|\mathcal{F}|$, and $\mathrm{wt}^d_{M/\!\!/ e}(\mathcal{F}') = d|\mathcal{F}'|$, so $d\varepsilon(M) = d|\mathcal{F}| > \beta^i\mu(q^{r(M)-2i} - 1)$ by hypothesis, and minimality of M gives $d\varepsilon(M /\!\!/ e) = d|\mathcal{F}'| = \tau^d(M /\!\!/ e) \leq \beta^i\mu(q^{r(M/\!\!/ e)-2i} - 1)$. Combining these two inequalities and using the fact that $\frac{\beta^i\mu}{d} \geq \lambda \geq 1$ gives

$$\varepsilon(M) > \frac{\beta^i\mu}{d}(q^{r(M)-2i} - 1) \geq \frac{\beta^i\mu}{d}(q^{r(M/\!\!/ e)+1-2i} - q) + 1 \geq q\varepsilon(M /\!\!/ e) + 1.$$

By Lemma 1.1.1 and a majority argument, M therefore has a $U_{2,q+2}$-restriction.

Case 2: Either $\mathcal{F}$ or $\mathcal{F}'$ contains a set of rank at least 2.

Suppose that $F \in \mathcal{F}$ satisfies $r_M(F) \geq 2$. By Lemma 6.4.3, F is d-thick (and therefore $(q+2)$-thick) in M, and by Lemma 6.4.5, we have $r_M(F) \leq a$. By Lemma 6.2.1, contracting points of $M|F$ until F has rank 2 gives a $(q+2)$-thick, rank-2 matroid, which clearly has a $U_{2,q+2}$-restriction. We need to contract at most $r(F) - 2 \leq a - 2$ such points, giving the set C we want. If $\mathcal{F}'$ contains a set of rank at least 2, then the argument is similar, except we need to contract e also.

Now, $M /\!\!/ C$ has a $U_{2,q+2}$ restriction L. Let $\{e_1, e_1'\}$ be a basis for L in $M /\!\!/ C$. By

Lemma 6.4.1, we have

$$\begin{aligned}
\tau^d(M /\!\!/ (C \cup L)) &\geq d^{-r_M(C \cup L)} \tau^d(M) \\
&\geq d^{-(a+1)} \beta^i \mu(q^{r(M)-2i} - 1) \\
&= 2q^2 \beta^{-1} \mu(q^{r(M)-2(i)} - 1) \\
&\geq \beta^{i-1} \mu(2q^{r(M)-2i+2} - 2q^2) \\
&> \beta^{i-1} \mu(q^{r(M)-2(i-1)} - 1),
\end{aligned}$$

where the last inequality follows from the fact that $q^{r(M)-2i+2} \geq q^2$. By the inductive hypothesis, $M /\!\!/ (C \cup L)$ therefore has a contraction-minor $N = M /\!\!/ (C \cup L \cup C')$ such that $r(N) > 2(i-1)$, and $\tau^d(N) > \mu(q^{r(N)-2(i-1)} - 1)$, and N has an $(i-1, q)$-stack restriction. Let $M' = M /\!\!/ (C \cup C')$. Now, L is a $U_{2,q+2}$-restriction of M', and $N = M' /\!\!/ L$ has an $(i-1, q)$-stack restriction, so M' has an (i, q)-stack restriction. Moreover, $r(N) = r(M') - 2$, so $r(M') > 2i$, and $\tau^d(M') \geq \tau^d(N) > \mu(q^{r(N)-2(i-1)} - 1) = \mu(q^{r(M')-2i} - 1)$. Therefore M' satisfies the claim. □

Now, let $M \in \mathcal{U}(a, b)$ be a matroid such that $\tau_a(M) \geq \alpha_{7.3.1}(a, b, m, q, \lambda) q^{r(M)}$; we show that M has the contraction-minor M' we want. By Lemma 6.4.5, we have $\tau_a(N) \leq \tau^d(N) \leq d^a \tau_a(N)$ for all minors N of M. So $\tau^d(M) \geq \mu \beta^m q^{r(M)} > \mu \beta^m (q^{r(M)} - 1)$. By the claim, M has a contraction-minor M' such that $r(M') > 2m$, and $\tau^d(M') > \mu(q^{r(M)-2m} - 1) \geq \mu q^{r(M)-2m-1}$, and M' has an (m, q)-stack restriction. Now, $\tau_a(M') \geq d^{-a} \tau^d(M) = d^{-a} q^{-2m-1} \mu q^{r(M')} = \lambda q^{r(M')}$, so M' is the desired contraction-minor. □

The next lemma refines the previous one, showing that we can assume the density of the minor we find to be 'skew' to its large stack restriction.

Lemma 7.3.2. *There is a real-valued function $\alpha_{7.3.2}(a, b, m, q, \lambda)$ so that, for any real number $\lambda > 0$, and integers a, b, m, q with $m \geq 0$, $1 \leq a < b$ and $q \geq 2$, if $M \in \mathcal{U}(a, b)$ is a matroid satisfying $\tau_a(M) \geq \alpha_{7.3.2}(a, b, m, q, \lambda) q^{r(M)}$, then there is a contraction-minor M' of M such that*

- *M' has an (m, q)-stack restriction S, and*
- *there is a set $X \subseteq M'$, skew to $E(S)$ in M', such that $\tau_a(M'|X) \geq \lambda q^{r(M'|X)}$.*

Proof. Let $\lambda > 0$ be a real number, and a, b, m, q be integers with $m \geq 0$, $1 \leq a < b$ and $q \geq 2$. Let $\lambda' = \binom{b-1}{a}^{4(a+1)m-a} \lambda$. We set $\alpha_{7.3.2}(a, b, m, q, \lambda) = \alpha_{7.3.1}(a, b, (a+1)m, q, \lambda')$.

If $M \in \mathcal{U}(a,b)$, and $\tau_a(M) > \alpha_{7.3.2}(a,b,m,q,\lambda)q^{r(M)}$, then by Lemma 7.3.1, there is a contraction-minor M_1 of M such that $\tau_a(M_1) \geq \lambda' q^{r(M_1)}$, and M_1 has an $((a+1)m,q)$-stack restriction S_1. By Lemma 7.2.3, there is a set $X \subseteq E(M_1)$ such that $\sqcap_{M_1}(X, E(S_1)) \leq a$, and

$$\tau_a(M_1|X) \geq \binom{b-1}{a}^{r(S_1)-a} \tau_a(M_1) \geq \binom{b-1}{a}^{2(a+1)m} \lambda q^{r(M_1)}.$$

By Lemma 7.2.2, there is a set $C \subseteq E(S_1)$ such that $(M_1 /\!\!/ C)|E(S_1)$ has an (m,q)-stack restriction S, and the sets X and $E(S)$ are skew in $M_1 /\!\!/ C$. Now, $r_{M_1}(C) \leq r(S_1) = 2(a+1)m$, so by Lemma 6.1.1, $\tau_a((M_1 /\!\!/ C)|X) \geq \binom{b-1}{a}^{-2(a+1)m} \tau_a(M_1|X) \geq \lambda q^{r(M_1)} \geq \lambda q^{r((M_1 /\!\!/ C)|X)}$, and $M' = M_1 /\!\!/ C$ satisfies the lemma. $\square$

7.4 Rooted Stacks

We constructed stacks as examples of matroids 'far' from being GF(q)-representable. We make this idea more concrete by showing that a matroid in $\mathcal{U}(a,b)$ with a spanning projective geometry restriction R over GF(q), as well as a large stack S whose basis $B(S)$ is contained in R, will have a large projective geometry minor over a much larger field than GF(q).

Theorem 7.4.1. *There is an integer-valued function $f_{7.4.1}(a,b,n,q)$ so that, for any prime power q, and integers n,a,b,s with $n \geq 1$, $1 \leq a < b$, and $s \geq f_{7.4.1}(a,b,n,q)$, if $M \in \mathcal{U}(a,b)$ is a matroid such that*

- *M has a* PG$(r(M)-1,q)$*-restriction R, and*
- *M has an (s,q)-stack restriction S, with $B(S) \subseteq E(R)$,*

then M has a PG$(n-1,q')$*-minor for some $q' \geq q^2$.*

Proof. Let q be a prime power, and n,a,b be integers with $n \geq 1$ and $1 \leq a < b$. Set $f_{7.4.1}(n,q,a,b)$ to be an integer $m > 1$, large enough so that

$$q^{2s} - 1 \geq (q^{2a}-1)s^{f_{6.0.8}(n,q^2-1,a,b)}(q^2-1)^s$$

for all $s \geq m$. Let $s \geq m$, and M be a matroid with a PG$(r(M)-1,q)$-restriction R, and an (s,q)-stack restriction S such that $B(S) = \{e_1, e_1', \ldots, e_s, e_s'\} \subseteq E(R)$. For each $0 \leq i \leq s$, let $M_i = M /\!\!/ \{e_1, e_1', \ldots, e_i, e_i'\}$. Since $B(S) \subseteq E(R)$, we have $\mathrm{si}(M_i|E(R)) \cong$

$\mathrm{PG}(r(M_i)-1,q)$, and the points $\{e_{i+1}, e'_{i+1}, \ldots, e_s, e'_s\}$ are the basis for an $(s-i,q)$-stack S_i that is a restriction of M_i. For each i, the points e_{i+1} and e'_{i+1} therefore span a line of M_i containing at least $q+2$ points, so there is a nonloop f_{i+1} of M_i that is not parallel to any nonloop of $M_i|E(R)$, such that $f_{i+1} \in \mathrm{cl}_{M_i}(\{e_{i+1}, e'_{i+1}\})$. By its definition, f_{i+1} is a loop of M_{i+1}.

Let $K = \{f_i : 1 \le i \le s\}$. Since $f_1, \ldots, f_i$ are loops of M_i for each i, and f_{i+1} is a nonloop of M_i, it follows that $r_{M_i}(K) = s-i$ for each i. Our first claim states that sets in R have limited local connectivity to K.

7.4.1.1. *Each set $X \subseteq E(R)$ satisfies $\sqcap_M(X,K) \le \frac{1}{2} r_M(X)$.*

Proof of claim: We show that, for each $X \subseteq E(R)$ and $0 \le i \le s$, we have $\sqcap_{M_i}(X,K) \le \frac{1}{2} r_{M_i}(X)$. When $i = s$, this is clear, as $r_{M_s}(K) = 0$. Fix $0 \le i < s$, and suppose inductively that the result holds for $i+1$. Let $L = \mathrm{cl}_{M_i}(\{e_{i+1}, e'_{i+1}\})$; note that $f_{i+1} \in L$, and $M_{i+1} = M_i /\!\!/ L$. We have $r_{M_i /\!\!/ L}(K) = r_{M_{i+1}}(K) = s-(i+1) = r_{M_i}(K)-1$, so $\sqcap_{M_i}(L,K) = 1$. By the inductive hypothesis, $\sqcap_{M_{i+1}}(X,K) \le \frac{1}{2} r_{M_{i+1}}(X) = \frac{1}{2} r_{M_i}(X) - \frac{1}{2} \sqcap_{M_i}(X,L)$. Now,

$$\begin{aligned}
\sqcap_{M_i}(X,K) &= \sqcap_{M_i /\!\!/ L}(X,K) + \sqcap_{M_i}(K,L) + \sqcap_{M_i}(X,L) - \sqcap_{M_i}(X \cup K, L) \\
&= \sqcap_{M_{i+1}}(X,K) + 1 + \sqcap_{M_i}(X,L) - \sqcap_{M_i}(X \cup K, L) \\
&\le \tfrac{1}{2} r_{M_i}(X) + 1 + \tfrac{1}{2} \sqcap_{M_i}(X,L) - \sqcap_{M_i}(X \cup K, L).
\end{aligned}$$

It thus suffices to show that $\sqcap_{M_i}(X \cup K, L) \ge \frac{1}{2} \sqcap_{M_i}(X,L) + 1$. If X is skew to L in M_i, then the right hand side is 1, and $\sqcap_{M_i}(X \cup K, L) \ge \sqcap_{M_i}(K,L) = 1$. If X is not skew to L in M_i, then by modularity of flats in a projective geometry, $\mathrm{cl}_{M_i}(X)$ contains a nonloop x of $M_i|(L \cap E(R))$, and $f_{i+1} \ne x$, because f_{i+1} is not a point of R. Therefore, $\mathrm{cl}_{M_i}(X \cup K)$ contains the basis $\{x, f_{i+1}\}$ of $M_i|L$, so $\sqcap_{M_i}(X \cup K, L) = r_{M_i}(L) = 2$, and $\frac{1}{2} \sqcap_{M_i}(X,L) + 1 \le \frac{1}{2} r_{M_i}(L) + 1 = 2$, so the inequality holds, giving the claim. $\square$

Let $F = \mathrm{cl}_M(B(S))$. We have $r(M|F) = 2s$, and $R|F \cong \mathrm{PG}(2s-1,q)$. The set K has rank s in M, so $r((M|F) /\!\!/ K) = s$. By the claim, we have $\tau_a((M|F) /\!\!/ K) \ge \tau_{2a}(M|F) \ge \frac{q^{2s}-1}{q-1} / \frac{q^{2a}-1}{q-1} = \frac{q^{2s}-1}{q^{2a}-1}$. The result now follows from definition of s, and Theorem 6.0.8.

$\square$

7.5 Spanning Projective Geometries

In this section, we consider matroids with a spanning projective geometry restriction. We first require a small result regarding single-element extensions of projective geometries that essentially follows from Lemmas 2.5.1 and 2.5.2.

Lemma 7.5.1. *Let q be a prime power. If M is a simple matroid such that $r(M) \geq 2$, and $f \in E(M)$ satisfies $M \backslash f \cong \mathrm{PG}(r(M)-1, q)$, then there is a set $Z \subseteq E(M) - \{f\}$ such that $M /\!\!/ Z$ is a rank-2 matroid containing $q+2$ points.*

The main lemma of this section states that, given a matroid in $\mathcal{U}(a, b)$ with a spanning projective geometry restriction R, we can either contract a small set so that every point is parallel to a point of R, or we can find a large-rank minor with a projective geometry restriction, a line of which spans an extra point. The second outcome will be useful to increase the size of a stack in the next section.

Lemma 7.5.2. *There is an integer-valued function $f_{7.5.2}(a, b, n)$ so that, for any prime power q and integers n, a, b with $n \geq 2$ and $1 \leq a < b$, if $M \in \mathcal{U}(a, b)$ is a matroid with a $\mathrm{PG}(r(M)-1, q)$-restriction R, then either*

(i) There is a set $K \subseteq E(M)$ such that $r(M /\!\!/ K) \geq n$, and points $e, e' \in E(R)$ such that $\mathrm{cl}_{M /\!\!/ K}(\{e, e'\})$ contains $q+2$ points of $M /\!\!/ K$, or

(ii) There is a set $C \subseteq E(M)$ such that $\mathrm{si}(M /\!\!/ C) \cong \mathrm{si}((M /\!\!/ C)|E(R))$, and $r_M(C) \leq f_{7.5.2}(a, b, n)$.

Proof. Let n, a, b be integers with $n \geq 2$ and $1 \leq a < b$. Set

$$f_{7.5.2}(a, b, n) = \max\left(n + a + 1, a\binom{b-1}{a}^{n}\right).$$

Let $M \in \mathcal{U}(a, b)$ be a matroid with a $\mathrm{PG}(r(M)-1, q)$-restriction R. If $r(M) \leq n + a + 1$, then (ii) trivially holds for M, as we can choose C to be a basis. Therefore, in establishing one of the two outcomes for M, we may assume that $r(M) \geq n + a + 2$.

Let F be a flat of R with $r_M(F) = r(M) - n - a \geq 2$, and let B_F be a basis for F. We have $r(M /\!\!/ F) = n + a$, so Theorem 5.6.2 gives $\tau_a(M /\!\!/ F) \leq \binom{b-1}{a}^n$. Let $\mathcal{X}$ be a cover of $M /\!\!/ F$ with flats of rank a such that $|\mathcal{X}| \leq \binom{b-1}{a}^n$. Each $X \in \mathcal{X}$ is a flat of M containing F, of rank $r_M(F) + a = r(M) - n$. For each $X \in \mathcal{X}$, let B_X be a basis for $(M /\!\!/ F)|X$. Each B_X is skew to F in M, so $\mathrm{PG}(r_M(F)-1, q) \cong M|F = (M /\!\!/ B_X)|F$. Note that $|B_X| = a$, and F is spanning in $(M /\!\!/ B_X)|X$ for each $X \in \mathcal{X}$.

If, for some $X \in \mathcal{X}$, there is a nonloop f of $(M /\!\!/ B_X)|X$, not parallel to any nonloop of $(M /\!\!/ B_X)|F$, then the matroid $(M /\!\!/ B_X)|(F \cup \{f\})$ is a simple, single-element extension of a projective geometry. By Lemma 7.5.1, there is a set $Z \subseteq F$ so that $(M /\!\!/ (B_X \cup Z))|(F \cup \{f\})$ is a rank-2 matroid containing $q+2$ points. At most one such point does not lie in F, so there is a basis $\{e, e'\} \subseteq F \subseteq E(R)$ of this matroid. Let $K = B_X \cup Z$. We have $K \subseteq X$, so $r(M/\!\!/K) \geq r(M) - r_M(X) = n$, and e and e' span a line of $M/\!\!/K$ containing $(q+2)$ points, giving the first outcome.

We may therefore assume, for all $X \in \mathcal{X}$, and every nonloop f of M contained in X, that f is either a loop of $M/\!\!/B_X$, or is parallel in $M/\!\!/B_X$ to a nonloop of $(M/\!\!/B_X)|F$. Let $C = \bigcup_{X \in \mathcal{X}}(B_X)$. Each B_X has size a, so $r_M(C) \leq a|\mathcal{X}| \leq a\binom{b-1}{a}^n \leq f_{7.5.2}(a, b, n)$. Moreover, $M/\!\!/C$ is a contraction-minor of $M/\!\!/B_X$ for every X, and every $e \in E(M)$ is in some $X \in \mathcal{X}$, so every $E(M)$ is either a loop, or parallel to a nonloop of $F \subseteq E(R)$ in the matroid $M/\!\!/C$. Therefore, the second outcome holds. $\square$

Lemma 7.5.3. *There is an integer-valued function $f_{7.5.3}(a, b, n, s)$ so that: for any prime power q and integers a, b, s, t, n such that $1 \leq a < b$, $s \geq 0$, $n \geq 2s+2$, and $t \geq f_{7.5.3}(a, b, n, s)$, if $M \in \mathcal{U}(a, b)$ is a matroid such that*

- *M has a $\mathrm{PG}(r(M)-1, q)$-restriction R, and*
- *M has an (s, q)-stack restriction S such that $B(S) \subseteq E(R)$, and*
- *M has a (t, q)-stack restriction T,*

then there is a minor M' of M such that $r(M') \geq n$, and M' has a $\mathrm{PG}(r(M')-1, q)$-restriction R', and an $(s+1, q)$-stack restriction S' such that $B(S') \subseteq E(R')$.

Proof. Let a, b, n, s be integers such that $1 \leq a < b$, $s \geq 0$ and $n \geq 2s+2$. Let $m = 2s + f_{7.5.2}(a, b, n)$, and $f_{7.5.3}(a, b, n, s) = (s+1)(m+1) + m + n$. Let $t \geq f_{7.5.3}(a, b, n, s)$ be an integer, and $M \in \mathcal{U}(a, b)$ be a matroid with a $\mathrm{PG}(r(M)-1, q)$-restriction R, an (s, q)-stack restriction S such that $B(S) \subseteq E(R)$, and a (t, q)-stack restriction T. Let R_0 be a $\mathrm{PG}(r(M_0)-1, q)$-restriction of $M/\!\!/B(S)$ such that $E(R_0) \subset E(R)$. We apply Lemma 7.5.2 to $M/\!\!/B(S)$, splitting into cases depending on which outcome of the lemma holds:

Case 1: 7.5.2(i) holds.

There is a set $K \subseteq E(M)$ such that $r(M/\!\!/(B_S \cup K)) \geq n$, and there are points $e, e' \in E(R_0)$ that span a line of $M/\!\!/(B_S \cup K)$ containing $q+2$ points. Let $M' = M/\!\!/K$; the set $B(S) \cup \{e, e'\}$ is an independent set of $M'|E(R)$, so there is a $\mathrm{PG}(r(M')-1, q)$-restriction R' of M' such that $B(S) \cup \{e, e'\} \subseteq E(R')$. Moreover, since $M'|E(S) = S$ is an (s, q)-stack, and e and e' span a line of $M'/\!\!/B(S)$ containing $q+2$ points, the set $B(S) \cup \{e, e'\}$ is a basis $B(S')$ for an $(s+1, q)$-stack restriction S' of M', with $B(S') \subseteq E(R')$, giving the lemma.

Case 2: 7.5.2(ii) holds.

There is a set $C_1 \subseteq E(M)$ such that $r_{M/\!\!/B(S)}(C_1) \leq f_{7.5.2}(a,b,n)$ and $\mathrm{si}(M/\!\!/(B(S) \cup C_1)) \cong \mathrm{si}(M/\!\!/(B(S) \cup C_1))|E(R_0))$. Since $E(R_0) \subseteq E(R)$, every nonloop of $M/\!\!/(B(S) \cup C_0)$ is therefore parallel to a nonloop of $(M/\!\!/(B(S) \cup C_0))|E(R)$. Let T_1 be an $((s+1)(m+1), q)$-stack restriction of T.

$$\begin{aligned}\sqcap_M(E(T_1), B(S) \cup C_1) &\leq r_M(B(S) \cup C_1)\\ &= r_M(B(S)) + r_{M/\!\!/B_S}(C_1)\\ &\leq 2s + f_{7.5.2}(a,b,n)\\ &= m,\end{aligned}$$

and T_1 is a $((s+1)(m+1), q)$-stack, so by Lemma 7.2.2, there exists $C_2 \subseteq E(T_1)$ such that $M' = (M/\!\!/(B(S) \cup C_1))/\!\!/C_2$ has an $(s+1, q)$-stack restriction. But M' is a contraction-minor of $M/\!\!/(B(S) \cup C_1)$, so every nonloop of M' is parallel to a nonloop of $M'|E(R)$, and therefore $M'|E(R)$ also has an $(s+1, q)$-stack restriction S'. If B' is a basis for $M'|E(R)$ containing $B(S')$, then there is a $\mathrm{PG}(r(M')-1, q)$-restriction R' of M' with basis B', and $B(S') \subseteq E(R')$ by construction. Moreover,

$$\begin{aligned}r(M') &= r(M) - r_M(B(S) \cup C_1 \cup C_2)\\ &\geq r(M) - r_M(B(S) \cup C_1) - r_M(C_2)\\ &\geq 2t - m - r(T_1)\\ &\geq 2((s+1)(m+1) + n + m) - m - 2(s+1)(m+1)\\ &\geq n,\end{aligned}$$

so M' satisfies the lemma. □

7.6 The Main Result

We are now almost ready to prove Theorem 6.0.7; we first combine the preceding material into a technical lemma:

Lemma 7.6.1. *There is a real-valued function $\alpha_{7.6.1}(a,b,n,q,s)$ so that, for any prime power q, and integers a,b,s,n with $1 \leq a < b$ and $n \geq 2s \geq 0$, if $M \in \mathcal{U}(a,b)$ is a matroid such that $\tau_a(M) \geq \alpha_{7.6.1}(a,b,n,q,s)q^{r(M)}$, then either*

- *M has a $\mathrm{PG}(n-1, q')$-minor for some $q' > q$, or*

- *M has a rank-n minor N with a* $\mathrm{PG}(n-1, q)$*-restriction R, and an (s, q)-stack restriction S such that $B(S) \subseteq E(R)$.*

Proof. Let q be a prime power, and a, b, s, n be integers with $1 \leq a < b$ and $n \geq 2s \geq 0$. Set $\alpha_{7.6.1}(a, b, n, q, s) = \alpha_{7.0.1}(a, b, n, q)$ if $s = 0$; if $s > 1$, let $t_s = f_{7.5.3}(a, b, n, s-1)$, and recursively set $\alpha_{7.6.1}(a, b, n, q, s) = \alpha_{7.3.2}(a, b, t_s, q, \alpha_{7.6.1}(a, b, \max(n, 2t_s), q, s-1))$

Let $M \in \mathcal{U}(a, b)$ be a matroid such that $\tau_a(M) \geq \alpha_{7.6.1}(a, b, n, q, s) q^{r(M)}$. If $s = 0$, then, since an empty matroid is an (s, q)-stack, the lemma follows from Corollary 7.0.1. Thus, suppose inductively that $s > 0$, and that $\alpha_{7.6.1}(\cdot, \cdot, \cdot, \cdot, s-1)$ satisfies the lemma. By Lemma 7.1.1, there is a round restriction M_1 of M with $\tau_a(M_1) \geq \alpha_{7.6.1}(a, b, n, q, s) q^{r(M_1)}$. By Lemma 7.6.1, there is a contraction-minor M_2 of M_1 having a (t_s, q)-stack restriction T, and there is a set $X \subseteq E(M_2)$, skew to $E(T)$ in M_2, such that $\tau_a(M_2|X) \geq \alpha_{7.6.1}(a, b, \max(n, 2t_s), q, s-1) q^{r(M_2|X)}$. By the inductive hypothesis, $M_2|X$ either has a $\mathrm{PG}(\max(n, 2t_s)-1, q')$-minor for some $q' > q$, or a rank-n minor N' with a $\mathrm{PG}(\max(n, 2t_s)-1, q)$-restriction R', and an $(s-1, q)$-stack restriction S' such that $B(S') \subseteq E(R')$. In the first case, the lemma holds, so we may assume the second. Let $N' = (M_2|X) /\!\!/ C \backslash D$, where N' is a spanning restriction of $M_2 /\!\!/ C$.

Since M_1 is round, so is $M_2 /\!\!/ C$, and since $C \subseteq X$ is skew to $E(T)$ in M_2, we have $(M_2 /\!\!/ C)|E(T) = M_2|E(T) = T$. We have $r(N') = \max(n, 2t_s) \geq r(T)$, so by Lemma 7.1.2, there is a minor M_3 of $M_2 /\!\!/ C$ that has N' as a spanning restriction, and T as a restriction. By definition of T and N', the matroid M_3 satisfies the conditions listed in Lemma 7.5.3, so there is a minor N of M_3 such that $r(N) \geq n$, and N has a $\mathrm{PG}(r(N)-1)$-restriction R, and an (s, q)-stack restriction S such that $B(S) \subseteq E(R)$. This gives the lemma. $\square$

Theorem 6.0.7, which we restate here, now follows from Lemmas 7.6.1 and 7.4.1 by setting $\alpha_{6.0.7}(a, b, n, q) = \alpha_{7.6.1}(a, b, n, q, f_{7.4.1}(a, b, n, q))$.

Theorem 6.0.7. *There is a real-valued function $\alpha_{6.0.7}(a, b, n, q)$ so that, for any integers a, b, n, q with $n \geq 1$, $q \geq 2$ and $1 \leq a < b$, if $M \in \mathcal{U}(a, b)$ is a matroid such that $\tau_a(M) \geq \alpha_{6.0.7}(a, b, n, q) q^{r(M)}$, then M has a* $\mathrm{PG}(n-1, q')$*-minor for some $q' > q$.*

We can now restate and prove Theorem 5.4.3.

Theorem 7.6.2. *If $a \geq 1$ is an integer, and $\mathcal{M}$ is a minor-closed class of matroids, then either:*

1. *$\tau_a(M) \leq r(M)^{c_{\mathcal{M}}}$ for all $M \in \mathcal{M}$, or*

2. *There is a prime power q such that $\tau_a(M) \leq c_{\mathcal{M}} q^{r(M)}$ for all $M \in \mathcal{M}$, and $\mathcal{M}$ contains all* $\mathrm{GF}(q)$*-representable matroids, or*

3. *$\mathcal{M}$ contains all rank-*$(a+1)$ *uniform matroids.*

Proof. We may assume that the third outcome does not hold; let $b > a$ be an integer such that $\mathcal{M} \subseteq \mathcal{U}(a, b)$. As $U_{a+1,b}$ is $\mathrm{GF}(q)$-representable whenever $q \geq b - 1$ (see [19]), $\mathrm{PG}(a, q') \notin \mathcal{M}$ for all $q' \geq b - 1$, as such a projective geometry has a $U_{a+1,b}$-restriction.

If, for some integer $n > a$, we have $\tau_a(M) < r(M)^{f_{6.0.6}(a,b,n)}$ for all $M \in \mathcal{M}$ of rank at least 2, then the first outcome holds. We may therefore assume that, for all $n > a$, there exists a matroid $M_n \in \mathcal{M}$ such that $r(M_n) \geq 2$ and $\tau_a(M_n) \geq r(M_n)^{f_{6.0.6}(a,b,n)}$.

By Theorem 6.0.6, it follows that for all $n > a$, there exists a prime power q'_n such that $\mathrm{PG}(n-1, q'_n) \in \mathcal{M}$. We have $q'_n < b - 1$ for all n, so there are finitely many possible q'_n, and so there is a prime power $q_0 < b - 1$ such that $\mathrm{PG}(n-1, q_0) \in \mathcal{M}$ for arbitrarily large n, giving $\mathcal{L}(q_0) \subseteq \mathcal{M}$.

Let q be maximal such that $\mathcal{L}(q) \subseteq \mathcal{M}$. Since $\mathrm{PG}(a, q') \notin \mathcal{M}$ for all $q' \geq b - 1$, the value q is well-defined, and moreover there is some n such that $\mathrm{PG}(n-1, q') \notin \mathcal{M}$ for all $q' > q$. Theorem 6.0.7 therefore gives $\tau_a(M) \leq \alpha_{6.0.7}(a, b, n, q) q^{r(M)}$ for all $M \in \mathcal{M}$, giving the second outcome. □

Chapter 8

Further Work

We wrap the thesis up in this chapter by discussing future directions that our work could be taken.

8.1 Exponential Density

In Chapter 1, we gave two conjectures on exponentially dense minor-closed classes, which we now restate:

Conjecture 1.5.14. *Let $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids. There exists an integer $k \geq 0$ such that every extremal matroid M in $\mathcal{M}$ of sufficiently large rank is, up to simplification, a k-element projection of a projective geometry over* $\mathrm{GF}(q)$.

Conjecture 1.5.16. *Let $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids. There exists an integer $k \geq 0$, and an integer d with $0 \leq d \leq \frac{q^{2k}-1}{q^2-1}$ so that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+k} - 1}{q - 1} - qd$$

for all sufficiently large n.

These conjectures present a clear direction to extend the work in Part I of this thesis. Conjecture 1.5.16 seems the more approachable, and could yield to similar techniques to those in Chapter 4, using Conjecture 1.5.14 as a guideline. The idea is that one should prove an unavoidable minor theorem analogous to Theorem 4.2.3, showing that a matroid M of sufficiently large rank with density exceeding $\frac{q^{r(M)+k}-1}{q-1}$, for some k, should have a large-rank $(k+1)$-element projection N of a projective geometry as a minor. This minor should have density roughly $\frac{q^{r(N)+k+1}-1}{q-1}$. As the Growth Rate Theorem tells us that this process can repeat only a bounded number of times for a given class $\mathcal{M}$, this unavoidable minor theorem would imply that growth rate functions must 'settle' at a given k, eventually taking the form in Conjecture 1.5.16 for this k. See the proof of Theorem 1.5.8 for a formal argument along these lines.

We have neglected here to mention the constant d, which certainly will surface in the details. In fact, we conjecture that the range of values for d is best-possible:

Conjecture 8.1.1. *Let q be a prime power, and d and $k \geq 0$ be integers with $0 \leq d \leq \frac{q^{2k}-1}{q^2-1}$. There is a minor-closed class of matroids $\mathcal{M}$ such that $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$ for all $n \geq k+1$.*

We saw in Theorem 2.6.2 that, to find the aforementioned projection N, it is enough to find a minor with a spanning projective geometry, and a rank-$(k+1)$ flat disjoint from this geometry. In Chapter 4, we did this in one case by constructing a minor with a spanning projective geometry over $\mathrm{GF}(q)$, as well as an $(s, q, k+1)$-constellation-restriction for large s. Such a constellation contains a large number of $(k+1)$-matchings of q-long lines, which was enough to find the required flat. It may be possible, even in a much more general setting, to show that a single matching of q-long lines is enough to find this flat:

Conjecture 8.1.2. *Let q be a prime power, and $k \geq 0$ be an integer. If M is a matroid with a $\mathrm{PG}(r(M)-1, q)$-restriction R, and M has a matching $\mathcal{L}$ of size k such that each $L \in \mathcal{L}$ is a q-long line of M, then there is a rank-k flat of M, disjoint from $E(R)$.*

Resolution of this conjecture would provide a simpler proof of the material in Chapter 4, as well as a powerful tool for the proof of Conjecture 1.5.16.

Conjecture 1.5.14 looks harder; while Theorem 1.5.3 characterises the high-rank extremal matroids in $\mathcal{U}(\ell)$ for any ℓ, and the same argument can be applied to find them for arbitrary base-q exponentially dense subclasses of $\mathcal{U}(q^2-1)$, we do not have nontrivial characterisations for many other classes, even those in Chapter 4, where they are almost certainly the matroids $\mathrm{PG}^{(k)}(n, q)$. A full solution to Conjecture 1.5.16, as well as careful analysis of the extremal matroids in a class whose growth rate function is given by a specific k and d, may yield an answer.

Continuing the idea that projective geometries are 'dominant' in exponentially dense classes, to the point where they determine growth rate functions of some classes more or less exactly, we can conjecture that this remains essentially true with respect to other measures of density, starting with the a-covering number τ_a. We commented already that $\tau_a(\mathrm{PG}(n-1, q))$ is nontrivial to compute; however, results such as the following coarse analogue of Theorem 1.0.1 may hold:

Conjecture 8.1.3. *Let a and b be integers with $1 \leq a < b$, and let $\epsilon > 0$ be a real number. If M is a matroid of sufficiently large rank with no $U_{a+1,b}$-minor, and q is the largest prime power such that $U_{a+1,b}$ is not $\mathrm{GF}(q)$-representable, then $\tau_a(M) \leq (1+\epsilon)\tau_a(\mathrm{PG}(r(M)-1, q))$.*

In other words, up to an eventually arbitrarily small multiplicative factor, a matroid with no $U_{a+1,b}$-minor is no denser than the largest projective geometry with no $U_{a+1,b}$-minor.

We could perhaps take this further by considering other measures of density. Similar conjectures could be made for the parameter τ^d of the last two chapters; another example is an alternative generalisation of ε. If M is a matroid, and $a \geq 1$ is an integer, then let $\nu_a(M)$ denote the number of rank-a flats of M. (Thus, $\tau_1(M) = \nu_1(M) = \varepsilon(M)$). Unlike τ_a, the parameter ν_a for $a \geq 2$ is unbounded on (almost) any minor-closed class whose growth rate function is unbounded, but it is still possible that projective geometries exemplify exponential density with respect to ν when excluding a rank-2 matroid:

Conjecture 8.1.4. *Let $a \geq 1$ be a integer, and q be a prime power. If M is a matroid of sufficiently large rank with no $U_{2,q+2}$-minor, then $\nu_a(M) \leq \nu_a(\mathrm{PG}(r(M)-1, q))$.*

Even when $a = 2$, this problem seems to be hard; the rank-3 case corresponds to the following difficult problem in finite geometry:

Problem 8.1.5. *Let q be a prime power. If M is a rank-3 matroid with no $U_{2,q+2}$-minor, then how many lines can M contain?*

In this case, the lines of M are also the hyperplanes. The following problem, related to Conjecture 8.1.4, thus generalises the above question:

Problem 8.1.6. *Let $\ell \geq 2$ and $r \geq 2$ be integers. What is the maximum number of hyperplanes in a rank-r matroid with no $U_{2,\ell+2}$-minor?*

Since each hyperplane of M is spanned by $r-1$ points, Theorem 1.3.1 gives a very crude upper bound of $\binom{n}{r-1}$, where $n = \frac{\ell^r-1}{\ell-1}$. While this bound has been used in the literature (for example, [17]), the true answer is likely much smaller; when ℓ is a prime power q, it is possibly as small as $\frac{q^r-1}{q-1}$, the number of hyperplanes of a projective geometry over $\mathrm{GF}(q)$.

8.2 Quadratic Density

As we commented in Chapter 1, the prospects for exactly determining the growth rate functions or extremal matroids of linearly dense classes are not good. Theorem 1.2.3 tells us that, even in the very special case of the class of graphic matroids with no $M(K_t)$-minor, the extremal members are random graphs. (In fact, it is further true that arbitrarily large extremal graphs in this class can be constructed to have low branch-width.) This unstructured behaviour makes statements analogous to Conjecture 1.5.14, and associated assertions about growth rate function, very unlikely.

The quadratic setting, however, seems more promising. As was stated in Conjecture 5.3.2, it appears that relatively well-understood classes of frame matroids play the dominant role in the structure of quadratically dense classes, just as classes of representable matroids seem to do in the exponential case. The analogues of representable matroids should be classes $\mathcal{D}(\Gamma)$ of Γ-Dowling matroids, where Γ is a finite group; these are the graph-like matroids arising from Γ that were discussed in Chapter 5. One can show that $U_{2,|\Gamma|+3} \notin \mathcal{D}(\Gamma)$, and that $h_{\mathcal{D}(\Gamma)}(n) = n + |\Gamma|\binom{n}{2}$ for each $n \geq 0$.

In general, it seems that extremal matroids in classes with quadratic growth rate function, and no $U_{2,t+3}$-minor should all be constructed by applying a bounded number of lifts and projections of matroids in $\mathcal{D}(\Gamma)$ for some group Γ with $|\Gamma| \leq t$. The growth rate function for such a class should resemble $h_{\mathcal{D}(\Gamma)}$ in the same sense that the function in Conjecture 1.5.16 resembles $h_{\mathcal{L}(q)}$. In some cases, something stronger could be true:

Conjecture 8.2.1. *Let $t \geq 2$ be an integer, and $\mathbb{K}$ be a field of characteristic zero. If $\mathcal{M}$ is a minor-closed class of $\mathbb{K}$-representable matroids such that $U_{2,t+3} \notin \mathcal{M}$, then $h_{\mathcal{M}}(n) \leq n + t\binom{n}{2}$ for all sufficiently large n.*

In other words, if $\mathbb{K}$ is a field over which projective geometries are not representable, then minor-closed classes of $\mathbb{K}$-representable matroids excluding some line are no denser than the Γ-Dowling matroids that exclude the same line. This conjecture would provide a good route to an increased understanding of the behaviour of quadratic growth rate functions in general.

8.3 Structure

The ultimate goal in our study of minor-closed classes is to obtain true structural descriptions, proving results such as Conjecture 5.3.2. The work of Robertson and Seymour on the Graph Minors Structure Theorem [39], as well as its extension by Geelen, Gerards and Whittle to the binary matroids, suggest which approach we should be taking. A major milestone in both these bodies of work was a 'Grid Theorem', which finds a large grid minor in an object of sufficiently large branch-width; the following is a rephrasing of the Grid Theorem for matroids [11], which we stated as Theorem 5.2.2. Here, the graph G_n is the $n \times n$ planar grid:

Theorem 8.3.1 (Grid Theorem for Matroids)**.** *For all integers $n \geq 1$ and $\ell \geq 2$, there exists an integer t such that, if M is a matroid of branch-width at least t, with no $U_{2,\ell+2}$-minor and no $U_{\ell,\ell+2}$-minor, then M has an $M(G_n)$-minor.*

A much more general version of this theorem was conjectured by Johnson, Robertson and Seymour (see [9]). Here, $B(G_n)$ denotes the bicircular matroid associated with the $n \times n$ grid:

Conjecture 8.3.2 (Grid Conjecture)**.** *For all integers $n \geq 1$, there is an integer t such that, if M is a matroid of branch-width at least t, then M or M^* has a $U_{n,2n}$-minor, an $M(G_n)$-minor, or a $B(G_n)$-minor.*

A proof of this conjecture would be an major step towards a general structure theory. The proof of Theorem 8.3.1 relied heavily on the 'linear-quadratic' part of the Growth Rate Theorem (the statement that any minor-closed class excluding some line is either at most linearly dense, or contains all graphic matroids). Analogously, a linear-quadratic divide for minor-closed classes with respect to τ_a would be instrumental in a proof of the Grid Conjecture. This is one of the two remaining unresolved parts of Conjecture 5.4.2, which we restate to conclude this thesis.

Conjecture 5.4.2 (Growth Rate Conjecture)**.** *Let $a \geq 1$ be an integer, and $\mathcal{M}$ be a minor-closed class of matroids, not containing all rank-$(a+1)$ uniform matroids. Either:*

1. *$\tau_a(M) \leq c_{\mathcal{M}} r(M)$ for all $M \in \mathcal{M}$, or*
2. *$\tau_a(M) \leq c_{\mathcal{M}} r(M)^2$ for all $M \in \mathcal{M}$, and $\mathcal{M}$ contains either all graphic matroids, or all bicircular matroids, or*
3. *there is a prime power q such that $\tau_a(M) \leq c_{\mathcal{M}} q^{r(M)}$ for all $M \in \mathcal{M}$, and $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids.*

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Notation

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