Qualitative Properties of Stochastic Hybrid Systems and Applications

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Hybrid systems with or without stochastic noise and with or without time delay are addressed and the qualitative properties of these systems are investigated. The main contribution of this thesis is distributed in three parts.

In Part I, nonlinear stochastic impulsive systems with time delay (SISD) with variable impulses are formulated and some of the fundamental properties of the systems, such as existence of local and global solution, uniqueness, and forward continuation of the solution are established. After that, stability and input-to-state stability (ISS) properties of SISD with fixed impulses are developed, where Razumikhin methodology is used. These results are then carried over to discussed the same qualitative properties of large scale SISD. Applications to automated control systems and control systems with faulty actuators are used to justify the proposed approaches.

Part II is devoted to address ISS of stochastic ordinary and delay switched systems. To achieve a variety stability-like results, multiple Lyapunov technique as a tool is applied. Moreover, to organize the switching among the system modes, a newly developed initial-state-dependent dwell-time switching law and Markovian switching are separately employed.

Part III deals with systems of differential equations with piecewise constant arguments with and without random noise. These systems are viewed as a special type of hybrid systems. Existence and uniqueness results are first obtained. Then, comparison principles are established which are later applied to develop some stability results of the systems.

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Dedication

To you, who is far and nigh, who passionately taught me the true love of knowledge, and who is not with me in these moments, to the memory of my beloved father,

to my dearest family, and

to the humanity,

I dedicate this work.

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Notations and Symbols

$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \cdots\}$.
$\mathbb{R}$	the real number set.
$\mathbb{R}_+$	nonnegative real number set.
$\mathbb{R}^n$	n-dimensional vector space.
x^T	transpose of a vector x.
$\|x\|$	norm of a vector x.
$S(\varrho)$	$\{x \in \mathbb{R}^n \big\| \|x\| \leq \varrho$, for some $\varrho \geq 0\}$.
$\|A\|$	norm of a matrix A.
$\max(\vee), (\min \wedge)$	maximum, (minimum).
sup	supremum, the least upper bound.
inf	infimum, the greatest lower bound.
A^T	transpose of a matrix A.
A^{-1}	inverse of a (nonsingular) matrix A.
$\lambda(A)$	eigenvalue of a matrix A.
$\text{Re}[\lambda(A)]$	the real part of an eigenvalue of a matrix A.
$\lambda_{\max}(A)$	maximum eigenvalue of a symmetric matrix A.
$\lambda_{\min}(A)$	minimum eigenvalue of a symmetric matrix A.
$A > 0$	the real symmetric matrix A is positive definite.
I	identity (unit) matrix.

$\mathrm{tr}A$	the trace of a matrix A.
$A \setminus B$	the difference between two sets, i.e., $A \setminus B = \{x \mid x \in A \,\text{and}\; x \notin B\}$.
$\dot{x}(t)$	time derivative of a time-varying vector x.
D^+V	right-hand side derivative of a function V.
$\\|\phi\\|_r$	$\sup_{-r\leq s\leq 0} \\|\phi(s))\\|$.
$x(t^+)$	$\lim_{t\to t^+} x(t)$.
Δx	$x(t^+) - x(t)$.
$o(\cdot)$	order of magnitude.
$f \circ g$	the composite function of f and g, i.e., $(f \circ g)(x) = f(g(x))$.
$\mathcal{C}([a,b];\mathbb{R}^n)$	the space of continuous functions mapping $[a,b]$ into $\mathbb{R}^n$.
$\mathcal{PC}([a,b];\mathbb{R}^n)$	the space of piecewise continuous function mapping $[a,b]$ into $\mathbb{R}^n$.
$\mathcal{C}^m(\mathcal{D};\mathbb{R}^n)$	the family of continuously m-times differentiable $\mathbb{R}^n$-valued functions defined on $\mathcal{D}$.
$\mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathcal{D};\mathbb{R}_+)$	the family of $\mathbb{R}_+$-valued functions $V(t,x)$ defined on $\mathbb{R}_+ \times \mathcal{D}$, which are continuously once differentiable in $t \in \mathbb{R}_+$ and twice in $x \in \mathcal{D}$.
$\mathcal{L}_{ad}(\Omega; L^p[a,b])$	the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process $f(t)$ such that $\int_a^b \\|f(t)\\|^p dt < \infty$ (a.s.) for all $t \in [a,b]$.
$\mathcal{M}^p([a,b];\mathbb{R}^n)$	the family of processes $f(t)$ in $\mathcal{L}_{ad}(\Omega; L^p[a,b])$ such that $\mathbb{E}\int_a^b \\|f(t)\\|^p dt < \infty$.
$\mathcal{L}^p_{\mathcal{F}_0}([-r,0];\mathbb{R}^n)$	the family of $\mathcal{F}_0$-adapted $\mathcal{PC}([-r,0];\mathbb{R}^n)$-valued random variable ϕ such that $\mathbb{E}[\\|\phi\\|^p] < \infty$.
$L^p_{\mathcal{F}_0}([-r,0];\mathbb{R}^n)$	the family of $\mathcal{F}_0$-adapted $\mathcal{C}([-r,0];\mathbb{R}^n)$-valued random variable ϕ such that $\mathbb{E}[\\|\phi\\|^p] < \infty$.
1_A	indicator function of a set A, i.e., $1_A(x) = 1$ if $x \in A$ or otherwise 0.

Chapter 1

Introduction

Conventionally, the term "hybrid systems" means systems having behaviour characterized by continuous and discrete components interacting with each other, or between them (the continuous and discrete parts) with environmental factors. The hybrid paradigm is an adequate tool to cover a diversified applications in natural sciences and engineering systems, ranging from room heating systems to control systems with a high-level supervisor, from air traffic control to automated highway systems, from population growth dynamics to epidemical disease models, from secure communications to neural networks. The study of hybrid systems has created a fascinating discipline binding mathematics to various application fields.

Although hybrid systems have been with us for a long time, it was in 1980s that hybrid systems took a systematic configuration. Typically, the mathematical model of a hybrid system is (1) a combination of a set of continuous or discrete differential equations representing the evolution of the system and a set of difference equations representing jumps or impulsive actions in the system states. Here, the first set describes the continuous component of the system, while the second set describes the discrete part. (2) A hybrid system can also have a mix of a finite number of subsystems (or modes) and a control-based discrete logic to jump among the modes. The first category of hybrid systems is referred to as *impulsive systems* or *systems with impulsive differential equations*, wheres the second group is called *switched (or switching) systems*. Another class of hybrid systems are *impulsive switched systems*, in which impulses arise as a result of switching. These three hybrid system types are the main focus of this thesis. We should remark that, in the literature, but not in this thesis, a hybrid system is often meant to be a switched system.

The main characteristic of an impulsive system is that, at certain moments between the intervals of the continuous evolutions, the system process undergoes abrupt changes. The durations of these changes are sufficiently small when compared to the total duration of the process. These changes can be reasonably well-approximated by instantaneous changes of the state or *impulses*. The evolutionary process is then suitably modeled as an *impulsive system*.

The applications of impulsive systems are found in many areas, such as in mechanical and electrical engineering systems including pendulum and mass-spring systems, industrial robotics or electrical circuits, in aeronautics including impulse maneuver of a spacecraft, in biological systems including the function of the heart and biological neural networks, in pharmacokinetics including the maintenance of the drug levels in a body, in population dynamics including a specie maintenance through periodic shocks and harvesting, and in epidemical disease models including pulse vaccination.

Theoretically speaking, impulsive systems have richer properties than the corresponding non-impulsive ones. For instance, the initial value problem of an impulsive system may not have a solution even when the underlying non-impulsive system does; some other fundamental properties of the system, such as continuous dependence on the initial condition, continuation of solution, may be violated or needs new interpretation. On the other hand, under some conditions, impulses may be helpful in making the continuation of solutions possible [Lak89]. Another undesirable performance that an impulsive system may experience is the so-called beating phenomenon in which an impulsive hyper-surface is being visited infinitely many times. This challenge may happen when the impulses are state-dependent, but not constants.

Similar to other dynamical or control systems, among the most important properties of impulsive systems is stability. However, one cannot directly carry over the analysis of continuous system theory to impulsive systems. The reasons are twofold. On one hand, if a stable system is subject to frequent impulses, the system may lose its stability due to up or down jump discontinuities. On the other hand, if the impulses being applied to a system are well-timed in the sense that they follow a certain formalized impulsive logic, they may be helpful in recovering the aforesaid circumstances, and even play a stabilizing factor if the underlying system is unstable [Liu94].

The theory of impulsive differential equations is interesting in itself, and a reasonably great amount of research has been done on the analysis of such systems. For more information on the theory of impulsive systems, readers may refer to see [Bai89, Bai93, Had06, Lak89, Li05, Sam95, Yang01] and the references cited therein. Currently, the field of impulsive systems is very active since their applications are widespread. A part of this thesis is devoted to broaden the analysis and applications of impulsive system theory.

One kind of hybrid system, as stated earlier, is a switched system which is a composition of multi-dynamical systems with a monitoring device called a *switching logic*, also known as *switching law*, *switching rule*, or *switching signal*. The main objective of switching signal is to orchestrate the jumping among the system modes so as to accomplish a desired feature of the system. A peculiar feature of switched systems is the stability property. Although the system retains the classical stability properties, the methodology of determining the conditions for the stability of a switched system is a complicated, but very interesting, task in the sense that switching among asymptotically stable subsystems may produce instability if the switching moments are poorly chosen. By contrast, if the activation time designated to each subsystem is determined by a well-designed switching law, asymptotic stability of each individual mode may not be necessary for the stability of a switched system.

The importance of studying switched systems is threefold. Firstly, a large class of real life and engineering systems have behaviours which are intrinsically governed by multimodal dynamics, such as control systems, robots, thermostats in cooling or heating systems, prey-predator systems with different but finite prey sources, and epidemic disease models. The following switched SIR model with Pulse Treatment studied in [Ste09].

$$\begin{aligned}
\dot{S} &= \mu - \beta_i SI - \mu S, \qquad t \in (t_{k-1}, t_k],\\
\dot{I} &= \beta_i SI - gI - \mu I,\\
\dot{R} &= gI - \mu R,\\
S(t^+) &= S(t),\\
I(t^+) &= I(t) - p_i I(t),\\
R(t^+) &= R(t) + p_i I(t).
\end{aligned}$$

Secondly, many systems are asymptotically stabilized by several feedback control signals, rather than one signal. The following diagram illustrates a logic-based controller (or a supervisory controller).

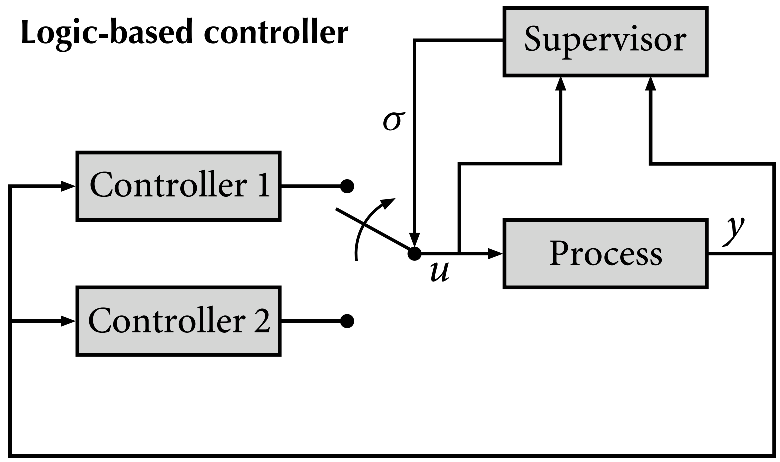

Figure 1.1: Logic-based supervisory controller.

Thirdly, switched system patterns can reduce the complexity of some systems. Switched systems have been an active research area in the last three decades. It has received much attention including books [Alu96, Ant95, Ant97, Gro93, Hes06, Li05, Lib03] and many research papers cited therein.

The study of switched systems is more challenging than that of single-mode systems. Nevertheless, there has been reasonable, increasing progress in this field. Most of the work has focused on designing an appropriate switching law to provide some stability properties. To the best of the author's knowledge, the earliest attempt appeared in [Nar94], where the authors proved exponential stability of linear time-invariant systems by using a common Lyapunov function. Also, in their work, the subsystem matrices are assumed to be asymptotically stable and piecewise commutative. This second condition is very restrictive which makes the approach not widely applicable. Later, in [Mor96], the authors showed that, when all individual modes are exponentially stable, the entire switched system preserves the same stability property provided that the running time between any two consecutive switchings called *dwell time* is sufficiently large. This dwell-time approach was later extended to a more general, relaxed one called *average dwell time* [Hes99]. These approaches were later employed to achieve the stability results for a larger class of switched systems in which some of the modes are unstable (see [Lib99, Hu99, Zha01]). In the general case, the stable modes should be activated longer to compensate the growth of unstable system states. In [Day99], the authors investigated the stability of a class of dynamical systems which undergo arbitrary switchings. In their work, the focus is on proving a converse Lyapunov function theorem for that class of systems. A more general approach than the dwell-time one is called *Markovian switching*, in which the switching signal is a Markov chain which takes values in a finite sets. In other words, the jump among the system modes follows a probabilistic or random rule [Mao06]. An interesting consequence in adopting this type of switching arises from involving the transition rates of the Markov chain in the calculation of dwell times. One can easily recognize that the stability requirement of the individual modes is neither sufficient, nor necessary for guaranteeing a stability property of a switched system.

The dwell-time and average dwell-time approaches have been widely applied to determine exponential stability of linear and nonlinear switched systems, with or without the presence of perturbation, and with or without time delay. Their limitation, however, is that they are independent of the system states even at the switching moments or at the initial time. In [DePe02], a *state-dependent dwell-time* approach was proposed to investigate asymptotic stability-like property for nonlinear switched systems subject to input disturbance. The interesting feature of this switching rule is that the dwell time is a function depending on the system states and comparison functions characterizing the considered qualitative property. This approach has inspired the author of this thesis to develop a new switching law called *initial-state-dependent dwell-time* approach to establish some stability-like properties of nonlinear switched systems with and without input disturbance. We should mention that, except in the first attempt, the more flexible multiple-Lyapunov-function approach is used in determining the stability conditions.

Ordinary differential equations have long played important roles in modeling many physical processes, and they will continue to serve as a fundamental tool in future investigations. A drawback of these models is that they are ruled by the so-called *Markovian principle* in which the future state of a dynamical system depends only on the present state, leading to *ordinary differential equations* (ODEs), and not on the past. In fact, they are approximations of some real systems. In those cases, more realistic models should involve some of the historical values of the state; this leads to *delay differential equations* (DDEs), also known as *(retarded) functional differential equations* (FDEs), or *differential equations with deviating arguments*. The early motivations for studying DDEs came from their applications in population dynamics when Volterra investigated the prey-predator model, and in Minorsky's study of ship stabilization and automatic steering. These studies indicate the importance of considering delay in the feedback mechanism [Min42]. Another motivation for studying state-delayed systems stems from the fact that the presence of delay, even in a first-order system, may not be trouble-free. It may cause undesirable performances, such as oscillations of large amplitudes, chaotic behaviour, losing uniqueness, or resulting in discontinuous solutions; whereas delay may make the continuation of a solution possible, or reduce the complexity of some systems [Bur05]. In other cases, a small delay may destabilize some systems, but a large delay may stabilize others. As a result, there have been many studies on delay systems in the past decades by researchers from different fields. One may be referred to see [Bel63, Dri77, Els73, Bel03, Kra63, Hal66, Hal71, Hal93], and other references dedicated to applications [Mac89, Gop92, Kua93].

If a hybrid system involves deviating arguments (or delayed states), we are led to a *hybrid system with time delay.* The system properties that have been received researchers' attentions are existence, uniqueness, and continuation of solution [Ball99b, Liu2000], stability [Liu01, Alw08a, Alw08b], stabilization by impulsive effects [Alw09a, Alw09b], and boundedness [Liu03]. Another challenge that may be caused by considering impulsive effects in delayed systems is the high discontinuity of the system vector fields. Namely, to have a well-behaved solution of a delay system, it is required that the initial state function be continuous. Due to the discontinuity feature of the solutions of impulsive systems, one may think, in the first place, to relax the continuity restriction on the initial function. Unfortunately, in doing so, the *delayed* state may be discontinuous everywhere, because the vector field, as a composite function depending on the delayed state, of the corresponding system cannot be conclusive as a continuous or even piecewise continuous function. This complicated situation, produced by a minor thing, can be ruled out by restricting the system vector fields to be in a class of piecewise continuous composite functions [Ball99b, Liu2000].

A special class of FDEs or hybrid systems are systems of *(differential) equations with piecewise constant arguments* (EPCA). From the perspective of functional differential equations, although the arguments can be delay, advanced or a mix of these two types, the past history is given at individual points, rather than intervals, which enables one to use the theory of ordinary differential equations, and not of FDEs. The hybridness is because the dynamics of these equations depend on both continuous and discrete arguments. This type of differential equation appears in the "sequential-continuous" disease models [Bus82]. EPCA also appeared as an attempt to extend FDEs with continuous arguments to equations with discontinuous ones. Typically, the vector fields in EPCA contain arguments which are constants on certain subintervals. Consequently, the corresponding solutions are continuous and generally differentiable everywhere except at the joining point between two consecutive subintervals where one-side derivatives exist. Furthermore, the continuity of the solution at such points produces a recursive relation or difference equation. Therefore, the initial data are given by a finite set, but not as a function as in the case of FDEs. In most of the work that has been done so far, the underlying difference equations are used to characterize some of the system properties, such as stability, oscillation, and periodic solutions. Moreover, another motivations for studying EPCA is that equations of the delay type can be used to find approximate solutions for differential equations with discrete delays.

The theory of EPCA was initially developed in [Coo94] and well discussed in the survey paper [Coo91] and book [Wie93]. Further properties and use of these equations were considered in some other works; for instance, oscillatory properties of first-order differential equations with retarded and advanced arguments are investigated in [Aft85], oscillatory and asymptotic properties of EPCA with delay arguments were discussed in [Gop92], a criterion for the existence of periodic solution of EPCA was developed in [Akh08b], and in [Cab04] the authors constructed a Green's function to the linear operator of the boundary-value EPCA and obtained some comparison results for the same differential equations. In [Coo94, Gyo91, Gyo08], EPCA were used to find numerical approximation of DDEs. Moreover, in [Yang09], the authors focused on numerical solutions of Runge-Kutta methods for first-order periodic EPCA. In this article, the solution was given by a numerical Green's function. A general type of EPCA, known as EPCAG (i.e., *equations with piecewise constant arguments of a general type*), in which a piecewise constant real function takes values over discrete subintervals instead of at the most-left endpoint of each subinterval, has appeared in some works [Akh08b, Akh08c]. In those works, the solutions of linear and quasi-linear EPCAG are determined by a unique initial datum at an initial moment, rather than by a countable set of initial data defining at discrete moments, or, as in the case of FDEs, by an initial function defined on some interval from the past history. Consequently, in either case, EPCA or EPCAG, FDEs reduce to ordinary ones.

In numerous mathematical models, we deal with systems whose states are driven by some inherent noise having a probabilistic (or stochastic), not deterministic, structure. Therefore, it is natural to incorporate this stochasticity in the design of these systems, leading to *stochastic systems* (SSs) or *systems with stochastic (or random) differential equations* (SDEs). From a practical perspective, systems of this type are more realistic compared to the deterministic ones in the sense that the former systems better match the available data used to design a mathematical model and accurately predict the future behaviour of a certain process. The theory of SSs (or SDEs), however, is more sophisticated than that of the deterministic systems. Consequently, many tools utilized in analyzing deterministic problems cannot be carried over to handle the corresponding stochastic problems. Stochastic integrals, for instance, may not be understood in the sense of the classical Leibniz-Newton calculus, but in the sense of Itô calculus, as will be discussed in Chapter 2. Another challenge produced by the randomness is that a solution of a SDE is given by a random process, consisting of an infinite sequence of the so-called *sample paths*, also known as *realizations or trajectories*, wheres a solution of a deterministic differential equation is represented by a single sample path. Moreover, analytic solutions of SDEs are very difficult and even impossible in some cases to obtain; therefore, the interest changes to approximate solutions [Klo99] or to the qualitative behaviour of these solutions. Due to the random (or probabilistic) behaviour of the SDE solutions, it is reasonable that the qualitative notions are redefined in a probabilistic sense. Among these properties is stability, which has received a fair amount of research including books [Gard88, Gih72, Kus67, Kha80, Mao08, Mao94, Moh84, Xie06] and many other references cited therein.

Considering random noise in a hybrid system with or without time delay leads to a *stochastic hybrid system with or without delay*. Systems of this type have received less attention due to some technical difficulties, especially those systems which are subject to impulsive effects. Among the available results are a book [Mao06], which concerns SDEs with Markovian switching, and research papers dealing with the problems of stability and stabilization of stochastic switched system [Mao07, Mao99, Yua05] and establishing some fundamental properties of stochastic impulsive systems, such as existence of global (or regular) solution, uniqueness, and stability [Liu07, Liu08].

One of the most important qualitative aspects that can be studied is *stability*. At the end of the nineteenth century, Aleksandr M. Lyapunov invented the direct method to study the stability of a system without prior knowledge of its solution. The method, which bears his name today, is the most effective technique provided the right auxiliary function, called a *Lyapunov function*, can be constructed to establish the stability property.

An alternative approach to Lyapunov stability is the *input-to-output stability* approach, in which the system output is directly related to the system input with no knowledge of the internal structure of the state equation. In other words, the system is viewed as a black box that can be accessed only through the input and output of the system. The bridge between these two different stability notions is *input-to-state stability* (ISS), where the system is described by a state space realization that includes a variable input function.

Roughly speaking, by ISS we mean that, assuming that the unforced system has asymptotically stable equilibrium point, if the system input is uniformly small, then the magnitude of system response is small regardless of the magnitude of system initial state. The ISS is an essential concept in analyzing stability-like aspects of nonlinear systems under input disturbance or noise. During the last two decays and due to its usefulness, ISS presented by Sontag [Son89, Son02] has become a central foundation of modern nonlinear feedback and design. It is a nonlinear generalization of finite L^2 gains and finite gains with respect to supremum norm. It is a key tool in systems with recursive design and co-prime factorizations. An implications of ISS is that when the input is identically zero it reduces to the classical asymptotic stability of the equilibrium state of the system. For further characterizations, implications, and applications of the ISS, readers may consult [Ang2000, Cai05, Kok99, Son89, Son95, Son96, Son98, Son02, Teel01, Teel03] and references cited therein. The ISS property of hybrid systems was addressed in [Cai05, Cai09, Che09, Hes05, Hua09].

In the design of safety-critical control systems, such as in aircraft and space vehicles, a hazard that may occur is the event of control component failures, such as actuator or sensor outages. Since failures are inevitable in the real world, it is necessary to design reliable controllers to achieve desired performance requirements of the plant, not only when the system operating properly, but also in the presence of actuator failures. Control systems that tolerate actuator and sensor outages are called *reliable control systems*. In the last three decades, the problem of designing reliable controllers has received much attention. In [Ack85], the author proposed a graphical approach for selecting from among stabilizing state-feedback gains to ensure reliable stability despite sensor failures. In [Vei92], a methodology was developed for the design of centralized and decentralized control systems which are reliable in providing guaranteed stability and H_∞ performance, not only when the control components are normal, but also in the presence of actuator and sensor failures. Later, in [Seo96], a robust reliable H_∞ control method was proposed, where a state feedback for linear systems with time-varying norm-bounded uncertainties and actuator failures is used. In [Wang99], a robust reliable control design was proposed for uncertain systems with time-delayed states and nonlinear disturbance. We should also point out that, in [Che05], the authors have addressed the problem of designing a robust reliable controller for a class of deterministic ordinary time-varing uncertain impulsive systems with actuator outage, using Lyapunov theorems which lead to solving an algebraic Riccati equation. The problem of designing a robust reliable control for control systems with time delay was addressed in [Gao01, Luo06, Wang2000]. In [Wang01], the authors considered the problem of robust reliable control for stochastic delay systems with nonlinear disturbances. The focus was on the design of a state feedback memoryless controllers such that, for all admissible uncertainties and actuator outages, the systems retain stochastic exponential stability, where the Lyapunov functional approach was used to analyze the stability property.

Motivated by what we have discussed, the main theme of this thesis deals with analyzing and broadening the theory of stochastic hybrid system with or without time delay, and with applying the theory of hybrid systems to further investigate systems with EPCA with or without random noise. The main contents of this thesis can be divided into three parts.

Part 1. Stochastic Impulsive Systems with Time Delay and Applications

This part concentrates on developing the essential foundations of the theory of stochastic impulsive systems with time delay (SISD) and some applications.

Chapter 2 serves as an introductory chapter to the rest of Part 1. It includes some basic definitions, system formulations, defining stochastic differential equations and developing the initial value problem of SISD where impulses occur at variable times.

In Chapter 3, we establish the existence of a well-behaved local solution, i.e., a solution that does not exhibit the beating phenomenon. Later in the same chapter, we develop sufficient conditions to ensure the global existence and uniqueness of a solution.

Having developed the regularity problems of the systems, in Chapter 4, we turn our attention to establish some stability results for the same systems, where the impulses occur at fixed times. Using Razumikhin technique, Lyapunov-type sufficient conditions are developed to prove some stability properties in the mean square (m.s.). We have also used the comparison principle approach to achieve the same qualitative properties.

Once the stability results have been proved, we extend our finding to large-scale SISD in Chapter 5. To do so, we adopt an efficient approach, in which the interconnected (or composite) system is decomposed into simpler, more manageable isolated subsystems and the rest will be viewed as perturbation. Lyapunov functions together with Razumikhin technique are used to prove the desired properties, which are later clarified by an application from control system.

Chapter 6 deals with the input-to-state stability (ISS) of nonlinear systems subject to input disturbance with bounded energy. Two approaches are proposed, an $(\varepsilon^u, \delta^u)$ (or the classical Lyapunov) and comparison principle techniques. An application to cascade (also known as feedforward or recursive) systems is presented to justify the effectiveness of these techniques. The systems in this chapter have deterministic hybridness-free ordinary differential equations. The material of this chapter will be used later in establishing the ISS properties of SISD and large scale SISD.

In Chapters 7 and 8, we consider SISD and large scale SISD subject to input disturbance and proved some ISS properties using the aforementioned approaches. Some interesting implications of these results are also given in these chapters.

Chapter 9 is devoted to the problem of designing a robust reliable control with state feedback for a class of uncertain stochastic impulsive systems with time delay. The uncertainties are time varying with bounded norms and the controllers have actuators with possible failures.

Part 2. Deterministic and Stochastic Switched Systems

This part is dedicated to deterministic and stochastic switched systems with and without time delay.

Chapter 10 serves as an introduction to the rest of this part. We state some definitions and theorems that will be used throughout this part of the thesis.

In Chapter 11, we designe a dwell-time-based switching signal to tackle the problems of stability and stabilization of uncertain impulsive switched systems with time delay.

The focus of Chapter 12 is on establishing some ISS properties in the mean square (m.s.) of ordinary stochastic switched systems. The switching rules used in this chapter are the newly developed *initial-state-dependent dwell-time* (τ_{isd}) condition and the *Markovian switching*. Systems with all stable modes are investigated first, then a more general case is considered, where unstable modes are included. Some special results are presented to show the usefulness of our proposed methodology.

In Chapter 13, the τ_{isd} switching signal is used to obtain some similar ISS properties of stochastic switched systems with time delay. Lyapunov-like theorems are proved using the Razumikhin technique.

Part 3. Differential Equations with Piecewise Constant Arguments (EPCA)

In this part of the thesis, we apply the theory of hybrid systems to further investigate the properties of systems with EPCA. Another motivation for adopting a hybrid system paradigm is to reduce the complexity of these systems. This part has three chapters.

In Chapter 14, we develop a comparison principle for systems with nonlinear EPCA, then this result will be utilized to establish some stability properties, where we use Lyapunov function approach. We also show that the piecewise arguments can play a stabilizing role in some cases where the underlying systems are unstable. Some special cases of linear EPCA with a general type (i.e., EPCAG) are considered.

Chapter 15 discusses systems with stochastic EPCA (SEPCA). We study the problems of existence of a global solution, uniqueness, and stability. In analyzing the stochastic qualitative characteristic, we follow two approaches. Namely, we extend the comparison principle developed in Chapter 14 and use Lyapunov functions together with Razumikhin technique.

Chapter 16 includes a conclusion of the thesis and some future works.

Part I

Stochastic Impulsive Systems with Time Delay and Applications

Chapter 2

Mathematical background

In this introductory chapter, we present some basic definitions and background to the rest of Part 1. Also, we introduce impulsive systems, and impulsive systems with time delay. We then define stochastic differential equations. Finally, we formulate stochastic impulsive systems with time delay.

2.1 Basic Definitions

Consider the following initial value problem

$$\begin{cases} \dot{x} = f(t, x), \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $t \geq t_0$ with $t_0 \in \mathbb{R}_+$, and $f : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n$ is continuous in t and locally Lipschitz in x with $\mathcal{D} \subset \mathbb{R}^n$ being the domain containing the origin $x = 0$. Assume that $f(t, x^*) = 0$ for all $t \geq t_0$. Then, the real root $x^* \in \mathbb{R}^n$ is called an *equilibrium point*, or *trivial solution* of system (2.1) if $x^* \equiv 0$. Throughout this thesis, we deal with $x^* \equiv 0$ (or $x \equiv 0$ for simplicity of notation), since any equilibrium points can be shifted to the origin. As stated earlier, from a practical perspective, a dynamical system must meet some essential requirements, and among them is the qualitative property of (Lyapunov) stability.

Definition 2.1. For any $t \geq t_0$, let $x(t) = x(t, t_0, x_0)$ be a solution of system (2.1). Then, the trivial solution $x \equiv 0$ of (2.1) is said to be

1. *stable* if, for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there is $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x_0\| < \delta \qquad \text{implies} \qquad \|x(t)\| < \varepsilon, \quad \forall t \geq t_0; \tag{2.2}$$

2. *uniformly stable* if it is stable and δ is independent of t_0;

3. *unstable* if it is not stable;

4. *asymptotically stable* if it is stable and there is a positive constant $c = c(t_0)$ such that, for all $\|x_0\| < c$, $\lim_{t\to\infty} x(t) = 0$;

5. *uniformly asymptotically stable* if it is uniformly stable and there is a positive constant c, independent of t_0, such that, for all $\|x_0\| < c$, $\lim_{t\to\infty} x(t) \to 0$, uniformly in t_0; that is, for any $\eta > 0$, there is $T = T(\eta) > 0$ such that, for all $\|x_0\| < c$,

$$\|x(t)\| < \eta, \qquad \forall t \geq t_0 + T(\eta);$$

6. *exponentially stable* if there are positive constants c, k, and λ such that

$$\|x(t)\| \leq k\|x_0\|e^{-\lambda(t-t_0)}, \quad \forall \|x_0\| < c.$$

Furthermore, the above stability properties are satisfied *globally* if (1,2,4,5,6) hold for any $x_0 \in \mathbb{R}^n$.

In analyzing the stability properties by the Lyapunov method, we define a class of functions which enjoy some positive definiteness features.

Definition 2.2. Let $\mathcal{D} \subset \mathbb{R}^n$ be an open set containing $x = 0$. A function $V : \mathcal{D} \to \mathbb{R}$ is said to be *positive semi-definite* if (i) $V(t, 0) = 0$ and (ii) $V(t, x) \geq 0$, for all $t \geq t_0$ and $x \in \mathcal{D} \setminus \{0\}$. It is said to be *positive definite* if the inequality in (ii) is replaced by (ii)′ $V(t, x) > 0$. Moreover, it is said to be *radially unbounded* (or *proper)* if it is positive definite and, for each fixed t, $\lim_{\|x\|\to\infty} V(t, x) = \infty$.

In analyzing the stability (or stability-like) properties, we usually introduce some special functions known as *comparison functions* [Kha02, Hah67].

Definition 2.3. A function $\alpha \in \mathcal{C}([0, a]; \mathbb{R}_+)$ is said to belong to class $\mathcal{K}$ (i.e., $\alpha \in \mathcal{K}$) if it is strictly increasing and $\alpha(0) = 0$. If, in addition, $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$, then α is said to belong to class $\mathcal{K}_\infty$.

Definition 2.4. A function $\beta \in \mathcal{C}([0, a) \times \mathbb{R}_+; \mathbb{R}_+)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(\cdot, s) \in \mathcal{K}$, and, for each fixed r, the mapping $\beta(r, \cdot)$ is decreasing and $\beta(r, s) \to 0$ as $s \to \infty$.

To motivate the notion of (asymptotic) input-to-state stability (ISS), consider the following nonlinear system

$$\begin{cases} \dot{x} = f(t, x, u), & t \geq t_0, \\ x(t_0) = x_0, \end{cases} \tag{2.3}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ with $t_0 \in \mathbb{R}_+$ and the input $u \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^m)$ with bounded energy (i.e., $\sup_{t\geq t_0} \|u(t)\| < \infty$). This system can be viewed as a perturbation of the unforced system

$$\dot{x} = f(t, x, 0), \tag{2.4}$$

with the same initial state. Assume now that the trivial solution of (2.4) is globally uniformly asymptotically stable. An immediate question that can be addressed is: what can be said about the qualitative behaviour of the nonlinear system (2.4) if it is perturbed by some bounded input disturbance u? Generally, the answer may not hold unless some further sufficient conditions are satisfied. The following definition summarizes these conditions.

Definition 2.5. System (2.3) is said to be *input-to-state stable* (ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for any initial state x_0 and bounded input u, the solution $x(t)$ exists, for all $t \geq t_0$, and satisfies

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma\Big(\sup_{t_0 \leq s \leq t} \|u(s)\|\Big). \tag{2.5}$$

In fact, this inequality can be written as follows

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma\Big(\sup_{t_0 \leq s \leq t} \|u(s)\|\Big), \quad \forall\, t_0 \leq t \leq t_0 + T,$$
$$\|x(t)\| \leq \gamma\Big(\sup_{t_0 \leq s \leq t} \|u(s)\|\Big), \quad \forall\, t \geq t_0 + T,$$

where $T \geq 0$. Evidently, for large enough T, the $\mathcal{KL}$ function β converges to zero asymptotically, and when $t \geq t_0 + T$, the solution will stay bounded by a class-$\mathcal{K}$ function γ, meaning that the solution of (2.3) has an ultimate bound γ, which is a ball with a radius depending on the input.

Clearly, from the inequality (2.5), if the input u is set to zero (i.e., $u(t) \equiv 0$ for all $t \geq t_0$), the ISS reduces to the globally uniformly asymptotic stability of the trivial solution of the unforced system (2.4).

Before stating some sufficient conditions regarding the stability properties, we need the following definition of upper right-hand derivative, which is also known as a *Dini derivative*, of the function V.

Definition 2.6. Let $J \subseteq \mathbb{R}_+$ and $\mathcal{D}$ be an open subset of $\mathbb{R}^n$. If $V : J \times \mathcal{D} \to \mathbb{R}_+$, then the *upper right-hand derivative* of V with respect to system (2.1) is defined by

$$D^+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}[V\big(t+h, x+hf(t,x)\big) - V(t,x)], \quad \forall (t,x) \in J \times \mathcal{D}.$$

If, moreover, V has continuous partial derivatives with respect to t and x, then we have

$$D^+V(t,x) = \dot{V}(t,x) = \frac{\partial V(t,x)}{\partial t} + \nabla_x V(t,x) \cdot f(t,x),$$

where $\nabla_x V$ is the gradient of V.

The following Lyapunov-type theorem gives sufficient conditions that ensure ISS, which can also prove the asymptotic stability property of $x \equiv 0$ of the unforced system (2.4).

Theorem 2.1. [Kha02] Let $x(t) = x(t, t_0, x_0)$ be a solution of (2.3). Assume that there exist class $\mathcal{K}_\infty$ functions a and b, a class $\mathcal{K}$ function ρ, and a positive-definite function c. Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that the following conditions holds:

$$b(\|x\|) \le V(t,x) \le a(\|x\|), \qquad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n;$$
$$\dot{V}(t,x,u) \le -c(x), \qquad \text{whenever} \qquad \|x\| \ge \rho(\|u\|),$$

for any $(t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$. Then, system (2.3) is ISS with $\gamma(\cdot) = b^{-1}\big(a(\rho(\cdot))\big)$. Particularly, if $u(t) \equiv 0$ for all $t \in \mathbb{R}_+$, then the trivial solution of the unforced system (2.4) is globally uniformly asymptotically stable.

2.2 Impulsive Systems

To formulate impulsive systems, consider the following control system

$$\begin{cases} \dot{x} = f(t,x) + u(t), \\ x(t_0) = x_0, \end{cases} \tag{2.6}$$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is the system input having the form

$$u(t) = \sum_{k=1}^{\infty} C_k x(t) \delta(t - \tau_k), \tag{2.7}$$

with C_k being the control gain matrix of an appropriate dimension and $\delta(\cdot)$ being the Dirac delta function defined by

$$\delta(t - \tau_k) = \begin{cases} 0, & t \neq \tau_k, \\ \text{undefined}, & t = \tau_k, \end{cases} \tag{2.8}$$

where τ_k forms a strictly increasing sequence $\{\tau_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \tau_k = \infty$. From (2.6) and (2.7) we get, after integrating over $[\tau_k, \tau_k + h]$,

$$x(\tau_k + h) - x(\tau_k) = \int_{\tau_k}^{\tau_k + h} \big(f(s, x(s)) + u(s)\big) ds,$$

where h is sufficiently small. As $h \to 0^+$, we obtain

$$\Delta x(t)\Big|_{\tau_k} = x(\tau_k^+) - x(\tau_k) = C_k x(\tau_k),$$

where $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, and $x(t_k) = x(t_k^-)$, i.e., the solution is assumed to be left-continuous. Apparently, the controller u has the effect of suddenly changing the state of system (2.6) at time instant τ_k; that is, u is an *impulsive controller.* Consequently, the closed-loop system (2.6) becomes

$$\begin{cases} \dot{x} = f(t, x), & t \neq \tau_k, \\ \Delta x(t) = C_k x(t), & t = \tau_k, \quad k = 1, 2, \cdots, \\ x(t_0^+) = x_0. \end{cases} \tag{2.9}$$

This system is called an *impulsive system* or *system with impulsive differential equations.* A general system is obtained when the right-hand side of the difference equation (or the impulsive amount) is given by a nonlinear function, say $\mathcal{I}(t, x(t))$ and the impulsive moments are state dependent, rather than constant, i.e., $t =$

$\tau_k(x)$. Even more general case is when impulses occur if a spatio-temporal relation $\kappa(t,x) = 0$ is satisfied. Then, the impulsive system (in the latter case) has the form

$$\dot{x}(t) = f(t, x(t)), \qquad \kappa(t,x) \neq 0, \tag{2.10a}$$

$$\Delta x(t) = \mathcal{I}(t, x(t)), \qquad \kappa(t,x) = 0, \tag{2.10b}$$

$$x(t_0) = x_0, \tag{2.10c}$$

where we have assumed that there is no an impulsive action at the initial time t_0, i.e., when $x(t_0^+) = x(t_0)$. The solution of this system evolves as follows: the system state starts when $\kappa(t_0, x_0) \neq 0$. Then, whenever $\kappa(t,x) \neq 0$, the system process is governed by the ordinary differential equation (2.10a) until $t = \tau_1$ such that $\kappa(\tau_1, x(\tau_1)) = 0$ is satisfied. At this moment, the process is subject to an impulse and instantly changes by some amount $\mathcal{I}(t, x(t))$, given by the difference equation in (2.10b), causing a jump discontinuity in the system state. For $t > \tau_1$, if the relation $\kappa(t,x) \neq 0$ holds, the process continues according to the differential equation in (2.10a) until an impulsive action occurs again. This continues in the same manner as long as the solution exists. Consequently, the resulting solution is either continuous or piecewise continuous with simple jump discontinuities at the moments of impulse t for which $\mathcal{I}(t, x(t)) \neq 0$.

Due to the difficulty in dealing with relations of the type $\kappa(t,x) = 0$, the interest deflects to a particular type of relation, where the set of points $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ for which $\kappa(t,x) = 0$ are assumed to be represented by a sequence of hyper-surfaces of the form $t = \tau_k(x)$, where generally $\tau_k \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}_+)$ for $k = 0, 1, 2, \cdots$, and $0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \cdots$, and $\lim_{k\to\infty} \tau_k(x) = \infty$ for each $x \in \mathbb{R}^n$. Therefore, the particular system can be written as

$$\dot{x}(t) = f(t, x(t)), \qquad t \neq \tau_k(x), \tag{2.11a}$$

$$\Delta x(t) = \mathcal{I}(t, x(t)), \qquad t = \tau_k(x), \tag{2.11b}$$

$$x(t_0) = x_0. \tag{2.11c}$$

In this case, the system is said to have impulses at variable times. Indicative features of this system are that solutions start at different points will be subject to impulses (or jump discontinuities) at different times. This problem breaks down the classical continuous dependence or stability since neighbouring solutions tend to undergo impulses at slightly different times. Also, a solution may hit the same hyper-surface several times or not at all, or intersect it more than once after intersecting other hyper-surfaces. The frequent interception of the same hyper-surface is called *pulse or beating phenomenon*. To avoid this circumstance, further restrictions have to be made on the impulsive hyper-surface, as will be seen in the following chapter.

If the functions τ_k's are constants (i.e., $\tau_k(x) = \tau_k$ for all k and x), system (2.11) is said to have impulses at fixed times, and all solutions undergo impulses at the same times.

Another challenging issue arising in impulsive systems, which makes the theory of ordinary differential equation is not directly applicable, is known as *confluence* (or solution merging), which happens when, for instance, two solutions start at different points merge after a certain impulse. The reason is that, for specific impulse amount represented by the function $\mathcal{I}$, the mapping $x + \mathcal{I}(\tau_k, x)$ is not one-to-one in x. On the other hand, if the mapping is not onto, the backward continuation of solutions would be impossible.

So far, we have assumed that the solutions of impulsive systems are left-continuous, instead, one may consider solutions to be right-continuous. Accordingly, system (2.11) is written as

$$\dot{x}(t) = f(t, x(t)), \qquad t \neq \tau_k(x(t^-)), \tag{2.12a}$$

$$\Delta x(t) = \mathcal{I}(t, x(t^-)), \qquad t = \tau_k(x(t^-)), \tag{2.12b}$$

$$x(t_0) = x_0. \tag{2.12c}$$

The choice of right-continuous is advantageous when time delay is involved in impulsive systems.

2.3 Delay Systems

As pointed out earlier, one of the main discrepancies between ordinary and delay differential equations is the initial data. In the ordinary case, the initial condition is given at a specific time, whereas for delay differential equations the initial data are generally continuous functions defined on a finite interval. To define the initial value problem of delay systems, we need some definitions.

Let $\mathcal{C}_r = \mathcal{C}([-r, 0], \mathbb{R}^n)$, with $r > 0$ representing a time delay, be the set of all continuous functions from $[-r, 0]$ to $\mathbb{R}^n$. If $\phi \in \mathcal{C}_r$, the r-norm of this function is defined by $\|\phi_t\|_r = \sup_{-r \le s \le 0} \|\phi(s)\|$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

Definition 2.7. Let $t^* \in \mathbb{R}$ and $a > 0$. If x is a function mapping $[t^* - r, t^* + a]$ into $\mathbb{R}^n$, then, for every $t \in [t^*, t^* + a]$, we define a new function x_t which maps $[-r, 0]$ into $\mathbb{R}^n$ by $x_t(s) = x(t + s)$, for all $s \in [-r, 0]$ (i.e., $x_t : [-r, 0] \to \mathbb{R}^n$).

Here, for each $t \in [t^* - r, t^*]$, $x_t(s)$ (or simply x_t) is the segment of the function x from $t^* - r$ to t^* that has been shifted to the interval $[-r, 0]$. A general nonlinear

DDE is described by

$$\dot{x}(t) = f(t, x_t), \tag{2.13a}$$

where f, which depends on both t and the new function $x_t \in \mathcal{C}_r$, is called a *functional*. An initial condition is given as a continuous function

$$x_{t_0}(s) = \phi(s), \quad s \in [-r, 0]. \tag{2.13b}$$

Thus, the *initial value problem of a delay system* is defined by (2.13).

One can similarly define the Dini derivative D^+V with respect to the delay system (2.13).

Definition 2.8. Let $J \subseteq \mathbb{R}_+$ and $\mathcal{D}$ be an open subset of $\mathbb{R}^n$. If $V : J \times \mathcal{D} \to \mathbb{R}_+$, then the *upper right-hand derivative* of V with respect to system (2.13) is defined by

$$D^+V(t, \psi(0)) = \lim_{h \to 0^+} \sup \frac{1}{h} \big[V\big(t + h, \psi(0) + hf(t, \psi)\big) - V(t, \psi(0)) \big],$$

for all $(t, \psi) \in J \times \mathcal{PC}([-r, 0]; \mathcal{D})$.
If, moreover, V has continuous partial derivatives with respect to its variables, then we have

$$D^+V(t, \psi(0)) = \dot{V}(t, \psi(0)) = \frac{\partial V(t, \psi(0))}{\partial t} + \nabla_{\psi(0)} V(t, \psi(0)) \cdot f(t, \psi).$$

2.4 Impulsive Systems with Time Delay

Incorporating impulsive effects of the variable time type in the delay system (2.13) leads to *impulsive system with time delay* (ISD). Due to the discontinuous behaviour of the system state, the functional and the initial functions should be defined on a larger class of piecewise continuous functions.

In the following, we define classes of functions, which are right-continuous on their domains and left-continuous except at simple jump discontinuities where the left-hand limits exist.

Definition 2.9. [Ball99a] For any $a, b \in \mathbb{R}$ with $a < b$ and for some set $\mathcal{D} \in \mathbb{R}^n$, define

$$\begin{aligned}
\mathcal{PC}\big([a,b];\mathcal{D}\big) =& \Big\{\psi : [a,b] \to \mathcal{D} \,\Big|\, \psi(t^+) = \psi(t),\ \forall t \in [a,b),\ \psi(t^-) \text{ exists in } \mathcal{D},\ \forall t \in (a,b], \\
& \text{and } \psi(t^-) = \psi(t) \text{ for all except at most a finite number of points} \\
& t \in (a,b]\Big\}, \\
\mathcal{PC}\big([a,b);\mathcal{D}\big) =& \Big\{\psi : [a,b) \to \mathcal{D} \,\Big|\, \psi(t^+) = \psi(t),\ \forall t \in [a,b),\ \psi(t^-) \text{ exists in } \mathcal{D},\ \forall t \in (a,b), \\
& \text{and } \psi(t^-) = \psi(t) \text{ for all except at most a finite number of points} \\
& t \in (a,b)\Big\}, \\
\mathcal{PC}\big([a,\infty);\mathcal{D}\big) =& \Big\{\psi : [a,\infty) \to \mathcal{D} \,\Big|\, \forall c > a, \psi\big|_{[a,c]} \in \mathcal{PC}([a,c];\mathcal{D})\Big\}.
\end{aligned}$$

The number of discontinuities is finite if the functions are defined on finite intervals; otherwise, i.e., on infinite interval, the number of discontinuities is countably infinite, which form an increasing sequence of points tending to infinity.

Let $\mathcal{PC}_r\big([-r,0];\mathbb{R}^n\big) = \{\phi : \phi \in \mathcal{PC}([-r,0];\mathbb{R}^n)\}$, and define the r-norm of $\phi \in \mathcal{PC}_r$ by $\|\phi\|_r = \sup_{-r\le s\le 0} \|\phi(s)\|$. If $x \in \mathcal{PC}([t_0 - r,\infty);\mathbb{R}^n)$ with $t_0 \in \mathbb{R}_+$, we define a function $x_t \in \mathcal{PC}([-r,0];\mathbb{R}^n)$ by $x_t(s) = x(t+s)$ for all $s \in [-r,0]$. Let $J \subseteq \mathbb{R}_+$ and $\mathcal{D} \subset \mathbb{R}^n$ be an open set. Then, a nonlinear ISD with impulses at variable times may have the form

$$\dot{x}(t) = f(t, x_t), \qquad t \neq \tau_k(x(t^-)), \tag{2.14a}$$

$$\Delta x(t) = \mathcal{I}(t, x_{t^-}), \qquad t = \tau_k(x(t^-), \tag{2.14b}$$

$$x_{t_0} = \phi(s), \qquad s \in [-r, 0], \tag{2.14c}$$

where $f : J \times \mathcal{PC}([-r, 0]; \mathcal{D}) \to \mathbb{R}^n$ and $\phi \in \mathcal{PC}([-r, 0]; \mathcal{D})$. If one is interested in impulses at fixed times, the corresponding ISD can be defined analogously.

2.5 Stochastic Differential Equations

In this section, we present some basic concepts that will be used throughout this part and the thesis in general. First of all, we start with introducing some notations and definitions from the probability theory. Then, we give the definition of stochastic processes which include the so-called Wiener (or Brownian motion) process. After that, we define a particularly important class of stochastic integrals, namely Itô integrals, which leads us to the main part of this section—stochastic differential equations.

2.5.1 Notations and Basic Definitions

Probability theory is a mathematical branch that deals with the analysis of random experiments, where the outcomes, which are called *elementary events* and traditionally denoted by ω, fully depend on chance. The (elementary) events can be grouped together to form a bigger set, say Ω, called a *sample space*. If the event ω is a possible outcome of a certain random experiment, we suitably write $\omega \in \Omega$.

Denote by $\mathcal{F}$ the family of all interesting events of Ω. For further purposes, $\mathcal{F}$ is required to be a σ-algebra (or σ-field), which is defined below.

Definition 2.10. A collection of subsets (or events) $\mathcal{F}$ of Ω is said to be a *σ-algebra on* Ω if the following conditions hold:

1. the empty subset $\emptyset \in \mathcal{F}$;

2. if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, where A^c stands for the complement of A;

3. if $\{A_i\}_{i\geq 1} \in \mathcal{F}$, then $\cup_{i\geq 1} A_i \in \mathcal{F}$.

A *measure space* can then be defined by the pair $(\Omega, \mathcal{F})$, and the elements of $\mathcal{F}$, in this case, are called *$\mathcal{F}$-measurable sets*. If $\mathcal{S}$ is a class of subsets of Ω, then one can find a smallest σ-algebra $\sigma(\mathcal{S})$ on Ω that contains $\mathcal{S}$. Particularly, if $\Omega = \mathbb{R}^d$ and $\mathcal{S}$ is the smallest class of all open set in $\mathbb{R}^d$, then $\mathcal{B}^d = \sigma(\mathcal{S})$ is called the *Borel σ-algebra* and its elements are called *Borel sets*. We can now introduce the concepts of a random variable and probability measure.

Definition 2.11. A real-valued function $X : \Omega \to \mathbb{R}$ is said to be a *random variable* or *$\mathcal{F}$-measurable* if $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$. Also, an $\mathbb{R}^d$-valued function $X(\omega) = (X_1(\omega), X_2(\omega), \cdots, X_d(\omega))^T$ is said to be $\mathcal{F}$-measurable if all the elements X_i are $\mathcal{F}$-measurable. Analogously, an $\mathbb{R}^{d\times m}$-valued function $X(\omega) = [X_{ij}(\omega)]_{d\times m}$ is said to be $\mathcal{F}$-measurable if all the elements X_{ij} are $\mathcal{F}$-measurable.

Definition 2.12. A function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is said to be a *probability measure* on the measurable space $(\Omega, \mathcal{F})$ if the following conditions hold:

1. $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$;

2. for any pairwise disjoint sequence or collection of subsets $\{A_i\}_{i\geq 1} \subset \mathcal{F}$ (i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$),

$$\mathbb{P}\big(\cup_{i\geq 1} A_i\big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Moreover, the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*. Also, the probability space is said to be *complete* if the σ-algebra is complete, i.e., $\mathcal{F} = \bar{\mathcal{F}}$, where $\bar{\mathcal{F}}$ is the completion of $\mathcal{F}$.

In this thesis, we will always assume that the probability space (or $\mathcal{F}$) is complete.

It is well known that the probabilistic behaviour of a random variable is completely and uniquely described by its *distribution function* $F(x)$, which is defined by

$$F(x) = \mathbb{P}\{\omega : X(\omega) \leq x\}, \qquad \text{for all } x \in \mathbb{R}.$$

Assume that X is a continuous random variable, then there exists a non-negative and integrable function $f(x)$ such that, for every x,

$$F(x) = \int_{-\infty}^{x} f(s)ds,$$

which implies that $f(x) = \frac{dF(x)}{dx}$, which is called the (probability)*density function* of X.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a random variable that is integrable with respect to the probability measure $\mathbb{P}$, then the *mathematical expectation*, also known as *mean* or *average value* of $x = X(\omega)$ with respect to $\mathbb{P}$ is a real number defined by

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\,\mathbb{P}(\omega) = \int_{-\infty}^{\infty} x\, dF(x),$$

the pth moment of X is defined by

$$\mathbb{E}[X^p] = \int_{\Omega} X^p(\omega)\, d\,\mathbb{P}(\omega) = \int_{-\infty}^{\infty} x^p\, dF(x),$$

where $p > 0$. Particularly, if $p = 2$, $\mathbb{E}[X^2]$ is the *mean square* (m.s.) of X. Also, the *variance* of X is defined by

$$V(X) = \mathbb{E}\big[X - \mathbb{E}[X]\big]^2,$$

and, if Y is another random variable, the *covariance* of X and Y is defined by

$$Cov(X, Y) = \mathbb{E}\big[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\big],$$

where all involved integrals exist.

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $X_1(\omega), X_2(\omega), \cdots$, a sequence of random variables, and $X(\omega)$ be defined on the given probability space. Then, the sequence $\{X_k(\omega)\}_{k \geq 1}$ is said to converge to $X(\omega)$ *with probability one* (w.p.1) or *almost surely* (a.s.) if

$$\mathbb{P}\big\{\omega : \lim_{k \to \infty} X_k(\omega) = X(\omega)\big\} = 1;$$

it is said to converge to $X(\omega)$ *in probability* or *stochastically* if, for every $\varepsilon > 0$,

$$\lim_{k \to \infty} \mathbb{P}\big\{\omega : |X_k(\omega) - X(\omega)| > \varepsilon\big\} = 0;$$

it is said to converge to $X(\omega)$ *in the pth moment* if

$$\lim_{k \to \infty} \mathbb{E}\big[|X_k(\omega) - X(\omega)|^p\big] = 0,$$

where all involved integrals exist, and it is said to converge to $X(\omega)$ *in the m.s.* if $p = 2$. Furthermore, if $\{X_k(\omega)\}_{k \geq 1}$ and $X(\omega)$ have distribution functions $F_k(x)$ and $F(x)$, respectively, then the sequence of the random variables is said to converge to $X(\omega)$ *in distribution* if

$$\lim_{k \to \infty} F_k(x) = F(x)$$

in every continuity point of $F(x)$.

2.5.2 Stochastic Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A *filtration* is a family (or a sequence) of increasing sub-σ-algebra $\{\mathcal{F}_t\}_{t\geq 0}$ of $\mathcal{F}$ (i.e., $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \leq t < s < \infty$). The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is said to be *right continuous* if $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$, and it is said to satisfy the *usual conditions* if it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets (i.e., any random event $A \in \mathcal{F}_0$ with $\mathbb{P}(A) = 0$). From now on, the complete probability space under consideration satisfies the usual conditions, and, in this case, we use the quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

Definition 2.13. A *stochastic process* $X(t)$ is a family of random variables $\{X_t(\omega); t \in I,\ \omega \in \Omega\}$ (which is also denoted by $X(t, \omega)$ for the same t and ω).

Throughout this thesis, we restrict ourselves to a parameter or (index) set $I \subseteq \mathbb{R}_+$ and state space Ω that is $\mathbb{R}$ or $\mathbb{R}^n$, unless stated otherwise. Apparently, a stochastic process is a function of two variables; for each fixed $t \in I$, $X_t(\omega)$ is a random variable (or $\mathbb{R}^n$-valued random variable), while, for each fixed $\omega \in \Omega$, $X_t(\omega)$ is real-valued (or $\mathbb{R}^n$-valued) function defined on I. The latter is called a *sample path* or *realization* of the stochastic process.

Let $X(t)$ be an $\mathbb{R}^d$-valued stochastic process. It is said to be *continuous* (respectively, *right continuous, left continuous*) if, for almost all $\omega \in \Omega$, $X_t(\omega)$ is continuous (respectively, right continuous, left continuous) for all $t \in \mathbb{R}_+$. It is said to be *cadlag* if it is right continuous and, for almost all $\omega \in \Omega$, the left limit $\lim_{s\to t} X_s(\omega)$ exists and is finite for all $t > 0$. It is said to be *integrable* if, for all $t \in \mathbb{R}_+$, $X_t(\omega)$ is an integrable random variable. It is said to be $\mathcal{F}_t$*-adapted* (or *non-anticipated*) if, for all $t \in \mathbb{R}_+$, it is $\mathcal{F}_t$-measurable. If $Y_t(\omega)$ is another stochastic process, then the two processes are said to be *indistinguishable* if

$$\mathbb{P}\big\{\omega : X_t(\omega) = Y_t(\omega),\, \forall t \in \mathbb{R}_+\big\} = 1.$$

Let $X(t)$ be an $\mathbb{R}^d$-valued cadlag $\mathcal{F}_t$-adapted process, and $\mathcal{D}$ be an open subset of $\mathbb{R}^d$. Then, the *first exit time of the process* $X(t)$ *from* $\mathcal{D}$ is defined by

$$\tau = \inf\{t \in \mathbb{R}_+ \big|\, X(t) \notin \mathcal{D}\},$$

where $\inf \emptyset = \infty$.

Like random variables, stochastic processes can be characterized by their moments, variance, and autocorrelation.

Definition 2.14. Let $X(t)$ be a continuous stochastic process. Then, the *mathematical expectation* (or *mean* or the *first moment*) of $X(t)$ is defined by

$$m(t) = \mathbb{E}[X(t)] = \int_{-\infty}^{\infty} x f(x,t)\, dx,$$

where f (or $f(x,t)$) is the probability density function of $x = X(t)$; the *second moment* (or the *mean square*) is defined by

$$m_2(t) = \mathbb{E}[X^2(t)] = \int_{-\infty}^{\infty} x^2 f(x,t)\, dx;$$

the *variance*,

$$Var[X(t)] = \mathbb{E}[(X(t) - m(t))^2] = m_2(t) - m^2(t),$$

and the *auto-correletion* is defined by

$$R(t_1,t_2) = \mathbb{E}[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1,t_1;x_2,t_2)\, dx_1 dx_2.$$

Definition 2.15. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A stochastic process $W(t)$ for all $t \in \mathbb{R}_+$ that is continuous (a.s.) and

$\mathcal{F}_t$-adapted is said to be *Wiener (or Brownian motion) process* if

1. $\mathbb{P}\{\omega : W(0) = 0\} = 1$;

2. for any $0 \leq s < t < \infty$, the increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$;

3. for any $t \in \mathbb{R}_+$ and $h > 0$, the increment $W(t+h) - W(t)$ is Gaussian (or normally) distributed with

$$\mathbb{E}[W(t+h) - W(t)] = \mu h;$$
$$\mathbb{E}[(W(t+h) - W(t))^2] = \sigma^2 h,$$

where the mean $\mu \in \mathbb{R}$ and the variance σ^2 is a positive constant. If $\mu = 0$ and $\sigma^2 = 1$, W is said to be a *standard Wiener process.*

Following the definition of distribution function F, the *jointly distribution function* of $X(t_1), \cdots, X(t_n)$ is defined by

$$F_{X(t_1),\cdots,X(t_n)}(x_1, \cdots, x_n) = \mathbb{P}\{X(t_1) \leq x_1, \cdots, X(t_n) \leq x_n\},$$

and, if F has partial derivatives at $x_1, \cdots, x_n$, then the corresponding probability density function of $(x_1, \cdots, x_n)$ is given by

$$f(x_1, \cdots, x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{X(t_1),\cdots,X(t_n)}(x_1, \cdots, x_n).$$

A stochastic process $X(t)$ is said to be *stationary* if and only if, for all time instants $t_1, \cdots, t_n$ and any time difference τ,

$$f_{X(t_1),\cdots,X(t_n)}(x_1, \cdots, x_n) = f_{X(t_1+\tau),\cdots,X(t_n+\tau)}(x_1, \cdots, x_n).$$

We conclude this subsection with a mathematically useful stochastic process called Gaussian white noise process.

Definition 2.16. A stochastic process $\mathcal{N}$ is said to be a *Gaussian white noise process* if and only if it is a stationary Gaussian process with mean zero and autocorrelation given by

$$R(\tau) = C\delta(\tau),$$

where C is a constant and δ is a Dirac delta or impluse function.

Clearly, the variance of the Gaussian white noise is $Var[\mathcal{N}(t)] = \infty$.

2.5.3 Stochastic Differential Equations

Consider a physical process described by the following ordinary differential equation

$$\frac{dx}{dt} = f(t,x). \tag{2.15}$$

If it is perturbed by some disturbance having a stochastic behaviour, say $\xi = \xi(t)$ for any t, then (2.15) may be written as

$$\frac{dX}{dt} = F(t,X,\xi). \tag{2.16}$$

Due to the random part, this differential equation cannot be interpreted as its ordinary counterpart in (2.15). To better understand the new situation, we consider the following special form of (2.16)

$$\frac{dX}{dt} = f(t,X) + g(t,X)\mathcal{N}(t), \tag{2.17}$$

with a deterministic *drift* coefficient $f(t,X)$ perturbed by a noise term $g(t,X)\mathcal{N}(t)$ with $\mathcal{N}$ being a Gaussian white noise process and the *diffusion* coefficient $g(t,X)$ is the noise intensity. Integrating (2.17) over $[t_0,t]$ yields

$$X(t) = X(t_0) + \int_{t_0}^{t} f(s,X(s))ds + \int_{t_0}^{t} g(s,X(s))\mathcal{N}(s)ds, \tag{2.18}$$

where the first integral is deterministic for almost every $\omega \in \Omega$, while the second one cannot be defined in any meaningful manner.

To cope with this difficulty, we replace the aforementioned integral by an integral of the form

$$\int_{t_0}^{t} g(s, X) dW(s), \tag{2.19}$$

where W is a Wiener process with the *formal* relationship with the Gaussian white noise process being given by $\dot{W}(t) = \mathcal{N}(t)$ and so $dW(t) = \mathcal{N}(t)dt$. The resulting integral in (2.19) cannot be defined as a Riemann-Stieltjes integral, because, for almost all $\omega \in \Omega$, the Wiener sample path $W(\omega)$ is nowhere differentiable and has unbounded variation over every time interval.

However, one can define this integral on a larger class of stochastic processes depending on the properties of Wiener process. This definition was first proposed by K. Itô, and the integral is now known as *Itô stochastic integral*.

Consider the integral of the form

$$\int_{a}^{b} g(s, \omega) dW(s, \omega), \tag{2.20}$$

where g is a stochastic process with appropriate conditions and W is a Wiener process, where we generally assume that the two processes are not mutually independent and $g(t, \omega)$ is not absolutely continuous for almost all $\omega \in \Omega$.

The core feature of the Itô integral is that the random function g is non-anticipative or adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, i.e., $g(t, \omega)$ can at most depend on the present and past, and not on the future, values of the Wiener process $W(t, \omega)$. More precisely, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space on which the Wiener process $W(t, \omega)$ is defined for all $t \in \mathbb{R}_+$ and

1. for every $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$ implies that $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$;

2. for all $t \in \mathbb{R}_+$, the random variable $W(t, \omega)$ is $\mathcal{F}_t$-measurable;

3. for $t_{i+1} > t_i \geq t$, the increment $W(t_{i+1}, \omega) - W(t_i, \omega)$ is independent of $\mathcal{F}_t$.

For $a, b \in \mathbb{R}_+$ with $a \leq b$, denote by $L^2[a, b]$ the class of all real-valued random processes (functions) $g(t)$ defined on $[a, b]$ and satisfying the following conditions:

4. for all $t \in [a, b]$, $g(t, \omega)$ is $\mathcal{F}_t$-measurable;

5. the integral

$$\int_a^b g^2(t, \omega) dt \tag{2.21}$$

is finite w.p.1.

To define the Itô (stochastic) integral, consider the partition $a = t_1 < t_2 < \cdots < t_{k+1} = b$, and let $g(t, \omega)$ be a step or simple function, i.e., $g(t, \omega) = g(t_i, \omega)$ for all $t \in [t_i, t_{i+1}]$, which is assumed to be $\mathcal{F}_{t_i}$-measurable, bounded random variable. Then, the Itô integral is defined by

$$\int_a^b g(t, \omega) dW(t) = \sum_{i=1}^{k} g(t_i, \omega)[W(t_{i+1}) - W(t_i)]. \tag{2.22}$$

Another way to define Itô integral is as a limit of a m.s. convergent sequence of simple processes. Let $g_n(t, \omega) \in L^2[a, b]$ be an arbitrary sequence of simple processes. Then, the Itô integral is defined by

$$\int_a^b g(t, \omega) dW(t) = \lim_{n \to \infty} \int_a^b g_n(t, \omega) dW(t) \tag{2.23}$$

in $L^2[a, b]$, i.e.,

$$\lim_{n \to \infty} \mathbb{E} \int_a^b |g(t, \omega) - g_n(t, \omega)|^2 dt = 0.$$

The Itô integral in (2.23) has some nice properties. Assuming that $g \in \mathcal{L}_{ad}([a, b]; \mathbb{R}^d)$, i.e., g is an $\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted process such that $\int_a^b \mathbb{E}\|g(t)\|^2 dt < \infty$, some of these properties are

1. $\mathbb{E}\int_a^b g(t)dW(t) = 0$;

2. $\mathbb{E}\left\| \int_a^b g(t)dW(t)\right\|^2 \leq \int_a^b \mathbb{E}\|g(t)\|^2\, dW(t)$.

Replacing the stochastic integral in (2.18) by the Itô integral results in the following *stochastic integral equation*

$$X(t) = X(t_0) + \int_{t_0}^{t} f(s, X(s))ds + \int_{t_0}^{t} g(s, X(s))dW(s), \qquad (2.24)$$

which is equivalent to the symbolic *stochastic differential equation (SDE) of Itô type*

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \qquad (2.25)$$

with the initial state $X(t_0) = X_0$. Before presenting the solution of this equation, we need to define the following class of random processes (functions).

Definition 2.17. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space. For any $\omega \in \Omega$, $a, b \in \mathbb{R}_+$, with $a < b$, and $p \geq 1$, a random process $f(t, \omega)$ is said to belong to class $\mathcal{L}_{ad}(\Omega, L^p[a, b])$ if it is $\mathcal{F}_t$-adapted and almost all its sample paths are pth integrable in the Riemann sense.

Definition 2.18. For any $t_0, T \in \mathbb{R}_+$, the $\mathbb{R}^n$-valued stochastic process $x(t) = x(t, t_0, x_0)$ is said to be a *solution* of n-dimensional initial value problem

$$dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), \qquad t \in [t_0, T], \qquad (2.26a)$$

$$x(t_0) = x_0, \qquad (2.26b)$$

where $W(t) = (W_1(t), \cdots, W_m(t))^T \in \mathbb{R}^m$ and x_0 is an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^n$-valued random variable such that $\mathbb{E}[\|x_0\|^2] < \infty$, if the the following properties hold:

1. $x(t)$ is continuous and $\mathcal{F}_t$-adapted;

2. the $\mathbb{R}^n$-valued $f \in \mathcal{L}_{ad}(\Omega, L^1[a,b])$ and the $\mathbb{R}^{n\times m}$-valued $g \in \mathcal{L}_{ad}(\Omega, L^2[a,b])$;

3. for any $t \in [t_0, T]$, $x(t)$ satisfies the SDE in (2.26a) w.p.1;

4. at $t = t_0$, x satisfies the initial condition in (2.26b) w.p.1.

Furthermore, a solution $x(t)$ is said to be *unique* if any other solution $y(t)$ is indistinguishable from $x(t)$, i.e.,

$$\mathbb{P}\big\{x(t) = y(t),\ \forall t \in [t_0, T]\big\} = 1.$$

When working on Itô SDEs, there arise some peculiarities, and amongst them is that if x is a solution of an Itô equation and $V(t, x(t))$ is a sufficiently smooth function, we cannot use the chain rule of the classical calculus to set up the SDE governing $V(t, x(t))$. Instead, we use the stochastic version of the chain rule, which is called *Itô formula*. Before stating the definition of Itô formula, we define $\mathcal{C}^{1,2}(\mathbb{R}^n;\mathbb{R})$ to be a class of functions, say V, such that $V_t = \partial V/\partial t$, V_x and V_{xx}, being the gradient and Hessian matrix of V, are all continuous functions.

Itô formula. For any $t_0 \in \mathbb{R}_+$ and $t \geq t_0$, let $x(t)$ be an $\mathbb{R}^n$-dimensional Itô process, i.e., $\mathbb{R}^n$-valued continuous adapted process satisfying

$$dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), \qquad \text{(a.s.)}, \tag{2.27}$$

where $f \in \mathcal{L}_{ad}(\Omega, L^1[a,b])$ and $g \in \mathcal{L}_{ad}(\Omega, L^2[a,b])$. Let $V \in \mathcal{C}^{1,2}(\mathbb{R}^n;\mathbb{R})$. Then, for any $t \geq t_0$, V is a real-valued Itô process satisfying

$$dV(t,x) = \mathcal{L}V(t,x)f(t,x)dt + V_x(t,x)g(t,x)dW(t), \qquad \text{(a.s.)}$$

where

$$\mathcal{L}V(t,x) = V_t(t,x) + V_x(t,x)f(t,x) + \frac{1}{2}\mathrm{tr}[g^T(t,x)V_{xx}(t,x)g(t,x)].$$

The operator $\mathcal{L}$ (or $\mathcal{L}V$ as a single notation) is also called the *averaged derivative* (or *infinitesimal diffusion operator*) at a point (t, x) and can be generally defined as

$$\mathcal{L}V(t,x) = \lim_{h \to 0^+} \frac{1}{h}\big[\mathbb{E}[V(t+h, x(t+h))] - V(t,x)\big].$$

As mentioned earlier, a more general system than (2.26) is when the system states are subject to time lag. This leads to *stochastic systems with time delay* or *systems with stochastic functional differential equations*, which are typically defined by

$$\begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), & t \in [t_0, T], \\ x_{t_0}(s) = \phi(s), & s \in [-r, 0], \end{cases} \tag{2.28}$$

for any $t_0, T \in \mathbb{R}_+$ with $T \geq t_0$.

We have stated clearly that one of the main discrepancies between ordinary and delay systems is the amount of the initial data, which, in the latter case, must be given over a certain period of time rather than at a specific time instance. Moreover, due to the randomness that drives the system states, the given initial condition function is generally defined as a stochastic process. Consequently, to define a solution of the initial value problem given in (2.28), it is natural to consider the initial function ϕ to be $\mathcal{F}_{t_0}$-measurable, continuous random variable mapping $[-r, 0]$ into $\mathbb{R}^n$ such that $\mathbb{E}[\|\phi\|_r^p] < \infty$ for some $p > 0$. The solution of (2.28) can then be defined similarly to that of (2.26) except, of course, $x(t)$ is defined over the interval $[t_0 - r, T]$ for all $T \in \mathbb{R}_+$ (or $[t_0 - r, t_0 + \alpha]$ for $\alpha > 0$).

Having defined the solution x of (2.28) and the Itô formula, we can present the definition of some stochastic properties of the trivial solution of (2.28).

Definition 2.19. For any $t \geq t_0$ with $t_0 \in \mathbb{R}_+$, let $x(t) = x(t, t_0, \phi)$ be a solution of system (2.28). Then, the trivial solution $x \equiv 0$ of (2.28) is said to be

1. *almost-surely stable (or stable w.p.1)* if, for any given ε, $\varepsilon' > 0$, and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon, \varepsilon', t_0)$ such that

$$\|\phi\|_r < \delta \qquad \text{implies} \qquad \mathbb{P}\{\omega : \sup_{t \geq t_0} \|x(t)\| > \varepsilon'\} < \varepsilon;$$

2. *pth moment stable* if, for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon, t_0)$ such that, for $p > 0$,

$$\|\phi\|_r^p < \delta \qquad \text{implies} \qquad \mathbb{E}[\sup_{t \geq t_0} \|x(t)\|^p] < \varepsilon;$$

3. *asymptotically stable* if, for any $\varepsilon \in (0, 1)$, there exists $\delta = \delta(\varepsilon, t_0)$ such that

$$\|\phi\|_r < \delta \qquad \text{implies} \qquad \mathbb{P}\{\omega : \lim_{t \to \infty} \sup \|x(t)\| = 0\} < 1 - \varepsilon;$$

4. *almost-surely asymptotically stable* if it is almost-surely stable and

$$\mathbb{P}\{\omega : \lim_{t \to \infty} \sup \|x(t)\| = 0\} = 1;$$

5. *pth moment asymptotically stable* if it is stable in the pth moment and

$$\lim_{t \to \infty} \mathbb{E}[\sup \|x(t)\|^p] = 0;$$

6. *pth moment exponentially stable* if there exist positive constants p, K, and λ such that, for any $t_0 \in \mathbb{R}_+$,

$$\|\phi\|_r^p < \delta \qquad \text{implies} \qquad \mathbb{E}[\|x(t)\|^p] \leq K \|\phi\|_r^p e^{-\lambda(t - t_0)}.$$

Moreover, the above stability properties are said to be satisfied *globally* if they hold for arbitrarily large δ. Also, they are said to hold *uniformly* if δ is chosen to be independent of t_0.

On the other hand, if the system states of (2.26) experience impulsive effects at fixed times, we are led to *stochastic impulsive systems* or *systems with stochastic impulsive differential equations*, which are generally given by

$$\begin{cases} dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), & t \neq \tau_k, \\ \Delta x(t) = \mathcal{I}(t, x(t^-)), & t = \tau_k, \\ x(t_0) = x_0. \end{cases} \tag{2.29}$$

This system was studied in [Liu07, Liu08]. The focus was on establishing the problems of existence and uniqueness of a global solution and some qualitative properties, such as asymptotic and exponential stability in the pth moment. In both results, the comparison principle approach was used to achieve the aforesaid system characteristics.

In analyzing regularity conditions of stochastic systems with and without time delays, or with and without impulsive effects, a very common practice is to assume that the system vector fields are (locally or globally) Lipschitz to assure a unique solution, and a linear growth condition to avoid the finite escape time that a solution may have. More specifically, these conditions are made to guarantee that a Picard successive iteration is convergent. However, employing Lyapunov technique, one can get unique solutions even if Lipschitz conditions are relaxed [Kha80, Mao06].

We conclude this section by presenting some inequalities [Mao06] that will be used throughout this thesis.

Definition 2.20. A function $\varphi : \mathbb{R} \to \mathbb{R}$ is said to be *convex* if the following hold

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y), \qquad \lambda \in (0,1).$$

It is said to be *concave* if $\leq$ is replaced by $\geq$.

Jensen's inequality. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function, and $x : \Omega \to \mathbb{R}$ is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[x] < \infty$, then

$$\varphi(\mathbb{E}[x]) \leq \mathbb{E}[\varphi(x)].$$

Tchebychev's inequality. If $x : \Omega \to \mathbb{R}^n$ is a random variable such that $\mathbb{E}[\|x\|^p] < \infty$, for some $p > 0$, then

$$\mathbb{P}\big\{\omega \in \Omega : \|x\| \geq \varepsilon\big\} \leq \frac{\mathbb{E}[\|x\|^p]}{\varepsilon^p}, \qquad \text{for some } \varepsilon > 0.$$

Hölder's inequality. Let x and y be $\mathbb{R}^n$-valued random processes. If $p, q \in (1, \infty)$ and $1/p + 1/q = 1$, then

$$\big|\mathbb{E}[x^T y]\big| \leq \mathbb{E}[\|x\|^p]^{1/p}\mathbb{E}[\|y\|^q]^{1/q}$$

holds provided that the pth moments on the right hand side are finite.

Bihari's inequality. [Bih56] For all $t \in [0, T]$ with $T > 0$, let $u(t) \geq 0$ be a Borel measurable function and $v(t) \geq 0$ be an integrable function. Suppose that $K : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function such that $K(t) > 0$ for all $t > 0$. If, for some $c > 0$,

$$u(t) \leq c + \int_0^t v(s)K(u(s))ds, \qquad \forall\, t \in [0, T],$$

then

$$u(t) \leq G^{-1}\Big(G(c) + \int_0^t v(s)ds\Big)$$

holds for all $t \in [0, T]$ such that

$$G(c) + \int_0^t v(s)ds \in \mathrm{Dom}(G^{-1}),$$

where $G(r) = \int_{0^+}^r \frac{ds}{K(s)}$, for $r > 0$, and G^{-1} is the inverse function of G.

2.6 Stochastic Impulsive System with Time Delay

In the previous section, we have described stochastic systems with time delay, and systems with stochastic impulsive differential equations. In this section, these systems are combined to lead us to consider *stochastic impulsive system with time delay* (SISD). Before formulating the latter system, and for convenient reading, we restate some of the notations that have been presented in previous sections.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $W(t) = (W_1(t), W_2(t), \cdots, W_m(t))^T$ be an m-dimensional Wiener process defined on the above probability space. Let $r > 0$ represent time delay and denote by $\mathcal{C}([-r, 0], \mathbb{R}^n)$ (and $\mathcal{PC}([-r, 0], \mathbb{R}^n)$) the space of continuous (piecewise continuous) functions ϕ mapping $[-r, 0]$ into $\mathbb{R}^n$. Moreover, if $x : [t - r, \infty) \to \mathbb{R}^n$, we define x_t by $x_t = x(t + s)$ for $s \in [-r, 0]$ and the corresponding r-norm is $\|x_t\|_r = \sup_{t-r\leq s\leq t} \|x(s)\|$. We also define $x_{t^-} \in \mathcal{PC}([-r, 0], \mathbb{R}^n)$ by $x_{t^-}(s) = x(t+s)$ for $-r \leq s < 0$ and $x_{t^-}(s) = x(t^-)$ for $s = 0$. We should mention that this does not mean $x_{t^-} = \lim_{s\to t^-} x_s$ because, if $x \in \mathcal{PC}([-r, 0], \mathbb{R}^n)$, the limit $\lim_{s\to t^-} x_s$ does not generally exist. For $p > 0$, let $\mathcal{L}^p_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$ be the set of all $\mathcal{F}_0$-measurable $\mathcal{PC}([-r, 0], \mathbb{R}^n)$-valued random variables $\phi = \{\phi(s) : -r \leq s \leq 0\}$ such that $\mathbb{E}[\|\phi\|^p_r] \leq c$, for some $c \geq 0$. We also assume that ϕ is independent of $W(t, \omega)$. For a given Wiener process $W(t, \omega)$ and filtration $\{\mathcal{F}_t |\, a \leq t \leq b\}$, we assume that $W(t, \omega)$ is $\mathcal{F}_t$-adapted (i.e. for each $t \in [a, b]$, $W(t, \omega)$ is $\mathcal{F}_t$-measurable) and for any $s \leq t$, the random variable $W(t, \omega) - W(s, \omega)$ is independent of the σ-algebra $\mathcal{F}_s$.

In the following, the definition of piecewise continuous functions introduced in

[Ball99b, Liu2000] is modified, since the solution of a stochastic initial value problem is a random process, rather than merely a deterministic function.

Definition 2.21. For $a, b \in \mathbb{R}$, with $a < b$, and $\mathcal{D} \subset \mathbb{R}^n$, a random process $\psi : [a, b] \times \Omega \to \mathcal{D}$ is said to be an element of the space $\mathcal{PC}([a, b] \times \Omega, \mathcal{D})$ (or $\mathcal{D}$-cadlag) if, for almost all $\omega \in \Omega$, $\psi(t^+, \omega) = \psi(t, \omega)\ \forall\, t \in [a, b)$ and $\psi(t^-, \omega)$ exists in $\mathcal{D}\ \forall\, t \in (a, b]$ and $\psi(t^-, \omega) = \psi(t, \omega)$ for all but at most a finite number of points $t \in (a, b]$. Furthermore, a random process $\psi : [a, \infty) \times \Omega \to \mathcal{D}$ is said to be an element of $\mathcal{PC}([a, \infty) \times \Omega, \mathcal{D})$ if, for almost all $\omega \in \Omega,\ c > a$, where $t \in [a, c]$, $\psi(t, \omega) \in \mathcal{PC}([a,\, c] \times \Omega,\, \mathcal{D})$.

Consider now the following nonlinear SDE with time delay

$$dx(t) = f(t, x_t)dt + g(t, x_t)\, dW(t), \quad t \in [a, b], \tag{2.30a}$$

where $x \in \mathbb{R}^n$ is the system state random process, $f \in \mathbb{R}^n$, and $g \in \mathbb{R}^{n\times m}$. The initial condition is given by

$$x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \tag{2.30b}$$

where $\phi \in \mathcal{L}^2_{\mathcal{F}_0}([-r, 0], \mathbb{R}^n)$ (i.e., the initial state is assumed to be $\mathcal{F}_0$-adapted, piecewise continuous with finite pth moment); thus, the corresponding stochastic integral equation is

$$x(t) = \phi(0) + \int_{t_0}^{t} f(s, x_s)ds + \int_{t_0}^{t} g(s, x_s)\, dW(s), \tag{2.31}$$

where $t \geq t_0$. The first integral is a Riemann integral almost surely (a.s.) and the second one is an Itô integral satisfying

$$\mathbb{E}\Big[\int_{t_0}^{t} g(s, x_s)\, dW(s)\Big] = 0, \quad \text{and} \quad \mathbb{E}\Big\|\int_{t_0}^{t} g(s, x_s)\, dW(s)\Big\|^2 = \int_{t_0}^{t} \mathbb{E}\|g(s, x_s)\|^2\, ds.$$

Considering impulse effects (of variable times) in (2.30a) leads to the following SISD

$$dx(t) = f(t, x_t)dt + g(t, x_t)\, dW(t), \quad t \neq \tau_k(x(t^-)), \tag{2.32a}$$

$$\Delta x(t) = \mathcal{I}(t, x_{t^-}), \qquad t = \tau_k(x(t^-)), \tag{2.32b}$$

where $\tau_k \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$ represents an impulsive hypersurface, for $k = 0, 1, 2, \cdots$, and satisfies $0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \cdots$ and $\lim_{k\to\infty} \tau_k(x) = \infty$ for $x \in \mathbb{R}^n$. The initial condition is given by

$$x_{t_0}(s) = \phi(s), \quad s \in [-r, 0]. \tag{2.32c}$$

We also assume that the solution of (2.32) is right-continuous (i.e., $x(t^+) = x(t)$). In difference equation (2.32b), $\Delta x = x(t) - x(t^-)$ and the functional $\mathcal{I}(\cdot)$ is the impulse amount, which is assumed to be $\mathcal{F}_{t_k}$-adapted.

In the following, we define the solution of the initial value problem (2.32).

Definition 2.22. For any $t_0 \in \mathbb{R}_+$ and $\alpha > 0$, an $\mathbb{R}^n$-valued random process $x \in \mathcal{PC}([t_0 - r, t_0 + \alpha]; \mathbb{R}^n)$ is said to be a solution of (2.32) if it satisfies the following conditions:

(i) the set of impulses $\mathbb{T} = \{t \in (t_0, t_0 + \alpha] \,\big|\, t = \tau_k(x(t^-)) \text{ for some } k\}$ is finite;

(ii) $x(t)$ is continuous for all $t \in (t_0, t_0 + \alpha]\backslash\mathbb{T}$ and $\mathcal{F}_t$-adapted;

(iii) the functionals $f \in \mathcal{L}_{ad}(\Omega, L[t_0, t_0 + \alpha])$ and $g \in \mathcal{L}_{ad}(\Omega, L^2[t_0, t_0 + \alpha])$;

(iv) for any $t \in (t_0, t_0+\alpha]$, $\phi \in \mathcal{L}^2_{\mathcal{F}_0}([-r, 0], \mathbb{R}^n)$, and $\mathcal{I}(t_k, x_{t_k^-})$ that is $\mathcal{F}_{t_k}$-adapted, the following equation

$$x(t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0] \\ \phi(0) + \int_{t_0}^{t} f(s, x_s)ds + \int_{t_0}^{t} g(s, x_s)\, dW(s) & \\ \qquad + \sum_{\{k: t_k \in (t_0, t]\}} \mathcal{I}(t_k, x_{t_k^-}), & t \in (t_0, t_0 + \alpha] \end{cases} \tag{2.33}$$

holds w.p.1;

(v) for any $t \in \mathbb{T}$, $x(t)$ satisfies the difference equation in (2.32b) w.p.1;

(vi) x satisfies the initial condition in (2.32c) w.p.1.

Remark 2.1. In fact, one can restate condition (ii) as follows:

(ii)$'$ for $\mathcal{D} \subset \mathbb{R}^n$, $x \in \mathcal{PC}([t_0 - r, t_0 + \alpha], \mathcal{D})$ and is $\mathcal{F}_t$-adapted.

We should also mention that, in the definition, we have restricted ourselves to the case where solutions undergo a finite number of impulses over any finite interval. However, letting $t \in (t_0, \infty)$, there would be a countably infinite number of impulses, which represent the simple jump discontinuities of x.

A special class of the SISD (2.32) is when the impulsive instances occur at fixed times, i.e.,

$$dx(t) = f(t, x_t)dt + g(t, x_t)\, dW(t), \quad t \neq \tau_k, \tag{2.34a}$$

$$\Delta x(t) = \mathcal{I}(t, x_{t^-}), \qquad t = \tau_k, \tag{2.34b}$$

$$x_{t_0}(s) = \phi(s), \quad s \in [-r, 0]. \tag{2.34c}$$

This system will be studied in later chapters.

We conclude this section with the following results, which have further use in the next chapter.

Let x and y be two $\mathbb{R}^n$-valued random processes having probability measures $\mathbb{P}_x$ and $\mathbb{P}_y$, respectively. Then, the *Prokhorov distance* between the (probability) measures is denoted by $\mathcal{D}(x, y) = \mathcal{D}(\mathbb{P}_x, \mathbb{P}_y)$. Moreover, if $\mathcal{D}(x, y) = 0$, then x and y have the same probability measure. Also, if $\mathbb{P}\big\{\omega \in \Omega : \lim_{n\to\infty} \|x_n(\omega) - x(\omega)\| = 0\big\} = 1$, then $\{x_n\}$ is a $\mathcal{D}$-Cauchy sequence. The converse of this fact is true in the following sense.

Skorokhod's Theorem. Let $\{x_n\}$ be a $\mathcal{D}$-Cauchy sequence of random variables. Then, one can construct another sequence of random variables $\{y_n\}$ and a random variable y such that

$$\mathcal{D}(x_n, y_n) = 0 \quad \text{and} \quad \mathbb{P}\Big\{\omega \in \Omega : \lim_{n\to\infty} \|y_n(\omega) - y(\omega)\| = 0\Big\} = 1.$$

Definition 2.23. A collection of sequences of random variables $Q = \{x_r \,|\, r \in \Lambda\}$, for some index set Λ, is said to be totally $\mathcal{D}$-bounded if every infinite sequence $\{x_{n_r}\} \subset Q$ has a $\mathcal{D}$-Cauchy subsequence.

Prokhorov's Theorem. Q is totally $\mathcal{D}$-bounded if and only if, for every $\varepsilon > 0$, there exists a compact set K_ε of $\mathbb{R}^n$ such that

$$\mathbb{P}\{x \in K_\varepsilon\} > 1 - \varepsilon,$$

for every $x \in Q$.

Lemma 2.1. Let $S \subset \mathbb{R}^n$, $a, b \in \mathbb{R}_+$ with $a < b$, and c, ε be positive constants. Then,

$$\begin{aligned} Q' = \Big\{x^{(n)} \in \mathcal{C}([a,b], S) \,\Big|\, & \mathbb{E}[\|x^{(n)}(t)\|^2] \le c \text{ and} \\ & \mathbb{E}[\|x^{(n)}(t_1) - x^{(n)}(t_2)\|^2] \le \varepsilon, \quad \forall\, n \in \mathbb{N},\, t_1, t_2 \in [a,b]\Big\} \end{aligned}$$

is totally $\mathcal{D}$-bounded subset of $\mathcal{C}([a,b], S)$.

Proof. By Tchebychev's inequality, one can find, for every $\varepsilon > 0$, $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ such that $\mathbb{P}\{\omega \in \Omega : \|x^{(n)}(t)\| > \gamma_1(\varepsilon)\} \leq \frac{\varepsilon}{2}$ and $\mathbb{P}\{\omega \in \Omega : \|x^{(n)}(t_1) - x^{(n)}(t_2)\| > \gamma_2(\varepsilon)\} \leq \frac{\varepsilon}{2}$. Hence, $\mathbb{P}\{\omega \in \Omega : \|x^{(n)}(t)\| > \gamma_1(\varepsilon)$ or $\|x^{(n)}(t_1) - x^{(n)}(t_2)\| > \gamma_2(\varepsilon)\} \leq \varepsilon$, which implies that $\mathbb{P}\{\omega \in \Omega : \|x^{(n)}(t)\| \leq \gamma_1(\varepsilon)$ or $\|x^{(n)}(t_1) - x^{(n)}(t_2)\| \leq \gamma_2(\varepsilon)\} > 1 - \varepsilon$ for every $x \in Q$. For some $\alpha > 0$, let

$$K_\varepsilon = \Big\{ x^{(n)} \in \mathcal{C}([t_0, t_0 + \alpha], S) \Big| \, \|x^{(n)}(t)\| \leq \gamma_1(\varepsilon) \text{ and } \\ \|x^{(n)}(t_1) - x^{(n)}(t_2)\| \leq \gamma_2(\varepsilon), \, \forall \, t_1, t_2 \in [t_0, t_0 + \alpha] \Big\}.$$

Clearly, $\mathbb{P}\{x \in K_\varepsilon\} > 1 - \varepsilon$. By Arzela-Ascoli's Theorem, the compactness of K_ε follows. Finally, applying Prokhorov's Theorem yields the totally $\mathcal{D}$-boundedness of the subset Q'.

Remark 2.2. Q' is a collection of sequences which are both uniformly bounded and equicontinuous in the m.s.

Chapter 3

Fundamental Properties of SISD

In this chapter, we consider the SISD constructed in chapter 2. The main interest is to establish the essential foundation of the theory of the system. Using Itô calculus, we develop results on the local and global existence, forward continuation, and uniqueness. As stated earlier, the system has impulses at variable times, the time delay is constant, and the random noise is approximated by a Wiener process. The material of this chapter has been published in [Alw10].

3.1 Local Results

We start this section with establishing a local existence result of the initial value problem (2.32). We will show how the solution evolves between two impulse hypersurfaces and then, under certain condition on the surfaces, if this solution starts initially at a hypersurface, it will depart this surface in mean. While, in most of the local existence results of stochastic systems, the vector field functions (or functionals in delay systems) are assumed to be bounded by a linear growth estimation, in this work, we deviate from this convention, and assume that the functionals are bounded by some time-varying random function having bounded integral. This general case makes the current result more efficient than the available approaches. The technique adopted to prove the following local result is a combination of the approaches used in developing the existence results for impulsive delay systems in [Ball99a] and stochastic systems in [Lad80]. One more thing to note is that, in [Ball99b, Liu2000], the functional f is assumed to be quasi-bounded. In fact, the time-varying estimate considered in this work already includes this condition.

Theorem 3.1. Let $J \subset \mathbb{R}_+$ and $\mathcal{D} \subset \mathbb{R}^n$ be an open set containing $\phi(0)$. Assume that $f \in \mathcal{L}_{ad}(\Omega, L[t_0, t_0 + \alpha])$ and $g \in \mathcal{L}_{ad}(\Omega, L^2[t_0, t_0 + \alpha])$, where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$, and are continuous in ψ. Moreover, there exists a (random) function $m(t)$ such that, for $(t, \psi) \in [t_0, t_0 + \beta] \times F$, for some positive $\beta \leq \alpha$ and compact set $F \subset \mathcal{D}$,

$$\|f(t, \psi)\|^2 \vee \|g(t, \psi)\|^2 \leq m(t), \qquad \text{(a.s.)} \tag{3.1}$$

where

$$\int_{t_0}^{t} m(s)\, ds < \infty, \qquad \text{(a.s.)}.$$

Then, for almost all $\omega \in \Omega$ and each $(t, \phi) \in J \times \mathcal{L}^2_{\mathcal{F}_0}([-r, 0], \mathbb{R}^n)$, there exists a (local) $\mathcal{F}_t$-adapted solution $x(t) = x(t, t_0, \phi)$ of (2.32) on $[t_0 - r, t_0 + \beta]$. Furthermore, assume that $\tau_k \in \mathcal{C}^2(\mathcal{D}, \mathbb{R}_+)$, for $k = 1, 2, \cdots$, and, whenever $t^* = \tau_k(x^*)$ for some $(t^*, x^*) \in J \times \mathcal{D}$ and some k, there exists a $\delta > 0$, where $[t^*, t^* + \delta] \subset J$, such that

$$\mathbb{E}[\mathcal{L}\tau_k(x(t))] \neq 1, \tag{3.2}$$

for all $t \in (t^*, t^* + \delta]$ and for all functions x that are $\mathcal{F}_t$-adapted $\mathcal{PC}([t^* - r, t^* + \delta], \mathcal{D})$, continuous on $(t^*, t^* + \delta]$ and satisfy $x(t^*) = x^*$ and $\mathbb{E}[\|x(s) - x^*\|^2] < \lambda$ for $s \in$

$[t^*, t^* + \delta]$ and $\lambda > 0$. Then, the solution x leaves the hypersurface $\tau_k(x)$ in mean, i.e., x exists on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$, for which x will not intersect any impulse hypersurface at any time $t \in (t_0, t_0 + \beta]$.

Proof. Let $(t, \phi) \in J \times \mathcal{L}^2_{\mathcal{F}_0}([-r, 0] \times \Omega, \mathbb{R}^n)$ and choose $\alpha > 0$ such that $[t_0, t_0 + \alpha] \subset J$. Since for almost all $\omega \in \Omega$, $\phi(0) \in \mathcal{D}$ and $\mathcal{D}$ is an open set, one can choose $\lambda > 0$ such that

$$F := F(z, \lambda) = \{z \in \mathbb{R}^n; \|z - \phi(0)\| \leq \lambda\} \subset \mathcal{D}. \tag{3.3}$$

Clearly, F is a compact set. Set

$$M(t) = \int_{t_0}^{t} m(s)\, ds, \qquad t \in [t_0, t_0 + \alpha].$$

Clearly, $M(t)$ is absolutely continuous (a.s.) with respect to t and nondecreasing. Also, $M(t_0) = 0$ and $M(t)$ is bounded (a.s.). Therefore, there is a positive number, say $\widetilde{M}$, such that

$$M(t) = \int_{t_0}^{t} m(s)\, ds \leq \widetilde{M}, \qquad t \in [t_0, t_0 + \alpha].$$

Let $\beta = \min\left\{\alpha, \frac{\lambda}{2\widetilde{M}} - 1\right\} > 0$. For $0 < \beta_1 < \beta$, define

$$Q = \Big\{x \in \mathcal{PC}([t_0 - r, t_0 + \beta_1], \mathcal{D}) \Big| \, x_{t_0} = \phi, \, x \text{ is continuous on } (t_0, t_0 + \beta_1] \\ \text{and } \mathcal{F}_t\text{-adapted, and } \|x(t) - \phi(0)\|^2 \leq \lambda \text{ (a.s.)}, \ \forall t \in (t_0, t_0 + \beta_1]\Big\}.$$

If $x \in Q$, (i.e., x is continuous on $[t_0, t_0 + \beta_1]$ and $\mathcal{F}_t$-adapted), then the composite functions $f(t, x_t)$ and $g(t, x_t)$ are adapted and almost surely integrable since $f(t, x_t) \in \mathcal{L}_{ad}(\Omega, L[t_0, t_0 + \beta_1])$ and $g(t, x_t) \in \mathcal{L}_{ad}(\Omega, L^2[t_0, t_0 + \beta_1])$.

For $n = 1, 2, 3, \cdots$, define the sequence of random processes

$$x^{(n)}(t) = \begin{cases} \phi(t-t_0), & t \in [t_0 - r, t_0], \\ \phi(0), & t \in (t_0, t_0+\beta/n], \\ \phi(0) + \int_{t_0}^{t-\beta/n} f(s, x_s^{(n)})\, ds & \\ \qquad + \int_{t_0}^{t-\beta/n} g(s, x_s^{(n)})\, dW(s), & t \in (t_0+\beta/n, t_0+\beta]. \end{cases} \tag{3.4}$$

By the above argument and $\phi \in \mathcal{L}^2_{\mathcal{F}_0}([-r,0],\mathbb{R}^n)$, the sequence $\{x^{(n)}\}$ is well defined and, for each n, $x^{(n)}(t)$ is $\mathcal{F}_t$-adapted. Moreover, for $t \in (t_0+\beta/n, t_0+2\beta/n]$, we have

$$\|x^{(n)}(t) - \phi(0)\| \leq \Big\| \int_{t_0}^{t-\beta/n} f(s, x_s^{(n)}) ds \Big\| + \Big\| \int_{t_0}^{t-\beta/n} g(s, x_s^{(n)}) dW(s) \Big\|.$$

Therefore, in view of (3.1),

$$\begin{aligned} \mathbb{E}\Big[\|x^{(n)}(t) - \phi(0)\|^2\Big] &\leq 2\Big\{ \mathbb{E}\Big\| \int_{t_0}^{t-\beta/n} f(s, x_s^{(n)}) ds \Big\|^2 + \mathbb{E}\Big\| \int_{t_0}^{t-\beta/n} g(s, x_s^{(n)}) dW(s) \Big\|^2 \Big\} \\ &\leq 2\Big\{ \frac{\beta}{n} \int_{t_0}^{t_0+\beta/n} \mathbb{E}\|f(s, x_s^{(n)})\|^2 ds + \int_{t_0}^{t_0+\beta/n} \mathbb{E}\|g(s, x_s^{(n)})\|^2 ds \Big\} \\ &\leq 2(\frac{\beta}{n} + 1)\widetilde{M} \leq \lambda, \end{aligned}$$

where we used $(a+b)^2 \leq 2(a^2+b^2)$ and Cauchy-Schwartz inequality. If a subsequence of $\{x^{(n)}\}$ is taken, then $\{x^{(n)}\} \in Q$ (a.s.) and by mathematical induction we can show that this is true for $t \in (t_0+k\beta/n, t_0+(k+1)\beta/n]$, for $k = 1, 2, \cdots, n-1$. Thus, for $n \geq 2$, $x^{(n)}$ belongs to Q. We also have, from (3.4),

$$\|x^{(n)}(t)\| \leq \|\phi(0)\| + \Big\| \int_{t_0}^{t-\beta/n} f(s, x_s^{(n)}) ds \Big\| + \Big\| \int_{t_0}^{t-\beta/n} g(s, x_s^{(n)}) dW(s) \Big\|.$$

Therefore,

$$\mathbb{E}\Big[\|x^{(n)}(t)\|^2\Big] \leq 3\Big\{\mathbb{E}\|\phi(0)\|^2 + \mathbb{E}\Big\|\int_{t_0}^{t-\beta/n} f(s, x_s^{(n)})ds\Big\|^2 + \mathbb{E}\Big\|\int_{t_0}^{t-\beta/n} g(s, x_s^{(n)})dW(s)\Big\|^2\Big\}$$

$$\leq 3\Big\{c_1 + \frac{\beta}{n}\int_{t_0}^{t_0+\beta/n} \mathbb{E}\|f(s, x_s^{(n)})\|^2 ds + \int_{t_0}^{t_0+\beta/n} \mathbb{E}\|g(s, x_s^{(n)})\|^2 ds\Big\}$$

$$\leq 3\Big\{c_1 + (\frac{\beta}{n} + 1)\widetilde{M}\Big\}.$$

Namely, we have

$$\mathbb{E}\Big[\|x^{(n)}(t)\|^2\Big] \leq \lambda', \tag{3.5}$$

where $\lambda' = 3\Big\{c_1 + (\frac{\beta}{n}+1)\widetilde{M}\Big\}$. By Tchebychev's inequality, one can find, for $\varepsilon > 0$, $\gamma_1(\varepsilon)$ such that

$$\mathbb{P}\Big\{\|x^{(n)}(t)\| > \gamma_1(\varepsilon)\Big\} \leq \frac{\mathbb{E}\Big[\|x^{(n)}(t)\|^2\Big]}{\gamma_1(\varepsilon)^2} \leq \frac{\lambda'}{\gamma_1(\varepsilon)^2} = \frac{\varepsilon}{2}.$$

Now, for each n, let $y^{(n)}$ denote the restriction of $x^{(n)}$ to $[t_0, t_0+\beta]$. Then, $y^{(n)}$ is continuous on $[t_0, t_0+\beta]$ and, moreover, for $t \in [t_0, t_0+\beta]$, we have

$$\mathbb{P}\Big\{\|y^{(n)}(t)\| > \gamma_1(\varepsilon)\Big\} \leq \frac{\varepsilon}{2}, \tag{3.6}$$

meaning that the sequence $\{y^{(n)}(t)\}$ is uniformly bounded (a.s.). We also have

$$y^{(n)}(t_1) - y^{(n)}(t_2) = \int_{t_2}^{t_1} f(s, y_s^{(n)})ds + \int_{t_2}^{t_1} g(s, y_s^{(n)})dW(s),$$

so that

$$\mathbb{E}\Big[\Big\|y^{(n)}(t_1) - y^{(n)}(t_2)\Big\|^2\Big] \leq 2\Big\{\mathbb{E}\Big\|\int_{t_2}^{t_1} f(s, y_s^{(n)})ds\Big\|^2 + \mathbb{E}\Big\|\int_{t_2}^{t_1} g(s, y_s^{(n)})\, dW(s)\Big\|^2\Big\}$$

$$\leq 2M^2(|t_1 - t_2| + 1) \leq \varepsilon',$$

namely,

$$\mathbb{E}\Big[\Big\|y^{(n)}(t_1) - y^{(n)}(t_2)\Big\|^2\Big] \leq \varepsilon',$$

which implies that, for a positive ε, there exists $\gamma_2(\varepsilon)$ such that

$$\mathbb{P}\Big\{\|y^{(n)}(t_1) - y^{(n)}(t_2)\| > \gamma_2(\varepsilon)\Big\} \leq \frac{\varepsilon}{2}, \tag{3.7}$$

which shows that the sequence $\{y^{(n)}\}$ is equicontinuous (a.s.).

Combining (3.6) and (3.7) yields

$$\mathbb{P}\Big\{\|y^{(n)}(t)\| \leq \gamma_1(\varepsilon) \text{ or } \|y^{(n)}(t_1) - y^{(n)}(t_2)\| \leq \gamma_2(\varepsilon)\Big\} > 1 - \varepsilon.$$

Set

$$K_\varepsilon = \Big\{y^{(n)} \in C([t_0, t_0 + \beta], \mathcal{D})\Big| \, \|y^{(n)}(t)\| \leq \gamma_1(\varepsilon) \text{ and } \|y^{(n)}(t_1) - y^{(n)}(t_2)\| \leq \gamma_2(\varepsilon)\Big\}.$$

The following part of the proof is aimed to prove the convergence of the SIE sequence in (3.4)[1]. Since K_ε is uniformly bounded and equicontinuous, by Arzela-Ascoli's Theorem [Lad80], it is a compact subset of $\mathcal{C}([t_0, t_0 + \beta], \mathcal{D})$. In addition, by Lemma 3.1, it satisfies $\mathbb{P}\{y^{(n)} \in K_\varepsilon\} > 1 - \varepsilon$. Thus, by Prokhorov's Theorem, the collection of continuous processes $\{y^{(n)}(t)\}$ is totally D-bounded. Thus, $\{(y^{(n)}(t), W^{(n)}(t), y_0^{(n)})\}$ is totally bounded, where $W^{(n)}(t) \equiv W(t)$ and $y_0^{(n)} \equiv \phi(0) =: y_0$. Therefore, one can find a D-Cauchy subsequence $\{(y^{(n_r)}(t), W^{(n_r)}(t), y_0^{(n_r)})\}$ of $\{(y^{(n)}(t), W^{(n)}(t), y_0^{(n)})\}$. By Skorohod's Theorem [Lad80], we can construct a sequence of random functions $(u^{(n_r)}(t), w^{(n_r)}(t), u_0^{(n_r)})$ and a random function $(u(t), w(t), u_0)$ such that

$$D\Big((y^{(n_r)}(t),\, B^{(n_r)}(t),\, y_0^{(n_r)}),\, (u^{(n_r)}(t),\, w^{(n_r)}(t),\, u_0^{(n_r)})\Big) = 0, \tag{3.8}$$

for $n_1, n_2, n_3, \cdots$, and

$$\mathbb{P}\Big\{(u^{(n_r)}(t), w^{(n_r)}(t), u_0^{(n_r)}) \to (u(t), w(t), u_0)\Big\} = 1, \tag{3.9}$$

[1]This part of the proof is inspired by that of Theorem 4.2.1 in [Lad80] except the equations there are delay-free. We reproduced it here for the proof to be self-contained.

as $r \to \infty$.

Notation. Denote the *superscript* n_r by the *subscript* r; for example, the subsequence $\{u^{(n_r)}(t)\}$ becomes $\{u_r(t)\}$.

The subsequence $\{u_r(t)\}$ is a D-Cauchy sequence. By the definition of totally D-bounded set, one can construct or find (n-indexed) D-Cauchy subsequence $\{u_r^n(t)\}$ of $\{u_r(t)\}$ and construct a subsequence $\{u^n(t)\}$ of the (restricted) solution sequence $\{y^{(n)}\}$ as follows

$$u_r^n(t) = \begin{cases} u_{r_0}, & t \in (t_0, t_0 + \beta/n], \\ u_{r_0} + \int_{t_0}^{t-\beta/n} f(s, u_{r_s}^n)\, ds & \\ \qquad + \int_{t_0}^{t-\beta/n} g(s, u_{r_s}^n)\, dw_r(s), & t \in (t_0 + \beta/n, t_0 + \beta], \end{cases}$$

for every $r = 1, 2, \cdots$, and

$$u^n(t) = \begin{cases} u_0, & t \in (t_0, t_0 + \beta/n], \\ u_0 + \int_{t_0}^{t-\beta/n} f(s, u_s^n)\, ds & \\ \qquad + \int_{t_0}^{t-\beta/n} g(s, u_s^n)\, dw(s), & t \in (t_0 + \beta/n, t_0 + \beta]. \end{cases}$$

Set

$$I_r(t) = \int_{t_0}^{t} f(s, u_{r_s})\, ds + \int_{t_0}^{t} g(s, u_{r_s})\, dw_r(s), \tag{3.10a}$$

$$I_r^n(t) = \int_{t_0}^{t-\beta/n} f(s, u_{r_s}^n)\, ds + \int_{t_0}^{t-\beta/n} g(s, u_{r_s}^n)\, dw_r(s), \tag{3.10b}$$

$$I(t) = \int_{t_0}^{t} f(s, u_s)\, ds + \int_{t_0}^{t} g(s, u_s)\, dw(s), \tag{3.10c}$$

$$I^n(t) = \int_{t_0}^{t-\beta/n} f(s, u_s^n)\, ds + \int_{t_0}^{t-\beta/n} g(s, u_s^n)\, dw(s), \tag{3.10d}$$

$$I_r^r(t) = \int_{t_0}^{t-\beta/r} f(s, u_{r_s}^r)\, ds + \int_{t_0}^{t-\beta/r} g(s, u_{r_s}^r)\, dw_r(s). \tag{3.10e}$$

From (3.10a) and (3.10b), we have

$$I_r^n(t) - I_r(t) = \overbrace{\int_{t_0}^{t-\beta/n} f(s, u_{r_s}^n)\,ds}^{l_0} - \int_{t_0}^{t} f(s, u_{r_s})\,ds$$
$$+ \underbrace{\int_{t_0}^{t-\beta/n} g(s, u_{r_s}^n)\,dw_r(s)}_{l_1} - \int_{t_0}^{t} g(s, u_{r_s})\,dw_r(s) \qquad (3.11)$$

The integrals l_0 and l_1 can be written as

$$\int_{t_0}^{t} f^n(s, u_{r_s}^n)\,ds, \qquad \int_{t_0}^{t} g^n(s, u_{r_s}^n)\,dw_r(s),$$

where $f^n(s, u_{r_s}^n)$ and $g^n(s, u_{r_s}^n)$ are sequences of step functions. As for f^n and g^n, we expect that they are at least piecewise continuous functions. Also, since the functionals f and g are continuous in the second argument and $u_r^n(t)$ is a D-Cauchy sequence which converges to $u_r(t)$, we have

$$\int_{t_0}^{t} \|f^n(s, u_{r_s}^n) - f(s, u_{r_s})\|^2\,ds \to 0$$

and

$$\int_{t_0}^{t} \|g^n(s, u_{r_s}^n) - g(s, u_{r_s})\|^2\,ds \to 0$$

in probability[2].

Therefore, the sequence of the deterministic integrals converges to

$$\int_{t_0}^{t} f(s, u_{r_s})\,ds,$$

[2]In fact, if a subsequence is taken, the convergence holds w.p.1.

and by the definition of Itô integral, we have

$$\int_{t_0}^{t} g(s, u_{r_s})\, dw_r(s) = \int_{t_0}^{t} g^n(s, u^n_{r_s})\, dw_r(s)$$

in probability. Hence $I^n_r(t)$ converges to $I_r(t)$ uniformly in probability as $n \to \infty$; namely, we have, for any $r = 1, 2, \cdots$ and given $\varepsilon > 0$,

$$\mathbb{P}\{\|I^n_r(t) - I_r(t)\| > \varepsilon\} < \varepsilon, \tag{3.12}$$

as $n \to \infty$. Similarly, from (3.10c) and (3.10d), we obtain

$$\mathbb{P}\{\|I^n(t) - I(t)\| > \varepsilon\} < \varepsilon. \tag{3.13}$$

From (3.10b) and (3.10d), we get

$$\mathbb{P}\{I^n_r(t) \to I^n(t)\} = 1, \tag{3.14}$$

as $r \to \infty$, because we have a sequence of stochastic integrals $\{I^n_r(t)\}_{r=1}^{\infty}$ which, by (3.9), converges to the stochastic integral $I^n(t)$ as $r \to \infty$. Also, (3.14) implies that, for any $\varepsilon > 0$, there exists a positive number r such that $r \geq r_0 = r_0(\varepsilon)$,

$$\|f(s, u^n_{r_s}) - f(s, u^n_s)\| < \sqrt{\frac{\varepsilon^3}{4\beta^2}},$$

and

$$\|g(s, u^n_{r_s}) - g(s, u^n_s)\| < \sqrt{\frac{\varepsilon^3}{4\beta}}.$$

Hence

$$\begin{aligned}\mathbb{E}[\|I^n_r(t) - I_r(t)\|^2] &\leq 2\mathbb{E}\Big[\beta \int_{t_0}^{t_0+\beta} \|f(s, u^n_{r_s}) - f(s, u^n_s)\|^2\, ds\Big] \\ &\quad + 2\mathbb{E}\Big[\int_{t_0}^{t_0+\beta} \|g(s, u^n_{r_s}) - g(s, u^n_s)\|^2\, ds\Big] \\ &\leq 4\mathbb{E}\Big[\int_{t_0}^{t_0+\beta} \frac{\varepsilon^3}{4\beta}\, ds\Big] = \varepsilon^3,\end{aligned}$$

and by Tchebychev's inequality, we get

$$\mathbb{P}\{\|I_r^n(t) - I^n(t)\| > \varepsilon\} < \varepsilon, \qquad r \geq r_0(\varepsilon). \tag{3.15}$$

We want now to show that

$$u(t) = \phi(0) + \overbrace{\int_{t_0}^{t} f(s, u_s)\, ds + \int_{t_0}^{t} g(s, u_s)\, dW(s)}^{=:I(t)}$$

holds. Note that

$$\begin{aligned}
&\mathbb{P}\Big\{\|u(t) - \phi(0) - I(t)\| > 6\varepsilon\Big\} \\
&\quad = \mathbb{P}\Big\{\|u(t) - \overbrace{u_r^r(t) + u_{r_0} + I_r^r(t)}^{=0} - \phi(0) - I(t) + I^n(t) - I^n(t) + I_r^n(t) - I_r^n(t) \\
&\qquad\quad + I_r(t) - I_r(t)\| > 6\varepsilon\Big\} \\
&\quad = \mathbb{P}\Big\{\|(u(t) - u_r^r(t)) + (u_{r_0} - \phi(0)) - (I(t) - I^n(t)) + (I_r^n(t) - I^n(t)) \\
&\qquad\quad + (I_r(t) - I_r^n(t)) + (I_r^r(t) - I_r(t))\| > 6\varepsilon\Big\} \\
&\quad \leq \mathbb{P}\Big\{\|y(t) - y_r^r(t)\| > \varepsilon\Big\} + \mathbb{P}\Big\{\|u_{r_0} - \phi(0)\| > \varepsilon\Big\} + \mathbb{P}\Big\{\|I(t) - I^n(t)\| > \varepsilon\Big\} \\
&\qquad\quad + \mathbb{P}\Big\{\|I_r^n(t) - I^n(t)\| > \varepsilon\Big\} + \mathbb{P}\Big\{\|I_r(t) - I_r^n(t)\| > \varepsilon\Big\} \\
&\qquad\quad + \mathbb{P}\Big\{\|I_r^r(t) - I_r(t)\| > \varepsilon\Big\} < 6\varepsilon,
\end{aligned}$$

namely

$$\mathbb{P}\Big\{\|u(t) - \phi(0) - I(t)\| > 6\varepsilon\Big\} < 6\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that

$$u(t) = \phi(0) + \int_{t_0}^{t} f(s, u_s)\, ds + \int_{t_0}^{t} g(s, u_s)\, dW(s),$$

with probability one. Hence, $y \equiv u$. Finally, define

$$x(t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ y(t), & t \in (t_0, t_0 + \beta]. \end{cases} \tag{3.16}$$

Thus, x is the required solution of (2.32). To complete the proof, we show that, under the assumption in (3.2), the solution x cannot continue along the hypersurface $t = \tau_k(x)$ when it starts on it. If it were not true, then there would exist some $\delta > 0$ such that $t = \tau_k(x(t))$ for all $t \in [t_0, t_0 + \delta]$. Thus, it follows that, by differentiating with respect to t, applying Itô formula, and taking the mathematical expectation,

$$1 = \mathbb{E}[\mathcal{L}\tau_k(x(t))], \quad t \in (t_0, t_0 + \delta),$$

which contradicts assumption (3.2). This completes the proof.

The hypotheses of Theorem 3.1 are generally made to assure that the initial value problem in (2.32) has a local solution (a.s.) evolving between any two hypersurfaces. Particularly, the boundedness condition (3.1) prevents a solution from exhibiting a finite escape time over any finite interval. While the restriction imposed on the hypersurfaces given in (3.2) guarantees that any solution begins on an arbitrary surface will not evolve along it even after a small period of time.

Under the same hypotheses on the functionals f and g (i.e., almost surely all guaranteed solutions do not have any finite escape time), a similar existence result can be extracted from Theorem 3.1, where the impulse moments are all constants as shown in (2.34). This special result has further use in later chapters when stability is discussed.

Corollary 3.1. Assume that the functionals f and g satisfy the conditions of Theorem 3.1. Then, for any $t_0 \in J$ and $\phi \in \mathcal{L}^2_{\mathcal{F}_0}([-r, 0] \times \Omega; \mathbb{R}^n)$, the SISD with fixed time impulses in (2.34) has a local solution defined on $[t_0 - r, t_0 + \beta]$ for some positive β.

Proof. Since the hypersurfaces are all constants, condition (3.2) holds. Thus, the desired conclusion follows when Theorem 3.1 is applied.

3.2 Forward Continuation

Having seen how the solution x grows between two hypersurfaces, regardless of where it initially begins, we are in a position to address the problem of forward continuation of solution of (2.32) which, at the same time, does not exhibit the beating phenomenon on an impulse hypersurface. Before establishing the main theorems of this section, we start with presenting the concept of *forward continuation* of a solution. Here, we restrict ourselves to the forward, but not backward, continuation because of the resulting difficulties in considering both the time delay and impulsive effects. It is also practically meaningful to consider increasing time.

Definition 3.1. Let x and y be solutions of the impulsive stochastic system (2.32) on the intervals J_1 and J_2, respectively, where $J_1 \subset J_2$ and both intervals have the same closed left endpoints. If $x(t)$ and $y(t)$ are indistinguishable for all $t \in J_1$ (i.e., $x(t) = y(t)$ (a.s.) $\forall\, t \in J_1$), then y is said to be a *proper forward continuation of* x, or simply *continuation of* x. In this case, a solution x defined on J_1 is said to be *continuable*; otherwise, it is said to be noncontinuable and J_1 is called the *maximal interval of existence of* x.

Before presenting the forward continuation problem (Theorem 3.2), we state Zorn's Lemma [Phi84], which will be used in the proof of the theorem.

Zorn's Lemma. Let $(X, \prec)$ be a partially ordered set such that, if for every totally ordered subset S of X there exists an element z in X for which $z \succ y$ for all $y \in S$ (i.e., z is an upper bound of S), then the partially ordered set X has a maximal element.

Theorem 3.2. Suppose that the functionals f and g satisfy the conditions in Theorem 3.1, $\tau_k \in \mathcal{C}^2(\mathcal{D}, \mathbb{R}_+)$, for some $k = 1, 2, \cdots$, and the limit $\lim_{k\to\infty} \tau_k(x) = \infty$ is uniform in x. Assume that

$$\mathbb{E}[\mathcal{L}\tau_k(\psi(0))] < 1, \tag{3.17}$$

for all $(t, \psi) \in J \times \mathcal{PC}([-r, 0], \mathcal{D})$ and $k = 1, 2, \cdots$. Furthermore, assume that

$$\begin{gathered} \psi(0) + \mathcal{I}(\tau_k(\psi(0)), \psi) \in \mathcal{D}, \\ \tau_k(\psi(0) + \mathcal{I}(\tau_k(\psi(0)), \psi)) \leq \tau_k(\psi(0)) \end{gathered} \tag{3.18}$$

hold almost surely for all $\psi \in \mathcal{PC}([-r, 0], \mathcal{D})$ for which $\psi(0^-) = \psi(0)$ (a.s.) and for all $k = 1, 2, \cdots$. Then, for every continuable solution x of (2.32), there exists a continuation y of x that is noncontinuable. Moreover, any solution x of (2.32) can intersect each impulse hypersurface at most once.

Proof. Let x be any solution of (2.32) that is defined on $[t_0 - r, t_0 + \beta_1)$ or $[t_0 - r, t_0 + \beta_1]$, where $0 < \beta_1 < \infty$. Denote by X the set of all solutions x with their continuations. For any $y, z \in X$, we define the partial ordering $\prec$ by $y \prec z$ if, for almost all $\omega \in \Omega$, either $y = z$ or z is a continuation of y. Let S be a totally ordered subset of X. Now for $y \in S$, we associate $\beta(y)$ such that $\beta_1 \leq \beta(y) \leq \infty$ and by which the solution y is defined on $[t_0 - r, t_0 + \beta(y))$ or $[t_0 - r, t_0 + \beta(y)]$.

Define

$$\beta_2 = \sup\{\beta(y) \,|\, y \in S\}.$$

Clearly, $\beta_1 \leq \beta_2 \leq \infty$ and y is defined on a subset of $[t_0 - r, t_0 + \beta_2]$ if $\beta_2 < \infty$ or $[t_0 - r, t_0 + \beta_2)$ if $\beta_2 = \infty$. At this stage, one considers two cases. The trivial case is when $\beta_2 < \infty$ and there is some solution y that is defined on $[t_0 - r, t_0 + \beta_2]$. Consequently, this solution y of (2.32) is an upper bound on S and at the same time it is the required solution continuation. In the other case, we will show that there is a solution z defined on $[t_0 - r, t_0 + \beta_2)$ such that, for all $y \in S$, $y \prec z$, i.e., z will be an upper bound on S. Hence, by Zorn's lemma, the set X has a maximal element. For this purpose, for $t \in [t_0 - r, t_0 + \beta_2)$, we define the following function

$$z(t) = y(t), \qquad \text{(a.s.)}, \tag{3.19}$$

where y is any solution in S for which $t < t_0 + \beta(y)$. The new function z is well-defined, it is right-continuous (i.e., $z(t^+) = z(t)$ (a.s.)) for all $t \in [t_0 - r, t_0 + \beta_2)$, the left limit $z(t^-)$ exists for all $t \in (t_0 - r, t_0 + \beta_2)$ and $z(t^-) = z(t)$ (a.s.) for all but at most a finite number of points in $(t_0 - r, t_0)$ [Ball99a]. Moreover, if z has a finite number of simple jump discontinuities in any finite interval of $(t_0, t_0 + \beta_2)$, then z is a solution of (2.32) (i.e., $z \in \mathcal{PC}([t_0 - r, t_0 + \beta_2))$ and $\mathcal{F}_t$-adapted). To show this is the only possible case, for $\beta_2 < \infty$, define

$$\mathbb{T} = \{t \in (t_0, t_0 + \beta_2) \,|\, t = \tau_k(z(t^-)) \text{ for some } k\}.$$

Then, except at these points, $z(t^-) = z(t)$ (a.s.). We first consider the case where $\mathbb{T}$ is finite. Under the assumptions on f and g, the functions $f(t, z_t)$ and $g(t, z_t)$ can only have a finite number of simple jump discontinuities on the interval $(t_0, t_0 + \beta_2)$ and, except at these points or at the points of $\mathbb{T}$, the solution z is continuous and has the solution form given in (2.33). This is because the functions $f(t, z_t)$ and $f(t, y_t)$ have the same properties. Thus, if $y \in \mathcal{PC}([t_0 - r, t_0 + \beta_2))$ and $\mathcal{F}_t$-adapted, so is z. A more challenging case is when $\beta_2 < \infty$ and $\mathbb{T}$ has an infinite number of discontinuities in $(t_0, t_0 + \beta_2)$. In this case, $\mathbb{T}$ has an increasing sequence of impulse times $\mathbb{T} = \{t_k\}_{k=1}^{\infty}$, where $t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_0 + \beta_2$ and $\lim_{k\to\infty} t_k = t_0 + \beta_2$. For $k = 1, 2, \cdots$, denote by j_k the index of the unique impulse hypersurface τ_{j_k} that the solution z reaches at t_k, i.e., $t_k = \tau_{j_k}(z(t_k^-))$. For some finite integer $N > 0$, if $j_k < N$, then z can reach only a finite number of impulse hypersurfaces. Since, as assumed, there is an increasing number of impulse times, the solution z must reach at least one impulse hypersurface more than once. In other words, $j_k = j_{k+m}$ and hence $t_k = \tau_{j_k}(z(t_k^-))$ and $t_{k+m} = \tau_{j_k}(z(t_{k+m}^-))$ for some positive integers k and m (i.e., the hypersurface τ_{j_k} is being hit at times t_k and t_{k+m}). This also implies that, if $y \in S$, then $t_k = \tau_{j_k}(y(t_k^-))$ and $t_{k+m} = \tau_{j_k}(y(t_{k+m}^-))$, where $t_{k+m} < t_0 + \beta(y)$. We will show that, according to our assumptions, this cannot happen for the solution y to reach the same hypersurface more than once.

For this purpose, for $i = 0, 1, 2, \cdots, m$, we define

$$h_{k+i}(t) = t - \tau_{j_{k+i}}(y(t)), \qquad \text{(a.s.)}, \tag{3.20}$$

for $t \in [t_0 - r, t_{k+m}]$. Note that $h_{k+i}(t_{k+i}^-) = 0$ for all i. Suppose for the sake of contradiction that, for some $0 \le i \le m-1$, we have $j_{k+i} > j_{k+i+1}$ and hence $\tau_{j_{k+i}}(\nu) > \tau_{j_{k+i+1}}(\nu)$ for all $\nu \in \mathcal{D}$. This implies [Ball99a]

$$h_{k+i+1}(t_{k+i}) \ge 0, \qquad \text{(a.s.)}. \tag{3.21}$$

On the other hand, differentiating $h_{k+i+1}(t)$ with respect to t, for all $t \in (t_{k+i}, t_{k+i+1})$, applying Itô formula, and taking the mathematical expectation give

$$D^+\mathbb{E}[h_{k+i+1}(t)] = 1 - \mathbb{E}[\mathcal{L}\tau_{j_{k+i+1}}(y(t))], \tag{3.22}$$

for all $t \in (t_{k+i}, t_{k+i+1})$. By (3.17), $h_{k_i+1}(t)$ is increasing over the interval (t_{k+1}, t_{k+i+1}), and the fact that $h_{k+i+1}(t_{k+i+1}^-) = 0$, we conclude that $h_{k+i+1}(t_{k+i}) < 0$ in mean, which contradicts with what we got in (3.21). Thus, $j_{k+i} < j_{k+i+1}$ and hence

$j_k < j_{k+1} < \cdots < j_{k+m}$, which also contradicts with our supposition $j_k = j_{k+m}$. Therefore, the solution y and hence z must intersect a given impulse hypersurface at most once in mean. This completes the proof.

In Theorem 3.2, our interest is in a solution, as introduced in Definition 2.17, intersecting consecutive hypersurfaces either a finite number of times in limited time period, or countably infinitely many times in an unlimited time period. This condition is represented by requiring $\lim_{k\to\infty} \tau_k(x) = \infty$ uniformly in x. Moreover, to prevent the solution from experiencing rhythmical beating phenomenon, the condition (3.17) must hold.

In the following corollary, we consider SISD with fixed impulsive times (i.e., $t = \tau_k$ for all k) and establish the same forward continuation result, i.e., there is always a maximal interval on which a solution of (2.34) can be defined .

Corollary 3.2. Suppose that the functionals f and g satisfy the conditions of Theorem 3.1. Also, assume that $\psi(0) + \mathcal{I}(\tau_k, \psi) \in \mathcal{D}$ (a.s.) for all $\psi \in \mathcal{D}$ for which $\psi(0^-) = \psi(0)$ and for all $k = 1, 2, \cdots$. Then, for every continuable solution x of (2.34), there exists a continuation y of x that is noncontinuable.

Proof. Since $t = \tau_k$ for all k, the conditions (3.17) and (3.18) hold for all k. Thus, by Theorem 3.2, it is guaranteed that a solution of (2.34) can be always defined on a maximal interval of existence.

Before developing the global existence result, we address the case where the solution is noncontinuable in the sense that the solution cannot be entirely contained in any compact set.

Theorem 3.3. Let x be a solution of (2.32) that is defined for all $t \in [t_0 - r, t_0 + \beta)$, where $0 < \beta < \infty$ and $[t_0, t_0 + \beta] \subset J$. If x is noncontinuable, then there is a sequence $\{s_k\}_{k=1}^{\infty}$, with $t_0 < s_1 < s_2 < \cdots < s_k < \cdots < t_0+\beta$ and $\lim_{k\to\infty} s_k = t_0+\beta$ (a.s.) such that $x(s_k) \notin F$, for any compact set $F \subset \mathcal{D}$.

Proof. Assume, for contradiction, that there is a compact set $F_1 \subset \mathcal{D}$ and $\beta_1 > 0$ for which $x(t) \in F_1$ for all $t \in [t_0+\beta_1, t_0+\beta)$. Let F_2 be the closure of the range of the solution x when t is restricted to $[t_0-r, t_0+\beta_1]$. Then, the set $F = F_1 \cup F_2 \subset \mathcal{D}$ is also compact, and $x(t) \in F$ for all $t \in [t_0-r, t_0+\beta)$. Now for any $t, \bar{t} \in [t_0+\beta_1, t_0+\beta)$, we have from (2.33)

$$\|x(t) - x(\bar{t})\| \leq \Big\| \int_{\bar{t}}^{t} f(s, x_s)ds \Big\| + \Big\| \int_{\bar{t}}^{t} g(s, x_s)dW(s) \Big\|. \tag{3.23}$$

Hence,

$$\begin{aligned}
\mathbb{E}\Big[\|x(t) - x(\bar{t})\|^2\Big] &\leq 2\Big\{\mathbb{E}\Big\| \int_{\bar{t}}^{t} f(s, x_s)ds \Big\|^2 + \mathbb{E}\Big\| \int_{\bar{t}}^{t} g(s, x_s)dW(s) \Big\|^2\Big\} \\
&\leq 2\Big\{[t-\bar{t}\,] \int_{\bar{t}}^{t} \mathbb{E}\|f(s, x_s)\|^2 ds + \int_{\bar{t}}^{t} \mathbb{E}\|g(s, x_s)\|^2 ds\Big\} \\
&\leq 2\widetilde{M}^2(|t-\bar{t}| + 1) < \varepsilon, \qquad (3.24)
\end{aligned}$$

for some arbitrary $\varepsilon > 0$ and $\widetilde{M} > 0$, which is guaranteed by Theorem 3.1. By Tchebychev's inequality, we obtain

$$\mathbb{P}\Big\{\|x(t) - x(\bar{t})\| > \eta\Big\} \leq \frac{\varepsilon}{\eta^2},$$

for some $\eta > 0$. Then, by Cauchy criterion the limit $\lim_{t\to(t_0+\beta)} x(t)$ exists with probability one and its limit point, say ζ, is in F. That is the solution x can be continued by defining $x(t_0+\beta) = \zeta$. But this contradicts with our supposition that x is noncontinuable. Thus, the conclusion of the Theorem follows.

3.3 Global Existence of a Solution

In this section, we address the global existence problem of the solution of (2.32). Although, in most of the available results concerning global existence of delay or stochastic systems, the vector fields are assumed to grow linearly to avoid any finite escape time that a solution may have, in the current results, the functionals f and g are assumed to be bounded by a nonlinear estimate (of the system state) described by a continuous increasing concave function κ. Consequently, this requires using a Bihari's inequality, a more general result than the well-known Gronwall-type inequalities.

Theorem 3.4. Suppose that $J = \mathbb{R}_+$, $\mathcal{D} = \mathbb{R}^n$, the functionals $f(t, \psi) \in \mathcal{L}_{ad}(\Omega, L[t_0, t_0 + \alpha])$ and $g(t, \psi) \in \mathcal{L}_{ad}(\Omega, L^2[t_0, t_0 + \alpha])$, where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$, and are continuous in ψ. Assume further that there are two measurable functions h_1, h_2 (or h_1, $h_2 \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}_+)$) and a continuous increasing concave function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t, \psi)\|^2 \vee \|g(t, \psi)\|^2 \leq h_1^2(t) + h_2^2(t)\kappa(\|\psi\|_r^2), \qquad \text{(a.s.)}, \qquad (3.25)$$

for all $(t, \psi) \in \mathbb{R}_+ \times \mathcal{L}^2_{\mathcal{F}_t}([-r, 0]; \mathbb{R}^n)$ (i.e., ψ is an $\mathcal{F}_t$-adapted piecewise continuous and $\mathbb{E}[\|\psi\|_r^2] < \infty$). Then, for each $(t, \phi) \in \mathbb{R}_+ \times \mathcal{L}^2_{\mathcal{F}_0}([-r, 0], \mathbb{R}^n)$, there exists a local $\mathcal{F}_t$-adapted solution $x = x(t,\, t_0,\, \phi(0))$ for (2.32) that can be continued to $[t_0 - r, \infty)$.

Proof. For all $(t, \phi) \in \mathbb{R}_+ \times \mathcal{L}^2_{\mathcal{F}_0}([-r, 0], \mathbb{R}^n)$, let $x = x(t, \phi(0))$ be a local solution of (2.32) that is guaranteed by Theorem 3.2. Assume, for contradiction, that for finite β the solution x is noncontinuable in the sense of Theorem 3.3. We will show that, according to our assumptions, this is not possible.

Let $a = \mathbb{E}[\|\phi(0)\|^2] + \mathbb{E}\Big[\Big(\sum_{\{k : t_k \in (t_0, t]\}} \Big\|I(t_k, x_{t_k^-})\Big\|\Big)^2\Big]$, $b = (\beta + 1)\beta\hbar^2$, where

$\hbar = \sup\{h_1(t) \,|\, \forall t \in [t_0, t_0+\beta]\}$, and $c = \mathbb{E}[\|\phi\|_r^2]$.

Then, for all $t \in (t_0, t_0+\beta)$,

$$\begin{aligned}\mathbb{E}[\|x(t)\|^2] &\le 4\Big\{\mathbb{E}[\|\phi(0)\|^2] + \mathbb{E}\Big[\Big(\sum_{\{k:t_k\in(t_0,t]\}}\Big\|I(t_k, x_{t_k^-})\Big\|\Big)^2\Big] + \beta\int_{t_0}^t \mathbb{E}\|f(s,x_s)\|^2 ds \\ &\quad + \int_{t_0}^t \mathbb{E}\|g(s,x_s)\|^2 ds\Big\} \\ &\le 4\Big\{a + b + (\beta+1)\int_{t_0}^t h_2^2(s)\kappa(\mathbb{E}[\|x_s\|_r^2])ds\Big\},\end{aligned}$$

which implies that

$$\begin{aligned}\mathbb{E}[\|x_t\|_r^2] &\le c + 4(a+b) + 4(\beta+1)\int_{t_0}^t h_2^2(s)\kappa(\mathbb{E}[\|x_s\|_r^2])\,ds \\ &= B + 4(\beta+1)\int_{t_0}^t h_2^2(s)\kappa(\mathbb{E}[\|x_s\|_r^2])\,ds,\end{aligned}$$

where $B = c + 4(a+b)$. Using Bihari's Lemma [Bih56, Mao94] yields

$$\mathbb{E}[\|x_t\|_r^2] \le G^{-1}\Big(G(B) + 4(\beta+1)\int_{t_0}^t h_2^2(s)ds\Big),$$

where

$$G(u) = \int_{0^+}^u \frac{ds}{\kappa(s)}, \qquad u > 0,$$

and $G(B) + 4(\beta+1)\int_{t_0}^t h_2^2(s)ds \in \mathrm{Dom}(G^{-1})$. If $B \to 0$, then $G(B) \to -\infty$, and hence $G^{-1} \to 0$. Namely, if $B \to 0$, $\mathbb{E}[\|x_t\|_r^2] \le 0 < \infty$.

Hence $\mathbb{E}[\|x(t)\|^2] < \infty$. This contradicts with that x is noncontinuable. Thus, the solution must be bounded when $t \to (t_0+\beta)^-$ and the global existence result follows. This completes the proof.

We should remark that, in Theorem 3.4, due to the generality of condition (3.25), we have excluded the time varying bound m on f and g. In fact, one can easily show this inclusion, because κ is continuous and hence bounded by a constant on any compact set containing x.

If one considers SISD with fixed impulses, i.e., system (2.34), then a similar result can be obtained, as shown in the next corollary; the proof is a direct result from Corollary 3.2 and hence Theorem 3.1.

Corollary 3.4. Suppose that $J = \mathbb{R}_+$, $\mathcal{D} = \mathbb{R}^n$, the functionals $f(t, \psi) \in \mathcal{L}_{ad}(\Omega, L[t_0, t_0+\alpha])$ and $g(t, \psi) \in \mathcal{L}_{ad}(\Omega, L^2[t_0, t_0+\alpha])$, where $\alpha > 0$ and $[t_0, t_0+\alpha] \subset J$, and are continuous in ψ. Assume further that there are two measurable functions h_1, h_2 (or h_1, $h_2 \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}_+)$) and a continuous increasing concave function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|f(t, \psi)\|^2 \vee \|g(t, \psi)\|^2 \le h_1^2(t) + h_2^2(t)\kappa(\|\psi\|_r^2)$ (a.s.), for all $(t, \psi) \in \mathbb{R}_+ \times \mathcal{L}^2_{\mathcal{F}_t}([-r, 0], \mathbb{R}^n)$ (i.e., ψ is $\mathcal{F}_t$-adapted piecewise continuous and $\mathbb{E}[\|\psi\|_r^2] < \infty$). Then, for each $(t, \phi) \in \mathbb{R}_+ \times \mathcal{L}^2_{\mathcal{F}_0}([-r, 0], \mathbb{R}^n)$, there exists a local $\mathcal{F}_t$-adapted solution $x = x(t,\, t_0,\, \phi(0))$ for (2.34) (i.e., systems with fixed impulses) that can be continued to $[t_0 - r, \infty)$.

Finally, we end the main contribution of this chapter by stating a sufficient condition on the functionals f and g to assure that the system (2.32) has a unique solution. Among these conditions is the Lipschitz condition.

Lipschitz condition. A functional f is said to satisfy *Lipschitz condition* if there exists a positive constant L such that

$$\|f(t, \psi_1) - f(t, \psi_2)\| \le L\|\psi_1 - \psi_2\|_r, \qquad \forall t \in [0, T], \quad T > 0,$$

for all ψ_1, ψ_2 in some compact set $F \subset \mathcal{D}$ where $\mathcal{D} \subset \mathbb{R}^n$ is an open subset.

Theorem 3.5. Suppose that the assumptions of Theorem 3.4 hold. Also, assume that the functionals $f(t, \psi)$ and $g(t, \psi)$ are locally Lipschitz in ψ. Then, system

(2.32) has a unique solution defined on $[t_0 - r, t_0 + \beta)$, where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$.

Proof. For all $t \in [t_0 - r, t_0 + \beta)$, let $x = x(t, t_0, \phi(0))$ and $y = y(t, t_0, \phi(0))$ be two solutions of (2.32) such that $x \not\equiv y$ (a.s.), for contradiction. This in turn implies that there is some $t \in (t_0, t_0 + \beta)$ such that $x(t) \neq y(t)$ (a.s.). Define the stopping time $t_1 = \inf\{t \in (t_0, t_0 + \beta) \,|\, x(t) \neq y(t)\}$. If t_1 is not an impulsive time (i.e., $t_1 \neq \tau_k(x(t_1^-))$ or equivalently $t_1 \neq \tau_k(y(t_1^-))$ for all k), then $x(t_1) = x(t_1^-) = y(t_1^-) = y(t_1)$ (a.s.); otherwise, $x(t_1) = x(t_1^-) + \mathcal{I}(t_1, x_{t_1^-}) = y(t_1^-) + \mathcal{I}(t_1, y_{t_1^-}) = y(t_1)$. Therefore, in both cases we have $x(t_1) = y(t_1)$ (a.s.). Let $\varepsilon > 0$ be sufficiently small such that $t_1 + \varepsilon < t_0 + \beta$ and the solutions x and y do not reach any hypersurface over $(t_1, t_1 + \varepsilon]$. Let $\delta > 0$ be a sufficiently small number such that $\delta < \varepsilon$ and $\delta(\delta+1)L^2 \leq \frac{1}{4}$, where $L > 0$, such that $\|f(t, \psi_1) - f(t, \psi_2)\| \vee \|g(t, \psi_1) - g(t, \psi_2)\| \leq L\|\psi_1 - \psi_2\|_r$, for all $t \in [t_0, t_1 + \epsilon]$ and all ψ_1, ψ_2 in some compact set $F \subset \mathcal{D}$ where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$. Then for all $t \in [t_1, t_1 + \delta]$, we have from (2.33)

$$
\begin{aligned}
\mathbb{E}[\|x(t) - y(t)\|^2] = 2\Big\{ &\mathbb{E}\Big\| \int_{t_1}^{t} (f(s, x_s) - f(s, y_s))\, ds \Big\|^2 \\
&+ \mathbb{E}\Big\| \int_{t_1}^{t} (g(s, x_s) - g(s, y_s))\, dW(s) \Big\|^2 \Big\} \\
\leq 2\Big\{ &\delta \int_{t_1}^{t} \mathbb{E}\|f(s, x_s) - f(s, y_s))\|^2\, ds \\
&+ \int_{t_1}^{t} \mathbb{E}\|g(s, x_s) - g(s, y_s))\|^2\, ds \Big\} \\
\leq 2(\delta + 1) &\int_{t_1}^{t} \mathbb{E}[L^2 \|x_s - y_s\|_r^2]\, ds \\
\leq 2(\delta + 1)L^2 &\int_{t_1}^{t} \sup_{u \in [t_1, s]} \mathbb{E}\|x(u) - y(u)\|^2 ds \\
\leq 2(\delta + 1)L^2 &\int_{t_1}^{t_1 + \delta} \sup_{u \in [t_1, t_1 + \delta]} \mathbb{E}\|x(u) - y(u)\|^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2(\delta+1)L^2\delta \sup_{u\in[t_1,t_1+\delta]} \mathbb{E}\|x(u)-y(u)\|^2 \\
&\leq \frac{1}{2} \sup_{u\in[t_1,t_1+\delta]} \mathbb{E}\|x(u)-y(u)\|^2
\end{aligned}
$$

for all $t \in [t_1, t_1+\delta]$. The last inequality implies that $\sup_{[t_1,t_1+\delta]} \mathbb{E}[\|x(t)-y(t)\|^2] = 0$. Since x and y are continuous functions for all $t \in [t_1, t_1+\delta]$, then

$$
\mathbb{P}\Big\{ \sup_{[t_1,t_1+\delta]} \|x(t)-y(t)\| > 0 \Big\} = 0, \tag{3.26}
$$

which implies that $x(t) = y(t)$ (a.s.) for all $t \in [t_1, t_1+\delta]$ [Gard88]. But this contradicts with our supposition that $x \not\equiv y$. Thus, it must be true that (2.32) has a unique solution. This completes the proof.

Considering SISD with fixed impulses, one can get the same uniqueness result, as the next corollary tells us. This proof is straightforward, thus it is omitted here.

Corollary 3.5. Assume that the SISD (2.34) satisfies the conditions of Theorem 3.5. Then, there exists a unique solution for (2.34) that is defined on $[t_0-r, t_0+\beta)$, where $0 < \beta \leq \infty$ and $[t_0, t_0+\beta) \subset J$.

3.4 Conclusion and Comments

A general nonlinear stochastic impulsive system with time delay, experiencing impulsive effects at variable times, was introduced in this chapter. We established a local existence result of stochastic systems over a space of piecewise continuous and $\mathcal{F}_t$-adapted functions. We should mention that, in proving the equicontinuity property of the solution sequence, one may get the same result by following another, but lengthy, approach and then employing Kolmogorov's Theorem for continuity. As mentioned earlier, the proof of the convergence of sequence of SIEs is inspired by that of Theorem 4.2.1 in [Lad80]; one can prove the same convergence result if the functionals satisfy Lipschitz condition. We also showed that, by imposing a further condition on the impulsive hypersurface, solutions leave this surface in mean. Due to some technical difficulties in backward extending a given solution of an impulsive system with or without time delay, we focused on forward continuation. We also showed that, under some conditions on the impulse function and impulses, solutions do not exhibit rhythmical beating upon a hypersurface. Supposing that the drift and diffusion coefficients are bounded by some nonlinear estimate, a global result has been achieved. One can get the same result if the coefficients are assumed to grow linearly. Finally, a unique solution was guaranteed if Lipschitz condition is imposed on the coefficients.

Chapter 4

Stability Properties for SISD

This chapter is devoted to establishing some stability properties of SISD with fixed impulses (2.34). In analyzing these results, we adopt two approaches, namely, an (ε, δ)-based and comparison principle techniques. In both cases, the interest is to develop Lyapunov-type sufficient conditions to assure the qualitative properties in the m.s., employing Razumikhin technique. Conventionally, in Razumikhin methodology, we consider Lyapunov functions $V(t, \psi(0))$ for all $t \geq 0$, but not functionals $V(s, \psi(s))$ for all $s \in [t-r, t]$, and examine their time derivatives, along the system trajectories, which are required to be non-positive or strictly negative for all the time *whenever* $V(t, \psi(0))$ is sufficiently larger than $V(s, \psi(s))$ for all $s \in [t-r, t]$. The material of this chapter forms the basis of [Alw-a, Alw-b].

For convenient reading, we consider again the SISD with fixed impulses

$$dx(t) = f(t, x_t)dt + g(t, x_t)\, dW(t), \quad t \neq \tau_k, \tag{4.1a}$$

$$\Delta x(t) = \mathcal{I}(t, x_{t^-}), \qquad t = \tau_k, \tag{4.1b}$$

$$x_{t_0}(s) = \phi(s), \qquad s \in [-r, 0], \tag{4.1c}$$

where f and g satisfy the existence of a unique solution conditions in Corollary 3.5. We should keep in mind that the solution x of interest is $\mathcal{F}_t$-adapted and belongs to $\mathcal{PC}([t_0 - r, t_0 + \alpha]; \mathcal{D})$ for some $\alpha > 0$ and open subset $\mathcal{D} \subset \mathbb{R}^n$. For the system to possess a trivial solution $x(t) \equiv 0 \in \mathcal{D}$, we assume that $f(t, 0) = 0 \in \mathbb{R}^n$ and $g(t, 0) = 0 \in \mathbb{R}^{n \times m}$ for all $t \in \mathbb{R}_+$ and $\mathcal{I}(\tau_k, 0) = 0 \in \mathbb{R}^n$ for all $\tau_k \in \mathbb{T}$.

Before analyzing the stability results of the trivial solution of (4.1), we introduce some assumptions and definitions that will be used in this chapter.

Definition 4.1. $S(\varrho) = \{z \in \mathbb{R}^n \big| \, \|z\| \leq \varrho \text{ (a.s.)}, \; \varrho > 0\}$.

To guarantee that the solution stays bounded (in the m.s.) after impulsive actions being applied, we assume the following:

Assumption A1. There exist $0 \leq \varrho_1 \leq \varrho$ such that, for all $\tau_k \in \mathbb{T}$ and x defined on $\mathcal{PC}([-r, 0]; \mathcal{D})$, if

$$\mathbb{E}[\|x(\tau_k^-)\|^2] < \varrho_1, \quad \text{then} \quad \mathbb{E}[\|x(\tau_k)\|^2] < \varrho.$$

We also assume that the impulsive moments satisfy the following assumption.

Assumption A2. For any $k \in \mathbb{N}$, we have

$$\tau_{\sup} = \sup\{\tau_k - \tau_{k-1}\} < \infty \qquad \text{and} \qquad \tau_{\inf} = \inf\{\tau_k - \tau_{k-1}\} > 0.$$

In the following definition, we present some function classes that will be used throughout this thesis.

Definition 4.2. A function α is said to belong to class-$\mathcal{K}_1$ if it is a class-$\mathcal{K}$ and convex; it is said to belong to class-$\mathcal{K}_2$ if it is a class-$\mathcal{K}$ and concave; it is said to belong to class-$\mathcal{K}_3$ if it belongs to $\mathcal{C}(\mathbb{R}_+; \mathbb{R}_+)$ such that $\alpha(0) = 0$, $\alpha(s) > 0$ for all $s > 0$ and it is nondecreasing.

In the following we state the concepts of m.s. stability of (4.1).

Definition 4.3. Let $\phi \in \mathcal{L}^2_{\mathcal{F}_0}([-r,0],\mathcal{D})$ and $x(t) = x(t, t_0, \phi)$, with $x \in \mathcal{PC}([t_0 - r, t_0 + \alpha]; \mathcal{D})$ for some $\alpha > 0$, be a solution of (4.1). Then, the trivial solution $x \equiv 0$ is said to be

(i) *stable in the m.s.*, if for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$\mathbb{E}[\|\phi\|_r^2] \leq \delta \qquad \text{implies} \qquad \mathbb{E}[\|x(t)\|^2] < \varepsilon, \qquad \forall t \geq t_0;$$

(ii) *uniformly stable in the m.s.*, if δ in (i) is independent of t_0;

(iii) *asymptotically stable in the m.s.*, if it is stable and for any $t_0 \in \mathbb{R}_+$, there exists $\eta = \eta(t_0) > 0$ such that

$$\mathbb{E}[\|\phi\|_r^2] \leq \eta \qquad \text{implies} \qquad \lim_{t\to\infty} x(t) = 0;$$

(iv) *uniformly asymptotically stable in the m.s.*, if it is uniformly stable in the m.s. and there exists some $\eta > 0$ such that, for every $\gamma > 0$, there exists $T = T(\eta, \gamma) > 0$ for which

$$\mathbb{E}[\|\phi\|_r^2] \leq \eta \qquad \text{implies} \qquad \mathbb{E}[\|x(t)\|^2] < \gamma, \quad \forall t \geq t_0 + T;$$

(v) *unstable in m.s.* if (i) fails to hold.

4.1 Analysis by a Scalar Lyapunov Function: (ε, δ)-based Approach

This section deals with establishing m.s. stability properties of (4.1) using (ε, δ)-based Lyapunov theorems together with Razumikhin technique. Generally, we will assume that the continuous system has m.s. stable (or asymptotically stable) trivial solution and the impulses tend to be thought of as small perturbation. The interest is to develop sufficient conditions such that the impulsive system retains the qualitative properties.

Theorem 4.1. Assume that the conditions of Corollary 3.5 and Assumptions A1 and A2 are satisfied, and there exist functions $a \in \mathcal{K}_2$, $b \in \mathcal{K}_1$, and a constant $d_k \geq 0$ with $d = \sum_{k=1}^{\infty} d_k < \infty$. Suppose that $V \in \mathcal{C}^{1,2}\big([-r,\infty) \times S(\varrho)\big); \mathbb{R}_+\big)$ satisfies

(i) for all $(t, \psi(0)) \in [-r, \infty) \times S(\varrho)$,

$$b(\|\psi(0)\|^2) \leq V(t, \psi(0)) \leq a(\|\psi(0)\|^2), \qquad \text{(a.s.)};$$

(ii) for all $t \neq \tau_k$ in $\mathbb{R}_+$ and $\psi \in \mathcal{PC}\big([-r, 0]; S(\varrho)\big)$,

$$\mathcal{L}V(t, \psi) \leq 0, \qquad \text{(a.s.)},$$

provided that $V(t+s, \psi(s)) \leq q(V(t, \psi(0)))$ for some $s \in [-r, 0]$, where q is a class-$\mathcal{K}_3$ function;

(iii) at any impulsive moment $\tau_k \in \mathbb{T}$ and $\psi \in \mathcal{PC}\big([-r, 0]; S(\varrho)\big)$,

$$V(\tau_k, \psi(0) + \mathcal{I}(\tau_k, \psi(\tau_k^-))) \leq \alpha(d_k) V(\tau_k^-, \psi(0)), \qquad \text{(a.s.)},$$

with $\psi(0^-) = \psi(0)$, where $(\tau_k, \psi(\tau_k^-)) \in \mathbb{R}_+ \times \mathcal{PC}\big([-r, 0]; S(\varrho_1)\big)$, $\prod_{k=1}^{\infty} \alpha(d_k) < \infty$, and $\alpha(d_k) > 1 \, \forall k$.

Then, the trivial solution $x \equiv 0$ is uniformly stable in the m.s.

Proof. From condition (i), we have for $s \in [0, \varrho]$, $b(s) \leq a(s)$; so that we can find two functions $\hat{b} \in \mathcal{K}_1$ and $\hat{a} \in \mathcal{K}_2$ such that $\hat{b}(s) \leq b(s) \leq a(s) \leq \hat{a}(s)$ for all

$s \in [0, \varrho]$. This implies

$$\hat{b}(\|\psi(0)\|^2) \leq V(t, \psi(0)) \leq \hat{a}(\|\psi(0)\|^2), \qquad \text{(a.s.)},$$

for all $t \in \mathbb{R}_+$ and $\psi \in \mathcal{PC}\big([-r, 0]; S(\varrho)\big)$.

Let $x(t) = x(t, t_0, \phi)$ be the unique solution of system (4.1), and $0 < \varepsilon \leq \varrho_1$. Define $\overline{d} = \prod_{k=1}^{\infty} \alpha(d_k)$. Then, $1 \leq \overline{d} < \infty$ because $d < \infty$. Choose $\delta = \delta(\varepsilon)$ so that $\delta < \hat{a}^{-1}\big(\hat{b}(\varepsilon)/\overline{d}\big)$ and clearly $0 < \delta < \varepsilon$.

Let $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer l and ϕ for which $\mathbb{E}[\|\phi\|_r^2] \leq \delta$.

We claim that the trivial solution is uniformly stable in the m.s. If our claim were not true, there would exist t^s such that, for all $t \in [t_0 - r, t^s)$, we have

$$\mathbb{E}[\|x(t)\|^2] < \varepsilon < \varrho_1,$$

and either

$$\mathbb{E}[\|x(t^s)\|^2] = \varepsilon,$$

which implies that

$$\mathbb{E}[\|x(t^s)\|^2] = \mathbb{E}[\|x_{t^s}\|_r^2] = \varepsilon,$$

or

$$\varepsilon < \mathbb{E}[\|x(t^s)\|^2], \quad \text{where} \quad t^s = \tau_k \quad \text{for some } k,$$

and by Assumption A1,

$$\varepsilon < \mathbb{E}[\|x(t^s)\|^2] < \varrho$$

since $\mathbb{E}[\|x_{t^{s-}}\|^2] \leq \varepsilon < \varrho_1$. Thus, in either case, $V(t, x(t))$ is defined for $t \in [t_0, t^s]$.

Moreover, from assumption (ii), we have

$$\mathcal{L}V(t, x_t) \leq 0.$$

Applying the Itô formula to the process $V(t, x(t))$ for $t \in [t_0, t^s]$ and taking the mathematical expectation yield

$$\begin{aligned}\mathbb{E}[V(t, x(t))] &\leq \mathbb{E}[V(s, x(s))] + \mathbb{E}\int_s^t \mathcal{L}V(u, x_u)du, \qquad \forall\, t_0 \leq s \leq t \leq t^s \\ &\leq \mathbb{E}[V(s, x(s))].\end{aligned}$$

Define $m(t) = \mathbb{E}[V(t, x(t))]$ for all $t \in [t_0, t^s]$. Then,

$$D^+m(t) = \lim_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \leq 0,$$

that is, the function $m(t)$ is non-increasing for all $t \in (t_0, t^s]$ between the impulse moments.

By the condition in (iii), we have

$$m(\tau_k) \leq \alpha(d_k)m(\tau_k^-), \qquad \forall\, t \in (t_0, t^s].$$

Since $m(t) = \mathbb{E}[V(t, x(t))] \leq \mathbb{E}[V(s, x(s))] = m(s)$, $m(t)$ is non-increasing for all $t \in [t_0, t^s]$ between impulses. If $t^s \in (t_0, t_l)$, then

$$\hat{b}(\mathbb{E}[\|x(t^s)\|^2]) \leq m(t^s) \leq m(t_0) < \hat{a}(\delta) < \frac{\hat{b}(\varepsilon)}{\bar{d}} \leq \hat{b}(\varepsilon).$$

On the other hand, let $t^s \in [\tau_k, \tau_{k+1})$ for some $k \geq l$. In this case we have,

$$m(t^s) \leq m(\tau_k), \qquad \text{because } m \text{ is nonincreasing } \forall\, t \leq t^s, \tag{4.2}$$

$$m(\tau_k^-) \leq m(t_0) < \hat{a}(\delta), \tag{4.3}$$

$$m(\tau_i^-) \leq m(\tau_{i-1}), \quad i = l+1, l+2, \cdots, k, \tag{4.4}$$

$$m(\tau_i) \leq \alpha(d_i)m(\tau_i^-), \quad i = l, l+1, l+2, \cdots, k. \tag{4.5}$$

By (4.5), we have

$$
\begin{aligned}
m(\tau_i) &\le \alpha(d_i) m(\tau_i^-) \\
&\le \alpha(d_i) m(\tau_{i-1}) && \text{by (4.4)} \\
&\le \alpha(d_i)\alpha(d_{i-1}) m(\tau_{i-1}^-) && \text{by (4.5)} \\
&\vdots \\
&\le \prod_{i=1}^{l} \alpha(d_i) m(t_0) \\
m(\tau_i) &\le \overline{d}\, m(t_0) \le \overline{d}\, \hat{a}(\delta). && \text{by (4.2)}
\end{aligned}
$$

Namely, $m(\tau_i) \le \overline{d}\, \hat{a}(\delta)$, which implies that

$$
m(t^s) \le m(\tau_i) \le \overline{d}\, \hat{a}(\delta),
$$

where the first inequality is from (4.2). We also have

$$
\hat{b}(\varepsilon) < \hat{b}(\mathbb{E}\|x(t^s)\|^2) \le m(t^s) < \overline{d}\, \hat{a}(\delta) < \hat{b}(\varepsilon).
$$

This is a contradiction. It turns out that $x \equiv 0$ is uniformly stable in m.s. This completes the proof.

In the existence and uniqueness results, we have assumed that the vector field functionals f and g are bounded above by a time varying integrable random function m over a compact segment of $\mathbb{R}_+$. To prove asymptotic stability, we need to strengthen our boundedness assumption on f and g to be valid over $\mathbb{R}_+$.

Definition 4.4. A functional $f : \mathbb{R}_+ \times \mathcal{PC}([-r, 0]; \mathcal{D}) \to \mathbb{R}^n$ is said to be *strongly quasi-bounded in the m.s.* if, for each compact set $F \subset \mathcal{D} \subset \mathbb{R}^n$, there exists a positive constant M such that $\mathbb{E}[\|f(t, \psi)\|^2] \le M$ for all $(t, \psi) \in \mathbb{R}_+ \times \mathcal{PC}\big([-r, 0]; F\big)$.

In the next theorem, we address the m.s. asymptotic stability result for (4.1). This qualitative property requires strengthening the infinitesimal diffusion operator $\mathcal{L}$ to be bounded above by a strictly negative estimate.

Theorem 4.2. Assume that the assumptions of Corollary 3.5 and A1 and A2 are satisfied, the functionals f and g are strongly quasi-bounded in m.s., there exist functions $a \in \mathcal{K}_2$, b, $c \in \mathcal{K}_1$, and a constant $d_k \geq 0$ with $d = \sum_{k=1}^{\infty} d_k < \infty$. Let $V \in \mathcal{C}^{1,2}\big([-r,\infty) \times S(\varrho)); \mathbb{R}_+\big)$ satisfy

(i) assumptions (i) and (iii) of Theorem 4.1;

(ii) for all $t \neq \tau_k \in \mathbb{R}_+$ and $\psi \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$,

$$\mathcal{L}V(t,\psi) \leq -c(\|\psi(0)\|^2), \qquad \text{(a.s.)},$$

provided that $V(t+s,\psi(s)) \leq q(V(t,\psi(0)))$ for some $s \in [-r,0]$, where q is a class-$\mathcal{K}_3$ function.

Then, the trivial solution $x \equiv 0$ of (4.1) is uniformly asymptotically stable in the m.s.

Proof. Let $\hat{c} \in \mathcal{K}_1$ such that $\hat{c}(s) \leq c(s)$ for all $s \in \mathbb{R}_+$. Given any $0 < \varepsilon \leq \varrho_1$. Let $\delta = \delta(\varepsilon)$ be the constant of uniform stability in the m.s. defined in Theorem 4.1. Choose a number $N > 0$ such that $N > \hat{m}_0 \overline{d} \bar{c}^{-1}\Big(\frac{1}{5}[\delta^2 - \delta(\frac{\delta}{4M}+1)]\Big)\frac{4M}{\delta}$.

Suppose that a solution $x = x(\sigma,\phi)$ with $\mathbb{E}[\|\phi\|^2] < \frac{\delta^2}{3}$ satisfies $\mathbb{E}[\|x_t\|_r^2] \geq \frac{\delta^2}{3}$ for any $t \geq \sigma$.

Assume that each interval of length r contains a t_k such that $\mathbb{E}[\|x(t_k)\|^2] \geq \frac{\delta^2}{3}$. Then, there exists a sequence $\{t_k\}$ such that

$$\sigma + (2k-1)r \leq t_k \leq \sigma + 2kr, \qquad\qquad k = 1, 2, \cdots,$$

and

$$\mathbb{E}[\|x(t_k)\|^2] \geq \frac{\delta^2}{3},$$

which implies that $\mathbb{E}[\|x_{t_k}\|_r^2] \geq \frac{\delta^2}{3}$.

Integrating the stochastic differential equation (4.1a) over $[t_k, t_k + \frac{\delta}{4M}]$ yields

$$x(t) = x_{t_k} + \int_{t_k}^{t_k+\frac{\delta}{4M}} f(t, x_t)dt + \int_{t_k}^{t_k+\frac{\delta}{4M}} g(t, x_t)dW(t),$$

from which we get

$$\mathbb{E}[\|x(t)\|^2] \leq 3\mathbb{E}[\|x_{t_k}\|_r^2] + 3\frac{\delta}{4M}\int_{t_k}^{t_k+\frac{\delta}{4M}} \mathbb{E}\|f(t, x_t)\|^2 dt + 3\int_{t_k}^{t_k+\frac{\delta}{4M}} \mathbb{E}\|g(t, x_t)\|^2 dt,$$

which gives

$$\mathbb{E}[\|x(t)\|^2] - 3\mathbb{E}[\|x_{t_k}\|_r^2] \leq \frac{3}{4}\frac{\delta}{M}M\frac{\delta}{4M} + 3M\frac{\delta}{4M},$$

i.e.,

$$\mathbb{E}[\|x(t)\|^2] - 3\mathbb{E}[\|x_{t_k}\|_r^2] < \delta(\frac{\delta}{4M} + 1). \tag{4.6}$$

Since we have assumed that $\mathbb{E}[\|x_{t_k}\|_r^2] \geq \frac{\delta^2}{3}$, then $-\mathbb{E}[\|x_{t_k}\|_r^2] \leq -\frac{\delta^2}{3}$ or $-3\mathbb{E}[\|x_{t_k}\|_r^2] \leq -\delta^2$. By adding $5\mathbb{E}[\|x(t)\|^2]$ to the both sides of the last inequality, we get

$$5\mathbb{E}[\|x(t)\|^2] - 3\mathbb{E}[\|x_{t_k}\|_r^2] < 5\mathbb{E}[\|x(t)\|^2] - \delta^2. \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$0 \leq 6(\mathbb{E}[\|x(t)\|^2] - \mathbb{E}[\|x_{t_k}\|_r^2]) \leq 5\mathbb{E}[\|x(t)\|^2] - \delta^2 + \delta(\frac{\delta}{4M} + 1),$$

which gives

$$\mathbb{E}[\|x(t)\|^2] \geq \frac{1}{5}[\delta^2 - \delta(\frac{\delta}{4M} + 1)] > 0, \quad \text{provided that} \quad \delta > (\frac{\delta}{4M} + 1),$$

from which we have

$$-\bar{c}\Big(\mathbb{E}[\|x(t)\|^2]\Big) \leq -\bar{c}\Big(\frac{1}{5}[\delta^2 - \delta(\frac{\delta}{4M}+1)]\Big).$$

Therefore, for $t \in [t_k, t_k + \frac{\delta}{4M}]$, we have

$$\mathbb{E}[\mathcal{L}V(t, x_t)] \leq -\hat{c}\Big(\mathbb{E}[\|x(t)\|^2]\Big) \leq -\hat{c}\Big(\frac{1}{5}[\delta^2 - \delta(\frac{\delta}{4M}+1)]\Big).$$

By Itô's formula, we have

$$\begin{aligned}\mathbb{E}[V(t, x(t))] &\leq \mathbb{E}[V(t_k, x(t_k))] + \mathbb{E}\int_{t_k}^{t_k+\frac{\delta}{4M}} \mathcal{L}V(t, x_t)dt \\ &\leq \mathbb{E}[V(t_k, x(t_k))] - \hat{c}\Big(\frac{1}{5}[\delta^2 - \delta(\frac{\delta}{4M}+1)]\Big)\frac{\delta}{4M},\end{aligned}$$

or

$$m(t) \leq m(t_k) - \hat{c}\Big(\delta^2 - \delta(\frac{\delta}{4M}+1)\Big)\frac{\delta}{4M},$$

where $m(t) = \mathbb{E}[V(t, x(t))]$, i.e., the function m decreases by $\hat{c}\Big(\frac{1}{5}[\delta^2 - \delta(\frac{\delta}{4M} + 1)]\Big)\frac{\delta}{4M} > 0$ over the interval $[t_k, t_k + \frac{\delta}{4M}]$.

To investigate the overall behaviour of the function $m(t)$ for all $t \geq t_0$, we define a new function, say $\hat{m}$, as follows

$$\hat{m}(t) = \begin{cases} m(t), & t \in [t_0, t_l), \\ \Big[\prod_{k=l}^{i} \alpha(d_k)\Big]^{-1} m(t), & t \in (t_i, t_{i+1}), \quad i = l, l+1, \cdots. \end{cases}$$

This shows that the function $\hat{m}$ decreases by $\overline{d}^{\,-1}\hat{c}\Big(\frac{1}{5}[\delta^2 - \delta(\frac{\delta}{4M}+1)]\Big)\frac{\delta}{4M} > 0$ over the interval $[t_k, t_k + \frac{\delta}{4M}]$ or $[t_k - \frac{\delta}{4M}, t_k]$, where $\bar{d} = \prod_{k=l}^{i} \alpha(d_k)$. This implies that

$$\hat{m}(t_0 + T) \leq \hat{m}(t_0) - N\overline{d}^{\,-1}\bar{c}\Big(\frac{1}{5}[\delta^2 - \delta(\frac{\delta}{4M}+1)]\Big)\frac{\delta}{4M}.$$

By our assumptions and choice of N, we conclude that

$$\hat{m}(t_0+T) \le a(\varrho_1) - N\overline{d}^{\,-1}\overline{c}\Big(\frac{1}{5}[\delta^2 - \delta(\frac{\delta}{4M}+1)]\Big)\frac{\delta}{4M} < 0,$$

which is a contradiction. Thus, it must be true that, under our assumptions, $\mathbb{E}[\|x(t)\|^2] < \varepsilon$ for all $t \ge t_0$, i.e., the trivial solution of (4.1) is uniformly asymptotically stable in the m.s. This completes the proof.

Assumptions (i) and (ii) in Theorem 4.1 and assumption (ii) and the first part of (i) in Theorem 4.2 are made to ensure that the Lyapunov function V is non-increasing and strictly decreasing, respectively, in the m.s., which implies that the continuous system is m.s. uniformly stable and asymptotically stable, respectively. To assure that the overall behaviour of V decreases for all time, we assume that V is non-increasing at the impulsive moments, because, otherwise, the reduction of V may not compensate the jump increases. This condition is summarized in assumption (iii) in Theorem 4.1 (and the second part of (i) in Theorem 4.2). We should also mention that the strongly quasi boundedness condition on f and g can be dropped if the upper bound of the operator $\mathcal{L}$ in Theorem 4.2 is replaced by the stronger condition

(ii)$'$ for all $t \neq \tau_k$ and $\psi \in \mathcal{PC}([-r,0]; S(\varrho))$, $\mathcal{L}V(t,\psi) \le -c(V(t,\psi(0)))$, (a.s.), provided that $V(t+s,\psi(s)) \le q(V(t,\psi(0)))$ for some $s \in [-r,0]$ and $q \in \mathcal{K}_3$.

Furthermore, it is obvious that these two theorems do not impose any restriction on the time delay. This makes our results efficient to delay differential equations.

Example 4.1. Consider the following impulsive system

$$\begin{aligned} dx &= \big(-4x + x(t-1)e^{-|x|}\big)dt - 0.1\sin x(t-1)dW, \quad t \neq \tau_k, \\ \Delta x(t) &= \frac{1}{k^2}\, x_{t^-}, \qquad\qquad t = \tau_k, \quad k = 1, 2, \cdots . \end{aligned}$$

Define $V(x) = x^2$ as a Lyapunov function candidate. Then, one can easily show that $\mathcal{L}V(x) \leq -c(x) < 0$ with $q = 2$, where $c(s) = 3s^2$. At $t = \tau_k$, we have

$$|x(\tau_k)| = |x(\tau_k^-) + \frac{1}{k^2} x_{\tau_k^-}| \leq |x(\tau_k^-)| + \frac{\sqrt{2}}{k^2}|x(\tau_k^-)| \leq (1 + \frac{\sqrt{2}}{k^2})|x(\tau_k^-)|,$$

and from which we have $V(x(\tau_k)) \leq \alpha(d_k)V(x(\tau_k^-))$, where $\alpha(d_k) = (1 + \sqrt{2}d_k)^2$ and $d_k = \frac{1}{k^2}$. We also have $\varrho_1 < \varrho/(1 + \sqrt{2}d_k)$. Choose $a(s) = b(s) = s^2$. Thus, the assumptions of Theorems 4.1 and 4.2 are satisfied, i.e., the trivial solution $x \equiv 0$ is asymptotically stable in the m.s. The simulation result of this example is shown in Figure 4.1.

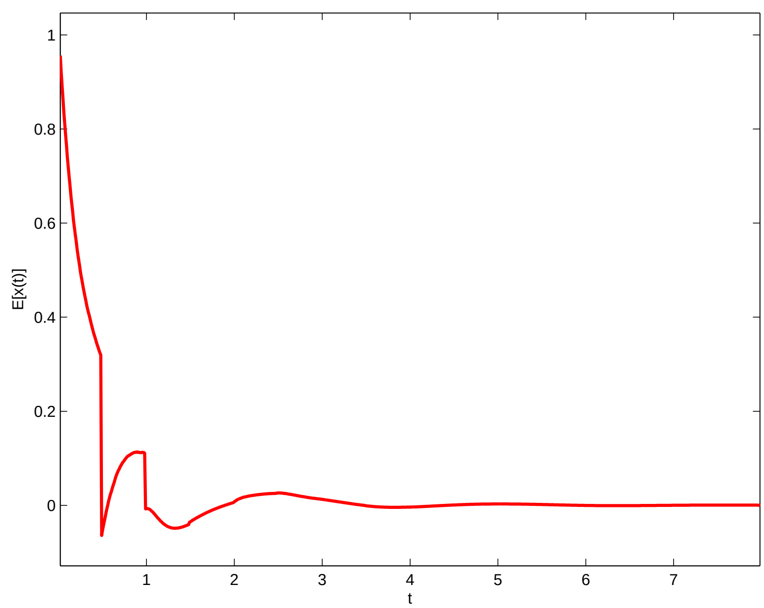

Figure 4.1: First moment asymptotic stability of $x \equiv 0$.

4.2 Analysis by Comparison Principle

The focus of this section is to develop a comparison principle for the SISD with fixed impulses (4.1) and then, by utilizing this technique, we establish some m.s.

stability properties of this system. Generally, the approach of comparison principle enables one to compare multivariable systems with a single variable system and hence the features of the latter system imply the corresponding features of the compared systems. As in the previous section, we employ Razumikhin methodology to write Lyapunov-like sufficient conditions guaranteeing the stability results. For further reading about the comparison principle of stochastic ordinary systems with impulsive differential equations, one may consult [Liu08].

Theorem 4.3. Assume that the assumptions of Corollary 3.5 and A1 and A2 are satisfied, and there exists a class-$\mathcal{K}_2$ function a. Let $V \in \mathcal{C}^{1,2}\big([-r,\infty)\times S(\varrho)\big);\mathbb{R}_+\big)$ satisfy

(i) $V(t,\psi(0)) \le a(\|\psi(0)\|^2) \le a(\|\psi\|_r^2)$, (a.s.), $\forall (t,\psi(0)) \in [-r,\infty)\times S(\varrho)$;

(ii) $\mathcal{L}V(t,\psi(t)) \le h(t,V(t,\psi(0)))$, (a.s.), $\forall t \ne \tau_k$ and $\psi \in \mathcal{PC}\big([-r,0];S(\varrho)\big)$ provided that $V(t+s,\psi(s)) \le q(V(t,\psi(0)))$ for all $s\in[-r,0]$, with q being a class-$\mathcal{K}_3$ function, where $h:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is continuous on $[\tau_{k-1},\tau_k)$, $h(t,z)$ is concave in z for any $t\in\mathbb{R}_+$, and, for each $x\in\mathbb{R}^n$ and $k\ge 1$,

$$\lim_{(t,y)\to(\tau_k^-,x)} h(t,y) = h(\tau_k^-,x)$$

exists;

(iii) $\forall\,\tau_k\in\mathbb{T}$ and $\psi\in\mathcal{PC}\big([t_0-r,\infty);S(\varrho)\big)$,

$$V(\tau_k,\psi(0)+\mathcal{I}(\tau_k,\psi(\tau_k^-))) \le \alpha_k(V(\tau_k^-,\psi(0^-))), \qquad \text{(a.s.)},$$

where $\psi(0^-)=\psi(0)$, $(\tau_k,\psi(\tau_k^-))\in\mathbb{R}_+\times\mathcal{PC}\big([-r,0];S(\varrho_1)\big)$, and $\alpha_k:\mathbb{R}_+\to\mathbb{R}_+$ is a non-decreasing, concave function;

(iv) the scalar impulsive system

$$\begin{cases} D^+v(t) = h(t, v(t)), & t \neq \tau_k, \\ v(t) = \alpha_k(v(t^-)), & t = \tau_k, \\ v(t_0) = v_0 \geq 0 \end{cases} \tag{4.8}$$

has a maximal solution $r(t) = r(t, t_0, v_0)$.

Then, $\mathbb{E}[V(t_0, x_0)] < v_0$ implies $\mathbb{E}[V(t, x(t))] < r(t)$ for all $t \geq t_0$.

Proof. Let $x(t) = x(t, t_0, \phi)$ be any solution of system (4.1). We have from (i)

$$\mathbb{E}[V(t, x(t))] \leq \mathbb{E}[a(\|x(t)\|^2)] \leq a(\mathbb{E}[\|x(t)\|^2]) < \infty.$$

Also, by Itô formula and condition (ii), we have, for all $t \in [\tau_{k-1}, \tau_k)$,

$$\begin{aligned} \mathbb{E}[V(t, x(t))] &= \mathbb{E}[V(\tau_{k-1}, x(\tau_{k-1})] + \mathbb{E}\int_{\tau_{k-1}}^{t} \mathcal{L}V(s, x_s)ds \\ &\leq \mathbb{E}[V(\tau_{k-1}, x(\tau_{k-1})] + \int_{\tau_{k-1}}^{t} \mathbb{E}[h(s, V(s, x(s)))]ds \\ &\leq \mathbb{E}[V(\tau_{k-1}, x(\tau_{k-1})] + \int_{\tau_{k-1}}^{t} h(s, \mathbb{E}[V(s, x(s))])ds, \end{aligned}$$

and from which we get

$$D^+m(t) \leq h(t, m(t)), \qquad t \neq \tau_k,$$

where $m(t) = \mathbb{E}[V(t, x(t))]$ for all $t \in [\tau_{k-1}, \tau_k)$. At the impulsive moments, we have from condition (iii), $m(\tau_k) \leq \alpha_k(m(\tau_k^-))$. Namely, we have

$$\begin{cases} D^+m(t) \leq h(t, m(t)), & t \neq \tau_k, \\ m(t) \leq \alpha_k(m(t^-)), & t = \tau_k, \\ m(t_0) = \mathbb{E}[V(t_0, x_0)]. \end{cases}$$

Therefore, comparing with (4.8) leads to

$$m(t) = \mathbb{E}[V(t, x(t))] < r(t) = v(t), \qquad \forall t \geq t_0.$$

This completes the proof.

In this following, we make use of this comparison result to show how the stability properties of the scalar impulsive system (4.8) imply those of the SISD (4.1).

Theorem 4.4. Assume that the conditions of Corollary 3.5 and Assumptions A1 and A2 hold, and there exist functions $a \in \mathcal{K}_2$ and $b \in \mathcal{K}_1$. Assume further that $V \in \mathcal{C}^{1,2}\big([-r,\infty) \times S(\varrho)); \mathbb{R}_+\big)$ satisfies

(i) for all $(t, \psi(0)) \in [-r, \infty) \times S(\varrho)$,

$$b(\|\psi(0)\|^2) \leq V(t, \psi(0)) \leq a(\|\psi(0)\|^2), \qquad \text{(a.s.)};$$

(ii) for all $t \neq \tau_k$ and $\psi \in \mathcal{PC}\big([-r, 0]; S(\varrho)\big)$,

$$\mathcal{L}V(t, \psi(t)) \leq h(t, V(t, \psi(0))), \qquad \text{(a.s.)},$$

provided that $V(t+s, \psi(s)) \leq q(V(t, \psi(0)))$ with $s \in [-r, 0]$, where q is a class-$\mathcal{K}_3$ function, $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous in its variables, $h(t, 0) = 0$ and $h(t, z)$ is concave in z for any $t \in \mathbb{R}_+$, and, for each $x \in \mathbb{R}^n$ and $k \geq 1$,

$$\lim_{(t,y)\to(\tau_k^-, x)} h(t, y) = h(\tau_k^-, x)$$

exists;

(iii) $\forall\, \tau_k \in \mathbb{T}$ and $\psi \in \mathcal{PC}\big([-r, 0]; S(\varrho)\big)$,

$$V(\tau_k, \psi(0) + \mathcal{I}_k(\tau_k, \psi(\tau^-))) \leq \alpha_k(V(\tau_k^-, \psi(0^-))), \qquad \text{(a.s.)},$$

where $\psi(0^-) = \psi(0)$, $(\tau_k, \psi(\tau_k^-)) \in \mathbb{R}_+ \times \mathcal{PC}\big([-r, 0]; S(\varrho_1)\big)$, and $\alpha_k : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing, concave function and $\alpha_k(0) = 0$.

Then, the stability properties of the scalar impulsive system (4.8) imply the corresponding properties of (4.1).

Proof. Let $0 < \varepsilon < \varrho_1 < \varrho$, and $t_0 \in \mathbb{R}_+$. Assume that the comparison system is stable. So that, for given $b(\varepsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$v_0 < \delta \qquad \text{implies} \qquad v(t, t_0, v_0) < b(\varepsilon), \quad \forall t \geq t_0,$$

where $v(t, t_0, v_0)$ is any solution of the comparison system.

Choose $v_0 = a(\|\phi\|_r^2)$ and $\delta_1 = \delta_1(\varepsilon) > 0$ for which $a(\delta_1) < b(\varepsilon)$. Define $\hat{\delta} = \min\{\delta, \delta_1\}$. We claim that, if $\mathbb{E}[\|\phi\|_r^2] \leq \hat{\delta}$, then

$$\mathbb{E}[\|x(t)\|^2] < \varepsilon, \qquad \forall t \geq t_0.$$

If our claim were not true, there would be a $\bar{t} \in [\tau_k, \tau_{k+1})$ for some k such that

$$\varepsilon \leq \mathbb{E}[\|x(\bar{t})\|^2],$$

and

$$\mathbb{E}[\|x(t)\|^2] < \varepsilon, \qquad \forall\, t \in [\tau_k, \bar{t}).$$

Also, this together with Assumption A1, i.e., $\mathbb{E}[\|x(\tau_k^-)\|^2] < \varepsilon < \varrho_1$ and

$$\mathbb{E}[\|x(\tau_k)\|^2] = \mathbb{E}[\|x(\tau_k^-) + \mathcal{I}(\tau_k, x_{\tau_k^-})\|^2] < \varrho,$$

implies the existence of a $\underline{t}$ such that $\tau_k < \underline{t} \leq \bar{t}$ satisfying

$$\varepsilon < \mathbb{E}[\|x(\underline{t})\|^2] < \varrho.$$

Define $m(t) = \mathbb{E}[V(t, x(t))]$ for all $t \in [t_0, \underline{t}]$. By Theorem 4.3, we get

$$m(t) < r(t, t_0, a(\mathbb{E}[\|\phi\|_r^2])), \qquad \forall\, t \in [t_0, \underline{t}],$$

where $r(t, t_0, a(\mathbb{E}[\|\phi\|_r^2]))$ is the maximal solution of the scalar comparison system.

Finally, by condition (i), we obtain

$$b(\varepsilon) \le m(\underline{t}) = \mathbb{E}[V(\underline{t}, x(\underline{t}))] \le r(\underline{t}, t_0, a(\mathbb{E}[\|\phi\|_r^2])) \le r(\underline{t}, t_0, a(\delta)) < b(\varepsilon),$$

which contradicts with our supposition. Therefore, it must be true that

$$\mathbb{E}[\|x(t)\|^2] < \varepsilon, \qquad \forall\, t \ge t_0.$$

As for the uniform property, it suffices to choose δ independent of t_0.

To prove the uniform attractivity, we choose $0 < \eta < \varrho_1 < \varrho$. Assume that the comparison system is uniformly attractive, i.e., for a given $b(\eta) > 0$, there exist $\delta > 0$ and a constant $T = T(\eta) > 0$ such that

$$v_0 \le \delta \qquad \text{implies} \qquad v(t, t_0, v_0) < b(\eta), \qquad \forall\, t \ge t_0 + T.$$

Following the argument used in proving the stability property, we obtain

$$b(\mathbb{E}[\|x(t)\|^2]) \le v(t, t_0, v_0) < b(\eta), \qquad \forall\, t \ge t_0 + T,$$

i.e., the system (4.1) is uniformly attractive in the m.s., which leads to the m.s. uniformly asymptotic stability property of $x \equiv 0$. This completes the proof.

We should remark that, using the efficient comparison principle, our theorem does not impose any restriction on the stability of continuous system. That is to say, as will be seen in the next corollary, the impulsive effects can have a stabilizing role even when the underlying continuous system is unstable. The requirement in this circumstance is that the impulses applied to the system be small enough to reduce the growth of the continuous part.

Corollary 4.1. In Theorem 4.4, assume that there exists a piecewise constant function $p : \mathbb{R}_+ \to \mathbb{R}_+$ and a class-$\mathcal{K}_2$ function c such that, for any $(t, \psi(0)) \in$

$\mathbb{R}_+ \times \mathcal{PC}([t-r,\infty); S(\varrho))$,

$$h(t, V(t,\psi(0))) = p(t)c(V(t,\psi(0))). \tag{4.9}$$

Suppose further that there exist $\gamma_k \geq 0$ and $\varrho_0 > 0$ such that, for all $z \in (0, \varrho_0)$ and any $k = 1, 2, \cdots$,

$$\int_{\tau_{k-1}}^{\tau_k} p(s)ds + \int_z^{\alpha_k(z)} \frac{ds}{c(s)} \leq -\gamma_k. \tag{4.10}$$

Then, the trivial solution $x \equiv 0$ of SISD (4.1) is uniformly stable in the m.s. If, moreover, $\sum_{k=1}^{\infty} \gamma_k = +\infty$, then $x \equiv 0$ is asymptotically stable in the m.s.

Proof. In light of Theorem 4.3, defining $m(t) = \mathbb{E}[V(t, x(t))]$ for any $t \geq t_0$ yields

$$\begin{cases} D^+ m(t) \leq p(t)c(m(t)), & t \neq \tau_k, \\ m(t) \leq \alpha_k(m(t^-)), & t = \tau_k, \\ m(t_0) = m_0 = \mathbb{E}[V(t_0, x_0)]. \end{cases} \tag{4.11}$$

Consider the following scalar impulsive comparison system

$$\begin{cases} D^+ v(t) = p(t)c(v(t)), & t \neq \tau_k, \\ v(t) = \alpha_k(v(t^-)), & t = \tau_k, \\ v(t_0) = v_0 > m_0. \end{cases} \tag{4.12}$$

We are now aiming to prove the stability properties of the comparison system (4.12), which, by Theorem 4.4, imply the corresponding properties of SISD (4.1).

Let $0 < \varepsilon < \varrho_0$ and $t_0 \in [\tau_1, \tau_2)$. Choose $\delta > 0$ for which $\delta < \min\{\varepsilon, \alpha_k(\varepsilon)\}$ and $0 \leq v_0 < \delta$. We claim that $v(t) < \varepsilon$ for all $t \in [t_0, \tau_2)$, where v is any solution of (4.12). If our claim were not true, then there would exist a $t^* \in [t_0, \tau_2)$ such that $v(t^*) \geq \varepsilon$. Integrating the differential inequality in (4.11) over (t_0, t^*) gives

$$\int_{v(t_0)}^{v(t^*)} \frac{ds}{c(s)} \leq \int_{t_0}^{t^*} p(s)ds, \tag{4.13}$$

where a variable substitution is performed. By our choice of t_0 and t^* and the positiveness of p, we have

$$\int_{t_0}^{t^*} p(s)ds \leq \int_{\tau_1}^{\tau_2} p(s)ds,$$

and by the early analysis,

$$\int_{v(t_0)}^{v(t^*)} \frac{ds}{c(s)} > \int_{\alpha_1(\varepsilon)}^{\varepsilon} \frac{ds}{c(s)}.$$

Therefore, (4.13) becomes

$$\int_{\tau_1}^{\tau_2} p(s)ds + \int_{\varepsilon}^{\alpha_1(\varepsilon)} \frac{ds}{c(s)} > 0,$$

which is a contradiction with (4.10), i.e., it must be true that $v(t) < \varepsilon$ for all $t \in [t_0, \tau_2)$ or $t \in [\tau_1, \tau_2)$.

Suppose that, for all $t \in [t_0, \tau_k)$ (or generally $t \in [\tau_{k-1}, \tau_k)$), $v(t) < \varepsilon$. Then, it follows from (4.11) that, for all $t \in [\tau_k, \tau_{k+1})$,

$$\int_{v(\tau_k)}^{v(t)} \frac{ds}{c(s)} \leq \int_{\tau_k}^{t} p(s)ds \leq \int_{\tau_k}^{\tau_{k+1}} p(s)ds. \tag{4.14}$$

Noting that $v(\tau_k) = \alpha_k(v(\tau_k^-))$, the last inequality becomes

$$\int_{v(\tau_k^-)}^{v(t)} \frac{ds}{c(s)} \leq \int_{\tau_k}^{\tau_{k+1}} p(s)ds + \int_{v(\tau_k^-)}^{\alpha(v(\tau_k^-))} \frac{ds}{c(s)} \leq -\gamma_k. \tag{4.15}$$

Thus, $v(t) \leq v(\tau_k^-) < \varepsilon$ for all $t \in [\tau_k, \tau_{k+1})$, and, by induction, $v(t) < \varepsilon$ for all $t \geq t_0$, i.e., the trivial solution $v \equiv 0$ is uniformly stable.

To prove asymptotic stability of $v \equiv 0$, let $\varepsilon = \varrho_0$ and choose $\delta_0 = \delta_0(\varrho) > 0$ such that $v_0 < \delta_0$ implies that $v(t) < \varrho_0$ for all $t \geq t_0$. We will prove that $\lim_{k\to\infty} v(\tau_k) = 0$. If this were not the case, there would exist an $\eta > 0$ such that $\lim_{k\to\infty} v(\tau_k) = \eta$. From (4.15), we get

$$\int_{v(\tau_k)}^{v(\tau_{k+1})} \frac{ds}{c(s)} = \frac{v(\tau_{k+1}) - v(\tau_k)}{c(\eta)} \leq -\gamma_k,$$

where

$$\frac{1}{c(\eta)} = \sup \left\{ \frac{1}{c(s)} \Big| \, \forall \, s \in [v(\tau_k), v(\tau_{k+1})] \right\},$$

which also implies, by consecutive induction, that

$$v(\tau_k) \le v(\tau_{k-1}) - c(\eta) \sum_{i=1}^{k} \gamma_k.$$

Letting k goes to infinity produces a contradiction. Therefore, it must be true that $\eta = 0$, which proves the asymptotic stability of $v \equiv 0$. Finally, applying Theorem 4.4 implies that $x \equiv 0$ is asymptotically stable. This completes the proof.

A similar result can be obtained if $p(t)$ in (4.9) is replaced by $-p(t)$ for all t or, particularly, $p(t) = \pm p$ and the impulsive condition (iii) of Theorem 4.4 is replaced by $\alpha_k \cdot V(\tau_k^-, \psi(0))$. In the latter case, the inequality in (4.10) reduces to

$$\pm \, p(\tau_k - \tau_{k-1}) + \ln \alpha_k \le -\gamma_k, \qquad \forall \, k. \tag{4.16}$$

Example 4.2. Consider the following SISD

$$\begin{aligned} dx &= \Big(-7x - 0.5y(t-1)e^{-x^2} \Big) dt, \\ dy &= \Big(-5y + \sin x(t-1) \Big) dt + \Big(-\frac{0.1x(t-1)}{1+y^2}) \Big) dW_2, \\ \Delta x(\tau_k) &= -2x(\tau_k^-), \\ \Delta y(\tau_k) &= 0.2y(\tau_k^- - 1). \end{aligned}$$

Define $V(x, y) = \frac{1}{2}(x^2 + y^2)$ as a Lyapunov function candidate. Then, after cumbersome calculation, we get $\mathcal{L}V(x, y) \le -6.98V(x, y)$, where $q = 2$, and, at $t = \tau_k$, we get $V(x(\tau_k), y(\tau_k)) \le \alpha_k V(x(\tau_k^-), y(\tau_k^-))$, where $\alpha_k = 6$. By (4.16), we find that $\tau_k - \tau_{k-1} = 0.6$ for all k. Thus, the trivial solution is asymptotically stable in the m.s. Figure 4.2 shows the simulation result.

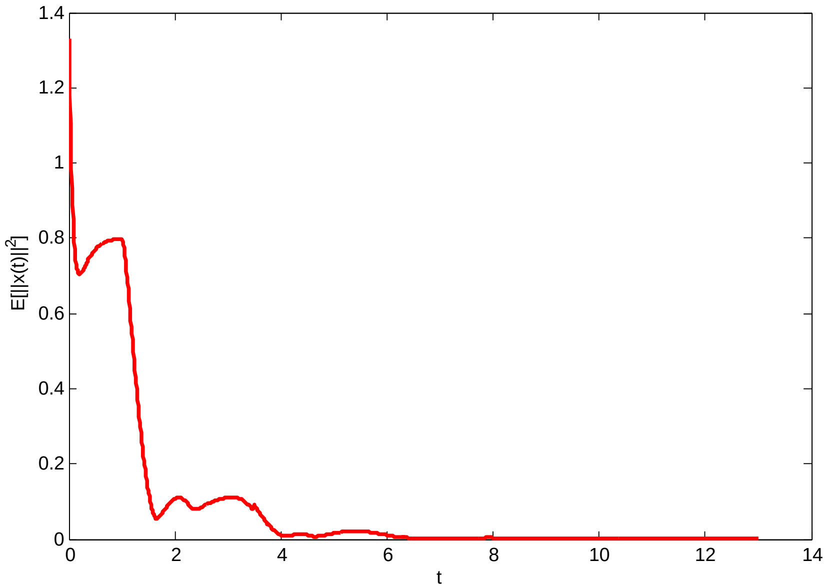

Figure 4.2: Mean square asmptotic stability of $(x, y)^T = (0, 0)$.

Example 4.3. Consider the following SISD

$$dx = \Big(4x - x^2(t-1)\Big)dt + 0.1x\,dW, \quad t \neq \tau_k,$$
$$\Delta x(t) = -\frac{k+2}{k+1}x(t^-), \qquad t = \tau_k, \quad k = 1, 2, \cdots.$$

Define $V(x) = \frac{1}{2}x^2$. Then, $\mathcal{L}V(x) \leq 5.55x^2$, i.e., the underlying non-impulsive system has an unstable trivial solution. Apply now the impulsive effect to get, at $t = \tau_k$, $V(x(\tau_k)) \leq \alpha_k V(x(\tau_k^-))$, where $\alpha_k = \frac{1}{(k+1)^2} < 1$. From (4.16), we get $\tau_k - \tau_{k-1} = 0.2$ for all k. The simulation result is shown in Figure 4.3, which shows the stabilizing effects of impulses.

4.3 Conclusions and Comments

Throughout this chapter, the focus was on SISD with fixed impulses and the main interest was to investigate some stability properties to time-delayed stochastic im-

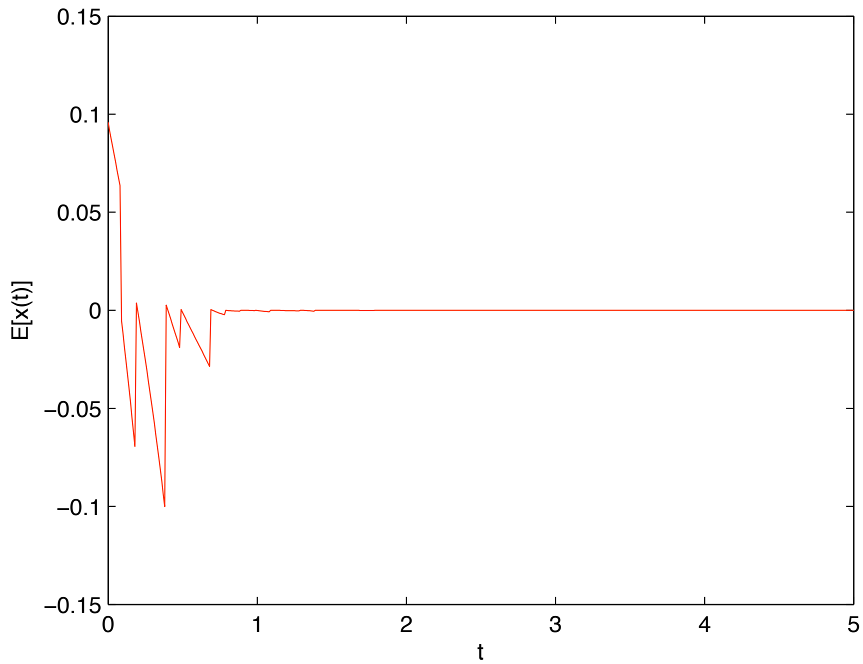

Figure 4.3: First moment asymptotic stability of $x \equiv 0$.

pulsive systems. We have adopted two different approaches to establish these properties, namely, an (ε, δ)-based (Section 4.1) and comparison principle (Section 4.2) approaches. In both cases, using Razumikhin methodology, we have developed some Lyapunov-like sufficient conditions. The latter technique, which is Lyapunov-function-based, is efficient to examine qualitative properties of delay systems, because it enables one to employ the theory of ordinary differential equations and it provides results that are independent of time delay.

Particularly, in Section 4.1, the continuous dynamic considered is stable and perturbed by impulsive actions with bounded total effects. It is noticed that the impulsive system can retain its stability property if the impulses are relatively small and infrequently applied to the system. In Section 4.2, it is shown that systems can preserve their stability properties even if they are disturbed by unbounded impulses. Moreover, it is evident that impulses can help in stabilizing systems which are originally unstable. This case requires the system to be subject to very frequent impulses in order to reduce the growth of continuous states.

The proposed results of this chapter will be further used in establishing some stability properties of large scale SISD, which is the main theme of the next chapter.

Chapter 5

Large Scale Stochastic Impulsive Systems with Time Delay

In this chapter, we consider large scale nonlinear stochastic systems with delayed states and subject to impulsive effects. Typically, a large scale system is described by a large number of variables, nonlinearities, and uncertainties. We will continue to apply the Razumikhin technique to develop Lyapunov-type sufficient conditions to guarantee some stability properties of these systems. Analyzing the qualitative properties of large scale systems can be achieved by different ways. An efficient approach to deal with such a complex system is to decompose the composite (or interconnected) the system into simpler, more manageable isolated subsystems at different hierarchical levels. Analyze each individual subsystem, namely, initially ignore the interconnection between the subsystems, then combine the available results together with interconnection, which is usually viewed as a perturbation, to draw a conclusion about the qualitative properties of the composite system. Conventionally, the individual impulsive subsystems are stable with a certain degree of stability. In order for the composite system to preserve the stability property, the perturbation has to be relatively smaller than the degree of stability of each subsystems. This type of relation between isolated subsystems and the interconnection is usually represented in a special type of matrices called *test matrices*.

The qualitative properties of the composite systems will be established in three sections. In Section 5.1, we examine the properties by a scalar Lyapunov function, which is a sum of the Lyapunov functions related to the isolated subsystems. In Section 5.2, the comparison principle developed in Chapter 4 is used to extend the stability results for the large scale systems, where a scalar Lyapunov function is considered for the analysis. In Section 5.3, we continue with a comparison principle in our stability analysis, but, rather than using a scalar function, we consider a vector Lyapunov function in describing the solution behaviour of the composite systems. The material of this chapter forms the basis of [Alw-c].

Before we state the main contribution of this chapter, we define what is meant to be a large scale system.

Typically, an interconnected or composite system with decomposition $\mathbb{D}_i$ may have the form

$$\mathbb{D}_i : \begin{cases} dw^i(t) = f_i(t, w_t^i)dt + g_i(t, w_t^1, w_t^2, \cdots, w_t^l)dt + \sum_{j=1}^{l} \sigma_{ij}(t, w_t^j)dW_j(t), & t \neq \tau_k, \\ \triangle w^i(t) = \mathcal{I}_i(t, w_{t^-}^i), & t = \tau_k, \\ w_{t_0}^i = \phi_i(s), \qquad s \in [-r, 0], \end{cases} \tag{5.1}$$

where $k \in \mathbb{N}$ and $i = 1, 2, \cdots l$ for some $l \in \mathbb{N}$. Here, we have w^i (or w_t^i) $\in \mathbb{R}^{n_i}$, which is an n_i-dimensional vector state (or, respectively, deviated state) and $n = \sum_i^l n_i$ for some $n_i \in \mathbb{N}$. We should emphasize that w equipped with superscript i is a system state, and not an outcome of the sample space Ω of a probability space.

$f_i : \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $g_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n_i}$, $\sigma_{ij} : \mathbb{R}_+ \times \mathbb{R}^{n_j} \to \mathbb{R}^{n_i \times m_j}$, $m = \sum_i^l m_i$ for some $m_i \in \mathbb{N}$, $\mathcal{I}_i : \mathbb{T} \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ with $\mathbb{T} = \{\tau_k \big| k = 1, 2, \cdots\}$ where τ_k represents constant impulsive moments and satisfies $0 < \tau_1 < \tau_2 < \cdots$, and $\lim_{k\to\infty} \tau_k = \infty$, and $\phi_i : [-r, 0] \to \mathbb{R}^{n_i}$.

Define the isolated (or unperturbed) subsystems $\mathbb{S}_i$ as follows

$$\mathbb{S}_i : \begin{cases} dw^i(t) = f_i(t, w_t^i)dt + \sigma_{ii}(t, w_t^i)dW_i(t), & t \neq \tau_k, \\ \triangle w^i(t) = \mathcal{I}_i(t, w_{t^-}^i), & t = \tau_k, \\ w_{t_0}^i = \phi_i(s), \qquad s \in [-r, 0]. & \end{cases} \tag{5.2}$$

For $x \in \mathbb{R}^n$, let $x^T = [(w^1)^T, (w^2)^T, \cdots, (w^l)^T]$ and $x_t^T = [(w_t^1)^T, (w_t^2)^T, \cdots, (w_t^l)^T]$, and define $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$f^T(t, x_t) = [f_1^T(t, w_t^1), f_2^T(t, w_t^2), \cdots, f_l^T(t, w_t^l)],$$

$g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{aligned} g^T(t, x_t) &= [g_1^T(t, x_t), g_2^T(t, x_t), \cdots, g_l^T(t, x_t)] \\ &= [g_1^T(t, w_t^1, w_t^2, \cdots, w_t^l), g_2^T(t, w_t^1, w_t^2, \cdots, w_t^l), \cdots, g_l^T(t, w_t^1, w_t^2, \cdots, w_t^l)], \end{aligned}$$

$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ by

$$\sigma(t, x_t) = [\sigma_{ij}(t, w_t^j)],$$

and $W : \mathbb{R}_+ \to \mathbb{R}^m$ by

$$W^T = [W_1, W_2, \cdots, W_l],$$

where, for any i, $W_i : \mathbb{R}_+ \to \mathbb{R}^{m_i}$. We also define the impulsive functional $\mathcal{I} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$\mathcal{I}^T(t, x_{t^-}) = [\mathcal{I}_1^T(t, w_{t^-}^1), \mathcal{I}_2^T(t, w_{t^-}^2), \cdots, \mathcal{I}_l^T(t, w_{t^-}^l)].$$

Adopting these notations, the impulsive composite (or interconnected) system with decomposition $\mathbb{D}_i$ can be written in the form $\mathbb{S}$

$$\mathbb{S}: \begin{cases} dx(t) = F(t, x_t)dt + \sigma(t, x_t)dW(t), & t \neq \tau_k, \\ \triangle x(t) = \mathcal{I}(t, x_{t^-}), & t = \tau_k, \\ x_{t_0} = \Phi(s), \qquad s \in [-r, 0], \end{cases} \tag{5.3}$$

where $F(t, x_t) = f(t, x_t) + g(t, x_t)$ is an $\mathcal{L}_{ad}\big(\Omega, L[t_0, t_0 + \alpha]\big)$ function for some $\alpha > 0$, $\sigma \in \mathcal{L}_{ad}\big(\Omega, L^2[t_0, t_0 + \alpha]\big)$, and $\Phi : [-r, 0] \to \mathbb{R}^n$. The initial function of the composite system is defined by $\Phi^T = [\phi_1^T, \phi_2^T, \cdots, \phi_l^T]$ and it is assumed to be in $\mathcal{L}^2_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$.

5.1 Analysis by a Scalar Lyapunov Function

In this section, we are concerned with establishing m.s. uniformly asymptotic stability of the trivial solution of the composite system (5.3). For the random noise intensity, we consider two cases, which are, in this first case (Theorem 5.1), $\sigma_{ii}(\cdot, \cdot) \neq 0$ and $\sigma_{ij}(\cdot, \cdot) = 0$ for all $i \neq j$, whilst, in Theorem 5.2, $\sigma_{ij}(\cdot, \cdot) \neq 0$ for all i, j.

As mentioned earlier, the individual isolated subsystems are assumed to have trivial solution $w^i = 0 \in \mathbb{R}^{n_i}$ that is uniformly asymptotically stable in the m.s., as discussed in Theorem 4.2. For convenience, we state the sufficient conditions guaranteeing the stability property in the following definition.

Definition 5.1. For all $i = 1, 2, \cdots, l$, the isolated subsystem $\mathbb{S}_i$ in (5.2) is said to possess **Property A** if Assumptions A1 and A2 (in Chapter 4) hold, the functionals f_i and σ_{ii} are strongly quasi-bounded in the m.s., there exist functions $a_i \in \mathcal{K}_2$,

b_i, $c_i \in \mathcal{K}_1$, and constants $\sigma_i < 0$ and $d_k \geq 0$ with $d = \sum_{k=1}^{\infty} d_k < \infty$, and $V^i \in \mathcal{C}^{1,2}\big([-r,\infty) \times S(\varrho)\big); \mathbb{R}_+\big)$ such that

(i) for all $(t, \psi^i(0)) \in [-r,\infty) \times S(\varrho)$,

$$b_i(\|\psi^i(0)\|^2) \leq V^i(t, \psi^i(0)) \leq a_i(\|\psi^i(0)\|^2), \qquad \text{(a.s.)};$$

(ii) for all $t \neq \tau_k$ in $\mathbb{R}_+$ and $\psi^i \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$,

$$\mathcal{L}_i V^i(t, \psi^i) \leq \sigma_i c_i(\|\psi^i(0)\|^2), \qquad \text{(a.s.)},$$

provided that $V^i(t+s, \psi^i(s)) \leq \bar{q} V(t, \psi^i(0))$ for some $\bar{q} > 1$ and $s \in [-r, 0]$;

(iii) for any $\tau_k \in \mathbb{T}$ and $\psi^i \in \mathcal{PC}\big([0-r,0]; S(\varrho)\big)$,

$$V^i\big(\tau_k, \psi^i(0) + \mathcal{I}_i(\tau_k, \psi^i(\tau_k^-))\big) \leq \alpha(d_k) V^i(\tau_k^-, \psi^i(0)), \qquad \text{(a.s.)},$$

where $\psi^i(0^-) = \psi^i(0)$ and $\prod_{k=1}^{\infty} \alpha(d_k) < \infty$ with $\alpha(d_k) > 1\, \forall k$.

In the following theorem, we state and prove the m.s. square asymptotic stability of the trivial solution of (5.3).

Theorem 5.1. Suppose that the composite system $\mathbb{S}$ in (5.3) satisfies the following conditions:

(i) every isolated subsystem $\mathbb{S}_i$ possesses Property A;

(ii) for any $i, j = 1, 2, \cdots, l$, there exists a positive constant b_{ij} such that

$$g_i^T(t, \psi^i) V^i_{\psi^i(0)}(t, \psi^i(0)) \leq c_i^{1/2}(\|\psi^i(0)\|^2) \sum_{j=1}^{l} \bar{q} b_{ij} c_j^{1/2}(\|\psi^j(0)\|^2),$$

where $\bar{q}$, c_i, γ are defined in (i);

(iii) the test matrix $S = [s_{ij}]_{l\times l}$ is negative definite, where

$$s_{ij} = \begin{cases} \alpha_i(\sigma_i + \bar{q}b_{ii}), & i = j, \\ \frac{1}{2}\bar{q}(\alpha_i b_{ij} + \alpha_j b_{ji}), & i \neq j, \end{cases}$$

for some positive constant α_i, and σ is defined in (i).

Then, the trivial solution $x \equiv 0$ of composite system $\mathbb{S}$ in (5.3) is uniformly asymptotically stable in the m.s.

Proof. Let $x(t) = x(t, t_0, \Phi)$ be the solution of the composite system $\mathbb{S}$. Define the composite Lyapunov function candidate

$$V(t, x(t)) = \sum_{i=1}^{l} \alpha_i V^i(t, w^i),$$

where, for any i, α_i is a positive constant and V^i is a positive-definite function related to the i^{th} isolated (unperturbed) subsystem $\mathbb{S}_i$ given in (5.2). From (i), there exist $b_i \in \mathcal{K}_1$ and $a_i \in \mathcal{K}_2$ such that, for any i,

$$b_i(\|w^i(t)\|^2) \leq V^i(t, w^i(t)) \leq a_i(\|w^i(t)\|^2) \leq a_i(\|w^i_t\|^2_r),$$

from which we have

$$\sum_{i=1}^{l} \alpha_i\, b_i(\|w^i(t)\|^2) \leq V(t, x(t)) \leq \sum_{i=1}^{l} \alpha_i\, a_i(\|w^i_t\|^2_r).$$

Clearly, the sum V is a positive definite, decreasing function. Thus, there exist $b \in \mathcal{K}_1$ and $a \in \mathcal{K}_2$ such that

$$b(\|x(t)\|^2) \leq V(t, x(t)) \leq a(\|x_t\|^2_r).$$

Since $\sigma_{ij}(t, w^j) \equiv 0$ for any $i \neq j$, the infinitesimal diffusion operator becomes

$$\mathcal{L}V^i(t, x) = \mathcal{L}_i V^i(t, w^i) + g_i^T(t, x_t) V^i_{w^i}(t, w^i),$$

and from which, together with conditions (i) and (ii), we get

$$\begin{aligned}
\mathcal{L}V(t,x) &= \sum_{i=1}^{l} \alpha_i \, \mathcal{L}V_i(t,x) \\
&= \sum_{i=1}^{l} \alpha_i \left\{ \mathcal{L}_i V^i(t,w^i) + g_i(t,x_t)^T V^i_{w^i}(t,w^i) \right\} \\
&\le \sum_{i=1}^{l} \alpha_i \Big\{ \sigma_i c_i(\|w^i\|^2) + c_i^{1/2}(\|w^i\|^2) \sum_{j=1}^{l} \bar{q} b_{ij} c_j^{1/2}(\|w^j\|^2) \Big\} \\
&= z^T S z,
\end{aligned}$$

where $z^T = \big(c_1^{1/2}(\|w^1\|^2), c_2^{1/2}(\|w^2\|^2), \cdots, c_l^{1/2}(\|w^l\|^2)\big)$, and S is the negative-definite matrix defined in (iii). It follows that the eigenvalues of S are strictly negative (i.e., $\lambda_M(S) < 0$). Therefore,

$$\mathcal{L}V(t,x) \le \lambda_M(S) \sum_{i=1}^{l} c_i(\|w^i\|^2),$$

i.e., $\mathcal{L}V(t,x)$ is negative definite, which implies that

$$\mathcal{L}V(t,x) \le -c(\|x(t)\|^2),$$

where c is a class-$\mathcal{K}_1$ function. At the impulsive moments $t = \tau_k$, we have

$$\begin{aligned}
V(\tau_k, x(\tau_k)) &= \sum_{i=1}^{l} \alpha_i V^i(\tau_k, w^i(\tau_k)) \\
&\le \sum_{i=1}^{l} \alpha_i \alpha_i(d_k) V^i(\tau_k^-, w^i(\tau_k^-)) \\
&\le \alpha_M(d_k) \sum_{i=1}^{l} \alpha_i V^i(\tau_k^-, w^i(\tau_k^-)), \qquad \alpha_M(d_k) = \max_i \{\alpha_i(d_k)\} \\
&= \alpha_M(d_k) V(\tau_k^-, x(\tau_k^-)).
\end{aligned}$$

Thus, the conditions of Theorem 4.2 are all satisfied; therefore composite system (5.3) is uniformly asymptotically stable in the m.s. This completes the proof.

In the next theorem, we consider that the isolated subsystems and the interconnection are both subject to random noise, i.e., $\sigma_{ij}(t, w_t^i) \neq 0$ for any $i, j = 1, 2, \cdots, l$.

Theorem 5.2. Suppose that composite system (5.3) satisfies following conditions:

(i) assumptions (i) and (ii) of Theorem 5.1 hold;

(ii) for any $i = 1, 2, \cdots, l$, there exists $e_i > 0$ such that

$$(y^i)^T V^i_{\psi^i(0)\psi^i(0)}(t, \psi^i(0)) y^i \leq \bar{q} e_i \|y^i(0)\|^2,$$

where $y^i = \sigma_{ij}(t, \psi_t^j)$, the i^{th} row of the matrix σ;

(iii) for any $\sigma_{ij}(t, \psi_t^j)$, $i, j = 1, 2, \cdots, l$, there exists $d_{ij} \geq 0$ such that

$$\|\sigma_{ij}(t, \psi^j)\|^2 \leq \bar{q} d_{ij} c_i(\|\psi^j(0)\|^2);$$

(iv) the matrix $S = [s_{ij}]_{l \times l}$ is negative definite, where

$$s_{ij} = \begin{cases} \alpha_i(\sigma_i + \bar{q} b_{ii}) + \frac{1}{2} \sum_{k=1, k \neq i} \bar{q} \alpha_k e_k d_{ki}, & i = j, \\ \frac{1}{2} \bar{q}(\alpha_i b_{ij} + \alpha_j b_{ji}), & i \neq j, \end{cases}$$

for some positive constant α_i for any i.

Then, composite system (5.3) is uniformly asymptotically stable in the m.s.

Proof. Let x be the solution of the composite system. Define the composite Lyapunov function candidate

$$V(t, x) = \sum_{i=1}^{l} \alpha_i V^i(t, w^i),$$

from which we get

$$\mathcal{L}V(t,x) = \sum_{i=1}^{l} \alpha_i \Big\{ \mathcal{L}_i V_i(t,w^i) + g_i^T(t,x_t) V^i_{w^i}(t,w^i)$$
$$+ \frac{1}{2} \sum_{j=1, i\neq j}^{l} \mathrm{tr}\big[\sigma_{ij}^T(t,w_t^j) V^i_{w^i w^i}(t,w^i) \sigma_{ij}(t,w_t^j)\big] \Big\}$$
$$\leq \sum_{i=1}^{l} \alpha_i \Big\{ \sigma_i c_i(\|w^i\|^2) + c_i^{1/2}(\|w^i\|^2) \sum_{j=1}^{l} \bar{q} b_{ij} c_j^{1/2}(\|w^j\|^2)$$
$$+ \frac{1}{2} \sum_{j=1, i\neq j}^{l} \bar{q} e_i \|\sigma_{ij}(t,w_t^j)\|^2 \Big\}$$
$$\leq \sum_{i=1}^{l} \alpha_i \Big\{ \sigma_i c_i(\|w^i\|^2)$$
$$+ c_i^{1/2}(\|w^i\|^2) \sum_{j=1}^{l} \bar{q} b_{ij} c_j^{1/2}(\|w^j\|^2) + \frac{1}{2} \sum_{j=1, i\neq j}^{l} \bar{q} e_i d_{ij} c_i(\|w^j\|^2) \Big\}$$
$$= z^T S z,$$

where $z^T = \big(c_1^{1/2}(\|w^1\|^2), c_2^{1/2}(\|w^2\|^2), \cdots, c_l^{1/2}(\|w^l\|^2)\big)$, and S is the negative-definite matrix defined in (iv). Thus, as in Theorem 5.1, there exists a class-$\mathcal{K}_1$ function c such that

$$\mathcal{L}V(t,x) \leq -c(\|x(t)\|^2),$$

and, at the impulsive moments $t = \tau_k$, we have

$$V(\tau_k, x(\tau_k)) \leq \alpha_M(d_k) V(\tau_k^-, x(\tau_k^-)).$$

Thus, the conditions of Theorem 4.2 are all satisfied; therefore composite system (5.3) is uniformly asymptotically stable in the m.s. This completes the proof.

In Theorems 5.1 and 5.2, we have assumed that the individual isolated subsystems possess Property A so as to guarantee their m.s. uniformly asymptotic stability. Also, assumption (ii) in Theorem 5.1 and the second part of assumption (i) in Theorem 5.2 describe the upper bound on the deterministic interconnection of the system. In Theorem 5.2, we have assumed that the system interconnection undergoes noisy perturbation, which is estimated by an upper bound given in assumption (ii) of Theorem 5.2. The relationship between the stability degree (or the decay rate) of each subsystem and their interconnections, deterministic and stochastic, is formed in the test matrix S, and the negative definiteness ensures that the stability margin of each individual is stronger than the interconnection.

In the following corollary, we state the sufficient conditions to assure m.s. exponential stability of the trivial solution of composite system (5.3). The proof is an immediate result of the two theorems, and is omitted here.

Corollary 5.1. In Theorem 5.1 or 5.2, for any $i = 1, 2, \cdots, l$ and $s > 0$, let $a_i(s) = a_i s^2$, $b_i(s) = b_i s^2$, and $c_i(s) = c_i s^2$, where a_i, b_i, and c_i are positive constants. If we choose

$$a(s) = \max_i \{\alpha_i a_i\} s^2, \qquad b(s) = \min_i \{\alpha_i b_i\} s^2, \qquad c(s) = \lambda_M(S) s^2,$$

the trivial solution of composite system (5.3) is exponentially stable in the m.s.

As an application of the proposed results, we consider an *indirect control system* in automatic control, which describes the longitudinal motion of an aircraft [Lef65, Mich77]. The control system under consideration is a modification of Example 4.6.1 in [Mich77], where we have involved time delay and impulsive effects.

Example 5.1. Consider the control SISD

$$\begin{cases} dx = Axdt + bf(y)dt + \sigma_{11}(x(t-1))dW_1 + \sigma_{12}(y)dW_2, & t \neq \tau_k, \\ dy = \big(-\zeta y - \xi f(y)\big)dt + a^T xdt + \sigma_{21}(x)dW_1 + \sigma_{22}(y(t-1))dW_2, & t \neq \tau_k, \end{cases} \tag{5.4}$$

where $x^T = (x_1, x_2, x_3, x_4)$, while $y \in \mathbb{R}$ is the controller (i.e., $n_1 = 4$, $n_2 = 1$), $A \in \mathbb{R}^{4\times 4}$, $b \in \mathbb{R}^4$, $\zeta, \xi \in \mathbb{R}$, $f \in \mathbb{R}$ is continuous for all $y \in \mathbb{R}$, $f(y) = 0$ if and only if $y = 0$, and $0 < yf(y) < k|y|^2$ for all $y \neq 0$ and $k > 0$, $a \in \mathbb{R}^4$, $\sigma_{11} \in \mathbb{R}^{4\times 4}$, $\sigma_{12} \in \mathbb{R}^{1\times 1}$, $\sigma_{21} \in \mathbb{R}^{4\times 1}$, $\sigma_{22} \in \mathbb{R}^{1\times 1}$, $W_1 \in \mathbb{R}^4$, and $W_2 \in \mathbb{R}$.

The isolated subsystems are

$$\mathbb{S}_i : \begin{cases} dx = Axdt + \sigma_{11}(x(t-1))dW_1, & t \neq \tau_k, \\ dy = \big(-\zeta y - \xi f(y)\big)dt + \sigma_{22}(y(t-1))dW_2, & t \neq \tau_k. \end{cases} \tag{5.5}$$

The impulses are given by the following difference equations

$$\begin{cases} \Delta x(\tau_k) = \mathcal{I}_1(\tau_k, x(\tau_k^-)) = \frac{1}{k^2}\big(-2x_1(\tau_k^-), -2x_2(\tau_k^-), 2x_3(\tau_k^-), 0\big)^T, \\ \Delta y(\tau_k) = \mathcal{I}_2(\tau_k, y(\tau_k^-)) = -\frac{1}{1+k^2} y(\tau_k^-). \end{cases} \tag{5.6}$$

Let $A = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & -10 \end{pmatrix}$,

$$\sigma_{11} = 0.01 \begin{pmatrix} \sin x_1(t-1) & 0 & \frac{x_2(t-1)}{1+x_4^2} & 0 \\ 0 & \frac{x_2(t-1)}{1+x_1^2} & 0 & -x_3^2(t-1) \\ 0 & 0 & x_3(t-1) & 0 \\ 0 & 0 & 0 & -x_4(t-1) \end{pmatrix},$$

$b^T = (1,1,1,1)$, $a^T = (1,1,1,1)$, $\zeta = 5$, $\xi = 2$, $\sigma_{12} = 0.01\frac{y}{1+y^2}$, $\sigma_{21}^T = 0.01(x_2, x_1, x_4, x_3)$, and $\sigma_{22} = 0.01 \sin y(t-1)$.

Let $V^1(x) = \|x\|^2$ and $V^2(y) = y^2$ be the Lyapunov function candidates for the isolated subsystems in (5.5). After cumbersome calculations, one may get $\mathcal{L}_1 V^1(x) \leq (-10 + 0.0002\bar{q})\|x\|^2$ and $\mathcal{L}_2 V^2(y) \leq (-2\zeta + 0.0001\bar{q})y^2 = (-10 + 0.0001\bar{q})y^2$ (i.e., $\sigma_1 = -10 + 0.0002\bar{q}$ and $\sigma_2 = -10 + 0.0001\bar{q}$). For the stability of the non-impulsive isolated subsystems, we take $\bar{q} = 2$. As for the interconnections, we have $V_x^{1^T}(x)g_1(x,y) = 2x^T\xi f(y) \leq 4k\|x\| \cdot |y|$ (i.e., $b_{12} = 4k$), $V_y^2(y)g_2(x,y) = 2ya^T x \leq 4\|x\| \cdot |y|$ (i.e., $b_{21} = 4$). The (noisy) interconnections are: $\sigma_{12}^T(y)V_{xx}^1\sigma_{12}(y) = 2\|\sigma_{12}(y)\|^2 \leq 0.0002y^2$ and $\sigma_{21}^T(x)V_{yy}^2\sigma_{21}(x) = 2\|\sigma_{21}(x)\|^2 \leq 0.0002\|x\|^2$ (i.e., $e_1 = e_2 = 2$ and $d_{12} = d_{21} = 0.0001$).

Let $V(x,y) = \alpha_1 V^1(x) + \alpha_2 V^2(y) = \|x\|^2 + y^2$ (i.e., $\alpha_1 = \alpha_2 = 1$) be the composite Lyapunov function candidate for composite system (5.4). Then, the matrix

$$S = \begin{pmatrix} -9.9997 & 2k+2 \\ 2k+2 & -7.9997 \end{pmatrix}$$

is negative definite if $k < 3.9998$. Let $f(y) = \frac{2y}{1+y^2}$. Clearly, if we choose $k = 2$, the required conditions are satisfied. Therefore, the condition $\mathcal{L}V((x,y)) \leq z^T S z < 0$ is satisfied, where $z^T = (\|x\|, |y|)$.

At the impulsive moments τ_k, we have

$$\begin{aligned} V(x(\tau_k), y(\tau_k)) &= \|x(\tau_k)\|^2 + y^2(\tau_k) \\ &\leq (1 + \frac{2}{k^2})\|x(\tau_k^-)\|^2 + (1 - \frac{5}{1+k^2})y^2(\tau_k^-) \\ &\leq \alpha_M(d_k)V(x(\tau_k^-), y(\tau_k^-)), \end{aligned}$$

where $\alpha_M(d_k) = 1 + \frac{2}{k^2}$. For any $i = 1, 2$, and $s > 0$, choose $a_i(s) = b_i(s) = s^2$. Then, the trivial solution $(x,y)^T \equiv (0,0) \in \mathbb{R}^5$ of composite SISD system (5.4)-(5.6) is exponential stable in the m.s. if $a(s) = b(s) = s^2$ and $c(s) = 2.9169s^2$ for all $s > 0$. The simulation result is shown in Figure 5.1.

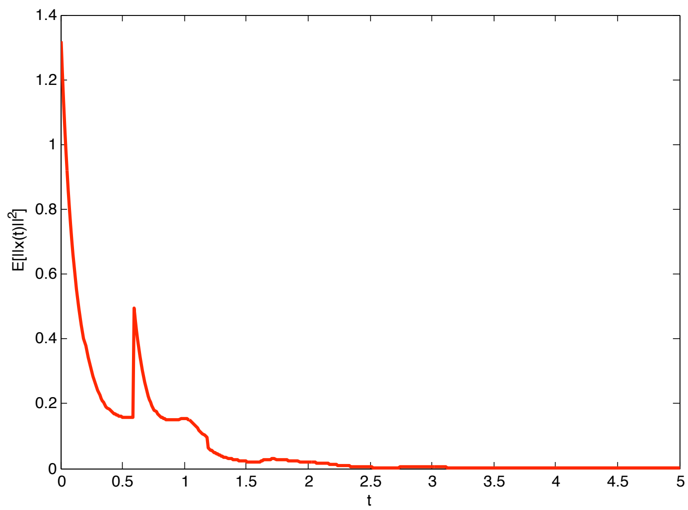

Figure 5.1: Mean square asymptotic stability of $(x, y)^T \equiv (0, 0)$.

5.2 Analysis by Comparison Principle

In this section, depending on the type of composite Lyapunov function candidate, we adopt two approaches to analyze the stability property using the comparison principle. In subsection 5.2.1, a scalar function is considered, as chosen in Section 5.1, while in subsection 5.2.2, we use a vector Lyapunov function.

5.2.1 Scalar Lyapunov Function Approach

In this subsection, we applied our stability results developed in Section 4.2 to large scale system (5.3). As done so far, we analyze each isolated subsystem, which is required here to meet the conditions of Theorem 4.4 and, under a certain restriction imposed on the interconnection, we draw conclusion regarding the overall system stability.

Theorem 5.3. Assume that the assumptions of Theorem 5.2 hold except that, provided that $V^i(t + s, \psi^i(s)) \leq \bar{q}V(t, \psi^i(0))$ for some $\bar{q} > 1$ and $s \in [-r, 0]$,

$$\mathcal{L}_i V^i(t,\psi^i) \le h_{1_i}(t, V^i(t,\psi^i(0))), \qquad \text{(a.s.)},$$

and

$$g_i^T(t,\psi)V^i_{\psi^i(0)}(t,\psi^i(0)) + \frac{1}{2}\sum_{j=1, i\neq j} \mathrm{tr}\big[\sigma_{ij}^T(t,\psi^j)V^i_{\psi^i(0)\psi^i(0)}(t,\psi^i(0))\sigma_{ij}(t,\psi^j)\big]$$
$$< h_{2_i}(t, V(t,\psi(0))),$$

where $\bar{h} \in \mathcal{C}\big([\tau_{k-1},\tau_k)\times\mathbb{R}_+;\mathbb{R}\big)$, $\bar{h}(t,u)$ is concave in u for all $t\in\mathbb{R}_+$ and

$$\lim_{(t,y)\to(\tau_k^-,x)} \bar{h}(t,y) = \bar{h}(\tau_k^-, x),$$

where $\bar{h}$ is both h_{1_i} and h_{2_i}. Then, the stability properties of the composite system (5.3) are implied by those of the following scalar comparison system

$$\begin{cases} D^+v = h(t,v), & t\neq\tau_k, \\ v(t) = \alpha_M(d_k)v(t^-), & t=\tau_k, \\ v(t_0) = v_0 \ge 0, \end{cases} \tag{5.7}$$

where h is a scalar function defined later.

Proof. Let $x^T = \big((w^1)^T, (w^2)^T, \cdots, (w^l)^T\big)$ be the solution of the composite system. Define the composite Lyapunov function candidate by

$$V(t,x) = \sum_{i=1}^{l} \alpha_i V^i(t,w^i).$$

Then, whenever $V(t,x_t) \le \bar{q}V(t,x)$,

$$\mathcal{L}V(t,x) = \sum_{i=1}^{l} \alpha_i \Big\{ \mathcal{L}_i V^i(t,w^i) + g_i(t,x_t)^T V^i_{w^i}(t,w^i)$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{j=1,i\neq j}^{l}\operatorname{tr}\left[\sigma_{ij}(t,w_t^j)V^i_{w^iw^i}(t,w^i)\sigma_{ij}(t,w_t^j)\right]\Big\} \\
\leq &\sum_{i=1}^{l}\alpha_i\Big\{h_{1_i}(t,V^i(t,w^i))+h_{2_i}(t,V^i(t,w^i))\Big\} \\
=: &h(t,V(t,x)), \qquad\qquad t\neq\tau_k.
\end{aligned}
$$

It follows that, after applying Itô formula to process V and taking the mathematical expectation,

$$
D^+m(t)\leq h(t,m(t)),
$$

and, at $t=\tau_k$, we have shown in Theorem 5.1 that

$$
m(\tau_k)\leq\alpha_M(d_k)m(\tau_k^-).
$$

In summary, we have

$$
\begin{cases}
D^+m\leq h(t,m(t)), & t\neq\tau_k,\\
m(t)\leq\alpha_M(d_k)m(t^-), & t=\tau_k,\\
m(t_0)\leq v_0,
\end{cases}
$$

which is compared with the scalar comparison system (5.7). To conclude the desired result, it suffices to apply Theorem 4.4. This completes the proof.

The next corollary is analogous to Corollary 4.1; thus we state it without a proof.

Corollary 5.2. In Theorem 5.3, let $p:\mathbb{R}_+\to\mathbb{R}_+$ be a piecewise continuous function and $c\in\mathcal{K}_2$ such that

$$
h(t,V(t,x))=p(t)c(V(t,x)),
$$

and

$$\int_{\tau_{k-1}}^{\tau_k} p(s)ds + \ln \alpha_M(d_k) \leq -\gamma_k, \qquad k = 1, 2, \cdots, \tag{5.8}$$

for some positive constant $\alpha_M(d_k)$. Then, if $\gamma_k \geq 0$, the composite system is uniformly stable in m.s., and if $\sum_{k=1}^{\infty} \gamma_k = +\infty$, the system is asymptotically stable in the m.s.

Example 5.2. Consider again the continuous control composite system given in (5.4) and same composite Lyapunov scalar function $V(x, y) = \|x\|^2 + y^2$. By the previous analysis, we have found

$$\begin{aligned}
&\mathcal{L}_1 V^1(x) \leq \sigma_1 V^1(x) = (-10 + 0.0002\bar{q})V^1(x), \\
&\mathcal{L}_2 V^2(y) \leq \sigma_2 V^2(y) = (-10 + 0.0001\bar{q})V^2(y), \\
&V^1(x) g_1^T(x, y) \leq 2k(V^1(x) + V^2(y)) = 2kV(x, y), \\
&V^2(y) g_2^T(x, y) \leq 2(V^1(x) + V^2(y)) = 2V(x, y), \\
&\sigma_{12}^T(y) V_{xx}^1 \sigma_{12}(y) \leq 0.0002V^2(y), \\
&\sigma_{21}^T(x) V_{yy}^2 \sigma_{21}(y) \leq 0.0002V^1(x),
\end{aligned}$$

that is, $h_{1_1}(V^1(x)) = \sigma_1 V^1(x)$, $h_{1_2}(V^2(y)) = \sigma_2 V^2(y)$, $h_{2_1}(V^1(x)) = (2k+2.0001)V^1(x)$, and $h_{2_2}(V^2(y)) = (2k + 2.0001)V^2(y)$. Therefore,

$$\begin{aligned}
h(V(x, y)) &= \sum_{i=1}^{l} \alpha_i \Big\{ h_{1_i}(V^i(t, w^i)) + h_{2_i}(V^i(t, w^i)) \Big\} \\
&= (\sigma_1 + 2k + 2.0001)V^1(x) + (\sigma_2 + 2k + 2.0002)V^2(y) \\
&\leq pV(x, y),
\end{aligned}$$

where $p = \sigma_1 + 2k + 2.0001 = -3.9997$, from which one has

$$\mathcal{L}V(x, y) \leq pV(x, y).$$

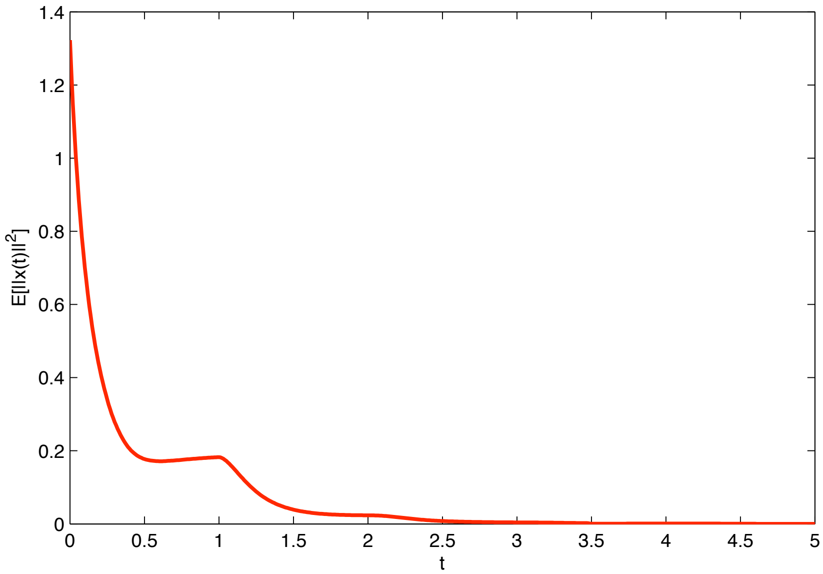

Figure 5.2: Mean square asymptotic stability of $(x, y)^T = (0, 0)$.

Consider now the following impulsive difference equations

$$\begin{cases} \Delta x(\tau_k) = -\frac{5}{4}x(\tau_k^-), \\ \Delta y(\tau_k) = -\frac{5}{4}y(\tau_k^-). \end{cases} \tag{5.9}$$

It follows that $V(x(\tau_k), y(\tau_k)) \leq \alpha_k V(x(\tau_k^-), y(\tau_k^-))$ where $\alpha_k = \frac{1}{16}$. Making use of condition (5.8), one obtains $\tau_k - \tau_{k-1} > 0.69$ for any k. Therefore, the trivial solution $(x, y)^T = (0, 0) \in \mathbb{R}^5$ of composite SISD (5.8)-(5.9) is exponentially stable in the m.s. The simulation result is shown in Figure 5.2.

Reconsider the control composite continuous system (5.4) with unstable state subsystem where

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & -10 \end{pmatrix}.$$

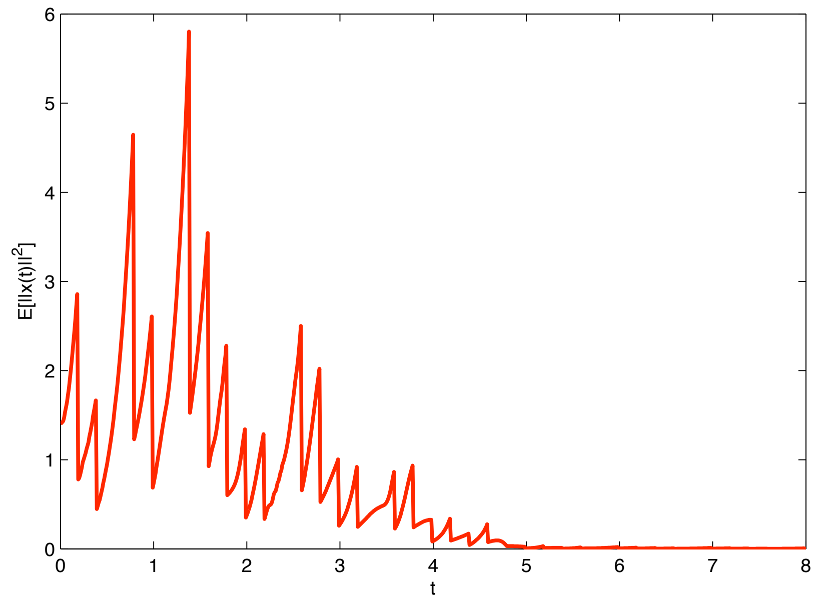

Figure 5.3: Mean square asymptotic stability of $(x, y)^T = (0, 0)$.

Following the same analysis, we obtain $\mathcal{L}_1 V^1(x) \leq (10 + 0.0001\bar{q})V^1(x)$; that is, the state isolated subsystem is unstable, while $\mathcal{L}_2 V^2(y) \leq -9.9998V^2(y)$. It follows that the composite system is unstable where $h(V(x, y), u) = 6.0005V(x, y) > 0$. Considering the stabilizing impulsive effects in (5.9) gives $\tau_k - \tau_{k-1} \leq 0.3$. Figure 5.3 shows the simulation result.

5.2.2 Vector Lyapunov Function Approach

In this subsection, we continue to develop a comparison principle for composite large scale SISD (5.3), where we use a *vector* Lyapunov function having components which are Lyapunov functions related to the isolated subsystems and the finding of Theorem 5.3 will be carried over to every individual subsystems. In other words, the comparison occurs between a vector of differential inequalities and a vector of differential equations whose solutions are known and enjoy some stability properties. As done in Section 5.1, for convenient theorem statement, we define Property B.

Definition 5.2. The isolated subsystem $\mathbb{S}_i$ (5.2) is said to possess **Property B** if Assumptions A1 and A2 hold, there exist functions $c_i \in \mathcal{K}_1$, and a_i which satisfies the conditions of $\bar{h}$ in Theorem 5.3, and $V^i \in \mathcal{C}^{1,2}\big([-r,\infty) \times S(\varrho)); \mathbb{R}_+\big)$, which is decrescent and satisfies

(i) for all $(t, \psi^i(0)) \in [-r,\infty) \times S(\varrho)$,

$$c_i(\|\psi^i(0)\|^2) \leq V^i(t, \psi^i(0)), \qquad \text{(a.s.)};$$

and, for all $t \neq \tau_k$ in $\mathbb{R}_+$ and $\psi^i \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$,

$$\mathcal{L}_i V^i(t, \psi^i) \leq a_i(t, V^i(t, \psi^i(0))), \qquad \text{(a.s.)},$$

provided that $V^i(t+s, \psi^i(s)) \leq \bar{q} V(t, \psi^i(0))$ for some $\bar{q} > 1$ and $s \in [-r, 0]$;

(ii) for any $\tau_k \in \mathbb{T}$ and $\psi^i \in \mathcal{PC}\big([t_0 - r,\infty); S(\varrho)\big)$,

$$V^i\big(\tau_k, \psi^i(0) + \mathcal{I}_i(\tau_k, \psi^i(\tau_k^-))\big) \leq \alpha(d_k) V^i(\tau_k^-, \psi^i(0)), \qquad \text{(a.s.)},$$

where $\psi^i(0^-) = \psi^i(0)$ and $\prod_{k=1}^{\infty} \alpha_k(d_k)$ with $\alpha(d_k) > 1$ for all k.

Definition 5.3. A function $g(t, u)$ (or $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$) is said to be *quasi monotone nondecreasing in* u if, for any $u, v \in \mathbb{R}^n$ such that $0 \leq u_j < v_j$ for all $i \neq j$ and $0 \leq u_i = v_i$, we have $g(t, u) < g(t, v)$ for any fixed t in $\mathbb{R}_+$.

In the following theorems, we state and prove a comparison principle and stability results for composite system (5.3).

Theorem 5.4. Assume that the following assumptions hold:

(i) every isolated subsystem $\mathbb{S}_i$ has Property B;

(ii) for any $i = 1, 2, \cdots, l$, there exist a function $\bar{b}_i(t, u) \in \mathcal{C}\big([\tau_{k-1}, \tau_k) \times \mathbb{R}_+; \mathbb{R}\big)$ and $\bar{b}_i$ is quasi monotone nondecreasing in u such that

$$g_i^T(t,\psi)V^i_{\psi^i(0)}(t,\psi^i(0)) + \frac{1}{2}\sum_{j=1, i\neq j}^{l} \mathrm{tr}\big[\sigma_{ij}^T(t,\psi^j)V^i_{\psi^i(0)\psi^i(0)}(t,\psi^i(0))\sigma_{ij}(t,\psi^j)\big]$$
$$< \bar{b}_i(t, V(t,\psi(0))),$$

where $V^T(t,x) = \big(V^1(t,w^1), V^2(t,w^2), \cdots, V^l(t,w^l)\big)$;

(iii) let $a^T(\cdot) = \big(a_1(\cdot), a_2(\cdot), \cdots, a_l(\cdot)\big) \in \mathcal{L}_{ad}\big(\Omega, L[t_0, t_0+\alpha]\big)$ and $\bar{b}^T(\cdot) = \big(\bar{b}_1(\cdot), \bar{b}_2(\cdot), \cdots, \bar{b}_l(\cdot)\big) \in \mathcal{L}_{ad}\big(\Omega, L^2[t_0, t_0+\alpha]\big)$, where $a_i(\cdot)$ and $\bar{b}_i(\cdot)$ are defined in assumptions (i) and (ii), respectively, and assume that the following inequalities hold

$$|a(t,v') + \bar{b}(t,v')|^2 \leq h_1(t) + h_2(t)\kappa(\|v'\|^2),$$
$$|a(t,v') + \bar{b}(t,v') - a(t,v'') - \bar{b}(t,v'')| \leq K\|v' - v''\|,$$

where $t \in \mathbb{R}_+$, h_1, h_2 are Borel measurable functions (or $\mathcal{PC}(\mathbb{R}_+, \mathbb{R}_+)$ functions), $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, increasing, concave function, v', $v'' \in \mathbb{R}^l_+$, and $K > 0$;

(iv) there exists an adapted function $p : \mathbb{R}^l \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\sup_{V(t,x)\leq v} \sum_{i,j=1}^{l} \|\sigma_{ij}^T(t,\psi^j)V_{\psi^i(0)^i(t,\psi^i(0))}\|^2 \leq p(t,v),$$

where

$$p(t,v) \leq h_1(t) + h_2(t)\kappa(\|v\|^2).$$

Then, provided that $V(t_0, x_0) < v_0$, $V(t, x(t)) < v(t)$, for all $t \geq t_0$, where $v = (v^1, v^2, \cdots, v^l)^T$ (i.e., $v \in \mathbb{R}^l$) is a solution of the vector stochastic impulsive differential equation

$$\begin{cases} dv = \big(a(t, v) + \bar{b}(t, v)\big)dt + \mathbb{V}dW(t), & t \neq \tau_k, \\ \Delta v(t) = \alpha_M(d_k)v(t^-), & t = \tau_k, \end{cases} \tag{5.10}$$

with $\mathbb{V} = [v_{ij}]_{l\times l}$ being a matrix random process such that

$$\|\mathbb{V}\|^2 \leq p(t, v),$$

and $\alpha_M(\cdot) = \max_i\{\alpha_i(\cdot); i = 1, 2, \cdots, l\}$.

Proof. Let x be the solution of impulsive system (5.3). Define

$$V^T(t, x(t)) = \big(V^1(t, w^1), V^2(t, w^2), \cdots, V^l(t, w^l)\big)$$

as a vector Lyapunov function candidate for the composite system, where V^i is the Lyapunov function related to the i^{th} isolated subsystem $\mathbb{S}_i$. Then, by the vector form of Itô formula, we have

$$dV^T(t, x(t)) = \big(dV^1(t, w^1), dV^2(t, w^2), \cdots, dV^l(t, w^l)\big),$$

where

$$dV^i(t, w^i) < \Big(a_i(t, V^i(t, w^i)) + \bar{b}_i(t, V^i(t, w^i))\Big)dt + \sum_{ij}^{l} v_{ij}dW_i(t),$$

where $v_{ij} = V^{iT}_{w^i}(t, w^i)\sigma_{ij}(t, w^j_t)$. It follows that the vector differential inequality is

$$dV(t, x(t)) < \Big(a(t, V(t, x(t))) + \bar{b}(t, V(t, x(t)))\Big)dt + \mathbb{V}dW(t),$$

for any $t \in [\tau_{k-1}, \tau_k)$, $k = 1, 2, \cdots$.

At the impulsive moments $t = \tau_k$, we have

$$
\begin{aligned}
&V^T(\tau_k, x(\tau_k)) \\
&\quad = \big(V^1(\tau_k, w^1(\tau_k)), V^2(t, w^2(\tau_k)), \cdots, V^l(t, w^l(\tau_k))\big) \\
&\quad \leq \Big(\alpha_1(d_k)V^1(\tau_k, w^1(\tau_k^-)), \alpha_2(d_k)V^2(\tau_k, w^2(\tau_k^-)), \cdots, \alpha_l(d_k)V^l(\tau_k, w^l(\tau_k^-))\Big) \\
&\quad \leq \alpha_M(d_k)\big(V^1(\tau_k^-, w^1(\tau_k^-)), V^2(\tau_k^-, w^2(\tau_k^-)), \cdots, V^l(\tau_k^-, w^l(\tau_k^-))\big) \\
&\quad = \alpha_M(d_k)V^T(\tau_k^-, x(\tau_k^-)).
\end{aligned}
$$

Particularly, for $t \in [\tau_0, \tau_1)$, we have $V^i(t_0, w^i(t_0)) < v_0$ and

$$
dV^i(t, w^i) - dv_i < \Big\{[a_i(t, V^i(t, w^i)) - a_i(t, v_i)] + [\bar{b}_i(t, V(t, x(t))) - \bar{b}_i(t, v(t))]\Big\}dt.
$$

Since the composite system satisfies the existence-uniqueness conditions, $V(t, x(t))$ is a continuous process w.p.1 for all $[\tau_0, \tau_1)$. Similar conclusion can be drawn for the process $v(t)$. Therefore, to ensure that, given $V(t_0, x_0) < v_0$, $V(t, x(t)) < v(t)$ w.p.1 for all $[\tau_0, \tau_1)$, it suffices to show that $dV^i(t, w^i) - dv^i(t) < 0$ whenever $V^i(t, w^i) = y^i(t)$. But this inequality is true because $\bar{b}_i$ is quasi monotone non-decreasing. Thus, we obtain that $V^i(t, w^i(t)) < v_i(t)$ for all $t \in [\tau_0, \tau_1)$, and at the impulsive moment τ_1, we have

$$
V^i\big(\tau_1, w^i(\tau_1)\big) - v_i(\tau_1) < \alpha_M(d_k)\Big[V^i(\tau_1^-, w^i(\tau_1^-)) - v_i(\tau_1^-)\Big] < 0,
$$

i.e.,

$$
V^i(\tau_1, w^i(\tau_1)) < v_i(\tau_1).
$$

Similarly, for any $k = 2, 3, \cdots$ and $t \in [\tau_{k-1}, \tau_k)$, we get $V^i(t, w^i(t)) < v_i(t)$ and at $t = \tau_k$, $V^i(\tau_k, w^i(\tau_k)) < v_i(\tau_k)$. Therefore, for all $t \geq t_0$ and $i = 1, 2, \cdots, l$, $V_i(t, w^i(t)) < v_i(t)$, from which we get the vector inequality

$$
V(t, x(t)) < v(t), \qquad \forall\, t \geq t_0.
$$

This completes the proof.

Theorem 5.5. Suppose that the assumptions of Theorem 5.4 hold, and there exist class-$\mathcal{K}_1$ functions α_1 and c, a function $\bar{h} \in \mathcal{C}\big([\tau_k, \tau_{k-1}) \times \mathbb{R}^l; \mathbb{R}_+\big)$, $z \in \mathbb{R}^l$, and $U \in \mathcal{C}^{1,2}\big([\tau_{k-1}, \tau_k) \times \mathbb{R}^l; \mathbb{R}_+\big)$, which is decrescent, $U(t, 0) = 0$, and satisfies

(i) for all $t \in \mathbb{R}_+$ and $v \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^l)$,

$$\alpha_1(\|v\|^2) \leq U(t, v), \qquad \text{(a.s.)},$$
$$z^T U_{vv}(t, v) z \leq \bar{h}(t, v)\|z\|^2, \qquad \text{(a.s.)},$$

and

$$U_t(t, v) + U_v(t, v)\big[a(t, v) + \bar{b}(t, v)\big] + \frac{1}{2}h(t, v)p(t, v) \leq -c(\|v\|), \qquad \text{(a.s.)}, ;$$

(ii) for any $\tau_k \in \mathbb{T}$ and $v \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^l)$,

$$U(\tau_k, v(\tau_k)) = \alpha(d_k) U(\tau_k^-, v(\tau_k^-)).$$

Then, comparison system (5.10) and, hence, composite SISD (5.3) have asymptotically stable trivial solutions in the m.s.

Proof. Let $v \geq 0$ be the solution vector of the comparison system (5.10). Apply Itô formula to process U to get

$$\mathcal{L}U(t, v) \leq -c(\|v\|),$$

which shows that, by the previous analysis, (5.10) has the desired stability property. As for composite system (5.3), we have shown in Theorem 5.4 that the vector inequality $V(t, x(t)) < v(t)$ holds for all $t \geq t_0$. It follows that

$$\alpha_1(\|x(t)\|^2) \leq \Big[\sum_{i=1}^{l} c_i^2(\|w^i\|^2)\Big]^{1/2} \leq \|V(t, x(t))\| < \|v(t)\|,$$

where $c \in \mathcal{K}_1$. Taking the mathematical expectation and then applying α_1^{-1} to both sides imply the desired result, i.e., $\mathbb{E}[\|x(t)\|^2] \leq \alpha_1^{-1}(\mathbb{E}[\|v(t)\|^2])$ for all $t \geq t_0$. This completes the proof.

Corollary 5.3. In Theorem 5.5, assume that there exists a positive constant c such that $c(s) = c\,s$ for all $s > 0$ and

$$\beta^T\big(a(t,v) + \bar{b}(t,v)\big) \leq -c\|v\|,$$

for some positive vector $\beta \in \mathbb{R}^l$. Then, system (5.10) possesses the same stability property.

Proof. Let $U(t,v) = \beta^T v > 0$ be a Lyapunov function candidate. Then, $U_v = \beta^T$ and $U_{vv} = 0 \in \mathbb{R}^{l\times l}$, from which $\mathcal{L}U(t,v) \leq -c\|v\|$. Applying the impulsive effects yields the desired result.

Example 5.3. Consider the composite system in (5.4) and the same Lyapunov functions. We have found $\mathcal{L}_1 V^1(x) \leq \sigma_1 V^1(x)$ and $\mathcal{L}_2 V^2(x) \leq \sigma_2 V^2(x)$, from which we get $a(V(x,y))^T = (a_1(V^1(x)), a_2(V^2(y))) = (\sigma_1 V^1(x), \sigma_2 V^2(y))$. From the interconnection, we have found $\bar{b}(V(x,y))^T = \big((2k{+}0.0001)V(x,y), 2.0001V(x,y)\big)$. Clearly, the functions a and b satisfy the conditions in (iii) of Theorem 5.4. As for the condition (iv), we have

$$\begin{aligned}
&\sup_{V\leq v}\sum_{i,j=1}^{l}\|\sigma_{ij}^T(w^i)V_{w^i}(w^i)^i\|^2\\
&\qquad= \sup_{V\leq v}\Big[\|\sigma_{11}^T(x(t-1))V_x^1(x)\|^2 + \|\sigma_{12}^T(y))V_x^1(x)\|^2\|\sigma_{21}^T(x)V_y^2(x)\|^2\\
&\qquad\qquad + \|\sigma_{22}^T(y(t-1)))V_y^2(x)\|^2\Big]\\
&\qquad\leq 4\sup_{V\leq v}\Big[\xi_1(V^1(x))^2 + 0.0004V^1(x)V^2(y) + \xi_2(V^2(y))^2\Big]
\end{aligned}$$

$$\leq 4 \sup_{V \leq v} \left[\xi_1 v_1^2 + 0.0004 v_1 v_2 + \xi_2 v_2^2 \right]$$

$$\leq 8\bar{\xi}\|v\|^2,$$

that is, $p(v) \leq 8\bar{\xi}\|v\|^2$, where $\bar{\xi} = \max\{\xi_1, \xi_2\}$, $\xi_1 = 1.0004$ and $\xi_2 = 1.0002$ with $\bar{q} = 2$.

Making use of the impulsive effect given in Example 5.1, we get

$$\begin{aligned} V^T(x(\tau_k), y(\tau_k)) &= (V^1 x(\tau_k)), V^2(y(\tau_k)) \\ &\leq (1 + \frac{1}{k^2})(V^1(x(\tau_k^-)), V^2(y(\tau_k^-))) = (1 + \frac{1}{k^2}) V^T(x(\tau_k^-), y(\tau_k^-)) \\ &\leq (1 + \frac{1}{k^2}) v^T(\tau_k^-) = v^T(\tau_k). \end{aligned}$$

Thus, by Theorem 5.4, $V(x(t), y(t)) \leq v(t)$, for all $t \geq t_0$.

As for the stability result, choose $U(v) = v_1 + v_2$, i.e., $\beta^T = (1, 1)$. It is easy to show that $\mathcal{L}U(v) \leq -5.9997U(v)$, where we have chosen $k = 2$. Also, $U(v(\tau_k)) = \alpha_M(d_k) U(v(\tau_k^-))$, where $\alpha_M(d_k) = 1 + \frac{1}{k^2}$. Therefore, the trivial solution of composite system (5.4) is asymptotically stable in the m.s.

5.3 Conclusion and Comments

In this chapter, we considered large scale SISD with fixed impulses. Our interest was to establish some qualitative properties by decomposing the system into smaller isolated subsystems and the rest that was treated as system perturbation. Assuming that the subsystems have asymptotic stable trivial solutions in the m.s., and the perturbation is estimated by an upper bound, which is smaller than the stability margin of the individual subsystems, we were able to conclude that the interconnected SISD has trivial solution that is asymptotically stable in the m.s.

Most of the stability results formulated in this chapter were based on our findings obtained in Chapter 4. Namely, we developed some standard Lyapunov theorems (Section 5.1) and established a comparison principle using scalar and vector Lyapunov functions (Section 5.2). We also showed that impulses can stabilize some unstable continuous systems. To demonstrate the effectiveness of the proposed theoretical results, we discussed the stability and stabilization problems of an automated indirect control system. Along the line of some of the proofs adopted here, one may consult [Mich77].

Chapter 6

Input-to-State Stability of Ordinary Differential Equations

This chapter deals with the concept of input-to-state (IS) stability of nonlinear ordinary differential equations. Throughout the relevant literature on IS stability, a common practice in proving the IS stability notion of a system is to define a function that is positive definite everywhere and vanishes at the equilibrium point (of the system). Furthermore, this function strictly decreases along the solution trajectories of the system whenever the solution magnitude is larger than some positive, increasing function depending on the input and vanishes at zero input. In this case, the system response decreases over a certain time period and eventually it lingers on at an ultimate bound depending on the input. These sufficient conditions were adopted to define the concept of IS stability [Son89, Son02]. Clearly, if the input is zero, the IS stability becomes asymptotic stability of the equilibrium point.

To have a better insight into the system behaviour, we propose another approach based on the parameters $(\varepsilon^u, \delta^u)$ to define and prove the IS stability of ordinary systems. We are mainly interested in developing Lyapunov-like sufficient conditions to establish asymptotic IS stability, i.e., the system response is stable and attracted in the presence of input. This property also implies that, if the input is zero, the equilibrium point becomes stable and attractive. To show the effectiveness of this approach, we apply the results to a recursive (or cascade) system. We will also develop comparison theorems to establish the same qualitative results. The material of this chapter forms the basis of [Alw-d].

Consider the nonlinear system

$$\dot{x} = f(t, x, u), \tag{6.1a}$$

$$x(t_0) = x_0, \tag{6.1b}$$

where $x \in \mathbb{R}^n$ is the system state, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the vector field which is piecewise continuous, $f(t, 0, 0) = 0$ for all $t \geq t_0$ with $t_0 \in \mathbb{R}_+$, and f is locally Lipschitz in x and u. We assume that the input $u : \mathbb{R}_+ \to \mathbb{R}^m$ is continuous or piecewise continuous and bounded for all $t \in \mathbb{R}_+$. We also assume that the unforced system

$$\dot{x} = f(t, x, 0), \tag{6.2a}$$

$$x(t_0) = x_0, \tag{6.2b}$$

has a trivial solution that is globally uniformly asymptotically stable (g.u.a.s.).

In the following, we give some definitions that will be used later.

Definition 6.1. Let $x(t) = x(t, t_0, x_0)$ be a solution of (6.1) and $\rho \in \mathcal{K}$. System (6.1) is said to be

(i) *input-to-state stable* (IS stable) with a gain ρ if, for every $\varepsilon^u > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta^u = \delta^u(t_0, \varepsilon^u) > 0$ such that

$$\rho(\|u\|) \le \|x_0\| \le \delta^u \qquad \text{implies} \qquad \rho(\|u\|) \le \|x(t)\| \le \varepsilon^u, \qquad \forall t \ge t_0;$$

(ii) *uniformly IS stable* with a gain ρ if δ^u in (i) is independent of t_0;

(iii) *input-to-state attractive* with a gain ρ if, for any $\eta^u > 0$ and $t_0 \in \mathbb{R}_+$, there exist $\delta^u > 0$ and $T^u = T^u(t_0, \eta^u)$ such that

$$\rho(\|u\|) \le \|x_0\| \le \delta^u \quad \text{implies} \quad \rho(\|u\|) \le \|x(t)\| \le \eta^u, \quad \forall t \ge t_0 + T^u;$$

(iv) *uniformly IS attractive* with a gain ρ if T^u in (iii) is independent of t_0;

(v) *uniformly asymptotic input-to-state stable* (aIS stable) with a gain ρ if (ii) and (iv) hold;

(vi) *exponentially input-to-state* (eIS) stable with a gain ρ if (v) holds, and, moreover, there exist two positive constants K and λ such that

$$\|x(t)\| \le K\|x_0\|e^{-\lambda(t-t_0)}, \qquad \forall\, t \ge t_0. \tag{6.3}$$

6.1 Analysis by $(\varepsilon^u, \delta^u)$ Approach

In this section, we state and prove some Lyapunov-type theorems regarding the uniform properties of IS stability and aIS stability.

Theorem 6.1. Let $V \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$. Suppose there exist class-$\mathcal{K}$ functions a, b, and ρ such that

(i) $b(\|x\|) \le V(t,x) \le a(\|x\|), \qquad$ for all $(t,x) \in \mathbb{R}_+ \times S(\varrho)$;

(ii) $\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \leq 0, \qquad \text{whenever } \|x\| \geq \rho(\|u\|),$

for all $(t,x,u) \in \mathbb{R}_+ \times S(\varrho) \times \mathbb{R}_+$.

Then, system (6.1) is uniformly IS stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be the solution of system (6.1). For a given $\varepsilon^u \in (0, \varrho)$ and any $t_0 \in \mathbb{R}_+$, choose $\delta^u = \delta^u(t_0, \varepsilon^u) > 0$ such that $\delta^u < a^{-1}(b(\varepsilon^u))$, which implies $0 < \delta^u < \varepsilon^u$.

We claim that our supposition guarantees that system (6.1) is IS stable. If this were not true, there would exist $t^* > t_0$ such that

$$\rho(\|u\|) \leq \varepsilon^u = \|x(t^*)\|, \tag{6.4}$$

and

$$\rho(\|u\|) \leq \|x(t)\| < \varepsilon^u, \qquad \text{for all } t \in [t_0, t^*).$$

From (ii), $V(t, x(t)) \leq V(t_0, x_0)$ for all $t \in [t_0, t^*]$. Define $m(t) = V(t, x(t))$ over $[t_0, t^*]$. Then, we have

$$m(t^*) \leq m(t_0) \leq a(\|x_0\|) \leq a(\delta^u) < b(\varepsilon^u)$$

by our choice of δ^u. On the other hand, by (6.4), we have

$$\bar{\rho}(\|u\|) \leq b(\varepsilon^u) = b(\|x(t^*)\|) \leq m(t^*), \qquad \bar{\rho} = b \circ \rho,$$

where $b \circ \rho$ stands for the composite function of b and ρ, i.e., $b \circ \rho(\cdot) = b(\rho(\cdot))$. The last inequality implies that

$$\rho(\|u\|) \leq \varepsilon^u < \varepsilon^u,$$

which is a contradiction. We conclude that the system is IS stable. To prove the uniformity property, it is sufficient to choose $\delta^u = \delta^u(\varepsilon^u) > 0$. This completes the proof.

Theorem 6.2. Assume that the assumptions of Theorem 6.1 hold except that the condition in (ii) is replaced by

$$\dot{V}(t,x) \leq -c(\|x\|), \qquad \text{whenever } \|x\| \geq \rho(\|u\|),$$

for all $(t,x,u) \in \mathbb{R}_+ \times S(\varrho) \times \mathbb{R}^m$, where $c \in \mathcal{K}$. Then, system (6.1) is uniformly aIS stable. If, in addition, $\varrho = \infty$ and $b(s) \to \infty$ as $s \to \infty$, the system satisfies the stability property globally.

Proof. Let $x(t)$ be the solution of (6.1). Obviously, the system is uniformly IS stable, i.e., there exists $\delta_0^u > 0$ such that, for a given $\varrho > 0$ and $t_0 \in \mathbb{R}_+$,

$$\rho(\|u\|) \leq \|x_0\| < \delta_0^u \qquad \text{implies} \qquad \rho(\|u\|) \leq \|x(t)\| < \varrho, \qquad \forall\, t \geq t_0.$$

Let $\varepsilon^u \in (0, \varrho)$ and choose $\delta^u(\varepsilon^u)$ as in Theorem 6.1. Choose $T^u = T(\varepsilon^u)$ such that $T^u = \frac{a(\delta_0^u)}{c(\eta^u)} + 1$ for some η^u.

We claim that, by the given information, the system satisfies the desired stability property, which implies the existence of a $t^* \in [t_0, t_0 + T^u]$ such that

$$\rho(\|u\|) \leq \|x(t^*)\| < \eta^u.$$

If this were not the case, then for all $t \in [t_0, t_0 + T^u]$,

$$\rho(\|u\|) \leq \eta^u \leq \|x(t)\| < \varrho.$$

Define $m(t) = V(t, x(t))$ over $[t_0, t_0 + T^u]$. Our assumption implies that

$$\begin{aligned} m(t_0 + T^u) &= m(t_0) - \int_{t_0}^{t_0+T^u} c(\|x(t)\|)dt \\ &\leq a(\|x_0\|) - c(\eta^u)T^u \\ &\leq a(\delta_0^u) - c(\eta^u)\Big(\frac{a(\delta_0^u)}{c(\eta^u)} + 1\Big) = -c(\eta^u) < 0, \end{aligned}$$

which is a contradiction. Therefore, it must be true that there is a $t^* \in [t_0, t_0+T^u]$ at which $\rho(\|u(t)\|) \le \|x(t^*)\| < \eta^u$. This conclusion, together with uniform property, proves the desired result.

In Theorem 6.2, if $c(s) = cs$, for some real number c and all $s > 0$, $a(s) = as$, and $b(s) = bs$, for some positive constants a, b, and all $s > 0$, the result reduces to exponential IS stability. Also, another immediate special result is when $u(t) \equiv 0$ for all $t \in \mathbb{R}_+$; that is, the result reduces to the classical uniform asymptotic stability of the trivial solution $x \equiv 0$.

6.2 Analysis by Comparison Principle

In this section, we continue to establish the same IS stability properties of system (6.1) by using a comparison principle. We start with comparing a solution of a system of differential inequality with a maximal solution of an auxiliary system of differential equations. Later, assuming that the auxiliary system enjoys some IS stability properties, we will be able to conclude the corresponding properties of the original system (6.1).

Theorem 6.3. Assume that system (6.1) has a unique solution x, and there exists a class-$\mathcal{K}$ function a. Let $V \in \mathcal{C}^1(\mathbb{R}_+ \times S(\varrho); \mathbb{R}_+)$ for some positive constant ϱ such that

(i) $\dot{V}(t,x) \le h(t,V,u)$, for all $(t,V,u) \in \mathbb{R}_+^2 \times \mathbb{R}^m$,

where $h : \mathbb{R}_+^2 \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function on its domain;

(ii) the scalar comparison system

$$\begin{cases} \dot{v} = h(t, v, u), \\ v(t_0) = v_0 \geq 0 \end{cases} \tag{6.5}$$

has a maximal solution $r(t) = r(t, t_0, v_0)$ for all $t \geq t_0$.

Then, $V(t, x) < r(t)$ for all $t \geq t_0$ whenever $V(t_0, x_0) < v_0$.

Proof. Let $x(t)$ be a unique solution of system (6.1). Define $m(t) = V(t, x(t))$ over $\mathbb{R}_+$, then from (i), we have

$$\dot{m}(t) \leq h(t, m(t), u(t)), \qquad \text{for all } t \geq t_0.$$

Given $m(t_0) = V(t_0, x_0) < v_0$, we claim that $m(t) < r(t)$ for all $t > t_0$. If this were not true, there would exist a $t^* > t_0$ such that

$$m(t^*) = r(t^*) = v(t^*), \quad \text{and} \quad m(t) < r(t) = v(t), \quad \text{for all} \quad t \in [t_0, t^*).$$

This implies that

$$\begin{aligned} \dot{m}(t^*) &> \dot{v}(t^*) \\ &= h(t^*, v(t^*), u(t^*)) \\ &= h(t^*, m(t^*), u(t^*)) \\ &\geq \dot{m}(t^*), \end{aligned}$$

which is a contradiction. This completes the proof.

Theorem 6.4. Suppose that system (6.1) has a unique solution, and there exist class-$\mathcal{K}$ functions a and b. Let $V \in \mathcal{C}^1(\mathbb{R}_+ \times S(\varrho); \mathbb{R}_+)$ such that

(i) $b(\|x\|) \leq V(t, x) \leq a(\|x\|), \qquad$ for all $(t, x) \in \mathbb{R}_+ \times S(\varrho)$;

(ii) $\dot{V}(t,x) \le h(t,V,u), \qquad \text{for all } (t,V,u) \in \mathbb{R}_+^2 \times \mathbb{R}^m,$

where $h : \mathbb{R}_+^2 \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function on its domain, and $h(t,0,0) = 0$.

Then, IS stability properties of comparison system (6.5) imply the corresponding properties of system (6.1).

Proof. Let $\varepsilon^u > 0$ and $t_0 \in \mathbb{R}_+$. Assume that the comparison system is IS stable. Then, there is a $\delta^{v,u} = \delta^{v,u}(t_0, \varepsilon^u)$ such that

$$0 < \rho(\|u\|) \le v_0 < \delta^{v,u} \quad \text{implies} \quad \rho(\|u\|) \le v(t,t_0,v_0) < b(\varepsilon), \quad \forall\, t \ge t_0,$$

where $v(t,t_0,v_0)$ is any solution of the comparison system and $\rho \in \mathcal{K}$.

Choose $v_0 = a(\|x_0\|)$ and δ^* such that $a(\delta^*) < b(\varepsilon^u)$. Define $\delta^{x,u} = \min\{\delta^{v,u}, \delta^*\}$. We claim that, if

$$\rho(\|u\|) \le \|x_0\| < \delta^{x,u}, \quad \text{then} \quad \rho(\|u\|) \le \|x(t)\| < \varepsilon^u, \qquad \forall\, t \ge t_0.$$

If our claim were not true, there would be a $t^* > t_0$ such that

$$\rho(\|u\|) \le \varepsilon^u \le \|x(t^*)\|, \quad \text{and} \quad \rho(\|u\|) \le \|x(t)\| < \varepsilon^u, \quad \forall\, t \in [t_0, t^*).$$

Define $m(t) = V(t, x(t))$ over $[t_0, t^*]$. Then, by Theorem 6.3, we have

$$m(t) < r(t) = v(t, v_0, a(\|x_0\|)), \qquad \forall\, t \in [t_0, t^*],$$

and

$$\bar{\rho}(\|u\|) \le b(\varepsilon^u) \le b(\|x(t^*)\|) \le m(t^*) < v(t^*, t_0, a(\|x_0\|)) \le v(t^*, t_0, a(\delta^{x,u})) < b(\varepsilon^u),$$

with $\bar{\rho} = b \circ \rho$, which is a contradiction. Therefore, it must be true that

$$\rho(\|u\|) \le \|x(t)\| < \varepsilon^u, \qquad \forall\, t \ge t_0.$$

To prove the uniform property, it suffices to choose $\delta^{v,u}$ independent of t_0.

As for the IS attractive property, we choose $\eta^u > 0$ and assume that comparison system (6.5) is IS attractive, i.e., for a given $b(\eta^u) > 0$, there exist a $\delta^u > 0$ and $T^u = T^u(\eta^u) > 0$ such that

$$\rho(\|u\|) \leq v_0 < \delta^u \quad \text{implies} \quad \rho(\|u\|) \leq v(t, t_0, v_0) < b(\eta^u), \quad \forall\, t \geq t_0 + T^u,$$

from which we obtain

$$\bar{\rho}(\|u\|) \leq b(\|x(t)\|) \leq m(t) \leq v(t, t_0, v_0) < b(\eta^u), \qquad \forall\, t \geq t_0 + T^u,$$

i.e., system (6.1) is IS attractive. Thus, the system is uniformly aIS stable. This completes the proof.

In the following corollary, we consider two special cases of Theorem 6.4.

Corollary 6.1. Assume that the assumptions of Theorem 6.4 hold.

(i) If there exist class-$\mathcal{K}$ functions c and γ such that

$$h(t, V(t, x(t)), u(t)) \leq -c(V(t, x(t))) + \gamma(\|u\|),$$

then system (6.1) is uniformly aIS stable;

(ii) if there exist positive constants a, b, and c such that $a(s) = as$, $b(s) = bs$, and $c(s) = cs$ for all $s > 0$, then system (6.1) is eIS stable.

Proof. **(i)** From the given condition, we have

$$\dot{m}(t) \leq -c(m(t)) + \gamma\big(\|u\|\big),$$

where $m(t) = V(t, x(t))$, or

$$\dot{m}(t) \leq -\bar{c}(m(t)),$$

whenever $c(m(t)) \geq \frac{1}{\theta}\gamma\big(\|u\|\big)$, for some $\theta \in (0,1)$, where $\bar{c}(\cdot) = (1-\theta)c(\cdot)$, which implies that $m(t) \geq c^{-1}\Big(\frac{1}{\theta}\gamma\big(\|u\|\big)\Big) =: \bar{\rho}\big(\|u\|\big)$. Therefore, the differential inequality may be compared with

$$\dot{v}(t) = -\bar{c}(v(t)),$$

whenever $v(t) \geq m(t) \geq \bar{\rho}\big(\|u\|\big)$. By the classical stability theorems for nonlinear systems, for a given $\eta^v > 0$ and $t_0 \in \mathbb{R}_+$, there exist a $\delta^v > 0$ and $T^v = T^v(\eta^v) > 0$ such that

$$v(t) \leq \eta^v, \qquad \forall\, t \geq t_0 + T^v,$$

whenever $v(t) \geq \bar{\rho}\big(\|u\|\big)$, or

$$\bar{\rho}\big(\|u\|\big) \leq v(t) \leq \eta^v, \qquad \forall\, t \geq t_0 + T^v.$$

That is, comparison system (6.5) is uniformly aIS stable. Hence, by Theorem 6.4, system (6.1) has the same stability property.

(ii) We have $\dot{m}(t) \leq -\bar{c}\, m(t)$ whenever $m(t) \geq \frac{1}{c\,\theta}\gamma\big(\|u\|\big)$. This implies that

$$\frac{1}{c\,\theta}\gamma\big(\|u\|\big) \leq m(t) \leq v(t) \leq K v_0 e^{-\bar{c}(t-t_0)},$$

i.e., the comparison system is eIS stable, which implies the desired result.

6.3 Application: Cascade Systems

To demonstrate the applicability of the proposed result, we consider the following cascade system

$$\dot{x} = f(t,x,y), \qquad x(t_0) = x_0, \tag{6.6}$$

$$\dot{y} = g(t,y), \qquad y(t_0) = y_0, \tag{6.7}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$. The question of interest is that, under what conditions on (6.6) and (6.7), the cascade system has globally uniformly asymptotically stable (g.u.a.s.) equilibrium point? The following theorem reveals the answer.

Theorem 6.5. Consider the cascade system (6.6)-(6.7). Suppose that the trivial solution $y \equiv 0$ is g.u.a.s. If (6.6) is aIS stable with y being viewed as an input, then the trivial solution $z^T = (x, y) \equiv (0, 0)$ is g.u.a.s..

Proof. Assume that $y \equiv 0$ is g.u.a.s.; that is, for a given $\eta^y > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $T^y = T^y(\eta^y)$ such that

$$\|y(t)\| \leq \eta^y, \qquad \forall\, t \geq t_0 + T^y, \tag{6.8}$$

and, since the system (6.6) is aIS stable, we have

$$\|x(t)\| \leq \gamma\big(\sup_t y(t)\big), \qquad \forall\, t \geq t_0 + T^x, \tag{6.9}$$

where $\gamma \in \mathcal{K}$ and $T^x > T^y$, or

$$\|x(t)\| \geq \rho\big(\sup_t y(t)\big), \qquad \forall\, t \geq t_0 + T^x, \tag{6.10}$$

where $\rho \in \mathcal{K}$, and

$$\rho\big(\sup_t y(t)\big) \leq \|x(t)\| \leq \eta^x, \qquad \forall\, t \geq t_0 + T^x, \tag{6.11}$$

for any $\eta^x > 0$. Therefore, by (6.8) and (6.11), we obtain, for $t \geq t_0 + T^x$,

$$\rho\big(\sup_{t \geq t_0 + T^x} y(t)\big) \leq \|x(t)\| \leq \|z(t)\| \leq \|x(t)\| + \|y(t)\| \leq \eta^x + \eta^y. \tag{6.12}$$

Choosing $\eta^x = \rho\big(\sup_{t \geq t_0 + T^x} y(t)\big)$ implies that

$$\|z(t)\| \leq \eta^y, \qquad \forall\, t \geq t_0 + T^x,$$

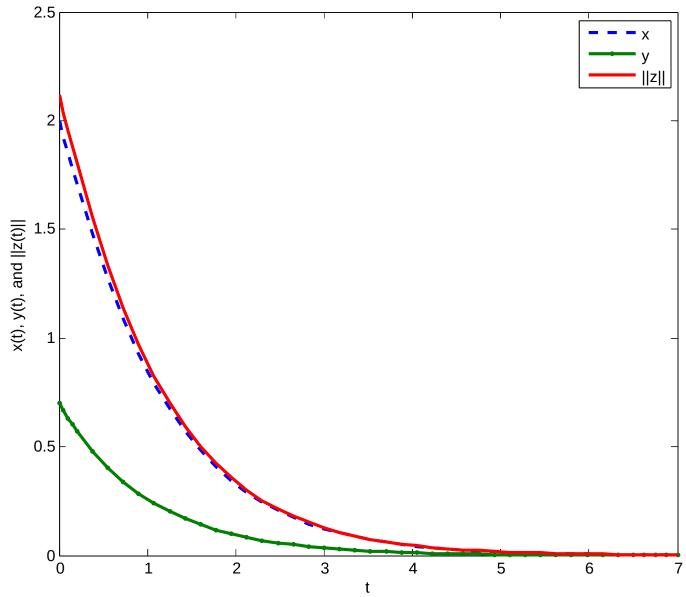

Figure 6.1: Simulation results of the cascade system.

i.e., $z^T = (0, 0)$ is uniformly asymptotically stable. This completes the proof.

The following example elaborates this result.

Example 6.1. Consider again the cascade system (6.6)-(6.7) with the following specific vector fields: $f(x, y) = -(x + y^2)$ and $g(y) = -2y$, and the initial states $x(0) = 2$ and $y(0) = 0.7$. Clearly, $y = 0$ is g.u.a.s., which leads to that the system $\dot{x} = -x + 0.49e^{-4t}$, with $x(0) = 2$, is aIS stable with input $y(t) = 0.7e^{-2t}$. Furthermore, defining $V(x) = \frac{1}{2}x^2$ as an IS stable Lyapunov function yields $|x(t)| \geq \sqrt{2}y(t)$, where we take $\theta = 1/2 < 1$, i.e., the gain is $\rho(r) = \sqrt{2}r$. Also, it is easy to show that, if we choose $\eta^y = 0.01$, then $T^x = 6$. Finally, taking $\eta^x = \rho(\sup_{t\geq 6} y(t))$ implies that the cascade system has the required stability property. The simulation results are shown in Figure 6.1.

6.4 Conclusion

In this chapter, the qualitative notion of input-to-stae stability proposed by Sontag was re-presented in the same line of defining the classical concept of stability. That is to say, we used an $(\varepsilon^u, \delta^u)$-based approach so as to have a closer insight into the system behaviour. This method was also adopted to develop a comparison principle to achieve the same stability-like concept. To justify the effectiveness of this theoretical result, we applied it to a cascade system to prove the uniform asymptotic stability of the corresponding equilibrium point.

Chapter 7

Input-to-State Stability for SISD

In Chapter 5, we discussed some stability properties of SISD, which were later extended to analyze the properties of large scale SISD. In Chapter 6, we developed results for studying the input-to-state stability concept of ordinary systems. We intent to further investigate the input-to-state stability properties of systems whose states undergo impulsive effects, time lag, and random noise in this chapter. In particular, assuming that the SISD have m.s. uniformly asymptotically stable equilibrium point at the origins, we want to apply the IS stability results of the last chapter to examine the system states after being perturbed by input disturbance with bounded energy. The material of this chapter forms the basis of [Alw-e].

7.1 Input-to-State Stability of SISD

The focus of this chapter is on establishing input-to-state stability properties for SISD with fixed impulses. We adopt the two proposed approaches presented in Chapter 6, namely, the $(\varepsilon^u, \delta^u)$-based and comparison principle techniques. Using

Razumikhin method, Lyapunov-type sufficient conditions are developed to prove the stability properties in the m.s.

Before presenting the main contributions of this chapter, we introduce some materials that will be used later.

Consider the following nonlinear time-delayed stochastic differential equation with input $u \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^q)$

$$dx(t) = f(t, x_t, u(t))dt + g(t, x_t, u(t))\, dW(t), \quad t \in [a, b], \tag{7.1}$$

where $x \in \mathbb{R}^n$ is the system state random process, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$, which belongs to $\mathcal{L}_{ad}(\Omega, L([t_0, t_0 + \alpha]))$, and $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^{n\times m}$, which belongs to $\mathcal{L}_{ad}(\Omega, L^2([t_0, t_0 + \alpha]))$.

Considering impulse effects with fixed times in (7.1) leads to the following stochastic impulsive system with time delay and input

$$dx(t) = f(t, x_t, u(t))dt + g(t, x_t, u(t))\, dW(t), \qquad t \neq \tau_k, \tag{7.2a}$$

$$\Delta x = \mathcal{I}(t, x_{t^-}, u(t^-)), \qquad t = \tau_k. \tag{7.2b}$$

The initial condition is given by

$$x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \tag{7.2c}$$

where $\phi \in \mathcal{L}^2_{\mathcal{F}_0}([-r, 0] \times \Omega, \mathbb{R}^n)$, τ_k represents constant impulsive moments, for $k = 0, 1, 2, \cdots$, and satisfies $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$, and $\lim_{k\to\infty} \tau_k = \infty$. We also assume that the solution of (7.2) is right-continuous (i.e., $x(t^+) = x(t)$). In difference equation (7.2b), $\Delta x = x(t) - x(t^-)$ and the functional $\mathcal{I} : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$, where $\mathbb{T} = \{\tau_k \,\big|\, k = 0, 1, 2, \cdots\}$, is the impulse amount, which is assumed to be $\mathcal{F}_{t_k}$-adapted. Furthermore, for the system to admit a trivial solution, we assume that, for almost all sample paths in Ω, $f(t,0,0)=0\in\mathbb{R}^n$, $g(t,0,0)=0\in\mathbb{R}^{n\times m}$ for all $t\geq t_0$, and $\mathcal{I}(\tau_k,0,0)=0\in\mathbb{R}^n$ for all $\tau_k\in\mathbb{T}$.

Definition 7.1. A functional f is said to be strongly quasi-bounded in the m.s., if for each compact set $F\subset\mathcal{D}\subset\mathbb{R}^n$, there exists a positive constant M such that $\mathbb{E}[\|f(t,\psi,u)\|^2]\leq M$ for all $(t,\psi,u)\in\mathbb{R}_+\times\mathcal{PC}\big([-r,0];F\big)\times\mathbb{R}^q$.

Assumption B1. There exist $0\leq\varrho_1\leq\varrho$ such that, for all $\tau_k\in\mathbb{T}$, x defined on $\mathcal{PC}[t-r,\infty)$ for all $t\geq t_0\in\mathbb{R}_+$, $u\in\mathcal{PC}(\mathbb{R}_+;\mathbb{R}^q)$, and $\rho\in\mathcal{K}$ such that, if

$$\rho(\|u\|)\leq\mathbb{E}[\|x(\tau_k^-)\|^2]<\varrho_1,\quad\text{ then }\quad\mathbb{E}[\|x(\tau_k)\|^2]<\varrho.$$

Assumption B1 is made to guarantee that the solution stays bounded (in the m.s.) after impulses. Also, the solution is allowed to cross the ultimate bound of u after an impulsive effect.

Definition 7.2. Let $\phi\in\mathcal{L}^2_{\mathcal{F}_0}([-r,0]\times\Omega,\mathbb{R}^n)$, $x(t)=x(t,t_0,\phi)$ be a solution of (7.2), and $\rho\in\mathcal{K}$. Then, system (7.2) is said to be

(i) *input-to-state (IS) stable in the m.s.* with a gain ρ if, for every $\varepsilon^u>0$ and $t_0\in\mathbb{R}_+$, there exists a $\delta^u=\delta^u(t_0,\varepsilon^u)>0$ such that

$$\rho(\|u\|)\leq\mathbb{E}[\|\phi\|_r^2]\leq\delta^u\qquad\text{implies}\qquad\rho(\|u\|)\leq\mathbb{E}[\|x(t)\|^2]<\varepsilon^u,\qquad\forall t\geq t_0;$$

(ii) *uniformly IS stable in the m.s.* with a gain ρ if δ^u in (i) is independent of t_0;

(iii) *IS attractive in the m.s.* with a gain ρ if, for any $\eta^u>0$ and $t_0\in\mathbb{R}_+$, there exist a $\delta^u>0$ and $T^u=T^u(t_0,\eta^u)$ such that

$$\rho(\|u\|)\leq\mathbb{E}[\|\phi\|_r^2]\leq\delta^u\quad\text{implies}\quad\rho(\|u\|)\leq\mathbb{E}[\|x(t)\|^2]<\eta^u,\quad\forall t\geq t_0+T^u;$$

(iv) *uniformly IS attractive in the m.s.* with a gain ρ if T^u in (iii) is independent of t_0;

(v) *uniformly asymptotic input-to-state (aIS) stable in the m.s.* with a gain ρ if (ii) and (iv) hold;

(vi) *exponentially input-to-state (eIS) stable in the m.s.* with a gain ρ if (v) holds, and, moreover, there exist two positive constants K and λ such that

$$\|x(t)\| \leq K\mathbb{E}[\|\phi\|_r^2]e^{-\lambda(t-t_0)}, \qquad \forall\, t \geq t_0. \tag{7.3}$$

Remark 7.1. Immediate implications of this definition are stated in the following.

1. Clearly, for zero input, the above definitions reduce to the classical uniformly asymptotic stability in the m.s. of the *trivial solution* of (7.2) with zero input.

2. If $g = 0 \in \mathbb{R}^{n\times m}$, $r = 0 \in \mathbb{R}$ and $\mathcal{I} = 0 \in \mathbb{R}^n$, Definition 7.2 reduces to Definition 6.1.

3. If $g = 0 \in \mathbb{R}^{n\times m}$, $r = 0 \in \mathbb{R}$, Definition 7.2 reduces to that of impulsive system subject to input disturbance

$$\begin{aligned} \dot{x} &= f(t, x, u), && t \neq \tau_k, \\ \Delta x(t) &= \mathcal{I}(t, x(t^-)), && t = \tau_k, \\ x(t_0) &= x_0. \end{aligned}$$

Due to the dependence of the functionals f and g on the input u, the Itô formula should be modified accordingly.

Itô formula. For $t_0 \in \mathbb{R}_+$ and all $t \geq t_0$, let $x(t)$ be an n-dimensional Itô process, i.e., $\mathbb{R}^n$-valued continuous adapted process satisfying

$$dx(t) = f(t, x_t, u(t))\, dt + g(t, x_t, u(t))\, dW(t), \qquad \text{(a.s.)},$$

where f and g are as defined before. Let $V \in \mathcal{C}^{1,2}\big(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+\big)$. Then, for all $t \geq t_0$, $V(t, x(t))$ is an Itô process with its stochastic differential equation given by

$$dV(t, x(t)) = \mathcal{L}V(t, x_t, u(t))dt + V_x(t, x(t))g(t, x_t, u(t))\, dW(t), \qquad \text{(a.s.)},$$

where

$$\mathcal{L}V(t, x_t, u(t)) = \\ V_t(t, x(t)) + V_x(t, x)f(t, x_t, u(t)) + \frac{1}{2}\mathrm{tr}\Big(g^T(t, x_t, u(t))V_{xx}(t, x(t))g(t, x_t, u(t))\Big),$$

and $V_x(t, x(t))$ and $V_{xx}(t, x(t))$ are the gradient and Hessian matrix of $V(t, x(t))$.

Evidently, the diffusion infinitesimal operator $\mathcal{L}$ depends on u, although the process V is input-free.

7.2 Analysis by an $(\varepsilon^u, \delta^u)$ Approach

In this section, we state and prove some IS stability properties of system (7.2), using the technique developed in Section 6.1. We should mention that, in this section, the impulsive functional is input-free.

Theorem 7.1. For any solution x of (7.2), assume that Assumptions B1 and A2 hold, and there exist functions $a \in \mathcal{K}_2$, $b \in \mathcal{K}_1$, $c \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}_+)$, $\gamma \in \mathcal{K}$, and a constant $d_k \geq 0$ with $d = \sum_{k=1}^{\infty} d_k < \infty$. Suppose that $V \in \mathcal{C}^{1,2}\big([-r, \infty) \times S(\varrho)); \mathbb{R}_+\big)$ satisfies

(i) for all $(t, \psi(0)) \in [-r, \infty) \times S(\varrho)$,

$$b(\|\psi(0)\|^2) \leq V(t, \psi) \leq a(\|\psi(0)\|^2), \qquad \text{(a.s.)};$$

(ii) for all $t \neq \tau_k$, $\psi \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$, and $u \in \mathcal{PC}(\mathbb{R}_+;\mathbb{R}^q)$,

$$\mathcal{L}V(t,\psi,u) \leq -c(\|\psi(0)\|^2) + \gamma(\|u\|), \qquad \text{(a.s.)},$$

provided that $V(t+s,\psi(s)) \leq q(V(t,\psi(0)))$ for some $s \in [-r,0]$, where q is a class-$\mathcal{K}_3$ function;

(iii) for any $\tau_k \in \mathbb{T}$ and $\psi \in \mathcal{PC}\big([r,0]; S(\varrho)\big)$,

$$V(\tau_k, \psi(0) + \mathcal{I}(\tau_k, \psi(\tau_k^-))) \leq \alpha(d_k) V(\tau_k^-, \psi(0)), \qquad \text{(a.s.)},$$

where $\psi(0^-) = \psi(0)$, $(\tau_k, \psi(\tau_k^-)) \in \mathbb{R}_+ \times \mathcal{PC}\big([-r,0]; S(\varrho_1)\big)$, $\prod_{k=1}^{\infty} \alpha(d_k) < \infty$ with $\alpha(d_k) > 1$ for all k.

Then, system (7.2) is uniformly IS stable in the m.s.

Proof. Following the same analysis as in Theorem 5.1, let $x(t) = x(t,t_0,\phi)$ be the unique solution of system (7.2), and $0 < \varepsilon^u \leq \varrho_1$. Define $\overline{d} = \prod_{k=1}^{\infty} \alpha(d_k)$. Then, $1 \leq \overline{d} < \infty$ because $d < \infty$. Choose $\delta^u = \delta^u(\varepsilon^u)$ so that $\delta^u < \hat{a}^{-1}\big(\hat{b}(\varepsilon^u)/\overline{d}\big)$ and clearly $0 < \delta^u < \varepsilon^u$. Let $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer l and ϕ for which $\rho(\|u\|) \leq \mathbb{E}[\|\phi\|_r^2] \leq \delta^u$.

We claim that the system is uniformly IS stable in the m.s. If this were not the case, then there would be a t^s at which, for all $t \in [t_0 - r, t^s)$,

$$\rho(\|u\|) \leq \mathbb{E}[\|x(t)\|^2] < \varepsilon^u < \varrho_1,$$

and either

$$\rho(\|u\|) \leq \mathbb{E}[\|x(t^s)\|^2] = \varepsilon^u,$$

which implies that

$$\rho(\|u\|) \leq \mathbb{E}[\|x(t^s)\|^2] = \mathbb{E}[\|x_{t^s}\|_r^2] = \varepsilon^u,$$

or

$$\rho(\|u\|) \le \varepsilon^u < \mathbb{E}[\|x(t^s)\|^2], \quad \text{where} \quad t^s = \tau_k \quad \text{for some } k.$$

By Assumption B1,

$$\rho(\|u\|) \le \varepsilon^u < \mathbb{E}[\|x(t^s)\|^2]| < \varrho,$$

since $\mathbb{E}[\|x_{t^{s-}}\|^2] \le \varepsilon^u < \varrho_1$. Thus, in either case, $V(t,x)$ is defined for $t \in [t_0, t^s]$.

Moreover, from assumption (ii), we have

$$\|x\| \ge \rho(\|u\|) \qquad \text{implies} \qquad \mathcal{L}V(t, x_t, u) \le 0,$$

where $\rho(\cdot) = [c^{-1}(\gamma(\cdot))]^{1/2}$. Applying Itô formula to process $V(t, x(t))$ for $t \in [t_0, t^s]$ and taking the mathematical expectation give

$$\begin{aligned}\mathbb{E}[V(t,x(t))] &\le \mathbb{E}[V(s,x(s))] + \mathbb{E}\int_s^t \mathcal{L}V(w, x_w, u)dw, \qquad \forall\, t_0 \le s \le t \le t^s \\ &\le \mathbb{E}[V(s,x(s))].\end{aligned}$$

Define $m(t) = \mathbb{E}[V(t,x(t))]$ for all $t \in [t_0, t^s]$. Then,

$$D^+m(t) = \lim_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \le 0, \qquad \text{whenever} \quad \|x\| \ge \rho(\|u\|),$$

i.e., the function $m(t)$ is non-increasing for all $t \in (t_0, t^s]$ between the impulse moments. By the condition in (iii), we have

$$m(\tau_k) \le \alpha(d_k) m(\tau_k^-), \qquad \forall\, t \in (t_0, t^s].$$

Since $m(t) = \mathbb{E}[V(t,x(t))] \le \mathbb{E}[V(s,x(s))] = m(s)$, $m(t)$ is non-increasing for all $t \in [t_0, t^s]$ between impulses. If $t^s \in (t_0, t_l)$, then

$$\hat{b}[\rho(\|u\|)] \le \hat{b}(\mathbb{E}[\|x(t^s)\|^2]) \le m(t^s) \le m(t_0) < \hat{a}(\delta^u) < \frac{\hat{b}(\varepsilon^u)}{\bar{d}} \le \hat{b}(\varepsilon^u).$$

On the other hand, let $t^s \in [\tau_k, \tau_{k+1})$ for some $k \geq l$. In this case, whenever $\|x\| \geq \rho(\|u\|)$, we have, by the same argument followed in Theorem 5.1,

$$m(\tau_i) \leq \overline{d}\, m(t_0) \leq \overline{d}\, \hat{a}(\delta^u), \tag{7.4}$$

which implies that

$$m(t^s) \leq m(\tau_i) \leq \overline{d}\, \hat{a}(\delta^u).$$

We also have

$$\bar{\rho}(\|u\|) \leq \hat{b}(\varepsilon^u) < \hat{b}(\mathbb{E}\|x(t^s)\|^2) \leq m(t^s) < \overline{d}\, \hat{a}(\delta^u) < \hat{b}(\varepsilon^u),$$

where $\bar{\rho}(\cdot) = \hat{b} \circ \rho(\cdot)$, which is a contradiction. Therefore, it must be true that system (7.2) is uniformly IS stable in m.s. This completes the proof.

In the following theorem, we prove the aIS stability in the m.s. of the system (7.2). This property requires strengthening the upper bound estimation of the diffusion operator, as the following theorem tells us.

Theorem 7.2. For any solution x of (7.2), assume that Assumptions B1 and A2 hold, the functionals f and g are strongly quasi-bounded in m.s., and there exist functions $a \in \mathcal{K}_2$, b, $c \in \mathcal{K}_1$, and a constant $d_k \geq 0$ with $d = \sum_{k=1}^{\infty} d_k < \infty$. Let $V \in \mathcal{C}^{1,2}\big([-r, \infty) \times S(\varrho)); \mathbb{R}_+\big)$ satisfy

(i) assumptions (i) and (iii) of Theorem 7.1;

(ii) for all $t \neq \tau_k \in \mathbb{R}_+$ and $\psi \in \mathcal{PC}\big([-r, 0]; S(\varrho)\big)$,

$$\mathcal{L}V(t, \psi, u) \leq -c(\|\psi(0)\|) + \gamma(\|u(t)\|), \qquad \text{(a.s.)},$$

provided that $V(t+s, \psi(s)) \leq q(V(t, \psi(0)))$ for some $s \in [-r, 0]$, where q is a class-$\mathcal{K}_3$ function.

Then, system (7.2) is uniformly aIS stable in the m.s.

Proof. Since the solution is uniformly IS stable, given any $0 < \varepsilon^u \leq \varrho_1$, choose $\delta^u = \delta^u(\varepsilon^u)$, as done in Theorem 7.1.

For $0 < \theta < 1$, the inequality in (ii) can be written as

$$\|\psi(0)\| \geq \rho(\|u\|) \qquad \text{implies} \qquad \mathcal{L}V(t,\psi,u) \leq -\bar{c}(\|\psi(0)\|), \qquad \text{(a.s.)},$$

where $\rho(\cdot) = [\hat{c}^{-1}(\frac{1}{\theta}\gamma(\cdot))]^{1/2}$ and $\bar{c}(\cdot) = (1-\theta)\hat{c}(\cdot)$.

Choose a number $N > 0$ such that $N > \hat{m}_0 \overline{d} \bar{c}^{-1}\Big(\frac{1}{5}[\delta^{u^2} - \delta^u(\frac{\delta^u}{4M}+1)]\Big)\frac{4M}{\delta^u}$.

Suppose now a solution $x = x(\sigma,\phi)$ with $\rho(\|u\|) \leq \mathbb{E}[\|\phi\|^2] < \frac{\delta^{u^2}}{3}$ satisfies $\mathbb{E}[\|x_t\|_r^2] \geq \frac{\delta^{u^2}}{3}$ for any $t \geq \sigma$.

Assume that each interval of length r contains t_k such that $\mathbb{E}[\|x(t_k)\|^2] \geq \frac{\delta^{u^2}}{3}$. Then, there exists a sequence $\{t_k\}$ such that

$$\sigma + (2k-1)r \leq t_k \leq \sigma + 2kr, \qquad\qquad k = 1, 2, \cdots,$$

and

$$\mathbb{E}[\|x(t_k)\|^2] \geq \frac{\delta^{u^2}}{3}.$$

Integrating the stochastic differential equation (7.1) over $[t_k, t_k + \frac{\delta^u}{4M}]$ yields

$$x(t) = x_{t_k} + \int_{t_k}^{t_k+\frac{\delta^u}{4M}} f(t,x_t,u(t))dt + \int_{t_k}^{t_k+\frac{\delta^u}{4M}} g(t,x_t,u(t))dW(t),$$

from which we get, as concluded in Theorem 5.1,

$$-\bar{c}\Big(\mathbb{E}[\|x(t)\|^2]\Big) \leq -\bar{c}\Big(\frac{1}{5}[\delta^{u^2} - \delta^u(\frac{\delta^u}{4M}+1)]\Big),$$

so that for $t \in [t_k, t_k + \frac{\delta^u}{4M}]$, whenever $\|x(t)\| \geq \rho(\|u\|)$,

$$\mathbb{E}[\mathcal{L}V(t,x_t,u)] \leq -\bar{c}\Big(\mathbb{E}[\|x(t)\|^2]\Big) \leq -\bar{c}\Big(\frac{1}{5}[\delta^{u^2} - \delta^u(\frac{\delta^u}{4M}+1)]\Big).$$

By Itô formula we have, whenever $\|x(t)\| \geq \rho(\|u\|)$,

$$\mathbb{E}[V(t,x(t))] \leq \mathbb{E}[V(t_k,x(t_k))] + \mathbb{E}\int_{t_k}^{t_k+\frac{\delta^u}{4M}} \mathcal{L}V(t,x_t,u(t))dt$$

or

$$m(t) \leq m(t_k) - \bar{c}\Big(\delta^{u^2} - \delta^u(\frac{\delta^u}{4M}+1)\Big)\frac{\delta^u}{4M},$$

where $m(t) = \mathbb{E}[V(t,x_t)]$, that is the function m decreases by $\bar{c}\Big(\frac{1}{5}[\delta^{u^2} - \delta^u(\frac{\delta^u}{4M}+1)]\Big)\frac{\delta^u}{4M} > 0$ over the interval $[t_k, t_k + \frac{\delta^u}{4M}]$.

To investigate the overall behaviour of function $m(t)$ for all $t \geq t_0$, we define new function, say $\hat{m}$, as follows

$$\hat{m}(t) = \begin{cases} m(t), & t \in [t_0, t_l), \\ \Big[\prod_{k=l}^{i}\alpha(d_k)\Big]^{-1} m(t), & t \in (t_i, t_{i+1}), \quad i = l, l+1, \cdots. \end{cases}$$

This shows that function $\hat{m}$ decreases by $\overline{d}^{\,-1}\bar{c}\Big(\frac{1}{5}[\delta^{u^2} - \delta^u(\frac{\delta^u}{4M}+1)]\Big)\frac{\delta^u}{4M} > 0$ over the interval $[t_k, t_k + \frac{\delta^u}{4M}]$ or $[t_k - \frac{\delta^u}{4M}, t_k]$, where $\bar{d} = \prod_{k=l}^{i}\alpha(d_k)$. This implies that

$$\rho(\|u\|) \leq \hat{m}(t_0+T) \leq \hat{m}(t_0) - N\overline{d}^{\,-1}\bar{c}\Big(\frac{1}{5}[\delta^{u^2} - \delta^u(\frac{\delta^u}{4M}+1)]\Big)\frac{\delta^u}{4M}.$$

By our assumption and our choice of N, we conclude that

$$\rho(\|u\|) \leq \hat{m}(t_0+T) \leq a(\varrho_1) - N\overline{d}^{\,-1}\bar{c}\Big(\frac{1}{5}[\delta^{u^2} - \delta^u(\frac{\delta^u}{4M}+1)]\Big)\frac{\delta^u}{4M} < 0,$$

which is a contradiction. Thus, it must be true that, under our assumptions, $\rho(\|u\|) \leq \mathbb{E}[\|x(t)\|^2] < \varepsilon^u$ for all $t \geq t_0$, i.e., system (7.2) is uniformly aIS stable in the m.s. This completes the proof.

Example 7.1. Consider the following impulsive system with input disturbance

$$\begin{aligned} dx &= \big(-4x + x(t-1)e^{-|x|} + 0.5u(t)\big)dt - 0.1\sin x(t-1)dW, \quad t \neq t_k, \\ \Delta x(t) &= \frac{1}{k^2}x_{t^-}, \qquad t = t_k, \quad k = 1, 2, \cdots. \end{aligned}$$

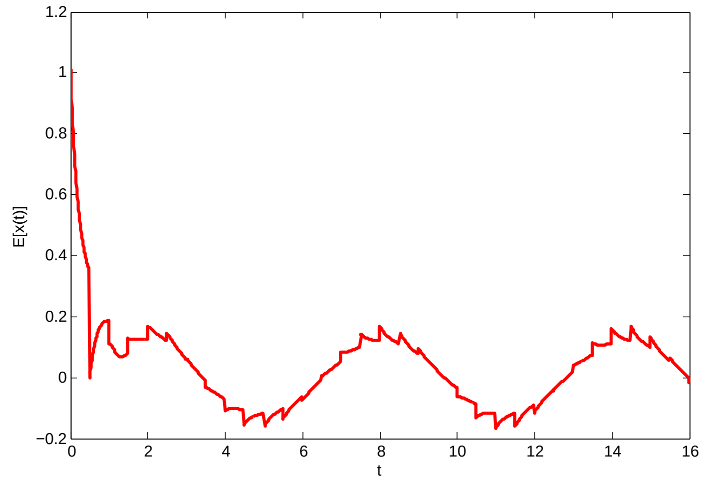

Figure 7.1: First moment aIS stability with $u(t) = \sin(t)$.

We showed in Example 5.1 that the trivial solution of the unforced system is uniformly asymptotically stable in the m.s. To examine the IS stability properties, consider the input function $u(t) = \sin(t)$. Then, whenever $|x| \geq [\frac{1}{2\theta}\sin(t)]^{1/2}$, $\mathcal{L}V(x_t, u) \leq -c(x)$ with $c(s) = 2.5s^2$, where we have chosen $q = 2$ and $\theta = 1/2$. Thus, by Theorem 7.2, the system is uniformly aIS stable in the m.s. The simulation result of this example is shown in Figures 7.1.

7.3 Input-to-State Stability of Large Scale SISD

In this section, we continue to examine the IS stability properties of system (7.2). We carry over the technique of Section 7.1 to build up a comparison principle, which will be used to prove the qualitative results. We are mainly interested in developing some Lyapunov-type theorems. Also, in these theorems, impulses not necessarily have bounded total effects and, moreover, they are assumed to be perturbed by input disturbance.

Theorem 7.3. For any solution x of (7.2), assume that Assumptions B1 and A2 hold, and there exists a class-$\mathcal{K}_2$ function a. Let $V \in \mathcal{C}^{1,2}\big([-r,\infty) \times S(\varrho)\big); \mathbb{R}_+\big)$ satisfy

(i) $V(t,\psi(0)) \leq a(\|\psi(0)\|^2)$, (a.s.), for all $(t,\psi(0)) \in [-r,\infty) \times S(\varrho)$;

(ii) $\mathcal{L}V(t,\psi(t),u(t)) \leq h(t,V(t,\psi(0),u(t)))$, (a.s.), for all $t \neq \tau_k$ in $\mathbb{R}_+$, $\psi \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$, and $u \in \mathcal{PC}(\mathbb{R}_+;\mathbb{R}^q)$, provided that $V(t+s,\psi(s)) \leq q(V(t,\psi(0)))$, where q is a class-$\mathcal{K}_3$ function, $h : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$ is continuous on $[\tau_{k-1},\tau_k)$, $h(t,z,u)$ is concave in z for any $t \in \mathbb{R}_+$, and, for each $x \in \mathbb{R}^n$ and $k \geq 1$,

$$\lim_{(t,y,v)\to(\tau_k^-,x,u)} h(t,y,v) = h(\tau_k^-,x,u)$$

exists;

(iii) $\forall\, \tau_k \in \mathbb{T}$ and $\psi \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$,

$$V(\tau_k, \psi(0) + \mathcal{I}_k(\tau_k, \psi(\tau_k^-), u(\tau_k^-))) \leq \alpha_k(V(\tau_k^-, \psi(0))) + \gamma(\|u(\tau_k^-)\|), \quad \text{(a.s.)},$$

where $\psi(0^-) = \psi(0)$, $(\tau_k, \psi(\tau_k^-)) \in (t_0,\infty) \times \mathcal{PC}\big([-r,0]; S(\varrho_1)\big)$, $\gamma \in \mathcal{K}$ and $\alpha_k : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing, concave function;

(iv) the scalar impulsive system

$$\begin{cases} D^+v(t) = h(t,v(t),u(t)), & t \neq \tau_k, \\ v(t) = \alpha_k(v(t^-)) + \gamma(\|u(t^-)\|), & t = \tau_k, \\ v(t_0) = v_0 \geq 0 \end{cases} \tag{7.5}$$

has a maximal solution $r(t) = r(t,t_0,v_0)$.

Then, $\mathbb{E}[V(t_0, x_0)] < v_0$ implies $\mathbb{E}[V(t, x(t))] < r(t)$ for all $t \geq t_0$.

Proof. Let $x(t) = x(t, t_0, \phi)$ be any solution of system (7.2). From (i) with the aid of existence results, we have

$$\mathbb{E}[V(t, x(t))] \leq \mathbb{E}[a(\|x(t)\|^2)] \leq a(\mathbb{E}[\|x(t)\|^2]) < \infty.$$

Also, by Itô formula and condition (ii), we have, for all $t \in [\tau_{k-1}, \tau_k)$,

$$\begin{aligned}
\mathbb{E}[V(t, x(t))] &= \mathbb{E}[V(\tau_{k-1}, x(\tau_{k-1})] + \mathbb{E}\int_{\tau_{k-1}}^{t} \mathcal{L}V(s, x_s, u(s))ds \\
&\leq \mathbb{E}[V(\tau_{k-1}, x(\tau_{k-1})] + \int_{\tau_{k-1}}^{t} h\big(s, \mathbb{E}[V(s, x(s))], u(s)\big)ds,
\end{aligned}$$

from which we get

$$D^+ m(t) \leq h(t, m(t), u(t)), \qquad t \neq \tau_k,$$

where $m(t) = \mathbb{E}[V(t, x(t))]$ for all $t \in [\tau_{k-1}, \tau_k)$ and all k. At the impulsive moments, we have, from condition (iv),

$$m(\tau_k) \leq \alpha_k(m(\tau_k^-)) + \gamma(\|u(\tau_k^-)\|).$$

In summary, we have obtained

$$\begin{cases}
D^+ m(t) \leq h(t, m(t), u(t)), & t \neq \tau_k, \\
m(t) \leq \alpha_k(m(t^-)) + \gamma(\|u(t^-)\|), & t = \tau_k, \\
m(t_0) = \mathbb{E}[V(t_0, x_0)].
\end{cases}$$

We claim that

$$m(t) = \mathbb{E}[V(t, x(t))] < r(t) = v(t), \qquad \forall t \geq t_0,$$

is true. If this were not the case, without loss of generality, there would be a $t_1 > t_0$, where $t_1 \neq \tau_k$, such that $m(t_1) = v(t_1)$, and for all $t \in [t_0, t_1)$, $m(t) < v(t)$. This implies that

$$\begin{aligned} D^+ m(t_1) &> D^+ v(t_1) \\ &= h(t_1, v(t_1), u(t_1)) \\ &= h(t_1, m(t_1), u(t_1)) \\ &= D^+ m(t_1). \end{aligned}$$

Contradiction. Thus, it must be true that $m(t) < v(t)$ for all $t \neq \tau_k$. Finally, at $t = \tau_k \in \mathbb{T}$, we have

$$m(\tau_k) \leq \alpha_k(m(\tau_k^-)) + \gamma(\|u(\tau_k^-)\|) < \alpha_k(v(\tau_k^-)) + \gamma(\|u(\tau_k^-)\|) = v(\tau_k).$$

This completes the proof.

Having proved the required comparison principle, we are in a position to establish the qualitative results.

Theorem 7.4. For any solution x of system (7.2), assume that Assumptions B1 and A2 hold, and there exist functions $a \in \mathcal{K}_2$ and $b \in \mathcal{K}_1$. Assume further that $V \in \mathcal{C}^{1,2}\big([-r, \infty) \times S(\varrho)\big); \mathbb{R}_+\big)$ satisfies

(i) for all $(t, \psi(0)) \in [-r, \infty) \times S(\varrho)$,

$$b(\|\psi(0)\|^2) \leq V(t, \psi(0)) \leq a(\|\psi(0)\|^2), \qquad \text{(a.s.)};$$

(ii) for all $t \neq \tau_k$, $\psi \in \mathcal{PC}\big([-r, 0]; S(\varrho)\big)$, and $u \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^q)$,

$$\mathcal{L}V(t, \psi(t), u(t)) \leq h(t, V(t, \psi(0)), u(t)), \qquad \text{(a.s.)},$$

provided that $V(t+s,\psi(s)) \le q(V(t,\psi(0)))$, where q is a class-$\mathcal{K}_3$ function, $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^q \to \mathbb{R}$ is continuous in its variables, $h(t,0,0) = 0$ and $h(t,z,u)$ is concave in z for any $t \in \mathbb{R}_+$, and, for each $x \in \mathbb{R}^n$ and $k \ge 1$,

$$\lim_{(t,y,w)\to(\tau_k^-,x,u)} h(t,y,w) = h(\tau_k^-,x,u)$$

exists;

(iii) $\forall\, \tau_k \in \mathbb{T}$ and $\psi \in \mathcal{PC}\big([-r,0];S(\varrho)\big)$,

$$V(\tau_k,\psi(0)+\mathcal{I}_k(\tau_k,\psi(\tau_k^-),u(\tau_k^-))) \le \alpha_k(V(\tau_k^-,\psi(0))) + \gamma(\|u(\tau_k^-)\|), \quad \text{(a.s.)},$$

where $\psi(0^-) = \psi(0)$, $(\tau_k,\psi(\tau_k^-)) \in (t_0,\infty)\times\mathcal{PC}\big([-r,0];S(\varrho_1)\big)$, $\gamma \in \mathcal{K}$ and α_k is a non-decreasing, concave function.

Then, the IS stability properties of the scalar impulsive system (7.5) imply the corresponding properties of (7.2).

Proof. Let $0 < \varepsilon^u < \varrho_1 < \varrho$ and $t_0 \in \mathbb{R}_+$. Assume that comparison system (7.5) is IS stable. Therefore, for given $b(\varepsilon^u) > 0$ and $t_0 \in \mathbb{R}_+$, choose $\delta^{v,u} = \delta^{v,u}(t_0,\varepsilon^u) > 0$ such that

$$0 < \rho(\|u\|) \le v_0 < \delta^{v,u} \qquad \text{implies} \qquad \rho(\|u\|) \le v(t,t_0,v_0) < b(\varepsilon^u), \quad \forall t \ge t_0,$$

for any solution $v(t) = v(t,t_0,v_0)$ of comparison system (7.5).

Choose $v_0 = a(\|\phi\|_r^2)$ and $\delta_1 = \delta_1(\varepsilon) > 0$ for which $a(\delta_1) < b(\varepsilon^u)$. Define $\delta^{x,u} = \min\{\delta^{v,u},\delta_1\}$. We claim that, if $\rho(\|u\|) \le \mathbb{E}[\|\phi\|_r^2] \le \delta^{x,u}$, then

$$\rho(\|u\|) \le \mathbb{E}[\|x(t)\|^2] < \varepsilon^u, \qquad \forall t \ge t_0.$$

If our claim were not true, there would be a $\bar{t} \in [\tau_k,\tau_{k+1})$ for some k such that

$$\rho(\|u\|) \le \varepsilon^u \le \mathbb{E}[\|x(\bar{t})\|^2],$$

and

$$\rho(\|u(t)\|) \le \mathbb{E}[\|x(t)\|^2] < \varepsilon^u, \qquad \forall\, t \in [\tau_k, \bar{t}).$$

Also, this together with Assumption B1, i.e., $\mathbb{E}[\|x(\tau_k^-)\|^2] < \varepsilon^u < \varrho_1$ and

$$\mathbb{E}[\|x(\tau_k)\|^2] = \mathbb{E}[\|x(\tau_k^-) + \mathcal{I}(\tau_k, x_{\tau_k^-}, u(\tau_k^-)\|^2] < \varrho$$

implies the existence of a $\underline{t}$ such that $\tau_k < \underline{t} \le \bar{t}$ satisfying

$$\rho(\|u\|) \le \varepsilon^u < \mathbb{E}[\|x(\underline{t})\|^2] < \varrho.$$

Define $m(t) = \mathbb{E}[V(t, x(t))]$ for all $t \in [t_0, \underline{t}]$. Then, by Theorem 7.3, we get

$$m(t) < r(t, t_0, a(\mathbb{E}[\|\phi\|_r^2])), \qquad \forall\, t \in [t_0, \underline{t}],$$

where $r(t, t_0, a(\mathbb{E}[\|\phi\|_r^2]))$ is the maximal solution of the scalar comparison system. Finally, by the condition (i), we obtain

$$\bar{\rho}(\|u\|) \le b(\varepsilon^u) \le m(\underline{t}) = \mathbb{E}[V(\underline{t}, x(\underline{t}))] < r(\underline{t}, t_0, a(\mathbb{E}[\|\phi\|_r^2])) < r(\underline{t}, t_0, a(\delta^{x,u})) < b(\varepsilon^u),$$

where $\bar{\rho}(\cdot) = b \circ \rho(\cdot)$, which is a contradiction. Thus, it must be true that

$$\rho(\|u\|) \le \mathbb{E}[\|x(t)\|^2] < \varepsilon^u, \qquad \forall\, t \ge t_0.$$

If $\delta^{v,u}$ is chosen independently of t_0, then system (7.2) is uniformly IS stable. To prove the uniform IS attractivity, we choose $0 < \eta^u < \varrho_1 < \varrho$. Assume that comparison system (7.5) is uniformly IS attractive, i.e., for a given $b(\eta^u) > 0$, there exist $\delta > 0$ and constant $T^u = T^u(\eta^u) > 0$ such that

$$\rho(\|u\|) \le v_0 \le \delta \qquad \text{implies} \qquad \rho(\|u\|) \le v(t, t_0, v_0) < b(\eta^u), \qquad \forall\, t \ge t_0 + T^u.$$

Following the argument used in proving the IS stability property, we obtain

$$b(\rho(\|u\|)) \le b(\mathbb{E}[\|x(t)\|^2]) \le v(t, t_0, v_0) < b(\eta^u), \qquad \forall\, t \ge t_0 + T^u,$$

i.e., the system (7.2) is uniformly IS attractive in the m.s., which leads to the uniform aIS stability property in the m.s. of the system. This completes the proof.

Corollary 7.1. In Theorem 7.4, assume that there exist a positive constant p, $c \in \mathcal{K}_1$, and $\gamma \in \mathcal{K}$ such that, for any $(t, \psi(0)) \in \mathbb{R}_+ \times \mathcal{PC}([t-r, \infty); S(\varrho))$,

$$h(t, \psi(0), u(t)) = p\, c(V(t, \psi(0))) + \gamma(\|u(t)\|).$$

Suppose further that there exist $\zeta_k \geq 0$ and $\varrho_0 > 0$ such that, for all $z \in (0, \varrho_0)$ and any $k \in \mathbb{N}$,

$$\bar{p}(\tau_k - \tau_{k-1}) + \int_z^{\alpha_k(z)+\gamma(\|u(\tau_k^-)\|)} \frac{ds}{c(s)} \leq -\zeta_k. \tag{7.6}$$

Then, composite system (7.2) is uniformly IS stable in the m.s. If, moreover, $\sum_{k=1}^{\infty} \zeta_k = +\infty$, the system is aIS stable in the m.s.

Proof. Defining $m(t) = \mathbb{E}[V(t, x(t))]$ for any $t \geq t_0$ yields

$$\begin{cases} D^+ m(t) \leq pc(m(t)) + \gamma(\|u(t)\|), & t \neq \tau_k, \\ m(t) \leq \alpha_k(m(t^-)) + \gamma(\|u(t^-)\|), & t = \tau_k, \\ m(t_0) = m_0 = \mathbb{E}[V(t_0, x_0)]. \end{cases} \tag{7.7}$$

Consider the following impulsive comparison system

$$\begin{cases} D^+ v(t) = pc(v(t)) + \gamma(\|u(t)\|), & t \neq \tau_k, \\ v(t) = \alpha_k(v(t^-)) + \gamma(\|u(t^-)\|), & t = \tau_k, \\ v(t_0) = v_0 > m_0. \end{cases} \tag{7.8}$$

We want to prove that comparison system (7.8) is uniformly aIS stable in the m.s., and, by the comparison principle result, the SISD with input (7.2) has the same qualitative property.

We claim that, under our supposition, system (7.8) is uniformly IS stable in the m.s., i.e., given $0 < \varepsilon^u < \varrho_0$ and $t_0 \in [\tau_1, \tau_2)$, one can choose δ^u such that $\delta^u < \min\{\varepsilon^u, \alpha_k(\varepsilon)\}$ and $\rho(\|u\|) \le v_0 < \delta^u$ imply that $\rho(\|u\|) \le v(t) < \varepsilon^u$ for all $t \in [t_0, \tau_2)$. If it were not true, there would be a $t^* \in [t_0, \tau_2)$ such that $\rho(\|u\|) \le \varepsilon^u < v(t^*)$. Also, one may write the differential inequality in (7.7) as

$$D^+ m(t) \le \bar{p} c(m(t)), \qquad \text{whenever } m(t) \ge c^{-1}\big(\frac{1}{p\theta}\gamma(\|u\|)\big), \tag{7.9}$$

where $\bar{p} = (1+\theta)p$ for some $\theta > 0$. It follows that, after integration over (t^*, t_0),

$$\int_{v(t_0)}^{v(t^*)} \frac{ds}{c(s)} \le \int_{t_0}^{t^*} \bar{p} ds \le \bar{p}(\tau_2 - \tau_1), \tag{7.10}$$

whenever $m(t) \ge c^{-1}\big(\frac{1}{p\theta}\gamma(\|u\|)\big) =: \rho(\|u(t)\|)$. Since $v(\tau_k) = \alpha_k(v(\tau_k^-)) + \gamma(\|u(\tau_k^-)\|)$ for all k, we have

$$\int_{v(t_0)}^{v(t^*)} \frac{ds}{c(s)} > \int_{\alpha_1(\varepsilon)+\gamma(\|u(\tau_1^-)\|)}^{\varepsilon} \frac{ds}{c(s)}.$$

It follows that

$$\bar{p}(\tau_2 - \tau_1) + \int_{\varepsilon}^{\alpha_1(\varepsilon)+\gamma(\|u(\tau_1^-)\|)} \frac{ds}{c(s)} > 0,$$

which contradicts with (7.6). By the same argument followed in proving Corollary 4.1, we obtain

$$\int_{v(\tau_k^-)}^{v(t)} \frac{ds}{c(s)} \le \bar{p}(\tau_{k+1} - \tau_k) + \int_{v(\tau_k^-)}^{\alpha(v(\tau_k^-))+\gamma(\|u(\tau_k^-)\|)} \frac{ds}{c(s)} \le -\zeta_k, \tag{7.11}$$

whenever $\rho(\|u(t)\|) < m(t)$, which implies that $v(t) \le v(\tau_k^-) < \varepsilon$ for all $t \in [\tau_k, \tau_{k+1})$. By the comparison result, we have $\rho(\|u(t)\|) < m(t) < v(t) \le v(\tau_k^-) < \varepsilon$ for all $t \in [\tau_k, \tau_{k+1})$. By induction, $\rho(\|u(t)\|) < v(t) < \varepsilon$ for all $t \ge t_0$, i.e., (7.8) is uniformly IS stable in the m.s.

The proof of uniformly aIS stability in the m.s. is analogous to that of Corollary 4.1. The proof is complete.

Example 7.2. Consider the following impulsive system with input

$$
\begin{aligned}
dx &= \Big(-7x - 0.5y(t-1)e^{-x^2} \Big) dt, \\
dy &= \Big(-5y + \sin x(t-1) \Big) dt + \Big(-\frac{0.1x(t-1)}{1+y^2} + u(t) \Big) dW_2, \\
\Delta x(t_k) &= -2x(t_k^-) + \eta \sin(t_k^-), \\
\Delta y(t_k) &= 0.2y(t_k^- - 1).
\end{aligned}
$$

Define $V(x,y) = \frac{1}{2}(x^2+y^2)$. We showed in Example 4.2 that the trivial solution $x \equiv 0$ of the unforced system is asymptotically stable in the m.s. To investigate the aIS stability property, we choose $u(t) = \sin(t)$ and $u(\tau_k^-) = \sin(\tau_k^-)$. Then, one can show that $h(V,u) = -2.96V$, whenever $\|(x,y)^T\| \geq \sqrt{2}|\sin(t)|$, where we have taken $q = 2$ and $\theta = 1/2$. Also, at $t = \tau_k$, we have $V(x(\tau_k), y(\tau_k)) \leq 6V(x(\tau_k^-), y(\tau_k^-)) + \eta^2 \sin^2(\tau_k^-)$. Taking $\eta = 0.05$ gives $\tau_k - \tau_{k-1} = 0.95$ for all k. That is, the system is aIS in the m.s. Figure 7.2 shows the simulation result of the system.

Example 7.3. Consider the following system

$$
\begin{aligned}
dx &= \Big(-x + u(t)[5x - x^2(t-1)] \Big) dt + 0.1x dW, \quad && t \neq t_k, \\
\Delta x(t) &= -\frac{k+2}{k+1} x(t^-) + 0.01 \sin(t^-), && t = t_k, \quad k = 1, 2, \cdots .
\end{aligned}
$$

Clearly, the unforced system has an asymptotically stable equilibrium point at the origin. Set $u(t) = 1$ for all t. Define $V(x) = \frac{1}{2}x^2$. Then, $\mathcal{L}V(x) \leq 5.55x^2$, i.e., the non-impulse system is not IS stable. On the other hand, at $t = \tau_k$, we have $V(x(\tau_k)) \leq \alpha_k V(x(\tau_k^-)) + 0.01$, where $\alpha_k = \frac{1}{(k+1)^2} < 1$. We also get, by Corollary 7.1, $\tau_k - \tau_{k-1} = 0.2$ for all k. The simulation result of this system is given in Figure 7.5, which shows the stabilizing effects of impulses.

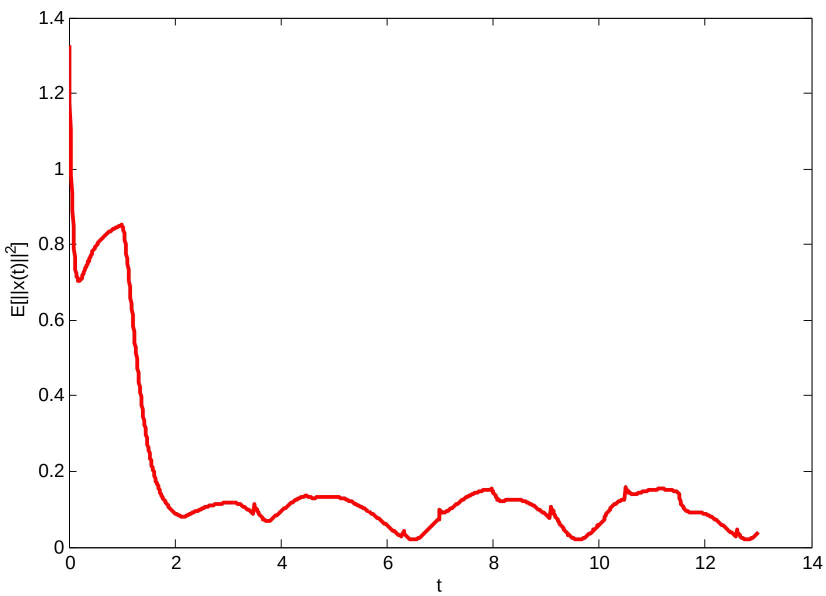

Figure 7.2: Mean square aIS stability with $u(t) = \sin(t)$.

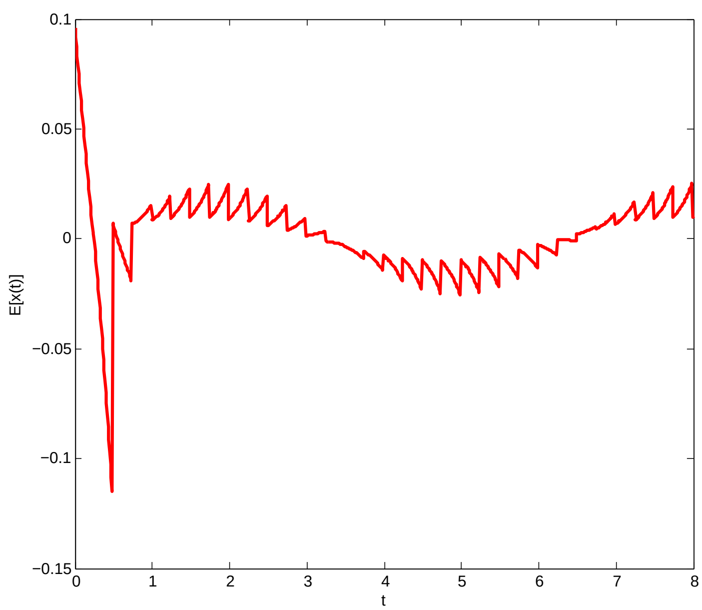

Figure 7.3: First moment aIS stability with $u(t) = \sin(t)$.

7.4 Conclusions and Comments

In this chapter, we investigated some input-to-state stability properties for SISD with fixed impulses and input disturbance. We used two different approaches to establish these properties, namely, an $(\varepsilon^u, \delta^u)$-based and comparison principle techniques developed in Chapter 6. In the first technique, the continuous dynamics were stable and the total effects of impulses was bounded, while, in the second technique, unbounded impulses and both stable and unstable continuous dynamics were considered. Our focus was on developing Lyapunov-like sufficient condition theorems, using Razumikhin methodology.

Chapter 8

Input-to-State Stability of Large Scale SISD

In this chapter, we consider large-scale nonlinear stochastic systems with time delay and subject to impulsive effects and disturbance input. As stated earlier, the random noise is described by Wiener process, the time delay is finite, the input has bounded energy, and the impulsive actions occur at constant times, not of state dependent type due to some technical difficulties. Also, for the same reasoning, throughout this chapter, the impulses are considered input-free. The focus is to explore m.s. asymptotic IS stability properties of the system. We will continue to apply the approaches developed in Chapters 6 and 7 to establish the qualitative properties. Also, we use Razumikhin technique and comparison principle to develop Lyapunov-like sufficient conditions. In analyzing the qualitative properties, we decompose the interconnected system into smaller isolated subsystems, which are assumed to be uniformly asymptotically IS stable in the m.s. and the rest will be viewed as perturbation, which is required to be small in magnitude compared with degree of stability of each isolated subsystems. The material of this chapter forms the basis of [Alw-f].

Consider the *forced* interconnected or composite system with decomposition $\mathbb{D}_i^u$

$$\mathbb{D}_i^u : \begin{cases} dw^i(t) = f_i(t, w_t^i, u)dt + g_i(t, w_t^1, w_t^2, \cdots, w_t^l, u)dt \\ \qquad\qquad + \sum_{j=1}^{l} \sigma_{ij}(t, w_t^j, u)dW_j(t), & t \neq \tau_k, \\ \triangle w^i(t) = \mathcal{I}_i(t, w_{t^-}^i), & t = \tau_k, \\ w_{t_0}^i = \phi_i(s), \qquad s \in [-r, 0], \end{cases} \tag{8.1}$$

where $k \in \mathbb{N}$ and $i = 1, 2, \cdots l$ for some $l \in \mathbb{N}$. Here, w^i or $w_t^i \in \mathbb{R}^{n_i}$ are n_i-dimensional vector state or, respectively, its deviated state, $n = \sum_i^l n_i$ for some $n_i \in \mathbb{N}$ and u is a $\mathcal{PC}(\mathbb{R}_+; \mathbb{R}^q)$ function. $f_i : \mathbb{R}_+ \times \mathbb{R}^{n_i} \times \mathbb{R}^q \to \mathbb{R}^{n_i}$, $g_i : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^{n_i}$, $\sigma_{ij} : \mathbb{R}_+ \times \mathbb{R}^{n_j} \times \mathbb{R}^q \to \mathbb{R}^{n_i \times m_j}$ and $m = \sum_i^l m_i$ for some $m_i \in \mathbb{N}$, $\mathcal{I}_i : \mathbb{T} \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ with $\mathbb{T} = \{\tau_k \big| k = 1, 2, \cdots\}$ where τ_k represents constant impulsive moments and satisfies $0 < \tau_1 < \tau_2 < \cdots$, and $\lim_{k \to \infty} \tau_k = \infty$, and $\phi_i : [-r, 0] \to \mathbb{R}^{n_i}$.

The *forced* isolated subsystems $\mathbb{S}_i^u$ can be defined as

$$\mathbb{S}_i^u : \begin{cases} dw^i(t) = f_i(t, w_t^i, u)dt + \sigma_{ii}(t, w_t^i, u)dW_i(t), & t \neq \tau_k, \\ \triangle w^i(t) = \mathcal{I}_i(t, w_{t^-}^i), & t = \tau_k, \\ w_{t_0}^i = \phi_i(s), \qquad s \in [-r, 0]. \end{cases} \tag{8.2}$$

Also, for $x \in \mathbb{R}^n$, let $x^T = [(w^1)^T, (w^2)^T, \cdots, (w^l)^T]$ and $x_t^T = [(w_t^1)^T, (w_t^2)^T, \cdots, (w_t^l)^T]$ and define the functionals $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ by

$$f^T(t, x_t, u) = [f_1^T(t, w_t^1, u), f_2^T(t, w_t^2, u), \cdots, f_l^T(t, w_t^l, u)],$$

$g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ by

$$\begin{aligned} g^T(t, x_t, u) &= [g_1^T(t, x_t, u), \cdots, g_l^T(t, x_t, u)] \\ &= [g_1^T(t, w_t^1, w_t^2, \cdots, w_t^l, u), \cdots, g_l^T(t, w_t^1, w_t^2, \cdots, w_t^l, u)], \end{aligned}$$

$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^{n\times m}$ by

$$\sigma(t, x_t, u) = [\sigma_{ij}(t, w_t^j, u)],$$

and $W : \mathbb{R}_+ \to \mathbb{R}^m$ by

$$W^T = [W_1, W_2, \cdots, W_l],$$

where, for any i, $W_i : \mathbb{R}_+ \to \mathbb{R}^{m_i}$. We also define the impulsive functional $\mathcal{I} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mathcal{I}^T(t, x_{t^-}) = [\mathcal{I}_1^T(t, w_{t^-}^1), \mathcal{I}_2^T(t, w_{t^-}^2), \cdots, \mathcal{I}_l^T(t, w_{t^-}^l)].$$

Accordingly, the impulsive composite (or interconnected) system with decomposition $\mathbb{D}_i^u$ can be defined as

$$\mathbb{S}^u : \begin{cases} dx(t) = F(t, x_t, u)dt + \sigma(t, x_t, u)dW(t), & t \neq \tau_k, \\ \triangle x(t) = \mathcal{I}(t, x_{t^-}), & t = \tau_k, \\ x_{t_0} = \Phi(s), \qquad s \in [-r, 0], \end{cases} \tag{8.3}$$

where $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is defined by $F(t, x_t, u) = f(t, x_t, u) + g(t, x_t, u)$, which is an $\mathcal{L}_{ad}\big(\Omega, L[t_0, t_0 + \alpha]\big)$ function for some $\alpha > 0$, σ is as defined above, which is an $\mathcal{L}_{ad}\big(\Omega, L^2[t_0, t_0 + \alpha]\big)$ function, and $\Phi : [-r, 0] \to \mathbb{R}^n$ is defined by $\Phi^T = [\phi_1^T, \phi_2^T, \cdots, \phi_l^T]$, which is an $\mathcal{L}^2_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$ function.

8.1 ISS Properties by a Scalar Lyapunov Function

This section deals with the m.s. uniformly aIS stability properties of composite SISD (8.3). We first start with summarizing the conditions guaranteeing the m.s. uniformly aIS stability property of each isolated subsystem.

Definition 8.1. The isolated subsystem $\mathbb{S}_i^u$ is said to possess **Property C** if Assumptions B1 and A2 hold, the functionals f_i and σ_{ii} are strongly quasi-bounded in the m.s., there exist functions $a_i \in \mathcal{K}_2$, b_i, $c_i \in \mathcal{K}_1$, $\gamma \in \mathcal{K}$ and constants $\sigma_i < 0$ and $d_k \geq 0$ with $d = \sum_{k=1}^{\infty} d_k < \infty$, and $V^i \in \mathcal{C}^{1,2}\big([-r,\infty) \times S(\varrho)); \mathbb{R}_+\big)$ such that

(i) for all $(t, \psi^i(0)) \in [-r,\infty) \times S(\varrho)$,

$$b_i(\|\psi^i(0)\|^2) \leq V^i(t, \psi^i(0)) \leq a_i(\|\psi^i(0)\|^2), \qquad \text{(a.s.)};$$

(ii) for all $t \neq \tau_k$, $\psi^i \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$, and $u \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^q)$,

$$\mathcal{L}_i V^i(t, \psi^i, u) \leq \sigma_i c_i(\|\psi^i(0)\|^2) + \gamma(\|u\|), \qquad \text{(a.s.)},$$

provided that $V^i(t+s, \psi^i(s)) \leq \bar{q} V(t, \psi^i(0))$ for some $\bar{q} > 1$ and $s \in [-r, 0]$;

(iii) for any $\tau_k \in \mathbb{T}$ and $\psi^i \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$,

$$V^i\big(\tau_k, \psi^i(0) + \mathcal{I}_i(\tau_k, \psi^i(\tau_k^-))\big) \leq \alpha(d_k) V^i(\tau_k^-, \psi^i(0)), \qquad \text{(a.s.)},$$

where $\psi(0^-) = \psi(0)$, $\prod_{k=1}^{\infty} \alpha(d_k) < \infty$ with $\alpha(d_k) > 1$ for all k.

Remark 8.1. One can re-write the diffusion operator inequality in (ii) as

$$\mathcal{L}_i V^i(t, \psi^i, u) \leq \bar{\sigma}_i c_i(\|\psi^i(0)\|^2), \qquad \text{(a.s.)},$$

whenever $\|\psi^i(0)\| \geq \big[c_i^{-1}\big(\frac{1}{\theta\sigma_i}\|\gamma(\|u\|)\big)\big]^{1/2} =: \rho_{1_i}(\|u\|)$, where $\bar{\sigma}_i = \sigma_i + \theta < 0$, for some $\theta > 0$, and $\rho_{1_i} \in \mathcal{K}$.

In the following theorem, we state and prove the aIS property of the solution of comparison system (8.3).

Theorem 8.1. Suppose that composite system (8.3) satisfies the following conditions:

(i) every isolated subsystem $\mathbb{S}_i^u$ possesses Property C;

(ii) for any $i, j = 1, 2, \cdots, l$, there exists a positive constant b_{ij} such that

$$g_i^T(t, \psi^i, u)V^i_{\psi^i(0)}(t, \psi^i(0)) \leq c_i^{1/2}(\|\psi^i(0)\|^2)\sum_{j=1}^{l}\bar{q}b_{ij}c_j^{1/2}(\|\psi^j(0)\|^2),$$

whenever $\|\psi^i(0)\| \geq \rho_{2_i}(\|u\|)$, where $\bar{q}$, c_i, and γ are defined in (i) and ψ^k is the k^{th} component of ψ;

(iii) for any vector $(y^i)^T$, $i = 1, 2, \cdots, l$, there exists $e_i > 0$ such that

$$(y^i)^T V^i_{\psi^i(0)\psi^i(0)}(t, \psi^i(0))y^i \leq e_i\|y^i(0)\|^2,$$

with $y^i = \sigma_{ij}(t, \psi^i, u)$ being the i^{th} row of matrix σ;

(iv) for any $\sigma_{ij}(t, \psi^j_t, u)$, $i, j = 1, 2, \cdots, l$, there exists $d_{ij} \geq 0$ such that

$$\|\sigma_{ij}(t, \psi^j, u)\|^2 \leq \bar{q}d_{ij}c_i(\|\psi^j(0)\|^2),$$

whenever $\|\psi^i(0)\| \geq \rho_{3_i}(\|u\|)$;

(v) the matrix $S = [s_{ij}]_{l\times l}$ is negative definite, where

$$s_{ij} = \begin{cases} \alpha_i(\bar{\sigma}_i + \bar{q}b_{ii}) + \frac{1}{2}\sum_{k=1,k\neq i}\bar{q}\alpha_k e_k d_{ki}, & i = j, \\ \frac{1}{2}\bar{q}(\alpha_i b_{ij} + \alpha_j b_{ji}), & i \neq j, \end{cases}$$

for some positive constant α_i for any i.

Then, composite system (8.3) is uniformly aIS stable in the m.s.

Proof. Let x be the solution of composite system (8.3). Define the composite Lyapunov function candidate by

$$V(t, x) = \sum_{i=1}^{l}\alpha_i V^i(t, w^i),$$

from which we get

$$\begin{aligned}
\mathcal{L}V(t,x,u) &= \sum_{i=1}^{l} \alpha_i \Big\{ \mathcal{L}_i V_i(t,w^i,u) + g_i^T(t,x_t,u) V^i_{w^i}(t,w^i) \\
&\qquad + \frac{1}{2} \sum_{j=1, i\neq j}^{l} \mathrm{tr}\big[\sigma_{ij}^T(t,w_t^j,u) V^i_{w^i w^i}(t,w^i)\sigma_{ij}(t,w_t^j,u)\big] \Big\} \\
&\leq \sum_{i=1}^{l} \alpha_i \Big\{ \bar{\sigma}_i c_i(\|w^i\|^2) + c_i^{1/2}(\|w^i\|^2) \sum_{j=1}^{l} \bar{q} b_{ij} c_j^{1/2}(\|w^j\|^2) \\
&\qquad + \frac{1}{2} \sum_{j=1, i\neq j}^{l} \bar{q} e_i \|\sigma_{ij}(t,w_t^j)\|^2 \Big\} \\
&\leq \sum_{i=1}^{l} \alpha_i \Big\{ \bar{\sigma}_i c_i(\|w^i\|^2) + c_i^{1/2}(\|w^i\|^2) \sum_{j=1}^{l} \bar{q} b_{ij} c_j^{1/2}(\|w^j\|^2) \\
&\qquad + \frac{1}{2} \sum_{j=1, i\neq j}^{l} \bar{q} e_i d_{ij} c_i(\|w^j\|^2) \Big\} \\
&= z^T S z,
\end{aligned}$$

whenever

$$\|x\| \geq \rho(\|u\|) := \max\{\max_i \rho_{1_i}(\|u\|), \max_i \rho_{2_i}(\|u\|), \max_i \rho_{3_i}(\|u\|)\},$$

where $z^T = \big(c_1^{1/2}(\|w^1\|^2), c_2^{1/2}(\|w^2\|^2), \cdots, c_l^{1/2}(\|w^l\|^2)\big)$, and S is the negative-definite matrix defined in (v). It follows that the eigenvalues of S are strictly negative. Therefore,

$$\mathcal{L}V(t,x_t,u) \leq \lambda_M(S) \sum_{i=1}^{l} c_i(\|w^i\|^2), \qquad \text{whenever } \|x\| \geq \rho(\|u\|),$$

i.e., $\mathcal{L}V(t,x_t,u)$ is negative definite, which implies that

$$\mathcal{L}V(t,x,u) \leq -c(\|x(t)\|^2), \qquad \text{whenever } \|x\| \geq \rho(\|u\|),$$

where c is a class-$\mathcal{K}_1$ function. At the impulsive moments $t = \tau_k$, we have

$$\begin{aligned} V(\tau_k, x(\tau_k)) &= \sum_{i=1}^{l} \alpha_i V^i(\tau_k, w^i(\tau_k)) \\ &\le \sum_{i=1}^{l} \alpha_i \alpha_i(d_k) V^i(\tau_k^-, w^i(\tau_k^-)) \\ &\le \alpha_M(d_k) \sum_{i=1}^{l} \alpha_i V^i(\tau_k^-, w^i(\tau_k^-)), \qquad \alpha_M(d_k) = \max_i \{\alpha_i(d_k)\} \\ &= \alpha_M(d_k) V(\tau_k^-, x(\tau_k^-)). \end{aligned}$$

Thus, the conditions of Theorem 7.2 are satisfied; therefore composite SISD (8.3) is uniformly aIS stable in the m.s. This completes the proof.

In Theorem 8.2, as clarified earlier, Property C is assumed to insure the m.s. uniformly aIS stability property of each isolated subsystem. Assumptions (ii) and (iii) represent the upper bound estimations on the deterministic and noisy perturbations (or interconnection). Moreover, to guarantee the stability feature of composite SISD, these perturbations have to be sufficiently small, as described by the test matrix S in assumption (v).

The following corollary states some special cases of Theorem 8.1.

Corollary 8.1. In Theorem 8.1,

1. if $u(t) \equiv 0$ for all $t \in \mathbb{R}_+$, the trivial solution $x \equiv 0$ of the corresponding *unforced* composite system in (8.3) is asymptotically stable in the m.s.;

2. for any $i = 1, 2, \cdots, l$ and $s > 0$, let $a_i(s) = a_i s^2$, $b_i(s) = b_i s^2$, and $c_i(s) = c_i s^2$ for any i; consequently the functions shown in the proof can be chosen as

$$a(s) = \max_i \{\alpha_i a_i\} s^2, \qquad b(s) = \min_i \{\alpha_i b_i\} s^2, \qquad c(s) = \lambda_M(S) s^2.$$

Then, composite system (8.3) is eIS stable in the m.s. If, moreover, $u(t) \equiv 0$ for all $t \in \mathbb{R}_+$, then the trivial solution $x \equiv 0$ is exponentially stable in the m.s.

Example 8.1. Consider the following control system, which is a modification of the control system presented in Example 5.1,

$$\begin{cases} dx = Axdt + bf(y)dt + \sigma_{11}(x(t-1))dW_1 + \sigma_{12}(y)dW_2, & t \neq \tau_k, \\ dy = \big(-\zeta y - \xi f(y) + u\big)dt + a^T xdt + \sigma_{21}(x)dW_1 + \sigma_{22}(y(t-1))dW_2, & t \neq \tau_k, \end{cases} \tag{8.4}$$

where $x^T = (x_1, x_2, x_3, x_4)$, $y \in \mathbb{R}$ is the controller (i.e., $n_1 = 4$, $n_2 = 1$), $A \in \mathbb{R}^{4\times 4}$, $b \in \mathbb{R}^4$, $\zeta, \xi \in \mathbb{R}$, $f \in \mathbb{R}$ is continuous for all $y \in \mathbb{R}$, $f(y) = 0$ if and only if $y = 0$, and $0 < yf(y) < k|y|^2$ for all $y \neq 0$ and $k > 0$, $u \in \mathbb{R}$, $a \in \mathbb{R}^4$, $\sigma_{11} \in \mathbb{R}^{4\times 4}$, $\sigma_{12} \in \mathbb{R}^{1\times 1}$, $\sigma_{21} \in \mathbb{R}^{4\times 1}$, $\sigma_{22} \in \mathbb{R}^{1\times 1}$, $W_1 \in \mathbb{R}^4$, and $W_2 \in \mathbb{R}$.

The impulses are given by the following difference equations

$$\begin{cases} \Delta x(\tau_k) = \mathcal{I}_1(\tau_k, x(\tau_k^-)) = \frac{1}{k^2}\big(-2x_1(\tau_k^-), -2x_2(\tau_k^-), 2x_3(\tau_k^-), 0\big)^T, \\ \Delta y(\tau_k) = \mathcal{I}_2(\tau_k, y(\tau_k^-)) = -\frac{1}{1+k^2} y(\tau_k^-). \end{cases} \tag{8.5}$$

The isolated subsystems are

$$\begin{cases} dx = Axdt + \sigma_{11}(x(t-1))dW_1, & t \neq \tau_k, \\ dy = \big(-\zeta y - \xi f(y) + u\big)dt + \sigma_{22}(y(t-1))dW_2, & t \neq \tau_k. \end{cases} \tag{8.6}$$

We showed in Example 5.1 that the trivial solution $z^T = (x, y) = (0, 0) \in \mathbb{R}^5$ is exponentially stable in the m.s.

Consider now the input $u(t) = \sin(t)$ in the control system. Then, $\mathcal{L}_2 V^2(y) \leq (-2\zeta + \bar{q} + \theta) < 0$ provided that $|y| \geq \frac{2}{\theta}|\sin(t)|$, where $\theta = 1/2$ and $\bar{q} = 2$. Therefore,

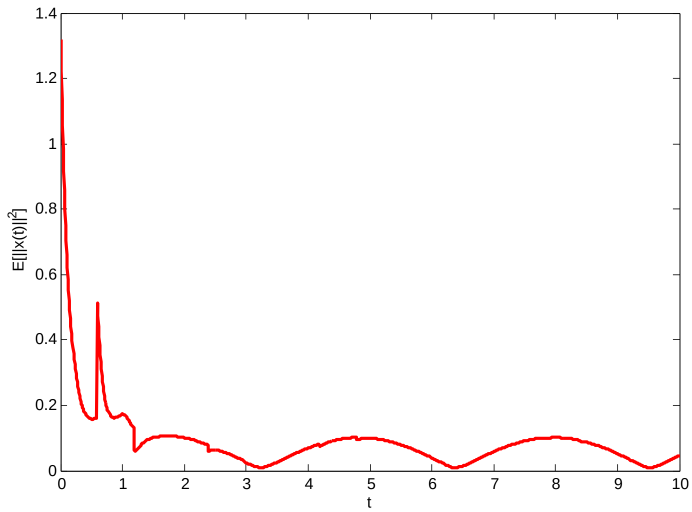

Figure 8.1: Mean square aIS stability.

provided that $\|(x,y)^T\| \geq |y| \geq 4|\sin(t)|$, $\mathcal{L}V((x,y),u) \leq -2.9169\|(x,y)^2\| < 0$. This, together with the impulsive condition, implies that composite system (8.4) with the impulsive effect in (8.5) is eIS stable in the m.s. The simulation result of this system is shown in Figure 8.1.

8.2 ISS Properties by Comparison Principle

In this section, we continue to prove the qualitative properties of composite SISD with fixed impulses by using the comparison principle developed in previous chapters.

8.2.1 Analysis by a Scalar Lyapunov Function

As achieved previously, in this subsection, we consider a scalar Lyapunov function to establish our results.

Theorem 8.2. Assume that the assumptions of Theorem 8.1 hold except that, provided that $V^i(t+s,\psi^i(s)) \le \bar{q}V(t,\psi^i(0))$ for some $\bar{q}>1$ and $s\in[-r,0]$,

$$\mathcal{L}_iV^i(t,\psi^i,u(t)) \le h_{1_i}(t,V^i(t,\psi^i(0),u(t))),$$

and

$$g_i^T(t,\psi,u(t))V^i_{\psi^i(0)}(t,\psi^i(0)) + \frac{1}{2}\sum_{j=1,i\neq j} \mathrm{tr}[\sigma_{ij}^T(t,\psi^j,u(t))V^i_{\psi^i(0)\psi^i(0)}(t,\psi^i(0))\sigma_{ij}(t,\psi^j,u(t)) < h_{2_i}(t,V(t,\psi(0)),u(t)),$$

where $\bar{h}\in\mathcal{C}\big([\tau_{k-1},\tau_k)\times\mathbb{R}_+\times\mathbb{R}^q;\mathbb{R}\big)$, $\bar{h}(t,v,u)$ is concave in v for all $t\in\mathbb{R}_+$ and $u\in\mathcal{PC}(\mathbb{R}_+;\mathbb{R}^q)$, and

$$\lim_{(t,y,v)\to(\tau_k^-,x,u)} \bar{h}(t,y,v) = \bar{h}(\tau_k^-,x,u),$$

where $\bar{h}$ is both h_{1_i} and h_{2_i}. Then, IS stability properties of composite system (8.3) are implied by those of the following scalar comparison system

$$\begin{cases} D^+v = h(t,v,u), & t\neq\tau_k, \\ v(t) = \alpha_M(d_k)v(t^-), & t=\tau_k, \\ v(t_0)=v_0\ge 0, \end{cases} \tag{8.7}$$

where h is a scalar function defined later.

Proof. For any solution $x^T = \big((w^1)^T,(w^2)^T,\cdots,(w^l)^T\big)$ of the composite system, define the composite Lyapunov function candidate by

$$V(t,x)=\sum_{i=1}^{l}\alpha_iV^i(t,w^i),$$

where, for all $i=1,2,\cdots,l$, $\alpha_i>0$ and V^i is the Lyapunov function related to the

i^{th} isolated subsystems $\mathbb{S}_i^u$. Then, whenever $V(t, x_t) \leq \bar{q}V(t, x)$,

$$\begin{aligned}
\mathcal{L}V(t, x_t, u) = & \sum_{i=1}^{l} \alpha_i \Big\{ \mathcal{L}_i V^i(t, w^i, u) + g_i(t, x_t, u)^T V^i_{w^i}(t, w^i) \\
& \qquad + \frac{1}{2} \sum_{j=1, i\neq j}^{l} \mathrm{tr}\big[\sigma_{ij}(t, w_t^j, u) V^i_{w^i w^i}(t, w^i) \sigma_{ij}(t, w_t^j, u)\big] \Big\} \\
\leq & \sum_{i=1}^{l} \alpha_i \Big\{ h_{1_i}(t, V^i(t, w^i), u) + h_{2_i}(t, V^i(t, w^i), u) \Big\} \\
=: & h(t, V(t, x, u)), \qquad\qquad t \neq \tau_k.
\end{aligned}$$

It follows that, after applying Itô formula and taking the mathematical expectation,

$$D^+ m(t) \leq h(t, m(t), u(t)),$$

and, at $t = \tau_k$, we have shown in Theorem 8.1 that

$$m(t) \leq \alpha_M(d_k) m(t^-).$$

In summary, we have

$$\begin{cases} D^+ m \leq h(t, m(t), u(t)), & t \neq \tau_k, \\ m(t) \leq \alpha_M(d_k) m(t^-), & t = \tau_k, \\ m(t_0) \leq u_0, \end{cases} \tag{8.8}$$

which is compared with the scalar comparison system (8.7). By Theorem 7.4, provided that (8.7) is uniformly aIS stable, composite system (8.3) is uniformly aIS stable in the m.s. This completes the proof.

We should remark that, in Theorem 7.4, the difference equation is input-dependent, i.e., generally, $\gamma(\|u(\tau_k^-)\|) \not\equiv 0$.

In the following corollary, we state a special case of Theorem 8.2, which is also similar to Corollary 7.1.

Corollary 8.2. In Theorem 8.2, assume that there exist $p \in \mathbb{R}$, $c \in \mathcal{K}_1$, and $\gamma \in \mathcal{K}$ such that

$$h(t, V(t,x), u) = pc(V(t,x)) + \gamma(\|u\|).$$

Suppose further that there exist ζ_k and $\varrho_0 > 0$ such that, for all $z \in (0, \varrho_0)$ and $k = 1, 2, \cdots$, the following inequality

$$p(\tau_k - \tau_{k-1}) + \ln \alpha_M(d_k) \leq -\gamma_k, \qquad k = 1, 2, \cdots,$$

holds. Then, if $\zeta_k \geq 0$, the composite system is uniformly IS stable in m.s., and if $\sum_{k=1}^{\infty} \zeta_k = +\infty$, the system is aIS stable in the m.s.

Example 8.2. Consider composite system (8.4). By the previous analysis, we have, for the same Lyapunov function candidates,

$$\mathcal{L}_2 V^2(y) \leq (-10 + 0.0001\bar{q} + \theta)V^2(y), \qquad \text{whenever} \quad |y| \geq \frac{2}{\theta}|\sin(t)|,$$

and

$$\mathcal{L}V(x_t, u) \leq -3.9997V((x,y)) =: h(V(x,y)),$$

whenever $\|(x,y)^T\| \geq |y| \geq 4|\sin(t)|$.

Consider the impulsive difference equations

$$\begin{cases} \Delta x(\tau_k) = -\frac{5}{4}x(\tau_k^-), \\ \Delta y(\tau_k) = -\frac{5}{4}y(\tau_k^-). \end{cases} \tag{8.9}$$

Then, $V(x(\tau_k), y(\tau_k)) \leq \alpha_k V(x(\tau_k^-), y(\tau_k^-))$, where $\alpha_k = \frac{1}{16}$. Making use of the condition in Corollary 8.2, one may obtain $\tau_k - \tau_{k-1} > 0.69$ for any k, which means that the conditions of Corollary 7.2 are satisfied. Therefore, the impulsive

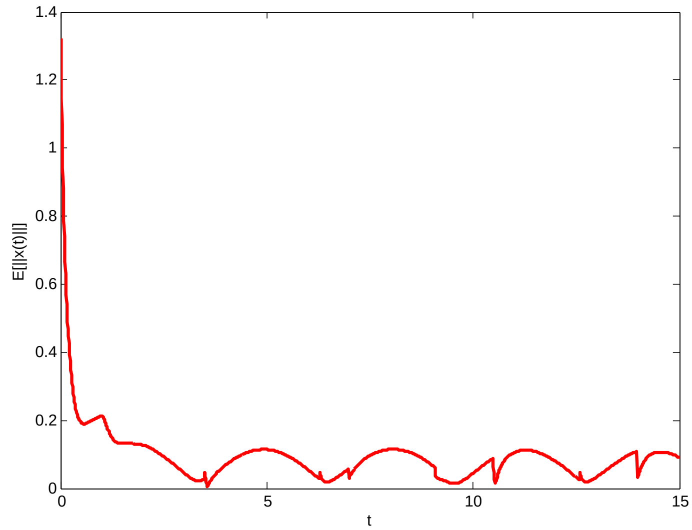

Figure 8.2: Mean square aIS stability.

composite system given in (8.4) and (8.9) is eIS stable in the m.s. The simulation result is shown in Figure 8.2.

According to Corollary 8.2, one can show that the given impulses stabilize the unstable system (with input) (8.4), where matrix A in this case is given by

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & -10 \end{pmatrix}.$$

Then, one may get $\mathcal{L}_1 V^1(x) \leq (10 + 0.0001\bar{q})V^1(x)$, which shows that the isolated subsystem is unstable, while $\mathcal{L}_2 V^2(y) \leq -9.4998V^2(y)$, whenever $|y| \geq \frac{2}{\theta}|\sin(t)|$. Putting these together, one may get $h(V(x,y),u) = 7.0005V(x,y) > 0$, which shows that the non-impulsive composite system is unstable. By considering the stabilizing impulsive effects, we obtain $\tau_k - \tau_{k-1} \leq 0.33$. Figure 8.3 shows the simulation result.

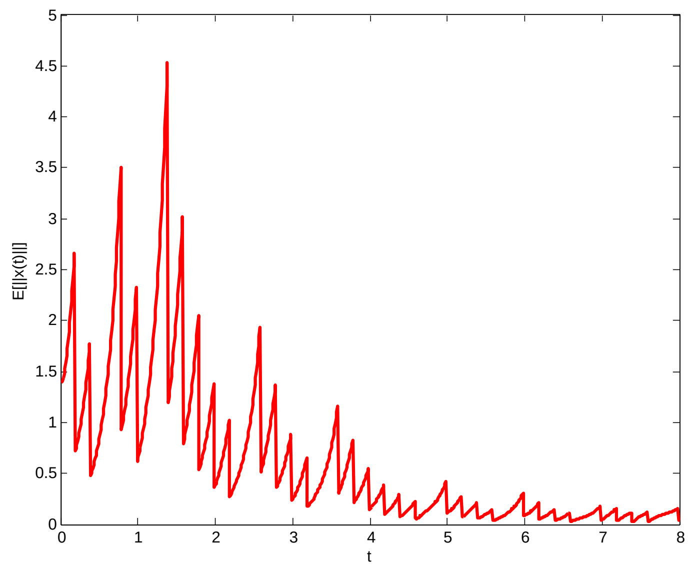

Figure 8.3: Mean square aIS stability.

8.2.2 Analysis by a Vector Lyapunov Function

In this subsection, we want to develop a comparison principle and then prove the stability properties of the composite SISD with fixed impulses (8.3). Our analysis is based on using a vector Lyapuonv function and decomposing the large scale system into subsystems with smaller scales. Before stating the main theorems, we present the following definition, which will be used in the rest of this subsection.

Definition 8.2. The isolated subsystem $\mathbb{S}_i^u$ in (8.2) is said to possess **Property D** if Assumptions A2 and B1 hold, there exist functions $c_i \in \mathcal{K}_1$, a_i which satisfies the conditions of $\bar{h}$ in Theorem 8.2, and $V^i \in \mathcal{C}^{1,2}\big([-r,\infty) \times S(\varrho)\big);\mathbb{R}_+\big)$, which is decrescent and satisfies

(i) for all $(t, \psi^i(0)) \in [-r,\infty) \times S(\varrho)$,

$$c_i(\|\psi^i(0)\|^2) \leq V^i(t, \psi^i(0)), \qquad \text{(a.s.)},$$

and, for all $t \neq \tau_k$, $\psi^i \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$, and $u \in \mathcal{PC}(\mathbb{R}_+;\mathbb{R}^q)$,

$$\mathcal{L}_i V^i(t, \psi^i, u) \leq a_i(t, V^i(t, \psi^i(0)), u(t)), \qquad \text{(a.s.)},$$

provided that $V^i(t+s, \psi^i(s)) \leq \bar{q} V(t, \psi^i(0))$ for some $\bar{q} > 1$ and $s \in [-r, 0]$;

(ii) for any $\tau_k \in \mathbb{T}$ and $\psi^i \in \mathcal{PC}\big([-r,0]; S(\varrho)\big)$,

$$V^i\big(\tau_k, \psi^i(0) + \mathcal{I}_i(\tau_k, \psi^i(\tau_k^-))\big) \le \alpha(d_k) V^i(\tau_k^-, \psi^i(0)), \qquad \text{(a.s.)},$$

where $\psi^i(0^-) = \psi^i(0)$ and $\prod_{k=1}^{\infty} \alpha(d_k) < \infty$ with $\alpha(d_k) > 1$ for all k.

In the following theorems, we state and prove a comparison principle and IS stability results for composite system (8.3).

Theorem 8.3. Assume that the following assumptions hold:

(i) every isolated subsystem $\mathbb{S}_i^u$ has Property D;

(ii) for any $i = 1, 2, \cdots, l$, there exist a function $\bar{b}_i \in \mathcal{C}\big([\tau_{k-1}, \tau_k) \times \mathbb{R}_+ \times \mathbb{R}^q; \mathbb{R}\big)$ and $\bar{b}_i$ is quasi monotone nondecreasing such that

$$g_i^T(t, \psi, u) V^i_{\psi^i(0)}(t, \psi^i(0)) + \frac{1}{2} \sum_{j=1, i \ne j}^{l} \mathrm{tr}[\sigma_{ij}^T(t, \psi^j, u) V^i_{\psi^i(0)\psi^i(0)}(t, \psi^i(0)) \sigma_{ij}(t, \psi^j, u)$$
$$< \bar{b}_i(t, V(t, \psi(0)), u),$$

where $V^T(t, x) = \big(V^1(t, w^1), V^2(t, w^2), \cdots, V^l(t, w^l)\big)$;

(iii) let $a^T(\cdot) = \big(a_1(\cdot), a_2(\cdot), \cdots, a_l(\cdot)\big) \in \mathcal{L}_{ad}\big(\Omega, L[t_0, t_0 + \alpha]\big)$ and $\bar{b}^T(\cdot) = \big(\bar{b}_1(\cdot), \bar{b}_2(\cdot), \cdots, \bar{b}_l(\cdot)\big) \in \mathcal{L}^2_{ad}\big(\Omega, L^2[t_0, t_0 + \alpha]\big)$, where $a_i(\cdot)$ and $\bar{b}_i(\cdot)$ are defined in assumptions (i) and (ii), respectively, and assume that the following inequalities hold:

$$|a(t, v', u') + \bar{b}(t, v', u')|^2 \le h_1(t) + h_2(t)\kappa(\|v'\|^2),$$
$$|a(t, v', u') + \bar{b}(t, v', u') - a(t, v'', u'') - b(t, v'', u'')| \le K\big(\|v' - v''\| + \|u' - u''\|\big),$$

where $t \in \mathbb{R}_+$, h_1 and h_2 are $\mathcal{PC}(\mathbb{R}_+, \mathbb{R}_+)$ functions, $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, increasing, concave function, v' and $v'' \in \mathbb{R}^l_+$, u' and $u'' \in \mathbb{R}^q$, and $K > 0$;

(iv) there exists a function $p : \mathbb{R}^l \times \mathbb{R}_+ \times \mathbb{R}^q \to \mathbb{R}$ such that

$$\sup_{V(t,x)\leq v} \sum_{i,j=1}^{l} \|\sigma_{ij}^T(t,\psi^j,u)V_{\psi^i(0)^i(t,\psi^i(0))}(t,\psi^i(0))\|^2 \leq p(t,v,u),$$

where

$$p(t,v,u) \leq h_1(t) + h_2(t)\kappa(\|v\|^2).$$

Then, $V(t_0,x_0) < v_0$ implies that $V(t,x(t)) < v(t)$, for all $t \geq t_0$, where $v = (v^1, v^2, \cdots, v^l)^T$ is a solution of the vector stochastic impulsive differential equation

$$\begin{cases} dv = [a(t,v,u) + \bar{b}(t,v,u)]dt + \mathbb{V}dW(t), & t \neq \tau_k, \\ \Delta v(t) = \alpha_M(d_k)v(t^-), & t = \tau_k, \end{cases} \tag{8.10}$$

with $\mathbb{V} = [v_{ij}]_{l\times l}$ being a matrix random process such that

$$\|\mathbb{V}\|^2 \leq p(t,v,u),$$

and $\alpha_M(\cdot) = \max_i\{\alpha_i(\cdot), i = 1, 2, \cdots, l\}$.

Proof. For any solution x of composite SISD (8.3), define the vector Lyapunov function for the composite system

$$V^T(t,x(t)) = \big(V^1(t,w^1), V^2(t,w^2), \cdots, V^l(t,w^l)\big),$$

where V^i is the Lyapunov function related to the i^{th} isolated subsystem $\mathbb{S}_i^u$. Then, by the vector form of the Itô formula, we have

$$dV^T(t,x(t)) = \big(dV^1(t,w^1), dV^2(t,w^2), \cdots, dV^l(t,w^l)\big),$$

where

$$dV^i(t,w^i) < [a_i(t,V^i(t,w^i),u) + \bar{b}_i(t,V^i(t,w^i),u)]dt + \sum_{ij}^{l} v_{ij}dW_i(t),$$

where $v_{ij} = V_{w^i}^{iT}(t, w^i)\sigma_{ij}(t, w_t^j, u)$. It follows that the vector differential inequality is

$$dV(t, x(t)) < \big[a(t, V(t, x(t)), u(t)) + \bar{b}(t, V(t, x(t)), u(t))\big]dt + \mathbb{V}dW(t),$$

for any $t \in [\tau_{k-1}, \tau_k)$ and $k \in \mathbb{N}$.

By the same argument followed in proving Theorem 5.4, we have, at the impulsive moments $t = \tau_k$,

$$V^T(\tau_k, x(\tau_k)) = \alpha_M(d_k)V^T(\tau_k^-, x(\tau_k^-)),$$

and, for all $t \geq t_0$ and $i = 1, 2, \cdots, l$, $V_i(t, w^i(t)) < v_i(t)$. It follows that

$$V(t, x(t)) < v(t), \qquad \forall\, t \geq t_0.$$

Finally,

$$\begin{aligned} \|\mathbb{V}\|^2 &= \sum_{i,j=1}^{l} \|v_{ij}\|^2 = \sum_{ij}^{l} \|V_{w^i}^{iT}(t, w^i)\sigma_{ij}(t, w_t^j, u)\|^2 \\ &\leq \sup_{V(t,x)<v(t)} \sum_{i,j=1}^{l} \|V_{w^i}^{iT}(t, w^i)\sigma_{ij}(t, w_t^j, u)\|^2 \\ &\leq p(t, v, u). \end{aligned}$$

This completes the proof.

In the following theorem, we state and prove the stability result.

Theorem 8.4. Suppose that the assumptions of Theorem 8.3 hold, and there exist $\alpha_1 \in \mathcal{K}_2$, $c \in \mathcal{K}_1$, a function $\bar{h} \in C\big([\tau_k, \tau_{k-1}) \times \mathbb{R}^l; \mathbb{R}_+\big)$, $z \in \mathbb{R}^l$, and $U \in \mathcal{C}^{1,2}\big([\tau_k, \tau_{k-1}) \times \mathbb{R}^l : \mathbb{R}_+\big)$ which is decrescent, $U(t, 0) = 0$, and satisfies

(i) for all $t \in \mathbb{R}_+$ and $v \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^l)$,

$$\alpha_1(\|v\|^2) \leq U(t, v), \qquad \text{(a.s.)},$$
$$z^T U_{vv}(t, v) z \leq \bar{h}(t, v)\|z\|^2, \qquad \text{(a.s.)},$$

and

$$U_t(t, v) + U_v(t, v)\big[a(t, v, u) + \bar{b}(t, v, u)\big] + \frac{1}{2}h(t, v)p(t, v, u) \leq -c(\|v\|), \quad \text{(a.s.)},$$

whenever $\|v\| > V^i(t, w^i) \geq \rho(\|u\|)$, where $\rho \in \mathcal{K}$ and for any i;

(ii) for any $\tau_k \in \mathbb{T}$ and $v \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^l)$,

$$U(\tau_k, v(\tau_k)) = \alpha(d_k) U(\tau_k^-, v(\tau_k^-)).$$

Then, the IS stability properties of comparison system (8.10) imply the corresponding properties of composite SISD (8.3).

Proof. Let $v \geq 0$ be the solution vector of comparison system (8.10). By Itô formula, we obtain

$$\mathcal{L}U(t, v, u) \leq -c(\|v\|), \qquad \text{whenever} \quad \|v\| \geq \rho(\|u\|).$$

By the previous analysis, we conclude that (8.10) is aIS stable in the m.s. As for composite system (8.3), we have shown in Theorem 8.3 that the vector inequality $V(t, x(t)) < v(t)$ holds for all $t \geq t_0$, from which, together with the condition in (i), we obtain $\|v\| > \|V(t, x)\| \geq V^i(t, w^i) \geq \rho(\|u\|)$. It follows that

$$\alpha_1(\|x(t)\|^2) \leq \Big[\sum_{i=1}^{l} c_i^2(\|w^i\|^2)\Big]^{1/2} \leq \|V(t, x(t))\| < \|v(t)\|,$$

where $\alpha_1 \in \mathcal{K}_1$. Taking the mathematical expectation and then applying α_1^{-1} to both sides imply the desired result. This completes the proof.

In the following corollary, we consider a special case of Theorem 8.4.

Corollary 8.3. In Theorem 8.4, assume that there exists a positive constant c such that $c(s) = c\,s$ for all $s \geq 0$, and, whenever $\|v\| \geq \rho(u)$,

$$\beta^T\big(a(t,v,u) + \bar{b}(t,v,u)\big) \leq -c\|v\|,$$

for some positive vector $\beta \in \mathbb{R}^l$. Then, system (8.10) is aIS stable in the m.s.

Proof. Let v be the solution of (8.10) and define $U(t,v) = \beta^T v$ as a Lyapunov function candidate. Then, $U_v = \beta^T$ and $U_{vv} = 0 \in \mathbb{R}^{l\times l}$. It follows that $\mathcal{L}U(t,v,u) \leq -c\|v\|$, whenever $\|v\| \geq \rho(u)$, which implies the required result.

Example 8.3. Consider composite system (8.4) and the same Lyapunov functions. We showed in Example 5.3 that $a(V(x,y)) = (a_1(V^1(x)), a_2(V^2(y)))^T = (\sigma_1 V^1(x), \sigma_2 V^2(y))^T$ and $\bar{b}(V(x,y)) = \big((2k{+}0.0001)V(x,y), 2.0001V(x,y)\big)^T$. Clearly, functions a and $\bar{b}$ satisfy the conditions in (iii) of Theorem 8.3. As for condition (iv), we have

$$\sup_{V\leq v}\sum_{i,j=1}^{l}\|\sigma_{ij}^T(w^i)V_{w^i}(w^i)^i\|^2 \leq 8\bar{\xi}\|v\|^2,$$

i.e., $p(v) \leq 8\bar{\xi}\|v\|^2$, where $\bar{\xi} = \max\{\xi_1, \xi_2\}$, $\xi_1 = 1.0004$, and $\xi_2 = 1.0002$ with $\bar{q} = 2$.

Also, at the impulsive moments given in Example 5.1, we get

$$V^T(x(\tau_k), y(\tau_k)) \leq (1 + \frac{1}{k^2})v^T(\tau_k^-) = v^T(\tau_k).$$

Thus, by Theorem 8.3, $V(x(t), y(t)) < v(t)$, for all $t \geq t_0$, which proves the comparison result. As for the stability result, we choose $U(v) = v_1 + v_2$, i.e., $\beta^T = (1,1)$, which gives $\mathcal{L}U(v) \leq -5.9997U(v)$, where we have chosen $k = 2$, and

$U(v(\tau_k)) = \alpha_M(d_k)U(v(\tau_k^-))$, where $\alpha_M(d_k) = 1+\frac{1}{k^2}$. Therefore, the trivial solution of unforced composite system (8.4) is asymptotically stable in the m.s.

Consider now the input $u(t) = \sin(t)$ in the isolated control subsystem. Then, one can easily find $\mathcal{L}_1 V^1(x) \leq \sigma_1 V^1(x)$ and $\mathcal{L}_2 V^2(y) \leq \sigma_2^* V^2(y) + u^2$, where $\sigma_2^* = -9 + 0.0001\bar{q}$, from which we get $a(V(x,y),u) = (\sigma_1 V^1(x), \sigma_2^* V^2(y) + u^2)^T$ and $\bar{b}(V(x,y),u) = \bar{b}(V(x,y))$, which satisfy the conditions in (iii) of Theorem 8.3. At the impulsive times τ_k, we have, for $i = 1, 2$,

$$\begin{aligned} V^i(w^i(\tau_k)) - v_i(\tau_k) &\leq \alpha_i(d_k)\big(V^i(w^i(\tau_k^-)) - v_i(\tau_k^-)\big) \\ &\leq \alpha_M(d_k)\big(V^i(w^i(\tau_k^-)) - v_i(\tau_k^-)\big) \\ &< 0. \end{aligned}$$

As concluded earlier, $V(t,x) < v(t)$ for all $t \geq t_0$. The comparison result is complete. As for the stability property, we have, from the diffusion operator of the isolated control subsystem,

$$\mathcal{L}_2 V^2(y) \leq -\bar{\sigma}_2^* V^2(y),$$

where $\bar{\sigma}_2^* = \sigma_2^* - 1/2$, whenever $V^2(y) = y^2 > 2u^2$ (i.e., $|y| > \sqrt{2}|u| = \rho(|u|)$), which implies that $\|v\| > \|V(x,y)\| \geq \rho(|u|)$. At the impulsive moments τ_k, we have

$$U(v(\tau_k)) = \beta^T v(\tau_k) = \beta^T \alpha_M(d_k) v(\tau_k^-) = \alpha_M(d_k)\beta^T y(\tau_k^-) = \alpha_M(d_k) U(y(\tau_k^-)).$$

Therefore, by Theorem 8.4, composite SISD (8.4) and (8.5) is uniformly aIS stable in the m.s.

8.3 Conclusion

In this chapter, we considered a large scale nonlinear stochastic impulsive systems with time delay and input. The main interest was to establish m.s. asymptotic IS stability of the system. We started with developing Lyapunov-type theorems using Razumikhin technique. Later, the focus was on establishing a comparison principle to achieve the same stability property. We also applied the theoretical proposed results to an automated control system.

Chapter 9

Reliable Robust Control for Uncertain SISD

This chapter deals with the problem of designing a robust reliable control for a class of uncertain stochastic impulsive systems with time delay. The uncertainties are assumed to be time-varying and norm-bounded, the time delay is constant, and the nonlinear disturbances are unknown, but have linear-growth-type bounds. The actuators are categorized into two sets. One set has actuators, which are susceptible to failure, while the other set is robust to failures and never fails. Particularly, the interest is to design a state feedback controller such that, for all admissible uncertainties and actuator failures occurring in a prespecified subset of actuators, the plant preserves exponential stability in the mean square and independently of the time delay. Using Razumikhin technique, Lyapunov-like sufficient conditions are developed to guarantee the stability property, which leads to solving a modified algebraic Riccati equation. The material of this chapter forms the basis of [Alw-g].

Consider the following stochastic control system with time delay and impulsive effects

$$dx(t) = [(A + \Delta A(t))x + (\bar{A} + \Delta \bar{A}(t))x_t + Bu + f(x_t)]dt$$
$$+ g(x_t)\, dW(t), \qquad t \neq \tau_k, \tag{9.1a}$$
$$\Delta x(t) = C_k x(t^-), \qquad t = \tau_k, \tag{9.1b}$$
$$x_{t_0}(s) = \phi(s), \qquad s \in [-r, 0], \tag{9.1c}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^q$ is the control input of the form Kx, with $K \in \mathbb{R}^{n\times q}$ being a control matrix gain, $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^{n\times m}$ are disturbance functions, for any $k \in \mathbb{N}$, C_k is a matrix of real numbers, which represents the impulse intensity, $\phi \in \mathbb{R}^n$ is the initial state function, which is assumed to be in $\mathcal{L}^2_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$, τ_k represents constant impulsive moments, which satisfies $0 = \tau_0 < \tau_1 < \cdots < \tau_k < \cdots$, and $\lim_{k\to\infty} \tau_k = \infty$. In difference equation (9.1b), $\Delta(t) = x(t^+) - x(t^-)$, where $x(t^-)$ (and $x(t^+)$) is the state just before (and after) the impulsive action. We also assume that the solution is right continuous (i.e., $x(t^+) = x(t)$). $A, \bar{A}$, and B are real constant matrices of appropriate dimensions, and ΔA and $\Delta \bar{A}$ are real-valued matrices, which are piecewise continuous functions representing parameter uncertainties with bounded norms. To guarantee that SISD (9.1) has unique regular solution, we assume that functionals $f \in \mathcal{L}_{ad}(\Omega, L[a, b])$ and $g \in \mathcal{L}_{ad}(\Omega, L^2[a, b])$ satisfy the Lipschitz condition. We also assume that $f(0) = 0 \in \mathbb{R}^n$ and $g(0) = 0 \in \mathbb{R}^{n\times m}$ to ensure that the system admits a trivial solution. A symmetric matrix P is said to be positive definite if the scalar $x^T Px > 0$ for all nonzero $x \in \mathbb{R}^n$ and $x^T Px = 0$ for $x = 0$. Denote by $\lambda_{\min}(P)$ (and $\lambda_{\max}(P)$) the smallest (and largest) eigenvalue of P.

The following definition and assumption will be needed throughout this chapter.

Definition 9.1. The trivial solution of system (9.1) is said to be *robustly globally* *exponentially stable in the m.s.* if there exist positive constants λ and K such that, if $\phi \in \mathcal{L}^2_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$, then

$$\mathbb{E}[\|x(t)\|^2] \leq K\mathbb{E}[\|\phi\|_r^2]e^{-\lambda(t-t_0)}, \qquad \forall\, t \geq t_0,$$

for any solution $x(t) = x(t, t_0, \phi)$ of (9.1).

Assumption A3. For all $t \in \mathbb{R}_+$, the admissible parameter uncertainties are defined by

$$\Delta A = D\mathcal{U}(t)H \quad \text{and} \quad \Delta \bar{A} = \bar{D}\mathcal{V}(t)\bar{H},$$

where $D, \bar{D}, H$, and $\bar{H}$ are known real constant matrices with appropriate dimensions that give the structure of the uncertainties, and $\mathcal{U}$ and $\mathcal{V}$ are unknown real time-varying matrices containing the uncertain parameters in the linear parts and satisfy

$$\|\mathcal{U}(t)\| \leq 1 \quad \text{and} \quad \|\mathcal{V}(t)\| \leq 1,$$

respectively.

As for the reliability with respect to actuator failures, it is common practice that the m control actuators are categorized into two groups. Let the set of actuators that are susceptible to failures be denoted by $\Sigma \subseteq \{1, 2, \cdots, m\}$, where the actuators may fail. The other set of actuators, which are robust to failures and are needed to stabilize the system under consideration, is denoted by $\bar{\Sigma} \subseteq \{1, 2, \cdots, m\} - \Sigma$, where the actuators never fail. This means that, in the stabilization problem, the elements of Σ are redundant, but useful in improving the performance of the control systems, while the elements of $\bar{\Sigma}$ are required to stabilize the system. Consider the decomposition of the control matrix

$$B = B_{\Sigma} + B_{\bar{\Sigma}},$$

where B_{Σ} is the control matrix associated with the set Σ, and $B_{\bar{\Sigma}}$ is the control matrix associated with the complementary subset of the control input, i.e., B_{Σ} and $B_{\bar{\Sigma}}$ are generated by zeroing out the columns corresponding to Σ and $\bar{\Sigma}$, respectively. Let $\sigma \in \Sigma$ correspond to a particular subset of the susceptible actuators that experience a failure, and assume that the controller failures are modeled as the control input failures $u_i = 0$ for all $i \in \sigma$. The decomposition becomes

$$B = B_{\sigma} + B_{\bar{\sigma}},$$

where B_{σ} and $B_{\bar{\sigma}}$ have the same definitions of B_{Σ} and $B_{\bar{\Sigma}}$.

As mentioned earlier, our interest is to design a state feedback controller of the form

$$u(t) = Kx(t), \tag{9.2}$$

which robustly globally exponentially stabilizes SISD (9.1) in the m.s. for all admissible uncertainties and all actuator failures occurring with the pre-specified subset Σ.

Since the control input u is applied to the system plant only through the normal actuators and the outputs of the faulty actuators are assumed to be zero, the closed-loop control system is

$$dx(t) = [(A + \Delta A(t) + B_{\bar{\sigma}}K)x + (\bar{A} + \Delta\bar{A}(t))x_t + f(x_t)]dt$$
$$+ g(x_t)\, dW(t), \qquad t \neq \tau_k, \tag{9.3a}$$
$$\Delta x(t) = C_k x(t^-), \qquad t = \tau_k, \tag{9.3b}$$
$$x_{t_0}(s) = \phi(s), \qquad s \in [-r, 0]. \tag{9.3c}$$

9.1 Stability and Stabilization of Uncertain SISD

In this section, we state and prove the main contribution of this chapter. We start with proving some matrix inequalities that will be used in the proofs of the main theorems.

Lemma 9.1. For any arbitrary $\varepsilon_1 > 0$ and a positive-definite matrix P, we have

$$2x^T P\bar{A}x_t \leq x^T\Big(\varepsilon_1 P\bar{A}\bar{A}^T P + \frac{\bar{q}}{\varepsilon_1}I\Big)x,$$

where $\bar{q} > 1$ such that $V(x_t) \leq \bar{q}V(x)$ with V being a positive-definite function.

Proof. Let $\varepsilon_1 > 0$. Since

$$\Big(\sqrt{\varepsilon}x^T(P\bar{A}) - \frac{1}{\sqrt{\varepsilon}}x_t^T\Big)\Big(\sqrt{\varepsilon}x^T(P\bar{A}) - \frac{1}{\sqrt{\varepsilon}}x_t^T\Big)^T \geq 0,$$

then

$$\begin{aligned}0 &\leq \varepsilon_1 x^T P\bar{A}\bar{A}^T Px + \frac{1}{\varepsilon_1}x_t^T x_t - x^T(P\bar{A})x_t - x_t(P\bar{A})^T x \\ &= \varepsilon_1 x^T P\bar{A}\bar{A}^T Px + \frac{1}{\varepsilon_1}x_t^T x_t - 2x^T(P\bar{A})x_t.\end{aligned}$$

It follows that

$$\begin{aligned}2x^T(P\bar{A})x_t &\leq \varepsilon_1 x^T P\bar{A}\bar{A}^T Px + \frac{1}{\varepsilon_1}x_t^T x_t \\ &\leq \varepsilon_1 x^T P\bar{A}\bar{A}^T Px + \frac{\bar{q}}{\varepsilon_1}x^T x \\ &= x^T\Big(\varepsilon_1 P\bar{A}\bar{A}^T Px + \frac{\bar{q}}{\varepsilon_1}I\Big)x.\end{aligned}$$

Lemma 9.2. For any arbitrary $\varepsilon_2 > 0$, $\bar{q} > 1$ as defined in Lemma 9.1, and a positive-definite matrix P, we have

$$2x^T P(\Delta\bar{A})x_t = 2x^T P(\bar{D}\mathcal{V}(t)\bar{H})x_t \leq \varepsilon_2 x^T P\bar{D}\bar{D}^T Px + \frac{\bar{q}}{\varepsilon_2}\|\bar{H}\|^2 x^T x.$$

Proof. Let $\varepsilon_2 > 0$. Then,

$$\begin{aligned}0 &\leq \Big(\sqrt{\varepsilon_2}x^T P\bar{D}\mathcal{V}(t) - \frac{1}{\sqrt{\varepsilon_2}}x_t^T\bar{H}^T\Big)\Big(\sqrt{\varepsilon_2}x^T P\bar{D}\mathcal{V}(t) - \frac{1}{\sqrt{\varepsilon_2}}x_t^T\bar{H}^T\Big)^T \\ &= \varepsilon_2 x^T P(\mathcal{V}(t)\mathcal{V}^T(t))\bar{D}\bar{D}^T Px + \frac{1}{\varepsilon_2}x_t^T\bar{H}^T\bar{H}x_t - x^T P\bar{D}\mathcal{V}(t)\bar{H}x_t - x_t^T\bar{H}^T\mathcal{V}^T(t)\bar{D}^T Px.\end{aligned}$$

It follows that

$$\begin{aligned}2x^T P\bar{D}\mathcal{V}(t)\bar{H}x_t &\leq \varepsilon_2 x^T P\bar{D}\bar{D}^T Px + \frac{1}{\varepsilon_2}x_t^T\bar{H}^T\bar{H}x_t \\ &\leq \varepsilon_2 x^T P\bar{D}\bar{D}^T Px + \frac{\bar{q}}{\varepsilon_2}\|\bar{H}\|^2 x^T x \\ &= x^T\Big(\varepsilon_2 P\bar{D}\bar{D}^T P + \frac{\bar{q}}{\varepsilon_2}\|\bar{H}\|^2 I\Big)x.\end{aligned}$$

Lemma 9.3. For any arbitrary $\varepsilon_4 > 0$, $\bar{q} > 1$ as defined in Lemma 9.1, and a positive-definite matrix P, we have

$$f^T(x_t)Px + x^T Pf(x_t) \leq x^T\Big(\varepsilon_4\bar{q}\|U\|^2 I + \frac{1}{\varepsilon_4}P^2\Big)x,$$

where U is a matrix such that

$$\|f(x_t)\|^2 \leq \|U\|^2\|x_t\|^2.$$

Proof. Let $\varepsilon_4 > 0$. Then,

$$\begin{aligned}0 &\leq \Big(\sqrt{\varepsilon_4}f^T(x_t) - \frac{1}{\varepsilon_4}x^T P\Big)\Big(\sqrt{\varepsilon_4}f^T(x_t) - \frac{1}{\varepsilon_4}x^T P\Big)^T \\ &= \varepsilon_4 F^T(x_t)F(x_t) + \frac{1}{\varepsilon_4}x^T PPx - F^T(x_t)Px - x^T Pf(x_t).\end{aligned}$$

This implies that

$$
\begin{aligned}
F^T(x_t)Px + x^T Pf(x_t) &\le \varepsilon_4 \|f(x_t)\|^2 + \frac{1}{\varepsilon_4} x^T P^2 x \\
&\le \varepsilon_4 \|U\|^2 \|x_t\|^2 + \frac{1}{\varepsilon_4} x^T P^2 x \\
&\le \varepsilon_4 \bar{q} \|U\|^2 \|x\|^2 + \frac{1}{\varepsilon_4} x^T P^2 x \\
&\le \varepsilon_4 \bar{q} \|U\|^2 x^T x + \frac{1}{\varepsilon_4} x^T P^2 x \\
&\le x^T \Big(\varepsilon_4 \bar{q} \|U\|^2 I + \frac{1}{\varepsilon_4} P^2 \Big) x.
\end{aligned}
$$

The following theorem, which is a theoretical basis in the design of reliable robust control systems, guarantees the m.s. robust global exponential stability of the trivial solution of SISD (9.1) independently of the time delay. This result is achieved if the algebraic Riccati-like equation stated in the theorem is solvable for a positive-definite matrix P.

Theorem 9.1. Let the controller gain K be given. Assume that Assumptions A1-A3 hold, there exist positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4, and a positive-definite matrix P such that the following algebraic Riccati-like matrix inequality

$$
\begin{aligned}
&\Big(A + BK\Big)^T P + P\Big(A + BK\Big) + P\Big(\varepsilon_3 DD^T + \varepsilon_1 \bar{A}\bar{A}^T + \varepsilon_2 \bar{D}\bar{D}^T + \frac{1}{\varepsilon_4} I\Big)P \\
&+ \bar{q}\Big(\frac{1}{\varepsilon_1} + \frac{\|H\|^2}{\varepsilon_2} + \varepsilon_4 \|U\|^2 + \gamma\Big)I + \frac{1}{\varepsilon_3} H^T H + \alpha P = 0
\end{aligned}
$$

holds, where $\bar{q}$ is defined in Lemma 9.1, $\alpha > 0$, and $\gamma > 0$ such that

$$
\mathrm{tr}[g^T(x_t)Pg(x_t)] \le 2\gamma \bar{q} x^T Px.
$$

Suppose further that there exists a positive constant

$$
\beta = \lambda_{\max}[(I + C_k)^T P(I + C_k)]/\lambda_{\min}(P) \tag{9.4}
$$

such that the inequality

$$\ln\beta - \nu(\tau_k - \tau_{k-1}) \le 0 \tag{9.5}$$

holds, where $0 < \nu < \alpha$ and $k = 1, 2, \cdots$. Then, uncertain SISD (9.1) is robustly globally exponentially stabilized in the m.s. by the state feedback control law given in (9.2).

Proof. Let x be the solution of SISD (9.1) and $V(x) = x^T P x$ be a Lyapunov function candidate. Then,

$$\begin{aligned}
\mathcal{L}V(x) &= [(A+\Delta A)x + (\bar{A}+\Delta\bar{A})x_t + BKx + f(x_t)]^T Px \\
&\qquad + x^T P[(A+\Delta A)x + (\bar{A}+\Delta\bar{A})x_t + BKx + f(x_t)] \\
&\qquad + \frac{1}{2}\mathrm{tr}[g^T(x_t)Pg(x_t)] \\
&\le x^T[A^TP + PA + 2K^TB^TP]x + 2x^TP(\Delta A)x + 2x^TP\bar{A}x_t \\
&\qquad + 2x^TP(\Delta\bar{A})x_t + f^T(x_t)Px + x^TPf(x_t) + \gamma\bar{q}x^TPx.
\end{aligned}$$

By Lemmas 1-3 and the fact that [Li97]

$$2x^TP(\Delta A)x = 2x^T(D\mathcal{U}(t)H)Px \le x^T\Big(\varepsilon_3 PDD^TP + \frac{1}{\varepsilon_3}H^TH\Big)x,$$

for some $\varepsilon_3 > 0$, we get

$$\begin{aligned}
\mathcal{L}V(x) &\le x^T\Big((A+BK)^TP + P(A+BK)\Big)x + x^T\Big(\varepsilon_3 PDD^TP + \frac{1}{\varepsilon_3}H^TH\Big)x \\
&\qquad + x^T\Big(\varepsilon_1 P\bar{A}\bar{A}^TP + \frac{\bar{q}}{\varepsilon_1}I\Big)x + x^T\Big(\varepsilon_2 P\bar{D}\bar{D}^TP + \frac{\bar{q}}{\varepsilon_2}\|\bar{H}\|^2 I\Big)x \\
&\qquad + x^T\Big(\varepsilon_4\bar{q}\|U\|^2 I + \frac{1}{\varepsilon_4}P^2\Big)x + \gamma\bar{q}x^TPx \\
&= x^T\Big((A+BK)^TP + P(A+BK) + \varepsilon_3 PDD^TP + \frac{1}{\varepsilon_3}H^TH \\
&\qquad + \varepsilon_1 P\bar{A}\bar{A}^TP + \frac{\bar{q}}{\varepsilon_1}I + \varepsilon_2 P\bar{D}\bar{D}^TP + \frac{\bar{q}}{\varepsilon_2}\|\bar{H}\|^2 I \\
&\qquad + \varepsilon_4\bar{q}\|U\|^2 I + \frac{1}{\varepsilon_4}P^2 + \gamma\bar{q}P\Big)x
\end{aligned}$$

$$\leq -\alpha x^T P x = -\alpha V(x).$$

Applying the Itô formula to process $V(x)$ and taking the mathematical expectation give

$$D^+ m(t) \leq -\alpha m(t), \qquad t \in (\tau_{k-1}, \tau_k),$$

where $m(t) = \mathbb{E}[V(x(t))]$ for all $t \neq \tau_k$.

At $t = \tau_k$, we have

$$V(x(\tau_k)) = x^T(\tau_k) P x(\tau_k) = x^T(\tau_k^-)\big(I + C_k\big)^T P \big(I + C_k\big) x(\tau_k^-) \leq \beta V(x(\tau_k^-)),$$

which implies that

$$m(\tau_k) \leq \beta m(\tau_k^-).$$

Then, one may get

$$m(t) \leq m_0 \prod_{t<\tau_k<t} \beta e^{-\alpha(t-t_0)},$$

where $m_0 = m(t_0) = \mathbb{E}[V(x_0)]$. Applying the condition in (9.5), we get

$$m(t) \leq m_0 e^{-(\alpha-\nu)(t-t_0)}.$$

Therefore, uncertain SISD (9.1) is globally exponentially stabilized by the robust state feedback control law (9.2). This completes the proof.

Remark 9.1. The solvability condition of the algebraic Riccati-like equation is made to ensure that the positive-definite matrix V is strictly decreasing in the m.s. between the impulsive moments. Moreover, the decay rate of V is greater than the jumps caused by applying the impulsive effects (see Figures 9.1 and 9.2). This condition is summarized in (9.5). The positive tuning parameters ε_i (for

$i = 1, 2, 3, 4$) are presented to reduce the conservativeness of the matrix inequalities proved in Lemmas 9.1-9.3. We should also remark that Theorem 9.1 does not impose any restriction on the impulses and time delay.

Having proved the key-role theorem, we are in a position to propose the robust reliable control design, which provides robust global stability in the presence of actuator outages. As mentioned earlier, the outputs of the faulty actuators are assumed to be zero.

Theorem 9.2. Assume that Assumptions A1-A3 and the impulse condition in Theorem 9.1 hold, and there exist positive constant parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, and ε_5, and a positive-definite matrix P such that the following Riccati-like matrix inequality

$$A^T P + PA + P\Big(- \varepsilon_5 B_{\bar{\Sigma}} B_{\bar{\Sigma}}^T + \varepsilon_3 DD^T + \varepsilon_1 \bar{A}\bar{A}^T + \varepsilon_2 \bar{D}\bar{D}^T + \frac{1}{\varepsilon_4} I \Big) P$$
$$+ \bar{q}\Big(\frac{1}{\varepsilon_1} + \frac{\|\bar{H}\|^2}{\varepsilon_2} + \varepsilon_4 \|U\|^2\Big) I + \frac{1}{\varepsilon_3} H^T H + \gamma \bar{q} P + \alpha P = 0$$

holds. Then, uncertain SISD (9.1) is robustly globally exponentially stabilized in the m.s. by the state feedback control law $u = Kx$, where

$$K = -\frac{1}{2}\varepsilon_5 B^T P, \tag{9.6}$$

for any admissible uncertainties and all actuator failures corresponding to $\sigma \subseteq \Sigma$.

Proof. Since the control input u is applied to the system plant only through the normal actuators and the outputs of the faulty actuators are assumed to be zero, we have, from (9.6), $BK = -\frac{1}{2}\varepsilon_5 B_{\bar{\sigma}} B_{\bar{\sigma}}^T P$.

Let x be the solution of system (9.1) and $V(x) = x^T P x$ be a Lyapunov function candidate. Then, as in Theorem 9.1, we have

$$
\begin{aligned}
&(A+BK)^TP + P(A+BK) + \varepsilon_3 PDD^TP + \frac{1}{\varepsilon_3}H^TH + \varepsilon_1 P\bar{A}\bar{A}^TP + \frac{\bar{q}}{\varepsilon_1}I \\
&\quad + \varepsilon_2 P\bar{D}\bar{D}^TP + \frac{\bar{q}}{\varepsilon_2}\|\bar{H}\|^2 I + \varepsilon_4\bar{q}\|U\|^2 I + \frac{1}{\varepsilon_4}P^2 + \gamma\bar{q}P \\
&= A^TP + PA - \varepsilon_5 PB_{\bar{\sigma}}B_{\bar{\sigma}}^TP + \varepsilon_3 PDD^TP + \frac{1}{\varepsilon_3}H^TH + \varepsilon_1 P\bar{A}\bar{A}^TP + \frac{\bar{q}}{\varepsilon_1}I \\
&\quad + \varepsilon_2 P\bar{D}\bar{D}^TP + \frac{\bar{q}}{\varepsilon_2}\|\bar{H}\|^2 I + \varepsilon_4\bar{q}\|U\|^2 I + \frac{1}{\varepsilon_4}P^2 + \gamma\bar{q}P \\
&\leq A^TP + PA - \varepsilon_5 PB_{\bar{\Sigma}}B_{\bar{\Sigma}}^TP + \varepsilon_3 PDD^TP + \frac{1}{\varepsilon_3}H^TH + \varepsilon_1 P\bar{A}\bar{A}^TP + \frac{\bar{q}}{\varepsilon_1}I \\
&\quad + \varepsilon_2 P\bar{D}\bar{D}^TP + \frac{\bar{q}}{\varepsilon_2}\|\bar{H}\|^2 I + \varepsilon_4\bar{q}\|U\|^2 I + \frac{1}{\varepsilon_4}P^2 + \gamma\bar{q}P \\
&= -\alpha P < 0,
\end{aligned}
$$

where we have used the fact $B_{\bar{\Sigma}}B_{\bar{\Sigma}}^T \leq B_{\bar{\sigma}}B_{\bar{\sigma}}^T$ [Vei92] in the second last inequality. Thus, by Theorem 9.1, we conclude the desired result.

In the following, we demonstrate the proposed approach through a numerical example with simulations. We consider two cases. In Case 1, the control components (or actuators) are operating properly and, in Case 2, there is a failure in the second actuator. In both cases, the state feedback control law guarantees the stabilization requirement.

Example 9.1. Consider uncertain SISD (9.1) where $x^T = (x_1, x_2)$,

$$
A = \begin{bmatrix} 0.1 & 0.3 \\ 0 & -15 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0.2 & 1 \\ 0.5 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -0.1 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} -4 & 0.6 \\ 0 & 1 \end{bmatrix},
$$

$$
D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad H = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \mathcal{U}(t) = \sin(t), \quad f(x(t-1)) = \begin{bmatrix} -0.1x_1(t-1) \\ 0.05x_2(t-1) \end{bmatrix},
$$

$$\bar{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{H} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \mathcal{V}(t) = \cos(t), \quad g(x(t-1)) = 0.01 \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix},$$

$\varepsilon_1 = 1, \quad \varepsilon_2 = 0.1, \quad \varepsilon_3 = 0.4, \quad \varepsilon_4 = 0.2, \quad \gamma = 0.01, \quad \alpha = 2, \quad \bar{q} = 2,$

$$U = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad \text{and} \quad C_k = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \text{ for all } k.$$

Case 1: When there is no actuator failure, we have, from the algebraic Riccati-like equation,

$$P = \begin{bmatrix} 1.4721 & -0.2012 \\ -0.2012 & 1.1115 \end{bmatrix}.$$

It follows that $\beta = 3.4398$ and, after taking $\nu = 1.5 < \alpha$, $\tau_k - \tau_{k-1} > 0.62$, for all k. The simulation result is shown in Figure 9.1, where the initial function is $\phi(s) = 1 - s$ for all $s \in [-1, 0]$.

Case 2. When there is a failure in actuator 2, i.e.,

$$\bar{\Sigma} = \{2\}, \quad \text{and} \quad B_{\bar{\Sigma}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

we have

$$P = \begin{bmatrix} 1.9701 & -0.2566 \\ -0.2566 & 0.9861 \end{bmatrix}, \quad K = \begin{bmatrix} -0.4925 & 0.0642 \\ 0 & 0 \end{bmatrix},$$

where $\varepsilon_5 = 0.05$. It follows that $\tau_k - \tau_{k-1} > 1.07$ for all k. The simulation result is shown in Figure 9.2.

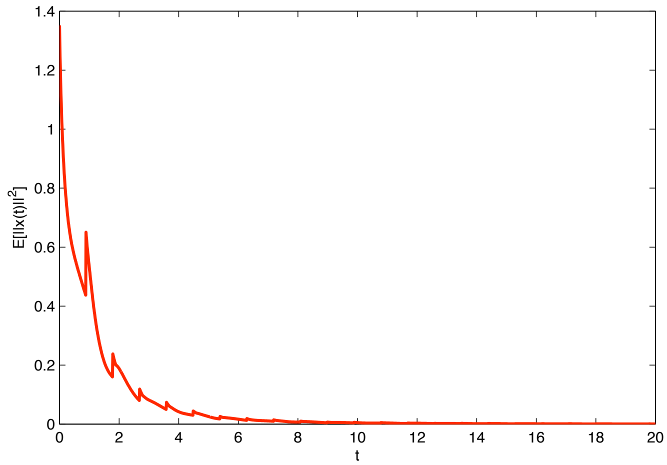

Figure 9.1: Mean square exponential stability: normal actuators.

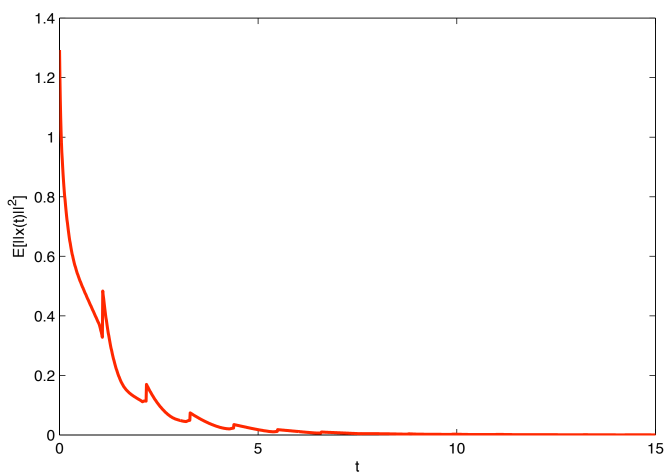

Figure 9.2: Mean square exponential stability: a failure in actuator 2.

9.2 Conclusions.

In this chapter, we addressed the problem of reliable robust controller for uncertain SISD. The focus was on the design of such a controller to guarantee stability, not only when the control components are operationally normal, but also when there is a failure in some prespecified subset of actuators. Furthermore, the outputs of the faulty actuators were assumed to be zero. The proposed approach is efficiently applicable to impulsive systems with deviated states. We applied the Razumikhin criterion, where Lyapunov functions were used in analyzing the stability property, which led to solving a Ricatti-like matrix equation.

Part II

Deterministic and Stochastic Hybrid Systems

Chapter 10

Mathematical Background

As described in the introductory chapter of this thesis, a switched system is a combination of a finite number of subsystems and a control-based switching logic to organize the switching among the subsystems. In this chapter, we focus on a mathematical formulation of such a system, including defining what is meant to be a switching signal or law. Then, we present some definitions of switched systems with time delay and are subject to some random noise represented by a Wiener process. We will also introduce some solution and stability definitions of stochastic switched systems under a given switching signal. Finally, we conclude this introductory chapter with impulsive switched systems, i.e., switched systems experience jump discontinuities in their states.

Consider the following controlled system

$$\dot{x} = f(t, x) + u(t), \tag{10.1}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^n$, where $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the system state, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the system vector field, and $u \in \mathbb{R}^n$ is the system input of the form

$$u(t) = \sum_{k=1}^{\infty} C_k x(t) l_k(t), \tag{10.2}$$

where C_k is a control gain matrix with appropriate dimensions and $l_k(\cdot)$ is the ladder function, which is defined by

$$l_k = \begin{cases} 1, & t_{k-1} \leq t < t_k, \\ 0, & \text{otherwise.} \end{cases} \tag{10.3}$$

Controller (10.2) can be written as

$$u(t) = C_k x(t), \qquad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N},$$

meaning that the controller $u(t)$ switches its values at every time instant $t = t_k$, i.e., u is a *switching controller*. Accordingly, closed-loop system (10.1) becomes

$$\begin{cases} \dot{x} = f(t, x) + C_k x, & t \in [t_{k-1}, t_k), \\ x(t_0) = x_0. \end{cases} \tag{10.4}$$

This system is called *switched (or switching) system*.

Typically, a nonlinear switched system takes the form

$$\begin{cases} \dot{x} = f_{\sigma(t)}(t, x), & \qquad t \geq t_0, \\ x(t_0) = x_0, \end{cases} \tag{10.5}$$

where $\sigma(t) : [t_0, \infty) \to \mathcal{S} = \{1, 2, \cdots, N\}$, for some $N \in \mathbb{N}$, is a piecewise constant function called *switching signal*, also known as a *switching law* or *switching rule*, and takes values in the compact set $\mathcal{S}$, which is also named by the finite state space. The role of σ is to switch among the vector fields on the right-hand side of (10.5), i.e., f_i for all $i \in \mathcal{S}$, so as to accomplish a certain desired task. The solution of

(10.5) is generally equipped with a proper switching signal, i.e., it is represented by the pair (x, σ) to emphasize the switching signal in use.

As in systems and control theory, one of the most important problems in switched systems is the search for conditions assuring stability. The basic, but very interesting, problems in stability of switched systems are introduced in [Lib99] and classified into the following three categories.

Problem A. (Stability under arbitrary switching) Finding sufficient conditions to guarantee asymptotic stability of a switched system for an arbitrary switching signal.

Problem B. (Stability by a constrained switching) Identifying the switching signals for which a switched system is asymptotically stable.

Problem C. (Stabilizability) Constructing a switching signal that makes a switched system asymptotically stable.

Problems A and B are usually considered under the hypotheses that the individual subsystems are asymptotically stable, while Problem C is considered under the assumption that the individual subsystems are unstable. Throughout this thesis, we are mainly concerned with Problems B and C.

We have mentioned earlier that switched systems inherit the stability properties of the fundamental theory of single mode systems. However, a possible strange behaviour is that switching among all asymptotically stable subsystems does not necessarily guarantee the stability of switched system. The remedy to this undesirable situation is to design a logic-based switching law in order to control the transition among the involved modes. It has been shown in [Mor96, Lib99, Hes99] that, if the running time of each single mode is sufficiently large to allow the switching effect to diminish, then it ensures that the entire switched system preserves the same stability property. This type of switching is often named by *slow* or *constrained switching* and the running time between any two successive switching moments, say t_k for any $k \in \mathbb{N}$, is called *dwell time* and is denoted by τ. This type of switching signals can be represented by

$$\mathcal{S}_{\text{inf}}(\tau) = \{\tau \big| \inf t_k - t_{k-1} \geq \tau, \, \forall k \in \mathbb{N}\}, \tag{10.6}$$

for some $\tau > 0$.

From a practical perspective, it may not be suitable to activate every individual mode over a time period τ to accomplish the asymptotic stability property. Instead, to achieve the same qualitative property, as proposed in [Hes99], the *average dwell time*, denoted by τ_{ave}, can be taken sufficiently large. This type of switching signals, denoted by $\mathcal{S}_{\text{ave}}(\tau, N_0)$, is defined as follows: for any $T \geq t \geq t_0$,

$$N_\sigma(T, t) \leq N_0 + \frac{T - t}{\tau_{\text{ave}}}, \tag{10.7}$$

where $N_\sigma(T, t)$ represents the number of switching moments of σ in the interval (t, T) and N_0 is the chatter bound.

A more general class of switching signal than $\mathcal{S}_{\text{inf}}(\tau)$ is called *Markovian switching*, in which the signal σ is a right-continuous Markov chain (or process), which takes values in a finite state space $\mathcal{S}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$; that is, jumps among the system modes follow a probabilistic rule defined by

$$\mathbb{P}\{r(t+h) = j | r(t) = i\} = \begin{cases} \gamma_{ij} h + o(h), & \text{if} \quad i \neq j, \\ 1 + \gamma_{ii} h + o(h), & \text{if} \quad i = j, \end{cases} \tag{10.8}$$

where $h > 0$. Here, $\gamma_{ij} > 0$ is the transition rate from i to j if $i \neq j$, and $\gamma_{ii} = -\sum_{j=1, j\neq i}^{N} \gamma_{ij}$ and $o(h)$ is such that $\lim_{h \to 0} \frac{o(h)}{h} = 0$.

Conventionally, if the switching signal is represented by a Markov process, the corresponding switched system (10.5) has the form

$$\begin{cases} \dot{x}(t) = f(t, x(t), \sigma(t)), & t \geq t_0, \\ x(t_0) = x_0, \quad \sigma(t_0) = \sigma_0, \end{cases} \tag{10.9}$$

for some initial state $\sigma_0 \in \mathcal{S}$.

In the nonlinear switched system (10.5), if we consider time delay and random noise, we are led to the following nonlinear *stochastic switched systems with time delay* (SSSD)

$$\begin{cases} dx(t) = f_{\sigma(t)}(t, x_t)dt + g_{\sigma(t)}(t, x_t)dW(t), & t \geq t_0, \\ x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \end{cases} \tag{10.10}$$

where $f_\sigma : \mathbb{R}_+ \times \mathcal{C}([-r, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is assumed to belong to the function class $\mathcal{L}_{ad}(\Omega; L[a, b])$ for some $a, b \in \mathbb{R}_+$ with $a < b$, $g_\sigma : \mathbb{R}_+ \times \mathcal{C}([-r, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times m}$ represents the noise intensity, which belongs to the function class $\mathcal{L}_{ad}(\Omega; L^2[a, b])$, $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}^m$ is m-dimensional Wiener process defined on the complete probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$, and $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ is the initial function, which belongs to a class of $\mathcal{F}_t$-measurable $\mathcal{C}([-r, 0]; \mathbb{R}^n)$ random variable ϕ with $\mathbb{E}[\|\phi\|_r^p] < \infty$. The latter function class is denoted by $L^p_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$ for some $p > 0$.

In the following, we present the definition of a solution of SSSD.

Definition 10.1. For any $t \in [t_0, T]$, with $t_0, T \in \mathbb{R}_+$ and $t_0 < T$, and $\mathbb{R}^n$-valued random process $x(t) = x(t, t_0, \phi)$, the pair $(x(t), \sigma(t))$ is said to be a *solution* of SSSD in (10.10) if it has the following properties:

1. $x(t)$ is continuous and adapted with respect to the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$;

2. $f_{\sigma(t)}(t, x_t) \in \mathcal{L}_{ad}(\Omega; L[t_0, T])$ and $g_{\sigma(t)}(t, x_t) \in \mathcal{L}_{ad}(\Omega; L^2[t_0, T])$;

3. the stochastic integral equation

$$x(t) = \phi + \int_{t_0}^{t} f_{\sigma(s)}(s, x_s)ds + \int_{t_0}^{t} g_{\sigma(s)}(s, x_s)dW(s) \tag{10.11}$$

holds w.p.1, where $x(t) = \phi(t)$ for all $t \in [-r, 0]$.

For simplicity of notation, we denote the solution of (10.10) by the process x. To avoid any confusion between the domains of the solution x and switching signal σ, we state it clearly that x is defined for all $t \geq -r$, while σ is defined over $\mathbb{R}_+$.

A solution $x(t)$ is said to be unique if any other solution $y(t)$ is indistinguishable form $x(t)$ for all $t \geq -r$.

Classical hypotheses that ensure the existence of a unique solution of SSSD are that the vector fields satisfy a linear growth condition, and Lipschitz condition in the second variable. The following theorem summarizes these conditions [Mao06].

Theorem 10.1. Let $\sigma : \mathbb{R}_+ \to \mathcal{S}$ be a switching signal. Assume that there exist a positive constant C such that functionals f_σ and g_σ satisfy the following conditions:

$$\|f_{\sigma(t)}(t, \psi)\|^2 + \|g_{\sigma(t)}(t, \psi)\|^2 \leq C(1 + \|\psi\|_r^2), \tag{10.12}$$

for all $t \in \mathbb{R}_+$ and $\psi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$, and

$$\|f_{\sigma(t)}(t, \psi_1) - f_{\sigma(t)}(t, \psi_2)\|^2 + \|g_{\sigma(t)}(t, \psi_1) - g_{\sigma(t)}(t, \psi_2)\|^2 \leq C\|\psi_1 - \psi_2\|_r^2, \tag{10.13}$$

for all $t \in \mathbb{R}_+$ and $\psi_1, \psi_2 \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$. Then, there exists a unique solution x defined for all $t \geq -r$ with the initial function $\phi \in L^p_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$. Furthermore, the solution x satisfies

$$\mathbb{E}\Big[\sup_{-r \leq t \leq T} \|x(t)\|^2\Big] < \infty, \qquad \text{for all } T > 0. \tag{10.14}$$

Once again, if the switching signal σ is a Markov process, which is assumed to be independent of the Wiener process, the corresponding SSSD can be written as

$$\begin{cases} dx(t) = f(t, x_t, \sigma(t))dt + g(t, x_t, \sigma(t))dW(t), & t \geq t_0, \\ x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \\ \sigma(t_0) = \sigma_0, \end{cases} \tag{10.15}$$

where $f : \mathbb{R}_+ \times \mathcal{C}([-r, 0]; \mathbb{R}^n) \times \mathcal{S} \to \mathbb{R}^n$, $g : \mathbb{R}_+ \times \mathcal{C}([-r, 0]; \mathbb{R}^n) \times \mathcal{S} \to \mathbb{R}^{n \times m}$ and $\sigma_0 \in \mathcal{S}$. The solution x of SSSD in (10.15) can be similarly defined as the solution of (10.10) except that the stochastic integral is slightly modified as follows:

$$x(t) = \phi + \int_{t_0}^{t} f(s, x_s, \sigma(s))ds + \int_{t_0}^{t} g(s, x_s, \sigma(s))dW(s), \tag{10.16}$$

which is required to hold w.p.1. We should also modify the assumption guaranteeing the existence of a unique solution, as stated in the following theorem [Mao06].

Theorem 10.2. Let $\sigma : \mathbb{R}_+ \to \mathcal{S}$ be a switching signal that is represented by a Markov process. Assume that there exist a positive constant C such that the functionals f and g satisfy the following conditions:

$$\|f(t, \psi, \sigma(t))\|^2 + \|g(t, \psi, \sigma(t))\|^2 \leq C(1 + \|\psi\|_r^2), \tag{10.17}$$

for all $t \in \mathbb{R}_+$ and $\psi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$, and

$$\|f(t, \psi_1, \sigma(t)) - f(t, \psi_2, \sigma(t))\|^2 + \|g(t, \psi_1, \sigma(t)) - g(t, \psi_2, \sigma(t))\|^2 \leq C\|\psi_1 - \psi_2\|_r^2, \tag{10.18}$$

for all $t \in \mathbb{R}_+$ and ψ_1, $\psi_2 \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$. Then, there exists a unique solution x defined for all $t \geq -r$ with the initial function $\phi \in L^p_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$. Furthermore, the solution x satisfies

$$\mathbb{E}\Big[\sup_{-r \leq t \leq T} \|x(t)\|^2\Big] < \infty, \qquad \text{for all } T > 0. \tag{10.19}$$

After having obtained some qualitative properties of SISD in previous chapters, we introduced an important diffusion operator ($\mathcal{L}$, or $\mathcal{L}V$ as a single notation) associated with the underlying stochastic differential equation and then examined its estimated upper bound along the trajectories of the system solutions. In SSSD, we continue to present such an operator. However, due to the deterministic or probabilistic nature of the switching signal σ, the operator can be defined accordingly. Particularly, if σ is of a deterministic type, then we define $\mathcal{L}_i$ (or $\mathcal{L}V_i$) as before, where i is such that $\sigma = i \in \mathcal{S}$; that is, $\mathcal{L}_i$ (or $\mathcal{L}V_i$) is the operator of the solution process of the i^{th} subsystem associated with the $\mathcal{C}^{1,2}$-function V_i, which is designated to the same subsystem. If σ, on the other hand, is a Markov process, one has to take into account the transition rates of this jump process when writing this operator. In the following definition, we state the generalized Itô formula [Mao06].

Definition 10.2. (Generalized Itô Formula) If $x(t)$ (or $(x(t), \sigma(t))$) is an Itô process governed by (10.15), and $V(t, x(t), i) \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}; \mathbb{R}_+)$ with $\sigma = i \in \mathcal{S}$, then $V(t, x(t), i)$ is an Itô process with its differential equation given by

$$dV(t, x(t), i) = \mathcal{L}V(t, x(t), i)dt + V_x(t, x(t), i)g(t, x(t), i)dW(t), \tag{10.20}$$

where

$$\begin{aligned} \mathcal{L}V(t, x(t), i) =& V_t(t, x(t), i) + V_x(t, x(t), i)f(t, x(t), i) \\ &+ \frac{1}{2}\text{tr}[g^T(t, x(t), i)V_{xx}(t, x(t), i)g(t, x(t), i)] \\ &+ \sum_{j=1}^{N} \gamma_{ij} V(t, x(t), j). \end{aligned} \tag{10.21}$$

In analyzing a certain switched system, it may be convenient to specify the switching signal σ in $\mathcal{S}$ to indicate the system mode in action, and the subinterval on which the selected mode is being activated. If, for instance, we have chosen a switching law, say Θ, then generally, we use i_k to refer to the i^{th} mode, for any $i \in \mathcal{S}$, and k^{th} subinterval $[t_{k-1}, t_k)$, for any $k \in \mathbb{N}$. Also, we denote by $\{t_k\}_{k\in\mathbb{N}}$ the switching sequence or signal, which is generated by the switching law Θ. Furthermore, whenever investigating a system property, we always assume that the switching sequence is strictly increasing and $\lim_{k\to\infty} t_k = \infty$, so long as $t \in \mathbb{R}_+$, to avoid a problem trivialness. The second issue of importance is that any mode cannot be activated on any two successive subintervals $[t_{k-1}, t_k)$ and $[t_k, t_{k+1})$, and the switching sequence in this case is usually called *minimal*. Consequently, following the above particular notation, SSSD in (10.15) is simply written as follows:

$$\begin{cases} dx(t) = f(t, x_t, i)dt + g(t, x_t, i)dW(t), \qquad t \in [t_{k-1}, t_k), \\ x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \\ \sigma(t_0) = \sigma_0, \end{cases} \tag{10.22}$$

and, by the same manner, after replacing the subscript σ by the mode number i, we write the SSSD in (10.10).

One more issue about switched systems is the stability definition. In fact, it can be formulated parallel to that of a single-mode system except that, in switched systems, we should highlight the switching law under consideration. In the following, we state some stochastic stability properties of the trivial solution of SSSD in (10.15), which of course imply the corresponding definitions of the other special systems.

Definition 10.3. For any $t_0 \in \mathbb{R}_+$ and a given switching law σ with an initial state σ_0, let $(x(t), \sigma(t))$ be any solution of (10.15), where $x(t) = x(t, t_0, \phi) \in \mathcal{C}([t_0 - r, t_0 + \alpha]; \mathbb{R}^n)$, for some $\alpha > 0$, with $\phi \in L^p_{\mathcal{F}_0}\mathcal{C}([-r, 0]; \mathbb{R}^n)$. Then, the trivial solution of (10.15) is said to be

1. *stable in the pth moment* if, for any given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a

$\delta = \delta(t_0, \varepsilon) > 0$ such that

$$\mathbb{E}[\|\phi\|_r^p] < \delta \qquad \text{implies} \qquad \mathbb{E}[\|x(t)\|^p] < \varepsilon, \quad \forall t \geq t_0;$$

2. *uniformly stable in the pth moment* if it is stable in the pth moment and $\delta = \delta(\varepsilon)$;

3. *asymptotically stable in the pth moment* if it is stable in the pth moment and there exists an $\eta = \eta(t_0) > 0$ such that

$$\mathbb{E}[\|\phi\|_r^p] < \eta \qquad \text{implies} \qquad \lim_{t\to\infty} \mathbb{E}[\|x(t)\|^p] = 0;$$

4. *uniformly asymptotically stable in the pth moment* if it is uniformly stable in the pth moment and there exists $\eta > 0$ such that, for a given $\gamma > 0$, there exists $T = T(\eta, \gamma) > 0$ such that

$$\mathbb{E}[\|\phi\|_r^p] < \eta \qquad \text{implies} \qquad \mathbb{E}[\|x(t)\|^p] < \gamma, \quad \forall t \geq t_0 + T;$$

5. *exponentially stable in the pth moment* if there exist positive constants K and λ such that

$$\mathbb{E}[\|x(t)\|^p] \leq K\mathbb{E}[\|\phi\|_r^p]e^{-\lambda(t-t_0)}, \qquad \text{whenever} \quad \mathbb{E}[\|\phi\|_r^p] < \eta.$$

Moreover, the above stability properties are said to hold *globally* if δ and η are chosen arbitrarily large.

Having familiarized ourselves with impulsive and switched systems, we are in a position to define another type of hybrid systems, namely, *impulsive switched systems*, also known as *switched systems with impulsive effects*. The impulses arise when a switched system transits from one mode to another. Such systems have applications in biology, pulse vaccination, and engineering. An early study that formulated this system and developed some of its qualitative results was in [Lak98].

Later, this type of systems was appeared in some other works including papers [Wan04, Gua05] and a book [Li05].

A nonlinear deterministic ordinary impulsive switched system can have the following form

$$\dot{x}(t) = f_{\sigma(t)}(t, x(t)), \qquad t \neq t_k, \tag{10.23a}$$

$$\Delta x(t) = \mathcal{I}(t, x(t^-)), \qquad t = t_k, \tag{10.23b}$$

$$x(t_0) = x_0, \tag{10.23c}$$

where $\sigma : [t_0, \infty) \to \mathcal{S}$ for any $t_0 \in \mathbb{R}_+$ is the switching signal that is a piecewise constant function. The discontinuities of σ, which represent the impulsive moments and at the same time switching moments, form a strictly increasing sequence $\mathbb{T} = \{t_k\}_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} t_k = \infty$. As elaborated above, if one is interested in labeling a system mode which is operating on the k^{th} subinterval, we will write $\sigma = i_k$ for any $i_k \in \mathcal{S}$. It follows that, the differential equation (10.23a) is written as follows:

$$\dot{x}(t) = f_{i_k}(t, x(t)), \qquad t \in [t_{k-1}, t_k).$$

We next define a solution of the initial value problem in (10.23).

Definition 10.4. For any $t \geq t_0$ with $t_0 \in \mathbb{R}_+$, $x \in \mathcal{PC}([t_0 - r, t_0 + \alpha]; \mathbb{R}^n)$, for some $\alpha > 0$, and a given switching signal σ, the pair $(x(t), \sigma(t))$ is said to be a *solution* of the impulsive switched system in (10.23) if

1. $x(t)$ is continuous for all $t \in \mathbb{R}_+$ except at the switching (or impulsive) moments $\mathbb{T} = \{t_k\}_{k \in \mathbb{N}}$ (i.e., $\forall t \in \mathbb{R}_+ \setminus \mathbb{T}$);

2. the derivative of x exists and continuous for all $t \neq t_k$, and at t_k the right-hand derivative exists;

3. the right-hand derivative of x satisfies the differential equation in (10.23a) for all $t \in \mathbb{R}_+ \setminus \mathbb{T}$;

4. x satisfies the difference equation (10.23b) for all $t \in \mathbb{T}$;

5. x satisfies the initial condition in (10.23c).

Finally, it could be of special interest to write the general form of the above solution, which is, after using the so-called method of steps,

$$x(t) = x_0 + \int_{t_0}^{t} f_{i_k}(s, x(s))ds + \sum_{\{k: t_0 < t_k \leq t\}} \mathcal{I}(t_k, x(t_k^-)), \tag{10.24}$$

for all $t \geq t_0$.

Chapter 11

Robust Stability and Stabilization

This chapter deals with robust stability and stabilization of uncertain time-delayed switched systems experiencing impulsive effects. The nominal ordinary version of this system (i.e., system without uncertainties and zero time lag) was introduced in the last chapter. The focus here is on uncertainties of the structured type. We study linear and weakly nonlinear systems that incorporate stable and unstable subsystems, and others consist of all unstable subsystems. The technique of multiple Lyapunov functions and dwell-time approach are used to investigate some stability properties. We also develop a switching rule to stabilize impulsive switched systems incorporating all unstable subsystems. Numerical examples are also presented to illustrate the effectiveness of the proposed approach and gain better insight into the systems. The material of this chapter has been published in [Alw09a].

The organization of this chapter is as follows: in Section 11.1, we formulate the system under consideration and introduce the material that is required to tackle the problem. Our main results are given in Section 11.2; a linear system is first studied, then a special case of nonlinear system is considered. Particular results, where systems have all unstable subsystems, are also presented in the same section.

11.1 Problem Formulation

A general impulsive switched system with time delay (ISSD) is given by

$$\dot{x}(t) = f_{\sigma(t)}(t, x(t), x_t), \qquad t \neq t_k, \tag{11.1a}$$

$$\Delta x(t) = \mathcal{I}(x(t^-), x_{t^-}), \qquad t = t_k, \tag{11.1b}$$

$$x_{t_0}(s) = \phi(s), \qquad s \in [-r, 0], \tag{11.1c}$$

where $x \in \mathbb{R}^n$ is the state vector of the system and $\sigma(t) : [t_0, \infty) \to \mathcal{S}$ is the switching signal, which takes values in the compact set $\mathcal{S} = \{1, 2, \cdots, N\}$, for some $N \in \mathbb{N}$. The discontinuities of $\sigma(t)$, representing the switching-impulsive moments which occur simultaneously, form a strictly increasing sequence $\mathbb{T} = \{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_k = \infty$. Throughout this chapter, we assume that σ belongs to a class of deterministic dwell-time-based switching signals. We also assume that the solution is right-continuous (i.e., $x(t_k) = x(t_k^+) = \lim_{h\to 0^-} x(t_k + h)$). In difference equation (11.1b), $\Delta x = x(t_k) - x(t_k^-)$ and $\mathcal{I}(\cdot)$ represents the impulse amount at time instant t_k. For the existence of a unique solution, we assume further that, for any $i_k \in \mathcal{S}$, functional f_{i_k} is a piecewise continuous function and it is Lipschitz in the state variables. Moreover, to ensure that the system admits a trivial solution, for all $i_k \in \mathcal{S}$, $f_{i_k}(t, 0, 0) = 0$ for all $t \in [t_{k-1}, t_k)$ and $\mathcal{I}(t_k^-, 0) = 0$ for all $t_k \in \mathbb{T}$. Let $x_t = x(t+s)$, where $s \in [-r, 0]$ and r is a positive constant that represents the time delay, $x(t) \in \mathcal{PC}([t_0 - r, T]; \mathbb{R}^n)$, for some $T > 0$, be the solution of (11.1), where the initial condition $x_{t_0} = \phi \in \mathcal{PC}([-r, 0]; \mathbb{R}^n)$, and $\|x_t\|_\tau = \sup_{t-\tau\leq s\leq t} \|x(s)\|$, with $\|\cdot\|$ being the Euclidean norm.

The following notations will be used in this chapter. Let $\mathbb{R}^{n\times n}$ denote the set of all $n\times n$ matrices. Denote by A^T the transpose of a matrix A, $\lambda(A)$ the eigenvalues of an $n\times n$ matrix A, $\text{Re}[\lambda(A)]$ the real part of $\lambda(A)$, and $\|A\| = \max_{1\le j\le n}(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ the norm of the $n \times n$ matrix $A = (a_{ij})$. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is said to be positive definite if the scalar $x^T Px > 0$ for all nonzero $x \in \mathbb{R}^n$ and $x^T Px = 0$ for $x = 0$. Denote by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ the minimum and maximum eigenvalues of matrix P, respectively. For a single mode linear time-invariant system $\dot{x} = Ax$, if A is a Hurwitz matrix[1], then there exist positive-definite matrices P and Q satisfying the following Lyapunov equation

$$A^T P + PA = -Q. \tag{11.2}$$

Defining $V(x) = x^T Px$ yields

$$\lambda_{\min}(P)\|x\|^2 \le V(x) \le \lambda_{\max}(P)\|x\|^2.$$

If there is a $V_i(x)$, for any $i \in \mathcal{S} = \{1, 2, \cdots, N\}$, it follows that

$$V_j(x) \le \mu V_i(x), \quad \forall i, j \in \mathcal{S},$$

where $\mu = \lambda_M/\lambda_m \ge 1$ with $\lambda_M = \max\{\lambda_{\max}(P_i), \forall i \in \mathcal{S}\}$, and $\lambda_m = \min\{\lambda_{\min}(P_i), \forall i \in \mathcal{S}\}$.

Before we tackle the stability problem, the following definition and lemmas are needed.

Definition 11.1 [Ball99a] For any $t \ge t_0$, let $x(t) = x(t, t_0, \phi)$, with $\phi \in \mathcal{PC}([-r, 0]; \mathcal{D})$ for some open set $\mathcal{D} \subset \mathbb{R}^n$, be a solution of (11.1). Then, the trivial solution of system (11.1) is said to be

[1] An $n \times n$ matrix A is said to be Hurwitz if all its eigenvalues have negative real parts (i.e. $\text{Re}[\lambda(A)] < 0$).

1. *stable* if, for any given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$\|\phi\|_r \leq \delta \qquad \text{implies} \qquad \|x(t)\| \leq \varepsilon, \quad \forall t \geq t_0;$$

2. *unstable* if it is not stable;

3. *asymptotically stable* if it is stable and there exists $\delta = \delta(t_0) > 0$ such that

$$\|\phi\|_r \leq \delta \qquad \text{implies} \qquad \lim_{t \to \infty} x(t) = 0;$$

4. *exponentially stable* if there exist positive constants c, k, and λ such that

$$\|\phi\|_r \leq \delta \quad \text{implies} \quad \|x(t)\| \leq k\|\phi\|_r e^{-\lambda(t-t_0)}, \quad \forall \|\phi\|_r < c.$$

The above stability properties are said to hold *globally* if ϕ is chosen arbitrarily large or $\mathcal{D} = \mathbb{R}^n$.

Lemma 11.1. [Alw08a, Hal66] Let $u : [t_0 - \tau, \infty) \to R_+$ be continuous function, and satisfy the following delay differential inequality

$$\dot{u}(t) \leq \alpha u(t) + \beta \sup_{\theta \in [t-r,t]} u(\theta), \qquad t \in [t_0, \infty).$$

(i) If $\alpha < 0$, $\beta > 0$, and $\alpha + \beta < 0$, then there exist positive constants k and ζ such that

$$u(t) \leq k e^{-\zeta(t-t_0)}, \qquad t \geq t_0;$$

(ii) if $\alpha > 0$ and $\beta > 0$, then there exist positive constants ξ and k such that

$$u(t) \leq k e^{\xi(t-t_0)}, \qquad t \geq t_0, \tag{11.3}$$

where $\xi = \alpha + \beta$ and $k = \sup_{\theta \in [t_0-r,t_0]} u(\theta)$.

Lemma 11.2. [Li97] Let F, Ξ, and H be real matrices of appropriate dimensions with $\|\Xi\| < 1$. Then, for any $\varepsilon > 0$, the following inequality holds.

$$F\Xi H + H^T \Xi^T F^T \leq \varepsilon^{-1} F F^T + \varepsilon H^T H.$$

11.2 Uncertain Impulsive Switched Systems with Time Delay

In this section, we address the problems of stability and stabilization of uncertain impulsive switched systems with time delay (UISSD). As stated earlier, the uncertainties are of structured type. Two systems will be considered, linear and weakly nonlinear systems.

11.2.1 Linear UISSD

Linear UISSD can have the following form

$$\dot{x}(t) = [A_{i_k} + \Delta A_{i_k}]x(t) + [B_{i_k} + \Delta B_{i_k}]x(t-r), \qquad t \in [t_{k-1}, t_k), \qquad (11.4a)$$

$$\Delta x(t) = B_k x(t^-) + C_k x(t^- - r), \qquad t = t_k, \qquad (11.4b)$$

where $i_k \in \mathcal{S} = \{1, 2, \cdots, N\}$, A_{i_k} and B_{i_k} are $n \times n$ known real constant matrices. The admissible uncertainties in this chapter are defined in the following assumption.

Assumption A4. Assume that the uncertainties satisfy the following properties:

$$\Delta A_{i_k} = H_{i_k}\Xi_{i_k}(t)F_{i_k} \quad \text{and} \quad \Delta B_{i_k} = J_{i_k}\Gamma_{i_k}(t)K_{i_k},$$

where H_{i_k}, F_{i_k}, J_{i_k}, and K_{i_k} are known real constant matrices with appropriate dimensions that give the structure of the uncertainties, and $\Xi_{i_k}(t)$ and $\Gamma_{i_k}(t)$ are unknown real time-varying matrices satisfying $\|\Xi_{i_k}(t)\| \leq 1$ and $\|\Gamma_{i_k}(t)\| \leq 1$, respectively.

In the following theorem, for any admissible uncertainty, we state Lyapunov-based sufficient conditions to guarantee *robustly exponential stability* of the uncertain system composing of unstable and stable subsystems. Let $\mathcal{S} = \mathcal{S}_u \cap \mathcal{S}_s$, where, for convenience, $\mathcal{S}_u = \{1, 2, \cdots, r\}$ and $\mathcal{S}_s = \{r+1, r+2, \cdots, N\}$ are the sets of indices of unstable and stable subsystems, respectively.

Theorem 11.1. The trivial solution of system (11.4) is robustly globally exponentially stable (with respect to any admissible uncertainty defined in A4) if the following assumptions hold:

(i) **(1)** for $i_k \in \mathcal{S}_u$,

$$Re[\lambda(A_{i_k})] > 0, \quad Re[\lambda(A_{i_k} + B_{i_k})] > 0, \quad \text{and} \quad \Big(2\gamma + \frac{\lambda^*_{i_k} + \beta^*_{i_k} + \beta^{**}_{i_k}}{\lambda_m}\Big) > 0.$$

(2) for $i_k \in \mathcal{S}_s$, A_{i_k} is Hurwitz and

$$-\Big(\frac{\lambda_{\min}(Q_{i_k}) - \lambda^*_{i_k} - \beta^*_{i_k} - \beta^{**}_{i_k}}{\lambda_M}\Big) + \frac{\beta^*_{i_k} + \beta^{**}_{i_k}}{\lambda_m} < 0,$$

where $\beta^*_{i_k} = \|P_{i_k} B_{i_k}\|$, $\beta^{**}_{i_k} = \|P_{i_k} J_{i_k} K_{i_k}\|$, and $\lambda^*_{i_k} = \lambda_{\max}[\varepsilon^{-1} F_{i_k} F^T_{i_k}$ $+ \varepsilon (P_{i_k} H_{i_k})^T (P_{i_k} H_{i_k})]$ with ε being a positive constant;

(ii) let $\lambda^+ = \max\{\xi_{i_k} : i_k \in \mathcal{S}_u\}, \lambda^- = \min\{\zeta_{i_k} : i_k \in \mathcal{S}_s\}$, and $T^+(t_0, t)$ and $T^-(t_0, t)$ be the total activation time of the unstable and stable modes, respectively. For any t_0, assume that the switching law guarantees that

$$\inf_{t \geq t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*}, \tag{11.5}$$

where $\lambda^* \in (0, \lambda^-)$. Furthermore, there exists $0 < \nu < \zeta_{i_k}$ such that

(**1**) for $i_k \in \mathcal{S}_u$ and $k = 1, 2, \cdots, l$,

$$\ln \mu(\alpha_k + \psi_k) - \nu(t_k - t_{k-1}) \leq 0; \tag{11.6}$$

(**2**) for $i_k \in \{l+1, l+2, \cdots, N-1\}$ and $k = l+1, l+2, \cdots, N-1$,

$$\ln \mu(\alpha_k + \psi_k e^{\zeta_{i_k} r}) + \zeta_{i_k} r - \nu(t_k - t_{k-1}) \leq 0, \tag{11.7}$$

where $\alpha_k = (\alpha^*_k + \psi^*_k)/\lambda_m, \quad \psi_k = (\psi^*_k + \gamma^*_k)/\lambda_m, \quad \alpha^*_k = \lambda^2_{\max}[I + B_k]\lambda_M,$ $\psi^*_k = P_M \|I + B_k\| \cdot \|C_k\|$, $\gamma^*_k = \lambda_{\max}[C_k]^2 \lambda_M$, and $P_M = \max\{\|P_{i_k}\|, \forall i_k \in \mathcal{S}\}$.

Proof: For $i_k \in \mathcal{S}$, let $V_{i_k}(x) = x^T P_{i_k} x$. Then, the total derivative of V_{i_k} along the trajectories of (11.4) is

$$\begin{aligned}\dot{V}_{i_k}(x) =& x^T[A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k}]x + x^T[\Delta A_{i_k}^T P_{i_k} + P_{i_k}\Delta A_{i_k}]x \\ &+ 2x^T P_{i_k}[B_{i_k} + \Delta B_{i_k}]x(t-r).\end{aligned}$$

For $i_k \in \mathcal{S}_s$, we have

$$\begin{aligned}\dot{V}_{i_k}(x) \leq& -\lambda_{\min}(Q_{i_k})\|x\|^2 + x^T[F_{i_k}^T E_{i_k}^T(t)(P_{i_k}H_{i_k})^T + (P_{i_k}H_{i_k})E_{i_k}(t)F_{i_k}]x \\ &+ 2x^T P_{i_k} B_{i_k} x(t-r) + 2x^T P_{i_k} J_{i_k}\Gamma_{i_k}(t) K_{i_k} x(t-r) \\ \leq& -\lambda_{\min}(Q_{i_k})\|x\|^2 + x^T[\varepsilon^{-1}F_{i_k}F_{i_k}^T + \varepsilon(P_{i_k}H_{i_k})^T(P_{i_k}H_{i_k})]x \\ &+ \|P_{i_k}B_{i_k}\|(\|x(t)\|^2 + \|x_t\|_r^2) + \|P_{i_k}J_{i_k}\Gamma_{i_k}(t)K_{i_k}\|(\|x(t)\|^2 + \|x_t\|_r^2)\end{aligned}$$

$$\begin{aligned}\leq& -\lambda_{\min}(Q_{i_k})\|x\|^2 + x^T[\varepsilon^{-1}F_{i_k}F_{i_k}^T + \varepsilon(P_{i_k}H_{i_k})^T(P_{i_k}H_{i_k})]x \\ &+ \|P_{i_k}B_{i_k}\|(\|x(t)\|^2 + \|x_t\|_r^2) + \|P_{i_k}J_{i_k}K_{i_k}\|(\|x(t)\|^2 + \|x_t\|_r^2) \\ \leq& -\lambda_{\min}(Q_{i_k})\|x\|^2 + \lambda_{\max}[\varepsilon^{-1}F_{i_k}F_{i_k}^T + \varepsilon(P_{i_k}H_{i_k})^T(P_{i_k}H_{i_k})]\|x(t)\|^2 \\ &+ \|P_{i_k}B_{i_k}\|(\|x(t)\|^2 + \|x_t\|_r^2) + \|P_{i_k}J_{i_k}K_{i_k}\|(\|x(t)\|^2 + \|x_t\|_r^2) \\ \leq& -\Big(\frac{\lambda_{\min}(Q_{i_k}) - \lambda_{i_k}^* - \beta_{i_k}^* - \beta_{i_k}^{**}}{\lambda_M}\Big)V_{i_k}(x) + \frac{\beta_{i_k}^* + \beta_{i_k}^{**}}{\lambda_m}\|V_{i_{k_t}}\|_r.\end{aligned}$$

By item (i) of Lemma 11.1, with $\alpha_{i_k} = \frac{\lambda_{\min}(Q_{i_k}) - \lambda_{i_k}^* - \beta_{i_k}^* - \beta_{i_k}^{**}}{\lambda_M}$ and $\beta_{i_k} = \frac{\beta_{i_k}^* + \beta_{i_k}^{**}}{\lambda_m}$, there exists a $\zeta_{i_k} > 0$ such that

$$V_{i_k}(x) \leq \|V_{i_{k_{t_{k-1}}}}\|_r e^{-\zeta_{i_k}(t-t_{k-1})}.$$

For $i_k \in \mathcal{S}_u$, we have

$$\begin{aligned}\dot{V}_{i_k}(x) \leq& 2\gamma x^T P_{i_k} x + \lambda_{i_k}^*\|x\|^2 + \|P_{i_k}B_{i_k}\|(\|x(t)\|^2 + \|x_t\|_r^2) \\ &+ \|P_{i_k}J_{i_k}K_{i_k}\|(\|x(t)\|^2 + \|x_t\|_\tau^2) \\ \leq& \Big(2\gamma + \frac{\lambda_{i_k}^* + \beta_{i_k}^* + \beta_{i_k}^{**}}{\lambda_m}\Big)V_{i_k}(x) + \frac{\beta_{i_k}^* + \beta_{i_k}^{**}}{\lambda_m}\|V_{i_{k_t}}\|_r,\end{aligned}$$

where $\gamma > 0$ such that $\text{Re}([A_{i_k} - \gamma I]) < 0$. Using Lemma 11.1 (ii), with $\beta_{1i_k} = 2\gamma + (\lambda^*_{i_k} + \beta^*_{i_k} + \beta^{**}_{i_k})/\lambda_m$ and $\beta_{2i_k} = (\beta^*_{i_k} + \beta^{**}_{i_k})/\lambda_m$, there exists a $\xi_{i_k} > 0$ such that

$$V_{i_k}(x) \leq \|V_{i_{k t_{k-1}}}\|_r e^{\xi_{i_k}(t-t_{k-1})}.$$

At the impulsive moments $t = t_k$, we have

$$\begin{aligned}
V_{i_k}(x(t_k)) =& x^T(t_k)P_{i_k}x(t_k) \\
=& [(I+B_k)x(t_k^-) + C_k x(t_k^- - r)]^T P_{i_k}[(I+B_k)x(t_k^-) + C_k x(t_k^- - r)] \\
=& x^T(t_k^-)[(I+B_k)^T P_{i_k}(I+B_k)]x(t_k^-) + 2x^T(t_k^-)(I+B_k)^T P_{i_k} C_k x(t_k^-) \\
& + x^T(t_k^- - r)C_k P_{i_k} C_k x(t_k^- - r)
\end{aligned}$$

$$\begin{aligned}
\leq& \lambda_{\max}[(I+B_k)^T P_{i_k}(I+B_k)]x^T(t_k^-)x(t_k^-) \\
& + \|I+B_k\|.\|P_{i_k}\|.\|C_k\|(\|x(t_k^-)\|^2 + \|x_{t_k^-}\|_r^2) \\
& + \lambda_{\max}[C_k^T P_{i_k} C_k]x^T(t_k^- - r)x(t_k^- - r) \\
\leq& \overbrace{\lambda^2_{\max}[I+B_k]^T\lambda_M}^{=:\alpha^*_k}\|x(t_k^-)\|^2 + \overbrace{\|I+B_k\|.P_M.\|C_k\|}^{=:\psi^*_k}(\|x(t_k^-)\|^2 + \|x_{t_k^-}\|_r^2) \\
& + \overbrace{\lambda_{\max}[C_k]^2\lambda_M}^{=:\gamma^*_k}\|x_{t_k^-}\|_r^2 \\
=& (\alpha^*_k + \psi^*_k)\|x(t_k^-)\|^2 + (\psi^*_k + \gamma^*_k)\|x_{t_k^-}\|_r^2 \\
\leq& \alpha_k V_{i_k}(x(t_k^-)) + \psi_k\|V_{i_{k t_k^-}}\|_r,
\end{aligned}$$

where $\alpha_k = (\alpha^*_k + \psi^*_k)/\lambda_m$ and $\psi_k = (\psi^*_k + \gamma^*_k)/\lambda_m$.

For simplicity, let us activate an unstable and stable modes on $[t_0, t_1)$ and $[t_1, t_2)$, respectively. Then, we have

$$V_1(x(t)) \leq \|V_{1_{t_0}}\|_r e^{\xi_1(t-t_0)},$$

$$V_2(t) \leq \|V_{2_{t_1}}\|_r e^{-\zeta_2(t-t_1)}. \tag{11.8}$$

The norm in (11.8) is calculated as follows:

$$V_2(t_1) \leq \alpha_1 V_2(t_1^-) + \psi_1 \|V_{2_{t_1^-}}\|_r,$$

where

$$V_2(t_1^-) \leq \mu V_1(t_1^-) \leq \mu \|V_{1_{t_0}}\|_r e^{\xi_1 (t_1 - t_0)},$$

and

$$\|V_{2_{t_1^-}}\|_r \leq \mu \|V_{1_{t_0}}\|_r e^{\xi_1 (t_1 - t_0)}.$$

Then,

$$V_2(t_1) \leq \mu(\alpha_1 + \psi_1) \|V_{1_{t_0}}\|_r e^{\xi_1 (t_1 - t_0)}.$$

Therefore, inequality (11.8) becomes

$$V_2(t) \leq \mu(\alpha_1 + \psi_1) \|V_{1_{t_0}}\|_\tau e^{\xi_1 (t_1 - t_0)} e^{-\zeta_2 (t - t_1)}.$$

Generally, one may have the following estimate

$$V_N(t) \leq \prod_{k, i_k = 1}^{l} \mu(\alpha_k + \psi_k) e^{\xi_{i_k} (t_k - t_{k-1})} \times \prod_{k, i_k = l+1}^{N-l-1} \mu(\alpha_k + \psi_k e^{\zeta_{i_k} r}) e^{\zeta_{i_k} r} e^{-\zeta_{i_k} (t_k - t_{k-1})}$$
$$\times \|V_{1_{t_0}}\|_\tau e^{-\zeta_N (t - t_{N-1})}.$$

Using assumption (ii), we have

$$V_N(t) \leq \|V_{1_{t_0}}\|_r e^{-(\lambda^* - \nu)(t - t_0)}.$$

Therefore,

$$\|x(t)\| \leq K \|\phi\|_r e^{-(\lambda^* - \nu)(t - t_0)},$$

where $K = \sqrt{\mu}$. This shows that the trivial solution of system (11.4) is robustly globally exponentially stable. This completes the proof.

In Theorem 11.1, assumption (i) is introduced to describe the continuous parts of the system, where in item (i) all modes are unstable and in item (ii) the modes are stable. Assumption (ii) is concerned with the switching signal, which tells us that, in order to guarantee exponential stability or to compensate the growth in unstable modes, the stable modes must be activated longer than the unstable ones.

Example 11.1. Consider system (11.4) where $\mathcal{S} = \{1, 2\}$, $r = 1$ and the matrices are given below.

1. Unstable Mode.

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0.1 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0.3 & 0.1 \\ 0 & 0.3 \end{pmatrix},$$

$$\Xi_1(t) = \sin(t) I_2, F_1 = I_2, J_1 = I_2, \Gamma_1(t) = \cos(t) I_2, \text{ and } K_1 = \begin{pmatrix} 0.3 & 0 \\ 0.1 & 0.3 \end{pmatrix}.$$

2. Stable Mode.

$$A_2 = \begin{pmatrix} -2 & 0 \\ 0.1 & -3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.2 & 0 \\ 0.1 & 0.1 \end{pmatrix}, \quad H_2 = I_2, \quad \Xi_2(t) = \sin(t) I_2,$$

$$F_2 = \begin{pmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{pmatrix}, \quad J_2 = 0.1 I_2, \quad \Gamma_2(t) = \cos(t) I_2, \quad K_2 = I_2.$$

The impulsive actions are $B_k = -0.3 I_2$ and $C_k = 0.3 I_2$.

Taking $\gamma = 3$, $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1.7 \end{pmatrix}$, and $Q_2 = 2I_2$ give $P_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.425 \end{pmatrix}$,

$P_2 = \begin{pmatrix} 0.5 & 0.01 \\ 0.01 & 0.3337 \end{pmatrix}$, $\mu = 1.503 > 1$, and $\zeta = 1.21$. With little effort, one can check that assumptions (i) and (ii) are fulfilled, where $\varepsilon = 1$ and $\nu = 1$. The impulsive parameters are $\alpha_k^* = 0.2453$, $\psi_k^* = 0.1051$, $\gamma_k^* = 0.0451$, $\alpha_k = 1.0521$, and $\psi_k = 0.4509$. The unstable and stable dwell time are $\tau^u = 1$ and $\tau^s = 3.3$, respectively. Figure 11.1 shows the convergence of the solutions to the equilibrium state of the system.

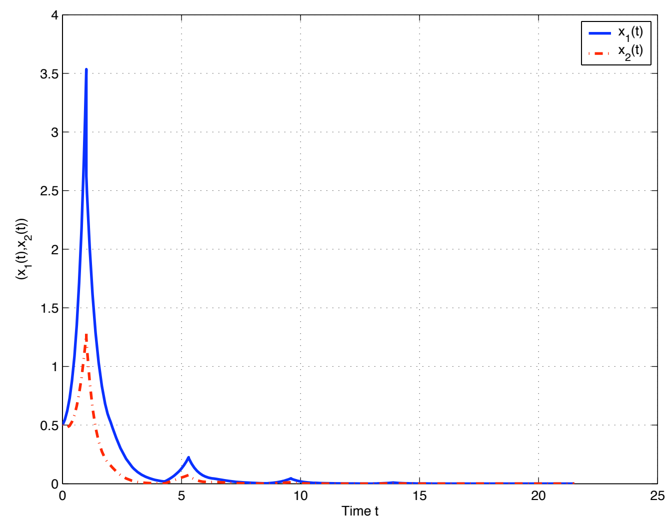

Figure 11.1: System with unstable and stable subsystems: $\phi(t) = t + 0.5$.

Having established exponential stability of uncertain systems with unstable and stable subsystems, in the next theorem we consider systems consisting of all unstable subsystems, and present sufficient conditions to guarantee stability and asymptotical stability of these systems; in other words, we want to show that impulses can play as a stabilizer.

Theorem 11.2. Consider the system given in (11.4) with uncertainties being defined in Assumption A4, where $\mathcal{S} = \mathcal{S}_u$. Assume that the following assumptions are satisfied:

(i) assumption (i)(1) of Theorem 11.1 holds;

(ii) there exists a constant $\vartheta \geq 1$ such that

$$\ln\big(\vartheta\mu(\alpha_k + \psi_k)\big) + \xi_{i_k}(t_{k+1} - t_k) \leq 0,$$

where α_k, ψ_k, and ξ_{i_k} are defined in Theorem 11.1.

Then, $\vartheta = 1$ implies that the trivial solution of system (11.4) is robustly stable, and $\vartheta > 1$ implies that the trivial solution of system (11.4) is robustly asymptotically stable.

Proof. For each $i_k \in \mathcal{S}$, let $V_{i_k}(t) = x^T P_{i_k} x$. Then, from Theorem 11.1, we have the following

$$V_{i_k}(x) \leq \|V_{i_k t_{k-1}}\|_r e^{\xi_{i_k}(t-t_{k-1})},$$

and, at $t = t_k$, we have

$$\begin{aligned} V_{i_k}(x(t_k)) &= x^T(t_k) P_{i_k} x(t_k) \\ &\leq \alpha_k V_{i_k}(x(t_k^-)) + \psi_k \|V_{i_{k t_k^-}}\|_r. \end{aligned}$$

Hence,

$$\begin{aligned} V_{i_k}(t) \leq & \|V_{1_{t_0}}\|_r e^{\xi_1(t_1-t_0)} \mu(\alpha_1 + \psi_1) e^{\xi_2(t_2-t_1)} \\ & \times \mu(\alpha_2 + \psi_2) e^{\xi_2(t_3-t_2)} \cdots \mu(\alpha_k + \psi_k) e^{\xi_{i_k}(t_{k+1}-t_k)} \\ = & \|V_{1_{t_0}}\|_r \frac{1}{\vartheta^k} e^{\xi_1(t_1-t_0)} \vartheta\mu(\alpha_1 + \psi_1) e^{\xi_2(t_2-t_1)} \\ & \times \vartheta\mu(\alpha_2 + \psi_2) e^{\xi_2(t_3-t_2)} \cdots \vartheta\mu(\alpha_k + \psi_k) e^{\xi_{i_k}(t_{k+1}-t_k)}, \end{aligned}$$

and, by (ii), we get

$$V_{i_k}(t) \le \|V_{1_{t_0}}\|_\tau \frac{1}{\vartheta^k} e^{\xi_1(t_1-t_0)}.$$

Therefore,

$$\|x(t)\| \le \frac{K}{\sqrt{\vartheta^k}} \|\phi\|_r e^{\xi_1(t_1-t_0)/2}.$$

Apparently, if $\vartheta = 1$, system (11.4) is robustly stable, and if $\vartheta > 1$ and $k \to \infty$, system (11.4) is robustly asymptotically stable.

Example 11.2. Consider the uncertain system (11.4) with $r = 1$ and the subsystems are given by

1. First Mode.

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_1 = I_2, \quad H_1 = 0.1I_2, \quad \Xi_1(t) = \cos(10\pi t)I_2,$$

$$F_1 = \begin{pmatrix} 1 & 0.2 \\ 0 & 1 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0.2 & 0 \\ 0.1 & 0.2 \end{pmatrix}, \quad \Gamma_1(t) = \sin(10\pi t)I_2, \quad K_1 = I_2.$$

2. Second Mode.

$$A_2 = I_2, \quad B_2 = I_2, \quad H_2 = 0.1I_2, \quad \Xi_2(t) = \sin(10\pi t)I_2, \quad F_2 = 0.3I_2,$$

$$J_2 = 0.1I_2, \quad \Gamma_2(t) = \cos(10\pi t)I_2, \quad K_2 = \begin{pmatrix} 1 & -0.1 \\ 0 & 1 \end{pmatrix}.$$

The impulsive actions are $B_k = -0.99I_2$ and $C_k = 0.1I_2$.

Take $\gamma = 2$, $Q_1 = I_2$, and $Q_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ to get $P_1 = \begin{pmatrix} 0.5 & 0.1667 \\ 0.1667 & 0.3333 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1.25 & 0.25 \\ 0.25 & 0.5 \end{pmatrix}$, $\mu = 5.7557$. Taking $\varepsilon = 2$ gives $\xi_1 = 9.8899$, $\xi_2 = 11.7792$,

$\vartheta^* = 2.4947$. Let $\vartheta = 1.5 \in [1, \vartheta^*)$. The dwell time of the first and second subsystems are, respectively, 0.07 and 0.06. As expected, the dwell times are very small since the subsystems are both unstable. Figure 11.2 illustrates this result.

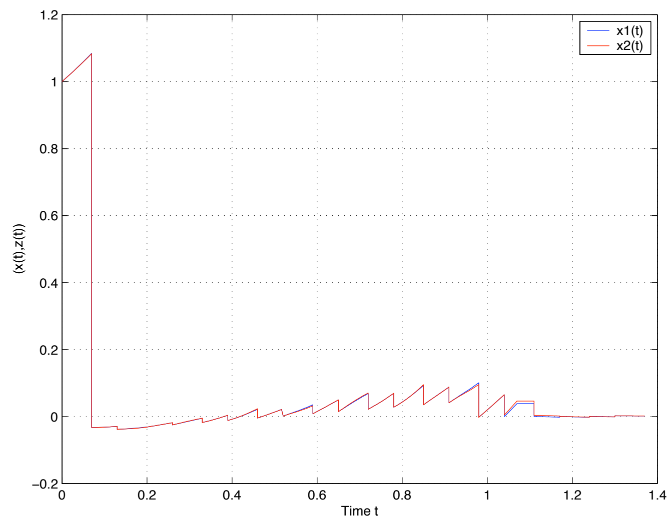

Figure 11.2: System with all unstable subsystems: $\phi(t) = t + 1$.

11.2.2 Weakly Nonlinear UISSD

Consider the following system

$$\dot{x}(t) = f_{i_k}(t, x, x_t), \qquad t \in [t_{k-1}, t_k), \tag{11.9a}$$

$$\Delta x(t) = R_k(x(t^-), x_{t^-}), \qquad t = t_k, \tag{11.9b}$$

where $f_{i_k} = (A_{i_k} + \Delta A_{i_k})x + g_{i_k}(t, x, x_t)$ and the uncertainty ΔA_{ik} is defined in Assumption A4. We also assume that system (11.9) has a unique equilibrium point at the origin, i.e., $g_{i_k}(t, 0, 0) = 0$ for all $t \geq t_0$ and $R_k(0, 0) = 0$ for any $t_k \in \mathbb{T}$. In the following theorem, we give sufficient conditions that ensure robust exponential stability of uncertain system (11.9).

Theorem 11.3. Consider the uncertain system (11.9) with $\mathcal{S} = \mathcal{S}_u \cup \mathcal{S}_s$. Then, the trivial solution of system (11.9) is robustly globally exponentially stable if the following assumptions hold:

(i) **(1)** for $i_k \in \mathcal{S}_u$,

$$Re[\lambda(A_{i_k})] > 0 \quad \text{and} \quad \Big(2\gamma + \frac{\lambda^*_{i_k} + a^*_{i_k}}{\lambda_m}\Big) > 0;$$

(2) for $i_k \in \mathcal{S}_s$, A_{i_k} is Hurwitz and

$$-\Big(\frac{\lambda_{\min}(Q_{i_k}) - \lambda^*_{i_k} - a^*_{i_k}}{\lambda_M}\Big) + \frac{b^*_{i_k}}{\lambda_m} < 0,$$

where $\lambda^*_{i_k}$ is defined in Theorem 11.1, and a_{i_k} and b_{i_k} are positive constants such that

$$2x^T P_{i_k} g_{i_k}(t, x, x_t) \leq a_{i_k}\|x(t)\|^2 + b_{i_k}\|x_t\|^2_r;$$

(ii) there exist positive constants d_k and e_k such that

$$\begin{aligned}2x^T P_{i_k} R_k(x(t_k^-), x_{t_k^-}) + R_k^T(x(t_k^-), x_{t_k^-}) P_{i_k} R_k(x(t_k^-), x_{t_k^-})\\ \leq d_k\|x(t_k^-)\|^2 + e_k\|x_{t_k^-}\|^2_r;\end{aligned}$$

(iii) assumption (ii) of Theorem 11.1 with $\alpha_k = 1 + \frac{d_k}{\lambda_m}$ and $\psi_{i_k} = \frac{e_k}{\lambda_m}$.

Proof: For each $i_k \in \mathcal{S}$, let $V_{i_k}(x) = x^T P_{i_k} x$ be a Lyapunov function candidate. Then, the time derivative of V_{i_k} along the trajectories of (11.9) is

$$\begin{aligned}\dot{V}_{ik}(x) &= x^T[A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k}]x + x^T[\Delta A_{i_k}^T P_{i_k} + P_{i_k}\Delta A_{i_k}]x\\ &\quad + 2x^T P_{i_k} g_{i_k}(t, x, x_t)\\ &\leq x^T[A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k}]x + \lambda^*_{i_k}\|x\|^2 + a_{i_k}\|x\|^2 + b_{i_k}\|x_t\|^2_r.\end{aligned}$$

By a similar argument followed in proving Theorem 11.1, for any $i_k \in \mathcal{S}_u$, there exits a positive constant ξ_{i_k} such that

$$V_{i_k} \leq \|V_{i_{k_{t_{k-1}}}}\|_\tau e^{\xi_{i_k}(t-t_{k-1})},$$

and, for any $i_k \in \mathcal{S}_s$, there exists a positive constant ζ_{i_k} such that

$$V_{i_k} \leq \|V_{i_{k_{t_{k-1}}}}\|_\tau e^{-\zeta_{i_k}(t-t_{k-1})}.$$

At $t = t_k$, and for any $i_k \in \mathcal{S}$, we have

$$\begin{aligned}
V_{i_k}(x(t_k)) &= x^T(t_k)P_{i_k}x(t_k) \\
&= [x(t_k^-) + R_k(x(t_k^-)), x_{t_k^-})]^T P_{i_k}[x(t_k^-) + R_k(x(t_k^-)), x_{t_k^-})] \\
&= x^T(t_k^-)P_{i_k}x(t_k^-) + 2x^T P_{i_k} R_k(x(t_k^-), x_{t_k^-}) \\
&\quad + R_k^T(x(t_k^-), x_{t_k^-})P_{i_k}R_k(x(t_k^-), x_{t_k^-}) \\
&\leq V_{i_k}(x(t_k^-)) + d_k\|x(t_k^-)\|^2 + e_k\|x_{t_k^-}\|_r^2 \\
&\leq (1 + \frac{d_k}{\lambda_m})V_{i_k}(t_k^-) + \frac{e_k}{\lambda_m}\|V_{i_{k_{t_k^-}}}\|_r \\
&= \alpha_k V_{i_k}(t_k^-) + \psi_k\|V_{i_{k_{t_k^-}}}\|_r.
\end{aligned}$$

Hence, there exists a positive constant K such that

$$\|x(t)\| \leq K\|\phi\|_r e^{-(\lambda^*-\nu)(t-t_0)},$$

where K, λ^*, and ν are as defined in Theorem 11.1. This completes the proof.

For better understanding, we study the following example.

Example 11.3. Consider system (11.9) and the nonlinear subsystems are

1. Unstable Subsystem

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t) + 0.01x_2(t)\cos t + 0.1x_1(t-1)\cos x_2(t), \\
\dot{x}_2(t) &= x_2(t) - 0.02x_1(t)\cos t + 0.1\ln(1 + x_1^2(t)).
\end{aligned}$$

2. Stable Subsystem

$$\dot{x}_1(t) = -x_1(t) + 0.2\ln(1 + x_2^2(t)),$$
$$\dot{x}_2(t) = -2x_2(t) + 0.05x_2(t)\sin t + 0.1x_1(t-1).$$

The impulsive functions are

$$R_k(x(t_k^-), x_{t_k}^-) = \begin{pmatrix} 0.1x_1(t_k^-)\sin x_2(t_k^-) \\ -x_2(t_k^-) + 2x_1(t_k^- - 1) \end{pmatrix}.$$

Taking $Q_1 = I_2$, $\gamma = 2$, and $Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ give $P_1 = P_2 = 0.5I_2$, and hence $V_1(x) = V_2(x) = \frac{1}{2}(x_1^2 + x_2^2)$. With little effort, we get the following constants: $a_1 = 0.1$, $b_1 = 0.05$, $\lambda_1^* = 0.0001$, the growth rate $\xi = 2.1$, $a_2 = 0.2$, $b_2 = 0.05$, $\lambda_2^* = 0.0125$, and the decay rate $\zeta = 1.306$; for all k, $d_k = 8/5$, $e_k = 4$, $\alpha_k = 21/5$, and $\psi_k = 8$. For $\nu = 1.2$, the dwell times are $\tau^u = 2$ and $\tau^s = 4.5$. The simulation result is shown in Figure 11.3.

As for systems that consist of all unstable subsystems, we consider the following nonlinear uncertain system with nonlinear impulse

$$\dot{x}(t) = (A_{i_k} + \Delta A_{i_k})x(t) + g_{i_k}(t, x(t), x_t), \qquad t \in [t_{k-1}, t_k), \tag{11.10a}$$
$$\Delta x(t) = B_k x(t^-) + R_k(x(t^-), x_{t^-}), \qquad t = t_k, \tag{11.10b}$$

where $B_k \in R^{n\times n}$ is a real constant matrix and ΔA_{i_k} is defined in Assumption A4. In the following theorem, we introduce sufficient conditions that guarantee stability and asymptotic stability of the system experiencing impulsive effects given in (11.10b).

Theorem 11.4. Consider the uncertain impulsive switched system (11.10). Assume that the following assumptions are satisfied:

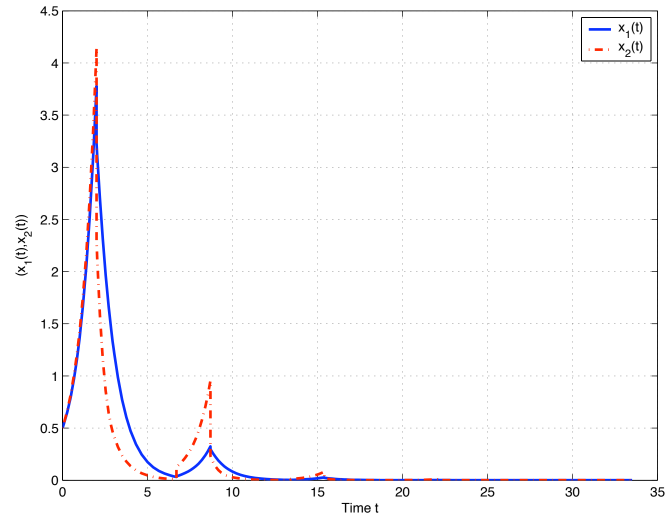

Figure 11.3: System with unstable and stable modes: $\phi(t) = t + 0.5$.

(i) assumption (i)(1) of Theorem 11.3 holds;

(ii) there exist positive constants a_{i_k} and b_{i_k} such that

$$2x^T P_{i_k} g_{i_k}(t, x, x_t) \leq a_{i_k}\|x\|^2 + b_{i_k}\|x_t\|_r^2;$$

(iii) there exists a constant $\vartheta \geq 1$ such that

$$\ln(\vartheta\mu(\alpha_k + \psi_k)) + \xi_{i_k}(t_{k+1} - t_k) \leq 0, \tag{11.11}$$

where μ is defined before, $\alpha_k = (\lambda_{max}^2[I + B_k] + d_k)/\lambda_m$ and $\psi_k = e_k/\lambda_m$, d_k and e_k are positive constants such that the following inequality holds

$$\begin{aligned} 2x^T(t_k^-)[I + B_k]^T P_i R_k(x(t_k^-), x_{t_k^-}) + R_k(x(t_k^-), x_{t_k^-})^T P_i R_k(x(t_k^-), x_{t_k^-}) \\ \leq d_k\|x(t_k^-)\|^2 + e_k\|x_{t_k^-}\|_r^2, \end{aligned}$$

and $\xi_{i_k} = (2\gamma\lambda_m + \lambda_{i_k}^* + a_{i_k} + b_{i_k})/2\lambda_m$ is the growth rate of the i_kth subsystem, with $\lambda_{i_k}^*$ being defined in Theorem 11.1.

Then, $\vartheta = 1$ implies that the trivial solution of system (11.10) is robustly stable, and $\vartheta > 1$ implies that the trivial solution of system (11.10) is robustly asymptotically stable.

Proof. For each i_k, define $V_{i_k}(x) = x^T P_{i_k} x$. Then, from Theorem 11.3, we have

$$V_{i_k} \leq \|V_{i_{k_{t_{k-1}}}}\|_r e^{\xi_{i_k}(t-t_{k-1})},$$

and, at $t = t_k$,

$$\begin{aligned} V(x(t_k)) &= x^T(t_k) P_{i_k} x(t_k) \\ &= [(I + B_{i_k})x(t_k) + R_{i_k}(x(t_k^-), x_k^-)]^T P_{i_k}[(I + B_{i_k})x(t_k) + R_{i_k}(x(t_k^-), x_k^-)] \\ &\leq x^T(t_k^-)(I + B_k)^T P_{i_k}(I + B_k)x(t_k^-) + 2x^T(t_k^-)(I + B_k)^T P_{i_k} R_k(x(t_k^-), x_k^-) \\ &\quad + R_k^T(x(t_k^-), x_k^-) P_{i_k} R_k(x(t_k^-), x_k^-) \\ &\leq \lambda_{\max}[(I + B_k)^T P_{i_k}(I + B_k)x(t_k^-)]\|x(t_k^-)\|^2 + d_k\|x(t_k^-)\|^2 + e_k\|x_{t_k^-}\|_r^2 \\ &\leq \lambda_{\max}^2(I + B_k)\lambda_M\|x(t_k^-)\|^2 + d_k\|x(t_k^-)\|^2 + e_k\|x_{t_k^-}\|_r^2 \\ &\leq \alpha_k V_{i_k}(t_k^-) + \psi_k\|V_{i_{k_{t_k^-}}}\|_r. \end{aligned}$$

By the same argument followed in proving Theorem 11.2, we have

$$V_{i_k}(x) \leq \|V_{1_{t_0}}\|_\tau \frac{1}{\vartheta^k} e^{\xi_1(t_1-t_0)},$$

from which we obtain

$$\|x(t)\| \leq \frac{K}{\sqrt{\vartheta^k}}\|\phi\|_r e^{\xi_1(t_1-t_0)/2}.$$

Clearly, if $\vartheta = 1$, system (11.10) is robustly stable, and, if $\vartheta > 1$ and $k \to \infty$, system (11.10) is robustly asymptotically stable.

Example 11.4. Consider impulsive system (11.10) and the nonlinear subsystems are

1. First Mode

$$\dot{x}_1(t) = 3x_1(t) + 0.01\sin(5\pi t)x_1(t) + x_2(t-1)e^{-|x_1(t)|},$$
$$\dot{x}_2(t) = 0.01\sin(5\pi t)x_2(t) + \sin x_1(t-1).$$

2. Second Mode

$$\dot{x}_1(t) = 2x_1(t) + 0.01\cos(5\pi t)x_1(t) + x_1(t)\cos x_2(t-1),$$
$$\dot{x}_2(t) = 5x_2(t) + 0.01\cos(5\pi t)x_2(t) + x_2(t-1).$$

The impulsive functions are

$$R_k(x(t_k^-), x_{t_k}^-) = \begin{pmatrix} -0.1x_1(t_k^-) + 0.1x_2(t_k^- - 1) \\ -0.1\sin x_1(t_k^- - 1) \end{pmatrix}.$$

Taking $\gamma = 6$, $\varepsilon = 1$, $Q_1 = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$, and $Q_2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ give $P_1 = P_2 = 0.5I_2$, and hence $V_1(x) = V_2(x) = \frac{1}{2}(x_1^2 + x_2^2)$. One can easily get the following constants: for the first subsystem, $a_1 = b_1 = 0.5$, $\lambda_1^* = 0.0125$, the growth rate $\xi_1 = 7.0125$, and $\vartheta_1^* = 14.2857$; for the second subsystem, $a_2 = 1$, $b_2 = 0.5$, $\lambda_2^* = 0.0125$, the growth rate $\xi_2 = 7.5125$, and $\vartheta_2^* = \vartheta_1^*$; for all k, $\alpha_k = 0.04$ and $\psi_k = 0.03$. For $\vartheta = 4 \in [1, \vartheta^*)$, the dwell times of the first and second subsystems are $\tau = 0.18$ and $\tau = 0.16$, respectively. The simulation result is shown in Figure 11.4.

According to the results of Theorems 11.1 and 11.2, one can notice that the convergence of solutions to the equilibrium state is influenced by the size of decay and growth rates of subsystems and impulsive amounts. Consequently, assumptions (ii) and (iii) of Theorems 11.1 and 11.3, respectively, can be refined by relaxing the condition of activating stable subsystems longer than unstable ones. The following corollary illustrates this result.

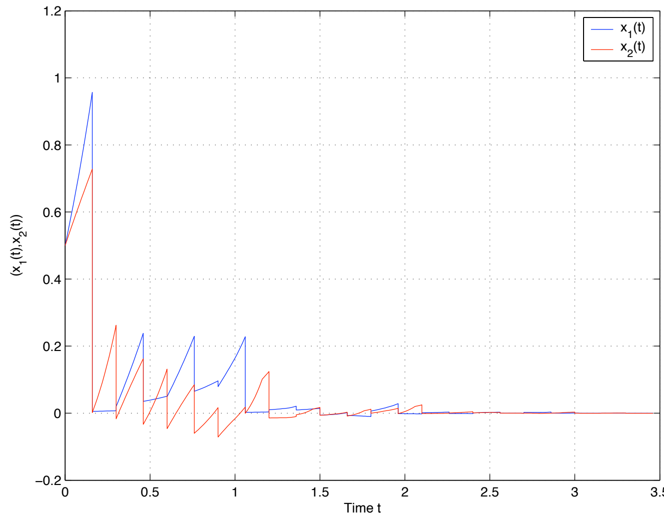

Figure 11.4: Systems with all unstable subsystems: $\phi(t) = -t + 0.5$.

Corollary 11.1. Consider uncertain system (11.10) with $\mathcal{S} = \mathcal{S}_u \cup \mathcal{S}_s$. The trivial solution is robustly asymptotically stable if the following assumptions hold:

(i) assumptions (i) and (ii) of Theorem 11.3 are satisfied;

(ii)(1) for $i_k \in \mathcal{S}_u$ and $k = 1, 2, \cdots, l$,

$$\ln \mu(\alpha_k + \psi_k) + \xi_{i_k}(t_k - t_{k-1}) \leq 0;$$

(ii) (2) for $i_k \in \{l+1, l+2, \cdots, N-1\}$ and $k = l+1, l+2, \cdots, N-1$,

$$\ln \mu(\alpha_k + \psi_k e^{\zeta_{i_k} r}) + \zeta_{i_k} r - \zeta_{i_k}(t_k - t_{k-1}) \leq 0,$$

where α_k and ψ_{i_k} are defined in Theorem 11.4, and ξ_{i_k} and ζ_{i_k} are the growth and decay rates of unstable and stable subsystems, respectively.

11.3 Conclusion

Throughout this chapter, we discussed a time-delayed impulsive switched system with uncertainties of the structured type. The focus was on establishing the problems of robust stability and stabilization of this system by designing switching laws to organize the switching among either a mix of stable and unstable or all unstable subsystems. In the latter case, we showed how helpful impulses can stabilize a switched system with all unstable subsystems. In the stability analysis, we considered a deterministic dwell-time-based approach together with the technique of multiple Lyapunov functions.

Chapter 12

Input-to-State Stability of Stochastic Switched Systems

This chapter concerns with studying the sensitivity of nonlinear switched systems, whose states are driven by Wiener process, to bounded disturbances or controls. Lyapunov-based sufficient conditions are established to guarantee input-to-state (ISS) properties in the pth moment. The first case in which system switches among a family of ISS modes is studied. Then, a more general class is considered, in which unstable subsystems perturbed by bounded disturbance are taken into account. Switching among the system modes is controlled by two separated switching rules, a new criterion called *initial-state-dependent dwell-time (τ_{isd}) condition*, and Markovian switching. Implications of our results are stated and some numerical examples are presented to justify the proposed theoretical results.

The τ_{isd} technique is inspired by the state-dependent approach proposed in [DePe02]. The features of the new approach are (1) like state-dependent approach, the dwell time depends on the comparison functions that distinguish each subsystem, so long as multiple Lyapunov technique is adopted, (2) in contrast to state-dependent approach, which requires the knowledge of the state at the switching times, it is easier to work with in the sense that it can be determined *a priori*, because it depends on the magnitude of the initial state only, (3) unlike constant dwell-time condition which may result in divergence if it is adopted to nonlinear or, in some cases, to linear systems, τ_{isd} approach generates a sequence of state magnitudes, evaluated at the switching moments, that is convergent to a limit set depending on the ISS-gain of the system, or convergent to zero in the case of input-free (or unforced) systems; briefly, assuming that the solution exists all the time, in considering the τ_{isd} condition, it is guaranteed that the solution converges when time goes to infinity, and (4) it can be applied to a family of all ISS subsystems and even a larger class, where some of the systems are unstable.

The switched system under consideration has a finite family of subsystems having ISS property. We also consider the case, in which some of these subsystem are unstable, i.e., the unforced subsystems are unstable. This case occurs if the modes are being viewed as stable closed-loop and some of the controllers are unavailable leading to instability. In both cases, the state is excited by a random noise that is represented by a Wiener (or Brownian motion) process. As mentioned earlier, the focus of this chapter is to establish some results on asymptotic input-to-state stability (aISS) in pth moment.

The result has some implications. If we do not consider the random noise effect, the stability property reduces to aISS for nonlinear deterministic switched systems. If the system is subject to random noise, but not to input disturbance, the result reduces to the pth moment asymptotic stability of the equilibrium point of the unforced stochastic switched system, and it implies asymptotic stability if the system is noise-free. Moreover, if the system has a single mode, the results reduce to the pth moment aISS property for nonlinear stochastic systems, i.e., a generalization of the standard aISS concept introduced by Sontag. We should mention that the authors of [Spil03] analyzed such systems in which the input is a random function. They established some results on the necessary conditions for the stochastic ISS notion. The material of this chapter forms the basis of [Alw-h].

Consider the following stochastic switched system with input

$$dx(t) = f_{\sigma(t)}(t, x(t), u(t))dt + g_{\sigma(t)}(t, x(t), u(t))dW(t), \quad t \geq t_0, \tag{12.1a}$$

$$x(t_0) = x_0, \tag{12.1b}$$

where the state vector $x \in \mathbb{R}^n$ is assumed to be a right-continuous Itô process, the input $u : [t_0, \infty) \to \mathbb{R}^l$ is an essentially bounded function with $\|u(t)\|_\infty \leq 1$, where $\|u(t)\|_\infty := \text{ess.}\sup_{t \geq t_0} \|u(t)\|$, the switching signal $\sigma(t) : [t_0, \infty) \to \mathcal{S}$ is a piecewise constant function taking values in a finite compact set $\mathcal{S} = \{1, 2, \cdots, N\}$ for some $N \in \mathbb{N}$. If switching among the elements of $\mathcal{S}$ occurs randomly, we assume that the switching signal $\sigma(\cdot)$ is a right-continuous Markov chain taking values in $\mathcal{S}$ with the generator $\Gamma = [\gamma_{ij}]_{N \times N}$ and its evolution is governed by the following probability transitions

$$\mathbb{P}\{\sigma(t+h) = j \big| \sigma(t) = i\} = \begin{cases} \gamma_{ij} h + o(h), & \text{if} \quad i \neq j, \\ 1 + \gamma_{ii} h + o(h), & \text{if} \quad i = j, \end{cases}$$

where $h > 0$, γ_{ij} is the transition rate from mode i to mode j with $\gamma_{ij} \geq 0$, when $i \neq j$, and $\gamma_{ii} = -\sum_{j=1, j\neq i}^{N} \gamma_{ij}$, and $o(h)$ is such that $\lim_{h \to 0} o(h)/h = 0$. The switching signal $\sigma(\cdot)$ is assumed to be independent of $W(\cdot)$. Let $\{t_k\}_{k \in \mathbb{N}}$ (with $t_k \in \mathbb{R}_+$) be a strictly increasing sequence of switching times satisfying $\lim_{k \to \infty} t_k = \infty$. For any i_k, or, for simplicity of notation, $i \in \mathcal{S}$, the functions $f_i : [t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$, $g_i : [t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{n \times m}$, belonging to $\mathcal{L}_{ad}(\Omega, L^p[t_{k-1}, t_k))$ with $p = 1$ and $p = 2$, respectively, are assumed to be smooth enough to guarantee a unique solution, and $f_i(t,0,0) = 0 \in \mathbb{R}^n$ and $g_i(t,0,0) = 0 \in \mathbb{R}^{n\times m}$; that is, the unforced system (12.1) admits a trivial solution $x \equiv 0$. We also assume the initial state x_0 to be $\mathcal{F}_0$-measurable with finite pth moment (i.e., $\mathbb{E}[\|x_0\|^p] < \infty$).

In Definition 10.2, we stated the generalized Itô formula of a switched system ruled by Markovian switching. One can similarly states the formula if the switched system undergoes input disturbance, and, consequently, the infinitesimal diffusion operator $\mathcal{L}$ will have the following form

$$\begin{aligned}\mathcal{L}V(t,x(t),u(t),i) =& V_t(t,x(t),i) + V_x(t,x(t),i)f(t,x(t),u(t),i)\\ &+ \frac{1}{2}\mathrm{tr}[g^T(t,x(t),u(t),i)V_{xx}(t,x(t),i)g(t,x(t),u(t),i)]\\ &+ \sum_{j=1}^{N}\gamma_{ij}V(t,x(t),j), \end{aligned} \tag{12.2}$$

where V is a $\mathcal{C}^{1,2}([t_{k-1},t_k)\times\mathbb{R}^n;\mathbb{R}_+)$ function mapping the pair $(x(t),i)$ into $V(t,x(t),i)$, and γ_{ij} is the transition rate defined above.

A common practice in proving the existence of a regular (global) solution for a stochastic system is to assume that the vector field functions f and g to grow linearly over the entire space. This restrictive condition can be relaxed if Lyapunov technique is used instead [Gard88, Kha80, Mao06]. In these references it was shown that the solution is regular if a *local* solution exists and the infinitesimal diffusion operator $\mathcal{L}$ (for a fixed mode number with $u \equiv 0$) is either non-positive [Gard88] or bounded by some nonnegative linear estimate of V (i.e., for some positive constant c, $\mathcal{L}V \le cV$) [Kha80, Mao06]. In the following lemma, which is inspired by Theorem 3.4.1 in [Kha80] and Theorem 3.19 in [Mao06], we consider the Lyapunov approach to prove the existence of a regular solution, where the operator is bounded by a nonnegative nonlinear estimate of V (i.e., where $\mathcal{L}V(t,x,u) \le \alpha(V(t,x))$ for some $\alpha \in \mathcal{K}_2$). As will be seen, the lemma has a further use in Theorem 12.2 regarding the estimate of $\mathbb{E}[V(t, x(t))]$. We should also remark that the result of this lemma remains correct if the system is input-free.

Lemma 12.1. Assume that a unique solution $x(t) = x(t, t_0, x_0)$ of the initial value problem

$$\begin{aligned} dx(t) &= f(t, x(t), u(t))dt + g(t, x(t), u(t))dW(t), \\ x(t_0) &= x_0 \end{aligned}$$

exists for all $t \in [t_0, \tau_\infty)$, where τ_∞ is the explosion time. Let $V \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ such that it is radially unbounded (i.e., $\lim_{\|x\|\to\infty} \left[\inf_{t\geq t_0} V(t, x)\right] = \infty$, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$) and

$$\mathcal{L}V(t, x, u) \leq \alpha(V(t, x)),$$

where α is a class-$\mathcal{K}_2$ function. Then, the solution $x(t)$ is unique and defined for all $t \geq t_0$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be a local solution of the system. We claim that $\tau_\infty = \infty$. If our claim were not true, there would exist positive constants ε and T such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon.$$

Define a sequence of stopping times τ_l, $\forall\, l \geq 1$, of process x from the ball $\|x\| > l$, i.e.,

$$\tau_l = \inf\{t \geq t_0 : \|x(t)\| > l\},$$

such that $\tau_l \to \tau_\infty$ (a.s.). This implies that, for sufficiently large l^*,

$$\mathbb{P}\{\tau_l \leq T\} > \varepsilon', \qquad \text{for some } \varepsilon' < \varepsilon, \quad l \geq l^*.$$

For any $t \in [t_0, T]$ and $l \geq l^*$, let $\tau_l(t) = \min\{\tau_l, t\}$. Apply the generalized Itô formula to process $V(\tau_l(t), x(\tau_l(t)))$ and then take the mathematical expectation to get

$$\begin{aligned}\mathbb{E}[V(\tau_l(t), x(\tau_l(t)))] &= \mathbb{E}[V(t_0, x_0)] + \mathbb{E}\int_{t_0}^{\tau_l(t)} \mathcal{L}V(s, x(s), u(s))ds \\ &\leq \mathbb{E}[V(t_0, x_0)] + \mathbb{E}\int_{t_0}^{t} \mathcal{L}V(\tau_l(s), x(\tau_l(s)), u(\tau_l(s)))ds \\ &\leq \mathbb{E}[V(t_0, x_0)] + \int_{t_0}^{t} \alpha\big(\mathbb{E}[V(\tau_l(s), x(\tau_l(s)))]\big)ds.\end{aligned}$$

By Bihari's inequality [Mao94], we get

$$\begin{aligned}\mathbb{E}[V(\tau_l(t), x(\tau_l(t)))] &= G^{-1}\Big[G\big(\mathbb{E}[V(t_0, x_0)]\big) + (t - t_0)\Big] \\ &\leq G^{-1}\Big[G\big(\mathbb{E}[V(t_0, x_0)]\big) + (T - t_0)\Big] < \infty,\end{aligned}$$

where $G(s) = \int_1^s \frac{dt}{\alpha(t)}$, G^{-1} is the inverse function of G, and $G\big(\mathbb{E}[V(t_0, x_0)]\big) + (T - t_0) \in \text{Domain}(G^{-1})$. From the above inequality, we see $\mathbb{E}[V(t, x(t)] < \infty$ for any $t \in [t_0, T]$.

On the other hand,

$$\mathbb{E}\big[1_{\{\tau_l \leq T\}} V(\tau_l, x(\tau_l))\big] \leq G^{-1}\Big[G\big(\mathbb{E}[V(t_0, x_0)]\big) + (T - t_0)\Big],$$

where 1_A is the indicator function of a set A, i.e., $1_A(x) = 1$ if $x \in A$ and otherwise 0. Define

$$\eta_l = \inf\{V(t, x) : \|x\| \geq l,\, t \geq t_0\}.$$

Thus,

$$G^{-1}\Big[G\big(\mathbb{E}[V(t_0, x_0)]\big) + (T - t_0)\Big] \geq \eta_l \mathbb{P}\{\tau_l \leq T\} \geq \varepsilon' \eta_l.$$

Letting $l \to \infty$ implies contradiction, because V is radially unbounded; therefore, it must be true that

$$\mathbb{P}\{\tau_l \geq T\} = 1.$$

The uniqueness follows from the definition of x up to equivalence, i.e., if y is another solution, then

$$\mathbb{P}\{\|x(t) - y(t)\| = 0, \quad t_0 \leq t \leq \sigma_\infty\} = 1.$$

This completes the proof..

Definition 12.1. For any $t_0 \in \mathbb{R}_+$, $t \geq t_0$, and $x_0 \in \mathbb{R}^n$, let $x(t) = x(t, t_0, x_0)$ be a solution of system (12.1). Then, system (12.1) is said to be *uniformly asymptotically input-to-state stable (aISS) in the pth moment* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for any u and $p \geq 1$, the solution satisfies

$$\mathbb{E}[\|x(t)\|^p] \leq \beta\big(\mathbb{E}[\|x_0\|^p], t - t_0\big) + \gamma(\|u(t)\|_\infty), \quad \forall t > t_0.$$

It is said to be *exponentially input-to-state stable (eISS) in the pth moment* if in addition $\beta\big(\mathbb{E}[\|x_0\|^p], t - t_0\big) \leq K\mathbb{E}[\|x_0\|^p]e^{-\lambda(t-t_0)}$, for some positive constants K and λ.

Remark 12.1. Immediate implications of the above definition are, e.g., for zero input, it reduces to the uniform pth moment asymptotic (or exponential) stability of the trivial solution of unforced system, and for non-zero input and $g \equiv 0$, it reduces to the standard definition of uniform ISS, which in turn implies the uniform asymptotic (or exponential) stability for zero input.

12.1 Initial-state-dependent dwell-time condition

In this section, we state and prove our Lyapuonv-based sufficient conditions to ensure aIS stability in the pth moment of the solution of the switched system (12.1). In Theorem 12.1, we consider a switched system with all stable modes and show the convergence of the solution trajectories to a ball of radius depending on the input magnitude. We also consider a more general case (Theorem 12.2), in which some of the modes are unstable.

Theorem 12.1. For any $i \in \mathcal{S}$, $t \in [t_{k-1}, t_k)$, $x \in \mathbb{R}^n$, and $p \geq 1$, there exists $V_i \in \mathcal{C}^{1,2}\big([t_{k-1}, t_k) \times \mathbb{R}^n; \mathbb{R}_+\big)$ with $V_i(t, 0) = 0$, which satisfies the following assumptions:

(i) there exist a concave function $\alpha_{1_i} \in \mathcal{K}_\infty$ and a convex function $\alpha_{2_i} \in \mathcal{K}_\infty$ such that

$$\alpha_{2_i}(\|x\|^p) \leq V_i(t, x) \leq \alpha_{1_i}(\|x\|^p), \qquad \text{(a.s.)}; \tag{12.3}$$

(ii) there exist $\alpha^*_{3_i} \in \mathcal{K}_1$ and $\gamma \in \mathcal{K}$ such that

$$\mathcal{L}V_i(t, x(t), u(t)) \leq -\alpha_{3_i}(\|x\|^p), \qquad \text{(a.s.)}, \tag{12.4}$$

provided that $\|x\|^p > \alpha^{*^{-1}}_{3_i}\Big(\frac{1}{\nu}\gamma(\|u\|_\infty)\Big) =: \rho_i(\|u\|_\infty)$ (a.s.), where $0 < \nu < 1$ and $\alpha_{3_i}(\cdot) = (1 - \nu)\alpha^*_{3_i}(\cdot)$;

(iii) for all $k = 1, 2, \cdots$, the τ_{isd} condition

$$t_k - t_{k-1} \geq \ln \frac{\theta_{2_i}\big(a_{k-1}\mathbb{E}[\|x_0\|^p]\big)}{\theta_{1_i}\big(a_k\mathbb{E}[\|x_0\|^p]\big)} > 0 \tag{12.5}$$

holds, where a_k are positive real numbers with $a_0 = 1$, $a_k < a_{k-1}$, and $\lim_{k\to\infty} a_k = 0$, and θ_{i_1} and θ_{2_i} are some class-$\mathcal{K}_\infty$ functions.

Then, system (12.1) is pth moment aISS with the ultimate bound (or ISS gain) $\rho_M = \max\{\rho_i = \alpha_{3_i}^{*^{-1}} \circ \gamma^* \ : i \in \mathcal{S}\}$ where $\gamma^*(\cdot) = \frac{1}{\nu}\gamma(\cdot)$.

Remark 12.2.

1. Assumptions (i) and (ii) are made to ensure the aISS property in the pth moment of each subsystem. The function V_i satisfying these assumptions is called *stochastic* ISS Lyapunov function related to the i^{th} subsystem.

2. The idea behind the dwell-time-based condition in (iii) is to generate a sequence of solution trajectories at the switching times that converges (in the pth moment) to a limit set with a radius depending on the ultimate bound of the input.

Proof of Theorem 12.1. For all $t \geq t_0$, let $x(t)$ be the solution of (12.1), and for any $t \in [t_{k-1}, t_k)$, let $V_i(t, x(t))$ be a Lyapunov function related to the i^{th} mode. By (ii), we define $m_i(t) = \mathbb{E}[V_i(t, x(t))]$ for all $t \in [t_{k-1}, t_k)$. Applying the Itô formula to $V_i(t)$ and taking the mathematical expectation give

$$\begin{aligned} m_i(t) &= m_i(t_{k-1}) + \mathbb{E}\int_{t_{k-1}}^{t} \mathcal{L}V_i(s, x(s), u(s))ds \\ &\leq m_i(t_{k-1}) - \mathbb{E}\int_{t_{k-1}}^{t} \alpha_{3_i}(\|x(s)\|^p)ds \\ &\leq m_i(t_{k-1}) - \int_{t_{k-1}}^{t} \alpha_{3_i}(\mathbb{E}[\|x(s)\|^p])ds \\ &\leq m_i(t_{k-1}) - \int_{t_{k-1}}^{t} \alpha_{3_i}(\alpha_{1_i}^{-1}(m_i(s))ds \\ &= m_i(t_{k-1}) - \int_{t_{k-1}}^{t} \alpha_i(m_i(s))ds, \end{aligned}$$

or

$$D^+ m_i(t) \leq -\alpha_i(m_i(t)),$$

where $\alpha_i = \alpha_{3_i} \circ \alpha_{1_i}^{-1}$. Then, by a classical stability result [Kha02], there exists a class-$\mathcal{KL}$ function β_i^* such that

$$m_i(t) \leq \beta_i^*(m_i(t_{k-1}), t - t_{k-1}),$$

or, by (i),

$$\begin{aligned}\alpha_{2_i}(\mathbb{E}[\|x(t)\|^p]) \leq & m_i(t) \leq \beta_i^*(m_i(t_{k-1}), t - t_{k-1}) \leq \beta_i^*(\alpha_{1_i}(\mathbb{E}[\|x(t_{k-1})\|^p]), t - t_{k-1}) \\ & =: \beta_i(\mathbb{E}[\|x(t_{k-1})\|^p], t - t_{k-1}) \\ & \leq \theta_{1_i}^*\Big[\theta_{2_i}^*(\mathbb{E}[\|x(t_{k-1})\|^p])e^{-(t-t_{k-1})}\Big],\end{aligned}$$

which implies that

$$\begin{aligned}\mathbb{E}[\|x(t)\|^p] &\leq \alpha_{2_i}^{-1}\Big\{\theta_{1_i}^*\Big[\theta_{2_i}^*(\mathbb{E}[\|x(t_{k-1})\|^p])e^{-(t-t_{k-1})}\Big]\Big\} \\ &= \theta_{1_i}^{-1}\Big[\theta_{2_i}(\mathbb{E}[\|x(t_{k-1})\|^p])e^{-(t-t_{k-1})}\Big], \qquad (12.6)\end{aligned}$$

where $\theta_{1_i}(r) := \theta_{1_i}^{*^{-1}}(\alpha_{2_i}(r))$, $\theta_{2_i} := \theta_{2_i}^*(r)$, and $\theta_{1_i}^*$ and $\theta_{2_i}^*$ are $\mathcal{K}_\infty$ functions, which are guaranteed by [DePe02, Son98].

Specifically, for $i = 1$ and $k = 1$ (i.e., $t \in [t_0, t_1)$), we have from (12.6)

$$\mathbb{E}[\|x(t)\|^p] \leq \theta_{1_1}^{-1}\Big[\theta_{2_1}(\mathbb{E}[\|x(t_0)\|^p])e^{-(t-t_0)}\Big],$$

and, at the switching instant $t = t_1$ (i.e., after $t_1 - t_0 > 0$), we have

$$\mathbb{E}[\|x(t_1)\|^p] \leq \theta_{1_1}^{-1}\Big[\theta_{2_1}(\mathbb{E}[\|x(t_0)\|^p])e^{-(t_1-t_0)}\Big].$$

By the τ_{isd} condition (12.5), we get

$$\mathbb{E}[\|x(t_1)\|^p] \leq a_1\mathbb{E}[\|x_0\|^p],$$

which implies, that for $i = 2$ and $t \in [t_1, t_2)$,

$$\begin{aligned}\mathbb{E}[\|x(t)\|^p] &\leq \theta_{1_2}^{-1}\Big[\theta_{2_2}\mathbb{E}[\|x(t_1)\|^p])e^{-(t-t_1)}\Big] \\ &\leq \theta_{1_2}^{-1}\Big[\theta_{2_2}(a_1\mathbb{E}[\|x_0\|^p])e^{-(t-t_1)}\Big].\end{aligned}$$

By the same argument, for $t \in [t_{k-1}, t_k)$, we get

$$\mathbb{E}[\|x(t)\|^p] \leq \theta_{1_i}^{-1}\Big[\theta_{2_i}(a_{k-1}\mathbb{E}[\|x_0\|^p])e^{-(t-t_{k-1})}\Big],$$

whenever $\|x(t)\| > [\rho_i(\|u\|_\infty)]^{1/p}$ (a.s.), and, at $t = t_k$, $\mathbb{E}[\|x(t_k)\|^p] \leq a_k\mathbb{E}[\|x_0\|^p]$. Since $\lim_{k\to\infty} a_k = 0$, the system states will approach (in the pth moment) the ultimate bound $[\rho(\|u\|_\infty)]^{1/p}$, where $\rho = \max_i\{\rho_i\}$; that is, the switched system (12.1) is aISS in the pth moment. This completes the proof.

Implications of this result are stated in the following corollary, whose proofs are straightforward and are omitted here.

Corollary 12.1. In Theorem 12.1, if

1. $\alpha_{1_i}(s) = \alpha_{1_i}s$, $\alpha_{2_i}(s) = \alpha_{2_i}s$, and $\alpha^*_{3_i}(s) = \alpha^*_{3_i}s$, for all $s > 0$, where α_{1_i}, α_{2_i}, and $\alpha^*_{3_i}$ are positive constants, the above aISS properties reduce to eISS.

2. $u(t) \equiv 0$ for any $t \in \mathbb{R}_+$, aISS reduces to the uniform global asymptotic stability (GAS) in the pth moment of the trivial solution of the nonlinear stochastic switched system

$$\begin{aligned} dx(t) &= f_{\sigma(t)}(t, x(t))dt + g_{\sigma(t)}(t, x(t))dW(t), \\ x(t_0) &= x_0. \end{aligned}$$

3. $g \equiv 0$ and $u \not\equiv 0$, the aISS property reduces to the standard aISS of the nonlinear switched system

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(t, x(t), u(t)), \\ x(t_0) &= x_0, \end{aligned}$$

where $\mathcal{L}V = \dot{V} = V_t + V_x f(t, x, u)$.

4. $g \equiv 0$ and $u \equiv 0$, the aISS property reduces to GAS of the nonlinear switched system

$$\dot{x}(t) = f_{\sigma(t)}(t, x(t)),$$
$$x(t_0) = x_0,$$

where $\mathcal{L}V = \dot{V} = V_t + V_x f(t, x)$.

In the following example, we illustrate these results.

Example 12.1. Consider the following switched system with input

$$dx = (-a_i x + u(t))dt + u(t)\sin x\, dW(t),$$

where $i \in \mathcal{S} = \{1, 2\}$ and a_i is a positive real number. Let $V_i(x) = \frac{1}{4}x^4$ be a Lyapunov function candidate related to the i^{th} subsystem and $\alpha_{1_i}(x) = \alpha_{2_i}(x) = V_i(x)$. Then

$$\begin{aligned}\mathcal{L}V_i(x, u) &\leq -a_i x^4 + |x|^3|u| + \frac{3}{2}x^4 \\ &\leq -a_i x^4 + a_i\theta x^4 - a_i\theta x^4 + |x|^3|u| + \frac{3}{2}x^4, \qquad 0 < \theta < 1, \\ &\leq -\alpha_{3_i} V(x),\end{aligned}$$

provided that $|x| \geq |u|/(a_i\theta - 3/2)$ with $a_i\theta > 3/2$, where $\alpha_{3_i} = 4a_i(1-\theta) > 0$. Thus, the subsystems are both aISS in the fourth moment. Letting $a_1 = 4$, $a_2 = 8$, and $\theta = 1/2$ gives $\alpha_{3_1} = 8$ and $\alpha_{3_2} = 14$. By Theorem 12.1, we have $m_i(t) \leq m_i(t_{k-1})e^{-\alpha_{3_i}(t-t_{k-1})} \leq e^{-(t-t_{k-1})}$, where $m_i(t) = \mathbb{E}[V_i(x(t))]$ for any $t \in [t_{k-1}, t_k)$. This also implies that $\theta_{1_i}(r) = \theta_{2_i}(r) = r$, from which we obtain $\mathbb{E}[x^4] \leq \mathbb{E}[x^4(t_{k-1})] \leq e^{-(t-t_{k-1})}$. Therefore, the τ_{isd} condition implies that $t_k - t_{k-1} \geq \ln(\frac{a_{k-1}}{a_k})$, where we choose $a_k = \frac{1}{k+1}$, $k = 1, 2, \cdots$. In Figure 12.1, we show the simulation result of the second moment of the solution, where the input function

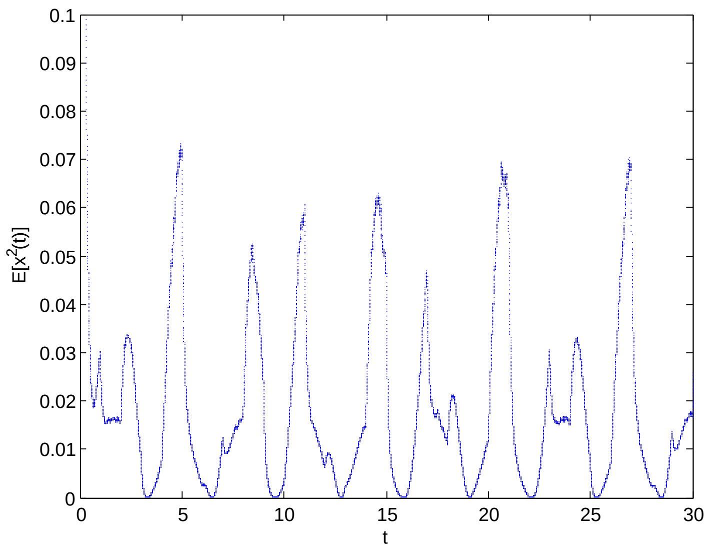

Figure 12.1: Mean square aISS with $u(t) = \sin(t)$.

$u(t) = \sin(t)$. We should also remark that, for better insight into the solution curve, the display is over the rectangle $t \in [0, 30]$ and $x \in [0, 0.1]$. With the same sinusoidal input function, Figure 12.2 shows that the system is aISS in the first moment. Figure 12.3 illustrates the first moment aISS property of the system, where disturbance input is given by the hyperbolic function $u(t) = 1/(1+t)$. Figure 12.4 shows the classical asymptotic stability (in the first moment) property of the equilibrium point $x \equiv 0$ for the stochastic switched system, i.e., when $u(t) \equiv 0$. The standard aISS property of the deterministic switched system is shown in Figure 12.5.

In the following theorem, we state and prove the pth moment aISS property of switched system with stable and unstable subsystems. For convenience of notation, let $\mathcal{S}_s = \{1, 2, \cdots, N_s\}$ and $\mathcal{S}_u = \{1, 2, \cdots, N_u\}$, with $N_s + N_u = N$, be the index sets of stable and unstable subsystems, respectively, and $\mathcal{S} = \mathcal{S}_s \cup \mathcal{S}_u$.

Theorem 12.2. Consider system (12.1) with $\mathcal{S} = \mathcal{S}_s \cup \mathcal{S}_u$. Assume that the following assumptions hold:

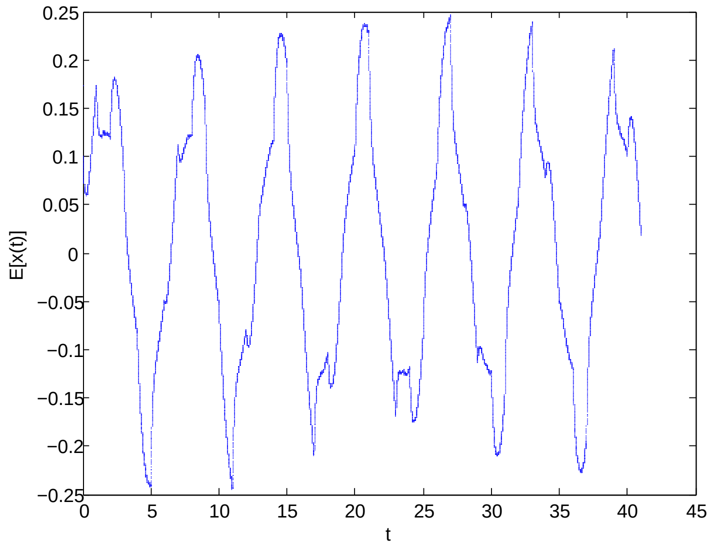

Figure 12.2: First moment aISS with $u(t) = \sin(t)$.

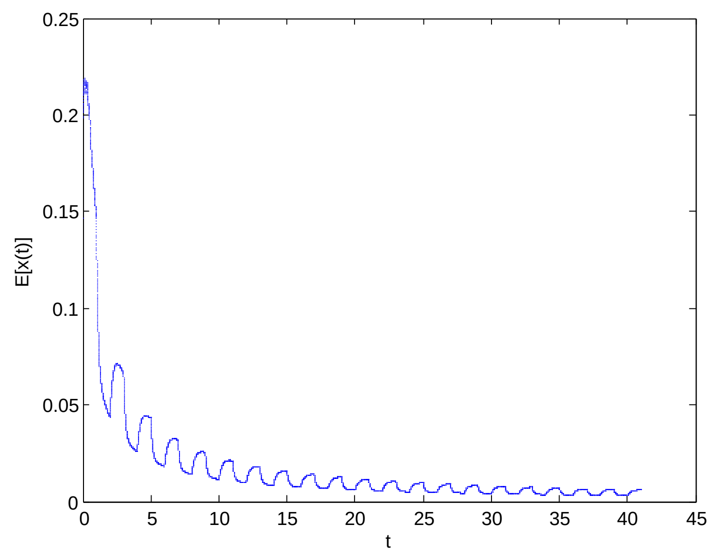

Figure 12.3: First moment aISS property with $u(t) = 1/(1 + t)$.

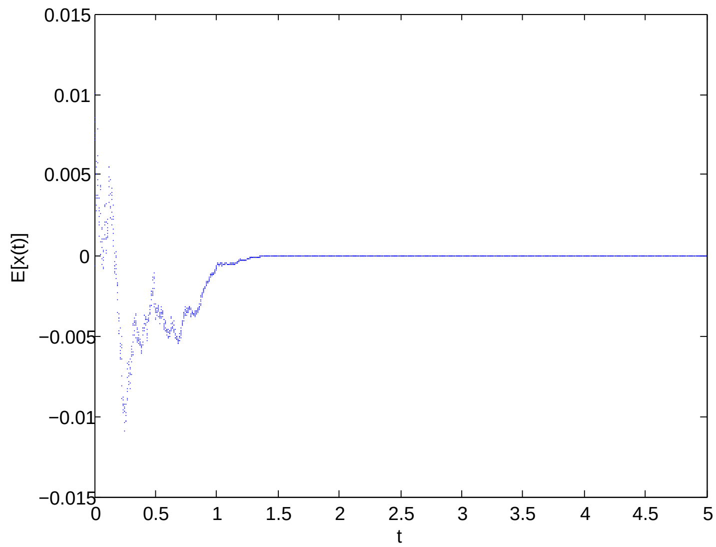

Figure 12.4: First moment asymptotic stability of $x \equiv 0$ (i.e., $u(t) = 0$).

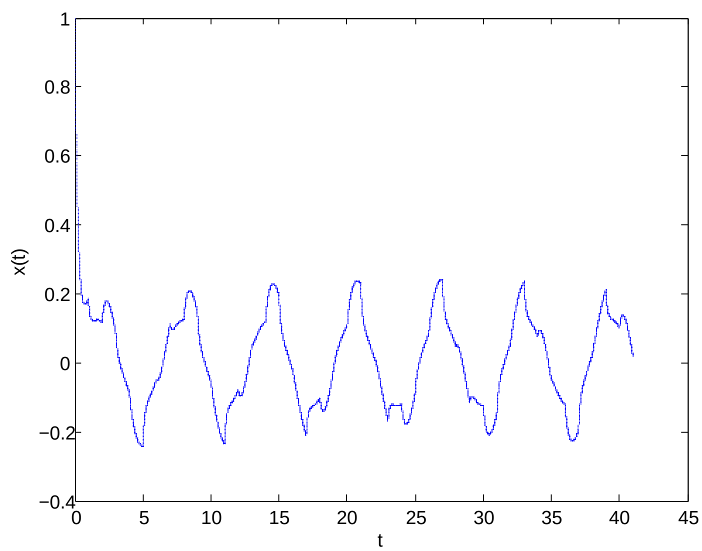

Figure 12.5: aISS property with $u(t) = \sin(t)$ and $g \equiv 0$.

(i) for each $i \in \mathcal{S}$, there exist class-$\mathcal{K}_\infty$ functions α_{1_i}, α_{2_i}, and ρ_i, where α_{1_i} is concave and α_{2_i} is convex, a class-$\mathcal{K}_1$ function α_{3_i}, and a class-$\mathcal{K}_2$ function $\bar{\alpha}_{3_i}$ such that

(1) $\alpha_{2_i}(\|x\|^p) \leq V_i(t,x) \leq \alpha_{1_i}(\|x\|^p), \qquad \text{(a.s.)};$

(2) $\mathcal{L}V_i(t,x,u) \leq -\alpha_{3_i}(\|x\|^p), \qquad \text{(a.s.)}, \quad \forall i \in \mathcal{S}_s, \quad \text{whenever } \|x\|^p > \rho_i(\|u\|_\infty);$

(3) $\mathcal{L}V_i(t,x,u) \leq \bar{\alpha}_{3_i}(\|x\|^p), \qquad \text{(a.s.)}, \quad \forall i \in \mathcal{S}_u,$

where $V_i(t,x(t)) \in \mathcal{C}^{1,2}([t_{k-1},t_k) \times \mathbb{R}^n;\mathbb{R}_+)$ with $V_i(t,0)=0$;

(ii) the following τ_{isd} condition holds

(1) for all $i \in \mathcal{S}_s = \{1,2,3,\cdots,N_s\}$ and $k = 1,3,5,\cdots$,

$$t_1 - t_0 \geq \ln \frac{\theta_{2_1}\big(\mathbb{E}[\|x_0\|^p]\big)}{\theta_{1_i}\big(a_1\mathbb{E}[\|x_0\|^p]\big)} > 0,$$
$$t_3 - t_2 \geq \ln \frac{\theta_{2_3}\big(a_1A_1\mathbb{E}[\|x_0\|^p]\big)}{\theta_{1_3}\big(a_2\mathbb{E}[\|x_0\|^p]\big)} > 0,$$
$$\cdots$$

(2) for all $i \in \mathcal{S}_u = \{1,2,3,\cdots,N_u\}$ and $k = 2,4,6,\cdots$

$$0 < t_2 - t_1 \leq G_2\Big[\alpha_{2_2}\big(a_1A_1\mathbb{E}[\|x_0\|^p]\big)\Big] - G_2\Big[\alpha_{1_2}\big(a_1\mathbb{E}[\|x_0\|^p]\big)\Big],$$
$$0 < t_4 - t_3 \leq G_4\Big[\alpha_{2_4}\big(a_2A_2\mathbb{E}[\|x_0\|^p]\big)\Big] - G_4\Big[\alpha_{1_4}\big(a_2\mathbb{E}[\|x_0\|^p]\big)\Big],$$
$$\cdots$$

where $0 < a_k < a_kA_k \leq a_{k-1}$ with $a_0 = 1$, and $\theta_{1_i}(r) := \theta_{1_i}^{*^{-1}}(\alpha_{1_i}(r))$ and $\theta_{2_i} := \theta_{2_i}^*(r)$ are functions of class $\mathcal{K}_\infty$, and G_2, $G_4, \cdots$ are functions defined in Lemma 12.1.

Then, switched system (12.1) is pth moment aISS with the ultimate bound $\rho_M := \max_i \rho_i$.

Remark 12.3. In the proof of this theorem, we adopt the case, where the switching among the stable and unstable modes occurs alternatively for convenience.

Proof. For all $t \geq t_0$, let $x(t)$ be the solution of (12.1), and, for any $t \in [t_{k-1}, t_k)$ and $i \in \mathcal{S}$, we take $V_i(t, x(t)) \in \mathcal{C}^{1,2}([t_{k-1}, t_k) \times \mathbb{R}^n; \mathbb{R}_+)$. For $i = 1$, we run a stable mode for all $t \in [t_0, t_1)$. Then, we have, from Theorem 12.1,

$$\mathbb{E}[\|x(t)\|^p] \leq \theta_{1_1}^{-1}\Big[\theta_{2_1}(\mathbb{E}[\|x_0\|^p])e^{-(t-t_0)}\Big],$$

and, at $t = t_1$, we have, by the stable τ_{isd} condition (i.e., by (ii) (1)),

$$\mathbb{E}[\|x(t_1)\|^p] \leq a_1\mathbb{E}[\|x_0\|^p].$$

If an unstable mode is activated for $t \in [t_1, t_2)$, we have, by Lemma 12.1 and (i),

$$\begin{aligned}\mathbb{E}[\|x(t)\|^p] &\leq \alpha_{2_2}^{-1}\Big\{G_2^{-1}\Big(G_2(\alpha_{1_2}(\mathbb{E}[\|x(t_1)\|^p])) + (t - t_1)\Big)\Big\}\\ &\leq \alpha_{2_2}^{-1}\Big\{G_2^{-1}\Big(G_2(\alpha_{1_2}(a_1\mathbb{E}\|x_0\|^p])) + (t - t_1)\Big)\Big\},\end{aligned}$$

and, at $t = t_2$, it implies, by the unstable τ_{isd} condition (i.e., by (ii) (2)),

$$\mathbb{E}[\|x(t_2)\|^p] \leq a_1A_1\mathbb{E}[\|x_0\|^p].$$

By the same manner, one generates a sequence of states at the switching times

$$\mathbb{E}[\|x(t_k)\|^p] \leq a_k\mathbb{E}[\|x_0\|^p] \qquad \text{and} \qquad \mathbb{E}[\|x(t_{k+1})\|^p] \leq a_kA_k\mathbb{E}[\|x_0\|^p].$$

Since, for all $k \in \mathbb{N}$, $a_k < a_kA_k \leq a_{k-1}$, $\lim_{k\to\infty} a_k = 0$ and $\mathbb{E}[\|x_0\|^p] < \infty$, $\lim_{k\to\infty}\mathbb{E}[\|x(t_k)\|^p] = 0$, i.e., when $t \to \infty$ the solution (in the pth moment) will linger on at the ultimate bound of the system input. This completes the proof of the pth moment aISS of switched system (12.1).

Example 12.2. Consider the switched system with the following unstable mode

$$dx = (-ax^3 + xu(t))dt + \sqrt{2a}x^2 dW(t),$$

and the stable mode

$$dx = (-ax^3 - bx + u(t))dt + \sqrt{2a}x^2 dW(t),$$

where a and b are positive constants. Here, $\mathcal{S} = \mathcal{S}_u \cup \mathcal{S}_s = \{1, 2\}$.

For any $i \in \mathcal{S}$, let $V_i(x) = \frac{1}{2}x^2$. Then, for $i = 1 \in \mathcal{S}_u$, we have

$$\mathcal{L}V_1(x, u) = x^2 u(t) = x^2,$$

where $u(t) = 1$ for all $t \in [t_{k-1}, t_k)$, for some k, i.e., the subsystem is unstable. This also implies that $D^+\mathbb{E}[V_1(x(t))] = 2\mathbb{E}[V_1(x(t))]$, and, by Lemma 12.1, we get

$$\mathbb{E}[V_1(x(t))] = \mathbb{E}[V_1(x(t_{k-1}))]e^{2(t-t_{k-1})},$$

i.e., $G(r) = \ln(r)$ and $G^{-1}(r) = e^r$. If we choose $\alpha_{1_1}(x) = \alpha_{2_1}(x) = V_1(x)$, we obtain

$$\mathbb{E}[x^2(t)] = \mathbb{E}[x^2(t_{k-1})]e^{2(t-t_{k-1})}.$$

Analogously, for $i = 2 \in \mathcal{S}_s$, we have $\mathcal{L}V_2(x, u) \leq -\alpha_{3_2}V_2(x)$, provided that $|x| \geq |u|/b\theta$, where $\alpha_{3_2} = 2b(1-\theta) > 0$ and $0 < \theta < 1$, which implies that $\mathbb{E}[V_2(x)] \leq \mathbb{E}[V_2(x(t_{k-1}))]e^{-\alpha_{3_2}(t-t_{k-1})} \leq \mathbb{E}[V_2(x(t_{k-1}))]e^{-2(t-t_{k-1})}$, if we choose $b = 2$ and $\theta = 1/2$, i.e., $\theta_{1_2}(r) = \theta_{2_2}(r) = r$. Let $\alpha_{1_2}(x) = \alpha_{2_2}(x) = V_2(x)$. Then,

$$\mathbb{E}[x^2(t)] \leq \mathbb{E}[x^2(t_{k-1})]e^{-(t-t_{k-1})}, \qquad t \in [t_{k-1}, t_k).$$

As for the dwell time, let us first run a stable mode (i.e., $k = 1$). Then, from the stable dwell-time condition, we get, if $a_1 = 1/2$, $t_1 - t_0 \geq \ln 2 = 0.7$, and, for $k = 2$,

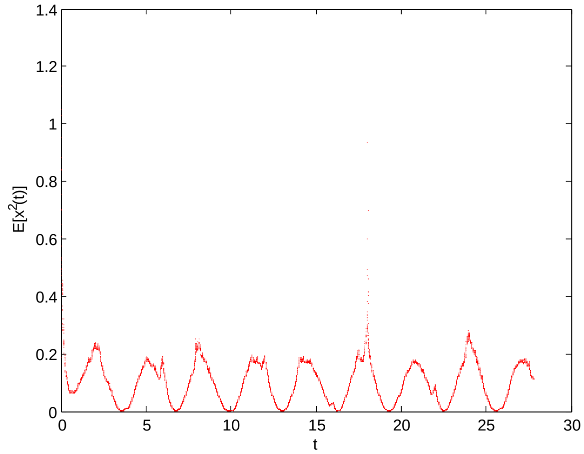

Figure 12.6: Mean square aISS with $u(t) = \sin(t)$.

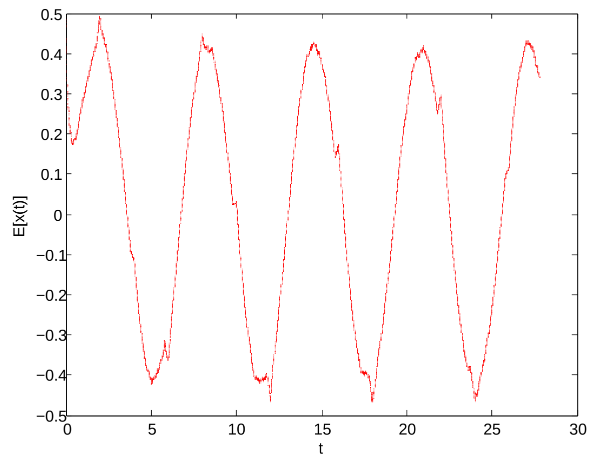

Figure 12.7: First moment aISS with $u(t) = \sin(t)$.

we run the unstable mode with $A_1 = 1.5 > 1$, which gives $t_2 - t_1 \leq \frac{1}{2} \ln 1.5 = 0.2$. By the same manner, we get, for $k = 3$, $t_3 - t_2 \geq \ln \frac{a_1 A_1}{a_2}$, where $a_1 A_1 > a_2$, which implies $a_2 < 3/4$, so that taking $a_2 = 1/4$ gives $t_3 - t_2 \geq 1.1$. For $k = 4$, we get $1 < A_2 \leq 2$, so that taking $A_2 = 1.5$ gives $t_4 - t_3 \leq 0.2$. Figures 12.6 and 12.7 show the second and first moment aISS of the switched system under sinusoidal disturbance input function $u(t) = \sin(t)$.

12.2 Markovian Switching

In this section, we continue to examine ISS properties of the switched system with input discussed in last section. The interest is to develop Lyapunov sufficient conditions to ensure the qualitative properties. In the analysis, we use the Markovian switching and multiple Lyapunov function technique.

Consider the nonlinear system with Markovian switching

$$\begin{cases} dx(t) = f(t, x(t), u(t), \sigma(t))dt + g(t, x(t), u(t), \sigma(t))dW(t), \\ x(t_0) = x_0, \qquad \sigma(t_0) = \sigma_0 \in \mathcal{S}. \end{cases} \tag{12.7}$$

The switching signal $\sigma(t)$ is a Markov process taking values in a finite state space $\mathcal{S} = \{1, 2, \cdots, N\}$ (i.e., $\sigma(t) : [t_0, \infty) \to \mathcal{S}$). In the following theorem, we state the sufficient conditions that guarantee the pth moment eISS of forced system (12.7).

Theorem 12.3. For any $i \in \mathcal{S}$, assume that the following assumptions hold:

(i) there exist constants $K > 0$, $\alpha_i > 0$, $\rho_i \geq 0$, and $\sigma_i \geq 0$ such that

$$\|f(t, x, 0, i)\| \leq K\|x\|, \qquad \|x^T f(t, x, 0, i)\| \leq \alpha_i \|x\|^2,$$
$$\|g(t, x, 0, i)\| \leq \rho_i\|x\|, \qquad \|x^T g(t, x, 0, i)\| \leq \sigma_i \|x\|^2;$$

(ii) the functions f and g are locally Lipschitz in u, for all t and x, i.e., there exist positive constants c_3 and c_4 such that

$$\|f(t, x, u, i) - f(t, x, 0, i)\| \leq c_3\|u\|,$$
$$\|g(t, x, u, i) - g(t, x, 0, i)\| \leq c_4\|u\|;$$

(iii) there exist positive constants λ, c_1, and c_2 such that

$$c_1\|x\|^p \leq V(t, x, i) \leq c_2\|x\|^p, \tag{12.8}$$

$$\mathcal{L}V(t, x, u, i) \leq -\lambda\|x\|^p, \tag{12.9}$$

whenever $\|x\| > \rho(\|u\|_\infty)$, where $V \in \mathcal{C}^{1,2}([t_0,\infty)\times\mathbb{R}^n\times\mathcal{S};\mathbb{R}_+)$ and ρ is a class-$\mathcal{K}$ function.

Then, system (12.7) is pth moment eISS for $0 < p < \min\{2, (3c_4+4\sigma_i)/(c_4+2\sigma_i)\}$ with Lyapunov exponent being not larger than $-\lambda/c_2$.

Proof. For any $t \geq t_0$, let $x(t) = x(t,t_0,x_0)$ be the solution of (12.7). For any $i \in \mathcal{S}$ and $\beta_i > 0$, define $V(t,x(t),i) = \beta_i\|x(t)\|^p$ as a Lyapunov function candidate related to the i^{th} mode. Then,

$$
\begin{aligned}
&\mathcal{L}V(t,x,u,i)\\
&\quad = p\beta_i\|x\|^{p-2}x^T f(t,x,u,i) + \frac{1}{2}p\beta_i\|x\|^{p-2}\|g(t,x,u,i)\|^2\\
&\qquad -\frac{1}{2}p(2-p)\beta_i\|x\|^{p-4}\|x^T g(t,x,u,i)\|^2 + \sum_{j=1}^{N}\gamma_{ij}\beta_j\|x\|^p\\
&\quad = p\beta_i\|x\|^{p-2}x^T\Big[f(t,x,u,i) - f(t,x,0,i) + f(t,x,0,i)\Big]\\
&\qquad +\frac{1}{2}p\beta_i\|x\|^{p-2}\|g(t,x,u,i) - g(t,x,0,i) + g(t,x,0,i)\|^2\\
&\qquad -\frac{1}{2}p(2-p)\beta_i\|x\|^{p-4}\|x^T[g(t,x,u,i) - g(t,x,0,i) + g(t,x,0,i)]\|^2\\
&\qquad +\sum_{j=1}^{N}\gamma_{ij}\beta_i\|x\|^p\\
&\quad \leq p\beta_i\|x\|^{p-2}\Big\{\|x^T[f(t,x,u,i) - f(t,x,0,i)]\| + \|x^T f(t,x,0,i)\|\Big\}\\
&\qquad +\frac{1}{2}p\beta_i\|x\|^{p-2}\Big\{\|g(t,x,u,i) - g(t,x,0,i)\|^2 + \|g(t,x,0,i)\|^2\\
&\qquad +2\|g(t,x,u,i) - g(t,x,0,i)\|\cdot\|g(t,x,0,i)\|\Big\}\\
&\qquad +\frac{1}{2}p(2-p)\beta_i\|x\|^{p-4}\Big\{\|x^T[g(t,x,u,i) - g(t,x,0,i)]\|^2 + \|x^T g(t,x,0,i)\|^2\\
&\qquad +2\|x^T[g(t,x,u,i) - g(t,x,0,i)]\|\cdot\|x^T g(t,x,0,i)\|\Big\} + \sum_{j=1}^{N}\gamma_{ij}\beta_j\|x\|^p\\
&\quad \leq p\beta_i\|x\|^{p-2}\|x^T[f(t,x,u,i) - f(t,x,0,i)]\| + p\beta_i\|x\|^{p-2}\|x^T f(t,x,0,i)\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}p\beta_i\|x\|^{p-2}\|g(t,x,u,i)-g(t,x,0,i)\|^2+\frac{1}{2}p\beta_i\|x\|^{p-2}\cdot\|g(t,x,0,i)\|^2 \\
&+2\frac{1}{2}p\beta_i\|x\|^{p-2}\|g(t,x,u,i)-g(t,x,0,i)\|\cdot\|g(t,x,0,i)\|\Big\} \\
&+\frac{1}{2}p(2-p)\beta_i\|x\|^{p-4}\|x^T[g(t,x,u,i)-g(t,x,0,i)]\|^2 \\
&+\frac{1}{2}p(2-p)\beta_i\|x\|^{p-4}\|x^Tg(t,x,0,i)\|^2 \\
&+2\frac{1}{2}p(2-p)\beta_i\|x\|^{p-4}\|x^T[g(t,x,u,i)-g(t,x,0,i)]\|\cdot\|x^Tg(t,x,0,i)\| \\
&+\sum_{j=1}^N\gamma_{ij}\beta_j\|x\|^p \\
\le\ & p\beta_ic_3\|x\|^{p-1}\|u\|+p\beta_i|\alpha_i|\|x\|^p+\frac{1}{2}p\beta_ic_4^2\|x\|^{p-2}\|u\|^2+\frac{1}{2}p\beta_i\rho_i^2\|x\|^p \\
&+p\beta_i\rho_ic_4\|x\|^{p-1}\|u\|_\infty+\frac{1}{2}p(2-p)\beta_ic_4^2\|x\|^{p-2}\|u\|_\infty^2+\frac{1}{2}p(2-p)\beta_i\sigma_i^2\|x\|^p \\
&+p(2-p)\beta_i\sigma_ic_4\|x\|^{p-2}\|u\|_\infty+\sum_{j=1}^N\gamma_{ij}\beta_i\|x\|^p \\
\le\ & \Big\{p\beta_i|\alpha_i|+\frac{1}{2}p\beta_i\rho_i^2+\frac{1}{2}p(2-p)\beta_i\sigma_i^2+\sum_{j=1}^N\gamma_{ij}\beta_j\Big\}\|x\|^p \\
&+\Big\{\beta_ic_3\|x\|^{p-1}+\frac{1}{2}p\beta_ic_4^2\|x\|^{p-2}+p\beta_i\rho_ic_4\|x\|^{p-1}+\frac{1}{2}p(2-p)\beta_ic_4^2\|x\|^{p-2} \\
&+p(2-p)\beta_i\sigma_ic_4\|x\|^{p-2}\Big\}\|u\|_\infty \\
=\ & \Big\{p|\alpha_i|+\frac{1}{2}p\rho_i^2+\frac{1}{2}p(2-p)\sigma_i^2\Big\}\beta_i+\sum_{j=1}^N\gamma_{ij}\beta_j\|x\|^p \\
&+\Big\{\Big[c_3+p\rho_ic_4\Big]\|x\|^{p-1}+\Big[\frac{1}{2}pc_4^2+\frac{1}{2}p(2-p)c_4^2+p(2-p)\sigma_ic_4\Big]\|x\|^{p-2}\Big\}\beta_i\|u\|_\infty \\
=\ & \left\{\Big[p|\alpha_i|+\frac{1}{2}p\rho_i^2+\frac{1}{2}p(2-p)\sigma_i^2\Big]\beta_i+\sum_{j=1}^N\gamma_{ij}\beta_j\right\}\|x\|^p \\
&+\Big\{\Big[c_3+p\rho_ic_4\Big]\|x\|^{p-1}+\Big[-(0.5c_4+\sigma_i)p+(1.5c_4+2\sigma_i)\Big]c_4p\|x\|^{p-2}\Big\}\beta_i\|u\|_\infty \\
=\ & -\beta_i^*\|x\|^p+2M(\|x\|)\|u\|_\infty \\
\le\ & -\lambda^*\|x\|^p+2M(\|x\|)\|u\|_\infty, \qquad (12.10)
\end{aligned}
$$

where $\lambda^* = \min\{-\beta_i^* : i \in \mathcal{S}\}$ with

$$\beta_i^* = -\beta_i \left[p|\alpha_i| + \frac{1}{2}p\rho_i^2 + \frac{1}{2}p(2-p)\sigma_i^2\right] + \sum_{j=1}^{N} \gamma_{ij}\beta_j < 0,$$

and

$$M(\|x\|) = \max\left\{\beta_i\big[c_3 + p\rho_i c_4\big]\|x\|^{p-1}, c_4 p\beta_i\big[-(0.5c_4+\sigma_i)p + (1.5c_4 + 2\sigma_i)\big]\|x\|^{p-2}\right\}.$$

To use $\lambda^*\|x\|^p$ to dominate $2M(\|x\|)\|u\|_\infty$, we write the last inequality in (12.10) as

$$\begin{aligned}
\mathcal{L}V(t,x,u,i) &\le -(\lambda^* - \nu)\|x\|^p - \nu\|x\|^p + 2M(\|x\|)\|u\|_\infty, \qquad 0 < \nu < \lambda^*, \\
&\le -(\lambda^* - \nu)\|x\|^p \\
&= -\lambda\|x\|^p,
\end{aligned}$$

where $\lambda := \lambda^* - \nu > 0$, provided that $\nu\|x\|^p > 2M(\|x\|)\|u\|_\infty$ or

$$\begin{cases}
\|x\| > 2\beta_i/\nu \cdot [c_3 + p\rho_i c_4]\,\|u\|_\infty, \qquad \text{if} \quad M(\|x\|) = \beta_i\,[c_3 + p\rho_i c_4]\,\|x\|^{p-1}, \\
\|x\| > \{2\beta_i p c_4/\nu \cdot [-(0.5c_4 + \sigma_i)p + (1.5c_4 + 2\sigma_i)]\,\|u\|_\infty\}^{1/2}, \\
\qquad\qquad \text{if} \quad M(\|x\|) = c_4 p\beta_i\,[-(0.5c_4 + \sigma_i) + (1.5c_4 + 2\sigma_i)]\,\|x\|^{p-2}.
\end{cases} \tag{12.11}$$

Applying the generalized Itô formula to $e^{\lambda t/c_2}V(t,x,i)$ and taking the mathematical expectation yield

$$\begin{aligned}
&\mathbb{E}[e^{\frac{\lambda}{c_2}t}V(t,x,i)] \\
&\quad = \mathbb{E}[V(t_0,x_0,\sigma_0)]e^{\frac{\lambda}{c_2}t_0} + \mathbb{E}\Big[\int_{t_0}^{t} e^{\frac{\lambda}{c_2}s}[\frac{\lambda}{c_2}V(s,x,i) + \mathcal{L}V(s,x,i)]ds\Big] \\
&\quad \le \mathbb{E}[V(t_0,x_0,\sigma_0)]e^{\frac{\lambda}{c_2}t_0} + \mathbb{E}\Big[\int_{t_0}^{t} e^{\frac{\lambda}{c_2}s}[\frac{\lambda}{c_2}V(s,x,i) - \frac{\lambda}{c_2}V(s,x,i)]ds\Big] \\
&\quad = \mathbb{E}[V(t_0,x_0,\sigma_0)]e^{\frac{\lambda}{c_2}t_0}.
\end{aligned}$$

By (12.8), the last inequality becomes

$$\mathbb{E}[e^{\frac{\lambda}{c_2}t}c_1\|x\|^p] \le c_2\mathbb{E}[\|x_0\|^p]e^{\frac{\lambda}{c_2}t_0},$$

from which we obtain

$$c_1 e^{\frac{\lambda}{c_2}t}\mathbb{E}[\|x\|^p] \le c_2\mathbb{E}[\|x_0\|^p]e^{\frac{\lambda}{c_2}t_0},$$

or

$$\begin{aligned}\mathbb{E}[\|x\|^p] &\le \frac{c_2}{c_1}\mathbb{E}[\|x_0\|^p]e^{-\frac{\lambda}{c_2}(t-t_0)}\\ &= K\mathbb{E}[\|x_0\|^p]e^{-\frac{\lambda}{c_2}(t-t_0)}, \qquad K = c_2/c_1.\end{aligned}$$

This result shows that system (12.7) is pth moment eISS with the ultimate bound given in (12.11) and Lyapunov exponent $-\lambda/c_2$.

Example 12.3. Consider the following switched system with input

Mode 1.

$$dx = \frac{a}{1+t}(x + u(t))dt + b(\sin x + u(t))dW(t),$$

Mode 2.

$$dx = c(xe^{-|x|} + u(t))dt + b(x + u(t)\ln|1+x|)dW(t),$$

and the probability transition matrix

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

where a, b, c, and d are some constants to be chosen later.

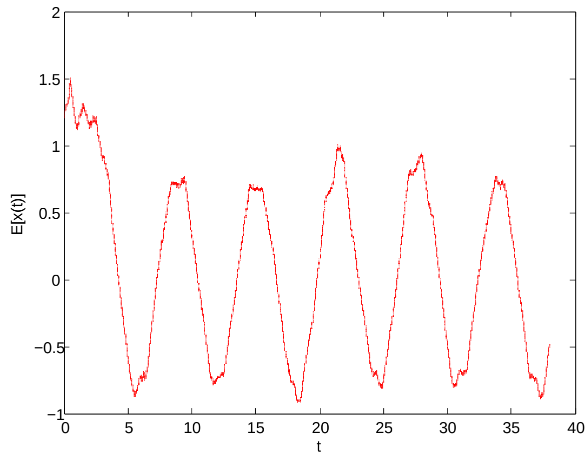

Figure 12.8: First moment aISS with $a = c = -1$ and $u(t) = \sin(t)$.

Clearly, the vector field functions satisfy assumptions (i) and (ii) of Theorem 12.3, where $K_1 = \alpha_1 = c_{3_1} = |a|$, $K_2 = \alpha_2 = c_{3_2} = |c|$ with $c \in \{-1, 1\}$, $\rho_1 = \sigma_1 = c_{4_1} = |b|$, $\rho_2 = \sigma_2 = c_{4_2} = |d|$, $c_3 = \max\{|a|, |c|\}$, and $c_4 = \max\{|b|, |d|\}$. For $i = 1, 2$, let $V(x, i) = \beta_i |x|^p$ with $0 < p < \min\{2, (\frac{3}{2}c_4 + 2\sigma_i)/(\frac{1}{2}c_4 + \sigma_i)\}$. Taking $|a| = |b| = |c| = |d| = p = 1$ yields $\beta_1^* = -3\beta_1 + \beta_2$ and $\beta_2^* = \beta_1 - 3\beta_2$, and upon choosing $\beta_1 = \beta_2 = 1$, we obtain $\lambda^* = \min\{\beta_1, \beta_2\} = -2$. Therefore, if $\nu = 1 \leq -\lambda^*$, $\mathcal{L}V(x, u, i) \leq -|x| < 0$, provided that $|x| > 4|u|_\infty$. By our choice of the probability transition matrix $\Gamma = [\gamma_{ij}]_{2\times 2}$, we get $\pi_1 = \pi_2 = 0.5$, the time spent in the first and second modes. Figures 12.8-12.10 illustrate the first moment aISS property with $u(t) = \sin(t)$ (Figures 12.8 and 12.9) and $u(t) = e^{-t}$ (Figure 12.10). In Figures 12.8 and 12.10, the switching occurred between two stable modes, while in Figure 12.9, between stable and unstable modes.

12.3 Conclusion

Nonlinear switched system property of aISS in the pth moment was established in this chapter. The main interest was to develop sufficient conditions to guarantee

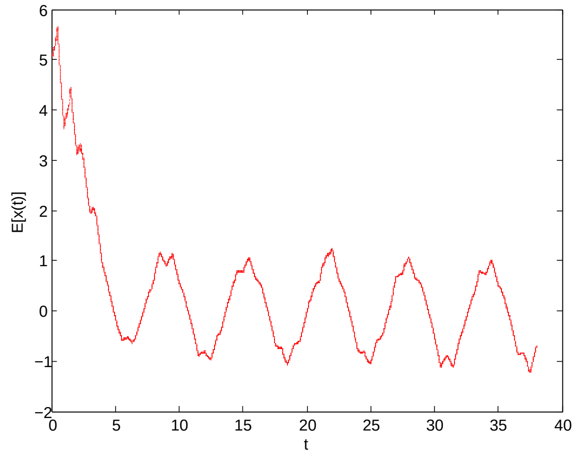

Figure 12.9: First moment aISS with $a = -1, c = 1$ and $u(t) = \sin(t)$.

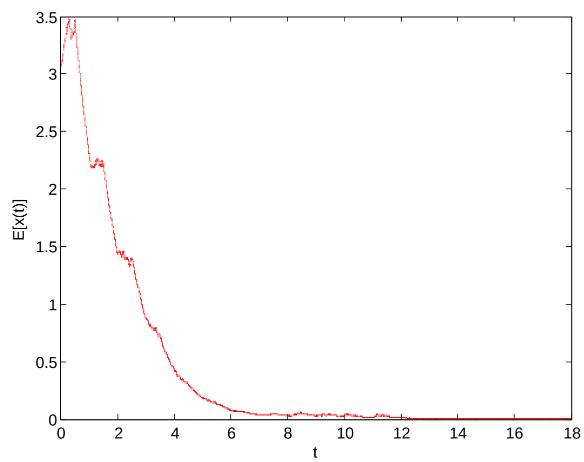

Figure 12.10: First moment aISS with $a = c = -1$ and $u(t) = e^{-t}$.

the system property. We should remark that, throughout the chapter, the entire switched system is subject to the same input disturbance. Therefore, one may consider a more general case, in which there are more than one inputs or controllers. In the first part of this chapter, we applied the initial-state-dependent dwell-time condition to control the switching among the system modes. Two cases were discussed, namely, systems with all stable modes, and with stable and unstable modes. The latter case required generalizing a lemma, in which Bihari's lemma, rather than Bellman-Grownwall lemma, plays an important role. We showed that the result of Theorem 12.1 has some implications that can be applied to some special cases, such as random-noise-free systems and non-zero input disturbance, or the other way around, or applied to switched systems under no effect of these two types of perturbations. In fact, one can also derive some analogous implications from Theorems 12.2 and 12.3. In Section 12.2, i.e., Theorem 12.3, we stated and proved the eISS property for systems according to Markovian switching rule. We also showed that in Theorems 12.2 and 12.3, as known in analyzing stability of switched systems, stability of each single mode is not necessary for the stability of the entire switched system. The ISS property, in this case, is warranted if stable modes are activated longer than unstable ones.

Chapter 13

Input-to-State Stability of Stochastic Switched Systems with Time Delay

In this chapter, we consider stochastic switched systems with time delay. The focus is on establishing the problem of pth moment asymptotic input-to-state stability of the systems. Particularly, we continue to apply the initial-state-dependent dwell-time τ_{isd} condition proposed in last chapter to organize the switching among the system modes. As noticed in the last chapter, by adopting the τ_{isd} switching law, we generate a convergent sequence of solutions evaluated at the switching instants. In fact, as will be seen, due to the type of sufficient conditions developed in this chapter, we make a slight change in the structure of this condition, where the input disturbance is now involved. To analyze the results, we seek Lyapunov-type sufficient conditions, where Razumikhin method is exploited to enable us to use Lyapunov functions. The results of this chapter will be developed in two steps; we first embark on systems with all stable modes. Then, we consider systems including unstable modes. The material of this chapter forms the basis of [Alw-i].

Consider the following switched system

$$dx(t) = f_{\sigma(t)}(t, x_t, u(t))dt + g_{\sigma(t)}(t, x_t, u(t))dW(t), \qquad t \geq t_0, \tag{13.1a}$$

$$x_{t_0}(s) = \phi(s), \qquad s \in [-r, 0], \tag{13.1b}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^l)$ is an external input with $\|u\|_\infty < \infty$, $\phi(t) \in L^p_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$, for some $p > 0$, is the initial condition, $\sigma(t) : [t_0, \infty) \to \mathcal{S} = \{1, 2, \cdots, N\}$ is the switching signal, and $f_i(t, 0, 0) = 0$ and $g_i(t, 0, 0) = 0$ for all $t \geq t_0$ and $i \in \mathcal{S}$, where $i = i_k$, for simplicity of notation. We also denote by $\{t_k\}_{k\in\mathbb{N}}$ a strictly increasing sequence of switching times with $\lim_{k\to\infty} t_k = \infty$.

In the following, we state the definition of asymptotic and exponential input-to-state stability of system (13.1).

Definition 13.1. For any $t_0 \in \mathbb{R}_+$, $t \geq t_0$ and $\phi(t) \in L^p_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^n)$, let $x(t) = x(t, t_0, \phi)$ be a solution of (13.1). Then, the system is said to be *uniformly asymptotically ISS (aISS) in the pth moment* if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for any input u, the solution x satisfies

$$\mathbb{E}[\|x(t)\|^p] \leq \beta\big(\mathbb{E}[\|\phi\|_r^p], t - t_0\big) + \gamma(\|u\|_\infty). \tag{13.2}$$

It is said to be *exponentially ISS (eISS) in the pth moment* if in addition $\beta\big(\mathbb{E}[\|\phi\|_r^p], t - t_0\big) \leq K\mathbb{E}[\|\phi\|_r^p]e^{-\lambda(t-t_0)}$, for some positive constants λ and K.

13.1 Switched System with Stable Modes

This section concentrates on developing conditions that guarantee the ISS property using the initial-state-dependent dwell-time (τ_{isd}) condition.

Theorem 13.1. For any $i \in \mathcal{S}$, $k \in \mathbb{N}$, and $u \in \mathcal{PC}(\mathbb{R}_+;\mathbb{R}^l)$, assume that there exist $c_i > 0$, $\alpha_{1_i} \in \mathcal{K}_2$, $\alpha_{2_i} \in \mathcal{K}_1$, and $\gamma_i \in \mathcal{K}$. Suppose further that $V_i \in \mathcal{C}^{1,2}([t_{k-1}, t_k) \times \mathcal{D};\mathbb{R}_+)$, with $\mathcal{D}$ being an open subset of $\mathbb{R}^n$, and $V_i(t, 0) = 0$ such that the following conditions hold:

(i) $\alpha_{2_i}(\|\psi(0)\|^p) \leq V_i(t, \psi(0)) \leq \alpha_{1_i}(\|\psi(0)\|^p)$, (a.s.), $\forall t \in [t_{k-1}, t_k)$ and $\psi(0) \in \mathcal{C}([-r, 0];\mathbb{R}^n)$;

(ii) $\mathcal{L}V(t, \psi, u) \leq -c_i V(t, \psi(0)) + \gamma_i(\|u(t)\|)$, (a.s.), $\forall t \in [t_{k-1}, t_k)$ and $\psi \in \mathcal{C}([-r, 0];\mathbb{R}^n)$, whenever $V_i(t + s, \psi(s)) \leq \bar{q}_i V_i(t, \psi(0))$ for some $\bar{q}_i > 1$ and $s \in [-r, 0]$;

(iii) the dwell-time τ_{isd} condition satisfies

$$\begin{aligned} t_1 - t_0 &\geq \frac{1}{c_1}\Big\{ \ln\Big[2\alpha_{1_1}(\mathbb{E}[\|\phi\|_r^p])e^{rc_1}\Big] - \ln\Big[\alpha_{2_1}\Big(a_1\mathbb{E}[\|\phi\|_r^p]\Big)\Big]\Big\}, \\ t_k - t_{k-1} &\geq \frac{1}{c_i}\Big\{ \ln\Big[2\alpha_{1_i}\Big(a_{i-1}\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_{i-1}}^{-1}\big(\frac{2}{c_{i-1}}\gamma_{i-1}(\|u\|_\infty)\big)\Big)\Big]e^{rc_i} \\ &\qquad - \ln\Big(\alpha_{2_i}(a_i\mathbb{E}[\|\phi\|_r^p])\Big)\Big\}, \end{aligned} \tag{13.3}$$

where $a_i < a_{i-1} < 1$, for any $i = 2, 3, \cdots$, such that $\lim_{i\to\infty} a_{i-1} = 0$.

Then, system (13.1) is aISS in the pth moment.

Proof. Let x be a solution of (13.1) and define $V_i(t, x)$ as a Lyapunov function candidate related to the i^{th} subsystem. Also, define $m_i(t) = \mathbb{E}[V_i(t, x(t))]$ for all $t \in [t_{k-1}, t_k)$. Then, from (ii) and Itô formula, we have

$$\begin{aligned} m_i(t) &= m_i(t_{k-1}) + \mathbb{E}\int_{t_{k-1}}^{t} \mathcal{L}V_i(s, x_s, u(s))ds \\ &\leq m_i(t_{k-1}) - \int_{t_{k-1}}^{t} \big(c_i\mathbb{E}[V_i(s, x(s))] + \gamma_i(\|u(s)\|)\big)ds, \end{aligned}$$

which gives

$$D^+m_i(t) \leq -c_i m_i(t) + \gamma_i(\|u(t)\|).$$

It follows that, for all $t \in [t_{k-1}, t_k)$,

$$\begin{aligned}
m_i(t) &\leq m_i(t_{k-1})e^{-c_i(t-t_{k-1})} + \int_{t_{k-1}}^{t} e^{-c_i(s-t_{k-1})}\gamma_i(\|u(s)\|)\, ds \\
&\leq \alpha_{1_i}(\mathbb{E}[\|x_{t_{k-1}}\|_r^p])e^{-c_i(t-t_{k-1})} + \gamma_i(\|u\|_\infty)\int_{t_{k-1}}^{t} e^{-c_i(s-t_{k-1})}\, ds \\
&\leq \alpha_{1_i}(\mathbb{E}[\|x_{t_{k-1}}\|_r^p])e^{-c_i(t-t_{k-1})} + \gamma_i(\|u\|_\infty)\Big[\frac{1}{c_i}\Big(1 - e^{-c_i(t-t_{k-1})}\Big)\Big] \\
&\leq \alpha_{1_i}(\mathbb{E}[\|x_{t_{k-1}}\|_r^p])e^{-c_i(t-t_{k-1})} + \frac{1}{c_i}\gamma_i(\|u\|_\infty).
\end{aligned}$$

Namely, for all $k \in \mathbb{N}$ and $t \in [t_{k-1,} t_k)$, we have

$$\alpha_{2_i}(\mathbb{E}[\|x(t)\|^p]) \leq m_i(t) \leq \alpha_{1_i}(\mathbb{E}[\|x_{t_{k-1}}\|_r^p])e^{-c_i(t-t_{k-1})} + \frac{1}{c_i}\gamma_i(\|u\|_\infty). \tag{13.4}$$

Particularly, for $i = 1$ and $t \in [t_0, t_1)$ (i.e., $k = 1$), we have, from (13.4)

$$m_1(t) \leq \alpha_{1_i}(\mathbb{E}[\|\phi\|_r^p])e^{-c_1(t-t_0)} + \frac{1}{c_1}\gamma_1(\|u\|_\infty), \tag{13.5}$$

and, for $i = 2$ and $t \in [t_1, t_2)$, we have

$$m_2(t) \leq \alpha_{1_2}(\mathbb{E}[\|x_{t_1}\|_r^p])e^{-c_2(t-t_1)} + \frac{1}{c_2}\gamma_2(\|u\|_\infty), \tag{13.6}$$

where $\mathbb{E}[\|x_{t_1}\|_r^p]$ can be found as follows: from (13.5) and $\alpha_{2_1}(\mathbb{E}[\|x(t)\|^p]) \leq m_1(t)$, for all $t \in [t_0, t_1)$, we have

$$\begin{aligned}
\mathbb{E}[\|x(t)\|^p] &\leq \alpha_{2_1}^{-1}\Big(\alpha_{1_1}(\mathbb{E}[\|\phi\|_r^p])e^{-c_1(t-t_0)} + \frac{1}{c_1}\gamma_1(\|u\|_\infty)\Big) \\
&\leq \alpha_{2_1}^{-1}\Big(2\alpha_{1_1}(\mathbb{E}[\|\phi\|_r^p])e^{-c_1(t-t_0)}\Big) + \alpha_{2_1}^{-1}\Big(2\frac{1}{c_1}\gamma_1(\|u\|_\infty)\Big),
\end{aligned}$$

and, after taking the supremum norm over the interval $[t_1 - r, t_1]$, we get

$$\mathbb{E}[\|x_{t_1}\|_r^p] \leq \alpha_{2_1}^{-1}\Big(2\alpha_{1_1}(\mathbb{E}[\|\phi\|_r^p])e^{rc_1}e^{-c_1(t_1-t_0)}\Big) + \alpha_{2_1}^{-1}\Big(\frac{2}{c_1}\gamma_1(\|u\|_\infty)\Big).$$

By the dwell-time condition (i.e., after $t_1 - t_0 > 0$, as given in (iii)), we get

$$\mathbb{E}[\|x_{t_1}\|_r^p] \leq a_1 \mathbb{E}[\|\phi\|_r^p] + \alpha_{2_1}^{-1}\Big(\frac{2}{c_1}\gamma_1(\|u\|_\infty)\Big),$$

where $0 < a_1 < 1$. Therefore, inequality (13.6) becomes, with the aid of (i),

$$\alpha_{2_2}(\mathbb{E}[\|x(t)\|^p]) \leq m_2(t) \leq \alpha_{1_2}\Big(a_1\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_1}^{-1}\Big[\frac{2}{c_1}\gamma_1(\|u\|_\infty)\Big]\Big)e^{-c_2(t-t_1)} + \frac{1}{c_2}\gamma_2(\|u\|_\infty), \tag{13.7}$$

or

$$\mathbb{E}[\|x(t)\|] \leq \alpha_{2_2}^{-1}\Big[\alpha_{1_2}\Big(a_1\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_2}^{-1}\big(\frac{2}{c_1}\gamma_1(\|u\|_\infty)\big)\Big)e^{-c_2(t-t_1)} + \frac{1}{c_2}\gamma_2(\|u\|_\infty)\Big],$$

from which, we obtain

$$\mathbb{E}[\|x_{t_2}\|_r^p] \leq \alpha_{2_2}^{-1}\Big[2\alpha_{1_2}\Big(a_1\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_1}^{-1}\big(\frac{2}{c_1}\gamma_1(\|u\|_\infty)\big)\Big)e^{rc_2}\cdot e^{-c_2(t_2-t_1)}\Big] + \alpha_{2_2}^{-1}\Big(2\frac{1}{c_2}\gamma_2(\|u\|_\infty)\Big),$$

and, after $t_2 - t_1 > 0$, we get [1]

$$\mathbb{E}[\|x_{t_2}\|_r^p] \leq a_2\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_2}^{-1}\Big(2\frac{1}{c_2}\gamma_2(\|u\|_\infty)\Big),$$

so that, for $i = 3$ and $t \in [t_2, t_3)$, we have

$$\alpha_{2_3}(\mathbb{E}[\|x(t)\|^p]) \leq m_3(t) \leq \alpha_{1_3}\Big(a_2\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_2}^{-1}\big(\frac{2}{c_2}\gamma_2(\|u\|_\infty)\big)\Big)e^{-c_3(t-t_2)} + \frac{1}{c_3}\gamma_3(\|u\|_\infty).$$

---

[1]The dwell time τ_{isd}

$$t_2 - t_1 \geq \frac{1}{c_2}\Big\{\ln\Big[2\alpha_{1_2}\Big(a_1\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_1}^{-1}\big(\frac{2}{c_1}\gamma_1(\|u\|_\infty)\big)\Big)e^{rc_2}\Big] - \ln\Big(\alpha_{2_2}(a_2\mathbb{E}[\|\phi\|_r^p])\Big)\Big\}.$$

Generally, for any $i \in \mathcal{S}$ and $t \in [t_{k-1}, t_k)$, we have

$$m_i(t) \leq \alpha_{1_i}\Big(a_{i-1}\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_{(i-1)}}^{-1}\big(\frac{2}{c_{i-1}}\gamma_{i-1}(\|u\|_\infty)\big)\Big)e^{-c_i(t-t_{k-1})} + \frac{1}{c_i}\gamma_i(\|u\|_\infty), \tag{13.8}$$

and, after $t_k - t_{k-1} > 0$, we get

$$\mathbb{E}[\|x_{t_{k-1}}\|_r^p] \leq a_i\mathbb{E}[\|\phi\|_r^p] + \alpha_1^{-1}\Big(\frac{2}{c_i}\gamma_i(\|u\|_\infty)\Big).$$

In fact, from (13.8) with the aid of (i), we obtain

$$\begin{aligned}\mathbb{E}[\|x(t)\|^p] \leq\ & \alpha_{2_i}^{-1}\big(2\alpha_{1_i}(a_i\mathbb{E}[\|\phi\|_r^p])e^{-c_i(t-t_{k-1})}\big) \\ & + \alpha_{2_i}^{-1}\Big(2\alpha_{2_{(i-1)}}^{-1}\big(\frac{2}{c_{i-1}}\gamma_{i-1}(\|u\|_\infty)\big)e^{-c_i(t-t_{k-1})} + \frac{2}{c_i}\gamma_i(\|u\|_\infty)\Big),\end{aligned} \tag{13.9}$$

where the first term on the right hand side of the inequality is a $\mathcal{KL}$ function, say $\beta(a_{i-1}\mathbb{E}[\|\phi\|_r^p], t - t_{k-1})$, which approaches zero when $t \to \infty$, since this in turn implies that $\lim_{i\to\infty} a_{i-1} = 0$ by the definition of a_i. The second term is a class-$\mathcal{K}$ function, say $\gamma(\|u\|_\infty)$, which becomes zero only when $u \equiv 0$. This shows that system (13.1) is aISS in the pth moment.

In the following corollary, we state some special results of Theorem 13.1, where the proofs are direct conclusions and will be omitted here.

Corollary 13.1. In Theorem 13.1, if

1. the random intensity $g_i \equiv 0$ for all $i \in \mathcal{S}$, the result reduces to the uniform aISS of the nonlinear deterministic switched system

$$\begin{aligned}\dot{x}(t) &= f_{\sigma(t)}(t, x_t, u(t)), && t \geq t_0,\\ x_{t_0}(s) &= \phi(s), && s \in [-r, 0];\end{aligned}$$

2. the input $u \equiv 0$, the result reduces to the pth moment uniform aISS of the stochastic switched system

$$dx(t) = f_{\sigma(t)}(t, x_t)dt + g_{\sigma(t)}(t, x_t)dW(t), \qquad t \geq t_0,$$
$$x_{t_0}(s) = \phi(s), \qquad s \in [-r, 0];$$

3. $\alpha_{1_i}(s) = \alpha_{1_i}s$ and $\alpha_{2_i}(s) = \alpha_{2_i}s$ for all $i \in \mathcal{S}$ and $s > 0$, the result reduces to uniform eISS in the pth moment.

13.2 Systems with Stable and Unstable Modes

In this section, we extend the result of Theorem 13.1 to a system that consists of stable and unstable modes, i.e., $\mathcal{S} = \mathcal{S}_u \cup \mathcal{S}_s$.

Theorem 13.2. For any $i \in \mathcal{S} = \mathcal{S}_s \cup \mathcal{S}_u$, $k \in \mathbb{N}$, and $u \in \mathcal{PC}(\mathbb{R}_+; \mathbb{R}^l)$, assume that there exist $\alpha_{1_i} \in \mathcal{K}_1$, $\alpha_{2_i} \in \mathcal{K}_2$, $\gamma_i \in \mathcal{K}$, $c_i > 0$, and $d_i > 0$. Suppose further that $V_i(t, \psi(0)) \in \mathcal{C}^{1,2}([t_{k-1}, t_k) \times \mathcal{D}; \mathbb{R}_+)$ and $V_i(t, 0) = 0$ such that the following conditions hold:

(i) $\alpha_{2_i}(\|\psi(0)\|^p) \leq V_i(t, \psi(0)) \leq \alpha_{1_i}(\|\psi(0)\|^p)$, (a.s.), $\forall t \in [t_{k-1}, t_k)$ and $\psi(0) \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$;

(i) (1) $\forall\, i \in \mathcal{S}_s$, $\mathcal{L}V(t, \psi, u) \leq -c_i V(\psi(0)) + \gamma_i(\|u(t)\|)$, (a.s.), $\forall t \in [t_{k-1}, t_k)$ and $\psi \in \mathcal{C}([t-r, t]; \mathbb{R}^n)$, whenever $V_i(t+s, \psi(s)) \leq \bar{q}_i V_i(t, \psi(0))$ for some $\bar{q}_i > 1$ and $s \in [-r, 0]$;

(ii) (2) $\forall\, i \in \mathcal{S}_u$, $\mathcal{L}V(t, \psi, u) \leq d_i V(t, \psi(0)) + \gamma_i(\|u(t)\|)$, (a.s.), $\forall t \in [t_{k-1}, t_k)$ and $\psi \in C([-r, 0]; \mathbb{R}^n)$, whenever $V_i(t + s, \psi(s)) \leq \bar{q}_i V_i(t, \psi(0))$ for some $\bar{q}_i > 1$ and $s \in [-r, 0]$;

(iii) the τ_{isd} condition satisfies

(1) for $i \in \mathcal{S}_u$

$$t_1 - t_0 \leq \frac{1}{c_1}\Big(\ln \big(A_1\alpha_{2_1}(\mathbb{E}[\|\phi\|_r^p])\big) - \ln \big(2\alpha_{1_1}(\mathbb{E}[\|\phi\|_r^p])\big)\Big),$$

$$t_k - t_{k-1} \leq \frac{1}{d_i}\Big[\ln \Big(\alpha_{2_i}\big(A_i \prod_{j=1}^{i-1} a_{i-1}A_{i-2}\mathbb{E}[\|\phi\|_r^p]\big)\Big),$$

$$- \ln \Big(\alpha_{1_i}\big(2 \prod_{j=1}^{i-1} a_{i-1}A_{i-2}\mathbb{E}[\|\phi\|_r^p]\big) + \alpha_{2_{i-1}}^{-1}\big(\frac{2}{c_{i-1}}\gamma_{i-1}(\|u\|_\infty)\big)\Big)\Big],$$

where $i = 3, 5, \cdots, N-1$;

(2) for $i \in \mathcal{S}_s = \{2, 4, \cdots, N\}$,

$$t_k - t_{k-1} \leq \frac{1}{c_i}\Big[\ln \Big(2\alpha_{1_i}\big(A_{i-1} \prod_{j=2}^{i} a_jA_{j-1}\mathbb{E}[\|\phi\|_r^p]\big)$$

$$+ \alpha_{2_{i-1}}^{-1}\big(\frac{2}{d_{i-1}}\gamma_{i-1}(\|u\|_\infty)e^{d_{i-1}(t_k-t_{k-1})}\big)e^{rc_i}\Big)$$

$$- \ln \Big(\alpha_{2_i}\big(\prod_{j=2}^{i} a_jA_{j-1}\mathbb{E}[\|\phi\|_r^p\big)\Big)\Big],$$

where a_i and A_i are positive constants such that $a_iA_i < 1$ for any $i \in \mathcal{S}$ (or $\lim_{i\to\infty}(a_iA_i) = 0$).

Then, system (13.1) is aISS in the pth moment.

Proof. Let $x(t) = x(t, t_0, \phi)$ be a solution of (13.1). Define $V_i(t, x(t))$ as a Lyapunov function candidate related to the i^{th} subsystem. We also define $m_i(t) = \mathbb{E}[V_i(t, x(t))]$ for all $t \in [t_{k-1}, t_k)$.

In light of Theorem 13.1, for $i \in \mathcal{S}_u$ and $t \in [t_{k-1}, t_k)$, we have from (ii) (2) and (i)

$$\alpha_{2_i}(\mathbb{E}[\|x(t)\|^p]) \leq m_i(t) \leq \alpha_{1_i}(\mathbb{E}[\|x_{t_{k-1}}\|_r^p])e^{d_i(t-t_{k-1})} + \frac{1}{d_i}\gamma_i(\|u\|_\infty)e^{d_i(t-t_{k-1})}, \tag{13.10}$$

while, for $i \in \mathcal{S}_s$ and $t \in [t_{k-1}, t_k)$, we have, from (ii) (1) and (i),

$$\alpha_{2_i}(\mathbb{E}[\|x(t)\|^p]) \leq m_i(t) \leq \alpha_{1_i}(\mathbb{E}[\|x_{t_{k-1}}\|_r^p])e^{-c_i(t-t_{k-1})} + \frac{1}{c_i}\gamma_i(\|u\|_\infty). \tag{13.11}$$

Particularly, for $i = 1$ and $t \in [t_0, t_1)$, if an unstable subsystem is activated, we have

$$\alpha_{2_1}(\mathbb{E}[\|x(t)\|^p]) \leq \alpha_{1_1}(\mathbb{E}[\|\phi\|_r^p])e^{d_1(t-t_0)} + \frac{1}{d_1}\gamma_1(\|u\|_\infty)e^{d_1(t-t_0)}, \tag{13.12}$$

and if, for $i = 2$ and $t \in [t_1, t_2)$, we run a stable subsystem, we get

$$\alpha_{2_2}(\mathbb{E}[\|x(t)\|^p]) \leq m_2(t) \leq \alpha_{1_2}(\mathbb{E}[\|x_{t_1}\|_r^p])e^{-c_2(t-t_1)} + \frac{1}{c_2}\gamma_2(\|u\|_\infty), \tag{13.13}$$

where $\mathbb{E}[\|x_{t_1}\|_r^p]$ is found as follows: from (13.12), we have

$$\mathbb{E}[\|x(t)\|^p] \leq \alpha_{2_1}^{-1}\Big(2\alpha_{1_1}(\mathbb{E}[\|\phi\|_r^p])e^{d_1(t-t_0)}\Big) + \alpha_{2_1}^{-1}\Big(\frac{2}{d_1}\gamma_1(\|u\|_\infty)e^{c_1(t-t_0)}\Big),$$

from which, after taking the supremum norm over the interval $[t_1 - r, t_1]$,

$$\mathbb{E}[\|x_{t_1}\|_r^p] \leq \alpha_{2_1}^{-1}\Big(2\alpha_{1_1}(\mathbb{E}[\|\phi\|_r^p])e^{d_1(t_1-t_0)}\Big) + \alpha_{2_1}^{-1}\Big(\frac{2}{d_1}\gamma_1(\|u\|_\infty)e^{c_1(t_1-t_0)}\Big).$$

This also implies that by the dwell-time condition, after $t_1 - t_0 > 0$,

$$\mathbb{E}[\|x_{t_1}\|_r^p] \leq A_1\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_1}^{-1}\Big(\frac{2}{d_1}\gamma_1(\|u\|_\infty)e^{d_1(t_1-t_0)}\Big).$$

Therefore, (13.13) becomes

$$\begin{aligned}\alpha_{2_2}(\mathbb{E}[\|x(t)\|^p]) \leq \alpha_{1_2}\Big[A_1\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_1}^{-1}\Big(\frac{2}{d_1}\gamma_1(\|u\|_\infty)e^{d_1(t_1-t_0)}\Big)\Big]e^{-c_2(t-t_1)} \\ + \frac{1}{c_2}\gamma_2(\|u\|_\infty),\end{aligned} \tag{13.14}$$

or

$$\begin{aligned}\mathbb{E}[\|x(t)\|^p] \leq \alpha_{2_2}^{-1}\Big[2\alpha_{1_2}\Big(A_1\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_1}^{-1}(\frac{2}{d_1}\gamma_1(\|u\|_\infty)e^{d_1(t_1-t_0)})\Big)e^{-c_2(t-t_1)}\Big] \\ + \alpha_{2_2}^{-1}\Big(\frac{2}{c_2}\gamma_2(\|u\|_\infty)\Big),\end{aligned} \tag{13.15}$$

and

$$\mathbb{E}[\|x_{t_2}\|_r^p] \le \alpha_{2_2}^{-1}\Big[2\alpha_{1_2}\Big(A_1\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_1}^{-1}\big(\frac{2}{d_1}\gamma_1(\|u\|_\infty)e^{d_1(t_1-t_0)}\big)\Big)e^{-c_2(t_2-t_1)}e^{rc_2}\Big] + \alpha_{2_2}^{-1}\Big(\frac{2}{c_2}\gamma_2(\|u\|_\infty)\Big). \tag{13.16}$$

After $t_2 - t_1 > 0$, which can be found as in Theorem 13.1, we have

$$\mathbb{E}[\|x_{t_2}\|_r^p] \le a_2 A_1 \mathbb{E}[\|\phi\|_r^p] + \alpha_{2_2}^{-1}\Big(\frac{2}{c_2}\gamma_2(\|u\|_\infty)\Big).$$

For $i = 3$ and $k = 3$, i.e., $t \in [t_2, t_3)$, we run an unstable subsystem

$$m_3(t) \le \alpha_{1_3}(\|x_{t_2}\|_r)e^{d_3(t-t_2)} + \frac{1}{d_3}\gamma_3(\|u\|_\infty)e^{d_3(t-t_2)}, \tag{13.17}$$

where the upper bound of $\|x_{t_2}\|_r$ is given in (13.16); therefore by (i), (13.17) becomes

$$\alpha_{2_3}(\mathbb{E}[\|x(t)\|^p]) \le \alpha_{1_3}\Big(a_2A_1\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_2}^{-1}\big(\frac{2}{c_2}\gamma_2(\|u\|_\infty)\big)\Big)e^{d_3(t-t_2)} + \frac{1}{d_3}\gamma_3(\|u\|_\infty)e^{d_3(t-t_2)}. \tag{13.18}$$

For $i = 4$ and $t \in [t_3, t_4)$ we have, after running a stable mode,

$$m_4(t) \le \alpha_{1_4}(\mathbb{E}[\|x_{t_3}\|_r^p])e^{-c_4(t-t_3)} + \frac{1}{c_4}\gamma_4(\|u\|_\infty); \tag{13.19}$$

similarly, we find

$$\mathbb{E}[\|x_{t_3}\|_r^p] \le A_3 a_2 A_1 \mathbb{E}[\|\phi\|_r^p] + \alpha_{2_3}^{-1}\Big(\frac{2}{d_3}\gamma_3(\|u\|_\infty)e^{d_3(t_3-t_2)}\Big).$$

Generally, if, for $i = N-1$ and $t \in [t_{k-2}, t_{k-1})$, we run an unstable subsystem, we have

$$m_{N-1}(t) \le \alpha_{1_{N-1}}\Big(\prod_{j=1}^{N-1} a_{j-1}A_{j-2}\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_{(N-1)-1}}^{-1}\Big[\frac{2}{c_{(N-1)-1}}\gamma_{(N-1)-1}(\|u\|_\infty)\Big]\Big) \times e^{d_{N-1}(t_-t_{k-2})} + \frac{1}{d_{N-1}}\gamma_{N-1}(\|u\|_\infty)e^{d_{N-1}(t-t_{k-2})},$$

and if, for $i = N$ and $t \in [t_{k-1}, t_k)$, we run a stable subsystem, we have

$$\begin{aligned}
m_N(t) &\le \alpha_{1_N}\Big(A_{N-1}\prod_{j=1}^{N-1} a_{j-1}A_{j-2}\mathbb{E}[\|\phi\|_r^p] + \alpha_{2_{N-1}}^{-1}\Big[\frac{2}{d_{N-1}}\gamma_{N-1}(\|u\|_\infty)\Big]\Big)e^{-c_N}(t-t_{k-1}) \\
&\quad + \frac{1}{c_N}\gamma_N(\|u\|_\infty) \\
&\le \alpha_{1_N}\Big(2A_{N-1}\prod_{j=1}^{N-1} a_{j-1}A_{j-2}\|\phi\|_r\Big)e^{-c_N}(t-t_{k-1}) \\
&\quad + \alpha_{1_N}\Big(2\alpha_{2_{N-1}}^{-1}\Big[\frac{2}{d_{N-1}}\gamma_{N-1}(\|u\|_\infty)\Big]\Big)e^{-c_N}(t-t_{k-1}) + \frac{1}{c_N}\gamma_N(\|u\|_\infty).
\end{aligned}$$

The first term on the right-hand side of the inequality is a class-$\mathcal{KL}$ function, which vanishes when $N \to \infty$, by the definition of sequence $\{a_iA_{i-1}\}$ for any i. The second term is a class-$\mathcal{K}$ function, which is generally bounded and vanishes if $u \equiv 0$. Therefore, when time evolves, the initial state becomes very small and the solution will eventually be bounded by a class-$\mathcal{K}$ function, which depends on the external input force u. To complete the proof, it suffices to make use of (i).

13.3 Conclusion

In this chapter, we considered a stochastic switched system with time delay and input disturbance, where the main interest was to establish some input-to-state stability properties of the system. Using Razumikihin method, we developed some Lyapunov-like theorems, where the initial-state-dependent dwell-time condition proposed in Chapter 12 was used, after modification by taking into account the input disturbance, to control switching among the system modes. Also, we considered two mode cases; in the first case, switching occurs between all stable modes, while in the second case, the system consists of stable and unstable modes.

Part III

Differential Equations with Piecewise Constant Arguments-Hybrid System Approach

Chapter 14

Comparison Principle and Stability Results for EPCA

By EPCA we mean differential equations with piecewise constant arguments over certain intervals. The arguments can be delay, advanced, or a mix of these two types. The dynamics of these differential equations generally depend on both continuous and discrete arguments, which results in discontinuities of system vector fields. This type of discontinuity enables us to study such systems under hybrid (or particularly switched) system umbrella. Using switched system approach will allow us to apply the theory of continuous differential equations on every subinterval, which will motivate the concept of dwell time. From the functional differential equation theory perspective, EPCA are special equations, where the state history is given at certain individual points, rather than on intervals, which allow us to employ the theory of ordinary differential equations, but not delay differential equations. The material of this chapter forms the basis of [Alw-j].

Typically, nonlinear EPCA have the form

$$\dot{x}(t) = f(t, x(t), x(\gamma(t))), \tag{14.1}$$

where the argument γ is a piecewise constant function defined on intervals with a certain length, and it may be defined by $\gamma(t) = [t], [t-n], t-n[t], [t+1]$, for any t and a positive integer n, where $[\cdot]$ is the greatest-integer function [Coo84, Coo91, Wie93].

A general type of EPCA, (EPCAG), in which the piecewise constant real function γ takes values over discrete subintervals instead of at the most-left endpoint of each subinterval, have appeared in some works [Akh08b, Akh08c].

On the other hand, in the system studied in [Lak98], the differential equations have the form

$$\dot{x}(t) = f\big(t, x(t), \lambda_k(x_k)\big), \quad t \in [t_k, t_{k+1}],$$

where, for some non-negative integer k, $x_k = x(t_k)$, and λ_k are some continuous functions, and the system state experiences impulsive effects due to switching in the arguments λ_k and x_k. In that work, the focus was on establishing comparison and stability results for this impulsive switched system.

In this chapter, the purpose is to develop a comparison principle for this system. Then, by employing this result, together with the use of Lyapunov-function approach, we establish some stability properties of the system. The organization of this chapter is as follows. In Section 14.1, we formulate the problem and define some concepts that will be used in the rest of this chapter. The main contribution will be given in Section 14.2. Some special cases will also be introduced. Some numerical examples are presented in Section 14.3. Finally, we conclude our work in Section 14.4.

14.1 Problem Formulation and Preliminaries

For non-negative integers k, define $\{t_k\}_{k=0}^{\infty}$ and $\{\xi_k\}_{k=0}^{\infty}$ as sequences of non-negative real numbers such that $t_0 \in \mathbb{R}_+$ and $\lim_{k\to\infty} t_k = \infty$. Generally, ξ_k is defined such that $t_{k-1} < \xi_k \leq t_k$, for any $k \in \mathbb{N}$ and $\xi_0 = t_0$.

Consider the following EPCA

$$\dot{x}(t) = f\big(t, x(t), \lambda_{\varrho(t)}(x(\gamma(t)))\big), \tag{14.2a}$$

where $x \in \mathbb{R}^n$ is the system state, and, for all $t \geq t_0$, $\varrho(t)$ and $\gamma(t)$ take values in $\{k\}_{k=0}^{\infty}$ and $\{\xi_k\}_{k=0}^{\infty}$, respectively. More specifically, for $t \in [t_k, t_{k+1}]$, we define $\varrho(t) = k$ and $\gamma(t) = \xi_k$. These piecewise constant functions, ϱ and γ, represent the switching signals with roles of switching between the vector field function arguments λ_k and the values of its state argument x, respectively. Obviously, if, for all k, λ_k is an identity function, EPCA (14.2a) reduces to (14.1). We should note that, for $k = 0$, we have $\xi_0 = t_0$, $t \in [t_0, t_1]$, and the differential equation in (14.2a) is an ordinary one; then, for $k > 0$ and $t \in [t_k, t_{k+1}]$, the system state is allowed to be fed back with some historic data at individual moments $\xi_k \in (t_{k-1}, t_k]$. In addition, since the solution depends on the past history through an individual point, the initial state, in contrast to the case of functional differential equation, is given at a specific time, rather than over an interval, i.e.,

$$x(t_0) = x_0, \tag{14.2b}$$

for some $x_0 \in \mathbb{R}^n$.

In the following, we define the solution of the initial-value problem (IVP) (14.2).

Definition 14.1. A function $x : (\alpha,\, \beta) \to \mathbb{R}$ is said to be a *solution* of (14.2) if the following conditions hold:

(i) $x(t)$ is continuous for all $t \in (\alpha, \beta)$;

(ii) the derivative of $x(t)$ exists and is continuous at $t \neq \xi_k$, $t \in (\alpha, \beta)$ $(k = 1, 2, 3, \cdots)$, and, at $t = \xi_k$, one-sided derivative exists;

(iii) the derivative of $x(t)$, wherever exists, satisfies the EPCA in (14.2a);

(iv) $x(t)$ satisfies the initial condition in (14.2b) at $t = t_0$.

System (14.2) may be rewritten in the form

$$\dot{x}(t) = f\big(t, x(t), \lambda_k(x_{\xi_k})\big), \qquad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \cdots, \tag{14.3a}$$

$$x(t_0) = x_0, \tag{14.3b}$$

where $x_{\xi_k} = x(\xi_k)$ and $\lambda_k(x_{\xi_k}) = \lambda_k(x(\xi_k))$ are constants. Throughout this chapter, we assume that function $f(t, x, y)$ is continuous in its variables, i.e., $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, and globally Lipschitz in x and y.

As mentioned earlier, the dependence of the solution $x(t)$ of IVP (14.2) or (14.3) on the initial state at $t = t_0$ allows us to employ the theory of ordinary differential equations. For instance, for $k = 0$ and $t \in (t_0, t_1)$, the IVP

$$\dot{x}(t) = f(t, x(t), \lambda_0(x_{\xi_0})),$$

$$x(t_0) = x_0,$$

with $\xi_0 = t_0$, has a unique solution, say $x_0(t)$, $\forall t \in (t_0, t_1)$, and $\lim_{t \to t_1^-} x_0(t) = x_0(t_1^-) \in \mathbb{R}^n$. Similarly, for $k = 1$ and $t \in [t_1, t_2)$, we have the IVP[1]

$$\dot{x}(t) = f(t, x(t), \lambda_1(x_{\xi_1})),$$

$$x(t_1) = x_0(t_1^-),$$

[1] We should remark that, in the unified notation of the solution x, the initial condition $x(t_1) = x_0(t_1^-)$ becomes $x(t_1) = x(t_1^-)$, by our definition of x.

which has a unique solution $x_1(t)$, $\forall t \in [t_1, t_0)$ and $\lim_{t\to t_2^-} x_1(t) = x_1(t_2^-)$. By induction, for any k and $t \in [t_k, t_{k+1})$, $x_k(t)$ is a unique solution and $\lim_{t\to t_{k+1}^-} x_k(t)$ exists. Define the solution $x(t)$ by

$$x(t) = \begin{cases} x_0, & t = t_0, \\ x_0(t, t_0, x_0), & t \in (t_0, t_1), \\ x_1(t, t_1, x_1), & t \in (t_1, t_2), \text{ where } x_1 = x_0(t_1^-, t_0, x_0), \\ \cdots \\ x_k(t, t_k, x_k), & t \in (t_k, t_{k+1}), \text{ where } x_k = x_{k-1}(t_k^-, t_{k-1}, x_{k-1}), \\ \cdots . \end{cases}$$

Since $\lim_{t\to t_{k+1}^-} x(t)$ exists for any k, the solution x must exist over a right-maximal interval $[t_0, \infty)$. We have the following result.

Proposition 14.1. For $k = 0, 1, \cdots$, let $\varrho(t) : [t_k, t_{k+1}) \to \{k\}_{k=0}^{\infty}$ and $\gamma(t) : [t_k, t_{k+1}) \to \{\xi_k\}_{k=0}^{\infty}$, where ξ_k is as defined earlier. Assume that $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ and $f(t, x, y)$ is globally Lipschitz in x and y. Then, the IVP (14.2) or (14.3) has a unique solution x defined over the right-maximal interval $[t_0, \infty)$.

The scalar initial-value problem can be defined analogously.

$$\dot{u}(t) = g(t, u(t), \sigma_k(u_{\xi_k})), \tag{14.4a}$$

$$u(t_0) = u_0, \tag{14.4b}$$

where $u \in \mathbb{R}_+$, $u_{\xi_k} = u(\xi_k)$, $\sigma_k \in \mathcal{C}(\mathbb{R}_+; \mathbb{R})$ and $g \in \mathcal{C}(\mathbb{R}_+^2 \times \mathbb{R}; \mathbb{R})$.

Assuming that $f(t, 0, \lambda_k(0)) = 0$ and $g(t, 0, \sigma_k(0)) = 0$, for all $t \in \mathbb{R}_+$, systems (14.3) and (14.4) admit trivial solutions $x \equiv 0 \in \mathbb{R}^n$ and $u \equiv 0 \in \mathbb{R}$, respectively.

Definition 14.2. Let $x, y \in \mathbb{R}^n$ and $t \in [t_k, t_{k+1})$, for $k = 0, 1, 2, \cdots$. Then, if $V \in \mathcal{C}([t_k, t_{k+1}) \times \mathbb{R}^n; \mathbb{R}_+)$, the *upper right-hand derivative of* V is defined as follows:

$$D^+V(t,x,y) = \lim_{h\to 0^+} \sup \frac{1}{h}\Big[V(t,x+hf(t,x,\lambda_k(y))) - V(t,x)\Big]. \tag{14.5}$$

Moreover, if $V \in \mathcal{C}^1([t_k,t_{k+1}) \times \mathbb{R}^n; \mathbb{R}_+)$, then

$$D^+V(t,x,y) = \frac{\partial V(t,x)}{\partial t} + \nabla V(t,x) \cdot f(t,x,\lambda_k(y)). \tag{14.6}$$

14.2 Main Results

In this section, we will state and prove our main results. We first develop a comparison principle for nonlinear EPCA; then we make use of this result to establish some stability properties for the system. We will also consider some special case of EPCA and EPCAG. In Theorems 14.1 and 14.2, ξ_k is as defined in Section 14.1.

Theorem 14.1. Assume that

(i) for $k = 0,1,2,\cdots$, $V \in \mathcal{C}([t_k,t_{k+1}) \times \mathbb{R}^n; \mathbb{R}_+)$, $V(t,x)$ is locally Lipschitz in x, and

$$D^+V(t,x,V_{\xi_k}) \le g(t,V(t,x),\sigma_k(V_{\xi_k}))), \qquad t \in (t_k,t_{k+1}),$$

where $V_{\xi_k} = V(\xi_k, x(\xi_k))$;

(ii) the maximal solution $\vartheta(t,t_0,u_0)$ of the scalar EPCA (14.4) exists on $[t_0,\infty)$.

Then, for any solution $x(t) = x(t,t_0,x_0)$ of (14.3), $V(t_0,x_0) \le u_0$ implies $V(t,x(t)) \le \vartheta(t,t_0,u_0)$ for $t \ge t_0$.

Proof. Define $m(t) = V(t,x(t))$ for any solution $x(t) = x(t,t_0,x_0)$ that is defined on $[t_0,\infty)$. Then, we have

$$D^+m(t) \le g(t,m(t),\sigma_k(m_{\xi_k})), \qquad t \in (t_k,t_{k+1}),$$

where $m_{\xi_k} = m(\xi_k)$. Particularly, for $t \in [t_0, t_1]$, we have, by the classical comparison principle [Lak69],

$$m(t) \leq \vartheta_0(t, t_0, u_0), \qquad t \in [t_0, t_1],$$

where $\vartheta_0(t, t_0, u_0)$ is the maximal solution of

$$\begin{aligned} \dot{u}(t) &= g(t, u(t), \sigma_0(u_{\xi_0})), \\ u(t_0) &= u_0. \end{aligned}$$

For $t \in [t_1, t_2]$, we have

$$m(t) \leq \vartheta_1(t, t_1, u_1) = \vartheta_1(t, t_1, \vartheta_0(t_1, t_0, u_0)), \qquad u_1 = u(t_1) = \vartheta_0(t_1, t_0, u_0),$$

where $\vartheta_1(t, t_1, u_1)$ is the maximal solution of

$$\begin{aligned} \dot{u}(t) &= g(t, u(t), \sigma_1(u_{\xi_1})), \\ u(t_1) &= u_1. \end{aligned}$$

Generally, one may get

$$m(t) \leq \vartheta_k(t, t_k, u_k), \qquad t \in [t_k, t_{k+1}],$$

where $\vartheta_k(t, t_k, u_k)$ is the maximal solution of

$$\begin{aligned} \dot{u}(t) &= g(t, u(t), \sigma_k(u_{\xi_k})), \\ u(t_k) &= u_k. \end{aligned}$$

Define $u(t)$ by

$$u(t) = \begin{cases} u_0, & t = t_0, \\ \vartheta_0(t, t_0, u_0), & t \in (t_0, t_1], \\ \vartheta_1(t, t_1, u_1), & t \in (t_1, t_2], \text{ where } u_1 = \vartheta_0(t_1, t_0, u_0), \\ \cdots \\ \vartheta_k(t, t_k, u_k), & t \in (t_k, t_{k+1}], \text{ where } u_k = \vartheta_{k-1}(t_k, t_{k-1}, u_{k-1}), \\ \cdots . \end{cases}$$

Then, for $t \geq t_0$, we get

$$m(t) \leq u(t).$$

Since $\vartheta(t, t_0, u_0)$ is the maximal solution of the scalar EPCA (14.4), then, for $t \geq t_0$

$$m(t) \leq \vartheta(t, t_0, u_0).$$

The proof is complete.

In the following corollary and example, we consider some special cases of EPCA and EPCAG.

Corollary 14.1. Suppose that the conditions in Theorem 14.1 hold. Let $k = 0, 1, 2, \cdots$ and $t \in [t_k, t_{k+1}]$. If we choose that

(i) $g(t, u, \sigma_k(u_{\xi_k})) = \beta_k u_{\xi_k}$, with β_k being a constant for all k, then

(1) for $\xi_k = t_k$,

$$V(t, x(t)) \leq \begin{cases} \Big[1 + \beta_0(t - t_0)\Big] V(t_0, x_0), & k = 0, \quad t \in (t_0, t_1], \\ \Big[1 + \beta_k(t - t_k)\Big] \prod_{j=1}^{k} \Big[1 + \beta_{j-1}(t_j - t_{j-1})\Big] V(t_0, x_0), \\ \qquad k = 1, 2, \cdots, \quad t \in (t_k, t_{k+1}], \end{cases}$$

where for $k = 0, 1, 2, \cdots$, $t_k < t_{k+1}$ for $\beta_k > 0$, and $t_{k+1} < t_k - \frac{1}{\beta_k}$ for $\beta_k < 0$;

(2) for $t_{k-1} < \xi_k \leq t_k$, where $k = 1, 2, 3, \cdots$ and $\xi_0 = t_0$,

$$V(t, x(t)) = V_0(t, x(t)) \leq \Big[1 + \beta_0(t - t_0)\Big] V_0(t_0, x_0),$$

for any $t \in [t_0, t_1)$ such that $t_1 - t_0 < -\frac{1}{\beta_0}$, and

$$V(t, x(t)) = V_k(t, x(t)) \leq V_{k-1}(t_k, x(t_k)) + \beta_k(t - t_k)V_{k-1}(\xi_k, x(\xi_k)),$$

for any $t \in [t_k, t_{k+1})$ such that, for any $k = 1, 2, 3, \cdots$, $t_{k+1} - t_k < -\frac{C_k}{\beta_k C_{\xi_k}}$ where $C_k = V_{k-1}(t_{k-1}, x(t_{k-1}))$ and $C_{\xi_k} = V_{k-1}(\xi_k, x(\xi_k))$;

(ii) $g(t, u, \sigma_k(u_{\xi_k})) = \alpha u(t) + \beta_k u_{\xi_k}$, with α and β_k being constants for any k, then

(1) for $\xi_k = t_k$,

$$V(t, x(t)) \leq \begin{cases} \Big[(1 + \frac{\beta_0}{\alpha})e^{\alpha(t-t_0)} - \frac{\beta_0}{\alpha}\Big] V(t_0, x_0), & k = 0, \quad t \in (t_0, t_1], \\ \Big[(1 + \frac{\beta_k}{\alpha})e^{\alpha(t-t_k)} - \frac{\beta_k}{\alpha}\Big] \prod_{j=1}^{k} \Big[(1 + \frac{\beta_{j-1}}{\alpha})e^{\alpha(t_j - t_{j-1})} - \frac{\beta_{j-1}}{\alpha}\Big] & \\ \quad \times V(t_0, x_0), & k = \mathbb{N}, \quad t \in (t_k, t_{k+1}], \end{cases}$$

provided that, for $k = 0, 1, 2, \cdots$,

$$\begin{cases} \text{for any } t_{k+1} > t_k, & \text{when } \alpha > 0, \beta_k > 0, \text{ or} \\ & \text{when } \alpha < 0,\ \beta_k > 0 \text{ with} \\ & \alpha > -\beta_k > 0, \\ t_{k+1} < t_k + \frac{1}{\alpha} \ln \left[\frac{\beta_k}{\alpha}\left(1 + \frac{\beta_k}{\alpha}\right)^{-1}\right], & \text{when } \alpha > 0,\ \beta_k < 0 \text{ with} \\ & \frac{\beta_k}{\alpha}\left(1 + \frac{\beta_k}{\alpha}\right)^{-1} > 1; \end{cases}$$

(2) for $t_{k-1} < \xi_k \le t_k$, where $k = 1, 2, 3, \cdots$, and $\xi_0 = t_0$,

$$V(t, x(t)) = V_0(t, x(t)) \le \Big[e^{\alpha(t-t_0)} + \frac{\beta_0}{\alpha}\Big(e^{\alpha(t-t_0)} - 1\Big)\Big] \times V_0(t_0, x_0), \qquad \forall t \in [t_0, t_1),$$

and

$$V(t, x(t)) = V_k(t, x(t)) \le e^{\alpha(t-t_k)} V_{k-1}(t_k, x(t_k)) + \frac{\beta_k}{\alpha}\Big[e^{\alpha(t-t_k)} - 1\Big] \times V_{k-1}(\xi_k, x(\xi_k)), \qquad \forall t \in [t_k, t_{k-1}),$$

provided that, for $k = 0, 1, 2, \cdots$,

$$t_{k+1} > t_k + \frac{1}{\alpha} \ln T_k,$$

where $\alpha > 0$ and $\beta_k > 0$, or, when $\alpha < 0$ and $\beta_k > 0$ with $V_{k-1}(t_k, x(t_k)) + \frac{\beta_k}{\alpha} V_{k-1}(\xi_k, x(\xi_k)) < 0$, or

$$t_{k+1} < t_k + \frac{1}{\alpha} \ln T_k,$$

when $\alpha > 0$, $\beta_k < 0$ with $V_{k-1}(t_k, x(t_k)) + \frac{\beta_k}{\alpha} V_{k-1}(\xi_k, x(\xi_k)) < 0$, where $T_k = \frac{\beta_k}{\alpha} V_{k-1}(\xi_k, x(\xi_k)) \Big(V_{k-1}(t_k, x(t_k)) + \frac{\beta_k}{\alpha} V_{k-1}(\xi_k, x(\xi_k))\Big)^{-1} > 1$;

(iii) $g(t, u, \sigma_k(u_{\xi_k})) = \alpha u(t) + h(t, u, \sigma_k(u_{\xi_k}))$ with $\alpha \in \mathbb{R}$, $h \in \mathcal{C}\big(\mathbb{R}_+ \times \mathbb{R}^2; \mathbb{R}_+\big)$, and $h(t, u, v)$ is globally Lipschitz in u and v, then

$$V(t, x(t)) \le e^{\alpha(t-t_0)} V(x_0) + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} e^{\alpha(t-s)} h(s, V(s, x(s)), \sigma_k(V_{\xi_{j-1}}))ds + \int_{t_k}^{t} e^{\alpha(t-s)} h(s, V(s, x(s)), \sigma_k(V_{\xi_k}))ds.$$

Proof (i)(1) For $t \in [t_k, t_{k+1}]$, since $u_{\xi_k} = u_{t_k}$, the solution of $\dot{u}(t) = \beta_k u_{\xi_k}$ is

$$u(t) = \Big[1 + \beta_k (t - t_k)\Big] u_k.$$

Particularly, for $k = 0$ and $t \in [t_0, t_1]$,

$$u(t) = \Big[1 + \beta_0(t - t_0)\Big]u_0,$$

and, for $k = 1$ and $t \in [t_1, t_2]$,

$$u(t) = \Big[1 + \beta_1(t - t_1)\Big]\Big[1 + \beta_0(t_1 - t_0)\Big]u_0.$$

By induction, we get

$$u(t) = \begin{cases} \Big[1 + \beta_0(t - t_0)\Big]u_0, & k = 0, \quad t \in (t_0, t_1], \\ \Big[1 + \beta_k(t - t_k)\Big]\prod_{j=1}^{k}\Big[1 + \beta_{j-1}(t_j - t_{j-1})\Big]u_0, & \\ & k = 1, 2, \cdots, \quad t \in (t_k, t_{k+1}]. \end{cases}$$

To complete the proof, we use our comparison result.

(i)(2) For any k and $t \in [t_k, t_{k+1})$, we have

$$u(t) = u(t_k) + \beta_k(t - t_k)u(\xi_k).$$

For $k = 0$, $\xi_0 = t_0$ and

$$u(t) = \Big[1 + \beta_0(t - t_0)\Big]u_0 =: u_0(t),$$

where the R.H.S. is positive if $t < t_0 - 1/\beta_0$; therefore, for $k = 1$ and $t \in [t_1, t_2)$, we get

$$u(t) = u_0(t_1) + \beta_1(t - t_1)u_0(\xi_1) =: u_1(t).$$

By induction, one may get

$$u(t) = u_k(t) = u_{k-1}(t_k) + \beta_k(t - t_k)u_{k-1}(\xi_k), \qquad t \in [t_k, t_{k+1}), \quad k = 1, 2, 3, \cdots,$$

from which, we get the general form in (i)(2).

(ii)(1) For $t \in (t_k, t_{k+1}]$, we have

$$\dot{u}(t) = \alpha u(t) + \beta_k u_{\xi_k},$$

and its solution is given by

$$u(t) = \Big[e^{\alpha(t-t_k)} + \frac{\beta_k}{\alpha}\Big(e^{\alpha(t-t_k)} - 1\Big)\Big]u_k, \tag{14.7}$$

from which we obtain

$$u(t) = \begin{cases} \Big[(1+\frac{\beta_0}{\alpha})e^{\alpha(t-t_0)} - \frac{\beta_0}{\alpha}\Big]u_0, & k=0, \quad t \in (t_0, t_1], \\ \Big[(1+\frac{\beta_k}{\alpha})e^{\alpha(t-t_k)} - \frac{\beta_k}{\alpha}\Big]\prod_{j=1}^{k}\Big[(1+\frac{\beta_{j-1}}{\alpha})e^{\alpha(t_j-t_{j-1})} - \frac{\beta_{j-1}}{\alpha}\Big]u_0, & \\ & k=1,2,\cdots, \quad t \in (t_k, t_{k+1}], \end{cases}$$

where α and β_k are defined in (ii). Applying the comparison principle leads us to the required result. The proof of (ii)(2) can be obtained is a similar way used in (i)(2) and is omitted here.

(iii) For $k = 0, 1, 2, \cdots$ and $t \in [t_k, t_{k+1}]$, we have

$$\dot{u}(t) = \alpha u(t) + h(t, u(t), \sigma_k(u_{\xi_k})),$$

and its solution is given by

$$u(t) = e^{\alpha(t-t_k)}u_k + \int_{t_k}^{t} e^{\alpha(t-s)}h(s, u(s), \sigma_k(u_{\xi_k}))\, ds.$$

For $t \in [t_0, t_1]$, we have

$$u(t) = e^{\alpha(t-t_0)}u_0 + \int_{t_0}^{t} e^{\alpha(t-s)}h(s, u(s), \sigma_0(u_{\xi_0}))\, ds,$$

and, at $t = t_1$,

$$u_1 = e^{\alpha(t_1 - t_0)} u_0 + \int_{t_0}^{t_1} e^{\alpha(t_1 - s)} h(s, u(s), \sigma_0(u_{\xi_0}))\, ds.$$

For $t \in [t_1, t_2]$, we have

$$\begin{aligned} u(t) &= e^{\alpha(t-t_1)} u_1 + \int_{t_1}^{t} e^{\alpha(t-s)} h(s, u(s), \sigma_1(u_{\xi_1}))\, ds \\ &= e^{\alpha(t-t_1)} \Big\{ e^{\alpha(t_1-t_0)} u_0 + \int_{t_0}^{t_1} e^{\alpha(t_1-s)} h(s, u(s), \sigma_0(u_{\xi_0}))\, ds \Big\} \\ &\quad + \int_{t_1}^{t} e^{\alpha(t-s)} h(s, u(s), \sigma_1(u_{\xi_1}))\, ds \\ &= e^{\alpha(t-t_0)} u_0 + \int_{t_0}^{t_1} e^{\alpha(t-s)} h(s, u(s), \sigma_0(u_{\xi_0}))\, ds + \int_{t_1}^{t} e^{\alpha(t-s)} h(s, u(s), \sigma_1(u_{\xi_1}))\, ds. \end{aligned}$$

For $t \in [t_2, t_3]$, we have

$$\begin{aligned} u(t) &= e^{\alpha(t-t_0)} u_0 + \int_{t_0}^{t_1} e^{\alpha(t-s)} h(s, u(s), \sigma_0(u_{\xi_0}))\, ds + \int_{t_1}^{t_2} e^{\alpha(t-s)} h(s, u(s), \sigma_1(u_{\xi_1}))\, ds \\ &\quad + \int_{t_2}^{t} e^{\alpha(t-s)} h(s, u(s), \sigma_2(u_{\xi_2}))\, ds. \end{aligned}$$

By induction, for $t \in [t_k, t_{k+1}]$,

$$\begin{aligned} u(t) = e^{\alpha(t-t_0)} u_0 &+ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} e^{\alpha(t-s)} h(s, u(s), \sigma_{j-1}(u_{\xi_{j-1}}))\, ds \\ &+ \int_{t_k}^{t} e^{\alpha(t-s)} h(s, u(s), \sigma_k(u_{\xi_k}))\, ds, \end{aligned}$$

and, for $t \geq t_0$, we have

$$u(t) = e^{\alpha(t-t_0)} u_0 + \sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} e^{\alpha(t-s)} h(s, u(s), \sigma_{j-1}(u_{\xi_{j-1}}))\, ds,$$

and, by the comparison result, we get

$$V(t, x(t)) \leq e^{\alpha(t-t_0)} V(t_0, x_0) + \sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} e^{\alpha(t-s)} h(s, V(s, x(s)), \sigma_{j-1}(V_{\xi_{j-1}}))\, ds.$$

The proof is complete.

Remark 14.1. In some special cases of the function g, such as those in Corollary 14.1(i) and (ii), one can consider EPCAG, in which $\xi_k \in [t_k, t_{k+1})$, rather than at the most left-end point t_k, for any k. In the following example, we state these results.

Example 14.1. Suppose that the conditions in Theorem 14.1 hold where $\xi_k \in (t_k, t_{k+1})$ and $t \in [t_k, t_{k+1}]$ for any k. If we choose that

(i) $g(t, u, \sigma_k(u_{\xi_k})) = \beta_k u_{\xi_k}$, with β_k being a constant for all k, then

$$V(t,x(t)) \leq \begin{cases} \left[\frac{\beta_0}{1-\beta_0(\xi_0-t_0)}(t-t_0)+1\right]V(t_0,x_0), & k=0, \quad t\in(t_0,t_1], \\ \left[\frac{\beta_k}{1-\beta_k(\xi_k-t_k)}(t-t_k)+1\right]\prod_{j=1}^{k}\left[\frac{\beta_{j-1}}{1-\beta_{j-1}(\xi_{j-1}-t_{j-1})}(t_j-t_{j-1})+1\right] & \\ \quad \times V(t_0,x_0), & k\in\mathbb{N}, \quad t\in(t_k,t_{k+1}], \end{cases}$$

provided that, for $k = 0, 1, 2, \cdots$,

$$\begin{cases} \xi_k < t_k + \frac{1}{\beta_k}, & \text{for } \beta_k > 0, \\ \xi_k > t_{k+1} + \frac{1}{\beta_k}, & \text{for } \beta_k < 0; \end{cases}$$

(ii) $g(t, u, \sigma_k(u_{\xi_k})) = \alpha u(t) + \beta_k u_{\xi_k}$, with α and β_k being constants, then

$$V(t,x(t)) \leq \begin{cases} \left[e^{\alpha(t-t_0)} + \frac{\beta_0 e^{\alpha(\xi_0-t_0)}}{\alpha[1-\frac{\beta_0}{\alpha}(e^{\alpha(\xi_0-t_0)}-1)]}\left(e^{\alpha(t-t_k)}-1\right)\right]V(t_0,x_0), & \\ \qquad k=0, \quad t\in(t_0,t_1], & \\ \left[e^{\alpha(t-t_k)} + \frac{\beta_k e^{\alpha(\xi_k-t_k)}}{\alpha[1-\frac{\beta_k}{\alpha}(e^{\alpha(\xi_k-t_k)}-1)]}\left(e^{\alpha(t-t_k)}-1\right)\right]\prod_{j=1}^{k}\left[e^{\alpha(\xi_{j-1}-t_{j-1})}\right. & \\ \quad \left.+\frac{\beta_{j-1} e^{\alpha(\xi_{j-1}-t_{j-1})}}{\alpha[1-\frac{\beta_{j-1}}{\alpha}(e^{\alpha(\xi_{j-1}-t_{j-1})}-1)]}\left(e^{\alpha(t_j-t_{j-1})}-1\right)\right]V(t_0,x_0), & \\ \qquad k=1,2,\cdots, \quad t\in(t_k,t_{k+1}], & \end{cases}$$

provided that, for $k = 0, 1, 2, \cdots$,

$$\begin{cases} \xi_k < t_k + \frac{1}{\alpha} \ln\Big(1 + \frac{\alpha}{\beta_k}\Big), & \text{when } \alpha > 0,\ \beta_k > 0, \text{ or} \\ & \text{when } \alpha < 0,\ \beta_k > 0 \text{ with } \beta_k > -\alpha > 0, \\ \xi_k > \frac{1}{\alpha} \ln\Big(e^{\alpha t_{k+1}} + \frac{\alpha}{\beta_k} e^{\alpha t_k}\Big) \text{ and} & \\ t_{k+1} > \frac{1}{\alpha} \ln\Big(-\frac{\alpha}{\beta_k} e^{\alpha t_k}\Big), & \text{when } \alpha > 0,\ \beta_k < 0. \end{cases}$$

With Theorem 14.1 in hand, we are in a position to establish some stability results for the nonlinear EPCA.

Theorem 14.2. Let the conditions in Theorem 14.1 hold, and assume further that

$$b(\|x\|) \le V(t, x) \le a(\|x\|)$$

is satisfied, where a and b are class-$\mathcal{K}$ functions. Then, the stability properties of the trivial solution ($u \equiv 0$) of scalar EPCA (14.4) imply the corresponding stability properties of the trivial solution ($x \equiv 0$) of (14.3).

Proof. Let $t_0 \in \mathbb{R}_+$ and $\varepsilon > 0$ be given. Suppose that $u \equiv 0$ is stable. Then, for given $b(\varepsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ for which we have

$$0 \le u_0 \le \delta_1 \quad \text{implies} \quad u(t, t_0, u_0) \le b(\varepsilon), \qquad t \ge t_0,$$

where $u(t, t_0, u_0)$ is any solution of (14.4). Choose $\delta_2 = \delta_2(\varepsilon)$ such that $a(\delta_2) < b(\varepsilon)$. Define $\delta = \min\{\delta_1, \delta_2\}$. We claim that, if $\|x_0\| < \delta$, then $\|x(t)\| < \varepsilon$, for $t \ge t_0$, where $x(t) = x(t, t_0, x_0)$ is any solution of (14.3). If our claim were not true, then there would exist a $t^* > t_0$ and $t_k < t^* \le t_{k+1}$ for which $\|x_0\| < \delta$ and

$$\|x(t)\| < \varepsilon \qquad \text{for} \quad t_0 \le t \le t_k, \tag{14.8}$$
$$\|x(t)\| \ge \varepsilon \qquad \text{for} \quad t_k \le t^* \le t_{k+1}.$$

From (14.8), we have $\|x(t_k)\| < \varepsilon$. Hence, we can find a $\widetilde{t}$ such that $t_k < \widetilde{t} \le t^*$, at which

$$\varepsilon \le \|x(\widetilde{t})\|.$$

Let $u_0 = a(\|x_0\|) < \delta_1$, and define $m(t) = V(t, x(t))$, for $t_0 < t \le \widetilde{t}$. By Theorem 14.1,

$$V(t, x(t)) \le \vartheta(t, t_0, a(\|x_0\|)), \qquad t_0 \le t \le \widetilde{t},$$

where $\vartheta(t, t_0, a(\|x_0\|))$ is the maximal solution of scalar system (14.4). Then, we obtain

$$b(\varepsilon) \le b(\|x(\widetilde{t})\|) \le V(\widetilde{t}, x(\widetilde{t})) \le \vartheta(\widetilde{t}, t_0, a(\|x_0\|)) < b(\varepsilon),$$

which is a contradiction. This shows that $x \equiv 0$ is stable. If, moreover, δ is independent of t_0, then $x \equiv 0$ is uniformly stable.

To prove asymptotic stability of $x \equiv 0$, it suffices to show attractivity of this solution. Suppose that $u \equiv 0$ is asymptotically stable. Then, it implies that $x \equiv 0$ is stable, i.e., for each $\varepsilon > 0$, there exists a $\delta = \delta(t_0, \varepsilon)$ such that

$$\|x_0\| < \delta \qquad \text{implies} \qquad \|x(t)\| < \varepsilon, \quad \forall t \ge t_0.$$

Since $u \equiv 0$ is attractive, given $b(\varepsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exist $\delta_0^* = \delta_0^*(t_0) > 0$ and $T = T(t_0, \varepsilon) > 0$ such that

$$0 \le u_0 \le \delta_0^* \qquad \text{implies} \qquad u(t, t_0, u_0) < b(\varepsilon), \quad \forall t \ge t_0 + T.$$

Choose a $\widetilde{\delta}$ such that $a(\widetilde{\delta}) < \delta_0^*$. Define $\rho = \min\{\delta_0^*, \widetilde{\delta}\}$ and let $\|x_0\| < \rho$. Then, as we did in proving the stability of $x \equiv 0$, we can get

$$b(\|x(t)\|) \le V(t, x(t)) \le \vartheta(t, t_0, a(\|x_0\|)) < b(\varepsilon),$$

from which $\|x(t)\| < \varepsilon$ for all $t \geq t_0 + T$, i.e., $x \equiv 0$ is attractive. Hence, $x \equiv 0$ is asymptotically stable. If T is independent of t_0, then $x \equiv 0$ is uniformly asymptotically stable.

Corollary 14.2. In Theorem 14.2, let $g(t, u(t), \sigma_k(u_{\xi_k})) = \beta_k u_{\xi_k}$, with β_k being a constant for all k.

(i) In the case $\xi_k = t_k$,

(1) if $\beta_k > 0$ for any k and the infinite series

$$\sum_{j=1}^{\infty} \beta_{j-1}(t_j - t_{j-1}) \tag{14.9a}$$

converges, then $x \equiv 0$ is uniformly stable;

(2) if $\beta_k < 0$ for any k and, in addition to assumption (i)(1), for any j,

$$0 < t_j - t_{j-1} < -\frac{1}{\beta_{j-1}}, \tag{14.9b}$$

then $x \equiv 0$ is uniformly asymptotically stable.

(ii) In the case $\beta_k < 0$, and $t_{k-1} < \xi_k \leq t_k$ for any $k = 0, 1, 2, \cdots$ and $\xi_0 = t_0$, if $u_k(t) \leq L$ for some positive constant L, where $u_k(t)$ is defined in Corollary 14.1 for any k and $t \in [t_k, t_{k+1})$, then $u \equiv 0$ is uniformly stable; if, in addition, $u_k(t) \leq L_k$ for any k and $t \in [t_k, t_{k+1})$, and $\sum_{k=0}^{\infty} L_k < \infty$, then the trivial solution $u \equiv 0$ and hence $x \equiv 0$ are uniformly asymptotically stable. Particularly, one may define $L = \sup\{L_k : k = 0, 1, 2, \cdots\}$.

Proof (i)(1) The solution of the scalar EPCA

$$\dot{u}(t) = \beta_k u_{\xi_k}, \qquad t \in [t_k, t_{k+1}], \quad k = 0, 1, 2, \cdots,$$
$$u(t_0) = u_0$$

is given by

$$u(t) = \Big(1 + \beta_k(t - t_k)\Big) \prod_{j=1}^{k} \Big[1 + \beta_{j-1}(t_j - t_{j-1})\Big] u_0.$$

By (14.9a), the product $\prod_{j=1}^{\infty} \Big[1+\beta_{j-1}(t_j - t_{j-1})\Big]$ converges. Therefore, defining $M = \prod_{j=1}^{\infty} \Big[1 + \beta_{j-1}(t_j - t_{j-1})\Big] < \infty$ yields

$$u(t, t_0, u_0) = Mu_0 < M\sigma, \qquad \text{for some } \sigma > 0 \text{ such that } u_0 < \sigma,$$

meaning that the trivial solution $u \equiv 0$ is uniformly stable, which implies, by Theorems 14.2, the uniform stability of the trivial solution $x \equiv 0$. In particular, for $k = 0, 1, 2, \cdots$, one may choose that $\beta_k = \frac{1}{2^k}$, $t_{k+1} - t_k < \delta$ for some $\delta > 0$.

(i)(2) Assumption (14.9b) is equivalent to $0 < 1 + \beta_{j-1}(t_j - t_{j-1}) < 1$, so that let $1 + \beta_{j-1}(t_j - t_{j-1}) = \frac{1}{e}$, for example, then M approaches zero; this proves the uniform asymptotic stability of $u \equiv 0$ and $x \equiv 0$.

(ii) The proof is straightforward and is omitted here.

Remark 14.2. It is worth noting that the assumption $0 < u_k(t) \leq L_k$, for any k and $t \in [t_k, t_{k+1})$, is equivalent to

$$\frac{L_k - C_k}{\beta_k C_{\xi_k}} < t - t_k < \frac{-C_k}{\beta_k C_{\xi_k}},$$

where C_k and C_{ξ_k} are defined in Corollary 14.1.

Corollary 14.3. In Theorem 14.3, let $g(t, u(t), \sigma_k(u_{\xi_k})) = \alpha u(t) + \beta_k u_{\xi_k}$, where $\alpha > 0$, $\beta_k < 0$, and $\xi_k = t_k$ for $k = 1, 2, \cdots$. Then, the trivial solution $x \equiv 0$ is uniformly stable if the infinite series

$$\sum_{j=1}^{\infty} \Big[(1 + \frac{\beta_{j-1}}{\alpha})e^{\alpha(t_j - t_{j-1})} - \frac{\beta_{j-1}}{\alpha}\Big] \tag{14.10}$$

converges. Furthermore, if, in addition, the terms in corresponding infinite product are all less than unity, then $x \equiv 0$ is uniformly asymptotically stable.

Proof. Since the infinite series in (14.10) converges, so does the infinite product

$$M = \prod_{j=1}^{\infty} \Big[(1 + \frac{\beta_{j-1}}{\alpha}) e^{\alpha(t_j - t_{j-1})} - \frac{\beta_{j-1}}{\alpha} \Big].$$

Then,

$$u(t) < M\sigma,$$

for some positive σ for which $u_0 < \sigma$; that is, $u \equiv 0$ is uniform stability. Employing our comparison result, the uniform stability of $x \equiv 0$ will be a consequence of this stability property. Finally, by our assumption, if, for instance, every term in the infinite product is less than or equal to $1/e$, then

$$u(t) = \prod_{j=1}^{\infty} \Big[(1 + \frac{\beta_{j-1}}{\alpha}) e^{\alpha(t_j - t_{j-1})} - \frac{\beta_{j-1}}{\alpha} \Big] u_0 \to 0,$$

that is, $u \equiv 0$ and accordingly $x \equiv 0$ are uniformly asymptotically stable.

Remark 14.3.

(i) The interesting finding of Corollary 14.3 is that the system has unstable ordinary part, which is stabilized by negative piecewise constant given at an individual point in each subinterval.

(ii) Assuming each term in the product is equal to or less than some positive constant $c < 1$ results in, for $\xi_k = t_k$,

$$t_{k+1} - t_k > \frac{1}{\alpha} \ln \Big[\Big(c + \frac{\beta_k}{\alpha} \Big) \Big(1 + \frac{\beta_k}{\alpha} \Big)^{-1} \Big],$$

where $\left(c + \frac{\beta_k}{\alpha} \right) \left(1 + \frac{\beta_k}{\alpha} \right)^{-1} > 1$ so long as $\left(1 + \frac{\beta_k}{\alpha} \right) < 0$ and $c < 1$.

Corollary 14.4. In Theorem 14.2, let $g(t, u(t), \sigma_k(u_{\xi_k})) = -\omega(u) + \beta_k u_{\xi_k}$ with $\omega \in \mathcal{K}$, $\beta_k \geq 0$, and $\xi_k = t_k$ for all k. Then, $x \equiv 0$ is uniformly asymptotically stable provided that the series $\sum_{j=1}^{\infty} \beta_j(t_j - t_{j-1})$ converges.

Proof. Since $D^+V(x, V_{\xi_k}) \leq -\omega(V(x)) + \beta_k V_{\xi_k}$ implies

$$D^+V(x, V_{\xi_k}) \leq \beta_k V_{\xi_k},$$

then it follows from Corollary 14.3 that $u \equiv 0$ of the scalar EPCA

$$\dot{u}(t) = -\omega(u(x)) + \beta_k u_{\xi_k}, \tag{14.11a}$$

$$u(t_0) = u_0 \tag{14.11b}$$

is uniformly stable. Thus, for a fixed $\rho > 0$, there is a $\sigma = \sigma(\rho) > 0$ such that

$$0 \leq u_0 \leq \sigma \quad \text{implies} \quad u(t, t_0, u_0) < \rho, \quad t \geq t_0, \tag{14.12}$$

for any solution of (14.11). Let $\varepsilon \in (0, \rho)$ be given and $\delta = \delta(\varepsilon)$. In the rest of the proof, we need to show that $u \equiv 0$ is attractive; it suffices to show that there exists a $T = T(\varepsilon) > 0$ such that

$$u(t^*, t_0, u_0) < \delta = \delta(\varepsilon), \tag{14.13}$$

for any $t^* \in [t_0, t_0 + T]$ and any solution $u(t, t_0, u_0)$ of (14.11) that satisfies (14.12).

Since $\sum_{j=1}^{\infty} \beta_j(t_j - t_{j-1})$ converges, define

$$M = \sum_{j=1}^{\infty} \beta_j(t_j - t_{j-1}) < \infty.$$

Choose $T_1 = T_1(\varepsilon) > 0$ such that

$$T_1 > 2\rho M[\omega(\delta)]^{-1}. \tag{14.14}$$

Define

$$T = \max\Big\{T_1, \frac{2(\sigma+1)}{\omega(\delta)}\Big\}. \tag{14.15}$$

We claim that (14.13) is true for T given by (14.15). If this were not true, suppose, for contradiction, that there were a solution $u(t) = u(t, t_0, u_0)$ of (14.11) with $u_0 < \sigma$ such that

$$u(t) \geq \delta, \qquad t \in [t_0, t_0 + T]. \tag{14.16}$$

Integrating (14.11) over $[t_0, t_0 + T]$ yields

$$\begin{aligned} 0 \leq u(t_0+T) &= u_0 - \int_{t_0}^{t_0+T} \omega(u(s))\,ds + \sum_{j=1}^{k} \beta_{j-1} u_{\xi_{j-1}}(t_j - t_{j-1}) \\ &\quad + \beta_k u_{\xi_k}(t_0 + T - t_k) \\ &\leq \sigma - T\omega(\delta) + \rho M \\ &= \sigma - \frac{T\omega(\delta)}{2} - \frac{T\omega(\delta)}{2} + \rho M \\ &\leq \sigma - \frac{T\omega(\delta)}{2} < -1 < 0, \end{aligned}$$

which is a contradiction. Thus, (14.13) must be true, that is,

$$u(t^*, t_0, u_0) < \delta,$$

for any solution of $u(t, t_0, u_0)$ of (14.11) with $u_0 < \sigma$. Hence, $u \equiv 0$ is uniformly attractive and consequently uniformly asymptotically stable, which in turn implies that $x \equiv 0$ is uniformly asymptotically stable.

Corollary 14.5. Let $g(t, u(t), \sigma_k(u_{\xi_k})) = \alpha u(t) + h(t, u(t), \sigma_k(u_{\xi_k}))$ with $\alpha < 0$. Then, $x \equiv 0$ is uniformly asymptotically stable provided that the series

$\sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} e^{\alpha(t-s)} h(s, u(s), \sigma_{j-1}(u_{\xi_{j-1}}))\, ds$
converges. In particular, $h(t, u(t), \sigma_k(u_{\xi_k})) = 0$ when k (or t) $\to \infty$.

Proof. The proof is straightforward since, from the solution

$$u(t) = e^{\alpha(t-t_0)} u_0 + \sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} e^{\alpha(t-s)} h(s, u(s), \sigma_{j-1}(u_{\xi_{j-1}}))\, ds,$$

we get $\lim_{t\to\infty} u(t) = 0$.

14.3 Numerical Examples

To illustrate our results, we discuss some examples.

Example 14.2. Consider the nonlinear EPCA

$$\begin{cases} \dot{x} = 2x + 2\beta_k e^{y^2} y_{\xi_k}, & t \in [t_k, t_{k+1}], \quad k = 0, 1, 2, \cdots, \\ \dot{y} = y + \beta_k (1 + x^2) x_{\xi_k}, \end{cases} \tag{14.17}$$

where $\beta_k = -3.5$ for all k. Clearly, the ordinary part is unstable. Let $V(x, y) = x + y$ for $x > 0$ and $y > 0$. Then, one may get

$$\dot{V} \le \alpha V + \beta_k V_{\xi_k},$$

where $\alpha = 2$. The solution of the differential inequality is given in Corollary 14.1(ii), and by Corollary 14.3 the trivial solution $x \equiv 0$ of (14.17) is uniformly asymptotically stable. If $\xi_k = t_k$, then $t_{k+1} \in (0.15, 0.34)$, where $c = 0.6$. Figure 14.1 shows the simulation result in the case $\xi_k = t_k$ for all k.

Example 14.3. Consider the nonlinear EPCA

$$\begin{cases} \dot{x} = y - x[1 + \theta(x^2 + y^2)], & t \in [t_k, t_{k+1}], \\ \dot{y} = -x - y[1 + \theta(x^2 + y^2)] + \frac{2y_{\xi_k}}{2^k}, \end{cases} \tag{14.18}$$

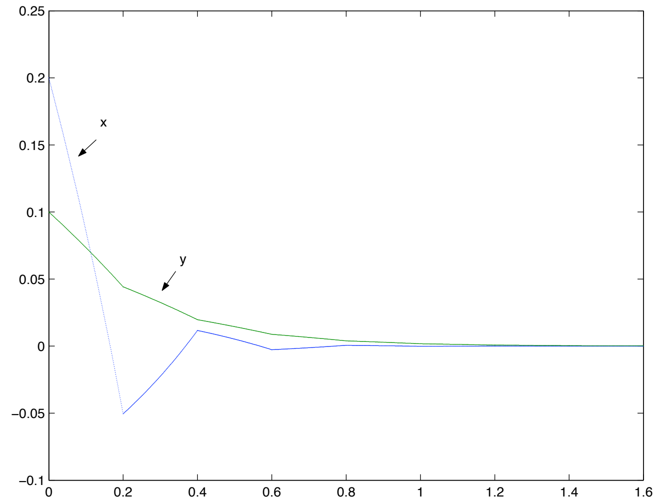

Figure 14.1: Uniform asymptotic stability of $(x, y)^T = (0, 0)$ in Example 14.2

for any $k = 0, 1, 2, \cdots$, where $0 < \theta \ll 1$. Let $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Then,

$$\begin{aligned}\dot{V}(x,y) &= xy - x^2[1 + \theta(x^2 + y^2)] - xy - y^2[1 + \theta(x^2 + y^2)] + \frac{2yy_{\xi_k}}{2^k} \\ &\leq -(x^2 + y^2) - \theta(x^2 + y^2)^2 + \frac{1}{2^k}(x^2 + y^2) + \frac{1}{2^k}(x^2_{\xi_k} + y^2_{\xi_k}) \\ &= -\theta V^2(x, y) + \beta_k V(x_{\xi_k}, y_{\xi_k}).\end{aligned}$$

Let $\omega(u) = \theta u^2$. Then, by Corollary 14.4, the trivial solution of (14.18) is uniformly asymptotically stable. Simulation result is shown in Figure 14.2, where $\theta = 0.01$, $\xi_k = t_k$, and $t_{k+1} - t_k = 1$ for all $k = 0, 1, 2, \cdots$.

Example 14.4. Consider the nonlinear EPCA

$$\begin{cases} \dot{x} = -x, & t \in [t_k, t_{k+1}], \quad k = 0, 1, 2, \cdots, \\ \dot{y} = -2y + \frac{x_{\xi_k} \sin y_{\xi_k}}{1+x^2} y e^{-t}. \end{cases} \tag{14.19}$$

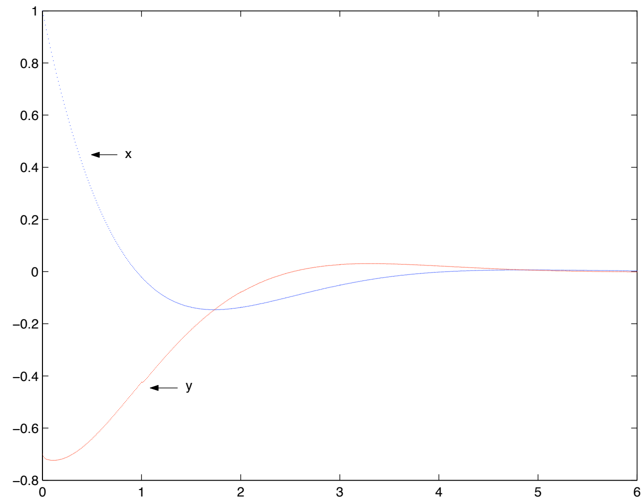

Figure 14.2: Uniform asymptotic stability of $(x, y)^T = (0, 0)$ in Example 14.3

Let $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Then,

$$\begin{aligned}
\dot{V}(x, y) &= -x^2 - 2y^2 + \frac{x_{\xi_k} \sin y_{\xi_k}}{1 + x^2} y^2 e^{-t} \\
&\leq -(x^2 + y^2) + \frac{1}{2}(y^4 + x_{\xi_k}^2)e^{-t} \\
&= -2V(x, y) + \Big(2V^2(x, y) + V(x_{\xi_k}, y_{\xi_k})\Big)e^{-t} \\
&= \alpha V + h(t, V, V_{\xi_k}),
\end{aligned}$$

where $\alpha = -2$ and $h(t, V, V_{\xi_k}) = (2V^2 + V_{\xi_k})e^{-t}$. By Corollary 14.5, the trivial solution of (14.19) is uniformly asymptotically stable. Figure 14.3 shows the asymptotic stability of the trivial solution of (14.19).

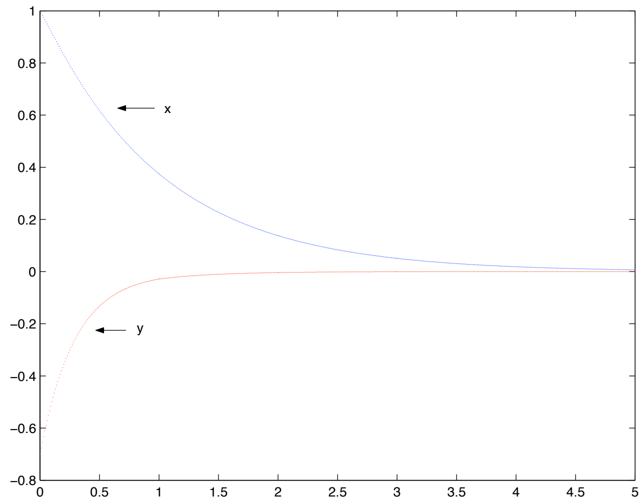

Figure 14.3: Uniform asymptotic stability of $(x, y)^T = (0, 0)$ in Example 14.4.

14.4 Conclusion

Systems of nonlinear EPCA, which treated as a switched system, were formulated. A comparison principle was developed and successfully used to establish some stability properties of the system. A Lyapunov-function criterion was used to analyze our stability results. Besides, some special cases of retarded EPCA and EPCAG were considered. We also showed that piecewise constant arguments contribute to stabilize unstable systems of ordinary differential equations.

Chapter 15

Existence, Uniqueness and Stability of Stochastic EPCA

In this chapter, we consider systems with stochastic EPCA (or SEPCA). Our interest is to establish some results on the existence of a unique solution. We then establish a comparison principle, which will be used later to develop some stability results by using Razumikhin methodology. The organization of this chapter is as follows: in Section 15.1, we state and prove the existence and uniqueness results. In Section 15.2, we develop Lyapunov-like sufficient conditions to guarantee the stability properties. The material of this chapter forms the basis of [Alw-k].

Consider the nonlinear systems with SEPCA of the form

$$dx(t) = f(t, x(t), \lambda_{\varrho(t)}(x(\gamma(t))))dt + g(t, x(t), \lambda_{\varrho(t)}(x(\gamma(t))))dW(t), \tag{15.1a}$$

$$x(t_0) = x_0, \tag{15.1b}$$

where $x \in \mathbb{R}^n$ is the system state and, for all $t \geq t_0$, $\varrho(t)$ and $\gamma(t)$ are piecewise constant functions taking values in the sets $K = \{k\}_{k=0}^{\infty}$ and $\Xi = \{\xi_k\}_{k=0}^{\infty}$, respectively, where $t_k \leq \xi_k < t_{k-1}$ for any $k = 0, 1, 2, \cdots$. As stated in Chapter 14, these functions represent the switching logics of the system switching between the piecewise constant argument λ_k and the values of its state argument x.

Accordingly, one may define system (15.1) as follows: for any $t \in [t_k, t_{k+1})$, $k \in K$,

$$dx(t) = f(t, x(t), \lambda_k(x(\xi_k)))dt + g(t, x(t), \lambda_k(x(\xi_k)))dW(t), \tag{15.2a}$$

$$x(t_0) = x_0, \tag{15.2b}$$

or equivalently

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), \lambda_k(x(\xi_k)))ds + \int_{t_0}^{t} g(s, x(s), \lambda_k(x(\xi_k)))dW(s). \tag{15.3}$$

The following definitions will be needed in this chapter.

Definition 15.1. For any α, $\beta \in \mathbb{R}$, an $\mathbb{R}^n$-valued stochastic process $x : (\alpha, \beta) \to \mathbb{R}$ is said to be a *solution* of (15.1) if the following hold:

(i) $x(t)$ is continuous and $\mathcal{F}_t$-adapted for all $t \in (\alpha, \beta)$;

(ii) $f(t, x(t), \lambda_k(x(\xi_k))) \in \mathcal{L}_{ad}(\Omega, L^1(\alpha, \beta))$ and $g(t, x(t), \lambda_k(x(\xi_k))) \in \mathcal{L}_{ad}(\Omega, L^2(\alpha, \beta))$;

(iii) the stochastic integral equation (15.3) holds w.p.1.

Definition 15.2. [Mao06] For all $t \in [a, b]$, an $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process $f(t)$ with $\int_a^b \|f(t)\|^p dt < \infty$ (a.s.) (i.e., $f \in \mathcal{L}_{ad}(\Omega; L^p[a, b])$) is said to be in $\mathcal{M}^p([a, b]; \mathbb{R}^n)$ if $\mathbb{E}\big[\int_a^b \|f(t)\|^p dt\big] < \infty$.

Definition 15.3. [Mao06] An $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted integrable process $X(t)$ is said to be a *martingale with respect to the filtration* $\{\mathcal{F}_t\}_{t \geq 0}$ if

$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s), \qquad \text{(a.s.)}, \quad \text{for all } 0 \leq s < t < \infty,$$

where $\mathbb{E}[X(t)|\mathcal{F}_s]$ stands for the conditional expectation of process $X(t)$ with respect to the filtration $\mathcal{F}_s$.

Doob's martingale inequality. [Mao06] For all $t \geq 0$, let $X(t)$ be an $\mathbb{R}^n$-valued martingale and $[a, b]$ be a bounded interval of $\mathbb{R}$. If $p > 1$ and $X(t) \in L^p(\Omega; \mathbb{R}^n)$, then

$$\mathbb{E}\Big[\sup_{a \leq t \leq b} \|X(t)\|^p\Big] \leq \Big(\frac{p}{p-1}\Big)^p \mathbb{E}[\|X(b)\|^p].$$

Borel-Cantelli's lemma. [Mao06] If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{F}$ and $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$, then

$$\mathbb{P}(\limsup_{k \to \infty} A_k) = 0.$$

15.1 Existence and Uniqueness Results

In this section, we discuss the existence of a unique solution of SEPCA given in (15.1) or (15.2). The technique followed in proving these results is to generate a convergent Cauchy sequence of solutions. For this purpose, we assume that the system vector fields are bounded by a linear growth estimates and satisfy the Lipschitz condition. The first condition is to avoid a finite escape time that a solution may have when time evolves. The second condition is made to be used in proving the convergence of the generated sequence and to guarantee the uniqueness of the solution. Along the same line of proving the existence and uniqueness results of ordinary stochastic differential equation, one may consult [Mao06].

Theorem 15.1. Assume the following assumptions hold:

(i) the vector fields functions f and g satisfy the linear growth condition, i.e., there exists a positive L_1 such that

$$\|f(t,x,y)\|^2 + \|g(t,x,y)\|^2 \leq L_1(1+\|x\|^2+\|y\|^2), \qquad \text{(a.s.)},$$

for all $(t,x,y) \in [t_k, t_{k+1}) \times \mathbb{R}^n \times \mathbb{R}^n$;

(ii) f and g satisfy a global Lipschitz condition, i.e., there exists a positive constant L_2 such that

$$\|f(t,x_1,y_1 - f(t,x_2,y_2))\|^2 + \|g(t,x_1,y_1) - g(t,x_2,y_2))\|^2$$
$$\leq L_2\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2, \qquad \text{(a.s.)},$$

for all $(t,x,y) \in [t_k, t_{k+1}) \times \mathbb{R}^n \times \mathbb{R}^n$.

Then, system (15.1) or (15.2) has a unique solution for all $t \geq t_0$.

Before we prove this theorem, the following lemma is needed.

Lemma 15.1. For any $k \in K$, assume that the linear growth condition holds. Then, solution x cannot grow faster than the following exponential estimate

$$\mathbb{E}\left(\sup_{t_k \leq t \leq t_{k+1}} \|x(t)\|^2\right) \leq (1+c_k)e^{3L_1(t_{k+1}-t_k+4)(t_{k+1}-t_k)},$$

where $c_k = 3\mathbb{E}[\|x_0\|^2] + 3L_1(t_{k+1} - t_k + 4)(t_{k+1} - t_k)\mathbb{E}[\|\lambda_k(x_{\xi_k})\|^2] < \infty$. In other words, $x(t) \in \mathcal{M}^2([t_k, t_{k+1}); \mathbb{R}^n)$ with $0 < t_{k+1} - t_k \leq \theta < \infty$ for any k.

Proof. Choose k arbitrarily, and, for any $l \geq 1$, define a sequence of stopping times

$$\tau_l = t_{k+1} \wedge \inf\{t \in [t_k, t_{k+1}) : \|x(t)\| \geq l\},$$

where $\lim_{l\to\infty} \tau_l = t_{k+1}$ (a.s.). For simplicity of notation, we set $x_l(t) = x(t \wedge \tau_l)$ for all $t \in [t_k, t_{k+1})$. Then, from system (15.1), we get

$$x_l(t) = x_k + \int_{t_k}^t f(s, x_l(s), \lambda_k(x_{\xi_k}))1_{[t_k,\tau_l]}ds + \int_{t_k}^t g(s, x_l(s), \lambda_k(x_{\xi_k}))1_{[t_k,\tau_l]}dW(s),$$

where 1_A is the indicator function of a set A. In virtue of (i) and using Doob's martingale inequality to the stochastic Itô integral, one may get

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_k \le t \le t} \|x_l(t)\|^2\right) \\
&\quad \le 3\mathbb{E}[\|x_k\|^2] + 3L_1(t_{k+1} - t_k) \int_{t_k}^{t} (1 + \mathbb{E}[\|x_l(s)\|^2] + \mathbb{E}[\|\lambda_k(x_{\xi_k})\|^2]) ds \\
&\qquad + 12L_1 \int_{t_k}^{t} (1 + \mathbb{E}[\|x_l(s)\|^2] + \mathbb{E}[\|\lambda_k(x_{\xi_k})\|^2]) ds \\
&\quad \le 3\mathbb{E}[\|x_k\|^2] + 3L_1(t_{k+1} - t_k + 4) \int_{t_k}^{t} (1 + \mathbb{E}[\|x_l(s)\|^2]) ds \\
&\qquad + 3L_1(t_{k+1} - t_k + 4)(t_{k+1} - t_k)\mathbb{E}[\|\lambda_k(x_{\xi_k})\|^2],
\end{aligned}$$

which implies

$$\begin{aligned}
&1 + \mathbb{E}\left(\sup_{t_k \le t \le t} \|x_l(t)\|^2\right) \\
&\quad \le 1 + c_k + 3L_1(t_{k+1} - t_k + 4) \int_{t_k}^{t} (1 + \mathbb{E}[\|x_l(s)\|^2]) ds \\
&\quad \le 1 + c_k + 3L_1(t_{k+1} - t_k + 4) \int_{t_k}^{t} (1 + \mathbb{E}[\sup_{t_k \le t \le \tau_l} \|x_l(s)\|^2]) ds.
\end{aligned}$$

By the Gronwall inequality, we get

$$\mathbb{E}\left(\sup_{t_k \le t \le t} \|x_l(t)\|^2\right) \le (1 + c_k) e^{3L_1(t_{k+1} - t_k + 4)(t_{k+1} - t_k)}.$$

The desired result is implied by letting $l \to \infty$. This completes the proof.

Proof of Theorem 15.1. The proof given here is over $[t_0, t_1)$ since the rest will be similar. Define the sequence $x_n(t)$, with $x_0(t) = x_0$, by the following iteration

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s), \lambda_k(x_{n-1_{\xi_0}})) ds + \int_{t_0}^{t} g(s, x_{n-1}(s), \lambda_k(x_{n-1_{\xi_0}})) dW(s), \tag{15.4}$$

where $x_{j_{\xi_0}} = x_j(\xi_0) = x_j(t_0)$. By Lemma 15.1, $x_0 \in \mathcal{M}^2([t_k, t_{k+1}); \mathbb{R}^n)$ and by mathematical induction, we can see that $x_n(t) \in \mathcal{M}^2([t_k, t_{k+1}); \mathbb{R}^n)$ as follows:

$$\mathbb{E}[\|x_n(t)\|^2] \leq C_1 + 3L_1(t + t_1) \int_{t_0}^{t} \mathbb{E}[\|x_{n-1}(s)\|^2] ds,$$

where $C_1 = 3\mathbb{E}[\|x_0\|^2] + 3L_1 t_1(1 + t_1) \left(1 + \mathbb{E}[\|\lambda_k(x_{n-1_{\xi_0}})\|^2]\right) < \infty$, where we used the fact $t_1 - t_0 < t_1$. This also implies that, for an arbitrary j,

$$\begin{aligned}
\max_{1 \leq n \leq j} \mathbb{E}[\|x_n(t)\|^2] &\leq C_1 + 3L_1(t + t_1) \int_{t_0}^{t} \max_{1 \leq n \leq j} \mathbb{E}[\|x_{n-1}(s)\|^2] ds \\
&\leq C_1 + 3L_1(t + t_1) \int_{t_0}^{t} \left(\mathbb{E}[\|x_0\|^2] + \max_{1 \leq n \leq j} \mathbb{E}[\|x_n(s)\|^2] \right) ds \\
&= C_2 + 3L_1(t + t_1) \int_{t_0}^{t} \max_{1 \leq n \leq j} \mathbb{E}[\|x_n(s)\|^2] ds,
\end{aligned}$$

where $C_2 = C_1 + 3L_1 t_1(1 + t_1)\mathbb{E}[\|x_0\|^2]$. By the Gronwall inequality

$$\max_{1 \leq n \leq j} \mathbb{E}[\|x_n(t)\|^2] \leq C_2 e^{3L_1 t_1(1+t_1)}.$$

Since j is arbitrary, we get

$$\mathbb{E}[\|x_n(t)\|^2] \leq C_2 e^{3L_1 t_1(1+t_1)}, \tag{15.5}$$

i.e., for all n, $x_n \in \mathcal{M}^2([t_k, t_{k+1}); \mathbb{R}^n)$, that is, $x_n(t)$ is bounded over $[t_0, t_1)$.

Now, we want to prove that this sequence is convergent. Note that

$$\begin{aligned}
\|x_1(t) - x_0(t)\|^2 &= \|x_1(t) - x_0\|^2 \\
&\leq 2\Big\| \int_{t_0}^{t} f(s, x_0, \lambda_k(x_{0_{\xi_0}})) ds \Big\|^2 + 2\Big\| \int_{t_0}^{t} g(s, x_0, \lambda_k(x_{0_{\xi_0}})) dW(s) \Big\|^2,
\end{aligned}$$

which implies, after taking the mathematical expectation,

$$\begin{aligned}
&\mathbb{E}[\|x_1(t) - x_0(t)\|^2] \\
&\quad \leq 2L_1 \left[(t_1 - t_0)(1 + (t_1 - t_0))\right] \left(1 + \mathbb{E}[\|x_0\|^2] + \mathbb{E}[\|\lambda_k(x_{\xi_0})\|^2]\right) = C,
\end{aligned}$$

i.e., $\mathbb{E}[\|x_1(t) - x_0(t)\|^2] \leq C$, where

$$C = 2L_1\left[(t_1 - t_0)(1 + (t_1 - t_0))\right]\left(1 + \mathbb{E}[\|x_0(t)\|^2] + \mathbb{E}[\|\lambda_k(x_{\xi_0})\|^2]\right).$$

We will show by mathematical induction that, for any $n \geq 0$ and $t \in [t_0, t_1)$,

$$\mathbb{E}[\|x_{n+1}(t) - x_n(t)\|^2] \leq \frac{C[M(t - t_0)]^n}{n!}, \tag{15.6}$$

with $M = 2L_2(t_1 - t_0 + 1)$. Obviously, the relation is true for $n = 0, 1$. Assume that it is also true for some $n \geq 0$. As for the case of $n + 1$, we have

$$\begin{aligned}
&\|x_{n+2}(t) - x_{n+1}(t)\|^2 \\
&\qquad \leq 2\left\| \int_{t_0}^{t} \left(f(s, x_{n+1}(s), \lambda_k(x_{n+1_{\xi_0}})) - f(s, x_n(s), \lambda_k(x_{n_{\xi_0}})) \right) ds \right\|^2 \\
&\qquad\quad + 2\left\| \int_{t_0}^{t} \left(g(s, x_{n+1}(s), \lambda_k(x_{n+1_{\xi_0}})) - g(s, x_n(s), \lambda_k(x_{n_{\xi_0}})) \right) dW(s) \right\|^2.
\end{aligned}$$

Taking the mathematical expectation and using the Lipschitz condition give

$$\begin{aligned}
\mathbb{E}[\|x_{n+2}(t) - x_{n+1}(t)\|^2] &\leq 2L_2(t - t_0 + 1)\mathbb{E} \int_{t_0}^{t} \Big(\|x_{n+1}(s) - x_n(s)\|^2 \\
&\qquad + \|\lambda_k(x_{n+1_{\xi_0}}) - \lambda_k(x_{n_{\xi_0}})\|^2 \Big) ds \\
&= M \int_{t_0}^{t} \mathbb{E}[\|x_{n+1}(s) - x_n(s)\|^2] ds \\
&\leq M \int_{t_0}^{t} \frac{C[M(s - t_0)]^n}{n!} ds = \frac{C[M(t - t_0)]^{n+1}}{(n + 1)!},
\end{aligned}$$

because $\lambda_k(x_{n+1_{\xi_0}}) - \lambda_k(x_{n_{\xi_0}}) = 0$ for any $n \geq 0$. For instance, for $n = 0$, we have

$$\lambda_k(x_{1_{\xi_0}}) - \lambda_k(x_{0_{\xi_0}}) = \lambda_k(x_1(t_0)) - \lambda_k(x_0(t_0)) = 0 \qquad \text{(a.s.)}.$$

This is because $x_0(t) = x_0$ for all t, and, by the solution sequence (15.4), we have $x_1(t_0) = x_0(t_0) = x_0$. Thus, the relation is true for $n + 1$.

To prove x_n is a Cauchy sequence, replace n by $n-1$ and consider

$$\begin{aligned}
&\sup_{t_0\le t\le t_1}\|x_{n+1}(t)-x_n(t)\|^2\\
&\qquad\le 2\sup_{t_0\le t\le t_1}\left\|\int_{t_0}^{t}[f(s,x_n(s),\lambda_k(x_{n_{\xi_0}}))-f(s,x_{n-1}(s),\lambda_k(x_{n-1_{\xi_0}}))]ds\right\|^2\\
&\qquad\quad+2\sup_{t_0\le t\le t_1}\left\|\int_{t_0}^{t}[g(s,x_n(s),\lambda_k(x_{n_{\xi_0}}))-g(s,x_{n-1}(s),\lambda_k(x_{n-1_{\xi_0}}))]dW(s)\right\|^2,
\end{aligned}$$

which implies, after taking the mathematical expectations and using the Doob's martingale inequality

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le t\le t_1}\|x_{n+1}(t)-x_n(t)\|^2\right)\\
&\le 2L_2(t_1-t_0+4)\int_{t_0}^{t_1}\mathbb{E}\left[\|x_n(s)-x_{n-1}(s)\|^2+\|\lambda_k(x_{n_{\xi_0}})-\lambda_k(x_{n-1_{\xi_0}})\|^2\right]ds\\
&=2L_2(t_1-t_0+4)\int_{t_0}^{t_1}\mathbb{E}\left[\|x_n(s)-x_{n-1}(s)\|^2\right]ds, \qquad (15.7)
\end{aligned}$$

because $\lambda_k(x_{n_{\xi_0}})-\lambda_k(x_{n-1_{\xi_0}})=0$ for any $n\ge 1$. For instance, for $n=1$, we have

$$\lambda_k(x_{1_{\xi_0}})-\lambda_k(x_{0_{\xi_0}})=\lambda_k(x_1(t_0))-\lambda_k(x_0(t_0))=0 \qquad \text{(a.s.)}.$$

This is because $x_0(t)=x_0$ for all t, and, by the solution sequence (15.4), we have $x_1(t_0)=x_0(t_0)=x_0$. Therefore, from (15.7), it follows that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{t_0\le t\le t_1}\|x_{n+1}(t)-x_n(t)\|^2\Big)&\le 4M\int_{t_0}^{t_1}\frac{4C[M(s-t_0)]^{n-1}}{(n-1)!}ds\\
&=\frac{4C[M(t_1-t_0)]^n}{n!},
\end{aligned}$$

from which, we get

$$\mathbb{P}\Big\{\sup_{t_0\le t\le t_1}\|x_{n+1}(t)-x_n(t)\|^2>\frac{1}{2^n}\Big\}\le\frac{4C[M(t_1-t_0)]^n}{n!}.$$

Since series $\sum_{n=0}^{\infty}\frac{4C[M(t_1-t_0)]^n}{n!}$ is convergent, by the Borel-Cantelli's lemma, we have

$$\sup_{t_0\le t\le t_1}\|x_{n+1}(t)-x_n(t)\|^2\le\frac{1}{2^n}.$$

It follows that, w.p.1, the partial sums

$$x_n(t) = x_0(t) + \sum_{j=0}^{n-1} (x_{j+1}(t) - x_j(t))$$

are convergent over $[t_0, t_1]$. Therefore, we conclude that sequence $x_n(t)$ is Cauchy, i.e., there exists a limit point x such that $\lim_{n\to\infty} x_n(t) = x(t)$, which implies that, for all $t \in [t_0, t_1)$,

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), \lambda_k(x_{\xi_0}))ds + \int_{t_0}^{t} g(s, x(s), \lambda_k(x_{\xi_0}))dW(s). \tag{15.8}$$

Similarly, one can show this relation holds for any $t \in [t_k, t_{k+1})$. We should mention that the inequality in (15.8) is still true for any k because by defining the general form of the solution sequence for any $t \in [t_k, t_{k+1})$, we have

$$\begin{aligned} x_n(t) = x_0(t_k) &+ \int_{t_k}^{t} f(s, x_{n-1}(s), \lambda_k(x_{n-1_{\xi_k}}))ds \\ &+ \int_{t_k}^{t} g(s, x_{n-1}(s), \lambda_k(x_{n-1_{\xi_k}}))dW(s), \end{aligned} \tag{15.9}$$

where $x_{j_{\xi_k}} = x_j(\xi_k) = x_j(t_k)$; for instance, if $n = 2$, we obtain

$$\begin{aligned} \lambda_k(x_{2_{\xi_k}})) - \lambda_k(x_{1_{\xi_k}})) &= \lambda_k(x_2(t_k)) - \lambda_k(x_1(t_k)) \\ &= \lambda_k(x_0(t_k)) - \lambda_k(x_0(t_k)) \\ &= 0, \end{aligned}$$

w.p.1. Due to the continuity of solution x, $\lim_{t\to t_{k+1}^-} x(t) = x(t_{k+1})$. Thus, the constructed solution is continuous and $\mathcal{F}_t$-adapted for all $t \geq t_0$. Furthermore, from (15.6), for all $t \geq t_0$, sequence $x_n(t)$ is Cauchy in L^2, which implies that $\lim_{n\to\infty} x_n(t) = x(t)$ in L^2. It follows that, by letting $n \to \infty$ in (15.5),

$$\mathbb{E}[\|x(t)\|^2] \leq C_2 e^{3L_1 t_1 (1+t_1)}, \quad \text{for all } t \geq t_0,$$

i.e., $x \in \mathcal{M}^2(\mathbb{R}_+; \mathbb{R}^n)$. Next, we will show that x satisfies the stochastic integral equation in (15.3), for all $t \in [t_k, t_{k+1}]$ and every k, as follows:

$$
\begin{aligned}
&\mathbb{E}\Big\| \int_{t_0}^{t} f(s, x_n(s), \lambda_k(x_{n_{\xi_k}}))ds - f(s, x(s), \lambda_k(x_{\xi_k}))ds \Big\|^2 \\
&\qquad + \mathbb{E}\Big\| \int_{t_0}^{t} g(s, x_n(s), \lambda_k(x_{n_{\xi_k}}))dW(s) - g(s, x(s), \lambda_k(x_{\xi_k}))dW(s) \Big\|^2 \\
&\quad \leq L_2(t_{k+1} - t_0 + 1) \int_{t_0}^{t_{k+1}} \mathbb{E}\|x_n(s) - x(s)\|^2 ds \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Therefore, by letting $n \to \infty$ in (15.4), we get the required result. Finally, to prove the uniqueness, assume that there is another solution, say $y(t)$. Then,

$$
\begin{aligned}
x(t) - y(t) = &\int_{t_0}^{t} \big(f(s, x(s), \lambda_k(x_{\xi_k})) - f(s, y(s), \lambda_k(y_{\xi_k}))\big)ds \\
&+ \int_{t_0}^{t} \big(f(s, x(s), \lambda_k(x_{\xi_k})) - f(s, y(s), \lambda_k(y_{\xi_k}))\big)dW(s),
\end{aligned}
$$

which implies that, after applying the Hölder's inequality, Doob's martingale inequality, and Lipschitz condition,

$$
\mathbb{E}\Big[\sup_{t_0 \leq s \leq t} \|x(s) - y(s)\|^2\Big] \leq 2L_2(t_{k+1} + 4) \int_{t_0}^{t} \mathbb{E}[\sup_{t_0 \leq u \leq s} \|x(u) - y(u)\|^2] ds.
$$

By the Gronwall inequality, we obatin

$$
\mathbb{E}\Big[\sup_{t_0 \leq s \leq t} \|x(s) - y(s)\|^2\Big] = 0.
$$

Thus, processes x and y are indistinguishable for all t. Hence, system (15.1) has a unique solution $x(t)$ for all $t \geq t_0$. This completes the proof.

15.2 Stability Results

Having established the existence of a unique solution, this section deals with the stability properties of the trivial solution of system (15.1). Our results are based on developing Lyapunov-like sufficient conditions by using a comparison principle (Subsection 15.2.1) and Razumikhin technique (Subsection 15.2.2).

15.2.1 Analysis by Comparison Principle

In this subsection, as achieved in the last chapter, we develop a comparison principle that will be used later to prove some stability results.

Theorem 15.2. Assume that the following assumptions hold:

(i) for $k \in K$, $V \in \mathcal{C}^{1,2}([t_k, t_{k+1}) \times \mathbb{R}^n; \mathbb{R}_+)$, V is bounded below, and satisfies

$$\mathcal{L}V(t, x, y) \leq h(t, x, \sigma_k(y)), \qquad \text{(a.s.)}, \qquad t \in [t_k, t_{k+1}),$$

where the function h is concave and nondecreasing in x and σ_k with σ_k being a concave function;

(ii) the scalar comparison system

$$\begin{aligned} \dot{u}(t) &= h(t, u(t), \sigma_k(u_{\xi_k})), \qquad t \in [t_k, t_{k+1}), \\ u(t_0) &= u_0 \end{aligned} \tag{15.10}$$

has a maximal solution $\nu(t, t_0, u_0)$ for all $t \geq t_0$.

Then, for any solution x of (15.1), $\mathbb{E}[V(t_0, x_0)] \leq u_0$ implies $\mathbb{E}[V(t, x)] \leq \nu(t, t_0, u_0)$ for any $t \geq t_0$.

Proof. For any $k \in K$ and $t \in [t_k, t_{k+1})$, let $x(t)$ be the solution of system (15.1) that is guaranteed by Theorem 15.1. Let τ_{k_l}, or, for simplicity τ_l, for $l \geq 1$, be the first exit time of the process from the ball

$$B_l(x) = \{x \in \mathbb{R}^n : \|x\| \leq l\},$$

i.e., $\tau_l = \inf\{t \in [t_k, t_{k+1}) : \|x(t)\| > l\}$.

Define $\tau_l(t) = \min\{\tau_l, t\}$. Then, by Itô formula, we have, for any $t \in [t_k, \tau_l(t)]$,

$$\begin{aligned}\mathbb{E}[V(\tau_l(t), x(\tau_l(t)))] &= \mathbb{E}[V(t_k, x(t_k))] + \mathbb{E}\int_{t_k}^{\tau_l(t)} \mathcal{L}V(s, x(s), \sigma_k(V_{\xi_k}))ds \\ &\leq \mathbb{E}[V(t_k, x(t_k))] + \mathbb{E}\int_{t_k}^{\tau_l(t)} h(s, V(s, x(s)), \sigma_k(V_{\xi_k}))ds,\end{aligned}$$

where $V_{\xi_k} = V(\xi_k, x(\xi_k))$. Define $m(t) = \mathbb{E}[V(s, x(s))]$ for all $t_k \leq s \leq \tau_l(t)$. Thus, by the properties of h and σ_k, the last inequality becomes

$$m(t) \leq m(t_k) + \int_{t_k}^{s} h(r, m(r), \sigma_k(m_{\xi_k}))dr, \qquad t_k \leq r \leq s \leq \tau_l(t),$$

where $m_{\xi_k} = m(\xi_k) = \mathbb{E}[V(\xi_k, x(\xi_k))]$.

By Theorem 14.1, we obatin

$$m(t) \leq \nu_k(\,t, t_k, m_{\xi_k}), \qquad t \in [t_k, \tau_l(t)],$$

and, by letting $l \to \infty$, we obtain, for all $t \in [t_k, t_{k+1})$, $m(t) \leq \nu_k(\,t, t_k, m_{\xi_k})$. Particularly, for $t \in [t_0, t_1)$, we have

$$m(t) \leq \nu_0(\,t, t_0, m_{\xi_0}) = \nu_0(\,t, t_0, m(t_0)) \leq \nu_0(\,t, t_0, u_0) =: \nu(t, t_0, u_0),$$

where $\nu_0(\cdot, \cdot, \cdot)$ is the maximal solution of the scalar comparison system (15.10) for $t \in [t_0, t_1)$ with $m(t_0) = \mathbb{E}[V(t_0, x(t_0))] \leq u_0$, as given initially.

For $t \in [t_1, t_2)$, we have

$$\begin{aligned}m(t) &\leq \nu_1(\,t, t_1, m_{\xi_1}) = \nu_1(\,t, t_1, m(t_1)) = \nu_1(\,t, t_1, \nu_1(\,t_1, t_0, u_0)) \\ &=: \nu(t, t_0, u_0),\end{aligned}$$

or

$$m(t) \leq \nu(t, t_0, u_0), \qquad t \in [t_0, t_2).$$

In general, one obtains

$$m(t) = \mathbb{E}[V(t, x(t))] \leq \nu(t, t_0, u_0), \qquad t \geq t_0,$$

where $\nu(t, t_0, u_0)$ is the maximal solution of the comparison system (15.10) for all $t \geq t_0$. This completes the proof.

In the following theorem, we prove some stability properties of the trivial solution of (15.1).

Theorem 15.3. Assume that the conditions of Theorem 15.2 hold. Suppose also that there exist two functions $b \in \mathcal{K}_1$ and $a \in \mathcal{K}_2$ such that

$$b(\|x\|^2) \leq V(t, x) \leq a(\|x\|^2), \qquad \text{(a.s.)}. \tag{15.11}$$

Then, the stability properties of the trivial solution $u \equiv 0$ of system (15.10) imply the stability properties (in the m.s.) of the trivial solution $x \equiv 0$ of system (15.1).

Proof. Assume that the trivial solution $u \equiv 0$ of the comparison system (15.10) is stable. Then, for every $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ for which

$$\nu(t, t_0, u_0) < b(\varepsilon), \qquad \text{whenever} \qquad u_0 \leq \delta, \qquad \forall\, t \geq t_0 \geq 0, \tag{15.12}$$

where $\nu(t, t_0, u_0)$ is the maximal solution of the scalar comparison system (15.10).

To investigate the stability at t_0, we choose $\delta = \delta(t_0, \varepsilon) \leq \delta_1$ (for the same ε) with $a(\delta_1) < b(\varepsilon)$ and let $u_0 = a(\mathbb{E}[\|x_0\|^2]) \leq \delta_1$. Now, let $\mathbb{E}[\|x_0\|^2] \leq \delta$. Then, from (15.11), we obtain

$$b(\mathbb{E}[\|x(t_0)\|^2]) \leq \mathbb{E}[V(t_0, x_0)] \leq a(\mathbb{E}[\|x_0\|^2]) \leq a(\delta) \leq b(\varepsilon),$$

i.e., $\mathbb{E}[\|x_0\|^2] \leq \varepsilon$, whenever $\mathbb{E}[\|x_0\|^2] \leq \delta$.

Under the given assumptions, we claim that the trivial solution $x \equiv 0$ of SEPCA (15.1) is stable in the m.s. for all $t > t_0$, i.e., for the assigned ε and δ, the following statement

$$\mathbb{E}[\|x_0\|^2] \le \delta \qquad \text{implies} \qquad \mathbb{E}[\|x(t)\|^2] < \varepsilon, \qquad \forall\, t > t_0$$

holds. If our claim were not true, there would be a $t^* > t_k > t_0$, specifically $t_k < t^* \le t_{k+1}$, such that $\mathbb{E}[\|x_0\|^2] \le \delta$ and

$$\mathbb{E}[\|x(t)\|^2] < \varepsilon, \qquad t_k \le t < t^*, \tag{15.13}$$

$$\mathbb{E}[\|x(t^*)\|^2] = \varepsilon. \tag{15.14}$$

Recall that, by Theorem 15.2, we have shown $\mathbb{E}[V(t, x(t))] \le \nu(t, t_0, u_0)$ for all $t \ge t_0$. This, together with (15.12), implies

$$\mathbb{E}[V(t^*, x(t^*))] \le \nu(t^*, t_0, u_0) = \nu(t^*, t_0, a(\mathbb{E}[\|x_0\|^p])) < b(\varepsilon).$$

We also have, by (15.11) and (15.14),

$$b(\varepsilon) = b(\mathbb{E}[\|x(t^*)\|^2]) \le \mathbb{E}[V(t^*, x(t^*))].$$

Combining the last two inequality results in a contradiction. Therefore, our claim must be true, i.e., the trivial solution $x \equiv 0$ is stable in the m.s. for all $t \ge t_0$. As for the uniformity property, it suffices to choose δ independently of t_0.

To prove the m.s. asymptotic stability property of $x \equiv 0$, we need only to establish attractivity of this solution. Assume that $u \equiv 0$ is asymptotic stable, which implies the existence of $\delta_2 = \delta(t_0)$ and $T = T(t_0, \varepsilon) > 0$, for any given ε, such that

$$u_0 \le \delta_2 \qquad \text{implies} \qquad \nu(t, t_0, u_0) < b(\varepsilon), \qquad \forall\, t \ge t_0 + T.$$

Following the same argument of the first part, we choose $u_0 = a(\mathbb{E}[\|x_0\|^2]) \leq \delta_2$ and $\delta_3 < \delta_2$ such that $\mathbb{E}[\|x_0\|^2] \leq \delta_3$. Then,

$$b(\mathbb{E}[\|x(t)\|^2]) \leq \mathbb{E}[V(t, x(t))] \leq \nu(t, t_0, a(\mathbb{E}[\|x_0\|^2])) \leq b(\varepsilon),$$

i.e., $\mathbb{E}[\|x(t)\|^2] \leq \varepsilon$ for all $t \geq t_0 + T$. We have proved that $x \equiv 0$ is asymptotic stability in the m.s. Furthermore, choosing $T = T(\varepsilon)$ leads to the uniformity property.

In the following, we illustrate our theoretical result through a numerical example with simulation.

Example 15.1. Consider the following SEPCA

$$\begin{aligned} dx &= \big(-x[\lambda + \theta(x^2 + y^2) + \beta_k x_{\xi_k}]\big)dt + axdW_1, \\ dy &= bydt - x^2 dW_1 + \gamma_{\xi_k} y_{\xi_k} e^{-x^2} dW_2. \end{aligned} \tag{15.15}$$

Taking $V(x, y) = \frac{1}{2}(x^2 + y^2)$ as a Lyapunov function candidate implies

$$\begin{aligned} \mathcal{L}V &\leq -(\lambda + \frac{\beta_k^2}{2} + \frac{a^2}{2})x^2 + by^2 + \frac{\beta_k^2}{2} x_{\xi_k}^2 + \frac{\gamma_k^2}{2} y_{\xi_k}^2 \\ &\leq \frac{\theta^*}{2}(x^2 + y^2) + \frac{1}{2}\xi_k (x_{\zeta_k}^2 + y_{\xi_k}^2) \\ &= \theta^* V(x, y) + \zeta_k V_{\xi_k}, \end{aligned}$$

where $\theta^* = 2\min\{-(\lambda + \frac{\beta_k^2}{2} + \frac{a^2}{2}), b\} < 0$ and $\zeta_k = \max\{\beta_k^2, \gamma_k^2\} > 0$. Choose $\lambda = 2$, $\theta = 1$, $a = 1$, $b = -1$, $\beta_k = \gamma_k = 1/2^k$, and $a = b = V = \frac{1}{2}\|(x, y)\|^2$. Clearly, the trivial solution of the comparison system is asymptotically stable. This conclusion can be checked with Corollary 14.5, where $w(s) = s > 0$, $\beta_k = \zeta_k$, and $t_k - t_{k-1} = 1$ for any k. We deduce that $(x, y)^T = (0, 0)$ is asymptotically stable in the m.s. Figures 15.1 and 15.2 show the simulation results of the mean and m.s. of the solution.

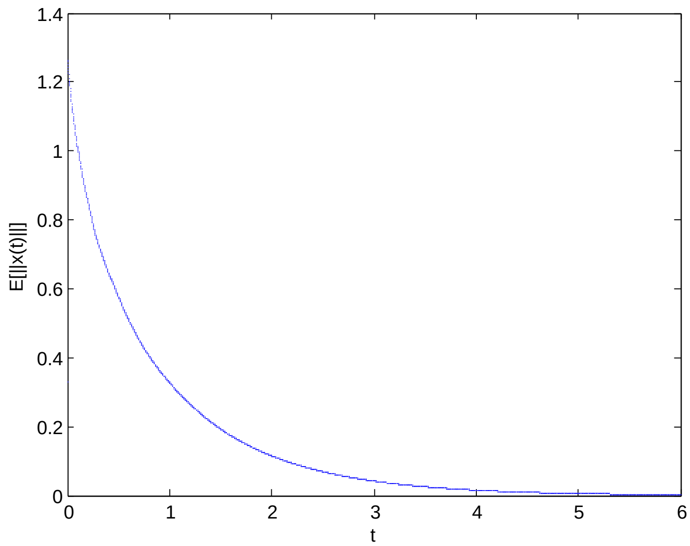

Figure 15.1: First moment asymptotic stability of $(x, y)^T = (0, 0)$.

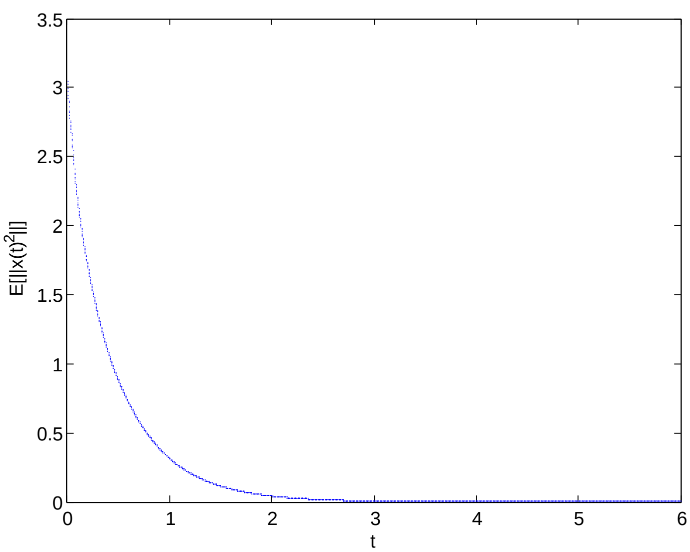

Figure 15.2: Mean square asymptotic stability of $(x, y)^T = (0, 0)$.

15.2.2 Analysis by Razumikhin Technique

In this subsection, we continue to investigate some stability properties of the trivial solution of (15.1), where we use Razumikhin method to state Lyapunov-like theorems. Before stating the stability conditions, the following lemma is needed.

Lemma 15.2. Assume that the conditions that guarantee the existence of a unique solution of system (15.1) with $\lambda_k(s) = s$, for all s, and $\sigma(t) = t_k$, for all $t \in [t_k, t_{k+1})$, hold. Then, for any $t \geq t_0$,

$$\mathbb{E}[\|x(\beta(t))\|^2] \leq K(t^*, L_2)\mathbb{E}[\|x(t)\|^2],$$

where $K(t^*, L_2) = \frac{3}{1-3t^*(t^*+1)L_2-t^*L_2C_1} > 1$ with t^* being such that $t_{k+1} - t_k < t^*$.

Proof. For a fixed k and any $t \in [t_k, t_{k+1})$, consider the stochastic integral equation

$$x(t) = x(t_k) + \int_{t_k}^{t} f(s, x(s), x(t_k))ds + \int_{t_k}^{t} g(s, x(s), x(t_k))dW(s),$$

which implies

$$\begin{aligned}
\mathbb{E}[\|x(t)\|^2] = 3\mathbb{E}[\|x(t_k)\|^2] + 3t^* &\int_{t_k}^{t} \mathbb{E}[\|f(s, x(s), x(t_k))\|^2]ds \\
&+3\int_{t_k}^{t} \mathbb{E}[\|g(s, x(s), x(t_k))\|^2]ds \\
&\leq 3\mathbb{E}[\|x(t_k)\|^2] + 3t^* \int_{t_k}^{t} \mathbb{E}[L_2(\|x(s)\|^2 + \|x(t_k)\|^2)]ds \\
&+3\int_{t_k}^{t} \mathbb{E}[L_2(\|x(s)\|^2 + \|x(t_k)\|^2)]ds \\
&\leq 3\mathbb{E}[\|x(t_k)\|^2] + 3(t^*+1)\left(L_2t^*\mathbb{E}[\|x(t_k)\|^2] + L_2\int_{t_k}^{t} \mathbb{E}[\|x(s)\|^2]ds\right),
\end{aligned}$$

i.e.,

$$\mathbb{E}[\|x(t)\|^2] = 3\left[1 + t^*(t^*+1)L_2\right]\mathbb{E}[\|x(t_k)\|^2] + 3L_2(t^*+1)\int_{t_k}^{t} \mathbb{E}[\|x(s)\|^2]ds.$$

By the Gronwall inequality, we get

$$\mathbb{E}[\|x(t)\|^2] \leq C_1 \mathbb{E}[\|x(t_k)\|^2],$$

where $C_1 = 3\left[1 + t^*(t^*+1)L_2\right] e^{3L_2(t^*+1)}$.

On the other hand, one gets

$$\begin{aligned}
\mathbb{E}[\|x(t_k)\|^2] &\leq 3\mathbb{E}[\|x(t)\|^2] + 3t^* \int_{t_k}^{t} \mathbb{E}[\|f(s, x(s), x(t_k))\|^2] ds \\
&\quad + 3\int_{t_k}^{t} \mathbb{E}[\|g(s, x(s), x(t_k))\|^2] ds \\
&\leq 3\mathbb{E}[\|x(t)\|^2] + 3(t^*+1)\left[t^* L_2 \mathbb{E}[\|x(t_k)\|^2] + L_2 \int_{t_k}^{t} \mathbb{E}[\|x(s)\|^2] ds\right] \\
&\leq 3\mathbb{E}[\|x(t)\|^2] + 3t^*(t^*+1)L_2\mathbb{E}[\|x(t_k)\|^2] + L_2 \int_{t_k}^{t} C_1 \mathbb{E}[\|x(t_k)\|^2] ds \\
&= 3\mathbb{E}[\|x(t)\|^2] + 3t^*(t^*+1)L_2\mathbb{E}[\|x(t_k)\|^2] + t^* L_2 C_1 \mathbb{E}[\|x(t_k)\|^2] \\
&= 3\mathbb{E}[\|x(t)\|^2] + \left(3t^*(t^*+1)L_2 + t^* L_2 C_1\right) \mathbb{E}[\|x(t_k)\|^2],
\end{aligned}$$

from which we get

$$\mathbb{E}[\|x(t_k)\|^2] \leq K(t^*, L_2)\mathbb{E}[\|x(t)\|^2],$$

where $K(t^*, L_2) = 3/\left[1 - 3t^*(t^*+1)L_2 - t^* L_2 C_1\right] > 1$.

Theorem 15.4. Consider the SEPCA in (15.1) with $\lambda_k(s) = s$, for all s, and $\sigma(t) = t_k$, for all $t \in [t_k, t_{k+1})$. Assume that there exist $b \in \mathcal{K}_1$ and $a \in \mathcal{K}_2$. Let $V \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times S(\varrho); \mathbb{R}_+)$ such that the following conditions holds:

(i) $b(\|x\|^2) \leq V(t, x) \leq a(\|x\|^2)$, (a.s.), $\forall (t, x) \in \mathbb{R}_+ \times S(\varrho)$;

(ii) for any $t \in (t_k, t_{k+1})$ and $x, y \in S(\varrho)$,

$$\mathcal{L}V(t, x, y) \leq 0, \qquad \text{(a.s.)}, \quad \text{whenever} \quad V(\beta(t), y) \leq V(t, x).$$

Then, the trivial solution $x \equiv 0$ of (15.1) is uniformly stable in the m.s.

Proof. Let x be a solution of SEPCA (15.1), and $V \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times S(\varrho); \mathbb{R}_+)$, $t_0 = t_k$, for any k. For a given $0 < \varepsilon < \rho$, choose $\delta > 0$ such that $a(\delta) < b(\varepsilon)$ and

$$\mathbb{E}[V(t_0, x(t_0))] < a(\delta) < b(\varepsilon), \qquad \text{whenever} \qquad \mathbb{E}[\|x_0\|^p] \leq \delta,$$

where $x_0 = x(t_0)$.

By (ii), one can define $m(t) = \mathbb{E}[V(t, x(t))]$ for any t. We claim that for any $t \geq t_0$, $m(t) \leq m(t_0)$. If our claim were not true, there would exist $\underline{t}, \bar{t} \in [t_0, \tau_l]$ such that $t_0 \leq \underline{t} < \bar{t} < \tau_l$ and

$$m(\underline{t}) = m(t_0),$$

and

$$m(t) > m(t_0), \qquad t \in (\underline{t}, \bar{t}]. \tag{15.16}$$

By the Mean-Value Theorem, there exists $t^* \in (\underline{t}, \bar{t})$ such that

$$\dot{m}(t^*) = \frac{m(\bar{t}) - m(\underline{t})}{\bar{t} - \underline{t}} > 0, \tag{15.17}$$

by (15.16), which implies

$$m(t^*) > m(t_0).$$

On the other hand, by (ii), $\dot{m}(t^*) \leq 0$, where $t^* < \tau_l$, which contradicts with $\dot{m}$ being positive as shown in (15.17). Therefore, it must be true that

$$m(t) \leq m(t_0), \qquad \forall t \in [t_0, \tau_l],$$

and, by letting $l \to \infty$, the last inequality holds for $t \in [t_k, t_{k+1})$.

Therefore, we have shown that

$$m(t) \leq m(t_0) = \mathbb{E}[V(t_0, x(t_0))] < b(\varepsilon).$$

On the other hand, by (i), we have

$$b(\mathbb{E}[\|x(t)\|^2]) \leq m(t) = \mathbb{E}[V(t, x(t))] < b(\varepsilon),$$

which implies that

$$\mathbb{E}[\|x(t)\|^2] < \varepsilon, \qquad \text{whenever} \qquad \mathbb{E}[\|x_0\|^2] \leq \delta,$$

which shows that, for any k and $t_0 = t_k$, the trivial solution is uniformly stable in the m.s.

To complete the proof, we show that the trivial solution is uniformly stable for any t_0 differs from t_k. For the same choice of ε, δ, and $\delta = \delta_1/K(t^*, L_2)$, we choose the solution $x(t)$ of SEPCA to satisfy $\mathbb{E}[\|x(t_0)\|^2] < \delta$, which, by Lemma 15.2, implies $\mathbb{E}[\|x(t_k)\|^2] < K(t^*, L_2)\mathbb{E}[\|x(t)\|^2] < \delta_1$ (where $t = t_0$ and by our choice of δ). Thus, if $\mathbb{E}[\|x(t_k)\|^2] < \delta_1$, $m(t_k) < b(\varepsilon)$ yields the required results by the earlier discussion of the case $t_0 = t_k$. The proof is complete.

Theorem 15.5. Suppose that the assumptions in Theorem 15.4 hold except that the condition in assumption (ii) is replaced by
(ii)$'$ for any $t \in (t_k, t_{k+1})$ and $x, y \in S(\varrho)$,

$$\mathcal{L}V(t, x, y) \leq -w(\|x(t)\|^2), \qquad \text{(a.s.)},$$

whenever $V(\beta(t), y) \leq V(t, x)$, where w is a class-$\mathcal{K}_1$ function.

Assume further that there exists a continuous nondecreasing convex function ψ, for which $\psi(s) > s$ for all $s > 0$. Then, the trivial solution $x \equiv 0$ is uniformly asymptotically stable in the m.s.

Proof. Let $x(t) = x(t, t_0, x_0)$ be the solution of (15.1). Since $\mathcal{L}V(t, x, y) < 0$, by Theorem 15.4, the trivial solution is uniformly stable in the m.s.

We need to prove that $x \equiv 0$ is attractive. For a fixed k, let $t_0 = t_k$, and, for a given $0 < \varepsilon < \varrho_1 < \varrho$, choose $\delta > 0$ such that $a(K(t^*, L_2)\delta) = b(\varrho_1)$. This implies that, if $\mathbb{E}[\|x(t_k)\|^2] < \delta$, $\mathbb{E}[\|x(t)\|^2] < \varrho_1$ because $b(\mathbb{E}[\|x(t)\|^2]) \leq m(t)$, and by Theorem 15.4, $m(t) \leq a(\delta) < a(K(t^*, L_2)\delta)$ for all $t \geq t_k$. Next, we prove the existence of a $T = T(\varepsilon)$ for which, if $\mathbb{E}[\|x(t_k)\|^2] < \delta$, then $\mathbb{E}[\|x(t)\|^2] < \varepsilon$ for all $t \geq t_k + T$.

Define $\gamma = \inf\{w(s) : a^{-1}(b(\varepsilon)) \leq s \leq \varrho_1\}$ and $\delta_1 = K(t^*, L_2)\delta$. Then, by the properties of ψ, there exists an $a > 0$ such that $\psi(s) - s > a$, for all $b(\varepsilon) < s < a(\delta)$.

Let N be the smallest positive number for which $b(\varepsilon) + Na \geq a(\delta_1)$.

Choose

$$r_k = k(\frac{a(\delta_1)}{\gamma} + t^*) + t_k, \qquad k = 1, 2, \cdots, N.$$

We will prove (by mathematical induction) that

$$m(t) \leq b(\varepsilon) + (N - k)a, \qquad k = 0, 1, \cdots, N.$$

Clearly, for $k = 0$, we have

$$m(t) \leq a(\delta_1) \leq b(\varepsilon) + Na,$$

i.e., the inequality is correct for $k = 0$. Assume that it is correct for some k. Now we want to prove the validity of the relation for the case $k + 1$, i.e.,

$$m(t) \leq b(\varepsilon) + (N - k)a, \qquad t \geq r_{k+1}.$$

Set $I_k = [\beta(r_k) + t^*, r_{k+1}]$. We claim that there is some $t' \in I_k$ such that

$$m(t') \leq b(\varepsilon) + (N - (k + 1))a.$$

If not, we would have

$$m(t) > b(\varepsilon) + (N-k-1)a, \qquad \forall\, t \in I_k.$$

On the other hand, since we assumed that the relation is correct for the k case, i.e.,

$$m(t) \leq b(\varepsilon) + (N-k)a, \qquad t \geq r_k,$$

we have

$$m(\beta(r_k)) \leq m(t) \leq b(\varepsilon) + (N-k)a, \qquad \forall\, t \geq \beta(r_k) + t^*.$$

From the properties of the ψ function, we have

$$\psi(m(t)) \geq m(t) + a > b(\varepsilon) + (N-k)a \geq m(\beta(t)).$$

Since $a^{-1}(b(\varepsilon)) \leq \mathbb{E}[\|x(t)\|^2] \leq \rho_1$ for all $t \in I_k$, it follows

$$D^+ m(t) \leq -w(\mathbb{E}[\|x(t)\|^2]) \leq -\gamma,$$

which implies that

$$\begin{aligned} m(r_{k+1}) &\leq m(\beta(r_k) + t^*) - \gamma(r_{k+1} - \beta(r_k) - t^*) \\ &< a(\delta_1) - \gamma(r_{k+1} - \beta(r_k) - t^*) = 0, \end{aligned} \tag{15.18}$$

which is a contradiction. Thus, it must be true that

$$m(t') \leq b(\varepsilon) + (N-(k+1))a.$$

We want to prove that

$$m(t) \leq b(\varepsilon) + (N-(k+1))a, \qquad \forall\, t \in [t', \infty).$$

We claim it is true. If not, there would be a $t'' \in (t', \infty)$ such that

$$m(t'') > b(\varepsilon) + (N - (k+1))a > m(t'),$$

which means that we can find a $\bar{t} \in (t', t'')$ such that $\bar{t} \neq t_k$ and

$$D^+ m(\bar{t}) > 0,$$

and

$$m(\bar{t}) > b(\varepsilon) + (N - (k+1))a.$$

If there is no such $\bar{t}$, then, for all $t \in (t', t'')$ with $t \neq t_k$,

$$D^+ m(t) \leq 0,$$

or

$$m(\bar{t}) \leq b(\varepsilon) + (N - (k+1))a. \tag{15.19}$$

Now, if $D^+ m(t) \leq 0$, it follows that $m(t'') \leq m(t')$, which is a contradiction. If $m(\bar{t}) \leq b(\varepsilon) + (N - (k+1))a$, it follows that $m(t) \leq m(t'')$, for all (t', t'') with $t \neq t_k$, which is a contradiction. Thus, there must exist $\bar{t}$ satisfying (15.19).

From the properties of ψ, we have

$$\psi(m(\bar{t})) > m(\bar{t}) > b(\varepsilon) + (N - k)a \geq m(\beta(\bar{t})),$$

which implies that $D^+ m(\bar{t}) \leq -\gamma < 0$, which is a contradiction. Thus, it must be true that

$$m(t) \leq b(\varepsilon) + (n - k - 1)a, \qquad \forall\, t \geq r_{k+1}.$$

Particularly, for $k = N$, we have

$$m(t) \leq b(\varepsilon), \qquad \forall\, t \geq r_N = N(\frac{a(\delta_1)}{\gamma} + t^*) + t_0,$$

from which we get

$$\mathbb{E}[\|x(t)\|^2] < \varepsilon,$$

for all $t \geq t_k + T$ where $T = N(\frac{a(\delta_1)}{\gamma} + t^*)$. This completes the proof of uniform asymptotic stability in the m.s. of $t_0 = t_k$ for a fixed k. As for the case $t_0 \neq t_k$, one can adopt the analysis of Theorem 15.4 to achieve the required result.

Example 15.2. Consider the SEPCA

$$\begin{aligned} dx &= axdt + (y + \beta_k x_{\xi_k})dW_1, \\ dy &= (by + \gamma_k y_{\xi_k})dt + \ln|1 + x|dW_1 - y^2 dW_2. \end{aligned}$$

Taking $V(x, y) = \frac{1}{2}(x^2 + y^2)$ as a Lyapunov function candidate yields

$$\begin{aligned} \mathcal{L}V\big((x, y), (x_{\xi_k}, y_{\xi_k})\big) &\leq (a + 1)x^2 + (b + \frac{3}{2})y^2 + \frac{1}{2}(x_{\xi_k}^2 + y_{\xi_k}^2) \\ &\leq \theta^* V(x, y) + V(x_{\xi_k}, y_{\xi_k}) \\ &\leq (\theta^* + q)V(x, y), \end{aligned}$$

where $\theta^* = 2\min\{a + 1, b + \frac{3}{2}\} < 0$, $q > 1$ such that $\theta^* + q < 0$. Choosing $a = -2$, $\beta_k = \frac{1}{2^k}$, $b = -7$, and $\gamma_k = \frac{1}{3^k}$ results in $\theta^* < 0$ and $\theta^* + q < 0$ for $q = 2 > 1$. Let $a(\|(x, y)\|^2) = b(\|(x, y)\|^2) = V(x, y) = \frac{1}{2}\|(x, y)\|^2$ and $\psi(s) = qs$. Then, by Theorem 15.5, the trivial solution is uniformly asymptotically stable in the m.s. Figures 15.3 and 15.4 show the simulation results of the mean and m.s. of the solution.

15.3 Conclusion

In this chapter, we considered systems with SEPCA, which were treated as a hybrid (or switched) system. The focus was on establishing some existence and uniqueness

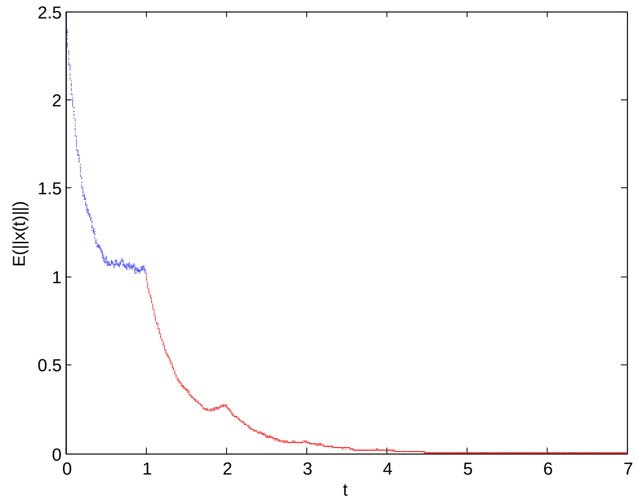

Figure 15.3: First moment asymptotic stability of $(x, y)^T = (0, 0)$.

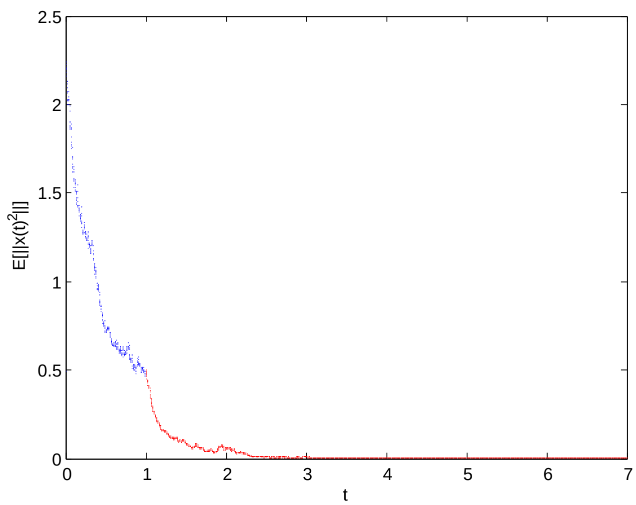

Figure 15.4: Mean square asymptotic stability of $(x, y)^T = (0, 0)$.

results. Then, we investigated some stability properties. As for the existence result, we assumed that the vector fields were bounded above by some linear growth estimation. Therefore, we can modified this result by considering the (nonlinear) sufficient condition adopted in the early existence result for the SISD. The second part of this chapter dealt with developing stability results, where we used comparison principle and Razumikhin techniques to obtain some sufficient conditions to guarantee the stability properties in the m.s. We should mention that the results of Section 15.3 are a modification of the same technique used in [Akh09] to prove the stability properties of deterministic EPCA.

Chapter 16

Conclusions and Future Research

Hybrid systems, including impulsive, switched, impulsive switched systems, are adequate as a tool to model many physical processes subject to abrupt changes (or impulses) in their states, mode switching, a mix of the two aspects, or switching in a state argument at a certain moment. They become even more useful if time delays are considered in their evolutionary behaviour. Moreover, to have a more realistic description of a physical process, some environmentally influencing random factors (or noises) must be taken into account. When random effects are considered in a hybrid system with deviating states, we have a stochastic hybrid system with time delay. The main objective of this thesis is to enrich the research area of stochastic hybrid systems with or without delayed states.

In this chapter, we highlight the contributions of this thesis and suggest some future research problems that are related to hybrid systems with or without time lags and stochastic noise.

In Chapter 3, we established the essence of the theory of stochastic impulsive systems with time delay (SISD), i.e., the existence of unique forward continuable solution. To have a better insight into the system, one may extend the the theoretical foundation by addressing the problems of continuous dependence of the solution on the initial data, and maximal and minimal solutions of the system.

To further study the system, m.s. stability and input-to-state stability (ISS) properties were developed in Chapters 4 and 7, where a partial result of Chapter 7 depends on the ISS property proposed in Chapter 6. In analyzing these qualitative notions, we employed Razumikhin methodology, which required defining a suitable Lyapunov function. These results have also been applied to tackle the same qualitative properties of large scale SISD. To justify these theoretical results, we applied them to some control systems with faulty actuators and systems describing the longitudinal motion of an aircraft. This approach can be applied to some large electric networks, electric power systems, or neural systems in biology. Also, the proposed theoretical results can be adopted to tackle problem of output regulation (or servomechanism) for stochastic hybrid systems. This problem deals with designing a feedback controller to achieve asymptotic tracking (or disturbance rejection) for a class of reference inputs (or disturbances in uncertain systems) and to maintain closed-loop stability.

In Part II of the thesis, we considered switched systems. In Chapter 11, we discussed deterministic switched systems with impulsive effects. The main interest was to design a dwell-time switching signal to establish some stability results using multiple Lyapunov function technique. This result can be further generalized to include some perturbation of stochastic structure. In Chapters 12 and 13, we focused on stochastic switched systems with and without time delay. The main contribution of these chapters was to develop a new switching law called initial-state-dependent dwell time in order to investigate some ISS properties of the systems (in a probabilistic sense). Therefore, one may extend the dependence of the switching law on the system states, but not only on the initial states. In Chapter 12, we used the Markov process as a switching law to jump among the system modes to obtain m.s. exponential stability of the system.

Part III was devoted to broaden the applicability of the theory of switched systems to deterministic and stochastic differential equations with piecewise constants arguments (EPCA). As stated earlier, the delay type of EPCA can be used to find approximate solutions of delay differential equations with discrete delays. Therefore, due to the difficulties in evaluating analytical solutions, it is worthwhile to conduct research on applying the proposed approach to obtain the same finding.

On the other hand, throughout this thesis, we remarked that the random noise is approximated by a Wiener process. In fact, in practice, there are some types of noises described, for instance, by a Poisson process. Consequently, one may consider other processes and address the mentioned problems. Furthermore, it is known that the Wiener process, as an integrator, belongs to a class of martingale processes, which is a subclass of semi-martingale processes. It is of practical and theoretical importance to consider these processes (or integrators) in hybrid systems and investigate many features of the systems.

Moreover, when dealing with hybrid systems of either type, as presented here, the interest was to apply the theory of ordinary differential equations. In fact, there are other, but complicated and interesting at the same time, approaches to handle switched or impulsive systems. For example, in the first case, the finite set of *differential equations* $\dot{x} = f_i(t, x)$, for some i, is replaced by a single *differential inclusion* $\dot{x} \in F(t, x)$. While in an impulsive system, the *differential and difference equations* are alternatively represented by a *measure differential inclusion* $dx \in \mathcal{F}(t, x)dt$. In both cases, F and $\mathcal{F}$ are set-valued mappings, and not single points. These two approaches require a rigorous background, which is beyond the scope of this thesis, since most of the classical theory cannot be applied to these types of systems.

Bibliography

[Ack85] J. Ackremann. *Sampled-Data Control Systems.* Springer-Verlag, New York, 1985.

[Aft85] A.R. Aftabizadeh and J. Wiener. Oscillatory properties of first order linear functional differential equations. *Applicable Analysis*, **20**, pp.165-187, 1985.

[Akh08b] M.U. Akhmet. Stability of differential equations with piecewise constant arguments of generalized type. *Nonlinear Analysis*, **68**, pp. 794-803, 2008.

[Akh08c] M.U. Akhmet and C. Büyükadali. On periodic solutions of differential equations with piecewise constant argument. *Computers and Mathematics with Applications*, **56**, pp. 2034-2042, 2008.

[Akh09] M.U. Akhmet and D. Aruğaslan. Lyapunov-Razumikhin method for differential equations with piecewise constant argument. *Discrete and Continuous Dynamical Systems*, **25**(2), pp. 457-466, 2009.

[Alw06] M.S. Alwan. *Stability of Hybrid Singularly Perturbed Systems with Time Delay*, Master's Thesis, University of Waterloo, Ontario, Canada, 2006.

[Alw08a] M.S. Alwan and X.Z. Liu. On stability of linear and weakly nonlinear switched systems with time delay. *Mathematical and Computer Modelling*, **48** pp. 1150-1157, 2008.

[Alw08b] M.S. Alwan, X.Z. Liu, and B. Ingalls. Exponential stability of singularly switched systems with time delay. *Nonlinear Analysis: Hybrid Systems*, **2**, pp. 913-921, 2008.

[Alw09a] M.S. Alwan, X.Z. Liu, and W.-C. Xie. *Advances in Nonlinear Analysis: Theory Methods and Applications.* (Editors) S. Sivasundaram, Devi J. Vasundhara, Z. Drici, and F. McRae. Chapter 21: Robust stability and stabilization of impulsive hybrid systems with time delay, pp. 135-251, 2009.

[Alw09b] M.S. Alwan and X.Z. Liu. Stability of singularly perturbed switched systems with time delay and impulsive effects. *Nonlinear Analysis: Theory, Method and Applications*, **71**(9), pp. 4297-4308, 2009.

[Alw10] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Existence, continuation, and uniqueness problems of stochastic impulsive systems with time delay. *Journal of the Franklin Institute*, **347**, pp. 1317-1333, 2010.

[Alw-a] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Mean square stability of nonlinear stochastic impulsive systems with time delay. Presented at the International Conference on Applied Mathematics, Modeling and Computational Science (AMMCS-2011), Waterloo, Ontario, Canada (Accepted).

[Alw-b] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Stability properties of stochastic impulsive systems with time delay.

[Alw-c] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Stability results for large scale stochastic impulsive systems with time delay.

[Alw-d] M.S. Alwan, X.Z. Liu, and W.-C. Xie. On input-to-state stability and application to cascade systems.

[Alw-e] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Input-to-state stability results for nonlinear stochastic impulsive systems with time delay.

[Alw-f] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Input-to-state stability of large-scale stochastic impulsive systems with time delay.

[Alw-g] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Robust reliable control for uncertain stochastic impulsive systems with time delay.

[Alw-h] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Input-to-state stability of nonlinear stochastic switched systems.

[Alw-i] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Input-to-state stability of nonlinear stochastic switched systems with time delay.

[Alw-j] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Comparison principle and stability results for nonlinear differential equations with piecewise constant arguments.

[Alw-k] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Existence, uniqueness, and stability results for stochastic differential equations with piecewise constant arguments.

[Alu96] R. Alur, T. Henzinger, and E. Sontag (Editors.). *Hybrid Systems III-Verfication and Control*. Berlin, Germany: Springer-Verlag, Lecture Notes in Computer Science, 1996.

[Ang2000] D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Transactions on Automatic Control*, **45**(6), pp. 1082-1097, 2000.

[Ant95] P. Antsaklis, W. Kohn, N. Nerode, and S. Sastry (Editors.). *Hybrid Systems*. Berlin, Germany: Springer-Verlag, Vol. 999, Lecture Notes in Computer Science, 1995.

[Ant97] P. Antsaklis, W. Kohn, N. Nerode, and S. Sastry (Editors.). *Hybrid Systems*. Berlin, Germany: Springer-Verlag, Vol. 1273, Lecture Notes in Computer Science, 1997.

[Bai89] D.D. Bainov and P.S. Simeonov. *Systems with Impulsive Effects*. Ellis Horwood Ltd, England, 1989.

[Bai93] D.D. Bainov and P.S. Simeonov. *Impulsive Differential Equations: Periodic Solutions and Applications*. Longman Scientific and Technical Group, England, 1993.

[Ball99a] G.H. Ballinger. *Qualitative Theory of Impulsive Delay Differential Equations*, Ph.D. Thesis, University of Waterloo, Ontario, Canada, 1999.

[Ball99b] G.H. Ballinger and X.Z. Liu. Existence and uniqueness results for impulsive delay differential equations. *Dynamics of Continuous, Discrete, and Impulsive Systems*, **6**, pp. 579-591, 1990.

[Bel03] A. Bellen and M. Zennaro. *Numerical Methods for Delay Differential Equations*. Oxford University Press, New York, 2003.

[Bel63] R. Bellman and K.L. Cooke. *Differential-Difference Equations*. Academic Press, New York, 1963.

[Bih56] I. Bihari. A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Mathematica Hungarica*, **7**, pp. 81-94, 1956.

[Bur05] T.A. Burton. *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*. Dover, 2005.

[Bus82] S. Busenberg and K.L. Cooke. Models of vertically transmitted with sequential-continuous dynamics. *Nonlinear Phenomena in Mathematical Sciences*. (Editor) V. Lakshmikantham, Academic Press, New York, pp. 179-187, 1982.

[Cai05] C. Cai and A.R. Teel. Results on input-to-state stability for hybrid systems. *Proceeding of the* 44^{th} *IEEE Conference on Decision and Control*, pp. 5403-5408, Spain, Dec., 2005.

[Cai09] C. Cai and A.R. Teel. Characterization of input-to-state stability for hybrid systems. *Systems and Control Letters*, **58**(1), pp. 47-53, 2009.

[Cab04] A. Cabada, J.B. Ferreiro, and J.J. Nieto. Green's function and comparison principle for first-order periodic differential equations with piecewise constant arguments. *Journal of Mathematical Analysis and Applications*, **291**, pp.690-697, 2004.

[Che09] W.H. Chen and W.X. Zheng. Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with time delays. *Automatica*, **45** (3-4), pp. 1481-1488, 2009.

[Che05] X.-M. Cheng, W.-H. Gui, and Z.-J. Gan. Robust reliable control for a class of time-varying uncertain impulsive systems. *Journal of Central South University of Technology*, **12**(1), pp. 199-202, 2005.

[Coo84] K.L. Cooke and J. Wiener. Retarded differential equations with piecewise constant delays. *Journal of Mathematical Analysis and Applications*, **99**, pp. 265-297, 1984.

[Coo91] K.L. Cooke and J. Wiener. A survey of differential equations with piecewise constants arguments. *Delay Differential Equations and Dynamical Systems.* (Editors) S. Busenberg and M. Martelli (Lecture Notes in Mathematics), **1475**, Springer-Verlag-Heidelberg-New York, pp.1-15, 1991.

[Coo94] K.L. Cooke and I. Györi. Numerical approximation of the solutions of delay differential equations on an infinite interval using piecewise constant arguments. *Computers and Mathematics with Applications*, **28**, pp. 81-92, 1994.

[Day99] W.P. Dayawansa and C.F. Martin. A Converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Transactions on Automatic Control*, **44**(4), pp. 751-759, 1999.

[DePe02] C. De Persis, R. De Santis, and A.S. Morse. Nonlinear switched systems with state dependent dwell-time. *Proceeding of the* 41*st IEEE Conference on Decision and Control*, pp. 4419-4424, Las Vegas, Nevada, 2002.

[Dri77] R.D. Driver. *Ordinary and Delay Differential Equations.* Springer-Verlag, New York Inc., 1977.

[Els73] L.E. El'sgol'ts and S.B. Norkin. *Introduction to the Theory and Application of Differential Equations with Deviating Arguments.* Academic Press, New York, 1973.

[Gao01] J. Gao, B. Huang, Z. Wang, and D.G. Fisher. Robust reliable control for a class of uncertain nonlinear systems with time-varying multistate time delays. *International Journal of Systems Sciences*, **32**(7), pp. 817-824, 2001.

[Gard88] T.C. Gard. *Introduction to Stochastic Differential Equations.* Marcel Dekker, 1988.

[Gih72] I.I. Gihman, and A.V. Skorohod. *Stochastic Differential Equations*. Springer-Verlag, New York Inc., 1972. Translated from Russian by Kenneth Wickwire, Ergebnisse der Mathematik und ihrer Grenzgeiete, Band 72.

[Gop92] K. Gopalsamy. *Stability and Oscillations in Delay Differential Equations of Population Dynamics*. Kluwer Academic Puplishers, Netherlands, 1992.

[Gro93] R.L. Grossman, N. Nerode, A.P. Ravn, and H. Rischel (Editors). *Hybrid Systems*. Berlin, Germany: Springer-Verlag, Vol. 736, Lecture Notes in Computer Science, 1993.

[Gua05] Z-H. Guan, D.J. Hill and X. Shen. On hybrid impulsive and switching systems and application to nonlinear control. *IEEE Transactions on Automatic Control*, **50**(7), pp. 1058-1062, 2005.

[Gyo91] I. Györi. On approximation of the solutions of delay differential equations by using piecewise constant arguments. *International Journal of Mathematical Sciences*, **14**, pp. 111-126, 1991.

[Gyo08] I. Györi and F. Hartung. On numerical approximation using differential equations with piecewise-constant arguments. *Periodica Mathematica Hungarica*, **56**(1), pp. 55-69, 2008.

[Had06] W. Haddad, V. Chllaboina, and S.G. Nersesov. *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*, Princeton University Press, 2006.

[Hah67] W. Hahn. *Stability of Motion*. Translated by Arne P. Baartz. Springer-Verlag, New York, Inc., 1967.

[Hal66] A. Halanay. *Differential Equations: Stability, Oscillations, Time Lags.* Academic Press, Inc., New York, 1966.

[Hal71] J.K. Hale. *Functional Differential Equations.* Springer-Verlag, New York, 1971.

[Hal93] J.K. Hale and S.M.V. Lunel. *Introduction to Functional Differential Equations.* Springer-Verlag, New York, 1993.

[Hes99] J.P. Hespanha and A.S. Morse. Stability of switched systems with average dwell-time. *Proceeding of the* 38^{th} *IEEE Conference on Decision and Control*, pp. 2655-2660, Phoenix, AR, 1999.

[Hes05] J.P. Hespanha, D. Liberzon, and A.R. Teel. On input-to-state stability of impulsive systems. *Proceeding of the* 44^{th} *IEEE Conference on Decision and Control*, pp. 3992-3997, Spain, Dec. 2005.

[Hes06] J. Hespanha and A. Tiwari. *Hybrid Systems: Computation and Control:* 9^{th} *International Workshop. HSCC 2006*, Santa Barbara, CA, USA. Lecture Notes in Computer Science and General Issues, Springer, 2006

[Hu99] B. Hu, X. Xu, A.N. Michel, and P.J. Antsaklis. Stability analysis for a class of nonlinear switched systems. *Proceeding of the* 38^{th} *IEEE Conference on Decision and Control*, pp. 4374-4379, Phoenix, AR, 1999.

[Hua09] L. Huang and X. Mao. On input-to-state stability of stochastic retarded systems with Markovian switching. *IEEE Transactions on Automatic Control*, **54**(8), pp. 1898-1902, 2009.

[Kha80] R.Z. Khasminiskii. *Stochastic Stability of Differential Equations*, Sijthof & Noordhoff International Publishers, The Netherlands, 1980.

[Kha04] A. Khadra. *Impulsive Control and Synchronization of Chaos-Generating-Systems with Aplications to Secure Communication*, Ph.D. Thesis, University of Waterloo, Ontario, Canada, 2004.

[Kha02] H.K. Khalil. *Nonlinear Systems*. Prentice-Hall, Inc., NJ, 2002.

[Kim05] S. Kim. *Switching Systems with Delayed Feedback Control*, Ph.D. Thesis, University of Waterloo, Ontario, Canada, 2005.

[Klo99] P.E. Kloeden, and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, 1999.

[Kok99] P. Kokotovic, and M Arcak. Constructive nonlinear control: Progress in the $90's$. In *Proceeding of the* 14*th IFAC World Congress,* the plenary and Index Volume, Beijing, China, pp. 49-77, 1999.

[Kra63] N.N. Krasovskii and J.L. Brenner. *Stability of Motion: Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay.* Stanford University Press, Stanford, CA, 1963.

[Kua93] Y. Kuang. *Delay Differential Equations with Applications in Population Dynamics*. Academic Press Inc., CA, 1993.

[Kus67] H.J. Kushner. *Stochastic Stability and Control*. Academic Press Inc., New York, 1967.

[Lad80] G.S. Ladde, and V. Lakshmikantham. *Random Differential Inequalities.* Academic Press, New York, 1980.

[Lak69] V. Lakshmikantham and S. Leela. *Differential and Integrals Inequalities.* Academic Press, New York, 1969.

[Lak89] V. Lakshmiknatham, D.D. Bainov, and P.S. Simeonov. *Theory of Impulsive Differential Equations*. World Scientific, Singapore, 1989.

[Lak98] V. Lakshmikantham and X. Z. Liu. Impulsive hybrid systems and stability theory. *Dynamic Systems and Applications*, **7**, pp. 1-10, 1998.

[Lef65] S. Lefschetz. *Stability of Nonlinear Control Systems*. Academic Press Inc., New York, 1965

[Leo94] S.J. Leon. *Linear Algebra with Applications*. Macmillan Publishing Company, Inc., New York, 1994.

[Li97] X. Li and C.E. Souza. Criteria for robust stability and stabilization of uncertain linear systems with state delay. *Automatica*, **33**, pp. 1657-1662, 1997.

[Li03] Y. Li. *Impulsive Synchronization of Chaotic Systems and it's Applications to Communication Security*, Master's Thesis, University of Waterloo, Ontario, Canada, 2003.

[Li05] Z. Li, Y. Soh, and C. Wen (Editors). *Switched and Impulsive Systems: Analysis, Design and Applications*. Lecture Notes in Control and Information Sciences. Springer, Berlin/Heibelberg, 2005.

[Lib99] D. Liberzon and A.S. Morse. Basic problems in stability and design of switched systems. *IEEE Control System Magazine*, **19**(5), pp. 59-70, 1999.

[Lib03] D. Liberzon. *Switching in Systems and Control*. Brikhauser, Boston, 2003.

[Liu94] X. Liu. Stability results for impulsive systems with applications to population growth models. *Dynamics and Stability of Systems*, **9**(2), pp. 163-174, 1994.

[Liu2000] X.Z. Liu and G.H. Ballinger. Existence, uniqueness, and boundedness results for impulsive delay differential equations. *Applicable Analysis*, **74**(1-2), pp. 71-93, 2000.

[Liu01] X.Z. Liu and G.H. Ballinger. Uniform asymptotic stability of impulsive delay differential equations. *Computers and Mathematics with Applications*, **41**, pp. 903-915, 2001.

[Liu03] X.Z. Liu and G.H. Ballinger. Boundedness for impulsive delay differential equations and applications to population growth models. *Nonlinear Analysis*, **53**(7-8), pp. 1041-1062, 2003.

[Liu05] X. Liu, B. Xu, and D. Peng. Delay-dependent stability criteria for impulsive differential systems with delay. *2005 International Conference on Control and Automation (ICCA2005)*, Budapest, Hungary, June 27-29, 2005.

[Liu07] B. Liu, X.Z. Liu, and X. Liao, Existence and uniqueness and stability of solutions for stochastic impulsive systems. *Journal of Systems Science Complexity*, **20**, pp. 149-158, 2007.

[Liu08] B. Liu. Stability of solutions of stochastic impulsive systems via comparison approach. *IEEE Transactions on Automatic Control*, **53**(9), pp. 2128-2133, 2008.

[Luo06] X. Luo, L. Yang, H. Yang, and X. Guan. Robust reliable control for neutral delay systems. *Proceeding of the* 6^{th} *World Congress on Intelligent Control and Automation*, June 21-23, Dalian, China, pp. 144-148, 2006.

[Mac89] N. MacDonald. *Biological Delay Systems: Linear Stability Theory.* Cambridge University Press, Cambridge, 1989.

[Mao91] X. Mao, *Stability of Stochastic Differential Equations with Respect to Semimartingales*, Longman Scientific and Technical, 1991.

[Mao94] X. Mao. *Exponential Stability of Stochastic Differential Equations*. Marcel Dekker, 1994.

[Mao99] X. Mao. Stability of stochastic differential equations with Markovian switching. *Stochastic Processes and Their Applications*, **79**, pp. 45-67, 1999.

[Mao06] X. Mao and C. Yuan. *Stochastic Differential Equations with Markovian Switching*. Imperial College Press, 2006.

[Mao07] X. Mao, G.G. Yin, and C. Yuan. Stabilization and destabilization of hybrid systems of stochastic differential equations. *Automatica*, **43**, pp. 264-273, 2007.

[Mao08] X. Mao. *Stochastic Differential Equations and Applications*. Horwood Publishing Limited, England, 2008.

[Mich77] A.N. Michel and R.K. Miller. *Qualitative Analysis of Large Scale Dynamical Systems*. Academic Press Inc, New York, 1977.

[Min42] N. Minorsky. Self-excited oscillations in dynamical systems. *Journal of Applied Mechanics*, **9**, pp. 65-71, 1942.

[Moh84] S.E.A. Mohammed. *Stochastic Functional Differential Equations*. Vol.99 of Research Notes in Mathematics. Pitman, Boston, 1984.

[Mor96] A.S. Morse. Supervisory control of families of linear set-point controllers-part I: exact matching. *IEEE Transactions on Automatic Control*. **41**(10), pp. 1413-1431, 1996.

[Mor97] A.S. Morse (Editor). *Control Using Logic-Based Switching*. Lecture Notes in Control and Information Sciences, **222**, Springer, New York, 1997.

[Nar94] K.S. Narendra and J. Balakrishnan. A common Lyapunov function for stable LTI systems with commuting A-matrices. *IEEE Transactions on Automatic Control*, **39**(12), pp. 2469-2471, 1994.

[Ste09] P. Stechlinski. A Study of Infectious Disease Models with Switching, Master's Thesis, University of Waterloo, Ontario, Canada, 2009.

[Phi84] E.R. Philips. *An Introduction to Analysis and Integration Theory*. Dover Publication, New York, 1984.

[Sam95] A.M. Samoilenko and N.A. Perestyuk. *Impulsive Differential Equations*. World Scientific Publishing Co., Inc, River Edge, NJ, 1995.

[Seo96] C.-J. Seo and B.K. Kim. Robust reliable H_∞ control for linear systems with parameter uncertainty and actuator failure. *Automatica*, **32**, pp. 465-467, 1996.

[Son89] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, **34**(4), pp. 435-443, 1989.

[Son95] E.D. Sontag and Y. Wang. On characterization of input-to-state stability property. *Systems and Control Lettaers*, **24**, pp. 351-359, 1995.

[Son96] E.D. Sontag and Y. Wang. New characterization of input-to-state stability property. *IEEE Transactions on Automatic Control*, **34**(41), pp. 1283-1294, 1996.

[Son98] E.D. Sontag. Comments on integral variants of ISS. *Systems and Control Letters*, **34**, pp. 93-100, 1998.

[Son02] E.D. Sontag. The ISS philosophy as a unifying framework for stability-like behavior. *In: Lectures Notes Control Theory*, **2**, Springer Verlag, pp. 443-468, 2002.

[Son04] Y. Song, Z. Xiang, Q. Chen, and W. Hu. Dynamical output feedback control of discrete switched system with time delay. *2004 IEEE Proceeding of the* 5^{th} *World Congress on Intelligent Control and Automation*, pp. 1088-1091, Hangzhou, China, 2004.

[Spil03] J. Spiliotis and J.Tsinias. Notions of exponential robust stochastic stability ISS and their Lyapunov characterizations. *International Journal of Robust and Nonlinear Control*, **13**, pp. 173-187, 2003.

[Teel01] A.R. Teel, L. Moreau, and D. Nešic. A note on the robustness of ISS stability. *Proceeding of the* 40^{th} *IEEE on Decision and Control*, pp. 875-880, Florida, Dec. 2001.

[Teel03] A.R. Teel, L. Moreau, and D. Nesic. A unified framework for input-to-state stability in systems with two time scales. *IEEE Transactions on Automatic Control*, **48**(9), pp. 1526-1544, 2003.

[Vei92] R.J. Veillette, J.V. Medanic, and W.R. Perkins. Design of reliable control systems. *IEEE Transactions on Automatic Control*, **37**(3), pp. 290-304, 1992.

[Wan04] R. Wang, X. Liu, and Z. Guan. Robustness and stability analysis for a class of nonlinear switched systems with impulse effects. *Dynamic Systems and Applications*, **14**, pp. 233-248, 2004.

[Wan07] Q. Wang. *Stability and Boundedness of Impulsive Systems with Time Delay*, Ph.D. Thesis, University of Waterloo, Ontario, Canada, 2007.

[Wang99] Z. Wang, B. Huang, and H. Unbehauen. Robust reliable control for a class of uncertain nonlinear state-delayed systems. *Automatica*, **35**, pp. 955-963, 1999.

[Wang01] Z. Wang, B. Huang, and K.J. Burnham. Stochastic reliable control of a class of uncertain time-delay systems with unknown nonlinearities. *IEEE Transactions on Circuits and Systems-1: Fundamental Theory and Applications*, **48**(5), pp. 646-650, 2001.

[Wang2000] J. Wang and H. Shao. Delay-dependent robust and reliable H_∞ control for uncertain time-delay systems with actuator failurs. *Journal of the Franklin Institute*, **337**, pp. 781-791, 2000.

[Wie93] J. Wiener. *Generalized Solutions of Functional Differential Equations*. World Scientific, Singapore, 1993.

[Xie90] L. Xie and C.E. Souza. Robust H_∞ control for linear time-invariant systems with norm bounded uncertainty in the input matrix. *Systems and Control Letters*, **14**, pp. 389-396, 1990.

[Xie06] W.-C. Xie. *Dyanmic Stability of Structure*. Cambridge University Press, New York, 2006.

[Yang01] T. Yang. *Impulsive Control Theory*. Springer, 2001.

[Yang09] Z.W. Yang, M.M. Liu, and J.J. Nieto. Runge-Kutta methods for first-order periodic boundary value differential equations with piecewise constant arguments. *Journal of Computational and Applied Mathematics*, **233**, pp. 990-1004, 2009.

[Yua05] C. Yuan and J. Lygeros. Stabilization of a class of stochastic differential equation with Markovian switching. *Systems and Control Letters*, **54**, pp. 819-833, 2005.

[Zer98] M. Žefran and J.W. Burdick. Design of switching controllers for systems with changing dynamics. *Proceeding of the* 37^{th} *IEEE Conference on Decision and Control*, pp. 2113-2118, Tampa, FL, 1998.

[Zha04] Y. Zhang. *Stability of Hybrid Systems with Time Delay*, Ph.D. Thesis, University of Waterloo, Ontario, Canada, 2004.

[Zha01] G. Zhai, B. Hu, K. Yasuda, and A.N. Michel. Stability analysis of switched systems with stable and unstable subsystems: an average dwell-time approach. *International Journal of Systems Science*, **32**(8), pp. 1055-1061, 2001.