Analysis of some risk models involving dependence

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The seminal paper by Gerber and Shiu (1998) gave a huge boost to the study of risk theory by not only unifying but also generalizing the treatment and the analysis of various risk-related quantities in one single mathematical function - the *Gerber-Shiu expected discounted penalty function*, or *Gerber-Shiu function* in short. The Gerber-Shiu function is known to possess many nice properties, at least in the case of the classical compound Poisson risk model. For example, upon the introduction of a dividend barrier strategy, it was shown by Lin et al. (2003) and Gerber et al. (2006) that the Gerber-Shiu function with a barrier can be expressed in terms of the Gerber-Shiu function without a barrier and the expected value of discounted dividend payments. This result is the so-called *dividends-penalty identity*, and it holds true when the surplus process belongs to a class of Markov processes which are skip-free upwards. However, one stringent assumption of the model considered by the above authors is that all the interclaim times and the claim sizes are independent, which is in general not true in reality. In this thesis, we propose to analyze the Gerber-Shiu functions under various dependent structures. The main focus of the thesis is the risk model where claims follow a *Markovian arrival process* (MAP) (see, e.g., Latouche and Ramaswami (1999) and Neuts (1979, 1989)) in which the interclaim times and the claim sizes form a chain of dependent variables. The first part of the thesis puts emphasis on certain dividend strategies. In Chapter 2, it is shown that a matrix form of the dividends-penalty identity holds true in a MAP risk model perturbed by diffusion with the use of integro-differential equations and their solutions. Chapter 3 considers the *dual* MAP risk model which is a reflection of the ordinary MAP model. A threshold dividend strategy is applied to the model and various risk-related quantities are studied. Our methodology is based on an existing connection between the MAP risk model and a fluid queue (see, e.g., Asmussen et al. (2002), Badescu et al. (2005), Ramaswami (2006) and references therein).

The use of fluid flow techniques to analyze risk processes opens the door for further research as

to what types of risk model with dependency structure can be studied via probabilistic arguments. In Chapter 4, we propose to analyze the Gerber-Shiu function and some discounted joint densities in a risk model where each pair of the interclaim time and the resulting claim size is assumed to follow a *bivariate phase-type distribution*, with the pairs assumed to be independent and identically distributed (i.i.d.). To this end, a novel fluid flow process is constructed to ease the analysis.

In the classical Gerber-Shiu function introduced by Gerber and Shiu (1998), the random variables incorporated into the analysis include the time of ruin, the surplus prior to ruin and the deficit at ruin. The later part of this thesis focuses on generalizing the classical Gerber-Shiu function by incorporating more random variables into the so-called *penalty function*. These include the surplus level immediately after the second last claim before ruin, the minimum surplus level before ruin and the maximum surplus level before ruin. In Chapter 5, the focus will be on the study of the generalized Gerber-Shiu function involving the first two new random variables in the context of a *semi-Markovian risk model* (see, e.g., Albrecher and Boxma (2005) and Janssen and Reinhard (1985)). It is shown that the generalized Gerber-Shiu function satisfies a matrix defective renewal equation, and some discounted joint densities involving the new variables are derived. Chapter 6 revisits the MAP risk model in which the generalized Gerber-Shiu function involving the maximum surplus before ruin is examined. In this case, the Gerber-Shiu function no longer satisfies a defective renewal equation. Instead, the generalized Gerber-Shiu function can be expressed in terms of the classical Gerber-Shiu function and the Laplace transform of a first passage time that are both readily obtainable.

In a MAP risk model, the interclaim time distribution must be phase-type distributed. This leads us to propose a generalization of the MAP risk model by allowing for the interclaim time to have an arbitrary distribution. This is the subject matter of Chapter 7. Chapter 8 is concerned with the *generalized Sparre Andersen risk model* with *surplus-dependent premium rate*, and some ordering properties of certain ruin-related quantities are studied. Chapter 9 ends the thesis by some concluding remarks and directions for future research.

Acknowledgments

I would like to first thank my supervisors Professors David Landriault and Gordon Willmot. I am really fortunate to get their supervision since they are really exceptional both professionally and personally. They not only gave me quality academic advice but more importantly educated me to do the right thing and be a better person. Professor David Landriault is always like a brother to me; whereas Professor Gordon Willmot strongly encouraged me to work independently near the completion of the thesis, which I believe to be helpful to my career. Thanks also go to Professors Jun Cai, Steve Drekic, Qi-Ming He and Qihe Tang for their helpful comments and suggestions which improved the representation of this work. The financial support from the Natural Sciences and Engineering Research Council of Canada is also gratefully acknowledged.

Dedication

This is dedicated to the ones I love - Jae-Kyung Woo, Fanny Yun Lan Lee, and Anthony Kwok Keung Cheung. My lovely wife Jae-Kyung provided me with care, love and support which have made this work possible. Unfortunately my mother Fanny Lee, who passed away in 2004, does not have a chance to witness the completion of this thesis. Nonetheless, I am sure she would be very happy if she knew I was under good hands during my stay in Waterloo. Thanks also go to my father who supported me not only financially but also for the pursue of my PhD study.

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Chapter 1

Introduction and preliminaries

1.1 Background

If we denote the surplus process of an insurance company by $\{U_t\}_{t\geq 0}$, it is typically modelled as

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i \ , \quad t \geq 0 \ , \tag{1.1}$$

where $u = U_0$ is the initial surplus and $c > 0$ is the incoming premium rate per unit time. Furthermore, $\{Y_i\}_{i=1}^{\infty}$ is a sequence of positive random variables with Y_i representing the size of the i-th claim; while the claim number process $\{N_t\}_{t\geq 0}$ is defined through the positive interclaim times $\{V_i\}_{i=1}^{\infty}$ with V_1 being the time of the first claim and V_i for $i = 2, 3, \ldots$ the time between the $(i-1)$-th claim and the i-th claim, i.e. $N_t = \sup\{i \in \mathbb{N} : \sum_{j=1}^{i} V_j \leq t\}$. Also, we let $T = \inf\{t \geq 0 : U_t < 0\}$ be the time of ruin with $T = \infty$ if ruin does not occur. Then, in the case of ruin, U_{T^-} and $|U_T|$ represent the surplus prior to ruin and the deficit at ruin respectively.

Note that the above definition of the surplus process $\{U_t\}_{t\geq 0}$ is very general in terms of the distributions of the interclaim times and the claim sizes, since nothing is specified about $\{Y_i\}_{i=1}^{\infty}$

and $\{V_i\}_{i=1}^{\infty}$ (or $\{N_t\}_{t\geq 0}$). In particular, if $\{Y_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ are both *independent and identically distributed* (i.i.d.) sequences independent of each other, then the model (1.1) represents a *Sparre Andersen risk model.* If it is further assumed that any arbitrary V_i is exponentially distributed, then the model (1.1) further reduces to the *classical compound Poisson risk model.* See, e.g., Cramér (1955), Gerber (1979), Grandell (1991), Seal (1969) and Sparre Andersen (1957) for the very first treatments of the above two special cases.

The seminal paper by Gerber and Shiu (1998) introduced the *Gerber-Shiu expected discounted penalty function* (or simply *Gerber-Shiu function*) defined by

$$\phi_\delta(u) = E\left[e^{-\delta T} w(U_{T^-}, |U_T|) 1\{T < \infty\} | U_0 = u\right] \ , \quad u \geq 0 \ , \tag{1.2}$$

in the context of the classical compound Poisson risk model, where $\delta \geq 0$, $w : \mathbb{R}^2 \to \mathbb{R}$ is the so-called *penalty function* assumed to satisfy some mild integrable conditions, and $1\{A\}$ is the indicator function of the event A. Here δ can either be viewed as a force of interest or a Laplace transform argument. Gerber and Shiu (1998) showed that $\phi_\delta(u)$ satisfies a defective renewal equation whose solution can be expressed in terms of a compound geometric tail (see, e.g., Lin and Willmot (1999) and Resnick (1992)). The introduction of the Gerber-Shiu function is indeed an extremely clever idea which not only unifies but also generalizes the treatment of the three risk-related quantities T, U_{T^-} and $|U_T|$. Plenty of information can be extracted from the Gerber-Shiu function by assuming a specific penalty function $w(.,.)$. For example, one of the most simplest cases is where $w(.,.) \equiv 1$, then $\phi_\delta(u)$ reduces to the Laplace transform (with argument δ) of the time of ruin. By further assuming that $\delta = 0$, the ruin probability is obtained. Another important choice of penalty function would be $w(x, y) = e^{-s_1 x - s_2 y}$, which leads to the trivariate Laplace transform of $(T, U_{T^-}, |U_T|)$. Analytic Laplace transform inversions with respect to (δ, s_1, s_2) yield the trivariate density of $(T, U_{T^-}, |U_T|)$, as illustrated by Landriault and Willmot (2009). Other information which can be extracted from $\phi_\delta(u)$ includes (but is not limited to) the moments of T, U_{T^-} and $|U_T|$ (see Lin and Willmot (2000)). We remark that in many cases, the extraction of specific information from the Gerber-Shiu function is not trivial, even if it is clear that the information in question is contained in the Gerber-Shiu function.

Now if we return to the model (1.1) in general, the Gerber-Shiu function (1.2) can also be defined accordingly. However, depending on the specific assumptions on $\{Y_i\}_{i=1}^{\infty}$ and/or $\{V_i\}_{i=1}^{\infty}$ of some particular models, the definition (1.2) might have to be modified. For example, if the surplus process has an underlying *Markovian environment* (see, e.g., Asmussen (2000)), it might be convenient to define the Gerber-Shiu function with the additional information of the state at ruin, conditional on the initial state of the Markov chain. To avoid confusion, we delay these definitions until later chapters where specific models are encountered.

Note also that modifications such as *Brownian motion* (see, e.g., Dufresne and Gerber (1991), Gerber (1970), Gerber and Landry (1998) and Tsai and Willmot (2002)), *dividend barrier* (see, e.g., Gerber (1979) and Lin et al. (2003)) and *dividend threshold* (see, e.g., Lin and Pavlova (2006) and Lin and Sendova (2008)) are absent in the model (1.1), and can indeed be incorporated into it. Furthermore, a *dual* version of (1.1) can also be defined by reflection of the sample paths, and such a reflected process would be suitable for companies which incur expenses at constant rate over time and earn occasional gains that are random in nature (see, e.g., Avanzi et al. (2007) and Seal (1969)). Again, we delay the definitions of such modifications, if any, to later chapters when they are needed. In the next section, we primarily focus on the various specific dependency structures imposed on $\{Y_i\}_{i=1}^{\infty}$ and/or $\{V_i\}_{i=1}^{\infty}$ pertaining to the model (1.1).

1.2 Various dependency structures

1.2.1 MAP risk model

In a *MAP risk model*, the claim number process $\{N_t\}_{t\geq 0}$ follows a *Markovian arrival process* (MAP). The MAP risk model is the main subject of this thesis. Under a MAP risk model, $\{N_t\}_{t\geq 0}$ has representation MAP$(\mathbf{a}, \mathbf{G}_0, \mathbf{G}_1)$ of order m. A MAP is a two-dimensional Markov process on the state space $\mathbb{N} \times \{1, \ldots, m\}$ for which the first dimension reflects the evolution of the total number of claims over time while the second refers to the evolution of an underlying irreducible homogeneous *continuous-time Markov chain* (CTMC) $\{J_t\}_{t\geq 0}$ with finite state space $E = \{1, \ldots, m\}$. For such a process, transitions of the MAP are subdivided into two categories:

- transitions of the CTMC from state i to state j ($j \neq i$) without an accompanying claim (**type 1**); and
- transitions of the CTMC from state i to state j (with the possibility of $i = j$) with an accompanying claim (**type 2**).

Transitions of type 1 are governed by the generator $\mathbf{G}_0$ for which its (i, j)-th element $G_{0,ij} \geq 0$ corresponds to the instantaneous rate of transition from state i to state $j \neq i$ in E without an accompanying claim. Type-2 transitions are governed by the generator $\mathbf{G}_1$ for which its (i, j)-th element $G_{1,ij} \geq 0$ corresponds to the instantaneous rate of transition from state i to state j in E with an accompanying claim. The diagonal elements of $\mathbf{G}_0$ are assumed to be negative and such that the sum of the elements on each row of the matrix $\mathbf{G}_0 + \mathbf{G}_1$ is zero. We denote by $\mathbf{a}$ the initial probability vector of the underlying CTMC at time 0. For a detailed treatment of MAPs, we refer the reader to, e.g., Latouche and Ramaswami (1999) and Neuts (1979, 1989).

For type-2 transitions, it is further assumed that the distribution of the accompanying claim size may depend on the state of the CTMC immediately before and after the transition. Thus, for a type-2 transition of the CTMC from state i to state j, the accompanying claim size is assumed to have density $p_{ij}(.)$ and cumulative distribution function (c.d.f.) $P_{ij}(.)$ with finite mean μ_{ij}. To ensure that the surplus process $\{U_t\}_{t\geq 0}$ defined by (1.1) drifts to infinity in the long run, the *positive security loading condition* has to be satisfied. In our setup, this condition is given by

$$\sum_{i=1}^{m} \pi_i \sum_{j=1}^{m} G_{1,ij}\mu_{ij} < c \ , \tag{1.3}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ represents the stationary probabilities of the CTMC $\{J_t\}_{t\geq 0}$, and can be solved from the system

$$\begin{cases} \boldsymbol{\pi}(\mathbf{G}_0 + \mathbf{G}_1) = \mathbf{0} \ . \\ \boldsymbol{\pi}\mathbf{1} = 1 \ . \end{cases} \tag{1.4}$$

Here $\mathbf{0}$ is a zero column vector and $\mathbf{1}$ is a column vector of ones, both of appropriate dimension. Indeed, the left-hand side of (1.3) represents the long run average claim per unit time, and therefore condition (1.3) guarantees that on average the premium income is sufficient to cover the claim cost.

It is instructive to note that the MAP risk model contains various well-known models as special cases. For example, it contains the Sparre Andersen risk model with phase-type interclaim times. More specifically, if $\{V_i\}_{i=1}^{\infty}$ is an i.i.d. sequence having phase-type distribution with representation PH($\boldsymbol{\alpha}, \mathbf{G}$), then we simply let $\mathbf{a} = \boldsymbol{\alpha}$, $\mathbf{G}_0 = \mathbf{G}$ and $\mathbf{G}_1 = -\mathbf{G}\mathbf{1}\mathbf{a}$. In addition, by letting $\mathbf{G}_0$ and $\mathbf{G}_1$ be diagonal respectively, the MAP risk model reduces to the *semi-Markovian risk model* and the *Markov-modulated risk model* considered by Albrecher and Boxma (2005) and Asmussen (2000) respectively. Interested readers are also referred to Ahn and Badescu (2007) for the study of Gerber-Shiu function in the MAP risk model with phase-type claims via connection to a fluid queue.

1.2.2 Generalized Sparre Andersen risk model

In the *generalized Sparre Andersen risk model*, we assume that the bivariate random vectors $\{(V_i, Y_i)\}_{i=1}^{\infty}$ form an i.i.d. sequence, so that $\{cV_i - Y_i\}_{i=1}^{\infty}$ is also an i.i.d. sequence implying that the surplus process $\{U_t\}_{t\geq 0}$ retains the Sparre Andersen random walk structure. We may also refer the generalized Sparre Andersen risk model as the *Sparre Andersen type risk model*. In such a model, the positive security loading condition which ensures that the surplus process $\{U_t\}_{t\geq 0}$ goes to infinity in the long run is given by

$$E[Y] < cE[V] \ , \tag{1.5}$$

where (V, Y) denotes an arbitrary pair of (V_i, Y_i). See, e.g., Prabhu (1998, Theorems 3 and 7).

Asymptotics for ruin probabilities in the above model were examined by Albrecher and Teugels (2006) when claims are light-tailed, while the Gerber-Shiu function was studied by Cheung et al. (2010c).

With regards to the generalized Sparre Andersen risk model with specific distributional assumptions on the generic bivariate random vector (V, Y), in this thesis we shall only consider the case where (V, Y) is assumed to follow a *bivariate phase-type distribution* via a novel connection to a fluid flow process (see Chapter 4). This methodology is in contrast to the analytic methods employed by the papers in the previous paragraph. Interested readers are also referred to, e.g., Boudreault et al. (2006) and Cossette et al. (2008) for certain generalized Sparre Andersen risk model under other specific (and tractable) distributional assumptions on the bivariate random vector (V, Y).

1.2.3 Semi-Markovian risk model

One of the very first *semi-Markovian risk models* was proposed by Janssen and Reinhard (1985). However, the model described there is too general for detailed analysis to be done. From now on, when we refer to a semi-Markovian risk model, we mean the following model described here. Let ϱ_0 be the environmental state at time 0 and ϱ_i be the environmental state immediately following the i-th claim. We assume that $\{\varrho_i\}_{i=0}^{\infty}$ is a homogeneous and irreducible *discrete-time Markov chain* (DTMC) on the state space $E = \{1, 2, \ldots, m\}$ with one-period transition probability matrix $\mathbf{P} = [p_{ij}]_{i,j=1}^{m}$. Furthermore, for $i = 1, 2, \ldots$, $V_i|\varrho_{i-1} = j$ is assumed to have density $k_j(.)$, c.d.f. $K_j(.)$ and mean κ_j, while $Y_i|\varrho_i = j$ has density $b_j(.)$, c.d.f. $B_j(.)$, survival function $\overline{B}_j(.)$ and mean μ_j. Conditional on $\{\varrho_i\}_{i=0}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ are all mutually independent. Combining all the above assumptions, it follows that, for $i = 1, 2, \ldots$ and $j, k \in E$,

$$\Pr\{Y_i \leq y, V_i \leq t, \varrho_i = k | \varrho_{i-1} = j\} = K_j(t) p_{jk} B_k(y) \ , \quad t, y \geq 0 \ . \tag{1.6}$$

We remark that when the $k_j(.)$'s are exponential densities, (1.6) reduces to Eq. (2) of Albrecher and Boxma (2005), and hence the model (1.1) becomes the semi-Markovian risk model considered by them. The positive security loading condition under the semi-Markovian risk model is given by

$$\sum_{j=1}^{m} \pi_j (c\kappa_j - \mu_j) > 0 \ , \tag{1.7}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ is the stationary distribution of the Markov chain $\{\varrho_i\}_{i=0}^{\infty}$ satisfying

$$\begin{cases} \boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \ . \\ \boldsymbol{\pi}\mathbf{1} = 1 \ . \end{cases} \tag{1.8}$$

See, e.g., Reinhard (1984, Theorem 4).

1.2.4 Similarities and differences between the dependent risk models

While it is trivial that the classical compound Poisson risk model (in which $\{Y_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ are mutually independent i.i.d. sequences with any arbitrary V_i being exponentially distributed) is contained in all of the above-mentioned risk models involving different dependency structures, one should keep in mind that the three risk models of our interest are not special cases of each other. There are notable similarities as well as important differences between them. Certain similarities are first summarized as follows.

1. The dependency structures in the MAP risk model in Section 1.2.1 and the semi-Markovian risk model in Section 1.2.3 are both induced by an underlying Markov chain. As mentioned earlier, they both contain Albrecher and Boxma (2005)'s semi-Markovian risk model as special case.

2. Both the generalized Sparre Andersen risk model in Section 1.2.2 and the semi-Markovian risk model in Section 1.2.3 allow for the modelling of arbitrary inter-arrival times between successive claims.

Therefore, the semi-Markovian risk model appears to have the characteristics of both the MAP risk model and the generalized Sparre Andersen risk model. Nonetheless, the following differences should also be noted.

1. While the generalized Sparre Andersen risk model and the semi-Markovian risk model allow for arbitrary interclaim times, in a MAP risk model the interclaim times are phase-type distributed.

2. In the semi-Markovian risk model, any dependency between the claim sizes and the interclaim times is modelled via a Markov chain. This is in contrast to the generalized Sparre Andersen risk model in the generic pair (V, Y) is allowed to have arbitrary dependency

structure. Moreover, the interclaim times form an i.i.d. sequence in the generalized Sparre Andersen risk model, while the same is not true of the semi-Markovian risk model.

With regards to the motivation of the different risk models, the use of an underlying Markov chain in both the MAP risk model and the semi-Markovian risk model allows the modelling of different insurance claim frequencies and/or severities under different economic environments (e.g. 'normal' or 'dangerous' state). On the other hand, the generalized Sparre Andersen risk model is suitable when a given interclaim time possibly has an impact on distribution of the resulting claim size, which is evident in the context of earthquake insurance (see Boudreault et al. (2006)).

1.3 Generalizations of the Gerber-Shiu function

It has been more than ten years since the seminal paper by Gerber and Shiu (1998) was published. Over the past ten years, researchers have been performing the Gerber-Shiu analysis for various increasingly complex models, but very limited research has been done on generalizing the Gerber-Shiu function itself. Until recently, several generalizations have been made to incorporate more information into the Gerber-Shiu function so as to study the behaviour of the surplus process **before** the time of ruin T (as opposed to U_{T^-} and $|U_T|$ which are defined **at** time T). For example, Cai et al. (2009b) generalized the Gerber-Shiu function by applying a 'cost' to every point along the sample path until ruin through the use of a *cost function* (instead of only applying a 'penalty' at ruin through the penalty function). In addition, Cheung et al. (2010b) incorporated additionally the surplus level immediately after the second last claim before ruin into the penalty function in the classical compound Poisson risk model, while Biffis and Morales (2010) incorporated the minimum surplus level before ruin into the penalty in a Lévy insurance risk model using a fluctuation identity given by Doney and Kyprianou (2006). Cheung et al. (2010c) considered a penalty function involving both of the above two variables in the generalized Sparre Andersen risk model described in Section 1.2.2, while Cheung and Landriault (2010) used a penalty function which involves the maximum surplus prior to ruin in a risk model with taxation (see Albrecher and Hipp (2007) for the descriptions of a tax model).

To introduce certain generalizations of the Gerber-Shiu function (1.2), we first define the sequence $\{R_n\}_{n=0}^{\infty}$ such that

$$R_n = u + \sum_{i=1}^{n} (cV_i - Y_i) \ , \quad n = 0, 1, \ldots . \tag{1.9}$$

Clearly, R_n represents the surplus level immediately after the n-th claim for $n = 0, 1, \ldots$, with the usual assumption that the zero-th claim occurs at time 0. Then, we are interested in the quantity R_{N_T-1}, which is the surplus level immediately after the second last claim before ruin. A generalization of the Gerber-Shiu function (1.2), as suggested by Cheung et al. (2010b), would be

$$\phi_\delta(u) = E\left[e^{-\delta T} w(U_{T^-}, |U_T|, R_{N_T-1}) 1\{T < \infty\} | U_0 = u\right] \ , \quad u \geq 0 \ . \tag{1.10}$$

Cheung et al. (2010b) demonstrated that the Gerber-Shiu function (1.10) can be applied to find the distribution of the last interclaim time $V_{N_T} = (U_{T^-} - R_{N_T-1})/c$, and showed that it is stochastically smaller than a generic interclaim time variable in the context of the classical compound Poisson risk model, which agrees with intuition (see Chapter 8 for generalizations of such a result). Furthermore, the Gerber-Shiu function (1.10) can also be used to find the joint distribution of V_{N_T} together with the claim causing ruin $Y_{N_T} = U_{T^-} + |U_T|$. In general, one expects the pair (V_{N_T}, Y_{N_T}) to be dependent even if independence is assumed between the sequences $\{Y_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$.

Cheung et al. (2010c) extended the Gerber-Shiu function (1.10) by further incorporating the minimum surplus level before ruin $X_T = \min_{0 \leq s < T} U_s$ into the penalty function. Such an extension now reads

$$\phi_\delta(u) = E\left[e^{-\delta T} w(U_{T-}, |U_T|, X_T, R_{N_T-1}) 1\{T < \infty\} | U_0 = u\right] \ , \quad u \geq 0 \ . \tag{1.11}$$

It was shown in Cheung et al. (2010c) that the above Gerber-Shiu function under the generalized Sparre Andersen risk model satisfies a defective renewal equation, generalizing certain results in, e.g., Boudreault et al. (2006), Cossette et al. (2008) and Willmot (2007), where additional distributional and/or dependency assumptions are made on $\{Y_i\}_{i=1}^{\infty}$ and/or $\{V_i\}_{i=1}^{\infty}$. Cheung et al. (2010c) also showed that the distribution of the last ladder height $X_T + |U_T|$ can be obtained from the Gerber-Shiu function (1.11).

Opposite to the minimum surplus before ruin X_T, one might be interested to study the maximum surplus before ruin $Z_T = \max_{0 \leq s < T} U_s$, whose marginal distribution was studied by, e.g., Li and Dickson (2006). As far as the Gerber-Shiu function is concerned, we shall consider the generalization (see Cheung and Landriault (2010))

$$\phi_\delta(u) = E\left[e^{-\delta T} w(U_{T-}, |U_T|, Z_T) 1\{T < \infty\} | U_0 = u\right] \ , \quad u \geq 0 \ . \tag{1.12}$$

The Gerber-Shiu function (1.12) allows for the study of the largest distance of the surplus process up to and including the time of ruin, namely $Z_T + |U_T|$, which would help the understanding of the variability of the surplus process in case of ruin.

The study of the Gerber-Shiu functions (1.11), (1.12) and (1.10) will be the main subject of Chapters 5, 6 and 8 respectively under different risk models. The classical variables $(T, U_{T-}, |U_T|)$, the afore-mentioned new variables (X_T, R_{N_T-1}, Z_T) as well as their related quantities for a typical sample path are graphically depicted in Figure 1.1 below.

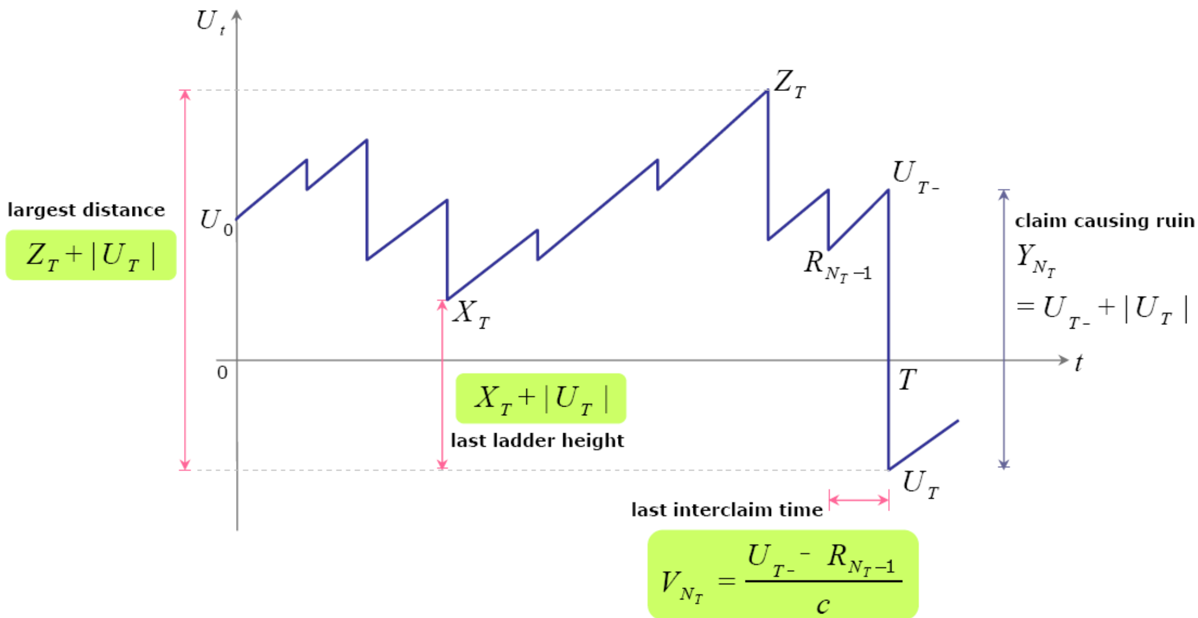

Figure 1.1: New variables and their related quantities

1.4 Mathematical preliminaries

1.4.1 Notations and operators

Notations introduced earlier in this chapter will be adopted throughout the entire thesis unless specified otherwise. For example, $1\{A\}$ is the indicator function of the event A, and T is the time of ruin. Furthermore, we shall always assume w to be the penalty function (with the appropriate number of arguments depending on whether (1.2), (1.10), (1.11) or (1.12) is referred to) which satisfies some mild integrable conditions. Matrices will be denoted in **boldface**, and we shall assume $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{I}$ to be the zero matrix (or vector), a column vector of ones, and the identity matrix respectively, all of appropriate dimension.

There are also several operators which will be used throughout the thesis. First, the Dickson-Hipp operator $\mathcal{T}_r$ (see, e.g., Dickson and Hipp (2001)) is defined as, for any integrable real-valued function $f(.)$ on $(0, \infty)$ and any complex number r with $\mathrm{Re}(r) \geq 0$,

$$\mathcal{T}_r f(y) = \int_y^\infty e^{-r(x-y)} f(x)\, dx \;, \quad y \geq 0 \;. \tag{1.13}$$

The Dickson-Hipp operator is known to possess several nice properties. For example, from Li and Garrido (2004, Section 3, properties 2 and 6), we know that, for any complex numbers $r_1 \neq r_2$,

$$\mathcal{T}_{r_1}\mathcal{T}_{r_2} f(y) = \mathcal{T}_{r_2}\mathcal{T}_{r_1} f(y) = \frac{\mathcal{T}_{r_1} f(y) - \mathcal{T}_{r_2} f(y)}{r_2 - r_1} \;, \quad y \geq 0 \;, \tag{1.14}$$

while for distinct complex numbers $r_1, r_2, \ldots, r_k$, (1.14) can be extended to

$$\mathcal{T}_{r_k} \ldots \mathcal{T}_{r_2}\mathcal{T}_{r_1} f(y) = (-1)^{k-1} \sum_{l=1}^{k} \frac{\mathcal{T}_{r_l} f(y)}{\tau_k'(r_l)} \;, \quad y \geq 0 \;, \tag{1.15}$$

where $\tau_k(r) = \prod_{l=1}^{k}(r - r_l)$. The two properties (1.14) and (1.15) will be used later in the thesis.

Apart from the Dickson-Hipp operator, we also define the convolution operator $*$, for any functions $f(.)$ and $g(.)$ on $(0, \infty)$,

$$(f * g)(y) = \int_0^y f(y-x) g(x)\, dx = \int_0^y g(y-x) f(x)\, dx = (g * f)(y) \;, \quad y \geq 0 \;. \tag{1.16}$$

Furthermore, the first divided difference of any function $f(.)$ with respect to two distinct numbers x_1 and x_2 is defined to be

$$f[x_1, x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \;, \tag{1.17}$$

while the second divided difference with respect to three distinct numbers x_1, x_2 and x_3 is defined to be

$$f[x_1, x_2, x_3] = \frac{f[x_1, x_2] - f[x_1, x_3]}{x_2 - x_3} \;, \tag{1.18}$$

and one can move on to define the k-th divided difference of the function $f(.)$. By noting the similarity between the double Dickson-Hipp operator (1.14) and the divided difference (1.17), one can see that a result similar to (1.15) also holds true for the k-th divided difference of a function. Since this will not be used explicitly in this thesis, we omit the details. We also remark that the notion of divided difference can be extended from scalar to matrix quantities (see, e.g., Lu and Tsai (2007)), and this will be considered in Chapter 5.

Finally, the Laplace transform of a function $f(.)$ on $(0, \infty)$ is defined to be, for any complex number s with $\mathrm{Re}(s) \geq 0$,

$$\widetilde{f}(s) = \int_0^{\infty} e^{-sx} f(x)\, dx \ , \quad y \geq 0 \ . \tag{1.19}$$

For the rest of the thesis, we shall use the notation ' $\sim$ ' to denote the Laplace transform of a function. Furthermore, when the Laplace transform of a random variable is referred to, it is understood that we mean the Laplace transform with respect to the density of the random variable (since we are mostly concerned with continuous random variables in this thesis). Note that the Laplace transform is in fact a special case of the Dickson-Hipp operator defined by (1.13) since $\mathcal{T}_r f(0) = \widetilde{f}(r)$ for $\mathrm{Re}(r) \geq 0$.

The following conventions will be adopted throughout the thesis. The Dickson-Hipp operator (Laplace transform) of a matrix.vector is simply the matrix/vector containing the Dickson-Hipp operator (Laplace transform) of its individual elements. Also, the derivative of a matrix/vector represents the same matrix/vector whose elements are singly differentiated.

1.4.2 Fluid flow process, related quantities and its connection to risk process

A detailed review of the fluid process used to analyze risk process has been given in, e.g., Badescu et al. (2007a). To keep this thesis self-contained, a brief review is presented here.

Underlying the fluid level process $\{F(t)\}_{t\geq 0}$ in this thesis is an irreducible homogeneous CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ which defines an environmental process. The states of this process are also referred to as 'phases'. The CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ is assumed to have finite state space $S = S_0 \cup S_1 \cup S_2$ and infinitesimal generator

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{00} & \mathbf{Q}_{01} & \mathbf{Q}_{02} \\ \mathbf{Q}_{10} & \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{20} & \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{pmatrix}, \tag{1.20}$$

where for $i, j = 0, 1, 2$, the submatrix $\mathbf{Q}_{ij}$ is an $|S_i| \times |S_j|$ matrix containing the (r, s)-th elements of the infinitesimal generator $\mathbf{Q}$ for all $r \in S_i$ and all $s \in S_j$. The partition of the state space S into S_0, S_1 and S_2 is as follows:

- during a sojourn of $\{J^{(F)}(t)\}_{t\geq 0}$ in S_1 (S_2), the fluid process $\{F(t)\}_{t\geq 0}$ increases (decreases) at a constant rate of c; and
- during a sojourn of $\{J^{(F)}(t)\}_{t\geq 0}$ in S_0, the fluid process $\{F(t)\}_{t\geq 0}$ remains constant.

Therefore, the fluid flow process of our interest is given by the bivariate process $\{F(t), J^{(F)}(t)\}_{t\geq 0}$. For simplicity, for the remainder of this thesis, unless we would like to put emphasis on the dependence of the fluid level $\{F(t)\}_{t\geq 0}$ on the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$, we would simply write $\{F(t)\}_{t\geq 0}$ instead of $\{F(t), J^{(F)}(t)\}_{t\geq 0}$.

Associated to the above fluid flow process $\{F(t), J^{(F)}(t)\}_{t\geq 0}$ is the reflected fluid flow process $\{F^r(t), J^{(F)}(t)\}_{t\geq 0}$ in which the fluid level $\{F^r(t)\}_{t\geq 0}$ increases (decreases) at a constant rate of c

during a sojourn of the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ in S_2 (S_1), and the fluid level remains constant during a sojourn of the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ in S_0. Then, we define ${}^z_a\sigma(x,y)$ (${}^z_a\sigma^r(x,y)$) to be the first passage time of the fluid level process $\{F(t)\}_{t\geq 0}$ ($\{F^r(t)\}_{t\geq 0}$) from level x to level y while avoiding a visit to $[0,a]\cup[z,\infty)$ enroute. The arguments a and/or z will be suppressed whenever they are not helpful. For example, we shall write ${}_0\sigma(0,x)$ instead of ${}^x_0\sigma(0,x)$.

The key to the analysis of the fluid flow process $\{F(t)\}_{t\geq 0}$ is the quantity $\sigma(0,0)$, which is the first return time of the fluid to level 0 given that the process starts at level 0 at time 0. Then we define the $|S_1|\times|S_2|$ matrix $\mathbf{\Psi}(\delta)$ with (i,j)-th element by

$$[\mathbf{\Psi}(\delta)]_{ij} = E\left[e^{-\delta\sigma(0,0)}1\{J^{(F)}(\sigma(0,0))=j\}|J^{(F)}(0)=i\right] \ , \quad i\in S_1; j\in S_2 \ . \tag{1.21}$$

Similarly, the $|S_2|\times|S_1|$ matrix $\mathbf{\Psi}^r(\delta)$ is defined by

$$[\mathbf{\Psi}^r(\delta)]_{ij} = E\left[e^{-\delta\sigma^r(0,0)}1\{J^{(F)}(\sigma^r(0,0))=j\}|J^{(F)}(0)=i\right] \ , \quad i\in S_2; j\in S_1 \ . \tag{1.22}$$

We remark that Ahn and Ramaswami (2005) provided an algorithm which converges quadratically fast to compute $\mathbf{\Psi}(\delta)$ defined by (1.21). The quantity $\mathbf{\Psi}^r(\delta)$ can simply be computed in an identical way by reversing the roles of S_1 and S_2. It turns out that all other related quantities in the fluid flow process $\{F(t)\}_{t\geq 0}$ that will be used in this thesis can be expressed solely in terms of the matrices $\mathbf{\Psi}(\delta)$ and/or $\mathbf{\Psi}^r(\delta)$ (apart from some other known parameters of the model).

In addition, for the process $\{F(t)\}_{t\geq 0}$, we also define, for $i,j=1,2$, the $|S_i|\times|S_j|$ matrix ${}^z_a\widehat{\mathbf{f}}_{ij}(x,y,\delta)$ of the Laplace-Stieltjes transform (LST) of the first passage time ${}^z_a\sigma(x,y)$ with (k,l)-th element given by

$$[{}^z_a\widehat{\mathbf{f}}_{ij}(x,y,\delta)]_{kl} = E\left[e^{-\delta\ {}^z_a\sigma(x,y)}1\{J^{(F)}({}^z_a\sigma(x,y))=l\}|J^{(F)}(0)=k\right] \ , \quad k\in S_i; l\in S_j \ . \tag{1.23}$$

Similarly, ${}^z_a\widehat{\mathbf{f}}^r_{ij}(x,y,\delta)$ represents the corresponding LST of ${}^z_a\sigma^r(x,y)$ in the process $\{F^r(t),J^{(F)}(t)\}_{t\geq 0}$.

Note that there are some subtle relationships between the LSTs of the ordinary process and the reflected process. For example, it can be probabilistically argued that ${}^{x}\widehat{\mathbf{f}}_{22}(x,0,\delta) = {}_{0}\widehat{\mathbf{f}}^{r}_{22}(0,x,\delta)$. In addition, the matrix $\mathbf{\Psi}(\delta)$ can be expressed as $\mathbf{\Psi}(\delta) = \widehat{\mathbf{f}}_{12}(0,0,\delta)$. For future use we shall also write ${}^{x}\mathbf{\Psi}(\delta) = {}^{x}\widehat{\mathbf{f}}_{12}(0,0,\delta)$ and ${}^{x}\mathbf{\Psi}^{r}(\delta) = {}_{0}\widehat{\mathbf{f}}_{21}(x,x,\delta)$.

In what follows, we present the LSTs of various first passage times that will be used in this thesis and provide the references from which the matrices can be computed. As mentioned earlier, all these quantities can be expressed in terms of $\mathbf{\Psi}(\delta)$ and/or $\mathbf{\Psi}^{r}(\delta)$.

1. $\widehat{\mathbf{f}}^{r}_{11}(x,0,\delta)$ can be obtained indirectly from $\widehat{\mathbf{f}}_{22}(x,0,\delta)$ for $x \geq 0$ by reversing the roles of S_1 and S_2, where $\widehat{\mathbf{f}}_{22}(x,0,\delta)$ can be computed by Ahn and Ramaswami (2005, Theorem 3(c)).

2. $\widehat{\mathbf{f}}_{12}(x,0,\delta) = \mathbf{\Psi}(\delta)\widehat{\mathbf{f}}_{22}(x,0,\delta)$, for $x \geq 0$, is given by Ramaswami (2005, Theorem 3(a)), where $\widehat{\mathbf{f}}_{22}(x,0,\delta)$ follows from item 1.

3. ${}_{0}\widehat{\mathbf{f}}_{11}(0,x,\delta)$, for $x \geq 0$, is computed by Ramaswami (2006, Theorem 1 and Lemma 2).

4. ${}^{x}\mathbf{\Psi}(\delta)$, for $x \geq 0$, is expressed in terms of ${}_{0}\widehat{\mathbf{f}}_{11}(0,x,\delta)$ by Ramaswami (2006, Theorem 2), which is item 3 above. ${}^{x}\mathbf{\Psi}^{r}(\delta)$ can be obtained by reversing the roles of S_1 and S_2 (see Ramaswami (2006, Theorem 4)).

5. ${}_{0}\widehat{\mathbf{f}}_{11}(x,y,\delta) = [\mathbf{I} - {}^{y-x}\mathbf{\Psi}(\delta)\, {}^{x}\mathbf{\Psi}^{r}(\delta)]^{-1}\, {}_{0}\widehat{\mathbf{f}}_{11}(0,y-x,\delta)$, for $0 \leq x < y$, is given by Ahn et al. (2007, Theorem 1 (b)), with ${}_{0}\widehat{\mathbf{f}}_{11}(0,y-x,\delta)$ computed by item 3 above and ${}^{y-x}\mathbf{\Psi}(\delta)$ (and ${}^{x}\mathbf{\Psi}^{r}(\delta)$) computed by item 4.

6. ${}^{x}\widehat{\mathbf{f}}_{22}(x,0,\delta) = {}_{0}\widehat{\mathbf{f}}^{r}_{22}(0,x,\delta)$, for $x \geq 0$, is given by Ramaswami (2006, Theorem 3), while ${}_{0}\widehat{\mathbf{f}}^{r}_{22}(0,x,\delta)$ can be obtained from ${}_{0}\widehat{\mathbf{f}}_{11}(0,x,\delta)$ (item 3) by reversing the roles of S_1 and S_2.

7. ${}^{y}\widehat{\mathbf{f}}_{12}(x,0,\delta)$ and ${}^{y}\widehat{\mathbf{f}}_{22}(x,0,\delta)$, for $0 \leq x < y$, are both expressed in terms of ${}^{y-x}\mathbf{\Psi}(\delta)$ (and ${}^{x}\mathbf{\Psi}^{r}(\delta)$) (item 4) and ${}^{x}\widehat{\mathbf{f}}_{22}(x,0,\delta)$ (item 6) by Ahn et al. (2007, Theorem 1 (a)).

For other fluid flow quantities, interested readers are referred to Ahn et al. (2007), Ahn and Ramaswami (2005, 2006) and Ramaswami (2006).

Applications of fluid flow process in analyzing MAP risk model can be found in, e.g., Ahn and Badescu (2007), Ahn et al. (2007), Asmussen et al. (2002), Badescu et al. (2005, 2007a,b), Badescu and Landriault (2008), Ramwasami (2006) and references therein. In such an existing connection between fluid and risk processes, it is assumed that the set S_0 is empty. By extracting the times spent in S_2 and then pasting the rest of $\{F(t)\}_{t\geq 0}$ together, one essentially obtains the surplus process $\{U_t\}_{t\geq 0}$ of the MAP risk model. See Figure 1.2. We also refer interested readers to Badescu et al. (2005) and Ramaswami (2006) for construction of the generator $\mathbf{Q}$ (in (1.20)) in terms of the generators $\mathbf{G}_0$, $\mathbf{G}_1$ and the parameters of the claim size densities $p_{ij}(.)$'s.

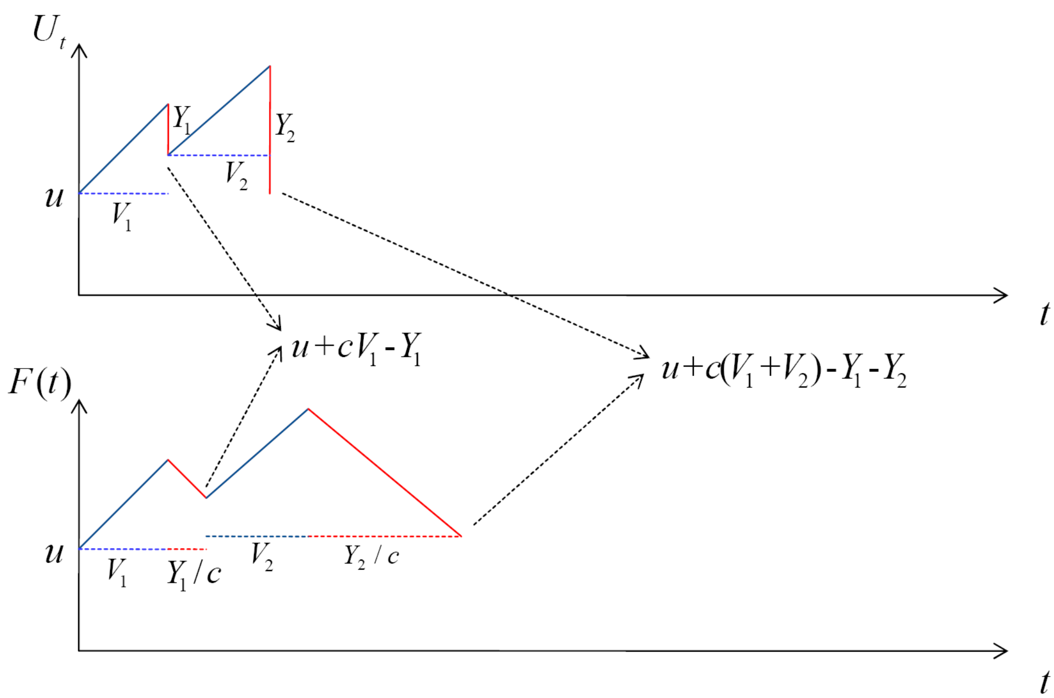

Figure 1.2: Existing connection between $\{U_t\}_{t\geq 0}$ and $\{F(t)\}_{t\geq 0}$

We remark that in order to use fluid flow techniques to analyze the MAP risk model, all the densities $p_{ij}(.)$'s have to be phase-type distributed. Furthermore, since every decrease of the surplus process (due to a claim), say of size y, is represented by a decreasing segment of length y/c in the corresponding fluid flow process, the generators $\mathbf{Q}_{21}$ and $\mathbf{Q}_{22}$ always depend on c, and we shall emphasize this dependency by writing $\mathbf{Q}_{21,c}$ and $\mathbf{Q}_{22,c}$ instead of $\mathbf{Q}_{21}$ and $\mathbf{Q}_{22}$ respectively. In general we have $\mathbf{Q}_{21,c} = c\mathbf{Q}_{21,1}$ and $\mathbf{Q}_{22,c} = c\mathbf{Q}_{22,1}$. See Badescu et al. (2007a, Examples 2.1 and 2.2). It is also important to note that the CTMCs $\{J^{(F)}(t)\}_{t\geq 0}$ and $\{J_t\}_{t\geq 0}$ (underlying the fluid flow process and the MAP risk process respectively) are different. In general, $\{J^{(F)}(t)\}_{t\geq 0}$ has many more states than $\{J_t\}_{t\geq 0}$, due to the fact that in constructing the generator $\mathbf{Q}$ of $\{J^{(F)}(t)\}_{t\geq 0}$ one has to keep track of both the states of $\{J_t\}_{t\geq 0}$ as well as the 'phases' of the phase-type claim sizes. In addition, when ruin-related quantities are analyzed via fluid flow process they are usually defined with respect to the states in the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$, as we shall see in later chapters.

An important observation which is central to the whole connection between fluid process and risk process was made by Ramaswami (2006). He observed that the portion of the time spent by the fluid flow process $\{F(t)\}_{t\geq 0}$ in S_1 during a first passage time $\sigma(x, y)$ is given by $\sigma(x, y)/2 + (y - x)/(2c)$. This property will be implicitly used many times in later chapters.

1.5 Organization of the thesis

This thesis is organized as follows. In Chapter 2, in the context of a MAP risk model perturbed by diffusion, the classical Gerber-Shiu function (1.2) and the moments of the total discounted dividends are derived, and it is shown that a matrix form of the *dividends-penalty identity* (see Gerber et al. (2006) and Lin et al. (2003)) holds true. Our methodology relies on the use of integro-differential equations and their solutions. A barrier level which depends on the environmental states is also considered. Chapter 3 considers the *dual* MAP risk model under a threshold dividend strategy. The Laplace transform of the time of ruin and the moments of discounted dividends are studied via the connection to a fluid queue described above. In Chapter 4, we propose to analyze the Gerber-Shiu function and some discounted joint densities in the generalized Sparre Andersen risk model where the generic pair $(V, Y/c)$ is assumed to follow a *bivariate phase-type distribution*. A novel fluid flow process is constructed in our analysis such that the set S_0 is non-empty (unlike the existing connection). In Chapter 5, we focus on the study of the generalized Gerber-Shiu function (1.11) in the semi-Markovian risk model described in Section 1.2.3. It is shown that the generalized Gerber-Shiu function satisfies a matrix defective renewal equation, and some discounted joint densities involving the new variables are derived. Chapter 6 revisits the MAP risk model in which (a special case of) the generalized Gerber-Shiu function (1.12) is examined. It is shown that such a Gerber-Shiu function can be expressed in terms of the classical Gerber-Shiu function (1.2) and the Laplace transform of a first passage time. While the traditional methods (i.e. conditioning on the time and amount of the first claim to obtain integral equations) are mainly used in Chapters 5 and 6, at the end of both chapters, the discounted joint densities of various ruin-related quantities are also derived using the existing connection to fluid process. In Chapter 7, a generalization of the MAP risk model is proposed by allowing for the interclaim time to have an arbitrary distribution (instead of being phase-type distributed), and the classical Gerber-Shiu function (with a specific form of penalty function) is considered. Chapter 8 revisits the generalized Sparre Andersen risk model but further generalizes the premium income to allow for *surplus-dependent premium rate*. Some ordering properties of the last interclaim time V_{N_T} and the claim causing ruin Y_{N_T} are studied in relation to the generic interclaim time V and the generic claim size Y using the generalized Gerber-Shiu function (1.10). The Gerber-Shiu function (1.10) itself is also considered in more detail in the compound Poisson risk model involving a dividend threshold or credit interest. Chapter 9 ends the thesis with some concluding remarks and directions for future research.

It is important to note that while efforts have been made to keep the notations as consistent as possible, due to the different models and quantities considered in this thesis, the reader is recommended to treat the remaining chapters as being independent of each other.

Chapter 2

Perturbed MAP risk model with a dividend barrier

2.1 Introduction

In this chapter, we consider the MAP risk model described in Section 1.2.1. Furthermore, to account for small fluctuations in the level of surplus a diffusion component is included in the surplus process $\{U_t\}_{t\geq 0}$. Specifically, we assume that the surplus process is perturbed by a *Brownian motion* with mean 0 and volatility $\sigma_i > 0$ whenever the CTMC $\{J_t\}_{t\geq 0}$ is in state i. Under the above descriptions, the surplus process $\{U_t\}_{t\geq 0}$ is now modified to give (in contrast to (1.1))

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i + \int_0^t \sigma_{J_s}\, dW_s \ , \quad t \geq 0 \ , \tag{2.1}$$

where $u = U_0$ is the initial surplus, and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion independent of $\{Y_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$. Since ruin can occur due to the diffusion component, for notational convenience we slightly rewrite the time of ruin as $T = \inf\{t \geq 0 : U_t \leq 0\}$.

It is further assumed that a *dividend barrier strategy* is applied to the surplus process $\{U_t\}_{t\geq 0}$ defined by (2.1). For such a dividend strategy, the insurer pays the overflow as dividends to the shareholders whenever the surplus level attains a fixed barrier level $b > 0$; otherwise, no dividends are paid (see, e.g., Gerber (1979), Gerber and Shiu (2004a) and Lin et al. (2003)). We then denote the barrier-modified surplus process by $\{U_t^{(b)}\}_{t\geq 0}$. To give a formal definition for $\{U_t^{(b)}\}_{t\geq 0}$, we require the process of running maximum $\{Z_t\}_{t\geq 0}$ corresponding to the barrier process $\{U_t\}_{t\geq 0}$, which is defined through $Z_t = \max_{0\leq s\leq t} U_s$. Then the total (non-discounted) dividends paid until time t under a barrier strategy is given by

$$D_t = (Z_t - b)_+ \ , \quad t \geq 0 \ . \tag{2.2}$$

Then, the surplus process of interest in this chapter, $\{U_t^{(b)}\}_{t\geq 0}$, satisfies

$$U_t^{(b)} = U_t - D_t \ , \quad t \geq 0 \ . \tag{2.3}$$

Pertaining to the surplus process (2.3) is the time to ruin $T_b = \inf\{t \geq 0 : U_t^{(b)} \leq 0\}$ which is finite almost surely (a.s.). Another important quantity of interest in the barrier model is the total discounted dividends paid until ruin under a force of interest $\delta \geq 0$, which is defined by

$$D_\delta(u;b) = \int_0^{T_b} e^{-\delta s}\, dD_s \, \Big| U_0^{(b)} = u \ , \quad 0 \leq u \leq b \ . \tag{2.4}$$

Furthermore, we let $M_{\delta,i}(s,u;b)$ be the moment generating function of $D_\delta(u;b)$ conditional on the initial state $J_0 = i$ of the CTMC, i.e., for $i \in E$,

$$M_{\delta,i}(s,u;b) = E\left[e^{sD_\delta(u;b)} | J_0 = i\right] = 1 + \sum_{n=1}^{\infty} \frac{s^n}{n!} V_{\delta,i,n}(u;b) \ , \quad 0 \leq u \leq b \ , \tag{2.5}$$

where $V_{\delta,i,n}(u;b) = E[\{D_\delta(u,b)\}^n | J_0 = i]$ is the n-th moment of $D_\delta(u,b)$. We shall adopt the usual convention that $V_{\delta,i,0}(u;b) = 1$, i.e. the zero-th moment of $D_\delta(u,b)$ is 1. In addition, for convenience, we shall denote the first moment by $V_{\delta,i,1}(u;b) = V_{\delta,i}(u;b)$. The m-dimensional column vectors $\mathbf{V}_{\delta,n}(u;b) = (V_{\delta,1,n}(u;b), \ldots, V_{\delta,m,n}(u;b))^T$ and $\mathbf{V}_{\delta}(u;b) = (V_{\delta,1}(u;b), \ldots, V_{\delta,m}(u;b))^T$ are also defined.

In this chapter, one of our interests is to derive a result equivalent to the *dividends-penalty identity* of Gerber et al. (2006) in a perturbed MAP risk model. To this end, we have to define the Gerber-Shiu function for the process $\{U_t^{(b)}\}_{t\geq 0}$. However, as the Gerber-Shiu function under a barrier strategy will be expressed in terms of the Gerber-Shiu function in a barrier-free process (see, e.g., Lin et al. (2006) and Li and Lu (2008)), we first define the Gerber-Shiu function for the process $\{U_t\}_{t\geq 0}$. Due to the diffusion component in (2.1), contributions to the Gerber-Shiu function shall be broken down by the cause of ruin: oscillation or a claim. If ruin is due to oscillation, both the surplus prior to ruin U_{T^-} and the deficit at ruin $|U_T|$ are simply 0. For that purpose, we define the Gerber-Shiu function as, for $i \in E$,

$$\phi_{\delta,i}(u) = w_0 \phi_{\delta,i}^d(u) + \phi_{\delta,i}^c(u) \ , \quad u \geq 0 \ , \tag{2.6}$$

where w_0 is the fixed penalty at ruin if the ruin is caused by oscillation,

$$\phi_{\delta,i}^d(u) = E\left[e^{-\delta T} 1\{T < \infty, U_T = 0\} | U_0 = u, J_0 = i\right] \ , \quad u \geq 0 \ , \tag{2.7}$$

is the Laplace transform of the time of ruin in $\{U_t\}_{t\geq 0}$ due to oscillation, and

$$\phi_{\delta,i}^c(u) = E\left[e^{-\delta T} w(U_{T^-}, |U_T|) 1\{T < \infty, U_T < 0\} | U_0 = u, J_0 = i\right] \ , \quad u \geq 0 \ , \tag{2.8}$$

is the contribution to the Gerber-Shiu function in $\{U_t\}_{t\geq 0}$ due to a claim. For later use we define the m-dimensional column vectors $\mathbf{\Phi}_{\delta}(u) = (\phi_{\delta,1}(u), \ldots, \phi_{\delta,m}(u))^T$, $\mathbf{\Phi}_{\delta}^d(u) = (\phi_{\delta,1}^d(u), \ldots, \phi_{\delta,m}^d(u))^T$ and $\mathbf{\Phi}_{\delta}^c(u) = (\phi_{\delta,1}^c(u), \ldots, \phi_{\delta,m}^c(u))^T$. As far as the Gerber-Shiu functions in the barrier-free process are concerned, it is common to assume either $\delta > 0$ or the positive security loading condition holds.

In the perturbed MAP risk model, this condition is identical to the unperturbed model and is given by (1.3).

Similarly, for $\{U_t^{(b)}\}_{t\geq 0}$ we define, for $i \in E$,

$$\phi_{\delta,i}(u;b) = w_0 \phi^d_{\delta,i}(u;b) + \phi^c_{\delta,i}(u;b) \ , \quad 0 \leq u \leq b \ , \tag{2.9}$$

$$\phi^d_{\delta,i}(u;b) = E\left[e^{-\delta T_b} 1\{U_{T_b}^{(b)} = 0\} | U_0^{(b)} = u, J_0 = i\right] \ , \quad 0 \leq u \leq b \ , \tag{2.10}$$

$$\phi^c_{\delta,i}(u;b) = E\left[e^{-\delta T_b} w(U_{T_b^-}^{(b)}, |U_{T_b}^{(b)}|) 1\{U_{T_b}^{(b)} < 0\} | U_0^{(b)} = u, J_0 = i\right] \ , \quad 0 \leq u \leq b \ , \tag{2.11}$$

the column vectors $\mathbf{\Phi}_\delta(u;b) = (\phi_{\delta,1}(u;b), \ldots, \phi_{\delta,m}(u;b))^T$,$\mathbf{\Phi}^d_\delta(u;b) = (\phi^d_{\delta,1}(u;b), \ldots, \phi^d_{\delta,m}(u;b))^T$ and $\mathbf{\Phi}^c_\delta(u;b) = (\phi^c_{\delta,1}(u;b), \ldots, \phi^c_{\delta,m}(u;b))^T$. Note that the event $\{T_b < \infty\}$ has been dropped out from the indicator functions in both the definitions (2.10) and (2.11) because such an event occurs a.s. as discussed before.

As pointed out in Section 1.4.2, MAP risk models have been mainly analyzed via connecting the surplus process to a fluid flow process. Such arguments are most probabilistic, and results are usually expressed in terms of the Laplace transforms of various first passage times in the fluid flow model which are known in the literature. However, a main drawback of using such *matrix analytic methods* (MAMs) to analyze MAP risk models is the assumption that the claim size densities $p_{ij}(.)$ are all phase-type. This excludes, for example, heavy-tailed claim size distributions, from the analysis. In this chapter, we show that the use of a purely analytic approach allows us to analyze MAP risk models with arbitrary claim size distributions. See, e.g., Badescu (2008).

The chapter is structured as follows: in Sections 2.2 and 2.3, we analyze respectively the moments of the discounted dividend payments and the Gerber-Shiu function for the surplus process $\{U_t^{(b)}\}_{t\geq 0}$. In addition, it is also shown in Section 2.3 that a relationship similar to the dividends-penalty identity also holds for the class of perturbed MAP risk processes, extending some results of Li and Lu (2008) derived in the context of a Markov-modulated risk model. In Section 2.4, all these ruin-related quantities are revisited in the surplus process (2.3) with the only exception that the barrier level enforced at a given time t depends on the state of the CTMC $\{J_t\}_{t\geq 0}$ at that time. Section 2.5 is an appendix about the barrier-free model.

2.2 Discounted dividend payments

2.2.1 Expected discounted dividend payments

We shall follow the heuristic arguments in Gerber and Landry (1998, Section 3) to derive a system of integro-differential equations for $V_{\delta,i}(u;b)$. Considering a very small time interval of length h leads to, for $i \in E$,

$$V_{\delta,i}(u;b) = (1+G_{0,ii}h)(1-\delta h)E[V_{\delta,i}(u+ch+\sigma_i W_h;b)] + \sum_{j=1,j\neq i}^{m} G_{0,ij} h V_{\delta,j}(u;b) + \sum_{j=1}^{m} G_{1,ij} h \int_0^u V_{\delta,j}(u-y;b) p_{ij}(y)\, dy + o(h) \ , \quad 0 < u < b \ . \tag{2.12}$$

According to Gerber and Landry (1998) and Tsai and Willmot (2002), one also has that

$$E[V_{\delta,i}(u+ch+\sigma_i W_h;b)] = V_{\delta,i}(u;b) + \left[c V'_{\delta,i}(u;b) + \frac{\sigma_i^2}{2} V''_{\delta,i}(u;b) \right] h + o(h) \ , \quad 0 < u < b \ . \tag{2.13}$$

Note that here we have assumed that $V_{\delta,i}(u;b)$ is twice differentiable in u for $0 < u < b$. For detailed discussion of differentiability of various ruin functions, the reader is referred to, e.g., Cai (2004), Cai and Yang (2005), Loisel (2005), Wang and Wu (2001) and Zhu and Yang (2009).

By substituting (2.13) into (2.12), dividing both sides by h and letting $h \to 0$, one arrives at, for $i \in E$,

$$\frac{\sigma_i^2}{2} V''_{\delta,i}(u;b) + cV'_{\delta,i}(u;b) - \delta V_{\delta,i}(u;b) + \sum_{j=1}^{m} G_{0,ij} V_{\delta,j}(u;b) + \sum_{j=1}^{m} G_{1,ij} \int_0^u V_{\delta,j}(u-y;b) p_{ij}(y)\, dy = 0 \ ,$$

$$0 < u < b \ . \qquad (2.14)$$

The boundary conditions associated to the above system of integro-differential equations are given by, for $i \in E$,

$$V_{\delta,i}(0;b) = 0 \ , \qquad (2.15)$$

and

$$V'_{\delta,i}(b;b) = 1 \ . \qquad (2.16)$$

Indeed, (2.15) holds since ruin occurs immediately a.s. for the surplus process $\{U_t^{(b)}\}_{t \geq 0}$ with zero initial surplus due to the diffusion component. The condition (2.16) is a special case of (2.42) at $n = 1$ and the reader is referred to the proof in Section 2.2.2.

To determine the form of the solution for $V_{\delta,i}(u;b)$, we consider the system of integro-differential equations, for $i \in E$,

$$\frac{\sigma_i^2}{2} v''_{\delta,i}(u) + cv'_{\delta,i}(u) - \delta v_{\delta,i}(u) + \sum_{j=1}^{m} G_{0,ij} v_{\delta,j}(u) + \sum_{j=1}^{m} G_{1,ij} \int_0^u v_{\delta,j}(u-y) p_{ij}(y)\, dy = 0 \ , \quad u \geq 0 \ . \ (2.17)$$

Since (2.17) forms a system of second order integro-differential equations, it is clear that the initial conditions $(v_{\delta,1}(0), \ldots, v_{\delta,m}(0))$ and $(v'_{\delta,1}(0), \ldots, v'_{\delta,m}(0))$ uniquely determine its solution $(v_{\delta,1}(u), \ldots, v_{\delta,m}(u))$. Thus, for a given $j \in E$, let $\mathbf{v}^A_{\delta,\cdot,j}(u) = (v^A_{\delta,1,j}(u), \ldots, v^A_{\delta,m,j}(u))$ and $\mathbf{v}^B_{\delta,\cdot,j}(u) = (v^B_{\delta,1,j}(u), \ldots, v^B_{\delta,m,j}(u))$ be the particular solutions of (2.17) with initial conditions $\mathbf{v}^A_{\delta,\cdot,j}(0) = \mathbf{e}_j$, $(\mathbf{v}^A_{\delta,\cdot,j})'(0) = \mathbf{0}$ and $\mathbf{v}^B_{\delta,\cdot,j}(0) = \mathbf{0}$, $(\mathbf{v}^B_{\delta,\cdot,j})'(0) = \mathbf{e}_j$ respectively. Here $\mathbf{e}_j$ is an m-dimensional row vector with the only non-null entry of 1 at the j-th position. According to Lakshmikantham and Rao (1995, Theorem 2.1.1), the set of $2m$ solutions $\{\mathbf{v}^A_{\delta,\cdot,j}(u)\}_{j=1}^m$ and $\{\mathbf{v}^B_{\delta,\cdot,j}(u)\}_{j=1}^m$ are linearly independent, and hence the general solution of (2.17), namely $(v_{\delta,1}(u), \ldots, v_{\delta,m}(u))$, can be expressed as, for $i \in E$,

$$v_{\delta,i}(u) = \sum_{j=1}^{m} v_{\delta,j}(0) v^A_{\delta,i,j}(u) + \sum_{j=1}^{m} v'_{\delta,j}(0) v^B_{\delta,i,j}(u) \ , \quad u \geq 0 \ . \tag{2.18}$$

It follows that, the solution to the system (2.14) is given by, for $i \in E$,

$$V_{\delta,i}(u;b) = \sum_{j=1}^{m} V_{\delta,j}(0;b) v^A_{\delta,i,j}(u) + \sum_{j=1}^{m} V'_{\delta,j}(0;b) v^B_{\delta,i,j}(u) \ , \quad 0 \leq u \leq b \ . \tag{2.19}$$

Incorporating the boundary conditions (2.15) into the above representation leads to, for $i \in E$,

$$V_{\delta,i}(u;b) = \sum_{j=1}^{m} V'_{\delta,j}(0;b) v^B_{\delta,i,j}(u) \ , \quad 0 \leq u \leq b \ . \tag{2.20}$$

Using the form of the solution (2.20), the boundary condition (2.16) can be rewritten as, for $i \in E$,

$$V'_{\delta,i}(b;b) = \sum_{j=1}^{m} V'_{\delta,j}(0;b) (v^B_{\delta,i,j})'(b) = 1 \ . \tag{2.21}$$

Letting $\mathbf{v}^B_\delta(u) = [v^B_{\delta,i,j}(u)]^m_{i,j=1}$, (2.21) can be re-expressed as, using a matrix representation,

$$\mathbf{V}'_\delta(0;b) = [(\mathbf{v}^B_\delta)'(b)]^{-1}\mathbf{1} \ . \tag{2.22}$$

Combining (2.20) and (2.22), the expected discounted dividend payments admits the representation

$$\mathbf{V}_\delta(u;b) = \mathbf{v}^B_\delta(u)[(\mathbf{v}^B_\delta)'(b)]^{-1}\mathbf{1} \ , \quad 0 \leq u \leq b \ . \tag{2.23}$$

The above expression gives the first moment of $D_\delta(u,b)$ conditional on the initial state J_0 of the underlying CTMC. Pre-multiplying both sides of (2.23) by the initial probability vector $\mathbf{a}$ yields the first unconditional moment of $D_\delta(u,b)$. In what follows, results are given in conditional form only. It is understood that the general unconditional counterpart can always be obtained by pre-multiplying the conditional representation of a given ruin-related quantity by $\mathbf{a}$.

Remark 1 *(2.23) can be viewed as a complement to the (scalar) result derived by Gerber et al. (2006, Eq. (6)) for a class of Markov surplus processes which are skip-free upwards (i.e. there are only downward jumps but not upward jumps). However, their probabilistic proof further allows the possibility of, for instance, surplus-dependent premium (see Chapter 8).*

Returning to (2.23), one notices that the m particular solutions $\{\mathbf{v}^B_{\delta,\cdot,j}(u)\}_{j=1}^m$ play a key role in the representation of $\mathbf{V}_\delta(u;b)$. Thus, further details are provided next for the evaluation of $\mathbf{v}^B_\delta(u)$. Taking Laplace transforms on both sides of (2.17) with $v_{\delta,i}(.)$ replaced by $v^B_{\delta,i,j}(.)$, one finds, for $i,j \in E$,

$$\left(\frac{\sigma_i^2}{2}s^2 + cs - \delta\right)\widetilde{v}^B_{\delta,i,j}(s) + \sum_{k=1}^m G_{0,ik}\widetilde{v}^B_{\delta,k,j}(s) + \sum_{k=1}^m G_{1,ik}\widetilde{p}_{ik}(s)\widetilde{v}^B_{\delta,k,j}(s) = \frac{\sigma_i^2}{2}(v^B_{\delta,i,j})'(0) \ . \tag{2.24}$$

With $\widetilde{\mathbf{v}}^B_\delta(s) = [\widetilde{v}^B_{\delta,i,j}(s)]_{i,j=1}^m$, we note that $(\mathbf{v}^B_\delta)'(0) = \mathbf{I}$ and therefore (2.24) can be expressed in terms of matrices as

$$\mathbf{A}_\delta(s)\widetilde{\mathbf{v}}^B_\delta(s) = \operatorname{diag}\left\{\frac{\sigma_1^2}{2}, \ldots, \frac{\sigma_m^2}{2}\right\} \ , \tag{2.25}$$

where

$$\mathbf{A}_\delta(s) = \mathbf{H}_\delta(s) + \mathbf{G}_0 + \widetilde{\mathbf{G}}_p(s) \ , \tag{2.26}$$

with $\mathbf{H}_\delta(s) = \operatorname{diag}\{\sigma_1^2 s^2/2 + cs - \delta, \ldots, \sigma_m^2 s^2/2 + cs - \delta\}$ and $\widetilde{\mathbf{G}}_p(s) = [G_{1,ij}\widetilde{p}_{ij}(s)]_{i,j=1}^m$. Let $\operatorname{adj}\mathbf{A}_\delta(s)$ be the adjoint matrix of $\mathbf{A}_\delta(s)$ and $[\operatorname{adj}\mathbf{A}_\delta(s)]_{ij}$ be its (i,j) element. It follows that

$$\widetilde{v}^B_{\delta,i,j}(s) = \frac{\sigma_j^2}{2}\frac{[\operatorname{adj}\mathbf{A}_\delta(s)]_{ij}}{\det\mathbf{A}_\delta(s)} \ . \tag{2.27}$$

If the Laplace transforms $\widetilde{p}_{ij}(s)$'s are all ratios of two polynomials in s, then $[\operatorname{adj}\mathbf{A}_\delta(s)]_{ij}$ and $\det\mathbf{A}_\delta(s)$ are all also rational functions in s and therefore the right-hand side of (2.27) can be resolved into partial fractions. This allows analytic inversion of the Laplace transforms, as illustrated below.

For example, for the remainder of this subsection, we assume that for $i, j \in E$,

$$\widetilde{p}_{ij}(s) = \frac{q_{1,ij}(s)}{q_{2,ij}(s)} \ , \tag{2.28}$$

where $q_{1,ij}(s)$ is a polynomial of degree less than r_{ij} and $q_{2,ij}(s)$ is a polynomial of degree exactly r_{ij}. Furthermore, for each fixed $i, j \in E$, $q_{1,ij}(s)$ and $q_{2,ij}(s)$ have no common factor and $q_{1,ij}(0)/q_{2,ij}(0) = 1$. It is clear that the so-called *Lundberg's fundamental equation*

$$\det \mathbf{A}_\delta(s) = 0 \ , \tag{2.29}$$

has $2m + r$ solutions, say $\{\rho_i\}_{i=1}^{2m+r}$, with $r = \sum_{i=1}^m \sum_{j=1}^m r_{ij}$. Letting $q_2(s) = \prod_{i=1}^m \prod_{j=1}^m q_{2,ij}(s)$, (2.27) can be rewritten as

$$\widetilde{v}^B_{\delta,i,j}(s) = \frac{\sigma_j^2}{2} \frac{\varpi_{1,ij}(s)}{\varpi_2(s)} \ , \tag{2.30}$$

where $\varpi_{1,ij}(s) = q_2(s)[\operatorname{adj} \mathbf{A}_\delta(s)]_{ij}$ is a polynomial of degree less than $2m + r$ and $\varpi_2(s) = q_2(s) \det \mathbf{A}_\delta(s)$ is of degree $2m+r$. Assuming that the solutions $\{\rho_i\}_{i=1}^{2m+r}$ are distinct and choosing an arbitrary real number κ such that $\kappa \neq \rho_i$ for $i = 1, \ldots, 2m + r$, (2.30) can be rewritten as

$$\widetilde{v}^B_{\delta,i,j}(s) = \frac{\sigma_j^2}{2\varpi_2(\kappa)} \sum_{l=1}^{2m+r} \varpi_{1,ij}(\rho_l) \vartheta_l(\kappa) \frac{\kappa - \rho_l}{s - \rho_l} \tag{2.31}$$

using the Lagrange interpolating polynomial, where $\vartheta_l(s) = \prod_{k=1,k\neq l}^{2m+r} \frac{s-\rho_k}{\rho_l-\rho_k}$. Inverting (2.31) with respect to s yields

$$v^B_{\delta,i,j}(u) = \frac{\sigma_j^2}{2\varpi_2(\kappa)} \sum_{l=1}^{2m+r} \varpi_{1,ij}(\rho_l) \vartheta_l(\kappa)(\kappa - \rho_l) e^{\rho_l u} \ , \quad u \geq 0 \ . \tag{2.32}$$

Combining (2.23) and (2.32), a closed-form expression for $\mathbf{V}_\delta(u; b)$ can readily be found when the claim size densities have a rational Laplace transform. We also refer the reader to Chapter 9 for discussion of the roots of Lundberg's fundamental equations.

2.2.2 Higher-order moments of discounted dividends

As for the higher moments of the discounted dividends, one could follow Gerber and Shiu (2004a) or the arguments in Section 2.2.1 to derive a system of integro-differential equations for the moment generating functions defined by (2.5). By doing so we arrive at, for $i \in E$,

$$\frac{\sigma_i^2}{2}\frac{\partial^2}{\partial u^2}M_{\delta,i}(s,u;b) + c\frac{\partial}{\partial u}M_{\delta,i}(s,u;b) - \delta s\frac{\partial}{\partial s}M_{\delta,i}(s,u;b) + \sum_{j=1}^{m} G_{0,ij}M_{\delta,j}(s,u;b)$$
$$+ \sum_{j=1}^{m} G_{1,ij}\int_0^u M_{\delta,j}(s,u-y;b)p_{ij}(y)\,dy = 0 \ , \quad 0 < u < b \ . \tag{2.33}$$

In addition, the boundary conditions are given by, for $i \in E$,

$$M_{\delta,i}(s,0;b) = 1 \ , \tag{2.34}$$

and

$$\left.\frac{\partial M_{\delta,i}(s,u;b)}{\partial u}\right|_{u=b} = sM_{\delta,i}(s,b;b) \ . \tag{2.35}$$

Note that the condition (2.34) is trivial since ruin occurs immediately a.s. for $\{U_t^{(b)}\}_{t\geq 0}$ with zero initial surplus and hence $D_\delta(u;b) = 0$ a.s.. The condition (2.35) can be obtained via the heuristic argument used by Gerber and Shiu (2004b). For the sake of completeness it is given as follows. Define, for $i \in E$,

$$\Theta_{\delta,i}(s,u;b) = E[\vartheta(sD_\delta(u,b))|J_0 = i] \ , \quad 0 \leq u \leq b \ , \tag{2.36}$$

where $\vartheta(.)$ is a non-negative differentiable function. For the surplus process $\{U_b(t), t \geq 0\}$, we consider two situations with initial surplus b and $b-h$ respectively for some small positive h. In both situations, it is almost certain that the surplus process will be at the barrier level b shortly (and before ruin). However, a dividend of h would have been paid by then for the case where $U_b(0) = b$, but not for $U_b(0) = b - h$. Now being at level b, both processes evolve identically going forward. Thus the approximation $D_\delta(b; b) \approx h + D_\delta(b-h; b)$ holds and therefore

$$\vartheta(sD_\delta(b;b)) - \vartheta(sD_\delta(b-h;b)) \approx sh\vartheta'(sD_\delta(b;b)) \ . \tag{2.37}$$

Taking expectation conditional on the initial state $J_0 = i$ leads to, for $i \in E$,

$$\Theta_{\delta,i}(s,b;b) - \Theta_{\delta,i}(s,b-h;b) \approx shE[\vartheta'(sD_\delta(b,b))|J_0 = i]. \tag{2.38}$$

Division by h followed by taking limit as $h \to 0$ yields

$$\left.\frac{\partial\Theta_{\delta,i}(s,u;b)}{\partial u}\right|_{u=b} = sE[\vartheta'(sD_\delta(b,b))|J_0 = i] \ . \tag{2.39}$$

It can be easily seen that (2.35) is a special case of (2.39) with the choice of $\vartheta(x) = e^x$.

Using the expression (2.5) and equating the coefficients of s^n on both sides of (2.33) for $n = 1, 2, \ldots$, one obtains, for $i \in E$,

$$\begin{aligned}\frac{\sigma_i^2}{2}V''_{\delta,i,n}(u;b) + cV'_{\delta,i,n}(u;b) - n\delta V_{\delta,i,n}(u;b) + \sum_{j=1}^{m} G_{0,ij}V_{\delta,j,n}(u;b)& \\ + \sum_{j=1}^{m} G_{1,ij}\int_0^u V_{\delta,j,n}(u-y;b)p_{ij}(y)\,dy = 0 \ , \quad 0 < u < b \ .&\end{aligned} \tag{2.40}$$

Note that (2.40) is of the same form as (2.14) with δ replaced by $n\delta$. Similarly, the boundary conditions (2.34) and (2.35) reduce to, for $i \in E$,

$$V_{\delta,i,n}(0;b) = 0 \ , \tag{2.41}$$

and

$$V'_{\delta,i,n}(b;b) = nV_{\delta,i,n-1}(b;b) \ , \tag{2.42}$$

respectively. We omit the rather repetitive arguments and state that the solution of the system (2.40) subject to the boundary conditions (2.41) and (2.42) can be expressed as

$$\mathbf{V}_{\delta,n}(u;b) = n\mathbf{v}^B_{\delta,n}(u)[(\mathbf{v}^B_{\delta,n})'(b)]^{-1}\mathbf{V}_{\delta,n-1}(b;b) \ , \quad 0 \leq u \leq b \ , \tag{2.43}$$

$\mathbf{v}^B_{\delta,n}(u)$ is simply $\mathbf{v}^B_{\delta}(u)$ with all the calculations performed at a force of interest $n\delta$ instead of δ. From (2.43), it is clear that the closed-form expression for $\mathbf{V}_{\delta,n}(u;b)$ is given by

$$\mathbf{V}_{\delta,n}(u;b) = n!\mathbf{v}^B_{\delta,n}(u)[(\mathbf{v}^B_{\delta,n})'(b)]^{-1}\mathbf{v}^B_{\delta,n-1}(b)[(\mathbf{v}^B_{\delta,n-1})'(b)]^{-1}\dots\mathbf{v}^B_{\delta,1}(b)[(\mathbf{v}^B_{\delta,1})'(b)]^{-1}\mathbf{1} \ , \quad 0 \leq u \leq b \ . \tag{2.44}$$

Remark 2 *An equation of the form (2.44) for the higher-order moments of the discounted dividend payments before ruin have been shown to hold in the Markov-modulated risk model and in the Sparre Andersen model with phase-type interclaim time distribution (see Li and Lu (2007) and Cheung (2007) respectively).*

2.3 Gerber-Shiu function and dividends-penalty identity

Following the same arguments used to obtain (2.14), the systems of integro-differential equations for $\phi^d_{\delta,i}(u;b)$ and $\phi^c_{\delta,i}(u;b)$ are obtained as, for $i \in E$,

$$\frac{\sigma_i^2}{2}(\phi^d_{\delta,i})''(u;b) + c(\phi^d_{\delta,i})'(u;b) - \delta\phi^d_{\delta,i}(u;b) + \sum_{j=1}^m G_{0,ij}\phi^d_{\delta,j}(u;b) + \sum_{j=1}^m G_{1,ij}\int_0^u \phi^d_{\delta,j}(u-y;b)p_{ij}(y)\,dy = 0 \ , \quad 0 < u < b \ , \tag{2.45}$$

and

$$\frac{\sigma_i^2}{2}(\phi_{\delta,i}^c)''(u;b) + c(\phi_{\delta,i}^c)'(u;b) - \delta\phi_{\delta,i}^c(u;b) + \sum_{j=1}^m G_{0,ij}\phi_{\delta,j}^c(u;b)$$
$$+ \sum_{j=1}^m G_{1,ij}\left[\int_0^u \phi_{\delta,j}^c(u-y;b)p_{ij}(y)\,dy + \omega_{ij}(u)\right] = 0 \ , \quad 0 < u < b \ , \tag{2.46}$$

respectively, where

$$\omega_{ij}(u) = \int_u^\infty w(u, y-u)p_{ij}(y)\,dy \ , \quad u \geq 0 \ . \tag{2.47}$$

Given that ruin occurs immediately a.s. for the surplus process $\{U_t^{(b)}\}_{t\geq 0}$ with a zero initial surplus, we have that, for $i \in E$,

$$\phi_{\delta,i}^d(0;b) = 1 \ ; \quad \phi_{\delta,i}^c(0;b) = 0 \ . \tag{2.48}$$

Following the same heuristic arguments in obtaining (2.35) (i.e. by comparing two situations with initial surplus b and $b-h$ respectively for some small positive h), it can also be argued that, for $i \in E$,

$$(\phi_{\delta,i}^d)'(b;b) = (\phi_{\delta,i}^c)'(b;b) = 0 \ . \tag{2.49}$$

Such an argument was indeed used in Gerber et al. (2006) in deriving their Eq. (8).

Note that the systems of integro-differential equations (2.45) and (2.46) also hold true for the barrier-free counterparts $(\phi_{\delta,1}^d(u), \ldots, \phi_{\delta,m}^d(u))$ and $(\phi_{\delta,1}^c(u), \ldots, \phi_{\delta,m}^c(u))$ respectively in the risk model without barrier, with domain extended from $0 < u < b$ to $u > 0$. Thus, $(\phi_{\delta,1}^c(u), \ldots, \phi_{\delta,m}^c(u))$ can be viewed as a particular solution of (2.46). We also point out that (2.17) is the homogeneous version of (2.46) for which $\{\mathbf{v}_{\delta,\cdot,j}^A(u)\}_{j=1}^m$ and $\{\mathbf{v}_{\delta,\cdot,j}^B(u)\}_{j=1}^m$ are $2m$ linearly independent solutions. From the general theory of integro-differential equations, it follows that (see, e.g., Lakshmikantham and Rao (1995, p.50)), for $i \in E$,

$$\phi^c_{\delta,i}(u;b) = \phi^c_{\delta,i}(u) + \sum_{j=1}^{m} \left[\phi^c_{\delta,j}(0;b) - \phi^c_{\delta,j}(0)\right] v^A_{\delta,i,j}(u) + \sum_{j=1}^{m} \left[(\phi^c_{\delta,j})'(0;b) - (\phi^c_{\delta,j})'(0)\right] v^B_{\delta,i,j}(u) \ ,$$

$$0 \le u \le b \ . \quad (2.50)$$

From the boundary conditions (2.48) (which also hold true for $\phi^d_{\delta,i}(0)$ and $\phi^c_{\delta,i}(0)$), (2.50) can be simplified to, for $i \in E$,

$$\phi^c_{\delta,i}(u;b) = \phi^c_{\delta,i}(u) + \sum_{j=1}^{m} \left[(\phi^c_{\delta,j})'(0;b) - (\phi^c_{\delta,j})'(0)\right] v^B_{\delta,i,j}(u) \ , \quad 0 \le u \le b \ . \quad (2.51)$$

A matrix representation of (2.51) is given by

$$\mathbf{\Phi}^c_\delta(u;b) = \mathbf{\Phi}^c_\delta(u) + \mathbf{v}^B_\delta(u)\mathbf{\Upsilon} \ , \quad 0 \le u \le b \ , \quad (2.52)$$

where $\mathbf{\Upsilon} = ((\phi^c_{\delta,1})'(0;b) - (\phi^c_{\delta,1})'(0), \dots, (\phi^c_{\delta,m})'(0;b) - (\phi^c_{\delta,m})'(0))^T$. To determine $\mathbf{\Upsilon}$, we use the boundary conditions (2.49) which lead to

$$(\mathbf{\Phi}^c_\delta)'(b) + (\mathbf{v}^B_\delta)'(b)\mathbf{\Upsilon} = \mathbf{0} \ . \quad (2.53)$$

Solving for $\mathbf{\Upsilon}$ in (2.53) followed by substitution into (2.52) yields

$$\mathbf{\Phi}^c_\delta(u;b) = \mathbf{\Phi}^c_\delta(u) + \mathbf{v}^B_\delta(u)[(\mathbf{v}^B_\delta)'(b)]^{-1}(\mathbf{\Phi}^c_\delta)'(b) \ , \quad 0 \le u \le b \ . \quad (2.54)$$

Note that a similar line of logic leads to the following representation for $\mathbf{\Phi}^d_\delta(u;b)$, namely,

$$\mathbf{\Phi}^d_\delta(u;b) = \mathbf{\Phi}^d_\delta(u) + \mathbf{v}^B_\delta(u)[(\mathbf{v}^B_\delta)'(b)]^{-1}(\mathbf{\Phi}^d_\delta)'(b) \ , \quad 0 \le u \le b \ . \quad (2.55)$$

Combining (2.6), (2.54) and (2.55), one easily finds

$$\mathbf{\Phi}_\delta(u;b) = \mathbf{\Phi}_\delta(u) + \mathbf{v}_\delta^B(u)[(\mathbf{v}_\delta^B)'(b)]^{-1}\mathbf{\Phi}'_\delta(b) \ , \quad 0 \leq u \leq b \ . \tag{2.56}$$

Note from (2.54), (2.55) and (2.56) that $\mathbf{\Phi}_\delta^c(u;b)$, $\mathbf{\Phi}_\delta^d(u;b)$ and $\mathbf{\Phi}_\delta(u;b)$ are all expressed in terms their barrier-free counterparts $\mathbf{\Phi}_\delta^c(u)$, $\mathbf{\Phi}_\delta^d(u)$ and $\mathbf{\Phi}_\delta(u)$ respectively, apart from the matrix $\mathbf{v}_\delta^B(u)$ which has been discussed in Section 2.2.1. The matrices $\mathbf{\Phi}_\delta^c(u)$ and $\mathbf{\Phi}_\delta^d(u)$ will be given in the Appendix at the end of this chapter.

Remark 3 *In a compound Poisson risk model perturbed by diffusion, both (2.23) and (2.56) hold with the matrices and vectors replaced by scalars. It follows that the dividends-penalty identity, can be established between the Gerber-Shiu function and the expected discounted dividend payments (see Gerber et al. (2006)). However, while both (2.23) and (2.56) hold in the perturbed MAP risk model, their matrix representations do not allow us to express* $\mathbf{\Phi}_\delta(u;b)$ *in terms of* $\mathbf{V}_\delta(u;b)$*. Nevertheless, a comparison of (2.23) and (2.56) allows us to identify the matrix* $\mathbf{v}_\delta^B(u)[(\mathbf{v}_\delta^B)'(b)]^{-1}$ *as a key to the determination of both* $\mathbf{V}_\delta(u;b)$ *and* $\mathbf{\Phi}_\delta(u;b)$.

2.4 A barrier dependent on the environmental process: The two-state case

In this section, a different barrier strategy is applied to the perturbed MAP risk model described by (2.1). In the spirit of Zhu and Yang (2008) (who considered a threshold-type strategy), we consider a dividend barrier strategy for which the barrier level effective at a given time, say t, depends on the state of the CTMC $\{J_t\}_{t\geq 0}$ at time t. Let b_i be the barrier level effective whenever the CTMC $\{J_t\}_{t\geq 0}$ is in some state $i \in E$. We assume that the b_i's are distinct. The rationale of such a model is that, it makes more sense for an insurance company to set a higher barrier level when it is in a more ‘dangerous’ state (where the claims are ‘larger’, the interclaim times are ‘shorter’, and/or the diffusion component is larger reflecting more uncertainties), so that dividends are paid only if the surplus reaches a more secure level and more capital is available to deal with possible adverse claims experience. Once the economic environment returns to a ‘normal’ (or ‘non-dangerous’) state, the excess reserves might be released as lump sum dividends in addition to the resuming of ‘normal’ dividend payments at a lower barrier level.

In what follows, we are interested in the study of certain ruin-related quantities corresponding to the surplus process $\{U_t^{(\mathbf{b})}\}_{t\geq 0}$ under the above descriptions. Here $\mathbf{b} = (b_1, \ldots, b_m)$ is a row vector containing the set of barrier levels. We assume without loss of generality that the environmental states are such that $b_i < b_j$ for $i < j$. As an illustration, we only consider a two-state model. Comments on how the corresponding quantities in a multi-state model can be found in Section 2.4.4.

2.4.1 Expected discounted dividend payments

Let $V_{\delta,i}(u;\mathbf{b})$ be the expected discounted dividends in the surplus process $\{U_t^{(\mathbf{b})}\}_{t\geq 0}$ with an initial surplus of u and an initial state of the CTMC $J_0 = i$. We assume that, at the time of a transition in the Markovian process $\{J_t\}_{t\geq 0}$, the excess of the surplus over the new barrier level, if positive, will be paid out entirely as a dividend, i.e. we have that, for $i = 1, 2$,

$$V_{\delta,i}(u;\mathbf{b}) = V_{\delta,i}(b_i;\mathbf{b}) + u - b_i \ , \quad u > b_i \ . \tag{2.57}$$

Following again the same arguments used to derive (2.14), one readily obtains

$$\begin{aligned} &\frac{\sigma_1^2}{2}V''_{\delta,1}(u;\mathbf{b}) + cV'_{\delta,1}(u;\mathbf{b}) - \delta V_{\delta,1}(u;\mathbf{b}) + G_{0,11}V_{\delta,1}(u;\mathbf{b}) + G_{0,12}V_{\delta,2}(u;\mathbf{b}) \\ &+ G_{1,11}\int_0^u V_{\delta,1}(u-y;\mathbf{b})p_{11}(y)\,dy + G_{1,12}\int_0^u V_{\delta,2}(u-y;\mathbf{b})p_{12}(y)\,dy = 0 \ , \quad 0 < u < b_1 \ , \end{aligned} \tag{2.58}$$

and

$$\frac{\sigma_2^2}{2}V''_{\delta,2}(u;\mathbf{b})+cV'_{\delta,2}(u;\mathbf{b})-\delta V_{\delta,2}(u;\mathbf{b})+G_{0,21}V_{\delta,1}(u;\mathbf{b})+G_{0,22}V_{\delta,2}(u;\mathbf{b})$$
$$+G_{1,21}\int_0^u V_{\delta,1}(u-y;\mathbf{b})p_{21}(y)\,dy+G_{1,22}\int_0^u V_{\delta,2}(u-y;\mathbf{b})p_{22}(y)\,dy=0\ ,\quad 0<u<b_2\ . \quad (2.59)$$

Clearly, (2.58) and (2.59) have common domain $0<u<b_1$. Hence, using the linearly independent solutions $\{\mathbf{v}^A_{\delta,\cdot,j}(u)\}_{j=1}^2$ and $\{\mathbf{v}^B_{\delta,\cdot,j}(u)\}_{j=1}^2$ which satisfied (2.17) (with $m=2$), it follows that, for $i=1,2$,

$$V_{\delta,i}(u;\mathbf{b})=V'_{\delta,1}(0;\mathbf{b})v^B_{\delta,i,1}(u)+V'_{\delta,2}(0;\mathbf{b})v^B_{\delta,i,2}(u)\ ,\quad 0\le u\le b_1\ . \quad (2.60)$$

It is clear that, for $i=1$, the boundary condition (2.16) is converted to

$$V'_{\delta,1}(b_1;\mathbf{b})=V'_{\delta,1}(0;\mathbf{b})(v^B_{\delta,1,1})'(b_1)+V'_{\delta,2}(0;\mathbf{b})(v^B_{\delta,1,2})'(b_1)=1\ , \quad (2.61)$$

while, for $i=2$, the quantity $V'_{\delta,2}(b_1;\mathbf{b})$ given by

$$V'_{\delta,2}(b_1;\mathbf{b})=V'_{\delta,1}(0;\mathbf{b})(v^B_{\delta,2,1})'(b_1)+V'_{\delta,2}(0;\mathbf{b})(v^B_{\delta,2,2})'(b_1) \quad (2.62)$$

is yet to be determined. Letting $\mathbf{V}_\delta(u;\mathbf{b})=(V_{\delta,1}(u;\mathbf{b}),V_{\delta,2}(u;\mathbf{b}))^T$, a matrix representation of (2.61) and (2.62) is given by

$$\mathbf{V}'_\delta(0;\mathbf{b})=[(\mathbf{v}^B_\delta)'(b_1)]^{-1}\begin{pmatrix}1\\ V'_{\delta,2}(b_1;\mathbf{b})\end{pmatrix}\ . \quad (2.63)$$

Substituting (2.63) into the matrix form of (2.60) leads to

$$\mathbf{V}_\delta(u;\mathbf{b})=\mathbf{v}^B_\delta(u)[(\mathbf{v}^B_\delta)'(b_1)]^{-1}\begin{pmatrix}1\\ V'_{\delta,2}(b_1;\mathbf{b})\end{pmatrix}\ ,\quad 0\le u\le b_1\ . \quad (2.64)$$

Note that $V'_{\delta,2}(b_1;\mathbf{b})$ is unknown, which implies that (2.64) alone does not fully characterize $\mathbf{V}_\delta(u;\mathbf{b})$ for $0 \le u \le b_1$.

Now we consider (2.59) for an initial surplus $b_1 \le u < b_2$. By letting $\xi(u) = V_{\delta,2}(u+b_1;\mathbf{b})$ for $0 \le u \le b_2 - b_1$, it is immediate that

$$\frac{\sigma_2^2}{2}\xi''(u) + c\xi'(u) - \delta\xi(u) + G_{0,22}\xi(u) + G_{1,22}\int_0^u \xi(u-y)p_{22}(y)\,dy + \alpha(u) = 0\ , \quad 0 \le u < b_2 - b_1\ , \tag{2.65}$$

where

$$\begin{aligned}
\alpha(u) = {} & G_{0,21}V_{\delta,1}(u+b_1;\mathbf{b}) + G_{1,21}\int_0^{u+b_1} V_{\delta,1}(y;\mathbf{b})p_{21}(u+b_1-y)\,dy \\
& + G_{1,22}\int_0^{b_1} V_{\delta,2}(y;\mathbf{b})p_{22}(u+b_1-y)\,dy \\
= {} & \left[G_{0,21} + G_{1,21}P_{21}(u)\right]V_{\delta,1}(b_1;\mathbf{b}) + G_{1,21}\int_0^{b_1} V_{\delta,1}(y;\mathbf{b})p_{21}(u+b_1-y)\,dy \\
& + G_{0,21}u + G_{1,21}\int_0^u (u-y)p_{21}(y)\,dy + G_{1,22}\int_0^{b_1} V_{\delta,2}(y;\mathbf{b})p_{22}(u+b_1-y)\,dy\ , \quad u \ge 0\ ,
\end{aligned} \tag{2.66}$$

and (2.57) has been used.

To find the solution of the non-homogeneous integro-differential equation (2.65), we rely on the use of one of its particular solutions, namely, $\chi(u)$ for $u \ge 0$, with boundary conditions $\chi(0) = \chi'(0) = 0$. By taking Laplace transform on both sides of (2.65) with $\xi(.)$ replaced by $\chi(.)$, we obtain

$$\widetilde{\chi}(s) = \frac{-\widetilde{\alpha}(s)}{\frac{\sigma_2^2}{2}s^2 + cs - \delta + G_{0,22} + G_{1,22}\widetilde{p}_{22}(s)}\ , \tag{2.67}$$

where the Laplace transform of $\alpha(.)$ is readily found to be, from (2.66),

$$\widetilde{\alpha}(s) = \left[\frac{G_{0,21}}{s} + \frac{G_{1,21}}{s}\widetilde{p}_{21}(s)\right]\left[V_{\delta,1}(b_1;\mathbf{b}) + \frac{1}{s}\right] + G_{1,21}\left\{V_{\delta,1}(\cdot;\mathbf{b}) * \mathcal{T}_s p_{21}\right\}(b_1)$$
$$+ G_{1,22}\left\{V_{\delta,2}(\cdot;\mathbf{b}) * \mathcal{T}_s p_{22}\right\}(b_1) \ , \qquad (2.68)$$

using the Dickson-Hipp and the convolution operators (see (1.13) and (1.16)).

Also, we require the identification of two linearly independent solutions $\varphi_1(u)$ and $\varphi_2(u)$ for $u \geq 0$ of the homogeneous equation associated to (2.65), namely,

$$\frac{\sigma_2^2}{2}\varphi''(u) + c\varphi'(u) - \delta\varphi(u) + G_{0,22}\varphi(u) + G_{1,22}\int_0^u \varphi(u-y)p_{22}(y)\,dy = 0 \ , \quad u \geq 0 \ . \qquad (2.69)$$

Remark 4 *Note that an homogeneous integro-differential equation of the form (2.69) has already been studied in the context of the classical compound Poisson risk model perturbed by diffusion. Readers are referred to Li (2006) for more details on the form of two linear independent solutions.*

By defining the initial conditions $\varphi_i^{(j)}(0) = 1\{i = j+1\}$ for $i = 1, 2$ and $j = 0, 1$, it follows from the general theory of integro-differential equations that

$$V_{\delta,2}(u + b_1;\mathbf{b}) = \xi(u) = \chi(u) + V_{\delta,2}(b_1;\mathbf{b})\varphi_1(u) + V'_{\delta,2}(b_1;\mathbf{b})\varphi_2(u) \ , \quad 0 \leq u \leq b_2 - b_1 \ . \qquad (2.70)$$

Incorporating the boundary condition $V'_{\delta,2}(b_2;\mathbf{b}) = 1$ (which is equivalent to (2.16) with $i = 2$), differentiation of (2.70) with respect to u at $u = b_2 - b_1$ yields

$$\varphi_1'(b_2 - b_1)V_{\delta,2}(b_1;\mathbf{b}) + \varphi_2'(b_2 - b_1)V'_{\delta,2}(b_1;\mathbf{b}) = 1 - \chi'(b_2 - b_1) \ . \qquad (2.71)$$

Combining the second equation in (2.64) at $u = b_1$, namely,

$$V_{\delta,2}(b_1;\mathbf{b}) = \mathbf{e}_j \mathbf{v}_\delta^B(b_1)[(\mathbf{v}_\delta^B)'(b_1)]^{-1}\begin{pmatrix} 1 \\ V'_{\delta,2}(b_1;\mathbf{b}) \end{pmatrix} , \tag{2.72}$$

to (2.71), we have a system of two linear equations for the unknown quantities $V_{\delta,2}(b_1;\mathbf{b})$ and $V'_{\delta,2}(b_1;\mathbf{b})$. Note that $\chi'(b_2 - b_1)$ (in (2.71)) does depend on $V'_{\delta,2}(b_1;\mathbf{b})$ via its non-homogeneous term $\alpha(u)$. The solution of the above system leads to a complete characterization of $V_{\delta,1}(u;\mathbf{b})$ and $V_{\delta,2}(u;\mathbf{b})$ via (2.64) and (2.70).

2.4.2 Higher moments of discounted dividends

Let $V_{\delta,i,n}(u;\mathbf{b})$ be the n-th moment of the total discounted dividends in the surplus process $\{U_t^{(\mathbf{b})}\}_{t\geq 0}$ with initial surplus of u and initial state $J_0 = i$. A binomial expansion readily leads to, for $i = 1, 2$,

$$V_{\delta,i,n}(u;\mathbf{b}) = \sum_{k=0}^{n} \binom{n}{k} (u - b_i)^{n-k} V_{\delta,i,k}(b_i;\mathbf{b}) , \quad u > b_i . \tag{2.73}$$

By the same arguments used to derive (2.64), it can be shown that

$$\mathbf{V}_{\delta,n}(u;\mathbf{b}) = \mathbf{v}_{\delta,n}^B(u)[(\mathbf{v}_{\delta,n}^B)'(b_1)]^{-1}\begin{pmatrix} nV_{\delta,1,n-1}(b_1;\mathbf{b}) \\ V'_{\delta,2,n}(b_1;\mathbf{b}) \end{pmatrix} , \quad 0 \leq u \leq b_1 , \tag{2.74}$$

where $\mathbf{V}_{\delta,n}(u;\mathbf{b}) = (V_{\delta,1,n}(u;\mathbf{b}), V_{\delta,2,n}(u;\mathbf{b}))^T$. Given the form of (2.74), it is clear that the moments $\mathbf{V}_n(u;\mathbf{b})$ have to be determined recursively in terms of n. Letting $\xi_n(u) = V_{\delta,2,n}(u + b_1;\mathbf{b})$, one also knows that, analogous to (2.65),

$$\frac{\sigma_2^2}{2}\xi_n''(u) + c\xi_n'(u) - n\delta\xi_n(u) + G_{0,22}\xi_n(u) + G_{1,22}\int_0^u \xi_n(u-y)p_{22}(y)\,dy + \alpha_n(u) = 0 , \quad 0 \leq u < b_2 - b_1 , \tag{2.75}$$

where

$$\alpha_n(u) = \sum_{k=0}^{n} \binom{n}{k} \left[G_{0,21} u^{n-k} + G_{1,21} \int_0^u (u-y)^{n-k} p_{21}(y)\, dy \right] V_{\delta,1,k}(b_1; \mathbf{b})$$

$$+ G_{1,21} \int_0^{b_1} V_{\delta,1,n}(y; \mathbf{b}) p_{21}(u + b_1 - y)\, dy + G_{1,22} \int_0^{b_1} V_{\delta,2,n}(y; \mathbf{b}) p_{22}(u + b_1 - y)\, dy \ , \quad u \geq 0 \ . \tag{2.76}$$

The solution of the integro-differential equation (2.75) satisfies

$$V_{\delta,2,n}(u + b_1; \mathbf{b}) = \xi_n(u) = \chi_n(u) + V_{\delta,2,n}(b_1; \mathbf{b}) \varphi_{1,n}(u) + V'_{\delta,2,n}(b_1; \mathbf{b}) \varphi_{2,n}(u) \ , \quad 0 \leq u \leq b_2 - b_1 \ . \tag{2.77}$$

where for $u \geq 0$, $\chi_n(u)$ is a particular solution of (2.75) with $\chi_n(0) = \chi'_n(0) = 0$ while $\varphi_{1,n}(u)$ and $\varphi_{2,n}(u)$ are two linearly independent solutions of the homogeneous version of (2.75) with initial conditions $\varphi_{i,n}^{(j)}(0) = 1\{i = j + 1\}$ for $i = 1, 2$ and $j = 0, 1$. Applying the boundary condition $V'_{\delta,2,n}(b_2; \mathbf{b}) = nV_{\delta,2,n-1}(b_2; \mathbf{b})$ to the representation (2.77) leads to

$$\varphi'_{1,n}(b_2 - b_1) V_{\delta,2,n}(b_1; \mathbf{b}) + \varphi'_{2,n}(b_2 - b_1) V'_{\delta,2,n}(b_1; \mathbf{b}) = nV_{\delta,2,n-1}(b_2; \mathbf{b}) - \chi'_n(b_2 - b_1) \tag{2.78}$$

The solution of the system of two linear equations which consists of the second equation of (2.74) at $u = b_1$ and (2.78) together with the form of the solutions in (2.74) and (2.77) leads to a recursive procedure for the evaluation of $\mathbf{V}_{\delta,n}(u; \mathbf{b})$ in terms of n.

We point out that the Laplace transform of $\chi_n(u)$ has a representation of the form (2.67) with δ replaced by $n\delta$ and $\widetilde{\alpha}(s)$ replaced by

$$\widetilde{\alpha}_n(s) = \sum_{k=0}^{n} \frac{n!}{k!} \frac{1}{s^{n-k+1}} \left[G_{0,21} + G_{1,21} \widetilde{p}_{21}(s) \right] V_{\delta,1,k}(b_1; \mathbf{b}) + G_{1,21} \left\{ V_{\delta,1,n}(\cdot; \mathbf{b}) * \mathcal{T}_s p_{21} \right\}(b_1)$$

$$+ G_{1,22} \left\{ V_{\delta,2,n}(\cdot; \mathbf{b}) * \mathcal{T}_s p_{22} \right\}(b_1) \ . \tag{2.79}$$

2.4.3 Gerber-Shiu function

Note that the quantities corresponding to (2.6), (2.7) and (2.8) in the surplus process $\{U_t^{(\mathbf{b})}\}_{t\geq 0}$ are respectively denoted by $\phi_{\delta,i}(u;\mathbf{b})$, $\phi_{\delta,i}^d(u;\mathbf{b})$ and $\phi_{\delta,i}^c(u;\mathbf{b})$. Clearly, for $i=1,2$, we have $\phi_{\delta,i}(u;\mathbf{b})=\phi_{\delta,i}(b_i;\mathbf{b})$, $\phi_{\delta,i}^d(u;\mathbf{b})=\phi_{\delta,i}^d(b_i;\mathbf{b})$ and $\phi_{\delta,i}^c(u;\mathbf{b})=\phi_{\delta,i}^c(b_i;\mathbf{b})$ for $u>b_i$. To determine $\phi_{\delta,i}^d(u;\mathbf{b})$, analogous to (2.64), one easily arrives at

$$\mathbf{\Phi}_\delta^d(u;\mathbf{b})=\mathbf{\Phi}_\delta^d(u)+\mathbf{v}_\delta^B(u)[(\mathbf{v}_\delta^B)'(b_1)]^{-1}\begin{pmatrix} -(\phi_{\delta,1}^d)'(b_1) \\ (\phi_{\delta,2}^d)'(b_1;\mathbf{b})-(\phi_{\delta,1}^d)'(b_1)\end{pmatrix}\ , \quad 0\leq u\leq b_1\ , \tag{2.80}$$

where $\mathbf{\Phi}_\delta^d(u;\mathbf{b})=(\phi_{\delta,1}^d(u;\mathbf{b}),\phi_{\delta,2}^d(u;\mathbf{b}))^T$.

Note that $(\phi_{\delta,2}^d)'(b_1;\mathbf{b})$ in (2.80) is unknown and needs to be determined. Letting $\xi^d(u)=\phi_{\delta,2}^d(u+b_1;\mathbf{b})$, one has

$$\frac{\sigma_2^2}{2}(\xi^d)''(u)+c(\xi^d)'(u)-\delta\xi^d(u)+G_{0,22}\xi^d(u)+G_{1,22}\int_0^u \xi^d(u-y)p_{22}(y)\,dy+\alpha^d(u)=0\ ,\quad 0\leq u<b_2-b_1\ , \tag{2.81}$$

where

$$\begin{aligned}\alpha^d(u)=\ &[G_{0,21}+G_{1,21}P_{21}(u)]\,\phi_{\delta,1}^d(b_1;\mathbf{b})+G_{1,21}\int_0^{b_1}\phi_{\delta,1}^d(y;\mathbf{b})p_{21}(u+b_1-y)\,dy\\ &+G_{1,22}\int_0^{b_1}\phi_{\delta,2}^d(y;\mathbf{b})p_{22}(u+b_1-y)\,dy\ ,\quad u\geq 0\ .\end{aligned} \tag{2.82}$$

Again, the solution of the integro-differential equation (2.81) satisfies

$$\phi_{\delta,2}^d(u+b_1;\mathbf{b})=\xi^d(u)=\zeta^d(u)+\left[\phi_{\delta,2}^d(b_1;\mathbf{b})-\zeta^d(0)\right]\varphi_1(u)+\left[(\phi_{\delta,2}^d)'(b_1;\mathbf{b})-(\zeta^d)'(0)\right]\varphi_2(u)\ ,\quad 0\leq u\leq b_2-b_1\ , \tag{2.83}$$

where for $u\geq 0$, $\zeta^d(u)$ is a particular solution of (2.81) with $\zeta^d(0)=(\zeta^d)'(0)=0$. Given that

$(\phi^d_{\delta,2})'(b_2;\mathbf{b})=0$, the representation (2.83) yields

$$\varphi_1'(b_2-b_1)\left[\phi^d_{\delta,2}(b_1;\mathbf{b})-\zeta^d(0)\right]+\varphi_2'(b_2-b_1)\left[(\phi^d_{\delta,2})'(b_1;\mathbf{b})-(\zeta^d)'(0)\right]=-(\zeta^d)'(b_2-b_1) \quad (2.84)$$

Therefore, the constants $\phi^d_{\delta,2}(b_1;\mathbf{b})$ and $(\phi^d_{\delta,2})'(b_1;\mathbf{b})$ can be solved from the system of equations which consists of (2.80) at $u=b_1$ and (2.84). A complete representation of $\mathbf{\Phi}^d_{\delta}(u;\mathbf{b})$ is then obtained by combining (2.80) and (2.83).

Similarly, for $\phi^c_{\delta,i}(u;\mathbf{b})$, all the results derived for $\phi^d_{\delta,i}(u;\mathbf{b})$ still hold true, except for (2.82). Therefore, one can simply replace the superscript 'd' by 'c' from (2.80) to (2.84), but with (2.82) replaced by

$$\begin{aligned}\alpha^c(u)=\ &[G_{0,21}+G_{1,21}P_{21}(u)]\,\phi^c_{\delta,1}(b_1;\mathbf{b})+G_{1,21}\int_0^{b_1}\phi^c_{\delta,1}(y;\mathbf{b})p_{21}(u+b_1-y)\,dy\\ &+G_{1,22}\int_0^{b_1}\phi^c_{\delta,2}(y;\mathbf{b})p_{22}(u+b_1-y)\,dy+G_{1,21}\omega_{21}(u+b_1)+G_{1,22}\omega_{22}(u+b_1)\ ,\quad u\geq 0\ ,\end{aligned} \quad (2.85)$$

Finally, for $i=1,2$, one obtains the Gerber-Shiu function $\phi_{\delta,i}(u;\mathbf{b})$ via $\phi_{\delta,i}(u;\mathbf{b})=w_0\phi^d_{\delta,i}(u;\mathbf{b})+\phi^c_{\delta,i}(u;\mathbf{b})$.

Remark 5 *For a detailed numerical example regarding the two-state model with different barrier levels, interested readers are referred to the original paper Cheung and Landriault (2009) from which this chapter is adapted.*

2.4.4 Analysis for an arbitrary number of environmental states

The above analysis assumes that the CTMC has only two environmental states. The choice of a 2-state environment process has been primarily due to its simple mathematical tractability which is easier to understand. However, it is clear that the technique used in the two-state model can be readily extended to a CTMC with an arbitrary finite number of states. We highlight the procedure for the expected discounted dividend payments $\mathbf{V}_\delta(u;\mathbf{b}) = (V_{\delta,1}(u;\mathbf{b}), \ldots, V_{\delta,m}(u;\mathbf{b}))^T$ here.

- **Step 1**: Consider surplus values in $(0, b_1)$, where the complete set of integro-differential equations for $\mathbf{V}_\delta(u;\mathbf{b})$ holds. Find the form of the solution on $(0, b_1)$ and incorporate the first derivative condition $V'_{\delta,1}(b_1;\mathbf{b}) = 1$ into it.

- **Step 2**: Consider surplus values in (b_i, b_{i+1}). Only a subset of the original integro-differential equations holds for $\mathbf{V}_\delta(u;\mathbf{b})$. Find the form of the solution on (b_i, b_{i+1}) and incorporate the first derivative condition $V'_{\delta,i+1}(b_{i+1};\mathbf{b}) = 1$.

- **Step 3**: Repeat Step 2 until all the values of i in $\{1, 2, \ldots, m-1\}$ have been considered. Combine all the first derivative conditions and solve the resulting system of linear equations.

An application of this procedure leads to a complete characterization of $\mathbf{V}_\delta(u;\mathbf{b})$. For other ruin-related quantities of interest, we simply have to use the appropriate boundary conditions.

2.5 Appendix: The barrier-free model

The aim of this Appendix is to give explicit expressions for the Laplace transform of the quantities $\mathbf{\Phi}^d_\delta(u)$ and $\mathbf{\Phi}^c_\delta(u)$ when $m = 2$. Since all the techniques and procedures are almost identical to those given in Lu and Tsai (2007), details are omitted and only results are given here.

With $m = 2$, $\mathbf{A}_\delta(s)$ defined by (2.26) is given by

$$\mathbf{A}_\delta(s) = \begin{pmatrix} \frac{\sigma_1^2}{2}s^2 + cs - \delta + G_{0,11} + G_{1,11}\widetilde{p}_{11}(s) & G_{0,12} + G_{1,12}\widetilde{p}_{12}(s) \\ G_{0,21} + G_{1,21}\widetilde{p}_{21}(s) & \frac{\sigma_1^2}{2}s^2 + cs - \delta + G_{0,22} + G_{1,22}\widetilde{p}_{22}(s) \end{pmatrix}. \quad (2.86)$$

For $\delta > 0$, an application of Rouché's Theorem reveals that $\det \mathbf{A}_\delta(s) = 0$ has two positive real roots which we shall denote by ρ_1 and ρ_2, and these are the only roots on the right half of the complex plane. If the positive security condition (1.3) is assumed, then we have that $\rho_1 \to 0^+$ as $\delta \to 0^+$.

Analogous to Eq. (4.11) and Eq. (4.12) of Lu and Tsai (2007), we arrive at

$$\begin{aligned}\widetilde{\mathbf{\Phi}}^c_\delta(s) = \frac{(s-\rho_1)(s-\rho_2)}{\det \mathbf{A}_\delta(s)} \Bigg\{ &\operatorname{adj} \mathbf{A}_\delta[s,\rho_1,\rho_2]\mathbf{B}^c_\delta(\rho_2) - \operatorname{adj} \mathbf{A}_\delta[s,\rho_1] \begin{pmatrix} \sum_{j=1}^2 G_{1,1j}\widetilde{\omega}_{1j}[s,\rho_2] \\ \sum_{j=1}^2 G_{1,2j}\widetilde{\omega}_{1j}[s,\rho_2] \end{pmatrix} \\ &- \operatorname{adj} \mathbf{A}_\delta(\rho_1) \begin{pmatrix} \sum_{j=1}^2 G_{1,1j}\widetilde{\omega}_{1j}[s,\rho_1,\rho_2] \\ \sum_{j=1}^2 G_{1,2j}\widetilde{\omega}_{1j}[s,\rho_1,\rho_2] \end{pmatrix} \Bigg\}, \end{aligned} \tag{2.87}$$

and

$$\widetilde{\mathbf{\Phi}}^d_\delta(s) = \frac{(s-\rho_1)(s-\rho_2)}{\det \mathbf{A}_\delta(s)} \Bigg\{ \operatorname{adj} \mathbf{A}_\delta[s,\rho_1,\rho_2]\mathbf{B}^d_\delta(\rho_2) + \operatorname{adj} \mathbf{A}_\delta[s,\rho_1] \begin{pmatrix} \frac{\sigma_1^2}{2} \\ \frac{\sigma_2^2}{2} \end{pmatrix} \Bigg\}, \tag{2.88}$$

where

$$\mathbf{B}^c_\delta(\rho_2) = \big[\operatorname{adj} \mathbf{A}_\delta[\rho_1,\rho_2]\big]^{-1} \operatorname{adj} \mathbf{A}_\delta(\rho_1) \begin{pmatrix} \sum_{j=1}^2 G_{1,1j}\widetilde{\omega}_{1j}[\rho_1,\rho_2] \\ \sum_{j=1}^2 G_{1,2j}\widetilde{\omega}_{1j}[\rho_1,\rho_2] \end{pmatrix}, \tag{2.89}$$

and

$$\mathbf{B}^d_\delta(\rho_2) = -\big[\operatorname{adj} \mathbf{A}_\delta[\rho_1,\rho_2]\big]^{-1} \operatorname{adj} \mathbf{A}_\delta(\rho_1) \begin{pmatrix} \frac{\sigma_1^2}{2} \\ \frac{\sigma_2^2}{2} \end{pmatrix}. \tag{2.90}$$

The reader is referred to P.13 for the notion of divided differences.

Chapter 3

The dual MAP risk model with a dividend threshold

3.1 Introduction

In this chapter, we consider a reflection of the MAP risk model described in Section 1.2.1. This results in the *dual MAP risk model*. The surplus process of such a model is denoted by $\{U_t^{\text{dual}}\}_{t\geq 0}$ with dynamics

$$U_t^{\text{dual}} = u - ct + \sum_{i=1}^{N_t} Y_i \ , \quad t \geq 0 \ . \tag{3.1}$$

Here the properties of $\{N_t\}_{t\geq 0}$ (and hence $\{V_i\}_{i=1}^{\infty}$) and $\{Y_i\}_{i=1}^{\infty}$ are exactly as described in Section 1.2.1. For example, $\{N_t\}_{t\geq 0}$ still follows a MAP, and $\{Y_i\}_{i=1}^{\infty}$ are positive random variables resulting from type-2 transitions in the MAP. However, the physical interpretations of various attributes have now to be modified. In (3.1), $u = U_0^{\text{dual}}$ is still the initial surplus of the company, but $c > 0$ is now interpreted as the constant rate of expenses incurred per unit time. Furthermore, $\{Y_i\}_{i=1}^{\infty}$ is now the sequence of random gains experienced by the company, with Y_i representing the size of the i-th gain. Note that under the dual MAP risk model, ruin is caused by the expenses incurred, and in case of ruin, both the surplus prior to ruin and the deficit at ruin are zero. The time of ruin is thus defined to be the first time that the process $\{U_t^{\text{dual}}\}_{t\geq 0}$ hits level 0, and is given by $T^{\text{dual}} = \inf\{t \geq 0 : U_t^{\text{dual}} = 0\}$ with $T^{\text{dual}} = \infty$ if ruin does not occur. Note that ruin occurs immediately with zero initial surplus. In addition, the Gerber-Shiu function reduces to (a constant multiple of) the Laplace transform of the time of ruin, since the penalty applied at ruin is constant in this case (owing to the constant nature of the surplus prior to ruin and the deficit at ruin).

According to Avanzi et al. (2007), a dual risk model is appropriate for describing the surplus process of companies which are involved in invention or discovery, such as pharmaceutical and petroleum companies. The characteristics of these companies are such that, they are paying expenses over time, while occasional gains from invention or discovery would bring upward jumps to the surplus process, and we can think of each upward jump to be the net present value of future income as a result of an invention or discovery. In addition, a dual model might also be appropriate for settings involving annuity or pension fund (see, e.g., Seal (1969, p.116)). For a detailed study of the model (3.1) under an independent set-up, we refer readers to, e.g., Mazza and Rullière (2004) and Seal (1969).

This chapter considers a modification of the dual MAP risk model described above by incorporating a dividend strategy into it. However, in the dual model with a dividend barrier (see, e.g., Avanzi and Gerber (2008), Avanzi et al. (2007) and Cheung and Drekic (2008)), the ruin probability is 1, which is practically undesirable. The purpose of this chapter is to propose a threshold-type dual model (see also Ng (2009) for the study of a dual compound Poisson model) in which the ruin probability may or may not be 1. Under a threshold modification, the surplus process is denoted by $\{U^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t)\}_{t\geq 0}$ with dynamics

$$dU^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t) = \begin{cases} -c_1\,dt + d\left(\sum_{i=1}^{N_t} Y_i\right) \ , & 0 \leq U^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t) < b_1 \ . \\ -c_2\,dt + d\left(\sum_{i=1}^{N_t} Y_i\right) \ , & b_1 \leq U^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t) < b_1 + b_2 = \infty \ . \end{cases} \tag{3.2}$$

In the model $\{U^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t)\}_{t\geq 0}$, the level b_1 is the so-called threshold level. It is assumed that the company pays dividends to shareholders at rate d_1 (d_2) and therefore the surplus decreases at rate c_1 (c_2) with c_i = expense rate + d_i whenever the surplus level is below (above) b_1. To ease our analysis, we also define the threshold-free process $\{U^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t)\}_{t\geq 0}$ given by

$$dU^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t) = -c_2\,dt + d\left(\sum_{i=1}^{N_t} Y_i\right) \ , \quad 0 \leq U^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t) < b_2 = \infty \ . \tag{3.3}$$

We remark that the process $\{U^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t)\}_{t\geq 0}$ with $U^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(0) = u$ and $c_2 = c$ is identical to the surplus process $\{U^{\text{dual}}_t\}_{t\geq 0}$ described by (3.1). But the seemingly more complicated notations here indeed ease presentation later on.

Under the threshold dividend strategy described above, the positive security loading condition for the process $\{U^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t)\}_{t\geq 0}$ is identical to that of the threshold-free process $\{U^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t)\}_{t\geq 0}$, and is given by (1.3) with the inequality sign reversed and c replaced by c_2. This guarantees the two processes to have ruin probability less than 1.

For $i = 1, 2$, pertaining to the surplus process $\{U^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(t)\}_{t\geq 0}$, is the time of ruin $T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u) = \inf\{t \geq 0 : U^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(t) = 0 | U^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(0) = u\}$. In addition, we define, for $i = 1, 2$, the discounted (at a force of interest $\delta > 0$) dividend random variable for the surplus process $\{U^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(t)\}_{t\geq 0}$ to be

$$D^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u) = \sum_{j=i}^{2} d_j \int_0^{T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)} e^{-\delta t} 1\left\{\sum_{k=i}^{j-1} b_k \leq U^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(t) < \sum_{k=i}^{j} b_k\right\} dt \ . \tag{3.4}$$

We remark that in defining the above random variables, their dependence on the initial surplus u is emphasized. This proves to be helpful later on when sample paths analysis is performed.

The study of the Laplace transform of $T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)$ and the moments of $D^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u)$ will be the focus of this chapter. Our methodology used, in contrast to Chapter 2, will be an existing connection to a fluid flow process as described in Section 1.4.2.

Remark 6 *In the classical approach of conditioning on the time and amount of the first gain (see, e.g., Ng (2009)) used in the study of ruin-related quantities in dual risk models, the resulting integral and/or integro-differential equations no longer contain convolution-type integrals. Convolution-type integrals, which usually arise in standard (i.e. non-dual) risk models, are relatively easy to deal with in comparison to the integrals arising in dual risk models. Therefore, the study of dual risk models is not an easy problem if the classical approach is applied. However, as we shall see, with the use of fluid flow methodology, simply a reflected fluid flow process is required to study the dual MAP risk model without any further complications.*

Corresponding to the surplus processes (3.2) and (3.3) are the reflected fluid flow processes $\{F^r_{\underline{c}_1,\underline{b}_1}(t)\}_{t\geq 0}$ and $\{F^r_{\underline{c}_2,\underline{b}_2}(t)\}_{t\geq 0}$ defined by

$$dF^r_{\underline{c}_1,\underline{b}_1}(t) = \big(1\{J^{(F)}(t)\in S_2\} - 1\{J^{(F)}(t)\in S_1\}\big)\begin{cases} c_1\,dt\ , & 0\leq F^r_{\underline{c}_1,\underline{b}_1}(t) < b_1\ , \\ c_2\,dt\ , & b_1\leq F^r_{\underline{c}_1,\underline{b}_1}(t) < b_1+b_2=\infty\ , \end{cases} \tag{3.5}$$

and

$$dF^r_{\underline{c}_2,\underline{b}_2}(t) = \big(1\{J^{(F)}(t)\in S_2\} - 1\{J^{(F)}(t)\in S_1\}\big)\,c_2\,dt\ , \quad 0\leq F^r_{\underline{c}_2,\underline{b}_2}(t) < b_2=\infty\ , \tag{3.6}$$

respectively. Note that for $i=1,2$, the reflected fluid process $\{F^r_{\underline{c}_i,\underline{b}_i}(t)\}_{t\geq 0}$ is related to the ordinary (i.e. non-reflected) fluid process $\{F_{\underline{c}_i,\underline{b}_i}(t)\}_{t\geq 0}$ (see Badescu and Landriault (2008, Eq. (2.4))) by

$$dF^r_{\underline{c}_i,\underline{b}_i}(t) = -dF_{\underline{c}_i,\underline{b}_i}(t)\ , \quad t\geq 0\ . \tag{3.7}$$

Furthermore, for $i=1,2$, the quantity ${}^{z}_{a}\tau^r_{\underline{c}_i,\underline{b}_i}(x,y)$ (${}^{z}_{a}\tau_{\underline{c}_i,\underline{b}_i}(x,y)$) denotes the first passage time of $\{F^r_{\underline{c}_i,\underline{b}_i}(t)\}_{t\geq 0}$ ($\{F_{\underline{c}_i,\underline{b}_i}(t)\}_{t\geq 0}$) from level x to level y while avoiding a visit to the levels $[0,a]\cup[z,\infty)$ enroute. As in Section 1.4.2, the arguments a and/or z will be suppressed whenever they are not helpful. See also Section 1.4.2 regarding the details of the LSTs of various related first passage times.

Remark 7 *In Section 1.4.2, the dependence on c of the LSTs is suppressed because there is only one drift rate c. However, for the threshold model in this chapter, there are two rates c_1 and c_2 at which the fluid level can be increasing/decreasing. Therefore, we add an additional subscript for the LSTs. For example, at a rate of c, the LSTs previously denoted by $\widehat{\mathbf{f}}^r_{11}(x,0,\delta)$ and $\mathbf{\Psi}^r(\delta)$ are now denoted by $\widehat{\mathbf{f}}^r_{11,c}(x,0,\delta)$ and $\mathbf{\Psi}^r_c(\delta)$ respectively.*

The ideas in this chapter mainly come from Badescu and Landriault (2008). In their paper, a multi-threshold ordinary MAP risk model was considered, and sample paths analysis was used to decompose the time of ruin and the dividend payments into several pieces along disjoint time intervals under different scenarios. Then expectation was taken with respect to the variable of interest, taking advantage of the fact that given the state of the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$, quantities defined on disjoint intervals are independent. This chapter is organized as follows. In Section 3.2, the Laplace transform of the time of ruin is derived, while Sections 3.3 and 3.4 consider respectively the moments of discounted dividends with and without ruin. Such separation of the moments into cases where ruin occurs or not is believed to be first contributed by Badescu and Landriault (2008).

3.2 Laplace transform of time of ruin

Recall from Section 1.4.2 that whenever MAP risk process is analyzed by a connection to a fluid flow process, the ruin-related quantities have to be defined with respect to the states of the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ (not $\{J_t\}_{t\geq 0}$). Then, for $i = 1, 2$, we define the $|S_1| \times |S_1|$ matrix of the LST of the distribution of $T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)$ by $\boldsymbol{\rho}^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u)$, whose (j,k)-th element is given by

$$[\boldsymbol{\rho}^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u)]_{jk} = E\left[e^{-\delta T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)} 1\{T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u) < \infty, J^{(F)}(\tau^r_{\underline{c}_i,\underline{b}_i}(u,0)) = k\} | J^{(F)}(0) = j\right] , \quad u \geq 0 \ . \quad (3.8)$$

Note that ruin occurs immediately at the initial state with zero initial surplus and therefore $\boldsymbol{\rho}^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(0) = \mathbf{I}$.

For the threshold-free surplus process $\{U^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t)\}_{t\geq 0}$, it is trivial that

$$\boldsymbol{\rho}^{\text{dual}}_{\delta,\underline{c}_2,\underline{b}_2}(u) = e^{-\frac{\delta u}{2c_2}} \,\widehat{\mathbf{f}}^r_{11,c_2}\left(u, 0, \frac{\delta}{2}\right) \ , \quad u \geq 0 \ . \tag{3.9}$$

In the case of $\{U^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t)\}_{t\geq 0}$, using similar arguments as in the proof of Theorem 3.1 of Badescu et al. (2007a), it can be shown that

$$\boldsymbol{\rho}^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u) = e^{-\frac{\delta b_1}{2c_1}} \,\boldsymbol{\rho}^{\text{dual}}_{\delta,\underline{c}_2,\underline{b}_2}(u-b_1) \left[\mathbf{I} - {}^{b_1}\boldsymbol{\Psi}_{c_1}\left(\frac{\delta}{2}\right) \boldsymbol{\Psi}^r_{c_2}\left(\frac{\delta}{2}\right)\right]^{-1} {}_0\widehat{\mathbf{f}}_{11,c_1}\left(0, b_1, \frac{\delta}{2}\right) \ , \quad u > b_1 \ , \tag{3.10}$$

and

$$\begin{aligned}\boldsymbol{\rho}^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u) = e^{-\frac{\delta u}{2c_1}} \Bigg\{ & {}_0\widehat{\mathbf{f}}_{11,c_1}\left(b_1 - u, b_1, \frac{\delta}{2}\right) + {}^{b_1}\widehat{\mathbf{f}}_{12,c_1}\left(b_1 - u, 0, \frac{\delta}{2}\right) \boldsymbol{\Psi}^r_{c_2}\left(\frac{\delta}{2}\right) \\ & \times \left[\mathbf{I} - {}^{b_1}\boldsymbol{\Psi}_{c_1}\left(\frac{\delta}{2}\right) \boldsymbol{\Psi}^r_{c_2}\left(\frac{\delta}{2}\right)\right]^{-1} {}_0\widehat{\mathbf{f}}_{11,c_1}\left(0, b_1, \frac{\delta}{2}\right) \Bigg\} \ , \quad 0 \leq u \leq b_1 \ . \end{aligned} \tag{3.11}$$

The ruin probability can be obtained as a special case of (3.10) and (3.11) by letting $\delta = 0$.

3.3 Dividend moments with ruin

Analogous to Eq. (3.1) in Badescu and Landriault (2008), for $i = 1, 2$ and $l, n \in \mathbb{N}$, we define the $|S_1| \times |S_1|$ matrix $\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_i,\underline{b}_i}(u)$ which represents the matrix of the *generalized moments of the* *discounted dividend payments with ruin occurrence*, with its (j,k)-th element given by

$$[\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_i,\underline{b}_i}(u)]_{jk} = E\left[e^{-l\delta T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)}(D^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u))^n 1\{T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u) < \infty, J^{(F)}(\tau^r_{\underline{c}_i,\underline{b}_i}(u,0)) = k\}|J^{(F)}(0) = j\right] ,$$
$$u \geq 0 \ . \quad (3.12)$$

Adopting the same abbreviations as in Badescu and Landriault (2008), for the remainder of this chapter we shall write (3.12) as, for $i = 1, 2$ and $l, n \in \mathbb{N}$,

$$[\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_i,\underline{b}_i}(u)]_{jk} = E^{(\text{ruin})}_{jk}\left[e^{-l\delta T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)}(D^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u))^n\right] , \quad u \geq 0 \ . \tag{3.13}$$

For the threshold-free surplus process $\{U^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t)\}_{t\geq 0}$, it follows from an identical argument used to obtain Eq. (3.3) in Proposition 1 of Badescu and Landriault (2008) that, for $l, n \in \mathbb{N}$,

$$\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_2,\underline{b}_2}(u) = \left(\frac{d_2}{\delta}\right)^n \sum_{h=0}^{n} \binom{n}{h} (-1)^h \boldsymbol{\rho}^{\text{dual}}_{(l+h)\delta,\underline{c}_2,\underline{b}_2}(u) , \quad u \geq 0 \ . \tag{3.14}$$

Next we consider the quantity $\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_1,\underline{b}_1}(u)$ for the surplus process $\{U^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t)\}_{t\geq 0}$. First, for $u > b_1$, the corresponding fluid flow process $\{F^r_{\underline{c}_1,\underline{b}_1}(t)\}_{t\geq 0}$ has to make a transition from (u, S_1) to (b_1, S_1) in order for ruin to occur. In other words, $\tau^r_{\underline{c}_1,\underline{b}_1}(u,0)$ has the same distribution as $\tau^r_{\underline{c}_2,\underline{b}_2}(u-b_1,0) + (\tau^r_{\underline{c}_1,\underline{b}_1})^*(b_1,0)$, which in turn implies that $T^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u)$ has the same distribution as $T^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(u-b_1)+(T^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1)$. Here we assume $(\tau^r_{\underline{c}_1,\underline{b}_1})^*(b_1,0) \overset{d}{=} \tau^r_{\underline{c}_1,\underline{b}_1}(b_1,0)$ and $(T^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1) \overset{d}{=} T^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(b_1)$. Therefore, the random variable $D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u)$ can be decomposed into

$$D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u) \overset{d}{=} D^{\text{dual}}_{\delta,\underline{c}_2,\underline{b}_2}(u-b_1) + e^{-\delta T^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(u-b_1)}(D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1})^*(b_1) , \tag{3.15}$$

where $(D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1})^*(b_1) \overset{d}{=} D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(b_1)$. Following the same ideas used in deriving Eq. (3.16) in Badescu and Landriault (2008), one can show that, for $l, n \in \mathbb{N}$,

$$\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_1,\underline{b}_1}(u) = \sum_{\xi=0}^{n} \binom{n}{\xi} \mathbf{W}^{\text{dual}}_{l+\xi,n-\xi,\underline{c}_2,\underline{b}_2}(u-b_1)\ \mathbf{W}^{\text{dual}}_{l,\xi,\underline{c}_1,\underline{b}_1}(b_1)\ , \quad u > b_1\ . \tag{3.16}$$

Second, for $0 \le u \le b_1$, the fluid flow process $\{F^r_{\underline{c}_1,\underline{b}_1}(t)\}_{t\ge 0}$ must either reach level b_1 in S_2 or reach level 0 in S_1 first. In the former case, for ruin to occur, the process must make a transition back to level b_1 in S_1. Hence, $\tau^r_{\underline{c}_1,\underline{b}_1}(u,0) \stackrel{d}{=} {}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0) + \tau^r_{\underline{c}_2,\underline{b}_2}(0,0) + (\tau^r_{\underline{c}_1,\underline{b}_1})^*(b_1,0)$, which means $T^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u) \stackrel{d}{=} {}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)/2 - (b_1-u)/(2c_1) + \tau^r_{\underline{c}_2,\underline{b}_2}(0,0)/2 + (T^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1)$. Now, we can decompose $D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u)$ into

$$\begin{aligned} D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u) \stackrel{d}{=}\ & d_1 \overline{a}_{\overline{{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)/2-(b_1-u)/(2c_1)}|} \\ & + e^{-\delta\left[{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)/2-(b_1-u)/(2c_1)\right]} \left[d_2 \overline{a}_{\overline{\tau^r_{\underline{c}_2,\underline{b}_2}(0,0)/2}|} + e^{-\delta \tau^r_{\underline{c}_2,\underline{b}_2}(0,0)/2} (D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1})^*(b_1) \right]\ . \end{aligned} \tag{3.17}$$

For the latter case, it is clear that $\tau^r_{\underline{c}_1,\underline{b}_1}(u,0) \stackrel{d}{=} {}_0\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,b_1)$. This immediately yields $T^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u) \stackrel{d}{=} {}_0\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,b_1)/2 + u/(2c_1)$, and so

$$D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u) \stackrel{d}{=} d_1 \overline{a}_{\overline{{}_0\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,b_1)/2+u/(2c_1)}|}\ . \tag{3.18}$$

Combining the above two cases with representations (3.17) and (3.18) respectively, we can apply the same procedure as in Proposition 2 of Badescu and Landriault (2008) (but omit the rather tedious but straightforward algebra) to arrive at, for $l, n \in \mathbb{N}$,

$$\begin{aligned} \mathbf{W}^{\text{dual}}_{l,n,\underline{c}_1,\underline{b}_1}(u) =& \sum_{\xi=0}^{n} \binom{n}{\xi} \overline{\mathbf{W}}^{\text{dual}}_{l,n,\xi,\underline{c}_1,\underline{b}_1}(u)\ \mathbf{W}^{\text{dual}}_{l,n-\xi,\underline{c}_1,\underline{b}_1}(b_1) \\ & + \left(\frac{d_1}{\delta}\right)^n \sum_{h=0}^{n} \binom{n}{h} (-1)^h e^{-\frac{(l+h)\delta u}{2c_1}}\ {}_0\widehat{\mathbf{f}}_{11,c_1}\left(b_1-u, b_1, \frac{(l+h)\delta}{2}\right)\ , \quad 0 \le u \le b_1\ , \end{aligned} \tag{3.19}$$

where for $\xi = 0, 1, \ldots, n$ and $l, n \in \mathbb{N}$,

$$\overline{\mathbf{W}}^{\text{dual}}_{l,n,\xi,\underline{c}_1,\underline{b}_1}(u) = \sum_{h=0}^{\xi} \binom{\xi}{h} \left(\frac{d_1}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} e^{\frac{(l+n-x)\delta(b_1-u)}{2c_1}} \; {}^{b_1}\widehat{\mathbf{f}}_{12,c_1}\left(b_1 - u, 0, \frac{(l+n-x)\delta}{2}\right)$$
$$\times \left(\frac{d_2}{\delta}\right)^{\xi-h} \sum_{y=0}^{\xi-h} \binom{\xi-h}{y} (-1)^{\xi-h-y} \; \mathbf{\Psi}^r_{c_2}\left(\frac{(l+n-h-y)\delta}{2}\right) \;, \quad 0 \le u \le b_1 \;, \tag{3.20}$$

is an explicit formula enabling the computation of the $|S_1| \times |S_1|$ matrix $\overline{\mathbf{W}}^{\text{dual}}_{l,n,\xi,\underline{c}_1,\underline{b}_1}(u)$. In particular, putting $u = b_1$ into (3.19) and solving for $\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_1,\underline{b}_1}(b_1)$ yields, for $l, n \in \mathbb{N}$,

$$\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_1,\underline{b}_1}(b_1) = \left[\mathbf{I} - \overline{\mathbf{W}}^{\text{dual}}_{l,n,0,\underline{c}_1,\underline{b}_1}(b_1)\right]^{-1} \left[\sum_{\xi=1}^{n} \binom{n}{\xi} \overline{\mathbf{W}}^{\text{dual}}_{l,n,\xi,\underline{c}_1,\underline{b}_1}(b_1) \; \mathbf{W}^{\text{dual}}_{l,n-\xi,\underline{c}_1,\underline{b}_1}(b_1)\right.$$
$$\left. + \left(\frac{d_1}{\delta}\right)^n \sum_{h=0}^{n} \binom{n}{h} (-1)^h e^{-\frac{(l+h)\delta b_1}{2c_1}} \; {}_0\widehat{\mathbf{f}}_{11,c_1}\left(0, b_1, \frac{(l+h)\delta}{2}\right)\right] , \tag{3.21}$$

which is a recursion in n for the evaluation of $\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_1,\underline{b}_1}(b_1)$ with starting value $\mathbf{W}^{\text{dual}}_{l,0,\underline{c}_1,\underline{b}_1}(b_1) = \boldsymbol{\rho}^{\text{dual}}_{l\delta,\underline{c}_1,\underline{b}_1}(b_1)$. Note that the term $\overline{\mathbf{W}}^{\text{dual}}_{l,n,0,\underline{c}_1,\underline{b}_1}(b_1)$ in (3.21) can be simplified (using (3.20)) to, for $l, n \in \mathbb{N}$,

$$\overline{\mathbf{W}}^{\text{dual}}_{l,n,0,\underline{c}_1,\underline{b}_1}(b_1) = \; {}^{b_1}\mathbf{\Psi}_{c_1}\left(\frac{(l+n)\delta}{2}\right) \mathbf{\Psi}^r_{c_2}\left(\frac{(l+n)\delta}{2}\right) \; . \tag{3.22}$$

To conclude this section, for $0 \le u \le b_1$, (3.19) together with (3.20) and (3.21) characterize $\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_1,\underline{b}_1}(u)$, while for $u > b_1$, $\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_1,\underline{b}_1}(u)$ is computed via (3.16) with the help of (3.14) and (3.21). The 'ordinary' moments of $D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u)$ can then be obtained from $\mathbf{W}^{\text{dual}}_{l,n,\underline{c}_1,\underline{b}_1}(u)$ by letting $l = 0$.

3.4 Dividend moments without ruin

For $i = 1, 2$ and $n \in \mathbb{N}$, we define the $|S_1|$ column vector $\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_i,\underline{b}_i}(u)$ with j-th element given by

$$[\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_i,\underline{b}_i}(u)]_j = E\left[(D^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u))^n 1\{T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u) = \infty\}|J^{(F)}(0) = j\right] \ , \quad u \geq 0 \ . \tag{3.23}$$

Similar to (3.13), we adopt the abbreviation, for $i = 1, 2$ and $n \in \mathbb{N}$,

$$[\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_i,\underline{b}_i}(u)]_j = E_j\left[(D^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u))^n 1\{T^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u) = \infty\}\right] \ , \quad u \geq 0 \ . \tag{3.24}$$

Then, for the threshold-free surplus process $\{U^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t)\}_{t\geq 0}$, it is easy to see that, for $n \in \mathbb{N}$,

$$\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_2,\underline{b}_2}(u) = \left(\frac{d_2}{\delta}\right)^n \left[\mathbf{1} - \boldsymbol{\rho}^{\text{dual}}_{0,\underline{c}_2,\underline{b}_2}(u)\mathbf{1}\right] \ , \quad u \geq 0 \ , \tag{3.25}$$

holds by using identical arguments in obtaining Eq. (3.27) in Badescu and Landriault (2008).

Next, for the surplus process $\{U^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t)\}_{t\geq 0}$, we first consider $u > b_1$. The corresponding fluid flow process $\{F^r_{\underline{c}_1,\underline{b}_1}(t)\}_{t\geq 0}$ can either visit or not visit level b_1 in S_1 and therefore (3.24) with $i = 1$ can be expressed as, for $n \in \mathbb{N}$,

$$\begin{aligned}
[\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_1,\underline{b}_1}(u)]_j = {} & E_j\left[(D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u))^n 1\{T^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(u - b_1) < \infty, (T^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1) = \infty\}\right] \\
& + E_j\left[(D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u))^n 1\{T^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(u - b_1) = \infty\}\right] \ , \quad u > b_1 \ .
\end{aligned} \tag{3.26}$$

For the former case represented by the first term on the right-hand side of (3.26), the random variable $D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u)$ has the same representation as (3.15) and hence the above equation leads to, for $n \in \mathbb{N}$,

$$\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_1,\underline{b}_1}(u) = \sum_{\xi=0}^{n} \binom{n}{\xi} \mathbf{W}^{\text{dual}}_{\xi,n-\xi,\underline{c}_2,\underline{b}_2}(u-b_1)\ \boldsymbol{\chi}^{\text{dual}}_{\xi,\underline{c}_1,\underline{b}_1}(b_1) + \boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_2,\underline{b}_2}(u-b_1)\ , \quad u > b_1. \tag{3.27}$$

For $0 \le u \le b_1$, the corresponding fluid flow process $\{F^r_{\underline{c}_1,\underline{b}_1}(t)\}_{t\ge 0}$ must first reach level b_1 in S_2 before reaching level 0 in S_1 to avoid ruin. After reaching level b_1 in S_2, the process either visits or does not visit level b_1 in S_1. Thus, one has, for $n \in \mathbb{N}$,

$$\begin{aligned}[\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_1,\underline{b}_1}(u)]_j = {} & E_j\left[(D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u))^n 1\{{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)<\infty, \tau^r_{\underline{c}_2,\underline{b}_2}(0,0)<\infty, (T^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1)=\infty\}\right] \\ & + E_j\left[(D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u))^n 1\{{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)<\infty, \tau^r_{\underline{c}_2,\underline{b}_2}(0,0)=\infty\}\right]\ , \quad 0 \le u \le b_1\ .\end{aligned} \tag{3.28}$$

The first term on the right-hand side of (3.28) represents the former case where the variable $D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u)$ has an identical decomposition as (3.17), while the second term represents the latter case with $D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u)$ given by

$$D^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u) \overset{d}{=} d_1 \overline{a}_{\overline{{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)/2-(b_1-u)/(2c_1)|}} + e^{-\delta\left[{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)/2-(b_1-u)/(2c_1)\right]}\left(\frac{d_2}{\delta}\right)\ . \tag{3.29}$$

Following the same line of logic in obtaining (3.19), one finds that (3.28) reduces to, for $n \in \mathbb{N}$,

$$\begin{aligned}\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_1,\underline{b}_1}(u) = {} & \sum_{\xi=0}^{n}\binom{n}{\xi}\overline{\mathbf{W}}^{\text{dual}}_{0,n,\xi,\underline{c}_1,\underline{b}_1}(u)\ \boldsymbol{\chi}^{\text{dual}}_{n-\xi,\underline{c}_1,\underline{b}_1}(b_1) \\ & + \sum_{\xi=0}^{n}\binom{n}{\xi}\left(\frac{d_1}{\delta}\right)^{\xi}\left(\frac{d_2}{\delta}\right)^{n-\xi}\sum_{h=0}^{\xi}\binom{\xi}{h}(-1)^{\xi-h}e^{\frac{(n-h)\delta(b_1-u)}{2c_1}} \\ & \times\ {}^{b_1}\widehat{\mathbf{f}}_{12,c_1}\left(b_1-u,0,\frac{(n-h)\delta}{2}\right)\left[\mathbf{1}-\boldsymbol{\Psi}^r_{c_2}(0)\mathbf{1}\right]\ , \quad 0 \le u \le b_1\ .\end{aligned} \tag{3.30}$$

Letting $u = b_1$ in the above equation together with the use of (3.22) yields, for $n \in \mathbb{N}$,

$$\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_1,\underline{b}_1}(b_1) = \left[\mathbf{I} - {}^{b_1}\boldsymbol{\Psi}_{c_1}\left(\frac{n\delta}{2}\right)\boldsymbol{\Psi}^r_{c_2}\left(\frac{n\delta}{2}\right)\right]^{-1}\left\{\sum_{\xi=1}^{n}\binom{n}{\xi}\overline{\mathbf{W}}^{\text{dual}}_{0,n,\xi,\underline{c}_1,\underline{b}_1}(b_1)\ \boldsymbol{\chi}^{\text{dual}}_{n-\xi,\underline{c}_1,\underline{b}_1}(b_1)\right.$$
$$\left. + \sum_{\xi=0}^{n}\binom{n}{\xi}\left(\frac{d_1}{\delta}\right)^{\xi}\left(\frac{d_2}{\delta}\right)^{n-\xi}\sum_{h=0}^{\xi}\binom{\xi}{h}(-1)^{\xi-h}\ {}^{b_1}\boldsymbol{\Psi}_{c_1}\left(\frac{(n-h)\delta}{2}\right)\left[\mathbf{1} - \boldsymbol{\Psi}^r_{c_2}(0)\mathbf{1}\right]\right\}, \tag{3.31}$$

which provides a recursive scheme for computing $\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_1,\underline{b}_1}(b_1)$, with starting value $\boldsymbol{\chi}^{\text{dual}}_{0,\underline{c}_1,\underline{b}_1}(b_1) = \mathbf{1} - \boldsymbol{\rho}^{\text{dual}}_{0,\underline{c}_1,\underline{b}_1}(b_1)\mathbf{1}$.

In conclusion, for $0 \leq u \leq b_1$, $\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_1,\underline{b}_1}(u)$ can be evaluated via (3.30) together with (3.20) and (3.31), while for $u > b_1$, $\boldsymbol{\chi}^{\text{dual}}_{n,\underline{c}_1,\underline{b}_1}(u)$ is determined by (3.14), (3.25), (3.27) and (3.31).

Chapter 4

Generalized Sparre Andersen risk model with bivariate phase-type distribution

4.1 Introduction

In Section 1.2.2, the generalized Sparre Andersen risk model has been introduced. In this chapter, we turn our attention to one of its particular cases in which the bivariate random vectors $\{(V_i, Y_i)\}_{i=1}^{\infty}$ form an i.i.d. sequence, with an arbitrary pair of V_i and Y_i related through a bivariate phase-type distribution. Indeed, we shall assume that the pair $(V, Y/c)$ (instead of (V, Y)) follows a *bivariate phase-type distribution*, which will be discussed in detail in Section 4.2. As in the univariate case, we point out that the class of bivariate phase-type distributions is dense in the set of distributions defined on $\mathbb{R}^+ \times \mathbb{R}^+$ (see Assaf et al. (1984, Corollary 1)).

For the surplus process (1.1) in the generalized Sparre Andersen risk model with $(V, Y/c)$ having a bivariate phase-type distribution, our main objective in this chapter is to analyze a subclass of Gerber-Shiu functions (see (1.2)), namely, those for which the penalty function depends on the deficit at ruin $|U_T|$ only, i.e. $w(x,y) = w_2(y)$. To this end, we define the Gerber-Shiu function of our interest as

$$\phi_{2,\delta}(u) = E\left[e^{-\delta T} w_2(|U_T|) 1\{T < \infty\} | U_0 = u\right] \ , \quad u \geq 0 \ . \tag{4.1}$$

This special class of Gerber-Shiu functions has been considered by, e.g., Landriault and Willmot (2008) and Willmot (2007). For the Gerber-Shiu function $\phi_{2,\delta}(u)$, we always assume $\delta > 0$ or the positive security loading condition (1.5) holds. In this chapter, we propose to analyze $\phi_{2,\delta}(u)$ using a novel connection to a particular fluid flow model described in Section 4.3.

The chapter is organized as follows. Section 4.2 gives a review of the bivariate phase-type distribution. In Section 4.3, the construction of a particular fluid flow model is presented and its connection to the surplus process $\{U_t\}_{t\geq 0}$ is established. The main results of this chapter are provided in Section 4.4 in which an explicit expression for the Laplace transform of the time of ruin T is obtained and the distribution of the deficit at ruin $|U_T|$ is shown to be phase-type. Finally, Section 4.5 deals with the analysis of some discounted joint distributions of certain ruin-related quantities including the surplus immediately prior to ruin U_{T^-} as well as the surplus level immediately after the second last claim before ruin $R_{N_T - 1}$.

4.2 Bivariate phase-type distributions

We consider the class of bivariate phase-type distributions introduced by Assaf et al. (1984). For purposes of completeness, this class of distributions is discussed in detail here. Suppose that the time-homogeneous CTMC $\{J_t\}_{t\geq 0}$ has finite state space $E = \{1, \ldots, m\} \cup \{\Delta\}$, infinitesimal generator $\mathbf{A}$ and initial probability vector $\mathbf{a}$. Let Γ_1 and Γ_2 be two non-empty stochastically closed subsets of E. Note that a set is said to be stochastically closed if once the CTMC enters it, the CTMC never leaves. Also, define $\Delta = \Gamma_1 \cap \Gamma_2$ to be a non-empty subset of E. We further assume that the CTMC $\{J_t\}_{t\geq 0}$ is defined such that both subsets Γ_1 and Γ_2 are visited at least once a.s..

Let Θ_1 and Θ_2 be the times of the first visit of $\{J_t\}_{t\geq 0}$ into Γ_1 and Γ_2 respectively, i.e., for $i = 1, 2$,

$$\Theta_i = \inf\{t \geq 0 : J_t \in \Gamma_i\} \; . \tag{4.2}$$

We assume, without loss of generality, that $\Delta = \Gamma_1 \cap \Gamma_2$ contains only one state, since both random variables Θ_1 and Θ_2 must have been realized upon absorption into Δ. The state space E is then partitioned into the following four subsets: $E_0 = \Gamma_1^c \cap \Gamma_2^c$, $E_1 = \Gamma_1 \cap \Gamma_2^c$, $E_2 = \Gamma_1^c \cap \Gamma_2$ and $\Delta = \Gamma_1 \cap \Gamma_2$. In addition, $|E_0| = m_0$, $|E_1| = m_1$ and $|E_2| = m_2$ such that $m_0 + m_1 + m_2 = m$. The infinitesimal generator $\mathbf{A}$ can be written as

$$\mathbf{A} = \begin{pmatrix} \mathbf{T} & -\mathbf{T1} \\ \mathbf{0} & 0 \end{pmatrix} \; , \tag{4.3}$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_{00} & \mathbf{T}_{01} & \mathbf{T}_{02} \\ \mathbf{0} & \mathbf{T}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_{22} \end{pmatrix} \; . \tag{4.4}$$

Note that

- $\mathbf{T}_{ii}$ is an $m_i \times m_i$ matrix with negative diagonal elements and non-negative off-diagonal elements containing the rate of transition in E_i; and
- $\mathbf{T}_{ij}$ $(i \neq j)$ is an $m_i \times m_j$ matrix with non-negative elements containing the rate of exit from any state in E_i to any state in E_j.

Due to the form of (4.4), the column vector $-\mathbf{T1}$ in (4.3) can be rewritten as

$$-\mathbf{T1} = \begin{pmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \mathbf{t}_2 \end{pmatrix} \; , \tag{4.5}$$

where for $i = 0, 1, 2$, $\mathbf{t}_i$ is an m_i-dimensional column vector. We shall call the joint distribution of (Θ_1, Θ_2) a *bivariate phase-type distribution* with representation $BPH(\mathbf{a}, \mathbf{T})$.

It is worth noting that such a class was further generalized by Kulkarni (1989) whose definition is based on the total accumulated reward until absorption in a finite-state CTMC. However, the class defined by Assaf et al. (1984) is dense in the set of distributions defined on $\mathbb{R}^+ \times \mathbb{R}^+$, and is practically sufficient for modelling purposes.

The class of bivariate phase-type distributions by Assaf et al. (1984) contains various well-known bivariate distributions as special cases, notably Marshall and Olkin (1967)'s bivariate exponential distribution and Freund (1961)'s extension of the exponential distribution (see, e.g., Assaf et al. (1984, Example 5.2) and Cai and Li (2007, Example 4.3) for their bivariate phase-type representations respectively). We remark that while the former distribution only allows for positive correlation between the two variables, the latter distribution can be used to model both positive and negative dependence structures. Interested readers are referred to Assaf et al. (1984, Section 5) for other special cases of bivariate phase-type distributions.

Note that bivariate phase-type distributions may have singular components along $\Theta_1 = 0$, $\Theta_2 = 0$ and/or $\Theta_1 = \Theta_2$. In our context, given that the claim sizes and the interclaim times are assumed to be positive random variables, the possible singular components along $\Theta_1 = 0$ and $\Theta_2 = 0$ are removed by assuming the initial probability vector $\mathbf{a}$ to be of the form $\mathbf{a} = (\mathbf{a}_0, \mathbf{0})$ where $\mathbf{a}_0$ is a row vector with m_0 non-negative elements satisfying $\mathbf{a}_0\mathbf{1} = 1$.

In addition, it seems unreasonable to assume that the interclaim time and (a constant multiple of) the claim size are equal with a positive probability. Such a possibility can be discarded by assuming that $\mathbf{t}_0 = \mathbf{0}$. However, in most cases, no significant simplification occurs in the subsequent analysis when $\mathbf{t}_0 = \mathbf{0}$. Thus, we will keep the vector $\mathbf{t}_0$ without assuming it is a zero vector unless specified otherwise.

4.3 Connection with a particular fluid queue

To analyze the surplus process $\{U_t\}_{t\geq 0}$, suppose we would like to apply the existing connection between risk models and fluid queues described in Section 1.4.2 where the set S_0 is assumed empty. Recall that in such existing construction, the fluid process $\{F(t), J^{(F)}(t)\}_{t\geq 0}$ has a rate of increase/decrease of c and alternates between periods of increasing and decreasing fluid level. Then the joint distribution of the length a period of increasing fluid level and its subsequent period of decreasing fluid level has to follow the generic pair $(V, Y/c)$ under the assumption of a generalized Sparre Andersen risk model. Indeed, this also suggests that it seems more natural to model $(V, Y/c)$ (instead of (V, Y)) as a bivariate phase-type distribution, since V and Y/c are of the same units and can both be interpreted as the length of time.

In the fluid flow literature, the use of MAMs to analyze a fluid process $\{F(t)\}_{t\geq 0}$ is well documented and typically requires the introduction of an underlying time-homogeneous CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ which makes the bivariate process $\{F(t), J^{(F)}(t)\}_{t\geq 0}$ Markovian. In our context, under the assumption that $(V, Y/c)$ has a bivariate phase-type distribution, it is believed that such an underlying CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ does not exist. Even if such an underlying CTMC exists, its definition will likely be a challenging task.

In what follows, we propose to apply the **general** fluid flow model where the generator of the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ is given by (1.20) **AND** the set S_0 is **NOT** empty. For the remainder of this chapter, we shall forget about the existing connection between risk process and fluid queue given in Section 1.4.2. Instead, we shall show that the risk process $\{U_t\}_{t\geq 0}$ under the assumption that the generic pair $(V, Y/c)$ has identical distribution as the bivariate phase-type random vector (Θ_1, Θ_2) can be analyzed by specifying the generator $\mathbf{Q}$ of $\{J^{(F)}(t)\}_{t\geq 0}$ in terms of the parameters of the distribution of (Θ_1, Θ_2). To this end, defining the fluid process $\{F(t)\}_{t\geq 0}$ together with its underlying CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ such that the bivariate process $\{F(t), J^{(F)}(t)\}_{t\geq 0}$ is Markovian represents the key step to analyze the risk process $\{U_t\}_{t\geq 0}$ via MAMs.

Note that for $i = 1, 2, \ldots$, every pair of $(V_i, Y_i/c)$ is generated by a different sample path of the CTMC $\{J_t\}_{t\geq 0}$. To avoid confusion, we shall refer to the sample path of $\{J_t\}_{t\geq 0}$ generating the pair $(V_i, Y_i/c)$ as the *i-th sample path of* $\{J_t\}_{t\geq 0}$. The construction of the fluid level process $\{F(t)\}_{t\geq 0}$ with $F(t) = u$ is done as follows:

1. the fluid process $\{F(t)\}_{t\geq 0}$ remains constant as long as the 1st sample path of $\{J_t\}_{t\geq 0}$ remains in $E_0 = \Gamma_1^c \cap \Gamma_2^c$, i.e. $\{J_t\}_{t\geq 0}$ has not yet visited any states in $\Gamma_1 \cup \Gamma_2$;

2. the fluid process $\{F(t)\}_{t\geq 0}$ starts to decrease (increase) at a rate c once the 1st sample path of $\{J_t\}_{t\geq 0}$ enters $E_1 = \Gamma_1 \cap \Gamma_2^c$ ($E_2 = \Gamma_1^c \cap \Gamma_2$) (given that $V_1 < (>) Y_1/c$);

3. at the time that the 1st sample path of $\{J_t\}_{t\geq 0}$ enters $\Delta = \Gamma_1 \cap \Gamma_2$, the fluid process $\{F(t)\}_{t\geq 0}$ stops its increasing/decreasing/constant pattern; and

4. from this newly established fluid level, we repeat Steps 1-3 by replacing the 1st sample path of $\{J_t\}_{t\geq 0}$ by successively the 2nd, 3rd, $\ldots$ sample path of $\{J_t\}_{t\geq 0}$.

Figure 4.1 depicts graphically the novel connection between $\{U_t\}_{t\geq 0}$ and $\{F(t)\}_{t\geq 0}$ (in contrast to the existing connection as in Figure 1.2 for the study of MAP risk model).

From the construction of the fluid process $\{F(t)\}_{t\geq 0}$, it is immediate that for $i = 1, 2, \ldots,$, the level of the surplus process $\{U_t\}_{t\geq 0}$ immediately after the payment of the i-th claim corresponds to the fluid level of the process $\{F(t)\}_{t\geq 0}$ at the end of the i-th sample path of $\{J_t\}_{t\geq 0}$. Thus, the ruin probability for the surplus process $\{U_t\}_{t\geq 0}$ coincides with the probability that the fluid process $\{F(t)\}_{t\geq 0}$ hits level 0 at least once. However, with regards to the time of ruin T, it is not true that the first passage time to level 0 of $\{F(t)\}_{t\geq 0}$ corresponds to the time of ruin T in the surplus process $\{U_t\}_{t\geq 0}$. Indeed, when the fluid process $\{F(t)\}_{t\geq 0}$ increases or remains constant, the surplus process $\{U_t\}_{t\geq 0}$ increases at a rate c, whereas time does not evolve in $\{U_t\}_{t\geq 0}$ when $\{F(t)\}_{t\geq 0}$ decreases. Hence, the time of ruin T in the surplus process $\{U_t\}_{t\geq 0}$ is equivalent to the

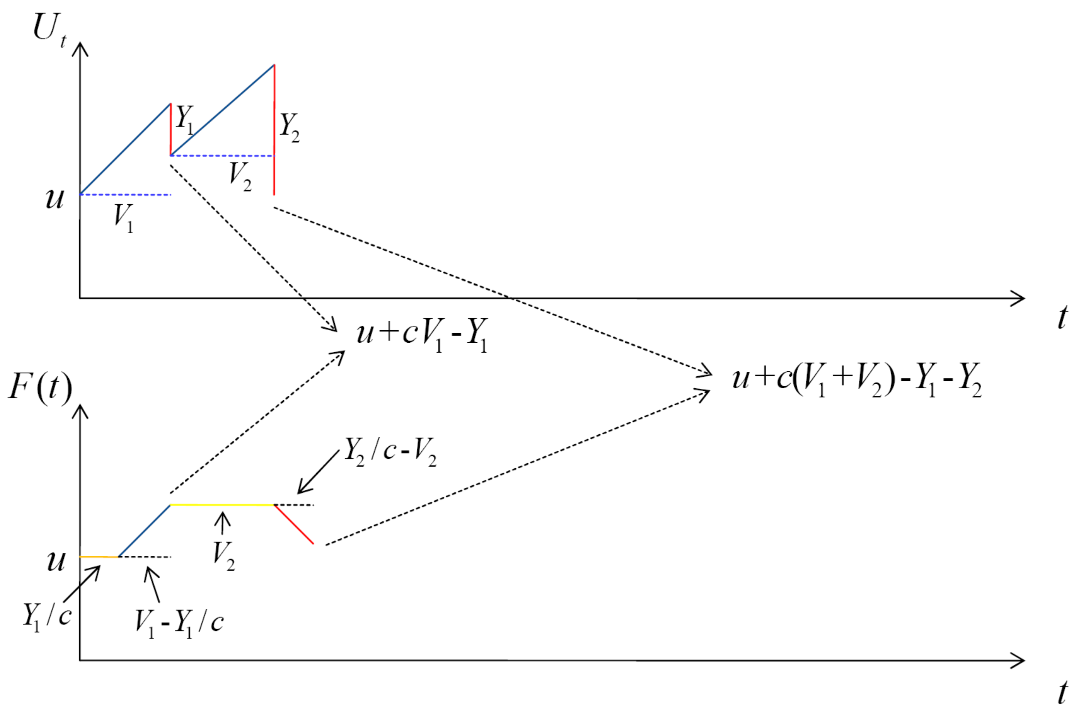

Figure 4.1: Novel connection between $\{U_t\}_{t\geq 0}$ and $\{F(t)\}_{t\geq 0}$

total amount of time the process $\{F(t)\}_{t\geq 0}$ takes to reach level 0, removing periods of time for which the fluid decreases over that first passage time.

Having explained the logic behind the construction of the bivariate process $\{F(t), J^{(F)}(t)\}_{t\geq 0}$, now it remains to formally define the sub-matrices $\mathbf{Q}_{ij}$'s of (1.20) in terms of the sub-matrices $\mathbf{T}_{ij}$'s of (4.4) for $i = 0, 1, 2$. At time 0, we consider the 1st sample path of $\{J_t\}_{t\geq 0}$ starting in some states in $E_0 = \Gamma_1^c \cap \Gamma_2^c$. Recall that the fluid level remains constant as long as $\{J_t\}_{t\geq 0}$ is in E_0. Furthermore, in case where $\mathbf{t}_0 \neq \mathbf{0}$, it is possible that the 1st sample path of CTMC $\{J_t\}_{t\geq 0}$ enters Δ directly from E_0 and we shall move on to consider the 2nd sample path of $\{J_t\}_{t\geq 0}$. Note that the transition rates of $\{J_t\}_{t\geq 0}$ within E_0 are governed by $\mathbf{T}_{00}$. Together with the fact that absorption directly into $\Delta = \Gamma_1 \cap \Gamma_2$ is governed by $\mathbf{t}_0$ and the initial probability of being in the states of E_0 regarding the 2nd sample path of $\{J_t\}_{t\geq 0}$ is governed by $\mathbf{a}_0$, we arrive at $S_0 = E_0$, and

$$\mathbf{Q}_{00} = \mathbf{T}_{00} + \mathbf{t}_0\mathbf{a}_0 \ . \tag{4.6}$$

When the process $\{J_t\}_{t\geq 0}$ first leaves E_0, it may enter $E_1 = \Gamma_1 \cap \Gamma_2^c$ (governed by rate matrix $\mathbf{T}_{01}$), $E_2 = \Gamma_1^c \cap \Gamma_2$ (governed by rate matrix $\mathbf{T}_{02}$), or $\Delta = \Gamma_1 \cap \Gamma_2$. The last case has already been accounted for by (4.6), so we only have to consider the first two cases. Suppose it enters E_1. From the time of entrance, as long as $\{J_t\}_{t\geq 0}$ does not enter Δ, the fluid level decreases at a rate c. It follows that $S_2 = E_1$. We also recall that the transitions within E_1 are governed by the matrices $\mathbf{T}_{11}$. Combining the above observations, it is immediate that

$$\mathbf{Q}_{02} = \mathbf{T}_{01} \qquad \text{and} \qquad \mathbf{Q}_{22} = \mathbf{T}_{11} \ . \tag{4.7}$$

When the fluid level is decreasing at rate c and the CTMC $\{J_t\}_{t\geq 0}$ is in E_1, apart from staying in E_1 the CTMC $\{J_t\}_{t\geq 0}$ can only enter Δ upon leaving the set E_1. Such transition of $\{J_t\}_{t\geq 0}$ into Δ is governed by $\mathbf{t}_1$. Once a transition into Δ occurs, both random variables from the bivariate phase-type distribution have been generated and we immediately move on to the next sample path of $\{J_t\}_{t\geq 0}$ via the initial probability vector $\mathbf{a_0}$. From the above descriptions, we have that

$$\mathbf{Q}_{20} = \mathbf{t}_1 \mathbf{a}_0 \ . \tag{4.8}$$

Furthermore, by noting that under our construction of the bivariate process $\{F(t), J^{(F)}(t)\}_{t\geq 0}$, S_1 cannot be reached from S_2 (and actually vice versa) without passing through S_0, we arrive at

$$\mathbf{Q}_{21} = \mathbf{0} \ . \tag{4.9}$$

Similarly, if the process $\{J_t\}_{t\geq 0}$ enters E_2 when it first leaves E_0, identical arguments lead to

$$\mathbf{Q}_{01} = \mathbf{T}_{02} \ , \qquad \mathbf{Q}_{11} = \mathbf{T}_{22} \ , \qquad \mathbf{Q}_{10} = \mathbf{t}_2 \mathbf{a}_0 \qquad \text{and} \qquad \mathbf{Q}_{12} = \mathbf{0} \ . \tag{4.10}$$

The characterization of the generator $\mathbf{Q}$ in (1.20) is now complete according to (4.6) - (4.10), i.e.

$$\mathbf{Q} = \begin{pmatrix} \mathbf{T}_{00} + \mathbf{t}_0\mathbf{a}_0 & \mathbf{T}_{02} & \mathbf{T}_{01} \\ \mathbf{t}_2\mathbf{a}_0 & \mathbf{T}_{22} & \mathbf{0} \\ \mathbf{t}_1\mathbf{a}_0 & \mathbf{0} & \mathbf{T}_{11} \end{pmatrix}. \tag{4.11}$$

4.4 Laplace transform of time of ruin T and distribution of deficit at ruin $|U_T|$

In this section, we aim at analyzing the particular Gerber-Shiu function $\phi_{2,\delta}(u)$ defined by (4.1). Such an analysis will be conducted via the connection of the surplus process $\{U_t\}_{t\geq 0}$ to the fluid level process $\{F(t)\}_{t\geq 0}$. We recall here some crucial observations resulting from Section 4.3:

- for $i = 1, 2, \ldots$, the surplus level immediately after the payment of the i-th claim corresponds to the fluid level of the process $\{F(t)\}_{t\geq 0}$ at the end of the i-th sample path of $\{J_t\}_{t\geq 0}$; and
- with respect to time, time evolves in the surplus process $\{U_t\}_{t\geq 0}$ only when the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ is either in S_0 or S_1 (in contrast to the process $\{F(t)\}_{t\geq 0}$ where time evolves independently of the state of the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$).

Clearly, the analysis of the Gerber-Shiu function $\phi_{2,\delta}(u)$ in the risk process $\{U_t\}_{t\geq 0}$ requires a freeze in the clock time whenever the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ is in the set of phases S_2. Then, we consider the evolution of the fluid process $\{F(t)\}_{t\geq 0}$ and its underlying CTMC $\{J^{(F)}(t)\}_{t\geq 0}$. Let ν be the time taken by $\{F(t)\}_{t\geq 0}$ to become empty for the first time, i.e. $\nu = \inf\{t \geq 0 : F(t) < 0\}$, with $\nu = \infty$ if $F(t) \geq 0$ for all $t \geq 0$.

Remark 8 *Note that using the notations of fluid flow process defined in Section 1.4.2, one sees that* $(\nu | F(0) = u) \overset{d}{=} \sigma(u, 0)$ *for* $u \geq 0$*. However, the new notation* ν *is defined to ease presentation* *here, as we shall see later.*

Also, for $i = 0, 1, 2$, let $\nu_i = \int_0^{\nu} 1\{J^{(F)}(s) \in S_i\}\, ds$ be the time spent in the set of phases S_i during the first passage time ν, with $\nu_i = \infty$ if $\nu = \infty$. Then, for $i = 0, 1, 2$, $j \in S_i$ and $k \in S_2$, we define the $|S_i| \times |S_2|$ matrix $\boldsymbol{\Upsilon}_{i,u}(x)$ with (j, k)-th element defined by

$$[\boldsymbol{\Upsilon}_{i,u}(x)]_{jk} = \Pr\{\nu < \infty, \nu_0 + \nu_1 \leq x, J^{(F)}(\nu) = k | F(0) = u, J^{(F)}(0) = j\} , \quad x \geq 0 . \tag{4.12}$$

We also define the LST, for $i = 0, 1, 2$, $\widehat{\boldsymbol{\Upsilon}}_{i,u}(\delta) = \int_0^{\infty} e^{-\delta x}\, d\boldsymbol{\Upsilon}_{i,u}(x)$. Our objective is to establish some relationships between the LSTs. To this end, we condition on the first transition of the process $\{J^{(F)}(t)\}_{t \geq 0}$ into another set of phases and readily obtain

$$\widehat{\boldsymbol{\Upsilon}}_{0,u}(\delta) = (\delta \mathbf{I} - \mathbf{Q}_{00})^{-1} \left[\mathbf{Q}_{01} \widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta) + \mathbf{Q}_{02} \widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta) \right] , \quad u \geq 0 , \tag{4.13}$$

$$\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta) = \int_0^{\infty} e^{(\mathbf{Q}_{11} - \delta \mathbf{I}) y} \mathbf{Q}_{10} \widehat{\boldsymbol{\Upsilon}}_{0,u+cy}(\delta)\, dy = \frac{1}{c} \int_u^{\infty} e^{(\mathbf{Q}_{11} - \delta \mathbf{I}) \frac{y-u}{c}} \mathbf{Q}_{10} \widehat{\boldsymbol{\Upsilon}}_{0,y}(\delta)\, dy , \quad u \geq 0 , \tag{4.14}$$

and

$$\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta) = e^{\mathbf{Q}_{22} \frac{u}{c}} + \int_0^{\frac{u}{c}} e^{\mathbf{Q}_{22} y} \mathbf{Q}_{20} \widehat{\boldsymbol{\Upsilon}}_{0,u-cy}(\delta)\, dy = e^{\mathbf{Q}_{22} \frac{u}{c}} + \frac{1}{c} \int_0^{u} e^{\mathbf{Q}_{22} \frac{u-y}{c}} \mathbf{Q}_{20} \widehat{\boldsymbol{\Upsilon}}_{0,y}(\delta)\, dy, \quad u \geq 0 . \tag{4.15}$$

In order to simplify the notations, we introduce the matrix $\mathbf{Q}^*(\delta)$ defined by

$$\mathbf{Q}^*(\delta) = \begin{pmatrix} \mathbf{Q}^*_{11}(\delta) & \mathbf{Q}^*_{12}(\delta) \\ \mathbf{Q}^*_{21}(\delta) & \mathbf{Q}^*_{22}(\delta) \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{10}(\delta \mathbf{I} - \mathbf{Q}_{00})^{-1} \mathbf{Q}_{01} & \mathbf{Q}_{10}(\delta \mathbf{I} - \mathbf{Q}_{00})^{-1} \mathbf{Q}_{02} \\ \mathbf{Q}_{20}(\delta \mathbf{I} - \mathbf{Q}_{00})^{-1} \mathbf{Q}_{01} & \mathbf{Q}_{20}(\delta \mathbf{I} - \mathbf{Q}_{00})^{-1} \mathbf{Q}_{02} \end{pmatrix} . \tag{4.16}$$

It is instructive to observe that for $i, j = 1, 2$, the block matrix $\mathbf{Q}^*_{ij}(\delta)$ gives the Laplace transform of the time spent by the underlying CTMC $\{J^{(F)}(t)\}_{t \geq 0}$ in S_0, given the state before entering S_0 is S_i and the state after leaving S_0 is S_j.

Now, differentiating (4.14) with respect to u and then making use of (4.13) and (4.16), one finds that

$$\begin{aligned}\frac{\partial}{\partial u}\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta) &= -\frac{1}{c}\left[\mathbf{Q}_{10}\widehat{\boldsymbol{\Upsilon}}_{0,u}(\delta) + (\mathbf{Q}_{11}-\delta\mathbf{I})\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta)\right] \\ &= -\frac{1}{c}\left\{\left[\mathbf{Q}_{10}(\delta\mathbf{I}-\mathbf{Q}_{00})^{-1}\mathbf{Q}_{01} + (\mathbf{Q}_{11}-\delta\mathbf{I})\right]\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta) + \mathbf{Q}_{10}(\delta\mathbf{I}-\mathbf{Q}_{00})^{-1}\mathbf{Q}_{02}\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta)\right\} \\ &= -\frac{1}{c}\left\{\left[\mathbf{Q}_{11}-\delta\mathbf{I}+\mathbf{Q}^*_{11}(\delta)\right]\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta) + \mathbf{Q}^*_{12}(\delta)\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta)\right\}\ , \quad u \ge 0\ . \end{aligned} \tag{4.17}$$

Note that, under $\delta > 0$ or the positive security loading condition (1.5),

$$\lim_{u\to\infty}\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta) = \mathbf{0}\ . \tag{4.18}$$

Similarly, the differentiation of (4.15) with respect to u followed by the use of (4.13) and (4.16) yields

$$\begin{aligned}\frac{\partial}{\partial u}\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta) &= \frac{1}{c}\left[\mathbf{Q}_{22}\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta) + \mathbf{Q}_{20}\widehat{\boldsymbol{\Upsilon}}_{0,u}(\delta)\right] \\ &= \frac{1}{c}\left\{\mathbf{Q}_{22}\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta) + \mathbf{Q}_{20}(\delta\mathbf{I}-\mathbf{Q}_{00})^{-1}\left[\mathbf{Q}_{01}\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta) + \mathbf{Q}_{02}\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta)\right]\right\} \\ &= \frac{1}{c}\left\{\left[\mathbf{Q}_{22}+\mathbf{Q}^*_{22}(\delta)\right]\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta) + \mathbf{Q}^*_{21}(\delta)\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta)\right\}\ , \quad u \ge 0\ , \end{aligned} \tag{4.19}$$

with the trivial boundary condition

$$\widehat{\boldsymbol{\Upsilon}}_{2,0}(\delta) = \mathbf{I}\ . \tag{4.20}$$

From (4.17), (4.18), (4.19) and (4.20), we observe that $\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta)$ and $\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta)$ satisfy a Feynman-Kac equation (see, e.g., Asmussen et al. (2002, Theorem 2)).

For a moment we turn our attention to another random time pertaining to the fluid level process $\{F(t)\}_{t\ge0}$. Starting with an initial fluid level u, we let η be the time taken for the fluid level to return to its initial level for the first time. It is clear that η is indeed independent of the initial level u. We then define $\eta = \inf\{t \ge 0 : F(t) < 0 | F(0) = 0\}$, with $\eta = \infty$ if $F(t) \ge 0$ for all

$t \geq 0$. Also, for $i = 0, 1, 2$, let $\eta_i = \int_0^\eta 1\{J^{(F)}(s) \in S_i\}\, ds$ be the time spent in the set of phases S_i during the first return time η, with $\eta_i = \infty$ if $\eta = \infty$. Again we would like to remove the time that the CTMC $\{J^{(F)}(t)\}_{t \geq 0}$ is in S_2, and therefore for $j \in S_1$ and $k \in S_2$ we define $\mathbf{\Lambda}(x)$ to be the $|S_1| \times |S_2|$ matrix with (j,k)-th element defined by

$$[\mathbf{\Lambda}(x)]_{jk} = \Pr\{\eta < \infty, \eta_0 + \eta_1 \leq x, J^{(F)}(\eta) = k | F(0) = 0, J^{(F)}(0) = j\} \ , \quad x \geq 0 \ , \tag{4.21}$$

with its LST given by $\widehat{\mathbf{\Lambda}}(\delta) = \int_0^\infty e^{-\delta x}\, d\mathbf{\Lambda}(x)$. Note also that $\widehat{\mathbf{\Lambda}}(\delta) = \widehat{\mathbf{\Upsilon}}_{1,0}(\delta)$.

Now if we return to the quantity $\widehat{\mathbf{\Upsilon}}_{1,u}(\delta)$, note that starting with initial fluid level u, for the fluid level to become zero, the skip-free property of the fluid process $\{F(t)\}_{t \geq 0}$ implies that the process must revisit level u at least once and ultimately make a transition from level u to level 0. Hence,

$$\widehat{\mathbf{\Upsilon}}_{1,u}(\delta) = \widehat{\mathbf{\Lambda}}(\delta) \widehat{\mathbf{\Upsilon}}_{2,u}(\delta) \ , \quad u \geq 0 \ . \tag{4.22}$$

By substitution of (4.22) into (4.19), one arrives at

$$\frac{\partial}{\partial u} \widehat{\mathbf{\Upsilon}}_{2,u}(\delta) = \frac{1}{c} \left[\mathbf{Q}_{22} + \mathbf{Q}^*_{22}(\delta) + \mathbf{Q}^*_{21}(\delta) \widehat{\mathbf{\Lambda}}(\delta) \right] \widehat{\mathbf{\Upsilon}}_{2,u}(\delta) \ , \quad u \geq 0 \ . \tag{4.23}$$

(4.23) together with the boundary condition (4.20) leads to

$$\widehat{\mathbf{\Upsilon}}_{2,u}(\delta) = e^{\left[\mathbf{Q}_{22} + \mathbf{Q}^*_{22}(\delta) + \mathbf{Q}^*_{21}(\delta) \widehat{\mathbf{\Lambda}}(\delta) \right] \frac{u}{c}} \ , \quad u \geq 0 \ . \tag{4.24}$$

Finally, using (4.24), (4.22) becomes

$$\widehat{\mathbf{\Upsilon}}_{1,u}(\delta) = \widehat{\mathbf{\Lambda}}(\delta) e^{\left[\mathbf{Q}_{22} + \mathbf{Q}^*_{22}(\delta) + \mathbf{Q}^*_{21}(\delta) \widehat{\mathbf{\Lambda}}(\delta) \right] \frac{u}{c}} \ , \quad u \geq 0 \ . \tag{4.25}$$

The substitution of (4.24) and (4.25) into (4.17) yields

$$\begin{aligned}
&\widehat{\boldsymbol{\Lambda}}(\delta)\left[\mathbf{Q}_{22}+\mathbf{Q}_{22}^{*}(\delta)+\mathbf{Q}_{21}^{*}(\delta)\widehat{\boldsymbol{\Lambda}}(\delta)\right]e^{\left[\mathbf{Q}_{22}+\mathbf{Q}_{22}^{*}(\delta)+\mathbf{Q}_{21}^{*}(\delta)\widehat{\boldsymbol{\Lambda}}(\delta)\right]\frac{u}{c}} \\
&=-\left\{\left[\mathbf{Q}_{11}-\delta\mathbf{I}+\mathbf{Q}_{11}^{*}(\delta)\right]\widehat{\boldsymbol{\Lambda}}(\delta)+\mathbf{Q}_{12}^{*}(\delta)\right\}e^{\left[\mathbf{Q}_{22}+\mathbf{Q}_{22}^{*}(\delta)+\mathbf{Q}_{21}^{*}(\delta)\widehat{\boldsymbol{\Lambda}}(\delta)\right]\frac{u}{c}}\ , \quad u\geq 0\ , \qquad (4.26)
\end{aligned}$$

which implies

$$\left[\mathbf{Q}_{11}-\delta\mathbf{I}+\mathbf{Q}_{11}^{*}(\delta)\right]\widehat{\boldsymbol{\Lambda}}(\delta)+\widehat{\boldsymbol{\Lambda}}(\delta)\left[\mathbf{Q}_{22}+\mathbf{Q}_{22}^{*}(\delta)\right]+\widehat{\boldsymbol{\Lambda}}(\delta)\mathbf{Q}_{21}^{*}(\delta)\widehat{\boldsymbol{\Lambda}}(\delta)+\mathbf{Q}_{12}^{*}(\delta)=\mathbf{0}\ . \qquad (4.27)$$

(4.27) satisfied by the LST $\widehat{\boldsymbol{\Lambda}}(\delta)$ is known as a *Riccati equation* (see, e.g., Abou-Kandil et al. (2003, Chapter 2)). Several numerical algorithms have been proposed in the literature to obtain solutions of a Riccati equation. We also refer interested readers to Badescu et al. (2005), Bean et al. (2005) and Guo (2001). Once $\widehat{\boldsymbol{\Lambda}}(\delta)$ has been determined, then $\widehat{\boldsymbol{\Upsilon}}_{1,u}(\delta)$ and $\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta)$ can be determined by (4.25) and (4.24) respectively, and $\widehat{\boldsymbol{\Upsilon}}_{0,u}(\delta)$ can be calculated from (4.13).

Remark 9 *An alternative proof of the fact that $\widehat{\boldsymbol{\Lambda}}(\delta)$ satisfies the Riccati equation (4.27) can be found in Badescu et al. (2009).*

Now we have all the necessary components for one of the main results in this chapter regarding the risk process $\{U_t\}_{t\geq 0}$. The Laplace transform of the time of ruin T, denoted by

$$\rho_\delta(u)=E\left[e^{-\delta T}1\{T<\infty\}|U_0=u\right]\ , \quad u\geq 0\ , \qquad (4.28)$$

has representation

$$\rho_\delta(u)=\mathbf{a}_0(\delta\mathbf{I}-\mathbf{Q}_{00})^{-1}\left[\mathbf{Q}_{01}\widehat{\boldsymbol{\Lambda}}(\delta)+\mathbf{Q}_{02}\right]\widehat{\boldsymbol{\Upsilon}}_{2,u}(\delta)\mathbf{1}\ , \quad u\geq 0\ . \qquad (4.29)$$

The probabilistic proof of (4.29) is as follows. Starting at time 0 in S_0 with the initial probability vector $\mathbf{a}_0$, the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ stays within the set S_0 for a period of time whose Laplace transform is given by $(\delta\mathbf{I}-\mathbf{Q}_{00})^{-1}$. Upon leaving the set S_0, the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ either enters S_1 (governed by $\mathbf{Q}_{01}$) or S_2 (governed by $\mathbf{Q}_{02}$). If the transition is made into S_1, in order for ruin to occur, the fluid flow $\{F(t)\}_{t\geq 0}$ has to first return to the same level u in S_2 and then hit level 0 in S_2, giving rise to $\widehat{\mathbf{\Lambda}}(\delta)\widehat{\mathbf{\Upsilon}}_{2,u}(\delta)$. On the other hand, if the transition is made into S_2, the fluid flow has to make a first passage from u to 0 for ruin to occur, explaining the term $\widehat{\mathbf{\Upsilon}}_{2,u}(\delta)$. Combining these two cases and adding up all the possible phases at ruin results in (4.29).

A direct consequence of (4.29) is the ruin probability, which can be retrieved by letting $\delta = 0$. It is given by

$$\Pr\{T < \infty | U_0 = u\} = \mathbf{a}_0(-\mathbf{Q}_{00})^{-1}\left[\mathbf{Q}_{01}\widehat{\mathbf{\Lambda}}(0) + \mathbf{Q}_{02}\right]\widehat{\mathbf{\Upsilon}}_{2,u}(0)\mathbf{1} \ , \quad u \geq 0 \ . \tag{4.30}$$

Note that (4.30) can also be obtained from Ahn and Ramaswami (2005, Theorem 3) with $s = 0$ by recalling that the ruin probability for the surplus process $\{U_t\}_{t\geq 0}$ coincides with the probability that the fluid process $\{F(t)\}_{t\geq 0}$ eventually hits level 0 at least once.

The (defective) distribution of the deficit at ruin also arises as a direct consequence of the Markov property exhibited by the process $\{F(t), J^{(F)}(t)\}_{t\geq 0}$. The deficit at ruin $|U_T|$ in the bivariate phase-type risk model $\{U_t\}_{t\geq 0}$ has a phase-type distribution with representation given by $PH(\mathbf{a}_0(-\mathbf{Q}_{00})^{-1}[\mathbf{Q}_{01}\widehat{\mathbf{\Lambda}}(0) + \mathbf{Q}_{02}]\widehat{\mathbf{\Upsilon}}_{2,u}(0), \mathbf{Q}_{22}/c)$. To see this, note that starting at level u, the fluid process $\{F(t)\}_{t\geq 0}$ has to hit level 0 in S_2 for ruin to occur. The distribution of the phase in S_2 of the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ at the time of hitting is given by $\mathbf{a}_0(-\mathbf{Q}_{00})^{-1}[\mathbf{Q}_{01}\widehat{\mathbf{\Lambda}}(0)+\mathbf{Q}_{02}]\widehat{\mathbf{\Upsilon}}_{2,u}(0)$. Since $\mathbf{Q}_{22}$ corresponds to the intensity matrix of a descending period, the correction factor of c arises because we are interested in the intensity matrix with respect to level instead of time.

Because the time of ruin T and the deficit at ruin $|U_T|$ are conditionally independent given the phases of the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$, the Gerber-Shiu function $\phi_{2,\delta}(u)$ can also be obtained in integral form immediately, and is given by

$$\phi_{2,\delta}(u) = \mathbf{a}_0(\delta\mathbf{I} - \mathbf{Q}_{00})^{-1}\left[\mathbf{Q}_{01}\widehat{\mathbf{\Lambda}}(\delta) + \mathbf{Q}_{02}\right]\widehat{\mathbf{\Upsilon}}_{2,u}(\delta)\int_0^{\infty} e^{\mathbf{Q}_{22}\frac{y}{c}}\left(\frac{\mathbf{q}_2}{c}\right) w_2(y)\,dy\ , \quad u \geq 0\ , \quad (4.31)$$

where $\mathbf{q}_2 = -\mathbf{Q}_{22}\mathbf{1}$.

4.5 Discounted joint density of $(U_{T^-}, |U_T|, R_{N_T-1})$

In this section, we investigate how some discounted joint distributions involving the surplus immediately prior to ruin U_{T^-} can be analyzed via the connection to the particular fluid queue discussed in Section 4.3. The discounted joint density of the surplus prior to ruin U_{T^-} and the deficit at ruin $|U_T|$ has been studied by many authors, e.g., Gerber and Shiu (1997), Li and Garrido (2005), and Ren (2007). In this section, we further make the practical assumption that ties between the variables V and Y/c are not possible, i.e. we assume $\mathbf{t}_0 = \mathbf{0}$.

In Section 4.4, the Laplace transform of the time of ruin T, the distribution of the deficit at ruin $|U_T|$ and the Gerber-Shiu function $\phi_{2,\delta}(u)$ in the bivariate phase-type risk model $\{U_t\}_{t\geq 0}$ are expressed in terms of some particular quantities in the fluid process $\{F(t)\}_{t\geq 0}$. However, a similar connection for the surplus prior to ruin U_{T^-} turns out to be highly non-trivial to establish. Indeed, from the construction of the process $\{F(t)\}_{t\geq 0}$, the initial upward segment of $\{U_t\}_{t\geq 0}$ (before the first claim) is translated in the fluid process $\{F(t)\}_{t\geq 0}$ to either a level segment (if $V_1 < Y_1/c$) or a combination of a level segment followed by an upward segment (if $V_1 > Y_1/c$). Thus, it is clear that the construction of $\{F(t)\}_{t\geq 0}$ does not allow us to directly associate U_{T^-} to any fluid level of $\{F(t)\}_{t\geq 0}$. However, we already pointed out that for $i = 1, 2, \ldots$, the fluid level of $\{F(t)\}_{t\geq 0}$ at the end of the i-th sample path of $\{J_t\}_{t\geq 0}$ corresponds to the surplus level of $\{U_t\}_{t\geq 0}$

immediately after the payment the i-th claim, which is exactly R_i according to the definition (1.9). As a consequence, it appears possible to analyze U_{T^-} by keeping track of the variable R_{N_T-1}, the surplus level immediately after the second last claim before ruin with the definition $R_0 = u$ if ruin is caused by the first claim (see Section 1.3). By using sample paths arguments, we shall obtain an expression for the discounted joint distribution of the triplet $(U_{T^-}, |U_T|, R_{N_T-1})$. Given that the contributions to this discounted joint distribution have different functional forms based on whether ruin is caused by the first claim or any of its subsequent claims (see, e.g., Cheung et al. (2010b) in the context of a compound Poisson risk model), we introduce two $|S_0| \times 1$ column vectors, namely $\mathbf{h}^*_{1,\delta}(x, y|u)$ and $\mathbf{h}^*_{2,\delta}(x, y, v|u)$, whose i-th elements are respectively

$$\begin{aligned}
&[\mathbf{h}^*_{1,\delta}(x, y|u)]_i \, dx \, dy \\
&= E\left[e^{-\delta T} 1\{N_T = 1, U_{T^-} \in (x, x + dx), |U_T| \in (y, y + dy)\} | F(0) = u, J^{(F)}(0) = i\right] , \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x > u \geq 0; y > 0 , \qquad (4.32)
\end{aligned}$$

and

$$\begin{aligned}
&[\mathbf{h}^*_{2,\delta}(x, y, v|u)]_i \, dx \, dy \, dv \\
&= E\left[\begin{array}{c} e^{-\delta T} 1\{N_T > 1, U_{T^-} \in (x, x + dx)\} \\ 1\{|U_T| \in (y, y + dy), R_{N_T-1} \in (v, v + dv)\} \end{array} \middle| F(0) = u, J^{(F)}(0) = i\right] , \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y > 0; x > v > 0; u \geq 0 . \qquad (4.33)
\end{aligned}$$

To identify expressions for $\mathbf{h}^*_{1,\delta}(x, y|u)$ and $\mathbf{h}^*_{2,\delta}(x, y, v|u)$, we have to first define some new quantities in the fluid process $\{F(t)\}_{t\geq 0}$ and its reflected version $\{F^r(t)\}_{t\geq 0}$ (see Section 1.4.2). For $0 \leq x < y$, let the $|S_1| \times |S_1|$ matrix ${}_0\widehat{\mathbf{g}}_{11}(x, y, \delta)$ be the LST (with argument δ) of the total time spent by $\{J^{(F)}(t)\}_{t\geq 0}$ in S_0 and S_1 during a first passage of $\{F(t), J^{(F)}(t)\}_{t\geq 0}$ from (x, S_1) to (y, S_1) avoiding level 0 enroute. In addition, analogous to $\widehat{\mathbf{\Lambda}}(\delta)$ defined in Section 4.4, for $y \geq 0$

we define the $|S_2| \times |S_1|$ matrix ${}^{y}\widehat{\mathbf{\Lambda}}^{r}(\delta)$ to be the LST (with argument δ) of the total time spent by $\{J^{(F)}(t)\}_{t\geq 0}$ in S_0 and S_1 during a first passage of $\{F^r(t), J^{(F)}(t)\}_{t\geq 0}$ from $(0, S_2)$ to $(0, S_1)$ avoiding level y enroute.

Remark 10 *The expressions of the quantities corresponding to ${}_{0}\widehat{\mathbf{g}}_{11}(x, y, \delta)$ and ${}^{y}\widehat{\mathbf{\Lambda}}^{r}(\delta)$ have been derived in the context where all the time spent by $\{J^{(F)}(t)\}_{t\geq 0}$ in S_0, S_1 and S_2 is accounted for (see, e.g., Ahn et al. (2007, Theorem 1) and Ramaswami (2006, Theorem 4) respectively and references therein). Their notations are ${}_{0}\widehat{\mathbf{f}}_{11}(x, y, \delta)$ and ${}^{y}\mathbf{\Psi}^{r}(\delta)$ respectively according to Section 1.4.2. Interested readers are also referred to Badescu et al. (2009, Appendix II) for the computation of ${}_{0}\widehat{\mathbf{g}}_{11}(x, y, \delta)$ and ${}^{y}\widehat{\mathbf{\Lambda}}^{r}(\delta)$.*

By sample paths analysis, we have that

$$\mathbf{h}^{*}_{1,\delta}(x, y|u) = \frac{1}{c^2}\; e^{(\mathbf{Q}_{00}-\delta\mathbf{I})\frac{x-u}{c}}\mathbf{Q}_{02}e^{\mathbf{Q}_{22}\frac{u+y}{c}}\mathbf{q}_2\;, \quad x > u \geq 0; y > 0\;, \tag{4.34}$$

and

$$\mathbf{h}^{*}_{2,\delta}(x, y, v|u) = (\delta\mathbf{I} - \mathbf{Q}_{00})^{-1}[\mathbf{Q}_{01}\mathbf{r}_{1,\delta}(x, y, v|u) + \mathbf{Q}_{02}\mathbf{r}_{2,\delta}(x, y, v|u)]\;, \quad y > 0; x > v > 0; u \geq 0\;, \tag{4.35}$$

where

$$\begin{aligned} &\mathbf{r}_{1,\delta}(x, y, v|u) \\ &= \begin{cases} \frac{1}{c}\; {}_{0}\widehat{\mathbf{g}}_{11}(u, v, \delta)\left[\mathbf{I} - \widehat{\mathbf{\Lambda}}(\delta)\; {}^{v}\widehat{\mathbf{\Lambda}}^{r}(\delta)\right]^{-1}\left[\mathbf{Q}_{10} + \widehat{\mathbf{\Lambda}}(\delta)\mathbf{Q}_{20}\right]\mathbf{h}^{*}_{1,\delta}(x, y|v)\;, & x > v > u \geq 0; y > 0\;, \\ \widehat{\mathbf{\Lambda}}(\delta)\mathbf{r}_{2,\delta}(x, y, v|u)\;, & x > v; y > 0; 0 < v < u\;, \end{cases} \end{aligned} \tag{4.36}$$

and

$$
\mathbf{r}_{2,\delta}(x,y,v|u)
$$

$$
= \begin{cases} {}^{u}\widehat{\mathbf{\Lambda}}^{r}(\delta)\mathbf{r}_{1,\delta}(x,y,v|u) \ , & x > v > u \geq 0; y > 0 \ . \\ \frac{1}{c}\widehat{\mathbf{\Upsilon}}_{2,u-v}(\delta)\left[\mathbf{I} - {}^{v}\widehat{\mathbf{\Lambda}}^{r}(\delta)\widehat{\mathbf{\Lambda}}(\delta)\ \right]^{-1}\left[\mathbf{Q}_{20} + {}^{v}\widehat{\mathbf{\Lambda}}^{r}(\delta)\mathbf{Q}_{10}\right]\mathbf{h}^{*}_{1,\delta}(x,y|v) \ , & x > v; y > 0; 0 < v < u \ . \end{cases} \tag{4.37}
$$

To prove (4.34), note that since the first claim causes ruin, the surplus prior to ruin $U_{T^-} = x$ has to be greater than the initial surplus $U_0 = u$. For U_{T^-} to be x and $|U_T|$ to be y,

- the surplus process $\{U_t\}_{t\geq 0}$ has to first reach level x from level u without a claim. This is translated into a level segment of duration $(x-u)/c$ in the associated fluid flow process $\{F(t)\}_{t\geq 0}$. Accounting for the time spent by $\{J^{(F)}(t)\}_{t\geq 0}$ in S_0 during this level segment yields a contribution of $e^{(\mathbf{Q}_{00}-\delta\mathbf{I})(x-u)/c}$ to $\mathbf{h}^{*}_{1,\delta}(x,y|u)$;

- then, the first claim occurs within $c^{-1}\,dx$ after reaching level x in the surplus process $\{U_t\}_{t\geq 0}$. In order for ruin to occur upon this first claim, the fluid process $\{F(t)\}_{t\geq 0}$ has to make a transition from a level segment to a decreasing segment, giving rise to $c^{-1}\mathbf{Q}_{02}$;

- but the duration of the above decreasing period in $\{F(t)\}_{t\geq 0}$ has to be $(u+y)/c$ to ensure a deficit at ruin of $|U_T| = y$. Given that the time spent by $\{F(t)\}_{t\geq 0}$ in S_2 is factored out, this yields a contribution of $e^{\mathbf{Q}_{22}(u+y)/c}$ to $\mathbf{h}^{*}_{1,\delta}(x,y|u)$; and

- finally, the fluid process $\{F(t)\}_{t\geq 0}$ should stop its descending pattern within $c^{-1}\,dy$ after reaching $-y$ which results in a contribution of $c^{-1}\mathbf{q_2}$ to $\mathbf{h}^{*}_{1,\delta}(x,y|u)$ by summing up all the phases in S_0.

Now, let us look at the expression for $\mathbf{h}^{*}_{2,\delta}(x,y,v|u)$ give by (4.35). Note that $\mathbf{h}^{*}_{2,\delta}(x,y,v|u)$ is non-zero only if $x > v$ (given that $U_{T^-} > R_{N_T-1}$ a.s.). First, the term $(\delta\mathbf{I} - \mathbf{Q}_{00})^{-1}$ corresponds to the Laplace transform of the time that the CTMC $\{J^{(F)}(t)\}_{t\geq 0}$ first leaves S_0 (given that $J^{(F)}(0) \in$

S_0). Upon this first exit from S_0, $\{J^{(F)}(t)\}_{t\geq 0}$ enters either S_1 (governed by $\mathbf{Q}_{01}$) if $V_1 > Y_1/c$ or S_2 (governed by $\mathbf{Q}_{02}$) if $V_1 < Y_1/c$. If the first exit is made into S_i, we denote, for $i = 1, 2$, the discounted (by the time remaining until ruin) joint density of the triplet $(U_{T^-}, |U_T|, R_{N_T-1})$ (with the event that the first claim does not cause ruin) by $\mathbf{r}_{i,\delta}(x, y, v|u)$. This explains the form of (4.35).

For $i = 1, 2$, we shall give a detailed probabilistic proof of the expressions $\mathbf{r}_{i,\delta}(x, y, v|u)$ for the case $x > v > u \geq 0; y > 0$ (i.e. the first equations of (4.36) and (4.37)). We first examine the quantity $\mathbf{r}_{1,\delta}(x, y, v|u)$. For R_{N_T-1} to be v ($> u$), the surplus process $\{U_t\}_{t\geq 0}$ has to first reach level v from level u before ruin. Equivalently, the fluid level process $\{F(t)\}_{t\geq 0}$, starting with level u in S_1, has to first attain level v in S_1 avoiding level 0 enroute. The LST of the total time spent in S_0 and S_1 during this first passage time is ${}_0\widehat{\mathbf{g}}_{11}(u, v, \delta)$. Being at level v in S_1 for the first time, it is possible to revisit level v in S_1 an arbitrary number (≥ 0) of times prior to ruin. The LST of the time spent by $\{J^{(F)}(t)\}_{t\geq 0}$ in S_0 and S_1 before the last visit of $\{F(t)\}_{t\geq 0}$ to level v in S_1 is given by $[\mathbf{I} - \widehat{\mathbf{\Lambda}}(\delta)\ {}^{v}\widehat{\mathbf{\Lambda}}^{r}(\delta)]^{-1}$. Now, having the fluid process $\{F(t)\}_{t\geq 0}$ at level v in S_1 for the last time, R_{N_T-1} can be v via two scenarios:

- the fluid process $\{F(t)\}_{t\geq 0}$ should stop its ascending pattern within $c^{-1}\,dv$ after reaching v for the last time in S_1, and this results in a contribution of $c^{-1}\mathbf{Q}_{10}$ to $\mathbf{r}_{1,\delta}(x, y, v|u)$; or
- the fluid process $\{F(t)\}_{t\geq 0}$ continues its ascending pattern, returns to level x_1 this time in S_2 and then stop its descending pattern within $c^{-1}\,dv$ after reaching v in S_2, which provides a contribution of $\widehat{\mathbf{\Lambda}}(\delta)c^{-1}\mathbf{Q}_{20}$ to $\mathbf{r}_{1,\delta}(x, y, v|u)$.

Note that in both cases, the fluid process $\{F(t)\}_{t\geq 0}$ reaches level v in S_0. Given that ruin has to occur at the time of the next claim with a surplus prior to ruin of x and a deficit at ruin of y, this yields a final contribution of $\mathbf{h}^*_{1,\delta}(x, y|v)$.

For $\mathbf{r}_{2,\delta}(x, y, v|u)$, note that the fluid process $\{F(t)\}_{t\geq 0}$, being at level u in S_2, must return to level u in S_1 avoiding level 0 enroute for R_{N_T-1} to be v, since $v > u$. The LST of the total time spent in S_0 and S_1 during this first passage time is exactly ${}^{u}\widehat{\mathbf{\Lambda}}^{r}(\delta)$. Being back at level u in S_1, the remaining contribution is easily seen to be $\mathbf{r}_{1,\delta}(x, y, v|u)$.

The formulae provided for $\mathbf{r}_{1,\delta}(x, y, v|u)$ and $\mathbf{r}_{2,\delta}(x, y, v|u)$ for the case $x > v; y > 0; 0 < v < u$ can be obtained probabilistically along the same line of logic.

From the results (4.34) - (4.37), it is immediate that the discounted joint density of U_{T^-} and $|U_T|$, denoted by $\mathbf{h}_\delta(x, y|u)$ and having as its i-th element, for $i \in S_0$,

$$[\mathbf{h}_\delta(x, y|u)]_i \, dx \, dy = E\left[e^{-\delta T} 1\{U_{T^-} \in (x, x + dx), |U_T| \in (y, y + dy)\}|F(0) = u, J^{(F)}(0) = i\right] ,$$
$$x, y > 0; u \geq 0 . \quad (4.38)$$

is given by

$$\mathbf{h}_\delta(x, y|u) = \mathbf{h}^*_{1,\delta}(x, y|u) + \int_0^x \mathbf{h}^*_{2,\delta}(x, y, v|u) \, dv \ , \quad x, y > 0; u \geq 0 \ . \quad (4.39)$$

Remark 11 *From Cheung et al. (2010c), the discounted joint density of the triplet* $(U_{T^-}, |U_T|, X_T)$ *can be directly obtained from the discounted joint density* $\mathbf{a}_0\mathbf{h}_\delta(x, y|0)$ *(recall from Section 1.3 that* $X_T = \min_{0 \leq s < T} U_s$ *is the minimum surplus level before ruin). Furthermore, the discounted joint density of the quadruple* $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ *can be obtained from* $\mathbf{a}_0\mathbf{h}^*_{1,\delta}(x, y|0)$ *and* $\mathbf{a}_0\mathbf{h}^*_{2,\delta}(x, y, v|0)$.

Chapter 5

Semi-Markovian risk model: The minimum surplus prior to ruin

5.1 Introduction

The chapter focuses on the class of semi-Markovian risk models introduced in Section 1.2.3. The generalized Gerber-Shiu function (1.11) (with slight modifications to suit the model) involving both the minimum surplus level before ruin X_T and the surplus level immediately after the second last claim before ruin R_{N_T-1} will be considered.

For the surplus process $\{U_t\}_{t\geq 0}$ defined in (1.1) with dynamics (1.6), the Gerber-Shiu function with a generalized penalty function which incorporates the quadruple $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ is defined as, for $i, j \in E$,

$$\phi_{\delta,ij}(u) = E\left[e^{-\delta T} w(U_{T^-}, |U_T|, X_T, R_{N_T-1}) 1\{T < \infty, \varrho_{N_T} = j\} | U_0 = u, \varrho_0 = i\right] \ , \quad u \geq 0 \ . \quad (5.1)$$

As mentioned in Section 1.3, the extended version (5.1) of the traditional Gerber-Shiu function

(1.2) allows the analysis of the last ladder height before ruin $|U_T| + X_T$ and the last interclaim time prior to ruin $V_{N_T} = (U_{T^-} - R_{N_T-1})/c$ among other quantities of possible interest (see Cheung et al. (2010b, c, d)). These are not possible with the classical Gerber-Shiu function. For later use, we also define the matrix version of (5.1) to be $\mathbf{\Phi}_\delta(u) = [\phi_{\delta,ij}(u)]_{i,j=1}^m$.

The remainder of this chapter is organized as follows. In Section 5.2, it is shown that the generalized Gerber-Shiu function $\mathbf{\Phi}_\delta(u)$ satisfies a matrix defective renewal equation, and its solution is then derived. In Section 5.3, we identify the discounted joint distribution of $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ and the discounted marginal distribution of the last ladder height before ruin. Special cases of the risk model defined in (1.1) with dynamics (1.6) are considered in more detail in Sections 5.4 and 5.5. Section 5.6 discusses the discounted joint distribution of $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ in a MAP risk model (described in Section 1.2.1) via the connection to the fluid flow process in Section 1.4.2.

5.2 Matrix defective renewal equation and its solution

As pointed out by Landriault and Willmot (2009), the joint density of T, U_{T^-} and $|U_T|$ takes different form depending on whether ruin occurs on the first claim ($N_T = 1$) or on any subsequent claim to the first ($N_T > 1$). This is also the case for the joint distribution of T, U_{T^-}, $|U_T|$ and R_{N_T-1} for similar reasons (see Cheung et al. (2010c)).

For ruin occurring on the first claim, the joint density of $(U_{T^-}, |U_T|)$ at (x, y) together with $\varrho_1 = j$ given that $U_0 = u$ and $\varrho_0 = i$ is easily found to be, for $i, j \in E$,

$$h^*_{1,ij}(x, y|u) = \frac{1}{c} k_i \left(\frac{x-u}{c} \right) p_{ij} b_j(x+y) \ , \quad x > u \geq 0; y > 0 \ . \tag{5.2}$$

In this case, it is clear that $T = (x-u)/c$ and $R_{N_T-1} = R_0 = u$. In matrix notation, one rewrites

(5.2) as

$$\mathbf{h}_1^*(x,y|u) = \frac{1}{c}\mathbf{k}\left(\frac{x-u}{c}\right)\mathbf{P}\ \mathbf{b}(x+y)\ , \quad x > u \geq 0; y > 0\ . \tag{5.3}$$

where the matrices $\mathbf{h}_1^*(x,y|u) = [h_{1,ij}^*(x,y|u)]_{i,j=1}^m$, $\mathbf{k}(t) = \text{diag}\{k_1(t),\ldots,k_m(t)\}$ and $\mathbf{b}(y) = \text{diag}\{b_1(y),\ldots,b_m(y)\}$ are defined. If ruin occurs on claims subsequent to the first, for $i,j \in E$ we denote the joint density of $(T, U_{T^-}, |U_T|, R_{N_T-1})$ at (t,x,y,v) together with $\varrho_{N_T} = j$, given that $U_0 = u$ and $\varrho_0 = i$, by $h_{2,ij}^*(t,x,y,v|u)$ for $t,y > 0; x > v > 0; u \geq 0$. For future reference, it is convenient to define the discounted (with respect to T) densities, for $i,j \in E$,

$$h_{1,\delta,ij}^*(x,y|u) = e^{-\frac{\delta(x-u)}{c}} h_{1,ij}^*(x,y|u)\ , \quad x > u \geq 0; y > 0\ , \tag{5.4}$$

and

$$h_{2,\delta,ij}^*(x,y,v|u) = \int_0^\infty e^{-\delta t} h_{2,ij}^*(t,x,y,v|u)\, dt\ , \quad y > 0; x > v > 0; u \geq 0\ . \tag{5.5}$$

The matrix versions of the above two discounted densities are denoted respectively by $\mathbf{h}_{1,\delta}^*(x,y|u) = [h_{1,\delta,ij}^*(x,y|u)]_{i,j=1}^m$ and $\mathbf{h}_{2,\delta}^*(x,y,v|u) = [h_{2,\delta,ij}^*(x,y,v|u)]_{i,j=1}^m$. We remark that the above discounted densities have been studied extensively in the classical compound Poisson model by Cheung et al. (2010b), and in the bivariate phase-type risk model in Chapter 4. It is also clear from (5.2) and (5.4) that

$$\mathbf{h}_{1,\delta}^*(x,y|u) = \mathbf{h}_{1,\delta}^*(x-u, y+u|0)\ , \quad x > u \geq 0; y > 0\ . \tag{5.6}$$

Our goal is to derive a matrix defective renewal equation for the Gerber-Shiu function $\mathbf{\Phi}_\delta(u)$. By conditioning on the first drop of the surplus level below its initial level u (which may occur upon the first claim or its subsequent claims) and keeping track of the underlying environmental states of the process $\{\varrho_i\}_{i=0}^\infty$, one arrives at, for $i,k \in E$,

$$\phi_{\delta,ik}(u) = \sum_{j=1}^m \int_0^u \left[\int_0^\infty h_{1,\delta,ij}^*(x,y|0)\, dx + \int_0^\infty \int_0^x h_{2,\delta,ij}^*(x,y,v|0)\, dv\, dx\right] \phi_{\delta,jk}(u-y)\, dy + \alpha_{\delta,ik}(u)\ ,$$

$$u \geq 0\ , \tag{5.7}$$

where, for $i, k \in E$,

$$\alpha_{\delta,ik}(u) = \int_u^\infty \int_0^\infty w(x+u, y-u, u, u) h^*_{1,\delta,ik}(x, y|0)\, dx\, dy \\ + \int_u^\infty \int_0^\infty \int_0^x w(x+u, y-u, u, v+u) h^*_{2,\delta,ik}(x, y, v|0)\, dv\, dx\, dy \ , \quad u \geq 0 \ . \tag{5.8}$$

In matrix form, (5.7) and (5.8) can respectively be rewritten as

$$\mathbf{\Phi}_\delta(u) = \int_0^u \left[\int_0^\infty \mathbf{h}^*_{1,\delta}(x, y|0)\, dx + \int_0^\infty \int_0^x \mathbf{h}^*_{2,\delta}(x, y, v|0)\, dv\, dx \right] \mathbf{\Phi}_\delta(u-y)\, dy + \boldsymbol{\alpha}_\delta(u) \ , \quad u \geq 0 \ , \tag{5.9}$$

and

$$\boldsymbol{\alpha}_\delta(u) = \int_u^\infty \int_0^\infty w(x+u, y-u, u, u) \mathbf{h}^*_{1,\delta}(x, y|0)\, dx\, dy \\ + \int_u^\infty \int_0^\infty \int_0^x w(x+u, y-u, u, v+u) \mathbf{h}^*_{2,\delta}(x, y, v|0)\, dv\, dx\, dy \ , \quad u \geq 0 \ , \tag{5.10}$$

where $\boldsymbol{\alpha}_\delta(u) = [\alpha_{\delta,ik}(u)]_{i,k=1}^m$. Letting

$$\mathbf{f}_\delta(y) = \int_0^\infty \mathbf{h}^*_{1,\delta}(x, y|0)\, dx + \int_0^\infty \int_0^x \mathbf{h}^*_{2,\delta}(x, y, v|0)\, dv\, dx \ , \quad y > 0 \ , \tag{5.11}$$

be the matrix version of the (defective) ladder height density, (5.9) becomes

$$\mathbf{\Phi}_\delta(u) = \int_0^u \mathbf{f}_\delta(y) \mathbf{\Phi}_\delta(u-y)\, dy + \boldsymbol{\alpha}_\delta(u) \ , \quad u \geq 0 \ , \tag{5.12}$$

which is a Markov renewal equation (see, e.g., Asmussen (2003) and Çinlar (1969)).

To further verify the 'defective' nature of the above Markov renewal equation, we need the following definitions. A square matrix is said to be *substochastic* if all of its entries are non-negative, and the sum of each row is less than or equal to 1. For a matrix to be *strictly substochastic*, it further requires that the sum of at least one row to be (strictly) less than 1. Then, we consider the m-dimensional square matrix $\boldsymbol{\Upsilon}_\delta$ defined by

$$\boldsymbol{\Upsilon}_\delta = \int_0^\infty \mathbf{f}_\delta(y)\, dy \tag{5.13}$$

with (i,j)-th element (according to (5.1), (5.10), (5.11) and (5.12) with $w(.,.,.,.) \equiv 1$ and $u = 0$)

$$[\boldsymbol{\Upsilon}_\delta]_{ij} = E\left[e^{-\delta T} 1\{T < \infty, \varrho_{N_T} = j\} | U_0 = 0, \varrho_0 = i\right] \ . \tag{5.14}$$

It is clear that for any $i \in E$, the sum of the i-th row of $\boldsymbol{\Upsilon}_\delta$, namely $E\left[e^{-\delta T} 1\{T < \infty\} | U_0 = 0, \varrho_0 = i\right]$, is less than 1 under either $\delta > 0$ or the positive security loading condition (1.7), implying that $\boldsymbol{\Upsilon}_\delta$ is strictly substochastic. Therefore, (5.12) can be viewed as a matrix version of a defective renewal equation.

Next we pay special attention to the particular Gerber-Shiu function $\boldsymbol{\Phi}_\delta(u)$ with $w(.,.,.,.) \equiv 1$ that we shall denote by $\boldsymbol{\Theta}_\delta(u)$. As we shall see, the solution $\boldsymbol{\Phi}_\delta(u)$ to (5.12) can be expressed in terms of the particular Gerber-Shiu function $\boldsymbol{\Theta}_\delta(u)$. For the scalar case, we refer interested readers to Lin and Willmot (1999). From (5.10), (5.11) and (5.12), $\boldsymbol{\Theta}_\delta(u)$ satisfies

$$\boldsymbol{\Theta}_\delta(u) = \int_0^u \mathbf{f}_\delta(y) \boldsymbol{\Theta}_\delta(u-y)\, dy + \int_u^\infty \mathbf{f}_\delta(y)\, dy \ , \quad u \geq 0 \ . \tag{5.15}$$

Taking Laplace transforms on both sides of (5.15) yields

$$\widetilde{\boldsymbol{\Theta}}_\delta(s) = \widetilde{\mathbf{f}}_\delta(s) \widetilde{\boldsymbol{\Theta}}_\delta(s) + \frac{\boldsymbol{\Upsilon}_\delta - \widetilde{\mathbf{f}}_\delta(s)}{s} \ . \tag{5.16}$$

Simple algebraic manipulations of (5.16) lead to

$$\begin{aligned}
\widetilde{\boldsymbol{\Theta}}_\delta(s) &= [\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s)]^{-1} \frac{\boldsymbol{\Upsilon}_\delta - \widetilde{\mathbf{f}}_\delta(s)}{s} \\
&= [\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s)]^{-1} \frac{\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s) - (\mathbf{I} - \boldsymbol{\Upsilon}_\delta)}{s} \\
&= \frac{\mathbf{I} - [\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s)]^{-1}(\mathbf{I} - \boldsymbol{\Upsilon}_\delta)}{s} \\
&= \frac{\boldsymbol{\Upsilon}_\delta - \{[\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s)]^{-1} - \mathbf{I}\}(\mathbf{I} - \boldsymbol{\Upsilon}_\delta)}{s} \ ,
\end{aligned} \tag{5.17}$$

Note that the inverse $[\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s)]^{-1}$ is known to exist given that the DTMC $\{\varrho_i\}_{i=0}^\infty$ is irreducible and $\widetilde{\mathbf{f}}_\delta(s)$ is strictly substochastic, since for $s \geq 0$,

$$\widetilde{\mathbf{f}}_\delta(s) = \int_0^\infty e^{-sy} \mathbf{f}_\delta(y)\, dy \leq \int_0^\infty \mathbf{f}_\delta(y)\, dy = \boldsymbol{\Upsilon}_\delta \ . \tag{5.18}$$

Given that $\boldsymbol{\Upsilon}_\delta = \boldsymbol{\Theta}_\delta(0)$, it is immediate from (5.17) that

$$\boldsymbol{\Theta}_\delta(u) = \sum_{n=1}^\infty \overline{\mathbf{F}}_\delta^{*n}(u)(\mathbf{I} - \boldsymbol{\Upsilon}_\delta) \ , \quad u \geq 0 \ , \tag{5.19}$$

where $\overline{\mathbf{F}}_\delta^{*n}(u) = \int_u^\infty \mathbf{f}_\delta^{*n}(y)\, dy$ is the survival function associated to the (defective) density $\mathbf{f}_\delta^{*n}(u)$, the n-fold (matrix) convolution of $\mathbf{f}_\delta(u)$. Note that $\boldsymbol{\Theta}_\delta(u)$ can be viewed as a matrix version of a compound geometric tail.

Now, taking the Laplace transform on both sides of (5.12) followed by some simple manipulations yields

$$\begin{aligned}
\widetilde{\boldsymbol{\Phi}}_\delta(s) &= [\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s)]^{-1} \widetilde{\boldsymbol{\alpha}}_\delta(s) \\
&= \widetilde{\boldsymbol{\alpha}}_\delta(s) + \left\{[\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s)]^{-1} - \mathbf{I}\right\} \widetilde{\boldsymbol{\alpha}}_\delta(s) \\
&= \widetilde{\boldsymbol{\alpha}}_\delta(s) + \left(\left\{[\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s)]^{-1} - \mathbf{I}\right\} (\mathbf{I} - \boldsymbol{\Upsilon}_\delta)\right) (\mathbf{I} - \boldsymbol{\Upsilon}_\delta)^{-1} \widetilde{\boldsymbol{\alpha}}_\delta(s) \ ,
\end{aligned} \tag{5.20}$$

where, from (5.17), one observes that $\{[\mathbf{I} - \widetilde{\mathbf{f}}_\delta(s)]^{-1} - \mathbf{I}\}(\mathbf{I} - \boldsymbol{\Upsilon}_\delta)$ corresponds to the Laplace transform of

$$-\boldsymbol{\Theta}'_\delta(u) = \sum_{n=1}^{\infty} \mathbf{f}_\delta^{*n}(u)(\mathbf{I} - \boldsymbol{\Upsilon}_\delta) \ , \quad u \geq 0 \ . \tag{5.21}$$

Inverting the Laplace transforms in (5.20) leads to

$$\boldsymbol{\Phi}_\delta(u) = \boldsymbol{\alpha}_\delta(u) - \int_0^u \boldsymbol{\Theta}'_\delta(u-y)(\mathbf{I} - \boldsymbol{\Upsilon}_\delta)^{-1}\boldsymbol{\alpha}_\delta(y)\, dy \ , \quad u \geq 0 \ , \tag{5.22}$$

which is the general solution to the matrix defective renewal equation (5.12). If $\boldsymbol{\alpha}_\delta(y)$ is differentiable, an alternative representation for $\boldsymbol{\Phi}_\delta(u)$ is found by applying integrating by parts to (5.22). Omitting the details here, one easily finds

$$\boldsymbol{\Phi}_\delta(u) = (\mathbf{I} - \boldsymbol{\Upsilon}_\delta)^{-1}\boldsymbol{\alpha}_\delta(u) - \boldsymbol{\Theta}_\delta(u)(\mathbf{I} - \boldsymbol{\Upsilon}_\delta)^{-1}\boldsymbol{\alpha}_\delta(0) - \int_0^u \boldsymbol{\Theta}_\delta(u-y)(\mathbf{I} - \boldsymbol{\Upsilon}_\delta)^{-1}\boldsymbol{\alpha}'_\delta(y)\, dy \ , \quad u \geq 0 \ . \tag{5.23}$$

By examining (5.10), (5.11), (5.13) and (5.19), one observes that all the quantities $\boldsymbol{\alpha}_\delta(u)$ $\boldsymbol{\Upsilon}_\delta$ and $\boldsymbol{\Theta}_\delta(u)$ are functions of $\mathbf{h}^*_{1,\delta}(x,y|0)$ and $\mathbf{h}^*_{2,\delta}(x,y,v|0)$. It follows from (5.22) (or (5.23)) that the generalized Gerber-Shiu function $\boldsymbol{\Phi}_\delta(u)$ is fully characterized by the discounted densities $\mathbf{h}^*_{1,\delta}(x,y|0)$ and $\mathbf{h}^*_{2,\delta}(x,y,v|0)$. An expression for $\mathbf{h}^*_{1,\delta}(x,y|0)$ has been found in (5.4) together with (5.2). However, the same cannot be said of $\mathbf{h}^*_{2,\delta}(x,y,v|0)$. In general, it is not easy to find an expression for $\mathbf{h}^*_{2,\delta}(x,y,v|0)$. An explicit expression might be derived by assuming that the interclaim time densities $k_j(.)$'s and/or the claim size densities $b_j(.)$'s come from a specific class of distributions (see Sections 5.4 and 5.5 for more details).

Remark 12 *Although the solution (5.22) (or (5.23)) is exact, closed form solution is in general difficult to obtain, due to the convolutions appearing in (5.19). Interested readers are referred to, e.g., Li and Luo (2005), Miyazawa (2002) and Wu (1999) for approximations, asymptotics and two-sided bounds for a matrix defective renewal equation.*

5.3 Discounted joint density of $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ and last ladder height $|U_T| + X_T$

In this section, it is shown that the discounted joint density of $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ can be expressed in terms of the discounted densities $\mathbf{h}^*_{1,\delta}(x, y|0)$ and $\mathbf{h}^*_{2,\delta}(x, y, v|0)$. The last ladder height before ruin $|U_T| + X_T$ is also discussed. Let $\boldsymbol{\Phi}_{1234,\delta}(u)$ and $\boldsymbol{\Phi}_{124,\delta}(u)$ be the Gerber-Shiu function with penalty function $w(x, y, z, v) = e^{-s_1x-s_2y-s_3z-s_4v}$ and $w(x, y, z, v) = e^{-s_1x-s_2y-s_4v}$ respectively. From (5.10) and (5.12), it is clear that

$$\boldsymbol{\Phi}_{1234,\delta}(u) = \int_0^u \mathbf{f}_\delta(y)\boldsymbol{\Phi}_{1234,\delta}(u-y)\, dy + e^{-s_3u}\boldsymbol{\alpha}_{124,\delta}(u) \ , \quad u \geq 0 \ , \tag{5.24}$$

and

$$\boldsymbol{\Phi}_{124,\delta}(u) = \int_0^u \mathbf{f}_\delta(y)\boldsymbol{\Phi}_{124,\delta}(u-y)\, dy + \boldsymbol{\alpha}_{124,\delta}(u) \ , \quad u \geq 0 \ , \tag{5.25}$$

where

$$\begin{aligned}\boldsymbol{\alpha}_{124,\delta}(u) = &\int_0^\infty \int_u^\infty e^{-s_1x-s_2y-s_4u}\mathbf{h}^*_{1,\delta}(x-u, y+u|0)\, dx\, dy \\ &+ \int_0^\infty \int_u^\infty \int_u^x e^{-s_1x-s_2y-s_4v}\mathbf{h}^*_{2,\delta}(x-u, y+u, v-u|0)\, dv\, dx\, dy \ , \quad u \geq 0 \ . \end{aligned}\tag{5.26}$$

From (5.22), the solution to (5.24) can be expressed as

$$
\begin{aligned}
&\boldsymbol{\Phi}_{1234,\delta}(u) \\
&= e^{-s_3 u}\boldsymbol{\alpha}_{124,\delta}(u) - \int_0^u \boldsymbol{\Theta}'_\delta(u-z)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1} e^{-s_3 z}\boldsymbol{\alpha}_{124,\delta}(z)\,dz \\
&= \int_0^\infty \int_u^\infty e^{-s_1 x - s_2 y - s_3 u - s_4 u} \left[\mathbf{h}^*_{1,\delta}(x-u, y+u|0)\right]\,dx\,dy \\
&\quad + \int_0^\infty \int_u^\infty \int_u^x e^{-s_1 x - s_2 y - s_3 u - s_4 v} \left[\mathbf{h}^*_{2,\delta}(x-u, y+u, v-u|0)\right]\,dv\,dx\,dy \\
&\quad - \int_0^u \int_0^\infty \int_z^\infty e^{-s_1 x - s_2 y - s_3 z - s_4 z} \left[\boldsymbol{\Theta}'_\delta(u-z)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\mathbf{h}^*_{1,\delta}(x-z, y+z|0)\right]\,dx\,dy\,dz \\
&\quad - \int_0^u \int_0^\infty \int_z^\infty \int_z^x e^{-s_1 x - s_2 y - s_3 z - s_4 v} \left[\boldsymbol{\Theta}'_\delta(u-z)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\mathbf{h}^*_{2,\delta}(x-z, y+z, v-z|0)\right]\,dv\,dx\,dy\,dz\ , \\
&\qquad u \geq 0\ . \qquad (5.27)
\end{aligned}
$$

Given that $\varrho_0 = i$ and $U_0 = u$, the quadruple $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ has discounted densities (with environmental state at ruin $\varrho_{N_T} = j$) on the subspace of $\mathbb{R}^4$. By the uniqueness of Laplace transforms, the above quantity is given by extracting the (i,j)-th element of the following matrices:

1. $\mathbf{h}^{**}_{12,\delta}(x,y|u) = \mathbf{h}^*_{1,\delta}(x-u, y+u|0)$ on $\{(x,y,z,v)|\ x>u, y>0, z=u, v=u\}$: contribution from ruin occurring on the first claim;

2. $\mathbf{h}^{**}_{124,\delta}(x,y,v|u) = \mathbf{h}^*_{2,\delta}(x-u, y+u, v-u|0)$ on $\{(x,y,z,v)|\ x>u, y>0, z=u, u<v<x\}$: contribution from the case where ruin occurs on the first drop in surplus below its initial excluding the first claim;

3. $\mathbf{h}^{**}_{123,\delta}(x,y,z|u) = -\boldsymbol{\Theta}'_\delta(u-z)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\mathbf{h}^*_{1,\delta}(x-z, y+z|0)$ on $\{(x,y,z,v)|\ x>z, y>0, 0<z<u, v=z\}$: contribution from the case where an arbitrary number (≥ 1) of drops bringing the surplus to level z followed by ruin on the next claim; and

4. $\mathbf{h}^{**}_{\delta}(x,y,z,v|u) = -\boldsymbol{\Theta}'_\delta(u-z)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\mathbf{h}^*_{2,\delta}(x-z, y+z, v-z|0)$ on $\{(x,y,z,v)|\ z<v<x, y>0, 0<z<u\}$: contribution from an arbitrary number (≥ 1) of drops bringing the surplus to level z followed by ruin occurring on the next drop in surplus but excluding the

next claim.

Again, since both the quantities $\mathbf{\Upsilon}_\delta$ and $\mathbf{\Theta}'_\delta(u)$ are functions of $\mathbf{h}^*_{1,\delta}(x,y|0)$ and $\mathbf{h}^*_{2,\delta}(x,y,v|0)$, the discounted joint density of $(U_{T^-},|U_T|,X_T,R_{N_T-1})$ can be expressed in terms of the discounted densities $\mathbf{h}^*_{1,\delta}(x,y|0)$ and $\mathbf{h}^*_{2,\delta}(x,y,v|0)$ only. Same comments made just before Remark 12 apply.

Assuming that an expression for $\mathbf{h}^*_{2,\delta}(x,y,v|0)$ has been identified, a complete characterization for the general Gerber-Shiu function $\mathbf{\Phi}_\delta(u)$ can be obtained via the discounted densities of the quadruple $(U_{T^-},|U_T|,X_T,R_{N_T-1})$ as

$$\begin{aligned}\mathbf{\Phi}_\delta(u) = &\int_0^\infty\int_u^\infty w(x,y,u,u)\mathbf{h}^{**}_{12,\delta}(x,y|u)\,dx\,dy\\ &+\int_0^\infty\int_u^\infty\int_u^x w(x,y,u,v)\mathbf{h}^{**}_{124,\delta}(x,y,v|u)\,dv\,dx\,dy\\ &+\int_0^u\int_0^\infty\int_z^\infty w(x,y,z,z)\mathbf{h}^{**}_{123,\delta}(x,y,z|u)\,dx\,dy\,dz\\ &+\int_0^u\int_0^\infty\int_z^\infty\int_z^x w(x,y,z,v)\mathbf{h}^{**}_{\delta}(x,y,z,v|u)\,dv\,dx\,dy\,dz\ ,\quad u\geq 0\ .\end{aligned} \tag{5.28}$$

One can easily verify that (5.28) is indeed consistent with (5.22).

Remark 13 *Along the same line of logic used in the derivation of the discounted joint density of* $(U_{T^-},|U_T|,X_T,R_{N_T-1})$*, one can also prove that the discounted joint density of* $(U_{T^-},|U_T|,X_T)$ *depends solely on the discounted joint density of* $(U_{T^-},|U_T|)$ *(see Cheung et al. (2010c)). The details are omitted here.*

From the use of our generalized penalty function, one may be interested to analyze various quantities related to the last ladder height before ruin $|U_T|+X_T$ by a choice of penalty function of the form $w(x,y,z,v)=w_5(y+z)$ (see Cheung et al. (2010c)). Let, for $i,j\in E$,

$$\phi_{5,\delta,ij}(u) = E\left[e^{-\delta T}w_5(|U_T|+X_T)1\{T<\infty,\varrho_{N_T}=j\}|U_0=u,\varrho_0=i\right]\ ,\quad u\geq 0\ , \tag{5.29}$$

and define $\mathbf{\Phi}_{5,\delta}(u) = [\phi_{5,\delta,ij}(u)]_{i,j=1}^{m}$. From (5.10), (5.11) and (5.12), it is clear that

$$\mathbf{\Phi}_{5,\delta}(u) = \int_0^u \mathbf{f}_\delta(y)\mathbf{\Phi}_{5,\delta}(u-y)\,dy + \int_u^\infty w_5(y)\mathbf{f}_\delta(y)\,dy\ , \quad u \geq 0\ , \tag{5.30}$$

whose solution is given by, with the application of (5.23),

$$\begin{aligned}\mathbf{\Phi}_{5,\delta}(u) = (\mathbf{I}-\mathbf{\Upsilon}_\delta)^{-1}\int_u^\infty w_5(y)\mathbf{f}_\delta(y)\,dy - \mathbf{\Theta}_\delta(u)(\mathbf{I}-\mathbf{\Upsilon}_\delta)^{-1}\int_0^\infty w_5(y)\mathbf{f}_\delta(y)\,dy \\ + \int_0^u \mathbf{\Theta}_\delta(u-y)(\mathbf{I}-\mathbf{\Upsilon}_\delta)^{-1}w_5(y)\mathbf{f}_\delta(y)\,dy\ , \quad u \geq 0\ .\end{aligned} \tag{5.31}$$

By further assuming $w_5(y) = e^{-s_5 y}$, we can invert (5.31) with respect to s_5 analytically to get the discounted (defective) density of the last ladder height before ruin, namely

$$\mathbf{f}_{5,\delta}(u,y) = \begin{cases} \left[\mathbf{\Theta}_\delta(u-y) - \mathbf{\Theta}_\delta(u)\right](\mathbf{I}-\mathbf{\Upsilon}_\delta)^{-1}\mathbf{f}_\delta(y)\ , & y < u\ . \\ \left[\mathbf{I} - \mathbf{\Theta}_\delta(u)\right](\mathbf{I}-\mathbf{\Upsilon}_\delta)^{-1}\mathbf{f}_\delta(y)\ , & y > u\ . \end{cases} \tag{5.32}$$

Note that (5.32) expressed the discounted density of the last ladder height before ruin $\mathbf{f}_{5,\delta}(u,y)$ in terms of the generic discounted ladder height density $\mathbf{f}_\delta(y)$ only, as both $\mathbf{\Upsilon}_\delta$ and $\mathbf{\Theta}_\delta(u)$ are functions of $\mathbf{f}_\delta(y)$ only (see (5.13) and (5.19)).

Remark 14 *In the scalar case (i.e. $m = 1$) or even more generally in the generalized Sparre Andersen risk model described in Section 1.2.2, it can be proved that the proper distribution (after normalizing with an appropriate constant) of the last ladder height before ruin $|U_T| + X_T$ is larger than a generic ladder height in likelihood ratio ordering (which implies stochastic ordering). See Cheung et al. (2010c, d). Unfortunately, in the semi-Markovian model in general, it appears not easy to define what distributions to compare since the ladder height can depend on the initial state and the state at ruin.*

5.4 Analysis with exponential interclaim times

In this section, we consider the semi-Markovian risk model (1.1) with dynamics (1.6) in which the interclaim time densities are all exponential, i.e. $k_j(t) = \lambda_j e^{-\lambda_j t}$ for $j \in E$. This exactly corresponds to the risk model studied by Albrecher and Boxma (2005).

5.4.1 Discounted joint density of $(U_{T^-}, |U_T|, R_{N_T-1})$

As discussed in Sections 5.2 and 5.3, in principle it is sufficient to determine $\mathbf{h}^*_{2,\delta}(x, y, v|0)$ in order to compute the Gerber-Shiu function $\mathbf{\Phi}_\delta(u)$. With that in mind, let us consider the Gerber-Shiu function $\mathbf{\Phi}_{124,\delta}(u)$ with penalty function $w(x, y, z, v) = e^{-s_1 x - s_2 y - s_4 v}$. By conditioning on the time and amount of the first claim, we have that, for $i, k \in E$,

$$\begin{aligned}
\phi_{124,\delta,ik}(u) &= \int_0^\infty \lambda_i e^{-(\lambda_i+\delta)t} \sum_{j=1}^m p_{ij} \int_0^{u+ct} b_j(y) \phi_{124,\delta,jk}(u + ct - y)\, dy\, dt \\
&\quad + \int_0^\infty \lambda_i e^{-(\lambda_i+\delta)t} p_{ik} \int_{u+ct}^\infty b_k(y) e^{-s_1(u+ct)-s_2(y-u-ct)-s_4 u}\, dy\, dt \\
&= \frac{\lambda_i}{c} \sum_{j=1}^m p_{ij} \int_u^\infty e^{-\frac{\lambda_i+\delta}{c}(t-u)} \int_0^t b_j(y) \phi_{124,\delta,jk}(t - y)\, dy\, dt \\
&\quad + \frac{\lambda_i}{c} p_{ik} e^{-(s_1+s_4)u} \int_u^\infty e^{-\left(\frac{\lambda_i+\delta}{c}+s_1\right)(t-u)}\ \mathcal{T}_{s_2} b_k(t)\, dt\ , \quad u \geq 0\ . \qquad (5.33)
\end{aligned}$$

Differentiating on both sides of (5.33) with respect to u, one obtains the integro-differential equation, for $i, k \in E$,

$$\begin{aligned}
\phi'_{124,\delta,ik}(u) &= \frac{\lambda_i + \delta}{c} \phi_{124,\delta,ik}(u) - \frac{\lambda_i}{c} \sum_{j=1}^m p_{ij} \int_0^u b_j(y) \phi_{124,\delta,jk}(u - y)\, dy \\
&\quad - \frac{\lambda_i}{c} p_{ik} e^{-(s_1+s_4)u}\ \mathcal{T}_{s_2} b_k(u) - \frac{\lambda_i}{c} p_{ik} s_4 e^{-(s_1+s_4)u}\ \mathcal{T}_{\frac{\lambda_i+\delta}{c}+s_1} \mathcal{T}_{s_2} b_k(u)\ , \quad u \geq 0\ . \qquad (5.34)
\end{aligned}$$

By taking Laplace transforms on both sides of (5.34) and rearranging terms, we arrive at, for $i, k \in E$,

$$\begin{aligned} &\left(s - \frac{\lambda_i + \delta}{c}\right) \widetilde{\phi}_{124,\delta,ik}(s) + \frac{\lambda_i}{c} \sum_{j=1}^{m} p_{ij} \widetilde{b}_j(s) \widetilde{\phi}_{124,\delta,jk}(s) \\ &= \phi_{124,\delta,ik}(0) - \frac{\lambda_i}{c} p_{ik} \mathcal{T}_{s+s_1+s_4} \mathcal{T}_{s_2} b_k(0) - \frac{\lambda_i}{c} p_{ik} s_4 \mathcal{T}_{s+s_1+s_4} \mathcal{T}_{\frac{\lambda_i+\delta}{c}+s_1} \mathcal{T}_{s_2} b_k(0) \ . \end{aligned} \tag{5.35}$$

One can rewrite (5.35) in matrix form as

$$\mathbf{A}_\delta(s) \widetilde{\mathbf{\Phi}}_{124,\delta}(s) = c \mathbf{\Phi}_{124,\delta}(0) - \mathbf{\Lambda} \mathbf{P} \mathcal{T}_{s+s_1+s_4} \mathbf{\Omega}(0) - s_4 \mathbf{\Lambda} \left[p_{ik} \mathcal{T}_{s+s_1+s_4} \mathcal{T}_{\frac{\lambda_i+\delta}{c}+s_1} \mathcal{T}_{s_2} b_k(0) \right]_{i,k=1}^{m} \tag{5.36}$$

with

$$\mathbf{A}_\delta(s) = (cs - \delta)\mathbf{I} - \mathbf{\Lambda} + \mathbf{\Lambda} \mathbf{P} \ \widetilde{\mathbf{b}}(s) \ , \tag{5.37}$$

where $\mathbf{\Lambda} = \operatorname{diag}\{\lambda_1, \ldots, \lambda_m\}$ and $\mathbf{\Omega}(u) = \operatorname{diag}\{\mathcal{T}_{s_2} b_1(u), \ldots, \mathcal{T}_{s_2} b_m(u)\}$. Letting $\mathbf{A}^*_\delta(s)$ be the adjoint matrix of $\mathbf{A}_\delta(s)$, one can express (5.36) as

$$\widetilde{\mathbf{\Phi}}_{124,\delta}(s) = \frac{\mathbf{A}^*_\delta(s) \{c \mathbf{\Phi}_{124,\delta}(0)\} - \mathbf{\Gamma}_{s_1,s_4}(s) - \mathbf{\Delta}_{s_1,s_4}(s)}{\det \mathbf{A}_\delta(s)} \ , \tag{5.38}$$

where

$$\mathbf{\Gamma}_{s_1,s_4}(s) = \mathbf{A}^*_\delta(s) \mathbf{\Lambda} \mathbf{P} \mathcal{T}_{s+s_1+s_4} \mathbf{\Omega}(0) \ , \tag{5.39}$$

and

$$\mathbf{\Delta}_{s_1,s_4}(s) = s_4 \mathbf{A}^*_\delta(s) \mathbf{\Lambda} \left[p_{jk} \mathcal{T}_{s+s_1+s_4} \mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1} \mathcal{T}_{s_2} b_k(0) \right]_{j,k=1}^{m} \ . \tag{5.40}$$

From Albrecher and Boxma (2005, Proposition 2.1), we know that the Lundberg's fundamental equation

$$\det \mathbf{A}_\delta(s) = 0 \ , \tag{5.41}$$

has m solutions with non-negative real parts, say $\{\rho_i\}_{i=1}^m$, when $\delta > 0$ or the positive security loading condition $\sum_{j=1}^m \pi_j(c/\lambda_j - \mu_j) > 0$ holds. To determine $\mathbf{\Phi}_{124,\delta}(0)$ in (5.38), we follow the ideas of Li and Lu (2008, Section 2.2). Assuming that every element of $\widetilde{\mathbf{\Phi}}_{124,\delta}(\rho_i)$ is finite, it follows from (5.38) that for $i \in E$,

$$\mathbf{A}^*_\delta(\rho_i)\{c\mathbf{\Phi}_{124,\delta}(0)\} = \mathbf{\Gamma}_{s_1,s_4}(\rho_i) + \mathbf{\Delta}_{s_1,s_4}(\rho_i) \ . \tag{5.42}$$

Repeated applications of divided differences yields

$$\mathbf{\Phi}_{124,\delta}(0) = \frac{1}{c}\{\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_m]\}^{-1}\{\mathbf{\Gamma}_{s_1,s_4}[\rho_1,\ldots,\rho_m] + \mathbf{\Delta}_{s_1,s_4}[\rho_1,\ldots,\rho_m]\} \ , \tag{5.43}$$

where $\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_m]$, $\mathbf{\Gamma}_{s_1,s_4}[\rho_1,\ldots,\rho_m]$ and $\mathbf{\Delta}_{s_1,s_4}[\rho_1,\ldots,\rho_m]$ are the m-th divided differences of the matrices $\mathbf{A}^*_\delta(s)$, $\mathbf{\Gamma}_{s_1,s_4}(s)$ and $\mathbf{\Delta}_{s_1,s_4}(s)$ respectively (see Section 1.4.1). Analogous to Eq. (2.6) of Li and Lu (2008), we have that

$$\mathbf{\Gamma}_{s_1,s_4}[\rho_1,\ldots,\rho_m] = \sum_{i=1}^m \mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]\mathbf{\Lambda}\mathbf{P}(-1)^{m-i} \ \mathcal{T}_{\rho_i+s_1+s_4}\ldots\mathcal{T}_{\rho_m+s_1+s_4}\mathbf{\Omega}(0) \ , \tag{5.44}$$

and

$$\mathbf{\Delta}_{s_1,s_4}[\rho_1,\ldots,\rho_m] = s_4\sum_{i=1}^m \mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]\mathbf{\Lambda}(-1)^{m-i}\left[p_{jk}\mathcal{T}_{\rho_i+s_1+s_4}\ldots\mathcal{T}_{\rho_m+s_1+s_4}\mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1}\mathcal{T}_{s_2}b_k(0)\right]_{j,k=1}^m \ . \tag{5.45}$$

Using Property 6 in Section 3 of Li and Garrido (2004) (see (1.15)) followed by a change in the order of summation, one arrives at

$$\begin{aligned}\mathbf{\Gamma}_{s_1,s_4}[\rho_1,\ldots,\rho_m] &= \sum_{i=1}^m \mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]\mathbf{\Lambda}\mathbf{P}\sum_{l=i}^m \frac{\mathcal{T}_{\rho_l+s_1+s_4}\mathbf{\Omega}(0)}{\tau_i'(\rho_l)} \\ &= \sum_{l=1}^m\left\{\sum_{i=1}^l \frac{\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]}{\tau_i'(\rho_l)}\right\}\mathbf{\Lambda}\mathbf{P}\mathcal{T}_{\rho_l+s_1+s_4}\mathbf{\Omega}(0) \ ,\end{aligned} \tag{5.46}$$

and

$$\begin{aligned}\mathbf{\Delta}_{s_1,s_4}[\rho_1,\ldots,\rho_m] &= s_4\sum_{i=1}^{m}\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]\mathbf{\Lambda}\sum_{l=i}^{m}\frac{1}{\tau'_i(\rho_l)}\left[p_{jk}\mathcal{T}_{\rho_l+s_1+s_4}\mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1}\mathcal{T}_{s_2}b_k(0)\right]_{j,k=1}^{m}\\ &= s_4\sum_{l=1}^{m}\left\{\sum_{i=1}^{l}\frac{\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]}{\tau'_i(\rho_l)}\right\}\mathbf{\Lambda}\left[p_{jk}\mathcal{T}_{\rho_l+s_1+s_4}\mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1}\mathcal{T}_{s_2}b_k(0)\right]_{j,k=1}^{m}\quad, \end{aligned}\tag{5.47}$$

where $\tau_i(s)=\prod_{n=i}^{m}(s-\rho_n)$ and thus for $i\in E$,

$$\tau'_i(\rho_l)=\prod_{n=i,n\neq l}^{m}(\rho_l-\rho_n)\ .\tag{5.48}$$

Simple manipulations of (5.47) lead to

$$\begin{aligned}&\mathbf{\Delta}_{s_1,s_4}[\rho_1,\ldots,\rho_m]\\ &=\sum_{l=1}^{m}\left\{\sum_{i=1}^{l}\frac{\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]}{\tau'_i(\rho_l)}\right\}\mathbf{\Lambda}\left[p_{jk}\left(s_4+\rho_l-\frac{\lambda_j+\delta}{c}\right)\mathcal{T}_{\rho_l+s_1+s_4}\mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1}\mathcal{T}_{s_2}b_k(0)\right]_{j,k=1}^{m}\\ &\quad-\sum_{l=1}^{m}\left\{\sum_{i=1}^{l}\frac{\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]}{\tau'_i(\rho_l)}\right\}\mathbf{\Lambda}\left[p_{jk}\left(\rho_l-\frac{\lambda_j+\delta}{c}\right)\mathcal{T}_{\rho_l+s_1+s_4}\mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1}\mathcal{T}_{s_2}b_k(0)\right]_{j,k=1}^{m}\\ &=\sum_{l=1}^{m}\left\{\sum_{i=1}^{l}\frac{\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]}{\tau'_i(\rho_l)}\right\}\mathbf{\Lambda}\left[p_{jk}\mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1}\mathcal{T}_{s_2}b_k(0)\right]_{j,k=1}^{m}-\sum_{l=1}^{m}\left\{\sum_{i=1}^{l}\frac{\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]}{\tau'_i(\rho_l)}\right\}\mathbf{\Lambda}\mathbf{P}\mathcal{T}_{\rho_l+s_1+s_4}\mathbf{\Omega}(0)\\ &\quad-\sum_{l=1}^{m}\left\{\sum_{i=1}^{l}\frac{\mathbf{A}^*_\delta[\rho_1,\ldots,\rho_i]}{\tau'_i(\rho_l)}\right\}\mathbf{\Lambda}\left[p_{jk}\left(\rho_l-\frac{\lambda_j+\delta}{c}\right)\mathcal{T}_{\rho_l+s_1+s_4}\mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1}\mathcal{T}_{s_2}b_k(0)\right]_{j,k=1}^{m}\quad,\end{aligned}\tag{5.49}$$

where the last line follows from Property 2 in Section 3 of Li and Garrido (2004) (see (1.14)). By substituting (5.46) and (5.49) into (5.43), it is immediate that

$$\begin{aligned}\mathbf{\Phi}_{124,\delta}(0) &= \frac{1}{c}\sum_{l=1}^{m}\mathbf{C}_{l,\delta}\mathbf{\Lambda}\left[p_{jk}\mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1}\mathcal{T}_{s_2}b_k(0)\right]_{j,k=1}^{m}\\ &\quad-\frac{1}{c}\sum_{l=1}^{m}\mathbf{C}_{l,\delta}\mathbf{\Lambda}\left[p_{jk}\left(\rho_l-\frac{\lambda_j+\delta}{c}\right)\mathcal{T}_{\rho_l+s_1+s_4}\mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1}\mathcal{T}_{s_2}b_k(0)\right]_{j,k=1}^{m}\quad.\end{aligned}\tag{5.50}$$

where, for $l \in E$,

$$\mathbf{C}_{l,\delta} = \{\mathbf{A}^*_\delta[\rho_1, \ldots, \rho_m]\}^{-1} \left\{ \sum_{i=1}^{l} \frac{\mathbf{A}^*_\delta[\rho_1, \ldots, \rho_i]}{\tau'_i(\rho_l)} \right\} . \tag{5.51}$$

Thus, it remains to invert (5.50) with respect to s_1, s_2 and s_4 to obtain $\mathbf{h}^*_{2,\delta}(x, y, v|0)$ (and $\mathbf{h}^*_{1,\delta}(x, y|0)$ as a by-product). Note that

$$\begin{aligned} \mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1} \mathcal{T}_{s_2} b_k(0) &= \int_0^\infty e^{-\left(\frac{\lambda_j+\delta}{c}+s_1\right)x} \int_x^\infty e^{-s_2(y-x)} b_k(y)\, dy\, dx \\ &= \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y} \left[e^{-\frac{\lambda_j+\delta}{c}x} b_k(x+y) \right] dy\, dx \ , \end{aligned} \tag{5.52}$$

and

$$\begin{aligned} \mathcal{T}_{\rho_l+s_1+s_4} \mathcal{T}_{\frac{\lambda_j+\delta}{c}+s_1} \mathcal{T}_{s_2} b_k(0) &= \int_0^\infty e^{-(\rho_l+s_1+s_4)v} \int_v^\infty e^{-\left(\frac{\lambda_j+\delta}{c}+s_1\right)(x-v)} \int_x^\infty e^{-s_2(y-x)} b_k(y)\, dy\, dx\, dv \\ &= \int_0^\infty \int_v^\infty \int_0^\infty e^{-s_1 x - s_2 y - s_4 v} \left[e^{-\rho_l v - \frac{\lambda_j+\delta}{c}(x-v)} b_k(x+y) \right] dy\, dx\, dv \ . \end{aligned} \tag{5.53}$$

Substitution of (5.52) and (5.53) into (5.50) followed by Laplace transform inversions yields

$$\mathbf{h}^*_{1,\delta}(x, y|0) = \frac{1}{c} \left(\sum_{l=1}^{m} \mathbf{C}_{l,\delta} \right) \mathbf{\Lambda} e^{-(\mathbf{\Lambda}+\delta\mathbf{I})\frac{x}{c}} \mathbf{P}\ \mathbf{b}(x+y) \ , \quad x, y > 0 \ , \tag{5.54}$$

and

$$\mathbf{h}^*_{2,\delta}(x, y, v|0) = \frac{1}{c^2} \sum_{l=1}^{m} e^{-\rho_l v} \mathbf{C}_{l,\delta} \mathbf{\Lambda} \left[\mathbf{\Lambda} + (\delta - c\rho_l)\mathbf{I} \right] e^{-(\mathbf{\Lambda}+\delta\mathbf{I})\frac{x-v}{c}} \mathbf{P}\ \mathbf{b}(x+y) \ , \quad y > 0; x > v > 0 \ . \tag{5.55}$$

We remark that from (5.2) and (5.4), it is clear that

$$\mathbf{h}^*_{1,\delta}(x, y|0) = \frac{1}{c} \mathbf{\Lambda} e^{-(\mathbf{\Lambda}+\delta\mathbf{I})\frac{x}{c}} \mathbf{P}\ \mathbf{b}(x+y) \ , \quad x, y > 0 \ . \tag{5.56}$$

Indeed, (5.54) is consistent with (5.56) since $\sum_{l=1}^{m} \mathbf{C}_{l,\delta} = \mathbf{I}$. To see this, we note from (5.48) and

(5.51) that

$$\begin{aligned}\sum_{l=1}^{m} \mathbf{C}_{l,\delta} &= \sum_{l=1}^{m} \{\mathbf{A}_\delta^*[\rho_1, \ldots, \rho_m]\}^{-1} \left\{ \sum_{i=1}^{l} \frac{\mathbf{A}_\delta^*[\rho_1, \ldots, \rho_i]}{\prod_{n=i,n\neq l}^{m} (\rho_l - \rho_n)} \right\} \\ &= \{\mathbf{A}_\delta^*[\rho_1, \ldots, \rho_m]\}^{-1} \sum_{i=1}^{m} \mathbf{A}_\delta^*[\rho_1, \ldots, \rho_i] \sum_{l=i}^{m} \frac{1}{\prod_{n=i,n\neq l}^{m} (\rho_l - \rho_n)} \\ &= \mathbf{I} \ , \end{aligned} \tag{5.57}$$

where the last line follows from the fact that

$$\sum_{l=i}^{m} \frac{1}{\prod_{n=i,n\neq l}^{m} (\rho_l - \rho_n)} = \begin{cases} 0, & i = 1, 2, \ldots, m-1 \ . \\ 1, & i = m \ . \end{cases} \tag{5.58}$$

See, e.g., Klugman et al. (2008, Eq. (3.19)).

Interestingly, using (5.56), one can rewrite (5.55) as

$$\mathbf{h}_{2,\delta}^*(x, y, v|0) = \sum_{l=1}^{m} e^{-\rho_l v} \mathbf{D}_{l,\delta} \mathbf{h}_{1,\delta}^*(x - v, y + v|0) \ , \quad y > 0; x > v > 0 \ , \tag{5.59}$$

where the matrix $\mathbf{D}_{l,\delta}$ is defined as, for $l \in E$,

$$\mathbf{D}_{l,\delta} = \frac{1}{c} \mathbf{C}_{l,\delta} [\mathbf{\Lambda} + (\delta - c\rho_l)\mathbf{I}] \ . \tag{5.60}$$

Finally, the ladder height matrix $\mathbf{f}_\delta(y)$ can easily be obtained by letting $s_1 = s_4 = 0$ in (5.50) and then inverting the resulting equation with respect to s_2. One can verify that

$$\mathbf{f}_\delta(y) = \frac{1}{c} \sum_{l=1}^{m} \mathbf{C}_{l,\delta} \mathbf{\Lambda} \mathbf{P} \ \mathcal{T}_{\rho_l} \mathbf{b}(y) \ , \quad y > 0 \ , \tag{5.61}$$

and therefore by (5.13) we have

$$\boldsymbol{\Upsilon}_\delta = \frac{1}{c}\sum_{l=1}^{m} \mathbf{C}_{l,\delta}\boldsymbol{\Lambda}\mathbf{P}\ \mathcal{T}_{\rho_l}\overline{\mathbf{B}}(0)\ , \tag{5.62}$$

where $\overline{\mathbf{B}}(y) = \text{diag}\{\overline{B}_1(y), \overline{B}_2(y), \ldots, \overline{B}_m(y)\}$.

5.4.2 Gerber-Shiu function $\boldsymbol{\Phi}_{124,\delta}(u)$ involving R_{N_T-1}

In this subsection, we consider the Gerber-Shiu function $\boldsymbol{\Phi}_{124,\delta}(u)$ now with a general penalty function not depending on the fourth argument, i.e. $w(x, y, z, v) = w_{124}(x, y, v)$. It is clear that $\boldsymbol{\Phi}_{124,\delta}(u)$ satisfies the matrix defective renewal equation (5.25) with

$$\begin{aligned}\boldsymbol{\alpha}_{124,\delta}(u) = &\int_0^\infty \int_u^\infty w_{124}(x, y, u)\mathbf{h}^*_{1,\delta}(x-u, y+u|0)\, dx\, dy \\ &+ \int_0^\infty \int_u^\infty \int_u^x w_{124}(x, y, v)\mathbf{h}^*_{2,\delta}(x-u, y+u, v-u|0)\, dv\, dx\, dy\ , \quad u \geq 0\ .\end{aligned} \tag{5.63}$$

Letting

$$\boldsymbol{\Pi}_\delta(u) = \int_0^\infty \int_u^\infty w_{124}(x, y, u)\mathbf{h}^*_{1,\delta}(x-u, y+u|0)\, dx\, dy\ , \quad u \geq 0\ , \tag{5.64}$$

and using (5.59), (5.63) can be rewritten as

$$\begin{aligned}\boldsymbol{\alpha}_{124,\delta}(u) &= \boldsymbol{\Pi}_\delta(u) + \sum_{l=1}^{m} \mathbf{D}_{l,\delta} \int_0^\infty \int_u^\infty \int_u^x w_{124}(x, y, v)e^{-\rho_l(v-u)}\mathbf{h}^*_{1,\delta}(x-v, y+v|0)\, dv\, dx\, dy \\ &= \boldsymbol{\Pi}_\delta(u) + \sum_{l=1}^{m} \mathbf{D}_{l,\delta} \int_u^\infty e^{-\rho_l(v-u)} \left[\int_0^\infty \int_v^\infty w_{124}(x, y, v)\mathbf{h}^*_{1,\delta}(x-v, y+v|0)\, dx\, dy\right] dv \\ &= \boldsymbol{\Pi}_\delta(u) + \sum_{l=1}^{m} \mathbf{D}_{l,\delta} \int_u^\infty e^{-\rho_l(v-u)}\boldsymbol{\Pi}_\delta(v)\, dv \\ &= \boldsymbol{\Pi}_\delta(u) + \sum_{l=1}^{m} \mathbf{D}_{l,\delta}\mathcal{T}_{\rho_l}\boldsymbol{\Pi}_\delta(u)\ , \quad u \geq 0\ .\end{aligned} \tag{5.65}$$

Now using (5.22) with $\boldsymbol{\alpha}_\delta(u)$ replaced by $\boldsymbol{\alpha}_{124,\delta}(u)$ obtained in (5.65), one finds that

$$\begin{aligned}
\boldsymbol{\Phi}_{124,\delta}(u) &= \boldsymbol{\Pi}_\delta(u) + \sum_{l=1}^m \mathbf{D}_{l,\delta}\mathcal{T}_{\rho_l}\boldsymbol{\Pi}_\delta(u) - \int_0^u \boldsymbol{\Theta}'_\delta(u-v)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\boldsymbol{\Pi}_\delta(v)\,dv \\
&\quad - \int_0^u \boldsymbol{\Theta}'_\delta(u-y)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\left[\sum_{l=1}^m \mathbf{D}_{l,\delta}\mathcal{T}_{\rho_l}\boldsymbol{\Pi}_\delta(y)\right]dy \\
&= \boldsymbol{\Pi}_\delta(u) + \int_u^\infty \left[\sum_{l=1}^m e^{-\rho_l(v-u)}\mathbf{D}_{l,\delta}\right]\boldsymbol{\Pi}_\delta(v)\,dv - \int_0^u \left[\boldsymbol{\Theta}'_\delta(u-v)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\right]\boldsymbol{\Pi}_\delta(v)\,dv \\
&\quad - \int_0^u \left[\int_0^v \boldsymbol{\Theta}'_\delta(u-y)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\sum_{l=1}^m e^{-\rho_l(v-y)}\mathbf{D}_{l,\delta}\,dy\right]\boldsymbol{\Pi}_\delta(v)\,dv \\
&\quad - \int_u^\infty \left[\int_0^u \boldsymbol{\Theta}'_\delta(u-y)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\sum_{l=1}^m e^{-\rho_l(v-y)}\mathbf{D}_{l,\delta}\,dy\right]\boldsymbol{\Pi}_\delta(v)\,dv\ , \quad u \ge 0\ . \qquad (5.66)
\end{aligned}$$

Clearly, $\boldsymbol{\Phi}_{124,\delta}(u)$ can be expressed as

$$\boldsymbol{\Phi}_{124,\delta}(u) = \boldsymbol{\Pi}_\delta(u) + \int_0^\infty \boldsymbol{\Xi}_\delta(u,v)\boldsymbol{\Pi}_\delta(v)\,dv\ , \quad u \ge 0\ , \qquad (5.67)$$

where

$$\boldsymbol{\Xi}_\delta(u,v) = \begin{cases} -\boldsymbol{\Theta}'_\delta(u-v)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1} - \int_0^v \boldsymbol{\Theta}'_\delta(u-y)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\left[\sum_{l=1}^m e^{-\rho_l(v-y)}\mathbf{D}_{l,\delta}\right]dy\ , & 0 < v < u\ . \\ \sum_{l=1}^m e^{-\rho_l(v-u)}\mathbf{D}_{l,\delta} - \int_0^u \boldsymbol{\Theta}'_\delta(u-y)(\mathbf{I}-\boldsymbol{\Upsilon}_\delta)^{-1}\left[\sum_{l=1}^m e^{-\rho_l(v-y)}\mathbf{D}_{l,\delta}\right]dy\ , & v > u \ge 0\ . \end{cases} \qquad (5.68)$$

When $m = 1$, the risk model reduces to the classical compound Poisson risk model. It is not hard to see that (5.67) and (5.68) reduce to Eq. (17) and Eq. (18) in Cheung et al. (2010b) respectively. A probabilistic interpretation of (5.67) is provided next. First, using (5.6), (5.64) becomes

$$\boldsymbol{\Pi}_\delta(u) = \int_0^\infty \int_u^\infty w_{124}(x,y,u)\mathbf{h}^*_{1,\delta}(x,y|u)\,dx\,dy\ , \quad u \ge 0\ , \qquad (5.69)$$

which implies that $\boldsymbol{\Pi}_\delta(u)$ is the contribution to the Gerber-Shiu function $\boldsymbol{\Phi}_{124,\delta}(u)$ from the case where ruin occurs upon the first claim. Thus, the integral term $\int_0^\infty \boldsymbol{\Xi}_\delta(u,v)\boldsymbol{\Pi}_\delta(v)\,dv$ in (5.67) shall be the contribution to $\mathbf{\Phi}_{124,\delta}(u)$ due to ruin occurring as a result of at least two claims. Indeed, from an initial surplus of u, the surplus process drops to level v immediately after an arbitrary number (≥ 1) of claims before ruin (with contribution $\mathbf{\Xi}_\delta(u,v)$ to $\mathbf{\Phi}_{124,\delta}(u)$) followed by the subsequent claim possibly causing ruin (with contribution $\mathbf{\Pi}_\delta(v)$ to $\mathbf{\Phi}_{124,\delta}(u)$). Integrating over the possible values of v, this gives rise to the integral term $\int_0^\infty \mathbf{\Xi}_\delta(u,v)\mathbf{\Pi}_\delta(v)\,dv$.

To further justify our interpretation of $\mathbf{\Xi}_\delta(u,v)$, let us revisit the Gerber-Shiu function $\mathbf{\Phi}_{124,\delta}(u)$ by separating cases where ruin occurs on the first claim or not, namely,

$$\begin{aligned}\mathbf{\Phi}_{124,\delta}(u) &= \int_0^\infty \int_u^\infty w_{124}(x,y,u)\mathbf{h}^*_{1,\delta}(x,y|u)\,dx\,dy + \int_0^\infty \int_0^\infty \int_0^x w_{124}(x,y,v)\mathbf{h}^*_{2,\delta}(x,y,v|u)\,dv\,dx\,dy \\ &= \mathbf{\Pi}_\delta(u) + \int_0^\infty \int_0^\infty \int_0^x w_{124}(x,y,v)\mathbf{h}^*_{2,\delta}(x,y,v|u)\,dv\,dx\,dy\ , \quad u \geq 0\ . \end{aligned} \tag{5.70}$$

On the other hand, by substituting (5.69) into the integral term in (5.67) followed by a change of order of integration, one obtains

$$\mathbf{\Phi}_{124,\delta}(u) = \mathbf{\Pi}_\delta(u) + \int_0^\infty \int_0^\infty \int_0^x w_{124}(x,y,v)\mathbf{\Xi}_\delta(u,v)\mathbf{h}^*_{1,\delta}(x,y|v)\,dv\,dx\,dy\ , \quad u \geq 0\ . \tag{5.71}$$

Since (5.70) and (5.71) hold true for a general penalty function $w_{124}(x,y,v)$, a comparison of them implies that

$$\mathbf{h}^*_{2,\delta}(x,y,v|u) = \mathbf{\Xi}_\delta(u,v)\mathbf{h}^*_{1,\delta}(x,y|v)\ , \quad y>0; x>v>0; u\geq 0\ . \tag{5.72}$$

The above equation expresses the discounted joint density of $(U_{T^-}, |U_T|, R_{N_T-1})$ for ruin on claims subsequent to the first claim (with initial surplus u) in terms of the discounted joint density of $(U_{T^-}, |U_T|)$ for ruin on the first claim (with initial surplus v), generalizing Eq. (23) of Cheung et al. (2010b) proved in the context of the classical compound Poisson risk model. From the definition of $\mathbf{h}^*_{2,\delta}(x,y,v|u)$, the surplus level immediately after the second last claim prior to ruin has to be v and, from this new surplus level v, the next claim shall cause ruin which is represented by the term $\mathbf{h}^*_{1,\delta}(x,y|v)$ on the right-hand side of (5.72). Since $\mathbf{\Xi}_\delta(u,v)$ connects the above two densities according to (5.72), our previous interpretation of $\boldsymbol{\Xi}_\delta(u,v)$ is justified.

Remark 15 *Equations of the form (5.67) and (5.72) also hold true in the context of a generalized Sparre Andersen risk model with surplus-dependent premium rate. See Section 8.2.*

5.5 Analysis with exponential claim sizes

In this section, we consider another special case of the semi-Markovian risk model described in Section 5.1 in which the claim size densities $b_j(.)$'s are all exponential (i.e. $b_j(y) = \beta_j e^{-\beta_j y}$ for $j \in E$) while the interclaim time densities $k_j(.)$'s are kept general. If possible, our goal is to identify $\mathbf{h}^*_{2,\delta}(x,y,v|0)$ in this model. For that purpose, let $\boldsymbol{\Phi}_{124,\delta}(u) = [\phi_{124,\delta,ij}(u)]_{i,j=1}^m$ be the particular Gerber-Shiu function $\boldsymbol{\Phi}_\delta(u)$ with penalty function $w(x,y,z,v) = e^{-s_1 x - s_4 v} w_2(y)$. Contrary to Section 5.4, it is believed that the usual approach of conditioning on the time and the amount of the first claim alone does not lead to an expression for $\boldsymbol{\Phi}_{124,\delta}(0)$ or $\mathbf{h}^*_{2,\delta}(x,y,v|0)$ in the present case. Interested readers are referred to Cheung et al. (2010c), Landriault and Willmot (2008) and Willmot (2007) for risk models with arbitrary interclaim times in which a similar problem arises. In such cases, the way to approach the problem is to first identify the form of the solution to $\phi_{124,\delta,ij}(u)$ apart from some unknown constants by making use of a pre-assumed discounted density and the property of the claim size distributions. The unknown constants can typically be solved from a system of linear equations which arise by conditioning on the time and the amount of the first claim. Due to the similarity of the approach to Chapter 7, we omit the rather tedious algebra here and only state the results.

Let $\{-\gamma_i\}_{i=1}^m$ be the m roots with negative real parts to the generalized Lundberg equation

$$\det[\mathbf{I} - \boldsymbol{\varsigma}_\delta(s)] = 0 \ , \tag{5.73}$$

where

$$\varsigma_\delta(s) = \widetilde{\mathbf{k}}(\delta - cs)\mathbf{P}\ \widetilde{\mathbf{b}}(s) \ . \tag{5.74}$$

For simplicity, we assume that $\{\gamma_i\}_{i=1}^m$, $\{\beta_i\}_{i=1}^m$ and $\{s_1 + s_4 + \beta_i\}_{i=1}^m$ are all distinct. One finds that

$$\phi_{124,\delta,ij}(u) = \sum_{k=1}^m \vartheta_{ijk} e^{-\gamma_k u} + \eta_{ij} e^{-(s_1+s_4+\beta_j)u} \ , \quad u \geq 0 \ . \tag{5.75}$$

For $i, j, k \in E$, the coefficients ϑ_{ijk}'s and η_{ij}'s in the solution (5.75) are determined as follows.

- For each fixed $j \in E$, $\{\eta_{ij}\}_{i=1}^m$ satisfies the following system of equations:

$$\eta_{ij} = \widetilde{k}_i(\delta + c(s_1+s_4+\beta_j)) \sum_{l=1}^m p_{il} \frac{\beta_l}{\beta_l - (s_1 + s_4 + \beta_j)} \eta_{lj} + p_{ij}\beta_j \widetilde{w}_2(\beta_j)\widetilde{k}_i(\delta + c(s_1+\beta_j)) \ , \quad i \in E \ . \tag{5.76}$$

- Once the solution for the η_{ij}'s have been found, one solves for the ϑ_{ijk}'s via the system of equations

$$\vartheta_{ijk} = \widetilde{k}_i(\delta + c\gamma_k) \sum_{l=1}^m p_{il} \frac{\beta_l}{\beta_l - \gamma_k} \vartheta_{ljk} \ , \quad i, j, k \in E \ , \tag{5.77}$$

  and

$$\sum_{k=1}^m \frac{\vartheta_{ljk}}{\beta_l - \gamma_k} + \frac{\eta_{lj}}{\beta_l - (s_1 + s_4 + \beta_j)} = 0 \ , \quad j, l \in E \ . \tag{5.78}$$

Note that, for any fixed $j, k \in E$, one of the m equations ($i \in E$) in (5.77) is redundant. Therefore, combining the resulting $m^2(m-1)$ equations with the m^2 equations in (5.78) yields a system of m^3 linear equations to solve for all the ϑ_{ijk}'s. The above procedure results in a complete characterization of $\mathbf{\Phi}_{124,\delta}(u)$. Finally, choosing $w_2(y) = e^{-s_2 y}$, it is clear that $\mathbf{\Phi}_{124,\delta}(u)$ corresponds to the joint Laplace transform of the quadruple $(T, U_{T^-}, |U_T|, R_{N_T-1})$. Note that the coefficients ϑ_{ijk}'s and η_{ij}'s in (5.75) depend on s_1, s_2 and s_4. Therefore, it seems unlikely that in general the Laplace transform inversion with respect to the arguments s_1, s_2 and s_4 can be done analytically (even when $u = 0$).

5.6 Discounted joint density of $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ in the MAP by fluid queue

So far, we have analyzed the generalized Gerber-Shiu function $\mathbf{\Phi}_\delta(u)$ for the class of semi-Markovian risk models described in Section 1.2.3. We have shown that in principle it is sufficient to determine $\mathbf{h}^*_{2,\delta}(x, y, v|0)$ to compute $\mathbf{\Phi}_\delta(u)$. To find $\mathbf{h}^*_{2,\delta}(x, y, v|0)$ under further assumptions on the interclaim times or the claim sizes, the usual way is to condition on the time and amount of the first claim. However, one should note that this is possible because of an important property of our assumed structure of (1.6). The property is that, whenever something happens, there has to be an accompanying claim which is positive. This property allows us to keep track of the surplus level immediately after the first claim (if the claim does not cause ruin) and hence the quantity R_{N_T-1}. Unfortunately, if claims follow a MAP as described in Section 1.2.1, then it is possible to have a change in environment without a claim and the usual arguments of conditioning on the first event that occurs do not work anymore. This is because when a change in environment without a claim occurs before a claim, we lose track of the surplus level immediately after the previous claim. In this section, we shall derive the discounted joint density of $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ when claims follow a MAP using the connection to a fluid queue described in Section 1.4.1. Readers should, however, be reminded that the semi-Markovian risk model and the MAP risk model are not special cases of one another (see Section 1.2.4).

By analyzing all possible sample paths of the MAP risk process and the corresponding fluid flow process (see, e.g., Ahn et al. (2007), Ramaswami (2006), and Badescu et al. (2007a,b)), the discounted joint density of $(U_{T^-}, |U_T|, X_T, R_{N_T-1})$ given that the risk process has an initial surplus of $U_0 = u \geq 0$ and the fluid flow process starts in state $i \in S_1$ is given by the i-th element of the following $|S_1|$ column vectors:

1. for ruin on the first claim, on $\{(x,y,z,v)|\ x>u, y>0, z=u, v=u\}$,

$$\mathbf{h}^{***}_{12,\delta}(x,y|u) = e^{(\mathbf{Q}_{11}-\delta\mathbf{I})\frac{x-u}{c}} \left(\frac{\mathbf{Q}_{12}}{c}\right) e^{\mathbf{Q}_{22,1}(x+y)}\mathbf{q}_{21,1} \ , \tag{5.79}$$

2. for ruin on the first drop in surplus other than the first claim, on $\{(x,y,z,v)|\ x>u, y>0, z=u, u<v<x\}$,

$$\mathbf{h}^{***}_{124,\delta}(x,y,v|u) = e^{-\frac{\delta(v-u)}{2c}}\ {}_0\widehat{\mathbf{f}}_{11}\left(0, v-u, \frac{\delta}{2}\right)\mathbf{\Psi}\left(\frac{\delta}{2}\right)\left[\mathbf{I} - {}^{v-u}\mathbf{\Psi}^{r}\left(\frac{\delta}{2}\right)\mathbf{\Psi}\left(\frac{\delta}{2}\right)\right]^{-1} \times \mathbf{Q}_{21,1}\mathbf{h}^{***}_{12,\delta}(x,y|v) \ , \tag{5.80}$$

3. for an arbitrary number (≥ 1) of drops bringing the surplus to level z followed by ruin on the next claim, on $\{(x,y,z,v)|\ x>z, y>0, 0<z<u, v=z\}$,

$$\mathbf{h}^{***}_{123,\delta}(x,y,z|u) = e^{\frac{\delta(u-z)}{2c}}\ \widehat{\mathbf{f}}_{12}\left(u-z, 0, \frac{\delta}{2}\right)\mathbf{Q}_{21,1}\mathbf{h}^{***}_{12,\delta}(x,y|z) \ , \tag{5.81}$$

4. for an arbitrary number (≥ 1) of drops bringing the surplus to level z followed by ruin occurring on the next drop in surplus excluding the next claim, on $\{(x,y,z,v)|\ z<v<x, y>0, 0<z<u\}$,

$$\mathbf{h}^{***}_{\delta}(x,y,z,v|u) = e^{\frac{\delta(u-z)}{2c}}\ \widehat{\mathbf{f}}_{12}\left(u-z, 0, \frac{\delta}{2}\right)\mathbf{Q}_{21,1}\mathbf{h}^{***}_{124,\delta}(x,y,v|z) \ , \tag{5.82}$$

where $\mathbf{q}_{21,1} = \mathbf{Q}_{21,1}\mathbf{1}$.

We shall only prove (5.80) and (5.81). First consider the quantity $\mathbf{h}^{***}_{124,\delta}(x,y,v|u)$. The explanation for different components are as follows:

- the surplus process $\{U_t\}_{t\geq 0}$ has to first reach level v from level u without dropping below u, since the first drop in surplus has to result in ruin. This is translated into a first passage in the associated fluid flow process $\{F(t)\}_{t\geq 0}$ from level u to level v avoiding level u enroute. This results in a contribution of $e^{-\delta(v-u)/(2c)}\ {}_0\widehat{\mathbf{f}}_{11}(0, v-u, \delta/2)$ to $\mathbf{h}^{***}_{124,\delta}(x, y, v|u)$;

- then, the fluid flow process $\{F(t)\}_{t\geq 0}$ has to make a transition back to level v in the set of phases S_2, since the requirement of $R_{N_T-1} = v$ in $\{U_t\}_{t\geq 0}$ would mean level v has to be reached in S_2 in $\{F(t)\}_{t\geq 0}$. This results in the term $\mathbf{\Psi}(\delta/2)$;

- the fluid flow process $\{F(t)\}_{t\geq 0}$ then revisits level v in S_2 for an arbitrary number (≥ 0) of times avoiding level u enroute, resulting in $[\mathbf{I} -\ {}^{v-u}\mathbf{\Psi}^r(\delta/2)\mathbf{\Psi}(\delta/2)]^{-1}$; and

- finally, after visiting level v in S_2 for an arbitrary number of times, the fluid process $\{F(t)\}_{t\geq 0}$ makes a transition into S_1, accounting for the term $\mathbf{Q}_{21,1}$. Clearly, the next claim causes ruin and this explains the term $\mathbf{h}^{***}_{12,\delta}(x, y|v)$ given by (5.79).

For the quantity $\mathbf{h}^{***}_{123,\delta}(x, y, z|u)$ given by (5.81), the fluid process $\{F(t)\}_{t\geq 0}$ has to make a first passage from level u to level z which corresponds to the minimum surplus level in the surplus process $\{U_t\}_{t\geq 0}$. This gives rise to the term $e^{\delta(u-z)/(2c)}\ \widehat{\mathbf{f}}_{12}(u-z, 0, \delta/2)$. After reaching the minimum before ruin, the fluid process $\{F(t)\}_{t\geq 0}$ has to switch immediately from decreasing to increasing pattern which explains $\mathbf{Q}_{21,1}$. Because the next claim causes ruin, we again require the term $\mathbf{h}^{***}_{12,\delta}(x, y|v)$.

Chapter 6

MAP risk model: The maximum surplus prior to ruin

6.1 Introduction

In this chapter, we revisit the MAP risk model defined in Section 1.2.1 and consider a generalization of the Gerber-Shiu function where the penalty function involves the maximum surplus before ruin $Z_T = \max_{0 \leq s < T} U_s$. In our context, the general Gerber-Shiu function is defined to be, for $i, j \in E$,

$$\phi_{\delta,ij}(u) = E\left[e^{-\delta T} w_{123}(U_{T^-}, |U_T|, Z_T) 1\{T < \infty, J_T = j\} | U_0 = u, J_0 = i\right] \ , \quad u \geq 0 \ , \tag{6.1}$$

The above function $\phi_{\delta,ij}(u)$ is hard to solve in general. For the remainder of this chapter, we assume that the penalty function $w_{123}(.,.,.)$ admits the factorization $w_{123}(x, y, z) = w_{12}(x, y)w_3(z)$, i.e. we are interested in the special case of the Gerber-Shiu function (6.1) given by, for $i, j \in E$,

$$\phi_{\delta,ij}(u) = E\left[e^{-\delta T} w_{12}(U_{T^-}, |U_T|) w_3(Z_T) 1\{T < \infty, J_T = j\} | U_0 = u, J_0 = i\right] \ , \quad u \geq 0 \ . \tag{6.2}$$

For convenience, we further define the matrix of Gerber-Shiu functions $\mathbf{\Phi}_{\delta}(u) = [\phi_{\delta,ij}(u)]_{i,j=1}^{m}$. The classical Gerber-Shiu function (which does not involve the random variable Z_T) can be retrieved from (6.2) by letting $w_3(.) \equiv 1$, and is denoted by $\phi_{12,\delta,ij}(u)$. Then we have the matrix $\mathbf{\Phi}_{12,\delta}(u) = [\phi_{12,\delta,ij}(u)]_{i,j=1}^{m}$. Again we assume either $\delta > 0$ or the positive security loading condition (1.3).

We remark that the assumed factorization form of $w_{123}(.,.,.)$ is still general enough for practical purposes because, in principle, by letting $w_{12}(x,y) = e^{-s_1 x - s_2 y}$ and $w_3(z) = e^{-s_3 z}$ in (6.2), the discounted joint distribution of the triplet $(U_{T^-}, |U_T|, Z_T)$ can be obtained by Laplace transform inversion. Interested readers are also referred to Li and Dickson (2006) and Li and Lu (2008) respectively for the study of the marginal distribution of the maximum surplus level in some Sparre Andersen models and in the Markov-modulated risk model respectively. On the other hand, if one assumes $w_{12}(x,y) = e^{-s_4 y}$ and $w_3(z) = e^{-s_4 z}$ under $\delta = 0$, one obtains the Laplace transform (with argument s_4) of the largest distance of the surplus process $\{U_t\}_{t\geq 0}$ up to and including the time of ruin $Z_T + |U_T|$.

In general, we do not expect $\mathbf{\Phi}_{\delta}(u)$ to satisfy a (matrix) defective renewal equation like the previous chapter. However, it is possible to express $\mathbf{\Phi}_{\delta}(u)$ in terms of $\mathbf{\Phi}_{12,\delta}(u)$ and some other known quantities. Furthermore, it can be shown that $\mathbf{\Phi}_{\delta}(u)$ is closely related to the classical Gerber-Shiu function in the same MAP risk model under a dividend barrier strategy. The corresponding generalized Gerber-Shiu function under the barrier strategy is also obtained with little extra effort. These aspects will be the subject matter of Section 6.2. In Section 6.3, we consider the simplest case of the MAP risk model - the classical compound Poisson risk model, and show that how the results in Section 6.2 can be applied to find the discounted joint density of the triplet $(U_{T^-}, |U_T|, Z_T)$ through analytic Laplace transform inversion of an appropriate Gerber-Shiu function. The density of the largest distance $Z_T + |U_T|$ is given and numerical examples are illustrated. Section 6.4 revisits the MAP risk model, and the discounted joint distribution of $(U_{T^-}, |U_T|, Z_T, X_T)$ (which additionally involves the minimum surplus before ruin $X_T = \min_{0\leq s<T} U_s$) is studied via the existing connection to a fluid flow process given in Section 1.4.1.

6.2 Gerber-Shiu function $\mathbf{\Phi}_\delta(u)$ involving Z_T

6.2.1 Solution for $\mathbf{\Phi}_\delta(u)$

Note that from time 0 the surplus process $\{U_t\}_{t\geq 0}$ is at its running maximum until the first claim occurs. However, the first change in the environmental process $\{J_t\}_{t\geq 0}$ may or may not be accompanied by a claim. If it is not accompanied by a claim, the surplus will still be at its running maximum until the first claim. If the change is accompanied by a claim, the claim may or may not cause ruin. If the claim causes ruin, the maximum surplus level is identical to the surplus prior to ruin. If the claim does not cause ruin, there are two possibilities as the process further evolves:

- the surplus process reaches the previous maximum level (i.e. the level just prior to the first claim) before ruin; or
- the surplus process drops below 0 before it can reach the previous maximum level and hence the maximum surplus prior to ruin is equal to the level just prior to the first claim.

Therefore, by conditioning on the time of the first change in environment and keeping track of the underlying environmental states, one arrives at, for $i, l \in E$,

$$\begin{aligned}
\phi_{\delta,il}(u) = \int_0^\infty & e^{(G_{0,ii}-\delta)t} \left\{ \sum_{j=1,j\neq i}^{m} G_{0,ij}\phi_{\delta,jl}(u+ct) \right. \\
& \left. + \sum_{j=1}^{m}\sum_{k=1}^{m} G_{1,ij} \left[\int_0^{u+ct} p_{ij}(y)\chi_{\delta,jk}(u+ct-y; u+ct)\, dy \right] \phi_{\delta,kl}(u+ct) \right\} dt \\
& + \int_0^\infty e^{(G_{0,ii}-\delta)t} \sum_{j=1}^{m} G_{1,ij} \left[\int_0^{u+ct} p_{ij}(y)\varphi_{12,\delta,jl}(u+ct-y; u+ct)\, dy \right] w_3(u+ct)\, dt \\
& + \int_0^\infty e^{(G_{0,ii}-\delta)t} G_{1,il} \left[\int_{u+ct}^\infty p_{il}(y) w_{12}(u+ct, y-u-ct)\, dy \right] w_3(u+ct)\, dt \ , \quad u \geq 0 \ ,
\end{aligned} \tag{6.3}$$

where for $j,k \in E$,

$$\chi_{\delta,jk}(u;b) = E\left[e^{-\delta\tau_b}1\{\tau_b < T, J_{\tau_b} = k\}|U_0 = u, J_0 = j\right] \ , \quad 0 \le u \le b \ , \tag{6.4}$$

is the Laplace transform of the first passage time $\tau_b = \inf\{t \ge 0 : U_t = b\}$ avoiding ruin enroute, and for $j,l \in E$,

$$\varphi_{12,\delta,jl}(u;b) = E\left[e^{-\delta T}w_{12}(U_{T^-}, |U_T|)1\{T < \tau_b, J_T = l\}|U_0 = u, J_0 = j\right] \ , \quad 0 \le u \le b \ , \tag{6.5}$$

is the classical Gerber-Shiu function with the event that the surplus process does not up-cross level b before ruin occurs. For later use we also define the matrices $\boldsymbol{\chi}_\delta(u;b) = [\chi_{\delta,jk}(u;b)]_{j,k=1}^m$ and $\boldsymbol{\varphi}_{12,\delta}(u;b) = [\varphi_{12,\delta,jl}(u;b)]_{j,l=1}^m$.

By changing the variable of integration $x = u + ct$ in (6.3), one finds, for $i,l \in E$,

$$\begin{aligned}
&\phi_{\delta,il}(u) \\
&= \frac{1}{c}\int_u^\infty e^{\frac{G_{0,ii}-\delta}{c}(x-u)}\left\{\sum_{j=1,j\neq i}^m G_{0,ij}\phi_{\delta,jl}(x) + \sum_{j=1}^m\sum_{k=1}^m G_{1,ij}\left[\int_0^x p_{ij}(y)\chi_{\delta,jk}(x-y;x)\,dy\right]\phi_{\delta,kl}(x)\right\}dx \\
&\quad + \frac{1}{c}\int_u^\infty e^{\frac{G_{0,ii}-\delta}{c}(x-u)}\left[\sum_{j=1}^m G_{1,ij}\int_0^x p_{ij}(y)\varphi_{12,\delta,jl}(x-y;x)\,dy + G_{1,il}\omega_{il}(x)\right]w_3(x)\,dx \ , \quad u \ge 0 \ ,
\end{aligned} \tag{6.6}$$

where for $i,l \in E$,

$$\omega_{il}(x) = \int_x^\infty p_{il}(y)w_{12}(x, y-x)\,dy \ , \quad x \ge 0 \ . \tag{6.7}$$

Differentiating with respect (6.6) to u yields, for $i, l \in E$,

$$\begin{aligned}
&\phi'_{\delta,il}(u) \\
&= -\frac{G_{0,ii}-\delta}{c}\phi_{\delta,il}(u) - \frac{1}{c}\sum_{j=1,j\neq i}^{m} G_{0,ij}\phi_{\delta,jl}(u) - \frac{1}{c}\sum_{j=1}^{m}\sum_{k=1}^{m} G_{1,ij}\left[\int_0^u p_{ij}(y)\chi_{\delta,jk}(u-y;u)\,dy\right]\phi_{\delta,kl}(u) \\
&\quad - \frac{1}{c}\left[\sum_{j=1}^{m} G_{1,ij}\int_0^u p_{ij}(y)\varphi_{12,\delta,jl}(u-y;u)\,dy + G_{1,il}\omega_{il}(u)\right] w_3(u)\ , \quad u \ge 0\ . \qquad (6.8)
\end{aligned}$$

It is convenient to re-express the above system of integro-differential equations in matrix form as

$$\begin{aligned}
\boldsymbol{\Phi}'_\delta(u) &= \frac{\delta}{c}\boldsymbol{\Phi}_\delta(u) - \frac{1}{c}\mathbf{G}_0\boldsymbol{\Phi}_\delta(u) - \frac{1}{c}\left[\int_0^u \mathbf{G}_p(y)\boldsymbol{\chi}_\delta(u-y;u)\,dy\right]\boldsymbol{\Phi}_\delta(u) \\
&\quad - \frac{1}{c}\left[\int_0^u \mathbf{G}_p(y)\boldsymbol{\varphi}_{12,\delta}(u-y;u)\,dy + \mathbf{G}_\omega(u)\right] w_3(u)\ , \quad u \ge 0\ , \qquad (6.9)
\end{aligned}$$

where $\mathbf{G}_p(y) = [G_{1,ij}p_{ij}(y)]_{i,j=1}^m$ and $\mathbf{G}_\omega(u) = [G_{1,ij}\omega_{il}(u)]_{i,l=1}^m$.

In order to solve (6.9) for $\boldsymbol{\Phi}_\delta(u)$, for a moment we turn our attention to the matrix $\boldsymbol{\chi}_\delta(u;b)$ with elements defined by (6.4). It follows from the argument in Chapter 2 and Li and Lu (2007) that

$$\boldsymbol{\chi}_\delta(u;b) = \mathbf{v}_\delta(u)\,[\mathbf{v}_\delta(b)]^{-1}\ , \quad 0 \le u \le b\ , \qquad (6.10)$$

where the matrix $\mathbf{v}_\delta(u)$ satisfies the homogeneous integro-differential equation

$$\mathbf{v}'_\delta(u) = \frac{\delta}{c}\mathbf{v}_\delta(u) - \frac{1}{c}\mathbf{G}_0\mathbf{v}_\delta(u) - \frac{1}{c}\int_0^u \mathbf{G}_p(y)\mathbf{v}_\delta(u-y)\,dy\ , \quad u \ge 0\ . \qquad (6.11)$$

Furthermore, with the initial condition

$$\mathbf{v}_\delta(0) = \mathbf{I}\ , \qquad (6.12)$$

the Laplace transform of $\mathbf{v}_\delta(u)$ is given by

$$\widetilde{\mathbf{v}}_\delta(s) = \left[\left(s - \frac{\delta}{c}\right)\mathbf{I} + \frac{1}{c}\mathbf{G}_0 + \frac{1}{c}\widetilde{\mathbf{G}}_p(s)\right]^{-1} . \tag{6.13}$$

When the Laplace transforms $\widetilde{p}_{ij}(.)$'s are all ratios of two polynomials in s, each element of (6.13) is also a rational function in s and can therefore be resolved into partial fractions. This allows analytic inversion of the Laplace transforms. See Chapter 2.

Now, applying (6.10) and (6.11) to (6.9) leads to

$$\begin{aligned}
\mathbf{\Phi}'_\delta(u) &= \frac{\delta}{c}\mathbf{\Phi}_\delta(u) - \frac{1}{c}\mathbf{G}_0\mathbf{\Phi}_\delta(u) - \frac{1}{c}\left[\int_0^u \mathbf{G}_p(y)\mathbf{v}_\delta(u-y)\,dy\right][\mathbf{v}_\delta(u)]^{-1}\mathbf{\Phi}_\delta(u) \\
&\quad - \frac{1}{c}\left[\int_0^u \mathbf{G}_p(y)\boldsymbol{\varphi}_{12,\delta}(u-y;u)\,dy + \mathbf{G}_\omega(u)\right] w_3(u) \\
&= \frac{\delta}{c}\mathbf{\Phi}_\delta(u) - \frac{1}{c}\mathbf{G}_0\mathbf{\Phi}_\delta(u) + \left[\mathbf{v}'_\delta(u) - \frac{\delta}{c}\mathbf{v}_\delta(u) + \frac{1}{c}\mathbf{G}_0\mathbf{v}_\delta(u)\right][\mathbf{v}_\delta(u)]^{-1}\mathbf{\Phi}_\delta(u) \\
&\quad - \frac{1}{c}\left[\int_0^u \mathbf{G}_p(y)\boldsymbol{\varphi}_{12,\delta}(u-y;u)\,dy + \mathbf{G}_\omega(u)\right] w_3(u) \\
&= \mathbf{v}'_\delta(u)\,[\mathbf{v}_\delta(u)]^{-1}\mathbf{\Phi}_\delta(u) - \frac{1}{c}\left[\int_0^u \mathbf{G}_p(y)\boldsymbol{\varphi}_{12,\delta}(u-y;u)\,dy + \mathbf{G}_\omega(u)\right] w_3(u) , \quad u \geq 0 .
\end{aligned} \tag{6.14}$$

However, (6.14) also holds true for the classical Gerber-Shiu function $\mathbf{\Phi}_{12,\delta}(u)$ (i.e. with $w_3(.) \equiv 1$) and therefore by rearranging terms one obtains,

$$-\frac{1}{c}\left[\int_0^u \mathbf{G}_p(y)\boldsymbol{\varphi}_{12,\delta}(u-y;u)\,dy + \mathbf{G}_\omega(u)\right] = \mathbf{\Phi}'_{12,\delta}(u) - \mathbf{v}'_\delta(u)\,[\mathbf{v}_\delta(u)]^{-1}\mathbf{\Phi}_{12,\delta}(u) , \quad u \geq 0 . \tag{6.15}$$

Substituting (6.15) into (6.14) yields

$$\mathbf{\Phi}'_\delta(u) = \mathbf{v}'_\delta(u)\,[\mathbf{v}_\delta(u)]^{-1}\mathbf{\Phi}_\delta(u) + \left\{\mathbf{\Phi}'_{12,\delta}(u) - \mathbf{v}'_\delta(u)\,[\mathbf{v}_\delta(u)]^{-1}\mathbf{\Phi}_{12,\delta}(u)\right\} w_3(u) , \quad u \geq 0 . \tag{6.16}$$

Multiplying both sides of (6.16) by $[\mathbf{v}_\delta(u)]^{-1}$ followed by rearrangement of terms, we arrive at

$$
\begin{aligned}
&- \left\{[\mathbf{v}_\delta(u)]^{-1}\,\boldsymbol{\Phi}'_\delta(u) - [\mathbf{v}_\delta(u)]^{-1}\,\mathbf{v}'_\delta(u)\,[\mathbf{v}_\delta(u)]^{-1}\,\boldsymbol{\Phi}_\delta(u)\right\} \\
=& - \left\{[\mathbf{v}_\delta(u)]^{-1}\,\boldsymbol{\Phi}'_{12,\delta}(u) - [\mathbf{v}_\delta(u)]^{-1}\,\mathbf{v}'_\delta(u)\,[\mathbf{v}_\delta(u)]^{-1}\,\boldsymbol{\Phi}_{12,\delta}(u)\right\} w_3(u)\ , \quad u \geq 0\ . \qquad (6.17)
\end{aligned}
$$

Capitalizing on the matrix differentiation property that $(d/du)[\mathbf{v}_\delta(u)]^{-1} = -[\mathbf{v}_\delta(u)]^{-1}\mathbf{v}'_\delta(u)[\mathbf{v}_\delta(u)]^{-1}$ together with the product rule of matrix differentiation, one can rewrite (6.17) as

$$
-\frac{d}{du}\left\{[\mathbf{v}_\delta(u)]^{-1}\,\boldsymbol{\Phi}_\delta(u)\right\} = -w_3(u)\frac{d}{du}\left\{[\mathbf{v}_\delta(u)]^{-1}\,\boldsymbol{\Phi}_{12,\delta}(u)\right\}\ , \quad u \geq 0\ . \qquad (6.18)
$$

Since $\lim_{u\to\infty}\boldsymbol{\Phi}_\delta(u) = \mathbf{0}$, replacing u by z in (6.18) and integrating with respect to z from u to ∞, it follows that

$$
[\mathbf{v}_\delta(u)]^{-1}\,\boldsymbol{\Phi}_\delta(u) = -\int_u^\infty w_3(z)\frac{d}{dz}\left\{[\mathbf{v}_\delta(z)]^{-1}\,\boldsymbol{\Phi}_{12,\delta}(z)\right\} dz\ , \quad u \geq 0\ , \qquad (6.19)
$$

or equivalently,

$$
\boldsymbol{\Phi}_\delta(u) = -\int_u^\infty w_3(z)\frac{d}{dz}\left[\boldsymbol{\chi}_\delta(u;z)\boldsymbol{\Phi}_{12,\delta}(z)\right] dz\ , \quad u \geq 0\ . \qquad (6.20)
$$

Note that (6.20) expresses the general Gerber-Shiu function $\boldsymbol{\Phi}_\delta(u)$ in terms of the classical Gerber-Shiu function $\boldsymbol{\Phi}_{12,\delta}(u)$ and the Laplace transform of the first passage time $\boldsymbol{\chi}_\delta(u;z)$. While the quantity $\boldsymbol{\chi}_\delta(u;z)$ can be computed by (6.10) and (6.13), $\boldsymbol{\Phi}_{12,\delta}(u)$ can be evaluated along the same lines as in Lu and Tsai (2007) (see Section 2.5).

Remark 16 *Note that although our first step of conditioning on the first change in environment to obtain (6.3) involves the function $\varphi_{12,\delta,jl}(u;b)$, $\boldsymbol{\varphi}_{12,\delta}(u;b)$ does not appear in our main result (6.20). Indeed, one could see that $\boldsymbol{\varphi}_{12,\delta}(u;b)$ has been eliminated from (6.14) through the use of (6.15).*

6.2.2 Dividend barrier strategy

In this subsection, we examine how $\mathbf{\Phi}_\delta(u)$ can be expressed in terms of the classical Gerber-Shiu function in the same MAP model under a dividend barrier strategy. In addition, the general Gerber-Shiu function under a dividend barrier strategy is also derived.

Under a dividend barrier strategy, whenever the surplus attains a fixed barrier level $b > 0$, the insurer pays the entire premium income to the shareholder as dividends until the next claim occurs, and no dividends are paid when the surplus level is below b (see also Section 2.1). We denote the resulting surplus process by $\{U_t^{(b)}\}_{t\geq 0}$, which satisfies, for $t \geq 0$,

$$dU_t^{(b)} = \begin{cases} c\,dt - d\left(\sum_{i=1}^{N_t} Y_i\right) \ , & U_t^{(b)} < b \ . \\ -d\left(\sum_{i=1}^{N_t} Y_i\right) \ , & U_t^{(b)} = b \ . \end{cases} \tag{6.21}$$

The time of ruin, the surplus prior to ruin, the deficit at ruin and the maximum surplus level before ruin are given by $T_b = \inf\{t \geq 0 : U_t^{(b)} < 0\}$, $U_{T_b^-}^{(b)}$, $|U_{T_b}^{(b)}|$ and $Z_{T_b}^{(b)} = \max_{0\leq s<T_b} U_s^{(b)}$ respectively. Therefore, the general Gerber-Shiu function of our interest is given by, for $i, j \in E$,

$$\phi_{\delta,ij}(u;b) = E\left[e^{-\delta T_b} w_{12}(U_{T_b^-}^{(b)}, |U_{T_b}^{(b)}|) w_3(Z_{T_b}^{(b)}) 1\{T_b < \infty, J_{T_b} = j\} | U_0^{(b)} = u, J_0 = i\right] \ , \quad 0 \leq u \leq b \ , \tag{6.22}$$

with the classical Gerber-Shiu function retrieved by letting $w_3(.) \equiv 1$ and denoted by $\phi_{12,\delta,ij}(u;b)$. We also define the matrices of Gerber-Shiu functions $\mathbf{\Phi}_\delta(u;b) = [\phi_{\delta,ij}(u;b)]_{i,j=1}^m$ and $\mathbf{\Phi}_{12,\delta}(u;b) = [\phi_{12,\delta,ij}(u;b)]_{i,j=1}^m$.

Suppose we want to relate $\mathbf{\Phi}_\delta(u)$ to $\mathbf{\Phi}_{12,\delta}(u;b)$. We need an additional result - the dividends-penalty identity which was first introduced by Gerber et al. (2006). Gerber et al. (2006) derived the identity in the context of a Markov process which is skip-free upwards. In the present context, we require a matrix version of the identity, which can be adapted from Chapter 2 and Li and Lu

(2008). It is given by

$$\mathbf{\Phi}_{12,\delta}(u;b) = \mathbf{\Phi}_{12,\delta}(u) - \mathbf{v}_\delta(u)\left[\mathbf{v}_\delta'(b)\right]^{-1}\mathbf{\Phi}_{12,\delta}'(b)\ , \quad 0 \le u \le b\ . \tag{6.23}$$

By (6.17) and (6.18), one readily obtains

$$\begin{aligned}
-\frac{d}{du}\left\{\left[\mathbf{v}_\delta(u)\right]^{-1}\mathbf{\Phi}_\delta(u)\right\} &= -w_3(u)\left\{\left[\mathbf{v}_\delta(u)\right]^{-1}\mathbf{\Phi}_{12,\delta}'(u) - \left[\mathbf{v}_\delta(u)\right]^{-1}\mathbf{v}_\delta'(u)\left[\mathbf{v}_\delta(u)\right]^{-1}\mathbf{\Phi}_{12,\delta}(u)\right\} \\
&= w_3(u)\left[\mathbf{v}_\delta(u)\right]^{-1}\mathbf{v}_\delta'(u)\left[\mathbf{v}_\delta(u)\right]^{-1}\left\{\mathbf{\Phi}_{12,\delta}(u) - \mathbf{v}_\delta(u)\left[\mathbf{v}_\delta'(u)\right]^{-1}\mathbf{\Phi}_{12,\delta}'(u)\right\} \\
&= -w_3(u)\left\{\frac{d}{du}\left[\mathbf{v}_\delta(u)\right]^{-1}\right\}\mathbf{\Phi}_{12,\delta}(u;u)\ , \quad u \ge 0\ ,
\end{aligned} \tag{6.24}$$

where the last line follows from (6.23). Thus, we arrive at, along the same lines in obtaining (6.20),

$$\mathbf{\Phi}_\delta(u) = -\int_u^\infty w_3(z)\left[\frac{d}{dz}\boldsymbol{\chi}_\delta(u;z)\right]\mathbf{\Phi}_{12,\delta}(z;z)\,dz\ , \quad u \ge 0\ , \tag{6.25}$$

which expresses $\mathbf{\Phi}_\delta(u)$ in terms of $\mathbf{\Phi}_{12,\delta}(z;z)$, the classical Gerber-Shiu function in the same MAP model under a dividend barrier strategy, and the Laplace transform $\boldsymbol{\chi}_\delta(u;z)$.

Next we are going to find the expression for the general Gerber-Shiu function $\mathbf{\Phi}_\delta(u;b)$ with elements defined by (6.22). Given an initial surplus of u such that $0 \le u \le b$, note that if the surplus process $\{U_t^{(b)}\}_{t\ge0}$ reaches level b before ruin first occurring (with such a first passage time having Laplace transform $\boldsymbol{\chi}_\delta(u;b)$), then the maximum surplus level before ruin is simply b, and the discounted penalty applied to the surplus prior to ruin and the deficit at ruin at the time of hitting would simply be $\mathbf{\Phi}_{12,\delta}(b;b)$. On the other hand, if the process $\{U_t^{(b)}\}_{t\ge0}$ drops below 0 before it can ever reach level b, then the general Gerber-Shiu function is equivalent to $\boldsymbol{\varphi}_\delta(u;b) = [\varphi_{\delta,ij}(u;b)]_{i,j=1}^m$ with (i,j)-th element given by, for $i,j \in E$,

$$\varphi_{\delta,ij}(u;b) = E\left[e^{-\delta T}w_{12}(U_{T^-},|U_T|)w_3(Z_T)1\{T<\tau_b, J_T=j\}|U_0=u, J_0=i\right]\ , \quad 0 \le u \le b\ , \tag{6.26}$$

We remark that when $w_3(.) \equiv 1$, $\boldsymbol{\varphi}_\delta(u;b)$ reduces to $\boldsymbol{\varphi}_{12,\delta}(u;b)$.

Combining all the above, we arrive at

$$\boldsymbol{\Phi}_\delta(u;b) = \boldsymbol{\chi}_\delta(u;b) w_3(b) \boldsymbol{\Phi}_{12,\delta}(b;b) + \boldsymbol{\varphi}_\delta(u;b) \ , \quad 0 \le u \le b \ . \tag{6.27}$$

From (6.27), it is clear that it remains to determine $\boldsymbol{\varphi}_\delta(u;b)$ if we want to find $\boldsymbol{\Phi}_\delta(u;b)$. In the same way as (6.9) is obtained, we omit the details and arrive at

$$\begin{aligned}\boldsymbol{\varphi}'_\delta(u;b) = \frac{\delta}{c}\boldsymbol{\varphi}_\delta(u;b) - \frac{1}{c}\mathbf{G}_0\boldsymbol{\varphi}_\delta(u;b) - \frac{1}{c}\left[\int_0^u \mathbf{G}_p(y)\boldsymbol{\chi}_\delta(u-y;u)\,dy\right]\boldsymbol{\varphi}_\delta(u;b)\\ - \frac{1}{c}\left[\int_0^u \mathbf{G}_p(y)\boldsymbol{\varphi}_{12,\delta}(u-y;u)\,dy + \mathbf{G}_\omega(u)\right] w_3(u) \ , \quad 0 \le u \le b \ ,\end{aligned} \tag{6.28}$$

with trivial boundary condition

$$\boldsymbol{\varphi}_\delta(b;b) = 0 \ . \tag{6.29}$$

Analogous to (6.20), with the above boundary condition we ultimately obtain

$$\boldsymbol{\varphi}_\delta(u;b) = -\int_u^b w_3(z)\frac{d}{dz}\left[\boldsymbol{\chi}_\delta(u;z)\boldsymbol{\Phi}_{12,\delta}(z)\right] dz \ , \quad 0 \le u \le b \ . \tag{6.30}$$

This completes our characterization of $\boldsymbol{\varphi}_\delta(u;b)$ and hence that for $\boldsymbol{\Phi}_\delta(u;b)$. Similar to (6.25), an alternate representation for $\boldsymbol{\varphi}_\delta(u;b)$ in relation to $\boldsymbol{\Phi}_{12,\delta}(z;z)$ is given by

$$\boldsymbol{\varphi}_\delta(u;b) = -\int_u^b w_3(z)\left[\frac{d}{dz}\boldsymbol{\chi}_\delta(u;z)\right]\boldsymbol{\Phi}_{12,\delta}(z;z)\,dz \ , \quad 0 \le u \le b \ . \tag{6.31}$$

6.3 Example: Classical compound Poisson risk model

In this section, we consider the simplest case of the MAP risk model where $m = 1$, i.e. the classical compound Poisson risk model. In such a case, $\{N_t\}_{t\ge0}$ reduces to a Poisson process with rate $\lambda > 0$, and we have $G_{0,11} = -\lambda$ and $G_{1,11} = \lambda$. To simplify our notation, we denote the claim size density by $p(.) = p_{11}(.)$. Furthermore, all the matrix quantities in Section 6.2 reduce to scalar quantities, and we shall write $\mathbf{\Phi}_\delta(u) = \phi_\delta(u)$, $\mathbf{\Phi}_{12,\delta}(u) = \phi_{12,\delta}(u)$, $\boldsymbol{\chi}_\delta(u;b) = \chi_\delta(u;b)$, $\mathbf{G}_\omega(u) = \lambda\omega_{11}(u) = \lambda\omega(u)$ and $\mathbf{v}_\delta(u) = v_\delta(u)$. In this simplest model, we shall first give more explicit expression for the general Gerber-Shiu function $\phi_\delta(u)$. Then such an expression will be used to find the discounted joint density of $(U_{T^-}, |U_T|, Z_T)$, as well as the density of the largest distance $Z_T + |U_T|$ followed by numerical illustrations.

6.3.1 Gerber-Shiu function $\mathbf{\Phi}_\delta(u)$ involving Z_T

In the compound Poisson model, (6.19) reduces to

$$\phi_\delta(u) = v_\delta(u) \int_u^\infty w_3(z) \left[-\frac{d}{dz} \frac{\phi_{12,\delta}(z)}{v_\delta(z)} \right] dz \ , \quad u \geq 0 \ . \tag{6.32}$$

Suppose we want to evaluate $-(d/dz)[\phi_{12,\delta}(z)/v_\delta(z)]$ in the above equation explicitly. We first define ρ to be the unique non-negative root of the Lundberg's fundamental equation

$$cs - (\lambda + \delta) + \lambda \widetilde{p}(s) = 0 \ . \tag{6.33}$$

In addition, let $g_\delta(u)$ be the compound geometric density

$$g_\delta(u) = \sum_{n=1}^{\infty} (1 - \kappa_\delta)(\kappa_\delta)^n l_\delta^{*n}(u) \ , \quad u \geq 0 \ , \tag{6.34}$$

where

$$\kappa_\delta = \frac{\lambda}{c} \int_0^\infty \int_0^\infty e^{-\rho x} p(x + y) \, dx \, dy < 1 \ , \tag{6.35}$$

and $l_\delta^{*n}(u)$ is the density function of the n-fold convolution of the ladder height density

$$l_\delta(y) = \frac{\int_0^\infty e^{-\rho x} p(x+y)\, dx}{\int_0^\infty \int_0^\infty e^{-\rho x} p(x+y)\, dx\, dy} \ , \quad y \geq 0 \ , \tag{6.36}$$

with itself. It is known from Gerber and Shiu (1998) that $\phi_{12,\delta}(u)$ satisfies a defective renewal equation with solution given by (see, e.g., Lin and Willmot (1999))

$$\phi_{12,\delta}(u) = \alpha_\delta(u) + \frac{1}{1-\kappa_\delta} \int_0^u \alpha_\delta(u-y) g_\delta(y)\, dy \ , \quad u \geq 0 \ , \tag{6.37}$$

where

$$\alpha_\delta(u) = \frac{\lambda}{c} \int_u^\infty \int_0^\infty w_{12}(x,y) e^{-\rho(x-u)} p(x+y)\, dy\, dx \ , \quad u \geq 0 \ . \tag{6.38}$$

Differentiating (6.37) with respect to u yields

$$\phi'_{12,\delta}(u) = \alpha'_\delta(u) + \frac{1}{1-\kappa_\delta} \left[\alpha_\delta(0) g_\delta(u) + \int_0^u \alpha'_\delta(u-y) g_\delta(y)\, dy \right] \ , \quad u \geq 0 \ . \tag{6.39}$$

From the definition of $\alpha_\delta(u)$ in (6.38), it is easy to find that, using the scalar version of (6.7),

$$\alpha'_\delta(u) = \rho \alpha_\delta(u) - \frac{\lambda}{c} \int_0^\infty w_{12}(u,y) p(u+y)\, dy = \rho \alpha_\delta(u) - \frac{\lambda}{c} \omega(u) \ , \quad u \geq 0 \ . \tag{6.40}$$

Substituting (6.40) in (6.39), one obtains

$$\begin{aligned}
\phi'_{12,\delta}(u) &= \left[\rho \alpha_\delta(u) - \frac{\lambda}{c} \omega(u) \right] + \frac{1}{1-\kappa_\delta} \left\{ \alpha_\delta(0) g_\delta(u) + \int_0^u \left[\rho \alpha_\delta(u-y) - \frac{\lambda}{c} \omega(u-y) \right] g_\delta(y)\, dy \right\} \\
&= \rho \left[\alpha_\delta(u) + \frac{1}{1-\kappa_\delta} \int_0^u \alpha_\delta(u-y) g_\delta(y)\, dy \right] \\
&\quad - \frac{\lambda}{c} \omega(u) + \frac{1}{1-\kappa_\delta} \left[\alpha_\delta(0) g_\delta(u) - \frac{\lambda}{c} \int_0^u \omega(u-y) g_\delta(y)\, dy \right] \\
&= \rho \phi_{12,\delta}(u) + \frac{1}{1-\kappa_\delta} \left\{ \alpha_\delta(0) g_\delta(u) - \frac{\lambda}{c} \left[(1-\kappa_\delta) \omega(u) + \int_0^u g_\delta(u-y) \omega(y)\, dy \right] \right\} \ , \quad u \geq 0 \ .
\end{aligned} \tag{6.41}$$

Similar to $\phi_{12,\delta}(u)$, it can be shown readily that (the scalar version of) the integro-differential equation (6.11) with initial condition (6.12) can be transformed to a defective renewal equation with solution

$$v_\delta(u) = e^{\rho u} + \frac{1}{1-\kappa_\delta} \int_0^u e^{\rho(u-y)} g_\delta(y)\, dy \ , \quad u \geq 0 \ . \tag{6.42}$$

We remark that an alternative form of $v_\delta(u)$ was given in Bühlmann (1970, Section 6.4.9) or Lin et al. (2003, Section 4). It can be proved that their solution is in fact equal to the one given here. However, as we shall see later, the solution (6.42) allows us to simplify some expressions.

It follows from (6.42) that

$$v'_\delta(u) = \rho v_\delta(u) + \frac{1}{1-\kappa_\delta} g_\delta(u) \ , \quad u \geq 0 \ . \tag{6.43}$$

Applying (6.41) and (6.43), it is easy to see that

$$\begin{aligned}
& -\frac{d}{dz}\frac{\phi_{12,\delta}(z)}{v_\delta(z)} \\
&= \frac{v'_\delta(z)\phi_{12,\delta}(z) - v_\delta(z)\phi'_{12,\delta}(z)}{[v_\delta(z)]^2} \\
&= \frac{\frac{1}{1-\kappa_\delta} g_\delta(z)\phi_{12,\delta}(z) - \frac{v_\delta(z)}{1-\kappa_\delta}\left\{\alpha_\delta(0)g_\delta(z) - \frac{\lambda}{c}\left[(1-\kappa_\delta)\omega(z) + \int_0^z g_\delta(z-y)\omega(y)\, dy\right]\right\}}{[v_\delta(z)]^2} \\
&= \frac{1}{(1-\kappa_\delta)v_\delta(z)}\left\{\frac{g_\delta(z)}{v_\delta(z)}\phi_{12,\delta}(z) - \alpha_\delta(0)g_\delta(z) + \frac{\lambda}{c}\left[(1-\kappa_\delta)\omega(z) + \int_0^z g_\delta(z-y)\omega(y)\, dy\right]\right\}, \quad z \geq 0 \ .
\end{aligned} \tag{6.44}$$

Therefore, substitution of (6.44) into (6.32) yields

$$\begin{aligned}
& \phi_\delta(u) \\
&= \int_u^\infty \frac{w_3(z)v_\delta(u)}{(1-\kappa_\delta)v_\delta(z)}\left\{\frac{g_\delta(z)}{v_\delta(z)}\phi_{12,\delta}(z) - \alpha_\delta(0)g_\delta(z) + \frac{\lambda}{c}\left[(1-\kappa_\delta)\omega(z) + \int_0^z g_\delta(z-y)\omega(y)\, dy\right]\right\} dz, \\
&\qquad u \geq 0 \ ,
\end{aligned} \tag{6.45}$$

which is an explicit expression for $\phi_\delta(u)$.

6.3.2 Discounted joint density of $(U_{T^-}, |U_T|, Z_T)$

In what follows, we apply our results from previous subsection to find the discounted joint density of $(U_{T^-}, |U_T|, Z_T)$. To do so, we let $w_{12}(x, y) = e^{-s_1 x - s_2 y}$ and $w_3(z) = e^{-s_3 z}$, so that it is clear from (6.2) that $\phi_\delta(u)$ represents the joint Laplace transform of the quadruple $(T, U_{T^-}, |U_T|, Z_T)$, whereas $\phi_{12,\delta}(u)$ represents the joint Laplace transform of the triplet $(T, U_{T^-}, |U_T|)$. We aim at analytically inverting $\phi_\delta(u)$ with respect to s_1, s_2 and s_3, and this will result in the so-called discounted joint density of $(U_{T^-}, |U_T|, Z_T)$.

Landriault and Willmot (2009) showed that

$$\phi_{12,\delta}(u) = \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y} h_\delta(x, y|u)\, dx\, dy \ , \quad u \geq 0 \ , \tag{6.46}$$

where

$$h_\delta(x, y|u) = \begin{cases} \frac{\lambda}{c(1-\kappa_\delta)} p(x+y) \int_0^x e^{-\rho(x-v)} g_\delta(u-v)\, dv \ , & x < u \ . \\ \frac{\lambda}{c(1-\kappa_\delta)} p(x+y) \left[(1-\kappa_\delta) e^{-\rho(x-u)} + \int_0^u e^{-\rho(x-v)} g_\delta(u-v)\, dv \right] \ , & x > u \ . \end{cases} \tag{6.47}$$

is the discounted joint density of $(U_{T^-}, |U_T|)$ at (x, y) for an initial surplus of $U_0 = u$. With the help of (6.42), (6.47) can be rewritten as

$$h_\delta(x, y|u) = \begin{cases} \frac{\lambda}{c} \left[e^{-\rho x} v_\delta(u) - v_\delta(u-x) \right] p(x+y) \ , & x < u \ . \\ \frac{\lambda}{c} v_\delta(u) e^{-\rho x} p(x+y) \ , & x > u \ . \end{cases} \tag{6.48}$$

For our choice of penalty functions, substitution of (6.7), (6.38) and (6.46) into (6.45) leads to

$$\begin{aligned}\phi_\delta(u) = &\int_u^\infty \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y - s_3 z} \frac{\chi_\delta(u;z)}{1-\kappa_\delta} \left[\frac{g_\delta(z)}{v_\delta(z)} h_\delta(x,y|z) - \frac{\lambda}{c} e^{-\rho x} p(x+y) g_\delta(z) \right] dx\, dy\, dz \\ &+ \int_u^\infty \int_0^\infty e^{-s_1 z - s_2 y - s_3 z} \left[\frac{\lambda}{c} \chi_\delta(u;z) p(z+y) \right] dy\, dz \\ &+ \int_u^\infty \int_0^z \int_0^\infty e^{-s_1 x - s_2 y - s_3 z} \frac{\chi_\delta(u;z)}{1-\kappa_\delta} \left[\frac{\lambda}{c} g_\delta(z-x) p(x+y) \right] dy\, dx\, dz \ , \quad u \geq 0 \ . \end{aligned} \tag{6.49}$$

By the uniqueness of Laplace transforms, given an initial surplus of $U_0 = u$, the triplet $(U_{T^-}, |U_T|, Z_T)$ has discounted densities on the subspaces of $\mathbb{R}^3$ given by

1. on $\{(x,y,z) : y > 0, x = z > u\}$:

$$h^*_{12,\delta}(x,y|u) = \frac{\lambda}{c} \chi_\delta(u;x) p(x+y) \ , \tag{6.50}$$

and

2. on $\{(x,y,z) : y > 0, 0 < x < z, z > u\}$:

$$\begin{aligned} h^*_{123,\delta}(x,y,z|u) &= \frac{\chi_\delta(u;z)}{1-\kappa_\delta} \left[\frac{\lambda}{c} g_\delta(z-x) p(x+y) + \frac{g_\delta(z)}{v_\delta(z)} h_\delta(x,y|z) - \frac{\lambda}{c} e^{-\rho x} p(x+y) g_\delta(z) \right] \\ &= \frac{\lambda}{c} \frac{\chi_\delta(u;z)}{1-\kappa_\delta} \left\{ g_\delta(z-x) + \frac{g_\delta(z)}{v_\delta(z)} \left[e^{-\rho x} v_\delta(z) - v_\delta(z-x) \right] - e^{-\rho x} g_\delta(z) \right\} p(x+y) \\ &= \frac{\lambda}{c} \frac{\chi_\delta(u;z)}{1-\kappa_\delta} \left[g_\delta(z-x) - g_\delta(z) \frac{v_\delta(z-x)}{v_\delta(z)} \right] p(x+y) \ , \end{aligned} \tag{6.51}$$

where the second last line follows from (6.48).

In addition, on $\{(x,y,z) : y > 0, x > z > u\}$, the density

$$
\begin{aligned}
h_{123,\delta}^{\text{null}}(x,y,z|u) &= \frac{\chi_\delta(u;z)}{1-\kappa_\delta}\left[\frac{g_\delta(z)}{v_\delta(z)}h_\delta(x,y|z) - \frac{\lambda}{c}e^{-\rho x}p(x+y)g_\delta(z)\right] \\
&= \frac{\lambda}{c}\frac{\chi_\delta(u;z)}{1-\kappa_\delta}\left\{\frac{g_\delta(z)}{v_\delta(z)}\left[v_\delta(z)e^{-\rho x}p(x+y)\right] - e^{-\rho x}p(x+y)g_\delta(z)\right\} \\
&= 0 \ , \qquad (6.52)
\end{aligned}
$$

where the second last line again follows from (6.48). (6.52) is indeed expected to be 0 since the surplus prior to ruin cannot be greater than the maximum surplus before ruin.

We remark that the density $h_{12,\delta}^*(x,y|u)$ given by (6.50) is the contribution to the joint distribution of $(U_{T^-}, |U_T|, Z_T)$ from the case where the claim causing ruin occurs exactly when the surplus process reaches its maximum level before ruin.

6.3.3 Density of largest distance until ruin $Z_T + |U_T|$ and numerical illustrations

Having obtained the discounted joint density of $(U_{T^-}, |U_T|, Z_T)$ in (6.50) and (6.51), one readily obtains the (defective) density of the largest distance until ruin $Z_T + |U_T|$ (at v) given an initial surplus of $U_0 = u$, namely $h_4(v|u)$, as

$$
\begin{aligned}
h_4(v|u) &= \int_u^v h_{12,0}^*(z, v-z|u)\,dz + \int_u^v \int_0^z h_{123,0}^*(x, v-z, z|u)\,dx\,dz \\
&= \frac{\lambda}{c}\int_u^v \chi_0(u;z)\left\{p(v) + \frac{1}{1-\kappa_0}\int_0^z \left[g_0(z-x) - g_0(z)\frac{v_0(z-x)}{v_0(z)}\right]p(x+v-z)\,dx\right\}dz \\
&= \frac{\lambda}{c}\int_u^v \frac{v_0(u)}{v_0(z)}\left\{p(v) + \frac{1}{1-\kappa_0}\left[\int_0^z g_0(x)p(v-x)\,dx - \frac{g_0(z)}{v_0(z)}\int_0^z v_0(x)p(v-x)\,dx\right]\right\}dz \ , \\
&\qquad v > u \ . \qquad (6.53)
\end{aligned}
$$

We remark that the same result can also be obtained by performing analytic Laplace transform inversion with respect to s_4 on the right-hand side of (6.49) under the choice of penalty functions $w_{12}(x, y) = e^{-s_4 y}$ and $w_3(z) = e^{-s_4 z}$ with $\delta = 0$. The details are omitted here.

Although the (defective) density $h_4(v|u)$ given by (6.53) is expressed in integral form, the integrals involved can be computed via computer software such as *Mathematica*. To show the numerical tractability, the density $h_4(v|u)$ and some moment-based quantities in relation to $Z_T + |U_T|$ are obtained in the compound Poisson risk model under three different claim size distributions including the sum of two exponential density

$$p(y) = 2\left(\frac{3}{2}e^{-\frac{3}{2}y}\right) - 3e^{-3y} \ , \quad y > 0 \ , \tag{6.54}$$

the exponential density

$$p(y) = e^{-y} \ , \quad y > 0 \ , \tag{6.55}$$

and the mixture of two exponentials density

$$p(y) = \frac{1}{3}\left(\frac{1}{2}e^{-\frac{1}{2}y}\right) + \frac{2}{3}\left(2e^{-2y}\right) \ , \quad y > 0 \ . \tag{6.56}$$

While the means of the three claim size distributions are all 1, they possess different amount of variability which is evident in their respective standard deviations of 0.745, 1 and 1.414. Under all cases, the Poisson claim arrival rate is $\lambda = 1$ and the premium rate is assumed to be $c = 1.5$. As a result, it is easy to check that the positive security loading condition holds.

Note that the density $h_4(v|u)$ in (6.53) is expressed in terms of the functions $v_0(.)$ and $g_0(.)$ apart from the claim size density $p(.)$. Letting $\psi(u) = \Pr\{T < \infty | U_0 = u\}$ be the ruin probability, it is known from e.g. Lin et al. (2003) that $v_0(u)$ and the compound geometric density $g_0(u)$ can be expressed as $v_0(u) = [1 - \psi(u)]/[1 - \psi(0)]$ and $g_0(u) = -\psi'(u)$ respectively. Since the densities (6.54), (6.55) and (6.56) all belong to the class of the combinations of exponentials, $\psi(u)$ can be found based on the results in Gerber et al. (1987, Section 3). As a direct consequence, both $v_0(u)$ and $g_0(u)$ are both linear sums of exponential terms.

First, for an initial surplus of $u = 5$, the density $h_4(v|u)$ under the three claim size densities is plotted in Figure 6.1 below.

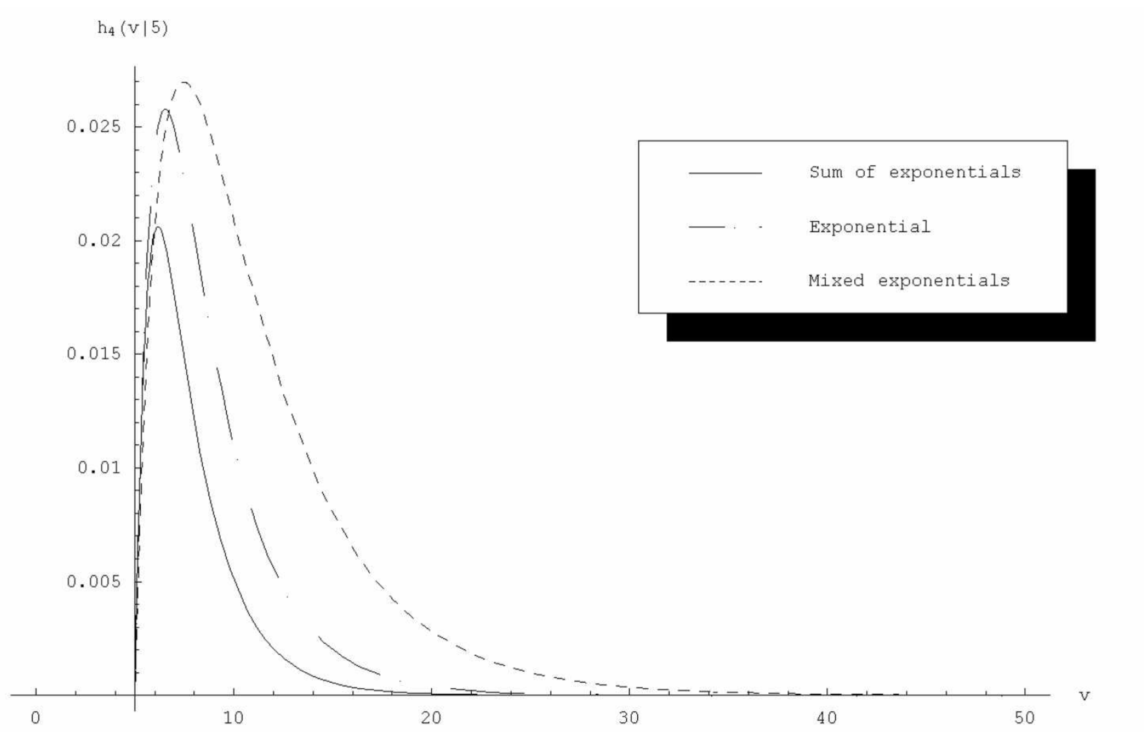

Figure 6.1: Plot of the density $h_4(v|5)$ of different claim size distributions

From Figure 1, one observes that the mode of the defective density $h_4(v|5)$ has the same ordering as the standard deviation of the claim size. Also, except for small values of v, the same is also true for $h_4(v|5)$ itself. This can be attributed to the fact that all else being equal, a larger variability in claim sizes leads to larger fluctuations in the surplus process, resulting in higher ruin probability and larger distance between the running maximum and the running minimum at the time of ruin.

Second, we also consider the moment-based quantities

$$\text{Mean} = E[Z_T + |U_T||T < \infty, U_0 = u] = \int_u^\infty v \left[\frac{h_4(v|u)}{\psi(u)}\right] dv \ , \tag{6.57}$$

$$\begin{aligned}\text{Standard deviation} &= SD[Z_T + |U_T||T < \infty, U_0 = u] \\ &= \sqrt{\int_u^\infty v^2 \left[\frac{h_4(v|u)}{\psi(u)}\right] dv - \left\{\int_u^\infty v \left[\frac{h_4(v|u)}{\psi(u)}\right] dv\right\}^2} \ ,\end{aligned} \tag{6.58}$$

and

$$\text{Coefficient of variation} = CV[Z_T + |U_T||T < \infty, U_0 = u] = \frac{SD[Z_T + |U_T||T < \infty, U_0 = u]}{E[Z_T + |U_T||T < \infty, U_0 = u]} \ , \tag{6.59}$$

all conditional on the event that ruin occurs. These three quantities at various initial surplus levels are given in Table 6.1 below for all three claim size distributions.

	Sum of exponentials			Exponential			Mixture of exponentials		
u	Mean	SD	CV	Mean	SD	CV	Mean	SD	CV
0	2.0813	1.7641	0.8476	2.6479	2.4050	0.9082	3.8769	4.0934	1.0558
5	7.8779	2.3194	0.2944	8.8027	3.0666	0.3484	11.1953	4.9898	0.4457
10	12.9564	2.3575	0.1820	13.9640	3.1452	0.2252	16.5754	5.1751	0.3122
15	17.9647	2.3615	0.1315	18.9933	3.1591	0.1663	21.7026	5.2351	0.2412
20	22.9656	2.3620	0.1028	23.9987	3.1617	0.1317	26.7472	5.2559	0.1965
25	27.9657	2.3620	0.0845	28.9998	3.1622	0.1090	31.7631	5.2633	0.1657
30	32.9658	2.3620	0.0717	34.0000	3.1623	0.0930	36.7688	5.2660	0.1432
100	102.9658	2.3620	0.0229	104.0000	3.1623	0.0304	106.7720	5.2675	0.0493

Table 6.1: Mean, standard deviation and coefficient of variation of $Z_T + |U_T|$

From Table 6.1, we observe that in general, the mean, standard deviation and coefficient of variation are larger when the claim size distribution has a larger standard deviation. It is also interesting to notice that in each of the three cases, the excess of the conditional mean of $Z_T+|U_T|$ over the initial surplus u appears to converge to a constant as the initial surplus u gets large. The same seems to be true for the standard deviation itself.

6.4 Discounted joint density of $(U_{T^-}, |U_T|, Z_T, X_T)$ by fluid queue

The techniques used in the previous sections in the chapter are analytic methods. In this chapter, we shall analyze the MAP risk model by a purely probabilistic approach through the connection to a fluid queue given in Section 1.4.1. We shall further incorporate the variable $X_T = \min_{0 \le s < T} U_s$ into our analysis, and the discounted joint density $(U_{T^-}, |U_T|, Z_T, X_T)$ will be considered.

To begin, we have to first define the discounted joint density of $(U_{T^-}, |U_T|, Z_T, X_T)$. Note that the contributions to this discounted joint density have different functional forms based on

1. whether ruin whether ruin is caused by the first claim or any of its subsequent claims (see, e.g., Chapter 5 and Landriault and Willmot (2009)); and
2. whether a claim causing ruin occurs immediately as the surplus reaches its maximum level before ruin (see Section 6.3.2).

Therefore, we have to introduce three column vectors to account for the discounted joint density of $(U_{T^-}, |U_T|, Z_T, X_T)$ as follows.

1. For ruin occurring on the first claim, the minimum surplus before ruin X_T is simply equal to the initial surplus whereas the maximum surplus before ruin Z_T equals the surplus prior to ruin U_{T^-}, and the time of ruin is $T = (U_{T^-} - u)/c$. In such a case, the discounted joint density of represented by the column vector $\mathbf{h}^{(1)}_{12,\delta}(x, y|u)$ with the i-th element $(i \in S_1)$ given

by, on $\{(x,y,z,v) : x = z > u, y > 0, v = u\}$,

$$
\begin{aligned}
&[\mathbf{h}^{(1)}_{12,\delta}(x,y|u)]_i \, dx \, dy \\
&= E\left[e^{-\delta T} 1\{N_T = 1, U_{T^-} = Z_T \in (x, x+dx), |U_T| \in (y, y+dy)\} \middle| F(0) = u, J^{(F)}(0) = i\right] \ .
\end{aligned}
\tag{6.60}
$$

2. For ruin occurring on claims subsequent to the first, it could possibly be caused by a claim occurring immediately when the surplus reaches its maximum before ruin. Then Z_T equals U_{T^-}, and the discounted joint density of represented by $\mathbf{h}^{(2)}_{124,\delta}(x,y,v|u)$ with the i-th element ($i \in S_1$) given by, on $\{(x,y,z,v) : x = z > u, y > 0, v < u\}$,

$$
\begin{aligned}
&[\mathbf{h}^{(2)}_{124,\delta}(x,y,v|u)]_i \, dx \, dy \, dv \\
&= E\left[\begin{array}{c} e^{-\delta T} 1\{N_T \geq 2, U_{T^-} = Z_T \in (x, x+dx)\} \\ 1\{|U_T| \in (y, y+dy), X_T \in (v, v+dv)\} \end{array} \middle| F(0) = u, J^{(F)}(0) = i\right] \ .
\end{aligned}
\tag{6.61}
$$

3. For ruin occurring on claims subsequent to the first, if the claim causing ruin does not occur when the surplus level reaches the maximum before ruin, then there is no simple relationship among the random variables. The discounted joint density of represented by $\mathbf{h}^{(2)}_{\delta}(x,y,z,v|u)$ with the i-th element ($i \in S_1$) given by, on $\{(x,y,z,v) : y > 0, v < x < z, z > u, v < u\}$,

$$
\begin{aligned}
&[\mathbf{h}^{(2)}_{\delta}(x,y,z,v|u)]_i \, dx \, dy \, dz \, dv \\
&= E\left[\begin{array}{c} e^{-\delta T} 1\{N_T \geq 2, U_{T^-} \in (x, x+dx), |U_T| \in (y, y+dy)\} \\ 1\{Z_T \in (z, z+dz), X_T \in (v, v+dv)\} \end{array} \middle| F(0) = u, J^{(F)}(0) = i\right] \ .
\end{aligned}
\tag{6.62}
$$

Now we are ready to derive the discounted joint density of $(U_{T^-}, |U_T|, Z_T, X_T)$ in the MAP risk model. For (6.60), it is easy to see that

$$\mathbf{h}^{(1)}_{12,\delta}(x,y|u) = e^{(\mathbf{Q}_{11}-\delta\mathbf{I})\frac{x-u}{c}} \left(\frac{\mathbf{Q}_{12}}{c}\right) e^{\mathbf{Q}_{22,1}(x+y)}\mathbf{q}_{21,1} \ , \tag{6.63}$$

where $\mathbf{q}_{21,1} = \mathbf{Q}_{21,1}\mathbf{1}$. For (6.61), one can show that

$$\begin{aligned}\mathbf{h}^{(2)}_{124,\delta}(x,y,v|u) = e^{-\frac{\delta(x-u)}{2c}} \ {}_{x-u}\mathbf{\Psi}\left(\frac{\delta}{2}\right) \ {}_{x-v}\widehat{\mathbf{f}}_{22}\left(u-v,0,\frac{\delta}{2}\right)\mathbf{Q}_{21,1} \ {}_{0}\widehat{\mathbf{f}}_{11}\left(0,x-v,\frac{\delta}{2}\right) \\ \times\left(\frac{\mathbf{Q}_{12}}{c}\right)e^{\mathbf{Q}_{22,1}(x+y)}\mathbf{q}_{21,1} \ .\end{aligned} \tag{6.64}$$

The proof of (6.64) is omitted and instead we give the proof for part of the next quantity which is the most complicated. For (6.62), we shall write

$$\mathbf{h}^{(2)}_{\delta}(x,y,z,v|u) = \mathbf{h}^{(2)}_{\max,\delta}(x,y,z,v|u) + \mathbf{h}^{(2)}_{\min,\delta}(x,y,z,v|u) \ , \tag{6.65}$$

where $\mathbf{h}^{(2)}_{\max,\delta}(x,y,z,v|u)$ is the contribution by the case where the maximum is attained before the minimum, and $\mathbf{h}^{(2)}_{\min,\delta}(x,y,z,v|u)$ represents the case where the minimum is reached before the maximum. We have that

$$\begin{aligned}\mathbf{h}^{(2)}_{\max,\delta}(x,y,z,v|u) = e^{-\frac{\delta(x-u)}{2c}} \ {}_{0}\widehat{\mathbf{f}}_{11}\left(u-v,z-v,\frac{\delta}{2}\right)\left(\frac{\mathbf{Q}_{12}}{c}\right) \ {}_{z-v}\widehat{\mathbf{f}}_{22}\left(z-v,0,\frac{\delta}{2}\right)\mathbf{Q}_{21,1} \\ \times \ {}_{0}\widehat{\mathbf{f}}_{11}\left(0,x-v,\frac{\delta}{2}\right)\left[\mathbf{I} - \ {}_{z-x}\mathbf{\Psi}\left(\frac{\delta}{2}\right) \ {}_{x-v}\mathbf{\Psi}^{r}\left(\frac{\delta}{2}\right)\right]^{-1}\left(\frac{\mathbf{Q}_{12}}{c}\right)e^{\mathbf{Q}_{22,1}(x+y)}\mathbf{q}_{21,1} \ ,\end{aligned} \tag{6.66}$$

and

$$\mathbf{h}^{(2)}_{\min,\delta}(x,y,z,v|u) = e^{-\frac{\delta(x-u)}{2c}} \; {}^{z-u}\boldsymbol{\Psi}\left(\frac{\delta}{2}\right) \; {}^{z-v}\widehat{\mathbf{f}}_{22}\left(u-v,0,\frac{\delta}{2}\right)\mathbf{Q}_{21,1}\; {}_0\widehat{\mathbf{f}}_{11}\left(0,z-v,\frac{\delta}{2}\right)\left(\frac{\mathbf{Q}_{12}}{c}\right)$$
$$\times {}^{z-x}\,\widehat{\mathbf{f}}_{22}\left(z-x,0,\frac{\delta}{2}\right) \; {}^{x-v}\boldsymbol{\Psi}^r\left(\frac{\delta}{2}\right)\left[\mathbf{I} - \; {}^{z-x}\boldsymbol{\Psi}\left(\frac{\delta}{2}\right) \; {}^{x-v}\boldsymbol{\Psi}^r\left(\frac{\delta}{2}\right)\right]^{-1}$$
$$\times\left(\frac{\mathbf{Q}_{12}}{c}\right)e^{\mathbf{Q}_{22,1}(x+y)}\mathbf{q}_{21,1}\;. \tag{6.67}$$

Now we give a probabilistic proof for (6.66) via sample paths argument. Recall again that $\mathbf{h}^{(2)}_{\max,\delta}(x,y,z,v|u)$ represents the case where ruin is caused by at least two claims, and the claim causing ruin does not occur when the surplus reaches its maximum. In addition, the maximum occurs before the minimum. In order for the surplus process $\{U_t\}_{t\geq 0}$ to reach the maximum level z from initial surplus u before the minimum level v, the fluid level process $\{F(t)\}_{t\geq 0}$ has to first reach level z from level u avoiding level v enroute, which is represented by the term ${}_0\widehat{\mathbf{f}}_{11}(u-v,z-v,\delta/2)$. Upon reaching the maximum, $\{F(t)\}_{t\geq 0}$ immediately switches from S_1 to S_2 and this accounts for the term $\mathbf{Q}_{12}/c$. Then $\{F(t)\}_{t\geq 0}$ has to reach the minimum level v in S_2 without reaching level z enroute, and this explains the term ${}^{z-v}\widehat{\mathbf{f}}_{22}(z-v,0,\delta/2)$. At the minimum level v, $\{F(t)\}_{t\geq 0}$ switches immediately to S_1, giving rise to the term $\mathbf{Q}_{21,1}$. Now being at minimum level v in S_1, $\{F(t)\}_{t\geq 0}$ has to reach level x, the surplus prior to ruin, in S_1 at least once avoiding level v enroute. This is represented by the term ${}_0\widehat{\mathbf{f}}_{11}(0,x-v,\delta/2)$. At level x in S_1, the fluid level $\{F(t)\}_{t\geq 0}$ further hits level x in S_1 an arbitrary number (≥ 0) of times avoiding the maximum level z and the minimum level v. This is accounted for by the term $[\mathbf{I} - \; {}^{z-x}\boldsymbol{\Psi}(\delta/2) \; {}^{x-v}\boldsymbol{\Psi}^r(\delta/2)]^{-1}$. Being at level x in S_1 for the last time, $\{F(t)\}_{t\geq 0}$ switches from S_1 to S_2 giving rise to $\mathbf{Q}_{12}/c$. Then $e^{\mathbf{Q}_{22,1}(x+y)}\mathbf{q}_{21,1}$ represents a claim of size $x+y$ and hence guarantees a deficit at ruin of exactly size y. We remark that the term $e^{-\delta(x-u)/(2c)}$ is an adjustment term arising when the downward segments of $\{F(t)\}_{t\geq 0}$ is removed to give the surplus process $\{U_t\}_{t\geq 0}$. See, e.g., Ramaswami (2006).

Chapter 7

Further generalizations of the MAP risk model

7.1 Introduction

In this chapter, we propose to further generalize the MAP risk model described in Section 1.2.1. Recall that, in a MAP risk model, there are two types of transitions, namely, type-1 transitions which occur without a claim, and type-2 transitions which occur with an accompanying claim. In what follows, we refer to either type of transition as a *system change*. It is instructive to note that from the construction of the MAP risk model, it is implicitly assumed that the times between two successive system changes are exponentially distributed. More precisely, for $i \in E$, given that the underlying state of the CTMC $\{J_t\}_{t\geq 0}$ at a given time is i, it is assumed that the time until the next system change is exponentially distributed with mean $-1/G_{0,ii}$. In addition, the probability that the system change is a transition of the CTMC $\{J_t\}_{t\geq 0}$ from state i to state j with (without) a claim is given by $-G_{1,ij}/G_{0,ii}$ ($-G_{0,ij}/G_{0,ii}$ for $j \neq i$).

Here, we propose to extend the class of MAP risk models by relaxing the assumptions on the distribution of the time between two system changes. More specifically, we shall assume an arbitrary distribution instead of an exponential one. Thus, such a generalized version of the MAP risk model has the added flexibility of allowing the selection of heavy-tailed distributions (e.g. Pareto or lognormal distribution) for the time between system changes.

We now introduce the risk model of interest in this chapter. Let ϱ_0 be the initial environmental state at time 0 and ϱ_i be the environmental state immediately after the i-th system change. It is assumed that the sequence $\{\varrho_i\}_{i=0}^{\infty}$ is an irreducible and time-homogeneous DTMC on the state space $E = \{1, 2, \ldots, m\}$. The one-period transition probability of the Markov chain $\{\varrho_i\}_{i=0}^{\infty}$ is $\mathbf{G} = \mathbf{Q} + \mathbf{P}$, where $\mathbf{Q} = [q_{ij}]_{i,j=1}^{m}$ and $\mathbf{P} = [p_{ij}]_{i,j=1}^{m}$ with $q_{ij}, p_{ij} \geq 0$ and $(\mathbf{Q} + \mathbf{P})\mathbf{1} = \mathbf{1}$. Note that the one-period transition probability $\mathbf{G}$ has been expressed as $\mathbf{Q} + \mathbf{P}$ given that, as in the MAP risk model, two types of system changes may occur: (1) a change in the environmental process without a claim; or (2) a possible change in the environment process accompanied by a claim. The transition probabilities of those scenarios are respectively contained in the matrices $\mathbf{Q}$ and $\mathbf{P}$. Note that the diagonal elements of $\mathbf{Q}$ are all zero due to the definition of a system change.

In order to express the surplus process $\{U_t\}_{t\geq 0}$ in the form of (1.1), we have to slightly modify the definitions of $\{N_t\}_{t\geq 0}$ (and hence $\{V_i\}_{i=1}^{\infty}$) and $\{Y_i\}_{i=1}^{\infty}$. For $i = 1, 2, \ldots$, let $T_0 = 0$ and T_i be the time of the i-th system change. Then we define $\{N_t\}_{t\geq 0}$ to be counting process of the system changes (instead of the claims), where $N_t = \sup\{i \in \mathbb{N} : T_i \leq t\}$. In addition, the sequence $\{V_i\}_{i=1}^{\infty}$ has to be modified such that $V_1 = T_1$ is the time of the first system change and for $i = 2, 3, \ldots$, $V_i = T_i - T_{i-1}$ is the time elapsed between the $(i-1)$-th and the i-th system changes. For $i = 1, 2, \ldots$, $V_i|\varrho_{i-1} = j$ is assumed to have density $k_j(.)$, c.d.f. $K_j(.)$ and mean κ_j. Furthermore, the sequence $\{Y_i\}_{i=1}^{\infty}$ now is such that for $i = 1, 2, \ldots$, Y_i is the 'claim size' associated with the i-th system change. We assume that $Y_i = 0$ if the i-th system change does not involve a claim while Y_i has density $f_{jk}(.)$, c.d.f. $F_{jk}(y) = 1 - \overline{F}_{jk}(y)$ and mean μ_{jk} if the i-th system change involves a claim and $\varrho_{i-1} = j$ while $\varrho_i = k$ for $j, k \in E$. Conditional on $\{\varrho_i\}_{i=0}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ are all mutually independent. Combining all the above assumptions, it follows that, for $i = 1, 2, \ldots$ and $j, k \in E$,

$$\Pr\{Y_i \le y, V_i \le t, \varrho_i = k | \varrho_{i-1} = j\} = K_j(t)[q_{jk} + p_{jk}F_{jk}(y)] \ , \quad t, y \ge 0 \ . \tag{7.1}$$

The definitions of the time of ruin T, the surplus prior to ruin U_{T^-} and the deficit at ruin $|U_T|$ are the same as in Section 1.1, and N_T is now the number of system changes until ruin.

Remark 17 *It is clear that the generalization of the MAP model described by (7.1) contains the semi-Markovian model described in Section 1.2.3 by letting* $\mathbf{Q}$ *be a zero matrix and assuming that the densities* $f_{jk}(.)$*'s do not depend on* j *for* $j, k \in E$.

Analogous to (1.7) (along with (1.8)), the positive security loading condition for the above model is

$$c \sum_{j=1}^{m} \pi_j \kappa_j > \sum_{j=1}^{m} \pi_j \sum_{k=1}^{m} p_{jk} \mu_{jk} \ , \tag{7.2}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ is the stationary distribution of the Markov chain $\{\varrho_i\}_{i=0}^{\infty}$ satisfying

$$\begin{cases} \boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{G} = \boldsymbol{\pi}(\mathbf{Q} + \mathbf{P}) \ . \\ \boldsymbol{\pi}\mathbf{1} = 1 \ . \end{cases} \tag{7.3}$$

The Gerber-Shiu function $\phi_i(u)$ in our model is given by, for $i \in E$,

$$\phi_{\delta,i}(u) = E\left[e^{-\delta T} w(U_{T^-}, |U_T|) 1\{T < \infty\} | U_0 = u, \varrho_0 = i\right] \ , \quad u \ge 0 \ . \tag{7.4}$$

Whenever the Gerber-Shiu function $\phi_{\delta,i}(u)$ is concerned, we assume either $\delta > 0$ or the positive security loading condition (7.2) holds. In this chapter, we are mainly interested in an important special case of the above Gerber-Shiu function with the choice of penalty function $w(x, y) =$

$e^{-s_1 x} w_2(y)$, i.e. for $i \in E$,

$$\varphi_{\delta,i}(u, s_1) = E\left[e^{-\delta T - s_1 U_{T^-}} w_2(|U_T|) 1\{T < \infty\} | U_0 = u, \varrho_0 = i\right] , \quad u \geq 0 . \tag{7.5}$$

We remark that the above particular case of Gerber-Shiu function has been studied extensively in the context of a Sparre Andersen model with general interclaim time by Willmot (2007) and Landriault and Willmot (2008). This chapter is organized as follows. In Section 7.2, when all the claim sizes are distributed as a combination of exponentials, the form of the solution for the Gerber-Shiu function (7.5) is identified apart from some unknown constants through a pre-assumed discounted joint density. In Section 7.3, the full characterization of (7.5) is obtained by showing that the unknown constants can be solved from a system of linear equations. Section 7.4 is concerned with numerical examples in which two special cases of the Gerber-Shiu function $\varphi_{\delta,i}(u, s_1)$, namely the ruin probability and the expected value of deficit at ruin, are studied.

7.2 Solution form of Gerber-Shiu function

Define $h_{\delta,ijk}(x, y)$ via, for $i, j, k \in E$,

$$\begin{aligned} &h_{\delta,ijk}(x, y)\, dx\, dy \\ &= E\left[e^{-\delta T} 1\{U_{T^-} \in (x, x + dx), |U_T| \in (y, y + dy), \varrho_{N_T - 1} = j, \varrho_{N_T} = k\} | U_0 = 0, \varrho_0 = i\right] , \quad x, y > 0 . \end{aligned} \tag{7.6}$$

Note that $h_{\delta,ijk}(x, y)$ is (a generalization of) the discounted joint density of U_{T^-} and $|U_T|$ with zero initial surplus. This quantity will play a crucial role in determining the solution form of the Gerber-Shiu function (7.5).

For the Gerber-Shiu function $\phi_{\delta,i}(u)$ defined by (7.4), by conditioning on the first drop in surplus and keeping track of the states of the Markov chain $\{\varrho_i\}_{i=0}^{\infty}$, we have that, for $i \in E$,

$$\phi_{\delta,i}(u) = \sum_{j=1}^{m}\sum_{k=1}^{m}\int_0^u \left[\int_0^\infty h_{\delta,ijk}(x,y)\,dx\right]\phi_{\delta,k}(u-y)\,dy \\ + \sum_{j=1}^{m}\sum_{k=1}^{m}\int_u^\infty \left[\int_0^\infty h_{\delta,ijk}(x,y)w(x+u,y-u)\,dx\right]dy\ , \quad u \geq 0\ . \tag{7.7}$$

Note that $h_{\delta,ijk}(x,y)$ can be decomposed as, for $i,j,k \in E$,

$$h_{\delta,ijk}(x,y) = h_{\delta,ij}(x)\frac{p_{jk}f_{jk}(x+y)}{\sum_{l=1}^{m} p_{jl}\overline{F}_{jl}(x)}\ , \quad x,y > 0\ , \tag{7.8}$$

where $h_{\delta,ij}(x)$ is such that, for $i,j, \in E$,

$$h_{\delta,ij}(x)\,dx = E\left[e^{-\delta T}1\{U_{T^-} \in (x,x+dx), \varrho_{N_T-1} = j\}|U_0 = 0, \varrho_0 = i\right]\ , \quad x > 0\ , \tag{7.9}$$

and can be interpreted as the discounted density of U_{T^-}. The probabilistic interpretation of (7.8) is as follows:

- the term $h_{\delta,ij}(x)$ explains that ruin occurs with the surplus prior to ruin U_{T^-} being x and the environmental state just before ruin ϱ_{N_T-1} being j, given zero initial surplus and an initial state of $\varrho_0 = i$;

- being at the surplus level x in state j, to ensure the deficit at ruin $|U_T|$ is y and the environmental state at ruin ϱ_{N_T} is k, a transition from state j to state k has to be made (instantly) with an accompanying claim of size $x+y$. This gives rise to the term $p_{jk}f_{jk}(x+y)$; and

- however, the second event is in fact conditional on the first item which tells us ruin occurs at the next instant (upon the next claim) originating from surplus level x and state j. Therefore the term $p_{jk}f_{jk}(x+y)$ should be further divided by the probability that a claim of size greater than x occurs when the Markov chain $\{\varrho_i\}_{i=0}^{\infty}$ is in state j. Taking into account all the possible destinations of transition from state j, such a probability is easily found to be $\sum_{l=1}^{m} p_{jl}\overline{F}_{jl}(x)$.

For further reference, it is helpful to write (7.8) as, for $i, j, k \in E$,

$$h_{\delta,ijk}(x,y) = h_{\delta,ij}(x)p^*_{jk}(x)\frac{f_{jk}(x+y)}{\overline{F}_{jk}(x)} \ , \quad x, y > 0 \ , \tag{7.10}$$

where, for $j, k \in E$,

$$p^*_{jk}(x) = \frac{p_{jk}\overline{F}_{jk}(x)}{\sum_{l=1}^{m} p_{jl}\overline{F}_{jl}(x)} \ , \quad x > 0 \ , \tag{7.11}$$

accounts for the transition probability from a state prior to ruin j to a state at ruin k given that the surplus prior to ruin is x and the state prior to ruin is j.

By substituting (7.10) into (7.7), one further obtains, for $i \in E$,

$$\begin{aligned}\phi_{\delta,i}(u) = &\sum_{j=1}^{m}\sum_{k=1}^{m}\int_0^u \left[\int_0^\infty h_{\delta,ij}(x)p^*_{jk}(x)\frac{f_{jk}(x+y)}{\overline{F}_{jk}(x)}\,dx\right]\phi_{\delta,k}(u-y)\,dy \\ &+ \sum_{j=1}^{m}\sum_{k=1}^{m}\int_u^\infty \left[\int_0^\infty h_{\delta,ij}(x)p^*_{jk}(x)\frac{f_{jk}(x+y)}{\overline{F}_{jk}(x)}w(x+u, y-u)\,dx\right]dy \ , \quad u \geq 0 \ .\end{aligned} \tag{7.12}$$

For the Gerber-Shiu function $\varphi_{\delta,i}(u, s_1)$ defined by (7.5), (7.12) reduces to, for $i \in E$,

$$\begin{aligned}\varphi_{\delta,i}(u,s_1) = &\sum_{j=1}^{m}\sum_{k=1}^{m}\int_0^u \left[\int_0^\infty h_{\delta,ij}(x)p^*_{jk}(x)\frac{f_{jk}(x+y)}{\overline{F}_{jk}(x)}\,dx\right]\varphi_{\delta,k}(u-y,s_1)\,dy \\ &+ \sum_{j=1}^{m}\sum_{k=1}^{m}\int_u^\infty \left[\int_0^\infty e^{-s_1(x+u)}h_{\delta,ij}(x)p^*_{jk}(x)\frac{f_{jk}(x+y)}{\overline{F}_{jk}(x)}\,dx\right]w_2(y-u)\,dy \ , \quad u \geq 0 \ .\end{aligned} \tag{7.13}$$

In what follows, we further assume that the claim size density $f_{jk}(.)$ is of the form, for $j, k \in E$,

$$f_{jk}(y) = \sum_{l=1}^{n_{jk}} A_{jkl}\beta_{jkl}e^{-\beta_{jkl}y} \ , \quad y > 0 \ . \tag{7.14}$$

Here the parameters $\beta_{jkl} > 0$ for $l = 1, 2, ..., n_{jk}$ are assumed to be distinct for each fixed $j, k \in E$, and moreover $\sum_{l=1}^{n_{jk}} A_{jkl} = 1$. It is known that the class of combinations of exponentials are is dense in the set of distributions defined on $\mathbb{R}^+$. See, e.g., Dufresne (2007) also for the fitting of this class of distributions.

Remark 18 *Recall that in this chapter we are interested in generalizing the MAP risk model to allow for arbitrary distributions between successive system changes. However, in order for full solutions to be obtained, such a generalization cannot be done without making distributional assumptions on the claim sizes (see Landriault and Willmot (2008) and Willmot (2007)). The choice of the class of combinations of exponentials as the claim size distribution is due to its denseness property as well as mathematical tractability. Indeed, the upcoming analysis can also be done if the claim sizes belong to the more general class of finite scale and shape mixture of Erlangs. But this would result in much more lengthy and tedious calculations without gaining additional insights. So we prefer the densities (7.14) for illustrative purposes.*

Obviously, the associated residual lifetime distribution of (7.14) is a different combination of the same exponentials given by, for $j, k \in E$,

$$\frac{f_{jk}(x+y)}{\overline{F}_{jk}(x)} = \sum_{l=1}^{n_{jk}} \varpi_{jkl}(x)\beta_{jkl}e^{-\beta_{jkl}y} \ , \quad x, y > 0 \ , \tag{7.15}$$

where for $l = 1, 2, \ldots, n_{jk}$ and $j, k \in E$,

$$\varpi_{jkl}(x) = \frac{A_{jkl}e^{-\beta_{jkl}x}}{\sum_{z=1}^{n_{jk}} A_{jkz}e^{-\beta_{jkz}x}} \ , \quad x > 0 \ . \tag{7.16}$$

Using (7.15), (7.13) can be rewritten as, for $i \in E$,

$$\begin{aligned}\varphi_{\delta,i}(u,s_1) =& \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{n_{jk}} \left[\int_0^\infty h_{\delta,ij}(x)p^*_{jk}(x)\varpi_{jkl}(x)\,dx\right]\int_0^u \beta_{jkl}e^{-\beta_{jkl}y}\varphi_{\delta,k}(u-y,s_1)\,dy \\ &+ \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{n_{jk}} \left[\int_0^\infty e^{-s_1(x+u)}h_{\delta,ij}(x)p^*_{jk}(x)\varpi_{jkl}(x)\,dx\right]\int_u^\infty \beta_{jkl}e^{-\beta_{jkl}y}w_2(y-u)\,dy\ , \\ & \qquad u \geq 0\ . \qquad (7.17)\end{aligned}$$

Letting, for $l = 1, 2, \ldots, n_{jk}$ and $i, j, k \in E$,

$$\chi_{\delta,ijkl}(s_1) = \int_0^\infty e^{-s_1x}h_{\delta,ij}(x)p^*_{jk}(x)\varpi_{jkl}(x)\,dx\ , \tag{7.18}$$

(7.17) can be reduced to, for $i \in E$,

$$\begin{aligned}\varphi_{\delta,i}(u,s_1) =& \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{n_{jk}} \chi_{\delta,ijkl}(0)\int_0^u \beta_{jkl}e^{-\beta_{jkl}y}\varphi_{\delta,k}(u-y,s_1)\,dy \\ &+ \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{n_{jk}} \chi_{\delta,ijkl}(s_1)\beta_{jkl}e^{-(s_1+\beta_{jkl})u}\widetilde{w}_2(\beta_{jkl})\ , \quad u \geq 0\ . \qquad (7.19)\end{aligned}$$

Taking the Laplace transform on both sides of (7.19) with respect to the argument u, one finds, for $i \in E$,

$$\widetilde{\varphi}_{\delta,i}(s,s_1) = \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{n_{jk}} \chi_{\delta,ijkl}(0)\frac{\beta_{jkl}}{\beta_{jkl}+s}\widetilde{\varphi}_{\delta,k}(s,s_1) + \xi_{\delta,i}(s,s_1)\ , \tag{7.20}$$

where, for $i \in E$,

$$\xi_{\delta,i}(s,s_1) = \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{n_{jk}} \chi_{\delta,ijkl}(s_1)\frac{\beta_{jkl}}{s_1+\beta_{jkl}+s}\widetilde{w}_2(\beta_{jkl})\ . \tag{7.21}$$

The use of matrices allows us to write (7.20) as

$$\widetilde{\boldsymbol{\varphi}}_\delta(s,s_1) = \mathbf{\Gamma}(s)\widetilde{\boldsymbol{\varphi}}_\delta(s,s_1) + \mathbf{\Xi}_\delta(s,s_1)\ , \tag{7.22}$$

implying

$$\widetilde{\boldsymbol{\varphi}}_\delta(s,s_1)=[\mathbf{I}-\boldsymbol{\Gamma}(s)]^{-1}\boldsymbol{\Xi}_\delta(s,s_1)\ , \tag{7.23}$$

where $\widetilde{\boldsymbol{\varphi}}_\delta(s,s_1)=(\widetilde{\varphi}_{\delta,1}(s,s_1),\ldots,\widetilde{\varphi}_{\delta,m}(s,s_1))^T$, $\boldsymbol{\Xi}_\delta(s,s_1)=(\xi_{\delta,1}(s,s_1),\ldots,\xi_{\delta,m}(s,s_1))^T$, and $\boldsymbol{\Gamma}(s)$ is an m-dimensional square matrix with (i,k)-th element given by

$$[\boldsymbol{\Gamma}(s)]_{ik}=\sum_{j=1}^{m}\sum_{l=1}^{n_{jk}}\chi_{\delta,ijkl}(0)\frac{\beta_{jkl}}{\beta_{jkl}+s}\ . \tag{7.24}$$

We remark that the inverse in (7.23) exists because the matrix $\boldsymbol{\Gamma}(s)$ is strictly substochastic (see Section 5.2 for the definition of a strictly substochastic matrix) for $s\geq 0$. To see this, we apply (7.18) at $s_1=0$ to (7.24) to arrive at, for $i\in E$,

$$\sum_{k=1}^{m}[\boldsymbol{\Gamma}(s)]_{ik}=\sum_{j=1}^{m}\sum_{k=1}^{m}\int_0^\infty h_{\delta,ij}(x)p^*_{jk}(x)\left[\sum_{l=1}^{n_{jk}}\varpi_{jkl}(x)\frac{\beta_{jkl}}{\beta_{jkl}+s}\right]dx\ . \tag{7.25}$$

Since the discounted densities $h_{\delta,ij}(x)$'s and the probabilities $p^*_{jk}(x)$'s are non-negative, and for each $j,k\in E$ the term inside the square bracket corresponds to the Laplace transform of the residual lifetime distribution (7.15), one has that, for $i\in E$,

$$\sum_{k=1}^{m}[\boldsymbol{\Gamma}(s)]_{ik}\leq\sum_{j=1}^{m}\sum_{k=1}^{m}\int_0^\infty h_{\delta,ij}(x)p^*_{jk}(x)\,dx\ . \tag{7.26}$$

According to the expressions (7.4) and (7.12) (both with $w(.,.)\equiv 1$ and $u=0$), the right-hand side of the above inequality equals $E\left[e^{-\delta T}1\{T<\infty\}|U_0=0,\varrho_0=i\right]$ which is less than 1 under either $\delta>0$ or the positive security loading condition (7.2). Hence we have proved that $\boldsymbol{\Gamma}(s)$ is strictly substochastic.

Letting

$$H(s)=\prod_{j=1}^{m}\prod_{k=1}^{m}\prod_{l=1}^{n_{jk}}(\beta_{jkl}+s)\ , \tag{7.27}$$

an alternative representation for $\widetilde{\boldsymbol{\varphi}}_{\delta}(s,s_1)$ in (7.23) is given by

$$\widetilde{\boldsymbol{\varphi}}_{\delta}(s,s_1)=\frac{H(s)\,\mathrm{adj}[\mathbf{I}-\boldsymbol{\Gamma}(s)]}{H(s)\det[\mathbf{I}-\boldsymbol{\Gamma}(s)]}\boldsymbol{\Xi}_{\delta}(s,s_1)\ . \tag{7.28}$$

From the definition of $\boldsymbol{\Gamma}(s)$, it is immediate that $H(s)\det[\mathbf{I}-\boldsymbol{\Gamma}(s)]$ is a polynomial in s of degree $n=\sum_{j=1}^{m}\sum_{k=1}^{m}n_{jk}$ with a leading coefficient of 1. It is also easy to conclude that the (i,k)-th element of the matrix $H(s)\,\mathrm{adj}[\mathbf{I}-\boldsymbol{\Gamma}(s)]$, given by $H(s)\,\mathrm{Cof}[\mathbf{I}-\boldsymbol{\Gamma}(s)]_{ki}$, is of the form $[\prod_{j=1}^{m}\prod_{l=1}^{n_{ji}}(\beta_{jil}+s)]L_{ik}(s)$ with $L_{ik}(s)$ a polynomial in s of degree at most $n-\sum_{j=1}^{m}n_{ji}$.

Let $\{\rho_k\}_{k=1}^{n}$ be the n zeros of $H(s)\det[\mathbf{I}-\boldsymbol{\Gamma}(s)]$. From Theorem 1 of Cheung et al. (2010a) (the original paper from which this chapter is adopted), all the n roots $\{\rho_k\}_{k=1}^{n}$ have negative real parts. One can rewrite (7.28) as

$$\widetilde{\boldsymbol{\varphi}}_{\delta}(s,s_1)=\frac{H(s)\,\mathrm{adj}[\mathbf{I}-\boldsymbol{\Gamma}(s)]\boldsymbol{\Xi}_{\delta}(s,s_1)}{\prod_{k=1}^{n}(s-\rho_k)}\ . \tag{7.29}$$

It follows that the i-th element of $\widetilde{\boldsymbol{\varphi}}_{\delta}(s,s_1)$, namely, $\widetilde{\varphi}_{\delta,i}(s,s_1)$, can be expressed as, for $i\in E$,

$$\widetilde{\varphi}_{\delta,i}(s,s_1)=\frac{\sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{l=1}^{n_{jz}}\frac{\beta_{jzl}}{s_1+\beta_{jzl}+s}\widetilde{w}_2(\beta_{jzl})H(s)\left\{\sum_{k=1}^{m}\mathrm{Cof}[\mathbf{I}-\boldsymbol{\Gamma}(s)]_{ki}\ \chi_{\delta,kjzl}(s_1)\right\}}{\prod_{k=1}^{n}(s-\rho_k)}\ . \tag{7.30}$$

For simplicity, it is assumed that for $l=1,2,\ldots,n_{jk}$ and $i,j,k\in E$, the ρ_i's and $-(s_1+\beta_{jkl})$'s are all distinct for a chosen $s_1\geq 0$. Clearly, (7.30) can be expressed via partial fractions as, for $i\in E$,

$$\widetilde{\varphi}_{\delta,i}(s,s_1)=\sum_{k=1}^{n}\frac{\vartheta_{ik}(s_1)}{s-\rho_k}+\sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{l=1}^{n_{jz}}\frac{\eta_{ijzl}(s_1)}{s_1+\beta_{jzl}+s}\ , \tag{7.31}$$

for some constants $\vartheta_{ik}(s_1)$'s and $\eta_{ijzl}(s_1)$ which need to be determined for $l=1,2,\ldots,n_{jz}$; $k=1,2,\ldots,n$ and $i,j,z\in E$. Thus, a Laplace transform inversion of (7.31) with respect to s yields, for $i \in E$,

$$\varphi_{\delta,i}(u,s_1) = \sum_{k=1}^{n} \vartheta_{ik}(s_1) e^{\rho_k u} + \sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{l=1}^{n_{jz}} \eta_{ijzl}(s_1) e^{-(s_1+\beta_{jzl})u} \ , \quad u \geq 0 \ . \tag{7.32}$$

If $s_1 = 0$, it can be proved (see Cheung et al. (2010a, Appendix)) that $\eta_{ijzl}(0) = 0$ for $l = 1, 2, \ldots, n_{jz}$ and $i, j, z \in E$, and hence the solution (7.32) simplifies to give, for $i \in E$,

$$\varphi_{\delta,i}(u,0) = \sum_{k=1}^{n} \vartheta_{ik}(0) e^{\rho_k u} \ , \quad u \geq 0 \ . \tag{7.33}$$

Interested readers are also referred to Landriault and Willmot (2008) and Willmot (2007) where a similar simplification arises.

7.3 Determination of constants

By conditioning on the time of the first system change V_1 and the environmental state ϱ_1 at this time, one finds that, for $i \in E$,

$$\begin{aligned}\phi_{\delta,i}(u) &= \int_0^\infty e^{-\delta t} k_i(t) \left[\sum_{j=1}^{m} q_{ij} \phi_{\delta,j}(u+ct) + \sum_{j=1}^{m} p_{ij} \int_0^{u+ct} f_{ij}(y) \phi_{\delta,j}(u+ct-y)\, dy \right] dt \\ &\quad + \int_0^\infty e^{-\delta t} k_i(t) \left[\sum_{j=1}^{m} p_{ij} \int_{u+ct}^{\infty} f_{ij}(y) w(u+ct, y-u-ct)\, dy \right] dt \ , \quad u \geq 0 \ . \end{aligned} \tag{7.34}$$

For its special case $\varphi_{\delta,i}(u,s_1)$, one easily deduces that, for $i \in E$,

$$\begin{aligned}\varphi_{\delta,i}(u,s_1) &= \int_0^\infty e^{-\delta t} k_i(t) \left[\sum_{j=1}^{m} q_{ij} \varphi_{\delta,j}(u+ct, s_1) + \sum_{j=1}^{m} p_{ij} \int_0^{u+ct} f_{ij}(y) \varphi_{\delta,j}(u+ct-y, s_1)\, dy \right] dt \\ &\quad + e^{-s_1 u} \int_0^\infty e^{-(\delta + c s_1)t} k_i(t) \left[\sum_{j=1}^{m} p_{ij} \int_0^\infty f_{ij}(u+ct+y) w_2(y)\, dy \right] dt \ , \quad u \geq 0 \ . \end{aligned} \tag{7.35}$$

By further assuming that the claim size has density (7.14), the above equation becomes, for $i \in E$,

$$
\begin{aligned}
&\varphi_{\delta,i}(u, s_1) \\
&= \int_0^\infty e^{-\delta t} k_i(t) \left\{ \sum_{j=1}^m q_{ij} \varphi_{\delta,j}(u+ct, s_1) + \sum_{j=1}^m p_{ij} \int_0^{u+ct} \left[\sum_{k=1}^{n_{ij}} A_{ijk} \beta_{ijk} e^{-\beta_{ijk} y} \right] \varphi_{\delta,j}(u+ct-y, s_1)\, dy \right\} dt \\
&\quad + e^{-s_1 u} \int_0^\infty e^{-(\delta+cs)t} k_i(t) \sum_{j=1}^m p_{ij} \sum_{k=1}^{n_{ij}} A_{ijk} \beta_{ijk} e^{-\beta_{ijk}(u+ct)} \left[\int_0^\infty e^{-\beta_{ijk} y} w_2(y)\, dy \right] dt \\
&= \int_0^\infty e^{-\delta t} k_i(t) \left\{ \sum_{j=1}^m q_{ij} \varphi_{\delta,j}(u+ct, s_1) + \sum_{j=1}^m p_{ij} \int_0^{u+ct} \left[\sum_{k=1}^{n_{ij}} A_{ijk} \beta_{ijk} e^{-\beta_{ijk} y} \right] \varphi_{\delta,j}(u+ct-y, s_1)\, dy \right\} dt \\
&\quad + \sum_{j=1}^m \sum_{k=1}^{n_{ij}} p_{ij} A_{ijk} \beta_{ijk} \widetilde{w}_2(\beta_{ijk}) \widetilde{k}_i(\delta + c(s_1 + \beta_{ijk})) e^{-(s_1+\beta_{ijk})u}\ , \quad u \geq 0\ . \qquad (7.36)
\end{aligned}
$$

From now on we shall assume that $s_1 \neq 0$, so that the solution form (7.32) holds true with the coefficients generally being non-zero. The case where $s_1 = 0$ will be treated separately at the end of the section. Then, substitution of (7.32) into (7.36) leads to, for $i \in E$,

$$
\begin{aligned}
&\sum_{l=1}^n \vartheta_{il}(s_1) e^{\rho_l u} + \sum_{x=1}^m \sum_{z=1}^m \sum_{l=1}^{n_{xz}} \eta_{ixzl}(s_1) e^{-(s_1+\beta_{xzl})u} \\
&= \sum_{j=1}^m q_{ij} \int_0^\infty e^{-\delta t} k_i(t) \left[\sum_{l=1}^n \vartheta_{jl}(s_1) e^{\rho_l(u+ct)} + \sum_{x=1}^m \sum_{z=1}^m \sum_{l=1}^{n_{xz}} \eta_{jxzl}(s_1) e^{-(s_1+\beta_{xzl})(u+ct)} \right] dt \\
&\quad + \sum_{j=1}^m p_{ij} \int_0^\infty e^{-\delta t} k_i(t) \left\{ \int_0^{u+ct} \left[\sum_{l=1}^n \vartheta_{jl}(s_1) e^{\rho_l(u+ct-y)} + \sum_{x=1}^m \sum_{z=1}^m \sum_{l=1}^{n_{xz}} \eta_{jxzl}(s_1) e^{-(s_1+\beta_{xzl})(u+ct-y)} \right] \right. \\
&\qquad\qquad \left. \times \left[\sum_{k=1}^{n_{ij}} A_{ijk} \beta_{ijk} e^{-\beta_{ijk} y} \right] dy \right\} dt \\
&\quad + \sum_{z=1}^m \sum_{l=1}^{n_{iz}} p_{iz} A_{izl} \beta_{izl} \widetilde{w}_2(\beta_{izl}) \widetilde{k}_i(\delta + c(s_1 + \beta_{izl})) e^{-(s_1+\beta_{izl})u}\ , \quad u \geq 0\ . \qquad (7.37)
\end{aligned}
$$

Straightforward integrations and grouping of terms give rise to

$$
\begin{aligned}
&\sum_{l=1}^{n} \vartheta_{il}(s_1) e^{\rho_l u} + \sum_{x=1}^{m}\sum_{z=1}^{m}\sum_{l=1}^{n_{xz}} \eta_{ixzl}(s_1) e^{-(s_1+\beta_{xzl})u} \\
&= \sum_{l=1}^{n} \left[\widetilde{k}_i(\delta - c\rho_l) \sum_{j=1}^{m} \left(q_{ij} + p_{ij} \sum_{k=1}^{n_{ij}} \frac{A_{ijk}\beta_{ijk}}{\beta_{ijk}+\rho_l} \right) \vartheta_{jl}(s_1) \right] e^{\rho_l u} \\
&+ \sum_{x=1}^{m}\sum_{z=1}^{m}\sum_{l=1}^{n_{xz}} \left\{ \widetilde{k}_i(\delta + c(s_1+\beta_{xzl})) \sum_{j=1}^{m} \left[q_{ij} + p_{ij} \sum_{k=1}^{n_{ij}} \frac{A_{ijk}\beta_{ijk}}{\beta_{ijk} - (s_1+\beta_{xzl})} \right] \eta_{jxzl}(s_1) \right\} e^{-(s_1+\beta_{xzl})u} \\
&- \sum_{j=1}^{m}\sum_{k=1}^{n_{ij}} \left\{ p_{ij} A_{ijk}\beta_{ijk} \widetilde{k}_i(\delta + c\beta_{ijk}) \left[\sum_{l=1}^{n} \frac{\vartheta_{jl}(s_1)}{\beta_{ijk}+\rho_l} + \sum_{x=1}^{m}\sum_{z=1}^{m}\sum_{l=1}^{n_{xz}} \frac{\eta_{jxzl}(s_1)}{\beta_{ijk} - (s_1+\beta_{xzl})} \right] \right\} e^{-\beta_{ijk}u} \\
&+ \sum_{z=1}^{m}\sum_{l=1}^{n_{iz}} p_{iz} A_{izl}\beta_{izl} \widetilde{w}_2(\beta_{izl}) \widetilde{k}_i(\delta + c(s_1+\beta_{izl})) e^{-(s_1+\beta_{izl})u} \ , \quad u \geq 0 \ . \qquad (7.38)
\end{aligned}
$$

Given that (7.38) holds true for all $u \geq 0$, equating the coefficients of $e^{\rho_l u}$ yields, for $i \in E$ and $l = 1, 2, \ldots, n$,

$$
\vartheta_{il}(s_1) = \widetilde{k}_i(\delta - c\rho_l) \sum_{j=1}^{m} \left(q_{ij} + p_{ij} \sum_{k=1}^{n_{ij}} \frac{A_{ijk}\beta_{ijk}}{\beta_{ijk}+\rho_l} \right) \vartheta_{jl}(s_1) \ . \qquad (7.39)
$$

For each fixed $l = 1, 2, \ldots, n$, (7.39) forms a system of m homogeneous linear equations in $\{\vartheta_{jl}(s_1)\}_{j=1}^{m}$. However, in general $\{\vartheta_{jl}(s_1)\}_{j=1}^{m}$ are not all equal to 0, which means that the above-mentioned system has non-trivial solution. This in turn implies that the coefficient matrix of the system has zero determinant. Thus, we have that, for $l = 1, 2, \ldots, n$,

$$
\det[\mathbf{I} - \boldsymbol{\varsigma}_\delta(\rho_l)] = 0 \ , \qquad (7.40)
$$

where the (i,j)-th element of the $m \times m$ matrix $\boldsymbol{\varsigma}_\delta(s)$ is given by $\widetilde{k}_i(\delta - cs)[q_{ij} + p_{ij}\widetilde{f}_{ij}(s)]$. Equivalently, we can say that $\{\rho_l\}_{l=1}^{n}$ satisfy the generalized Lundberg's equation

$$
\det[\mathbf{I} - \boldsymbol{\varsigma}_\delta(s)] = 0 \ . \qquad (7.41)
$$

According to Cheung et al. (2010a), (7.41) has exactly n roots with negative real parts. Therefore

$\{\rho_l\}_{l=1}^n$ must be these roots by recalling the fact that $H(s)\det[\mathbf{I}-\mathbf{\Gamma}(s)]$ has exactly n zeros (defined by $\{\rho_l\}_{l=1}^n$) and all of them have negative real parts. Since the matrix $\mathbf{\Gamma}(s)$ defined via (7.24) involves the unknown quantity $\chi_{\delta,ijkl}(0)$, $\{\rho_l\}_{l=1}^n$ should be identified as the n zeros of (7.41) with negative real parts.

By equating the coefficients of $e^{-(s_1+\beta_{xzl})u}$ in (7.38), one obtains, for $l=1,2,\ldots,n_{xz}; x,z\in E$ and $i=1,2,\ldots,x-1,x+1,\ldots,m$,

$$\eta_{ixzl}(s_1)=\widetilde{k}_i(\delta+c(s_1+\beta_{xzl}))\sum_{j=1}^m\left[q_{ij}+p_{ij}\sum_{k=1}^{n_{ij}}\frac{A_{ijk}\beta_{ijk}}{\beta_{ijk}-(s_1+\beta_{xzl})}\right]\eta_{jxzl}(s_1)\ , \tag{7.42}$$

and for $l=1,2,\ldots$ and $n_{xz}; x,z\in E$,

$$\begin{aligned}\eta_{xxzl}(s_1)=\widetilde{k}_x(\delta+c(s_1+\beta_{xzl}))\sum_{j=1}^m\left[q_{xj}+p_{xj}\sum_{k=1}^{n_{xj}}\frac{A_{xjk}\beta_{xjk}}{\beta_{xjk}-(s_1+\beta_{xzl})}\right]\eta_{jxzl}(s_1)\\ +p_{xz}A_{izl}\beta_{xzl}\widetilde{w}_2(\beta_{xzl})\widetilde{k}_x(\delta+c(s_1+\beta_{xzl}))\ .\end{aligned} \tag{7.43}$$

Therefore, for each fixed $l=1,2,\ldots$ and $n_{xz}; x,z\in E$, we have got a system of m linear equations ($m-1$ equations from (7.42) and 1 equation from (7.43)) to solve for $\{\eta_{ixzl}(s_1)\}_{i=1}^m$. Note that the linear system has coefficient matrix $\mathbf{I}-\boldsymbol{\varsigma}_\delta(-(s_1+\beta_{xzl}))$ which has non-zero determinant according to our assumption following (7.30) and the fact that $\{\rho_l\}_{l=1}^n$ are the only solutions to (7.41) with negative real parts. This guarantees that the solution $\{\eta_{ixzl}(s_1)\}_{i=1}^m$ to the linear system is unique, and we have a full characterization of all $\eta_{ixzl}(s_1)$'s.

Similarly, equating the coefficients of $e^{-\beta_{ijk}u}$ in (7.38) leads to, for $k=1,2,\ldots,n_{ij}$ and $i,j\in E$,

$$\sum_{l=1}^n\frac{\vartheta_{jl}(s_1)}{\beta_{ijk}+\rho_l}+\sum_{x=1}^m\sum_{z=1}^m\sum_{l=1}^{n_{xz}}\frac{\eta_{jxzl}(s_1)}{\beta_{ijk}-(s_1+\beta_{xzl})}=0\ . \tag{7.44}$$

Interestingly, (7.44) can also be written as, for $k = 1, 2, \ldots, n_{ij}$ and $i, j \in E$,

$$\widetilde{\varphi}_{\delta,j}(-\beta_{ijk}, s_1) = 0 \ . \tag{7.45}$$

Finally, for each fixed $l = 1, 2, \ldots, n$, we remove one of the m equations from (7.39) (i.e. remove from any $i \in E$), knowing that they are linearly dependent from (7.40). This creates $n(m-1)$ equations. These together with the n equations from (7.44) form a system of nm equations to solve for all the $\vartheta_{jl}(s_1)$'s.

As for the case $s_1 = 0$, we recall the simpler solution form (7.33). Omitting the details, similar arguments as in the case for $s_1 \neq 0$ leads to the following conclusion. (7.39) still holds true, and $\{\rho_l\}_{l=1}^n$ are the n roots with negative real parts to the Lundberg's fundamental equation (7.41). Furthermore, we have that, for $k = 1, 2, \ldots, n_{ij}$ and $i, j \in E$,

$$\sum_{l=1}^{n} \frac{\vartheta_{jl}(0)}{\beta_{ijk} + \rho_l} = \widetilde{w}_2(\beta_{ijk}) \ , \tag{7.46}$$

or equivalently,

$$\widetilde{\varphi}_{\delta,j}(-\beta_{ijk}, 0) = \widetilde{w}_2(\beta_{ijk}) \ . \tag{7.47}$$

Again, for each fixed $l = 1, 2, \ldots, n$, we remove one equation from (7.39) from $i \in E$, creating $n(m-1)$ equations. Together with the n equations from (7.46), a system of nm equations is obtained to solve for all the $\vartheta_{jl}(0)$'s.

7.4 Numerical illustrations: Ruin probability and expected deficit

In this section, we illustrate the tractability of the proposed methodology via the study of the ruin probability as well as the expected value of deficit at ruin, which are both special cases of the Gerber-Shiu function $\varphi_{\delta,i}(u,0)$. We would like to study whether the choice of the distributions for the waiting time between two successive system changes have a significant impact on the above two ruin-related quantities.

Let us consider a two-state model with probability matrices of system changes

$$\mathbf{Q} = \begin{pmatrix} 0 & 1/10 \\ 1/5 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{P} = \begin{pmatrix} 4/5 & 1/10 \\ 2/5 & 2/5 \end{pmatrix} . \tag{7.48}$$

Under these assumptions for the Markov chain $\{\varrho_i\}_{i=0}^{\infty}$, the long-run proportions of time in state 1 and 2 are found to be $\pi_1 = 3/4$ and $\pi_2 = 1/4$. Moreover, the claim size densities are assumed to be

$$f_{11}(y) = 2e^{-2y} , \qquad f_{12}(y) = \frac{1}{3}\left(\frac{1}{6}e^{-\frac{1}{6}y}\right) + \frac{2}{3}\left(\frac{2}{3}e^{-\frac{2}{3}y}\right) ,$$

$$f_{21}(y) = e^{-y} \qquad \text{and} \qquad f_{22}(y) = \frac{1}{5}\left(\frac{1}{10}e^{-\frac{1}{10}y}\right) + \frac{4}{5}\left(\frac{1}{5}e^{-\frac{1}{5}y}\right) , \quad y > 0 , \tag{7.49}$$

with means $\mu_{11} = 1/2$, $\mu_{12} = 3$, $\mu_{21} = 1$ and $\mu_{22} = 6$ respectively.

We consider three scenarios in which the distributions for the waiting time between two successive system changes vary. We choose the distribution of the waiting times to be of one of three types: exponential, gamma or Pareto (denoted by EXP(β), GAM(α,β) and PAR(α,θ)) having respective densities

$$k(t)=\beta e^{-\beta t}\ ,\qquad k(t)=\frac{\beta^{\alpha}t^{\alpha-1}e^{-\beta t}}{\Gamma(\alpha)}\qquad\text{and}\qquad k(t)=\frac{\alpha\theta^{\alpha}}{(t+\theta)^{\alpha+1}}\ ,\quad t>0\ . \tag{7.50}$$

The distributional assumptions on the k_1 and k_2 densities for the three scenarios are summarized in Table 7.1 below.

Scenario	Distribution of $k_1(.)$	Distribution of $k_2(.)$
1	EXP$(2/3)$	EXP(2)
2	GAM$(3/2, 1)$	GAM$(5/2, 5)$
3	PAR$(4, 9/2)$	PAR$(7, 3)$

Table 7.1: Distributional assumptions on waiting time densities $k_1(.)$ and $k_2(.)$

All the three $k_1(.)$ ($k_2(.)$) densities have mean $\kappa_1 = 3/2$ ($\kappa_2 = 1/2$). Under Scenarios 1-3, the variances of the $k_1(.)$ densities are 2.25, 1.5 and 4.5 respectively, whereas the variances of the $k_2(.)$ densities are found to be 0.25, 0.1 and 0.35. Finally, a premium rate of $c = 1.5$ is assumed so that the positive security loading condition (7.2) holds.

Note that one could consider Scenario 1, a MAP risk model, to be the baseline scenario. Scenarios 2 and 3 are respective generalizations of the MAP risk models with lower and higher variances for the times between successive system changes in both states. We remark that state 1 of the Markov chain $\{\varrho_i\}_{i=0}^{\infty}$ can be regarded as the 'normal' state while state 2 can be viewed as the 'dangerous' or 'infectious' state. Claims associated with state 1 have smaller mean than those associated with state 2. On the other hand, the mean waiting time until a system change in state 1 is larger than that in state 2, meaning that system changes (and also claims) are less frequent in state 1 than state 2.

First we study the ruin probability $\Pr(T < \infty | U_0 = u, \varrho_0 = i)$ of the surplus process $\{U_t\}_{t\geq 0}$, which is a special case of the Gerber-Shiu function $\varphi_{\delta,i}(u, 0)$ with $\delta = 0$ and $w_2(.) \equiv 1$. In Figure 7.1 below, the ruin probabilities under the three scenarios are plotted against the initial surplus

$U_0 = u$ for both initial states $\varrho_0 = 1$ and $\varrho_0 = 2$.

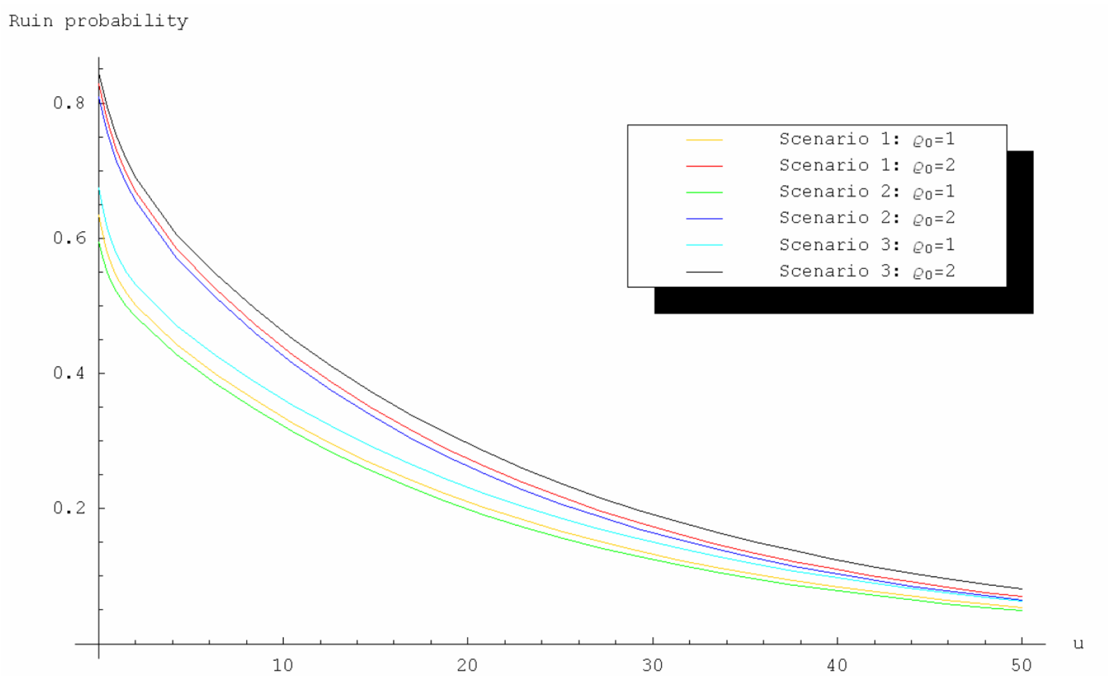

Figure 7.1: Plot of ruin probability under different waiting time distributions

One observes from Figure 7.1 that for a given scenario, the ruin probability with $\varrho_0 = 2$ is always greater than the ruin probability with $\varrho_0 = 1$ for all initial surplus levels, which is expected due to the infectious nature of state 2 of the Markov chain $\{\varrho_i\}_{i=0}^{\infty}$. Furthermore, for each fixed $i = 1, 2$, given $\varrho_0 = i$ the ruin probability increases with the variance of the $k_1(.)$ and $k_2(.)$ densities. This is consistent with an observation made by Landriault and Willmot (2008). Indeed, recall that the mean of the $k_1(.)$ and $k_2(.)$ densities are fixed under all three scenarios. A distribution with a large variance (e.g. Pareto distribution) is more likely to have extremes when compared to a distribution with identical mean but lower variance. Consequently, for a heavy-tailed distribution, a system change (and hence a claim) can occur shortly with a larger probability than a distribution with lighter tail. Thus, under Scenario 3, the occurrence of an early claim will be more likely to result in ruin.

Next, we study the expected deficit at ruin $E\left[|U_T|1\{T < \infty\}|U_0 = u, \varrho_0 = i\right]$ which is a special of $\varphi_{\delta,i}(u,0)$ with $\delta = 0$ and $w_2(y) = y$. A plot of the expected deficit against the initial surplus $U_0 = u$ under the three scenarios in Table 7.1 can be found in Figure 7.2.

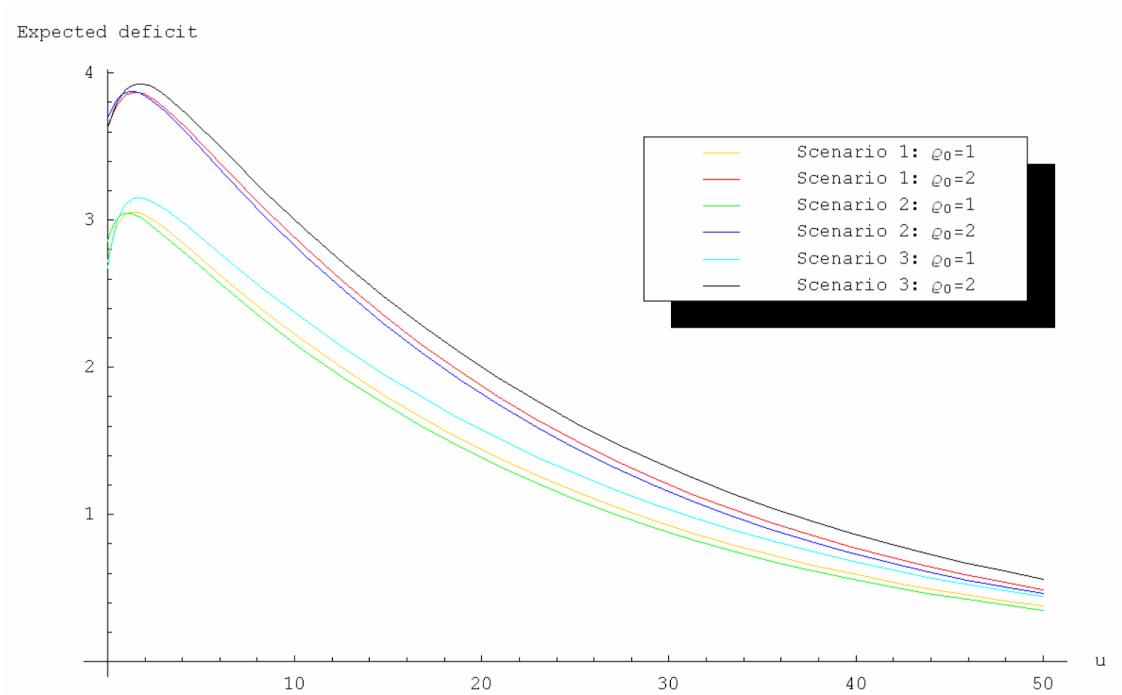

Figure 7.2: Plot of expected deficit under different waiting time distributions

Here again, the choice of the waiting time distributions has a significant impact on the values of the expected deficit at ruin. Also, we observe that the ordering of the six lines in Figure 2 (with the exception of small initial surplus levels) is identical to that for the ruin probability in Figure 7.1.

Chapter 8

Generalized Sparre Andersen risk model with surplus-dependent premium

8.1 Introduction

In this chapter, the generalized Sparre Andersen risk model considered in Section 1.2.2 is revisited, but we would like to further generalize the representation for the surplus process $\{U_t\}_{t\geq 0}$ in (1.1) by allowing the premium rate to depend on the surplus level. The motivation for a *surplus-dependent premium rate* is two-fold. First, as mentioned by Lin and Pavlova (2006), from the insurer's point of view, a higher surplus level allows the insurer to reduce premium to stay competitive. In contrast, in case of low surplus level, the insurer might need to charge a higher premium to avoid the possibility of insufficient funds. Second, from a mathematical point of view, the class of risk models with surplus-dependent premium rate includes a large variety of risk models which may involve dividend strategies and/or interest earnings.

The generalized Sparre Andersen risk model with *surplus-dependent premium rate* is described as follows. The aggregate claims process $\{\sum_{i=1}^{N_t} Y_i\}_{t\geq 0}$ is still defined via the claim number process $\{N_t\}_{t\geq 0}$ (or the sequence of interclaim times $\{V_i\}_{i=1}^{\infty}$) and the sequence of claims $\{Y_i\}_{i=1}^{\infty}$, with $\{(V_i, Y_i)\}_{i=1}^{\infty}$ forming an i.i.d. sequence of bivariate random vectors (see Section 1.2.2). With (V, Y) being a generic pair of (V_i, Y_i), we let $K(t) = 1 - \overline{K}(t) = \Pr\{V \leq t\}$ be the c.d.f. of V for $t \geq 0$. We further assume that $K(t)$ is differentiable and hence V has density $k(t) = K'(t)$. Since V and Y are possibly dependent, it is convenient to specify the joint distribution of (V, Y) by the product of the marginal density $k(t)$ and the conditional density of Y given V. To do so, we define the c.d.f. $P_t(y) = \Pr\{Y \leq y | V = t\} = 1 - \overline{P}_t(y)$ for $y \geq 0$. By assuming that $P_t(y)$ is differentiable in y for each fixed $t > 0$, its corresponding density is $p_t(y) = P_t'(y)$, so that the joint density of (V, Y) at (t, y) is given by $p_t(y)k(t)$. We remark that the traditional Sparre Andersen model can be recovered from the model considered here by assuming that $P_t(y)$ does not depend on t. With the surplus process of the insurance company being denoted by $\{U_t\}_{t\geq 0}$, by *surplus-dependent premium rate* we mean that the instantaneous premium rate at time $t \geq 0$ is assumed to be $c(U_t)$, where $c(.)$ is a positive deterministic function. Therefore $\{U_t\}_{t\geq 0}$ satisfies

$$dU_t = c(U_t)\,dt - d\sum_{i=1}^{N_t} Y_i \ , \quad t \geq 0 \ . \tag{8.1}$$

We further assume $c(.)$ satisfies the two technical conditions

$$\int_0^x \frac{dz}{c(z)} < \infty \ \text{ for any finite } \ x \geq 0 \qquad \text{and} \qquad \int_0^\infty \frac{dz}{c(z)} = \infty \ , \tag{8.2}$$

which will be used later in the derivation. See also Lin and Sendova (2008).

Indeed, apart from the simplest case of constant premium, the class of risk models with surplus-dependent premium rate includes many existing models as special cases. Two important examples are as follows.

- *Multi-threshold risk model*: In a risk model under a multi-threshold dividend strategy (see

Albrecher and Hartinger (2007) and Lin and Sendova (2008)) with n thresholds $\{b_i\}_{i=1}^n$ ($0 < b_1 < b_2 < \ldots < b_n < \infty$), it is assumed that when the surplus is between levels b_{i-1} and b_i, the incoming premium rate (net of any dividends) is c_i for $i = 1, 2, \ldots, n+1$ with $b_0 = 0$ and $b_{n+1} = \infty$. This can be retrieved from the present risk model by letting $c(x) = c_i$ for $b_{i-1} \leq x < b_i$. The case $n = 1$ with represents a (single-)threshold model (see Gerber and Shiu (2006), Lin and Pavlova (2006) and Zhou (2004)).

- *Risk model with credit interest and liquid reserves*: In this risk model, it is assumed that the insurer collects premium at a rate c. Whenever the surplus level is below a fixed threshold level η, the surplus is kept as liquid reserves and does not earn interest. On the other hand, whenever the surplus level exceeds η, the excess of the surplus over η earns credit interest at a rate $\varepsilon > 0$ (see Cai et al. (2009a) and Embrechts and Schmidli (1994)). In our context, we set $c(x) = c$ for $x < \eta$ and $c(x) = c + \varepsilon(x - \eta)$ for $x \geq \eta$. The special case $\eta = 0$ corresponds to the risk model with credit interest only (see, e.g., Cai and Dickson (2002), Gerber and Yang (2007) and Sundt and Teugels (1995)).

Under the above descriptions, we are interested in the generalized Gerber-Shiu function $\phi_\delta(u)$ defined by (1.10) involving the variable R_{N_T-1} which is the surplus immediately after the second last claim before ruin. However, with the premium rate being surplus-dependent, an alternative definition to the sequence $\{R_n\}_{n=0}^\infty$ (instead of (1.9)) is needed to retain the interpretation that R_n represents the surplus level immediately after the n-th claim for $n = 0, 1, \ldots$. To this end, we redefine the sequence $\{R_n\}_{n=0}^\infty$ as follows. With an initial surplus of $U_0 = u$, we suppose that the first claim occurs at some time $t > 0$. Furthermore, let $\gamma(u, s)$ denote the surplus level at time s for $0 \leq s < t$, with the definition that $\gamma(u, t) = \gamma(u, t^-)$ is the surplus level just before the first claim. Then,

$$\gamma(u,t) = u + \int_0^t c(\gamma(u,s))\, ds \ , \quad u \geq 0 \ . \tag{8.3}$$

We define the sequence of $\{R_n\}_{n=0}^{\infty}$ recursively via the function $\gamma(u,t)$ such that $R_0 = U_0 = u$ and

$$R_n = \gamma(R_{n-1}, V_n) - Y_n \ , \quad n = 1, 2, \dots \ . \tag{8.4}$$

Clearly, now we have the correct interpretation for $\{R_n\}_{n=0}^{\infty}$ and hence R_{N_T-1}.

As mentioned in Section 1.3, in the case of constant premium (i.e. $c(.) \equiv c$), the generalized Gerber-Shiu function $\phi_\delta(u)$ defined by (1.10) can be used to study the last interclaim time before ruin $V_{N_T} = (U_{T^-} - R_{N_T-1})/c$. In the current setting of surplus-dependent premium, the last interclaim time V_{N_T} can still be studied via $\phi_\delta(u)$ through the introduction of a new function as follows. With an initial surplus of $U_0 = u$, if $x = \gamma(u,t)$ denotes the surplus level just before the first claim, it can be verified that the time of the first claim must be $t = \vartheta(u,x)$, where

$$\vartheta(u,x) = \int_u^x \frac{dz}{c(z)} \ , \quad x > u \geq 0 \ . \tag{8.5}$$

Then the last interclaim time can be expressed as $V_{N_T} = \vartheta(R_{N_T-1}, U_{T^-})$. We remark that the technical conditions (8.2) are required for (8.5). Indeed, those conditions are required whenever there is a change of variable between surplus level and time unit.

In Section 8.2, a general representation for the generalized Gerber-Shiu function $\phi_\delta(u)$ (involving R_{N_T-1}) is derived in terms of a transition function which is independent of the penalty function. Such a representation is first used in Section 8.3 to derive some ordering properties of the last interclaim time $V_{N_T} = \vartheta(R_{N_T-1}, U_{T^-})$ and the claim causing ruin $Y_{N_T} = U_{T^-} + |U_T|$ in relation to the generic variables V and Y. It turns out that the exact solution of the above-mentioned transition function is not required for such purposes. Since the transition function characterizes the Gerber-Shiu function $\phi_\delta(u)$ itself, Section 8.4 deals with the transition function in more detail. Section 8.5 considers the classical compound Poisson risk model, and we illustrate how the transition function can be fully determined under a threshold dividend strategy or credit interest.

8.2 Gerber-Shiu function $\phi_\delta(u)$ involving R_{N_T-1}

As in Sections 4.5 and 5.2 (see also Cheung et al. (2010b,c)), we begin by introducing the joint distribution of the quadruple $(T, U_{T^-}, |U_T|, R_{N_T-1})$. According to the way the function $\vartheta(u, x)$ in (8.5) is defined, with an initial surplus of $U_0 = u$, if ruin occurs on the first claim, there is a one-to-one relationship between U_{T^-} and T given by $T = \vartheta(u, U_{T^-})$, and additionally $R_{N_T-1} = u$. Thus, it is sufficient to specify the joint distribution of $(U_{T^-}, |U_T|)$ at (x, y) for ruin upon the first claim. In order to have a deficit of $|U_T| = y$ after reaching level $U_{T^-} = x$, the first claim has to be of size $x + y$. By applying the joint density of (V_1, Y_1) (with a change of variable), such a joint (defective) density of $(U_{T^-}, |U_T|)$ is given by

$$h_1^*(x, y|u) = \frac{1}{c(x)} k(\vartheta(u, x)) p_{\vartheta(u,x)}(x + y) \ , \quad x > u \geq 0; y > 0 \ . \tag{8.6}$$

On the other hand, if ruin occurs on claims subsequent to the first, T and R_{N_T-1} are no longer simple functions of U_{T^-} and $|U_T|$, and we denote the joint (defective) density of $(T, U_{T^-}, |U_T|, R_{N_T-1})$ at (t, x, y, v) given $U_0 = u$ by $h_2^*(t, x, y, v|u)$, for $t, y > 0; x > v > 0; u \geq 0$. Then the discounted joint densities corresponding to $h_1^*(x, y|u)$ and $h_2^*(t, x, y, v|u)$ are given by

$$h_{1,\delta}^*(x, y|u) = e^{-\delta\vartheta(u,x)} h_1^*(x, y|u) \ , \quad x > u \geq 0; y > 0 \ , \tag{8.7}$$

and

$$h_{2,\delta}^*(x, y, v|u) = \int_0^\infty e^{-\delta t} h_2^*(t, x, y, v|u)\, dt \ , \quad y > 0; x > v > 0; u \geq 0 \ , \tag{8.8}$$

respectively.

In light of the relationship (5.72) in a semi-Markovian risk model, we would like to argue probabilistically that the discounted densities $h^*_{1,\delta}(.,.|.)$ and $h^*_{2,\delta}(.,.,.|.)$ are related by

$$h^*_{2,\delta}(x,y,v|u) = \tau_\delta(u,v)h^*_{1,\delta}(x,y|v) \ , \quad y>0; x>v>0; u\geq 0 \ , \tag{8.9}$$

where $\tau_\delta(.,.)$ is a non-negative transition function. In order for ruin to occur upon at least two claims and the surplus level after the second last claim before ruin to be v (as $h^*_{2,\delta}(x,y,v|u)$ suggests), the surplus process $\{U_t\}_{t\geq 0}$, starting with initial surplus $U_0 = u$, has to first make a transition from level u to level v after an arbitrary number (≥ 1) of claims. Such a transition is explained by the term $\tau_\delta(u,v)$. After reaching level v, the process restarts, and if the next claim causes ruin according to $h^*_{1,\delta}(x,y|v)$, then the triplet $(U_{T^-}, |U_T|, R_{N_T-1})$ will be exactly (x,y,v). Similar probabilistic interpretation has also been made at the end of Section 5.4. A proof of (8.9) (accompanied by a formal definition of $\tau_\delta(u,v)$) is given in Section 8.4.

Remark 19 *It is interesting to note that expression in the form of (8.9) is also evident in matrix form from (4.34) - (4.37) when $(V, Y/c)$ follows a bivariate phase-type distribution.*

An (almost) immediate consequence of (8.9) is a general representation for the generalized Gerber-Shiu function $\phi_\delta(u)$ defined by (1.10) which is central to the analysis in the remainder of this chapter. As in (5.70), since $\phi_\delta(u)$ is an expectation of a discounted penalty, it can simply be written as an integral of the penalty function with respect to the discounted densities $h^*_{1,\delta}(.,.|u)$ and $h^*_{2,\delta}(.,.,.|u)$ as

$$\begin{aligned}
\phi_\delta(u) &= \int_0^\infty \int_u^\infty w(x,y,u)h^*_{1,\delta}(x,y|u)\,dx\,dy + \int_0^\infty \int_0^\infty \int_v^\infty w(x,y,v)h^*_{2,\delta}(x,y,v|u)\,dx\,dy\,dv \\
&= \int_0^\infty \int_u^\infty w(x,y,u)h^*_{1,\delta}(x,y|u)\,dx\,dy + \int_0^\infty \tau_\delta(u,v) \int_0^\infty \int_v^\infty w(x,y,v)h^*_{1,\delta}(x,y|v)\,dx\,dy\,dv \\
&= \beta_\delta(u) + \int_0^\infty \tau_\delta(u,v)\beta_\delta(v)\,dv \ , \quad u\geq 0 \ ,
\end{aligned} \tag{8.10}$$

where the second last line follows from the substitution of (8.9), and $\beta_\delta(u)$ is defined by

$$\beta_\delta(u) = \int_0^\infty \int_u^\infty w(x,y,u) h^*_{1,\delta}(x,y|u)\, dx\, dy \ , \quad u \geq 0 \ . \tag{8.11}$$

It is clear from (8.11) that $\beta_\delta(u)$ is the contribution by ruin upon the first claim (see also (5.69)). Thus, the representation (8.10) is again an intuitive result complementing (5.67). We omit its probabilistic interpretation here since it is essentially identical to that of (5.67).

Note that the representation (8.10) holds true very generally, as it has been mentioned in Section 8.1 that the class of risk models with surplus-dependent premium considered in this chapter contains various risk models under dividend strategies and/or credit interest. The advantage of such a representation is that the dependence of $\phi_\delta(u)$ on the penalty function $w(.,.,.)$ only appears through $\beta_\delta(u)$, which is explicitly given by (8.11) (since $h^*_{1,\delta}(x,y|u)$ is known from (8.6) and (8.7)). It is clear from (8.10) that the generalized Gerber-Shiu function $\phi_\delta(u)$ can be characterized by the transition function $\tau_\delta(u,v)$, which is independent of the choice of penalty $w(.,.,.)$. Once $\tau_\delta(u,v)$ is determined, $\phi_\delta(u)$ follows accordingly. However, as mentioned earlier, as far as the ordering properties in the next section are concerned, solution for $\tau_\delta(u,v)$ is not required. Therefore we shall delay the formal definition of $\tau_\delta(u,v)$ as well as its evaluation under certain examples to Sections 8.4 and 8.5.

8.3 Ordering properties of ruin-related quantities

Since we are interested in the marginal distributions of the last interclaim time $V_{N_T} = \vartheta(R_{N_T-1}, U_{T^-})$ and the claim causing ruin $Y_{N_T} = U_{T^-} + |U_T|$, throughout this section we assume $\delta = 0$. Then using (8.6) followed by a change of variable $t = \vartheta(u, x)$, (8.11) becomes

$$\begin{aligned}\beta_0(u) &= \int_0^\infty \int_u^\infty w(x, y, u) \left[\frac{1}{c(x)} k(\vartheta(u, x)) p_{\vartheta(u,x)}(x + y) \right] dx\, dy \\ &= \int_0^\infty \int_{\gamma(u,t)}^\infty w(\gamma(u, t), y - \gamma(u, t), u) k(t) p_t(y)\, dy\, dt \ , \quad u \geq 0 \ . \end{aligned} \tag{8.12}$$

8.3.1 Last interclaim time $V_{N_T} = \vartheta(R_{N_T - 1}, U_{T^-})$

In this subsection, we aim at comparing the proper distribution of the last interclaim time before ruin with the distribution of a generic interclaim time V. To this end, we consider the Gerber-Shiu function $\phi_0(u)$ with penalty function $w(x, y, v) = e^{-s\vartheta(v,x)}$. For this choice of penalty function, (8.12) becomes

$$\beta_0(u) = \int_0^\infty \int_{\gamma(u,t)}^\infty e^{-s\vartheta(u,\gamma(u,t))} k(t) p_t(y)\, dy\, dt = \int_0^\infty e^{-st} k(t) \overline{P}_t(\gamma(u, t))\, dt \ , \quad u \geq 0 \ , \tag{8.13}$$

where the last equality follows from the definition (8.5) that $\vartheta(u, \gamma(u, t)) = t$. Therefore, (8.10) (with $\delta = 0$) becomes

$$E\left[e^{-sV_{N_T}} 1\{T < \infty\} | U_0 = u\right] = \int_0^\infty e^{-st} k(t) \left[\overline{P}_t(\gamma(u, t)) + \int_0^\infty \tau_0(u, v) \overline{P}_t(\gamma(v, t))\, dv \right] dt \ , \quad u \geq 0 \ . \tag{8.14}$$

Let $g_V(t|u)$ be the (proper) density of $(V_{N_T} | T < \infty)$ for an initial surplus of $U_0 = u$. One concludes that

$$g_V(t|u) = a_u(t) k(t) \ , \quad t > 0; u \geq 0 \ , \tag{8.15}$$

where

$$a_u(t) = \frac{1}{\psi(u)} \left[\overline{P}_t(\gamma(u, t)) + \int_0^\infty \tau_0(u, v) \overline{P}_t(\gamma(v, t))\, dv \right] \ , \quad t > 0; u \geq 0 \ , \tag{8.16}$$

and $\psi(u) = \Pr\{T < \infty | U_0 = u\}$ is the ruin probability.

From (8.15), we observe that if $a_u(t)$ is decreasing (i.e. non-increasing) in t for each fixed $u \geq 0$, then $(V_{N_T}|T < \infty)$ is smaller than a generic interclaim time variable V in likelihood ratio order, i.e.

$$(V_{N_T}|T < \infty) \leq_{LR} V \ , \tag{8.17}$$

for any initial surplus $U_0 = u \geq 0$. Define $\overline{G}_V(t|u) = \int_t^\infty g_V(x|u)\,dx$. (8.17) implies (see, e.g., Denuit et al. (2005))

$$\overline{G}_V(t|u) \leq \overline{K}(t) \ , \quad t > 0; u \geq 0 \ , \tag{8.18}$$

i.e. $(V_{N_T}|T < \infty)$ is stochastically smaller than V for any initial surplus $u \geq 0$. The reader is also referred to Schmidli (2010) for the conditional law of various risk processes given ruin occurs.

A closer look at (8.16) reveals that a sufficient condition for $a_u(t)$ to be decreasing in t for each fixed $u \geq 0$ (and hence the orderings (8.17) and (8.18)) is that $\overline{P}_t(\gamma(v,t))$ is decreasing in t for each fixed $v \geq 0$. Such a sufficient condition might not be that easy to check directly. Fortunately, it is obvious from (8.3) that $\gamma(v,t)$ is increasing (i.e. non-decreasing) in t for each fixed $v \geq 0$, and therefore a sufficient condition for all the above to hold true is that $\overline{P}_t(y)$ is decreasing in t for each fixed $y \geq 0$. We also remark that the above sufficient condition automatically holds true in the traditional Sparre Andersen model where $\{V_i\}_{i=1}^\infty$ and $\{Y_i\}_{i=1}^\infty$ are independent. In particular, Cheung et al. (2010b,c) respectively showed that (8.18) holds true when either the interclaim time or the claim size is exponentially distributed.

In the literature, the condition that $\overline{P}_t(y)$ is decreasing in t for each fixed $y \geq 0$ is well-documented, and it is equivalent to saying that the r.v. Y is *stochastically decreasing* in V, denoted by $SD(Y|V)$. The condition $SD(Y|V)$ is indeed a form of negative association for the pair (V, Y) which implies $\text{Cov}(V, Y) \leq 0$ (see, e.g., Joag-Dev and Proschan (1983), Lehmann (1966) and Shaked (1977)). In fact, the negative dependence between V and Y under the condition $SD(Y|V)$ explains intuitively why (8.18) holds true. A negative dependence means that a short interclaim time is likely to result in a large claim. In other words, insufficient premium income since the previous claim is likely to be followed by a large claim. Therefore, ruin is likely to be accompanied by a relatively short interclaim time and this explains (8.18).

Next we illustrate an example in which the sufficient condition $SD(Y|V)$ holds true. Consider the dependency structure proposed by Boudreault et al. (2006), i.e.

$$p_t(y) = e^{-\kappa t} f_1(y) + (1 - e^{-\kappa t}) f_2(y) \ , \quad t, y > 0 \ , \tag{8.19}$$

where $\kappa \geq 0$ is a dependence parameter and $f_i(.)$ $(i = 1, 2)$ is a proper density function with corresponding survival function $\overline{F}_i(.)$. If

$$\overline{F}_1(y) \geq \overline{F}_2(y) \ , \quad y \geq 0 \ , \tag{8.20}$$

then it is easy to see that $\overline{P}_t(y)$ is a decreasing function of t for each fixed $y \geq 0$, i.e. $SD(Y|V)$ holds. Thus, (8.20) is a sufficient condition for the ordering (8.17) (and hence (8.18)) to hold.

Remark 20 *One could only expect the upper bound for the survival function of the last interclaim time given by (8.18) to be a weak one. In cases where V and Y are independent, improved bound and/or two-sided bounds might be obtained depending on the reliability of the interclaim time distribution. See Cheung et al. (2010d) for details.*

8.3.2 Claim causing ruin $Y_{N_T} = U_{T^-} + |U_T|$

We now consider a penalty function of the form $w(x, y, v) = e^{-s(x+y)}$ which leads to the Laplace transform of the claim causing ruin Y_{N_T}. Hence, (8.12) can be reduced to, by a change of order of integration,

$$\beta_0(u) = \int_0^\infty \int_{\gamma(u,t)}^\infty e^{-sy} k(t) p_t(y) \, dy \, dt = \int_u^\infty e^{-sy} \left[\int_0^{\vartheta(u,y)} k(t) p_t(y) \right] dt \ , \quad u \geq 0 \ . \tag{8.21}$$

Thus, substitution into (8.10) (with $\delta = 0$) yields

$$
\begin{aligned}
&E\left[e^{-sY_{N_T}}1\{T<\infty\}|U_0=u\right] \\
&= \int_u^\infty e^{-sy}\left[\int_0^{\vartheta(u,y)} k(t)p_t(y)\,dt\right] dy + \int_0^\infty \tau_0(u,v)\int_v^\infty e^{-sy}\left[\int_0^{\vartheta(v,y)} k(t)p_t(y)\,dt\right] dy\,dv\ , \quad u \geq 0\ .
\end{aligned}
\tag{8.22}
$$

With a slight abuse of notation, use of Bayes theorem yields

$$
p_t(y)k(t) = p(y)k_y(t)\ , \quad t, y > 0\ , \tag{8.23}
$$

where $p(.)$ is the marginal density of the generic claim size Y and $k_y(t)$ is the conditional density of $(V|Y=y)$ at t. Substituting (8.23) into (8.22) followed by some simple manipulations leads to

$$
\begin{aligned}
&E\left[e^{-sY_{N_T}}1\{T<\infty\}|U_0=u\right] \\
&= \int_u^\infty e^{-sy}p(y)K_y(\vartheta(u,y))\,dy + \int_0^\infty \tau_0(u,v)\int_v^\infty e^{-sy}p(y)K_y(\vartheta(v,y))\,dy\,dv \\
&= \int_0^\infty e^{-sy}p(y)\left\{K_y(\vartheta(u,y)) + \int_0^y K_y(\vartheta(v,y))\tau_0(u,v)\,dv\right\} dy\ , \quad u \geq 0\ ,
\end{aligned}
\tag{8.24}
$$

where $K_y(t) = \int_0^t k_y(x)\,dx$ is the c.d.f. of $(V|Y=y)$ at t. It follows that the (proper) density of $(Y_{N_T}|T<\infty)$ at y for an initial surplus of $U_0 = u$, denoted by $g_Y(y|u)$, is

$$
g_Y(y|u) = \frac{1}{\psi(u)}b_u(y)p(y)\ , \quad y > 0; u \geq 0\ , \tag{8.25}
$$

where

$$
b_u(y) = \frac{1}{\psi(u)}\left[K_y(\vartheta(u,y)) + \int_0^y K_y(\vartheta(v,y))\tau_0(u,v)\,dv\right]\ , \quad y > 0; u \geq 0\ . \tag{8.26}
$$

Similar to the case for the last interclaim time $(V_{N_T}|T<\infty)$, one observes that if $b_u(y)$ is increasing in y for each fixed $u \geq 0$, then $(Y_{N_T}|T<\infty)$ is larger than a generic claim size r.v. Y in likelihood ratio order, i.e.

$$(Y_{N_T}|T < \infty) \geq_{LR} Y \ , \tag{8.27}$$

for any initial surplus $U_0 = u \geq 0$, which again implies stochastic ordering. Examination of (8.26) reveals that a sufficient condition for $b_u(y)$ to be increasing in y for each fixed $u \geq 0$ is that $K_y(\vartheta(v, y))$ is increasing in y for each fixed $v \geq 0$. This in turn has sufficient condition that $K_y(t)$ is increasing in y for each fixed $t \geq 0$, since from (8.5) $\vartheta(v, y)$ is increasing in y for each fixed $v \geq 0$. The latter condition is equivalent to $SD(V|Y)$, i.e. V is stochastically decreasing in Y. The same probabilistic interpretation as for the ordering (8.17) or (8.18) applies. Again the above-mentioned sufficient condition automatically holds true in the traditional Sparre Andersen model.

Parallel to the example given at the end of Section 8.3.1, we consider a reverse dependency structure compared to Boudreault et al. (2006) with

$$k_y(t) = e^{-\kappa^* y} g_1(t) + (1 - e^{-\kappa^* y}) g_2(t) \ , \quad t, y > 0 \ , \tag{8.28}$$

where $\kappa^* \geq 0$ and $g_i(.)$ $(i = 1, 2)$ is a (proper) density function with survival function $\overline{G}_i(.)$. In this dependent risk model, a sufficient condition for the ordering (8.27) to hold is

$$\overline{G}_1(t) \geq \overline{G}_2(t) \ , \quad t \geq 0 \ .$$

8.4 More on the transition function $\tau_\delta(u, v)$

As mentioned at the end of Section 8.2, the generalized Gerber-Shiu function $\phi_\delta(u)$ is characterized by the transition function $\tau_\delta(u, v)$ via the representation (8.10). This section aims at providing a formal definition of $\tau_\delta(u, v)$ which forms a basis for its evaluation. To do so, we provide a heuristic proof of (8.9). Using the Dirac delta function $\Delta(x)$ heuristically defined as

$$\Delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \tag{8.29}$$

satisfying $\int_{-\infty}^{+\infty} \Delta(x)\, dx = 1$, the discounted densities (8.7) and (8.8) admit the representations

$$\begin{aligned} h^*_{1,\delta}(x,y|u) &= E\left[e^{-\delta T}\Delta(U_{T^-}-x)\Delta(|U_T|-y)1\{N_T=1\}\middle|U_0=u\right] \\ &= E\left[e^{-\delta V_1}\Delta(\gamma(u,V_1)-x)\Delta(Y_1-(x+y))\right], \quad x>u\geq 0; y>0, \end{aligned} \tag{8.30}$$

and

$$\begin{aligned} h^*_{2,\delta}(x,y,v|u) = E\left[e^{-\delta T}\Delta(U_{T^-}-x)\Delta(|U_T|-y)\Delta(R_{N_T-1}-v)1\{N_T>1\}|U_0=u\right], \\ y>0; x>v>0; u\geq 0, \end{aligned} \tag{8.31}$$

respectively (see also (4.32) and (4.33)). Note that by conditioning on the number of claims causing ruin, $h^*_{2,\delta}(x,y,v|u)$ can be expressed as

$$\begin{aligned} h^*_{2,\delta}(x,y,v|u) &= \sum_{n=1}^{\infty} E\left[e^{-\delta T}\Delta(U_{T^-}-x)\Delta(|U_T|-y)\Delta(R_{N_T-1}-v)1\{N_T=n+1\}\middle|U_0=u\right] \\ &= \sum_{n=1}^{\infty} E\left[\begin{array}{c} e^{-\delta\sum_{j=1}^{n+1}V_j}\Delta(\gamma(R_n,V_{n+1})-x)\Delta(|\gamma(R_n,V_{n+1})-Y_{n+1}|-y) \\ \Delta(R_n-v)1\{N_T=n+1\} \end{array}\middle|U_0=u\right] \\ &= \sum_{n=1}^{\infty} E\left[\begin{array}{c} e^{-\delta\sum_{j=1}^{n+1}V_j}\Delta(\gamma(v,V_{n+1})-x)\Delta(Y_{n+1}-(x+y))\Delta(R_n-v) \\ 1\{R_i\geq 0, i=1,2,\ldots,n\} \end{array}\middle|U_0=u\right] \\ &= \sum_{n=1}^{\infty} E\left[e^{-\delta\sum_{j=1}^{n}V_j}\Delta(R_n-v)1\{R_i\geq 0, i=1,2,\ldots,n\}\middle|U_0=u\right] \\ &\quad \times E\left[e^{-\delta V_{n+1}}\Delta(\gamma(v,V_{n+1})-x)\Delta(Y_{n+1}-(x+y))\right], \quad y>0; x>v>0; u\geq 0, \end{aligned} \tag{8.32}$$

where the last line follows from the independence of (V_{n+1}, Y_{n+1}) on $\{(V_i, Y_i)\}_{i=1}^n$ together with the fact that for $n = 1, 2, \ldots,$ R_n only depends on $\{(V_i, Y_i)\}_{i=1}^n$ and $R_0 = U_0$. By further using the fact that (V_{n+1}, Y_{n+1}) has identical distribution as (V_1, Y_1), application of (8.30) to (8.32) leads to (8.9) with $\tau_\delta(u, v)$ given by

$$\tau_\delta(u,v) = \sum_{n=1}^{\infty} E\left[e^{-\delta \sum_{j=1}^n V_j} \Delta(R_n - v) 1\{R_i \geq 0, i = 1, 2, \ldots, n\} \Big| U_0 = u\right] \ , \quad v > 0; u \geq 0 \ . \tag{8.33}$$

Having obtained the formal definition of $\tau_\delta(u, v)$, we also outline the procedure of its determination as follows. The determination of $\tau_\delta(u, v)$ can be done through its Laplace transform (with respect to the argument v) defined by

$$\varphi_{\delta,r}(u) = \int_0^\infty e^{-rv} \tau_\delta(u, v) \, dv \ , \quad u \geq 0 \ , \tag{8.34}$$

which by substitution of (8.33) yields

$$\varphi_{\delta,r}(u) = \sum_{n=1}^{\infty} E\left[e^{-\delta \sum_{j=1}^n V_j} e^{-rR_n} 1\{R_i \geq 0, i = 1, 2, \ldots, n\} \Big| U_0 = u\right] \ , \quad u \geq 0 \ . \tag{8.35}$$

Then, $\varphi_{\delta,r}(u)$ can obtained by conditioning on the time t and the amount y of the first claim as

$$\varphi_{\delta,r}(u) = \int_0^\infty e^{-\delta t} \left\{ \int_0^{\gamma(u,t)} \left[e^{-r(\gamma(u,t)-y)} + \varphi_{\delta,r}(\gamma(u,t) - y)\right] p_t(y) \, dy \right\} k(t) \, dt \ . \tag{8.36}$$

Note that there is only contribution to $\varphi_{\delta,r}(u)$ if the first claim does not cause ruin. In such a case, the term $e^{-r(\gamma(u,t)-y)}$ is due to the process reaching level $\gamma(u, t) - y$ immediately after the first claim, while $\varphi_{\delta,r}(\gamma(u, t) - y)$ represents the future contribution to $\varphi_{\delta,r}(u)$ with the process restarting at level $\gamma(u, t) - y$. The integral equation (8.36) is usually solved by making additional distributional assumptions on the claim size and/or the interclaim time. In contrast, if we condition on the time and the amount of the first claim to directly get an integral equation for

$\phi_\delta(u)$, we do not expect that such an integral equation can be solved easily without making any further assumption on the form and/or differentiability of the penalty function $w(.,.,.)$ in cases where $c(.)$ is not constant.

To illustrate the generality of our approach in studying risk models with surplus-dependent premium, the next section considers the classical compound Poisson risk model under a threshold dividend strategy or credit interest. We refer the interested readers to Cheung et al. (2010b), Willmot and Woo (2010) and Woo (2010) for various (special cases of) generalized Sparre Andersen risk models in which $\tau_\delta(u,v)$ was obtained explicitly when $c(.)$ is constant.

8.5 Example: Classical compound Poisson risk model

In this entire section, we assume a classical compound Poisson risk model, i.e., $k(t) = \lambda e^{-\lambda t}$ and $p_t(y) = p(y)$. Under these assumptions, a change of variable $x = \gamma(u,t)$ in (8.36) followed by differentiation with respect to u leads to the integro-differential equation

$$\varphi'_{\delta,r}(u) = \frac{\lambda+\delta}{c(u)}\varphi_{\delta,r}(u) - \frac{\lambda}{c(u)}\int_0^u \varphi_{\delta,r}(u-y)p(y)\,dy - \frac{\lambda}{c(u)}\int_0^u e^{-r(u-y)}p(y)\,dy\ , \quad u \geq 0\ . \quad (8.37)$$

Next, we assume specific form of $c(.)$ in order to solve (8.37) for $\varphi_{\delta,r}(u)$ (and hence $\tau_\delta(u,v)$).

8.5.1 Dividend threshold

Under a threshold dividend strategy with fixed threshold level $b > 0$, whenever the surplus process $\{U_t\}_{t\geq 0}$ is above b (and ruin has not occurred), dividend is payable to the shareholders at rate $\alpha > 0$ out of the constant premium rate c received from the policyholders, otherwise no dividend is paid. According to Section 8.1, the (single-)threshold model can be retrieved from our present model with surplus-dependent premium by letting

$$c(x) = \begin{cases} c \ , & 0 \leq x < b \ . \\ c - \alpha \ , & x \geq b \ . \end{cases} \tag{8.38}$$

The positive security loading condition under the dividend threshold model is $\alpha < c - \lambda E[Y]$, and will be assumed here.

To emphasize the dependence of $\varphi_{\delta,r}(u)$ (and $\tau_\delta(u,v)$) on the threshold level b, we shall write $\varphi_{\delta,r}(u;b)$ instead of $\varphi_{\delta,r}(u)$ with $\varphi_{\delta,r}(u;b) = \int_0^\infty e^{-rv}\tau_\delta(u,v;b)\,dv$. Note also that $\varphi_{\delta,r}(u;b)$ is of different functional form depending on whether $0 \leq u < b$ or $u \geq b$, and therefore we shall write

$$\varphi_{\delta,r}(u;b) = \begin{cases} \varphi_{\delta,r,1}(u;b) = \int_0^\infty e^{-rv}\tau_{\delta,1}(u,v;b)\,dv \ , & 0 \leq u < b \ . \\ \varphi_{\delta,r,2}(u;b) = \int_0^\infty e^{-rv}\tau_{\delta,2}(u,v;b)\,dv \ , & u \geq b \ . \end{cases} \tag{8.39}$$

For later use, it will be convenient to denote the corresponding $\varphi_{\delta,r}$ function in a threshold-free model (i.e. $c(.) \equiv c$) by $\varphi_{\delta,r}(u;\infty) = \int_0^\infty e^{-rv}\tau_\delta(u,v;\infty)\,dv$. We remark that the quantity $\tau_\delta(u,v;\infty)$ has been given by Eq. (18) Cheung et al. (2010b), and therefore we shall regard such a quantity as known in the upcoming analysis.

Under our choice of $c(.)$ in (8.38), the integro-differential equation (8.37) is now expressed in a piecewise manner as

$$\varphi'_{\delta,r,1}(u;b) = \frac{\lambda+\delta}{c}\varphi_{\delta,r,1}(u;b) - \frac{\lambda}{c}\int_0^u \varphi_{\delta,r,1}(u-y;b)p(y)\,dy - \frac{\lambda}{c}\int_0^u e^{-r(u-y)}p(y)\,dy \ , \quad 0 \leq u < b \ , \tag{8.40}$$

and

$$\begin{aligned} \varphi'_{\delta,r,2}(u;b) = {} & \frac{\lambda+\delta}{c-\alpha}\varphi_{\delta,r,2}(u;b) - \frac{\lambda}{c-\alpha}\left[\int_0^{u-b} \varphi_{\delta,r,2}(u-y;b)p(y)\,dy + \int_{u-b}^u \varphi_{\delta,r,1}(u-y;b)p(y)\,dy\right] \\ & - \frac{\lambda}{c-\alpha}\int_0^u e^{-r(u-y)}p(y)\,dy \ , \quad u \geq b \ . \end{aligned} \tag{8.41}$$

Interestingly, the system comprising (8.40) and (8.41) is structurally identical to that given by Lin and Pavlova (2006, Eq. (3.1)) (with their $\zeta(u)$ replaced by $\int_0^u e^{-r(u-y)}p(y)\,dy = (e^{-r\cdot} * p)(u)$). In addition, by continuity one also has $\varphi_{\delta,r,1}(b^-;b) = \varphi_{\delta,r,2}(b;b)$ (and one can extend the domain of $\varphi_{\delta,r,1}(u;b)$ to include $u = b$). Therefore, we can directly apply the results in Lin and Pavlova (2006, Theorem 5.1) to state the solution of (8.40) and (8.41). To this end, we first define ρ_2 to be the unique non-negative root to the Lundberg's fundamental equation (6.33) with c replaced by $c-\alpha$. Also, let $\kappa_{\delta,2}$ and $l_{\delta,2}(.)$ be identical to κ_δ and $l_\delta(.)$ given by (6.35) and (6.36) respectively but with c replaced by $c-\alpha$ (and hence ρ replaced by ρ_2). Furthermore, $g_{\delta,2}(.)$ is the compound geometric density (6.34) with $\kappa_{\delta,2}$ and $l_{\delta,2}(.)$ in place of κ_δ and $l_\delta(.)$ respectively. Then, the solution of $\varphi_{\delta,r}(u;b)$ is given by

$$\varphi_{\delta,r,1}(u;b) = \varphi_{\delta,r}(u;\infty) + \eta_{\delta,r}(b)v_\delta(u)\ , \quad 0 \le u \le b\ , \tag{8.42}$$

and

$$\varphi_{\delta,r,2}(u;b) = \frac{1}{1-\kappa_{\delta,2}}\int_0^{u-b} \varpi_{\delta,r}(u-y;b)g_{\delta,2}(y)\,dy + \varpi_{\delta,r}(u;b)\ , \quad u > b\ , \tag{8.43}$$

where

$$\eta_{\delta,r}(b) = \frac{\kappa_{\delta,2}\int_0^b \varphi_{\delta,r}(b-y;\infty)l_{\delta,2}(y)\,dy - \varphi_{\delta,r}(b;\infty) + \frac{\lambda}{c-\alpha}\mathcal{T}_{\rho_2}(e^{-r\cdot} * p)(b)}{v_\delta(b) - \kappa_{\delta,2}\int_0^b v_\delta(b-y)l_{\delta,2}(y)\,dy}\ , \tag{8.44}$$

$$\varpi_{\delta,r}(u;b) = \kappa_{\delta,2}\int_{u-b}^u \varphi_{\delta,r,1}(u-y;b)l_{\delta,2}(y)\,dy + \frac{\lambda}{c-\alpha}\mathcal{T}_{\rho_2}(e^{-r\cdot} * p)(u)\ , \quad u > b\ , \tag{8.45}$$

and $v_\delta(u)$ is given by (6.42).

To perform Laplace transform inversion on (8.42) and (8.43) with respect to r, we note that

$$\begin{aligned}
\mathcal{T}_{\rho_2}(e^{-r\cdot} * p)(u) &= \int_u^\infty e^{-\rho_2(y-u)}\int_0^y e^{-rv}p(y-v)\,dv\,dy \\
&= \left(\int_0^u\int_u^\infty + \int_u^\infty\int_v^\infty\right) e^{-rv}e^{-\rho_2(y-u)}p(y-v)\,dy\,dv \\
&= \int_0^\infty e^{-rv}\chi_\delta(u,v)\,dv\ , \quad u \ge 0\ ,
\end{aligned} \tag{8.46}$$

where

$$\chi_\delta(u,v) = \begin{cases} \int_u^\infty e^{-\rho_2(y-u)} p(y-v)\,dy = \mathcal{T}_{\rho_2} p(u-v) \ , & v < u \ . \\ \int_v^\infty e^{-\rho_2(y-u)} p(y-v)\,dy = \widetilde{p}(\rho_2) e^{-\rho_2(v-u)} \ , & v > u \ . \end{cases} \tag{8.47}$$

Therefore, the numerator of (8.44) can be written as

$$\begin{aligned} &\kappa_{\delta,2} \int_0^b \varphi_{\delta,r}(b-y;\infty) l_{\delta,2}(y)\,dy - \varphi_{\delta,r}(b;\infty) + \frac{\lambda}{c-\alpha} \mathcal{T}_{\rho_2}(e^{-r\cdot} * p)(b) \\ &= \kappa_{\delta,2} \int_0^\infty e^{-rv} \int_0^b \tau_\delta(b-y,v;\infty) l_{\delta,2}(y)\,dy\,dv - \int_0^\infty e^{-rv} \tau_\delta(b,v;\infty)\,dv + \frac{\lambda}{c-\alpha} \int_0^\infty e^{-rv} \chi_\delta(b,v)\,dv \ , \end{aligned} \tag{8.48}$$

and inversion of (8.42) with respect to r leads to

$$\tau_{\delta,1}(u,v;b) = \tau_\delta(u,v;\infty) + \frac{\kappa_{\delta,2} \int_0^b \tau_\delta(b-y,v;\infty) l_{\delta,2}(y)\,dy - \tau_\delta(b,v;\infty) + \frac{\lambda}{c-\alpha} \chi_\delta(b,v)}{v_\delta(b) - \kappa_{\delta,2} \int_0^b v_\delta(b-y) l_{\delta,2}(y)\,dy} v_\delta(u) \ , \quad 0 \le u \le b \ . \tag{8.49}$$

Similarly, using (8.46), (8.45) can be written as

$$\varpi_{\delta,r}(u;b) = \int_0^\infty e^{-rv} \sigma_\delta(u,v;b)\,dv \ , \quad u > b \ , \tag{8.50}$$

where

$$\sigma_\delta(u,v;b) = \kappa_{\delta,2} \int_{u-b}^u \tau_{\delta,1}(u-y,v;b) l_{\delta,2}(y)\,dy + \frac{\lambda}{c-\alpha} \chi_\delta(u,v) \ , \quad u > b \ . \tag{8.51}$$

Hence, upon substitution of (8.50), inversion of (8.43) with respect to r yields

$$\tau_{\delta,2}(u,v;b) = \frac{1}{1-\kappa_{\delta,2}} \int_0^{u-b} \sigma_\delta(u-y,v;b) g_{\delta,2}(y)\,dy + \sigma_\delta(u,v;b) \ , \quad u > b \ . \tag{8.52}$$

To summarize this subsection, for $0 \le u \le b$, $\tau_{\delta,1}(u,v;b)$ is explicitly given by (8.49), while for $u > b$, $\tau_{\delta,2}(u,v;b)$ is given by (8.52). It is instructive to note that the expression (8.52) involves $\sigma_\delta(u,v;b)$, which depends on the function $\tau_{\delta,1}(u,v;b)$ via (8.51).

8.5.2 Credit interest

Instead of a dividend threshold, now we consider the case where the surplus process $\{U_t\}_{t\geq 0}$ is subject to credit interest (only). From the discussion in Section 8.1, this corresponds to a premium rate function of $c(x) = c + \varepsilon x$ for $x \geq 0$ where ε is the rate of credit interest earned. Then, the integro-differential equation (8.36) becomes

$$\varphi'_{\delta,r}(u) = \frac{\lambda+\delta}{c+\varepsilon u}\varphi_{\delta,r}(u) - \frac{\lambda}{c+\varepsilon u}\int_0^u e^{-r(u-y)}p(y)\,dy - \frac{\lambda}{c+\varepsilon u}\int_0^u \varphi_{\delta,r}(u-y)p(y)\,dy\ , \quad u \geq 0\ . \tag{8.53}$$

In general, it is not easy to solve (8.53) for $\varphi_{\delta,r}(u)$. In the context of compound Poisson risk models with credit interest, integro-differential equation of the form (8.53) is usually transformed into a Volterra integral equation whose solution can be approximated recursively by Picard's sequence (see, e.g., Cai and Dickson (2002) and Wu et al. (2007)). For illustrative purposes we assume exponential claim density $p(y) = \theta e^{-\theta y}$ for $y > 0$ where an explicit expression for $\tau_0(u,v)$ is derived.

Omitting some straightforward algebra, application of the operator $(d/du + \theta)$ to (8.53) leads to the differential equation

$$\varphi''_{\delta,r}(u) + \left(\theta + \frac{\varepsilon-\lambda-\delta}{c+\varepsilon u}\right)\varphi'_{\delta,r}(u) - \frac{\delta\theta}{c+\varepsilon u}\varphi_{\delta,r}(u) = -\frac{\lambda\theta}{c+\varepsilon u}e^{-ru}\ , \quad u \geq 0\ . \tag{8.54}$$

To solve the above equation for $\varphi_{\delta,r}(u)$, we further assume that $\delta = 0$. Then (8.54) reduces to

$$\varphi''_{0,r}(u) + \left(\theta + \frac{\varepsilon-\lambda}{c+\varepsilon u}\right)\varphi'_{0,r}(u) = -\frac{\lambda\theta}{c+\varepsilon u}e^{-ru}\ , \quad u \geq 0\ . \tag{8.55}$$

By Polyanin and Zaitsev (2003, Section 2.1.9 Solution 3), the solution to (8.55) is given by

$$\varphi_{0,r}(u) = C_1 + \int_0^u e^{-Q_1(x)}\left[C_2 + \int_0^x e^{Q_1(v)}Q_2(v)\,dv\right]dx\ , \quad u \geq 0\ , \tag{8.56}$$

where

$$Q_1(v) = \int_0^v \left(\theta + \frac{\varepsilon - \lambda}{c + \varepsilon z}\right) dz = \theta v + \left(1 - \frac{\lambda}{\varepsilon}\right) \ln\left(1 + \frac{\varepsilon}{c} v\right) \qquad \text{and} \qquad Q_2(v) = -\frac{\lambda\theta}{c + \varepsilon v} e^{-rv} \,, \quad v \geq 0 \,. \tag{8.57}$$

and C_1 and C_2 are constants to be determined by two boundary conditions. The first boundary condition can be obtained by letting $u \to \infty$ in (8.56) together with $\lim_{u\to\infty} \varphi_{0,r}(u) = 0$. This yields

$$0 = C_1 + C_2 \int_0^\infty e^{-Q_1(x)}\, dx + \int_0^\infty e^{-Q_1(x)} \int_0^x e^{Q_1(v)} Q_2(v)\, dv\, dx \,. \tag{8.58}$$

In addition, by putting $u = 0$ into the integro-differential equation (8.53), one arrives at

$$\varphi'_{0,r}(0) = \frac{\lambda}{c} \varphi_{0,r}(0) \tag{8.59}$$

which, according to (8.56), leads to the second boundary condition

$$C_2 = \frac{\lambda}{c} C_1 \,. \tag{8.60}$$

Solving (8.58) and (8.60) simultaneously yields

$$C_1 = -\frac{\int_0^\infty e^{-Q_1(x)} \int_0^x e^{Q_1(v)} Q_2(v)\, dv\, dx}{1 + \frac{\lambda}{c}\int_0^\infty e^{-Q_1(x)}\, dx} \qquad \text{and} \qquad C_2 = -\frac{\lambda}{c} \frac{\int_0^\infty e^{-Q_1(x)} \int_0^x e^{Q_1(v)} Q_2(v)\, dv\, dx}{1 + \frac{\lambda}{c}\int_0^\infty e^{-Q_1(x)}\, dx} \,, \tag{8.61}$$

and therefore (8.56) becomes

$$\varphi_{0,r}(u) = -\left[\frac{1 + \frac{\lambda}{c}\int_0^u e^{-Q_1(x)}\, dx}{1 + \frac{\lambda}{c}\int_0^\infty e^{-Q_1(x)}\, dx}\right] \int_0^\infty e^{-Q_1(x)} \int_0^x e^{Q_1(v)} Q_2(v)\, dv\, dx + \int_0^u e^{-Q_1(x)} \int_0^x e^{Q_1(v)} Q_2(v)\, dv\, dx \,, \quad u \geq 0 \,. \tag{8.62}$$

From the above expression, $\varphi_{0,r}(u)$ depends on r only through $Q_2(.)$. By noting that

$$\int_0^u e^{-Q_1(x)} \int_0^x e^{Q_1(v)} Q_2(v)\, dv\, dx = -\int_0^u e^{-rv} \left[\int_v^u e^{-Q_1(x)}\, dx\right] \frac{\lambda\theta}{c+\varepsilon v} e^{Q_1(v)}\, dv \ , \quad u \geq 0 \ , \quad (8.63)$$

inversion of (8.62) with respect to r leads to

$$\tau_0(u,v) = \left\{\left[\frac{1+\frac{\lambda}{c}\int_0^u e^{-Q_1(x)}\, dx}{1+\frac{\lambda}{c}\int_0^\infty e^{-Q_1(x)}\, dx}\right] \int_v^\infty e^{-Q_1(x)}\, dx - 1\{v<u\} \int_v^u e^{-Q_1(x)}\, dx\right\} \frac{\lambda\theta}{c+\varepsilon v} e^{Q_1(v)} \ . \quad (8.64)$$

A more explicit formula for $\tau_0(u,v)$ can be obtained by substituting into (8.64) the expression of $Q_1(v)$ given in (8.57). We then obtain

$$\begin{aligned} \tau_0(u,v) = &\left\{\left[\frac{1+\frac{\lambda}{c}\int_0^u e^{-\theta x}\left(1+\frac{\varepsilon}{c}x\right)^{-1+\frac{\lambda}{\varepsilon}}\, dx}{1+\frac{\lambda}{c}\int_0^\infty e^{-\theta x}\left(1+\frac{\varepsilon}{c}x\right)^{-1+\frac{\lambda}{\varepsilon}}\, dx}\right] \int_v^\infty e^{-\theta(x-v)} \left(\frac{c+\varepsilon x}{c+\varepsilon v}\right)^{-1+\frac{\lambda}{\varepsilon}}\, dx \right. \\ &\left. - 1\{v<u\} \int_v^u e^{-\theta(x-v)} \left(\frac{c+\varepsilon x}{c+\varepsilon v}\right)^{-1+\frac{\lambda}{\varepsilon}}\, dx\right\} \frac{\lambda\theta}{c+\varepsilon v} \ , \end{aligned} \quad (8.65)$$

which is explicitly expressed in terms of model parameters.

Remark 21 *From Cheung (2010), expressions in the form of (8.9) and (8.10) also hold true in the context of an absolute ruin model (see, e.g., Cai (2007) and Gerber and Yang (2007)). The methodology in the section can also be applied to obtain the corresponding $\tau_0(u,v)$ function.*

Chapter 9

Concluding remarks and future research

Chapters 2 and 3 of this thesis are mainly concerned with the study of various ruin-related quantities under different types of dividend strategies in the MAP risk model. More specifically, Chapter 2 is concerned with a dividend barrier strategy, which is possibly dependent on the underlying Markovian environment, while Chapter 3 studies a threshold-type strategy in the dual MAP risk model. In contrast, in Chapters 4, 5 and 6, Gerber-Shiu functions with a generalized penalty function involving additional variables and various discounted joint densities are considered. The surplus level immediately after the second last claim before ruin R_{N_T-1} is studied in Chapters 4 and 5. Interestingly, it appears that in Chapter 4 the study of such an additional variable is the only way to keep track of the surplus prior to ruin U_{T-} due to the methodology adopted (i.e. a novel connection to a fluid flow process). In Chapters 5 and 6, the minimum surplus level before ruin X_T and the maximum surplus level before ruin Z_T are studied respectively. Chapter 7 considers a generalization of the MAP model by relaxing its exponential distributional assumption between two system changes, whereas Chapter 8 studies some orderings of certain ruin-related quantities via the generalized Gerber-Shiu function involving R_{N_T-1}, even when premium rate can possibly be surplus-dependent.

In this thesis, two entirely different approaches have been used to study the MAP risk model. For example, in the entire Chapter 2 and Sections 6.1 and 6.2, the analysis is purely analytic. In these analyses, the usual convention is to condition on the time and amount of the first claim to obtain integral and/or integro-differential equations. In contrast, in the entire Chapter 3 and Sections 5.6 and 6.4, the analysis is purely probabilistic and is based on sample paths arguments via a connection to a fluid queue. It is important to understand the advantages and disadvantages of the two different approaches. For the analytic approach, the main advantage is that the results obtained usually hold true without any assumptions on the claim size distributions (see, e.g., Sections 5.4 and 6.2). In contrast, a significant drawback of using the fluid flow technique to analyze risk models is that all the interclaim times and claim sizes have to be phase-type distributed, which poses a problem in modelling heavy-tailed distributions. Nonetheless, one could observe from, e.g., Sections 5.6 and 6.4, that the use of fluid flow arguments allows certain discounted joint distributions to be obtained immediately. Central to the representation of these discounted joint densities is $\mathbf{\Psi}(\delta)$, the Laplace transform of the busy period in a fluid flow model, for which a quadratically convergent algorithm is available (see Ahn and Ramaswami (2005)). Such a quantity replaces the role of the roots of Lundberg's fundamental equation (see, e.g., (2.29), (5.41), (6.33) and (7.41)), which are usually required in the analytic analysis. The findings of these Lundberg's roots might cause numerical problems when the number of environmental states of the MAP model becomes large.

While the discounted joint density of $(U_{T^-}, |U_T|, Z_T, X_T)$ in the MAP risk model has already been studied in Section 6.4 via the existing connection to a fluid flow process, it is the author's belief that further research is required to study the Gerber-Shiu function involving both the minimum and maximum levels before ruin at the same time using analytic analysis in various risk models involving dependency. However, it appears that this could be a challenging task due to the huge difference in the two separate approaches used to analyze the minimum (Section 5.2) and the maximum (Section 6.2). To keep track of the minimum level, one has to focus on the drop in surplus below its initial level because a new minimum can only occur when there is a drop in surplus below the running minimum. A semi-Markovian type risk model described in Section 1.2.3 is believed to be one of the most general models in which the minimum level before ruin X_T (indeed together with R_{N_T-1}) can be studied using this argument. The rationale is that the variables X_T and R_{N_T-1} are surplus levels at claim instants (or at time 0), and in a semi-Markovian risk model the process restarts anew at claim instants. On the other hand, to study the maximum, it is important to keep track of whether the process up-crosses the running maximum again after a claim. Therefore, we require a Markov property such that one can keep track of the states at which an up-crossing occurs, and the MAP risk model turns out to be a good candidate. We remark that some joint distributions of various related variables have also been studied by Wei and Wu (2002) and Wu et al. (2003) in the context of the classical compound Poisson risk model, but the minimum level before ruin is not involved in their analyses.

Another direction for future research would be to study the Gerber-Shiu function (or discounted densities) involving the maximum surplus Z_T and/or the minimum surplus X_T before ruin in the MAP risk model under certain dividend strategies or credit/debit interest, using either probabilistic or analytic methods. While it is clear that the discounted joint density of the quadruple $(U_{T^-}, |U_T|, Z_T, X_T)$ can still be obtained using the sample paths arguments of Section 6.4 for a MAP risk model with barrier or threshold type dividend strategies, apparently (to the best of the author's knowlege) no research has been done on any related discounted densities under credit/debit interest using such probabilistic arguments. The author believes this is due to the difficulty in relating the 'clocks' between the risk process and the fluid flow process arising from the non-linearity of sample paths, since the key observation by Ramaswami (2006) regarding the two 'clocks' (see end of Chapter 1) no longer holds. The recent paper by Rabehasaina (2009) provided a breakthrough by finding the Laplace transform of the time of ruin in the MAP model with credit/debit interest. It might be interesting to see whether similar approach can be applied to study the discounted joint densities of various ruin-related quantities. With regards to analytic methods, in general we expect to be able to keep track of the maximum surplus before ruin Z_T upon the introduction of dividend strategies or credit/debit interest (see Cheung and Landriault (2010)). However, the minimum surplus before ruin X_T remains a difficult problem because the distribution of a drop in surplus now depends on the initial surplus level under the dividend/interest modifications.

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