Stability of Hybrid Singularly Perturbed Systems with Time Delay

by

Mohamad Sahib Alwan

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Mohamad Alwan

Abstract

Hybrid singularly perturbed systems (SPSs) with time delay are considered and exponential stability of these systems is investigated. This work mainly covers switched and impulsive switched delay SPSs . Multiple Lyapunov functions technique as a tool is applied to these systems. Dwell and average dwell time approaches are used to organize the switching between subsystems (modes) so that the hybrid system is stable. Systems with all stable modes are first discussed and, after developing lemmas to ensure existence of growth rates of unstable modes, these systems are then extended to include, in addition, unstable modes. Sufficient conditions showing that impulses contribute to yield stability properties of impulsive switched systems that consist of all unstable subsystems are also established. A number of illustrative examples are presented to help motivate the study of these systems.

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Dedication

To my family, and my wounded country, Iraq.

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Notations and Symbols

R	the real number set
R_+	nonnegative real number set
R^n	n-dimensional vector space
x^T	transpose of a vector x
$\|x\|$	norm of a vector x
$\|A\|$	norm of a matrix A
$\max, (\min)$	maximum, (minimum)
sup	supremum, the least upper bound
inf	infimum, the greatest lower bound
A^T	transpose of a matrix A
A^{-1}	inverse of a (nonsingular) matrix A
$\lambda(A)$	eigenvalue of a matrix A
$\mathrm{Re}[\lambda(A)]$	the real part of an eigenvalue of a matrix A
$\lambda_{\max}(A)$	maximum eigenvalue of a symmetric matrix A
$\lambda_{\min}(A)$	minimum eigenvalue of a symmetric matrix A
$A > 0$	the real symmetric matrix A is positive definite
I	identity (unit) matrix
$\dot{x}(t)$	time derivative of a time-varying vector x
$\dot{A}(t)$	time derivative of a time-varying matrix A
$C([a,b], R^n)$	continuous function set from $[a,b] \to R^n$
$PC([a,b], R^n)$	piecewise continuous function set from $[a,b] \to R^n$
$\|x\|_\tau$	$\sup_{t-\tau\le\theta\le t} \|x(\theta)\|$
$x(t^+)$	$\lim_{t\to t^+} x(t)$
Δx	$x(t^+) - x(t)$
$O(\cdot)$	order of magnitude

Chapter 1

Introduction

In a wide variety of areas in physics, chemistry, engineering, and, increasingly, in biology, physiology, and economics, it is necessary to build a mathematical model to represent a problem. Since differential and integral calculus were invented by Sir Isaac Newton (1642 – 1727) and Gottfried Wilhelm Leibnitz (1646 – 1716), dynamic mathematical models have involved the search for an unknown function which satisfies an equation where the rate of change (e.g. derivative) plays an important role. Such equations are called differential equations (DEs). The subject of differential equations is now considered one of the most effective branches for studying the physical world. Since Newton's time many phenomena such as planetary motion, population growth, and the spreading of cancer cells a human body, have been formulated as DEs. Early mathematicians who contributed to the field of DEs include brothers James (1654 – 1705) and John (1667-1748) Bernoulli, where the latter developed the well-known method of separation of variables and Count Jacope Ricatti (1676 – 1754) whose remarkable contribution of the equation still bearing his name attracted the attention of the Bernoulli family. Other developments were made by Leonhard Euler $(1707-1783)$ who worked on (1) the problem of reducing a special class of DEs of the second order to the first order using the substitution technique, and (2) linear equations with constant coefficients. The last equations were also solved by the mathematical figures Joseph Louis Lagrange $(1736-1813)$ and Pierre de Laplace $(1749-1827)$ by using multipliers (integrating factors) [Ince59]. For more information on contributions in the field of DEs, one may refer to [Kli72]. Ordinary differential equations involve unknown functions and their derivatives which depend on a single variable, often representing as time t. If the unknown functions depend on more than one independent variable, the equation is a partial differential equation. If the equation is perturbed by noise or an unknown random parameter, the equation is called a stochastic differential equation.

When a phenomenon is represented by DEs, a set of assumptions should be taken into account in order that the mathematical model satisfies certain properties such as existence, uniqueness, and the continuity of the solution with respect to initial conditions. The problem of existence and uniqueness of the first order DE dates back to the mathematician Augustin Louis Cauchy $(1789-1857)$. This work was generalized for a system of DEs by Rudolf Lipschitz $(1832-1903)$.

Although a wide class of DEs satisfy the above solution requirements, the solutions cannot generally be found explicitly. One may think of finding an approximate solution by using a very accurate numerical method. This is possible if we are interested in a certain feature in the solution, but generally the numerical solutions cannot provide us with other information such as how sensitive the solution is to a small perturbation due to the approximate data used to build the mathematical model or a change in the initial conditions, or how the solution behaves when time t goes to infinity. These questions motivate mathematicians to look for qualitative information about the solution rather than solving the equation explicitly; in other words, they have shifted the mathematical technique from the quantitative to the qualitative. The pioneers of the qualitative theory of differential equations are Henri Poincaré (1854 – 1912) and Aleksander Mikhailovich Lyapunov (1857 – 1918)[Bra69]. One of the qualitative aspects that can be studied is stability of a solution of a system of DEs. At the end of the nineteenth century, Lyapunov invented the direct method to study the stability of a system without previous knowledge of its solution. The method which bears his name today is the most effective technique provided that a researcher constructs the right auxiliary function, called a Lyapunov function, the main tool used to establish the stability property.

1.1 Singularly Perturbed Systems

In networks or in models of large-scale interconnected systems such as power systems, large economies, control systems, biochemical, or nuclear reactor models, one notices dynamics with different speeds or multiple time scales. As an example, consider a building that is divided into a large number of rooms, and each room is divided into small offices. Assume that the outside walls of the building are provided with *excellent* thermal insulations, the rooms are equipped with *good* insulators, while the offices are insulated *poorly*. Assume that at time t_0 there is a wide variation in the temperature readings of the offices in every single room. A few *hours* after t_0 , say, at t_1, the variation has become very little among the offices within every single room, while the temperature differences among the rooms are still large. *Days* after t_1, we notice that the temperature variations among the rooms have eventually disappeared. The difference in the time-scale of the temperature dynamics is attributed to the differing strength of connections (insulators). After a few hours, the poor insulation (connection) among the offices within a single room caused rapid decay in the temperature difference (fast dynamics), while the good insulation among the rooms led to very slow decay in the temperature difference (slow dynamics)[Sim61]. For other examples, one may refer to [Cho82] and some references therein.

Assume that the dynamics in the above large-scale model are classified as given in the following system;

$$\begin{aligned} \dot{x} &= f(t, x, z) \\ \dot{z} &= G(t, x, z) \end{aligned} \tag{1.1}$$

where the slow variable $x \in R^m$ and the fast variable $z \in R^n$. Here we assume that during the fast transients the slow dynamics remain approximately constant and that, over longer time, they become noticeable, while the fast dynamics have already reached their quasi-steady states. Therefore, as we shall see in Chapter 2, in short term studies, slow variables are considered constant, and fast variables eventually reach their quasi-steady state; in long term studies, the full model is formed by slow variables and the quasi-steady state of the fast variables, as shown in the following system;

$$\begin{aligned} \dot{x}_s &= f(t, x_s, z_s) \\ 0 &= G(t, x_s, z_s) \end{aligned} \tag{1.2}$$

where x_s and z_s are refered to as quasi-steady states. Clearly, the second equation has degenerated into an algebraic (or transcendental) equation, meaning that the time-varying variable is treated as constant ($\dot{z} = 0$). To remove this mathematical inconsistency, system (1.1) is treated as a two-time-scale singular perturbation problem with perturbation parameter ε, the ratio of the time-scales of the slow and fast phenomena [Cho82]. Re-scaling the time scale of system (1.1) yields the so-called singularly perturbed system or fast-slow system;

$$\begin{aligned} \dot{x} &= f(t, x, z) \\ \varepsilon \dot{z} &= g(t, x, z) \end{aligned} \tag{1.3}$$

where $g = \varepsilon G$.

Setting $\varepsilon = 0$, the dimension of the full state *reduces* from $m + n$ to m, and system (1.3) then has the following form

$$\begin{aligned} \dot{x} &= f(t, x, h(t, x)) \\ 0 &= g(t, x, z), \end{aligned}$$

where $h(t, x)$ is the solution of the algebraic or transcendental equation $0 = g(t, x, z)$. The result is the same as that of (1.2), but the derivation is now different. More details are given in Chapter 2.

The perturbation parameter ε, in fact, has different representations; for instance, in some power systems it indicates machine reactance, in a biochemical model ε might represent a small quantity of an enzyme, and in nuclear reactors model ε is due to the fast neutrons [Kok86a]. System properties that can be studied are stability, optimal controllability, observability, bifurcation phenomenon([Kok86a] and the references therein), detectability [Vu04].

1.2 Switched Systems

A large class of natural and engineering systems are inherently multimodal in the sense that their behavior is represented by several dynamical models. For example, a mobile robot is designed to respond to different environmental factors such as avoiding an obstacle or turning at a corner, etc. Here, switching from one mode to another mode is not previously predicted, but it is determined by environmental factors which usually are not part of the mathematical model. Such a system is called a switched system, a special kind of hybrid dynamical system that is composed of a family of continuous-time subsystems and a rule that controls the switching between them. Switched systems have various applications in the aircraft industry, air traffic control, and control of mechanical systems. For other motivations and examples, one may refer to [Mor97], [Zer98], [Day99], [Lib99], [Zha01] and [Li05].

The study of switched systems is more challenging than that of determinate systems and is still in its infancy. Nevertheless, there has been reasonable progress in this field. Most of the work has focused on designing an appropriate switching rule to stabilize the system. For example, Morse [Mor96] constructed a simple high-level controller called a *supervisor* which is capable of switching into feedback with a single-input/single-output process in order to force the output of the process to track a constant reference input. In the same paper it was shown that if the switched system has exponentially stable subsystems, then the entire system is exponentially stable provided that the *dwell time* (τ_D), the time between any two consecutive switchings, is sufficiently large. Later, Hespanha and Morse [Hes99] showed that a similar result holds when the average interval between the consecutive switchings is no less than τ_D, leading to the *average dwell time* concept. This approach was also used by Zhai *et al.* [Zha01] to prove the stability of a more general class of switched systems incorporating stable and unstable subsystems. Dayawansa and Martin [Day99] investigated the stability of a class of dynamical systems which undergo random switchings. In their work, the focus is to prove a converse Lyapunov theorem for this class of systems. Hu *et al.* [Hu99] showed that if the subsystems are linear time-invariant and the system matrices are commutative component wise and stable, then the entire switched system is globally exponentially stable under arbitrary switching laws. They also studied, under a certain switching law, the same stability property of systems with vanishing or non-vanishing perturbation.

1.3 Impulsive Systems

An impulsive system is a special kind of hybrid systems that consists of a differential system and a difference system that respectively describe continuous evolutions and discrete events occurring in a mathematical model of a physical system. Many evolutionary processes are characterized by the fact that at certain moments between intervals of continuous evolutions they undergo changes of state abruptly. The durations of these changes are often negligible when compared to the total duration of the process, so that these changes can be reasonably approximated as instantaneous changes of state, or *impulses*. These evolutionary processes are suitably modeled as *impulsive differential systems*, or simply *impulsive systems*. Generally, an impulsive system is characterized by a pair of equations, a system of ordinary differential equations that describes a continuous evolutionary process and a difference equation defining discrete impulsive actions. Impulsive systems have applications in various fields such as physics, biology, engineering, population dynamics, aeronautics (see [Bai89],[Lak89],[Bai93] and some references therein), and increasingly secure communications (see [Li03],[Kha04],[Li05] and some references therein).

The theory of impulsive differential equations is richer than the corresponding theory of differential equations without impulses. For instance, the initial value problem of such equations may not have solutions even when the corresponding differential equations do; some fundamental properties such as continuous dependence on initial condition, continuation of solutions, or stability may be violated or need new interpretation. On the other hand, under some conditions impulses stabilize some systems even when the underlying systems are unstable [Liu94],[Wan04], or make continuation of solutions possible. For more motivation, interested readers may refer to [Lak89]. The theory of impulsive differential equations is interesting in itself; consequently, it has attracted some researchers such as V. Lakshmikanthan, D.D. Bainov, P.S. Simeonov, X.Z. Liu, X. Shen, Z. Li. As a result, there are some works including books by Lakshmikanthan *et al.*[Lak89], Bainov and Simeonov [Bai89],[Bai93], Li *et al.*[Li05] and many references therein. The stability of impulsive systems has received a great deal of work including, in addition, papers by X.Z. Liu [Liu94], R. Wang [Wan04], Xiuxiang Liu [Liu05], Zhi-Hong Guan *et al.* [Gua05]. The field of impulsive systems is currently very active since the applications of this theory have been increasing.

1.4 Delay Systems

Ordinary and partial differential equations have long played important roles in modeling many physical processes and they will continue to serve as a fundamental tool in future investigations. A drawback of these models is that they are ruled by the principle of causality, meaning that the future state of the dynamical system depends only on the present state and not on the past. In fact, this is only a first approximations of some real systems. In those cases, more realistic models should include some of the historical values of the state; this leads us to delay differential equations (DDEs), also known as retarded functional differential equations, or differential equations with deviating argument. The early motivations for studying DDEs came from their applications in population dynamics when Volterra investigated the predator-prey model, and in Minorsky's study of ship stabilization and automatic steering. These studies indicate the importance of considering delay in the feedback mechanism [Min42]. Another motivation for studying time delayed systems is that the presence of delay, even in first order systems, may cause undesirable performance such as oscillations or chaotic behaviors. In some cases, *small* delay may destabilize some systems, but *large* delay may stabilize others. As a result, there are many studies of systems with time delay. A number of monographes have been devoted to the subject of DDEs. These works include books by Bellman and Cooke [Bel63] and Krasovkii [Kra63], texts by Halanay [Hal66], Hale [Hal71], Hale and Lunel [Hal93], Driver [Dri77], El'sgoll'ts and Norkin [Els73] and Bellen and Zennaro [Bel03]. Some other books dedicated to the applications of DDEs are MacDonald [Mac89], Gopalsamy [Gop92], and Kuang [Kua93]. The use of delay differential equations is currently very heavy due to the progress obtained in the understanding of the dynamics of many important time delayed systems.

Hybrid systems and time delay are important issues encountered in many fields; as a result, they have become the focus of some researchers. Among the recent works are papers by S. Yang *et al.* [Yan04], X. Liu *et al.*[Liu05] and some references therein, and Ph.D theses by G. Ballinger [Bal99], A. Khadra [Kha04], Y. Zhang [Zha04] and S. Kim [Kim05].

If a system exhibits singular perturbation and time delay, then we are led to consider time-delayed singularly perturbed systems. Such systems have attracted some researchers. For instance, Hasiao *et al.* [Hsi93] investigated the stability of time delayed singularly perturbed systems with a dither, a high frequency signal injected into the system to improve its performance. Shao and Rowland [Sha94] gave an upper bound for the perturbation parameter ε such that the singularly perturbed system with time delay in the slow state is stable. A perturbation on the time delay was also discussed in their work. Liu *et al.* [Liu03] investigated exponential stability of full singularly perturbed systems with time delay using vector delay inequalities and Lyapunov functions. Here in this document, this system is our focus; namely, we shall investigate exponential stability of switched and impulsive switched singularly perturbed systems with time delay.

The organization of the thesis is as follows; in Chapter 2, we give the required mathematical background including some definitions and theorems (Section 1) followed by discussion of interconnected systems and singularly perturbed systems (Sections 2 and 3, respectively). Switched and impulsive systems of ordinary differential equations are addressed in Sections 4 and 5, respectively. Finally, delay systems is discussed in Section 6. In Chapter 3, the focus is on the main result of this thesis. In Section 1, we first establish exponential stability of linear switched delay systems that consist of all stable modes by using multiple Lyapunov functions (MLFs) technique and, after developing lemmas to help us find growth rates of unstable modes, we extend these systems to include, in addition, unstable modes. In Section 2, we consider full singularly perturbed subsystems with a single constant time delay in both states. Two different cases are studied; in the first case, we investigate exponential stability of switched system with stable modes, while in the second case, a more general class is discussed where the system has stable and unstable modes. MLFs technique is again applied to these systems. Dwell time and average dwell time approaches are separately used to organize the switching among the modes. Finally, in Section 3, we consider a switched system with a special kind of nonlinear delay singularly perturbed modes. In Chapter 4, exponential stability of the system considered in Chapter 3 is investigated after imposing impulse effects. We also develop a switching rule to stabilize impulsive switched systems incorporating all unstable modes.

Chapter 2

Mathematical Background

This chapter is devoted to stating the required materials to achieve the goal of this Thesis. In Section 1, we shall state some useful definitions and theorems regarding the stability of ODEs. As known, there are different kinds of stability problems arising in the study of dynamical systems. In this work, the focus is on the stability of the equilibrium point in the sense of Lyapunov. Interconnected systems of ODEs is discussed in Section 2. In Section 3, we present the standard singularly perturbed system, and study the stability of this system. Switched and impulsive switched systems of a class of ODEs shall be discussed in Sections 4 and 5, respectively. In fact, we focus on a stabilizing problem of these systems by using the dwell time and average dwell time approaches. In Section 6, we describe some important tools in DDEs that lead us to the main purpose of this work. First, a class of piecewise continuous functions is defined and stability of impulsive delay systems is discussed. Second, linear time-varying singularly perturbed systems with a single constant delay are described and sufficient conditions that guarantee exponential stability are stated.

2.1 Basic Definitions and Theorems

Consider the following system of differential equations

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \qquad f : D \to R^n \tag{2.1}$$

where D is an open and connected subset of R^n, and f is a locally Lipschitz function mapping D into R^n. Most of the materials in this section are taken from Khalil [Kha02], unless otherwise mentioned.

Definition 2.1: *A point $x = x^*$ is said to be an equilibrium point of system (2.1) if it has the property that whenever the solution $x(t)$ of (2.1) starts at x^*, it remains at x^* for all future time.*

According to this definition, the equilibrium points of (2.1) are then the real roots of the equation

$$f(x^*) = 0.$$

For convenience, we will state all definitions and theorems for the case when the equilibrium point is at the origin ($x^* = 0$), since any equilibrium point can be shifted to the origin by a change of variables. In the sequel, we will assume that $f(x)$ satisfies $f(0) = 0$.

Definition 2.2: *The equilibrium point $x^* = 0$ of system (2.1) is said to be*

- *stable if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$\|x_0\| < \delta \quad \textit{implies} \quad \|x(t)\| < \varepsilon, \quad \forall t \geq 0$$

  where $x(t)$ is any solution of (2.1).

- *unstable if it is not stable.*
- *asymptotically stable if it is stable and there exists a constant* $\delta > 0$ *such that*

$$\|x_0\| < \delta \quad \textit{implies} \quad \lim_{t \to \infty} x(t) = 0.$$

Although asymptotic stability is the desirable property, the weakness of this concept is that it says nothing about how fast the trajectories march to the equilibrium point. There is a stronger form of asymptotic stability which is referred to as *exponential stability.*

Definition 2.3: *The equilibrium point* $x^* = 0$ *of system* (2.1) *is said to be locally exponentially stable if there exist positive constants* λ*, k, and c such that*

$$\|x(t)\| \leq k\|x_0\|e^{-\lambda t}, \quad \forall \|x_0\| < c. \tag{2.2}$$

It is said to be globally exponentially stable if c can be chosen arbitrarily large and (2.2) *holds for all* $x_0 \in R^n$.

Having defined the stability and asymptotic stability concepts, we use Lyapunov's approach to determining stability. The main idea behind this technique is to determine how a special class of functions behave along the solutions of system (2.1). Let us first define these functions.

Definition 2.4: *Let* D *be an open subset of* R^n *containing* $x = 0$*. A function* $V : D \to R$ *is said to be positive semi-definite on* D *if it satisfies the following conditions*

(*i*) $V(0) = 0$,

(*ii*) $V(x) \geq 0, \quad \forall x \in D - \{0\}$.

It is said to be positive definite on D if it satisfies (i) above and

$(ii)^*$ $\quad V(x) > 0, \quad \forall x \in D - \{0\}.$

It is said to be negative definite (semi-definite) on D if $-V$ is positive definite (semi-definite) on D.

Definition 2.5: *A positive definite function V defined on R^n is said to be radially unbounded (or proper) if the following condition holds.*

$$\lim_{\|x\| \to \infty} V(x) = \infty.$$

In the Lyapunov stability theorems, the focus is on the function V and its time derivative along the trajectories of the dynamical system under consideration. The time derivative of $V(x)$ along the trajectories of system (2.1) is (simply) denoted by $\dot{V}$ and defined as follows

$$\dot{V} = \nabla V \cdot f(x)$$

Theorem 2.1: *Let $x^* = 0$ be an equilibrium point for system (2.1). Let D be an open subset of R^n containing $x = 0$ and $V : D \to R$ be a continuously differentiable function defined on D such that*

(i) $\quad V(0) = 0,$

(ii) $\quad V(x) > 0, \quad \forall x \in D - \{0\},$

(iii) $\quad \dot{V} \leq 0, \quad \forall x \in D - \{0\}.$

Then, $x^* = 0$ is stable. If condition (iii) is replaced by

$(iii)^*$ $\quad \dot{V} < 0, \quad \forall x \in D - \{0\},$

then $x^* = 0$ is asymptotically stable. Moreover, if $D = R^n$ and V is radially unbounded, then $x^* = 0$ is globally asymptotically stable.

In the next definition, we define positive definite matrices which play an important role in defining Lyapunov functions.

Definition 2.6: [Leo94] *A real symmetric $n \times n$ matrix is said to be positive definite if and only if it has strictly positive eigenvalues.*

An important class of positive definite functions are the quadratic functions $V(x) = x^T P x$, where P is a positive definite matrix. Let $\lambda_{min}(P)$ and $\lambda_{max}(P)$ denote the minimum and maximum eigenvalues of P, respectively. Then, we have the following

$$\lambda_{min}(P)\|x\|^2 \leq V(x) = x^T P x \leq \lambda_{max}(P)\|x\|^2$$

This inequality is referred to as the *Rayleigh Inequality* [Mar03].

A special case of system (2.1) is when the vector field function $f(x)$ has the linear form Ax where A is a real $n \times n$ matrix; namely, we have

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{2.3}$$

which is called a *linear time-invariant (or autonomous) system.* The solution of (2.3) is given by

$$x(t) = e^{At} x_0.$$

An efficient technique to investigate the stability properties of system (2.3) is by determining the location of the eigenvalues of the matrix A, as shown in the following theorem.

Theorem 2.2: *The equilibrium point $x^* = 0$ of system* (2.3) *is stable if and only if the eigenvalues of A (λ_is) have non-positive real parts and for those with zero real parts and algebraic multiplicity q_i, $rank(A - \lambda_i I) = n - q_i$, where n represents the* *dimension of x. It is globally asymptotically stable if and only if all eigenvalues of A have strictly negative real part.*

Definition 2.7: *An $n \times n$ matrix is said to be Hurwitz (or stable) if all its eigenvalues have negative real part.*

The asymptotic stability property can also be characterized by using Lyapunov's method. Consider the following Lyapunov function candidate

$$V(x) = x^T P x$$

The derivative of $V(x)$ along the trajectories of (2.3) is given by

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x = -x^T Q x,$$

where Q is an $n \times n$ matrix given by

$$A^T P + PA = -Q. \tag{2.4}$$

If Q is positive definite, then by Theorem (2.1) the origin is an asymptotically stable equilibrium point. This result is summarized in the next theorems.

Theorem 2.3: *An $n \times n$ matrix A is Hurwitz if and only if , for any given positive definite matrix Q, there is a unique positive definite matrix P which satisfies* (2.4).

The matrix equation (2.4) is referred to as a *Lyapunov equation* which is solved for P for a given Q where

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt, \tag{2.5}$$

or alternatively it can be solved by using the *Kroneker product of matrices.*

Definition 2.8:[Rug96] *Let* $A = (a_{ij})$ *and* B *be two matrices of dimension* $p \times q$ *and* $r \times s$*, respectively. Then, the Kronecker product of* A *and* B *denoted by* $A \otimes B$ *is given by*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1q}B \\ \vdots & \vdots & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{pmatrix}_{pr \times qs} \tag{2.6}$$

The dimensions q and r are not necessarily the same. Using this definition, the Lyapunov equation (2.4) is then written as follows

$$[A^T \otimes I + I \otimes A^T]\text{vec}[P] = -\text{vec}[Q] \tag{2.7}$$

where

$$\text{vec}[P] = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}_{n^2 \times 1}$$

and P_i is the i^{th} column of P; similarly, we define vec[Q]. The resulting $n^2 \times n^2$ linear system (2.7) is solved for vec[P].

In the rest of this section, we analyze the stability of the time-varying system of the form

$$\dot{x}(t) = A(t)x(t). \tag{2.8}$$

where $A(t)$ is continuous and bounded. The solution of this system with the initial condition $x(t_0)$ is given by

$$x(t) = \Phi(t, t_0)x(t_0),$$

where $\Phi(t, t_0)$ is the *state transition matrix*.

Consider the following Lyapunov function candidate

$$V(t, x) = x^T P(t) x.$$

with $P(t)$ being continuously differentiable, symmetric, positive definite, and bounded. The time derivative of $V(t, x)$ along the trajectories of system (2.8) is given by

$$\begin{aligned}\dot{V}(t, x) &= \dot{x}^T P(t) x + x^T P(t) \dot{x} + x^T \dot{P}(t) x \\ &= x^T [A^T(t) P(t) + P(t) A(t) + \dot{P}(t)] = -x^T Q(t) x,\end{aligned}$$

where $Q(t)$ is a given matrix such that the following Lyapunov equation is satisfied.

$$A^T(t) P(t) + P(t) A(t) + \dot{P}(t) = -Q(t). \tag{2.9}$$

If $Q(t)$ is continuous, symmetric, positive definite, and bounded, then the origin of system (2.8) is globally exponentially stable. This result is summarized in the following theorem.

Theorem 2.4: *Let $x^* = 0$ be the exponentially stable equilibrium point of $\dot{x} = A(t)x$, with $A(t)$ being continuous and bounded. Let $Q(t)$ be a continuous, symmetric, positive definite, and bounded. Then, there exists a unique continuously differentiable, symmetric, positive definite, and bounded matrix $P(t)$ that satisfies (2.9).*

Here, the matrix $P(t)$ is given by

$$P(t) = \int_t^\infty \Phi^T(s, t) Q(s) \Phi(s, t) ds. \tag{2.10}$$

and the existence of such a matrix is conditioned with $x^* = 0$ being exponentially stable. Sufficient conditions for the equilibrium point of the non-autonomous system $\dot{x}(t) = f(t, x)$ to be exponentially stable are given in the next theorem.

Theorem 2.5: *Consider that* $x^* = 0$ *is an equilibrium point of* $\dot{x}(t) = f(t,x)$ *where* $t \in [0, \infty)$ *and* $x \in D \subset R^n$*. Let* $V(t,x)$ *be a continuously differentiable function such that*

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a \tag{2.11}$$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a \tag{2.12}$$

where k_1, k_2, k_3 *and* a *are positive constants. Then,* $x^* = 0$ *is exponentially stable. It is globally exponentially stable if the assumptions hold globally.*

The exponential stability of the system $\dot{x}(t) = A(t)x(t)$ can also be characterized based on the location of the eigenvalues of $A(t)$; the following theorem gives sufficient conditions to guarantee stability.

Theorem 2.6:[Rug96],[Kok86b] *Let* $A(t)$ *be continuously differentiable* $n \times n$ *matrix. Assume that there exist positive constants* α, β_1 *and* β_2 *such that, for all t,*
(i) $Re[\lambda(A(t))] \leq -\alpha < 0$.
(ii) $\|A(t)\| \leq \beta_1$.
(iii) $\|\dot{A}(t)\| \leq \beta_2$.
Then, $x^* = 0$ *is exponentially stable.*

In this case, the continuously differentiable, symmetric, positive definite, and bounded matrix $P(t)$ is the solution of

$$A^T(t)P(t) + P(t)A(t) = -I_n.$$

2.2 Interconnected Systems

It is common that some systems are modeled as an interconnection of lower order subsystems. Due to the high order of these interconnected systems, the stability analysis is more complicated. A proper way to deal with such complex systems is to decompose interconnected systems into small isolated subsystems and study the stability of each individual subsystem; namely, we initially ignore the interconnection between the subsystems. In the next step, we combine our results from the first step to draw a conclusion about the stability of the interconnected system. In the forthcoming analysis, we illustrate the two steps, and define a function that can be a proper Lyapunov function candidate for the interconnected system. We also introduce a special kind of matrix called an *M-matrix*, which plays an important role in studying the stability of large-scale interconnected systems. Let the nth order interconnected system have the form

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x), \quad i = 1, 2, \cdots, m \tag{2.13}$$

where $x_i \in R^{n_i}$, $\sum_{i=1}^{m} n_i = n$, and $x = \left(x_1^T, x_2^T, \cdots, x_m^T\right)^T$. Assume that, $\forall i$,

$$f_i(t, 0) = 0, \quad g_i(t, 0) = 0, \qquad \forall t \geq 0,$$

meaning that the origin $x = 0$ is an equilibrium point of the system (2.13). Ignoring the interconnection between the subsystems results in the following m isolated subsystems

$$\dot{x}_i = f_i(t, x_i). \tag{2.14}$$

Assume that the equilibrium points $x_i = 0$ of these subsystems are uniformly asymptotically stable, and that there are m Lyapunov functions $V_i(t, x_i)$.

Define the composite Lyapunov function for the collection of the m isolated subsystems by

$$V(t,x) = \sum_{i=1}^{m} d_i V_i(t,x_i) \tag{2.15}$$

where d_i are positive constant. Next consider (2.13) as a perturbation of (2.14). Take $V(t,x)$ be a Lyapunov function candidate for the interconnected system. The time derivative of V along the trajectories of (2.13) is

$$\dot{V}(t,x) = \sum_{i=1}^{m} d_i \Big[\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t,x_i) \Big] + \sum_{i=1}^{m} d_i \frac{\partial V_i}{\partial x_i} g_i(t,x) \tag{2.16}$$

The first term on the right-hand side is negative definite since $V_i s$ are Lyapunov functions for the m subsystems, while the second term is, generally, indefinite. Therefore, we assume that $[\partial V_i/\partial x_i] g_i$ is bounded by a nonnegative upper bound. To pursue the analysis mathematically, assume that $V_i(t,x_i)$ satisfies, for $\|x\| \le r$, (where $r > 0$)

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t,x_i) \le -\alpha_i \phi_i^2(x_i), \quad t \ge 0 \tag{2.17}$$

$$\Big\| \frac{\partial V_i}{\partial x_i} \Big\| \le \beta_i \phi_i(x_i) \tag{2.18}$$

where α_i and β_i are positive constants, and ϕ_i is a positive definite function. Suppose that the $g_i(t,x)$ satisfy, for $\|x\| \le r$,

$$\|g_i(t,x)\| \le \sum_{j=1}^{m} \gamma_{ij} \phi_j(x_j), \quad i = 1, 2, \cdots, m, \qquad t \ge 0 \tag{2.19}$$

where γ_{ij} are nonnegative constants. Then, the time derivative of $V(t,x)$ along the trajectories of the interconnected system satisfies

$$\dot{V}(t,x) \le \sum_{i=1}^{m} d_i \Big[-\alpha_i \phi_i^2(x_i) + \sum_{j=1}^{m} \beta_i \gamma_{ij} \phi_i(x_i) \phi_j(x_j) \Big]$$

The right-hand side is a quadratic in $\phi_1, \phi_2, \cdots, \phi_m$; that is,

$$\dot{V}(t,x) \leq -\frac{1}{2}\phi^T(DS + S^T D)\phi$$

where $\phi = (\phi_1, \phi_2, \cdots, \phi_m)^T$, $D = \text{diag}(d_1, d_2, \cdots, d_m)$ and S is an $n \times n$ matrix whose elements are given by

$$s_{ij} = \begin{cases} \alpha_i - \beta_i \gamma_{ij}, & i = j \\ \\ -\beta_i \gamma_{ij}, & i \neq j. \end{cases} \tag{2.20}$$

Clearly, the asymptotic stability of the interconnected system is guaranteed if the diagonal matrix D is chosen such that the matrix

$$DS + S^T D > 0. \tag{2.21}$$

The existence of such a diagonal matrix D is ensured if S is an M-matrix. The following definition and Lemma are also found in [Ara78],[Gop92].

Definition 2.9: *An $n \times n$ matrix S is said to be an M-matrix if its leading (successive) principal minors are positive:*

$$det \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1k} \\ s_{21} & s_{22} & \cdots & s_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ s_{k1} & s_{k2} & \cdots & s_{kk} \end{pmatrix} > 0, \quad k = 1, 2, \cdots, n$$

The following lemma gives the sufficient condition that guarantees the existence of D.

Lemma 2.1: *There exists a positive diagonal matrix D that satisfies* (2.21) *if and only if S is an M-matrix.*

Other equivalent definitions and properties about the M-matrix are found in the same references.

In fact, the diagonally dominant matrices with non-positive off-diagonal elements are M-matrices. Therefore, the M-matrix condition is equivalent to that the diagonal elements of S are larger as a whole than the off-diagonal (non-positive) elements. Physically, the diagonal elements of S represent the measures of the degree of stability for the isolated subsystems, and the off-diagonal elements of S represent the strength of the interconnection. Thus, the M-matrix condition can be read as follows; if the degrees of stability of the isolated subsystems are larger as a whole than the interconnections, then the origin of the interconnected system is uniformly asymptotically stable [Kha02]. The next theorem summarizes our findings.

Theorem 2.7: *Consider the interconnected system* (2.13)*. Assume that there exist positive definite functions $V_i(t, x_i)$ that satisfy* (2.17) *and* (2.18)*, and that $g_i(t, x)$ satisfies* (2.19)*. Assume that the matrix S defined by* (2.20) *is an M-matrix. Then, the origin is uniformly asymptotically stable. It is globally asymptotically stable if the assumptions hold globally and $V_i(t, x_i)$ are radially unbounded.*

2.3 Singularly Perturbed systems:

Consider the following time-varying singularly perturbed system

$$\dot{x}(t) = f(t, x(t), z(t), \varepsilon), \quad x(t_0) = \xi(\varepsilon) \tag{2.22}$$

$$\varepsilon \dot{z}(t) = g(t, x(t), z(t), \varepsilon), \quad z(t_0) = \eta(\varepsilon) \tag{2.23}$$

where f and g are continuously differentiable functions in their arguments $(t, x, z, \varepsilon) \in [t_0, \infty) \times D_x \times D_z \times [0, \varepsilon^*]$, with $D_x \subset R^n$ and $D_z \subset R^m$ being open and connected sets and $\xi(\varepsilon)$ and $\eta(\varepsilon)$ depend smoothly on ε, $(\varepsilon > 0)$. Setting $\varepsilon = 0$ in the full system (2.22)-(2.23) reduces the dimension from $n + m$ to n since the differential equation degenerates into an algebraic or transcendental equation

$$0 = g(t, x, z, 0) \tag{2.24}$$

Assume that for all t (2.24) has $k \geq 1$ isolated real roots given by

$$z = h_i(t, x), \quad i = 1, 2, \cdots, k \tag{2.25}$$

If this assumption holds, then system (2.22)-(2.23) is said to be in *standard form*. Substituting (2.25) into (2.22), at $\varepsilon = 0$, leads us to the i^{th} *reduced model*

$$\dot{x} = f(t, x, h_i(t, x), 0), \quad x(t_0) = \xi(0) =: \xi_0 \tag{2.26}$$

This model is also called the *quasi-steady state model* or *slow model*.

Let $x(t, \varepsilon)$ and $x_s(t)$ be the exact and reduced (or slow) solutions, respectively. Then, the difference

$$x(t, \varepsilon) - x_s(t) = O(\varepsilon)$$

holds uniformly for all $t \in [t_0, \infty)$ since at $t = t_0$ we have

$$x(t_0, \varepsilon) - x_s(t_0) = \xi(\varepsilon) - \xi(0) = O(\varepsilon).$$

When $x(t) = x_s(t)$, quasi-steady-state behavior of z is then given by

$$z_s(t) = h(t, x_s(t)) \tag{2.27}$$

which cannot be a uniform approximation of the exact solution $z(t, \varepsilon)$ since the difference between the initial state $\eta(\varepsilon)$ and its initial value $z_s(t_0) = h(t_0, \xi_0)$ may be large. Therefore, the best we can expect is that the estimate

$$z(t, \varepsilon) - z_s(t) = O(\varepsilon)$$

holds on the interval $[t_b, \infty)$ where $t_b > t_0$ [Kha02]. For z to reach its quasi-steady state $z_s(t)$, the solution along the initial *boundary-layer interval* $[t_0, t_b]$ should be customized for that purpose. In the following we show how this can be done.

For convenience, we shift the quasi-steady state of z to the origin by changing the variable as follows

$$y = z - h(t, x) \tag{2.28}$$

Using this transformation, the full system (2.22)-(2.23) will be

$$\dot{x} = f(t, x, y + h(t, x), \varepsilon), \quad x(t_0) = \xi(\varepsilon) \tag{2.29}$$

$$\begin{aligned} \varepsilon \dot{y} &= g(t, x, y + h(t, x), \varepsilon) - \varepsilon \frac{\partial h}{\partial t} \\ &\quad - \varepsilon \frac{\partial h}{\partial x} f(t, x, y + h(t, x)), \quad y(t_0) = \eta(\varepsilon) - h(t_0, \xi(\varepsilon)). \end{aligned} \tag{2.30}$$

Stretch the boundary-layer interval by defining the following new time variable.

$$s = \frac{1}{\varepsilon}(t - t_0) \tag{2.31}$$

The initial instant of this time which corresponds to the initial time $t = t_0$ is $s = 0$. Based on this definition, regardless of how short the time interval in t is, s goes to infinity as $\varepsilon \to 0$. In other words, if $\varepsilon \to 0$, the interval $t - t_0$ is stretched to an infinite interval in s. Furthermore, systems (2.29) and (2.30) become

$$\frac{dx}{ds} = \varepsilon f(t_0 + \varepsilon s, x, y + h(t_0 + \varepsilon s, x), \varepsilon), \quad x(0) = \xi(\varepsilon) \tag{2.32}$$

$$\begin{aligned} \frac{dy}{ds} &= g(t_0 + \varepsilon s, x, y + h(t_0 + \varepsilon s, x), \varepsilon) - \varepsilon \frac{\partial h}{\partial s} \\ &\quad - \varepsilon \frac{\partial h}{\partial x} f(t_0 + \varepsilon s, x, y + h(t_0 + \varepsilon s, x), \varepsilon), \quad y(0) = \eta(\varepsilon) - h(t_0, \xi(\varepsilon)) \end{aligned} \tag{2.33}$$

Setting $\varepsilon = 0$ freezes the variables at $t = t_0$ and $x = \xi_0$, and reduces (2.32) and (2.33) to the autonomous systems

$$\frac{dx}{ds} = 0, \qquad x(0) = \xi_0 \tag{2.34}$$

$$\frac{dy}{ds} = g(t_0, \xi_0, y + h(t_0, \xi_0), 0), \qquad y(0) = \eta_0 - h(t_0, \xi_0). \tag{2.35}$$

where $\eta_0 := \eta(0)$. If the equilibrium point of (2.35) $y = 0$ is asymptotically stable and the initial value $y(0)$ is in the region of attraction, then the solution of (2.35) will march to the quasi-steady state z_s during $[t_0, t_b]$. After t_b we need the solution of (2.35) to remain close to zero. In this case we allow the frozen parameters to take values in the region of the slowly varying parameters (t, x). Thus, system (2.35) can be written as follows

$$\frac{dy}{ds} = g(t, x, y + h(t, x), 0), \tag{2.36}$$

where (t, x) are treated as fixed parameters. System (2.36) is referred to as the *Boundary- layer model.*

From (2.34) the fast part of x is constant, and during the boundary-layer interval $[t_0, t_b]$, the fast part of z, say y_f, is given by the solution of the Boundary-layer model (2.36). Thus, for $t \in [t_0, T]$, z is represented by

$$z(t, \varepsilon) = y_f(s) + z_s(t) + O(\varepsilon) \tag{2.37}$$

The following theorem summarizes these results.

Theorem 2.8:[Kha02] *Consider the singularly perturbed system given by* (2.22) *and* (2.23)*. Let the real roots in* (2.25) *be isolated. Assume the following conditions hold for all*

$$[t, x, z - h(t, x), \varepsilon] \in [t_0, T] \times D_x \times D_y \times [0, \varepsilon]$$

where $D_x \in R^n$ is convex and $D_y \in R^m$ contains the origin;

- *The functions f and g, their first partial derivatives with respect to (x, z, ε), and the first partial derivative of g with respect to t are continuous; the function $h(t, x)$ and the Jacobian $[\partial g(t, z, 0)/\partial z]$ have continuous first partial derivatives with respect to their arguments; the initial data $\xi(\varepsilon)$ and $\eta(\varepsilon)$ are smooth functions of ε.*

- *The reduced model (2.26) has a unique solution $x_s(t) \in S$ for all $t \in [t_0, T]$ where S is a compact subset of D_x.*

- *The origin as an equilibrium point of the boundary-layer model (2.36) is exponentially stable, uniformly in (t, x); let $R_y \subset D_y$ be the region of attraction of (2.35) and Ω_y be a compact subset of R_y.*

Then, there exists a positive constant ε^* such that for all $\eta_0 - h(t_0, \xi_0) \in \Omega$ and $0 < \varepsilon < \varepsilon^*$, the singularly perturbed system (2.22)-(2.23) has a unique solution $x(t, \varepsilon)$, $z(t, \varepsilon)$ on $[t_0, T]$, and

$$x(t, \varepsilon) = x_s(t) + O(\varepsilon) \tag{2.38}$$

$$z(t, \varepsilon) = y_f(s) + h(t, x_s(t)) + O(\varepsilon) \tag{2.39}$$

hold uniformly for $t \in [t_0, T]$. Moreover, given any $t_b > t_0$, there exists $\varepsilon^{**} \leq \varepsilon^*$ such that

$$z(t, \varepsilon) = h(t, x_s(t)) + O(\varepsilon) \tag{2.40}$$

hold uniformly for $t \in [t_b, T]$ when $\varepsilon < \varepsilon^{**}$.

Having given the sufficient conditions to guarantee the existence of the solution, we study the stability of full system (2.22)-(2.23) by examining the stability of the reduced and boundary-layer models. In our stability analysis, we focus on the autonomous singularly perturbed system given by

$$\dot{x} = f(x, z) \tag{2.41}$$

$$\varepsilon \dot{z} = g(x, z) \tag{2.42}$$

We assume that the origin $(x = 0, z = 0)$ is an isolated equilibrium point, and the equation

$$0 = g(x, z)$$

has isolated real roots represented by

$$z = h(x).$$

For convenience, we shift the equilibrium point of the boundary-layer model to the origin $y = 0$ by introducing the following transformation

$$y = z - h(x).$$

Then, the singularly perturbed system is

$$\dot{x} = f(x, y + h(x)) \tag{2.43}$$

$$\varepsilon \dot{y} = g(x, y + h(x)) - \varepsilon \frac{\partial h}{\partial x} f(x, y + h(x)) \tag{2.44}$$

The corresponding reduced system

$$\dot{x} = f(x, h(x)) \tag{2.45}$$

has equilibrium point at $x = 0$ and the boundary-layer system

$$\frac{dy}{ds} = g(x, y + h(x)) \tag{2.46}$$

where $s = \frac{1}{\epsilon} t$ and x is treated as a fixed parameter has equilibrium point at $y = 0$.

In the following theorem, we state sufficient conditions to guarantee asymptotical stability of systems (2.41) and (2.42).

Theorem 2.9:[Kha02] *Consider the singularly perturbed system* (2.41) *and* (2.42). *Assume that there are Lyapunov functions* $V(x)$ *and* $W(x, y)$ *that satisfy the following*

$$\frac{\partial V}{\partial x} f(x, h(x)) \leq -\alpha_1 \psi_1^2(x) \tag{2.47}$$

$$\frac{\partial W}{\partial y} g(x, y + h(x)) \leq -\alpha_2 \psi_2^2(y) \tag{2.48}$$

$$W_1(y) \leq W(x, y) \leq W_2(y) \tag{2.49}$$

$$\frac{\partial V}{\partial x} [f(x, y + h(x)) - f(x, h(x))] \leq \beta_1 \psi_1^2(x) \psi_2(y) \tag{2.50}$$

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x, y + h(x)) \leq \beta_2 \psi_1(x) \psi_2(y) + \gamma \psi_2^2(y) \tag{2.51}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2,$ *and* γ *are positive constants, and* ψ_1, ψ_2 *are positive definite functions. Then, there is a positive constant* $\varepsilon^* = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma + \beta_1 \beta_2}$ *such that the origin* $(x = 0, z = 0)$ *is asymptotically stable for all* $0 < \varepsilon < \varepsilon^*$.

To illustrate these results, we take the following example which is found in [Kha02], [Kok87].

Example 2.1: A second-order model of an armature controlled DC motor is described as follows

$$\begin{aligned} J\frac{d\omega}{dt} &= ki \\ L\frac{di}{dt} &= -k\omega - Ri + u \end{aligned}$$

where i, u, R, and L are respectively the armature current, voltage, resistance, and inductance, J is the moment of inertia, ω is the angular speed, and ki and kw

are respectively the torque and back electromotive force developed with constant excitation flux. The first state equation is a mechanical torque equation, and the second one is an equation for the electric transient in the armature. In most DC motors, L is a small parameter which is often neglected (i.e. $\varepsilon = L$). This means that the motor's model where $\omega = x$ and $i = z$ is in the standard form of (2.22)-(2.23) whenever $R \neq 0$. In this case, when neglecting L, we get

$$0 = -k\omega - Ri + u$$

which has only one root given by

$$i = (u - k\omega)/R$$

Thus, the reduced model (2.26) is

$$J\dot{\omega} = -\frac{k^2}{R}\omega + \frac{k}{R}u$$

This (reduced) first-order equation is frequently used in designing DC motors [Kok87].

Quite often, a perturbation parameter ε is chosen as a dimensionless quantity, meaning that we non-dimensionalize the system first and neglect the resulting dimensionless quantity. To pursue our analysis, consider that a singularly perturbed system of a DC motor is given by

$$\begin{aligned} \dot{x} &= z, \qquad x(0) = 1 \\ \varepsilon\dot{z} &= -x - z, \qquad z(0) = 1 \end{aligned}$$

Set $\varepsilon = 0$ to get $h(x) = -x$, and the Boundary-Layer model is

$$\frac{dy}{ds} = -y(s), \qquad s = t/\varepsilon$$

which has globally exponentially stable equilibrium point at the origin. The reduced and Boundary-Layer problems are respectively

$$\begin{aligned} \dot{x}(t) &= -x(t), \qquad x(0) = 1 \\ \frac{dy}{ds} &= -y(s), \qquad y(0) = 2. \end{aligned}$$

Then, the solutions of the singularly perturbed system are

$$\begin{aligned} x(t,\varepsilon) &= e^{-t} + O(\varepsilon) \\ z(t,\varepsilon) &= 2e^{-t/\varepsilon} - e^{-t} + O(\varepsilon). \end{aligned}$$

For better understanding of this system, we study the stability problem and then demonstrate the exact and approximate solutions of $x(t)$ and $z(t)$ at different values of ε. By taking $V(x) = \frac{1}{2}x^2$ and $W(y) = \frac{1}{2}y^2$, a simple check shows that the positive constants of Theorem 2.9 are $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 1$ and $\gamma = 1$, and the positive definite functions are $\psi_1 = |x|$ and $\psi_2 = |y|$. Therefore, the origin is asymptotically stable for $\varepsilon < \varepsilon^* = 0.5$.

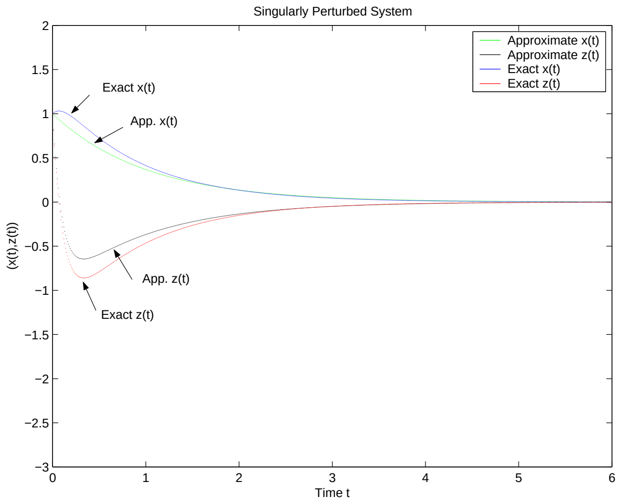
The exact solutions of the full singularly perturbed system at $\varepsilon = 0.1$ are

$$\begin{aligned} x(t) &= -0.2746e^{-8.873t} + 1.2746e^{-1.127t} \\ z(t) &= 2.4365e^{-8.873t} - 1.4365e^{-1.127t}. \end{aligned}$$

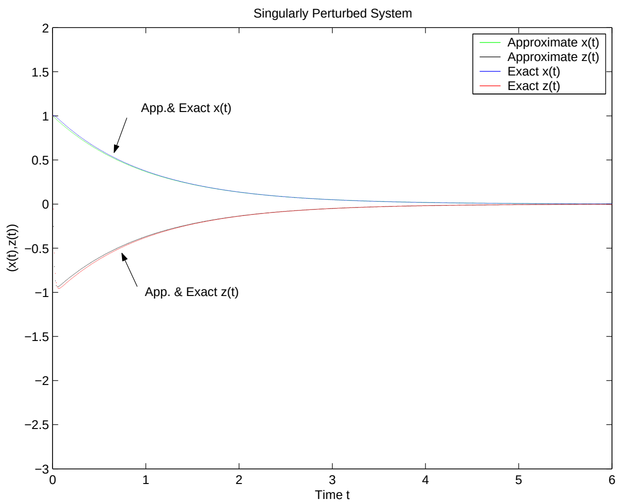
Figure 2.1(a) shows the simulation results at $\varepsilon = 0.1$. The trajectory of $z(t)$ apparently exhibits a two-time-scale behavior. It starts with a fast transient of $z(t,\varepsilon)$ from $\eta_0 = 1$ to $\bar{z}(t) = -e^{-t}$. After the decay of the transient, it remains close to $\bar{z}(t)$. Figure 2.1(b) displays results at $\varepsilon = 0.01$. The exact solutions of the singularly perturbed system are

$$\begin{aligned} x(t) &= -0.0275e^{-73.9898t} + 1.0275e^{-1.0102t} \\ z(t) &= 2.0437e^{-73.9898t} - 1.0347e^{-1.0102t}. \end{aligned}$$

Clearly, the Boundary-Layer interval is smaller than that of the first case, and the difference between the approximate and exact solutions $O(\varepsilon)$ has almost disappeared.



(a)



(b)

Figure 2.1: Exact and approximate solutions of singularly perturbed system (a) at $\varepsilon = 0.1$ (b) at $\varepsilon = 0.01$

2.4 Switched Systems

The basic problems in switched systems that can be considered are the following [Lib99].

Problem 1: *Find conditions to guarantee that a switched system is asymptotically stable for any switching signal.*

Problem 2: *Identify the switching signals for which the switched system is asymptotically stable.*

Problem 3: *Construct a switching signal that makes the switched system asymptotically stable.*

In this work, we focus on finding conditions to guarantee the exponential stability of switched system (Problem 1). Let us first analyze the stability of an ordinary system given by

$$\dot{x}(t) = A_i x(t), \quad t \in [t_{k-1}, t_k) \tag{2.52}$$

where $k = 1, 2, \cdots$, $t_{k-1} < t_k$, $\lim_{k\to\infty} t_k = \infty$, and A_i is an $n \times n$ constant matrix for all $i \in S = \{1, 2, \cdots, N\}$. We assume that the origin is an equilibrium point of system (2.52).

2.4.1 Systems with Stable Subsystems

In the following theorem we state sufficient conditions that guarantee exponential stability of system (2.52).

Theorem 2.10: [Lib99],[Zha01] *Consider the switched system* (2.52)*. Let* $A_i, i \in S$ *be a Hurwitz matrix. Then, the origin of* (2.52) *is exponentially stable if the follow-*

ing inequality holds.

$$\ln \mu - \nu(t_k - t_{k-1}) \leq 0, \qquad k = 1, 2, \cdots . \tag{2.53}$$

where $\mu = \frac{\lambda_M}{\lambda_m}$, $\lambda_M = \max\{\lambda_{\max}(P_i); i \in S\}$, $\lambda_m = \min\{\lambda_{\min}(P_i); i \in S\}$, P_i *is a positive definite matrix satisfying Lyapunov equation*

$$A_i^T P_i + P_i A_i = -Q_i, \tag{2.54}$$

for any positive definite matrix Q_i, *and* ν *is such that* $0 < \nu < \lambda_i$ *where* $\lambda_i = c_i/\lambda_M$, c_i *is a positive constant such that*

$$\frac{\partial V_i}{\partial x} A_i x \leq -c_i \|x\|^2. \tag{2.55}$$

Proof:

Define the Lyapunov function for the i^{th} subsystem by

$$V_i(x) = x^T P_i x, \tag{2.56}$$

which satisfies the following inequalities

$$\lambda_m \|x\|^2 \leq V_i(x) \leq \lambda_M \|x\|^2 \tag{2.57}$$

$$\frac{\partial V_i}{\partial x} A_i x \leq -c_i \|x\|^2 \tag{2.58}$$

Combining (2.57) and (2.58) yields

$$\dot{V}_i(x(t)) \leq -\lambda_i V_i(x(t))$$

where $\lambda_i = c_i/\lambda_M$, and the solution of this differential inequality is

$$V_i(x(t)) \leq V_i(x(t_{k-1})) e^{-\lambda_i (t - t_{k-1})} \tag{2.59}$$

From (2.57), we have, for any $i, j \in S$

$$V_j(x(t)) \leq \mu V_i(x(t)) \tag{2.60}$$

where $\mu = \frac{\lambda_M}{\lambda_m}$.

Activating mode 1 and 2 on the first and second intervals, respectively, we have

$$\begin{aligned} V_1(x(t)) &\leq e^{-\lambda_1(t-t_0)}V_1(x_0), \qquad t \in [t_0, t_1) \\ V_2(x(t)) &\leq e^{-\lambda_2(t-t_1)}V_2(x(t_1)), \qquad t \in [t_1, t_2) \\ &\leq e^{-\lambda_2(t-t_1)}\mu V_1(x(t_1)) \\ &\leq e^{-\lambda_2(t-t_1)}\mu e^{-\lambda_1(t_1-t_0)}V_1(x_0) \end{aligned}$$

Namely, we have

$$V_2(x(t)) \leq \mu e^{-\lambda_2(t-t_1)}e^{-\lambda_1(t_1-t_0)}V_1(x_0).$$

Generally, for $i \in S$ and $t \in [t_{k-1}, t_k)$ we have

$$V_i(x(t)) \leq \mu^{i-1}e^{-\lambda_i(t-t_{k-1})}e^{-\lambda_{i-1}(t_{k-1}-t_{k-2})} \cdots e^{-\lambda_1(t_1-t_0)}V_1(x_0). \tag{2.61}$$

Let $\lambda = \min\{\lambda_i; i \in S\}$. Then,

$$\begin{aligned} V_i(x(t)) &\leq \mu^{i-1}e^{-\lambda(t-t_0)}V_1(x_0) \\ &= \mu^{i-1}e^{-\nu(t-t_0)}e^{-(\lambda-\nu)(t-t_0)}V_1(x_0) \\ &\leq \mu^{i-1}e^{-\nu(t_k-t_0)}e^{-(\lambda-\nu)(t-t_0)}V_1(x_0) \\ &= \mu^{i-1}e^{-\nu(t_1-t_0)}e^{-\nu(t_2-t_1)} \cdots e^{-\nu(t_k-t_{k-1})}V_1(x_0)e^{-(\lambda-\nu)(t-t_0)} \end{aligned} \tag{2.62}$$

Namely, for $t \in [t_{k-1}, t_k)$

$$V_i(x(t)) \leq \mu e^{-\nu(t_1-t_0)}\mu e^{-\nu(t_2-t_1)} \cdots \mu e^{-\nu(t_k-t_{k-1})}V_1(x_0)e^{-(\lambda-\nu)(t-t_0)} \tag{2.63}$$

Making use of (2.53), we have for $t \in [t_0, \infty)$

$$V_i(x(t)) \leq V_1(x_0)e^{-(\lambda-\nu)(t-t_0)}$$

By (2.57), we have

$$\|x(t)\| \leq K\|x_0\|e^{-(\lambda-\nu)(t-t_0)/2}, \qquad t \geq t_0$$

where $K = \sqrt{\mu}$. This shows that the origin of the switched system is exponentially stable. In fact, one can write condition (2.53) as follows

$$t_k - t_{k-1} \geq \frac{\ln\mu}{\nu} =: T_D \qquad k \geq 1 \tag{2.64}$$

The fixed positive constant T_D is called *dwell time* [Mor96],[Lib99],[Hes99],[Zha01].

Theorem 2.10 says that, if the switched system has exponentially stable subsystems and the interval between any two consecutive discontinuities is larger than T_D, then the origin of the system is exponentially stable. Hespanha and Morse [Hes99] showed that a similar result still holds if the dwell time condition is not satisfied, but the average interval between consecutive discontinuities in no smaller than T_D. In this case, T_D is called the *average dwell time* [Hes99],[Hu99]. To follow this approach, let $N(t_0, t)$ representing the number of jumps in the open interval (t_0, t) satisfy

$$N(t_0, t) \leq N_0 + \frac{t - t_0}{T_a} \tag{2.65}$$

where N_0 and T_a are defined as the chatter bound and the average dwell time, respectively. Rewrite the inequality in (2.61) as follows

$$V_i(x) \leq e^{(i-1)\ln\mu - \lambda(t-t_0)}V_1(x_0)$$

Applying Condition (2.65) with $N_0 = \eta/\ln\mu, (\mu \neq 1)$, where η is an arbitrary constant, and $T_a = \ln\mu/(\lambda - \lambda^*)$, where $(\lambda^* < \lambda)$, leads to

$$V_i(x) \leq e^{\eta - \lambda^*(t-t_0)}V_1(x_0).$$

where the last inequality is found as follows;

$$
\begin{aligned}
(i-1)\ln\mu - \lambda(t-t_0) &\leq \Big(N_0 + \frac{t-t_0}{T_a}\Big)\ln\mu - \lambda(t-t_0) \\
&= \Big(\frac{\eta}{\ln\mu} + \frac{(\lambda-\lambda^*)(t-t_0)}{\ln\mu}\Big)\ln\mu - \lambda(t-t_0) \\
&= \eta - \lambda^*(t-t_0).
\end{aligned}
$$

2.4.2 Systems with Stable and Unstable Subsystems

Consider again the switched system (2.52) with $S = S_u \cup S_s$ where $S_u = \{1, 2, \cdots, l, \cdots r\}$ and $S_s = \{r+1, r+2, \cdots, m, \cdots, N\}$ represent the index sets of unstable and stable subsystems, respectively; that is,

$$
\begin{cases}
A_1, A_2, \cdots, A_r & : \ \text{unstable} \\
A_{r+1}, A_{r+2}, \cdots, A_N & : \ \text{stable}
\end{cases}
$$

Let $\lambda_i (i \in S)$ be positive constants such that

$$
\begin{cases}
A_i - \lambda_i I & : \ i \in S_u \\
A_i + \lambda_i I & : \ i \in S_s
\end{cases}
$$

are Hurwitz matrices. Then, there exists a symmetric positive definite matrix P_i such that

$$
\begin{cases}
(A_i - \lambda_i I)^T P_i + P_i (A_i - \lambda_i I) < 0 & : \ i \in S_u \\
(A_i + \lambda_i I)^T P_i + P_i (A_i + \lambda_i I) < 0 & : \ i \in S_s
\end{cases}
$$

For each $i \in S$, define

$$
V_i(x) = x^T P_i x
$$

Then,

$$
\dot{V}_i(x) \leq \begin{cases}
2\lambda_i V_i(x) & : \ i \in S_u \\
-2\lambda_i V_i(x) & : \ i \in S_s
\end{cases}
$$

Let us run, for instance, l unstable modes and run l times from an unstable one, and run $m-l$ stable modes and run $m-l-1$ times from a stable mode. Then, for $t \in [t_{k-1}, t_k)$

$$\begin{aligned} V_m(x) \;&\leq\; \mu^{m-1}\Big\{e^{2\lambda_1(t_1-t_0)} \cdots e^{2\lambda_l(t_l-t_{l-1})}\Big\} \\ &\quad \times\Big\{e^{-2\lambda_{l+1}(t_{l+1}-t_l)} \cdots e^{-2\lambda_{m-1}(t_{k-1}-t_{k-2})}\Big\}V_1(x_0)e^{-2\lambda_m(t-t_{k-1})} \end{aligned} \tag{2.66}$$

Let

$$\begin{aligned} \lambda^+ &= \max\{2\lambda_i : i = 1, 2, \cdots, l\} \\ \lambda^- &= \min\{2\lambda_i : i = l+1, l+2, \cdots, m\} \end{aligned}$$

and denote by $T^+(t_0, t)$ and $T^-(t_0, t)$ the total activation time of the unstable and stable modes, respectively. Then, for $t \in [t_{k-1}, t_k)$ we have

$$V_m(x) \leq \mu^{m-1}e^{\lambda^+T^+-\lambda^-T^-}$$

Choose $\lambda^* \in (0, \lambda^-)$, and assume that the switching law satisfies

$$\inf_{t\geq t_0} \frac{T^-(t_0,t)}{T^+(t_0,t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*}, \tag{2.67}$$

where this condition implies that for any $t \geq t_0$

$$\begin{aligned} (\lambda^+ + \lambda^*)T^+ &\leq (\lambda^- - \lambda^*)T^- \\ -\lambda^-T^- + \lambda^+T^+ &\leq -\lambda^*T^- - \lambda^*T^+ \\ &= -\lambda^*(T^- + T^+) \\ &= -\lambda^*(t - t_0) \end{aligned}$$

Therefore, applying condition (2.67) to (2.66) gives us

$$V_i(x(t)) \leq \mu^{m-1}e^{-\lambda^*(t-t_0)}V_1(x_0).$$

Assume that the dwell-time condition (2.53) holds. Then, we have for $t \in [t_0, \infty)$

$$V_i(x) \leq e^{-(\lambda^* - \nu)(t - t_0)} V_1(x_0).$$

We have proved the following

Theorem 2.11: *Consider the switched system* (2.52)*. Let $A_i (i \in S_u)$ be Hurwitz matrices, and $A_i (i \in S_s)$ be unstable matrices. Assume that Conditions* (2.53) *and* (2.67) *hold. Then, the origin of* (2.52) *is exponentially stable.*

2.5 Impulsive Systems

As pointed out earlier, an impulsive system consists of a system of ordinary differential equations that describes continuous evolutions and a system of difference equations defining the impulsive effects. In this section we describe and analyze the stability of impulsive systems.

Let $\Omega \in R^n$ be the phase space of an evolutionary process and assume Ω is an open set. Define

$$D = R_+ \times \Omega = \{(t, x) : t \in R_+ \quad \text{and} \quad x \in \Omega\}.$$

D is called the extended phase space. Let the evolutionary state of the process at time t be given by $x(t)$, and denote by P_t the point $(t, x(t))$ in D. Assume that the system of ordinary differential equations between moments of impulses has the form

$$\dot{x}(t) = f(t, x(t)) \tag{2.68}$$

and that impulses occur when a spatio-temporal relation $\kappa(t,x) = 0$ is satisfied. Let

$$M = \{(t,x) \in D : \kappa(t,x) = 0\} \tag{2.69}$$

denote the hypersurface of the equation $\kappa(t,x) = 0$, and $A : M \to D$, where $A(t,x) = (t, x + I(t,x))$, denote the function which describes the impulsive action. Here, we have $I(t,x) = \Delta x$, where $I : M \to R^n$. Therefore, the impulsive system has the following form

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)), \quad \kappa(t,x) \neq 0, \\ \Delta x &= I(t,x), \qquad \kappa(t,x) = 0. \end{aligned} \tag{2.70}$$

The solution of the impulsive system (2.70) is defined below.

Definition 2.10 *A function $x : (t_0, \beta) \to R^n$, where $0 \leq t_0 < \beta \leq \infty$, is said to be a solution of system* (2.70) *if the following conditions are satisfied.*
(i) $(t, x(t)) \in D$ *for* $t \in (t_0, \beta)$.
(ii) The right-hand limit $x(t_0^+) = \lim_{t \to t_0^+} x(t)$ *exists and* $(t_0, x(t_0^+)) \in D$.
(iii) $\forall t \in (t_0, \beta)$, *if* $\kappa(t, x(t)) \neq 0$ *then x is continuously differentiable at t and satisfies the differential equation* $\dot{x}(t) = f(t, x(t))$.
(iv) The set of moments of impulses $T = \{t \in (t_0, \beta) : \kappa(t, x(t)) = 0\}$ *is finite or consists of countable increasing sequence of points with limit* β.
(v) If the moment of impulse $t \in T$, *then the left-hand limit* $x(t^-) = \lim_{t \to t^-} x(t)$ *exists and* $x(t^-) = x(t)$ *for* $t \neq t_0$, *meaning that the solution is left-continuous, and* $x(t^+)$ *exists and* $x(t^+) = x(t) + I(t, x(t))$ *for* $t \neq \beta$.

In system (2.70), $\Delta x = x(t^+) - x(t)$. If $x(t)$ is a solution of (2.70), then we call the curve in D which is described by the points $P_t = (t, x(t))$ the *integral curve* *associated with* $x(t)$. To show how the solution of an evolutionary process behaves, let $T = \{t_k\}_{k=1}^{\infty}$ where $t_k < t_{k+1}$, for each k. For $t \in (t_0, t_1]$, let $x(t) = x(t; t_0, x(t_0^+))$ be any solution of (2.68) starting at $(t_0, x(t_0^+))$. The point $P_t = (t, x(t)) \in D$ begins its motion at the point $P_{t_0^+} = (t_0, x(t_0^+))$, and then moves along the curve $\{(t, x(t)) : t \geq t_0\}$ until $t = t_1$. At this moment, we have $\kappa(t_1, x(t_1)) = 0$, so an impulse occurs, and the function A immediately transfers the point $P_{t_1} = (t_1, x(t_1))$ into $P_{t_1^+} = A(t_1, x(t_1)) = (t_1, x(t_1) + I(t_1, x(t_1))) = (t_1, x(t_1^+))$. For $t > t_1$, the point $P_t = (t, x(t))$ moves further along the curve with $x(t) = x(t; t_1, x(t_1^+))$ until $t = t_2$ at which the function A transfers P_{t_2} into $P_{t_2^+}$. This process continues in the same manner for as long as $x(t)$ exists.

Consequently, the solution of impulsive system (2.70) is either continuous or piecewise continuous with simple jump discontinuities occurring at the moments of impulse t for which $I(t, x(t)) \neq 0$.

The initial value problem for the impulsive system (2.70) is given by

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & \kappa(t, x) \neq 0, \\ \Delta x = I(t, x), & \kappa(t, x) = 0. \\ x(t_0^+) = x_0. \end{cases} \tag{2.71}$$

If $\kappa(t_0, x_0) \neq 0$, meaning that there is no impulse at the initial time, then the initial condition may be written $x(t_0) = x_0$, and the solution of (2.71) may be defined at $t = t_0$.

Generally, a solution $x(t) = x(t; t_0, x_0)$ of (2.71) defined on an interval (t_0, β) and experiencing impulses at points $T = \{t_k\}_{k=1}^{\infty}$ with $t_k < t_{k+1}$ can be described as follows.

$$x(t;t_0,x_0) = \begin{cases} x(t;t_0,x_0), & t_0 < t \le t_1, \\ x(t;t_1,x(t_1^+)), & t_1 < t \le t_2, \\ \vdots & \\ x(t;t_k,x(t_k^+)), & t_k < t \le t_{k+1}, \\ \vdots & \end{cases} \tag{2.72}$$

where $x(t_k^+) = x(t_k) + I(t_k, x(t_k))$.

Because of some difficulties that may be caused by an arbitrary choice of the relation $\kappa(t,x) = 0$, we shall focus on a certain kind of impulses; namely, we consider simple systems for which the impulsive actions take place at fixed times $t_0 < t_1 < \cdots < t_k < \cdots$. In this case, the set M consists of a sequence of hyper-planes $t = t_k$ in D. These systems can then be written as follows.

$$\begin{cases} \dot{x}(t) = f(t,x(t)), & t \neq t_k, \\ \Delta x = I(t,x), & t = t_k. \end{cases} \tag{2.73}$$

Impulsive systems with impulses at variable times or autonomous systems with time-independent impulses are more difficult compared to impulsive systems with fixed moments of impulsive effects. In the first case, the set M consists of a sequence of hyper-surfaces represented by $t_k = \omega_k(x(t))$, for each k, with $\omega_k(x) < \omega_{k+1}(x)$ and $\lim_{k\to\infty} \omega_k(x) = \infty$. Such systems can be written as follows.

$$\begin{cases} \dot{x}(t) = f(t,x(t)), & t \neq \omega_k(x), \\ \Delta x = I(t,x), & t = \omega_k(x). \end{cases} \tag{2.74}$$

These systems have interesting properties; for example, solutions may experience an infinite number of impulses in a finite amount of time, or in addition solutions may not exist after reaching an impulse hyper-surface. The autonomous impulsive system can be written as follows

$$\begin{cases} \dot{x}(t) = f(x(t)), & x \in \Omega \backslash M, \\ \Delta x = I(x), & x \in M. \end{cases} \tag{2.75}$$

where $M = \{x \in \Omega : \kappa(x) = 0\}$. Solutions of (2.75) may experience an infinite number of impulses in finite time interval. For more details about these types of impulsive systems, readers may refer to [Lak89], and some references therein.

In the following definition we define stability of solutions of system (2.71) where $f(t, 0) = 0$ and $I(t, 0) = 0$, meaning that the origin in an equilibrium point of the system.

Definition2.11:[Bal95] *Consider system* (2.71)*. Let* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \bar{t}_0, y_0)$*,* $\bar{t}_0 \geq t_0$*, be solutions of* (2.71) *defined on* (t_0, ∞) *with initial conditions* $x(t_0^+) = x_0$ *and* $y(\bar{t}_0^+) = y_0$*, respectively. Then,* $x(t)$ *is said to be*
(i) stable in the sense of Lyapunov if for each $\varepsilon > 0$*, and* $\bar{t}_0 \geq t_0$*, there exists a positive constant* $\delta = \delta(\bar{t}_0, \varepsilon)$ *such that*

$$\|x(\bar{t}_0^+) - y_0\| < \delta \quad \textit{implies} \quad \|x(t) - y(t)\| < \varepsilon, \quad t > \bar{t}_0$$

(ii) unstable if (i) is not satisfied
(iii) attractive if for each $\bar{t}_0 \geq t_0$ *and* $\varepsilon > 0$*, there exist positive constants* $\delta = \delta(\bar{t}_0)$ *and* $T = T(\bar{t}_0, \varepsilon)$ *such that*

$$\|x(\bar{t}_0^+) - y_0\| < \delta \quad \textit{implies} \quad \|x(t) - y(t)\| < \varepsilon, \quad t > \bar{t}_0 + T$$

(iv) asymptotically stable if (i) and (iii) are satisfied
(v) uniformly asymptotically stable if (i) and (iii) are satisfied and the constants δ *and* T *are independent of* $\bar{t}_0$*.*

In the following theorem, we give the sufficient conditions that guarantee exponential stability (in the sense of Lyapunov) of a simple impulsive system given by

$$\begin{cases} \dot{x}(t) = Ax(t), \quad t \neq t_k, \\ \Delta x = B_k x, \quad (or \quad x(t_k^+) = [I + B_k]x(t_k)) \qquad t = t_k, \quad k = 1, 2, \cdots . \end{cases} \tag{2.76}$$

Theorem 2.12:[Wan04] *Assume that the eigenvalues of A have negative real parts. Then, the origin of system (2.76) is globally exponentially stable if the following inequality holds.*

$$\ln \alpha_k - \nu(t_k - t_{k-1}) \leq 0, \qquad k = 1, 2, \cdots \tag{2.77}$$

where $\alpha_k = \frac{\lambda_{\max}([I+B_k]^T P[I+B_k])}{\lambda_{\min}(P)}$ with P being a positive definite matrix satisfying

$$A^T P + PA = -Q$$

for any positive definite matrix Q, $0 < \nu < \xi$ and $\xi = \lambda_{\min}(Q)/\lambda_{\min}(P)$.

Proof:

For a given solution $x(.)$, define $v(t) =: V(x(t)) = x^T Px$. Then, the derivative of v along the trajectory of (2.76) is given by

$$\dot{v}(t) \leq -\xi v(t), \qquad t \in (t_{k-1}, t_k]$$

where $\xi = \lambda_{\min}(Q)/\lambda_{\min}(P)$, and

$$v(t) \leq v(t_{k-1}^+)e^{-\xi(t-t_{k-1})}, \qquad t \in (t_{k-1}, t_k]$$

while at $t = t_k^+$, we have

$$\begin{aligned} v(t_k^+) &= x(t_k^+)^T Px(t_k^+) \\ &= x(t_k)^T [I + B_k]^T P[I + B_k]x(t_k) \\ &\leq \lambda_{\max}([I + B_k]^T P[I + B_k])x(t_k)^T x(t_k) \\ &= \alpha_k v(t_k) \end{aligned}$$

Namely, we have

$$v(t_k^+) \le \alpha_k v(t_k) \tag{2.78}$$

where $\alpha_k = \frac{\lambda_{\max}([I+B_k]^T P[I+B_k])}{\lambda_{\min}(P)}$. Now, for $t \in (t_0, t_1]$, we have

$$v(t) \le v(t_0^+)e^{-\xi(t-t_0)}.$$

and

$$v(t_1^+) \le \alpha_1 v(t_1) \le \alpha_1 v(t_0^+)e^{-\xi(t_1-t_0)}.$$

Similarly, for $t \in (t_1, t_2]$, we have

$$v(t) \le v(t_0^+)\alpha_1 e^{-\xi(t_1-t_0)}e^{-\xi(t-t_1)} = v(t_0^+)\alpha_1 e^{-\xi(t-t_0)}.$$

Generally, we have for $t \in (t_k, t_{k+1}]$

$$\begin{aligned} v(t) &\le v(t_0^+)\alpha_1\alpha_2 \cdots \alpha_k e^{-\xi(t-t_0)} \\ &= v(t_0^+)\alpha_1\alpha_2 \cdots \alpha_k e^{-\xi(t-t_0)} \\ &= v(t_0^+)\alpha_1\alpha_2 \cdots \alpha_k e^{-\nu(t-t_0)}e^{-(\xi-\nu)(t-t_0)} \\ &= v(t_0^+)\alpha_1 e^{-\nu(t_1-t_0)}\alpha_2 e^{-\nu(t_2-t_1)} \cdots \alpha_k e^{-\nu(t_k-t_{k-1})}e^{-(\xi-\nu)(t-t_0)} \end{aligned}$$

By Assumption (2.77) we have

$$v(t) \le v(t_0^+)e^{-(\xi-\nu)(t-t_0)}, \qquad t \ge t_0$$

which implies that

$$\|x(t)\| \le K\|x(t_0^+)\|e^{-(\xi-\nu)(t-t_0)/2}, \qquad t \ge t_0$$

where $K = \sqrt{\mu}$; this shows that the origin is globally exponentially stable.

2.6 Delay Differential Equations

Let $C_\tau = C([-\tau, 0], R^n)$, with $\tau > 0$, representing a time delay, be the set of continuous functions from $[-\tau, 0]$ to R^n. If $\phi \in C_\tau$, the τ-norm of this function is defined by $\|\phi_t\|_\tau = \sup_{-\tau \leq \theta \leq 0} \|\phi(\theta)\|$, where $\|\cdot\|$ is the Euclidean norm on R^n.

Definition 2.12: *If x is a function mapping $[t-\tau, t]$ into R^n, a new function x_t mapping $[-\tau, 0]$ into R^n is defined as follows*

$$x_t(\theta) = x(t+\theta), \qquad \text{for} \quad \theta \in [-\tau, 0].$$

Here, $x_t(\theta)$ (or simply x_t) is the segment of the function x, from $t-\tau$ to t, that has been shifted to the interval $[-\tau, 0]$. A general delay differential equation is described as follows

$$\dot{x}(t) = f(t, x_t), \tag{2.79}$$

where f depends on both t and x_t. Since x_t is an element of $C([-\tau, 0], R^n)$, f is called a functional. Unlike the initial state of an ordinary differential equation, the initial state of system (2.79) is defined on the entire interval $[t_0 - \tau, t_0]$, not just t_0. Then, an initial condition is given as a continuous function

$$x_{t_0} = \phi(t), \quad t \in [t_0 - \tau, t_0]. \tag{2.80}$$

Thus, the *delay initial value problem* is given by

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \\ x_{t_0} &= \phi(t), \quad t \in [t_0 - \tau, t_0] \end{aligned}$$

Definition 2.13: *The equilibrium point $x(t) = 0$ of system (2.79) is said to be*

- *stable if, for a given* $\varepsilon > 0$*, there exists a* $\delta = \delta(\varepsilon, t_0) > 0$ *such that*

$$\|x_{t_0}\|_\tau < \delta \quad \textit{implies} \quad \|x(t)\| < \varepsilon, \quad \forall t \geq t_0 - \tau$$

- *unstable if it is not stable*
- *asymptotically stable if it is stable and there exists a* $\delta = \delta(t_0) > 0$ *such that*

$$\|x_{t_0}\|_\tau < \delta \quad \textit{implies} \quad \lim_{t\to\infty} x(t) = 0$$

- *locally exponentially stable if there exist positive constants* c*,* k*, and* λ *such that*

$$\|x(t)\| \leq k\|x_{t_0}\|_\tau e^{-\lambda(t-t_0)}, \quad \forall \|x_{t_0}\|_\tau < c. \tag{2.81}$$

- *It is said to globally exponentially stable if* c *can be chosen arbitrarily large and* (2.81) *holds for any* $x_{t_0} \in R^n$*.*

Before analyzing the stability of delay systems, we state the following proposition and lemma.

Proposition 2.1:[Hal66] *Consider the following delay differential inequality.*

$$\dot{u}(t) \leq f(t, u(t), \sup_{\theta\in[t-\tau,t]} u(\theta)), \qquad t \in [t_0, t_0 + a), \quad a > 0.$$

Assume that $y(t)$ *is a solution of the delay differential equation*

$$\dot{y}(t) = f(t, y(t), \sup_{\theta\in[t-\tau,t]} y(\theta)), \qquad t \in [t_0, t_0 + a)$$

such that

$$y(t) = u(t), \qquad t \in [t_0 - \tau, t_0]$$

Then,

$$u(t) \leq y(t), \qquad t \in [t_0, t_0 + a).$$

Lemma 2.2:[Hal66] *Assume that v is a continuous nonnegative function defined on $[t_0 - \tau, b)$ and satisfies*

$$\dot{v}(t) \leq -\alpha v(t) + \beta \sup_{\theta \in [t-\tau, t]} v(\theta), \qquad t \in [t_0, b)$$

where α and β are positive constant satisfying $\alpha > \beta > 0$. Then, there exists a positive constant ξ such that

$$v(t) \leq \sup_{\theta \in [t_0 - \tau, t_0]} v(\theta) e^{-\xi(t-t_0)}, \qquad t \in [t_0, b)$$

where ξ is a unique positive solution of

$$\xi = \alpha - \beta e^{\xi \tau}.$$

Consider now the following linear time-invariant delay system

$$\dot{x}(t) = Ax(t) + Bx(t - \tau), \tag{2.82}$$

where A and B are $n \times n$ constant matrices.

Theorem 2.13:*The origin of system* (2.82) *is exponentially stable if the matrix A is Hurwitz and the following inequality is satisfied*

$$-\frac{\lambda_{min}(Q) - \beta^*}{\lambda_{max}(P)} + \frac{\beta^*}{\lambda_{min}(P)} < 0 \tag{2.83}$$

where P and Q are positive definite matrices satisfying the Lyapunov equation

$$A^T P + PA = -Q$$

and $\beta^* = \|PB\|$.

Proof:

Define $V(x) = x^T P x$ as a Lyapunov function candidate for system (2.82). Then, the time derivative of V along the trajectories of system (2.82) is

$$\begin{aligned}
\dot{V} &= \Big(Ax(t) + Bx(t-\tau)\Big)^T Px + x^T P\Big(Ax(t) + Bx(t-\tau)\Big) \\
&= x^T(AP + PA)x + 2x^T PBx(t-\tau) \\
&\leq -x^T Qx + 2x^T PBx(t-\tau) \\
&\leq -\lambda_{min}(Q)\|x\|^2 + \|PB\|(\|x\|^2 + \|x(t-\tau)\|^2) \\
&= -(\lambda_{min}(Q) - \beta^*)\|x\|^2 + \beta^*\|x(t-\tau)\|^2 \\
&\leq -\frac{\lambda_{min}(Q) - \beta^*}{\lambda_{max}(P)} V(x) + \frac{\beta^*}{\lambda_{min}(P)}\|V_t\|_\tau .
\end{aligned}$$

Then, by Lemma 2.2 , where $\alpha = \frac{\lambda_{min}(Q)-\beta^*}{\lambda_{max}(P)}$ and $\beta = \frac{\beta^*}{\lambda_{min}(P)}$, there exists a positive constant ξ such that

$$V(x) \leq \|V_{t_0}\|_\tau e^{-\xi(t-t_0)}$$

Hence

$$\|x(t)\| \leq K\|x_{t_0}\|_\tau e^{-\xi(t-t_0)/2}$$

where $K = \sqrt{\mu}$; this shows that system (2.82) is exponentially stable.

2.6.1 Impulsive Delay Systems

Consider delay system (2.79) and the continuous initial condition (2.80)

$$\begin{aligned}
\dot{x}(t) &= f(t, x_t), \\
x_{t_0} &= \phi(t), \quad t \in [t_0 - \tau, t_0]
\end{aligned}$$

Adding impulses to system (2.79) will lead us to the consideration of piecewise continuous functions. Since the solutions will be piecewise continuous, the functional f must be defined on a class of piecewise continuous functions, and furthermore the continuous initial condition must be generalized to a piecewise continuous (initial) function. Before describing impulsive delay systems, we define some classes of piecewise continuous functions [Bal99]. Let $a, b \in R$ with $a < b$ and $D \subset R^n$. Define

$$\begin{aligned}
PC([a,b],D) &= \Big\{\psi : [a,b] \to D \Big| \psi(t^+) = \psi(t)\ \forall t \in [a,b), \psi(t^-) \text{ exists in} \\
&\quad D \quad \forall t \in (a,b] \text{ and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number} \\
&\quad \text{of points } t \in (a,b]\Big\}, \\
PC([a,b),D) &= \Big\{\psi : [a,b] \to D \Big| \psi(t^+) = \psi(t)\ \forall t \in [a,b), \psi(t^-) \text{ exists in} \\
&\quad D \quad \forall t \in (a,b) \text{ and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number} \\
&\quad \text{of points } t \in (a,b)\Big\}, \\
PC([a,\infty),D) &= \Big\{\psi : [a,b] \to D \Big| \forall c > a, \psi|_{[a,c]} \in PC([a,c],D)\Big\}.
\end{aligned}$$

Let $PC_\tau = \{\phi : \phi \in PC([-\tau,0],R^n)\}$ be the set of piecewise continuous functions mapping $[-\tau,0]$ into R^n. If $\phi \in PC_\tau$, the $\tau-$norm of ϕ is defined by $\|\phi\|_\tau = \sup_{-\tau\le\theta\le 0}\|\phi(\theta)\|$. If $x \in PC([t_0-\tau,\infty),R^n)$ where $t_0 \geq 0$, then we define $x_t \in PC([-\tau,0],R^n)$ by $x_t(\theta) = x(t+\theta)$ for $-\tau \le \theta \le 0$. Let J be an interval of the form $[a,b)$ with $0 \le a < b \le \infty$, and $D \subset R^n$ be an open set. Impulsive delay systems can be described as follows

$$\dot{x}(t) = f(t,x_t), \qquad \kappa(t,x(t^-)) \neq 0, \tag{2.84}$$

$$\Delta x = I(t,x(t^-)), \qquad \kappa(t,x(t^-)) = 0. \tag{2.85}$$

$$x(t) = \phi(t), \qquad t \in [t_0-\tau,t_0] \tag{2.86}$$

where $f : J \times PC([-\tau,0],D) \to R^n$, and $\phi \in PC([-\tau,0],D)$.

Before defining the solution of (2.84) – (2.86), we define the difference between two sets A and B as follows

$$A \backslash B = \{t : t \in A \text{ and } t \notin B\}$$

Definition 2.14:[Bal99] *A function $x \in PC([t_0 - \tau, t_0 + \alpha], D)$ where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$ is said to be a solution of* (2.84) *if*

(i) the set $T = \{t \in (t_0, t_0 + \alpha] : \kappa(t, x(t^-)) = 0\}$ of impulse times is finite (possible empty);

(ii) x is continuous at each $t \in (t_0, t_0 + \alpha]\backslash T$;

(iii) the derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0, t_0 + \alpha)$;

(iv) the right-hand derivative of x exists and satisfies the delay differential equation (2.84) *for all $t \in (t_0, t_0 + \alpha]\backslash T$; and*

(v) x satisfies the delay difference equation (2.85) *for all $t \in T$.*

Moreover, if in addition x satisfies (2.86)*, then x is said to be a solution of the initial impulsive delay system* (2.84) – (2.86)*.*

As a special case when impulses occur at fixed times (i.e. $T = \{t_k\}_{k=1}^{\infty}$) with $t_0 < t_1 < \cdots < t_k < \ldots$ and $\lim_{k\to\infty} t_k = \infty$, the solution of (2.84) – (2.86) is described as follows

$$x(t) = \begin{cases} x(t; t_0, \phi), & t \in [t_0 - \tau, t_1) \\ x(t; t_k, x_{t_k}), & t \in [t_k, t_{k+1}), \quad k = 1, 2, \ldots \end{cases} \tag{2.87}$$

where $x(t_k) = x(t_k^-) + I(t_k, x(t_k^-))$.

We end this subsection with stating the sufficient conditions that guarantee stability of the following system.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(t - \tau), \quad t \neq t_k \\ \Delta x &= B_k x(t^-), \quad t = t_k, \qquad k = 1, 2, \cdots \lim_{k\to\infty} t_k = \infty. \end{aligned}$$

Theorem 2.14 [Bal99] *Assume that the eigenvalues of A have negative real parts and let P be the solution of the Lyapunov equation $A^T P + PA = -I$. Suppose that $\|PB\| < 1/2$, $\Sigma_{k=1}^{\infty}\|B_k\| < \infty$, and $t^* = \inf_{k\in Z^+}\{t_k - t_{k-1}\} > 0$. Then the origin is uniformly asymptotically stable.*

2.6.2 Exponential Stability of Singularly Perturbed Systems with Time Delay

Linear time-varying singularly perturbed system with time delay can be described as follows

$$\begin{aligned} \dot{x} &= A_{11}(t)x + A_{12}(t)x_t + B_{11}(t)z + B_{12}(t)z_t \\ \varepsilon\dot{z} &= A_{21}(t)x + A_{22}(t)x_t + B_{21}(t)z \end{aligned} \tag{2.88}$$

where $x \in R^m$, $z \in R^n$ are respectively the slow and fast states of the system; $A_{ij}(t)$, $B_{1j}(t)$ and $B_{21}(t)(i, j = 1, 2)$ are continuous matrices with following dimensions $A_{1s} : m \times m$, $B_{1s} : m \times n$, $A_{2s} : n \times m$, $B_{21} : n \times n$; $A_{22}(t)$ and $B_{21}(t)$ are continuously differentiable, and $B_{21}(t)$ is nonsingular, and ε represents a small parameter. The delayed fast variable is not included for simplicity. Exponential stability of this system was investigated by Liu *et al.*[Liu03]. Here, we state the sufficient conditions that guarantee exponential stability of the system.

Definition 2.12: [Liu03] *The equilibrium point of system* (2.88) *is said to be exponentially stable if there exist positive constants K, and λ such that*

$$\|x(t)\| + \|z(t)\| \leq K\Big(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau\Big)e^{-\lambda(t-t_0)}, \quad t \geq t_0$$

for all $x(t)$ and $z(t)$, the solutions of system (2.88).

Theorem 2.15: [Liu03] *The origin of system* (2.88) *is globally exponentially stable if the following assumptions hold.*

A1) *There exist positive constants α, β such that*

$$\begin{array}{lll} Re[\lambda(A_{11}(t))] \leq -\alpha < 0, & \|A_{11}(t)\| \leq \beta, & \|\dot{A}_{11}(t)\| \leq \beta \\ Re[\lambda(B_{21}(t))] \leq -\alpha < 0, & \|B_{21}(t)\| \leq \beta, & \|\dot{B}_{21}(t)\| \leq \beta \\ \|B_{21}^{-1}(t)A_{21}(t)\| \leq \beta, & \|B_{21}^{-1}(t)A_{22}(t)\| \leq \beta. & \end{array}$$

A2) *There exist bounded functions $a_{rs}(t)$ and $b_{rs}(t)$ $(r, s = 1, 2)$ satisfying*

$$\begin{aligned} 2x^T P_1(t)[A_{12}(t)x_t + B_{11}(t)z + B_{12}(t)z_t] + x^T \dot{P}(t)x &\leq a_{11}(t)\|x\|^2 + a_{12}(t)\|x_t\|_\tau^2 \\ &\quad +b_{11}(t)\|(z-h)\|^2 \\ &\quad +b_{12}(t)\|(z-h)_t\|_\tau^2 \\ -2(z-h)^T P_2(t)\dot{h}(t) + (z-h)^T \dot{P}(t)(z-h) &\leq a_{21}(t)\|x\|^2 + a_{22}(t)\|x_t\|_\tau^2 \\ &\quad +b_{21}(t)\|(z-h)\|^2 \\ &\quad +b_{22}(t)\|(z-h)_t\|_\tau^2 \end{aligned}$$

where $h(t) = -B_{21}^{-1}(t)[A_{21}(t)x + A_{22}(t)x_t]$.

A3) *There exist positive constants ε^* and η such that $-\widetilde{A}(t)$ is an $M-$matrix and $\lambda(\widetilde{A}(t) + \widetilde{A}^T(t)) + 2\|\widetilde{B}(t)\| \leq -\eta < 0$ where*

$$\widetilde{A}(t) = \begin{pmatrix} -\frac{1-a_{11}(t)}{\lambda_{1M}} & \frac{b_{11}(t)}{\lambda_{2m}} \\ \\ \frac{a_{21}(t)}{\lambda_{1m}} & -\frac{1-\varepsilon^* b_{21}(t)}{\varepsilon^*\lambda_{2M}} \end{pmatrix} \quad and \quad \widetilde{B}(t) = \begin{pmatrix} \frac{a_{12}(t)}{\lambda_{1m}} & \frac{b_{12}(t)}{\lambda_{2m}} \\ \\ \frac{a_{22}(t)}{\lambda_{1m}} & \frac{b_{22}(t)}{\lambda_{2m}} \end{pmatrix}$$

where $\lambda_{rm} = min\{\lambda_{min}(P_r(t)); r = 1, 2\}$, and $\lambda_{rM} = max\{\lambda_{max}(P_r(t)); r = 1, 2\}$, and $P_r(t)(r = 1, 2)$ are respectively the solutions of the following Lyapunov equa-

tions.

$$A_{11}^T(t)P_1(t) + P_1(t)A_{11}(t) = -I_n$$
$$B_{21}^T(t)P_2(t) + P_2(t)B_{21}(t) = -I_m.$$

Assumption $A1$ is introduced to guarantee exponential stability of the uncoupled subsystems

$$\begin{aligned} \dot{x}(t) &= A_{11}(t)x(t) \\ \dot{z}(t) &= B_{21}(t)z(t). \end{aligned} \tag{2.89}$$

Viewing (2.88) as a perturbation of the uncoupled subsystems in (2.89), it is reasonable to have assumption $A2$, meaning that the interconnections are bounded. Finally, for the stability of the interconnected system (2.88), Assumptions $A1$ and $A2$ are not sufficient, so that an additional condition is required; that is, as pointed out in this chapter, the degrees of stability for the uncoupled subsystems are larger than the strength of the interconnections. This fact is represented by Assumption $A3$.

Chapter 3

Exponential Stability of Switched Delay Singularly Perturbed Systems

Having introduced the required material in Chapter 2, in this chapter we investigate exponential stability of switched singularly perturbed systems with time delay. In Section 3.1, we introduce some useful lemmas (Lemmas 3.1 and 3.3) that help us prove exponential stability of switched delay systems that consist of stable and unstable modes. These lemmas, in fact, are analogous to Lemmas 2.2 and 3.2 which apply to stable modes. In Section 3.2, linear singularly perturbed systems with time-varying matrices are discussed, while in Section 3.3 we study a special class of nonlinear systems. Multiple Lyapunov functions will be used as a tool to establish stability of the system. Examples are given to verify our theoretical results.

3.1 Exponential Stability of Switched Delay Systems

Linear time-invariant switched delay systems can be described as follows.

$$\begin{aligned}\dot{x}(t) &= A_i x(t) + B_i x(t-\tau), \qquad t \in [t_k, t_{k-1}) \\ x(t) &= \phi(t), \qquad t \in [t_0 - \tau, t_0]\end{aligned} \tag{3.1}$$

where $k = 1, 2, \cdots$, $\lim_{k\to\infty} t_k = \infty$, and A_i and B_i are $n \times n$ constant matrices for all $i \in S = \{1, 2, \cdots, N\}$. The stability of this system with stable subsystems was first investigated by Zhang and Liu [Zha04] by finding one *common Lyapunov function* that is valid for the family of stable modes given in (3.1). In this work, we use *multiple Lyapunov functions* as a tool for analyzing the stability of switched delay systems; that is, assuming that the subsystems in (3.1) are exponentially stable, there is *one* Lyapunov function for each subsystem. A more general switched system where the set S is extended to include in addition unstable subsystems will also be addressed in this section.

3.1.1 Systems with Stable Subsystems

In the following theorem, we give sufficient conditions that guarantee exponential stability of system (3.1).

Theorem 3.1: *The origin of system* (3.1) *is exponentially stable if the following conditions are satisfied*
A1) *For each* $i \in S$*, the delay subsystems in* (3.1) *satisfy the conditions of Theorem* 2.13.

A2) *There exists a positive constant* $0 < \nu < \xi_i$ *such that*

$$\ln \mu + \xi_i \tau - \nu(t_k - t_{k-1}) \leq 0.$$

where ξ_i *is defined in Lemma* 2.2.

Remark: Obviously, the presence of delay results in the dwell times of delay systems being longer than those of the corresponding ordinary systems.

Proof:

Let $V_i(x) = x^T P_i x (i \in S)$ be a Lyapunov function for the i^{th} subsystem. Then, the time derivative of V_i along the trajectories of (3.1) is

$$\dot{V}_i \leq -\frac{\lambda_{min}(Q_i) - \beta_i^*}{\lambda_{max}(P_i)} V_i(x) + \frac{\beta_i^*}{\lambda_{min}(P_i)} \|V_{i_t}\|_\tau$$

Then, by Lemma 2.2 , where $\alpha_i = \frac{\lambda_{min}(Q_i) - \beta_i^*}{\lambda_{max}(P_i)}$, $\beta_i = \frac{\beta_i^*}{\lambda_{min}(P_i)}$ and $\beta_i^* = \|P_i B_i\|$, there exists a positive constant ξ_i such that

$$V_i(x) \leq \|V_{i_{t_{k-1}}}\|_\tau e^{-\xi_i(t - t_{k-1})}$$

Following the analysis in Chapter 2 and after running modes 1 and 2, we have respectively

$$\begin{aligned} V_1(x) &\leq \|V_{1_{t_0}}\|_\tau e^{-\xi_1(t-t_0)}, \quad t \in [t_0, t_1) \\ V_2(x) &\leq \mu e^{\xi_1 \tau} e^{-\lambda_1(t_1 - t_0)} \|V_{1_{t_0}}\|_\tau e^{-\lambda_2(t-t_1)}, \quad t \in [t_1, t_2) \end{aligned}$$

Generally, we have for $t \in [t_{k-1}, t_k)$

$$\begin{aligned} V_i(x) &\leq \mu e^{\xi_1 \tau} e^{-\xi_1(t_1 - t_0)} \mu e^{\xi_2 \tau} e^{-\xi_2(t_2 - t_1)} \cdots \mu e^{\xi_{i-1} \tau} e^{-\xi_{i-1}(t_{k-1} - t_{k-2})} \\ &\quad \times \|V_{i_{t_0}}\|_\tau e^{-\xi_i(t - t_{k-1})} \end{aligned}$$

Let $\lambda = \min\{\xi_i : i \in S\}$. Then,

$$\begin{aligned} V_i(x) &\leq \mu e^{\xi_1 \tau} \mu e^{\xi_2 \tau} \cdots \mu e^{\xi_{i-1} \tau} \|V_{i_{t_0}}\|_\tau e^{-\lambda(t-t_0)} \\ &\leq \mu e^{\xi_1 \tau} e^{-\nu(t_1-t_0)} \mu e^{\xi_2 \tau} e^{-\nu(t_2-t_1)} \cdots \mu e^{\xi_{i-1} \tau} e^{-\nu(t_{k-1}-t_{k-2})} \|V_{i_{t_0}}\|_\tau e^{-(\lambda-\nu)(t-t_0)} \end{aligned}$$

By Assumption $A2$, we have

$$V_i(x) \leq \|V_{1_{t_0}}\|_\tau e^{-(\lambda-\nu)(t-t_0)}, \qquad t \geq t_0.$$

Then, there exists a positive constant K such that

$$\|x(t)\| \leq K \|x_{t_0}\|_\tau e^{-(\lambda-\nu)(t-t_0)/2}, \qquad t \geq t_0.$$

In the following example we illustrate the theoretical result of Theorem 3.1.

Example 3.1: Consider the switched delay system (3.1) where $S = \{1, 2\}$, $\tau = 1$,

$$A_1 = \begin{pmatrix} -12 & 1 \\ 1 & -9 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -8 & 2 \\ 0 & -7 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

and the initial condition is $\phi(t) = t + 1$. Take

$$Q_1 = \begin{pmatrix} 9 & 0 \\ 0 & 5 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 14 & 0 \\ 0 & 13 \end{pmatrix}.$$

Then,

$$P_1 = \begin{pmatrix} 0.3776 & 0.0314 \\ 0.0314 & 0.2813 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0.906 & 0.1238 \\ 0.1238 & 0.9286 \end{pmatrix}$$

Hence, $\mu = \lambda_M/\lambda_m = 3.8301$, $\alpha_1 = 10.5836$, $\alpha_2 = 11.481$, $\beta_1 = 3.3274$, $\beta_2 = 1.3136$, so that condition $-\alpha_i + \beta_i < 0$ is satisfied. The mode decay rates are $\xi_1 = 1.0523$ and $\xi_2 = 1.9787$. Taking $\nu = 1 < \xi_i$, then from Assumption $A2$ the dwell times are $T_{D_1} = 2.4$, and $T_{D_2} = 3.3$, respectively, and the switched system decay rate is $(\lambda - \nu)/2 = 0.00615$. But, if we take for instance $\nu = 0.5$, then the mode decay rates are respectively $T_{D_1} = 3.4$ and $T_{D_2} = 4.7$, and $(\lambda - \nu)/2 = 0.5524$. Clearly, the constant ν plays a role in the dwell time amount and system decay rates. Figure (1.1) shows that the solution of the switched system vanishes exponentially after running mode 1 and 2 on the first and second interval, respectively.

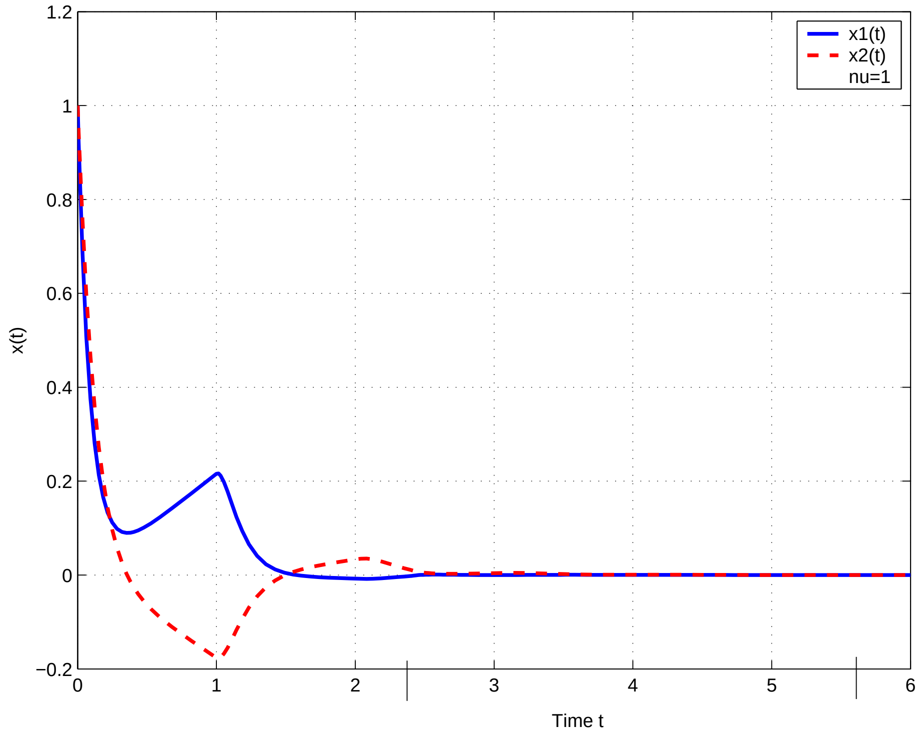


Figure 3.1: Switched delay system with stable modes

3.1.2 Systems with Stable and Unstable Subsystems

Consider again the switched delay systems (3.1)

$$\dot{x}(t) = A_i x(t) + B_i x(t-\tau),$$

where $S = S_u \cup S_s$. Before stating sufficient conditions that guarantee exponential stability of this system, we first present the following lemmas. These lemmas are important in the proof of our theorems.

Lemma 3.1: *For $a \in R$, with $a > 0$, and $t_0 \in R_+$, let $u : [t_0, t_0 + a) \to R_+$ satisfy the following delay differential inequality*

$$\dot{u}(t) \leq \alpha u(t) + \beta \sup_{\theta \in [t-\tau, t]} u(\theta), \qquad t \in [t_0, t_0 + a).$$

Assume that $\alpha + \beta > 0$. Then, there exist positive constants ξ and k such that

$$u(t) \leq k e^{\xi(t-t_0)}, \qquad t \in [t_0, t_0 + a) \tag{3.2}$$

where $\xi = \alpha + \beta$ and $k = \sup_{\theta \in [t_0 - \tau, t_0]} u(\theta)$.

Proof:

Claim $y(t) = k e^{\xi(t-t_0)}$ is a solution of the delay differential equation

$$\dot{y}(t) = \alpha y(t) + \beta \sup_{\theta \in [t-\tau, t]} y(\theta), \qquad t \in [t_0, t_0 + a) \tag{3.3}$$

with the initial condition

$$y(t) = u(t), \qquad t \in [t_0 - \tau, t_0].$$

To prove the claim, we check that

$$\begin{aligned} \dot{y}(t) &= \xi k e^{\xi(t-t_0)} \\ \sup_{\theta \in [t-\tau, t]} y(\theta) &= k e^{\xi(t-t_0)} = y(t) \end{aligned}$$

and (3.3) becomes

$$\xi k e^{\xi(t-t_0)} = \alpha k e^{\xi(t-t_0)} + \beta k e^{\xi(t-t_0)}$$

Hence, $y(t) = ke^{\xi(t-t_0)}$ is indeed a solution of (3.3) where ξ and k are defined above. By Proposition 2.1, we have

$$u(t) \le ke^{\xi(t-t_0)} = \sup_{\theta\in[t_0-\tau,t_0]} u(\theta)e^{\xi(t-t_0)}, \qquad t \in [t_0, t_0 + a).$$

Lemma 3.2: *For $a \in R$ with $a > 0$ and $t \in [t_0, t_0+a)$, $t_0 \in R_+$, let $A(t)$ and $B(t)$ be $n \times n$ matrices of continuous functions, $\alpha(t) = \lambda\Big(A(t) + A^T(t)\Big)$, $\|B(t)\| \le \beta_1$ and $\alpha(t) + \|B(t)\| \le \beta_2$, $(\beta_2 > 0)$. Assume that the following inequality is satisfied.*

$$\dot{y}(t) \le A(t)y(t) + B(t) \sup_{\theta\in[t-\tau,t]} y(\theta)$$

where $y(t) = \Big(y_1(t), y_2(t), \cdots y_n(t)\Big)^T \ge 0$ and
$\sup_{\theta\in[t-\tau,t]} y(\theta) = \Big(\sup_{\theta\in[t-\tau,t]} y_1(\theta), \sup_{\theta\in[t-\tau,t]} y_2(\theta), \cdots, \sup_{\theta\in[t-\tau,t]} y_n(\theta)\Big)^T$.
Then, there exists a $\xi > 0$ such that

$$\|y(t)\| \le \|y_{t_0}\|_\tau e^{\xi(t-t_0)}, \qquad t \in [t_0, t_0 + a)$$

where $\xi = (\beta_1 + \beta_2)/2$.

Proof:

Let $v(t) = y^T(t)y(t) = \|y(t)\|^2$. Then,

$$\begin{aligned}
\dot{v}(t) &= \dot{y}^T(t)y(t) + y^T(t)\dot{y}(t) \\
&\le \Big(A(t)y(t) + B(t) \sup_{\theta\in[t-\tau,t]} y(\theta)\Big)^T y(t) + y^T(t)\Big(A(t)y(t) + B(t) \sup_{\theta\in[t-\tau,t]} y(\theta)\Big) \\
&\le y^T(t)\Big(A^T(t) + A(t)\Big)y(t) + 2\|B(t)\|\|y(t)\|\|y_t\|_\tau \\
&\le \alpha(t)y^T(t)y(t) + \|B(t)\|\|y(t)\|^2 + \|B(t)\|\|y_t\|_\tau^2 \\
&= \Big(\alpha(t) + \|B(t)\|\Big)v(t) + \|B(t)\|\|v_t\|_\tau \\
&\le \beta_2 v(t) + \beta_1\|v_t\|_\tau.
\end{aligned}$$

By Lemma 3.1, there exists a $\xi > 0$ such that

$$v(t) \le \|v_{t_0}\|_\tau e^{2\xi(t-t_0)},$$

where $2\xi = \beta_2 + \beta_1$, and hence

$$\|y(t)\| \le \|y_{t_0}\|_\tau e^{\xi(t-t_0)}, \qquad t \in [t_0, t_0 + a).$$

Having proved these lemmas, we establish exponential stability of the switched delay system (3.1) with $S = S_u \cup S_s$.

Theorem 3.2: *The origin of switched delay system* (3.1)*, where* $S = S_u \cup S_s$*, is exponentially stable if the following assumptions are satisfied.*

A1-i) *For* $i \in S_u$*,*

$$Re[\lambda(A_i)] > 0, \quad \textit{and} \quad Re[\lambda(A_i + B_i)] > 0.$$

A1-ii) *For* $i \in S_s$*,*

$$Re[\lambda(A_i)] < 0, \quad \textit{and} \quad -\Big(\frac{\lambda_{\min}(Q_i) - \beta_i^*}{\lambda_M}\Big) + \frac{\beta_i^*}{\lambda_m} < 0$$

where $\beta_i^* = \|P_i B_i\|$*.*

A2) *For any* t_0*, the switching law guarantees that*

$$\inf_{t \ge t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \ge \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{3.4}$$

where $\lambda^* \in (0, \lambda^-)$*; furthermore, there exists* $0 < \nu < \lambda^*$ *such that*

(i) for $i \in \{1, 2, \cdots, l\}$

$$\ln \mu - \nu(t_k - t_{k-1}) \le 0, \qquad k = 1, 2, \cdots, l \tag{3.5}$$

(ii) for $i \in \{l+1, l+2, \cdots, m-1\}$

$$\ln \mu + \zeta_i \tau - \nu(t_k - t_{k-1}) \le 0, \qquad k = l+1, l+2, \cdots, m-1. \tag{3.6}$$

Assumption $(A1-i)$ means that for each $i \in S_u$ the i^{th} subsystem is unstable, while $(A1-ii)$ is made to ensure exponential stability of each subsystem in (3.1). Since (exponential) stability of each individual subsystem cannot guarantee the stability of switched (or generally hybrid) systems, this suggests finding a complementary condition, which is represented by Assumption $A2$. In fact, Condition (3.4) means that stable modes are required to be activated longer than unstable ones. Conditions (3.5) and (3.6) are to keep solutions down whenever modes are switched.

Proof:

For any $i \in S$, define $V_i(x) = x^T P_i x$. Then, the derivative of V_i along the trajectories of (3.1) is

$$\dot{V}_i(x) = x^T \Big(A_i^T P_i + P_i A_i \Big) x + 2x^T P_i B_i x(t-\tau)$$

For $i \in S_u$, we have

$$\begin{aligned}\dot{V}_i(x) &\leq 2\gamma x^T P_i x + \beta_i^* \|x\|^2 + \beta_i^* \|x_t\|^2 \\ &\leq \Big(2\gamma + \frac{\beta_i^*}{\lambda_m}\Big) V_i(x) + \frac{\beta_i^*}{\lambda_m} \|V_{i_t}\|_\tau\end{aligned}$$

where $\gamma > 0$ such that $Re([A_i - \gamma I]) < 0$. With the aid of Lemma 3.2, where $\beta_1 = 2\gamma + \beta_i^*/\lambda_m$ and $\beta_2 = \beta_i^*/\lambda_m$, there exists a $\xi_i > 0$ such that

$$V_i(x) \leq \|V_{i_{t_{k-1}}}\|_\tau e^{\xi_i (t-t_{k-1})}$$

Similarly, for $i \in S_s$, we have

$$\begin{aligned}\dot{V}_i(x) &\leq -\lambda_{\min}(Q_i)\|x\|^2 + \beta_i^* \|x\|^2 + \beta_i^* \|x_t\|^2 \\ &\leq -\Big(\frac{\lambda_{\min}(Q_i) - \beta_i^*}{\lambda_M}\Big) V_i(x) + \frac{\beta_i^*}{\lambda_m} \|V_{i_t}\|_\tau\end{aligned}$$

By Lemma 2.2, where $\alpha_i = \frac{\lambda_{\min}(Q_i) - \beta_i^*}{\lambda_M}$ and $\beta_i = \frac{\beta_i^*}{\lambda_m}$, there exists a $\zeta_i > 0$ such that

$$V_i(x) \leq \|V_{i_{t_{k-1}}}\|_\tau e^{-\zeta_i (t-t_{k-1})}$$

To obtain a general estimate, let us run l modes and switch l times from an unstable mode, and run *m-l* modes and switch *m-l-1* times from a stable mode. Then,

$$V_m(x) \leq \prod_{i=1}^{l} \mu e^{\xi_i(t_i-t_{i-1})} \times \prod_{j=l+1}^{m-l-1} \mu e^{\zeta_j \tau} e^{-\zeta_j(t_j-t_{j-1})} \times \|V_{1_{t_0}}\|_\tau e^{-\zeta_m(t-t_{m-1})}$$

By condition (3.4), we have

$$V_m(x) \leq \prod_{i=1}^{l} \mu \times \prod_{j=l+1}^{m-l-1} \mu e^{\zeta_j \tau} \times \|V_{1_{t_0}}\|_\tau e^{-\lambda^*(t-t_0)}$$

and, by (3.5) and (3.6) we get

$$V_m(x) \leq \|V_{1_{t_0}}\|_\tau e^{-(\lambda^*-\nu)(t-t_0)}, \qquad t \geq t_0.$$

Thus,

$$\|x(t)\| \leq K\|x_{t_0}\|_\tau e^{-(\lambda^*-\nu)(t-t_0)/2}, \qquad t \geq t_0, \quad K = \sqrt{\mu}$$

This shows that the origin of (3.1) is exponentially stable.

The following example illustrates these results.

Example 3.2: Consider system (3.1) where

$$A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

Let $\gamma = 1$,

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then,

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

We also have $\mu = 1.5$, the growth rate $\xi = 2.5$, the decay rate $\zeta = 1.3318$, and the dwell times of unstable and stable modes are respectively $T_{D_u} = 0.31$, and $T_{D_s} = 1.31$. Figure 3.2(a) shows the solution dying out exponentially. Obviously, solution $x_1(t)$ illustrates the requirement that stable modes be run longer than unstable ones.

In fact, one can accelerate the convergence of solutions to the origin by taking $T_{D_s} = 2$, for instance . Figure 3.2(b) shows this result.

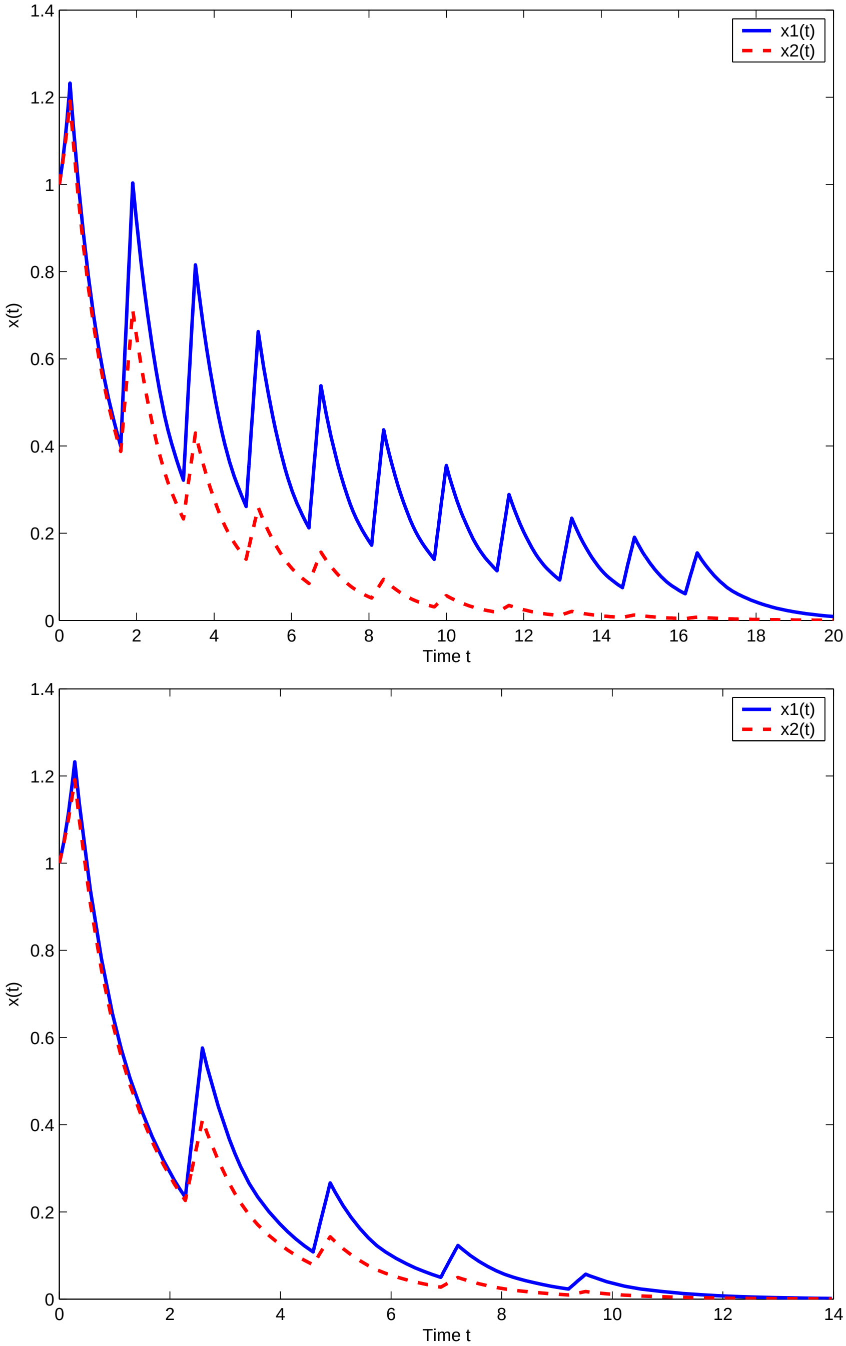


Figure 3.2: Switched delay system with unstable and stable modes

Lemma 3.2 in fact is an analog to Lemma 3.3 which is stated below and will be needed in proving our theorems.

Lemma 3.3: [Liu03] *For* $t \in [t_0, \infty)$*, let* $A(t)$ *and* $B(t)$ *be* $n \times n$ *matrices of continuous functions,* $A(t)$ *satisfy the assumptions of Theorem* 2.6*, and* $B(t)$ *be bounded. Furthermore, assume that*

*A*1) $\lambda(A^T(t) + A(t)) \leq -\alpha(t) < 0.$

*A*2) $-\alpha(t) + 2\|B(t)\| \leq -\beta < 0,$ *with* β *being a positive constant.*

*A*3) $\dot{y}(t) \leq A(t)y(t) + B(t)\sup_{t-\tau\leq\theta\leq t} y(\theta),$

where $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T \geq 0,$ *and*

$\sup_{t-\tau\leq\theta\leq t} y(\theta) = (\sup_{t-\tau\leq\theta\leq t} y_1(\theta), \sup_{t-\tau\leq\theta\leq t} y_2(\theta), \cdots, \sup_{t-\tau\leq\theta\leq t} y_n(\theta))^T.$

Then, there exists a positive constant ξ *such that*

$$\|y(t)\| \leq \|y_{t_0}\|_\tau e^{-\xi(t-t_0)}, \qquad t \geq t_0.$$

3.2 Linear Singularly Perturbed Systems

Linear time-varying switched singularly perturbed systems with time delay can be described as follows.

$$\begin{aligned} \dot{x} &= A_{11_i}(t)x + A_{12_i}(t)x_t + B_{11_i}(t)z + B_{12_i}(t)z_t, \\ \varepsilon\dot{z} &= A_{21_i}(t)x + A_{22_i}(t)x_t + B_{21_i}(t)z, \qquad t \in [t_{k-1}, t_k), \end{aligned} \tag{3.7}$$

where $i \in S = \{1, 2, \cdots, N\}$, $k = 1, 2, \cdots$ with $\lim_{k\to\infty} t_k = \infty$, $x \in R^m$, $z \in R^n$ are respectively the slow and fast states of the system; $A_{rs_i}(t)$, $B_{1s_i}(t)$ and $B_{21_i}(t)(r, s = 1, 2)$ are continuous matrices with following dimensions $A_{1s_i} : m \times m$, $B_{1s_i} : m \times n$,

$A_{2s_i} : n \times m$, $B_{21_i} : n \times n$; $A_{22_i}(t)$ and $B_{21_i}(t)$ are continuously differentiable, $B_{21_i}(t)$ is nonsingular, and ε represents a small parameter. The delayed fast variable is not included in the fast system for simplicity.

The continuous initial conditions of this system are

$$\begin{aligned} x(t) &= \phi_1(t), \\ z(t) &= \phi_2(t), \qquad t \in [t_0 - \tau, t_0] \end{aligned}$$

where τ represents time delay.

3.2.1 Systems with Stable Modes

Theorem 3.3: *The origin of system* (3.7) *is globally exponentially stable if the following assumptions hold.*

Assumptions: *Assume that for $i \in S$ and $t \in [t_0, \infty)$*

A1) *there exist positive constants α, β such that;*

$$\begin{array}{lll} Re[\lambda(A_{11_i}(t))] \le -\alpha < 0, & \|A_{11_i}(t)\| \le \beta, & \|\dot{A}_{11_i}(t)\| \le \beta \\ Re[\lambda(B_{21_i}(t))] \le -\alpha < 0, & \|B_{21_i}(t)\| \le \beta, & \|\dot{B}_{21_i}(t)\| \le \beta \end{array}$$
$$\|B_{21_i}^{-1}(t)A_{21_i}(t)\| \le \beta, \qquad \|B_{21_i}^{-1}(t)A_{22_i}(t)\| \le \beta.$$

A2) *there exist bounded functions $a_{rs_i}(t)$ and $b_{rs_i}(t)$, $(r, s = 1, 2)$, satisfying*

$$\begin{aligned} 2x^T P_{1_i}(t)[A_{12_i}(t)x_t + B_{11_i}(t)z + B_{12_i}(t)z_t] \quad + \quad & x^T \dot{P}_{1_i}(t)x \le a_{11_i}(t)\|x\|^2 \\ & + a_{12_i}(t)\|x_t\|_\tau^2 \\ & + b_{11_i}(t)\|(z - h_i)\|^2 \\ & + b_{12_i}(t)\|(z - h_i)_t\|_\tau^2 \end{aligned}$$

$$\begin{aligned} -2(z-h_i)^T P_{2_i}(t)\dot{h}_i + (z-h_i)^T \dot{P}_{2_i}(t)(z-h_i) \;\; &\leq \;\; a_{21_i}(t)\|x\|^2 \\ &\quad + a_{22_i}(t)\|x_t\|_\tau^2 \\ &\quad + b_{21_i}(t)\|(z-h_i)\|^2 \\ &\quad + b_{22_i}(t)\|(z-h_i)_t\|_\tau^2 \end{aligned}$$

where $h_i(t) = -B_{21_i}^{-1}(t)[A_{21_i}(t)x + A_{22_i}(t)x_t]$ *and* $P_{1_i}(t)$ *and* $P_{2_i}(t)$ *are respectively the solutions of the Lyapunov equations.*

$$\begin{aligned} A_{11_i}^T(t)P_{1_i}(t) + P_{1_i}(t)A_{11_i}(t) &= -I_m, \\ B_{21_i}^T(t)P_{2_i}(t) + P_{2_i}(t)B_{21_i}(t) &= -I_n, \end{aligned}$$

where I_m *and* I_n *are identity matrices.*

A3) *there exist positive constants* ε_i^* *and* η *such that* $-\widetilde{A_i}(t)$ *is an* $M-$*matrix and* $\lambda(\widetilde{A_i}(t) + \widetilde{A_i}^T(t)) + 2\|\widetilde{B_i}(t)\| \leq -\eta < 0$ *where*

$$\widetilde{A_i}(t) = \begin{pmatrix} -\frac{1-a_{11_i}(t)}{\lambda_{1M}} & \frac{b_{11_i}(t)}{\lambda_{2m}} \\ \\ \frac{a_{21_i}(t)}{\lambda_{1m}} & -\frac{1-\varepsilon_i^* b_{21_i}(t)}{\varepsilon_i^* \lambda_{2M}} \end{pmatrix} \quad \text{and} \quad \widetilde{B_i}(t) = \begin{pmatrix} \frac{a_{12_i}(t)}{\lambda_{1m}} & \frac{b_{12_i}(t)}{\lambda_{2m}} \\ \\ \frac{a_{22_i}(t)}{\lambda_{1m}} & \frac{b_{22_i}(t)}{\lambda_{2m}} \end{pmatrix}$$

where $\lambda_{rm} = \min\{\lambda_{min}(P_{r_i}), i \in S\}$, and $\lambda_{rM} = \max\{\lambda_{max}(P_{r_i}), i \in S\}$, $r = 1, 2$.

A4) *For* $i \geq 2$ *and* $k \geq 2$, *there exists a positive constant* $0 < \nu < \xi_i$ *such that*

$$\ln(2\mu) + \xi_j\tau - \nu(t_l - t_{l-1}) \leq 0, \qquad j = 1, 2, \cdots, i-1, \quad l = 1, 2, \cdots, k-1.$$

where ξ_i *is a unique positive solution of*

$$\xi_i = \lambda(\widetilde{A_i}^T + \widetilde{A_i}) + \|\widetilde{B_i}\| + \|\widetilde{B_i}\|e^{\xi_i\tau}$$

Proof:

For $i \in S$, define $V_i(x(t)) = x^T(t)P_{1_i}(t)x(t)$ and $W_i((z-h_i)(t)) = (z-h_i)^T(t)P_{2_i}(t)(z-h_i)(t)$ as Lyapunov functions for system (3.7).

Then, the derivative of $V_i(x(t))$ along the trajectories of the state $x(t)$ is given by

$$\begin{aligned}
\dot{V}_i(x(t)) &= \dot{x}^T P_{1_i}(t)x + x^T P_{1_i}(t)\dot{x} + x^T \dot{P}_{1_i}(t)x \\
&= x^T\Big(A_{11_i}^T(t)P_{1_i}(t) + P_{1_i}(t)A_{11_i}(t)\Big)x + 2x^T P_{1_i}(t)\Big(A_{12_i}(t)x_t + B_{11_i}(t)z \\
&\quad + B_{12_i}(t)z_t\Big) + x^T\dot{P}_{1_i}(t)x \\
&\leq -(1-a_{11_i}(t))\|x\|^2 + a_{12_i}(t)\|x_t\|_\tau^2 + b_{11_i}(t)\|(z-h_i)\|^2 \\
&\quad + b_{12_i}(t)\|(z-h_i)_t\|_\tau^2 \\
&\leq -\frac{1-a_{11_i}(t)}{\lambda_{1M}}V_i(x(t)) + \frac{b_{11_i}(t)}{\lambda_{2m}}W_i((z-h_i)(t)) + \frac{a_{12_i}(t)}{\lambda_{1m}}\|V_{i_t}\|_\tau \\
&\quad + \frac{b_{12_i}(t)}{\lambda_{2m}}\|W_{i_t}\|_\tau
\end{aligned}$$

Similarly, the derivative of $W_i((z-h_i)(t))$ along the trajectories of the state $z(t)$ is given by

$$\begin{aligned}
\dot{W}_i((z-h_i)(t)) &= (\dot{z}-\dot{h}_i)^T P_{2_i}(t)(z-h_i) + (z-h_i)^T P_{2_i}(t)(\dot{z}-\dot{h}_i) \\
&\quad + (z-h_i)^T\dot{P}_{2_i}(t)(z-h_i) \\
&= \Big(\frac{1}{\varepsilon}(\overbrace{A_{21_i}(t)x + A_{22_i}(t)x_t}^{-B_{21_i}(t)h_i} + B_{21_i}(t)z) - \dot{h}_i\Big)^T P_{2_i}(t)(z-h_i) \\
&\quad + (z-h_i)^T P_{2_i}(t) \times \Big(\frac{1}{\varepsilon}(\overbrace{A_{21_i}(t)x + A_{22_i}(t)x_t}^{-B_{21_i}(t)h_i} + B_{21_i}(t)z) - \dot{h}_i\Big) \\
&\quad + (z-h_i)^T\dot{P}_{2_i}(z-h_i) \\
&= \Big(\frac{1}{\varepsilon}(-B_{21_i}(t)h + B_{21_i}(t)z) - \dot{h}_i\Big)^T P_{2_i}(t)(z-h_i) + (z-h_i)^T P_{2_i}(t) \\
&\quad \times\Big(\frac{1}{\varepsilon}(-B_{21_i}(t)h + B_{21_i}(t)z) - \dot{h}_i\Big) + (z-h_i)^T\dot{P}_{2_i}(t)(z-h_i)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon}\Big(- h^T B_{21_i}^T(t) + z^T B_{21_i}^T(t)\Big) P_2(z-h_i)(t) - \dot{h}_i^T P_{2_i}(t)(z-h_i) \\
&\quad +(z-h_i)^T P_{2_i}(t)\frac{1}{\varepsilon}B_{21_i}(t)(z-h_i) - (z-h_i)^T P_{2_i}(t)\dot{h}_i \\
&\quad +(z-h_i)^T \dot{P}_{2_i}(t)(z-h_i) \\
&= \frac{1}{\varepsilon}(z-h_i)^T(B_{21_i}^T(t)P_{2_i} + P_{2_i}(t)B_{21_i}(t))(z-h_i) - 2(z-h_i)^T P_{2_i}(t)\dot{h}_i \\
&+ (z-h_i)^T \dot{P}_{2_i}(t)(z-h_i) \\
&\leq -\frac{1}{\varepsilon}\|(z-h_i)\|^2 + a_{21_i}(t)\|x\|^2 + a_{22_i}(t)\|x_t\|_\tau^2 + b_{21_i}(t)\|(z-h_i)\|^2 \\
&\quad +b_{22_i}(t)\|(z-h_i)_t\|_\tau^2 \\
&\leq \frac{a_{21_i}(t)}{\lambda_{1m}}V_i(x(t)) - \frac{1-\varepsilon_i b_{21_i}(t)}{\varepsilon_i\lambda_{2M}}W_i((z-h_i)(t)) + \frac{a_{22_i}(t)}{\lambda_{1m}}\|V_{i_t}\|_\tau \\
&\quad +\frac{b_{22_i}(t)}{\lambda_{2m}}\|W_{i_t}\|_\tau \\
&\leq \frac{a_{21_i}(t)}{\lambda_{1m}}V_i(x(t)) - \frac{1-\varepsilon_i^* b_{21_i}(t)}{\varepsilon_i^*\lambda_{2M}}W_i((z-h_i)(t)) + \frac{a_{22_i}(t)}{\lambda_{1m}}\|V_{i_t}\|_\tau \\
&\quad +\frac{b_{22_i}(t)}{\lambda_{2m}}\|W_{i_t}\|_\tau
\end{aligned}
$$

Namely, we have

$$
\begin{aligned}
\dot{V}(x(t)) &\leq -\frac{1-a_{11_i}(t)}{\lambda_{1M}}V_i(x(t)) + \frac{b_{11_i}(t)}{\lambda_{2m}}W_i((z-h_i)(t)) + \frac{a_{12_i}(t)}{\lambda_{1m}}\|V_{i_t}\|_\tau \\
&\quad +\frac{b_{12_i}(t)}{\lambda_{2m}}\|W_{i_t}\|_\tau \\
\dot{W}((z-h)i)(t)) &\leq \frac{a_{21_i}(t)}{\lambda_{1m}}V_i(x(t)) - \frac{1-\varepsilon_i^* b_{21_i}(t)}{\varepsilon_i^*\lambda_{2M}}W_i((z-h_i)(t)) + \frac{a_{22_i}(t)}{\lambda_{1m}}\|V_{i_t}\|_\tau \\
&\quad +\frac{b_{22_i}(t)}{\lambda_{2m}}\|W_{i_t}\|_\tau
\end{aligned}
$$

By Lemma 3.3, there exists a positive constant ξ_i such that

$$
\begin{aligned}
V_i(x(t)) &\leq (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{-\xi_i(t-t_{k-1})} \\
W_i((z-h_i)(t) &\leq (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{-\xi_i(t-t_{k-1})} \qquad (3.8)
\end{aligned}
$$

Recall that we have for any $i, j \in S$

$$\begin{aligned} V_j(x(t)) &\leq \mu_1 V_i(x(t)) \\ W_j((z-h_i)(t)) &\leq \mu_2 W_i((z-h_i)(t)) \end{aligned}$$

Let $\mu = \max\{\mu_1, \mu_2\}$. Then, we have

$$V_j(x(t)) \leq \mu V_i(x(t)) \tag{3.9}$$

$$W_j((z-h_i)(t)) \leq \mu W_i((z-h_i)(t)) \tag{3.10}$$

For convenience, we will carry out only one Lyapunov function V_i. Running the first and second modes on the first and second intervals respectively, we get

$$\begin{aligned} V_1(x(t)) &\leq (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\xi_1(t-t_0)} \\ V_2(x(t)) &\leq (\|V_{2_{t_1}}\|_\tau + \|W_{2_{t_1}}\|_\tau)e^{-\xi_2(t-t_1)} \end{aligned} \tag{3.11}$$

where the norms in the last inequalities are found as follows;
From (3.9), we have

$$V_2(x(t)) \leq \mu V_1(x(t)) \qquad \forall t \in [t_1, t_2)$$

This inequality holds for all $t \in [t_1, t_2)$, so that at $t = t_1$, we have

$$V_2(x(t_1)) \leq \mu V_1(x(t)) \leq \mu(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\xi_1(t-t_0)}$$

so that

$$\|V_{2_{t_1}}\|_\tau \leq \mu e^{\xi_1 \tau}(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\xi_1(t_1-t_0)}$$

Similarly,

$$\|W_{2_{t_1}}\|_\tau \leq \mu e^{\xi_1 \tau}(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\xi_1(t_1-t_0)}$$

Thus, inequality (3.11) becomes

$$V_2(x(t)) \leq 2\mu e^{\xi_1\tau}(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\xi_1(t_1-t_0)}e^{-\xi_2(t-t_1)}, \quad t \in [t_1, t_2)$$

Generally, for $t \in [t_{k-1}, t_k)$ we have

$$\begin{aligned} V_i(x(t)) &\leq 2\mu e^{\xi_1\tau}e^{-\xi_1(t_1-t_0)}2\mu e^{\xi_2\tau}e^{-\xi_2(t_2-t_1)}\cdots 2\mu e^{\xi_{i-1}\tau}e^{-\xi_{i-1}(t_{k-1}-t_{k-2})} \\ &\times (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\xi_i(t-t_{k-1})} \\ W_i((z-h_i)(t)) &\leq 2\mu e^{\xi_1\tau}e^{-\xi_1(t_1-t_0)}2\mu e^{\xi_2\tau}e^{-\xi_2(t_2-t_1)}\cdots 2\mu e^{\xi_{i-1}\tau}e^{-\xi_{i-1}(t_{k-1}-t_{k-2})} \\ &\times (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\xi_i(t-t_{k-1})} \end{aligned}$$

Let $\xi = \min\{\xi_j; j = 1, 2, \cdots, i\}$. Then,

$$\begin{aligned} V_i(x(t)) &\leq 2\mu e^{\xi_1\tau}2\mu e^{\xi_2\tau}\cdots 2\mu e^{\xi_{i-1}\tau}(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\xi(t-t_0)} \\ &= 2\mu e^{\xi_1\tau}2\mu e^{\xi_2\tau}\cdots 2\mu e^{\xi_{i-1}\tau}(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\nu(t-t_0)-(\xi-\nu)(t-t_0)} \\ &\leq 2\mu e^{\xi_1\tau}2\mu e^{\xi_2\tau}\cdots 2\mu e^{\xi_{i-1}\tau}(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-\nu(t_{k-1}-t_0)-(\xi-\nu)(t-t_0)} \\ &= 2\mu e^{\xi_1\tau}e^{-\nu(t_1-t_0)}2\mu e^{\xi_2\tau}e^{-\nu(t_2-t_1)}\cdots 2\mu e^{\xi_{i-1}\tau}e^{-\nu(t_{k-1}-t_{k-2})} \\ &\times\Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big)e^{-(\xi-\nu)(t-t_0)} \end{aligned} \tag{3.12}$$

Making use of Assumption $A4$, we get

$$\begin{aligned} V_i(x(t) &\leq (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-(\xi-\nu)(t-t_0)} \qquad (3.13) \\ W_i((z-h_i)(t)) &\leq (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau)e^{-(\xi-\nu)(t-t_0)}, \quad t \in [t_0, \infty) \end{aligned}$$

Now, from the fact that $V_1(x(t)) \leq \lambda_{1M}\|x(t)\|^2$ we have

$$\|V_{1_{t_0}}\|_\tau \leq \lambda_{1M}\|x_{t_0}\|_\tau^2, \tag{3.14}$$

and $W_1((z-h_i)(t)) \leq \lambda_{2M}\|(z-h_1)\|^2 \leq \lambda_{2M}(\|z\| + \|h_1\|)^2$ leads to

$$\begin{aligned} \|W_{1_{t_0}}\|_\tau &\leq \lambda_{2M}(\|z_{t_0}\|_\tau + \|h_{1_{t_0}}\|_\tau)^2, \\ &\leq \lambda_{2M}(\|z_{t_0}\|_\tau + \beta\|x_{t_0}\|_\tau)^2, \end{aligned} \tag{3.15}$$

where $\|h_{1_{t_0}}\| \leq \beta \|x_{t_0}\|_\tau$ and $\beta \geq \|B^{-1}_{21_i}(t) A_{12_i}(t)\|$.

Thus, (3.13) becomes

$$\begin{aligned} V_i(x(t)) &\leq \Big(\lambda_{1M}\|x_{t_0}\|_\tau^2 + \lambda_{2M}(\|z_{t_0}\|_\tau + \beta\|x_{t_0}\|_\tau)^2\Big) e^{-(\xi-\nu)(t-t_0)} \\ &\leq \rho\Big(\|x_{t_0}\|_\tau^2 + \|z_{t_0}\|_\tau^2 + 2\|x_{t_0}\|_\tau^2\|z_{t_0}\|_\tau^2 + \|x_{t_0}\|_\tau^2\Big) e^{-(\xi-\nu)(t-t_0)} \\ &\leq 2\rho(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau)^2 e^{-(\xi-\nu)(t-t_0)} \end{aligned} \tag{3.16}$$

where $\rho = \max\Big\{\lambda_{1M}, \lambda_{2M}\max\{1, \beta^2\}\Big\}$.

Together, the last inequality in (3.16) and the fact that $\|x(t)\|^2 \leq \frac{1}{\lambda_{1m}} V_i(x(t))$ give, for $t \in [t_0, \infty)$

$$\|x(t)\| \leq K_1(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau) e^{-(\xi-\nu)(t-t_0)/2} \tag{3.17}$$

where $K_1 = (2\rho/\lambda_{1m})^{1/2}$.

To find an upper bound to $z(t)$, we have

$$\|z\| - \|h_i\| \leq \|z - h_i\| \leq \frac{1}{\sqrt{\lambda_{2m}}} W_i^{1/2}$$

Then,

$$\|z\| \leq \|z - h_i\| + \|h_i\| \leq \frac{1}{\sqrt{\lambda_{2m}}} W_i^{1/2} + \|h_i\| \tag{3.18}$$

where, by the same technique used in finding the last inequality in (3.16),

$$W_i((z - h_i)(t)) \leq 2\rho(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau)^2 e^{-(\xi-\nu)(t-t_0)}$$

so that,

$$\frac{1}{\sqrt{\lambda_{2m}}} W_i(t)^{1/2} \leq (\frac{2\rho}{\lambda_{2m}})^{1/2}(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau) e^{-(\xi-\nu)(t-t_0)/2}, \tag{3.19}$$

and

$$\|h_i(t)\| \leq \beta\|x(t)\| + \beta\|x(t+\theta)\| \tag{3.20}$$

where

$$\|x(t)\| \leq K_1(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau)e^{-(\xi-\nu)(t-t_0)/2} \tag{3.21}$$

and, from (3.17), we have

$$\|x_t\|_\tau \leq K_1(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau)e^{-(\xi-\nu)(t-t_0)/2}e^{(\xi-\alpha)\tau/2}$$

Then, (3.20) becomes

$$\|h_i(t)\| \leq \beta K_1(1 + e^{(\xi-\alpha)\tau/2})(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau)e^{-(\xi-\nu)(t-t_0)/2} \tag{3.22}$$

Substituting (3.19) and (3.22) into (3.18) gives, for $t \in [t_0, \infty)$,

$$\|z(t)\| \leq K_2(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau)e^{-(\xi-\nu)(t-t_0)/2} \tag{3.23}$$

where

$$K_2 = (\frac{2\rho}{\lambda_{2m}})^{1/2} + \beta K_1(1 + e^{(\xi-\nu)\tau/2})$$

Finally, we have, for $t \in [t_0, \infty)$,

$$\|x(t)\| + \|z(t)\| \leq K(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau)e^{-(\xi-\nu)(t-t_0)/2},$$

where $K = K_1 + K_2$. This shows that system (3.7) is globally exponentially stable.

Now in case assumption $A4$ fails; i.e. there is no such a dwell time T_D, but the average dwell time T_a holds, assumption $A4$ should be refined as follows;

(A4)*** *Assume that, for any t_0, the switching law satisfies*

$$N(t, t_0) \leq N_0 + \frac{t - t_0}{T_a} \tag{3.24}$$

where $N(t, t_0)$ represents the number of switchings in (t, t_0), and N_0 and T_a are respectively the chatter bound and average dwell time to be defined.

To use this assumption, let

$$\xi' = \max\{\xi_j; j = 1, 2, \cdots, i-1; i \in \{2, 3, \cdots, N\}\}$$

Then, the first inequality in (3.12); namely,

$$V_i(x(t)) \;\; \leq \;\; 2\mu e^{\xi_1\tau} 2\mu e^{\xi_2\tau} \cdots 2\mu e^{\xi_{i-1}\tau} (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau) e^{-\xi(t-t_0)}$$

is written as follows;

$$\begin{aligned} V_i(x(t)) \;\; &\leq \;\; \left(2\mu e^{\xi' \tau}\right)^{i-1} (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau) e^{-\xi(t-t_0)} \\ &= \;\; (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau) e^{(i-1)\ln\varrho - \xi(t-t_0)}, \qquad \left(\varrho = 2\mu e^{\xi' \tau}\right) \end{aligned}$$

Applying assumption $(A4)^*$ with $N_0 = \eta/\ln\varrho$, where η is an arbitrary constant, and $T_a = \ln\varrho/(\xi - \xi^*)$, $(\xi > \xi^*)$ leads to

$$V_i(x(t)) \;\; \leq \;\; (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau) e^{\eta - \xi^*(t-t_0)}.$$

Similarly, we have

$$W_i((z - h_i)(t)) \;\; \leq \;\; (\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau) e^{\eta - \xi^*(t-t_0)}.$$

These inequalities give the same result.

As a special case of (3.7), consider the following system

$$\begin{aligned} \dot{x} \;\; &= \;\; A_{11_i} x + B_{12_i} z(t - \tau) \\ \varepsilon \dot{z} \;\; &= \;\; A_{22_i} x(t - \tau) + B_{21_i} z, \quad t \in [t_{k-1}, t_k), k = 1, 2, \cdots \end{aligned} \qquad (3.25)$$

Corollary 3.1: *The origin of system* (3.25) *is exponentially stable if the following assumptions hold.*

A1) *For $i \in S$, there exists a positive constant α such that*

$$Re[\lambda(A_{11_i})] \leq -\alpha < 0 \quad \textit{and} \quad Re[\lambda(B_{21_i})] \leq -\alpha < 0.$$

A2) *There exist positive constants* ε_i^* *and* η *such that* $-\widetilde{A_i}$ *is an* $M-$*matrix and* $\lambda(\widetilde{A_i} + \widetilde{A_i}^T) + 2\|\widetilde{B_i}\| \leq -\eta < 0$ *where*

$$\widetilde{A_i} = \begin{pmatrix} -\frac{\lambda_{min}(Q_{1_i}) - b_{12_i}}{\lambda_{1M}} & 0 \\ 0 & -\frac{\lambda_{min}(Q_{2_i}) - \varepsilon^* a_{22_i}}{\varepsilon^* \lambda_{2M}} \end{pmatrix}, \widetilde{B_i} = \begin{pmatrix} 0 & \frac{b_{12_i}}{\lambda_{2m}} \\ \frac{a_{22_i}}{\lambda_{1m}} & 0 \end{pmatrix},$$

where $b_{12_i} = \|P_{1_i} B_{12_i}\|, a_{22_i} = \|P_{2_i} B_{21_i}^{-1} A_{22_i} A_{11_i}\|$*, and* P_{1_i} *and* P_{2_i} *are respectively the solutions of Lyapunov equations*

$$\begin{aligned} A_{11_i}^T P_{1_i} + P_{1_i} A_{11_i} &= -Q_{1_i} \\ B_{21_i}^T P_{2_i} + P_{2_i} B_{21_i} &= -Q_{2_i} \end{aligned}$$

for any positive definite matrices Q_{1_i} *and* Q_{2_i}*.*

A3) *Assumption A4 of Theorem* 3.3 *holds.*

The proof of this corollary is a direct result from Theorem 3.3; thus it is omitted here. The following example illustrates our results.

Example 3.3: Consider system (3.25) with the continuous initial functions

$$\begin{aligned} x(t) &= t + 1 \\ z(t) &= t + 1, \quad t \in [-1, 0]. \end{aligned}$$

Mode 1: Stable mode

Let

$$A_{11_1} = \begin{pmatrix} -18 & 0 \\ -1 & -20 \end{pmatrix}, \quad B_{12_1} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

$$A_{22_1} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad , \quad B_{21_1} = \begin{pmatrix} -7 & -3 \\ -5 & -10 \end{pmatrix}.$$

Let $Q_{1_1} = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$ and $Q_{2_1} = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$.

Then, $P_{1_1} = \begin{pmatrix} 0.3333 & -0.0088 \\ -0.0088 & 0.3004 \end{pmatrix}$ and $P_{2_1} = \begin{pmatrix} 0.4481 & -0.2123 \\ -0.2123 & 0.4561 \end{pmatrix}$

and $\lambda(P_{1_1}) = 0.2982, 0.3355$ and $\lambda(P_{2_1}) = 0.2398, 0.6645$.

Mode 2: Stable mode

Let

$$A_{11_2} = \begin{pmatrix} -2.5 & 0 \\ 0 & -3 \end{pmatrix}, \quad B_{12_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$A_{22_2} = \begin{pmatrix} 0.1 & 0 \\ 2 & 0 \end{pmatrix} \quad , \quad B_{21_2} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}.$$

Let $Q_{1_2} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $Q_{2_2} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$.

Then, $P_{1_2} = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}$ and $P_{2_2} = \begin{pmatrix} 2.5 & 0 \\ 0 & 0.25 \end{pmatrix}$,

$\lambda(P_{1_2}) = 0.5, 0.6$, and $\lambda(P_{2_2}) = 0.25, 2.5$.

Hence, $\mu = \max\{\mu_1, \mu_2\} = 10.4253$.

$\beta_1 = \|\widetilde{B_1}\| = 8.8574$, $\beta_2 = \|\widetilde{B_2}\| = 2.5021$.

From assumption $A2$, we get $\varepsilon_1^* < 0.2017$, $\varepsilon_2^* < 0.2876$.

Take $\varepsilon_1^* = 0.2$, and $\varepsilon_2^* = 0.2$. Then, we have

$\alpha_1^* = \lambda(\widetilde{A_1}^T + \widetilde{A_1}) + \|\widetilde{B_1}\| = \{-101.8606, -27.8052\}$,

$\alpha_2^* = \lambda(\widetilde{A_2}^T + \widetilde{A_2}) + \|\widetilde{B_2}\| = \{-28.7738, -5.4979\}$.

The mode decay rates are $\xi_1 = 2.4183$, $\xi_2 = 2.3568$; therefore, taking $\nu = 2.3 < \{\xi_1, \xi_2\}$, the dwell times are respectively $T_{D_1} = 2.372$ and $T_{D_2} = 2.345$. We clearly notice that the dwell times are almost equal since the decay rates of the subsystems are almost the same.

Figure 3.3 shows that the solutions exponentially vanish after running the modes on the first and second intervals, respectively. As expected, since the modes are stable, the solutions of the switched system die out fast.

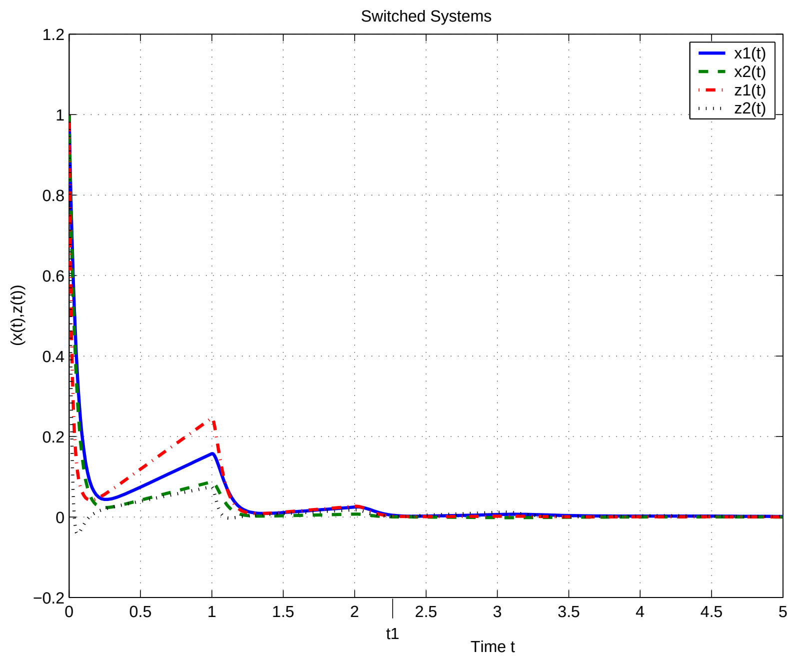


Figure 3.3: Switched delay system with stable linear modes

3.2.2 Systems with Stable and Unstable Modes

In this subsection, a more general system is considered where the family of modes is extended to include, in addition, unstable modes

Theorem 3.4: *The origin of system* (3.7) *is exponentially stable if the following assumptions hold.*

A1) *For* $t \in [t_0, \infty)$*, there exist positive constants* α, β *such that*

(i) for $i \in S_u$

$$\begin{array}{ccc} Re[\lambda(A_{11_i}(t))] > 0, & \|A_{11_i}(t)\| \leq \beta, & \|\dot{A}_{11_i}(t)\| \leq \beta \\ Re[\lambda(B_{21_i}(t))] \leq -\alpha < 0, & \|B_{21_i}(t)\| \leq \beta, & \|\dot{B}_{21_i}(t)\| \leq \beta \\ \|B_{21_i}^{-1}(t)A_{21_i}(t)\| \leq \beta, & \|B_{21_i}^{-1}(t)A_{22_i}(t)\| \leq \beta. & \end{array}$$

(ii) for $i \in S_s$

$$\begin{array}{ccc} Re[\lambda(A_{11_i}(t))] \leq -\alpha < 0, & \|A_{11_i}(t)\| \leq \beta, & \|\dot{A}_{11_i}(t)\| \leq \beta \\ Re[\lambda(B_{21_i}(t))] \leq -\alpha < 0, & \|B_{21_i}(t)\| \leq \beta, & \|\dot{B}_{21_i}(t)\| \leq \beta \\ \|B_{21_i}^{-1}(t)A_{21_i}(t)\| \leq \beta, & \|B_{21_i}^{-1}(t)A_{22_i}(t)\| \leq \beta, & \end{array}$$

A2) *there exist bounded functions* $a_{rs_i}(t)$ *and* $b_{rs_i}(t)$, $(r, s = 1, 2)$*, satisfying*

$$\begin{aligned} 2x^T P_{1_i}(t)[A_{12_i}(t)x_t + B_{11_i}(t)z + B_{12_i}(t)z_t] + x^T \dot{P}_{1_i}(t)x \leq\; & a_{11_i}(t)\|x\|^2 \\ & + a_{12_i}(t)\|x_t\|_\tau^2 \\ & + b_{11_i}(t)\|(z-h_i)\|^2 \\ & + b_{12_i}(t)\|(z-h_i)_t\|_\tau^2 \end{aligned}$$

$$\begin{aligned} -2(z-h_i)^T P_{2_i}(t)\dot{h}_i + (z-h_i)^T \dot{P}_{2_i}(t)(z-h_i) \leq\; & a_{21_i}(t)\|x\|^2 + a_{22_i}(t)\|x_t\|_\tau^2 \\ & + b_{21_i}(t)\|(z-h_i)\|^2 \\ & + b_{22_i}(t)\|(z-h_i)_t\|_\tau^2 \end{aligned}$$

where $h_i(t) = -B_{21_i}^{-1}(t)[A_{21_i}(t)x + A_{22_i}(t)x_t]$, *and* P_{1_i} *and* P_{2_i} *are respectively the solutions of Lyapunov equations*

$$\begin{aligned} A_{11_i}^T(t)P_{1_i}(t) + P_{1_i}(t)A_{11_i}(t) &= -I_m, \\ B_{21_i}^T(t)P_{2_i}(t) + P_{2_i}(t)B_{21_i}(t) &= -I_n. \end{aligned}$$

A3-i) *for* $i \in S_u$, *let* $\alpha(t) = \lambda(\widetilde{A}_i^T(t) + \widetilde{A}_i(t))$, *and* $\|\widetilde{B}_i(t)\| \leq \beta_1$ *where*

$$\widetilde{A}_i(t) = \begin{pmatrix} \frac{2\gamma + a_{11_i}(t)}{\lambda_{1m}} & \frac{b_{11_i}(t)}{\lambda_{2m}} \\ \\ \frac{a_{21_i}(t)}{\lambda_{1m}} & -\frac{1-\varepsilon_i b_{21_i}(t)}{\varepsilon_i \lambda_{2m}} \end{pmatrix}, \widetilde{B}_i(t) = \begin{pmatrix} \frac{a_{12_i}(t)}{\lambda_{1m}} & \frac{b_{12_i}(t)}{\lambda_{2m}} \\ \\ \frac{a_{22_i}(t)}{\lambda_{1m}} & \frac{b_{22_i}(t)}{\lambda_{2m}} \end{pmatrix},$$

and γ *is a positive constant such that the matrix* $A_{11_i} - \gamma I$ *has eigenvalues with negative real parts. Assume that* $\alpha(t) + \|\widetilde{B}_i(t)\| \leq \beta_2$, $(\beta_2 > 0)$.

A3-ii) *for* $i \in S_s$ *there exist positive constants* ε_i^* *and* η *such that* $-\widetilde{A}_i(t)$ *is an* $M-$*matrix and* $\lambda(\widetilde{A}_i(t) + \widetilde{A}_i^T(t)) + 2\|\widetilde{B}_i(t)\| \leq -\eta < 0$ *where*

$$\widetilde{A}_i(t) = \begin{pmatrix} -\frac{1-a_{11_i}(t)}{\lambda_{1M}} & \frac{b_{11_i}(t)}{\lambda_{2m}} \\ \\ \frac{a_{21_i}(t)}{\lambda_{1m}} & -\frac{1-\varepsilon_i^* b_{21_i}(t)}{\varepsilon_i^* \lambda_{2M}} \end{pmatrix} \quad \text{and} \quad \widetilde{B}_i(t) = \begin{pmatrix} \frac{a_{12_i}(t)}{\lambda_{1m}} & \frac{b_{12_i}(t)}{\lambda_{2m}} \\ \\ \frac{a_{22_i}(t)}{\lambda_{1m}} & \frac{b_{22_i}(t)}{\lambda_{2m}} \end{pmatrix}$$

A4) *Let*

$$\begin{aligned} \lambda^+ &= \max\{\xi_i : i \in S_u\}, \\ \lambda^- &= \min\{\zeta_i : i \in S_s\}, \end{aligned}$$

$T^+(t_0, t)$ *be the total activation time of the unstable modes,* $T^-(t_0, t)$ *be the total activation time of the stable modes, and, for any* t_0, *assume that the switching law guarantees that*

$$\inf_{t \geq t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{3.26}$$

where $\lambda^* \in (\lambda, \lambda^-)$ *and* $\lambda \in (0, \lambda^-)$. *Furthermore, there exists* $0 < \nu < \zeta_i$ *such that* (*i*) *for* $i \in \{1, 2, \cdots, l\}$ *and* $k = 1, 2, \cdots, l$

$$\ln(2\mu) - \nu(t_k - t_{k-1}) < 0 \tag{3.27}$$

(*ii*) *for* $i \in \{l+1, l+2, \cdots, m-1\}$ *and* $k = l+1, l+2, \cdots, m-1$

$$\ln(2\mu) + \zeta_i\tau - \nu(t_k - t_{k-1}) < 0. \tag{3.28}$$

Proof:

For each $i \in S$, define $V_i(x(t)) = x^T(t)P_{1_i}(t)x(t)$ and $W_i((z-h_i)(t)) = (z-h_i)^T(t)P_{2_i}(t)(z-h_i)(t)$.

Then, for $i \in S_u$, the derivative of $V_i(x(t))$ along the trajectories of the state $x(t)$ is given by

$$\begin{aligned}
\dot{V}_i(x(t)) &= \dot{x}^T P_{1_i}(t)x + x^T P_{1_i}(t)\dot{x} + x^T \dot{P}_{1_i}(t)x \\
&= x^T\Big(A_{11_i}^T(t)P_{1_i}(t) + P_{1_i}(t)A_{11_i}(t)\Big)x + 2x^T P_{1_i}(t)\Big(A_{12_i}(t)x_t + B_{11_i}(t)z \\
&\quad + B_{12_i}(t)z_t\Big) + x^T\dot{P}_{1_i}(t)x \\
&\le (2\gamma + \frac{a_{11_i}(t)}{\lambda_{1m}})V_i(x(t)) + \frac{b_{11_i}(t)}{\lambda_{2m}}W_i((z-h_i)(t)) + \frac{a_{12_i}(t)}{\lambda_{1m}}\|V_{i_t}\|_\tau \\
&\quad + \frac{b_{12_i}(t)}{\lambda_{2m}}\|W_{i_t}\|_\tau
\end{aligned}$$

Similarly, the derivative of $W_i((z-h_i)(t))$ along the trajectories of the state $z(t)$ is given by

$$\begin{aligned}
\dot{W}_i((z-h_i)(t)) &\le \frac{a_{21_i}(t)}{\lambda_{1m}}V_i(x(t)) - \frac{1-\varepsilon_i b_{21_i}(t)}{\varepsilon_i\lambda_{2M}}W_i((z-h_i)(t)) + \frac{a_{22_i}(t)}{\lambda_{1m}}\|V_{i_t}\|_\tau \\
&\quad + \frac{b_{22_i}(t)}{\lambda_{2m}}\|W_{i_t}\|_\tau
\end{aligned}$$

Then, by Lemma 3.2, there exists a positive constant $\xi_i, i \in S_u$ such that

$$\begin{aligned}
V_i(x(t)) &\le (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{\xi_i(t-t_{k-1})} \\
W_i((z-h_i)(t)) &\le (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{\xi_i(t-t_{k-1})}.
\end{aligned} \tag{3.29}$$

and by Lemma 3.3, there exists a positive constant ζ_i such that

$$\begin{aligned} V_i(x(t)) &\leq (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{-\zeta_i(t-t_{k-1})} \\ W_i((z-h_i)(t)) &\leq (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{-\zeta_i(t-t_{k-1})}. \end{aligned} \tag{3.30}$$

We shall show the analysis of finding an estimate of $V_i(x(t))$ for $t \in [t_{k-1}, t_k)$. This estimate can be found by the same spirit of the previous analysis. Without loss of generality, let us run l unstable mode and switch l times from an unstable mode, and run *m-l* stable modes and switch *m-l-1* times from a stable one. Then, we get

$$\begin{aligned} V_m(x(t)) &\leq \prod_{i=1}^{l} 2\mu e^{\xi_i(t_i-t_{i-1})} \times \prod_{j=l+1}^{m-l-1} 2\mu e^{\zeta_j\tau} e^{-\zeta_j(t_j-t_{j-1})} \\ &\quad \times \Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big)e^{-\zeta_m(t-t_{m-1})} \end{aligned} \tag{3.31}$$

Using condition (3.26), we get

$$V_m(x(t)) \leq \prod_{i=1}^{l} 2\mu \times \prod_{j=l+1}^{m-l-1} 2\mu e^{\zeta_j\tau} \times \Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big)e^{-\lambda^*(t-t_0)} \tag{3.32}$$

and, as we did in Chapter 2, we get

$$V_m(x(t)) \leq \Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big)e^{-(\lambda^*-\nu)(t-t_0)}, \qquad t \geq t_0.$$

Similarly, we have

$$W_m((z-h_i)(t)) \leq \Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big)e^{-(\lambda^*-\nu)(t-t_0)}, \qquad t \geq t_0.$$

By Theorem 3.3, there exists a positive constant K such that

$$\|x(t)\| + \|z(t)\| \leq K(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau)e^{-(\lambda^*-\nu)(t-t_0)/2}, \quad t \geq t_0.$$

This shows that the origin of system (3.7) is globally exponentially stable.

In fact one can use the average dwell time to achieve a similar result. To do so, from (3.32) we have

$$\begin{aligned} V_m(x(t)) &\leq \prod_{i=1}^{l+m-1} 2\mu e^{\zeta_j \tau} \times \Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big) e^{-\lambda^*(t-t_0)} \\ &\leq \prod_{i=1}^{l+m-1} 2\mu e^{\zeta^* \tau} \times \Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big) e^{-\lambda^*(t-t_0)} \\ &\leq \Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big) e^{(l+m-1)\ln\varrho - \lambda^*(t-t_0)} \end{aligned} \tag{3.33}$$

where $\varrho = 2\mu e^{\zeta^* \tau}$ and $\zeta^* = \max\{\zeta_i; i = 1, 2, \cdots, l+m-1\}$. By the same manner used in Chapter 2, we get

$$\begin{aligned} V_i(x(t)) &\leq \Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big) e^{\eta - \lambda^*(t-t_0)} \\ W_i((z-h_i)(t)) &\leq \Big(\|V_{1_{t_0}}\|_\tau + \|W_{1_{t_0}}\|_\tau\Big) e^{\eta - \lambda^*(t-t_0)} \end{aligned}$$

where η is an arbitrary constant. This shows that the origin is exponentially stable.

As a special case, we consider again system (3.25) with $S = S_u \cup S_s$.

Corollary 3.2: *The origin of system* (3.25) *where with* $S = S_u \cup S_s$ *is exponentially stable if the following assumptions hold.*

A1) *There exists a positive constant* σ *such that*

(i) for $i \in S_u$

$$Re[\lambda(A_{11_i})] > 0 \quad \textit{and} \quad Re[\lambda(B_{21_i})] \leq -\sigma < 0$$

(ii) for $i \in S_s$

$$Re[\lambda(A_{11_i})] \leq -\sigma < 0 \quad \textit{and} \quad Re[\lambda(B_{21_i})] \leq -\sigma < 0.$$

A2-i) *For* $i \in S_u$*, let* $\gamma > 0$ *be a positive constant such that the matrix* $A_{11_i} - \gamma I$ *has eigenvalues with negative real parts, and assume that* $\beta_{2_i} = \alpha_i + \beta_{1_i} > 0$*, where*

$\beta_{1_i} = \|\widetilde{B_i}\|$, $\alpha_i = \lambda(\widetilde{A_i}^T + \widetilde{A_i})$,

$$\widetilde{A_i} = \begin{pmatrix} 2\gamma + \frac{b_{12_i}}{\lambda_{1m}} & 0 \\ \\ 0 & -\frac{\lambda_{min}(Q_{2_i}) - \varepsilon a_{22_i}}{\varepsilon \lambda_{2M}} \end{pmatrix}, \quad and \quad \widetilde{B_i} = \begin{pmatrix} 0 & \frac{b_{12_i}}{\lambda_{2m}} \\ \\ \frac{a_{22_i}}{\lambda_{1m}} & 0 \end{pmatrix}$$

with $b_{12_i} = \|P_{1_i} B_{12_i}\|$ *and* $a_{22_i} = \|P_{2_i} B_{21_i}^{-1} A_{22_i} A_{11_i}\|$, *and* P_{1_i} *and* P_{2_i} *being respectively the solutions of Lyapunov equations*

$$\begin{aligned} A_{11_i}^T P_{1_i} + P_{1_i} A_{11_i} &= -Q_{1_i}, \\ B_{21_i}^T P_{2_i} + P_{2_i} B_{21_i} &= -Q_{2_i}. \end{aligned}$$

for any positive definite matrices Q_{1_i} *and* Q_{2_i}.

A2-ii) *for* $i \in S_s$, *there exist positive constants* ε_i^* *and* η *such that* $-\widetilde{A_i}$ *is an* $M-$*matrix and* $\lambda(\widetilde{A_i} + \widetilde{A_i}^T) + 2\|\widetilde{B_i}\| \leq -\eta < 0$ *where*

$$\widetilde{A_i} = \begin{pmatrix} -\frac{\lambda_{min}(Q_{1_i}) - b_{12_i}}{\lambda_{1M}} & 0 \\ \\ 0 & -\frac{\lambda_{min}(Q_{2_i}) - \varepsilon^* a_{22_i}}{\varepsilon^* \lambda_{2M}} \end{pmatrix}, \widetilde{B_i} = \begin{pmatrix} 0 & \frac{b_{12_i}}{\lambda_{2m}} \\ \\ \frac{a_{22_i}}{\lambda_{1m}} & 0 \end{pmatrix}$$

where b_{12_i}, a_{22_i}, Q_{1_i} *and* Q_{2_i} *are defined in* $(A2 - i)$.

A3) *Assumption A4 of Theorem 3.4 holds.*

The proof of this corollary is a direct result from Theorem 3.4; thus, it is omitted here. The following example illustrates these results.

Example 3.4: Consider system (3.25) with the initial conditions

$$\begin{aligned} x(t) &= t + 1 \\ z(t) &= t + 1, \quad t \in [-1, 0] \end{aligned}$$

Mode 1: Unstable mode

Let

$$A_{11_1} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, \quad B_{12_1} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

$$A_{22_1} = \begin{pmatrix} 1 & 9 \\ 2 & 0 \end{pmatrix}, \quad B_{21_1} = \begin{pmatrix} -2 & 0 \\ 0 & -10 \end{pmatrix}.$$

Let $Q_{1_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad Q_{2_1} = \begin{pmatrix} 9 & 0 \\ 0 & 5 \end{pmatrix}$, and $\gamma = 3$.

Then, $P_{1_1} = \begin{pmatrix} 1.25 & 0.25 \\ 0.25 & 0.25 \end{pmatrix}$ and $P_{2_1} = \begin{pmatrix} 2.25 & 0 \\ 0 & 0.25 \end{pmatrix}$

and $\lambda(P_{1_1}) = 0.191, 1.309$ and $\lambda(P_{2_1}) = 0.25, 2.25$.

Mode 2: Stable mode

Let

$$A_{11_2} = \begin{pmatrix} -5 & 0 \\ 0 & -9 \end{pmatrix}, \quad B_{12_2} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$A_{22_2} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \quad B_{21_2} = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}.$$

Let $Q_{1_2} = \begin{pmatrix} 10 & 0 \\ 0 & 9 \end{pmatrix}$ and $Q_{2_2} = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}$.

Then, $P_{1_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$ and $P_{2_2} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.25 \end{pmatrix}$,

$\lambda(P_{1_2}) = 0.5, 1$, and $\lambda(P_{2_2}) = 0.25, 0.5$.

Hence, $\mu = \max\{\mu_1, \mu_2\} = 9$.

Take $\varepsilon_1 = 0.31$. Then, we have

$\alpha_1 = \lambda(\widetilde{A_1}^T + \widetilde{A_1}) = \{3.5283, 52.5243\}$, $\beta_{1_1} = \|\widetilde{B_1}\| = 88.5581$, so that $\beta_{2_1} = \alpha_1 + \beta_{1_1} = \{92.0864, 141.0824\}$ and the growth rates of the unstable mode are $\xi = (\beta_{1_1} + \beta_{2_1})/2 = \{90.3222, 114.8202\}$.
From assumption $(A2 - ii)$, we get $\varepsilon_2^* < 0.3129$,
$\alpha_2^* = \lambda(\widetilde{A_2}^T + \widetilde{A_2}) + \|\widetilde{B_2}\| = \{-36.5, -8.2229\}$, where $\beta_2 = \|\widetilde{B_2}\| = 4$, so that the decay rates of the stable mode are $\zeta = \{2.1509, 0.0438\}$. Apparently, in this example, the growth rates of the unstable mode are much larger than the decay rates of the stable mode; therefore, activating the stable mode longer than the unstable one is strongly required.
Take $\nu = 2$. Then, the dwell time of the unstable and stable modes are respectively $T_{D_u} = 1.4$ and $T_{D_s} = 3$.

Figure 3.4 shows the switched system vanishing exponentially after running two modes, unstable and stable. The unstable mode was activated on the $1^{st}, 3^{rd}, 5^{th}$ and 7^{th} intervals, while the stable one was activated on the $2^{nd}, 4^{th}, 6^{th}$ and 8^{th} intervals. The peaks occur at the switching moments, when the switching is from the unstable mode into the stable. The solution $x_1(t)$ illustrates the requirement that the stable modes be activated longer than the unstable ones.

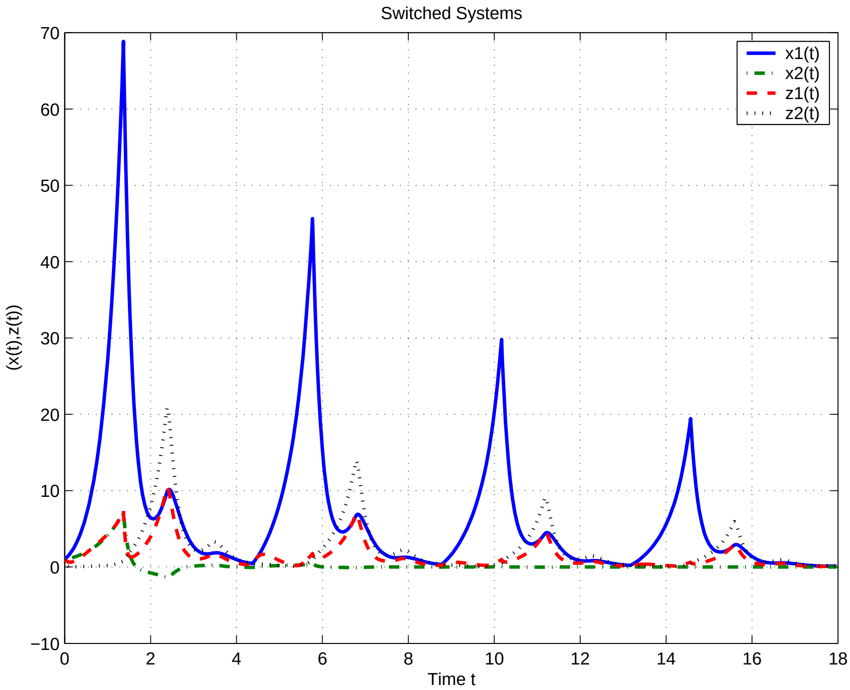


Figure 3.4: Switched delay system with unstable and stable linear modes

3.3 Nonlinear Singulary Perturbed Systems

In this section we study exponential stability of a nonlinear switched time-delayed singularly perturbed system of the form

$$\begin{aligned} \dot{x} &= f_i(x, x_t, z, z_t) \\ \varepsilon \dot{z} &= B_{21_i} z + B_i(x, x_t), \qquad t \in [t_{k-1}, t_k), \end{aligned} \tag{3.34}$$

where $x \in R^m$, $z \in R^n$, $i \in S = \{1, 2, \cdots, N\}$, $k = 1, 2, \cdots$ with $\lim_{k\to\infty} t_k = \infty$, $f_i(x, x_t, z, z_t) = A_{11_i} x + g_i(x, x_t, z, z_t)$. We assume that system (3.34) has a unique equilibrium point at the origin, the functions f_i and B_i are smooth enough to ensure that system (3.34) has a unique solution , and B_{21_i} is nonsingular. The delayed fast variable in not included in the fast subsystem for simplicity. Let the continuous initial functions be given by

$$\begin{aligned} x(t) &= \phi_1(t), \\ z(t) &= \phi_2(t), \qquad t \in [t_0 - \tau, t_0]. \end{aligned}$$

3.3.1 Systems with Stable Modes

Theorem 3.5: *The origin of system* (3.34) *is locally exponentially stable if the following assumptions hold.*

Assumptions: *Assume that, for $i \in S$,*

A1) *there exists a positive constant σ such that*

$$Re[\lambda(A_{11_i})] \leq -\sigma < 0 \quad \text{and} \quad Re[\lambda(B_{21_i}(t))] \leq -\sigma < 0.$$

A2) *there exist positive constants a_{rs_i} and b_{rs_i}, $(r, s = 1, 2)$ such that*

$$\begin{aligned} 2x^T P_{1_i} g_i(x, x_t, z, z_t) &\leq a_{11_i}\|x\|^2 + a_{12_i}\|x_t\|_\tau^2 + b_{11_i}\|z - h_i\|^2 + b_{12_i}\|(z - h_i)_t\|_\tau^2, \\ -2(z - h_i)^T P_{2_i} \dot{h}_i &\leq a_{11_i}\|x\|^2 + a_{12_i}\|x_t\|_\tau^2 + b_{11_i}\|z - h_i\|^2 + b_{12_i}\|(z - h_i)_t\|_\tau^2, \end{aligned}$$

where $h_i = -B_{21_i}^{-1} B_i(x, x_t)$, and P_{1_i}, P_{2_i} are the same as defined in Corollary 3.1

A3) *there exist positive constants ε_i^* and η such that $-\widetilde{A_i}$ is an M-matrix and $\lambda(\widetilde{A_i} + \widetilde{A_i}^T) + 2\|\widetilde{B_i}\| \leq -\eta < 0$ where*

$$\widetilde{A_i} = \begin{pmatrix} -\frac{\lambda_{min}(Q_{1_i}) - a_{11_i}}{\lambda_{1M}} & \frac{b_{11_i}}{\lambda_{2m}} \\ \\ \frac{a_{21_i}}{\lambda_{1m}} & -\frac{\lambda_{min}(Q_{2_i}) - \varepsilon_i^* b_{21_i}}{\varepsilon_i^* \lambda_{2M}} \end{pmatrix} \quad \text{and} \quad \widetilde{B_i} = \begin{pmatrix} \frac{a_{12_i}}{\lambda_{1m}} & \frac{b_{12_i}}{\lambda_{2m}} \\ \\ \frac{a_{22_i}}{\lambda_{1m}} & \frac{b_{22_i}}{\lambda_{2m}} \end{pmatrix}$$

where $\lambda_{rm} = \min\{\lambda_{min}(P_{r_i}), i \in S\}$, and $\lambda_{rM} = \max\{\lambda_{max}(P_{r_i}), i \in S\}$, $r = 1, 2$, and Q_{1_i}, Q_{2_i} are defined in Corollary 3.1

A4) *Assumption* A4 *of Theorem* 3.3 *holds.*

Proof:

For $i \in S$, define $V_i(x(t)) = x^T(x(t))P_{1_i}x(x(t))$ and $W_i((z-h_i)(t)) = (z-h_i)^T(t)P_{2_i}(z-h_i)(t)$.

Then, the time derivative of V_i along the trajectories of the state $x(t)$ is given by

$$\begin{aligned}
\dot{V}_i &= \dot{x}^T P_{1_i} x + x^T P_{1_i}\dot{x} \\
&= \Big(A_{11_i}x + g_i(x, x_t, z, z_t)\Big)^T P_{1_i}x + x^T P_{1_i}\Big(A_{11_i}x + g_i(x, x_t, z, z_t)\Big) \\
&= x^T\Big(A_{11_i}^T P_{1_i} + P_{1_i}A_{11_i}\Big)x + 2x^T P_{1_i} g_i(x, x_t, z, z_t) \\
&\leq -(\lambda_{min}(Q_{1_i}) - a_{11_i})\|x\|^2 + a_{12_i}\|x_t\|_\tau^2 + b_{11_i}\|(z-h_i)\|^2 + b_{12_i}\|(z-h_i)_t\|_\tau^2 \\
&\leq -\frac{\lambda_{min}(Q_{1_i}) - a_{11_i}}{\lambda_{1M}}V_i + \frac{b_{11_i}}{\lambda_{2m}}W_i + \frac{a_{12_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau + \frac{b_{12_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau.
\end{aligned}$$

Similarly, the time derivative of W_i along the trajectories of $z(t)$ is given by

$$\begin{aligned}
\dot{W}_i &= (\dot{z} - \dot{h}_i)^T P_{2_i}(z-h_i) + (z-h_i)^T P_{2_i}(\dot{z} - \dot{h}_i) \\
&= \Big(\frac{1}{\epsilon}(B_{21_i}z + B_i(x, x_t)) - \dot{h}_i\Big)^T P_{2_i}(z-h_i) + (z-h_i)^T P_{2_i}\Big(\frac{1}{\epsilon}(B_{21_i}z \\
&\quad + B_i(x, x_t)) - \dot{h}_i\Big) \\
&= \Big(\frac{1}{\epsilon}(B_{21_1}z - B_{21_i}h_i) - \dot{h}_i\Big)^T P_{2_i}(z-h_i) + (z-h_i)^T P_{2_i}\Big(\frac{1}{\epsilon}(B_{21_i}z \\
&\quad - B_{21_i}h_i) - \dot{h}_i\Big) \\
&= \Big(\frac{1}{\epsilon}B_{21_i}(z-h_i) - \dot{h}_i\Big)^T P_{2_i}(z-h_i) + (z-h_i)^T P_{2_i}\Big(\frac{1}{\epsilon}B_{21_i}(z-h_i) - \dot{h}_i\Big) \\
&= \frac{1}{\epsilon}(z-h_i)^T(B_{21_i}^T P_{2_i} + P_{2_i}B_{21_i})(z-h_i) - 2(z-h_i)^T P_{2_i}(t)\dot{h}_i \\
&\leq -\frac{1}{\epsilon}\|(z-h_i)\|^2 + a_{21_i}\|x\|^2 + a_{22_i}\|x_t\|_\tau^2 + b_{21_i}\|(z-h_i)\|^2 + b_{22_i}\|(z-h_i)_t\|_\tau^2 \\
&\leq \frac{a_{21_i}}{\lambda_{1m}}V_i - \frac{\lambda_{min}(Q_{2_i}) - \varepsilon_i b_{21_i}}{\varepsilon_i\lambda_{2M}}W_i + \frac{a_{22_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau + \frac{b_{22_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau \\
&\leq \frac{a_{21_i}}{\lambda_{1m}}V_i - \frac{\lambda_{min}(Q_{2_i}) - \varepsilon_i^* b_{21_i}}{\varepsilon_i^*\lambda_{2M}}W_i + \frac{a_{22_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau + \frac{b_{22_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau
\end{aligned}$$

By lemma 3.3, there is a ξ_i such that

$$\begin{aligned} V_i(x(t)) &\leq (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{-\xi_i(t-t_{k-1})} \\ W_i((z-h_i)(t)) &\leq (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{-\xi_i(t-t_{k-1})}. \end{aligned} \tag{3.35}$$

By Theorem 3.2, there exists a positive constant K such that

$$\|x(t)\| + \|z(t)\| \leq K\Big(\|x_{t_0}\|_\tau + \|z_{t_0}\|_\tau\Big)e^{-(\lambda^*-\nu)(t-t_0)} \qquad t \geq t_0.$$

To verify our theoretical results, consider the following example.

Example 3.5: Consider the following nonlinear switched system

Mode 1:

$$\begin{aligned} \dot{x}_1 &= -18\sin x_1 + x_2^2 + 3z_1(t-1) \\ \dot{x}_2 &= -\ln(1+x_1) - 20x_2 + 2z_2(t-1) \\ \varepsilon\dot{z}_1 &= x_1(t-1) + \sin x_2(t-1) - 7z_1 - 3z_2 \\ \varepsilon\dot{z}_2 &= 2x_1(t-1) - \cos x_2(t-1) - 5z_1 - 10z_2 \end{aligned}$$

Mode 2:

$$\begin{aligned} \dot{x}_1 &= -2.5x_1 + 3x_2^2 + z_1(t-1) + z_2^2(t-1) \\ \dot{x}_2 &= 3x_1^2 - 3x_2 - z_2(t-1) \\ \varepsilon\dot{z}_1 &= 0.1x_1(t-1) - z_1 + z_2^4 \\ \varepsilon\dot{z}_2 &= 2\ln(1+2x_1(t-1)) - 4z_2, \end{aligned}$$

The resulting linearized subsystems are given in Example 3.3. Figure 3.5 shows the solutions of the nonlinear system after running Mode 1 on the first interval, and Mode 2 on the second interval.

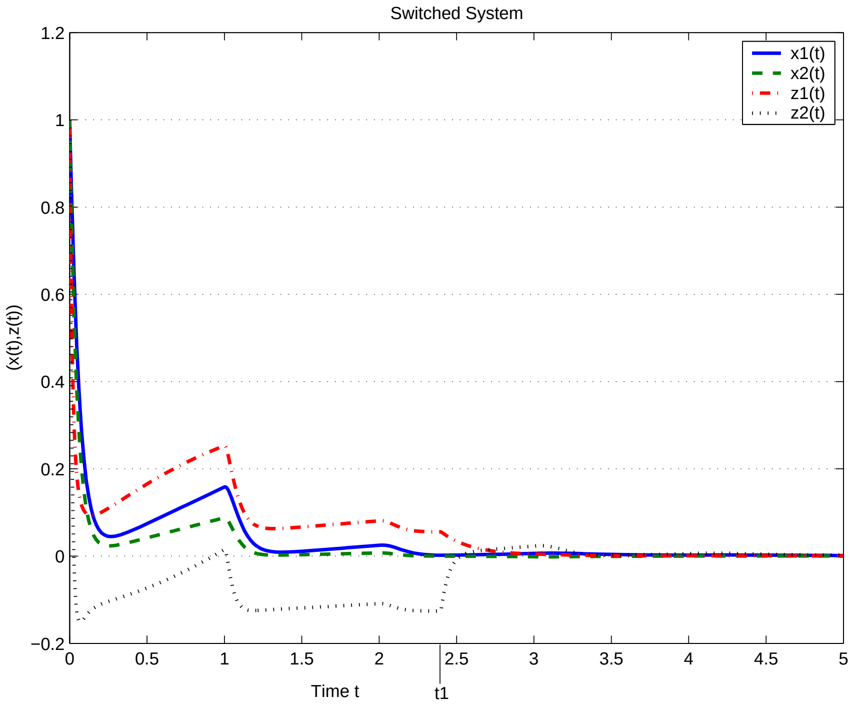


Figure 3.5: Switched system with nonliner stable nonlinear modes

3.3.2 Systems with Stable and Unstable Modes

Consider again the nonlinear system (3.34) with the same continuous initial functions and $S = S_u \cup S_s$.

Theorem 3.6: *The origin of system* (3.34) *with* $S = S_u \cup S_s$ *is exponentially stable if the following assumptions hold.*

A1-i) *For* $i \in S_u$*, there exists a positive constant* σ *such that*

$$Re[\lambda(A_{11_i})] > 0 \quad and \quad Re[\lambda(B_{21_i})] \leq -\sigma < 0.$$

A1-ii) *For $i \in S_s$, there exists a positive constant σ such that*

$$Re[\lambda(A_{11_i})] \leq -\sigma < 0 \quad and \quad Re[\lambda(B_{21_i})] \leq -\sigma < 0.$$

A2) *Assumption A2 of Theorem 3.5 holds.*

A3-i) *For $i \in S_u$, let $\gamma > 0$ be a positive constant such that the matrix $A_{11_i} - \gamma I$ has eigenvalues with negative real parts, and assume that $\beta_{2_i} = \alpha_i + \beta_{1_i} > 0$ where $\beta_{1_i} = \|\widetilde{B_i}\|$, $\alpha_i = \lambda(\widetilde{A_i}^T + \widetilde{A_i})$,*

$$\widetilde{A_i} = \begin{pmatrix} 2\gamma + \frac{a_{11_i}}{\lambda_{1m}} & \frac{b_{11_i}}{\lambda_{2m}} \\ \\ \frac{a_{21_i}}{\lambda_{1m}} & -\frac{\lambda_{min}(Q_{2_i}) - \varepsilon_i b_{21_i}}{\varepsilon_i \lambda_{2M}} \end{pmatrix} \quad and \quad \widetilde{B_i} = \begin{pmatrix} \frac{a_{12_i}}{\lambda_{1m}} & \frac{b_{12_i}}{\lambda_{2m}} \\ \\ \frac{a_{22_i}}{\lambda_{1m}} & \frac{b_{22_i}}{\lambda_{2m}} \end{pmatrix}.$$

A3-ii) *For $i \in S_s$, there exist positive constants ε_i^* and η such that $-\widetilde{A}_i$ is an $M-$matrix and $\lambda(\widetilde{A}_i + \widetilde{A}_i^T) + 2\|\widetilde{B_i}\| \leq -\eta < 0$ where*

$$\widetilde{A_i} = \begin{pmatrix} -\frac{\lambda_{min}(Q_{1_i}) - a_{11_i}}{\lambda_{1M}} & \frac{b_{11_i}}{\lambda_{2m}} \\ \\ \frac{a_{21_i}}{\lambda_{1m}} & -\frac{\lambda_{min}(Q_{2_i}) - \varepsilon_i^* b_{21_i}}{\varepsilon_i^* \lambda_{2M}} \end{pmatrix} \quad and \quad \widetilde{B_i} = \begin{pmatrix} \frac{a_{12_i}}{\lambda_{1m}} & \frac{b_{12_i}}{\lambda_{2m}} \\ \\ \frac{a_{22_i}}{\lambda_{1m}} & \frac{b_{22_i}}{\lambda_{2m}} \end{pmatrix}$$

with Q_{1_i} and Q_{2_i} being defined in Corollary 3.1.

A4) *Assumption A4 of Theorem 3.4 holds.*

Proof:

For each $i \in S$, define $V_i(x) = x^T P_{1_i} x$ and $W_i(z - h_i) = (z - h_i)^T P_{2_i}(z - h_i)$. Then, for the unstable modes, the derivative of V_i along the trajectories of the state x is given by

$$\begin{aligned}
\dot{V}_i &= \dot{x}^T P_{1_i} x + x^T P_{1_i} \dot{x} \\
&= [A_{11_i} x + g_i(x, x_t, z, z_t)]^T P_{1_i} x + x^T P_{1_i} [A_{11_i} x + g_i(x, x_t, z, z_t)] \\
&= x^T (A_{11_i}^T P_{1_i} + P_{1_i} A_{11_i}) x + 2x^T P_{1_i} g_i(x, x_t, z, z_t) \\
&\le (2\gamma + \frac{a_{11_i}}{\lambda_{1m}}) V_i + \frac{b_{11_i}}{\lambda_{2m}} W_i + \frac{a_{12_i}}{\lambda_{1m}} \|V_{i_t}\|_\tau + \frac{b_{12_i}}{\lambda_{2m}} \|W_{i_t}\|_\tau
\end{aligned}$$

Similarly, the derivative of W_i along the trajectories of the state z is given by

$$\begin{aligned}
\dot{W}_i &= (\dot{z} - \dot{h}_i)^T P_{2_i} (z - h_i) + (z - h_i)^T P_{2_i} (\dot{z} - \dot{h}_i) \\
&= [\frac{1}{\varepsilon}(B_{21_i} z + B_i(x, x_t)) - \dot{h}_i]^T P_{2_i} (z - h_i) + (z - h_i)^T P_{2_i} [\frac{1}{\varepsilon}(B_{21_i} z \\
&\quad + B_i(x, x_t)) - \dot{h}_i] \\
&= [\frac{1}{\varepsilon}(B_{21_1} z - B_{21_i} h_i) - \dot{h}_i]^T P_{2_i} (z - h_i) + (z - h_i)^T P_{2_i} [\frac{1}{\varepsilon}(B_{21_i} z - B_{21_i} h_i) - \dot{h}_i] \\
&= [\frac{1}{\varepsilon} B_{21_i} (z - h_i) - \dot{h}_i]^T P_{2_i} (z - h_i) + (z - h_i)^T P_{2_i} [\frac{1}{\varepsilon} B_{21_i} (z - h_i) - \dot{h}_i] \\
&= \frac{1}{\varepsilon} (z - h_i)^T (B_{21_i}^T P_{2_i} + P_{2_i} B_{21_i}) (z - h_i) - 2(z - h_i)^T P_{2_i}(t) \dot{h}_i \\
&\le -\frac{1}{\varepsilon} \lambda_{min}(Q_{2_i}) \|(z - h_i)\|^2 + a_{21_i} \|x\|^2 + a_{22_i} \|x_t\|_\tau^2 + b_{21_i} \|(z - h_i)\|^2 \\
&\quad + b_{22_i} \|(z - h_i)_t\|_\tau^2 \\
&\le \frac{a_{21_i}}{\lambda_{1m}} V_i - \frac{\lambda_{min}(Q_{2_i}) - \varepsilon_i b_{21_i}}{\varepsilon_i \lambda_{2M}} W_i + \frac{a_{22_i}}{\lambda_{1m}} \|V_{i_t}\|_\tau + \frac{b_{22_i}}{\lambda_{2m}} \|W_{i_t}\|_\tau
\end{aligned}$$

Then, by Lemma 3.2, there exist positive constants ξ_i with $\beta_1 = \beta_{1_i}$ and $\beta_2 = \beta_{2_i}$ such that

$$\begin{aligned}
V_i(t) &\le (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau) e^{\xi_i (t - t_{k-1})} \\
W_i(t) &\le (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau) e^{\xi_i (t - t_{k-1})}.
\end{aligned}$$

While for stable modes, by Lemma 3.3, there exists a positive constant ζ_i such that

$$\begin{aligned} V_i(x) &\leq (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{-\zeta_i(t-t_{k-1})} \\ W_i((z-h_i)) &\leq (\|V_{i_{t_{k-1}}}\|_\tau + \|W_{i_{t_{k-1}}}\|_\tau)e^{-\zeta_i(t-t_{k-1})}. \end{aligned}$$

The rest of the proof is similar to that of Theorem 3.4; thus, it is omitted here.

Example 3.6 Consider the following switched system

Mode 1:

$$\begin{aligned} \dot{x}_1 &= 2\ln(1+x_1) + 3x_2 + 3z_1(t-1) - \cos z_2(t-1) \\ \dot{x}_2 &= x_2 + \cos z_1(t-1) + 2z_2(t-1) \\ \varepsilon\dot{z}_1 &= x_1(t-1) + 9x_2(t-1) - 2z_1 - 3z_2^3 \\ \varepsilon\dot{z}_2 &= 2x_1(t-1) - 10z_2 \end{aligned}$$

Mode 2:

$$\begin{aligned} \dot{x}_1 &= -5x_1 + z_1(t-1) + 0.05z_2^2(t-1) \\ \dot{x}_2 &= x_1^2 - 9x_2 - 2z_2(t-1) \\ \varepsilon\dot{z}_1 &= \sin x_1(t-1) - 6z_1 - 2z_2^2 \\ \varepsilon\dot{z}_2 &= 2\ln(1+2x_1(t-1)) + z_1^2 - 6z_2, \end{aligned}$$

Figure 3.6 shows the solutions of the nonlinear system after running Mode 1 on the first and third intervals, and Mode 2 on the second and fourth intervals.

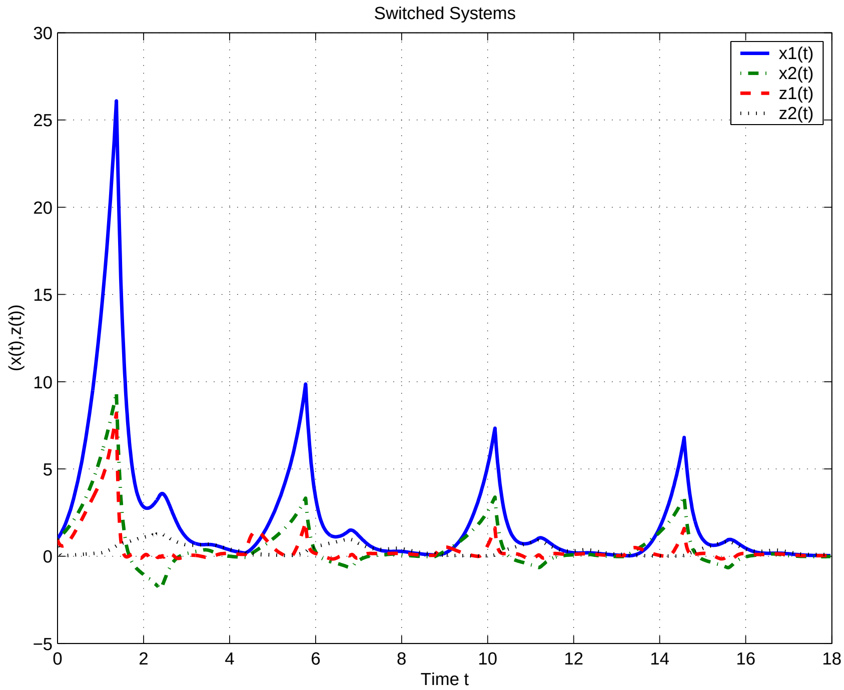


Figure 3.6: Switched delay system with unstable and stable nonlinear modes

Chapter 4

Stability of Impulsive Switched Singularly Perturbed Systems with Time Delay

In this chapter, we investigate stability of impulsive switched singularly perturbed systems with time delay. As we did in Chapter 3, we shall make use of multiple Lyapunov functions technique as a tool in our proofs of stability. Linear systems and a special case of nonlinear systems that consist of stable and unstable subsystems are studied. We shall also establish stability properties of systems incorporating unstable subsystems by impulsive effects. Illustrative examples are also given to verify our theoretical results.

4.1 Linear Systems

Before stating our theorems, we first describe the impulsive system that will be under considerations in this section. Switched time-delayed singularly perturbed systems with impulse effects occurring at fixed times can be written as follows.

$$
\begin{aligned}
\dot{x} &= A_{11_i}x + A_{12_i}x_t + B_{11_i}z + B_{12_i}z_t, \quad t \neq t_k \\
\varepsilon\dot{z} &= A_{21_i}x + A_{22_i}x_t + B_{21_i}z, \qquad t \neq t_k \\
\Delta x &= B_k x(t), \quad t = t_k \\
\Delta z &= C_k z(t), \quad t = t_k
\end{aligned} \tag{4.1}
$$

where $i \in S = S_u \cup S_s$, $x \in R^m$, $z \in R^n$ respectively represent the slow and fast states of the system, $k = 1, 2, \cdots$, the impulsive moments satisfy $t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty} t_k = \infty$. We assume that $x(t_k^-) = x(t_k)$, meaning that the solution is left-continuous. $\Delta y = y(t^+) - y(t)$. Let $A_{rs_i}, B_{1s_i}, B_{21_i} (r, s = 1, 2)$ be matrices with the following dimensions $A_{1s_i} : m \times m$, $B_{1s_i} : m \times n$, $A_{2s_i} : n \times m$, $B_{21_i} : n \times n$, and B_{21_i} be nonsingular. We assume that the cumulative effects of all of the impulses which are represented by B_k and C_k are finite (i.e, $\Sigma_{k=1}^{\infty}\|B_k\| < \infty$ and $\Sigma_{k=1}^{\infty}\|C_k\| < \infty$). The initial condition are

$$
\begin{aligned}
x(t) &= \phi_1(t), \qquad t \in [t_0 - \tau, t_0] \\
z(t) &= \phi_2(t), \qquad t \in [t_0 - \tau, t_0]
\end{aligned}
$$

where $\phi_1 \in PC([t_0 - \tau, t_0], R^m)$ and $\phi_2 \in PC([t_0 - \tau, t_0], R^n)$.

In the next theorem, we give sufficient conditions that guarantee stability of the equilibrium point $x^* = 0, z^* = 0$ of system (4.1) where $S = S_u \cup S_s$.

Theorem 4.1: *The origin of system* (4.1) *is exponentially stable if the following assumptions are satisfied.*

A1) *There exists a positive constant* α *such that*

(i) for $i \in S_u$

$$Re[\lambda(A_{11_i})] > 0 \quad , \quad Re[\lambda(B_{21_i})] \le -\alpha < 0$$

(ii) for $i \in S_s$

$$Re[\lambda(A_{11_i})] \le -\alpha < 0 \quad , \quad Re[\lambda(B_{21_i})] \le -\alpha < 0$$

A2) *For* $i \in S$ *and* $t \in (t_{k-1}, t_k]$, *there exist positive constants* a_{rs_i} *and* b_{rs_i} , $(r, s = 1, 2)$, *satisfying*

$$\begin{aligned} 2x^T P_{1_i}[A_{12_i}x_t + B_{11_i}z + B_{12_i}z_t] &\le a_{11_i}\|x\|^2 + a_{12_i}\|x_t\|_\tau^2 \\ &\quad + b_{11_i}\|(z - h_i)\|^2 + b_{12_i}\|(z - h_i)_t\|_\tau^2 \\ -2(z - h_i)^T P_{2_i}\dot{h}_i &\le a_{21_i}\|x\|^2 + a_{22_i}\|x_t\|_\tau^2 \\ &\quad + b_{21_i}\|(z - h_i)\|^2 + b_{22_i}\|(z - h_i)_t\|_\tau^2 \end{aligned}$$

where $h_i(t) = -B_{21_i}^{-1}[A_{21_i}x + A_{22_i}x_t]$, *and* P_{1_i}, P_{2_i} *are defined in Corollary* 3.1.

A3) *(i) For* $i \in S_u$, *let* γ^* *be a positive constant such that the matrix* $A_{11_i} - \gamma^* I$ *has eigenvalues with negative real parts, and assume that* $\beta_{2_i}^* = \alpha_i^* + \beta_{1_i}^* > 0$ *where* $\beta_{1_i}^* = \|\widetilde{B_i}\|$, $\alpha_i^* = \lambda(\widetilde{A_i}^T + \widetilde{A_i})$,

$$\widetilde{A_i} = \begin{pmatrix} \frac{2\gamma + a_{11_i}}{\lambda_{1m}} & \frac{b_{11_i}}{\lambda_{2m}} \\ \\ \frac{a_{21_i}}{\lambda_{1m}} & -\frac{\lambda_{min}(Q_{2_i}) - \varepsilon_i b_{21_i}}{\varepsilon_i \lambda_{2m}} \end{pmatrix} \quad \text{and} \quad \widetilde{B_i} = \begin{pmatrix} \frac{a_{12_i}}{\lambda_{1m}} & \frac{b_{12_i}}{\lambda_{2m}} \\ \\ \frac{a_{22_i}}{\lambda_{1m}} & \frac{b_{22_i}}{\lambda_{2m}} \end{pmatrix}$$

A3) *(ii) For* $i \in S_s$ *there exist positive constants* ε_i^* *and* η *such that* $-\widetilde{A}_i$ *is an*

$M-$*matrix and* $\lambda(\widetilde{A}_i + \widetilde{A}_i^T) + 2\|\widetilde{B}_i\| \leq -\eta < 0$ *where*

$$\widetilde{A}_i = \begin{pmatrix} -\frac{\lambda_{\min}(Q_{1_i}) - a_{11_i}}{\lambda_{1M}} & \frac{b_{11_i}}{\lambda_{2m}} \\ \\ \frac{a_{21_i}}{\lambda_{1m}} & -\frac{\lambda_{\min}(Q_{2_i}) - \varepsilon_i^* b_{21_i}}{\varepsilon_i^* \lambda_{2M}} \end{pmatrix} \quad and \quad \widetilde{B}_i = \begin{pmatrix} \frac{a_{12_i}}{\lambda_{1m}} & \frac{b_{12_i}}{\lambda_{2m}} \\ \\ \frac{a_{22_i}}{\lambda_{1m}} & \frac{b_{22_i}}{\lambda_{2m}} \end{pmatrix}$$

and Q_{1_i}, Q_{2_i} *being positive definite matrices defined in Corollary* 3.1.

A4) *Let*

$$\lambda^+ = \max\{\xi_i : i \in S_u\},$$
$$\lambda^- = \min\{\zeta_i : i \in S_s\},$$

$T^+(t_0, t)$ *be the total activation time of the unstable modes,* $T^-(t_0, t)$ *be the total activation time of the stable modes, and, for any* t_0*, assume that the switching law guarantees that*

$$\inf_{t \geq t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{4.2}$$

where $\lambda^* \in (\lambda, \lambda^-)$ *and* $\lambda \in (0, \lambda^-)$*. Furthermore, there exists* $0 < \nu < \zeta_i$ *such that (i) for* $i \in \{1, 2, \cdots, l\}$ *and* $k = 1, 2, \cdots, l$

$$\ln \mu(\alpha_k + \beta_k + \gamma_k + \psi_k) - \nu(t_k - t_{k-1}) \leq 0 \tag{4.3}$$

(ii) for $i \in \{l+1, l+2, \cdots, m-1\}$ *and* $k = l+1, l+2, \cdots, m-1$

$$\ln \mu(\alpha_k + \beta_k + \gamma_k + \psi_k e^{\zeta_i \tau}) + \zeta_i \tau - \nu(t_k - t_{k-1}) \leq 0. \tag{4.4}$$

where $\alpha_k = \mu \lambda^2_{\max}([I + B_k])$,
$\beta_k = \frac{\lambda_{2M}}{\lambda_{1m}}(\|U_k\| + r_k + s_k) r_k$,
$\gamma_k = \mu(\|U_k\| + r_k + s_k)\|U_k\|$,
$\psi_k = \frac{\lambda_{2M}}{\lambda_{1m}}(\|U_k\| + r_k + s_k) s_k$

$U_k = I + C_k$,
$r_k = \max\{\|R_{ik}\| : R_{ik} = [I + C_k]B_{21_i}^{-1}A_{21_i} - B_{21_i}^{-1}A_{21_i}[I + B_k] \quad \forall i \in S\}$, *and*
$s_k = \max\{\|S_{ik}\| : S_{ik} = [I + C_k]B_{21_i}^{-1}A_{22_i} - B_{21_i}^{-1}A_{22_i}[I + B_k] \quad \forall i \in S\}$.

Proof:

For $t \in (t_{k-1}, t_k]$, define $V_i(x(t)) = x^T(t)P_{1_i}x(t)$ and $W_i((z - h_i)(t)) = (z - h_i)^T(t)P_{2_i}(z - h_i)(t)$. Then, the time derivative of V_i and W_i along the trajectories of $x(t)$ and $z(t)$ are

(i) for $i \in S_u$

$$\begin{aligned}\dot{V}_i(x(t)) &\leq (2\gamma + \frac{a_{11_i}}{\lambda_{1m}})V_i(x) + \frac{b_{11_i}}{\lambda_{2m}}W_i((z - h_i)(t)) + \frac{a_{12_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau \\ &\quad + \frac{b_{12_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau \\ \dot{W}_i((z - h_i)) &\leq \frac{a_{21_i}}{\lambda_{1m}}V_i(x(t)) - \frac{\lambda_{\min}(Q_{2_i}) - \varepsilon_i b_{21_i}}{\varepsilon_i \lambda_{2M}}W_i((z - h_i)(t)) + \frac{a_{22_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau \\ &\quad + \frac{b_{22_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau\end{aligned}$$

(ii) for $i \in S_s$

$$\begin{aligned}\dot{V}_i(x(t)) &\leq -\frac{\lambda_{\min}(Q_{1_i}) - a_{11_i}}{\lambda_{1M}}V_i(x) + \frac{b_{11_i}}{\lambda_{2m}}W_i((z - h_i)(t)) + \frac{a_{12_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau \\ &\quad + \frac{b_{12_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau \\ \dot{W}_i((z - h_i)) &\leq \frac{a_{21_i}}{\lambda_{1m}}V_i(x(t)) - \frac{\lambda_{\min}(Q_{2_i}) - \varepsilon_i^* b_{21_i}}{\varepsilon_i^* \lambda_{2M}}W_i((z - h_i)(t)) + \frac{a_{22_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau \\ &\quad + \frac{b_{22_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau\end{aligned}$$

Then, by Lemma 3.2, there exists a positive constant ξ_i with $\beta_1 = \beta_{1_i}^*$ and $\beta_2 = \beta_{2_i}^*$, $i \in S_u$, such that

$$\begin{aligned}V_i(x(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau)e^{\xi_i(t - t_{k-1})} \\ W_i((z - h_i)(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau)e^{\xi_i(t - t_{k-1})}.\end{aligned}$$

and by Lemma 3.3, there exists a positive constant ζ_i, $i \in S_s$, such that

$$\begin{aligned} V_i(x(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau)e^{-\zeta_i(t-t_{k-1})} \\ W_i((z-h_i)(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau)e^{-\zeta_i(t-t_{k-1})}. \end{aligned}$$

At $t = t_k^+$, we have the following estimates

$$\begin{aligned} V_i(t_k^+) &= x(t_k^+)^T P_{1_i} x(t_k^+) \\ &= \{[I+B_k]x(t_k)\}^T P_{1_i} \{[I+B_k]x(t_k)\} \\ &= x^T(t_k)[I+B_k]^T P_{1_i}[I+B_k]x(t_k) \\ &\leq \lambda_{\max}\Big([I+B_k]^T P_{1_i}[I+B_k]\Big) x^T(t_k)x(t_k) \\ &= \lambda_{\max}^2(I+B_k)\lambda_{\max}(P_{1_i})\frac{1}{\lambda_{\min}(P_{1_i})} V_i(t_k) \\ &\leq \frac{\lambda_{1M}\lambda_{\max}^2(I+B_k)}{\lambda_{1m}} V_i(t_k) \\ &= \alpha_k V_i(t_k) \end{aligned}$$

Namely, we have

$$V_i(t_k^+) \leq \alpha_k V_i(t_k) \tag{4.5}$$

where $\alpha_k = \mu \lambda_{\max}^2(I+B_k)$.

We also have

$$\begin{aligned} W_i(t_k^+) &= \Big(z(t_k^+) - h_i(t_k^+)\Big)^T P_{2_i} \Big(z(t_k^+) - h_i(t_k^+)\Big) \\ &= \Big\{z(t_k^+) + B_{21_i}^{-1}[A_{21_i}x(t_k^+) + A_{22_i}x_{t_k^+}]\Big\}^T P_{2_i} \Big\{z(t_k^+) + B_{21_i}^{-1}[A_{21_i}x(t_k^+) \\ &\quad + A_{22_i}x_{t_k^+}]\Big\} \\ &= \Big\{[I+C_k]z(t_k) + B_{21_i}^{-1}[A_{21_i}[I+B_k]x(t_k) + A_{22_i}[I+B_k]x_{t_k}]\Big\}^T P_{2_i} \\ &\quad \times \Big\{[I+C_k]z(t_k) + B_{21_i}^{-1}[A_{21_i}[I+B_k]x(t_k) + A_{22_i}[I+B_k]x_{t_k}]\Big\} \end{aligned}$$

$$
\begin{aligned}
= \;& \Big\{[I+C_k]z(t_k) - [I+C_k]h_i(t_k) + [I+C_k]h_i(t_k) + B_{21_i}^{-1}[A_{21_i}[I+B_k]x(t_k) \\
& + A_{22_i}[I+B_k]x_{t_k}]\Big\}^T P_{2_i}\Big\{[I+C_k]z(t_k) - [I+C_k]h_i(t_k) + [I+C_k]h_i(t_k) \\
& + B_{21_i}^{-1}[A_{21_i}[I+B_k]x(t_k) + A_{22_i}[I+B_k]x_{t_k}]\Big\} \\
= \;& \Big\{[I+C_k]\Big(z(t_k) - h_i(t_k)\Big) + [I+C_k]\Big(- B_{21_i}^{-1}[A_{21_i}x(t_k) + A_{22_i}x_{t_k}]\Big) \\
& + B_{21_i}^{-1}A_{21_i}[I+B_k]x(t_k) + B_{21_i}^{-1}A_{22_i}[I+B_k]x_{t_k}\Big\}^T P_{2_i}\Big\{[I+C_k]\Big(z(t_k) - h_i(t_k)\Big) \\
& + [I+C_k]\Big(- B_{21_i}^{-1}[A_{21_i}x(t_k) + A_{22_i}x_{t_k}]\Big) + B_{21_i}^{-1}A_{21_i}[I+B_k]x(t_k) \\
& + B_{21_i}^{-1}A_{22_i}[I+B_k]x_{t_k}\Big\} \\
= \;& \Big\{[I+C_k]\Big(z(t_k) - h_i(t_k)\Big) - [I+C_k]B_{21_i}^{-1}A_{21_i}x(t_k) - [I+C_k]B_{21_i}^{-1}A_{22_i}x_{t_k} \\
& + B_{21_i}^{-1}A_{21_i}[I+B_k]x(t_k) + B_{21_i}^{-1}A_{22_i}[I+B_k]x_{t_k}\Big\}^T P_{2_i}\Big\{[I+C_k]\Big(z(t_k) - h_i(t_k)\Big) \\
& - [I+C_k]B_{21_i}^{-1}A_{21_i}x(t_k) - [I+C_k]B_{21_i}^{-1}A_{22_i}x_{t_k} + B_{21_i}^{-1}A_{21_i}[I+B_k]x(t_k) \\
& + B_{21_i}^{-1}A_{22_i}[I+B_k]x_{t_k}\Big\} \\
= \;& \Big\{\overbrace{[I+C_k]}^{:=U_k}\Big(z(t_k) - h_i(t_k)\Big) - \{\overbrace{[I+C_k]B_{21_i}^{-1}A_{21_i} - B_{21_i}^{-1}A_{21_i}[I+B_k]}^{:=R_{ik}}\}x(t_k) \\
& - \{\overbrace{[I+C_k]B_{21_i}^{-1}A_{22_i} - B_{21_i}^{-1}A_{22_i}[I+B_k]}^{:=S_{ik}}\}x_{t_k}\Big\}^T P_{2_i}\Big\{\overbrace{[I+C_k]}^{:=U_k}\Big(z(t_k) - h_i(t_k)\Big) \\
& \{\overbrace{[I+C_k]B_{21_i}^{-1}A_{21_i} - B_{21_i}^{-1}A_{21_i}[I+B_k]}^{:=R_{ik}}\}x(t_k) \\
& - \{\overbrace{[I+C_k]B_{21_i}^{-1}A_{22_i} - B_{21_i}^{-1}A_{22_i}[I+B_k]}^{:=S_{ik}}\}x_{t_k}\Big\} \\
= \;& \Big\{U_k\Big(z(t_k) - h_i(t_k)\Big) - R_{ik}x(t_k) - S_{ik}x_{t_k}\Big\}^T P_{2_i} \\
& \times\Big\{U_k\Big(z(t_k) - h_i(t_k)\Big) - R_{ik}x(t_k) - S_{ik}x_{t_k}\Big\} \\
= \;& \Big(z(t_k) - h_i(t_k)\Big)^T U_k^T P_{2_i} U_k\Big(z(t_k) - h_i(t_k)\Big) + x^T(t_k)R_{ik}^T P_{2_i} R_{ik}x(t_k) \\
& + x_{t_k}^T S_{ik}^T P_{2_i} S_{ik} x_{t_k} - 2\Big(z(t_k) - h_i(t_k)\Big)^T U_k^T P_{2_i} R_{ik} x(t_k) \\
& - 2\Big(z(t_k) - h_i(t_k)\Big)^T U_k^T P_{2_i} S_{ik} x_{t_k} - 2x^T(t_k) R_{ik}^T P_{2_i} S_{ik} x_{t_k}
\end{aligned}
$$

$$\begin{aligned}
&\leq \|U_k\|^2 \cdot \|P_{2_i}\| \cdot \|z(t_k) - h_i(t_k)\|^2 + \|R_{ik}\|^2 \cdot \|P_{2_i}\| \cdot \|x(t_k)\|^2 \\
&\quad + \|S_{ik}\|^2 \cdot \|P_{2_i}\| \cdot \|x_{t_k}\|_\tau^2 + \|U_k\| \cdot \|P_{2_i}\| \cdot \|R_{ik}\| \Big(\|z(t_k) - h_i(t_k)\|^2 + \|x(t_k)\|^2 \Big) \\
&\quad + \|U_k\| \cdot \|P_{2_i}\| \cdot \|S_{ik}\| \Big(\|z(t_k) - h_i(t_k)\|^2 + \|x_{t_k}\|_\tau^2 \Big) \\
&\quad + \|R_{ik}\| \cdot \|P_{2_i}\| \cdot \|S_{ik}\| \Big(\|x(t_k)\|^2 + \|x_{t_k}\|_\tau^2 \Big) \\
&= \|U_k\| \cdot \|P_{2_i}\| \Big(\|U_k\| + \|R_{ik}\| + \|S_{ik}\| \Big) \|z(t_k) - h_i(t_k)\|^2 \\
&\quad + \|R_{ik}\| \cdot \|P_{2_i}\| \Big(\|U_k\| + \|R_{ik}\| + \|S_{ik}\| \Big) \|x(t_k)\|^2 \\
&\quad + \|S_{ik}\| \cdot \|P_{2_i}\| \Big(\|U_k\| + \|R_{ik}\| + \|S_{ik}\| \Big) \|x_{t_k}\|_\tau^2 \\
&= \|P_{2_i}\| \Big(\|U_k\| + \|R_{ik}\| + \|S_{ik}\| \Big) \Big\{ \|U_k\| \|z(t_k) - h_i(t_k)\|^2 + \|R_{ik}\| \|x(t_k)\|^2 \\
&\quad + \|S_{ik}\| \|x_{t_k}\|_\tau^2 \Big\} \\
&\leq \lambda_{\max}(P_{2_i}) \Big(\|U_k\| + \|R_{ik}\| + \|S_{ik}\| \Big) \Big\{ \frac{\|U_k\|}{\lambda_{\min}(P_{2_i})} W_i(t_k) + \frac{\|R_{ik}\|}{\lambda_{\min}(P_{1_i})} V_i(t_k) \\
&\quad + \frac{\|S_{ik}\|}{\lambda_{\min}(P_{1_i})} \|V_{i_{t_k}}\|_\tau \Big\}
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda_{2M} \Big(\|U_k\| + r_k + s_k \Big) \Big\{ \frac{\|U_k\|}{\lambda_{2m}} W_i(t_k) + \frac{r_k}{\lambda_{1m}} V_i(t_k) + \frac{s_k}{\lambda_{1m}} \|V_{i_{t_k}}\|_\tau \Big\} \\
&= \gamma_k W_i(t_k) + \beta_k V_i(t_k) + \psi_k \|V_{i_{t_k}}\|_\tau
\end{aligned}$$

Namely, we have

$$W_i(t_k^+) \leq \beta_k V_i(t_k) + \gamma_k W_i(t_k) + \psi_k \|V_{i_{t_k}}\|_\tau \tag{4.6}$$

where $\beta_k = \lambda_{2M}\Big(\|U_k\| + r_k + s_k\Big)\frac{r_k}{\lambda_{1m}}$, $\gamma_k = \lambda_{2M}\Big(\|U_k\| + r_k + s_k\Big)\frac{\|U_k\|}{\lambda_{2m}}$, $\psi_k = \lambda_{2M}\Big(\|U_k\| + r_k + s_k\Big)\frac{s_k}{\lambda_{1m}}$, $r_k = \max\{\|R_{ik}\|; \forall i \in S\}$, and $s_k = \max\{\|S_{ik}\|; \forall i \in S\}$.

By running an unstable mode on the first interval and a stable one on the second interval, we have respectively.

$$\begin{aligned}
V_1(t) &\leq \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau \Big) e^{\xi_1(t-t_0)}, \\
V_2(t) &\leq \Big(\|V_{2_{t_1^+}}\|_\tau + \|W_{2_{t_1^+}}\|_\tau \Big) e^{-\zeta_2(t-t_1)}
\end{aligned} \tag{4.7}$$

where the norms in the second inequality are calculated as follows

$$V_2(t^+) \leq \alpha_1 V_2(t) \leq \alpha_1 \mu V_1(t)$$

Thus,

$$\begin{aligned}
\|V_{2_{t_1^+}}\|_\tau &\leq \alpha_1 \mu \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau \Big) e^{\xi_1(t_1 - t_0)} \\
\|W_{2_{t_1^+}}\|_\tau &\leq \mu(\beta_1 + \gamma_1 + \psi_1) \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau \Big) e^{\xi_1(t_1 - t_0)}
\end{aligned}$$

Substituting these inequalities into (4.7) to get

$$V_2(t) \leq \mu(\alpha_1 + \beta_1 + \gamma_1 + \psi_1) \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau \Big) e^{\xi_1(t_1 - t_0)} e^{-\zeta_2(t - t_1)}$$

Generally, after running l unstable modes and switching l times from an unstable modes, and running $m-l$ and switching $m-l-1$ times from a stable mode, we have

$$\begin{aligned}
V_m(t) \leq & \prod_{i=1}^{l} \mu(\alpha_i + \beta_i + \gamma_i + \psi_i) e^{\xi_i(t_i - t_{i-1})} \times \prod_{j=l+1}^{m-l-1} \mu(\alpha_j + \beta_j + \gamma_j \\
& + \psi_j e^{\zeta_j \tau}) e^{\zeta_j \tau} e^{-\zeta_j(t_j - t_{j-1})} \times \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau \Big) e^{-\zeta_m(t - t_{m-1})}
\end{aligned}$$

Making use of Assumption $A4$, we have

$$\begin{aligned}
V_m(t) &\leq \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau \Big) e^{-(\lambda^* - \nu)(t - t_0)} \\
W_m(t) &\leq \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau \Big) e^{-(\lambda^* - \nu)(t - t_0)}
\end{aligned}$$

By Theorem 3.3, there exists a positive constant K such that

$$\|x(t)\| + \|z(t)\| \leq K \Big(\|x_{t_0^+}\|_\tau + \|z_{t_0^+}\|_\tau \Big) e^{-(\lambda^* - \nu)(t - t_0)}, \quad t \geq t_0.$$

This shows that the origin of system (4.1) is exponentially stable.

To verify this result, consider the following example.

Example 4.1: Consider the following impulsive system

$$\begin{aligned}
\dot{x} &= A_{11_i}x + B_{12_i}z(t-\tau), \qquad t \neq t_k \\
\varepsilon\dot{z} &= A_{22_i}x(t-\tau) + B_{21_i}z, \qquad t \neq t_k, \quad k = 1, 2, \cdots \\
\Delta x &= B_k x, \qquad t = t_k \\
\Delta z &= C_k z, \qquad t = t_k
\end{aligned}$$

where the matrices A_{11_i}, A_{22_i}, B_{12_i}, B_{21_i} are given in Example 3.4, and the amount of impulses are given by

$$B_k = 2^{-k}\begin{pmatrix} -0.01 & -0.01 \\ 0 & 0.002 \end{pmatrix} \quad \text{and} \quad C_k = 2^{-k}\begin{pmatrix} 0 & -0.01 \\ 0.002 & 0 \end{pmatrix}.$$

The norms of B_k and C_k are respectively $\|B_k\| = 0.0142 \cdot 2^{-k}$ and $\|C_k\| = 0.01 \cdot 2^{-k}$ and so $\sum_{k=1}^{\infty} \|B_k\| = 0.0142 < \infty$ and $\sum_{k=1}^{\infty} \|C_k\| = 0.01 < \infty$, meaning that the cumulative effect of impulses is finite. We also notice that when time evolves the effect of later impulses becomes negligible since $\lim_{k\to\infty} \|B_k\| = 0$ and $\lim_{k\to\infty} \|C_k\| = 0$. Since the switched system without impulses is stable, then the diminishing effect of impulses is not enough to destabilize the system. Taking $k = 1, 2, 3, 4$, we have the following parameters

$\alpha_1 = 9.018, \quad \alpha_2 = 9.009, \quad \alpha_3 = 9.0045, \quad \alpha_4 = 9.0023.$

$\beta_k = 0 \quad \forall k = 1, 2, 3, 4.$

$\gamma_1 = 9.0896, \quad \gamma_2 = 9.0447, \quad \gamma_3 = 9.0224, \quad \gamma_4 = 9.0112.$

$\psi_1 = 0.0538, \quad \psi_2 = 0.0268, \quad \psi_3 = 0.0134, \quad \psi_4 = 0.0067.$

The dwell times are

$T_D = 2.2159$ (unstable mode).

$T_D = 2.2130$ (unstable mode).

$T_D = 3.2638$ (stable mode).

$T_D = 3.2576$ (stable mode).

Then, by Theorem 4.1, the origin is exponentially stable. Clearly, the dwell times of the unstable (and the stable) modes are shrinking since the magnitudes of $\|I + B_k\|$ and $\|I + C_k\|$ become smaller when k tends to infinity. Figure 4.1 shows the simulation results.

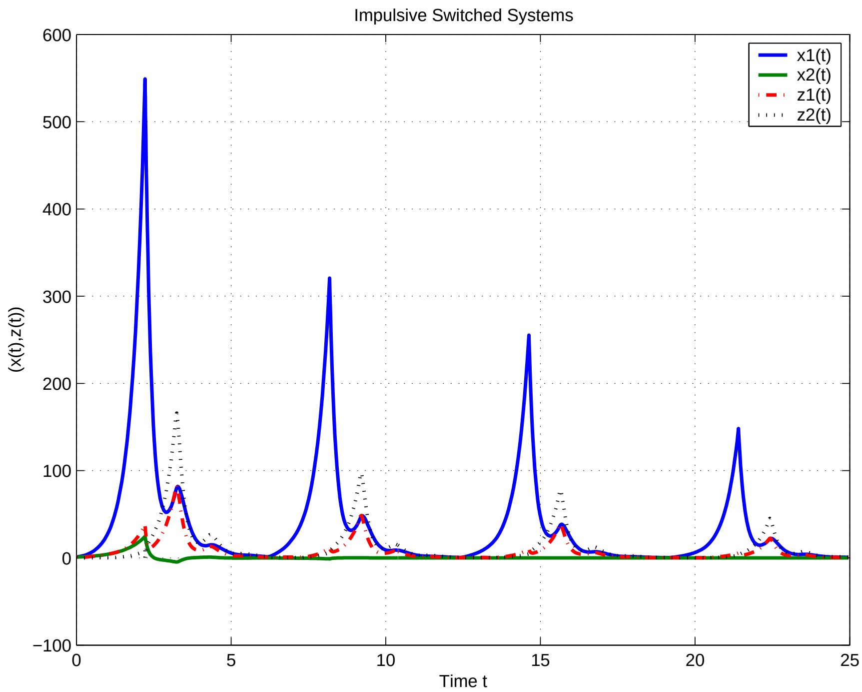


Figure 4.1: Impulsive Switched delay system with unstable and stable nonlinear modes

In the next theorem, we investigate the stability problem of system (4.1) where all subsystems are unstable. We shall show that impulses do contribute to yield stability properties of this system.

Theorem 4.2: *Consider the impulsive system* (4.1)*. Assume that the following assumptions are satisfied.*

A1) *Assumption* $(A1-i)$ *of Theorem* 4.1 *holds.*

A2) *Assumption* $A2$ *of Theorem* 4.1 *holds.*

A3) *Assumption* $(A3-i)$ *of Theorem* 4.1 *holds.*

A4) *There exists a constant* $\vartheta \geq 1$ *such that*

$$\ln\Big(\vartheta\mu(\alpha_i+\beta_i+\gamma_i+\psi_i)\Big)+\xi_i(t_{k+1}-t_k)\leq 0.$$

where α_i*,* β_i*,* γ_i*,* ψ_i *and* ξ_i *are defined in Theorem* 4.1*. Then,* $\vartheta = 1$ *implies that the origin of system* (4.1) *is stable, and* $\vartheta > 1$ *implies that the origin of system* (4.1) *is asymptotically stable.*

In fact Assumptions $(A1-A3)$ are made to ensure that the subsystems are unstable, while Assumption $A4$ is introduced to keep the solutions down whenever the subsystems are switched.

Proof:

For each $i \in S$ and $t \in (t_{k-1}, t_k]$, define $V_i(t) = x^T P_{1_i} x$ and $W_i(t) = (z-h_i)^T P_{2_i}(z-h_i)$. Then, the time derivative of V_i and W_i along the trajectories of system (4.1) are

$$\begin{aligned}
\dot{V}_i(t) &\leq (2\gamma+\frac{a_{11_i}}{\lambda_{1m}})V_i(x)+\frac{b_{11_i}}{\lambda_{2m}}W_i((z-h_i)(t))+\frac{a_{12_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau \\
&\quad +\frac{b_{12_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau \\
\dot{W}_i(t) &\leq \frac{a_{21_i}}{\lambda_{1m}}V_i(x(t))-\frac{\lambda_{\min}(Q_{2_i})-\varepsilon_i b_{21_i}}{\varepsilon_i\lambda_{2M}}W_i((z-h_i)(t))+\frac{a_{22_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau \\
&\quad +\frac{b_{22_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau
\end{aligned}$$

By Lemma 3.2, there exists a positive constant ξ_i such that

$$\begin{aligned} V_i(x(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau)e^{\xi_i(t-t_{k-1})} \\ W_i((z-h_i)(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau)e^{\xi_i(t-t_{k-1})}. \end{aligned}$$

By Theorem 4.1, we have at $t = t_k^+$

$$\begin{aligned} V_i(t_k^+) &\leq \alpha_k V_i(t_k) \\ W_i(t_k^+) &\leq \beta_k V_i(t_k) + \gamma_k W_i(t_k) + \psi_k \|V_{i_{t_k}}\|_\tau \end{aligned}$$

For $t \in (t_0, t_1]$, we have

$$V_1(t) \leq (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau)e^{\xi_1(t-t_0)}$$

and for $t \in (t_1, t_2]$, we have

$$V_2(t) \leq (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau)e^{\xi_1(t_1-t_0)}\mu(\alpha_1+\beta_1+\gamma_1+\psi_1)e^{\xi_2(t-t_1)}$$

Generally, for $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} V_i(t) &\leq (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau)e^{\xi_1(t_1-t_0)}\mu(\alpha_1+\beta_1+\gamma_1+\psi_1)e^{\xi_2(t_2-t_1)} \\ &\quad \times\mu(\alpha_2+\beta_2+\gamma_2+\psi_2)e^{\xi_2(t_3-t_2)}\cdots\mu(\alpha_k+\beta_k+\gamma_k+\psi_k)e^{\xi_i(t_{k+1}-t_k)} \\ &= (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau)\frac{1}{\vartheta^k}e^{\xi_1(t_1-t_0)}\vartheta\mu(\alpha_1+\beta_1+\gamma_1+\psi_1)e^{\xi_2(t_2-t_1)} \\ &\quad \times\vartheta\mu(\alpha_2+\beta_2+\gamma_2+\psi_2)e^{\xi_2(t_3-t_2)}\cdots\vartheta\mu(\alpha_k+\beta_k+\gamma_k+\psi_k)e^{\xi_i(t_{k+1}-t_k)} \\ &\leq (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau)\frac{1}{\vartheta^k}e^{\xi_1(t_1-t_0)} \end{aligned}$$

Namely, we have

$$V_i(t) \leq (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau)\frac{1}{\vartheta^k}e^{\xi_1(t_1-t_0)}$$

Similarly

$$W_i(t) \leq (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau)\frac{1}{\vartheta^k}e^{\xi_1(t_1-t_0)}$$

Following the technique used in Theorem 3.3, there exists a positive constant K such that

$$\|x(t)\| + \|z(t)\| \leq \frac{K}{\sqrt{\vartheta^k}}(\|x_{t_0^+}\|_\tau + \|z_{t_0^+}\|_\tau)e^{\xi_1(t_1-t_0)/2}.$$

Clearly, if $\vartheta = 1$, then system (4.1) is stable, and if $\vartheta > 1$ and $k \to \infty$, the system is asymptotically stable. The proof is completed.

The following example shows these results.

Example4.2: Consider the following subsystems

Mode 1:

$$\begin{aligned}\dot{x} &= x + 3z(t-1) \\ \varepsilon\dot{z} &= 2x(t-1) - 2z, \quad \varepsilon = 0.7,\end{aligned}$$

Mode 2:

$$\begin{aligned}\dot{x} &= x + 2z(t-1) \\ \varepsilon\dot{z} &= 4x(t-1) - 2z, \quad \varepsilon = 0.7,\end{aligned}$$

with the following impulses

$$\begin{aligned}\Delta x &= -0.97x(t) \\ \Delta z &= -0.9z(t),\end{aligned}$$

and $\|I + B_k\| = 0.03$ and $\|I + C_k\| = 0.1$ for all k.

Taking $\gamma = 2$, $Q_{1_1} = 2$, $Q_{2_1} = 2$, give us $P_{1_1} = 1$ and $P_{2_1} = 0.5$
and $Q_{1_2} = 3$, $Q_{2_2} = 1$, give us $P_{1_2} = 1.5$ and $P_{2_2} = 0.25$,
so that $\mu = \max\{\mu_1, \mu_2\} = \max\{1.5, 2\} = 2 > 1$.
We also get
$\alpha_1^* = \lambda(\widetilde{A_1}^T + \widetilde{A_1}) = \{-2.3571, 14\}$, $\beta_{1_1} = \|\widetilde{B_1}\| = 12$, so $\beta_{2_1} = \alpha_1^* + \beta_{1_1} =$

$\{26, 9.6424\}$ and the growth rates of the first subsystem are $\xi_1 = (\beta_{1_1} + \beta_{2_1})/2 = \{19, 10.8214\}$.

$\alpha_2^* = \lambda(\widetilde{A_2}^T + \widetilde{A_2}) = \{-0.9286, 14\}$, $\beta_{1_2} = \|\widetilde{B_2}\| = 12$, so $\beta_{2_2} = \alpha_2^* + \beta_{1_2} = \{26, 11.0714\}$ and the growth rates of the first subsystem are $\xi_2 = (\beta_{1_2} + \beta_{2_2})/2 = \{19, 11.5357\}$.

The impulse parameters are

$\alpha_k = 0.0018, \quad \beta_k = 0, \quad \gamma_k = 0.048, \quad \psi_k = 0.0672 \quad \forall k = 1, 2, \cdots.$

A simple check shows that Assumption $A4$ of Theorem 4.2 holds by choosing $\vartheta \in [1, 4.2735)$. Taking $\vartheta = 2$ for instance and

$$\begin{aligned} (i)\xi_1 &= 10.8214 \quad \text{gives} \quad t_{k+1} - t_k \leq 0.0702 \\ (ii)\xi_2 &= 11.5357 \quad \text{gives} \quad t_{k+1} - t_k \leq 0.0658. \end{aligned}$$

for all k. Then, by Theorem 4.2, the origin is asymptotically stable.

4.2 Nonlinear Systems

The nonlinear switched delay singularly perturbed systems with impulses at fixed times considered in this section is described as follows

$$\begin{aligned} \dot{x} &= f_i(x, x_t, z, z_t), \qquad t \neq t_k \\ \varepsilon \dot{z} &= B_{21_i} z + B_i(x, x_t), \quad t \neq t_k \\ \Delta x &= B_k x(t), \quad t = t_k \\ \Delta z &= C_k z(t), \quad t = t_k \end{aligned} \tag{4.8}$$

where $i \in S_u \cup S_s$, $f_i = A_{11_i} x + g_i(x, x_t, z, z_t)$, $k = 1, 2, \ldots$, with $\lim_{k\to\infty} t_k = \infty$, and the $n \times n$ matrix B_{12_i} is nonsingular. We assume that system (4.8) has a unique equilibrium point at the origin. The functions f_i, B_i are defined in [Bal99] to ensure that system (4.8) has a unique solution. The initial condition of this system is given by

$$\begin{aligned} x(t) &= \phi_1(t), \quad t \in [t_0 - \tau, t_0] \\ z(t) &= \phi_2(t), \quad t \in [t_0 - \tau, t_0] \end{aligned}$$

where $\phi_1 \in PC([t_0 - \tau, t_0], R^m)$ and $\phi_2 \in PC([t_0 - \tau, t_0], R^n)$.

In the next theorem we establish exponential stability of the origin of system (4.8).

Theorem 4.3: *The origin of system* (4.8) *is locally exponentially stable if the following assumptions are satisfied.*

A1) *Assumption A1 of Theorem 4.1 holds.*

A2-i) *Assumption A2 of Theorem 3.6 holds.*

A2-ii) *There exist positive constants a, b, c such that*

$$\begin{aligned} &2\Big(z(t_k) - h_i(t_k)\Big)^T [I + C_k]^T P_{2_i} \Big\{[I + C_k]h_i(t_k) - h_i(t_k^+)\Big\} \\ &+\Big\{[I + C_k]h_i(t_k) - h_i(t_k^+)\Big\}^T P_{2_i} \Big\{[I + C_k]h_i(t_k) - h_i(t_k^+)\Big\} \le a\|z(t_k) - h_i(t_k)\|^2 \\ &+b\|x(t_k)\|^2 + c\|x_{t_k}\|_\tau^2 \end{aligned} \tag{4.9}$$

where $h_i(t_k) = -B_{21_i} B_i(x, x_{t_k})$

A3) *Assumption A3 of Theorem 4.1.*

A4) *Let*

$$\begin{aligned} \lambda^+ &= \max\{\xi_i : i \in S_u\}, \\ \lambda^- &= \min\{\zeta_i : i \in S_s\}, \end{aligned}$$

$T^+(t_0, t)$ be the total activation time of the unstable modes, $T^-(t_0, t)$ be the total activation time of the stable modes, and, for any t_0, assume that the switching law

guarantees that

$$\inf_{t \geq t_0} \frac{T^-(t_0,t)}{T^+(t_0,t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{4.10}$$

where $\lambda^* \in (\lambda, \lambda^-)$ *and* $\lambda \in (0, \lambda^-)$*. Furthermore, there exists* $0 < \nu < \zeta_i$ *such that* (i) *for* $i \in \{1, 2, \cdots, l\}$ *and* $k = 1, 2, \cdots, l$

$$\ln \mu(\alpha_k + \beta_k + \gamma_k + \psi_k) - \nu(t_k - t_{k-1}) \leq 0 \tag{4.11}$$

(ii) *for* $i \in \{l+1, l+2, \cdots, m-1\}$ *and* $k = l+1, l+2, \cdots, m-1$

$$\ln \mu(\alpha_k + \beta_k + \gamma_k + \psi_k e^{\zeta_i \tau}) + \zeta_i \tau - \nu(t_k - t_{k-1}) \leq 0. \tag{4.12}$$

where $\alpha_k = \mu_1 \lambda^2_{\max}([I + B_k])$, $\beta_k = \frac{b}{\lambda_{1m}}$, $\gamma_k = \mu_2 \lambda^2_{\max}([I + C_k]) + a$ *and* $\psi_k = \frac{c}{\lambda_{1m}}$.

As seen, we don't have assumption $(A2 - ii)$ in Theorem 4.1 since the positive constant a, b, c are explicitly found in the linear case.

Proof:

For $t \in (t_{k-1}, t_k]$, define $V_i(x(t)) = x^T(t) P_{1_i} x(t)$ and $W_i((z - h_i)(t)) = (z - h_i)^T(t) P_{2_i} (z - h_i)(t)$. Then, the time derivative of V_i and W_i along the trajectories of $x(t)$ and $z(t)$ are

(i) for $i \in S_u$

$$\begin{aligned}
\dot{V}_i(x(t)) &\leq (2\gamma + \frac{a_{11_i}}{\lambda_{1m}}) V_i(x) + \frac{b_{11_i}}{\lambda_{2m}} W_i((z - h_i)(t)) + \frac{a_{12_i}}{\lambda_{1m}} \|V_{i_t}\|_\tau \\
&\quad + \frac{b_{12_i}}{\lambda_{2m}} \|W_{i_t}\|_\tau \\
\dot{W}_i((z - h_i)(t)) &\leq \frac{a_{21_i}}{\lambda_{1m}} V_i(x(t)) - \frac{1 - \varepsilon_i b_{21_i}}{\varepsilon_i \lambda_{2M}} W_i((z - h_i)(t)) + \frac{a_{22_i}}{\lambda_{1m}} \|V_{i_t}\|_\tau \\
&\quad + \frac{b_{22_i}}{\lambda_{2m}} \|W_{i_t}\|_\tau
\end{aligned}$$

(ii) for $i \in S_s$

$$\begin{aligned}
\dot{V}_i(x(t)) &\leq -\frac{\lambda_{\min}(Q_{1_i}) - a_{11_i}}{\lambda_{1M}} V_i(x) + \frac{b_{11_i}}{\lambda_{2m}} W_i((z-h_i)(t)) + \frac{a_{12_i}}{\lambda_{1m}} \|V_{i_t}\|_\tau \\
&\quad + \frac{b_{12_i}}{\lambda_{2m}} \|W_{i_t}\|_\tau \\
\dot{W}_i((z-h_i)) &\leq \frac{a_{21_i}}{\lambda_{1m}} V_i(x(t)) - \frac{\lambda_{\min}(Q_{2_i}) - \varepsilon_i^* b_{21_i}}{\varepsilon_i^* \lambda_{2M}} W_i((z-h_i)(t)) + \frac{a_{22_i}}{\lambda_{1m}} \|V_{i_t}\|_\tau \\
&\quad + \frac{b_{22_i}}{\lambda_{2m}} \|W_{i_t}\|_\tau
\end{aligned}$$

Then, by Lemma 3.2, there exists a positive constant $\xi_i, i \in S_u$ such that

$$\begin{aligned}
V_i(x(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau) e^{\xi_i (t - t_{k-1})} \\
W_i((z-h_i)(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau) e^{\xi_i (t - t_{k-1})}.
\end{aligned}$$

and by Lemma 3.3, there exists a positive constant ζ_i such that

$$\begin{aligned}
V_i(x(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau) e^{-\zeta_i (t - t_{k-1})} \\
W_i((z-h_i)(t)) &\leq (\|V_{i_{t_{k-1}^+}}\|_\tau + \|W_{i_{t_{k-1}^+}}\|_\tau) e^{-\zeta_i (t - t_{k-1})}.
\end{aligned}$$

At $t = t_k^+$, we have the following estimates

$$V_i(t_k^+) \leq \alpha_k V_i(t_k)$$

where $\alpha_k = \mu \lambda_{\max}^2 (I + B_k)$.

$$\begin{aligned}
W_i(t_k^+) &= \Big(z(t_k^+) - h_i(t_k^+)\Big)^T P_{2_i} \Big(z(t_k^+) - h_i(t_k^+)\Big) \\
&= \Big([I + C_k] z(t_k) - [I + C_k] h_i(t_k) + [I + C_k] h_i(t_k) - h_i(t_k^+)\Big)^T P_{2_i} \\
&\quad \Big([I + C_k] z(t_k) - [I + C_k] h_i(t_k) + [I + C_k] h_i(t_k) - h_i(t_k^+)\Big) \\
&= \Big(z(t_k) - h_i(t_k)\Big)^T [I + C_k]^T P_{2_i} [I + C_k] \Big(z(t_k) - h_i(t_k)\Big) \\
&\quad + 2\Big(z(t_k) - h_i(t_k)\Big)^T [I + C_k]^T P_{2_i} \Big\{[I + C_k] h_i(t_k) - h_i(t_k^+)\Big\} \\
&\quad + \Big\{[I + C_k] h_i(t_k) - h_i(t_k^+)\Big\}^T P_{2_i} \Big\{[I + C_k] h_i(t_k) - h_i(t_k^+)\Big\}
\end{aligned}$$

Making use of assumption $(A2-ii)$, we have

$$\begin{aligned}
W_i(t_k^+) &\leq \lambda_{\max}\Big([I+C_k]^T P_{2_i}[I+C_k]\Big)\|z(t_k)-h_i(t_k)\|^2 + a\|z(t_k)-h_i(t_k)\|^2 \\
&\quad + b\|x(t_k)\|^2 + c\|x_{t_k}\|_\tau^2 \\
&\leq \frac{\lambda_{2M}\lambda_{\max}^2([I+C_k])+a}{\lambda_{2m}} W_i(t_k) + \frac{b}{\lambda_{1m}} V_i(t_k) + \frac{c}{\lambda_{1m}}\|V_{i_{t_k^-}}\|_\tau \\
&= \beta_k V_i(t_k) + \gamma_k W_i(t_k) + \psi_k \|V_{i_{t_k}}\|_\tau
\end{aligned}$$

Namely, we have

$$W_i(t_k^+) \leq \beta_k V_i(t_k) + \gamma_k W_i(t_k) + \psi_k \|V_{i_{t_k}}\|_\tau$$

where $\beta_k = \frac{b}{\lambda_{1m}}$, $\gamma_k = \frac{\lambda_{2M}\lambda_{\max}^2([I+C_k])+a}{\lambda_{2m}}$, and $\psi_k = \frac{c}{\lambda_{1m}}$.

By Theorem 4.1, we have

$$\begin{aligned}
V_m(t) &\leq \prod_{i=1}^{l} \mu(\alpha_i+\beta_i+\gamma_i+\psi_i)e^{\xi_i(t_i-t_{i-1})} \times \prod_{j=l+1}^{m-l-1} \mu(\alpha_j+\beta_j+\gamma_j \\
&\quad + \psi_j e^{\zeta_j\tau})e^{\zeta_j\tau}e^{-\zeta_j(t_j-t_{j-1})} \times \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau\Big)e^{-\zeta_m(t-t_{m-1})}
\end{aligned}$$

Making use of Assumption $A4$, we have

$$\begin{aligned}
V_m(t) &\leq \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau\Big)e^{-(\lambda^*-\nu)(t-t_0)} \\
W_m(t) &\leq \Big(\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau\Big)e^{-(\lambda^*-\nu)(t-t_0)}
\end{aligned}$$

By Theorem 3.3, there exists a positive constant K such that

$$\|x(t)\| + \|z(t)\| \leq K\Big(\|x_{t_0^+}\|_\tau + \|z_{t_0^+}\|_\tau\Big)e^{-(\lambda^*-\nu)(t-t_0)}, \quad t \geq t_0.$$

This shows that the origin of system (4.8) is exponentially stable.

Our final result is to establish conditions for stability properties of the origin of the nonlinear system (4.8) where all subsystems are unstable.

Theorem 4.4: *Consider the impulsive system* (4.8)*. Assume that the following assumptions are satisfied.*

A1) *Assumption* $(A1-i)$ *of Theorem* 4.1 *holds.*

A2) *Assumption* $A2$ *of Theorem* 4.3 *holds.*

A3) *Assumption* $(A3-i)$ *of Theorem* 4.1 *holds.*

A4) *There exists a constant* $\vartheta \geq 1$ *such that*

$$\ln\Big(\vartheta\mu(\alpha_i+\beta_i+\gamma_i+\psi_i)\Big)+\xi_i(t_{k+1}-t_k)\leq 0.$$

where α_i*,* β_i*,* γ_i *and* ψ_i *are defined in Theorem* 4.3*, and* ξ_i *is defined in Theorem* 4.1*. Then,* $\vartheta=1$ *implies that the origin of system* (4.8) *is stable, and* $\vartheta>1$ *implies that the origin of system* (4.8) *is asymptotically stable.*

Proof:

For each $i\in S$ and $t\in(t_{k-1},t_k]$, define $V_i(t)=x^TP_{1_i}x$ and $W_i(t)=(z-h_i)^TP_{2_i}(z-h_i)$. Then, the time derivative of V_i and W_i along the trajectories of system (4.8) are

$$\begin{aligned}
\dot V_i(t) &\leq (2\gamma+\frac{a_{11_i}}{\lambda_{1m}})V_i(x)+\frac{b_{11_i}}{\lambda_{2m}}W_i((z-h_i)(t))+\frac{a_{12_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau\\
&\quad+\frac{b_{12_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau\\
\dot W_i(t) &\leq \frac{a_{21_i}}{\lambda_{1m}}V_i(x(t))-\frac{\lambda_{\min}(Q_{2_i})-\varepsilon_i b_{21_i}}{\varepsilon_i\lambda_{2M}}W_i((z-h_i)(t))+\frac{a_{22_i}}{\lambda_{1m}}\|V_{i_t}\|_\tau\\
&\quad+\frac{b_{22_i}}{\lambda_{2m}}\|W_{i_t}\|_\tau
\end{aligned}$$

By Lemma 3.2, there exists a positive constant ξ_i such that

$$\begin{aligned}
V_i(x(t)) &\leq (\|V_{i_{t^+_{k-1}}}\|_\tau+\|W_{i_{t^+_{k-1}}}\|_\tau)e^{\xi_i(t-t_{k-1})}\\
W_i((z-h_i)(t)) &\leq (\|V_{i_{t^+_{k-1}}}\|_\tau+\|W_{i_{t^+_{k-1}}}\|_\tau)e^{\xi_i(t-t_{k-1})}.
\end{aligned}$$

By Theorem 4.3, we have at $t = t_k^+$

$$
\begin{aligned}
V_i(t_k^+) &\leq \alpha_k V_i(t_k) \\
W_i(t_k^+) &\leq \beta_k V_i(t_k) + \gamma_k W_i(t_k) + \psi_k \|V_{i_{t_k}}\|_\tau
\end{aligned}
$$

For $t \in (t_k, t_{k+1}]$, we have

$$
\begin{aligned}
V_i(t) &\leq (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau) \frac{1}{\vartheta^k} e^{\xi_1(t_1-t_0)} \vartheta\mu(\alpha_1 + \beta_1 + \gamma_1 + \psi_1) e^{\xi_2(t_2-t_1)} \\
&\quad \times \vartheta\mu(\alpha_2 + \beta_2 + \gamma_2 + \psi_2) e^{\xi_2(t_3-t_2)} \cdots \vartheta\mu(\alpha_k + \beta_k + \gamma_k + \psi_k) e^{\xi_i(t_{k+1}-t_k)} \\
&\leq (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau) \frac{1}{\vartheta^k} e^{\xi_1(t_1-t_0)}
\end{aligned}
$$

Similarly

$$
W_i(t) \leq (\|V_{1_{t_0^+}}\|_\tau + \|W_{1_{t_0^+}}\|_\tau) \frac{1}{\vartheta^k} e^{\xi_1(t_1-t_0)}
$$

Following the technique used in Theorem 3.3, there exists a positive constant K such that

$$
\|x(t)\| + \|z(t)\| \leq \frac{K}{\sqrt{\vartheta^k}} (\|x_{t_0^+}\|_\tau + \|z_{t_0^+}\|_\tau) e^{\xi_1(t_1-t_0)/2}.
$$

Clearly, if $\vartheta = 1$, then system (4.8) is stable, and if $\vartheta > 1$ and $k \to \infty$, the system is asymptotically stable.

Chapter 5

Conclusions and Future Work

Hybrid systems are adequate as a tool to describe many physical processes that undergo abrupt changes in their states. Although the field of hybrid systems is somewhat in initial stages, it has become increasingly popular. One of the system qualitative properties that has received a great deal of work is the stability aspect of these systems. Singular perturbation is a very useful technique to handle many processes that exhibit multiple-time scales in some of their dynamics. A large class of networks or large-scale systems are modeled as singularly perturbed systems (SPSs). The stability notion of these systems is the interest of many researchers. In this thesis, we merge the two fields, hybrid systems and SPSs with time delay, which leads to hybrid SPSs with time delay.

In Chapter 2, we separately analyze the stability problem of these kinds of systems in order to lead readers of this document to better understanding Chapter 3 and 4.

Through the investigations in this thesis, many further research problems could be particularly interested. Some of them are straightforward extension of this work, but others might be more challenging.

In Chapter 3 we investigate exponential stability of switched SPSs with time delay. We first develop lemmas to help us find growth rates of unstable delay systems, and then apply them to linear switched delay systems and SPSs with time delay. While we consider one constant delay in slow and fast states of the SPSs, it would be interesting if one takes different delays, or even unbounded delays. It could also be that each subsystem has a different delay. Here, each singularly perturbed subsystems under consideration has one perturbation parameter. It would be more complicated to deal with subsystems having multiple parameters. It may include a more in-depth look at stability of each individual subsystem. Multiple Lyapunov functions technique is applied in stability analysis of these systems. Other approaches such as Lyapunov functional or Razumikhin type Lyapunov function could be used in examining similar results. As for the switched systems, we focus on systems that incorporate stable and unstable subsystems. One could study stability of these systems where all subsystems are unstable.

In Chapter 4, we establish stability of impulsive switched SPSs with time delay, and illustrate how impulses contribute to yield stability properties of systems that consist of all unstable modes. Difference equations considered in this work are simple, so that one could include the cross product terms or, in the nonlinear case, impulses represented by nonlinear functions. In this document, we discussed switched systems with impulsive actions at fixed times. In fact, one could consider other cases in which the impulses occur at variable times.

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