Synchronization of Complex-Valued Dynamical Networks

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Dynamical networks (DNs) have been broadly applied to describe natural and human systems consisting of a large number of interactive individuals. Common examples include Internet, food webs, social networks, neural networks, etc. One of the crucial and significant collective behaviors of DNs is known as synchronization. In reality, synchronization phenomena may occur either inside a network or between two or more networks, which are called "inner synchronization" and "outer synchronization", respectively. On the other hand, many real systems are more suitably characterized by complex-valued dynamical systems, such as quantum systems, complex Lorenz system, and complex-valued neural networks. The main focus of this thesis is on synchronization of complex-valued dynamical networks (CVDNs).

In this thesis, we firstly design a delay-dependent pinning impulsive controller to study synchronization of time-delay CVDNs. By taking advantage of the Lyapunov function in the complex field, some delay-independent synchronization criteria of CVDNs are established, which generalizes some existing synchronization results. Then, by employing the Lyapunov functional in the complex field, several delay-dependent sufficient conditions on synchronization of CVDNs with various sizes of delays are constructed. Moreover, we study synchronization of CVDNs with time-varying delays under distributed impulsive controllers. By taking advantage of time-varying Lyapunov function/ functional in the complex domain, several synchronization criteria for CVDNs with time-varying delays are derived in terms of complex-valued linear matrix inequalities (LMIs). Then, we propose a memory-based event-triggered impulsive control (ETIC) scheme with three levels of events in the complex field to investigate the synchronization problem of CVDNs with both discrete and distributed time delays, and we further consider an event-triggered pinning impulsive control (ETPIC) scheme combining the proposed ETIC and a pinning algorithm to study synchronization of time-delay CVDNs. Results show that the proposed ETIC scheme and ETPIC scheme can effectively synchronize CVDNs with the desired trajectory. Secondly, we study generalized outer synchronization of drive-response time-delayed CVDNs via hybrid control. A hybrid controller is proposed in the complex domain to construct response complex-valued networks. Some generalized outer synchronization criteria for drive-response CVDNs are established, which extend the existing generalized outer synchronization results to the complex field. Thirdly, we study the average-consensus problem of potential complex-valued multi-agent systems. A complex-variable hybrid consensus protocol is proposed, and time delays are taken into account in both the continuous-time protocol and the discrete-time protocol. Delay-dependent sufficient conditions are established to guarantee the proposed complex-variable hybrid consensus protocol can solve the average-consensus problem. Lastly, as a practical application for complex-valued net-

worked systems, the synchronization problem of master-slave complex-valued neural networks (CVNNs) is studied via hybrid control and delayed ETPIC, respectively. We also investigate the state estimation problem of CVNNs by designing the adaptive impulsive observer in the complex field.

Acknowledgements

First and foremost, I would like to thank my supervisor, Prof. Xinzhi Liu. Thank you for providing the guidance and support in my research and life. I would also like to thank my examining committee members, Prof. Brian Ingalls, Prof. Jun Liu, Prof. Sherman Shen and Prof. Albert C.J. Luo, for their invaluable feedback to enhance this thesis.

I would like to extend my thanks to all the members of my research group for the help and conversations during my program at Waterloo. I would also like to express my sincere gratitude to all my friends and family members for their encouragement.

Dedication

This thesis is dedicated to my parents.

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Chapter 1

Introduction

1.1 Motivations

Dynamical networks (DNs) have been broadly applied to describe natural and human systems consisting of a large number of interactive individuals over the past decades [1, 2, 3, 4]. A dynamical network generally consists of a large set of interconnected nodes, where each node is a basic unit with specific dynamic and detailed information contents [1, 4, 5]. In mathematics, a DN can be represented as a graph [3]. A graph $G = \{V, E\}$ is composed of a pair of sets, where V is a set of vertices (or nodes or points), and E is a set of edges (or links or lines), each of which connects two distinct vertices in V. Figure 1.1 shows the topological structure of a DN.

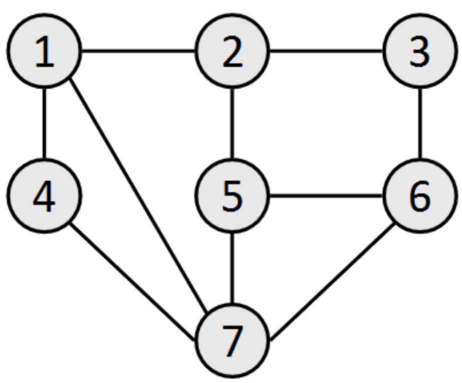

Figure 1.1: The topological structure of a DN.

Many real networks can be modeled by DNs, such as Internet, cellular and metabolic networks, food webs, telephone call graphs, transportation networks, electrical power grids, neural networks, and so on [1, 2, 3, 4]. The basic structure of a biological neural network is shown in Figure 1.2. A biological neural network consists of a series of interconnected neurons. A neuron mainly includes three portions: cell body, dendrites, and axon. Dendrites are the tree-like structure that receive signals from surrounding neurons, where each branch is connected to one neuron. Axon is a thin cylinder that transmits the signal from one neuron to others. At the end of the axon, the contact to the dendrites on other neurons are made through synapses.

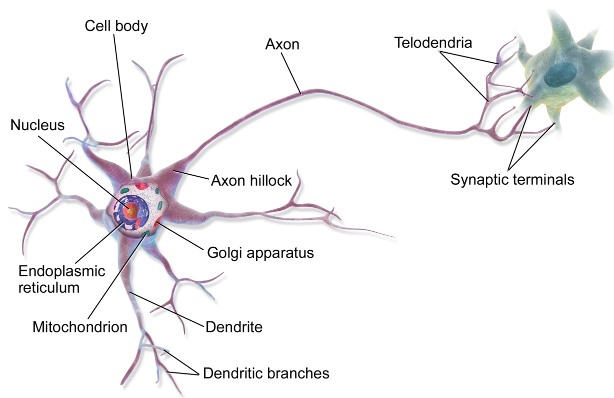

Figure 1.2: Biological neural network with neurons

Inspired by biological neural networks, W. S. McCulloch and W. Pitts [6] developed the first conceptual model of artificial neural networks (ANNs) in 1943. ANNs are computational models which can simulate the structure and functionality of the biological neural networks that constitute human brains. The basic structure of an ANN consists of an input layer, one or more hidden layers, and an output layer. Figure 1.3 shows a common structure of the ANN. The nodes of ANNs can be seen as processing units, which are called 'artificial neurons'. The artificial neurons receive input signals from the environment or other neurons, each of the input signals is weighted at the synaptic connection of the artificial neuron by a connection weight. The artificial neuron sums all the weighted input signals it receives, and the output is passed through an activation function, which is a nonlinear transform of the weighted sum [7, 8]. The input signals are called dendrites of the artificial neuron and the output signal is called the axon. Figure 1.4 shows a simple model of an artificial neuron. In recent years, ANNs have been intensively investigated by researchers and applied on a wide variety of tasks, including computer vision, speech recognition, machine translation, medical diagnosis, and data compression (see, [8]).

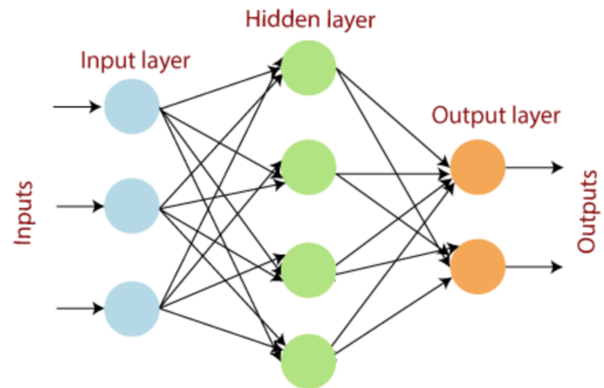

Figure 1.3: Artificial neural network

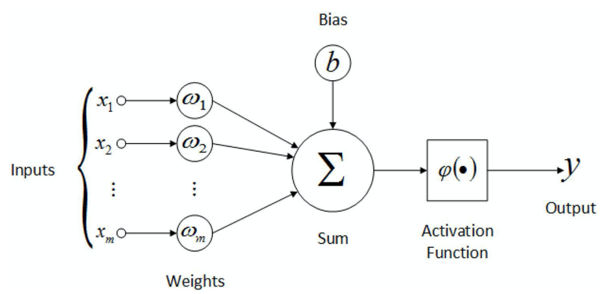

Figure 1.4: An artificial neuron with input signal x_i, connection weight w_i, activation function φ, and output signal y.

During the past few decades, the dynamical behaviors of neural networks have been paid considerable attention because of their wide applications in the fields of secure communication, image processing, pattern recognition, and associative memory (see, [9, 10, 11]).

Synchronization, as a typical and significant collective behavior of neural networks, has received a great deal of attention. Researchers noticed the fact that various parts of a biological entity operate in harmony seems to be a performance of self-organization in nature. One possible explanation of such self-organization is that some or all of the biological neurons are synchronized as time evolves so that the states of neurons differ from each other only by infinitesimal values. Recently, it has been discovered that ANNs can be designed to imitate such synchronization behaviors of biological neural networks or brains by firstly deducing the essential features of the biological neurons and their interconnections, then programming a computer to simulate these features. In recent years, many interesting results for studying the synchronization problem of neural networks have been reported (see, e.g., [12, 13, 14, 15, 16, 17]).

Recently, complex-valued neural networks (CVNNs) have received increasing research attention because complex numbers are often used in many practical fields, such as robotics, image processing, computer vision, filtering, remote sensing, speech recognition, and artificial neural information processing (see, [18, 19, 8, 20, 21, 22]), which suggests that ANNs built using complex numbers have potential applications in these domains. CVNNs are the ANNs that process information using complex-valued parameters and variables, that is, the inputs, outputs, connection weights, activation functions, and the states of the neurons are all complex-valued. The magnitude of the complex-valued states can be interpreted as the average firing rate (i.e. average number of spikes appearing during a short interval), which is similar to the real-valued case. The phase of the complex-valued states corresponds to the phase of neuronal rhythms, which are rhythmic patterns of neuronal spikes (i.e. peaks in the neuron's signal). Such rhythms are characterized by their average firing rate and their phase. Thus, the total input of a complex-valued artificial neuron no longer depends only on the firing rates of the input signals, but also on their relative timing.

Compared with real-valued neural networks (RVNNs), the flexibility in learning and self-organization is restricted in CVNNs. Thus, by employing CVNNs, the possibly harmful portion of the degree of freedom in learning or self-organization can be reduced for achieving better generalization characteristics. Furthermore, CVNNs can solve some problems, such as XOR problem and the detection of symmetry problem (see, [8, 23]), which cannot be solved with their real-valued counterparts. On the other hand, potential action may have different impulse patterns in the human brain, and the distance between impulses might be different. This suggests that introducing complex numbers representing phase and amplitude into ANNs to simulate and study the behaviors in human brains is reasonable. Therefore, it is necessary and significant to study the dynamical behaviors of CVNNs.

As we know, in conventional real-valued neural networks, the output of each neuron is represented only by its average firing rate. However, the relative phase between rhythmic signals might affect the resulting communication. Motivated by this speculation, synchronization was introduced into complex-valued ANNs and was used for segmenting images into separate objects (see, [24, 25]). On the other hand, the complex-valued recurrent neural network (CVRNN) was introduced for modeling the binding problem of artificial neuronal assemblies by adjusting the relative phase of the oscillations of different colors and shapes of the images (see, [26]). Both these works suggest that it is significant and meaningful to investigate the synchronization phenomenon of CVNNs.

It should be noted that most of the existing research concerning the dynamics of CVNNs have been carried out for stability and passivity analysis (see, e.g., [27, 28, 29, 30, 31, 32, 33, 34]). In [28], the global exponential stability of CVNNs with both time-varying delays and impulsive effects was discussed. In [31], the stability of switched delayed CVNNs with uncertainties was addressed. In [32], the global exponential stability of delayed CVNNs with discontinuous activation functions was studied. In [34], the passivity of memristor-based complex-valued recurrent neural networks with interval time-varying delays was investigated. However, only few published papers consider the synchronization problem of CVNNs.

On the other hand, many physical systems can be more effectively modeled by complex-variable chaotic systems. For example, rotating fluids and detuned laser can be described and simulated by complex-variable Lorenz system [35]. In [36], the complex-variable Chen and L$\ddot{u}$ systems are introduced. Since not only the complex-variable chaotic systems have broad applications in secure communications, but also complex variables show extraordinarily favorable in increasing the contents and security of the transmitted information, synchronization and control of dynamical networks coupled with complex-variable chaotic systems have also been studied and some valuable results have been obtained (see, e.g., [37, 38, 39, 40]).

The above discussions suggest that synchronization of complex-valued dynamical networks is a significant and open research topic, and could make potential contributions in the future.

Compared with real-valued networks, there are currently still many difficulties in investigating the synchronization of complex-valued dynamical networks. For example, in RVNNs, the activation functions are usually chosen to be smooth, bounded, and nonconstant. However, according to Liouville Theorem, every bounded entire function in the complex domain must be constant, which means the activation functions in CVNNs cannot be both bounded and analytic. Thus, how to choose activation functions in CVNNs is a challenging problem. Furthermore, complex-valued networks have higher dimensional dynamics and more complicated interaction manners (e.g., the coupling configuration matrix can be complex). Therefore, some properties and analytical techniques for studying real-valued networks cannot be directly used for complex-valued networks.

Due to the limited speed of signal processing and information transmission, time delays are ubiquitous not only within the network nodes but also in the coupling, which may result in chaos, oscillation, and instability of networks. Thus, it is necessary and reasonable to simultaneously consider the nodes' internal delay and the transmission delay for describing more realistic dynamical networks. The main focus of this thesis is to study the synchronization of complex-valued dynamical networks with time delays by using multiple types of control schemes.

1.2 Thesis Outline

The organization of this thesis is summarized as follows. In Chapter 1, the introduction of this thesis is given. Chapter 2 introduces the complex-valued time-delay dynamical network model, the synchronization problems related to complex-valued dynamical networks, and the mathematical background information of complex-valued impulsive system with time-delay. In Chapter 3, we investigate synchronization of complex-valued dynamical networks by using different control schemes, including impulsive control, pinning control, and event-triggered control. Based on the Lyapunov function/functional method, Razumikhin technique, and the linear matrix inequality (LMI) approach, we introduce several network synchronization results which are effective in the complex domain. In Chapter 4, by using the concepts of average impulsive interval (AII) and average impulsive gain (AIG), and the comparison principle, we introduce several outer synchronization results for complex-valued dynamical networks via hybrid control. In Chapter 5, we study the average-consensus problem of potential complex-valued multi-agent systems via hybrid protocols with time-delay. In Chapter 6, we study synchronization of master-slave time-delay CVNNs by using hybrid control and delayed event-triggered pinning impulsive control strategies, respectively. We also study the state estimation problem of CVNNs with time-delay by designing the adaptive impulsive observer in the complex field.

1.3 Contributions of the Thesis

The main contributions of this thesis are summarized as follows:

1. This thesis studies synchronization of CVDNs by using multiple types of control strategies. Several new synchronization criteria of CVDNs are established, which generalizes some existing network synchronization results.
2. In most of the existing works, the usual method for studying synchronization and control of CVDNs is to separate them into two real-valued dynamical networks (RVDNs) and apply the common analytical technique and control theory for both RVDNs to analyze dynamical behaviors of CVDNs. While in this thesis, we retain the complex nature of dynamical networks and investigate the synchronization problem by taking advantage of Lyapunov function/Lyapunov functional in the complex domain, which reduces the complexity of analysis and computation.
3. Time-delay is considered in CVDNs throughout this thesis. All the synchronization results presented in this thesis are applicable to CVDNs with various sizes of delays.
4. Most of the existing impulsive synchronization results for dynamical networks are based

on a time-triggered scheme, i.e., the impulsive instants are predesigned, and the impulsive controller will keep working even if the networked system is running smoothly, while the event-triggered impulsive control (ETIC) scheme is seldom considered for studying the network synchronization problem. In this thesis, we propose a novel type of memory-based ETIC scheme with three levels of events in the complex field to study the synchronization problem of time-delay CVDNs. By considering the advantages of pinning control, we further propose an event-triggered pinning impulsive control (ETPIC) scheme combining the ETIC scheme and a pinning algorithm to study synchronization of time-delay CVDNs. Results show that the proposed ETIC scheme and ETPIC scheme can effectively synchronize time-delay CVDNs with the desired trajectory.
5. To our best knowledge, the generalized outer synchronization problem of CVDNs is firstly studied in this thesis. A hybrid controller is proposed in the complex domain to construct the corresponding response complex-valued networks of the drive time-delay CVDN. In this thesis, some generalized outer synchronization criteria for drive-response CVDNs are established, which extend the existing generalized outer synchronization results for real-valued networks to the complex field.
6. As a practical application for complex-valued networked systems, the synchronization problem of master-slave CVNNs is studied in this thesis by using hybrid control and delayed ETPIC, respectively. In this thesis, we also investigate the state estimation problem of CVNNs by designing the adaptive impulsive observer and the updating law in the complex field.

1.4 Notations

In this section, we will introduce some notations that are commonly used in this thesis.

Throughout this thesis, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{N}^+$ represent the set of real numbers, complex numbers, nonnegative integers, and positive integers, respectively. Let $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the set of n-dimensional real vector space and complex vector space, respectively. $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ stand for the set of $n \times n$ real-valued matrices and complex-valued matrices, respectively. The notation $*$ denotes the conjugate transpose of a complex-valued vector or a matrix. For any complex number z, the notations z^R and z^I denote its real and imaginary parts respectively, $|z|$ denotes its modulus, and $\bar{z}$ represents its complex conjugate. For any $u, v \in \mathbb{C}^n$, $\langle u, v\rangle = u^*v$ represents the inner product in the complex vector space $\mathbb{C}^n$. For any matrix $A \in \mathbb{C}^{n\times n}$, if $A^* = A$, then A is called as a Hermitian matrix, and $A > 0$ shows that A is positive definite; $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest and the smallest eigenvalue of A, respectively, and $\|A\| = \sqrt{\lambda_{\max}(A^*A)}$ is the induced norm of A. Superscript " $\star$ " in Hermitian matrix stands for the conjugate transpose of block. $\lfloor x \rfloor$

denotes the integer part of the real number x, and $\otimes$ refers to the Kronecker product. Let $I_n \in \mathbb{R}^{n\times n}$ denote the $n \times n$ identity matrix. $\delta(\cdot)$ represents the Dirac delta function satisfying

$$\delta(x) = \begin{cases} 0, & \text{if } \ x \neq 0, \\ \infty, & \text{if } \ x = 0, \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(x)dx = 1.$$

For $a, b \in \mathbb{R}$ with $a < b$, $\mathcal{PC}([a, b], \mathbb{C}^n)$ stands for the set of functions $\psi : [a, b] \to \mathbb{C}^n$ that are continuous everywhere except at a finite number of points t_k, at which $\psi(t^+)$ and $\psi(t^-)$ exist and $\psi(t) = \psi(t^+)$. For constant $\tau > 0$, and $\psi \in \mathcal{PC}([-\tau, 0], \mathbb{C}^n)$, we define $\|\psi\|_\tau = \sup_{s\in[-\tau,0]} \|\psi(s)\|$.

Chapter 2

Background

2.1 Complex-Valued Dynamical Network Model

We will introduce the complex-valued dynamical network (CVDN) model in this section. Consider the CVDN with time-delay consisting of N nodes. The dynamics of the states of nodes can be described as follows:

$$\dot{z}_i = f_i(t, z_i, z_{it}) + g_i(t, \mathbf{z}) + h_i(t, \mathbf{z}_t), \qquad i = 1, 2, \ldots, N, \tag{2.1}$$

where $z_i \in \mathbb{C}^n$ denotes the state of the i-th node; define z_{it} as $z_{it} = z_i(t+s)$, for $s \in [-\tau, 0]$, and τ stands for the time-delay; $\mathbf{z} = (z_1^T, z_2^T, \ldots, z_N^T)^T$, and $\mathbf{z}_t = (z_{1t}^T, z_{2t}^T, \ldots, z_{Nt}^T)^T$. The function $f_i : \mathbb{R}^+ \times \mathcal{PC}(\mathbb{R}^+, \mathbb{C}^n) \times \mathcal{PC}([-\tau, 0], \mathbb{C}^n) \to \mathbb{C}^n$ represents the intrinsic dynamics of node i; the non-delayed and delayed coupling functions $g_i : \mathbb{R}^+ \times \mathcal{PC}(\mathbb{R}^+, \mathbb{C}^{nN}) \to \mathbb{C}^n$ and $h_i : \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathbb{C}^{nN}) \to \mathbb{C}^n$ describe the interactions between the i-th node and other nodes.

Remark 2.1.1. *Many types of CVNN models can be written in the form of* (2.1)*, such as complex-valued BAM neural network [41], complex-valued Hopfield network [42], complex-valued recurrent neural network [43], complex-valued Cohen–Grossberg neural network [44].*

2.2 Synchronization

In reality, synchronization phenomena may occur either inside a network or between two or more networks, which are called "inner synchronization" and "outer synchronization", respectively. Inner synchronization means that all nodes within a single network are synchronized with one another, while outer synchronization describes how the nodes in one network and the corresponding ones in other networks can behave in a synchronous way. Following outer synchronization being firstly studied by Li et al. [45], various types of outer synchronization for dynamical networks have been proposed, including complete synchronization (CS) [46], phase synchronization [47], lag synchronization [48], cluster synchronization [49], anti-synchronization [50], projective synchronization (PS) [51], function projective synchronization (FPS) [52], linear generalized synchronization (LGS) [53], and generalized synchronization (GS) [54].

As we know, complex-variable dynamical systems and networks have broad applications in various fields [8, 35, 55]. Recently, inner and outer synchronization of complex-variable systems and networks have been studied and some valuable results have emerged [37, 38, 39, 40, 56, 57, 58, 59]. In [57], the complex projective synchronization of drive-response complex-variable dynamical networks with complex coupling was investigated by using the impulsive control method. In [58], the complete synchronization for master-slave time-varying delayed CVNNs was studied via hybrid impulsive control. In [59], the complex function projective synchronization for drive-response complex-variable dynamical networks with coupling time delay was investigated by designing proper hybrid feedback controllers.

Next, we will introduce the formal definition of inner synchronization and outer synchronization of CVDNs, respectively.

Definition 2.2.1. *CVDN (2.1) is said to achieve synchronization if*

$$\lim_{t\to\infty} \|z_i(t) - z_j(t)\| = 0, \qquad i, j = 1, 2, \dots, N. \tag{2.2}$$

In particular, if the objective is to synchronize CVDN (2.1) with the goal dynamics $s(t)$, where $s(t)$ could be an equilibrium point, a periodic orbit, or a chaotic trajectory, then we have the following definition.

Definition 2.2.2. *Suppose that $s(t) \in \mathbb{C}^n$ is any smooth goal dynamics. CVDN (2.1) is said to be asymptotically synchronized onto the target state $s(t)$ if*

$$\lim_{t\to\infty} \|z_i(t) - s(t)\| = 0, \qquad i = 1, 2, \dots, N. \tag{2.3}$$

Remark 2.2.1. *Note that the nodes in CVDN (2.1) are nonidentical. In particular, if all functions f_i in (2.1) are the same, i.e., $f_i \equiv f, i = 1, \dots, N$, then (2.1) becomes a CVDN with identical nodes, and if the synchronous state $s(t)$ is chosen as a solution of an isolated node, namely $\dot{s} = f(t, s, s_t)$, then the synchronization problem of CVDN (2.1) becomes the synchronization problem of a CVDN with identical nodes.*

Next, we will give some formal definitions of outer synchronization for CVDNs. Consider CVDN (2.1) as the drive network. The corresponding response CVDN is given by:

$$\dot{\hat{z}}_i = \tilde{f}_i(t, \hat{z}_i, \hat{z}_{it}) + \tilde{g}_i(t, \hat{\mathbf{z}}) + \tilde{h}_i(t, \hat{\mathbf{z}}_t), \qquad i = 1, 2, \ldots, N, \tag{2.4}$$

where $\hat{z}_i \in \mathbb{C}^n$ denotes the response state vector of the i-th node; $\hat{z}_{it} = \hat{z}_i(t+s)$, for $s \in [-\tau, 0]$, and τ is the same as that in network (2.1); $\hat{\mathbf{z}} = (\hat{z}_1^T, \hat{z}_2^T, \ldots, \hat{z}_N^T)^T$, and $\hat{\mathbf{z}}_t = (\hat{z}_{1t}^T, \hat{z}_{2t}^T, \ldots, \hat{z}_{Nt}^T)^T$. The function $\tilde{f}_i : \mathbb{R}^+ \times \mathcal{PC}(\mathbb{R}^+, \mathbb{C}^n) \times \mathcal{PC}([-\tau, 0], \mathbb{C}^n) \to \mathbb{C}^n$ describes the intrinsic dynamics of the i-th node; $\tilde{g}_i : \mathbb{R}^+ \times \mathcal{PC}(\mathbb{R}^+, \mathbb{C}^{nN}) \to \mathbb{C}^n$ and $\tilde{h}_i : \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathbb{C}^{nN}) \to \mathbb{C}^n$ are the non-delayed and delayed coupling functions, respectively.

Definition 2.2.3. *The response CVDN (2.4) is said to achieve generalized synchronization with the drive CVDN (2.1) if*

$$\lim_{t \to \infty} \|z_i(t) - \phi_i(\hat{z}_i(t))\| = 0, \qquad i = 1, 2, \ldots, N,$$

where $\phi_i : \mathbb{C}^n \to \mathbb{C}^n$ $(i = 1, 2, \ldots, N)$ is a continuously differentiable map with an inverse map ϕ_i^{-1}.

Remark 2.2.2. *Definition 2.2.3 reduces to*
(i) linear generalized synchronization if the map $\phi_i(\hat{z}_i(t)) = P\hat{z}_i(t) + Q$, where P, Q are complex matrices with proper dimension;
(ii) function projective synchronization if $\phi_i(\hat{z}_i(t)) = \alpha(t)\hat{z}_i(t)$, where $\alpha(t) = \alpha^R(t) + j\alpha^I(t)$ is a continuously differentiable complex-valued scaling function;
(iii) projective synchronization if $\phi_i(\hat{z}_i(t)) = \alpha\hat{z}_i(t)$, where $\alpha \in \mathbb{C}$;
(iv) complete synchronization if $\phi_i(\hat{z}_i(t)) = \hat{z}_i(t)$;
(vi) anti-synchronization if $\phi_i(\hat{z}_i(t)) = -\hat{z}_i(t)$;
(vii) lag synchronization if the map $\phi_i(\hat{z}_i(t))$ is modified to $\hat{z}_i(t - \sigma)$, where σ denotes the time lag.

2.3 Stabilization

Stability of solutions of dynamical systems is one of the crucial issues in systems analysis. It has received lots of research interest within the dynamic systems and control community.

In recent years, complex-valued differential systems have been widely studied due to their potential applications in many scientific and engineering scenarios (e.g., quantum system, CVNN). Stability of complex-valued differential systems has attracted increasing research attention, and some stability results have been reported (see, e.g., [60, 61, 62]).

Actually, when investigating synchronization of dynamical networks, we always transform the network models into the equivalent synchronization error system, then study the asymptotic stability of the trivial state of the error system. It has been shown that the Lyapunov's stability method can be extended to analyze the stability of complex-valued differential systems by employing a positive definite function $V(t, z)$ in the complex fields. Thus, we can similarly convert CVDNs into equivalent complex-valued error dynamical systems, then the synchronization problem of CVDNs becomes the stability problem of the error system.

However, due to the complexities of node dynamics and topological structures of complex-valued networks, all the nodes cannot achieve the goal themselves. Therefore, external controllers need to be added onto the nodes of CVDNs for realizing network synchronization. In Chapter 3 and Chapter 4, several synchronization results for CVDNs are established by studying the stability of the trivial state of the complex-valued error dynamical system under various control schemes.

2.4 Complex-Valued Impulsive System with Delay

Consider the complex-valued impulsive system with time-delay

$$\begin{cases} \dot{z} = f(t, z, z_t), \qquad t \geq t_0,\ t \neq t_k, \\ \Delta z(t_k) = I_k(t_k, z(t_k^-)), \qquad k \in \mathbb{N}^+, \\ z_{t_0}(s) = \phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.5}$$

where $z \in \mathbb{C}^n$ is the state vector, z_t is defined by $z_t = z(t+s)$, $s \in [-\tau, 0]$ and τ is the time delay in (2.5), $f : \mathbb{R}^+ \times \mathcal{PC}(\mathbb{R}^+, \mathbb{C}^n) \times \mathcal{PC}([-\tau, 0], \mathbb{C}^n) \to \mathbb{C}^n$ is a continuous complex-valued function, $I_k : \mathbb{R}^+ \times \mathbb{C}^n \to \mathbb{C}^n$ denotes the impulsive input at each impulsive instant $t_k, k \in \mathbb{N}^+$, $\Delta z(t_k) = z(t_k^+) - z(t_k^-)$, and the impulsive time sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$, and $\lim_{k\to\infty} t_k = \infty$. $\phi \in \mathcal{PC}([-\tau, 0], \mathbb{C}^n)$ is the initial function of the system. Here, we assume for any initial condition $\phi \in \mathcal{PC}([-\tau, 0], \mathbb{C}^n)$, system (2.5) has at least one solution $z(t, t_0, \psi)$, which is denoted by $z(t)$ for simplicity. One may refer to [63] for the result of existence and uniqueness of solutions to complex-valued impulsive differential systems. Without loss of generality, we assume solutions of (2.5) are right continuous at each t_k, i.e., $z(t_k) = z(t_k^+)$.

In Chapter 3, we will introduce several inner synchronization results for CVDNs via impulsive control. The Lyapunov function/ functional method and the Razumikhin Technique are utilized to investigate exponential stability of the trivial solution of the complex-valued impulsive synchronization error system. Here, we will firstly give the definition of global exponential stability of complex-valued impulsive system (2.5) and definitions related to the Lyapunov function in the complex field.

Definition 2.4.1. *The trivial solution of system (2.5) is said to be globally exponentially stable (GES) if there exists positive constants* λ *and* M *such that for any solution* $z(t, t_0, \psi)$ *with the initial condition* $\phi \in \mathcal{PC}([-\tau, 0], \mathbb{C}^n)$*,*

$$\|z(t, t_0, \psi)\| \leq M\|\phi\|_\tau e^{-\lambda(t-t_0)}, \quad t \geq t_0. \tag{2.6}$$

Definition 2.4.2. $W : \mathbb{C}^n \to \mathbb{R}$ *is called a complex positive definite function if*
(i) $W(z) \geq 0$ *for all* $z \in \mathbb{C}^n$*;*
(ii) $W(z) = 0$ *if and only if* $z = 0$*.*

Definition 2.4.3. *A function* $V : [t_0, \infty) \times \mathbb{C}^n \to \mathbb{R}^+$ *is said to belong to the class* v_0 *if*
(i) V *is continuous on each of the sets* $[t_{k-1}, t_k) \times \mathbb{C}^n$*, and for each* $z \in \mathbb{C}^n, t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$*,* $\lim_{(t,y)\to(t_k^-,z)} V(t, y) = V(t_k^-, z)$ *exists;*
(ii) $V(t, z)$ *is locally Lipschitzian in* z*, and* $V(t, 0) \equiv 0$ *for all* $t \geq t_0$*;*
(iii) there exists a complex positive definite function $W(z)$ *such that* $V(t, z) \geq W(z)$ *for all* $(t, z) \in [t_0, \infty) \times \mathbb{C}^n$*.*

Definition 2.4.4. *Let* $V \in v_0$*, for any* $(t, \psi) \in [t_{k-1}, t_k) \times \mathcal{PC}([-\tau, 0], \mathbb{C}^n)$*, the upper right-hand derivative of* V *along a solution of system (2.5) is defined by*

$$D^+V(t, \psi(0)) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t + h, \psi(0) + hf(t, \psi(0), \psi)) - V(t, \psi(0))].$$

Chapter 3

Synchronization of CVDNs

Due to the complexities of node dynamics and topological structure of networks, it is usually difficult for a network to achieve synchronization by itself. Up to now, a variety of control schemes have been adopted to actuate network nodes to reach a common agreement, such as impulsive control [64], sampled-data control [65], pinning control [66], event-triggered control [67], adaptive control [68], intermittent control [69], sliding mode control [70], and so on. In this chapter, we study the synchronization problem of CVDNs by using multiple control strategies. In Section 3.2, we design a delay-dependent pinning impulsive controller to study synchronization of time-delay CVDNs. In Section 3.3, we investigate synchronization of CVDNs with time-varying coupling delay under distributed impulsive control. In Section 3.5, we propose a novel type of memory-based event-triggered impulsive control scheme with three levels of events to investigate synchronization of CVDNs with distributed delay. Then, in Section 3.6, we further consider an event-triggered pinning impulsive control scheme combining the proposed event-triggered impulsive control strategy in 3.5 and a pinning algorithm to synchronize CVDNs with distributed delay.

3.1 Impulsive Control Method

Consider a plant be the general complex-valued system with time-delay

$$\begin{cases} \dot{z} = f(t, z, z_t), \\ z_{t_0} = \phi, \end{cases} \tag{3.1}$$

where $z \in \mathbb{C}^n$ denotes the state vector, z_t is defined by $z_t(s) = z(t+s)$ for $s \in [-\tau, 0]$, and τ is the time-delay of system (3.1), $f : \mathbb{R}^+ \times \mathcal{PC}(\mathbb{R}^+, \mathbb{C}^n) \times \mathcal{PC}([-\tau, 0], \mathbb{C}^n) \to \mathbb{C}^n$ is the complex-valued nonlinear function, and $\phi \in \mathcal{PC}([-\tau, 0], \mathbb{C}^n)$ is the initial function.

For a given plant (3.1), the sequence $\{t_k, U_k(t_k, z(t_k))\}$ is called an impulsive control law of system (3.1), if there exits a set of control instants $\{t_k\}$, and control laws $U_k(t_k, z(t_k)) \in \mathbb{C}^n$ such that the solution of the following impulsive system

$$\begin{cases} \dot{z} = f(t, z, z_t), & t \neq t_k, \\ \Delta z(t_k) = U_k(t_k, z(t_k)), & k \in \mathbb{N}^+, \\ z_{t_0} = \phi \end{cases} \tag{3.2}$$

approaches a desired trajectory $z^*(t)$ as $k \to \infty$, where $U_k : \mathbb{R}^+ \times \mathbb{C}^n \to \mathbb{C}^n$ is the control input at each impulsive instant t_k, $k = 1, 2, \ldots$, $\Delta z(t_k) = z(t_k^+) - z(t_k^-)$, and the impulsive instants t_k satisfy $0 \leq t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$, $\lim_{k\to\infty} t_k = \infty$.

Remark 3.1.1. *When an impulsive control law is applied onto system (3.1), the system state will change abruptly at each impulsive instant t_k resulting in the state jump. The term $\Delta z(t_k)$ in system (3.2) represents the state jump at a certain impulsive instant t_k in the impulsive control process. $z(t_k^-)$ refers to the state before the jump, when the control input is added onto system (3.1) instantaneously at impulsive instant t_k, $z(t_k^+)$ represents the state after the jump, where $z(t_k^+) = z_(t_k^-) + U_k(t_k^-, z(t_k^-))$.*

According to the impulsive control law, one can add an impulsive controller $u(t, z)$ with the form of

$$u(t, z) = \sum_{k=1}^{\infty} U_k(t, z(t))\delta(t - t_k) \tag{3.3}$$

onto system (3.1) such that the solution of the following controlled system

$$\begin{cases} \dot{z} = f(t, z, z_t) + u(t, z), \\ z_{t_0} = \phi \end{cases} \tag{3.4}$$

reaches the target $z^*(t)$ as $t \to \infty$, where $\delta(\cdot)$ in (3.3) denotes the Dirac delta function.

Remark 3.1.2. *Note that under impulsive controller (3.3), system (3.4) can be converted into the equivalent impulsive system (3.2) based on the properties of Dirac delta function.*

Impulsive control is a kind of discontinuous control strategy. The main feature of impulsive control method is that the control action is exerted on the state of the system only at certain discrete moments. Over the past few decades, impulsive controllers have been designed to add into network nodes to achieve network synchronization (see, e.g., [64, 71, 72, 73, 74]). Compared with continuous control schemes, impulsive control is an efficient and economic control method for studying synchronization of CVDNs because it dramatically reduces the amount of information transmission and the bandwidth usage, and control cost is much lower. Impulsive control can also increase the robustness against the disturbance. In the recent years, many synchronization results for CVDNs have emerged by using impulsive control scheme (see, e.g., [39, 58, 75]).

3.2 Synchronization of Time-Delay CVDNs via Delay-Dependent Pinning Impulsive Control

This section studies synchronization of time-delay CVDNs via delay-dependent pinning impulsive control. In Subsection 3.3.2, we formulate the synchronization problem of time-delay CVDNs, a pinning impulsive controller is constructed to take time-delay effects into account, that is, the impulsive controllers acted on pinned nodes whose states at impulsive instants depend on both current and historical network states, and some preliminaries are introduced. In Subsection 3.2.4, by taking advantage of Lyapunov function in the complex field, some delay-independent synchronization criteria of time-delay CVDNs are established, which extends the network synchronization criteria to the complex domain. In Subsection 3.2.5, several delay-dependent synchronization criteria for CVDNs with various sizes of delays are introduced. Numerical examples are provided in Subsection 3.2.6 to illustrate the effectiveness of the theoretical results.

3.2.1 Delayed Impulsive Control

In the impulsive control process of the networked control systems, due to the finite speed of computation and information transmission, the sampling, processing, and transferring of impulsive information are difficult to instantaneously complete at discrete moments, which leads to the sampling delays at impulsive instants. Therefore, it is crucial to consider the time-delay effects in the impulsive controllers when impulsive control scheme is utilized to synchronize dynamical networks. The notion of delayed impulsive control was introduced to illustrate the cases that the states 'jump' at impulsive instants depend on not only the current states but also the historical states of systems. In the most recent years, an increasing interest has been devoted to study control problems of impulsive dynamical systems in which the impulses involve time-delay.

3.2.2 Pinning Impulsive Control

Networks are usually composed of a large number of high-dimensional nodes, controlling all nodes in a network is hard to implement and high-cost. To tackle this issue, the pinning control scheme was introduced in [76] by only controlling a small portion of nodes in a network, which can effectively reduce the cost of control. Recently, many pinning algorithms have been adopted to design proper controllers for achieving network synchronization (see, e.g., [77, 78, 79, 80, 81, 82]).

Since pinning control and impulsive control are both easier to implement and low-cost, the idea of combining pinning control and impulsive control (i.e., adding the impulsive controllers to a small fraction of network nodes) was proposed to study synchronization of dynamical networks. Compared with impulsive control or pinning control, pinning impulsive control can further reduce the cost of control and increase the robustness of synchronization of networks. In recent years, many interesting results on synchronization of dynamical networks via pinning impulsive control have been reported (see, e.g., [38, 83, 79, 84]). However, almost all the existing synchronization results via pinning impulsive control are derived for real-valued networks. To the best of our knowledge, no work has been considered for the synchronization problem of CVDNs by employing the pinning impulsive control method in which the impulses involve time-delay. In this section, we will establish several synchronization results for CVDNs by using a delay-dependent pinning impulsive control scheme.

3.2.3 Problem Formulation

Consider a time-delay CVDN consisting of N identical nodes as follows:

$$\dot{z}_i(t) = f(z_i(t), z_i(t-\tau_1)) + \sum_{j=1, j\neq i}^{N} c_{ij} A\big[h(z_j(t-\tau_2)) - h(z_i(t-\tau_2))\big], \tag{3.5}$$

for $i = 1, 2, \ldots, N$, where $z_i = (z_{i1}, z_{i2}, ...z_{in})^T \in \mathbb{C}^n$ is the state vector of the i-th node, $f : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is a complex-valued vector function representing the intrinsic dynamics of the corresponding network nodes, τ_1 denotes the node internal delay, $h : \mathbb{C}^n \to \mathbb{C}^n$ is the complex-valued nonlinear delayed coupling function, τ_2 represents the transmittal delay, $A \in \mathbb{R}^{n\times n}$ is the inner coupling matrix, $C = (c_{ij}) \in \mathbb{R}^{N\times N}$ is the outer coupling configuration matrix representing the coupling strength and the topological structure of the network, where c_{ij} is defined as follows: if there exists a directed link from node i to node j $(i \neq j)$, then $c_{ij} > 0$; otherwise, $c_{ij} = 0$.

Remark 3.2.1. *Note that the outer coupling configuration matrix C can be asymmetric, i.e., the corresponding graphs generated by matrix C can be directed. Furthermore, for most of the existing contributions about network synchronization, the diagonal entries of the outer coupling configuration matrix are assumed to satisfy the diffusively coupled condition: $c_{ii} = -\sum_{j=1,j\neq i}^{N} c_{ij}$, for $i = 1, 2, \ldots, N$, while the nodes in CVDN (3.5) may be non-diffusively coupled, that is, the outer coupling configuration matrix C may not be zero-row-sum.*

The objective is to synchronize CVDN (3.5) with the desired orbit $s(t)$, i.e.,

$$\lim_{t\to\infty} \|z_i(t) - s(t)\| = 0, \quad i = 1, 2, \ldots, N,$$

where $s(t) \in \mathbb{C}^n$ is a solution of an isolated node, namely, $\dot{s}(t) = f(s(t), s(t-\tau_1))$. The initial condition of the isolated node is given by $s(t_0 + \theta) = \varphi(\theta)$, $\theta \in [-\tau_1, 0]$.

To ensure CVDN (3.5) achieves synchronization, proper delay-dependent pinning impulsive controllers $u_i(t, z_i, s)$ are applied onto the nodes in CVDN (3.5), and the controlled network can be described by

$$\dot{z}_i(t) = f(z_i(t), z_i(t-\tau_1)) + \sum_{j=1,j\neq i}^{N} c_{ij} A\big[h(z_j(t-\tau_2)) - h(z_i(t-\tau_2))\big] + u_i(t, z_i, s), \quad (3.6)$$

for $i = 1, 2, \ldots, N$, where the delay-dependent pinning impulsive controllers $u_i(t) := u_i(t, z_i, s)$ are designed as follows

$$u_i(t) = \begin{cases} \sum_{k=1}^{\infty} [q_{1k}\big(z_i(t) - s(t)\big) + q_{2k}\big(z_i(t-d) - s(t-d)\big)]\delta(t-t_k), & i \in \mathcal{D}_k^l, \\ 0, \quad i \notin \mathcal{D}_k^l, \end{cases} \quad (3.7)$$

for $i = 1, 2, \ldots, N$, where q_{1k} and $q_{2k} \in \mathbb{R}$ are impulsive control gains to be determined at each impulsive instant t_k; the impulsive time sequence $\{t_k\}$ satisfies $t_0 < t_1 < t_2 < \ldots < t_k < \ldots$, and $\lim_{k\to\infty} t_k = \infty$; d denotes the impulse time-delay in controller u_i at t_k; $\delta(\cdot)$ is the Dirac delta function. Let $e_i(t) = z_i(t) - s(t)$ be the synchronization error state of node i ($i = 1, 2, \ldots, N$) at time t, and l denotes the number of nodes to be controlled at each impulsive instant t_k, then the index set $\mathcal{D}_k^l$ is defined as follows: at each impulsive instant $t_k, k \in \mathbb{N}^+$, one can reorder the synchronization error states $e_1(t_k), e_2(t_k), \ldots, e_N(t_k)$ such that $\|e_{p_1}(t_k)\| \geq \|e_{p_2}(t_k)\| \geq \ldots \geq \|e_{p_l}(t_k)\| \geq \left\|e_{p_{l+1}}(t_k)\right\| \geq \ldots \geq \|e_{p_N}(t_k)\|$. Particularly, if $\|e_{p_l}(t_k)\| = \left\|e_{p_{l+1}}(t_k)\right\|$, then let $p_l < p_{l+1}$. $\mathcal{D}_k^l = \{p_1, p_2, \ldots, p_l\}$ denotes the set of pinned nodes at impulsive instant t_k, and the number of nodes in set $\mathcal{D}_k^l$ is l.

Remark 3.2.2. *The pinning impulsive algorithm can be described as follows, nodes are selected to be impulsively controlled depending on the norm of the synchronization errors at distinct control instants. At each impulsive instant t_k, we only control the first l ($1 \leq l < N$) network nodes that have larger norm of error states. Since the error states are time-varying, the pinned nodes may not be the same at different impulsive instants.*

Under the delayed pinning impulsive controller (3.7), the controlled network (3.6) can be rewritten into the following impulsive system:

$$\begin{cases} \dot{z}_i(t) = f(z_i(t), z_i(t-\tau_1)) + \sum_{j=1, j\neq i}^{N} c_{ij} A\big[h(z_j(t-\tau_2)) - h(z_i(t-\tau_2))\big], \quad t \in [t_{k-1}, t_k), \\ \Delta z_i(t_k) = q_{1k}(z_i(t_k^-) - s(t_k)) + q_{2k}(z_i(t_k - d) - s(t_k - d)), \qquad i \in \mathcal{D}_k^l, \quad k \in \mathbb{N}^+, \\ \Delta z_i(t_k) = 0, \qquad i \notin \mathcal{D}_k^l, \end{cases} \tag{3.8}$$

for $i = 1, 2, \ldots, N$, where $\Delta z_i(t_k) = z_i(t_k^+) - z_i(t_k^-)$. Without loss of generality, we assume $z_i(t_k) = z_i(t_k^+)$ in the following discussion, which implies that the solutions of system (3.8) are right continuous. The initial condition of system (3.8) is denoted as $z_i(t_0 + \theta) = \phi_i(\theta)$ for $i = 1, 2, \ldots, N$, where $\phi_i \in \mathcal{PC}([-\tilde{\tau}, 0], \mathbb{C}^n)$, and $\tilde{\tau} = \max\{\tau_1, \tau_2, d\}$.
Since the synchronization errors are defined as $e_i(t) = z_i(t) - s(t) \in \mathbb{C}^n$, $i = 1, 2, ..., N$, then the complex-valued error dynamical system can be described as follows

$$\begin{cases} \dot{e}_i(t) = \tilde{f}(e_i(t), e_i(t-\tau_1)) + \sum_{j=1}^{N} c_{ij} A\tilde{h}(e_j(t-\tau_2)) - \sum_{j=1}^{N} c_{ij} A\tilde{h}(e_i(t-\tau_2)), \quad t \in [t_{k-1}, t_k), \\ \Delta e_i(t_k) = q_{1k} e_i(t_k^-) + q_{2k} e_i(t_k - d), \qquad i \in \mathcal{D}_k^l, \quad k \in \mathbb{N}^+, \\ e_i(t_k) = e_i(t_k^-), \qquad i \notin \mathcal{D}_k^l, \end{cases} \tag{3.9}$$

for $i = 1, 2, \ldots, N$, where $\tilde{f}(e_i(t), e_i(t-\tau_1)) = f(z_i(t), z_i(t-\tau_1)) - f(s(t), s(t-\tau_1))$, $\tilde{h}(e_i(t-\tau_2)) = h(z_i(t-\tau_2)) - h(s(t-\tau_2))$. The initial function of error system (3.9) is defined by $e_i(t_0+\theta) = \phi_i(\theta) - \varphi(\theta)$ for $\theta \in [-\tau_1, 0]$, and $e_i(t_0+\theta) = \phi_i(\theta)$ for $\theta \in [-\tilde{\tau}, -\tau_1]$, $i = 1, 2, \ldots, N$.

Then the inner synchronization problem of CVDN (3.5) is transformed into the stability problem of the zero solution of the error system (3.9).

We shall need the following assumptions and lemmas for the synchronization results.

Assumption 3.2.1. *Suppose that there exist positive constants L_1, L_2 such that*

$$\|f(u_1, v_1) - f(u_2, v_2)\| \leq L_1\|u_1 - u_2\| + L_2\|v_1 - v_2\|$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{C}^n$.

Assumption 3.2.2. *Suppose there exists positive constant* ρ *such that*

$$\|h(u) - h(v)\| \le \rho \|u - v\|$$

for all $u, v \in \mathbb{C}^n$.

Lemma 3.2.1. *For any vector* $X, Y \in \mathbb{C}^n$, *and positive constant* η, *if* H *is a positive definite Hermitian matrix, then*

$$X^*Y + Y^*X \le \eta X^*HX + 1/\eta Y^*H^{-1}Y.$$

Lemma 3.2.2. *[85] Consider the following impulsive differential inequality:*

$$\begin{cases} D^+u(t) \le au(t) + b_1[u(t)]_{\tau_1} + b_2[u(t)]_{\tau_2} + ... + b_h[u(t)]_{\tau_h}, & t \ne t_k,\ t \ge t_0, \\ u(t_k^+) \le p_k u(t_k^-) + q_k^1[u(t_k^-)]_{\sigma_1} + q_k^2[u(t_k^-)]_{\sigma_2} + ... + q_k^r[u(t_k^-)]_{\sigma_r}, & k \in \mathbb{N}^+, \\ u(s) = \phi(s), & s \in [t_0 - \tau, t_0], \end{cases}$$

where $a, b_i, p_k, q_k^j, \tau_i, \sigma_j$ *are constants and* $b_i \ge 0, p_k \ge 0, q_k^j \ge 0, \tau_i \ge 0, \sigma_j \ge 0$, $i = 1, 2, ..., h, j = 1, 2, ..., r$, $u \in \mathcal{PC}([t_0, \infty], \mathbb{R}^+)$, ϕ *is continuous on* $[t_0 - \tau, t_0]$, $[u(t)]_{\tau_i} = \sup_{t-\tau_i \le s \le t} u(s)$, *and* $[u(t_k^-)]_{\sigma_j} = \sup_{t_k - \sigma_j \le s < t_k} u(s)$, $k \in \mathbb{N}^+$. *Suppose that*

$$p_k + \sum_{j=1}^{r} q_k^j < 1, \tag{3.10}$$

$$a + \frac{\sum_{i=1}^{h} b_i}{p_k + \sum_{j=1}^{r} q_k^j} + \frac{\ln(p_k + \sum_{j=1}^{r} q_k^j)}{t_{k+1} - t_k} < 0. \tag{3.11}$$

Then there exists constants $M > 1$ *and* $\lambda > 0$ *such that*

$$u(t) \le M\|\phi(t_0)\|_\tau e^{-\lambda(t-t_0)}, \quad t \ge t_0,$$

where $\|\phi(t_0)\|_\tau = \sup_{t_0-\tau \le s \le t_0} \|\phi(s)\|$, $\tau = \max\{\tau_i, \sigma_j, i = 1, 2, ..., h, j = 1, 2, ..., r\}$.

Remark 3.2.3. *The result of Lemma 3.2.2 actually provides synchronization and stability criterion for impulsive differential systems with multiple time delays. The coefficient* a *appeared in the first term of the impulsive differential inequality is not required to be negative, it could be a positive constant. Note that* (3.11) *implies* (3.10) *holds if* $a > 0$.

Lemma 3.2.3. *For any vector $X, Y, Z \in \mathbb{C}^n$, the following inequality holds*

$$(X+Y+Z)^*(X+Y+Z) \leq (1+\varepsilon)X^*X + (1+\frac{1}{\varepsilon})(1+\xi)Y^*Y + (1+\frac{1}{\varepsilon})(1+\frac{1}{\xi})Z^*Z$$

for any $\varepsilon, \xi > 0$.

Proof. The result can be directly obtained by applying Lemma 3.2.1 twice with $H = I$. □

Lemma 3.2.4. *For $\varepsilon, \xi > 0$, and given constants $a, b, c \in \mathbb{R}$, define function*

$$H(\varepsilon, \xi) = (1+\varepsilon)a^2 + (1+\frac{1}{\varepsilon})(1+\xi)b^2 + (1+\frac{1}{\varepsilon})(1+\frac{1}{\xi})c^2,$$

then the function H attains its minimum $H_{\min} = (|a|+|b|+|c|)^2$ at $(\varepsilon, \xi) = (\frac{|b|+|c|}{|a|}, \frac{|c|}{|b|})$.

Proof. The result can be easily obtained by using the Extreme Value Theorem for multi-variable functions. □

3.2.4 Delay-Independent Synchronization Criteria

In this subsection, some delay-independent synchronization criteria for the pinning impulsively controlled time-delay CVDN (3.8) will be established by using the Lyapunov-Krasovskii function method and Lyapunov-Razumikhin techniques.

Theorem 3.2.1. *Suppose that Assumption 3.2.1 and Assumption 3.2.2 hold. The pinning impulsively controlled time-delay CVDN (3.8) can achieve synchronization if $\sum_{j=1}^{N} c_{ij} \leq 0$, for $i = 1, 2, ..., N$, and*

$$\alpha + \frac{L_2 + \beta}{a_k + b_k} + \frac{\ln(a_k + b_k)}{t_k - t_{k-1}} < 0, \qquad k \in \mathbb{N}^+, \tag{3.12}$$

where $a_k = 1 - \frac{l}{N}\left[1 - 2(1+q_{1k})^2\right], b_k = 2q_{2k}^2, \alpha = 2L_1 + L_2 + \rho\,\|C \otimes A\| - c\rho\|A\|, \beta = \rho\,\|C \otimes A\| - c\rho\|A\|$, and $c = \min_{1\leq i\leq N}\{\sum_{j=1}^{N} c_{ij}\}$.

Proof. Consider the following Lyapunov function candidate for the complex-valued error system (3.9)

$$V(t) = \sum_{i=1}^{N} e_i^*(t)e_i(t).$$

For $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$, differentiating V along the solution of the system (3.9), we have

$$\begin{aligned}
\dot{V} &= 2\sum_{i=1}^{N} \mathfrak{Re}\big[e_i^*(t)\dot{e}_i(t)\big] \\
&= 2\sum_{i=1}^{N} \mathfrak{Re}\big[e_i^*(t)\tilde{f}(e_i(t), e_i(t-\tau_1))\big] + 2\mathfrak{Re}\big[\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^*(t)c_{ij}A\tilde{h}(e_j(t-\tau_2))\big] \\
&- 2\mathfrak{Re}\big[\sum_{i=1}^{N}\big(\sum_{j=1}^{N} c_{ij}\big)e_i^*(t)A\tilde{h}(e_i(t-\tau_2))\big],
\end{aligned} \tag{3.13}$$

it follows from Cauchy-Schwartz inequality and Assumption 3.2.2 that

$$\begin{aligned}
&\mathfrak{Re}\big[e_i^*(t)A\tilde{h}(e_i(t-\tau_2))\big] \le \Big|\Big\langle e_i(t), A\tilde{h}(e_i(t-\tau_2))\Big\rangle\Big| \\
&\le \|e_i(t)\|\|A\|\|h(z_i(t-\tau_2)) - h(s(t-\tau_2))\| \le \rho\|A\|\|e_i(t)\|\|e_i(t-\tau_2)\|.
\end{aligned}$$

If $\sum_{j=1}^{N} c_{ij} \le 0$, for $i = 1, 2, .., N$, and denote $c = \min_{1\le i\le N}\{\sum_{j=1}^{N} c_{ij}\}$, then one can derive

$$\sum_{i=1}^{N}\big(\sum_{j=1}^{N} c_{ij}\big)\mathfrak{Re}\big[e_i^*(t)A\tilde{h}(e_i(t-\tau_2))\big] \ge \rho c\|A\|\sum_{i=1}^{N}\|e_i(t)\|\|e_i(t-\tau_2)\|,$$

hence, we can get

$$-2\mathfrak{Re}\big[\sum_{i=1}^{N}\big(\sum_{j=1}^{N} c_{ij}\big)e_i^*(t)A\tilde{h}(e_i(t-\tau_2))\big] \le -2\rho c\|A\|\sum_{i=1}^{N}\|e_i(t)\|\|e_i(t-\tau_2)\|. \tag{3.14}$$

Denote $E(t) = (e_1^T(t), e_2^T(t), \dots, e_N^T(t))^T$, and $\tilde{H}(E(t-\tau_2)) = (\tilde{h}^T(e_1(t-\tau_2)), \tilde{h}^T(e_2(t-\tau_2)), \dots, \tilde{h}^T(e_N(t-\tau_2)))^T$. It follows from (3.13), (3.14), Cauchy-Schwartz inequality, Assumption 3.2.1 and Assumption 3.2.2 that

$$\begin{aligned}
\dot{V} &\le 2\sum_{i=1}^{N}\Big|\Big\langle e_i(t), \tilde{f}(e_i(t), e_i(t-\tau_1))\Big\rangle\Big| + 2\mathfrak{Re}[E^*(t)\big(C\otimes A\big)\tilde{H}(E(t-\tau_2))] \\
&- 2\rho c\|A\|\sum_{i=1}^{N}\|e_i(t)\|\|e_i(t-\tau_2)\|
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\sum_{i=1}^{N}\|e_i(t)\|\|f(z_i(t),z_i(t-\tau_1))-f(s(t),s(t-\tau_1))\|+2\left|\Big\langle E(t),\big(C\otimes A\big)\tilde{H}(E(t-\tau_2))\Big\rangle\right| \\
&-2\rho c\|A\|\sum_{i=1}^{N}\|e_i(t)\|\|e_i(t-\tau_2)\| \\
&\leq 2\sum_{i=1}^{N}\big(L_1\|e_i(t)\|+L_2\|e_i(t-\tau_1)\|\big)\|e_i(t)\|+2\|E(t)\|\|C\otimes A\|\|\tilde{H}(E(t-\tau_2))\| \\
&-\rho c\|A\|\sum_{i=1}^{N}(\|e_i(t)\|^2+\|e_i(t-\tau_2)\|^2) \\
&\leq 2L_1\sum_{i=1}^{N}\|e_i(t)\|^2+L_2\sum_{i=1}^{N}(\|e_i(t)\|^2+\|e_i(t-\tau_1)\|^2)+(\rho\|C\otimes A\|-\rho c\|A\|)\sum_{i=1}^{N}(\|e_i(t)\|^2 \\
&+\|e_i(t-\tau_2)\|^2) \\
&\leq (2L_1+L_2+\rho\|C\otimes A\|-\rho c\|A\|)V(t)+L_2V(t-\tau_1)+(\rho\|C\otimes A\|-\rho c\|A\|)V(t-\tau_2).
\end{aligned}
$$

Let $\alpha=2L_1+L_2+\rho\|C\otimes A\|-\rho c\|A\|$, and $\beta=\rho\|C\otimes A\|-\rho c\|A\|$, then we have

$$
\dot{V}\leq \alpha V(t)+L_2[V(t)]_{\tau_1}+\beta[V(t)]_{\tau_2},\qquad t\in[t_{k-1},t_k),\ k\in\mathbb{N}^+. \tag{3.15}
$$

On the other hand, when $t=t_k$, $k\in\mathbb{N}^+$, it follows from (3.9) that $e_i(t_k)=(1+q_{1k})e_i(t_k^-)+q_{2k}e_i(t_k-d)$ for $i\in\mathcal{D}_k^l$, and $e_i(t_k)=e_i(t_k^-)$ for $i\notin\mathcal{D}_k^l$. Applying Lemma 3.2.1, we have

$$
\begin{aligned}
V(t_k)&=\sum_{i\in\mathcal{D}_k^l}e_i^*(t_k)e_i(t_k)+\sum_{i\notin\mathcal{D}_k^l}e_i^*(t_k)e_i(t_k) \\
&\leq (1+q_{1k})^2\sum_{i\in\mathcal{D}_k^l}e_i^*(t_k^-)e_i(t_k^-)+q_{2k}^2\sum_{i\in\mathcal{D}_k^l}e_i^*(t_k-d)e_i(t_k-d)+\sum_{i\in\mathcal{D}_k^l}\big[(1+q_{1k})^2e_i^*(t_k^-)e_i(t_k^-) \\
&+q_{2k}^2e_i^*(t_k-d)e_i(t_k-d)\big]+\sum_{i\notin\mathcal{D}_k^l}e_i^*(t_k^-)e_i(t_k^-) \\
&\leq 2(1+q_{1k})^2\sum_{i\in\mathcal{D}_k^l}e_i^*(t_k^-)e_i(t_k^-)+\sum_{i\notin\mathcal{D}_k^l}e_i^*(t_k^-)e_i(t_k^-)+2q_{2k}^2V(t_k-d).
\end{aligned} \tag{3.16}
$$

Let $a_k=1-\frac{l}{N}\big[1-2(1+q_{1k})^2\big]$, and $b_k=2q_{2k}^2$. Choose $q_{1k}\in(-\frac{\sqrt{2}}{2}-1,\frac{\sqrt{2}}{2}-1)$, $k\in\mathbb{N}^+$, then based on the pinning algorithm, we can obtain

$$
\begin{aligned}
&(1-a_k)\sum_{i\notin\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-) \le (1-a_k)(N-\ell)\max_{i\notin\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-)\\
&\le (1-a_k)(N-l)\min_{i\in\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-) \le \frac{(1-a_k)(N-l)}{l}\sum_{i\in\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-)\\
&= [a_k - 2(1+q_{1k})^2]\sum_{i\in\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-),
\end{aligned}
$$

hence we have

$$
\sum_{i\notin\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-) + 2(1+q_{1k})^2\sum_{i\in\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-) \le a_k\sum_{i\in\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-) + a_k\sum_{i\notin\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-).
$$

Then, it follows from (3.16) that

$$
V(t_k) \le a_k V(t_k^-) + b_k V(t_k - d), \qquad k \in \mathbb{N}^+. \tag{3.17}
$$

According to (3.15), (3.17), condition (3.12) and Lemma 3.2.2, we can conclude that there exists constants $M > 1$ and $\lambda > 0$ such that

$$
V(t) \le M\|V(t_0)\|_{\tilde{\tau}} e^{-\lambda(t-t_0)}, \quad t \ge t_0.
$$

This implies that the synchronization error $\|e_i(t)\| \to 0$ as $t \to \infty$, for $i = 1, 2, \cdots, N$, and the pinning impulsively controlled time-delay CVDN (3.8) achieves synchronization. □

Remark 3.2.4. *Since α, β, L_2 are all positive constants, (3.15) allows $\dot{V} > 0$, which means that underlying continuous-time dynamics might destroy the stability of the error systems and desynchronize the complex-valued network. This shows that the impulse-free CVDN (3.5) may not achieve synchronization. Condition (3.12) of Theorem 3.2.1 implies $a_k + b_k < 1$, and impulses play control role on stabilizing the synchronization error system. Theorem 3.2.1 shows that the time-delay CVDN (3.8) can achieve synchronization if the pinning impulsive controllers (3.7) are suitably designed to satisfy (3.12).*

In order to obtain a less conservative delay-independent synchronization criterion for CVDNs, we will state the following theorem using Razumikhin technique.

Theorem 3.2.2. *Suppose that Assumption 3.2.1 and Assumption 3.2.2 hold. The pinning impulsively controlled time-delay CVDN (3.8) can achieve synchronization if $\sum_{j=1}^N c_{ij} \le 0$,* *for* $i = 1, 2, ..., N$*, and there exists a constant* $q > 1$ *such that*

$$\frac{1}{b_k} > q > \frac{1}{a_k + b_k} > e^{[\alpha + (L_2+\beta)q](t_k - t_{k-1})}, \qquad k \in \mathbb{N}^+, \tag{3.18}$$

where $a_k = 1 - \frac{l}{N}\left[1 - 2(1+q_{1k})^2\right], b_k = 2q_{2k}^2, \alpha = 2L_1 + L_2 + \rho\,\|C \otimes A\| - c\rho\|A\|, \beta = \rho\,\|C \otimes A\| - c\rho\|A\|$*, and* $c = \min_{1\le i\le N}\{\sum_{j=1}^{N} c_{ij}\}$.

Proof. Consider the Lyapunov function candidate $V(t) = \sum_{i=1}^{N} e_i^*(t)e_i(t)$. If (3.18) holds, then there exists small enough constant $\lambda > 0$ such that

$$q > \frac{e^{\lambda\tilde{\tau}}}{a_k + b_k e^{\lambda\tilde{\tau}}} > \frac{1}{a_k + b_k e^{\lambda\tilde{\tau}}} > e^{[\alpha+(L_2+\beta)q+\lambda](t_k - t_{k-1})}, \quad k \in \mathbb{N}^+, \quad \text{and} \quad qe^{-\lambda\tilde{\tau}} > 1, \tag{3.19}$$

where $\tilde{\tau} = \max\{\tau_1, \tau_2, d\}$. Denote $\bar{q} = qe^{-\lambda\tilde{\tau}} > 1$, and define

$$W(t) = V(t)e^{\lambda(t-t_0)}, \qquad t \ge t_0 - \tilde{\tau}.$$

For $t \in [t_0 - \tilde{\tau}, t_0]$, we have

$$W(t) \le \sup_{t\in[t_0-\tilde{\tau},t_0]} V(t)e^{\lambda(t_0-t_0)} = \|V(t_0)\|_{\tilde{\tau}}.$$

Choose $M > 0$ such that $M > \bar{q} > 1$, then we have

$$W(t) \le M\|V(t_0)\|_{\tilde{\tau}}, \qquad t \in [t_0 - \tilde{\tau}, t_0]. \tag{3.20}$$

Claim that

$$W(t) \le M\|V(t_0)\|_{\tilde{\tau}}, \quad t \ge t_0. \tag{3.21}$$

We will firstly prove that

$$W(t) \le M\|V(t_0)\|_{\tilde{\tau}}, \quad t \in (t_0, t_1). \tag{3.22}$$

If (3.22) is not true, then there exists $t \in (t_0, t_1)$ such that $W(t) > M\|V(t_0)\|_{\tilde{\tau}}$. Define $t^* = \inf\{t \in (t_0, t_1) : W(t) > M\|V(t_0)\|_{\tilde{\tau}}\}$. By the continuity of $W(t)$, we can obtain

$$W(t^*) = M\|V(t_0)\|_{\tilde{\tau}}, \qquad W(t) < M\|V(t_0)\|_{\tilde{\tau}}, \qquad t \in (t_0, t^*). \tag{3.23}$$

According to (3.20) and (3.23), we can get

$$W(t) \leq M\|V(t_0)\|_{\tilde{\tau}}, \qquad t \in [t_0 - \tilde{\tau}, t^*]. \tag{3.24}$$

Define $t^{**} = \sup\{t \in (t_0, t^*) : W(t) \leq \frac{M}{\bar{q}}\|V(t_0)\|_{\tilde{\tau}}\}$, by the continuity of $W(t)$, we can obtain

$$W(t^{**}) = \frac{M}{\bar{q}}\|V(t_0)\|_{\tilde{\tau}}, \qquad W(t) > \frac{M}{\bar{q}}\|V(t_0)\|_{\tilde{\tau}}, \quad t \in (t^{**}, t^*],$$

which implies that

$$M\|V(t_0)\|_{\tilde{\tau}} \leq \bar{q}W(t), \qquad t \in [t^{**}, t^*]. \tag{3.25}$$

From (3.24) and (3.25), we have

$$W(t+s) \leq M\|V(t_0)\|_{\tilde{\tau}} \leq \bar{q}W(t), \qquad t \in [t^{**}, t^*], \quad s \in [-\tilde{\tau}, 0].$$

Then, for $t \in [t^{**}, t^*]$, $s \in [-\tilde{\tau}, 0]$,

$$V(t+s) = W(t+s)e^{-\lambda(t+s-t_0)} \leq \bar{q}W(t)e^{-\lambda(t+s-t_0)} \leq \bar{q}e^{\lambda\tilde{\tau}}V(t),$$

hence, we can conclude that

$$V(t+s) \leq qV(t), \qquad t \in [t^{**}, t^*], \quad s \in [-\tilde{\tau}, 0]. \tag{3.26}$$

We have proved in Theorem 3.2.1 that

$$\dot{V} \leq \alpha V(t) + L_2 V(t-\tau_1) + \beta V(t-\tau_2), \quad t \in [t_{k-1}, t_k),\ k \in \mathbb{N}^+,$$

From the definition of t^* and t^{**}, we have

$$\dot{V} \leq \alpha V(t) + L_2 V(t-\tau_1) + \beta V(t-\tau_2), \quad t \in [t^{**}, t^*].$$

According to (3.26) and choosing $s = -\tau_1$ and $s = -\tau_2$, respectively, gives,

$$\begin{aligned} V(t-\tau_1) &\leq qV(t), \quad t \in [t^{**}, t^*], \\ V(t-\tau_2) &\leq qV(t), \quad t \in [t^{**}, t^*]. \end{aligned}$$

Then, we can conclude that

$$\dot{V} \leq [\alpha + (L_2 + \beta)q]V(t), \qquad t \in [t^{**}, t^*],$$

and

$$\dot{W} = e^{\lambda(t-t_0)}[\lambda V(t) + \dot{V}(t)] \leq [\lambda + \alpha + (L_2 + \beta)q]W(t), \qquad t \in [t^{**}, t^*]. \tag{3.27}$$

Integrating (3.27) from t^{**} to t^*, gives

$$\begin{aligned} W(t^*) &\leq W(t^{**})e^{[\lambda+\alpha+(L_2+\beta)q](t^*-t^{**})} \leq W(t^{**})e^{[\lambda+\alpha+(L_2+\beta)q](t_1-t_0)} \\ &= \frac{M}{\bar{q}}\|V(t_0)\|_{\tilde{\tau}}e^{[\lambda+\alpha+(L_2+\beta)q](t_1-t_0)}. \end{aligned}$$

From (3.19), we have

$$\bar{q} = qe^{-\lambda\tilde{\tau}} > \frac{1}{a_1 + b_1 e^{\lambda\tilde{\tau}}} > e^{[\alpha+(L_2+\beta)q+\lambda](t_1-t_0)},$$

then we have

$$W(t^*) < M\|V(t_0)\|_{\tilde{\tau}},$$

which contradicts with (3.23). Therefore, (3.22) holds. It follows from (3.20) and (3.22) that

$$W(t) \leq M\|V(t_0)\|_{\tilde{\tau}}, \qquad t \in [t_0 - \tilde{\tau}, t_1). \tag{3.28}$$

Next, we shall prove

$$W(t) \leq M\|V(t_0)\|_{\tilde{\tau}}, \quad t \in [t_1, t_2). \tag{3.29}$$

When $t = t_1$, it follows from (3.17), (3.19) and (3.28) that

$$\begin{aligned} W(t_1) = V(t_1)e^{\lambda(t_1-t_0)} &\leq [a_1 V(t_1^-) + b_1 V(t_1 - d)]e^{\lambda(t_1-t_0)} \\ &= a_1 W(t_1^-) + b_1 e^{\lambda d} W(t_1 - d) \\ &\leq (a_1 + b_1 e^{\lambda\tilde{\tau}})M\|V(t_0)\|_{\tilde{\tau}} < M\|V(t_0)\|_{\tilde{\tau}}. \end{aligned}$$

In the following, we will show

$$W(t) \leq M\|V(t_0)\|_{\tilde{\tau}}, \qquad t \in (t_1, t_2). \tag{3.30}$$

If it is not true, there exists $t \in (t_1, t_2)$ such that $W(t) > M||V(t_0)||_{\tilde{\tau}}$. Let $\tilde{t}^* = \inf\{t \in (t_1, t_2) : W(t) > M\|V(t_0)\|_{\tilde{\tau}}\}$. By the continuity of $W(t)$, we have

$$W(\tilde{t}^*) = M\|V(t_0)\|_{\tilde{\tau}}, \tag{3.31}$$

and $W(t) < M\|V(t_0)\|_{\tilde{\tau}}$ for $t \in [t_1, \tilde{t}^*)$. Then, we can conclude that

$$W(t) \leq M\|V(t_0)\|_{\tilde{\tau}}, \quad t \in [t_0 - \tilde{\tau}, \tilde{t}^*] \tag{3.32}$$

Define $\tilde{t}^{**} = \sup\{t \in (t_1, \tilde{t}^*) : W(t) \leq \frac{M}{\bar{q}}\|V(t_0)\|_{\tilde{\tau}}\}$, by the continuity of $W(t)$, we can obtain $W(\tilde{t}^{**}) = \frac{M}{\bar{q}}\|V(t_0)\|_{\tilde{\tau}}$, and

$$W(t) > \frac{M}{\bar{q}}\|V(t_0)\|_{\tilde{\tau}}, \qquad t \in (\tilde{t}^{**}, \tilde{t}^*],$$

then we have

$$M\|V(t_0)\|_{\tilde{\tau}} \leq \bar{q}W(t), \qquad t \in [\tilde{t}^{**}, \tilde{t}^*]. \tag{3.33}$$

From (3.32) and (3.33), we get

$$W(t+s) \leq M\|V(t_0)\|_{\tilde{\tau}} \leq \bar{q}W(t), \qquad t \in [\tilde{t}^{**}, \tilde{t}^*], \quad s \in [-\tilde{\tau}, 0],$$

then we have

$$V(t+s) \leq qV(t), \qquad t \in [\tilde{t}^{**}, \tilde{t}^*], \quad s \in [-\tilde{\tau}, 0]. \tag{3.34}$$

From the proof of Theorem 3.2.1, we can get

$$\dot{V} \leq \alpha V(t) + L_2 V(t - \tau_1) + \beta V(t - \tau_2), \quad t \in [t_1, t_2),$$

then by the definition of $\tilde{t}^*, \tilde{t}^{**}$, we have

$$\dot{V} \leq \alpha V(t) + L_2 V(t - \tau_1) + \beta V(t - \tau_2), \qquad t \in [\tilde{t}^{**}, \tilde{t}^*].$$

Similar to the previous discussion, it follows from (3.34) that

$$\dot{V} \leq [\alpha + (L_2 + \beta)q]V(t), \qquad t \in [\tilde{t}^{**}, \tilde{t}^*],$$

and

$$\dot{W} \leq [\lambda + \alpha + (L_2 + \beta)q]W(t), \qquad t \in [\tilde{t}^{**}, \tilde{t}^*]. \tag{3.35}$$

Integrating (3.35) from $\tilde{t}^{**}$ to $\tilde{t}^{*}$, gives

$$\begin{aligned} W(\tilde{t}^{*}) &\leq W(\tilde{t}^{**})e^{[\lambda+\alpha+(L_2+\beta)q](\tilde{t}^{*}-\tilde{t}^{**})} \leq W(\tilde{t}^{**})e^{[\lambda+\alpha+(L_2+\beta)q](t_2-t_1)} \\ &= \frac{M}{\bar{q}}\|V(t_0)\|_{\tilde{\tau}}e^{[\lambda+\alpha+(L_2+\beta)q](t_2-t_1)}. \end{aligned}$$

From (3.19), we can get

$$\bar{q} = qe^{-\lambda\tilde{\tau}} > \frac{1}{a_2+b_2e^{\lambda\tilde{\tau}}} > e^{[\alpha+(L_2+\beta)q+\lambda](t_2-t_1)},$$

hence we have

$$W(\tilde{t}^{*}) < M\|V(t_0)\|_{\tilde{\tau}},$$

which contradicts with (3.31). Therefore, (3.30) holds. Then, we can conclude that

$$W(t) \leq M\|V(t_0)\|_{\tilde{\tau}}, \qquad t \in [t_0-\tilde{\tau}, t_2).$$

By simple mathematical induction, we can prove that (3.21) holds. Then we have

$$V(t) = W(t)e^{-\lambda(t-t_0)} \leq M||V(t_0)||_{\tilde{\tau}}e^{-\lambda(t-t_0)}, \qquad t \geq t_0.$$

This implies that the synchronization error $\|e_i(t)\| \to 0$ as $t \to \infty$, $i = 1, 2, .., N$. Thus, the pinning impulsively controlled time-delay CVDN (3.8) achieves synchronization. □

Remark 3.2.5. *Condition (3.18) of Theorem 3.2.2 also implies that $a_k + b_k < 1$, for any $k \in \mathbb{N}^+$. This also shows that the impulses are beneficial for the stability of the synchronization error system and play positive role on accelerating the synchronization process of the network. (3.18) also guarantees that the existence of λ such that (3.19) is satisfied.*

Remark 3.2.6. *From the proof of Theorem 3.2.1, (3.15) implies that the continuous-time dynamics may suppress the stability of the synchronization error system. Thus, the impulse sequence must be designed frequent enough and their amplitude must be suitably related to the growth rate of $V(t)$. Compared with the synchronization result in Theorem 3.2.1, condition (3.18) of Theorem 3.2.2 is less conservative by considering an adjustment parameter q.*

Remark 3.2.7. *In [38], the authors investigated synchronization criteria for delay-free CVDNs by adopting pinning impulsive control scheme, while sufficient delay-independent* *conditions for synchronization of CVDNs with both internal and coupling delays are firstly established in this subsection.*

Assume the impulses are uniformly distributed, i.e., $T = t_k - t_{k-1}$, $k \in \mathbb{N}^+$, and the impulsive control gains are designed to be the same at each impulsive instant t_k, say $q_{1k} = \bar{q}_1$, $q_{2k} = \bar{q}_2$, $k \in \mathbb{N}^+$. According to Theorem 3.2.1 and Theorem 3.2.2, we can obtain the following corollaries.

Corollary 3.2.1. *Suppose that Assumption 3.2.1 and Assumption 3.2.2 hold. The pinning impulsively controlled time-delay CVDN (3.8) can achieve synchronization if* $\sum_{j=1}^{N} c_{ij} \leq 0$, *for* $i = 1, 2, ..., N$, *and*

$$\alpha + \frac{L_2 + \beta}{\bar{a} + \bar{b}} + \frac{\ln(\bar{a} + \bar{b})}{T} < 0, \tag{3.36}$$

where $\bar{a} = 1 - \frac{l}{N}[1 - 2(1 + \bar{q}_1)^2], \bar{b} = 2\bar{q}_2^2, \alpha = 2L_1 + L_2 + \rho \|C \otimes A\| - c\rho\|A\|, \beta = \rho \|C \otimes A\| - c\rho\|A\|, c = \min_{1 \leq i \leq N}\{\sum_{j=1}^{N} c_{ij}\}, T = t_k - t_{k-1},\ k \in \mathbb{N}^+$.

Corollary 3.2.2. *Suppose that Assumption 3.2.1 and Assumption 3.2.2 hold. The pinning impulsively controlled time-delay CVDN (3.8) can achieve synchronization if* $\sum_{j=1}^{N} c_{ij} \leq 0$, *for* $i = 1, 2, ..., N$, *and if there exists a constant* $q > 1$ *such that*

$$\frac{1}{\bar{b}} > q > \frac{1}{\bar{a} + \bar{b}} > e^{[\alpha + (L_2 + \beta)q]T}, \tag{3.37}$$

where $\bar{a} = 1 - \frac{l}{N}[1 - 2(1 + \bar{q}_1)^2], \bar{b} = 2\bar{q}_2^2, \alpha = 2L_1 + L_2 + \rho \|C \otimes A\| - c\rho\|A\|, \beta = \rho \|C \otimes A\| - c\rho\|A\|, c = \min_{1 \leq i \leq N}\{\sum_{j=1}^{N} c_{ij}\}, T = t_k - t_{k-1},\ k \in \mathbb{N}^+$.

3.2.5 Delay-Dependent Synchronization Criteria

In this subsection, several delay-dependent synchronization criteria for the pinning impulsively controlled CVDN (3.8) with varies sizes of delays will be stated by employing the Lyapunov functional method.

In the following, we will firstly establish the delay-dependent synchronization criteria for the pinning impulsively controlled CVDN (3.8) with small delays under the assumption that $\tilde{\tau} < t_k - t_{k-1}$, $k \in \mathbb{N}^+$, where $\tilde{\tau} = \max\{\tau_1, \tau_2, d\}$.

Theorem 3.2.3. *Suppose that Assumption 3.2.1 and Assumption 3.2.2 are satisfied, then the pinning impulsively controlled CVDN (3.8) can achieve synchronization if* $\sum_{j=1}^{N} c_{ij} \leq 0$, *for* $i = 1, 2, ..., N$, *and* $\tilde{\tau} \leq t_k - t_{k-1}$, $k \in \mathbb{N}^+$, *and there exist a constant* $\gamma > 0$ *such that*

$$[a_k + b_k e^{-\alpha d} + \tau_1 + \tau_2] e^{(\gamma+\alpha)(t_k - t_{k-1})} \leq 1, \qquad k \in \mathbb{N}^+, \tag{3.38}$$

where $\alpha = 2L_1 + L_2^2 + 2 + 2\rho^2 \left\| C \otimes A \right\|^2 + 2c^2\rho^2 \|A\|^2$, $a_k = 1 - \frac{l}{N}[1 - 2(1 + q_{1k})^2]$, $b_k = 2q_{2k}^2$, $c = \min_{1 \leq i \leq N}\{\sum_{j=1}^N c_{ij}\}$.

Proof. Construct the Lyapunov functional candidate $V(t) = V_1(t) + V_2(t) + V_3(t)$ for the synchronization error system (3.9) with

$$\begin{aligned}
V_1(t) &= \sum_{i=1}^N e_i^*(t) e_i(t), \\
V_2(t) &= \omega \sum_{i=1}^N \int_{t-\tau_1}^t e_i^*(s) e_i(s) ds, \\
V_3(t) &= \omega \sum_{i=1}^N \int_{t-\tau_2}^t e_i^*(s) e_i(s) ds, \qquad 0 < \omega \leq 1.
\end{aligned} \tag{3.39}$$

For $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$, calculate the derivative of V along the trajectory of the error system (3.9), gives

$$\begin{aligned}
\dot{V} &= 2\sum_{i=1}^N \mathfrak{Re}\big[e_i^*(t)\tilde{f}(e_i(t), e_i(t-\tau_1))\big] + 2\mathfrak{Re}\big[\sum_{i=1}^N\sum_{j=1}^N e_i^*(t) c_{ij} A\tilde{h}(e_j(t-\tau_2))\big] \\
&\quad - 2\mathfrak{Re}\big[\sum_{i=1}^N \big(\sum_{j=1}^N c_{ij}\big) e_i^*(t) A\tilde{h}(e_i(t-\tau_2))\big] + \omega\sum_{i=1}^N [e_i^*(t)e_i(t) - e_i^*(t-\tau_1)e_i(t-\tau_1)] \\
&\quad + \omega\sum_{i=1}^N [e_i^*(t)e_i(t) - e_i^*(t-\tau_2)e_i(t-\tau_2)].
\end{aligned}$$

Denote $E(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T$, and $\tilde{H}(E(t-\tau_2)) = (\tilde{h}^T(e_1(t-\tau_2)), \tilde{h}^T(e_2(t-\tau_2)), \ldots, \tilde{h}^T(e_N(t-\tau_2)))^T$. It follows from (3.14), Cauchy-Schwartz inequality, Assumption 3.2.1 and Assumption 3.2.2 that

$$\dot{V} \leq 2\sum_{i=1}^N \|e_i(t)\| \|\tilde{f}(e_i(t), e_i(t-\tau_1))\| + 2\mathfrak{Re}[E^*(t)\big(C \otimes A\big)\tilde{H}(E(t-\tau_2))] + 2\omega\sum_{i=1}^N e_i^*(t)e_i(t)$$

$$
\begin{aligned}
&- 2\rho c\|A\| \sum_{i=1}^{N} \|e_i(t)\|\|e_i(t-\tau_2)\| - \omega \sum_{i=1}^{N} e_i^*(t-\tau_1)e_i(t-\tau_1) - \omega \sum_{i=1}^{N} e_i^*(t-\tau_2)e_i(t-\tau_2) \\
&\le (2L_1 + 2\omega) \sum_{i=1}^{N} e_i^*(t)e_i(t) + 2L_2 \sum_{i=1}^{N} \|e_i(t)\|\|e_i(t-\tau_1)\| + 2\left|\left\langle E(t), \big(C \otimes A\big)\tilde{H}(E(t-\tau_2))\right\rangle\right| \\
&- 2\rho c\|A\| \sum_{i=1}^{N} \|e_i(t)\|\|e_i(t-\tau_2)\| - \omega \sum_{i=1}^{N} e_i^*(t-\tau_1)e_i(t-\tau_1) - \omega \sum_{i=1}^{N} e_i^*(t-\tau_2)e_i(t-\tau_2) \\
&\le (2L_1 + 2\omega) \sum_{i=1}^{N} e_i^*(t)e_i(t) + \frac{L_2^2}{\omega} \sum_{i=1}^{N} e_i^*(t)e_i(t) + 2\rho\|C \otimes A\|\|E(t)\|\|E(t-\tau_2)\| \\
&- 2\rho c\|A\| \sum_{i=1}^{N} \|e_i(t)\|\|e_i(t-\tau_2)\| - \omega \sum_{i=1}^{N} e_i^*(t-\tau_2)e_i(t-\tau_2) \\
&\le (2L_1 + 2\omega + \frac{L_2^2}{\omega}) \sum_{i=1}^{N} e_i^*(t)e_i(t) + \frac{2\rho^2 \left\|C \otimes A\right\|^2}{\omega}\|E(t)\|^2 + \frac{\omega}{2}\|E(t-\tau_2)\|^2 \\
&+ \frac{2c^2\rho^2\|A\|^2}{\omega} \sum_{i=1}^{N} \|e_i(t)\|^2 + \frac{\omega}{2} \sum_{i=1}^{N} \|e_i(t-\tau_2)\|^2 - \omega \sum_{i=1}^{N} e_i^*(t-\tau_2)e_i(t-\tau_2) \\
&= (2L_1 + 2\omega + \frac{L_2^2}{\omega} + \frac{2\rho^2 \left\|C \otimes A\right\|^2}{\omega} + \frac{2c^2\rho^2\|A\|^2}{\omega}) \sum_{i=1}^{N} e_i^*(t)e_i(t) \\
&\le \left[2L_1 + 2\omega + \frac{1}{\omega}(L_2^2 + 2\rho^2 \left\|C \otimes A\right\|^2 + 2c^2\rho^2\|A\|^2)\right] V(t).
\end{aligned} \tag{3.40}
$$

Denote $\alpha = 2L_1 + 2\omega + \frac{1}{\omega}(L_2^2 + 2\rho^2 \left\|C \otimes A\right\|^2 + 2c^2\rho^2\|A\|^2)$ with $0 < \omega \le 1$. For small delays, choose $\omega = 1$ such that $\alpha = 2L_1 + L_2^2 + 2 + 2\rho^2 \left\|C \otimes A\right\|^2 + 2c^2\rho^2\|A\|^2$. Then, we can conclude that at

$$
V(t) \le V(t_{k-1})e^{\alpha(t-t_{k-1})}, \qquad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+. \tag{3.41}
$$

When $t = t_k$, $k \in \mathbb{N}^+$, similar to the proof of the impulse part in Theorem 3.2.1, we can obtain $V_1(t)$ satisfies (3.17), that is,

$$
V_1(t_k) \le a_k V_1(t_k^-) + b_k V_1(t_k - d), \quad k \in \mathbb{N}^+ \tag{3.42}
$$

with $a_k = 1 - \frac{\ell}{N}[1 - 2(1 + q_{1k})^2]$, $b_k = 2q_{2k}^2$.

In the following, we will use the method of mathematical induction to prove

$$V(t) \le Me^{-(\gamma+\alpha)(t_k-t_0)}e^{\alpha(t-t_0)}, \qquad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+. \tag{3.43}$$

According to (3.41), we have $V(t) \le V(t_0)e^{\alpha(t-t_0)}$, for $t \in [t_0, t_1)$. Then, there exists $\gamma > 0$ such that

$$V(t) \le Me^{-(\gamma+\alpha)(t_1-t_0)}e^{\alpha(t-t_0)}, \quad t \in [t_0, t_1)$$

with $M = V(t_0)e^{(\gamma+\alpha)(t_1-t_0)}$. Therefore, (3.43) holds for $k = 1$. Suppose (3.43) is true for $k = j$ ($j \ge 1$), i.e.,

$$V(t) \le Me^{-(\gamma+\alpha)(t_j-t_0)}e^{\alpha(t-t_0)}, \qquad t \in [t_{j-1}, t_j), \tag{3.44}$$

and if $\tilde{\tau} \le t_k - t_{k-1}$, $k \in \mathbb{N}^+$, then we can obtain from (3.42) and (3.44) that at $t = t_j$,

$$V_1(t_j) \le a_j V(t_j^-) + b_j V(t_j - d) \le [a_j + b_j e^{-\alpha d}]Me^{-(\gamma+\alpha)(t_j-t_0)}e^{\alpha(t_j-t_0)}.$$

Denote $V(s_0) = \sup_{s\in[t_j-\tau_1,t_j)} V(s)$ with $s_0 \in [t_j - \tau_1, t_j)$, then by the continuity of $V_2(t)$ and (3.44), we have

$$\begin{aligned} V_2(t_j) &= \int_{t_j-\tau_1}^{t_j} V_1(s)ds \le \tau_1 \sup_{s\in[t_j-\tau_1,t_j)} V_1(s) \le \tau_1 V(s_0) \\ &\le \tau_1 Me^{-(\gamma+\alpha)(t_j-t_0)}e^{\alpha(s_0-t_0)} \le \tau_1 Me^{-(\gamma+\alpha)(t_j-t_0)}e^{\alpha(t_j-t_0)}. \end{aligned}$$

Similarly, $V_3(t_j) \le \tau_2 Me^{-(\gamma+\alpha)(t_j-t_0)}e^{\alpha(t_j-t_0)}$. According to condition (3.38) of Theorem 3.2.3, we have

$$\begin{aligned} &V(t_j) \le [a_j + b_j e^{-\alpha d} + \tau_1 + \tau_2]Me^{-(\gamma+\alpha)(t_j-t_0)}e^{\alpha(t_j-t_0)} \\ &= [a_j + b_j e^{-\alpha d} + \tau_1 + \tau_2]e^{(\gamma+\alpha)(t_{j+1}-t_j)}Me^{-(\gamma+\alpha)(t_{j+1}-t_0)}e^{\alpha(t_j-t_0)} \le Me^{-(\gamma+\alpha)(t_{j+1}-t_0)}e^{\alpha(t_j-t_0)}, \end{aligned}$$

which implies that (3.43) holds for $t = t_j$. For $t \in (t_j, t_{j+1})$, according to (3.41), we have

$$V(t) \le V(t_j)e^{\alpha(t-t_j)} \le Me^{-(\gamma+\alpha)(t_{j+1}-t_0)}e^{\alpha(t_j-t_0)}e^{\alpha(t-t_j)} = Me^{-(\gamma+\alpha)(t_{j+1}-t_0)}e^{\alpha(t-t_0)}$$

Therefore, (3.43) holds for $t \in [t_j, t_{j+1})$, i.e., (3.43) is true for $k = j + 1$. By mathematical induction, we can prove that (3.43) is true, hence

$$V_1(t) \le V(t) \le Me^{-(\gamma+\alpha)(t_k-t_0)}e^{\alpha(t_k-t_0)} = Me^{-\gamma(t_k-t_0)}, \qquad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+.$$

This shows that as $t \to \infty$, $V_1(t) \to 0$, which implies that as $t \to \infty$, $\|e_i(t)\| \to 0$ for $i = 1, 2, \cdots, N$. Thus, the pinning impulsively controlled time-delay CVDN (3.8) achieves synchronization. □

Remark 3.2.8. *Theorem 3.2.3 extends the synchronization result of real-variable dynamical networks with small delays in [86] to complex-variable networks. Compared with [86], the network model introduced in this section is more general in the sense that the internal delay and coupling delay are different and the coupling terms are considered to be nonlinear. Furthermore, delayed impulses are also taken into account when designing pinning impulsive controllers. Theorem 3.2.3 shows that the synchronization result depends on the system parameter α, control parameters a_k, b_k, the size of delays, and the impulsive distances. The exponential term $b_k e^{-\alpha d}$ in (3.38) implies that the delayed impulses play a key role to synchronize all the states of nodes in complex-valued networks.*

Remark 3.2.9. *In Theorem 3.2.3, the constraint $\tilde{\tau} \leq t_k - t_{k-1}$ for all $k \in \mathbb{N}^+$ implies that the lower bound of the impulsive distances is $\tilde{\tau} = \max\{\tau_1, \tau_2, d\}$. Therefore, the complex-valued network synchronization result in Theorem 3.2.3 has restriction on designing the length of the impulsive intervals and may not be applicable for networks with relatively large delays. In other words, (3.38) is not a sufficient condition for complex-valued network synchronization if we get rid of the constraint $\tilde{\tau} \leq t_k - t_{k-1}$, $k \in \mathbb{N}^+$.*

Remark 3.2.10. *Note that Theorem 3.2.3 is valid only when $a_k + b_k e^{-\alpha d} + \tau_1 + \tau_2 < 1$, if $\tau_1 + \tau_2 \geq 1$, then (3.38) is not applicable even if $\tilde{\tau} \leq \inf_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\}$. we will state more general and applicable synchronization results for the pinning impulsively controlled CVDN (3.8) with relatively large delays (i.e., $\tau_1 + \tau_2 \geq 1$) and eliminate the restriction $\tilde{\tau} \leq \inf_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\}$ in Theorem 3.2.3.*

In the following, we will introduce two theorems with respect to CVDN (3.8) with large delays. In other words, the size of delays could exceed the length of impulse intervals (i.e. $\tilde{\tau} > \{t_k - t_{k-1}\}$ for some $k \in \mathbb{N}^+$), and $\tau_1 + \tau_2$ could be greater than 1.

Theorem 3.2.4. *Suppose that Assumption 3.2.1 and Assumption 3.2.2 are satisfied, then the pinning impulsively controlled CVDN (3.8) can achieve synchronization if $\sum_{j=1}^{N} c_{ij} \leq 0$, for $i = 1, 2, ..., N$, and there exist constants $0 < \omega \leq 1$ and $\lambda > 0$ such that*

$$[a_k + b_k e^{\lambda d} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}] e^{(\alpha+\lambda)(t_k - t_{k-1})} \leq 1, \qquad k \in \mathbb{N}^+, \tag{3.45}$$

where $\alpha = 2L_1 + 2\omega + \frac{1}{\omega}(L_2^2 + 2\rho^2 \|C \otimes A\|^2 + 2c^2\rho^2\|A\|^2)$, $a_k = 1 - \frac{\ell}{N}[1 - 2(1 + q_{1k})^2]$, $b_k = 2q_{2k}^2$, $c = \min_{1\leq i\leq N}\{\sum_{j=1}^{N} c_{ij}\}$.

Proof. Consider the same Lyapunov functional candidate as that of (3.39). Since $\lim_{k\to\infty} t_k = \infty$, there exists integer $p \geq 1$ such that $t_p - \tilde{\tau} \geq t_0$, where $\tilde{\tau} = \max\{\tau_1, \tau_2, d\}$. Then, there exists $\lambda > 0$ such that

$$V(t) = V(t)e^{\lambda(t-t_0)}e^{-\lambda(t-t_0)} \leq \tilde{M}e^{-\lambda(t-t_0)}, \qquad t \in [t_0, t_p) \tag{3.46}$$

with $\tilde{M} = \sup_{t\in[t_0,t_p)} V(t)e^{\lambda(t_p-t_0)}$. In the following, we will use the method of mathematical induction to show that

$$V(t) \leq \tilde{M}e^{-(\lambda+\alpha)(t_{k+1}-t_0)}e^{\alpha(t-t_0)}, \qquad t \in [t_k, t_{k+1}), \quad k \geq p. \tag{3.47}$$

Firstly, consider the base case when $k = p$, we can obtain from (3.42) and (3.46) that

$$V_1(t_p) \leq a_p V_1(t_p^-) + b_p V_1(t_p - d) \leq [a_p + b_p e^{\lambda d}]\tilde{M}e^{-\lambda(t_p-t_0)}.$$

Denote $V(\hat{s_0}) = \sup_{s\in[t_p-\tau_1,t_p)} V(s)$ with $\hat{s_0} \in [t_p - \tau_1, t_p)$, then by the continuity of $V_2(t)$ in (3.39) and (3.46), we have

$$V_2(t_p) \leq \omega\tau_1 \sup_{s\in[t_p-\tau_1,t_p)} V_1(s) \leq \omega\tau_1 V(\hat{s_0}) \leq \omega\tau_1\tilde{M}e^{-\lambda(\hat{s_0}-t_0)} \leq \omega\tau_1\tilde{M}e^{-\lambda(t_p-\tau_1-t_0)}.$$

Similarly, $V_3(t_p) \leq \omega\tau_2\tilde{M}e^{-\lambda(t_p-\tau_2-t_0)}$. Then, according to condition (3.45),

$$\begin{aligned}
V(t_p) &\leq [a_p + b_p e^{\lambda d} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}]\tilde{M}e^{-\lambda(t_p-t_0)} \\
&= [a_p + b_p e^{\lambda d} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}]\tilde{M}e^{-(\lambda+\alpha)(t_p-t_0)}e^{\alpha(t_p-t_0)} \\
&\leq [a_p + b_p e^{\lambda d} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}]e^{(\lambda+\alpha)(t_{p+1}-t_p)}\tilde{M}e^{-(\lambda+\alpha)(t_{p+1}-t_0)}e^{\alpha(t_p-t_0)} \\
&\leq \tilde{M}e^{-(\lambda+\alpha)(t_{p+1}-t_0)}e^{\alpha(t_p-t_0)},
\end{aligned}$$

which implies that (3.47) holds at $t = t_p$. For $t \in (t_p, t_{p+1})$, we have from (3.41) that

$$V(t) \leq \tilde{M}e^{-(\lambda+\alpha)(t_{p+1}-t_0)}e^{\alpha(t_p-t_0)}e^{\alpha(t-t_p)} = \tilde{M}e^{-(\lambda+\alpha)(t_{p+1}-t_0)}e^{\alpha(t-t_0)}$$

Thus, (3.47) holds for $t \in (t_p, t_{p+1})$, i.e., (3.47) is true for $k = p$. Suppose (3.47) is true for $k \leq j$ $(j > p)$, that is,

$$V(t) \leq \tilde{M}e^{-(\lambda+\alpha)(t_{k+1}-t_0)}e^{\alpha(t-t_0)}, \qquad t \in [t_k, t_{k+1}), \quad k \leq j \quad (j > p) \tag{3.48}$$

According to (3.42), we can obtain $V_1(t_{j+1}) \leq a_{j+1}V(t_{j+1}^-) + b_{j+1}V(t_{j+1} - d)$, and we will estimate $V(t_{j+1} - d)$ by considering the following two cases:

Case 1: If $t_{j+1}-d \in [t_0, t_p)$, then according to (3.46),

$$V(t_{j+1}-d) \leq \tilde{M}e^{-\lambda(t_{j+1}-d-t_0)},$$

Case 2: If $t_{j+1}-d \geq t_p$, then there exists a positive integer $\hat{k}$ ($p \leq \hat{k} \leq j$) such that $t_{j+1}-d \in [t_{\hat{k}}, t_{\hat{k}+1})$, then from (3.48),

$$\begin{aligned} V(t_{j+1}-d) &\leq \tilde{M}e^{-(\lambda+\alpha)(t_{\hat{k}+1}-t_0)}e^{\alpha(t_{j+1}-d-t_0)} \\ &\leq \tilde{M}e^{-(\lambda+\alpha)(t_{j+1}-d-t_0)}e^{\alpha(t_{j+1}-d-t_0)} = \tilde{M}e^{-\lambda(t_{j+1}-d-t_0)}, \end{aligned}$$

then we can conclude that $V(t_{j+1}-d) \leq \tilde{M}e^{\lambda d}e^{-\lambda(t_{j+1}-t_0)}$, hence

$$V_1(t_{j+1}) \leq [a_{j+1} + b_{j+1}e^{\lambda d}]\tilde{M}e^{-\lambda(t_{j+1}-t_0)}.$$

By the continuity of $V_2(t)$, we have

$$V_2(t_{j+1}) \leq \omega\tau_1 \sup_{s\in[t_{j+1}-\tau_1, t_{j+1})} V(s),$$

similarly, denote $V(s^*) = \sup_{s\in[t_{j+1}-\tau_1, t_{j+1})} V(s)$ with $s^* \in [t_{j+1}-\tau_1, t_{j+1})$, then we will estimate $V(s^*)$ by considering the following two cases:
Case 1: If $s^* \in [t_{j+1}-\tau_1, t_p)$, then we have from (3.46) that

$$V(s^*) \leq \tilde{M}e^{-\lambda(s^*-t_0)} \leq \tilde{M}e^{-\lambda(t_{j+1}-\tau_1-t_0)},$$

Case 2: If $s^* \geq t_p$, then there exists a positive integer $\hat{k}$ ($p \leq \hat{k} \leq j$) such that $s^* \in [t_{\hat{k}}, t_{\hat{k}+1})$, then from (3.48),

$$V(s^*) \leq \tilde{M}e^{-(\lambda+\alpha)(t_{\hat{k}+1}-t_0)}e^{\alpha(s^*-t_0)} \leq \tilde{M}e^{-(\lambda+\alpha)(t_{\hat{k}+1}-t_0)}e^{\alpha(t_{\hat{k}+1}-t_0)} \leq \tilde{M}e^{-\lambda(t_{j+1}-\tau_1-t_0)},$$

then we can conclude that $V(s^*) \leq \tilde{M}e^{-\lambda(t_{j+1}-\tau_1-t_0)}$, hence

$$V_2(t_{j+1}) \leq \omega\tau_1 e^{\lambda\tau_1}\tilde{M}e^{-\lambda(t_{j+1}-t_0)}.$$

Similarly, $V_3(t_{j+1}) \leq \omega\tau_2 e^{\lambda\tau_2}\tilde{M}e^{-\lambda(t_{j+1}-t_0)}$. Therefore, according to condition (3.45),

$$\begin{aligned} V(t_{j+1}) &\leq [a_{j+1} + b_{j+1}e^{\lambda d} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}]\tilde{M}e^{-\lambda(t_{j+1}-t_0)} \\ &= [a_{j+1} + b_{j+1}e^{\lambda d} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}]\tilde{M}e^{-(\lambda+\alpha)(t_{j+1}-t_0)}e^{\alpha(t_{j+1}-t_0)} \\ &= [a_{j+1} + b_{j+1}e^{\lambda d} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}]e^{(\lambda+\alpha)(t_{j+2}-t_{j+1})}e^{-(\lambda+\alpha)(t_{j+2}-t_0)}\tilde{M}e^{\alpha(t_{j+1}-t_0)} \end{aligned}$$

$$\leq \tilde{M}e^{-(\lambda+\alpha)(t_{j+2}-t_0)}e^{\alpha(t_{j+1}-t_0)},$$

which implies that (3.47) holds at $t = t_{j+1}$. For $t \in (t_{j+1}, t_{j+2})$, (3.41) implies that

$$V(t) \leq V(t_{j+1})e^{\alpha(t-t_{j+1})} \leq \tilde{M}e^{-(\lambda+\alpha)(t_{j+2}-t_0)}e^{\alpha(t_{j+1}-t_0)}e^{\alpha(t-t_{j+1})} = \tilde{M}e^{-(\lambda+\alpha)(t_{j+2}-t_0)}e^{\alpha(t-t_0)}.$$

This shows that (3.47) holds for $t \in (t_{j+1}, t_{j+2})$. Hence (3.47) holds for $t \in [t_{j+1}, t_{j+2})$, i.e., (3.47) is true for $k = j+1$. By mathematical induction, we can prove that (3.47) is true, then we can obtain

$$V_1(t) \leq V(t) \leq \tilde{M}e^{-(\lambda+\alpha)(t_{k+1}-t_0)}e^{\alpha(t_{k+1}-t_0)} = \tilde{M}e^{-\lambda(t_{k+1}-t_0)}, \qquad t \in [t_k, t_{k+1}), \quad k \geq p.$$

This shows that as $t \to \infty$, $V_1(t) \to 0$, which implies that as $t \to \infty$, $\|e_i(t)\| \to 0$ for $i = 1, 2, \cdots, N$. Thus, the pinning impulsively controlled time-delay CVDN (3.8) achieves synchronization. □

Remark 3.2.11. *It can be seen from Theorem 3.2.4 that the constraint $\tilde{\tau} \leq t_k - t_{k-1}, k \in \mathbb{N}^+$ in Theorem 3.2.3 has been removed, and there is no restriction on the lower bound of impulse intervals, thus Theorem 3.2.4 provides less conservative conditions on designing impulsive distances. For relatively large delays (i.e. $\tau_1 + \tau_2 > 1$), the parameter ω can adjust the value of $a_k + b_k e^{\lambda d} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}$ such that condition (3.45) in Theorem 3.2.4 can be satisfied. Hence, compared with Theorem 3.2.3, the obtained result in Theorem 3.2.4 is more general and applicable.*

Particularly, the size of the impulse delay might be larger than the length of the impulse intervals. In the following, we will introduce a sufficient delay-dependent synchronization criterion for pinning impulsively controlled CVDN (3.8) with relatively large impulse delay. For simplicity, we assume that all the impulses are uniformly distributed, let T denote the impulsive distance (i.e., $T = t_k - t_{k-1}, k \in \mathbb{N}^+$), and the impulsive control gains are consistently designed at each impulsive instant ($i.e., q_{1k} = q_1, q_{2k} = q_2, k \in \mathbb{N}^+$). Let ζ denote the number of impulses on the time interval $[t_k - d, t_k)$, that is,

$$\zeta = \begin{cases} \lfloor \frac{d}{T} \rfloor, & \text{if } \frac{d}{T} \notin \mathbb{N}, \\ \frac{d}{T} - 1, & \text{otherwise.} \end{cases}$$

According to the pinning algorithm, let ζ_i be the number of impulses added to the i-th node of the network on the time interval $[t_k - d, t_k)$, it is obvious that $\zeta_i \leq \zeta$ for $i = 1, 2, ..., N$.

Theorem 3.2.5. *Suppose that Assumption 3.2.1 and Assumption 3.2.2 are satisfied, then the pinning impulsively controlled CVDN (3.8) can achieve synchronization if $\sum_{j=1}^{N} c_{ij} \leq 0$,* *for* $i = 1, 2, ..., N$, *and there exists a constant* $0 < \omega \le 1$ *such that*

$$\ln[\gamma + \omega(\tau_1 + \tau_2)] < -\alpha T, \tag{3.49}$$

where

$\gamma = 1 - \frac{l}{N} + \left[\sqrt{\frac{l}{N}}(1+q_1+q_2) - q_2 d\big[\sqrt{2L_1^2 + 2L_2^2} - \rho\|A\| \min\{c_{ii}\}(\sqrt{Nl}+N)\big] - \sqrt{2}\zeta q_2 \sqrt{q_1^2 + q_2^2}\right]^2,$

$\alpha = 2L_1 + 2\omega + \frac{1}{\omega}(L_2^2 + 2\rho^2 \|C \otimes A\|^2 + 2c^2\rho^2\|A\|^2), \quad c = \min_{1\le i\le N}\{\sum_{j=1}^N c_{ij}\},$

$T = t_k - t_{k-1}, \quad k \in \mathbb{N}^+.$

Proof. Choose the Lyapunov functional candidate be the same as that of (3.39).

For $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$, we can obtain from (3.40) in the proof of Theorem 3.2.3 that $\dot{V} \le \alpha V$ with $\alpha = 2L_1 + 2\omega + \frac{1}{\omega}(L_2^2 + 2\rho^2 \|C \otimes A\|^2 + 2c^2\rho^2\|A\|^2)$, hence we can get

$$V(t) \le V(t_{k-1})e^{\alpha(t-t_{k-1})}, \qquad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+. \tag{3.50}$$

Assume that impulsive control gains are consistently designed at each impulsive instant $t = t_k, k \in \mathbb{N}^+$, then it follows from (3.9) that $e_i(t_k) = (1 + q_1)e_i(t_k^-) + q_2 e_i(t_k - d)$ for $i \in \mathcal{D}_k^l$, and $e_i(t_k) = e_i(t_k^-)$ for $i \notin \mathcal{D}_k^l$. For relatively large impulse delay d, integrating both sides of error system (3.9) from $t_k - d$ to t_k, gives:

$$\begin{aligned} e_i(t_k^-) - e_i(t_k - d) = &\int_{t_k-d}^{t_k} \Big(\tilde{f}(e_i(t), e_i(t-\tau_1)) + \sum_{j=1}^N c_{ij} A\tilde{h}(e_j(t-\tau_2)) \\ &- \sum_{j=1}^N c_{ij} A\tilde{h}(e_i(t-\tau_2))\Big)dt + \sum_{m\in M}[q_1 e_i(t_{k-m}^-) + q_2 e_i(t_{k-m} - d)], \end{aligned}$$

for $i = 1, 2, ..., N$, where $M = \{\sigma \in (1, 2, ..., \zeta) | i \in \mathcal{D}_{k-\sigma}^l\}$. Then, we have for $i \in \mathcal{D}_k^l$, $e_i(t_k) = X_i + Y_i + Z_i$ with

$$\begin{aligned} X_i &= (1 + q_1 + q_2)e_i(t_k^-), \\ Y_i &= -q_2 \int_{t_k-d}^{t_k} \Big(\tilde{f}(e_i(t), e_i(t-\tau_1)) + \sum_{j=1}^N c_{ij} A\tilde{h}(e_j(t-\tau_2)) - \sum_{j=1}^N c_{ij} A\tilde{h}(e_i(t-\tau_2))\Big)dt, \\ Z_i &= -q_2 \sum_{m\in M}[q_1 e_i(t_{k-m}^-) + q_2 e_i(t_{k-m} - d)]. \end{aligned}$$

According to Lemma 3.2.3, we have for any $\varepsilon_1, \xi_1 > 0$,

$$\begin{aligned}
&\sum_{i\in D_k^l} e_i^*(t_k)e_i(t_k) = \sum_{i\in D_k^l}(X_i+Y_i+Z_i)^*(X_i+Y_i+Z_i) \\
&\leq (1+\varepsilon_1)\sum_{i\in D_k^l} X_i^*X_i + (1+\frac{1}{\varepsilon_1})(1+\xi_1)\sum_{i\in D_k^l} Y_i^*Y_i + (1+\frac{1}{\varepsilon_1})(1+\frac{1}{\xi_1})\sum_{i\in D_k^l} Z_i^*Z_i.
\end{aligned} \tag{3.51}$$

Denote $W_{1i} = \tilde{f}(e_i(t), e_i(t-\tau_1)), W_{2i} = \sum_{j=1}^N c_{ij}A\tilde{h}(e_j(t-\tau_2)), W_{3i} = -\sum_{j=1}^N c_{ij}A\tilde{h}(e_i(t-\tau_2))$. According to Cauchy-Schwarz's inequality for square-integrable complex-valued functions, Assumption 3.2.1, Assumption 3.2.2 and Lemma 3.2.3, we have

$$\sum_{i\in \mathcal{D}_k^l} Y_i^*Y_i \leq q_2^2 d\sum_{i\in D_k^l}\int_{t_k-d}^{t_k}\big(W_{1i}+W_{2i}+W_{3i}\big)^*\big(W_{1i}+W_{2i}+W_{3i}\big)dt$$

$$\leq q_2^2 d\sum_{i\in \mathcal{D}_k^l}\int_{t_k-d}^{t_k}(1+\varepsilon_2)W_{1i}^*W_{1i} + (1+\frac{1}{\varepsilon_2})(1+\xi_2)W_{2i}^*W_{2i} + (1+\frac{1}{\varepsilon_2})(1+\frac{1}{\xi_2})W_{3i}^*W_{3i}dt$$

$$\leq q_2^2 d\int_{t_k-d}^{t_k}(1+\varepsilon_2)\sum_{i\in \mathcal{D}_k^l}\big(2L_1^2e_i^*(t)e_i(t)+2L_2^2e_i^*(t-\tau_1)e_i(t-\tau_1)\big) + (1+\frac{1}{\varepsilon_2})(1+\xi_2)\cdot$$

$$N\sum_{i\in \mathcal{D}_k^l}\sum_{j=1}^N c_{ij}^2\|A\|^2\|\tilde{h}(e_j(t-\tau_2))\|^2 + (1+\frac{1}{\varepsilon_2})(1+\frac{1}{\xi_2})N\sum_{i\in \mathcal{D}_k^l}\sum_{j=1}^N c_{ij}^2\|A\|^2\|\tilde{h}(e_i(t-\tau_2))\|^2dt$$

$$\leq q_2^2 d\int_{t_k-d}^{t_k}(1+\varepsilon_2)\big(2L_1^2\sum_{i=1}^N e_i^*(t)e_i(t) + 2L_2^2\sum_{i=1}^N e_i^*(t-\tau_1)e_i(t-\tau_1)\big) + (1+\frac{1}{\varepsilon_2})(1+\xi_2)\cdot$$

$$N\|A\|^2\rho^2\sum_{i\in \mathcal{D}_k^l}\sum_{j=1}^N c_{ij}^2\|e_j(t-\tau_2)\|^2 + (1+\frac{1}{\varepsilon_2})(1+\frac{1}{\xi_2})N\|A\|^2\rho^2\sum_{i\in \mathcal{D}_k^l}\big(\sum_{j=1}^N c_{ij}^2\big)\|e_i(t-\tau_2)\|^2dt$$

$$\leq q_2^2 d\int_{t_k-d}^{t_k}(1+\varepsilon_2)\big(2L_1^2V_1(t)+2L_2^2V_1(t-\tau_1)\big) + (1+\frac{1}{\varepsilon_2})(1+\xi_2)N\|A\|^2\rho^2 l\big(\min_{1\leq i\leq N}\{c_{ii}\}\big)^2\cdot$$

$$V_1(t-\tau_2) + (1+\frac{1}{\varepsilon_2})(1+\frac{1}{\xi_2})N\|A\|^2\rho^2 N\big(\min_{1\leq i\leq N}\{c_{ii}\}\big)^2V_1(t-\tau_2)dt$$

$$\leq q_2^2d^2\Bigg[(1+\varepsilon_2)2L_1^2\sup_{s\in[-d,0]}V_1(t_k^-+s) + (1+\varepsilon_2)2L_2^2\sup_{s\in[-d-\tau_1,0]}V_1(t_k^-+s) + \big[(1+\frac{1}{\varepsilon_2})(1+\xi_2)\cdot$$

$$N\|A\|^2\rho^2 l(\min\{c_{ii}\})^2 + (1+\frac{1}{\varepsilon_2})(1+\frac{1}{\xi_2})N^2\|A\|^2\rho^2(\min\{c_{ii}\})^2\big]\sup_{s\in[-d-\tau_2,0]}V_1(t_k^-+s)\Bigg]$$

$$\leq q_2^2 d^2 \Bigg[(1+\varepsilon_2)(2L_1^2+2L_2^2)+(1+\frac{1}{\varepsilon_2})(1+\xi_2)N\|A\|^2\rho^2(\min\{c_{ii}\})^2 l+(1+\frac{1}{\varepsilon_2})(1+\frac{1}{\xi_2})\cdot$$
$$N^2\|A\|^2\rho^2(\min\{c_{ii}\})^2\Bigg]\sup_{s\in[-d-\tilde{\tau},0]} V_1(t_k^-+s)$$

for any $\varepsilon_2, \xi_2 > 0$.
According to Lemma 3.2.4, we can obtain

$$\sum_{i\in\mathcal{D}_k^l} Y_i^* Y_i \leq q_2^2 d^2\big[\sqrt{2L_1^2+2L_2^2}-\rho\|A\|\min\{c_{ii}\}(\sqrt{Nl}+N)\big]^2 \sup_{s\in[-d-\tilde{\tau},0]} V_1(t_k^-+s), \quad (3.52)$$

with $(\varepsilon_2,\xi_2)=\left(\frac{-\rho\|A\|\min\{c_{ii}\}(\sqrt{Nl}+N)}{\sqrt{2L_1^2+2L_2^2}}, \sqrt{\frac{N}{l}}\right)$. On the other hand,

$$\begin{aligned}
&\sum_{i\in\mathcal{D}_k^l} Z_i^* Z_i \leq q_2^2\sum_{i\in\mathcal{D}_k^l}\zeta_i\sum_{m\in M}[q_1 e_i(t_{k-m}^-)+q_2 e_i(t_{k-m}-d)]^*[q_1 e_i(t_{k-m}^-)+q_2 e_i(t_{k-m}-d)]\\
&\leq 2q_2^2\sum_{i\in\mathcal{D}_k^l}\zeta_i\sum_{m\in M}[q_1^2 e_i^*(t_{k-m}^-)e_i(t_{k-m}^-)+q_2^2 e_i^*(t_{k-m}-d)e_i(t_{k-m}-d)]\\
&\leq 2q_2^2\zeta\sum_{m\in M}\big[q_1^2\sum_{i=1}^N e_i^*(t_{k-m}^-)e_i(t_{k-m}^-)+q_2^2\sum_{i=1}^N e_i^*(t_{k-m}-d)e_i(t_{k-m}-d)\big]\\
&= 2q_2^2\zeta\big[q_1^2\sum_{m\in M}V_1(t_{k-m}^-)+q_2^2\sum_{m\in M}V_1(t_{k-m}-d)\big]\\
&\leq 2q_2^2\zeta\big[q_1^2\zeta_i\sup_{s\in[-d,0]}V_1(t_k^-+s)+q_2^2\zeta_i\sup_{s\in[-2d,0]}V_1(t_k^-+s)\big]\\
&\leq 2\zeta^2 q_2^2(q_1^2+q_2^2)\sup_{s\in[-d-\tilde{\tau},0]}V_1(t_k^-+s). \qquad (3.53)
\end{aligned}$$

From (3.51)-(3.53), we can conclude that for any $\varepsilon_1, \xi_1 > 0$,

$$\begin{aligned}
&V_1(t_k)=\sum_{i\in\mathcal{D}_k^l} e_i^*(t_k)e_i(t_k)+\sum_{i\notin\mathcal{D}_k^l} e_i^*(t_k)e_i(t_k)\\
&\leq (1+\varepsilon_1)(1+q_1+q_2)^2\sum_{i\in\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-)+\big[(1+\frac{1}{\varepsilon_1})(1+\xi_1)q_2^2 d^2\big(\sqrt{2L_1^2+2L_2^2}-\min\{c_{ii}\}\cdot\\
&\rho\|A\|(\sqrt{Nl}+N)\big)^2+(1+\frac{1}{\varepsilon_1})(1+\frac{1}{\xi_1})2\zeta^2 q_2^2(q_1^2+q_2^2)\big]\sup_{s\in[-d-\tilde{\tau},0]}V_1(t_k^-+s)+\sum_{i\notin\mathcal{D}_k^l} e_i^*(t_k^-)e_i(t_k^-).
\end{aligned}$$

Similar to the proof of the pinning part in Theorem 3.2.1, we can obtain

$$V_1(t_k) \le \alpha V_1(t_k^-) + \beta \sup_{s\in[-d-\tilde{\tau},0]} V_1(t_k^- + s), \qquad k \in \mathbb{N}^+ \tag{3.54}$$

with $\alpha = 1 - \frac{l}{N}[1-(1+\varepsilon_1)(1+q_1+q_2)^2]$, and $\beta = (1+\frac{1}{\varepsilon_1})(1+\xi_1)q_2^2 d^2\big[\sqrt{2L_1^2+2L_2^2} - \min\{c_{ii}\}\rho\|A\|(\sqrt{Nl}+N)\big]^2 + (1+\frac{1}{\varepsilon_1})(1+\frac{1}{\xi_1})2\zeta^2 q_2^2(q_1^2+q_2^2)$. For any given $\varepsilon_1, \xi_1 > 0$, if there exists constant $0 < \omega \le 1$ such that

$$\ln[\alpha + \beta + \omega(\tau_1+\tau_2)] < -\alpha T, \tag{3.55}$$

then by the IVT, there exists unique $\lambda > 0$ such that

$$\ln[\alpha + \beta e^{\lambda(\tilde{\tau}+d)} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}] = -(\lambda+\alpha)T. \tag{3.56}$$

Since $\lim_{k\to\infty} t_k = \infty$, there exists a positive integer $p > 2$ such that $t_p - d - \tilde{\tau} \ge t_0$, where $\tilde{\tau} = \max\{\tau_1, \tau_2, d\}$.
Claim that

$$V(t) \le \tilde{M} e^{-(\lambda+\alpha)(t_{k+1}-t_0)} e^{\alpha(t-t_0)}, \qquad t \in [t_k, t_{k+1}), \quad k \ge p. \tag{3.57}$$

Similar to the proof of Theorem 3.2.4, we can obtain from (3.46), (3.54) and (3.56) that

$$\begin{aligned} V(t_p) &\le [\alpha + \beta e^{\lambda(\tilde{\tau}+d)} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}]\ \tilde{M} e^{-\lambda(t_p-t_0)} \\ &= e^{-(\lambda+\alpha)(t_{p+1}-t_p)} \tilde{M} e^{-(\lambda+\alpha)(t_p-t_0)} e^{\alpha(t_p-t_0)} = \tilde{M} e^{-(\lambda+\alpha)(t_{p+1}-t_0)} e^{\alpha(t_p-t_0)}, \end{aligned}$$

which implies that (3.57) holds at $t = t_p$. For $t \in (t_p, t_{p+1})$, it follows from (3.50) that $V(t) \le V(t_p)e^{\alpha(t-t_p)} \le \tilde{M} e^{-(\lambda+\alpha)(t_{p+1}-t_0)} e^{\alpha(t-t_0)}$. Then, (3.57) holds for $t \in [t_p, t_{p+1})$, i.e., (3.57) is true for $k = p$. Suppose (3.57) is true for $k \le j$ $(j > p)$, which implies

$$V(t) \le \tilde{M} e^{-(\lambda+\alpha)(t_{k+1}-t_0)} e^{\alpha(t-t_0)}, \qquad t \in [t_k, t_{k+1}), \quad k \le j \ (j > p). \tag{3.58}$$

According to (3.54), we have $V_1(t_{j+1}) \le \alpha V(t_{j+1}^-) + \beta \sup_{s\in[-d-\tilde{\tau},0]} V(t_{j+1}^- + s)$, and we will estimate $V(t_{j+1}^- + s)$ with $s \in [-d-\tilde{\tau}, 0]$ by considering the following two cases:
Case 1: $t_{j+1} + s \in [t_{j+1} - \tilde{\tau} - d, t_p)$ for some $s \in [-d-\tilde{\tau}, 0]$, then it follows from (3.46) that

$$V(t_{j+1}+s) \le \tilde{M} e^{-\lambda(t_{j+1}+s-t_0)} \le \tilde{M} e^{-\lambda(t_{j+1}-\tilde{\tau}-d-t_0)}.$$

Case 2: $t_{j+1} + s \ge t_p$ for some $s \in [-d-\tilde{\tau}, 0]$, then $t_{j+1} + s \in [t_{\hat{k}}, t_{\hat{k}+1})$ for some positive integer $p \le \hat{k} \le j$. According to (3.58), we have

$$V(t_{j+1}+s) \le \tilde{M}e^{-(\lambda+\alpha)(t_{\hat{k}+1}-t_0)}e^{\alpha(t_{j+1}+s-t_0)} \le \tilde{M}e^{-(\lambda+\alpha)(t_{j+1}+s-t_0)}e^{\alpha(t_{j+1}+s-t_0)}$$
$$\le \tilde{M}e^{-\lambda(t_{j+1}-\tilde{\tau}-d-t_0)}.$$

Hence, we can conclude that $V(t_{j+1}+s) \le \tilde{M}e^{\lambda(\tilde{\tau}+d)}e^{-\lambda(t_{j+1}-t_0)}$ for all $s \in [-d-\tilde{\tau}, 0]$, and

$$\sup_{s\in[-d-\tilde{\tau},0]} V(t_{j+1}^- + s) \le \tilde{M}e^{\lambda(\tilde{\tau}+d)}e^{-\lambda(t_{j+1}-t_0)}.$$

This implies $V_1(t_{j+1}) \le [\alpha + \beta e^{\lambda(\tilde{\tau}+d)}]\tilde{M}e^{-\lambda(t_{j+1}-t_0)}$. From the proof of Theorem 3.2.4, we have $V_2(t_{j+1}) \le \omega\tau_1 e^{\lambda\tau_1}\tilde{M}e^{-\lambda(t_{j+1}-t_0)}$, and $V_3(t_{j+1}) \le \omega\tau_2 e^{\lambda\tau_2}\tilde{M}e^{-\lambda(t_{j+1}-t_0)}$. According to (3.56), we have

$$V(t_{j+1}) \le [\alpha + \beta e^{\lambda(\tilde{\tau}+d)} + \omega\tau_1 e^{\lambda\tau_1} + \omega\tau_2 e^{\lambda\tau_2}]\tilde{M}e^{-\lambda(t_{j+1}-t_0)}$$
$$= e^{-(\lambda+\alpha)(t_{j+2}-t_{j+1})}\tilde{M}e^{-(\lambda+\alpha)(t_{j+1}-t_0)}e^{\alpha(t_{j+1}-t_0)} = \tilde{M}e^{-(\lambda+\alpha)(t_{j+2}-t_0)}e^{\alpha(t_{j+1}-t_0)},$$

which implies that (3.57) holds at $t = t_{j+1}$. For $t \in (t_{j+1}, t_{j+2})$, it follows from (3.50) that $V(t) \le V(t_{j+1})e^{\alpha(t-t_{j+1})} \le \tilde{M}e^{-(\lambda+\alpha)(t_{j+2}-t_0)}e^{\alpha(t-t_0)}$. This proves that (3.57) is true for $t \in (t_{j+1}, t_{j+2})$. Hence (3.57) holds for $t \in [t_{j+1}, t_{j+2})$, i.e., (3.57) is true for $k = j+1$. By mathematical induction, we can prove that (3.57) is true. Then, we have

$$V_1(t) \le \tilde{M}e^{-(\lambda+\alpha)(t_{k+1}-t_0)}e^{\alpha(t_{k+1}-t_0)} = \tilde{M}e^{-\lambda(t_{k+1}-t_0)}, \qquad t \in [t_k, t_{k+1}), \quad k \ge p.$$

This implies that $V_1(t) \to 0$ as $t \to \infty$, hence $\lim_{t\to\infty}\|e_i(t)\| = 0$ for $i = 1, 2, \cdots, N$. This shows that the pinning impulsively controlled time-delay CVDN (3.8) can achieve synchronization if (3.55) holds, where $\alpha = \alpha(\varepsilon_1, \xi_1) = 1 - \frac{l}{N}[1-(1+\varepsilon_1)(1+q_1+q_2)^2]$, and $\beta = \beta(\varepsilon_1, \xi_1) = (1+\frac{1}{\varepsilon_1})(1+\xi_1)q_2^2 d^2\big[\sqrt{2L_1^2+2L_2^2} - \rho\|A\|\min\{c_{ii}\}(\sqrt{N}l+N)\big]^2 + (1+\frac{1}{\varepsilon_1})(1+\frac{1}{\xi_1})2\zeta^2 q_2^2(q_1^2+q_2^2)$. Applying Lemma 3.2.4, we can find the minimum value of $\alpha+\beta$, and $\min_{\varepsilon_1,\xi_1>0}\{\alpha+\beta\} = 1 - \frac{l}{N} + \left[\sqrt{\frac{l}{N}}(1+q_1+q_2) - q_2 d\big[\sqrt{2L_1^2+2L_2^2} - \rho\|A\|\min\{c_{ii}\}(\sqrt{N}l+N)\big] - \sqrt{2}\zeta q_2\sqrt{q_1^2+q_2^2}\right]^2 = \gamma$. Then, we can conclude that the pinning impulsively controlled time-delay CVDN (3.8) can achieve synchronization if condition (3.49) is satisfied. □

Remark 3.2.12. *Theorem 3.2.4 shows the delay-dependent conditions for synchronization of CVDNs with large delays, the size of delay τ_1 or τ_2 or d could exceed the designed impulsive distances, while the synchronization result established in Theorem 3.2.5 focuses on the case for CVDNs with large impulse delay (i.e., $d > t_k - t_{k-1}$). Furthermore, Theorem 3.2.5 shows that as long as the pinning impulsive controllers* (3.7) *are designed to satisfy*

(3.49), *the pinning impulsively controlled CVDN* (3.8) *with large impulse delay can still achieve synchronization even if* $\tau_1 + \tau_2 \geq 1$.

3.2.6 Numerical Simulations

In this subsection, we consider two examples to illustrate the effectiveness of our theoretical results.

Example 3.2.1. *Consider the CVDN* (3.6) *consisting of 3 coupled identical nodes under the pinning impulsive controller* (3.7). *Choose the complex-valued dynamical function f as*

$$f(z_i(t), z_i(t-\tau_1)) = g(z_i(t)) + \tilde{g}(z_i(t-\tau_1))$$

with $g(z_i(t)) = (g_1(z_{i1}(t)), g_2(z_{i2}(t)))^T$, $\tilde{g}(z_i(t-\tau_1)) = (\tilde{g}_1(z_{i1}(t-\tau_1)), \tilde{g}_2(z_{i2}(t-\tau_1)))^T$, *and*

$$g_n(z_{in}(t)) = \frac{1}{5}(|z_{in}^R(t)| + j|z_{in}^I(t)|), \quad z_{in}(t) = z_{in}^R(t) + jz_{in}^I(t), \quad n = 1, 2,$$
$$\tilde{g}_n(z_{in}(t-\tau_1)) = \frac{1 - e^{-z_{in}^I(t-\tau_1)}}{1 + e^{-z_{in}^I(t-\tau_1)}} + j\frac{1}{1 + e^{-z_{in}^R(t-\tau_1)}}, \quad n = 1, 2, \quad i = 1, 2, 3.$$

For any $x, y \in \mathbb{C}$, *denote* $x = x^R + jx^I$, *and* $y = y^R + jy^I$. *By the mean value theorem, we have*

$$|\tilde{g}_n(x) - \tilde{g}_n(y)| \leq \left|\frac{1 - e^{-x^I}}{1 + e^{-x^I}} - \frac{1 - e^{-y^I}}{1 + e^{-y^I}}\right| + \left|\frac{1}{1 + e^{-x^R}} - \frac{1}{1 + e^{-y^R}}\right|$$
$$\leq \frac{1}{4}|x^I - y^I| + \frac{1}{2}|x^R - y^R| \leq \sqrt{\frac{3}{8}}|x - y|,$$

hence Assumption 3.2.1 is satisfied with $L_1 = \frac{1}{5}, L_2 = \sqrt{\frac{3}{8}}$. *Choose the inner coupling matrix as* $A = I_2$, *and the outer coupling configuration matrix C as*

$$C = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{pmatrix}$$

such that $\sum_{j=1}^{3} c_{ij} \leq 0$, *for* $i = 1, 2, 3$. *The nonlinear complex-valued delayed coupling function is given by* $h(z_i(t-\tau_2)) = (h_1(z_{i1}(t-\tau_2)), h_2(z_{i2}(t-\tau_2)))^T$ *with*

$$h_n(z_{in}) = 0.05\ \overline{z_{in}}, \qquad n = 1, 2, \quad i = 1, 2, 3,$$

then, Assumption 3.2.2 is satisfied with $\rho = 0.05$. *After calculation, we can get* $c = \min\{\sum_{j=1}^{3} c_{ij}\} = -2$, $\alpha = 2L_1 + L_2 + \rho\,\|C \otimes A\| - c\rho\|A\| = 1.335$, *and* $\beta = \rho\,\|C \otimes A\| - c\rho\|A\| = 0.3226$. *Let* $\tau_1 = 0.2$, $\tau_2 = 0.15$, $d = 0.12$, *and* $l = 1$ *(i.e., only 1 node in CVDN* (3.6) *will be controlled at each impulsive instant). Choose the impulsive control gains* $q_{1k} = \bar{q}_1 = -0.9, q_{2k} = \bar{q}_2 = 0.2$, *and the impulsive distance* $T = \{t_k - t_{k-1}\} = 0.1$, $k \in \mathbb{N}^+$. *By simple calculation, we have* $\bar{a} = 1 - \frac{l}{N}[1 - 2(1 + \bar{q}_1)^2] = 0.673, \bar{b} = 2\bar{q}_2^2 = 0.08$. *We can choose* $q = 1.4 > 1$ *such that condition* (3.37) *of Corollary 3.2.2 is satisfied. Corollary 3.2.2 implies that synchronization of the pinning impulsively controlled time-delay CVDN* (3.8) *is achieved. The initial conditions of* $s(t)$ *and* $z_i(t)$ *are chosen as* $\varphi(\theta) = [1-2j, -2+j]^T, \phi_1(\theta) = [-1+2j, 2-4j]^T, \phi_2(\theta) = [1+j, -1+3j]^T, \phi_3(\theta) = [3+4j, 1-j]^T$, *for* $\theta \in [-0.2, 0]$. *Figure 3.1 shows the trajectories of the real and imaginary parts of 2-dimensional synchronization errors. From the result of the simulations in Figure 3.1, it is clearly observed that both the real and the imaginary parts of all synchronization errors converge to zero as time gets large, which implies that the pinning impulsively controlled time-delay CVDN* (3.8) *achieves synchronization.*

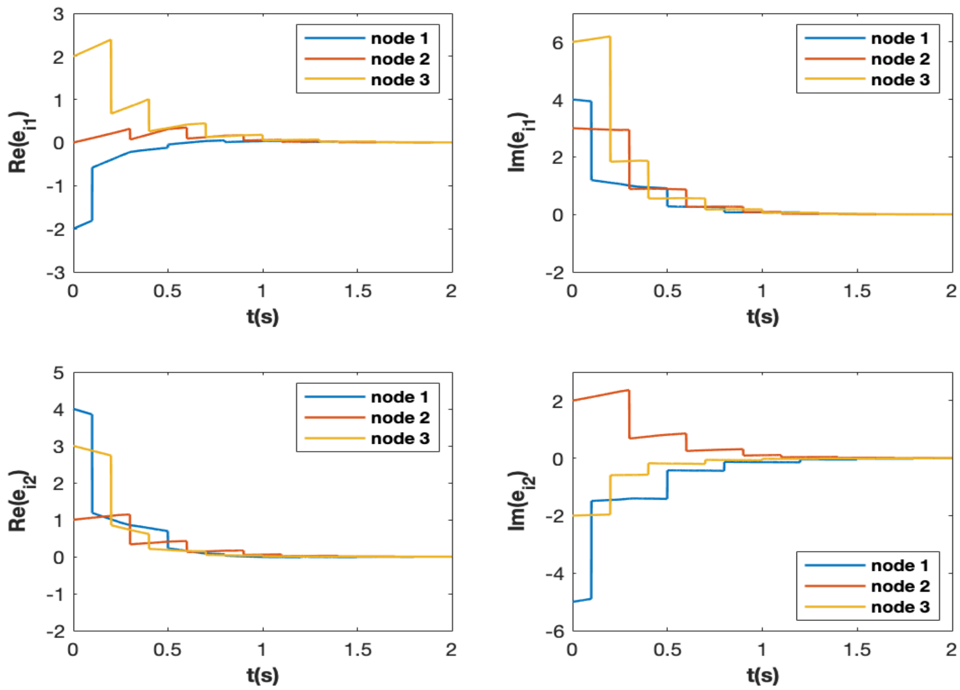

Figure 3.1: Trajectories of real and imaginary parts of 2-dimensional synchronization error system (3.9) with $\tau_1 = 0.2$, $\tau_2 = 0.15$, $d = 0.12$.

Example 3.2.2. *Consider time-delay CVDN* (3.5) *coupled with 5 identical nodes, each of which is a 2-dimensional complex-valued dynamical system. Choose the inner coupling matrix as* $A = I_2$*, and the outer coupling configuration matrix*

$$C = \begin{pmatrix} -1 & 0.1 & 0.2 & 0.3 & 0.2 \\ 0.3 & -1.2 & 0.5 & 0.1 & 0.2 \\ 0.1 & 0.3 & -0.8 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.5 & -1.5 & 0.3 \\ 0.4 & 0.1 & 0.3 & 0.2 & -1 \end{pmatrix}$$

satisfying $\sum_{j=1}^{5} c_{ij} \leq 0$*, for* $i = 1, 2, ..., 5$*. Since the outer coupling configuration matrix* C *is asymmetric, and all the off-diagonal entries of* C *are non-zero, hence the network is considered as a fully connected directed network. Figure* 3.2 *shows the network topology of time-delay CVDN* (3.5)*. The complex-valued dynamical function* $f(z_i(t), z_i(t-\tau_1))$ *and the nonlinear complex-valued coupling function* $h(z_i(t-\tau_2))$ *are chosen the same as those in Example* 3.2.1 *for* $i = 1, 2, ..., 5$*. Hence, Assumption* 3.2.1 *and Assumption* 3.2.2 *are satisfied with* $L_1 = 0.2$*,* $L_2 = \sqrt{\frac{3}{8}}$ *and* $\rho = 0.05$*. By simple calculation, we can obtain* $\min_i\{c_{ii}\} = -1.5$*,* $c = \min_i\{\sum_{j=1}^{5} c_{ij}\} = -0.4$*, and* $\|C \otimes A\| = 1.7145$*.*

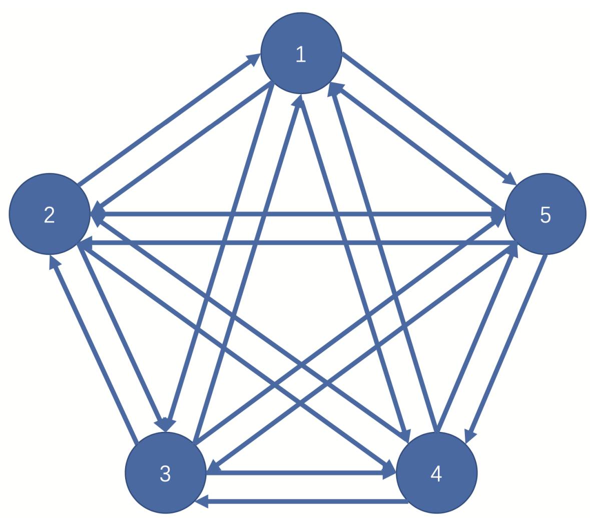

Figure 3.2: Network topology for Example 3.2.2.

Next, we consider the pinning impulsive controllers (3.7) *with* $l = 3$*, i.e., 3 nodes in CVDN* (3.5) *will be controlled at each impulsive instant. We will consider the following 3* *scenarios for the pinning impulsively controlled time-delay CVDN* (3.8) *with various sizes of delays.*

Case 1. *Let* $\tau_1 = 0.05$, $\tau_2 = 0.03$, *and* $d = 0.02$, *then* $\tilde{\tau} = \max\{\tau_1, \tau_2, d\} = 0.05s$. *Choose the impulsive control gains* $q_{1k} = -0.8, q_{2k} = 0.05$, *and the length of impulse intervals* $t_k - t_{k-1} = 0.08$, $k = 1, 2, \ldots$, *then we have* $\tilde{\tau} < t_k - t_{k-1}$. *We can choose* $\gamma = 0.1 > 0$ *such that the condition* (3.38) *of Theorem 3.2.3 holds. Thus, Theorem 3.2.3 implies that the pinning impulsively controlled CVDN* (3.8) *with small delays can achieve synchronization. The initial conditions of* $s(t)$ *and* $z_i(t)$ *are randomly chosen as* $\varphi(\theta) = [1-2j, -2+j]^T, \phi_1(\theta) = [-1+2j, 2-4j]^T, \phi_2(\theta) = [1+j, -1+3j]^T, \phi_3(\theta) = [3+4j, 1-j]^T, \phi_4(\theta) = [5-j, 2+3j]^T, \phi_5(\theta) = [3-5j, 0.5+1.5j]^T$, *for* $\theta \in [-0.05, 0]$. *Figure 3.3 shows the norm of synchronization errors. Figure 3.4 shows the time evolution of the real and imaginary parts of synchronization errors.*

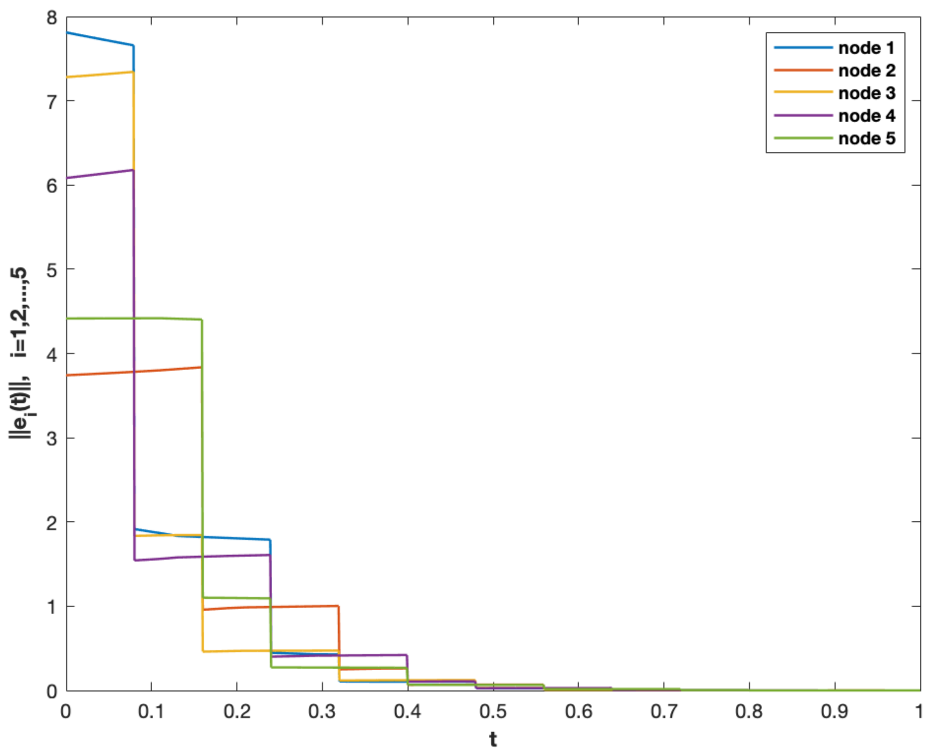

Figure 3.3: The norm of synchronization errors for time-delay CVDN (3.8) with $\tau_1 = 0.05$, $\tau_2 = 0.03$, $d = 0.02$.

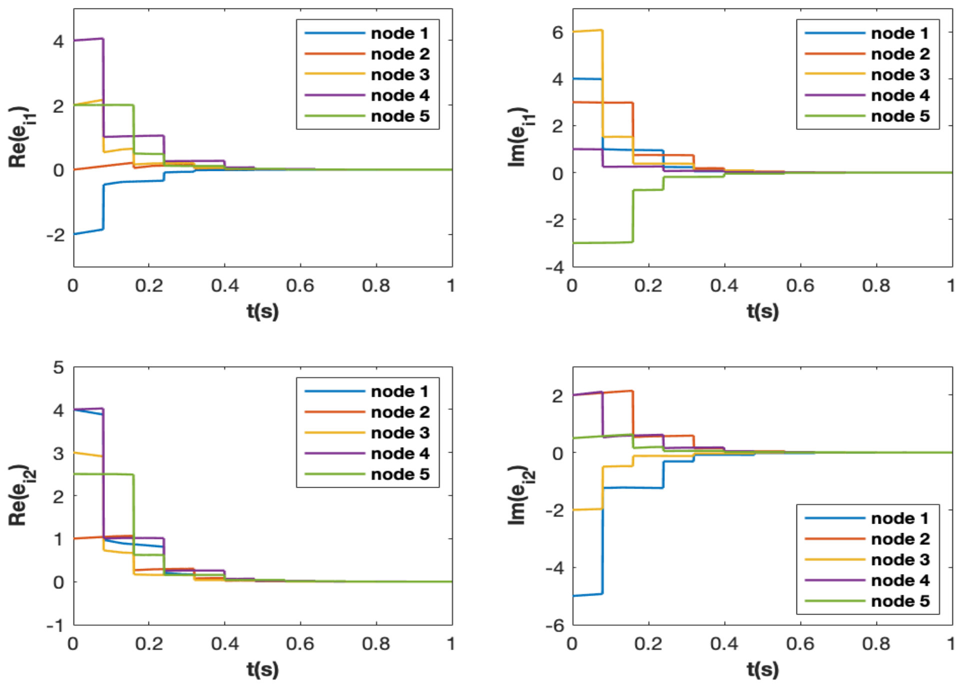

Figure 3.4: Trajectories of real and imaginary parts of 2-dimensional synchronization error system (3.9) with $\tau_1 = 0.05$, $\tau_2 = 0.03$, $d = 0.02$.

Case 2. *Let* $\tau_1 = 0.1$, $\tau_2 = 0.04$, *and* $d = 0.02$, *hence* $\tilde{\tau} = \max\{\tau_1, \tau_2, d\} = 0.1s$. *Choose the impulsive control gains* $q_{1k} = -0.8, q_{2k} = 0.1$, *and the impulsive distance* $t_k - t_{k-1} = 0.03$, $k = 1, 2, ...,$ *hence we have* $\tau_1, \tau_2 > t_k - t_{k-1}$ *and* $d < t_k - t_{k-1}$. *Then, we can choose* $\omega = 0.5 \leq 1$ *and* $\lambda = 0.1 > 0$ *such that the condition* (3.45) *of Theorem 3.2.4 holds. Theorem 3.2.4 implies that the pinning impulsively controlled CVDN* (3.8) *with relatively large delays can achieve synchronization. The initial functions of* $s(t)$ *and* $z_i(t)$ *are selected randomly by* $\varphi(\theta) = [\sin(\theta) - 2\cos(\theta)j, 2\cos(\theta) - \sin(\theta)j]^T, \phi_1(\theta) = [-\tanh(\theta)+(\sin(\theta)+0.1)j, -(\cos(\theta)+0.1)+\tanh(\theta)j]^T, \phi_2(\theta) = [-0.1\cos(\theta)+3j, 0.1\sin(\theta)+j]^T, \phi_3(\theta) = [2s+\cos(\theta)j, -1-\cos(\theta)j]^T, \phi_4(\theta) = [0.3\cos(0.6\theta)+2\sin(0.9\theta)j, -0.4\cos(6\theta)+\sin(3\theta)j]^T, \phi_5(\theta) = [\cos(\theta)-j, 2\sin(\theta)-1+2j]^T$ *for* $\theta \in [-0.1, 0]$. *The simulation results of the norm of synchronization errors and the time evolution of the real and imaginary parts of synchronization errors for the pinning impulsively controlled CVDN* (3.8) *with time delays* $\tau_1 = 0.1$, $\tau_2 = 0.04$, $d = 0.02$ *are shown in Figure 3.5* and *Figure 3.6, respectively.*
Case 3. *Let* $\tau_1 = 1.2$, $\tau_2 = 0.7$, *and* $d = 0.1$, $\tilde{\tau} = \max\{\tau_1, \tau_2, d\} = 1.2s$. *Choose the impulsive control gains* $q_1 = -0.4, q_2 = -0.01$, *and the impulsive distance* $T = t_k - t_{k-1} = 0.03$,

$k = 1, 2, \ldots$, hence $\tau_1, \tau_2, d > T$ with $\zeta = \lfloor \frac{d}{T} \rfloor = 3$, and there exists $\omega = 0.05 \leq 1$ such that condition (3.49) of Theorem 3.2.5 is satisfied. Theorem 3.2.5 implies that the pinning impulsively controlled CVDN (3.8) with relatively large impulse delay can achieve synchronization. The initial functions $\varphi(\theta)$ and $\phi_i(\theta)$ are chosen the same as those in Case 2 for $\theta \in [-1.2, 0]$. Figure 3.7 and Figure 3.8 show the simulation results related to synchronization errors of the pinning impulsively controlled CVDN (3.8) with time delays $\tau_1 = 1.2$, $\tau_2 = 0.7$, and $d = 0.1$.

Clearly, all synchronization errors converge to zero rapidly. The simulation results show that the synchronization criteria established in Theorem 3.2.3, Theorem 3.2.4, and Theorem 3.2.5 for CVDN (3.5) under pinning impulsive controllers (3.7) with various sizes of delays are effective.

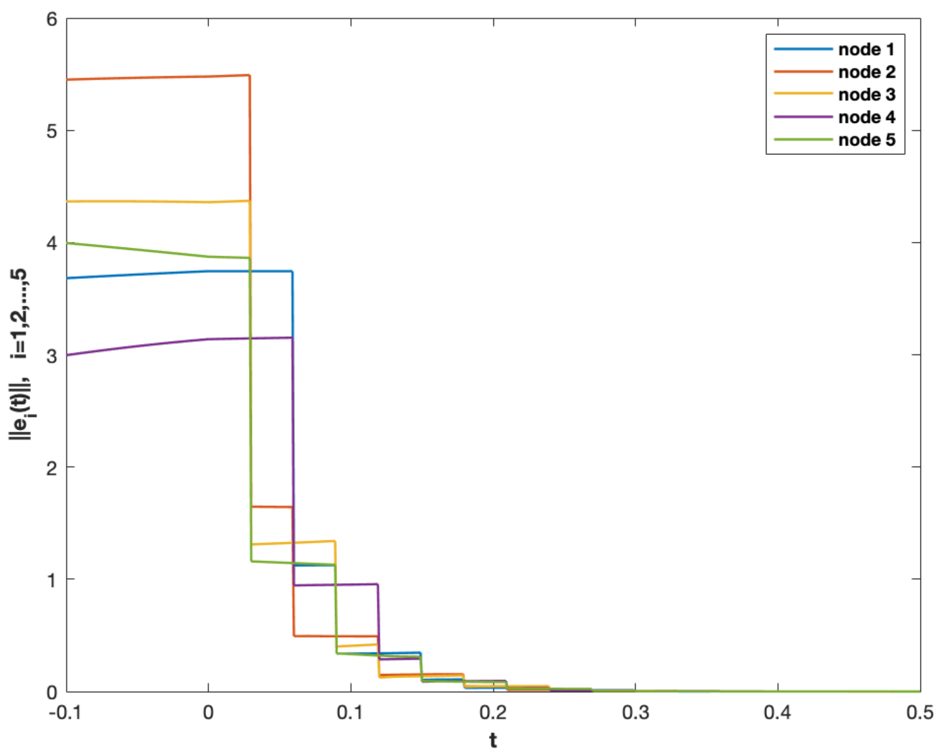

Figure 3.5: The norm of synchronization errors for time-delay CVDN (3.8) with $\tau_1 = 0.1$, $\tau_2 = 0.04$, $d = 0.02$.

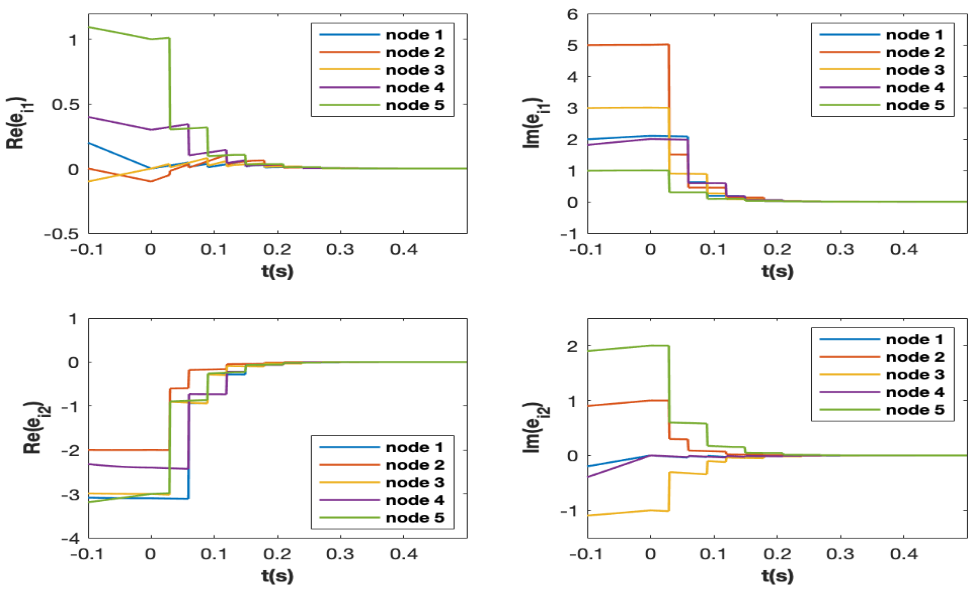

Figure 3.6: Trajectories of real and imaginary parts of 2-dimensional synchronization error system (3.9) with $\tau_1 = 0.1$, $\tau_2 = 0.04$, $d = 0.02$.

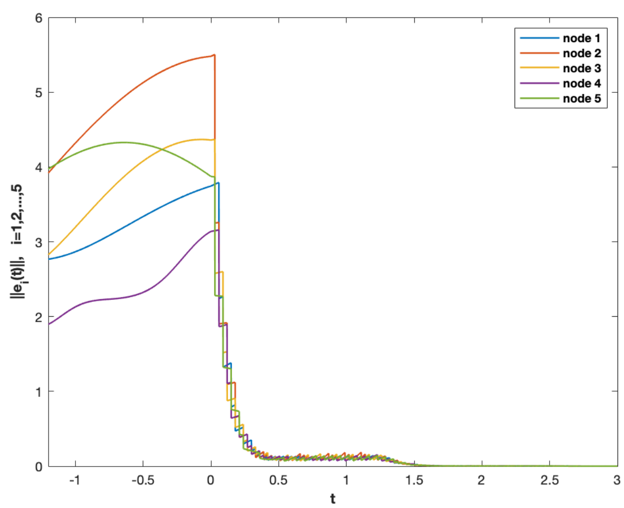

Figure 3.7: The norm of synchronization errors for time-delay CVDN (3.8) with $\tau_1 = 1.2$, $\tau_2 = 0.7$, $d = 0.1$.

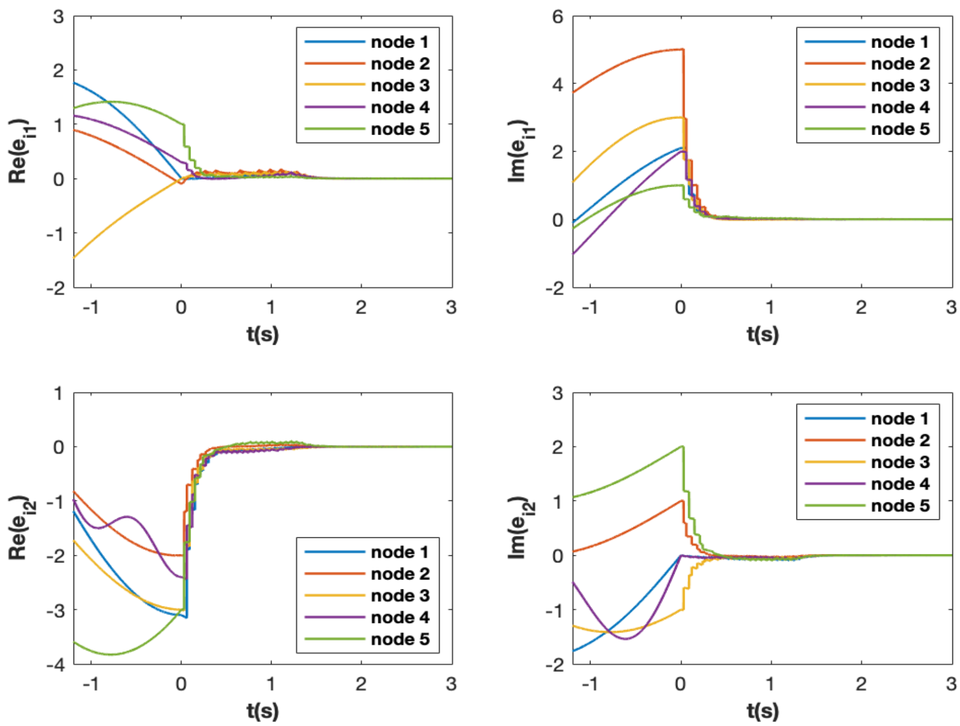

Figure 3.8: Trajectories of real and imaginary parts of 2-dimensional synchronization error system (3.9) with $\tau_1 = 1.2$, $\tau_2 = 0.7$, $d = 0.1$.

3.3 Synchronization of CVDNs with Time-Varying Coupling Delay via Distributed Impulsive Control

This section studies the synchronization problem of CVDNs with time-varying coupling delay under distributed impulsive controller. We consider two types of time-varying delays: 1) the delay is bounded but has no restriction on the delay derivative; 2) the delay is bounded and its derivative is strictly less than one. In Subsection 3.3.2, we formulate the problem of synchronization for CVDNs with time-varying coupling delay, and propose the distributed impulsive controller in the complex field. In Subsection 3.3.3, we present several LMI-based synchronization results for CVDNs by taking advantage of time-varying Lyapunov function/functional in the complex field.

3.3.1 Distributed Impulsive Control

There are two types of impulsive control strategies that are utilized to study the control problems of networked systems: decentralized impulsive control strategy and distributed impulsive control strategy. Decentralized impulsive control means that each node is controlled by a local impulsive control scheme, where only local state information of the node is used when designing impulsive controllers. In the decentralized impulsive control scheme, the local impulsive controller of each node ignores the state information of its neighboring nodes, thus this kind of impulsive control is more suitable for the weakly coupled dynamical networks. For networks with strong coupling, the distributed impulsive control scheme is efficient since such control scheme uses the state information of each node along with all its neighboring nodes at impulse instants, it may assure synchronization of the strongly coupled networks than the decentralized impulsive control. To our best knowledge, the distributed impulsive control strategy has not been used to study synchronization of CVDNs, even for the delay-free case. In this section, we investigate synchronization of CVDNs with time-varying delay under distributed impulsive controller.

3.3.2 Problem Formulation

Consider a CVDN coupled with N identical nodes with each node being a 1-dimensional complex-valued dynamical system, which is described as follows:

$$\dot{z}_i(t) = h(z_i(t)) + \sum_{j=1}^{N} a_{ij} f(z_j(t)) + \sum_{j=1}^{N} b_{ij} f(z_j(t-\tau(t))) + u_i(t), \quad i = 1, 2, \ldots, N, \tag{3.59}$$

where $z_i(t) \in \mathbb{C}$ represents the state variable of the i-th node, $h : \mathbb{C} \to \mathbb{C}$ is the nonlinear complex-valued dynamical function describing the node's intrinsic dynamics, $f : \mathbb{C} \to \mathbb{C}$ is the nonlinear complex-valued coupling function, $\tau(t)$ denotes the time-varying coupling delay satisfying $0 \leq \tau(t) \leq \tau$ for some positive scalar τ, $u_i(t)$ represents the control input of the i-th node, $A = (a_{ij}) \in \mathbb{C}^{N\times N}$ and $B = (b_{ij}) \in \mathbb{C}^{N\times N}$ represent the non-delay complex outer coupling configuration matrix and delay complex outer coupling configuration matrix, respectively, defined as: if there is a connection from node i to node j $(i \neq j)$ at time t (at time $t - \tau(t)$), then $a_{ij} \neq 0$ ($b_{ij} \neq 0$); otherwise $a_{ij} = 0$ ($b_{ij} = 0$), and the diagonal entries are given by $a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij}$ ($b_{ii} = -\sum_{j=1,j\neq i}^{N} b_{ij}$), for $i = 1, 2, \ldots, N$.

The objective is to investigate synchronization of CVDN (3.59) under the following distributed impulsive controller:

$$u_i(t) = \sum_{k=1}^{\infty}\sum_{j\in\mathcal{N}_i} l_{ij}\big(z_j(t) - z_i(t)\big)\delta(t-t_k), \qquad i = 1, 2, \ldots, N, \tag{3.60}$$

where $L = (l_{ij}) \in \mathbb{C}^{N\times N}$ is the control gain representing the complex coupling strength and the topological structure of the controller, $\mathcal{N}_i$ denotes the set of nodes which connects to the i-th node, $\{t_k\}$ is the impulsive time sequence satisfying $0 < t_1 < t_2 < ... < t_k < t_{k+1} < ...$, and $\lim_{k\to\infty} t_k = \infty$, and $\delta(\cdot)$ is the Dirac delta function. The gain matrix L satisfies the conditions that $l_{ii} = -\sum_{j=1,j\neq i}^{N} l_{ij}$, and $l_{ij} = 0$, if $j \notin \mathcal{N}_i$ and $j \neq i$.

Based on the properties of the Dirac delta function, CVDN (3.59) can be described by the following impulsive system:

$$\begin{cases} \dot{z}_i(t) = h(z_i(t)) + \sum_{j=1}^{N} a_{ij} f(z_j(t)) + \sum_{j=1}^{N} b_{ij} f(z_j(t-\tau(t))), \quad t \in [t_{k-1}, t_k), \\ \Delta z_i(t_k) = \sum_{j\in\mathcal{N}_i} l_{ij}(z_j(t_k^-) - z_i(t_k^-)), \qquad k \in \mathbb{N}^+, \\ z_i(t_0 + s) = \phi_i(s), \qquad i = 1, 2, \ldots, N, \end{cases} \tag{3.61}$$

where $\phi_i \in \mathcal{PC}([-\tau, 0], \mathbb{C})$ is the initial function. In the following discussion, all solutions of (3.61) are assumed to be right continuous, i.e., $z_i(t_k) = z_i(t_k^+),\ i = 1, 2, ..., N$.

Then system (3.61) can be rewritten into a matrix-form impulsive system:

$$\begin{cases} \dot{z}(t) = \bar{h}(z(t)) + A\bar{f}(z(t)) + B\bar{f}(z(t-\tau(t))), \qquad t \in [t_{k-1}, t_k), \\ \Delta z(t_k) = Lz(t_k^-), \qquad k \in \mathbb{N}^+, \\ z(t_0 + s) = \phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{3.62}$$

where $z = (z_1, z_2, ..., z_N)^T$, $\bar{h}(z) = (h(z_1), h(z_2), ..., h(z_N))^T$, $\bar{f}(z) = (f(z_1), f(z_2), ..., f(z_N))^T$, and $\phi = (\phi_1, \phi_2, ..., \phi_N)^T$.

Define the synchronization error as: $e_i(t) = z_i(t) - z_{i+1}(t),\ i = 1, 2, ..., N-1$. Let $e = (e_1, e_2, ..., e_{N-1})^T$. Define the two matrices G and M as following:

$$G = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times N}, \qquad M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{N\times(N-1)},$$

we can verify that the following properties hold:

$$Gz(t) = e(t), \quad A = AMG, \quad B = BMG, \quad L = LMG. \tag{3.63}$$

According to (3.62) and (3.63), we can obtain the following error system:

$$\begin{cases} \dot{e}(t) = H(e(t)) + (GAM)F(e(t)) + (GBM)F(e(t-\tau(t))), & t \in [t_{k-1}, t_k), \\ \Delta e(t_k) = GLMe(t_k^-) & k \in \mathbb{N}^+, \\ e(t_0 + s) = \psi(s), \quad s \in [-\tau, 0], \end{cases} \tag{3.64}$$

where $H(e) = \big(h(z_1)-h(z_2), h(z_2)-h(z_3), ..., h(z_{N-1})-h(z_N)\big)^T$, $F(e) = \big(f(z_1)-f(z_2), f(z_2)-f(z_3), ..., f(z_{N-1}) - f(z_N)\big)^T$, and $\psi(s) = G\phi(s)$.

In order to study synchronization of the distributed impulsively controlled CVDN (3.59), we will firstly present the following assumption and lemmas.

Assumption 3.3.1. *Suppose that there exists positive constants κ_1, κ_2 such that*

$$|h(u_1) - h(u_2)| \leq \kappa_1 |u_1 - u_2|, \qquad |f(v_1) - f(v_2)| \leq \kappa_2 |v_1 - v_2|$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{C}$.

Lemma 3.3.1. *Let $a, b \in \mathbb{C}^n$ and $P \in \mathbb{C}^{n\times n}$ be a positive definite Hermitian matrix, then $a^*b + b^*a \leq a^*P^{-1}a + b^*Pb$.*

Lemma 3.3.2. *[87] Given constants $p \in \mathbb{R}$, $q \geq 0$, $\delta > 1$, and function $f \in \mathcal{PC}([t_0 - \tau, +\infty), \mathbb{R}^+)$ satisfies*

$$\begin{cases} D^+ f(t) \leq pf(t) + q\bar{f}(t), & t \neq t_k, \quad t \geq t_0, \\ f(t_k) \leq \dfrac{f(t_k^-)}{\delta}, & k \in \mathbb{N}^+, \end{cases} \tag{3.65}$$

where $\bar{f}(t) = \sup_{-\tau \leq s \leq 0} f(t+s)$. If

$$p + q\delta < \frac{\ln \delta}{\sigma}, \quad \textit{where} \quad \sigma := \sup_{k\in\mathbb{N}} \{t_{k+1} - t_k\},$$

and suppose

$$0 < \lambda < \frac{\ln \delta}{\sigma} - p - q\delta e^{\lambda\tau}.$$

Then any solution of (3.65) *satisfies*

$$f(t) \leq \delta \bar{f}(t_0) e^{-\lambda(t-t_0)}, \qquad t \geq t_0.$$

3.3.3 Synchronization Results

In this subsection, we will present several LMI based synchronization results for CVDN (3.59) under the distributed impulsive controller (3.60). For CVDN (3.59), we consider the following two types of uncertain time-varying coupling delays:

Case 1. The time-varying delay $\tau(t)$ satisfies the following assumption:

Assumption 3.3.2. $\tau(t)$ *is a continuous function satisfying*

$$0 \leq \tau(t) \leq \tau \ \text{ for any } \ t \geq 0.$$

Case 2. The time-varying delay $\tau(t)$ satisfies the following assumption:

Assumption 3.3.3. $\tau(t)$ *is a continuously differentiable function satisfying*

$$0 \leq \tau(t) \leq \tau, \qquad \dot{\tau}(t) \leq r < 1 \ \text{ for any } \ t \geq 0.$$

We firstly present the synchronization result for CVDN (3.59) under distributed impulsive controller (3.60) and Assumption 3.3.2 by taking advantage of the Lyapunov function in the complex field.

Theorem 3.3.1. *Suppose that Assumption 3.3.1 and Assumption 3.3.2 are satisfied. Given positive constants α, β, and $\delta > 1$, if there exist an $(N-1) \times (N-1)$ positive definite Hermitian matrix P, three $(N-1) \times (N-1)$ positive diagonal matrices Q, R and S such that the following LMIs hold:*

$$\kappa_2^2 Q \leq \beta P, \tag{3.66}$$

$$\begin{bmatrix} \kappa_1^2 R + \kappa_2^2 S - \alpha P & P & P(GAM) & P(GBM) \\ \star & -R & 0 & 0 \\ \star & \star & -S & 0 \\ \star & \star & \star & -Q \end{bmatrix} \leq 0, \tag{3.67}$$

$$\begin{bmatrix} \frac{P}{\delta} & (I_{N-1} + GLM)^* P \\ \star & P \end{bmatrix} \geq 0, \tag{3.68}$$

and $\alpha + \beta\delta < \frac{\ln \delta}{\sigma}$, *where* $\sigma = \sup_{k \in \mathbb{N}} \{t_{k+1} - t_k\}$, *then CVDN (3.59) can achieve synchronization under the distributed impulsive controller (3.60).*

Proof. Applying the Schur complement on (3.67), yields:

$$\begin{bmatrix} \kappa_1^2 R + \kappa_2^2 S - \alpha P & P & P(GAM) \\ \star & -R & 0 \\ \star & \star & -S \end{bmatrix} + \begin{bmatrix} P(GBM) \\ 0 \\ 0 \end{bmatrix} Q^{-1} \begin{bmatrix} (GBM)^* P & 0 & 0 \end{bmatrix} \leq 0,$$

then we have,

$$\begin{bmatrix} \kappa_1^2 R + \kappa_2^2 S + P(GBM)Q^{-1}(GBM)^* P - \alpha P & P & P(GAM) \\ \star & -R & 0 \\ \star & \star & -S \end{bmatrix} \leq 0. \tag{3.69}$$

Consider the Lyapunov function for synchronization error system (3.64) as follows:

$$V(t) = e^*(t) P e(t).$$

For $t \in [t_{k-1}, t_k)$, calculate the derivative of $V(t)$ along the solution of error system (3.64), we can get from Lemma 3.3.1 that

$$\begin{aligned}
&\dot{V}(t) = \dot{e}^*(t) P e(t) + e^*(t) P \dot{e}(t) \\
&= H^*(e(t)) P e(t) + F^*(e(t))(GAM)^* P e(t) + F^*(e(t - \tau(t)))(GBM)^* P e(t) + e^*(t) P H(e(t)) \\
&+ e^*(t) P(GAM) F(e(t)) + e^*(t) P(GBM) F(e(t - \tau(t))) \\
&\leq H^*(e(t)) P e(t) + e^*(t) P H(e(t)) + F^*(e(t))(GAM)^* P e(t) + e^*(t) P(GAM) F(e(t)) \\
&+ F^*(e(t - \tau(t))) Q F(e(t - \tau(t))) + e^*(t) P(GBM) Q^{-1} (GBM)^* P e(t).
\end{aligned}$$

Let $Q = diag(q_1, q_2, ..., q_{N-1})$ be the positive diagonal matrix, then we can obtain from Assumption 3.3.1 that

$$\begin{aligned}
&F^*(e(t - \tau(t))) Q F(e(t - \tau(t))) = \sum_{i=1}^{N-1} q_i \big| f(z_i(t - \tau(t))) - f(z_{i+1}(t - \tau(t))) \big|^2 \\
&\leq \sum_{i=1}^{N-1} q_i \kappa_2^2 \big| e_i(t - \tau(t)) \big|^2 = \kappa_2^2 e^*(t - \tau(t)) Q e(t - \tau(t)).
\end{aligned}$$

Similarly, for positive diagonal matrices R and S, we can get

$$H^*(e(t))RH(e(t)) \leq \kappa_1^2 e^*(t)Re(t),$$
$$F^*(e(t))SF(e(t)) \leq \kappa_2^2 e^*(t)Se(t).$$

Hence, we have

$$\begin{aligned}
\dot{V}(t) &\leq e^*(t)\big[P(GBM)Q^{-1}(GBM)^*P\big]e(t) + F^*(e(t))(GAM)^*Pe(t) + e^*(t)P(GAM)F(e(t)) \\
&+ H^*(e(t))Pe(t) + e^*(t)PH(e(t)) + \kappa_2^2 e^*(t-\tau(t))Qe(t-\tau(t)) \\
&= e^*(t)\big[P(GBM)Q^{-1}(GBM)^*P - \alpha P\big]e(t) + H^*(e(t))Pe(t) + e^*(t)PH(e(t)) \\
&+ F^*(e(t))(GAM)^*Pe(t) + e^*(t)P(GAM)F(e(t)) + \alpha e^*(t)Pe(t) + \kappa_2^2 e^*(t-\tau(t))Qe(t-\tau(t))
\end{aligned}$$

Denote $\eta(t) = (e^T(t), H^T(e(t)), F^T(e(t)))^T$. From condition (3.66),(3.69) and Assumption 3.3.1, we can conclude that

$$\begin{aligned}
\dot{V}(t) &\leq \eta^*(t)\begin{bmatrix} P(GBM)Q^{-1}(GBM)^*P - \alpha P & P & P(GAM) \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix}\eta(t) + \alpha e^*(t)Pe(t) \\
&+ \kappa_2^2 e^*(t-\tau(t))Qe(t-\tau(t)) \\
&= \eta^*(t)\begin{bmatrix} P(GBM)Q^{-1}(GBM)^*P - \alpha P & P & P(GAM) \\ \star & -R & 0 \\ \star & \star & -S \end{bmatrix}\eta(t) + \alpha e^*(t)Pe(t) \\
&+ H^*(e(t))RH(e(t)) + F^*(e(t))SF(e(t)) + \kappa_2^2 e^*(t-\tau(t))Qe(t-\tau(t)) \\
&\leq \eta^*(t)\begin{bmatrix} P(GBM)Q^{-1}(GBM)^*P - \alpha P & P & P(GAM) \\ \star & -R & 0 \\ \star & \star & -S \end{bmatrix}\eta(t) + \alpha e^*(t)Pe(t) \\
&+ \kappa_1^2 e^*(t)Re(t) + \kappa_2^2 e^*(t)Se(t) + \beta e^*(t-\tau(t))Pe(t-\tau(t)) \\
&= \eta^*(t)\begin{bmatrix} \kappa_1^2 R + \kappa_2^2 S + P(GBM)Q^{-1}(GBM)^*P - \alpha P & P & P(GAM) \\ \star & -R & 0 \\ \star & \star & -S \end{bmatrix}\eta(t) + \alpha V(t) \\
&+ \beta V(t-\tau(t)) \\
&\leq \alpha V(t) + \beta[V(t)]_\tau.
\end{aligned}$$

From (3.68), we can apply Schur complement to obtain

$$\frac{P}{\delta} - (I_{N-1} + GLM)^*PP^{-1}P(I_{N-1} + GLM) \geq 0,$$

then, we have

$$(I_{N-1} + GLM)^* P (I_{N-1} + GLM) \leq \frac{P}{\delta}. \tag{3.70}$$

When $t = t_k$, $k \in \mathbb{N}^+$, we can get from (3.64) that $e(t_k) = (I_{N-1} + GLM)e(t_k^-)$, then it follows from (3.70) that

$$\begin{aligned} V(t_k) &= e^*(t_k)Pe(t_k) \\ &= e^*(t_k^-)(I_{N-1} + GLM)^* P(I_{N-1} + GLM)e(t_k^-) \\ &\leq \frac{1}{\delta} e^*(t_k^-)Pe(t_k^-) = \frac{1}{\delta} V(t_k^-). \end{aligned}$$

If $\alpha + \beta\delta < \frac{\ln\delta}{\sigma}$, where $\sigma = \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\}$, define $f(\lambda) = \frac{\ln\delta}{\sigma} - \alpha - \beta\delta e^{\lambda\tau} - \lambda$, then we have

$$f(0) = \frac{\ln\delta}{\sigma} - \alpha - \beta\delta > 0, \quad f(+\infty) < 0, \quad f'(\lambda) = -\tau\beta\delta e^{\lambda\tau} - 1 < 0.$$

By the IVT, there exists unique $\lambda^* > 0$ such that $f(\lambda^*) = 0$, hence there exists $\lambda \in (0, \lambda^*)$ such that $f(\lambda) > 0$, which implies $\lambda < \frac{\ln\delta}{\sigma} - \alpha - \beta\delta e^{\lambda\tau}$.

Applying Lemma 3.3.2, we can obtain:

$$V(t) \leq \delta \|V(t_0)\|_\tau e^{-\lambda(t-t_0)}, \qquad t \geq t_0.$$

Since $V(t) \geq \lambda_{\min}(P)\|e(t)\|^2$, and

$$\|V(t_0)\|_\tau = \sup_{s\in[-\tau,0]} e^*(t_0+s)Pe(t_0+s) \leq \lambda_{\max}(P) \sup_{s\in[-\tau,0]} \|e(t_0+s)\|^2 = \lambda_{\max}(P)\|e(t_0)\|_\tau^2,$$

then, we have

$$\|e(t)\| \leq \sqrt{\frac{\delta\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|e(t_0)\|_\tau e^{-\frac{\lambda}{2}(t-t_0)}, \qquad t \geq t_0.$$

This implies that $\|e(t)\| \to 0$ as $t \to \infty$, hence we can conclude that the CVDN (3.59) can achieve synchronization under the distributed impulsive controller (3.60). □

Next, we will establish the LMI based synchronization result for the controlled CVDN (3.59) under Assumption 3.3.2 by using the time-varying Lyapunov function method and Razumikhin technique. For this purpose, we will introduce several piecewise functions related to impulse time sequences. For any given impulse time sequence $\{t_k\}$, define

$$\bar{\rho}(t) = \frac{1}{t_k - t_{k-1}}, \quad \rho_{11}(t) = (t - t_{k-1})\bar{\rho}(t), \quad \rho_{12}(t) = (t_k - t)\bar{\rho}(t), \quad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+,$$

then it is obvious that $\rho_{11}(t) \in [0, 1)$, and $\rho_{12}(t) \in (0, 1]$ for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$, and

$$\rho_{11}(t_k) = \rho_{12}(t_k^-) = 0, \quad \rho_{11}(t_k^-) = \rho_{12}(t_k) = 1, \quad k \in \mathbb{N}^+.$$

If the impulse sequence $\{t_k\}$ is uniformly bounded, i.e., $m_1 \leq t_k - t_{k-1} \leq m_2$, $\forall k \in \mathbb{N}^+$, define $\rho_{21}(t) : \mathbb{R}^+ \to [0, 1]$ such that

$$\bar{\rho}(t) = \frac{1}{m_1}\rho_{21}(t) + \frac{1}{m_2}\rho_{22}(t),$$

where $\rho_{22}(t) = 1 - \rho_{21}(t)$. Furthermore, define

$$\rho_{3l}(t) = \begin{cases} \rho_{1l}(t - \tau(t)), & \text{if } t - \tau(t) \geq t_0, \\ \dfrac{1}{2}, & \text{if } t - \tau(t) < t_0, \end{cases} \qquad l = 1, 2.$$

Theorem 3.3.2. *Suppose that Assumption 3.3.1 and Assumption 3.3.2 hold. And suppose that the impulse sequence $\{t_k\}$ satisfies $m_1 \leq t_k - t_{k-1} \leq m_2$, $\forall k \in \mathbb{N}^+$. If there exist positive constants $\beta_j, j = 1, 2$, α_{ijl}, γ_{ijl}, $i, j, l = 1, 2$, and $(N-1) \times (N-1)$ positive definite Hermitian matrices $P_i, i = 1, 2$, such that the following LMIs hold:*

$$\begin{bmatrix} \phi_{ijl} & 0 & P_iGAM & P_i & P_iGBM \\ \star & \kappa_2^2 I_{N-1} - \beta_j P_l & 0 & 0 & 0 \\ \star & \star & -\gamma_{ijl} I_{N-1} & 0 & 0 \\ \star & \star & \star & -\alpha_{ijl} I_{N-1} & 0 \\ \star & \star & \star & \star & -I_{N-1} \end{bmatrix} < 0, \qquad i, j, l = 1, 2, \tag{3.71}$$

$$\begin{bmatrix} -P_1 & (I + GLM)^* P_2 \\ \star & -P_2 \end{bmatrix} \leq 0, \tag{3.72}$$

where $\phi_{ijl} = \frac{1}{m_j}(P_1 - P_2) + \kappa_1^2 \alpha_{ijl} I_{N-1} + \kappa_2^2 \gamma_{ijl} I_{N-1} + \beta_j P_i$, then the CVDN (3.59) can achieve synchronization under the distributed impulsive controller (3.60).

Proof. According to (3.71), there exist small enough constant $c > 0$ such that

$$\Omega_{ijl} = \begin{bmatrix} \phi_{ijl} + 2cP_i & 0 & P_iGAM & P_i & P_iGBM \\ \star & \kappa_2^2 I_{N-1} - e^{-2c\tau}\beta_j P_l & 0 & 0 & 0 \\ \star & \star & -\gamma_{ijl} I_{N-1} & 0 & 0 \\ \star & \star & \star & -\alpha_{ijl} I_{N-1} & 0 \\ \star & \star & \star & \star & -I_{N-1} \end{bmatrix} < 0.$$

Define

$$\Omega(t) = \sum_{i,j,l=1}^{2} \rho_{1i}(t)\rho_{2j}(t)\rho_{3l}(t)\Omega_{ijl},$$

and denote

$$P(t) = \sum_{i=1}^{2} \rho_{1i}(t)P_i, \quad \alpha(t) = \sum_{i,j,l=1}^{2} \rho_{1i}(t)\rho_{2j}(t)\rho_{3l}(t)\alpha_{ijl},$$
$$\beta(t) = \sum_{j=1}^{2} \rho_{2j}(t)\beta_j, \quad \gamma(t) = \sum_{i,j,l=1}^{2} \rho_{1i}(t)\rho_{2j}(t)\rho_{3l}(t)\gamma_{ijl}.$$

Then, we can rewrite $\Omega(t)$ as

$$\Omega(t) = \begin{bmatrix} \Omega_{11}(t) & 0 & P(t)GAM & P(t) & P(t)GBM \\ \star & \Omega_{22}(t) & 0 & 0 & 0 \\ \star & \star & -\gamma(t)I_{N-1} & 0 & 0 \\ \star & \star & \star & -\alpha(t)I_{N-1} & 0 \\ \star & \star & \star & \star & -I_{N-1} \end{bmatrix} < 0,$$

where $\Omega_{11}(t) = 2cP(t) + \bar{\rho}(t)(P_1 - P_2) + \alpha(t)\kappa_1^2 I_{N-1} + \gamma(t)\kappa_2^2 I_{N-1} + \beta(t)P(t), \Omega_{22}(t) = \kappa_2^2 I_{N-1} - e^{-2c\tau}\beta(t)P(t - \tau(t))$.

By Schur complement, we have

$$\begin{bmatrix} \widetilde{\Omega}_{11}(t) & 0 & P(t)GAM & P(t) \\ \star & \Omega_{22}(t) & 0 & 0 \\ \star & \star & -\gamma(t)I_{N-1} & 0 \\ \star & \star & \star & -\alpha(t)I_{N-1} \end{bmatrix} < 0, \qquad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+, \tag{3.73}$$

where $\widetilde{\Omega}_{11}(t) = \Omega_{11}(t) + P(t)GBM(GBM)^*P(t)$.

Choose the time-varying Lyapunov function candidate for the synchronization error system (3.64) as follows:

$$V(t,e) = e^*(t)P(t)e(t).$$

Denote $\lambda_1 = \max_{i=1,2}\{\lambda_{\max}(P_i)\}$, $\lambda_0 = \min_{i=1,2}\{\lambda_{\min}(P_i)\}$, then there exists a positive constant $\epsilon > 1$ such that $\lambda_1 < \epsilon\lambda_0$. Define

$$W(t) = e^{2c(t-t_0)}V(t,e(t)), \qquad t \in [t_0-\tau, +\infty).$$

For $t \in [t_0-\tau, t_0]$, denote $W(t) = W(t_0+s)$, for $s \in [-\tau, 0]$, we have

$$\begin{aligned} W(t_0+s) &= e^{2cs}e^*(t_0+s)P(t_0+s)e(t_0+s) \\ &\leq \sum_{i=1}^{2} \lambda_{\max}(P_i)\rho_{1i}(t_0+s)e^*(t_0+s)e(t_0+s) \\ &\leq \lambda_1 \sup_{s\in[-\tau,0]} \|\psi(s)\|^2 \leq \epsilon\lambda_0\|\psi\|_\tau^2, \end{aligned}$$

hence, we can get

$$W(t) \leq \epsilon\lambda_0\|\psi\|_\tau^2, \qquad t \in [t_0-\tau, t_0]. \tag{3.74}$$

In the following, we shall prove

$$W(t) \leq \epsilon\lambda_0\|\psi\|_\tau^2, \qquad t \geq t_0. \tag{3.75}$$

First, we will prove that

$$W(t) \leq \epsilon\lambda_0\|\psi\|_\tau^2, \qquad t \in (t_0, t_1). \tag{3.76}$$

Suppose that (3.76) is not true, then there exists $t \in (t_0, t_1)$ such that $W(t) > \epsilon\lambda_0\|\psi\|_\tau^2$. Define $t^* = \inf\{t \in (t_0,t_1) : W(t) > \epsilon\lambda_0\|\psi\|_\tau^2\}$. By the continuity of $W(t)$, we have

$$W(t^*) = \epsilon\lambda_0\|\psi\|_\tau^2, \qquad W(t) < \epsilon\lambda_0\|\psi\|_\tau^2, \qquad t \in (t_0, t^*),$$

and

$$\dot{W}(t^*) \geq 0. \tag{3.77}$$

Hence, from (3.74), we have

$$W(t) \le \epsilon \lambda_0 \|\psi\|_\tau^2, \qquad t \in [t_0 - \tau, t^*].$$

This shows that

$$W(t^*) \ge W(t^* + s), \qquad s \in [-\tau, 0],$$

which implies that for any $s \in [-\tau, 0]$, $0 \le W(t^*) - W(t^* + s)$. Let $s = -\tau(t^*) \in [-\tau, 0]$, according to Assumption 3.3.2, we can get

$$\begin{aligned}
0 &\le e^{2c(t^*-t_0)} e^*(t^*)P(t^*)e(t^*) - e^{2c(t^*-\tau(t^*)-t_0)} e^*(t^* - \tau(t^*))P(t^* - \tau(t^*))e(t^* - \tau(t^*)) \\
&\le e^{2c(t^*-t_0)}[e^*(t^*)P(t^*)e(t^*) - e^{-2c\tau} e^*(t^* - \tau(t^*))P(t^* - \tau(t^*))e(t^* - \tau(t^*))].
\end{aligned}$$

This implies that $e^*(t^*)P(t^*)e(t^*) - e^{-2c\tau} e^*(t^* - \tau(t^*))P(t^* - \tau(t^*))e(t^* - \tau(t^*)) \ge 0$. By the definition of $\beta(t)$, we have $\beta(t) > 0$ for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$, hence we can get at $t = t^*$,

$$0 \le \beta(t^*)[e^*(t^*)P(t^*)e(t^*) - e^{-2c\tau} e^*(t^* - \tau(t^*))P(t^* - \tau(t^*))e(t^* - \tau(t^*))]. \tag{3.78}$$

It follows from Assumption 3.3.1 that

$$H^*(e(t))H(e(t)) = \sum_{i=1}^{N-1} \big|h(z_i(t)) - h(z_{i+1}(t))\big|^2 \le \sum_{i=1}^{N-1} \kappa_1^2 \big|z_i(t) - z_{i+1}(t)\big|^2 = \kappa_1^2 e^*(t)e(t),$$

then we have $\kappa_1^2 e^*(t)e(t) - H^*(e(t))H(e(t)) \ge 0$. By the definition of $\alpha(t)$, we have $\alpha(t) > 0$ for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$, then we can get at $t = t^*$,

$$0 \le \alpha(t^*)[\kappa_1^2 e^*(t^*)e(t^*) - H^*(e(t^*))H(e(t^*))]. \tag{3.79}$$

Similarly, we have $F^*(e(t))F(e(t)) \le \kappa_2^2 e^*(t)e(t)$, and $\gamma(t) > 0$, $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$. Then at $t = t^*$,

$$0 \le \gamma(t^*)[\kappa_2^2 e^*(t^*)e(t^*) - F^*(e(t^*))F(e(t^*))]. \tag{3.80}$$

Since $\dot\rho_{11}(t) = \bar\rho(t)$, $\dot\rho_{12}(t) = -\bar\rho(t)$, then $\dot P(t) = \dot\rho_{11}(t)P_1 + \dot\rho_{12}(t)P_2 = \bar\rho(t)(P_1 - P_2)$. For $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$, differentiate W along the solution of error system (3.64), gives

$$\begin{aligned}
\dot W(t) &= 2ce^{2c(t-t_0)}V(t) + e^{2c(t-t_0)}\dot V(t) \\
&= 2ce^{2c(t-t_0)}e^*(t)P(t)e(t) + e^{2c(t-t_0)}\Big[\dot e^*(t)P(t)e(t) + e^*(t)\big[\dot P(t)e(t) + P(t)\dot e(t)\big]\Big]
\end{aligned}$$

$$
\begin{aligned}
&= e^{2c(t-t_0)}\Big[2ce^*(t)P(t)e(t) + \big[H^*(e(t)) + F^*(e(t))(GAM)^* + F^*(e(t-\tau(t)))(GBM)^*\big]\cdot \\
&P(t)e(t) + e^*(t)\dot{P}(t)e(t) + e^*(t)P(t)\big[H(e(t)) + (GAM)F(e(t)) + (GBM)F(e(t-\tau(t)))\big]\Big] \\
&= e^{2c(t-t_0)}\big[2ce^*(t)P(t)e(t) + H^*(e(t))P(t)e(t) + F^*(e(t))(GAM)^*P(t)e(t) \\
&+ F^*(e(t-\tau(t)))(GBM)^*P(t)e(t) + e^*(t)\bar{\rho}(t)(P_1 - P_2)e(t) + e^*(t)P(t)H(e(t)) \\
&+ e^*(t)P(t)(GAM)F(e(t)) + e^*(t)P(t)(GBM)F(e(t-\tau(t)))\big],
\end{aligned}
$$

then from (3.78),(3.79),(3.80), we have at $t = t^*$,

$$
\begin{aligned}
&\dot{W}(t^*) \le e^{2c(t^*-t_0)}\big[e^*(t^*)[2cP(t^*) + \bar{\rho}(t^*)(P_1 - P_2) + \alpha(t^*)\kappa_1^2 I_{N-1} + \gamma(t^*)\kappa_2^2 I_{N-1} + \beta(t^*)P(t^*)]\cdot \\
&e(t^*) + H^*(e(t^*))P(t^*)e(t^*) + e^*(t^*)P(t^*)H(e(t^*)) + F^*(e(t^*))(GAM)^*P(t^*)e(t^*) \\
&+ e^*(t^*)P(t^*)(GAM)F(e(t^*)) - \beta(t^*)e^{-2c\tau}e^*(t^*-\tau(t^*))P(t^*-\tau(t^*))e(t^*-\tau(t^*)) \\
&+ F^*(e(t^*-\tau(t^*)))(GBM)^*P(t^*)e(t^*) + e^*(t^*)P(t^*)(GBM)F(e(t^*-\tau(t^*))) \\
&- \alpha(t^*)H^*(e(t^*))H(e(t^*)) - \gamma(t^*)F^*(e(t^*))F(e(t^*))\big] \\
&\le e^{2c(t^*-t_0)}\big[e^*(t^*)[2cP(t^*) + \bar{\rho}(t^*)(P_1 - P_2) + \alpha(t^*)\kappa_1^2 I_{N-1} + \gamma(t^*)\kappa_2^2 I_{N-1} + \beta(t^*)P(t^*)]e(t^*) \\
&+ H^*(e(t^*))P(t^*)e(t^*) + e^*(t^*)P(t^*)H(e(t^*)) + F^*(e(t^*))(GAM)^*P(t^*)e(t^*) \\
&+ e^*(t^*)P(t^*)(GAM)F(e(t^*)) + e^*(t^*)P(t^*)GBM(GBM)^*P(t^*)e(t^*) \\
&+ \kappa_2^2 e^*(t^*-\tau(t^*))e(t^*-\tau(t^*)) - \beta(t^*)e^{-2c\tau}e^*(t^*-\tau(t^*))P(t^*-\tau(t^*))e(t^*-\tau(t^*)) \\
&- \alpha(t^*)H^*(e(t^*))H(e(t^*)) - \gamma(t^*)F^*(e(t^*))F(e(t^*))\big].
\end{aligned}
$$

Denote $\eta(t) = (e^T(t), e^T(t-\tau(t)), F^T(e(t)), H^T(e(t)))^T$, then at $t = t^*$, we have

$$
\dot{W}(t^*) \le e^{2c(t^*-t_0)}\eta^*(t^*)\begin{bmatrix} \widetilde{\Omega}_{11}(t^*) & 0 & P(t^*)GAM & P(t^*) \\ \star & \Omega_{22}(t^*) & 0 & 0 \\ \star & \star & -\gamma(t^*)I_{N-1} & 0 \\ \star & \star & \star & -\alpha(t^*)I_{N-1} \end{bmatrix}\eta(t^*).
$$

According to (3.73), we can obtain

$$
\dot{W}(t^*) < 0,
$$

which contradicts with (3.77). Thus, (3.76) holds. Next, we will prove

$$
W(t) \le \epsilon\lambda_0\|\psi\|_\tau^2, \qquad t \in [t_1, t_2). \tag{3.81}
$$

Applying the Schur complement to condition (3.72), gives

$$-P_1 - (I + GLM)^* P_2 (-P_2)^{-1} P_2 (I + GLM) \leq 0,$$

which implies that

$$(I + GLM)^* P_2 (I + GLM) \leq P_1.$$

At $t = t_1$, it follows from (3.64) and (3.76) that

$$\begin{aligned} W(t_1) &= e^{2c(t_1 - t_0)} e^*(t_1) P(t_1) e(t_1) \\ &= e^{2c(t_1 - t_0)} e^*(t_1^-)(I + GLM)^* P_2 (I + GLM) e(t_1^-) \\ &\leq e^{2c(t_1 - t_0)} e^*(t_1^-) P_1 e(t_1^-) \\ &= e^{2c(t_1 - t_0)} e^*(t_1^-) P(t_1^-) e(t_1^-) \\ &= W(t_1^-) \leq \epsilon \lambda_0 \|\psi\|_\tau^2, \end{aligned}$$

which implies that (3.81) holds at $t = t_1$. Then we will show (3.81) holds for $t \in (t_1, t_2)$. If it is not true, then there exist $t \in (t_1, t_2)$ such that $W(t) > \epsilon \lambda_0 \|\psi\|_\tau^2$. Define $t^{**} = \inf\{t \in (t_1, t_2) : W(t) > \epsilon \lambda_0 \|\psi\|_\tau^2\}$. By the continuity of $W(t)$, we have

$$W(t^{**}) = \epsilon \lambda_0 \|\psi\|_\tau^2, \qquad W(t) < \epsilon \lambda_0 \|\psi\|_\tau^2, \quad t \in (t_1, t^{**}),$$

and

$$\dot{W}(t^{**}) \geq 0. \tag{3.82}$$

Similar to the proof of claim (3.76), according to (3.73), we can show that $\dot{W}(t^{**}) < 0$, which contradicts with (3.82). Thus, (3.81) holds for $t \in (t_1, t_2)$. Hence, (3.81) is true. By simple mathematical induction, we can show that (3.75) is true, which implies

$$V(t, e(t)) \leq \epsilon \lambda_0 \|\psi\|_\tau^2 e^{-2c(t - t_0)}, \qquad \forall\, t \geq t_0.$$

Furthermore, we have

$$V(t, e(t)) \geq \sum_{i=1}^{2} \rho_{1i}(t) \lambda_{\min}(P_i) \|e(t)\|^2 \geq \lambda_0 \|e(t)\|^2,$$

hence

$$\|e(t)\| \leq \sqrt{\epsilon} \|\psi\|_\tau e^{-c(t - t_0)}, \qquad \forall\, t \geq t_0.$$

This shows that the zero solution of the error system (3.64) is GES. Thus, CVDN (3.59) can achieve synchronization under the distributed impulsive controller (3.60). □

In the following, we will investigate synchronization of CVDN (3.59) with time-varying delay $\tau(t)$ satisfying Assumption 3.3.3 under the distributed impulsive controller (3.60). The synchronization result is established by using the time-varying Lyapunov functional method and LMI approach.

Theorem 3.3.3. *Suppose that Assumption 3.3.1 and Assumption 3.3.3 hold. And suppose that the impulse sequence $\{t_k\}$ satisfies $m_1 \leq t_k - t_{k-1} \leq m_2$, $\forall k \in \mathbb{N}^+$. If for given constant $\mu \in (0, 1]$, there exist positive constants $\alpha_{ijl}, \gamma_{ijl}, i, j, l = 1, 2,$ and $(N-1) \times (N-1)$ positive definite Hermitian matrices $P_i, Q_i, i = 1, 2,$ such that the following LMIs hold:*

$$\begin{bmatrix} \phi_{ijl} & 0 & P_i GAM & P_i & P_i GBM \\ \star & \kappa_2^2 I_{N-1} - (1-r)Q_l & 0 & 0 & 0 \\ \star & \star & -\gamma_{ijl} I_{N-1} & 0 & 0 \\ \star & \star & \star & -\alpha_{ijl} I_{N-1} & 0 \\ \star & \star & \star & \star & -I_{N-1} \end{bmatrix} < 0, \qquad i, j, l = 1, 2, \tag{3.83}$$

$$\begin{bmatrix} -\mu P_1 & (I + GLM)^* P_2 \\ \star & -P_2 \end{bmatrix} \leq 0, \tag{3.84}$$

where $\phi_{ijl} = \frac{\ln \mu}{m_j} P_i + \frac{1}{m_j}(P_1 - P_2) + \kappa_1^2 \alpha_{ijl} I_{N-1} + \kappa_2^2 \gamma_{ijl} I_{N-1} + \frac{Q_i}{\mu}$, then CVDN (3.59) can achieve synchronization under the distributed impulsive controller (3.60).

Proof. According to (3.83), there exists a small enough constant $c > 0$ such that

$$\Omega_{ijl} = \begin{bmatrix} \widetilde{\phi_{ijl}} & 0 & P_i GAM & P_i & P_i GBM \\ \star & \kappa_2^2 I_{N-1} - (1-r)Q_l & 0 & 0 & 0 \\ \star & \star & -\gamma_{ijl} I_{N-1} & 0 & 0 \\ \star & \star & \star & -\alpha_{ijl} I_{N-1} & 0 \\ \star & \star & \star & \star & -I_{N-1} \end{bmatrix} < 0, \qquad i, j, l = 1, 2,$$

where $\widetilde{\phi_{ijl}} = \frac{\ln \mu}{m_j} P_i + \frac{1}{m_j}(P_1 - P_2) + 2cP_i + \kappa_1^2 \alpha_{ijl} I_{N-1} + \kappa_2^2 \gamma_{ijl} I_{N-1} + \frac{e^{2c\tau}}{\mu} Q_i$. Define

$$\Omega(t) = \sum_{i,j,l=1}^{2} \rho_{1i}(t) \rho_{2j}(t) \rho_{3l}(t) \Omega_{ijl},$$

and denote

$$P(t) = \sum_{i=1}^{2} \rho_{1i}(t)P_i, \qquad Q(t) = \sum_{i=1}^{2} \rho_{1i}(t)Q_i,$$

$$\alpha(t) = \sum_{i,j,l=1}^{2} \rho_{1i}(t)\rho_{2j}(t)\rho_{3l}(t)\alpha_{ijl}, \quad \gamma(t) = \sum_{i,j,l=1}^{2} \rho_{1i}(t)\rho_{2j}(t)\rho_{3l}(t)\gamma_{ijl},$$

and for given constant $\mu \in (0, 1]$, define

$$\zeta(t) = \mu^{\rho_{11}(t)}.$$

Then, $\Omega(t)$ can be rewritten as

$$\Omega(t) = \begin{bmatrix} \Omega_{11}(t) & 0 & P(t)GAM & P(t) & P(t)GBM \\ \star & \Omega_{22}(t) & 0 & 0 & 0 \\ \star & \star & -\gamma(t)I_{N-1} & 0 & 0 \\ \star & \star & \star & -\alpha(t)I_{N-1} & 0 \\ \star & \star & \star & \star & -I_{N-1} \end{bmatrix} < 0,$$

where $\Omega_{11}(t) = \ln \mu\bar{\rho}(t)P(t) + \bar{\rho}(t)(P_1 - P_2) + 2cP(t) + \kappa_1^2\alpha(t)I_{N-1} + \kappa_2^2\gamma(t)I_{N-1} + \frac{e^{2c\tau}}{\mu}Q(t)$, and $\Omega_{22}(t) = \kappa_2^2 I_{N-1} - (1 - r)Q(t - \tau(t))$. By Schur complement, we have

$$\begin{bmatrix} \overline{\Omega}_{11}(t) & 0 & P(t)GAM & P(t) \\ \star & \Omega_{22}(t) & 0 & 0 \\ \star & \star & -\gamma(t)I_{N-1} & 0 \\ \star & \star & \star & -\alpha(t)I_{N-1} \end{bmatrix} < 0, \qquad t \in [t_{k-1}, t_k), \ k \in \mathbb{N}^+, \tag{3.85}$$

where $\overline{\Omega}_{11}(t) = \Omega_{11}(t) + P(t)GBM(GBM)^*P(t)$.

Consider the time-varying Lyapunov functional candidate for synchronization error system (3.64) as $V(t, e_t) = V_1(t, e) + V_2(t, e_t)$ with

$$V_1(t, e) = \zeta(t)e^*(t)P(t)e(t),$$

$$V_2(t, e_t) = \int_{t-\tau(t)}^{t} e^{-2c(t-s-\tau)}e^*(s)Q(s)e(s)ds.$$

By the definition of $\zeta(t)$ and $P(t)$, we can verify that

$$\dot{\zeta}(t) = \mu^{\rho_{11}(t)} \ln \mu\dot{\rho}_{11}(t) = \mu^{\rho_{11}(t)} \ln \mu\bar{\rho}(t) = \zeta(t) \ln \mu\bar{\rho}(t),$$

and

$$\dot{P}(t) = \dot{\rho}_{11}(t)P_1 + \dot{\rho}_{12}(t)P_2 = \bar{\rho}(t)(P_1 - P_2).$$

For $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$, differentiate V along the solution of error system (3.64), according to Assumption 3.3.3, we have

$$\begin{aligned}
&\dot{V}(t) = -2cV(t) + \dot{\zeta}(t)e^*(t)P(t)e(t) + \zeta(t)\dot{e}^*(t)P(t)e(t) + \zeta(t)e^*(t)[\dot{P}(t)e(t) + P(t)\dot{e}(t)] \\
&+ 2cV(t) + \frac{d}{dt}\left[e^{-2c(t-\tau)}\int_{t-\tau(t)}^{t} e^{2cs}e^*(s)Q(s)e(s)ds\right] \\
&= -2cV(t) + \zeta(t)\ln\mu\bar{\rho}(t)e^*(t)P(t)e(t) + \zeta(t)\big[F^*(e(t))(GAM)^* + F^*(e(t-\tau(t)))(GBM)^* \\
&+ H^*(e(t))\big]P(t)e(t) + \zeta(t)e^*(t)\bar{\rho}(t)(P_1 - P_2)e(t) + \zeta(t)e^*(t)P(t)\dot{e}(t) + 2c\zeta(t)e^*(t)P(t)e(t) \\
&+ 2c\int_{t-\tau(t)}^{t} e^{-2c(t-s-\tau)}e^*(s)Q(s)e(s)ds - 2ce^{-2c(t-\tau)}\int_{t-\tau(t)}^{t} e^{2cs}e^*(s)Q(s)e(s)ds + e^{-2c(t-\tau)}\cdot \\
&\big[e^{2ct}e^*(t)Q(t)e(t) - (1-\dot{\tau}(t))e^{2c(t-\tau(t))}e^*(t-\tau(t))Q(t-\tau(t))e(t-\tau(t))\big] \\
&= -2cV(t) + \zeta(t)\Bigg[e^*(t)\big[\ln\mu\bar{\rho}(t)P(t) + \bar{\rho}(t)(P_1 - P_2) + 2cP(t)\big]e(t) + H^*(e(t))P(t)e(t) \\
&+ F^*(e(t))(GAM)^*P(t)e(t) + F^*(e(t-\tau(t)))(GBM)^*P(t)e(t) + e^*(t)P(t)H(e(t)) \\
&+ e^*(t)P(t)(GAM)F(e(t)) + e^*(t)P(t)(GBM)F(e(t-\tau(t)))\Bigg] + e^{2c\tau}e^*(t)Q(t)e(t) \\
&- (1-\dot{\tau}(t))e^{2c(\tau-\tau(t))}e^*(t-\tau(t))Q(t-\tau(t))e(t-\tau(t)) \\
&\le -2cV(t) + \zeta(t)\Bigg[e^*(t)\big[\ln\mu\bar{\rho}(t)e^*(t)P(t) + \bar{\rho}(t)(P_1 - P_2) + 2cP(t)\big]e(t) + H^*(e(t))P(t)e(t) \\
&+ F^*(e(t))(GAM)^*P(t)e(t) + F^*(e(t-\tau(t)))(GBM)^*P(t)e(t) + e^*(t)P(t)H(e(t)) \\
&+ e^*(t)P(t)(GAM)F(e(t)) + e^*(t)P(t)(GBM)F(e(t-\tau(t)))\Bigg] + e^{2c\tau}e^*(t)Q(t)e(t) \\
&- (1-r)e^*(t-\tau(t))Q(t-\tau(t))e(t-\tau(t)).
\end{aligned}$$

Since $\zeta(t) = \mu^{\rho_{11}(t)}$, and $\rho_{11}(t) \in [0,1)$, then we have $\zeta(t) \in (\mu, 1]$, and $e^{2c\tau}e^*(t)Q(t)e(t) - (1-r)e^*(t-\tau(t))Q(t-\tau(t))e(t-\tau(t)) \le \zeta(t)\big[\frac{e^{2c\tau}}{\mu}e^*(t)Q(t)e(t) - (1-r)e^*(t-\tau(t))Q(t-\tau(t))e(t-\tau(t))\big]$. Furthermore, it follows from Lemma 3.3.1 and Assumption 3.3.1 that

$$\begin{aligned}
&F^*(e(t-\tau(t)))(GBM)^*P(t)e(t) + e^*(t)P(t)(GBM)F(e(t-\tau(t))) \\
&\le F^*(e(t-\tau(t)))F(e(t-\tau(t))) + e^*(t)P(t)(GBM)(GBM)^*P(t)e(t)
\end{aligned}$$

$$\leq \kappa_2^2 e^*(t-\tau(t))e(t-\tau(t)) + e^*(t)P(t)(GBM)(GBM)^*P(t)e(t),$$

and

$$H^*(e(t))H(e(t)) \leq \kappa_1^2 e^*(t)e(t),$$
$$F^*(e(t))F(e(t)) \leq \kappa_2^2 e^*(t)e(t).$$

Since $\alpha(t), \gamma(t) > 0$ for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$, we can obtain

$$0 \leq \alpha(t)(\kappa_1^2 e^*(t)e(t) - H^*(e(t))H(e(t))),$$
$$0 \leq \gamma(t)(\kappa_2^2 e^*(t)e(t) - F^*(e(t))F(e(t))),$$

then, we can conclude that for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$,

$$\dot{V}(t) \leq -2cV(t) + \zeta(t)\Big[e^*(t)\big[\ln\mu\bar{\rho}(t)P(t) + \bar{\rho}(t)(P_1 - P_2) + 2cP(t) + \kappa_1^2\alpha(t)I_{N-1} + \kappa_2^2\gamma(t)\cdot$$
$$I_{N-1} + \frac{e^{2c\tau}}{\mu}Q(t) + P(t)(GBM)(GBM)^*P(t)\big]e(t) + H^*(e(t))P(t)e(t) + e^*(t)P(t)H(e(t)) +$$
$$F^*(e(t))(GAM)^*P(t)e(t) + e^*(t)P(t)(GAM)F(e(t)) + \kappa_2^2 e^*(t-\tau(t))e(t-\tau(t)) - (1-r)\cdot$$
$$e^*(t-\tau(t))Q(t-\tau(t))e(t-\tau(t)) - \alpha(t)H^*(e(t))H(e(t)) - \gamma(t)F^*(e(t))F(e(t))\Big].$$

Denote $\eta(t) = (e^T(t), e^T(t-\tau(t)), F^T(e(t)), H^T(e(t)))^T$. Then it follows from (3.85) that for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$,

$$\dot{V}(t) \leq -2cV(t) + \zeta(t)\eta^*(t)\begin{bmatrix} \overline{\Omega}_{11}(t) & 0 & P(t)GAM & P(t) \\ \star & \Omega_{22}(t) & 0 & 0 \\ \star & \star & -\gamma(t)I_{N-1} & 0 \\ \star & \star & \star & -\alpha(t)I_{N-1} \end{bmatrix}\eta(t)$$
$$\leq -2cV(t),$$

which implies that

$$V(t) \leq V(t_{k-1})e^{-2c(t-t_{k-1})}, \qquad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+. \tag{3.86}$$

Applying Schur complement to condition (3.84), gives $-\mu P_1 + (I + GLM)^*P_2P_2^{-1}P_2(I + GLM) \leq 0$, hence we can get

$$(I + GLM)^*P_2(I + GLM) \leq \mu P_1. \tag{3.87}$$

When $t = t_k$, $k \in \mathbb{N}^+$, we have $\rho_{11}(t_k) = 0$ and $\zeta(t_k) = 1$, then it follows from (3.64) and (3.87) that

$$\begin{aligned} V_1(t_k) &= \zeta(t_k)e^*(t_k)P(t_k)e(t_k) \\ &= e^*(t_k^-)(I + GLM)^* P_2(I + GLM)e(t_k^-) \\ &\leq \mu e^*(t_k^-)P_1 e(t_k^-) \\ &= \zeta(t_k^-)e^*(t_k^-)P(t_k^-)e(t_k^-) = V_1(t_k^-), \end{aligned}$$

by the continuity of $V_2(t)$, we can obtain $V_2(t_k) = V_2(t_k^-)$, then we have

$$V(t_k) \leq V(t_k^-), \qquad k \in \mathbb{N}^+. \tag{3.88}$$

For any $t \geq t_0$, there exist positive integer $\hat{k}$ such that $t \in [t_{\hat{k}-1}, t_{\hat{k}})$, it follows from (3.86) and (3.88) that

$$\begin{aligned} V(t) &\leq V(t_{\hat{k}-1})e^{-2c(t-t_{\hat{k}-1})} \\ &\leq V(t_{\hat{k}-1}^-)e^{-2c(t-t_{\hat{k}-1})} \\ &\leq V(t_{\hat{k}-2})e^{-2c(t_{\hat{k}-1}-t_{\hat{k}-2})}e^{-2c(t-t_{\hat{k}-1})} \\ &= V(t_{\hat{k}-2})e^{-2c(t-t_{\hat{k}-2})}. \end{aligned}$$

By iteration, we can show that

$$V(t) \leq V(t_0)e^{-2c(t-t_0)}, \qquad \forall\, t \geq t_0.$$

It is clear that $V(t) \to 0$ as $t \to \infty$. Furthermore, since

$$\begin{aligned} V(t) \geq V_1(t) &\geq \mu \sum_{i=1}^{2} \rho_{1i}(t)\lambda_{\min}(P_i)e^*(t)e(t) \\ &\geq \mu \min_{i=1,2}\{\lambda_{\min}(P_i)\} \sum_{i=1}^{2} \rho_{1i}(t)e^*(t)e(t) \\ &= \mu\lambda_0\|e(t)\|^2, \end{aligned}$$

then we can conclude that $\|e(t)\| \to 0$ as $t \to \infty$, which implies that $|z_i(t) - z_{i+1}(t)| \to 0$ as $t \to \infty$ for $i = 1, 2, ..., N-1$. Thus CVDN (3.59) achieves synchronization under the distributed impulsive controller (3.60). □

3.4 Event-triggered Mechanism

In networked control systems, the limited bandwidth resources will attack the transmission efficiency of networks. Recently, the time-triggered scheme has been adopted in networked control systems to transmit all sampled data to the controllers. Such kind of periodic sampling control scheme can ameliorate the utilization of limited network bandwidth resources and improve the efficiency of network transmission. However, the time-triggered scheme may lead to the waste of communication resources because not all the sampling signals are necessary. As an alternative scheme, event-triggered mechanism (ETM) has been proposed in networked control systems. In ETM, the sampled information is transmitted for control updates when well-designed event conditions are satisfied. Compared with the time-triggered scheme, event-triggered mechanism can effectively reduce the cost of information exchange between network nodes and save communication resources. Thus, event-triggered mechanism is an effective approach for moderating the transmission of data in networks.

3.4.1 Event-Triggered Impulsive Control (ETIC)

It should be noted that most of the impulsive synchronization results for networks are based on a time-triggered scheme, i.e., the impulsive instants are predesigned, and the impulsive controller will keep working even if the system runs smoothly. This kind of scheme would lead to unnecessary consumption of communication resources. In recent years, by integrating ETM into impulsive control, event-triggered impulsive control (ETIC) has attracted increasing attention from researchers [88, 89, 90, 91]. In such control strategy, the impulsive instants are determined by some well-designed event-triggered conditions, and the impulsive control is activated only when the events occur. Different from time-triggered impulsive control, ETIC can reduce the control cost and save limited resources of communication. In [89], authors studied synchronization of leader-follower coupled dynamical networks by proposing a new type of event-triggered impulsive control strategy. In [90], a novel ETIC scheme with three levels of event-triggering conditions was designed for studying uniform synchronization for chaotic dynamical systems. On the basis of the ETIC scheme in [90], we propose a novel type of memory-based ETIC scheme in the complex domain to study the synchronization problem of time-delay CVDNs in Section 3.5.

3.4.2 Event-Triggered Pinning Impulsive Control (ETPIC)

Due to the fact that pinning control has advantages of reducing the control cost and saving communication resources for networks consisting of a large number of nodes, event-triggered pinning impulsive control (ETPIC) combining pinning control and ETIC, i.e., a small fraction of network nodes are selected to be controlled at each event-triggered impulsive instant, can further reduce the cost of control and save network transmission and communication resources. In Section 3.6, we consider a ETPIC scheme in the complex field combining the ETIC scheme proposed in Section 3.5 and a pinning algorithm to further study synchronization of time-delay CVDNs.

3.5 Synchronization of CVDNs with Discrete and Distributed Time Delays via ETIC

This section studies synchronization problem of CVDNs with both discrete and distributed time delays. In Subsection 3.5.1, a memory-based ETIC scheme with three levels of events is proposed in the complex domain. The event-based impulsive controller relies on the cumulative information of network states in the complex domain. Sufficient conditions for the synchronization are constructed in Subsection 3.5.2. The result shows that the proposed memory-based ETIC scheme can effectively synchronize time-delay CVDNs with the desired trajectory. In Subsection 3.5.3, a numerical example is provided to demonstrate the theoretical result.

3.5.1 Problem Formulation

Consider a CVDN consisting of N identical coupled nodes with discrete and distributed time delays, which can be described as follows:

$$\dot{z}_i(t) = Az_i(t) + Bz_i(t-r) + D\int_{t-r}^{t} z_i(s)ds + \sum_{j=1, j\neq i}^{N} c_{ij}\big(z_j(t) - z_i(t)\big) + u_i(t), \qquad (3.89)$$

for $i = 1, 2, ..., N$, where $z_i = (z_{i1}, z_{i2}, ..., z_{in})^T \in \mathbb{C}^n$ is the state vector of the i-th node, r is the time-delay, $A, B,$ and $D \in \mathbb{C}^{n\times n}$ represent the non-delay complex-valued matrix, discrete-delay complex-valued matrix and distributed-delay complex-valued matrix, respectively. $u_i(t) \in \mathbb{C}^n$ denotes the control input of node i, and $C = (c_{ij})_{N\times N} \in \mathbb{C}^{N\times N}$ is the complex outer coupling configuration matrix representing the coupling strength and the topological structure of the network, where c_{ij} is defined as: if there is a connection from node i to node j $(i \neq j)$, then $c_{ij} \neq 0$ with $c_{ij} \in \mathbb{C}$; otherwise, $c_{ij} = 0$, and $c_{ii} \in \mathbb{C}$ is arbitrary. Let $\tilde{C} = diag\{\tilde{c}_1, ..., \tilde{c}_N\}$ with $\tilde{c}_i = \sum_{j=1}^{N} c_{ij}, i = 1, 2, ..., N,$ then $C - \tilde{C}$ is a complex-valued matrix with zero-row-sum. The initial condition of (3.89) is given by $z_i(t_0+\alpha)=\varphi_i(\alpha)$, for $i=1,2,...,N$, where $\varphi_i\in\mathcal{PC}([-r,0],\mathbb{C}^n)$.

The objective is to design the ETIC scheme $(u_i(t),\{t_k\})$ to exponentially synchronize CVDN (3.89) with the desired orbit $s(t)$, that is,

$$\lim_{t\to\infty}\|z_i(t)-s(t)\|=0,\qquad i=1,2,\cdots,N,$$

where $s(t)\in\mathbb{C}^n$ is a solution of an isolated node:

$$\dot{s}(t)=As(t)+Bs(t-r)+D\int_{t-r}^{t}s(\theta)d\theta. \tag{3.90}$$

The initial condition of (3.90) is given by $s(t_0+\alpha)=\psi(\alpha)$ for $\alpha\in[-r,0]$.

Let $e_i(t)=z_i(t)-s(t)$ $(i=1,2,...,N)$ be the synchronization errors. From (3.89) and (3.90), we can write the synchronization error system as:

$$\dot{e}_i(t)=Ae_i(t)+Be_i(t-r)+D\int_{t-r}^{t}e_i(s)ds+\sum_{j=1}^{N}c_{ij}e_j(t)-\tilde{c}_ie_i(t)+u_i(t), \tag{3.91}$$

for $i=1,2,...,N$. The initial condition of (3.91) is $e_i(t_0+\alpha)=\phi_i(\alpha)$ for $\alpha\in[-r,0]$, where $\phi_i(\alpha)=\varphi_i(\alpha)-\psi(\alpha)$ for $i=1,2,...,N$.

To ensure CVDN (3.89) achieves synchronization, the complex-valued memory-based impulsive controller $u_i(t)$ is designed in which the impulse sequence $\{t_k\}$ is determined by the Lyapunov-based ETM with three levels of events in [92] and [90], which will be shown later. Let $e(t)=(e_1^T(t),e_2^T(t),\cdots,e_N^T(t))^T$, the error system (3.91) can be rewritten as:

$$\dot{e}(t)=[I_N\otimes A+(C-\tilde{C})\otimes I_n]e(t)+(I_N\otimes B)e(t-r)+(I_N\otimes D)\int_{t-r}^{t}e(s)ds+u(t), \tag{3.92}$$

where $u(t)=(u_1^T(t),u_2^T(t),\cdots,u_N^T(t))^T$.

According to the Lyapunov-based ETM in [92] and [90], for the error system (3.92), choose a specific Lyapunov functional V as

$$V(t)=V_1(t)+V_2(t)+V_3(t) \tag{3.93}$$

with

$$V_1(t) = e^*(t)(I_N \otimes P)e(t),$$
$$V_2(t) = \omega \int_{t-r}^{t} e^*(s)(I_N \otimes Q)e(s)ds,$$
$$V_3(t) = \omega \int_0^r \int_{t-\theta}^{t} e^*(s)(I_N \otimes R)e(s)dsd\theta,$$

where $0 < \omega \leq 1$, and $P, Q, R \in \mathbb{C}^{n\times n}$ are positive definite Hermitian matrices to be determined. We design the following ETIC scheme with three levels of events to exponentially stabilize the error system (3.92).

ETIC Scheme: Taking three types of indices: the error threshold-value $\sigma_{\max} > 1$, the control-free index $0 < \sigma_{\min} < 1$, and the error check period $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r\}$ for $\mu > 0$. Choosing the Lyapunov-like functional V in the form of (3.93) with $0 < \omega \leq 1$, and positive definite Hermitian matrices P, Q and R satisfying the following LMI: for constant $\mu > 0$,

$$\Omega = \begin{bmatrix} \Omega_{11} & I_N \otimes (PB) & I_N \otimes (PD) \\ \star & -\omega(I_N \otimes Q) & 0 \\ \star & \star & -\frac{\omega}{r}(I_N \otimes R) \end{bmatrix} \leq 0, \tag{3.94}$$

where $\Omega_{11} = I_N \otimes (A^*P + PA) + [(C - \tilde{C}) + (C - \tilde{C})^*] \otimes P + \omega(I_N \otimes Q) + \omega r(I_N \otimes R) - \mu(I_N \otimes P)$. Then, the ETIC scheme $(u_i(t), \{t_k\})$ with three levels of events is designed as follows:

$$L_1 : \begin{cases} \text{if } \ \Gamma_{1k} := \{\exists t \in (t_{k-1}, t_{k-1} + \Delta] : V(t) \geq \sigma_{\max} V(t_{k-1}^+)\} \neq \emptyset, \\ \text{then, } t_k = \min\{t : t \in \Gamma_{1k}\}, \\ u_i(t) = \big[q_1 \int_{t_{k-1}}^{t} e_i(\theta)d\theta - e_i(t)\big]\delta(t - t_k), \qquad t \in (t_{k-1}, t_k], \end{cases} . \tag{3.95}$$

$$L_2 : \begin{cases} \text{if } \ \Gamma_{2k} := \{\forall t \in (t_{k-1}, t_{k-1} + \Delta] : V(t) < \sigma_{\max} V(t_{k-1}^+), \\ \qquad\qquad V(t_{k-1} + \Delta) \geq \sigma_{\min} V(t_{k-1}^+)\} \neq \emptyset, \\ \text{then, } t_k = t_{k-1} + \Delta, \\ u_i(t) = \big[q_2 \int_{t_{k-1}}^{t} e_i(\theta)d\theta - e_i(t)\big]\delta(t - t_k), \qquad t \in (t_{k-1}, t_k], \end{cases} \tag{3.96}$$

$$L_3:\begin{cases}\text{if}\quad \Gamma_{3k}:=\{\forall t\in(t_{k-1},t_{k-1}+\Delta]:V(t)<\sigma_{\max}V(t_{k-1}^{+}),\\ \qquad\qquad V(t_{k-1}+\Delta)<\sigma_{\min}V(t_{k-1}^{+})\}\neq\emptyset,\\ \text{then, } t_k=t_{k-1}+\Delta,\\ u_i(t)=0,\qquad t\in(t_{k-1},t_k],\end{cases}\tag{3.97}$$

for $i=1,2,...,N$, $k\in\mathbb{N}^+$, where Γ_{1k},Γ_{2k} and Γ_{3k} are the conditions for three levels of events, $q_m\in\mathbb{C}$ with $m\in\{1,2\}$ is the impulsive control gain of the event-based memory impulsive controller $u_i(t)$ to be designed, while the impulse sequence $\{t_k\}$ is determined by the three levels of Lyapunov-based event conditions, if the event occurs from level-1 (level-2) at $t=t_k$, $k\in\mathbb{N}^+$, then the corresponding impulsive control gain is q_1 (q_2).

Remark 3.5.1. *By considering the seriousness of events, the event conditions Γ_{1k},Γ_{2k} and Γ_{3k} in ETIC scheme (3.95)-(3.97) depend on three pre-set key indices: the error threshold value $\sigma_{\max}$, the control-free index $\sigma_{\min}$, and the error check period Δ. The first level of events is the most serious, accordingly, the most effective impulsive control is needed for network nodes; the second level of events is less serious than the first level of events but more serious than the third level of events, thus the corresponding impulsive control for network nodes might be less effective than that of level-1; the third level of events means that the error system (3.92) runs in an ideal situation, and there's no need to control the system, and it is control-free.*

Remark 3.5.2. *It can be seen from ETIC scheme (3.95)-(3.97) that at each instant t_k, $k\in\mathbb{N}^+$, only one kind of events occurs. If event from level-1 (L_1) or level-2 (L_2) occurs at $t=t_k$, then the impulsive controllers will be applied onto network nodes at triggered instant t_k, and the event-based impulsive controllers depend on the cumulative information of their state-dependent synchronization errors in the complex domain over the time interval from the last event-triggered instant t_{k-1} to the current event-triggered instant; if event from level-3 (L_3) occurs at $t=t_k$, then the system is control free, and there's no control input (i.e., $u_i(t_k)=0, i=1,2,...,N$).*

Remark 3.5.3. *ETIC scheme (3.95)-(3.97) guarantees that the length of intervals between every two consecutive triggered instants is at most Δ, which can avoid the situation that the event is not triggered during a long time period.*

Remark 3.5.4. *Impulsive control gains q_1,q_2 in ETIC scheme (3.95)-(3.97) are designed in the complex domain, which naturally contain 2-dimensional information. Compared with 2-dimensional real-valued control gains, the flexibility is restricted for a complex-valued control gain dealing with 1-dimensional degree of freedom information. Therefore, harmful portion of the degree of freedom can be reduced by designing complex-valued control gains.*

Remark 3.5.5. *Zeno behavior is a phenomenon in which an infinite number of events that occur in a finite time interval. Since the triggered instants are implicitly defined based on the event-triggered condition in ETIC* (3.95)*, it is necessary and important to guarantee the exclusion of Zeno behavior of error system* (3.92) *under the proposed ETIC scheme.*

Definition 3.5.1. *ETIC* (3.95)-(3.97) *is said to be non-Zeno with minimal dwell time if the time sequence* $\{t_k\}$ *satisfies: there exists constant* $\xi > 0$ *such that*

$$t_{k+1} - t_k \geq \xi > 0, \qquad \forall k \in \mathbb{N}.$$

Let Γ_m denote the set of events occur from level-m (L_m):

$$\Gamma_m = \bigcup_{k \in \mathbb{N}^+} \Gamma_{mk}, \quad m = 1, 2, 3.$$

Then from ETIC scheme (3.95)-(3.97), for $t \in (t_{k-1}, t_k]$, $u(t)$ has the form of

$$u(t) = \begin{cases} \big[q_m \int_{t_{k-1}}^{t} e(\theta)d\theta - e(t)\big]\delta(t - t_k), & \text{if } \Gamma_m \text{ occurs at } t = t_k, \quad m = 1, 2, \\ 0, \quad \text{if } \Gamma_3 \text{ occurs at } t = t_k, \ k \in \mathbb{N}^+. \end{cases} \tag{3.98}$$

Under (3.98), the error system (3.92) can be rewritten as:

$$\begin{cases} \dot{e}(t) = [I_N \otimes A + (C - \tilde{C}) \otimes I_n]e(t) + (I_N \otimes B)e(t - r) + (I_N \otimes D)\int_{t-r}^{t} e(s)ds, \quad t \neq t_k, \\ e(t^+) = q_m \int_{t_{k-1}}^{t_k} e(s)ds, \quad t = t_k, \ \text{ if } \Gamma_m \text{ occurs at } t = t_k, \quad m = 1, 2, \\ e(t^+) = e(t), \quad t = t_k, \ \text{ if } \Gamma_3 \text{ occurs at } t = t_k, \ k \in \mathbb{N}^+, \\ e(t_0 + \alpha) = \phi(\alpha), \quad \alpha \in [-r, 0], \end{cases} \tag{3.99}$$

where $\phi = (\phi_1^T, \phi_2^T, ..., \phi_N^T)^T$.

Lemma 3.5.1. *[93] A given Hermitian matrix* $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} < 0$*, where* $S_{11}^* = S_{11}, S_{12}^* = S_{21}$ *and* $S_{22}^* = S_{22}$*, is equivalent to any one of the following conditions:*

$$(1) \quad S_{22} < 0 \quad \textit{and} \quad S_{11} - S_{12}S_{22}^{-1}S_{21} < 0,$$

$$(2) \quad S_{11} < 0 \quad \textit{and} \quad S_{22} - S_{21}S_{11}^{-1}S_{12} < 0.$$

Lemma 3.5.2. *[94] For any positive definite Hermitian matrix $P \in \mathbb{C}^{n\times n}$, and for all scalar functions $w(s) : [a, b] \to \mathbb{C}^n$ with scalars $a < b$ such that the integrations concerned are well defined, then*

$$\left(\int_a^b w(s)ds\right)^* P\left(\int_a^b w(s)ds\right) \leq (b-a)\int_a^b w^*(s)Pw(s)ds.$$

3.5.2 Synchronization Results

In this subsection, we present some sufficient conditions for synchronization of CVDN (3.89) via ETIC (3.95)-(3.97), and the sufficient condition for non-Zeno behavior of the error system (3.99) is also established by utilizing the Lyapunov functional method.

Theorem 3.5.1. *Suppose that there exist positive constants μ, $\omega \leq 1$, and three positive definite Hermitian matrices P, Q and R satisfying LMI (3.94), then ETIC (3.95)-(3.97) with $V(t)$ in the form of (3.93) is non-Zeno, and the triggered time sequence $\{t_k\}$ satisfies*

$$\frac{\ln \sigma_{\max}}{\mu} \leq t_{k+1} - t_k \leq \Delta, \quad k \in \mathbb{N}. \tag{3.100}$$

Proof. According to LMI (3.94) and Lemma 3.5.1, we have

$$\begin{bmatrix} \Omega_{11} & I_N \otimes (PB) \\ \star & -\omega(I_N \otimes Q) \end{bmatrix} + (\frac{\omega}{r})^{-1} \begin{bmatrix} I_N \otimes (PD) \\ 0 \end{bmatrix} (I_N \otimes R)^{-1} \begin{bmatrix} I_N \otimes (PD)^* & 0 \end{bmatrix} \leq 0,$$

hence we can get

$$\begin{bmatrix} \Omega_{11} + \frac{r}{\omega}[I_N \otimes (PD)](I_N \otimes R)^{-1}[I_N \otimes (PD)^*] & I_N \otimes (PB) \\ \star & -\omega(I_N \otimes Q) \end{bmatrix} \leq 0. \tag{3.101}$$

Consider the Lyapunov functional candidate (3.93). For $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, differentiate V_1, V_2, and V_3 along the solution of error system (3.99), gives

$$\begin{aligned} \dot{V}_1(t) = {} & \dot{e}^*(t)(I_N \otimes P)e(t) + e^*(t)(I_N \otimes P)\dot{e}(t) \\ & = e^*(t)\big[\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big)^*(I_N \otimes P) + (I_N \otimes P)\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big)\big]e(t) \\ & + e^*(t-r)(I_N \otimes B)^*(I_N \otimes P)e(t) + e^*(t)(I_N \otimes P)(I_N \otimes B)e(t-r) + 2e^*(t)(I_N \otimes P)\cdot \\ & (I_N \otimes D)\int_{t-r}^t e(s)ds, \end{aligned}$$

$$\dot{V}_2(t) = \omega e^*(t)(I_N \otimes Q)e(t) - \omega e^*(t-r)(I_N \otimes Q)e(t-r),$$

and

$$\begin{aligned}
\dot{V}_3(t) &= \omega \int_0^r \left[\frac{d}{dt}\int_{t-\theta}^t e^*(s)(I_N \otimes R)e(s)ds\right]d\theta \\
&= \omega \int_0^r \big[e^*(t)(I_N \otimes R)e(t) - e^*(t-\theta)(I_N \otimes R)e(t-\theta)\big]d\theta \\
&= \omega r e^*(t)(I_N \otimes R)e(t) - \omega \int_{t-r}^t e^*(s)(I_N \otimes R)e(s)ds,
\end{aligned}$$

then from (3.93), we have

$$\begin{aligned}
&\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \\
&= e^*(t)\big[\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big)^*(I_N \otimes P) + (I_N \otimes P)\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big) \\
&+ \omega(I_N \otimes Q) + \omega r(I_N \otimes R)\big]e(t) + e^*(t-r)(I_N \otimes B)^*(I_N \otimes P)e(t) + e^*(t)(I_N \otimes P)\cdot \\
&(I_N \otimes B)e(t-r) - \int_{t-r}^t \big[\omega e^*(s)(I_N \otimes R)e(s) - 2e^*(t)(I_N \otimes P)(I_N \otimes D)e(s)\big]ds \\
&- \omega e^*(t-r)(I_N \otimes Q)e(t-r) \\
&= e^*(t)\big[\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big)^*(I_N \otimes P) + (I_N \otimes P)\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big) \\
&+ \omega(I_N \otimes Q) + \omega r(I_N \otimes R)\big]e(t) + e^*(t-r)(I_N \otimes B)^*(I_N \otimes P)e(t) + e^*(t)(I_N \otimes P)\cdot \\
&(I_N \otimes B)e(t-r) - \int_{t-r}^t \left[\sqrt{\omega}(I_N \otimes R)e(s) - \frac{1}{\sqrt{\omega}}(I_N \otimes D)^*(I_N \otimes P)e(t)\right]^*(I_N \otimes R)^{-1}\cdot \\
&\left[\sqrt{\omega}(I_N \otimes R)e(s) - \frac{1}{\sqrt{\omega}}(I_N \otimes D)^*(I_N \otimes P)e(t)\right]ds + \int_{t-r}^t \frac{1}{\omega}\Big[e^*(t)(I_N \otimes P)(I_N \otimes D)\cdot \\
&(I_N \otimes R)^{-1}(I_N \otimes D)^*(I_N \otimes P)e(t)\Big]ds - \omega e^*(t-r)(I_N \otimes Q)e(t-r) \\
&\le e^*(t)\big[\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big)^*(I_N \otimes P) + (I_N \otimes P)\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big) \\
&+ \omega(I_N \otimes Q) + \omega r(I_N \otimes R)\big]e(t) + e^*(t-r)(I_N \otimes B)^*(I_N \otimes P)e(t) + e^*(t)(I_N \otimes P)\cdot \\
&(I_N \otimes B)e(t-r) + \frac{r}{\omega}e^*(t)(I_N \otimes P)(I_N \otimes D)(I_N \otimes R)^{-1}(I_N \otimes D)^*(I_N \otimes P)e(t) \\
&- \omega e^*(t-r)(I_N \otimes Q)e(t-r).
\end{aligned}$$

For $\mu > 0$, we can get

$$\dot{V}(t) - \mu V_1(t) \leq e^*(t)\big[\big(I_N \otimes A + (C-\tilde{C}) \otimes I_n\big)^*(I_N \otimes P) + (I_N \otimes P)\big(I_N \otimes A + (C-\tilde{C}) \otimes I_n\big) + \omega(I_N \otimes Q) + \omega r(I_N \otimes R) - \mu(I_N \otimes P) + \frac{r}{\omega}(I_N \otimes P)(I_N \otimes D)(I_N \otimes R)^{-1}(I_N \otimes D)^* \cdot (I_N \otimes P)\big]e(t) + e^*(t-r)(I_N \otimes B)^*(I_N \otimes P)e(t) + e^*(t)(I_N \otimes P)(I_N \otimes B)e(t-r) - \omega e^*(t-r)(I_N \otimes Q)e(t-r)$$

Denote $\eta(t) = (e^T(t), e^T(t-r))^T$, based on the properties of Kronecker product, we have

$$\dot{V}(t) - \mu V_1(t) \leq \eta^*(t)\begin{bmatrix} \Omega_{11} + \frac{r}{\omega}[I_N \otimes (PD)](I_N \otimes R)^{-1}[I_N \otimes (PD)^*] & I_N \otimes (PB) \\ \star & -\omega(I_N \otimes Q) \end{bmatrix}\eta(t)$$

with $\Omega_{11} = I_N \otimes (A^*P + PA) + [(C-\tilde{C}) + (C-\tilde{C})^*] \otimes P + \omega(I_N \otimes Q) + \omega r(I_N \otimes R) - \mu(I_N \otimes P)$. It follows from (3.101) that $\dot{V}(t) \leq \mu V_1(t) \leq \mu V(t)$ for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$. Let $\Delta_k = t_{k+1} - t_k$, we have

$$V(t) \leq V(t_k^+)e^{\mu(t-t_k)} \leq V(t_k^+)e^{\mu\Delta_k}, \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{N}. \tag{3.102}$$

For any interval $(t_k, t_{k+1}], k \in \mathbb{N}$, if the event from L_1 occurs at $t = t_{k+1}$, then by the event-triggered condition in ETIC (3.95), we have $V(t_{k+1}) = \sigma_{\max}V(t_k^+)$. At $t = t_{k+1}$, it follows from (3.102) that $\Delta_k \geq \frac{\ln\sigma_{\max}}{\mu}$. If the event from L_2 or L_3 occurs at $t = t_{k+1}$, then it follows from ETIC (3.96) and (3.97) that $\Delta_k = \Delta$. By choosing $\Delta > \max\{\frac{\ln\sigma_{\max}}{\mu}, r\}$, it holds that $\Delta_k > \frac{\ln\sigma_{\max}}{\mu}$. Therefore, condition (3.100) holds, which implies ETIC (3.95)-(3.97) with $V(t)$ in the form of (3.93) is non-Zeno. □

Remark 3.5.6. *By using the linear matrix inequality approach, Theorem 3.5.1 shows the sufficient condition for excluding the Zeno behavior, and the lower bound for inter-execution time is $\frac{\ln\sigma_{\max}}{\mu}$, which implies no event will occur within such inter-execution time.*

Theorem 3.5.2. *Suppose that there exist positive constants μ, $\omega \leq 1$, and three positive definite Hermitian matrices P, Q and R such that (3.94) holds, and impulsive control gains q_1, q_2 are designed to satisfy*

$$|q_1| \leq |q_2| \leq \frac{1}{\Delta}\Big[\frac{\sigma_{\min}}{\sigma_{\max}} - \omega r(\kappa_1 + \frac{\kappa_2 r}{2})\Big]^{\frac{1}{2}}, \tag{3.103}$$

where $\kappa_1 = \frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}$, $\kappa_2 = \frac{\lambda_{\max}(R)}{\lambda_{\min}(P)}$, then CVDN (3.89) can achieve synchronization via ETIC (3.95)-(3.97) with $V(t)$ in the form of (3.93). Moreover, the convergence rate of *synchronization is* $\frac{-\ln\sigma_{\min}}{2\Delta(p+1)}$ *with* $p=\lfloor\frac{\mu r}{\ln\sigma_{\max}}\rfloor+1$.

Proof. Consider the Lyapunov functional candidate (3.93). According to (3.100) in Theorem 3.5.1 and ETIC (3.95)-(3.97), the number of events which occur on the interval $[t_0,t_0+r]$ is at most $\lfloor\frac{r}{\frac{\ln\sigma_{\max}}{\mu}}\rfloor$ if all the events occur from L_1 with $\Delta_k=t_{k+1}-t_k=\frac{\ln\sigma_{\max}}{\mu}$. Therefore, there exists $p=\lfloor\frac{\mu r}{\ln\sigma_{\max}}\rfloor+1\geq 1$ such that $t_p-r\geq t_0$. For $t\in[t_0,t_p]$, by the event conditions of ETIC (3.95)-(3.97) and (3.102), there exists an integer $\hat{k}$, $0\leq\hat{k}\leq p-1$, such that

$$V(t)\leq\sup_{t\in[t_0,t_p]}V(t)\leq\sigma_{\max}V(t_{\hat{k}}^{+})\leq\sigma_{\max}V(t_0)e^{\mu(t_{\hat{k}}-t_0)}.$$

Thus, for $\mu>0$, we have

$$V(t)\leq\sigma_{\max}V(t_0)e^{\mu(t_p-t_0)},\qquad t\in[t_0,t_p]. \tag{3.104}$$

For any interval $(t_k,t_{k+1}]$, $k\geq p-1$, we consider the following 3 cases:
Case 1. If the event from L_1 occurs at $t=t_{k+1}$ $(k\geq p-1)$, it follows from (3.99) that $e(t_{k+1}^{+})=q_1\int_{t_k}^{t_{k+1}}e(s)ds$. Applying Lemma 3.5.2 and the event condition of ETIC (3.95), gives

$$\begin{aligned}V_1(t_{k+1}^{+})&=e^*(t_{k+1}^{+})(I_N\otimes P)e(t_{k+1}^{+})=|q_1|^2\Big(\int_{t_k}^{t_{k+1}}e(s)ds\Big)^*(I_N\otimes P)\Big(\int_{t_k}^{t_{k+1}}e(s)ds\Big)\\&\leq|q_1|^2\Delta_k\int_{t_k}^{t_{k+1}}e^*(s)(I_N\otimes P)e(s)ds\leq|q_1|^2\Delta\int_{t_k}^{t_{k+1}}V_1(s)ds\\&\leq|q_1|^2\Delta\Delta_k\sup_{s\in(t_k,t_{k+1}]}V_1(s)\leq q_1^2\Delta^2\sup_{s\in(t_k,t_{k+1}]}V(s)\\&=|q_1|^2\Delta^2V(t_{k+1})=|q_1|^2\Delta^2\sigma_{\max}V(t_k^{+}),\end{aligned}$$

and by the continuity of $V_2(t)$, we have

$$\begin{aligned}V_2(t_{k+1}^{+})=V_2(t_{k+1})&=\omega\int_{t_{k+1}-r}^{t_{k+1}}e^*(s)(I_N\otimes Q)e(s)ds\\&\leq\omega\frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}\int_{t_{k+1}-r}^{t_{k+1}}e^*(s)(I_N\otimes P)e(s)ds\leq\kappa_1\omega r\sup_{s\in[-r,0]}V(t_{k+1}+s).\end{aligned}$$

Based on the ETIC scheme (3.95)-(3.97) and (3.100), the number of events that occur on the interval $[t_{k+1}-r,t_{k+1})$, $k\geq p-1$ is at most $\lfloor\frac{r}{\frac{\ln\sigma_{\max}}{\mu}}\rfloor=p-1$. Denote $\mathbb{N}_{-(p-1)}=$

$\{-(p-1), -(p-2), ..., -1, 0\}$. Then by the event conditions of ETIC scheme (3.95)-(3.97), we have for $k \geq p-1$,

$$\sup_{s\in[-r,0]} V(t_{k+1}+s) \leq \sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^+), \tag{3.105}$$

hence we can obtain

$$V_2(t_{k+1}^+) \leq \kappa_1 \omega r \sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^+).$$

Similarly, by the continuity of $V_3(t)$ and (3.105),

$$\begin{aligned}
V_3(t_{k+1}^+) = V_3(t_{k+1}) &= \omega \int_0^r \int_{t_{k+1}-\theta}^{t_{k+1}} e^*(s)(I_N \otimes R)e(s)dsd\theta \\
&\leq \omega \frac{\lambda_{\max}(R)}{\lambda_{\min}(P)} \int_0^r \int_{t_{k+1}-\theta}^{t_{k+1}} e^*(s)(I_N \otimes P)e(s)dsd\theta \\
&\leq \omega\kappa_2 \sup_{s\in[-r,0]} V_1(t_{k+1}+s) \int_0^r \theta d\theta \\
&\leq \frac{r^2}{2}\omega\kappa_2 \sup_{s\in[-r,0]} V(t_{k+1}+s) \leq \frac{r^2}{2}\omega\kappa_2\sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^+).
\end{aligned}$$

According to condition (3.103), we have for $k \geq p-1$,

$$\begin{aligned}
V(t_{k+1}^+) &\leq |q_1|^2\Delta^2\sigma_{\max}V(t_k^+) + (\kappa_1\omega r + \frac{r^2}{2}\omega\kappa_2)\sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^+) \\
&\leq (|q_1|^2\Delta^2 + \kappa_1\omega r + \frac{r^2}{2}\omega\kappa_2)\sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^+) \\
&\leq \sigma_{\min} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^+).
\end{aligned}$$

Case 2. If the event from L_2 occurs at $t = t_{k+1}$ $(k \geq p-1)$, it follows from (3.99) that $e(t_{k+1}^+) = q_2 \int_{t_k}^{t_{k+1}} e(s)ds$. It follows from Lemma 3.5.2 and the event condition of ETIC (3.96) that

$$\begin{aligned}
V_1(t_{k+1}^+) &= |q_2|^2\big(\int_{t_k}^{t_{k+1}} e(s)ds\big)^*(I_N \otimes P)\big(\int_{t_k}^{t_{k+1}} e(s)ds\big) \\
&\leq |q_2|^2\Delta \int_{t_k}^{t_{k+1}} e^*(s)(I_N \otimes P)e(s)ds
\end{aligned}$$

$$\leq |q_2|^2\Delta^2 \sup_{s\in(t_k,t_{k+1}]} V(s) < |q_2|^2\Delta^2\sigma_{\max}V(t_k^+).$$

Similar to Case 1, we have

$$V_2(t_{k+1}^+) \leq \kappa_1\omega r \sup_{s\in[-r,0]} V(t_{k+1}+s), \qquad V_3(t_{k+1}^+) \leq \frac{r^2}{2}\omega\kappa_2 \sup_{s\in[-r,0]} V(t_{k+1}+s).$$

By choosing $\Delta > \max\{\frac{\ln\sigma_{\max}}{\mu}, r\}$, according to ETIC (3.96), we can get $t_k = t_{k+1} - \Delta < t_{k+1} - r$, which implies that there's no event occurs on the interval $[t_{k+1}-r, t_{k+1})$. It follows from the event condition of ETIC (3.96) that

$$\sup_{s\in[-r,0]} V(t_{k+1}+s) < \sigma_{\max}V(t_k^+),$$

then we have

$$V_2(t_{k+1}^+) < \kappa_1\omega r\sigma_{\max}V(t_k^+),$$
$$V_3(t_{k+1}^+) < \frac{r^2}{2}\omega\kappa_2\sigma_{\max}V(t_k^+).$$

It follows from condition (3.103) that for $k \geq p-1$,

$$V(t_{k+1}^+) < (|q_2|^2\Delta^2 + \kappa_1\omega r + \frac{r^2}{2}\omega\kappa_2)\sigma_{\max}V(t_k^+) \leq \sigma_{\min}V(t_k^+) \leq \sigma_{\min}\cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^+).$$

Case 3. If the event from L_3 occurs at $t = t_{k+1}$ ($k \geq p-1$), according to ETIC (3.97), it is control free. Then by the event condition of (3.97), we have for $k \geq p-1$,

$$V(t_{k+1}^+) = V(t_{k+1}) < \sigma_{\min}V(t_k^+) \leq \sigma_{\min}\cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^+).$$

Let $z(k) = V(t_k^+)$, $k \in \mathbb{N}$. Combining all the three cases together, we can obtain

$$z(k+1) \leq \sigma_{\min}\bar{z}(k), \qquad k \geq p-1,$$

where $\bar{z}(k) = \max_{\theta\in\mathbb{N}_{-(p-1)}}\{z(k+\theta)\}$. By $\sigma_{\min} < 1$ and the proof of Theorem 3.3 and Theorem 4.2 in [95], we can get

$$z(k) \leq e^{-\alpha(k-p+1)}\cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} z(p-1+\theta), \qquad k \geq p-1 \tag{3.106}$$

with $\alpha = \frac{\ln(\frac{1}{\sigma_{\min}})^{\frac{(p-1)+1}{(p-1)+2}}}{(p-1)+1} = \frac{-\ln\sigma_{\min}}{p+1}$. For any $t > t_p$, there exist an integer $\hat{k} \geq p$ such that $t \in (t_{\hat{k}}, t_{\hat{k}+1}]$, and we have $t - t_p \leq (\hat{k}+1-p)\Delta$, which implies $\hat{k} - p + 1 \geq \frac{t-t_p}{\Delta}$. By $e^{-\alpha} < 1$ and (3.106), we have

$$\begin{aligned} V(t_{\hat{k}}^{+}) &\leq e^{-\alpha(\hat{k}-p+1)} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{p-1+\theta}^{+}) \\ &\leq e^{-(\frac{-\ln\sigma_{\min}}{p+1})(\frac{t-t_p}{\Delta})} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{p-1+\theta}^{+}), \end{aligned} \tag{3.107}$$

then for any $t > t_p$, $t \in (t_{\hat{k}}, t_{\hat{k}+1}], \hat{k} \geq p$, it follows form ETIC (3.95)-(3.97), (3.104) and (3.107) that

$$\begin{aligned} V(t) &\leq \sigma_{\max} V(t_{\hat{k}}^{+}) \leq \sigma_{\max} e^{-(\frac{-\ln\sigma_{\min}}{p+1})(\frac{t-t_p}{\Delta})} \sigma_{\max} V(t_0) e^{\mu(t_p-t_0)} \\ &= \sigma_{\max}^2 e^{(\mu-\frac{\ln\sigma_{\min}}{\Delta(p+1)})(t_p-t_0)} V(t_0) e^{\frac{\ln\sigma_{\min}}{\Delta(p+1)}(t-t_0)} \\ &\leq \sigma_{\max}^2 e^{(\mu p\Delta-\frac{p\ln\sigma_{\min}}{p+1})} V(t_0) e^{\frac{\ln\sigma_{\min}}{\Delta(p+1)}(t-t_0)} \\ &< \frac{\sigma_{\max}^2 e^{\mu p\Delta}}{\sigma_{\min}} V(t_0) e^{\frac{\ln\sigma_{\min}}{\Delta(p+1)}(t-t_0)}, \qquad t > t_p. \end{aligned} \tag{3.108}$$

For $\forall t \in [t_0, t_p]$, it follows from (3.104) that

$$V(t) \leq \sigma_{\max} V(t_0) e^{\mu(t_p-t_0)} \leq \sigma_{\max} V(t_0) e^{\mu p\Delta} \leq \frac{\sigma_{\max}^2 e^{\mu p\Delta}}{\sigma_{\min}} V(t_0) e^{\frac{\ln\sigma_{\min}}{\Delta(p+1)}(t-t_0)}, \tag{3.109}$$

combining (3.108) and (3.109), gives,

$$V_1(t) \leq V(t) \leq \frac{\sigma_{\max}^2 e^{\mu p\Delta}}{\sigma_{\min}} V(t_0) e^{\frac{\ln\sigma_{\min}}{\Delta(p+1)}(t-t_0)}, \qquad \forall t \geq t_0.$$

Moreover,

$$\begin{aligned} &V_1(t) \geq \lambda_{\min}(P)\|e(t)\|^2, \\ &V(t_0) \leq \lambda_{\max}(P)\|e(t_0)\|^2 + \lambda_{\max}(Q)\int_{t_0-r}^{t_0}\|e(s)\|^2 ds + \lambda_{\max}(R)\int_0^r\int_{t_0-\theta}^{t_0}\|e(s)\|^2 ds d\theta \\ &\leq \lambda_{\max}(P)\|e(t_0)\|^2 + \lambda_{\max}(Q) r\|e(t_0)\|_r^2 + \lambda_{\max}(R)\|e(t_0)\|_r^2\int_0^r\int_{t_0-\theta}^{t_0} ds d\theta \\ &\leq \big(\lambda_{\max}(P) + r\lambda_{\max}(Q) + \frac{r^2}{2}\lambda_{\max}(R)\big)\|e(t_0)\|_r^2, \end{aligned}$$

then we can conclude that

$$\|e(t)\| \leq M\|e(t_0)\|_r e^{\frac{\ln\sigma_{\min}}{2\Delta(p+1)}(t-t_0)} \qquad t \geq t_0,$$

where $M = \sigma_{\max} e^{\frac{\mu p\Delta}{2}} \sqrt{\frac{\lambda_{\max}(P)+r\lambda_{\max}(Q)+\frac{r^2}{2}\lambda_{\max}(R)}{\lambda_{\min}(P)\sigma_{\min}}}$. This shows that $||e(t)|| \to 0$ as $t \to \infty$. Thus, CVDN (3.89) achieves synchronization via ETIC (3.95)-(3.97), and the convergence rate of synchronization is $\frac{-\ln\sigma_{\min}}{2\Delta(p+1)}$ with $p = \lfloor \frac{\mu r}{\ln\sigma_{\max}} \rfloor + 1$. □

In particular, if LMI (3.94) has feasible solution for some $0 < \mu < \frac{\ln\sigma_{\max}}{r}$, then according to condition (3.100) in Theorem 3.5.1, we have $r < \frac{\ln\sigma_{\max}}{\mu} \leq t_{k+1} - t_k$ for all $k \in \mathbb{N}$, which implies that the size of time-delay r is smaller than the length of intervals between any two consecutive instants of events that occur. Thus, error system (3.99) can avoid Zeno behavior under ETIC (3.95)-(3.97). In the following, according to Theorem 3.5.1 and Theorem 3.5.2, some sufficient conditions for non-Zeno behavior and synchronization of CVDN with relatively small time-delay via ETIC (3.95)-(3.97) are established.

Corollary 3.5.1. *Suppose that there exist positive constants $\mu < \frac{\ln\sigma_{\max}}{r}$, $\omega \leq 1$, and three positive definite Hermitian matrices P, Q and R such that LMI (3.94) holds. If impulsive control gains q_1, q_2 are designed to satisfy (3.103), then CVDN (3.89) can achieve synchronization via ETIC (3.95)-(3.97) with $V(t)$ in the form of (3.93). Moreover, the convergence rate of synchronization is $\frac{-\ln\sigma_{\min}}{4\Delta}$, and ETIC (3.95)-(3.97) is non-Zeno.*

Proof. If LMI (3.94) has feasible solution for some $0 < \mu < \frac{\ln\sigma_{\max}}{r}$, then we have $\frac{\mu r}{\ln\sigma_{\max}} < 1$, thus the reslt can be directly obtained from Theorem 3.5.1 and Theorem 3.5.2 with $p = \lfloor \frac{\mu r}{\ln\sigma_{\max}} \rfloor + 1 = 1$. □

If we consider CVDN (3.89) only contains discrete delay but no distributed delay (i.e., $D = \mathbf{0}_{n\times n}$), then CVDN (3.89) is reduced to

$$\dot{z}_i = Az_i(t) + Bz_i(t-r) + \sum_{j=1, j\neq i}^{N} c_{ij}\big(z_j(t) - z_i(t)\big) + u_i(t), \quad i = 1, 2, ..., N. \tag{3.110}$$

The dynamics of an isolated node can be described as $\dot{s}(t) = As(t) + Bs(t-r)$. Define the synchronization error as $e_i(t) = z_i(t) - s(t)$, $i = 1, 2, ..., N$, and let $e = (e_1^T, e_2^T, ..., e_N^T)^T$. Since distributed delay is not considered in (3.110), then the Lyapunov-like functional V in ETIC (3.95)-(3.97) can be chosen in the form of (3.93) with $V_3(t) = 0$, i.e.,

$$V(t) = e^*(t)(I_N \otimes P)e(t) + \omega \int_{t-r}^{t} e^*(s)(I_N \otimes Q)e(s)ds \tag{3.111}$$

with $0 < \omega \le 1$, and $P, Q \in \mathbb{C}^{n\times n}$ are positive definite Hermitian matrices satisfying the following LMI: for $\mu > 0$,

$$\begin{bmatrix} \widetilde{\Omega}_{11} & I_N \otimes (PB) \\ \star & -\omega(I_N \otimes Q) \end{bmatrix} \le 0, \tag{3.112}$$

where $\widetilde{\Omega}_{11} = I_N \otimes (A^*P + PA) + [(C-\tilde{C}) + (C-\tilde{C})^*] \otimes P + \omega(I_N \otimes Q) - \mu(I_N \otimes P)$. According to Theorem 3.5.1 and Theorem 3.5.2, the following corollary establishes sufficient conditions for non-Zeno behavior and synchronization of CVDN (3.110) with discrete time-delay via ETIC scheme (3.95)-(3.97).

Corollary 3.5.2. *Suppose that there exist positive constants μ, $\omega \le 1$, and two positive definite Hermitian matrices P and Q such that LMI (3.112) holds. If impulsive control gains q_1, q_2 are designed to satisfy:*

$$|q_1| \le |q_2| \le \frac{1}{\Delta}\Big(\frac{\sigma_{\min}}{\sigma_{\max}} - \kappa\omega r\Big)^{\frac{1}{2}}, \tag{3.113}$$

where $\kappa = \frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}$, then CVDN (3.110) can achieve synchronization via ETIC (3.95)-(3.97) with $V(t)$ in the form of (3.111). Moreover, the convergence rate of synchronization is $\frac{-\ln\sigma_{\min}}{2\Delta(p+1)}$ with $p = \lfloor \frac{\mu r}{\ln \sigma_{\max}} \rfloor + 1$, and ETIC (3.95)-(3.97) is non-Zeno satisfying (3.100).

Proof. The proof is similar to that of Theorem 3.5.1 and Theorem 3.5.2 with $D = R = \mathbf{0}_{n\times n}$, thus it is omitted. □

3.5.3 Numerical Simulations

In this subsection, we will consider an example to demonstrate our theoretical result.

Example 3.5.1. *Consider CVDN (3.89) consisting of eight coupled nodes with parameters*

$$A = \begin{pmatrix} -1+j & 3j \\ 2-j & 0.6-2j \end{pmatrix}, \quad B = \begin{pmatrix} 1.5-2j & -j \\ 0 & -1+0.5j \end{pmatrix},$$

$$D = \begin{pmatrix} -0.6+0.8j & -2+j \\ -0.5j & 1-j \end{pmatrix},$$

and the complex outer coupling configuration matrix C is chosen as

$$C = \begin{pmatrix} 2-j & 0 & 0 & 0 & 0 & 0 & 1+j & 0 \\ 1+j & -1+j & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4+2j & 0 & 0 & 0 & -2+j & 0 \\ 0 & 0 & 0 & -j & 0 & 0 & 0 & -1-j \\ 0 & 1+j & 0 & 1+j & 0.5-j & 0 & 0 & 0 \\ 0 & 1 & 1-j & 0 & 0 & 3-2j & 0 & 0 \\ 0 & 0 & 0 & -1+2j & 0 & 1 & 1+j & 0 \\ 0 & 0 & 1-j & 0 & 0 & 0 & 0 & -2-3j \end{pmatrix},$$

and $\tilde{C} = diag(3, 1+2j, 2+3j, -1-2j, 2.5+j, 5-3j, 1+3j, -1-4j)$.

We consider the following two scenarios for CVDN (3.89) with different delay sizes:

Case 1. *For distributed time-delay $r = 0.1$, choose the three indices $\sigma_{\max}, \sigma_{\min}$ and Δ in ETIC scheme (3.95)-(3.97) as $\sigma_{\max} = 2, \sigma_{\min} = 0.8, \Delta = 1$. Let $\mu = 17.3$, $\omega = 0.01$, then the LMI (3.94) has the following feasible solution:*

$$P = \begin{pmatrix} 0.3213 & 0.0636+0.0013j \\ 0.0636-0.0013j & 0.5493 \end{pmatrix}, \quad Q = \begin{pmatrix} 58.5390 & 0.4536-9.5450j \\ 0.4536+9.5450j & 57.7506 \end{pmatrix},$$

$$R = \begin{pmatrix} 45.8834 & 0.4508+0.1330j \\ 0.4508-0.1330j & 47.2235 \end{pmatrix}.$$

The Lyapunov functional V in ETIC (3.95)-(3.97) is chosen as (3.93) with ω, P, Q, R shown above. Choose the impulsive control gains as $q_1 = 0.3+0.2j$, and $q_2 = 0.3-0.2j$, then condition (3.103) in Theorem 3.5.2 is satisfied. Theorem 3.5.2 implies that CVDN (3.89) achieves synchronization via ETIC (3.95)-(3.97), and the convergence rate of synchronization is $\frac{-\ln\sigma_{\min}}{8\Delta} = 0.028$. Moreover, it follows from Theorem 3.5.1 that ETIC (3.95)-(3.97) is non-Zeno satisfying $\frac{\ln\sigma_{\max}}{\mu} = 0.0401 \le t_{k+1} - t_k \le 1 = \Delta, k \in \mathbb{N}$.

The initial conditions $\varphi_i(\alpha)$ ($i = 1, 2, ..., 8$) of CVDN (3.89) are randomly chosen as $[3+j; 1.5-0.5j; -2+2j; -5j; -2+0.8j; 1+4j; -1-2j; -3+6j; 8+4j; 4-j; 6+3.5j; -2-j; 4-2j; 0.5+5j; 1.8-3j; -3.6+1.5j]^T$ for $\alpha \in [-0.1, 0]$, and the initial value of the isolated system (3.90) is chosen as $\psi(\alpha) = [1-j; -1+j]^T$ for $\alpha \in [-0.1, 0]$. Figure 3.9 shows the time evolution of real and imaginary parts of synchronization errors for CVDN (3.89) via ETIC (3.95)-(3.97) with $r = 0.1$, and the corresponding triggered time instants of three levels of events and release intervals are plotted in Figure 3.10. From the simulation results in Figure 3.9, it is clearly observed that CVDN (3.89) achieves synchronization.

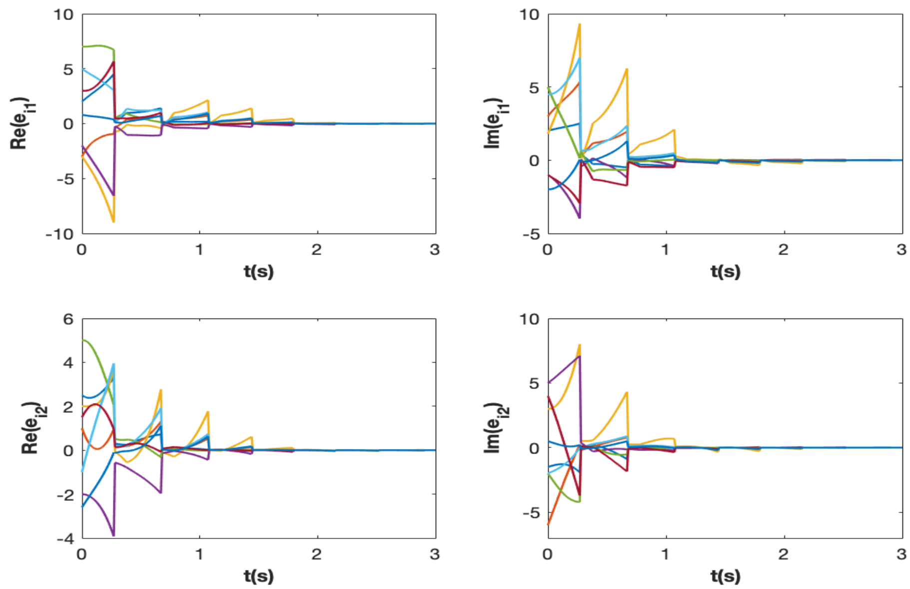

Figure 3.9: Trajectories of real and imaginary parts of 2-dimensional synchronization errors for CVDN (3.89) via ETIC (3.95)-(3.97) with $r = 0.1$.

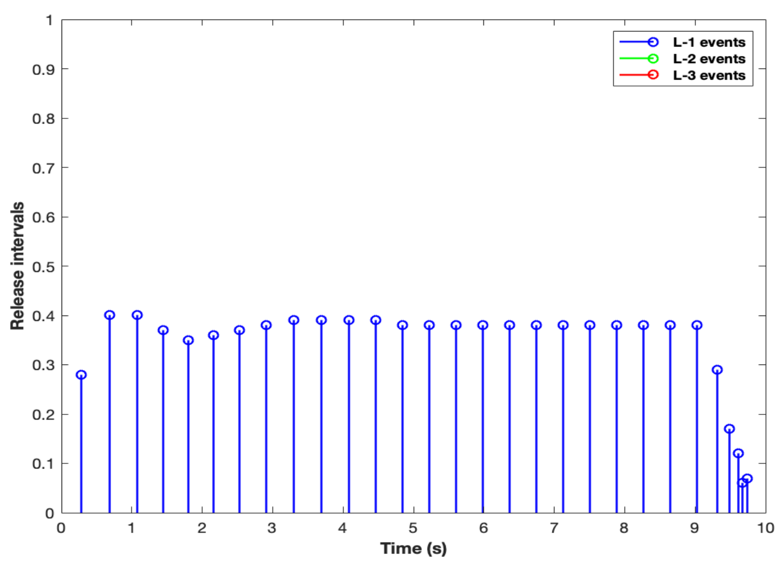

Figure 3.10: Event-triggered instants and release intervals for Example 3.5.1 with $r = 0.1$.

Case 2. *For distributed time-delay* $r = 0.04s$*, the three indices* $\sigma_{\max}, \sigma_{\min}$ *and* Δ *in ETIC scheme* (3.95)-(3.97) *are chosen the same as those in case 1:* $\sigma_{\max} = 2, \sigma_{\min} = 0.8, \Delta = 1$*. Let* $\mu = 17.3 < \frac{\ln \sigma_{\max}}{r}$*,* $\omega = 0.01$*, using MATLAB YALMIP toolbox, the LMI* (3.94) *has the following feasible solution:*

$$P = \begin{pmatrix} 0.4333 & 0.0796 - 0.1245j \\ 0.0796 + 0.1245j & 1.1136 \end{pmatrix}, \quad Q = \begin{pmatrix} 58.4324 & 0.1735 - 9.3917j \\ 0.1735 + 9.3917j & 58.0945 \end{pmatrix},$$

$$R = \begin{pmatrix} 74.9306 & 0.3834 + 0.0959j \\ 0.3834 - 0.0959j & 76.2814 \end{pmatrix}.$$

The Lyapunov functional V *in ETIC* (3.95)-(3.97) *is chosen as* (3.93) *with* ω, P, Q, R *shown above. Choose impulsive control gains* $q_1 = 0.3 + 0.2j$*, and* $q_2 = 0.3 - 0.2j$ *be the same as those in case 1, then* (3.103) *is satisfied. It follows from Corollary 3.5.1 that CVDN* (3.89) *achieves synchronization via ETIC* (3.95)-(3.97)*, and the convergence rate of synchronization is* $\frac{-\ln \sigma_{\min}}{4\Delta} = 0.056$*. Moreover, ETIC* (3.95)-(3.97) *is non-Zeno satisfying* $0.0401 \leq t_{k+1} - t_k \leq 1, k \in \mathbb{N}$*.*

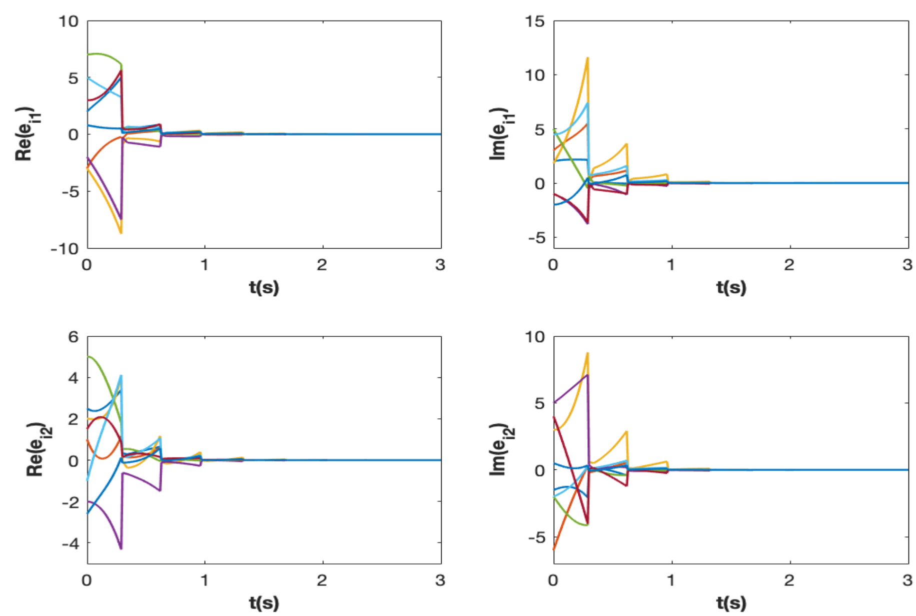

Figure 3.11: Trajectories of real and imaginary parts of 2-dimensional synchronization errors for CVDN (3.89) via ETIC (3.95)-(3.97) with $r = 0.04$.

The initial conditions $\varphi_i(\alpha)$, $\psi(\alpha)$ of CVDN (3.89) *and isolated system* (3.90) *are chosen the same as those in case 1 for $\alpha \in [-0.04, 0]$. Figure 3.11 shows the time evolution of real and imaginary parts of synchronization errors for CVDN* (3.89) *under ETIC* (3.95)-(3.97) *with $r = 0.04$, and the corresponding triggered time instants of three levels of events and release intervals are plotted in figure 3.12. It can be seen from the result of the simulation in Figure 3.11 that synchronization of CVDN* (3.89) *is achieved.*

Remark 3.5.7. *Note that in case 1, the LMI* (3.94) *has no feasible solution for $\mu < \frac{\ln \sigma_{\max}}{r} = 6.93$, which implies that Corollary 3.5.1 may not be applicable for networks with relatively large delays. Moreover, one can see that both real and imaginary parts of synchronization errors in Figure 3.11 converge to zero faster than those in Figure 3.9, which implies that the larger size of delay r may lead to the slower convergence speed of synchronization.*

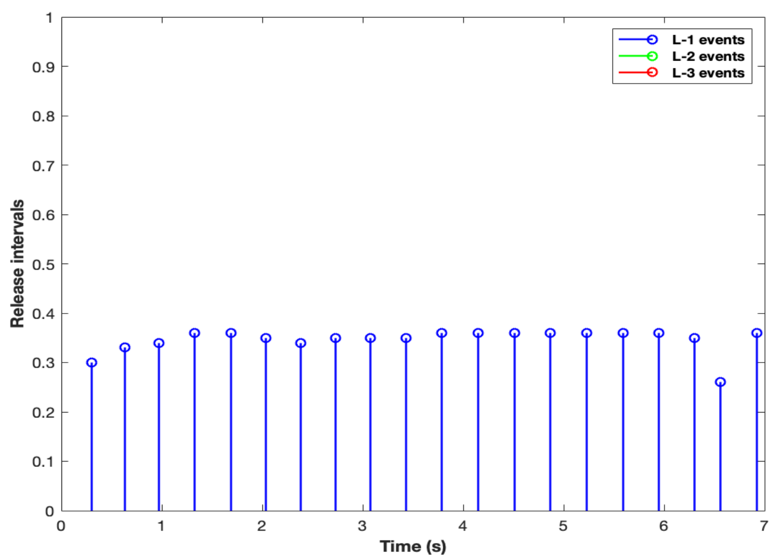

Figure 3.12: Event-triggered instants and release intervals for Example 3.5.1 with $r = 0.04$.

3.6 Synchronization of CVDNs with Discrete and Distributed Time Delays via ETPIC

The outline of this section is as follows. In Subsection 3.6.1, a memory-based ETPIC scheme is proposed in the complex domain by combining the ETIC scheme in Subsection

3.5.1 and a pinning algorithm. In Subsection 3.6.2, the non-Zeno result and sufficient conditions for synchronization of CVDN (3.89) are established. Results in Subsection 3.6.2 show that the proposed memory-based ETPIC scheme can successfully synchronize time-delay CVDN (3.89) with the desired orbit $s(t)$, where $s(t)$ is a solution of an isolated node described by (3.90). In Subsection 3.6.3, two numerical examples are provided to demonstrate the theoretical results.

3.6.1 Preliminaries

Consider CVDN (3.89) with discrete and distributed time delays. Based on the ETM with three levels of events introduced in subsection 3.5.1, we will give an ETPIC scheme to synchronize CVDN (3.89) with the desired orbit $s(t)$, where $s(t)$ is a solution of an isolated node described by (3.90). Define the synchronization error as $e_i(t) = z_i(t) - s(t)$, $i = 1, 2, ..., N$. From (3.89) and (3.90), the error dynamical system can be described by (3.91). Denote $e(t) = (e_1^T(t), e_2^T(t), ..., e_N^T(t))^T$.

ETPIC Scheme: Taking three types of indices: the error threshold-value $\sigma_{\max} > 1$, the control-free index $0 < \sigma_{\min} < 1$, and the error check period $\Delta > 0$ satisfying $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r\}$ for $\mu > 0$. Choosing the Lyapunov-like functional V as (3.93) with $0 < \omega \leq 1$, and positive definite Hermitian matrices P, Q and R satisfying LMI (3.94). Then, the ETPIC scheme $(u_i(t), \{t_k\})$ with three levels of events is designed as follows:

$$L_1 : \begin{cases} \text{if } \ \Gamma_{1k} := \{\exists t \in (t_{k-1}, t_{k-1} + \Delta] : V(t) \geq \sigma_{\max} V(t_{k-1}^+)\} \neq \emptyset, \\ \text{then, } t_k = \min\{t : t \in \Gamma_{1k}\}, \\ u_i(t) = \begin{cases} \big[q_1 \displaystyle\int_{t_{k-1}}^{t} e_i(\theta) d\theta - e_i(t)\big] \delta(t - t_k), & i \in \mathcal{D}_k^{l_1}, \\ 0, \quad i \notin \mathcal{D}_k^{l_1}, & t \in (t_{k-1}, t_k], \end{cases} \end{cases} \tag{3.114}$$

$$L_2 : \begin{cases} \text{if } \ \Gamma_{2k} := \{\forall t \in (t_{k-1}, t_{k-1} + \Delta] : V(t) < \sigma_{\max} V(t_{k-1}^+), \\ \qquad\qquad V(t_{k-1} + \Delta) \geq \sigma_{\min} V(t_{k-1}^+)\} \neq \emptyset, \\ \text{then, } t_k = t_{k-1} + \Delta, \\ u_i(t) = \begin{cases} \big[q_2 \displaystyle\int_{t_{k-1}}^{t} e_i(\theta) d\theta - e_i(t)\big] \delta(t - t_k), & i \in \mathcal{D}_k^{l_2}, \\ 0, \quad i \notin \mathcal{D}_k^{l_2}, & t \in (t_{k-1}, t_k], \end{cases} \end{cases} \tag{3.115}$$

$$L_3: \begin{cases} \text{if } \ \Gamma_{3k} := \{\forall t \in (t_{k-1}, t_{k-1}+\Delta] : V(t) < \sigma_{\max} V(t_{k-1}^+), \\ \qquad\qquad V(t_{k-1}+\Delta) < \sigma_{\min} V(t_{k-1}^+)\} \neq \emptyset, \\ \text{then, } t_k = t_{k-1}+\Delta, \\ u_i(t) = 0, \qquad i = 1, 2, \cdots, N, \quad t \in (t_{k-1}, t_k], \end{cases} \tag{3.116}$$

for $k \in \mathbb{N}^+$, where $q_m \in \mathbb{C}$ with $m \in \{1, 2\}$ is the complex-valued impulsive control gain of the event-based memory pinning impulsive controller $u_i(t)$. If the event from level-m ($m = 1, 2$) occurs at $t = t_k$, let the number of nodes to be controlled at $t = t_k$ be l_m. The index set $\mathcal{D}_k^{l_m}$ is defined as follows: if the event from level-m ($m = 1, 2$) occurs at $t = t_k$, we can reorder the complex-valued synchronization error states $e_1(t_k), e_2(t_k), ..., e_N(t_k)$ in norm such that $\|e_{p_1}(t_k)\| \geq \|e_{p_2}(t_k)\| \geq .... \geq \left\|e_{p_{l_m}}(t_k)\right\| \geq \left\|e_{p_{l_m+1}}(t_k)\right\| \geq ... \geq \|e_{p_N}(t_k)\|$. Particularly, if $\left\|e_{p_{l_m}}(t_k)\right\| = \left\|e_{p_{l_m+1}}(t_k)\right\|$, then let $p_{l_m} < p_{l_m+1}$. $\mathcal{D}_k^{l_m} = \{p_1, p_2, ..., p_{l_m}\}$ is the set of pinned nodes to be controlled at $t = t_k$, and the number of nodes in the set $\mathcal{D}_k^{l_m}$ is l_m.

Remark 3.6.1. *According to the ETPIC scheme (3.114)-(3.116), the first level of events is the most serious, and the second level of events is less serious than that of level-1. If the event from L_1 (L_2) occurs at $t = t_k$, based on the pinning algorithm, the impulse input for the pinned nodes contains the cumulative information of their state-dependent synchronization errors over the time interval from the last event-triggered instant t_{k-1} to the current instant that Γ_{1k} occurs, and the rest of the nodes are control free; if the event from L_3 occurs at $t = t_k$, which implies the system runs in an ideal situation and all the network nodes are control free at $t = t_k$. Note that if Γ_{1k} or Γ_{2k} occurs, the impulsive strength q_1 (q_2) is closely related to the number of the pinned nodes l_1 (l_2).*

Under ETPIC (3.114)-(3.116), the error system (3.91) can be rewritten as the following matrix-form impulsive system:

$$\begin{cases} \dot{e}(t) = \left[I_N \otimes A + (C - \tilde{C}) \otimes I_n\right] e(t) + (I_N \otimes B) e(t-r) + (I_N \otimes D) \displaystyle\int_{t-r}^{t} e(s) ds, \quad t \neq t_k, \\ e_i(t^+) = q_m \displaystyle\int_{t_{k-1}}^{t_k} e_i(s) ds, \quad t = t_k, \text{ if } \Gamma_m \text{ occurs at } t = t_k, \ m = 1, 2, \ i \in \mathcal{D}_k^{l_m}, \\ e_i(t^+) = e_i(t), \quad t = t_k, \text{ if } \Gamma_m \text{ occurs at } t = t_k, \ m = 1, 2, \ i \notin \mathcal{D}_k^{l_m}, \text{or } \Gamma_3 \text{ occurs at } t = t_k, \\ e(t_0 + \alpha) = \phi(\alpha), \quad \alpha \in [-r, 0], \end{cases} \tag{3.117}$$

where Γ_m denotes the set of events occur from level-m, $m = 1, 2, 3$, and $\phi = (\phi_1^T, \phi_2^T, ..., \phi_N^T)^T$ is the initial condition.

Next, we will introduce some useful lemmas.

Lemma 3.6.1. *For any vector $X, Y \in \mathbb{C}^n$, the following inequality holds for any $\varepsilon > 0$.*

$$(X+Y)^*(X+Y) \leq (1+\varepsilon)X^*X + (1+\frac{1}{\varepsilon})Y^*Y.$$

Lemma 3.6.2. *For $\varepsilon > 0$, and given constants $x, y \in \mathbb{R}$, define function*

$$f(\varepsilon) = (1+\varepsilon)x^2 + (1+\frac{1}{\varepsilon})y^2$$

then the function f attains its minimum $f_{\min} = (|x|+|y|)^2$ at $\varepsilon = \frac{|y|}{|x|}$.

Lemma 3.6.3. *For $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, and given constants $u, v, x, y, z \in \mathbb{R}$, define function $f(\eta_1, \eta_2, \eta_3, \eta_4) = (1+\eta_1)u^2 + (1+\frac{1}{\eta_1})(1+\eta_2)v^2 + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\eta_3)x^2 + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\eta_4)y^2 + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\frac{1}{\eta_4})z^2$, then the function f attains its minimum $f_{\min} = (|u|+|v|+|x|+|y|+|z|)^2$ at $(\eta_1, \eta_2, \eta_3, \eta_4) = (\frac{|v|+|x|+|y|+|z|}{|u|}, \frac{|x|+|y|+|z|}{|v|}, \frac{|y|+|z|}{|x|}, \frac{|z|}{|y|})$.*

Remark 3.6.2. *Lemma 3.6.2 and Lemma 3.6.3 can be easily derived by applying the extreme value theorem for single variable/multi-variable functions.*

3.6.2 Synchronization Results

In this subsection, we present some sufficient conditions for synchronization of CVDN (3.89) via ETPIC scheme (3.114)-(3.116), the non-Zeno result is also obtained. Moreover, we consider a special case in which the time-delay terms in CVDN (3.89) are excluded, and the synchronization results for the delay-free CVDN via ETPIC scheme (3.114)-(3.116) are also established.

Theorem 3.6.1. *Suppose that there exist positive constants μ, $\omega \leq 1$, and three positive definite Hermitian matrices P, Q and R such that (3.94) holds. If impulsive control gains q_1, q_2 are designed to satisfy*

$$|q_1| \leq \frac{\left[\frac{\sigma_{\min}}{\sigma_{\max}} - \omega\kappa_2 r - \frac{\omega\kappa_3 r^2}{2} - (\frac{N}{l_1}-1)\gamma_1^2\right]^{\frac{1}{2}}}{\Delta\left[\gamma_1 + \sqrt{\kappa_1}\Delta(\|A\| + \|B\| + r\|D\| + \sqrt{Nl_1}\max_{i,j}|c_{ij}| + \max_i|\tilde{c}_i|)\right]}, \tag{3.118}$$

$$|q_2| \leq \frac{\left[\frac{\sigma_{\min}}{\sigma_{\max}} - \omega\kappa_2 r - \frac{\omega\kappa_3 r^2}{2} - (\frac{N}{l_2}-1)\gamma_2^2\right]^{\frac{1}{2}}}{\Delta\left[\gamma_2 + \sqrt{\kappa_1}\Delta(\|A\| + \|B\| + r\|D\| + \sqrt{Nl_2}\max_{i,j}|c_{ij}| + \max_i|\tilde{c}_i|)\right]}, \tag{3.119}$$

where $\kappa_1 = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}, \kappa_2 = \frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}, \kappa_3 = \frac{\lambda_{\max}(R)}{\lambda_{\min}(P)},\ \gamma_1 = \sqrt{\frac{l_1\lambda_{\max}(P)}{l_1\lambda_{\min}(P)+(N-l_1)\lambda_{\max}(P)}}$, *and* $\gamma_2 = \sqrt{\frac{l_2\lambda_{\max}(P)}{l_2\lambda_{\min}(P)+(N-l_2)\lambda_{\max}(P)}}$, *then CVDN* (3.89) *can achieve synchronization via ETPIC* (3.114)-(3.116) *with* $V(t)$ *in form of* (3.93), *and the convergence rate of synchronization is* $\frac{-\ln\sigma_{\min}}{2\Delta(p+2)}$ *with* $p = \lfloor\frac{\mu r}{\ln\sigma_{\max}}\rfloor+1$. *Moreover, the ETPIC* (3.114)-(3.116) *is non-Zeno satisfying* (3.100).

Proof. Consider the Lyapunov functional candidate (3.93). By choosing $\Delta > \max\{\frac{\ln\sigma_{\max}}{\mu}, r\}$, it follows from the proof of Theorem 3.5.1 that (3.100) holds, which implies that ETPIC (3.114)-(3.116) is non-Zeno. For $t \in (t_k, t_{k+1}]$, $\forall k \in \mathbb{N}$, integrating both sides of error system (3.91) from t to t_{k+1}, gives,

$$e_i(t_{k+1}) - e_i(t) = \int_t^{t_{k+1}} \big[Ae_i(s) + Be_i(s-r) + D\int_{s-r}^{s} e_i(\theta)d\theta + \sum_{j=1}^{N} c_{ij}e_j(s) - \tilde{c}_i e_i(s)\big]ds,$$

for $i = 1, 2, ..., N$, hence we have

$$e_i(t) = e_i(t_{k+1}) - \int_t^{t_{k+1}} \big[Ae_i(s) + Be_i(s-r) + D\int_{-r}^{0} e_i(s+\theta)d\theta + \sum_{j=1}^{N} c_{ij}e_j(s) - \tilde{c}_i e_i(s)\big]ds. \tag{3.120}$$

Denote $\Delta_k = t_{k+1} - t_k$. Integrating both sides of (3.120) from t_k to t_{k+1}, we can obtain

$$\begin{aligned}\int_{t_k}^{t_{k+1}} e_i(t)dt =& \Delta_k e_i(t_{k+1}) - \int_{t_k}^{t_{k+1}}\int_t^{t_{k+1}} \big[Ae_i(s) + Be_i(s-r) + D\int_{-r}^{0} e_i(s+\theta)d\theta \\ &+ \sum_{j=1}^{N} c_{ij}e_j(s) - \tilde{c}_i e_i(s)\big]dsdt, \qquad i = 1, 2, ..., N, \quad k \in \mathbb{N}.\end{aligned} \tag{3.121}$$

According to (3.100), (3.102), and ETPIC (3.114)-(3.116), there exist a positive integer $p = \lfloor\frac{\mu r}{\ln\sigma_{\max}}\rfloor + 1 \geq 1$ such that $t_p - r \geq t_0$, and for $t \in [t_0, t_{p+1}]$, there exists an integer $\hat{k}$ $(0 \leq \hat{k} \leq p)$ such that

$$V(t) \leq \sup_{t\in[t_0,t_{p+1}]} V(t) \leq \sigma_{\max}V(t_{\hat{k}}^+) \leq \sigma_{\max}V(t_0)e^{\mu(t_{\hat{k}}-t_0)}.$$

For $\mu > 0$, we have

$$V(t) \leq \sigma_{\max}V(t_0)e^{\mu(t_p-t_0)}, \qquad t \in [t_0, t_{p+1}]. \tag{3.122}$$

.

For any interval $(t_k, t_{k+1}]$, $k \geq p$, we consider the following 3 cases:

Case 1. If the event from L_1 occurs at $t = t_{k+1}$ ($k \geq p$), and $i \in \mathcal{D}_{k+1}^{l_1}$, it follows from (3.117) and (3.121) that $e_i(t_{k+1}^+) = q_1 \int_{t_k}^{t_{k+1}} e_i(t)dt = X_i + Y_i$, where

$$X_i = q_1 \Delta_k e_i(t_{k+1}),$$
$$Y_i = -q_1 \int_{t_k}^{t_{k+1}} \int_t^{t_{k+1}} \big[Ae_i(s) + Be_i(s-r) + D\int_{-r}^0 e_i(s+\theta)d\theta + \sum_{j=1}^N c_{ij}e_j(s) - \tilde{c}_i e_i(s)\big] dsdt.$$

According to (3.93), we can rewrite $V_1(t)$ as $V_1(t) = \sum_{i=1}^N e_i^*(t)Pe_i(t)$, then we have

$$\begin{aligned} V_1(t_{k+1}^+) &= \sum_{i\in\mathcal{D}_{k+1}^{l_1}} e_i^*(t_{k+1}^+)Pe_i(t_{k+1}^+) + \sum_{i\notin\mathcal{D}_{k+1}^{l_1}} e_i^*(t_{k+1}^+)Pe_i(t_{k+1}^+) \\ &= \sum_{i\in\mathcal{D}_{k+1}^{l_1}} (X_i+Y_i)^*P(X_i+Y_i) + \sum_{i\notin\mathcal{D}_{k+1}^{l_1}} e_i^*(t_{k+1})Pe_i(t_{k+1}). \end{aligned}$$

Then it follows from Lemma 3.6.1 that for any $\varepsilon_1 > 0$, $k \geq p$,

$$\begin{aligned} V_1(t_{k+1}^+) \leq &\lambda_{\max}(P)(1+\varepsilon_1)\sum_{i\in\mathcal{D}_{k+1}^{l_1}} X_i^*X_i + \lambda_{\max}(P)(1+\frac{1}{\varepsilon_1})\sum_{i\in\mathcal{D}_{k+1}^{l_1}} Y_i^*Y_i \\ &+ \sum_{i\notin\mathcal{D}_{k+1}^{l_1}} e_i^*(t_{k+1})Pe_i(t_{k+1}). \end{aligned} \tag{3.123}$$

According to (3.100), we have

$$\sum_{i\in\mathcal{D}_{k+1}^{l_1}} X_i^*X_i \leq |q_1|^2\Delta^2 \sum_{i\in\mathcal{D}_{k+1}^{l_1}} e_i^*(t_{k+1})e_i(t_{k+1}). \tag{3.124}$$

Denote $W_i(s) = Ae_i(s)+Be_i(s-r)+D\int_{-r}^0 e_i(s+\theta)d\theta+\sum_{j=1}^N c_{ij}e_j(s)-\tilde{c}_ie_i(s)$. Applying the Cauchy-Schwarz inequality for square-integrable complex-valued functions twice, yields,

$$\sum_{i\in\mathcal{D}_{k+1}^{l_1}} Y_i^*Y_i = |q_1|^2 \sum_{i\in\mathcal{D}_{k+1}^{l_1}} \big(\int_{t_k}^{t_{k+1}}\int_t^{t_{k+1}} W_i(s)dsdt\big)^*\big(\int_{t_k}^{t_{k+1}}\int_t^{t_{k+1}} W_i(s)dsdt\big)$$

$$\leq |q_1|^2 \sum_{i\in\mathcal{D}_{k+1}^{l_1}} \Delta_k \int_{t_k}^{t_{k+1}} \big(\int_t^{t_{k+1}} W_i(s)ds)^* \big(\int_t^{t_{k+1}} W_i(s)ds\big)dt$$

$$\leq |q_1|^2 \Delta_k \sum_{i\in\mathcal{D}_{k+1}^{l_1}} \int_{t_k}^{t_{k+1}} (t_{k+1}-t)\big(\int_t^{t_{k+1}} W_i^*(s)W_i(s)ds\big)dt$$

$$\leq |q_1|^2 \Delta_k^2 \sum_{i\in\mathcal{D}_{k+1}^{l_1}} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} W_i^*(s)W_i(s)dsdt$$

$$\leq |q_1|^2 \Delta^3 \sum_{i\in\mathcal{D}_{k+1}^{l_1}} \int_{t_k}^{t_{k+1}} W_i^*(s)W_i(s)ds.$$

Then, applying Lemma 3.6.1 four times, we have for $\forall \eta_1, \eta_2, \eta_3, \eta_4 > 0$,

$$\sum_{i\in\mathcal{D}_{k+1}^{l_1}} Y_i^* Y_i \leq |q_1|^2 \Delta^3 \sum_{i\in\mathcal{D}_{k+1}^{l_1}} \int_{t_k}^{t_{k+1}} \big[(1+\eta_1)\big(Ae_i(s)\big)^*\big(Ae_i(s)\big) + (1+\frac{1}{\eta_1})(1+\eta_2)\cdot$$
$$\big(Be_i(s-r)\big)^*\big(Be_i(s-r)\big) + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\eta_3)\big(D\int_{-r}^0 e_i(s+\theta)d\theta\big)^*\big(D\int_{-r}^0 e_i(s+\theta)d\theta\big)$$
$$+(1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\eta_4)\big(\sum_{j=1}^N c_{ij}e_j(s)\big)^*\big(\sum_{j=1}^N c_{ij}e_j(s)\big) + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})\cdot$$
$$(1+\frac{1}{\eta_4})\big(\tilde{c}_i e_i(s)\big)^*\big(\tilde{c}_i e_i(s)\big)\big]ds$$
$$\leq |q_1|^2\Delta^3 \int_{t_k}^{t_{k+1}} \big[(1+\eta_1)\sum_{i\in\mathcal{D}_{k+1}^{l_1}} e_i^*(s)A^*Ae_i(s) + (1+\frac{1}{\eta_1})(1+\eta_2)\sum_{i\in\mathcal{D}_{k+1}^{l_1}} e_i^*(s-r)B^*Be_i(s-r)$$
$$+(1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\eta_3)\sum_{i\in\mathcal{D}_{k+1}^{l_1}} \big(\int_{-r}^0 e_i(s+\theta)d\theta\big)^* D^*D\big(\int_{-r}^0 e_i(s+\theta)d\theta\big) + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})\cdot$$
$$(1+\frac{1}{\eta_3})(1+\eta_4)N\sum_{i\in\mathcal{D}_{k+1}^{l_1}}\sum_{j=1}^N |c_{ij}|^2 e_j^*(s)e_j(s) + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\frac{1}{\eta_4})\sum_{i\in\mathcal{D}_{k+1}^{\ell_1}} |\tilde{c}_i|^2\cdot$$
$$e_i^*(s)e_i(s)\big]ds$$
$$\leq |q_1|^2\Delta^3\int_{t_k}^{t_{k+1}} \Big[(1+\eta_1)\lambda_{\max}(A^*A)\sum_{i=1}^N e_i^*(s)e_i(s) + (1+\frac{1}{\eta_1})(1+\eta_2)\lambda_{\max}(B^*B)\sum_{i=1}^N e_i^*(s-r)\cdot$$

$$e_i(s-r) + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\eta_3)\lambda_{\max}(D^*D)\sum_{i=1}^{N}\big(\int_{-r}^{0} e_i(s+\theta)d\theta\big)^*\big(\int_{-r}^{0} e_i(s+\theta)d\theta\big)$$

$$+ (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\eta_4)N \max_{i,j\in\{1,\dots,N\}} |c_{ij}|^2 l_1 \sum_{i=1}^{N} e_i^*(s)e_i(s) + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})\cdot$$

$$(1+\frac{1}{\eta_3})(1+\frac{1}{\eta_4}) \max_{i\in\{1,\dots,N\}} |\tilde{c}_i|^2 \sum_{i=1}^{N} e_i^*(s)e_i(s)\Big] ds$$

$$\leq \frac{|q_1|^2\Delta^3}{\lambda_{\min}(P)} \int_{t_k}^{t_{k+1}} \Big[(1+\eta_1)\|A\|^2 V_1(s) + (1+\frac{1}{\eta_1})(1+\eta_2)\|B\|^2 V_1(s-r) + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})\cdot$$

$$(1+\eta_3)\|D\|^2 r \int_{-r}^{0}\sum_{i=1}^{N} e_i^*(s+\theta)Pe_i(s+\theta)d\theta + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\eta_4)N\max_{i,j}|c_{ij}|^2\cdot$$

$$l_1 V_1(s) + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\frac{1}{\eta_4})\max_i |\tilde{c}_i|^2 V_1(s)\Big] ds$$

$$\leq \frac{|q_1|^2\Delta^3}{\lambda_{\min}(P)} \Big\{ \big[(1+\eta_1)\|A\|^2 + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\eta_4)Nl_1\max_{i,j}|c_{ij}|^2 + (1+\frac{1}{\eta_1})\cdot$$

$$(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\frac{1}{\eta_4})\max_i |\tilde{c}_i|^2\big] \int_{t_k}^{t_{k+1}} V(s)ds + (1+\frac{1}{\eta_1})(1+\eta_2)\|B\|^2 \int_{t_k}^{t_{k+1}} V(s-r)ds$$

$$+ (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\eta_3)\|D\|^2 r \int_{t_k}^{t_{k+1}}\int_{-r}^{0} V(s+\theta)d\theta ds\Big\}$$

$$\leq \frac{|q_1|^2\Delta^3}{\lambda_{\min}(P)} \Big[(1+\eta_1)\|A\|^2 + (1+\frac{1}{\eta_1})(1+\eta_2)\|B\|^2 + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\eta_3)\|D\|^2 r^2 +$$

$$(1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\eta_4)Nl_1\max_{i,j}|c_{ij}|^2 + (1+\frac{1}{\eta_1})(1+\frac{1}{\eta_2})(1+\frac{1}{\eta_3})(1+\frac{1}{\eta_4})\max_i|\tilde{c}_i|^2\Big]\cdot$$

$$\Delta_k \sup_{s\in[t_k-r,t_{k+1}]} V(s),$$

then it follows from Lemma 3.6.3 that

$$\sum_{i\in\mathcal{D}_{k+1}^{l_1}} Y_i^* Y_i \leq \frac{|q_1|^2\Delta^4}{\lambda_{\min}(P)}\big(\|A\| + \|B\| + r\|D\| + \sqrt{Nl_1}\max_{i,j}|c_{ij}| + \max_i|\tilde{c}_i|\big)^2 \sup_{s\in[t_k-r,t_{k+1}]} V(s)$$

with $(\eta_1,\eta_2,\eta_3,\eta_4) = \big(\frac{\|B\|+r\|D\|+\sqrt{Nl_1}\max_{i,j}|c_{ij}|+\max_i|\tilde{c}_i|}{\|A\|}, \frac{r\|D\|+\sqrt{Nl_1}\max_{i,j}|c_{ij}|+\max_i|\tilde{c}_i|}{\|B\|}, \frac{\sqrt{Nl_1}\max_{i,j}|c_{ij}|+\max_i|\tilde{c}_i|}{r\|D\|}, \frac{\max_i|\tilde{c}_i|}{\sqrt{Nl_1}\max_{i,j}|c_{ij}|}\big)$.

According to (3.100), the number of events that occur on the interval $[t_k - r, t_{k+1}]$, $k \geq p$ is at most $\lfloor \frac{r}{\frac{\ln \sigma_{\max}}{\mu}} \rfloor + 2 = p + 1$, then it follows from ETPIC (3.114)-(3.116) that

$$\sup_{s \in [t_k - r, t_{k+1}]} V(s) \leq \sigma_{\max} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t^+_{k+\theta})$$

with $\mathbb{N}_{-p} = \{-p, ..., -1, 0\}$. Then we can obtain

$$\sum_{i \in \mathcal{D}^{l_1}_{k+1}} Y^*_i Y_i \leq \frac{|q_1|^2 \Delta^4 \big(\|A\| + \|B\| + r\|D\| + \sqrt{Nl_1} \max_{i,j} |c_{ij}| + \max_i |\tilde{c}_i|\big)^2 \sigma_{\max}}{\lambda_{\min}(P)} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t^+_{k+\theta}). \tag{3.125}$$

According to (3.123), (3.124) and (3.125), we can conclude that for any $\varepsilon_1 > 0$, $k \geq p$,

$$V_1(t^+_{k+1}) \leq \alpha'_1 \sum_{i \in \mathcal{D}^{l_1}_{k+1}} e^*_i(t_{k+1}) P e_i(t_{k+1}) + \beta_1 \sigma_{\max} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t^+_{k+\theta}) + \sum_{i \notin \mathcal{D}^{l_1}_{k+1}} e^*_i(t_{k+1}) P e_i(t_{k+1}), \tag{3.126}$$

where $\alpha'_1 = \frac{(1+\varepsilon_1)\lambda_{\max}(P)|q_1|^2\Delta^2}{\lambda_{\min}(P)}$, $\beta_1 = \frac{(1+\frac{1}{\varepsilon_1})\lambda_{\max}(P)|q_1|^2\Delta^4(\|A\|+\|B\|+r\|D\|+\sqrt{Nl_1}\max_{i,j}|c_{ij}|+\max_i|\tilde{c}_i|)^2}{\lambda_{\min}(P)}$. For given $\varepsilon_1 > 0$, letting $\alpha_1 = \frac{l_1\lambda_{\min}(P)\alpha'_1 + (N-l_1)\lambda_{\max}(P)}{l_1\lambda_{\min}(P)+(N-l_1)\lambda_{\max}(P)}$, then according to the pinning algorithm, we can get

$$\begin{aligned}
&(1-\alpha_1) \sum_{i \notin \mathcal{D}^{l_1}_{k+1}} e^*_i(t_{k+1}) P e_i(t_{k+1}) \leq (1-\alpha_1)\lambda_{\max}(P) \sum_{i \notin \mathcal{D}^{l_1}_{k+1}} \|e_i(t_{k+1})\|^2 \\
&\leq (1-\alpha_1)\lambda_{\max}(P)(N-l_1) \max_{i \notin \mathcal{D}^{l_1}_{k+1}} \|e_i(t_{k+1})\|^2 \\
&\leq (1-\alpha_1)\lambda_{\max}(P)(N-l_1) \min_{i \in \mathcal{D}^{l_1}_{k+1}} \|e_i(t_{k+1})\|^2 \\
&\leq (1-\alpha_1)\lambda_{\max}(P) \frac{N-l_1}{l_1} \sum_{i \in \mathcal{D}^{l_1}_{k+1}} \|e_i(t_{k+1})\|^2 \\
&= \frac{\lambda_{\min}(P)(1-\alpha'_1)(N-l_1)\lambda_{\max}(P)}{l_1\lambda_{\min}(P)+(N-l_1)\lambda_{\max}(P)} \sum_{i \in \mathcal{D}^{l_1}_{k+1}} e^*_i(t_{k+1}) e_i(t_{k+1})
\end{aligned}$$

$$\leq \frac{(N-l_1)\lambda_{\max}(P) - \alpha_1^{'}(N-l_1)\lambda_{\max}(P)}{l_1\lambda_{\min}(P) + (N-l_1)\lambda_{\max}(P)} \sum_{i\in\mathcal{D}_{k+1}^{l_1}} e_i^*(t_{k+1})Pe_i(t_{k+1})$$
$$= (\alpha_1 - \alpha_1^{'}) \sum_{i\in\mathcal{D}_{k+1}^{l_1}} e_i^*(t_{k+1})Pe_i(t_{k+1}),$$

which implies that

$$\alpha_1^{'} \sum_{i\in\mathcal{D}_{k+1}^{l_1}} e_i^*(t_{k+1})Pe_i(t_{k+1}) + \sum_{i\notin\mathcal{D}_{k+1}^{l_1}} e_i^*(t_{k+1})Pe_i(t_{k+1}) \leq \alpha_1 V_1(t_{k+1}). \tag{3.127}$$

According to (3.126), (3.127) and the event condition of ETPIC (3.114), we have for any $\varepsilon_1 > 0$, $k \geq p$,

$$\begin{aligned} V_1(t_{k+1}^+) &\leq \alpha_1 V(t_{k+1}) + \beta_1\sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-p}} V(t_{k+\theta}^+) \\ &= \alpha_1\sigma_{\max} V(t_k^+) + \beta_1\sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-p}} V(t_{k+\theta}^+) \\ &\leq (\alpha_1+\beta_1)\sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-p}} V(t_{k+\theta}^+), \end{aligned}$$

where $\alpha_1+\beta_1 = (1+\varepsilon_1)\gamma_1^2|q_1|^2\Delta^2 + (1+\frac{1}{\varepsilon_1})\kappa_1|q_1|^2\Delta^4(\|A\|+\|B\|+r\|D\|+\sqrt{Nl_1}\max_{i,j}|c_{ij}|+\max_i|\tilde{c}_i|)^2 + (\frac{N}{l_1}-1)\gamma_1^2$. According to Lemma 3.6.2, we have for $k \geq p$,

$$\begin{aligned} V_1(t_{k+1}^+) \leq &\Big\{\big[\gamma_1|q_1|\Delta + \sqrt{\kappa_1}|q_1|\Delta^2(\|A\|+\|B\|+r\|D\|+\sqrt{Nl_1}\max_{i,j}|c_{ij}|+\max_i|\tilde{c}_i|)\big]^2 \\ &+ (\frac{N}{l_1}-1)\gamma_1^2\Big\}\sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-p}} V(t_{k+\theta}^+) \end{aligned}$$

with $\varepsilon_1 = \frac{\sqrt{\kappa_1}\Delta(\|A\|+\|B\|+r\|D\|+\sqrt{Nl_1}\max_{i,j}|c_{ij}|+\max_i|\tilde{c}_i|)}{\gamma_1}$.

By the continuity of $V_2(t)$, we have

$$\begin{aligned} V_2(t_{k+1}^+) &= \omega\int_{t_{k+1}-r}^{t_{k+1}} e^*(s)(I_N\otimes Q)e(s)ds \\ &\leq \omega\frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}\int_{t_{k+1}-r}^{t_{k+1}} V_1(s)ds \leq \kappa_2\omega r \sup_{s\in[-r,0]} V(t_{k+1}+s). \end{aligned}$$

According to (3.100), the number of events that occur on the interval $[t_{k+1}-r, t_{k+1}]$, $k \geq p$

is at most $\lfloor \frac{r}{\frac{\ln \sigma_{\max}}{\mu}} \rfloor + 1 = p$. It follows from ETPIC (3.114)-(3.116) that for $k \geq p$,

$$\sup_{s\in[-r,0]} V(t_{k+1}+s) \leq \sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^{+}),$$

hence for $k \geq p$,

$$V_2(t_{k+1}^{+}) \leq \kappa_2 \omega r \sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^{+}),$$

Similarly, by the continuity of $V_3(t)$, we have for $k \geq p$,

$$\begin{aligned}
V_3(t_{k+1}^{+}) &\leq \omega \frac{\lambda_{\max}(R)}{\lambda_{\min}(P)} \int_0^r \int_{t_{k+1}-\theta}^{t_{k+1}} V_1(s) ds d\theta \\
&\leq \omega \kappa_3 \sup_{s\in[-r,0]} V(t_{k+1}+s) \int_0^r \theta d\theta \\
&\leq \frac{r^2}{2} \omega \kappa_3 \sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-(p-1)}} V(t_{k+\theta}^{+}).
\end{aligned}$$

According to condition (3.118), we have for $k \geq p$,

$$\begin{aligned}
V(t_{k+1}^{+}) =& V_1(t_{k+1}^{+}) + V_2(t_{k+1}^{+}) + V_3(t_{k+1}^{+}) \\
&\leq \Big\{ \big[\gamma_1|q_1|\Delta + \sqrt{\kappa_1}|q_1|\Delta^2(\|A\| + \|B\| + r\|D\| + \sqrt{Nl_1}\max_{i,j}|c_{ij}| + \max_i|\tilde{c}_i|)\big]^2 \\
&+ (\frac{N}{l_1} - 1)\gamma_1^2 + \kappa_2 \omega r + \frac{r^2}{2}\omega\kappa_3 \Big\} \sigma_{\max} \cdot \max_{\theta\in\mathbb{N}_{-p}} V(t_{k+\theta}^{+}) \\
&\leq \sigma_{\min} \cdot \max_{\theta\in\mathbb{N}_{-p}} V(t_{k+\theta}^{+}).
\end{aligned}$$

Case 2. If the event from L_2 occurs at $t = t_{k+1}$ ($k \geq p$), and $i \in \mathcal{D}_{k+1}^{l_2}$, it follows from (3.117) and (3.121) that $e_i(t_{k+1}^{+}) = q_2 \int_{t_k}^{t_{k+1}} e_i(t)dt = \widetilde{X}_i + \widetilde{Y}_i$, where

$$\widetilde{X}_i = q_2 \Delta_k e_i(t_{k+1}),$$

$$\widetilde{Y}_i = -q_2 \int_{t_k}^{t_{k+1}} \int_{t}^{t_{k+1}} \big[Ae_i(s) + Be_i(s-r) + D\int_{-r}^{0} e_i(s+\theta)d\theta + \sum_{j=1}^{N} c_{ij}e_j(s) - \tilde{c}_i e_i(s)\big] ds dt.$$

Similar to the proof of case 1, we can obtain for $k \geq p$,

$$\begin{aligned} V_1(t_{k+1}^+) <& \Big\{ \big[\gamma_2 |q_2| \Delta + \sqrt{\kappa_1} |q_2| \Delta^2 (\|A\| + \|B\| + r\|D\| + \sqrt{N l_2} \max_{i,j} |c_{ij}| + \max_i |\tilde{c}_i|) \big]^2 \\ & + (\frac{N}{l_2} - 1)\gamma_2^2 \Big\} \sigma_{\max} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t_{k+\theta}^+), \end{aligned}$$

$$V_2(t_{k+1}^+) \leq \kappa_2 \omega r \sup_{s \in [-r,0]} V(t_{k+1} + s),$$

and

$$V_3(t_{k+1}^+) \leq \frac{r^2}{2} \omega \kappa_3 \sup_{s \in [-r,0]} V(t_{k+1} + s).$$

If the event from L_2 occurs at $t = t_{k+1}$, by choosing $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r\}$, we can obtain from ETPIC (3.115) that $t_k = t_{k+1} - \Delta < t_{k+1} - r$, which implies there's no event occurs on the interval $[t_{k+1} - r, t_{k+1})$. It follows from the event condition of ETPIC (3.115) that

$$\sup_{s \in [-r,0]} V(t_{k+1} + s) < \sigma_{\max} V(t_k^+),$$

then we can get

$$\begin{aligned} V_2(t_{k+1}^+) &< \kappa_2 \omega r \sigma_{\max} V(t_k^+) \leq \kappa_2 \omega r \sigma_{\max} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t_{k+\theta}^+), \\ V_3(t_{k+1}^+) &< \frac{r^2}{2} \omega \kappa_3 \sigma_{\max} V(t_k^+) \leq \frac{r^2}{2} \omega \kappa_3 \sigma_{\max} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t_{k+\theta}^+). \end{aligned}$$

According to condition (3.119), we can conclude that for $k \geq p$,

$$\begin{aligned} V(t_{k+1}^+) <& \Big\{ \big[\gamma_2 |q_2| \Delta + \sqrt{\kappa_1} |q_2| \Delta^2 (\|A\| + \|B\| + r\|D\| + \sqrt{N l_2} \max_{i,j} |c_{ij}| + \max_i |\tilde{c}_i|) \big]^2 \\ & + (\frac{N}{l_2} - 1)\gamma_2^2 + \kappa_2 \omega r + \frac{r^2}{2} \omega \kappa_3 \Big\} \sigma_{\max} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t_{k+\theta}^+) \\ & \leq \sigma_{\min} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t_{k+\theta}^+). \end{aligned}$$

Case 3. If the event from L_3 occurs at $t = t_{k+1}$ $(k \geq p)$, according to ETPIC (3.116), the error system is control free at $t = t_{k+1}$. By the event condition of ETPIC (3.116), we have for $k \geq p$,

$$V(t_{k+1}^+) = V(t_{k+1}) < \sigma_{\min} V(t_k^+) \leq \sigma_{\min} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t_{k+\theta}^+).$$

Combining all the three cases together, we can conclude that

$$V(t_{k+1}^+) \leq \sigma_{\min} \cdot \max_{\theta \in \mathbb{N}_{-p}} V(t_{k+\theta}^+), \qquad k \geq p.$$

Denote $z(k) = V(t_k^+)$ for $k \in \mathbb{N}$, then we have

$$z(k+1) \leq \sigma_{\min} \cdot \max_{\theta \in \mathbb{N}_{-p}} \{z(k+\theta)\}, \qquad k \geq p.$$

By $\sigma_{\min} < 1$ and the proof of Theorem 3.3 and Theorem 4.2 in [95], we can get

$$z(k) \leq e^{-\alpha(k-p)} \cdot \max_{\theta \in \mathbb{N}_{-p}} z(p+\theta), \qquad k \geq p, \tag{3.128}$$

where $\alpha = \frac{-\ln \sigma_{\min}}{p+2}$. Similar to the proof of Theorem 3.5.2, it follows from ETPIC (3.114)-(3.116), (3.122) and (3.128) that

$$V_1(t) \leq V(t) \leq \frac{\sigma_{\max}^2 e^{\mu p \Delta}}{\sigma_{\min}} V(t_0) e^{\frac{\ln \sigma_{\min}}{\Delta(p+2)}(t-t_0)}, \qquad t \geq t_0,$$

hence, we have

$$\|e(t)\| \leq M \|e(t_0)\|_r e^{\frac{\ln \sigma_{\min}}{2\Delta(p+2)}(t-t_0)}, \qquad t \geq t_0$$

with $M = \sigma_{\max} e^{\frac{\mu p \Delta}{2}} \sqrt{\frac{\lambda_{\max}(P) + r\lambda_{\max}(Q) + \frac{r^2}{2}\lambda_{\max}(R)}{\lambda_{\min}(P)\sigma_{\min}}}$. This shows that $||e(t)|| \to 0$ as $t \to \infty$. Thus, CVDN (3.89) achieves synchronization via ETPIC (3.114)-(3.116) with $V(t)$ in form of (3.93), and the convergence rate of synchronization is $\frac{-\ln \sigma_{\min}}{2\Delta(p+2)}$, where $p = \lfloor \frac{\mu r}{\ln \sigma_{\max}} \rfloor + 1$. □

Remark 3.6.3. *Since the feasible solution of LMI (3.94)) may not be unique, Theorem 3.6.1 shows that the design of impulsive control gains q_1 and q_2 is related to the choice of P, Q and R in Lyapunov functional (3.93), the error threshold-value $\sigma_{\max}$, the control-free index $\sigma_{\min}$, the error check period Δ, the size of delay r, and the number of the pinned nodes at each impulsive instant (i.e., l_1, l_2). Condition (3.118) and (3.119) imply that choosing smaller control free index $\sigma_{\min}$ or larger error threshold-value $\sigma_{\max}$ or error check period Δ requires stronger impulsive strength. Moreover, the smaller number of nodes are pinned at each impulsive instant, the stronger impulsive strength is required.*

Remark 3.6.4. *Theorem 3.6.1 shows that the size of delay may affect the convergence speed of synchronization, a larger size of delay may result in slower convergence speed of network synchronization.*

Remark 3.6.5. *In particular, if LMI (3.94) has feasible solution for some positive μ satisfying $\mu < \frac{\ln \sigma_{\max}}{r}$, then according to condition (3.100), we have $r < \frac{\ln \sigma_{\max}}{\mu} \leq t_{k+1} - t_k$ for all $k \in \mathbb{N}$, which implies that ETPIC scheme (3.114)-(3.116) is non-Zeno.*

In the following, according to Theorem 3.6.1, some sufficient conditions for non-Zeno behavior and synchronization of CVDN (3.89) via ETPIC (3.114)-(3.116) are established by considering the case that LMI (3.94) has feasible solution for $0 < \mu < \frac{\ln \sigma_{\max}}{r}$.

Corollary 3.6.1. *Suppose that there exist positive constants $\mu < \frac{\ln \sigma_{\max}}{r}$, $\omega \leq 1$, and three positive definite Hermitian matrices P, Q and R such that (3.94) holds. If impulsive control gains q_1, q_2 are designed to satisfy (3.118) and (3.119), then CVDN (3.89) can achieve synchronization via ETPIC (3.114)-(3.116) with $V(t)$ in form of (3.93), and the convergence rate of synchronization is $\frac{-\ln \sigma_{\min}}{6\Delta}$. Moreover, the ETPIC (3.114)-(3.116) is non-Zeno.*

Proof. If LMI (3.94) has feasible solution for some $0 < \mu < \frac{\ln \sigma_{\max}}{r}$, then we have $\frac{\mu r}{\ln \sigma_{\max}} < 1$, thus the result can be directly obtained from Theorem 3.6.1 with $p = \lfloor \frac{\mu r}{\ln \sigma_{\max}} \rfloor + 1 = 1$. □

Remark 3.6.6. *Note that the choice of μ in Corollary 3.6.1 is restricted by the delay size r and the error threshold-value $\sigma_{\max}$, if $\sigma_{\max}$ is chosen close to 1 or the size of time-delay r is large, then LMI (3.94) may not have feasible solution, which implies Corollary 3.6.1 may not be applicable for networks with relatively large delay.*

Remark 3.6.7. *If all the N network nodes are controlled at each impulsive instant (i.e., $l_1 = l_2 = N$), then for $t \in (t_{k-1}, t_k], k \in \mathbb{N}^+$, the controllers $u_i(t)$ in ETPIC scheme (3.114) and (3.115) are transformed into $u_i(t) = \big[q_m \int_{t_{k-1}}^{t} e_i(\theta)d\theta - e_i(t)\big]\delta(t - t_k), i = 1, 2, ..., N, m = 1, 2$, respectively, and ETPIC scheme (3.114)-(3.116) is changed to ETIC scheme (3.95)-(3.97) in Section 3.5. According to Theorem 3.6.1, we can obtain the following synchronization result for CVDN (3.89) via ETIC (3.95)-(3.97).*

Corollary 3.6.2. *Suppose that there exist positive constants μ, $\omega \leq 1$, and three positive definite Hermitian matrices P, Q and R such that (3.94) holds. If impulsive control gains q_1, q_2 are designed to satisfy*

$$|q_1| \leq |q_2| \leq \frac{\left[\frac{\sigma_{\min}}{\sigma_{\max}} - \omega\kappa_2 r - \frac{\omega\kappa_3 r^2}{2}\right]^{\frac{1}{2}}}{\Delta\sqrt{\kappa_1}\left[1 + \Delta(\|A\| + \|B\| + r\|D\| + N\max_{i,j}|c_{ij}| + \max_i|\tilde{c}_i|)\right]}, \quad (3.129)$$

where $\kappa_1 = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}, \kappa_2 = \frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}$, *and* $\kappa_3 = \frac{\lambda_{\max}(R)}{\lambda_{\min}(P)}$, *then CVDN* (3.89) *can achieve synchronization via ETIC* (3.95)-(3.97) *with* $V(t)$ *in the form of* (3.93), *and the convergence rate of synchronization is* $\frac{-\ln \sigma_{\min}}{2\Delta(p+2)}$ *with* $p = \lfloor \frac{\mu r}{\ln \sigma_{\max}} \rfloor + 1$. *Moreover, the ETIC* (3.95)-(3.97) *is non-Zeno satisfying* (3.100).

Proof. The result can be directly obtained from Theorem 3.6.1 with $l_1 = l_2 = N$. □

Next, we will study synchronization of the following delay-free CVDN:

$$\dot{z}_i(t) = Az_i(t) + \sum_{j=1, j\neq i}^{N} c_{ij}\big(z_j(t) - z_i(t)\big) + u_i(t), \qquad i = 1, 2, ..., N, \tag{3.130}$$

where (3.130) is a special case for CVDN (3.89) with $B = D = \mathbf{0}_{n\times n}$. The dynamics of an isolated node can be described as $\dot{s}(t) = As(t)$. Define the synchronization error as $e_i(t) = z_i(t) - s(t)$, then the error dynamical system can be written as

$$\dot{e}_i(t) = Ae_i(t) + \sum_{j=1}^{N} c_{ij}e_j(t) - \tilde{c}_i e_i(t) + u_i(t), \qquad i = 1, 2, ..., N. \tag{3.131}$$

Based on the ETPIC scheme (3.114)-(3.116) introduced in Subsection 3.6.1, choose a Lyapunov function V as $V(t) = \sum_{i=1}^{N} e_i^*(t)Pe_i(t)$, where P is a positive definite Hermitian matrix satisfying the following LMI: for positive constant μ,

$$\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big)^*(I_N \otimes P) + (I_N \otimes P)\big(I_N \otimes A + (C - \tilde{C}) \otimes I_n\big) - \mu(I_N \otimes P) \leq 0. \tag{3.132}$$

For achieving the synchronization, the ETPIC scheme $(u_i(t), \{t_k\})$ for CVDN (3.130) is designed as follows: Choose $\sigma_{\max} > 1$, $0 < \sigma_{\min} < 1$, and $\Delta > \frac{\ln \sigma_{\max}}{\mu}$. For $k \in \mathbb{N}^+$,

$$L_1 : \begin{cases} \text{if } \ \Gamma_{1k} := \{\exists t \in (t_{k-1}, t_{k-1} + \Delta] : \sum_{i=1}^{N} e_i^*(t)Pe_i(t) \geq \sigma_{\max} \sum_{i=1}^{N} e_i^*(t_{k-1}^+)Pe_i(t_{k-1}^+)\} \neq \emptyset, \\ \text{then, } t_k = \min\{t : t \in \Gamma_{1k}\}, \\ u_i(t) = \begin{cases} \big[q_1 \int_{t_{k-1}}^{t} e_i(\theta)d\theta - e_i(t)\big]\delta(t - t_k), & i \in \mathcal{D}_k^{l_1}, \\ 0, \quad i \notin \mathcal{D}_k^{l_1}, & t \in (t_{k-1}, t_k], \end{cases} \end{cases} \tag{3.133}$$

$$L_2:\begin{cases}\text{if }\ \Gamma_{2k}:=\{\forall t\in(t_{k-1},t_{k-1}+\Delta]:\sum_{i=1}^{N}e_i^*(t)Pe_i(t)<\sigma_{\max}\sum_{i=1}^{N}e_i^*(t_{k-1}^+)Pe_i(t_{k-1}^+),\\ \sum_{i=1}^{N}e_i^*(t_{k-1}+\Delta)Pe_i(t_{k-1}+\Delta)\geq\sigma_{\min}\sum_{i=1}^{N}e_i^*(t_{k-1}^+)Pe_i(t_{k-1}^+)\}\neq\emptyset,\\ \text{then, } t_k=t_{k-1}+\Delta,\\ u_i(t)=\begin{cases}\big[q_2\int_{t_{k-1}}^{t}e_i(\theta)d\theta-e_i(t)\big]\delta(t-t_k), & i\in\mathcal{D}_k^{l_2},\\ 0, \qquad i\notin\mathcal{D}_k^{l_2}, & t\in(t_{k-1},t_k],\end{cases}\end{cases}\tag{3.134}$$

$$L_3:\begin{cases}\text{if }\ \Gamma_{3k}:=\{\forall t\in(t_{k-1},t_{k-1}+\Delta]:\sum_{i=1}^{N}e_i^*(t)Pe_i(t)<\sigma_{\max}\sum_{i=1}^{N}e_i^*(t_{k-1}^+)Pe_i(t_{k-1}^+),\\ \sum_{i=1}^{N}e_i^*(t_{k-1}+\Delta)Pe_i(t_{k-1}+\Delta)<\sigma_{\min}\sum_{i=1}^{N}e_i^*(t_{k-1}^+)Pe_i(t_{k-1}^+)\}\neq\emptyset,\\ \text{then, } t_k=t_{k-1}+\Delta,\\ u_i(t)=0, \qquad i=1,2,...,N, \quad t\in(t_{k-1},t_k].\end{cases}\tag{3.135}$$

Let $e(t)=(e_1^T(t),e_2^T(t),\cdots,e_N^T(t))^T$, under ETPIC (3.133)-(3.135), the error system (3.131) can be rewritten as follows:

$$\begin{cases}\dot{e}(t)=\big[I_N\otimes A+(C-\tilde{C})\otimes I_n\big]e(t), \quad t\neq t_k,\\ e_i(t^+)=q_m\int_{t_{k-1}}^{t_k}e_i(s)ds, \quad t=t_k,\ \text{if } \Gamma_m \text{ occurs at } t=t_k,\ m=1,2,\ i\in\mathcal{D}_k^{l_m},\\ e(t_0)=\phi,\end{cases}\tag{3.136}$$

where $\phi=(\phi_1^T,\phi_2^T,...,\phi_N^T)^T$ is the initial condition with $\phi_i\in\mathbb{C}^n$.

According to Theorem 3.6.1, we will present some sufficient conditions for non-Zeno behavior and synchronization of delay-free CVDN (3.130) via ETPIC (3.133)-(3.135).

Theorem 3.6.2. *Suppose there exist positive scalar μ, and positive definite Hermitian matrix P such that LMI (3.132) holds. If impulsive control gains q_1,q_2 are designed to satisfy*

$$|q_1|\leq\frac{\big[\frac{\sigma_{\min}}{\sigma_{\max}}-(\frac{N}{l_1}-1)\gamma_1^2\big]^{\frac{1}{2}}}{\Delta\big[\gamma_1+\sqrt{\kappa}\Delta(\|A\|+\sqrt{Nl_1}\max_{i,j}|c_{ij}|+\max_i|\tilde{c}_i|)\big]},\tag{3.137}$$

$$|q_2| \le \frac{\left[\frac{\sigma_{\min}}{\sigma_{\max}} - (\frac{N}{l_2} - 1)\gamma_2^2\right]^{\frac{1}{2}}}{\Delta\left[\gamma_2 + \sqrt{\kappa}\Delta(\|A\| + \sqrt{Nl_2}\max_{i,j}|c_{ij}| + \max_i|\tilde{c}_i|)\right]}, \tag{3.138}$$

where $\kappa = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$, $\gamma_1 = \sqrt{\frac{l_1\lambda_{\max}(P)}{l_1\lambda_{\min}(P)+(N-l_1)\lambda_{\max}(P)}}$, *and* $\gamma_2 = \sqrt{\frac{l_2\lambda_{\max}(P)}{l_2\lambda_{\min}(P)+(N-l_2)\lambda_{\max}(P)}}$, *then CVDN* (3.130) *can achieve synchronization via ETPIC* (3.133)-(3.135) *with convergence rate* $\frac{-\ln\sigma_{\min}}{2\Delta}$. *Moreover, ETPIC* (3.133)-(3.135) *is non-Zeno satisfying* (3.100).

Proof. Choose the Lyapunov function as $V(t) = \sum_{i=1}^N e_i^*(t)Pe_i(t)$. Let $e(t) = (e_1^T(t), e_2^T(t), ..., e_N^T(t))^T$, then we can rewrite V as $V(t) = e^*(t)(I_N \otimes P)e(t)$. For $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, differentiate V along the solution of error system (3.136), it follows from LMI (3.132) that $\dot{V} \le \mu V$ for positive scalar μ. By choosing the error check period $\Delta > \frac{\ln\sigma_{\max}}{\mu}$, it follows from the proof of Theorem 3.5.1 that (3.100) holds, which implies ETPIC (3.133)-(3.135) is non-Zeno. For any interval $(t_k, t_{k+1}]$, $k \in \mathbb{N}$, we consider the following 3 cases:

Case 1. If the event from L_1 occurs at $t = t_{k+1}$, it follows from the proof of case 1 in Theorem 3.6.1 with $B = D = \mathbf{0}_{n\times n}$ and $V_2(t) = V_3(t) = 0$ that $V(t_{k+1}^+) \le \{[\gamma_1|q_1|\Delta + \sqrt{\kappa}|q_1|\Delta^2(\|A\| + \sqrt{Nl_1}\max_{i,j}|c_{ij}| + \max_i|\tilde{c}_i|)]^2 + (\frac{N}{l_1} - 1)\gamma_1^2\}\sigma_{\max}V(t_k^+)$. According to condition (3.137), we have $V(t_{k+1}^+) \le \sigma_{\min}V(t_k^+)$.

Case 2. If the event from L_2 occurs at $t = t_{k+1}$, according to the proof of case 2 in Theorem 3.6.1 with $B = D = \mathbf{0}_{n\times n}$, $V_2(t) = V_3(t) = 0$, we have $V(t_{k+1}^+) < \{[\gamma_2|q_2|\Delta + \sqrt{\kappa}|q_2|\Delta^2(\|A\| + \sqrt{Nl_2}\max_{i,j}|c_{ij}| + \max_i|\tilde{c}_i|)]^2 + (\frac{N}{l_2} - 1)\gamma_2^2\}\sigma_{\max}V(t_k^+)$, then it follows from condition (3.138) that $V(t_{k+1}^+) < \sigma_{\min}V(t_k^+)$.

Case 3. If the event from L_3 occurs at $t = t_{k+1}$, the error system is control free. By the event condition of ETPIC (3.135), we can obtain $V(t_{k+1}^+) = V(t_{k+1}) < \sigma_{\min}V(t_k^+)$.

Combing all the 3 cases together, we can conclude that for $\forall k \in \mathbb{N}$,

$$V(t_{k+1}^+) \le \sigma_{\min}V(t_k^+).$$

By iteration, we have

$$V(t_k^+) \le \sigma_{\min}^k V(t_0), \qquad k \in \mathbb{N}.$$

Then it follows from ETPIC (3.133)-(3.135) that

$$V(t) \le \sigma_{\max}V(t_k^+) \le \sigma_{\max}\sigma_{\min}^k V(t_0), \qquad t \in (t_k, t_{k+1}], \quad k \in \mathbb{N}. \tag{3.139}$$

For $\forall t \geq t_0$, there exist $\hat{k} \in \mathbb{N}$ such that $t \in (t_{\hat{k}}, t_{\hat{k}+1}]$, and we have $t - t_0 \leq (\hat{k}+1)\Delta$, which implies $\hat{k} \geq \frac{t-t_0}{\Delta} - 1$. By $\sigma_{\min} < 1$ and (3.139), we can get

$$V(t) \leq \sigma_{\max}\sigma_{\min}^{(\frac{t-t_0}{\Delta}-1)}V(t_0) = \frac{\sigma_{\max}}{\sigma_{\min}}V(t_0)e^{\frac{\ln \sigma_{\min}}{\Delta}(t-t_0)}, \qquad t \geq t_0.$$

Since $V(t) \geq \lambda_{\min}(P)\|e(t)\|^2$, and $V(t_0) \leq \lambda_{\max}(P)\|e(t_0)\|^2$, we can conclude that

$$\|e(t)\| \leq \sqrt{\frac{\kappa\sigma_{\max}}{\sigma_{\min}}}\|e(t_0)\|e^{\frac{\ln \sigma_{\min}}{2\Delta}(t-t_0)}, \qquad t \geq t_0.$$

This shows that $||e(t)|| \to 0$ as $t \to \infty$, hence $||e_i(t)|| \to 0$ as $t \to \infty$ for $i = 1, 2, ..., N$. Thus, CVDN (3.130) achieves synchronization via ETPIC (3.133)-(3.135) with convergence rate $\frac{-\ln \sigma_{\min}}{2\Delta}$. □

Remark 3.6.8. *Comparing Theorem 3.6.2 with Theorem 3.6.1, one can see that the delay-free CVDN has faster convergence speed, which implies that time-delay may reduce the convergence speed of synchronization.*

3.6.3 Numerical Simulations

In this subsection, we consider two examples to illustrate our theoretical results.

Example 3.6.1. *Consider CVDN (3.89) consisting of eight coupled nodes with parameters*

$$A = \begin{pmatrix} -1+j & 3j \\ 2-j & 0.6-2j \end{pmatrix}, \; B = \begin{pmatrix} 1.5-2j & -j \\ 0 & -1+0.5j \end{pmatrix}, \; D = \begin{pmatrix} -0.6+0.8j & -2+j \\ -0.5j & 1-j \end{pmatrix},$$

and the complex outer coupling configuration matrix C is given by

$$C = \begin{pmatrix} 2-j & 0 & 0 & 0 & 0 & 0 & 1+j & 0 \\ 1+j & -1+j & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+2j & 0 & 0 & 0 & -2+j & 0 \\ 0 & 0 & 0 & -j & 0 & 0 & 0 & -1-j \\ 0 & 1+j & 0 & 1+j & 0.5-j & 0 & 0 & 0 \\ 0 & 1 & 1-j & 0 & 0 & -1-2j & 0 & 0 \\ 0 & 0 & 0 & -1+2j & 0 & 1 & 1+j & 0 \\ 0 & 0 & 1-j & 0 & 0 & 0 & 0 & 2j \end{pmatrix},$$

and $\tilde{C} = diag(3, 1+2j, -1+3j, -1-2j, 2.5+j, 1-3j, 1+3j, 1+j)$.

Let distributed time-delay $r = 0.1$. Choose the three indices $\sigma_{\max}, \sigma_{\min}$ and Δ in ETPIC scheme (3.114)-(3.116) as $\sigma_{\max} = 1.5, \sigma_{\min} = 0.8, \Delta = 0.4$. Let $\mu = 20$, $\omega = 0.01$, using the MATLAB YALMIP toolbox, the LMI (3.94) has the following feasible solution:

$$P = \begin{pmatrix} 0.3343 & 0.0273 - 0.0061j \\ 0.0273 + 0.0061j & 0.4323 \end{pmatrix}, \quad Q = \begin{pmatrix} 38.0446 & 0.2466 - 4.2016j \\ 0.2466 + 4.2016j & 37.6242 \end{pmatrix},$$
$$R = \begin{pmatrix} 67.7697 & 0.4225 + 0.0084j \\ 0.4225 - 0.0084j & 69.1812 \end{pmatrix}.$$

The Lyapunov functional V in ETPIC (3.114)-(3.116) is chosen as (3.93) with ω, P, Q, R shown above. Let $l_1 = l_2 = 6$ (i.e., if the event from L_1 or L_2 occurs at $t = t_k$, then 6 nodes will be impulsively controlled at instant $t = t_k$), choose the impulsive control gains as $q_1 = 0.01 + 0.05j$, and $q_2 = 0.01 - 0.05j$, then conditions (3.118) and (3.119) in Theorem 3.6.1 are satisfied. Theorem 3.6.1 implies that CVDN (3.89) achieves synchronization via ETPIC (3.114)-(3.116), and the convergence rate of synchronization is $\frac{-\ln \sigma_{\min}}{14\Delta} = 0.04$. Moreover, ETPIC (3.114)-(3.116) is non-Zeno satisfying $\frac{\ln \sigma_{\max}}{\mu} = 0.0203 \leq t_{k+1} - t_k \leq 0.4 = \Delta,\ k \in \mathbb{N}$.

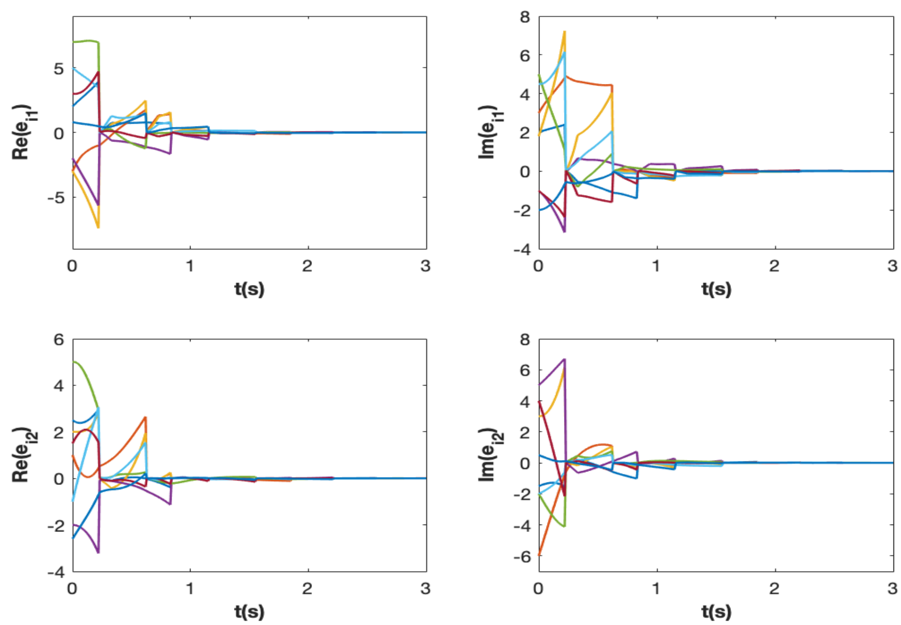

Figure 3.13: Trajectories of real and imaginary parts of 2-dimensional synchronization errors for CVDN (3.89) via ETPIC (3.114)-(3.116) with $r = 0.1$.

The initial conditions $\varphi_i(\alpha)$ *(*$i = 1, 2, ..., 8$*) of CVDN* (3.89) *are randomly chosen as* $[3+j; 1.5-0.5j; -2+2j; -5j; -2+0.8j; 1+4j; -1-2j; -3+6j; 8+4j; 4-j; 6+3.5j; -2-j; 4-2j; 0.5+5j; 1.8-3j; -3.6+1.5j]^T$ *for* $\alpha \in [-0.1, 0]$*, and the initial value of the isolated system* (3.90) *is chosen as* $\psi(\alpha) = [1-j; -1+j]^T$ *for* $\alpha \in [-0.1, 0]$*. Figure 3.13 shows the time evolution of real and imaginary parts of synchronization errors for CVDN* (3.89) *under ETPIC* (3.114)-(3.116) *with* $r = 0.1$*, and the corresponding triggered time instants of three levels of events and release intervals are plotted in Figure 3.14. From the result of the simulation in Figure 3.13, it is clear that CVDN* (3.89) *achieves synchronization.*

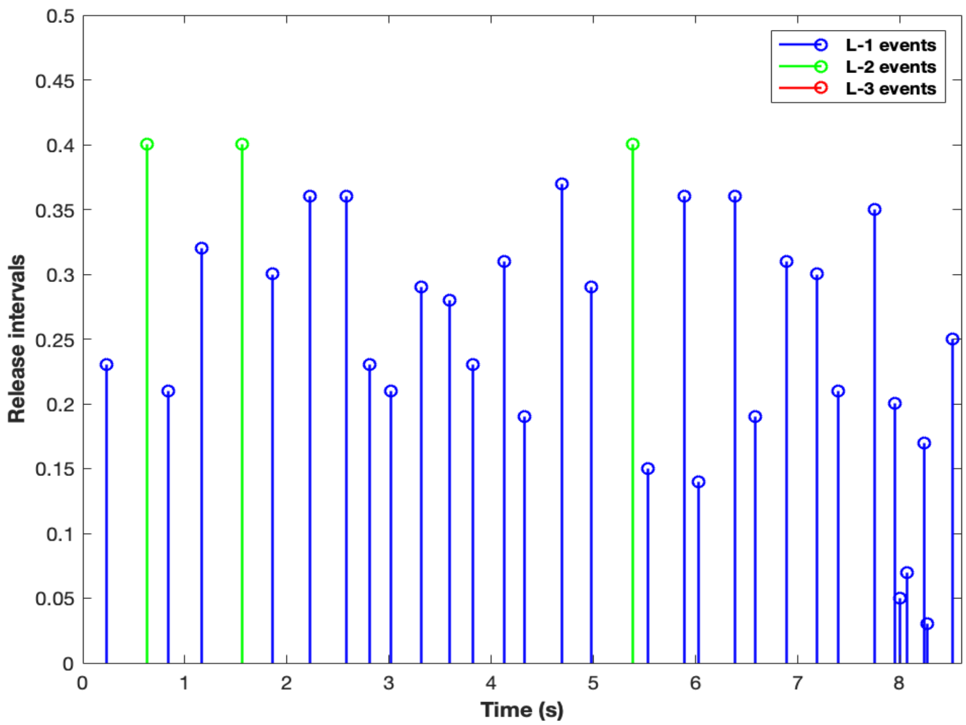

Figure 3.14: Triggered instants of three levels of events and release intervals for Example 3.6.1.

Example 3.6.2. *Consider delay-free CVDN* (3.130) *coupled with eight identical nodes with*

$$A = \begin{pmatrix} 0.25+0.75j & 0.2-j \\ 0.5+2j & -0.1+0.3j \end{pmatrix},$$

and the complex outer coupling configuration matrix C is chosen as

$$C = \begin{pmatrix} 1+j & -1+j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+j & 0.2+0.3j & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 1+j & 0.4-j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+j & 0.8+0.2j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+j & -1+j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+j & -0.6+0.15j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+j & 0.7+0.5j \\ 0.1+0.6j & 0 & 0.2-0.1j & 0 & 0 & 0 & 0 & 1+j \end{pmatrix},$$

and $\tilde{C} = diag(2j, 1.5+1.3j, 1.4, 1.8+1.2j, 2j, 0.4+1.15j, 1.7+1.5j, 1.3+1.5j)$.

Choose the three indices $\sigma_{\max}, \sigma_{\min}$ *and* Δ *in ETPIC scheme* (3.133)-(3.135) *as* $\sigma_{\max} = 1.2, \sigma_{\min} = 0.6, \Delta = 0.15$. *Let* $\mu = 15$, *solving the LMI* (3.132), *then the positive definite Hermitian matrix* P *in ETPIC scheme* (3.133)-(3.135) *is given by*

$$P = \begin{pmatrix} 8.0469 & 0.3270 - 1.6561j \\ 0.3270 + 1.6561j & 7.4577 \end{pmatrix}.$$

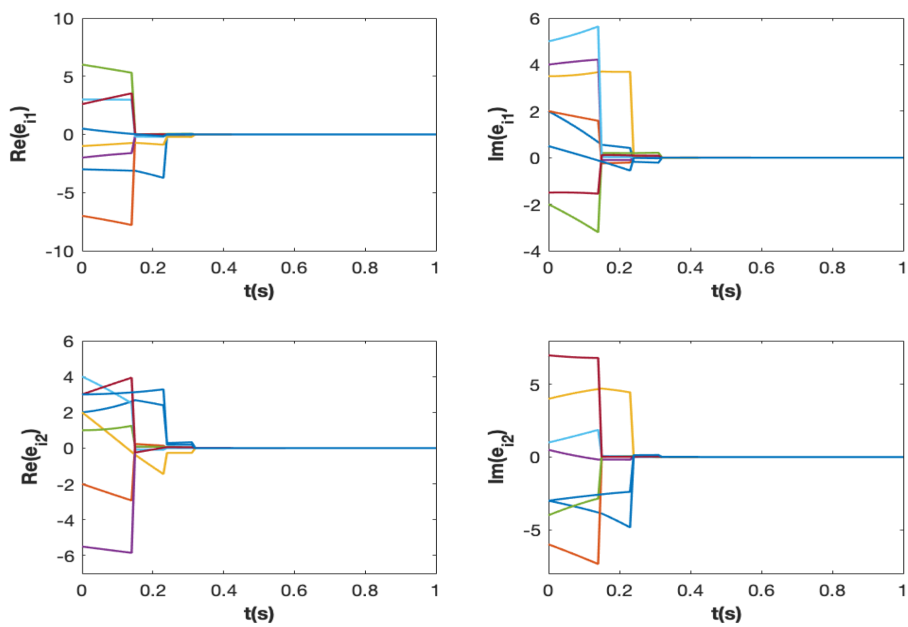

Figure 3.15: Trajectories of real and imaginary parts of 2-dimensional synchronization errors for delay-free CVDN (3.130) via ETPIC (3.133)-(3.135).

Let $l_1 = 6, l_2 = 5$ (i.e., 6 nodes will be controlled if the first level of events occur, and 5 nodes will be controlled if the second level of events occur), choose impulsive control gains as $q_1 = 0.2+0.6j$, and $q_2 = -0.08+0.2j$, then conditions (3.137) and (3.138) in Theorem 3.6.2 are satisfied. Theorem 3.6.2 implies that delay-free CVDN (3.130) achieves synchronization via ETPIC (3.133)-(3.135) with convergence rate $\frac{-\ln\sigma_{\min}}{2\Delta} = 1.703$, and ETPIC (3.133)-(3.135) is non-Zeno satisfying $\frac{\ln\sigma_{\max}}{\mu} = 0.012 \leq t_{k+1} - t_k \leq 0.15 = \Delta,\ k \in \mathbb{N}$.

The initial conditions for CVDN (3.130) with $i = 1, 2, ..., 8$ are randomly chosen as $[-2+j; 1-2j; -6+j; -3-5j; 2.5j; 1+5j; -1+3j; -6.5+1.5j; 7-3j; -3j; 4+4j; 3+2j; 3.6-2.5j; 2+8j; 1.5-0.5j; 2-2j]^T$, and initial condition of isolated system $\dot{s}(t) = As(t)$ is chosen as $[1-j; -1+j]^T$. Figure 3.15 shows trajectories of real and imaginary parts of synchronization errors for CVDN (3.130) under ETPIC (3.133)-(3.135), and the corresponding triggered time instants of three levels of events and release intervals are plotted in Figure 3.16. It can be seen from the result of the simulation in Figure 3.15 that synchronization of delay-free CVDN (3.130) is achieved.

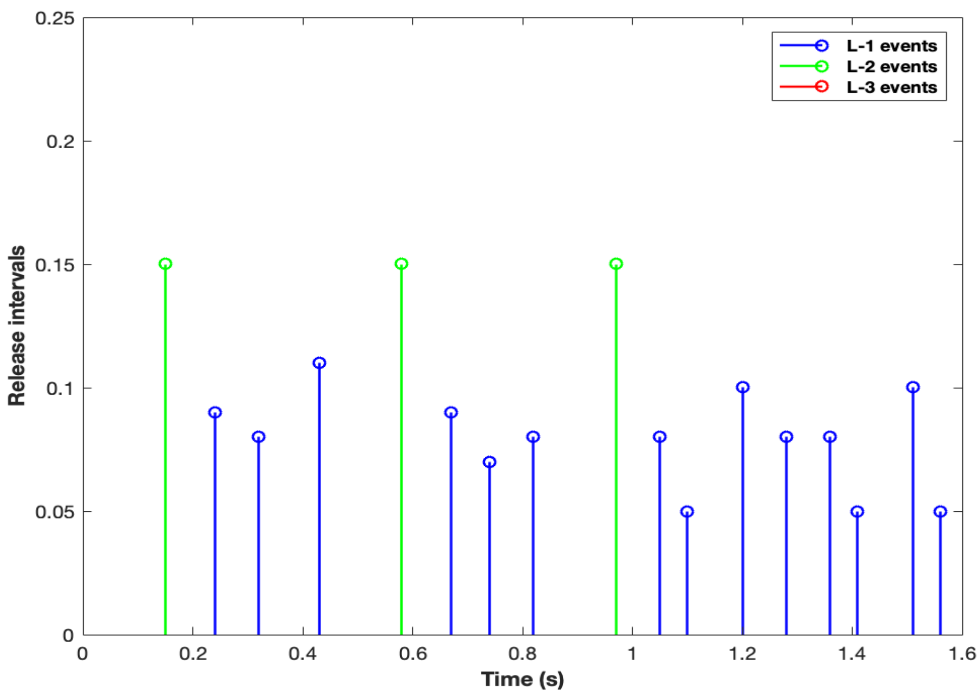

Figure 3.16: Triggered instants of three levels of events and release intervals for Example 3.6.2.

Chapter 4

Generalized Outer Synchronization of Time-Delay CVDNs

This Chapter studies generalized outer synchronization of drive-response time-delayed CVDNs via hybrid control. In Section 4.1, we formulate the generalized outer synchronization problem of drive-response CVDNs, and a hybrid controller is proposed in the complex domain to construct response complex-valued networks. In Section 4.2, some generalized outer synchronization criteria for drive-response CVDNs are established under the proposed hybrid controller, which extend the existing generalized outer synchronization results to the complex field. Results in Section 4.2 show that the proposed hybrid controller can effectively construct a corresponding response complex-valued network for achieving generalized outer synchronization with the drive time-delayed CVDN. In Section 4.3, numerical simulations are given to illustrate the effectiveness of the proposed control strategy.

4.1 Problem Formulation and Preliminaries

Consider a time-delay CVDN consisting of N nonidentical coupling nodes, which can be described as follows:

$$\dot{z}_i(t) = f_i(t, z_i(t-\tau_1)) + \sum_{j=1}^{N} b_{ij} A z_j(t) + \sum_{j=1}^{N} c_{ij} A z_j(t-\tau_2), \quad i = 1, 2, \ldots, N, \tag{4.1}$$

where $z_i = (z_{i1}, z_{i2}, \ldots, z_{in})^T \in \mathbb{C}^n$ is the state vector of the i-th node, $f_i : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$

denotes the nonlinear complex-valued analytic function describing the intrinsic dynamics of node i, τ_1 stands for the internal delay, and τ_2 represents the the transmission delay when processing information from the j-th node. $A \in \mathbb{C}^{n\times n}$ denotes the inner coupling matrix, $B = [b_{ij}]_{N\times N} \in \mathbb{C}^{N\times N}$ and $C = [c_{ij}]_{N\times N} \in \mathbb{C}^{N\times N}$ correspond to the current-state and delayed outer coupling configuration matrices, respectively, where b_{ij} (c_{ij}) is defined as: if there exists a directed link from node i to node j $(j \neq i)$ at time t (at time $t-\tau_2$), then the coupling strength $b_{ij} \neq 0$ $(c_{ij} \neq 0)$; otherwise, $b_{ij} = 0$ $(c_{ij} = 0)$, and the diagonal elements of matrices B and C are given by

$$b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij}, \quad c_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij}, \qquad i = 1, 2, \ldots, N.$$

The initial condition of system (4.1) is given by $z_i(t_0 + s) = \varphi_i(s),\ i = 1, 2, \ldots, N$, where $\varphi_i \in \mathcal{PC}([-\tau, 0], \mathbb{C}^n)$, and $\tau = \max\{\tau_1, \tau_2\}$.

For complex-valued analytic functions f_i, we make the following assumption:

Assumption 4.1.1. *Suppose that there exist positive constants L_i such that*

$$\|f_i(t,u) - f_i(t,v)\| \leq L_i \|u - v\|, \qquad i = 1, 2, \ldots, N$$

for all $t \in \mathbb{R}, u, v \in \mathbb{C}^n$. Denote $L = \max_{1\leq i\leq N}\{L_i\}$.

The system (4.1) is regarded as the drive network. Then, the corresponding controlled complex-valued response network is established as follows:

$$\dot{\hat{z}}_i(t) = u_i(t), \qquad i = 1, 2, \ldots, N, \tag{4.2}$$

where $\hat{z}_i = (\hat{z}_{i1}, \hat{z}_{i2}, \ldots, \hat{z}_{in})^T \in \mathbb{C}^n$ denotes the response state vector of the i-th node, and $u_i(t)$ is the control input of node i. The initial condition of the response system (4.2) is given by $\hat{z}_i(t_0 + s) = \psi_i(s),\ i = 1, 2, \ldots, N$, where $\psi_i \in \mathcal{PC}([-\tau, 0], \mathbb{C}^n)$.

The objective is to design proper hybrid controller $u_i(t)$ to construct the complex-valued response network (4.2) such that drive-response CVDNs (4.1)-(4.2) can achieve generalized outer synchronization, i.e.,

$$\lim_{t\to\infty} \|z_i(t) - \phi_i(\hat{z}_i(t))\| = 0, \qquad i = 1, 2, \ldots, N,$$

where $\phi_i : \mathbb{C}^n \to \mathbb{C}^n$ $(i = 1, 2, \ldots, N)$ is a continuously differentiable map with an inverse map ϕ_i^{-1}.

Let $\hat{z}_{il} = \hat{z}_{il}^R + j\hat{z}_{il}^I$ for $l = 1, 2, \ldots, n$, $\phi_i = (\phi_{i1}, \phi_{i2}, \ldots, \phi_{in})^T \in \mathbb{C}^n$, and

$$J_{\phi_i} = \frac{1}{2} \begin{pmatrix} \frac{\partial \phi_{i1}}{\partial \hat{z}_{i1}^R} - j\frac{\partial \phi_{i1}}{\partial \hat{z}_{i1}^I} & \frac{\partial \phi_{i1}}{\partial \hat{z}_{i2}^R} - j\frac{\partial \phi_{i1}}{\partial \hat{z}_{i2}^I} & \cdots & \frac{\partial \phi_{i1}}{\partial \hat{z}_{in}^R} - j\frac{\partial \phi_{i1}}{\partial \hat{z}_{in}^I} \\ \frac{\partial \phi_{i2}}{\partial \hat{z}_{i1}^R} - j\frac{\partial \phi_{i2}}{\partial \hat{z}_{i1}^I} & \frac{\partial \phi_{i2}}{\partial \hat{z}_{i2}^R} - j\frac{\partial \phi_{i2}}{\partial \hat{z}_{i2}^I} & \cdots & \frac{\partial \phi_{i2}}{\partial \hat{z}_{in}^R} - j\frac{\partial \phi_{i2}}{\partial \hat{z}_{in}^I} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_{in}}{\partial \hat{z}_{i1}^R} - j\frac{\partial \phi_{in}}{\partial \hat{z}_{i1}^I} & \frac{\partial \phi_{in}}{\partial \hat{z}_{i2}^R} - j\frac{\partial \phi_{in}}{\partial \hat{z}_{i2}^I} & \cdots & \frac{\partial \phi_{in}}{\partial \hat{z}_{in}^R} - j\frac{\partial \phi_{in}}{\partial \hat{z}_{in}^I} \end{pmatrix} \in \mathbb{C}^{n\times n}$$

be the complex Jacobian of the map $\phi_i(\hat{z}_i)$.

To enable the drive-response CVDNs (4.1) and (4.2) achieve generalized outer synchronization, we consider the following hybrid controller:

$$\begin{aligned} u_i(t) =& J_{\phi_i}^{-1}\big[f_i\big(t, \phi_i(\hat{z}_i(t-\tau_1))\big) + \sum_{j=1}^{N} b_{ij} A\phi_j(\hat{z}_j(t)) + \sum_{j=1}^{N} c_{ij} A\phi_j(\hat{z}_j(t-\tau_2))\big] \\ &+ \sum_{k=1}^{\infty} U_{ik}(z_i(t), \hat{z}_i(t))\delta(t - t_k^-), \end{aligned} \tag{4.3}$$

where $U_{ik}(z_i(t), \hat{z}_i(t)) = \phi_i^{-1}\big[(I_n + D_{ik})\phi_i(\hat{z}_i(t)) - D_{ik} z_i(t)\big] - \hat{z}_i(t)$, and $D_{ik} \in \mathbb{C}^{n\times n}$ is the impulsive gain matrix for the i-th node at impulsive instant t_k to be designed; $J_{\phi_i}^{-1}$ is the inverse matrix of the complex Jacobian J_{ϕ_i}; $\delta(\cdot)$ is the Dirac Delta function. The impulsive sequence $\zeta = \{t_1, t_2, t_3, \ldots\}$ satisfies $0 \le t_0 < t_1 < \ldots < t_k < \ldots$, and $\lim_{k\to\infty} t_k = \infty$.

Under the hybrid controller (4.3), the complex-valued response network (4.2) can be constructed as follows:

$$\begin{cases} \dot{\hat{z}}_i(t) = J_{\phi_i}^{-1}\big[f_i\big(t, \phi_i(\hat{z}_i(t-\tau_1))\big) + \sum_{j=1}^{N} b_{ij} A\phi_j(\hat{z}_j(t)) + \sum_{j=1}^{N} c_{ij} A\phi_j(\hat{z}_j(t-\tau_2))\big], \quad t \neq t_k, \\ \Delta\hat{z}_i(t) = U_{ik}(z_i(t^-), \hat{z}_i(t^-)), \qquad t = t_k, \quad k \in \mathbb{N}^+, \\ \hat{z}_i(t_0 + s) = \psi_i(s), \quad s \in [-\tau, 0] \end{cases} \tag{4.4}$$

for $i = 1, 2, \ldots, N$, where $\Delta\hat{z}_i(t_k) = \hat{z}_i(t_k^+) - \hat{z}_i(t_k^-)$. We assume solutions of system (4.4) are right continuous, i.e., $\hat{z}_i(t_k) = \hat{z}_i(t_k^+)$.

Define the generalized synchronization error as $e_i(t) = z_i(t) - \phi_i(\hat{z}_i(t)), i = 1, 2, \ldots, N$. Then from (4.1), (4.4) and the chain rule for complex-variable functions, we can obtain the following error dynamical system:

$$\begin{cases} \dot{e}_i(t) = \bar{F}_i(t, e_i(t-\tau_1)) + \sum_{j=1}^{N} b_{ij} A e_j(t) + \sum_{j=1}^{N} c_{ij} A e_j(t-\tau_2), & t \neq t_k, \\ e_i(t_k) = z_i(t_k^-) - \phi_i\big(\hat{z}_i(t_k^-) + U_{ik}(z_i(t_k^-), \hat{z}_i(t_k^-))\big), & k \in \mathbb{N}^+, \\ e_i(t_0 + s) = \varphi_i(s) - \phi_i(\psi_i(s)), \quad s \in [-\tau, 0] \end{cases} \tag{4.5}$$

for $i = 1, 2, \ldots, N$, where $\bar{F}_i(t, e_i(t-\tau_1)) = f_i(t, z_i(t-\tau_1)) - f_i\big(t, \phi_i(\hat{z}_i(t-\tau_1))\big)$.

Then the generalized outer synchronization problem of drive-response CVDNs (4.1) and (4.2) is transformed into the stability problem of the error system (4.5).

The following lemmas and definitions will be used to derive the main results in Section 4.2.

Lemma 4.1.1. *[87] Consider the following impulsive differential inequality*

$$\begin{cases} D^+ u(t) \leq p u(t) + q \bar{u}(t), & t \geq t_0, \ t \neq t_k, \\ u(t_k) \leq a_k u(t_k^-), & k \in \mathbb{N}^+, \end{cases} \tag{4.6}$$

where $u \in \mathcal{PC}([t_0 - \tau, +\infty), \mathbb{R}^+)$, and $\bar{u}(t) = \sup_{-\tau \leq s \leq 0} u(t+s)$. Suppose that $p \in \mathbb{R}, q \geq 0$, and let $a_0 := 1$, if there exists $\delta > 1$ such that $a_k \geq \frac{1}{\delta}$ for all $k \in \mathbb{N}$, and

$$p + q\delta < \frac{\ln \delta}{\sigma}, \qquad \textit{where} \ \ \sigma := \sup_{k \in \mathbb{N}} \{t_{k+1} - t_k\},$$

and suppose

$$0 < \lambda < \frac{\ln \delta}{\sigma} - p - q\delta e^{\lambda \tau}.$$

Then any solution of (4.6) satisfies

$$u(t) \leq \delta^{k+1} \bar{u}(t_0) \bigg(\prod_{m=0}^{k} a_m \bigg) e^{-\lambda(t - t_0)}, \qquad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}.$$

Lemma 4.1.2. *[96] Let $0 \leq \tau_i(t) \leq \tau$. Suppose that $u(t), v(t) \in \mathcal{PC}([-\tau, \infty), \mathbb{R})$ satisfy*

$$\begin{cases} D^+ u(t) \leq F(t, u(t), u(t - \tau_1(t)), \ldots, u(t - \tau_m(t))), & t \geq 0, \ t \neq t_k, \\ u(t_k) \leq I_k(u(t_k^-)), & k \in \mathbb{N}^+, \end{cases}$$

and

$$\begin{cases} D^+v(t) > F(t, v(t), v(t-\tau_1(t)), \ldots, v(t-\tau_m(t))), & t \geq 0,\ t \neq t_k, \\ v(t_k) \geq I_k(v(t_k^-)), \quad k \in \mathbb{N}^+, \end{cases}$$

where $F(t, u, \bar{u}_1, \ldots, \bar{u}_m) : \mathbb{R}^+ \times \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{m+1} \to \mathbb{R}$ *is nondecreasing in* $\bar{u}_i$ *for each fixed* $(t, u, \bar{u}_1, \ldots, \bar{u}_{i-1}, \bar{u}_{i+1}, \ldots, \bar{u}_m)$*,* $i = 1, 2, \ldots, m$*, and* $I_k(u) : \mathbb{R} \to \mathbb{R}$ *is nondecreasing in* u*. Then* $u(t) \leq v(t)$ *for* $-\tau \leq t \leq 0$ *implies that* $u(t) \leq v(t)$ *for* $t \geq 0$*.*

Definition 4.1.1. *[97]* T_a *is said to be the average impulsive interval (AII) of the impulsive sequence* $\zeta = \{t_1, t_2, \ldots\}$ *if there exists a positive integer* N_0 *such that*

$$\frac{T-t}{T_a} - N_0 \leq N_\zeta(t, T) \leq \frac{T-t}{T_a} + N_0, \qquad \forall T \geq t \geq t_0,$$

where $N_\zeta(t, T)$ *denotes the number of impulses of the impulsive sequence* ζ *on the interval* (t, T)*.*

Definition 4.1.2. *[98] The average impulsive gain (AIG) of the impulsive sequence* ζ *in* (4.6) *is defined as follows:*

$$\mu_a = \lim_{t\to\infty} \frac{|a_1| + |a_2| + \ldots + |a_{N_\zeta(t_0,t)}|}{N_\zeta(t_0, t)},$$

where $N_\zeta(t_0, t)$ *denotes the number of impulses of the impulsive sequence* ζ *on the interval* (t_0, t)*.*

4.2 Synchronization Results

In this section, by employing the Lyapunov function in the complex field and the comparison principle, some generalized outer synchronization criteria for drive-response CVDNs (4.1) and (4.2) are presented under the proposed hybrid controller (4.3). Furthermore, when constructing the impulsively controlled response network (4.4), the concepts of AII and AIG are used to deal with the situation that synchronizing impulses and desynchronizing impulses exist simultaneously in the impulsive sequence.

Theorem 4.2.1. *Suppose that Assumption 4.1.1 holds. Let* $\alpha = L + 2\|B \otimes A\| + \|C \otimes A\|$*,* $\beta = L + \|C \otimes A\|$*,* a_k *be the largest eigenvalue of* $(I_n + D_{ik})^*(I_n + D_{ik})$ *for* $k \in \mathbb{N}^+$*,* T_a

and μ_a *are the AII and AIG of the impulsive sequence* ζ*, respectively. If* $\mu_a < 1$*, and there exist constants* $\delta > 1, \lambda > 0$ *such that*

$$\alpha + \beta\delta < \frac{\ln\delta}{\sigma}, \qquad \textit{where} \ \ \sigma = \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\}, \tag{4.7}$$

$$a_k \geq \frac{1}{\delta} \qquad \textit{for all} \ \ k \in \mathbb{N}^+, \tag{4.8}$$

$$\lambda < \frac{\ln\delta}{\sigma} - \alpha - \beta\delta e^{\lambda\tau}, \tag{4.9}$$

and

$$\gamma = \lambda - \frac{\ln(\delta\mu_a)}{T_a} > 0, \tag{4.10}$$

then generalized outer synchronization of drive-response CVDNs (4.1) *and* (4.2) *can be achieved under the hybrid controller* (4.3)*. Moreover, the convergence rate of synchronization is* γ*.*

Proof. Consider the Lyapunov function candidate for the error system (4.5) as follows:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^*(t)e_i(t).$$

For $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, take derivative of $V(t)$ along the trajectory of error system (4.5), we have

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N} \mathfrak{Re}[e_i^*(t)\dot{e}_i(t)] \\
&= \sum_{i=1}^{N} \mathfrak{Re}\big[e_i^*(t)\big(f_i(t, z_i(t-\tau_1)) - f_i\big(t, \phi_i(\hat{z}_i(t-\tau_1))\big) + \sum_{j=1}^{N} b_{ij}Ae_j(t) + \sum_{j=1}^{N} c_{ij}Ae_j(t-\tau_2)\big)\big] \\
&\leq \sum_{i=1}^{N} \big|e_i^*(t)\big(f_i(t, z_i(t-\tau_1)) - f_i(t, \phi_i(\hat{z}_i(t-\tau_1)))\big)\big| + \mathfrak{Re}\Bigg[\sum_{i=1}^{N} e_i^*(t)\sum_{j=1}^{N} b_{ij}Ae_j(t)\Bigg] \\
&+ \mathfrak{Re}\Bigg[\sum_{i=1}^{N} e_i^*(t)\sum_{j=1}^{N} c_{ij}Ae_j(t-\tau_2)\Bigg].
\end{aligned}$$

Let $E(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T$. It follows from Cauchy-Schwarz inequality and Assumption 4.1.1 that

$$
\begin{aligned}
\dot{V}(t) \leq & \sum_{i=1}^{N} \|e_i(t)\| \|f_i(t, z_i(t-\tau_1)) - f_i(t, \phi_i(\hat{z}_i(t-\tau_1)))\| + \mathfrak{Re}\big[E^*(t)(B \otimes A)E(t)\big] \\
& + \mathfrak{Re}\big[E^*(t)(C \otimes A)E(t-\tau_2)\big] \\
& \leq \sum_{i=1}^{N} L_i \|e_i(t)\| \|e_i(t-\tau_1)\| + \big|E^*(t)(B \otimes A)E(t)\big| + \big|E^*(t)(C \otimes A)E(t-\tau_2)\big| \\
& \leq \frac{L}{2} \sum_{i=1}^{N} (\|e_i(t)\|^2 + \|e_i(t-\tau_1)\|^2) + \|B \otimes A\| \|E(t)\|^2 + \|E(t)\| \|C \otimes A\| \|E(t-\tau_2)\| \\
& \leq LV(t) + LV(t-\tau_1) + 2\|B \otimes A\| V(t) + \frac{\|C \otimes A\|}{2} (\|E(t)\|^2 + \|E(t-\tau_2)\|^2) \\
& = (L + 2\|B \otimes A\| + \|C \otimes A\|)V(t) + LV(t-\tau_1) + \|C \otimes A\| V(t-\tau_2) \\
& \leq \alpha V(t) + \beta \sup_{s \in [-\tau, 0]} V(t+s),
\end{aligned} \tag{4.11}
$$

where $\tau = \max\{\tau_1, \tau_2\}$.

On the other hand, for $t = t_k, k \in \mathbb{N}^+$, it follows from (4.5) that

$$
\begin{aligned}
V(t_k) = & \frac{1}{2} \sum_{i=1}^{N} \big[z_i(t_k^-) - \big((I_n + D_{ik})\phi_i(\hat{z}_i(t_k^-)) - D_{ik} z_i(t_k^-)\big)\big]^* \big[z_i(t_k^-) - \big((I_n + D_{ik})\phi_i(\hat{z}_i(t_k^-)) \\
& - D_{ik} z_i(t_k^-)\big)\big] \\
& = \frac{1}{2} \sum_{i=1}^{N} \big[(I_n + D_{ik}) e_i(t_k^-)\big]^* \big[(I_n + D_{ik}) e_i(t_k^-)\big] \\
& \leq \frac{\lambda_{\max}\big((I_n + D_{ik})^*(I_n + D_{ik})\big)}{2} \sum_{i=1}^{N} e_i^*(t_k^-) e_i(t_k^-) = a_k V(t_k^-).
\end{aligned} \tag{4.12}
$$

Let $a_0 = 1$, combining (4.11), (4.12) and conditions (4.7)-(4.9), it follows from Lemma 4.1.1 that

$$
V(t) \leq \delta^{k+1} \|V(t_0)\|_\tau \bigg(\prod_{m=0}^{k} a_m\bigg) e^{-\lambda(t-t_0)}, \qquad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}.
$$

Hence, applying the mean value inequality, we have for $\forall t \geq t_1$,

$$\begin{aligned}
V(t) &\leq \delta \|V(t_0)\|_\tau \prod_{t_0 < t_k \leq t} \big(a_k \delta\big) e^{-\lambda(t-t_0)} \\
&\leq \delta \|V(t_0)\|_\tau \big(a_1 \delta\big)\big(a_2 \delta\big) \ldots \big(a_{N_\zeta(t_0,t)} \delta\big) e^{-\lambda(t-t_0)} \\
&= \delta^{N_\zeta(t_0,t)+1} \|V(t_0)\|_\tau \big(a_1 a_2 \ldots a_{N_\zeta(t_0,t)}\big) e^{-\lambda(t-t_0)} \\
&\leq \delta^{N_\zeta(t_0,t)+1} \|V(t_0)\|_\tau \left(\frac{|a_1| + |a_2| + \ldots + |a_{N_\zeta(t_0,t)}|}{N_\zeta(t_0,t)} \right)^{N_\zeta(t_0,t)} e^{-\lambda(t-t_0)}.
\end{aligned}$$

According to the definition of AIG, there exists a large enough positive number T such that

$$V(t) \leq \delta^{N_\zeta(t_0,t)+1} \|V(t_0)\|_\tau \mu_a^{N_\zeta(t_0,t)} e^{-\lambda(t-t_0)}, \qquad \forall t > T. \tag{4.13}$$

From the definition of AII, we have

$$\frac{t-t_0}{T_a} - N_0 \leq N_\zeta(t_0,t) \leq \frac{t-t_0}{T_a} + N_0, \qquad \forall t \geq t_0, \tag{4.14}$$

where N_0 is a positive integer. By $\delta > 1$, it follows from (4.13) and (4.14) that

$$\begin{aligned}
V(t) &\leq \delta^{N_0+1} \delta^{\frac{t-t_0}{T_a}} \|V(t_0)\|_\tau \mu_a^{N_\zeta(t_0,t)} e^{-\lambda(t-t_0)} \\
&= \delta^{N_0+1} \|V(t_0)\|_\tau \mu_a^{N_\zeta(t_0,t)} e^{-(\lambda - \frac{\ln \delta}{T_a})(t-t_0)}, \qquad \forall t > T.
\end{aligned} \tag{4.15}$$

If $\mu_a < 1$, then it follows from (4.14) and (4.15) that

$$\begin{aligned}
V(t) &\leq \delta^{N_0+1} \|V(t_0)\|_\tau \mu_a^{(\frac{t-t_0}{T_a} - N_0)} e^{-(\lambda - \frac{\ln \delta}{T_a})(t-t_0)} \\
&= \frac{\delta^{N_0+1}}{\mu_a^{N_0}} \|V(t_0)\|_\tau e^{-\left(\lambda - \frac{\ln(\delta \mu_a)}{T_a}\right)(t-t_0)}, \qquad \forall t > T.
\end{aligned}$$

If the AII of the impulsive sequence ζ satisfies condition (4.10), then we have $V(t) \to 0$ as $t \to \infty$, which implies that $\|e_i(t)\| \to 0$ as $t \to \infty$ for $i = 1, 2, \ldots, N$. Therefore, the generalized outer synchronization of drive-response CVDNs (4.1) and (4.2) is achieved under the hybrid controller (4.3), and the convergence rate of synchronization is γ. □

Remark 4.2.1. *Zhang et al. [99] investigated generalized synchronization of delay-free complex dynamical networks by impulsive control, while in this section, both internal delay and transmission delay of networks are taken into account, and the generalized synchro-*

nization result is derived in the complex domain. Furthermore, when designing the hybrid controller (4.3) to realize generalized outer synchronization of networks, the impulsive sequence can simultaneously contain synchronizing impulses and desynchronizing impulses, which is a substantial extension of the traditional impulsive controller.

Remark 4.2.2. *In particular, if the map $\phi_i(\hat{z}_i(t)) = P\hat{z}_i(t) + Q$, where P, Q are complex matrices with proper dimension, then the proposed hybrid controller (4.3) reduces to $u_i(t) = P^{-1}\big[f_i\big(t, P\hat{z}_i(t-\tau_1)+Q\big) + \sum_{j=1}^{N} b_{ij}A(P\hat{z}_j + Q) + \sum_{j=1}^{N} c_{ij}A(P\hat{z}_j(t-\tau_2)+Q)\big] + \sum_{k=1}^{\infty} U_{ik}(z_i(t), \hat{z}_i(t))\delta(t-t_k^-)$, where $U_{ik}(z_i(t), \hat{z}_i(t)) = P^{-1}D_{ik}\big[P\hat{z}_i(t)+Q-z_i(t)\big]$. According to Theorem 4.2.1, the following linear generalized outer synchronization criteria can be obtained.*

Corollary 4.2.1. *Suppose that Assumption 4.1.1 holds. Let $\alpha = L + 2\|B\otimes A\| + \|C\otimes A\|$, $\beta = L + \|C \otimes A\|$, a_k be the largest eigenvalue of $(I_n + D_{ik})^*(I_n + D_{ik})$ for $k \in \mathbb{N}^+$, and T_a, μ_a are the AII and AIG of the impulsive sequence ζ, respectively. If $\mu_a < 1$, and there exist constants $\delta > 1, \lambda > 0$ such that (4.7)-(4.10) hold, then the drive-response CVDNs (4.1) and (4.2) can achieve linear generalized outer synchronization under the hybrid controller*

$$\begin{aligned} u_i(t) =& P^{-1}\big[f_i\big(t, P\hat{z}_i(t-\tau_1)+Q\big) + \sum_{j=1}^{N} b_{ij}A\big(P\hat{z}_j(t)+Q\big) + \sum_{j=1}^{N} c_{ij}A\big(P\hat{z}_j(t-\tau_2)+Q\big)\big] \\ &+ \sum_{k=1}^{\infty} P^{-1}D_{ik}\big(P\hat{z}_i(t)+Q-z_i(t)\big)\delta(t-t_k^-). \end{aligned} \tag{4.16}$$

Moreover, the convergence rate of synchronization is γ.

Proof. If $\phi_i(\hat{z}_i) = P\hat{z}_i + Q$, then we have $J_{\phi_i} = P$, and $\phi_i^{-1}(\hat{z}_i) = P^{-1}(\hat{z}_i - Q)$, the results can be directly derived by Theorem 4.2.1. □

Remark 4.2.3. *In [53], some sufficient conditions for the linear generalized outer synchronization of two complex dynamical networks with coupling delay are derived by using the adaptive control method, which is a kind of continuous control strategy. As far as we know, there are very few investigations focusing on the linear generalized synchronization problem of networks with time delays via impulsive control. Corollary 4.2.1 fills this gap and extends the linear generalized synchronization result to the complex domain.*

Remark 4.2.4. *If the map $\phi_i(\hat{z}_i(t)) = \alpha\hat{z}_i(t)$, where $\alpha \in \mathbb{C}$, then the proposed hybrid controller (4.3) reduces to $u_i(t) = \frac{1}{\alpha}f_i\big(t, \alpha\hat{z}_i(t-\tau_1)\big) + \sum_{j=1}^{N} b_{ij}A\hat{z}_j + \sum_{j=1}^{N} c_{ij}A\hat{z}_j(t-\tau_2) + \sum_{k=1}^{\infty} U_{ik}(z_i(t), \hat{z}_i(t))\delta(t-t_k^-)$, where $U_{ik}(z_i(t), \hat{z}_i(t)) = D_{ik}\big(\hat{z}_i(t) - \frac{1}{\alpha}z_i(t)\big)$. Theorem 4.2.1 yields the following projective outer synchronization criteria.*

Corollary 4.2.2. *Suppose that Assumption 4.1.1 holds. Let* $\alpha = L + 2\|B \otimes A\| + \|C \otimes A\|$, $\beta = L + \|C \otimes A\|$, a_k *be the largest eigenvalue of* $(I_n + D_{ik})^*(I_n + D_{ik})$ *for* $k \in \mathbb{N}^+$, *and* T_a, μ_a *are the AII and AIG of the impulsive sequence* ζ, *respectively. If* $\mu_a < 1$, *and there exist constants* $\delta > 1, \lambda > 0$ *such that* (4.7)-(4.10) *hold, then the the drive-response CVDNs* (4.1) *and* (4.2) *can achieve projective outer synchronization under the hybrid controller*

$$\begin{aligned} u_i(t) =& \frac{1}{\alpha} f_i\big(t, \alpha\hat{z}_i(t-\tau_1)\big) + \sum_{j=1}^{N} b_{ij} A\hat{z}_j(t) + \sum_{j=1}^{N} c_{ij} A\hat{z}_j(t-\tau_2) \\ &+ \sum_{k=1}^{\infty} D_{ik}\big(\hat{z}_i(t) - \frac{1}{\alpha} z_i(t)\big)\delta(t - t_k^-). \end{aligned} \tag{4.17}$$

Moreover, the convergence rate of synchronization is γ.

Proof. If $\phi_i(\hat{z}_i) = \alpha\hat{z}_i$, then we have $J_{\phi_i} = \alpha I_n$, and $\phi_i^{-1}(\hat{z}_i) = \frac{1}{\alpha}\hat{z}_i$, the conclusion follows from Theorem 4.2.1 directly, and the proof is complete. □

Remark 4.2.5. *In [57], projective synchronization for complex-variable drive-response dynamical networks was studied via impulsive control. However, the result in [57] implies that the impulsive sequence only contains synchronizing impulses, while the projective synchronization criteria derived in Corollary 4.2.2 are valid for the hybrid controller* (4.17) *containing synchronizing as well as desynchronizing impulses simultaneously, which can relax the restriction on impulsive sequences.*

Remark 4.2.6. *Note that the concepts of maximal impulsive interval and AII are both utilized in Theorem 4.2.1 to derive sufficient conditions for generalized outer synchronization of drive-response CVDNs. The conditions related to the impulsive sequence of the hybrid controller* (4.3) *in Theorem 4.2.1 is very conservative. In the following, a less conservative generalized outer synchronization result which only depend on the concept of AII is established by using the comparison principle for impulsive systems. Under the hybrid controller* (4.3), *the derived generalized outer synchronization criteria in Theorem 4.2.2 is valid for impulsive sequences with a wider range of impulsive intervals.*

Theorem 4.2.2. *Suppose that Assumption 4.1.1 holds. Let* $\alpha = L + 2\|B \otimes A\| + \|C \otimes A\|$, $\beta = L + \|C \otimes A\|$, a_k *be the largest eigenvalue of* $(I_n + D_{ik})^*(I_n + D_{ik})$ *for* $k \in \mathbb{N}^+$, *and* T_a, μ_a *are the AII and AIG of the impulsive sequence* ζ, *respectively. If* $\mu_a < 1$, *and*

$$\alpha + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0}\beta < 0, \tag{4.18}$$

then generalized outer synchronization of drive-response CVDNs (4.1) *and* (4.2) *can be achieved under the hybrid controller* (4.3)*. Moreover, the convergence rate of synchronization is* $\frac{\lambda}{2}$*, where* λ *is the unique positive root of the equation* $\lambda + \alpha + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0} L e^{\lambda \tau_1} + \mu_a^{-N_0} \|C \otimes A\| e^{\lambda \tau_2} = 0$.

Proof. Consider the following Lyapunov function candidate for the error dynamical system (4.5),

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} e_i^*(t) e_i(t).$$

It follows from the proof of Theorem 4.2.1 that

$$\dot{V}(t) \le \alpha V(t) + L V(t - \tau_1) + \|C \otimes A\| V(t - \tau_2), \quad t \ge t_0, \quad t \ne t_k,$$

and

$$V(t_k) \le a_k V(t_k^-), \quad k \in \mathbb{N}^+.$$

By using the comparison principle, we derive the following impulsive delay system

$$\begin{cases} \dot{v}(t) = \alpha v(t) + L v(t - \tau_1) + \|C \otimes A\| v(t - \tau_2) + \epsilon, \quad t \ge t_0, \quad t \ne t_k, \\ v(t_k) = a_k v(t_k^-), \quad k \in \mathbb{N}^+, \\ v(t_0 + s) = \dfrac{1}{2} \displaystyle\sum_{i=1}^{N} \|\varphi_i(s) - \phi_i(\psi_i(s))\|^2, \qquad -\tau \le s \le 0, \end{cases} \tag{4.19}$$

where $v(t)$ is a unique solution for any $\epsilon > 0$. Since $V(t) \le v(t)$ for $t_0 - \tau \le t \le t_0$, it follows from Lemma 4.1.2 that $V(t) \le v(t)$ for any $t \ge t_0$. By the formula for variation of parameters, $v(t)$ can be described by

$$v(t) = W(t, t_0) v(t_0) + \int_{t_0}^{t} W(t, s) \big[L v(s - \tau_1) + \|C \otimes A\| v(s - \tau_2) + \epsilon \big] ds, \quad t \ge t_0,$$

where $W(t, s)$, $t, s \ge t_0$ is the Cauchy matrix of the following linear impulsive system

$$\begin{cases} \dot{w}(t) = \alpha w(t), \quad t \ge t_0, \quad t \ne t_k, \\ w(t_k) = a_k w(t_k^-), \quad k \in \mathbb{N}^+. \end{cases}$$

According to the mean value inequality, the Cauchy matrix can be written as

$$W(t, s) = a_1 a_2 \cdots a_{N_{\zeta(s,t)}} e^{\alpha (t - s)}$$

$$\leq \left(\frac{|a_1| + |a_2| + \cdots + |a_{N_{\zeta(s,t)}}|}{N_{\zeta(s,t)}}\right)^{N_{\zeta(s,t)}} e^{\alpha(t-s)}, \qquad t \geq s \geq t_0.$$

From Definition 4.1.2, there exists a large enough positive number T such that

$$W(t,s) \leq \mu_a^{N_{\zeta(s,t)}} e^{\alpha(t-s)}$$

whenever $t > T$. If $\mu_a < 1$, it follows from Definition 4.1.1 that

$$W(t,s) \leq \mu_a^{\frac{t-s}{T_a} - N_0} e^{\alpha(t-s)} = \mu_a^{-N_0} e^{(\alpha + \frac{\ln \mu_a}{T_a})(t-s)}, \quad t > T.$$

Let $\xi = \frac{1}{2}\mu_a^{-N_0} \sup_{-\tau \leq s \leq 0} \sum_{i=1}^{N} \|\varphi_i(s) - \phi_i(\psi_i(s))\|^2$, then we have

$$\begin{aligned} v(t) \leq& \frac{1}{2}\sum_{i=1}^{N} \|\varphi_i(0) - \phi_i(\psi_i(0))\|^2 \mu_a^{-N_0} e^{(\alpha + \frac{\ln \mu_a}{T_a})(t-t_0)} + \int_{t_0}^{t} \mu_a^{-N_0} e^{(\alpha + \frac{\ln \mu_a}{T_a})(t-s)} \\ &\times \big[Lv(s-\tau_1) + \|C \otimes A\| v(s-\tau_2) + \epsilon\big] ds \\ \leq& \xi e^{(\alpha + \frac{\ln \mu_a}{T_a})(t-t_0)} + \int_{t_0}^{t} \mu_a^{-N_0} e^{(\alpha + \frac{\ln \mu_a}{T_a})(t-s)} \big[Lv(s-\tau_1) + \|C \otimes A\| v(s-\tau_2) + \epsilon\big] ds. \end{aligned} \tag{4.20}$$

Define $h(\lambda) = \lambda + \alpha + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0} L e^{\lambda \tau_1} + \mu_a^{-N_0} \|C \otimes A\| e^{\lambda \tau_2}$. It follows from condition (4.18) that $h(0) = \alpha + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0}(L + \|C \otimes A\|) < 0$. Furthermore, we have $h(\infty) > 0$ and $h'(\lambda) = 1 + \mu_a^{-N_0} L \tau_1 e^{\lambda \tau_1} + \mu_a^{-N_0} \|C \otimes A\| \tau_2 e^{\lambda \tau_2} > 0$ for $\lambda > 0$. Then by the IVT, $h(\lambda) = 0$ has a unique solution $\lambda > 0$.

If $\mu_a < 1$, it follows from condition (4.18) that $-\alpha \mu_a^{N_0} - \frac{\mu_a^{N_0} \ln \mu_a}{T_a} - \beta > 0$, and since $\epsilon > 0, \lambda > 0$, then according to the initial condition of the impulsive delay system (4.19), we can obtain

$$\begin{aligned} v(t) &\leq \frac{1}{2} \sup_{-\tau \leq s \leq 0} \sum_{i=1}^{N} \|\varphi_i(s) - \phi_i(\psi_i(s))\|^2 \\ &< \frac{1}{2}\mu_a^{-N_0} \sup_{-\tau \leq s \leq 0} \sum_{i=1}^{N} \|\varphi_i(s) - \phi_i(\psi_i(s))\|^2 + \frac{\epsilon}{-\alpha \mu_a^{N_0} - \frac{\mu_a^{N_0} \ln \mu_a}{T_a} - \beta} \\ &\leq \xi e^{-\lambda(t-t_0)} + \frac{\epsilon}{-\alpha \mu_a^{N_0} - \frac{\mu_a^{N_0} \ln \mu_a}{T_a} - \beta}, \qquad t_0 - \tau \leq t \leq t_0. \end{aligned} \tag{4.21}$$

In the following, we shall prove that

$$v(t) < \xi e^{-\lambda(t-t_0)} + \frac{\epsilon}{-\alpha\mu_a^{N_0} - \frac{\mu_a^{N_0}\ln\mu_a}{T_a} - \beta}, \qquad t > t_0. \tag{4.22}$$

Assume that (4.22) is not true, then there exists a $t^* > t_0$ such that

$$v(t^*) \geq \xi e^{-\lambda(t^*-t_0)} + \frac{\epsilon}{-\alpha\mu_a^{N_0} - \frac{\mu_a^{N_0}\ln\mu_a}{T_a} - \beta}, \tag{4.23}$$

and

$$v(t) < \xi e^{-\lambda(t-t_0)} + \frac{\epsilon}{-\alpha\mu_a^{N_0} - \frac{\mu_a^{N_0}\ln\mu_a}{T_a} - \beta}, \qquad t < t^*. \tag{4.24}$$

For the sake of simplicity, let $\chi = -\alpha\mu_a^{N_0} - \frac{\mu_a^{N_0}\ln\mu_a}{T_a} - \beta$. Then it follows from (4.20) and (4.24) that

$$\begin{aligned}
v(t^*) &< \xi e^{(\alpha+\frac{\ln\mu_a}{T_a})(t^*-t_0)} + \int_{t_0}^{t^*} \mu_a^{-N_0} e^{(\alpha+\frac{\ln\mu_a}{T_a})(t^*-s)} \Big[L\big(\xi e^{-\lambda(s-\tau_1-t_0)} + \frac{\epsilon}{\chi}\big) \\
&\quad + \|C\otimes A\|\big(\xi e^{-\lambda(s-\tau_2-t_0)} + \frac{\epsilon}{\chi}\big) + \epsilon \Big] ds \\
&\leq \xi e^{(\alpha+\frac{\ln\mu_a}{T_a})(t^*-t_0)} + \mu_a^{-N_0} e^{(\alpha+\frac{\ln\mu_a}{T_a})t^*} \Big[\int_{t_0}^{t^*} \xi L e^{\lambda(\tau_1+t_0)} e^{-(\lambda+\alpha+\frac{\ln\mu_a}{T_a})s} ds \\
&\quad + \int_{t_0}^{t^*} \xi \|C\otimes A\| e^{\lambda(\tau_2+t_0)} e^{-(\lambda+\alpha+\frac{\ln\mu_a}{T_a})s} ds + \int_{t_0}^{t^*} \big(\frac{L\epsilon}{\chi} + \frac{\|C\otimes A\|\epsilon}{\chi} + \epsilon\big) e^{-(\alpha+\frac{\ln\mu_a}{T_a})s} ds \Big] \\
&= \xi e^{(\alpha+\frac{\ln\mu_a}{T_a})(t^*-t_0)} + \xi e^{(\alpha+\frac{\ln\mu_a}{T_a})t^*} e^{\lambda t_0} \mu_a^{-N_0} (L e^{\lambda\tau_1} + \|C\otimes A\| e^{\lambda\tau_2}) \int_{t_0}^{t^*} e^{-(\lambda+\alpha+\frac{\ln\mu_a}{T_a})s} ds \\
&\quad + \mu_a^{-N_0} e^{(\alpha+\frac{\ln\mu_a}{T_a})t^*} \frac{\beta\epsilon + \chi\epsilon}{\chi} \int_{t_0}^{t^*} e^{-(\alpha+\frac{\ln\mu_a}{T_a})s} ds \\
&\leq \xi e^{(\alpha+\frac{\ln\mu_a}{T_a})(t^*-t_0)} + \xi e^{(\alpha+\frac{\ln\mu_a}{T_a})t^*} e^{\lambda t_0} \big[e^{-(\lambda+\alpha+\frac{\ln\mu_a}{T_a})t^*} - e^{-(\lambda+\alpha+\frac{\ln\mu_a}{T_a})t_0} \big] \\
&\quad + \frac{\epsilon}{\chi} e^{(\alpha+\frac{\ln\mu_a}{T_a})t^*} \big[e^{-(\alpha+\frac{\ln\mu_a}{T_a})t^*} - e^{-(\alpha+\frac{\ln\mu_a}{T_a})t_0} \big] \\
&= \xi e^{-\lambda(t^*-t_0)} + \frac{\epsilon}{\chi} - \frac{\epsilon}{\chi} e^{(\alpha+\frac{\ln\mu_a}{T_a})(t^*-t_0)} \\
&< \xi e^{-\lambda(t^*-t_0)} + \frac{\epsilon}{\chi},
\end{aligned}$$

which contradicts with (4.23). Therefore, (4.22) holds. Let $\epsilon \to 0$, then we have

$$\frac{1}{2}\sum_{i=1}^{N} e_i^*(t)e_i(t) = V(t) \le v(t) \le \xi e^{-\lambda(t-t_0)}, \qquad t \ge t_0.$$

Denote $e(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T$. From the definition of ξ, we have

$$\|e(t)\| \le \mu_a^{-\frac{N_0}{2}} \|e(t_0)\|_\tau e^{-\frac{\lambda}{2}(t-t_0)}, \qquad t \ge t_0,$$

which implies that $\|e_i(t)\| \to 0$ as $t \to \infty$ for $i = 1, 2, \ldots, N$. Thus, under the hybrid controller (4.3), generalized outer synchronization of drive-response CVDNs (4.1) and (4.2) is achieved with the convergence rate $\frac{\lambda}{2}$, where λ is the unique positive root of the equation $\lambda + \alpha + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0} L e^{\lambda\tau_1} + \mu_a^{-N_0}\|C \otimes A\| e^{\lambda\tau_2} = 0$. □

Remark 4.2.7. *It is worth noting that time delays have negative effects on generalized synchronization of networks. If the transmission delay τ_2 is not considered, then condition (4.18) in Theorem 4.2.2 becomes $L + 2\|B \otimes A\| + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0} L < 0$; if the internal delay τ_1 is not considered, then (4.18) becomes $\alpha + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0}\|C \otimes A\| < 0$. Furthermore, if the networks are considered to be delay-free, then (4.18) becomes $\frac{\ln \mu_a}{T_a} + 2(L + \|B \otimes A\|) < 0$. Compared (4.18) with the condition of the delay-free case, if $\mu_a < 1$, then smaller values of μ_a and T_a are required, which implies the control cost for the delayed case is higher.*

According to Theorem 4.2.2, some less conservative sufficient conditions for linear generalized outer synchronization and projective outer synchronization of drive-response CVDNs (4.1) and (4.2) under the hybrid controller (4.3) can be obtained.

Corollary 4.2.3. *Suppose that Assumption 4.1.1 holds. Let $\alpha = L + 2\|B \otimes A\| + \|C \otimes A\|$, $\beta = L + \|C \otimes A\|$, a_k be the largest eigenvalue of $(I_n + D_{ik})^*(I_n + D_{ik})$ for $k \in \mathbb{N}^+$, and T_a, μ_a are the AII and AIG of the impulsive sequence ζ, respectively. If $\mu_a < 1$, and (4.18) holds, then linear generalized outer synchronization of drive-response CVDNs (4.1) and (4.2) can be achieved under the hybrid controller (4.16) with the convergence rate $\frac{\lambda}{2}$, where λ is the unique positive root of the equation $\lambda + \alpha + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0} L e^{\lambda\tau_1} + \mu_a^{-N_0}\|C \otimes A\| e^{\lambda\tau_2} = 0$.*

Corollary 4.2.4. *Suppose that Assumption 4.1.1 holds. Let $\alpha = L + 2\|B \otimes A\| + \|C \otimes A\|$, $\beta = L + \|C \otimes A\|$, a_k be the largest eigenvalue of $(I_n + D_{ik})^*(I_n + D_{ik})$ for $k \in \mathbb{N}^+$, and T_a, μ_a are the AII and AIG of the impulsive sequence ζ, respectively. If $\mu_a < 1$, and (4.18) holds, then projective outer synchronization of drive-response CVDNs (4.1) and (4.2) can be achieved under the hybrid controller (4.17) with the convergence rate $\frac{\lambda}{2}$, where λ is the unique positive root of the equation $\lambda + \alpha + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0} L e^{\lambda\tau_1} + \mu_a^{-N_0}\|C \otimes A\| e^{\lambda\tau_2} = 0$.*

4.3 Numerical Examples

In this section, we present two examples to illustrate our theoretical results.

Example 4.3.1. *Consider the time-delay drive network (4.1) consisting of 4 coupled nodes with each node being a 2-dimensional complex-valued dynamical system, where* $z_i = (z_{i1}, z_{i2})^T$*, and* $z_{il} = z_{il}^R + jz_{il}^I$ *for* $i = 1, 2, 3, 4,\ l = 1, 2$*. The complex-valued nonlinear functions are given by*

$$f_i(t, z_i(t-\tau_1)) \equiv f(t, z_i(t-\tau_1)) = \begin{pmatrix} (0.4+0.1j)\tanh(z_{i1}^R(t-\tau_1) + j\sin(z_{i1}^I(t-\tau_1))) \\ (0.2-0.3j)\tanh(z_{i2}^R(t-\tau_1) + j\sin(z_{i2}^I(t-\tau_1))) \end{pmatrix}$$

for $i = 1, 2, 3, 4$*. Let* $g(z_i) = (\tanh(z_{i1}^R + j\sin(z_{i1}^I)), \tanh(z_{i2}^R + j\sin(z_{i2}^I)))^T$*, then* $f(t, z_i) = \Lambda g(z_i)$*, where* $\Lambda = \begin{pmatrix} 0.4+0.1j & 0 \\ 0 & 0.2-0.3j \end{pmatrix}$*. For any* $u, v \in \mathbb{C}^2$*, we have*

$$\begin{aligned} \|f_i(t,u) - f_i(t,v)\|^2 &= \|f(u) - f(v)\|^2 = (\Lambda g(u) - \Lambda g(v))^*(\Lambda g(u) - \Lambda g(v)) \\ &= (g(u) - g(v))^*\Lambda^*\Lambda(g(u) - g(v)) \\ &\le \|\Lambda\|^2\|u - v\|^2, \end{aligned}$$

thus Assumption 4.1.1 is satisfied with $L = \|\Lambda\| = 0.4123$*. Let* $\tau_1 = 0.01, \tau_2 = 0.006$*, choose the complex inner coupling matrix* $A = (0.1+0.1j)I_2$*, and the non-delayed and delayed complex outer coupling configuration matrices are respectively given by*

$$B = \begin{pmatrix} -2 & 1+j & 1-j & 0 \\ 0 & -1-j & 1+j & 0 \\ 0 & 1-j & -j & -1+2j \\ -1+j & 0 & 0 & 1-j \end{pmatrix}, \quad C = \begin{pmatrix} 0.2-0.3j & 0.2-j & 0.1+0.3j & -0.5+j \\ 0 & -0.4+0.8j & 0.2j & 0.4-j \\ 1-0.5j & 0 & 1-2.5j & -2+3j \\ -1-j & 0 & 0 & 1+j \end{pmatrix}.$$

By simple calculation, we have $\alpha = L + 2\|B\otimes A\| + \|C\otimes A\| = 2.1172$*, and* $\beta = L + \|C\otimes A\| = 1.1264$*.*

Consider (4.2) as the corresponding response network of the drive network (4.1). We will study three types of outer synchronization of drive-response CVDNs (4.1) and (4.2):

Type 1. *(Generalized Synchronization) Let* $\phi_i(\hat{z}_i) \equiv \phi(\hat{z}_i) = ((1+j)\hat{z}_{i1}, \hat{z}_{i1}^3 + \hat{z}_{i2})^T$*, where* $\hat{z}_i = (\hat{z}_{i1}, \hat{z}_{i2})^T \in \mathbb{C}^2$*,* $i = 1, 2, 3, 4$*. Then, we can get* $\phi_i^{-1}(\hat{z}_i) \equiv \phi^{-1}(\hat{z}_i) = (\frac{1-j}{2}\hat{z}_{i1}, \frac{1+j}{4}\hat{z}_{i1}^3 + \hat{z}_{i2})^T$*,* $J_{\phi_i} = \begin{pmatrix} 1+j & 0 \\ 3\hat{z}_{i1}^2 & 1 \end{pmatrix}$*, and* $J_{\phi_i}^{-1} = \begin{pmatrix} \frac{1-j}{2} & 0 \\ \frac{-3+3j}{2}\hat{z}_{i1}^2 & 1 \end{pmatrix}$*. Construct the complex-valued response network (4.2) under the hybrid controller (4.3). Choose the impulsive sequence* ζ *as* $t_{3k-2} = 0.12k - 0.08, t_{3k-1} = 0.12k - 0.05, t_{3k} = 0.12k$ *for* $k = 1, 2, \ldots,$ *and the corresponding impulsive control gain for each network nodes at each impulsive* *instant of the impulsive sequence* ζ *is chosen as* $D_{i,3k-2} = D_{3k-2} = (-0.7 + 0.4j)I_2$, $D_{i,3k-1} = D_{3k-1} = (-0.5 + 0.3j)I_2$, $D_{i,3k} = D_{3k} = (0.02 + 0.1j)I_2$ *for* $i = 1, 2, 3, 4$, $k = 1, 2, \ldots$. *After calculation, we can get* $\sigma = 0.05$, $T_a = 0.04$, $a_{3k-2} = 0.25$, $a_{3k-1} = 0.34$, $a_{3k} = 1.0504$, $k = 1, 2, \ldots$, *and* $\mu_a = 0.5468$. *We can choose* $\delta = 4$ *and* $\lambda = 20$ *such that conditions (4.7)-(4.10) in Theorem 4.2.1 hold. Then it follows from Theorem 4.2.1 that drive-response CVDNs* (4.1) *and* (4.2) *can achieve generalized outer synchronization under the hybrid controller* (4.3). *The initial condition of the complex-valued drive network (4.1) is chosen randomly as* $\varphi_i(s) = [1 + 2j, 3 + 4j]^T$ *for* $i = 1, 2, 3, 4, s \in [-0.01, 0]$, *and the initial condition of the complex-valued response network (4.2) is given by* $[\psi_1^T(s), \psi_2^T(s), \psi_3^T(s), \psi_4^T(s)] = [2-j, 1-2j, 0.5+j, 6+3j, -3+2j, 2.5+0.8j, 4-2j, -2+4j]$ *for* $s \in [-0.01, 0]$. *Figure 4.1 shows the time evolution of real and imaginary parts of 2-dimensional generalized synchronization errors for drive-response CVDNs* (4.1) *and* (4.2) *under the hybrid controller (4.3), it can be seen from the results of simulations in Figure 4.1 that both real and imaginary part of error variables converge to zero as time gets large, which implies that generalized outer synchronization of drive-response CVDNs (4.1) and (4.2) is achieved.*

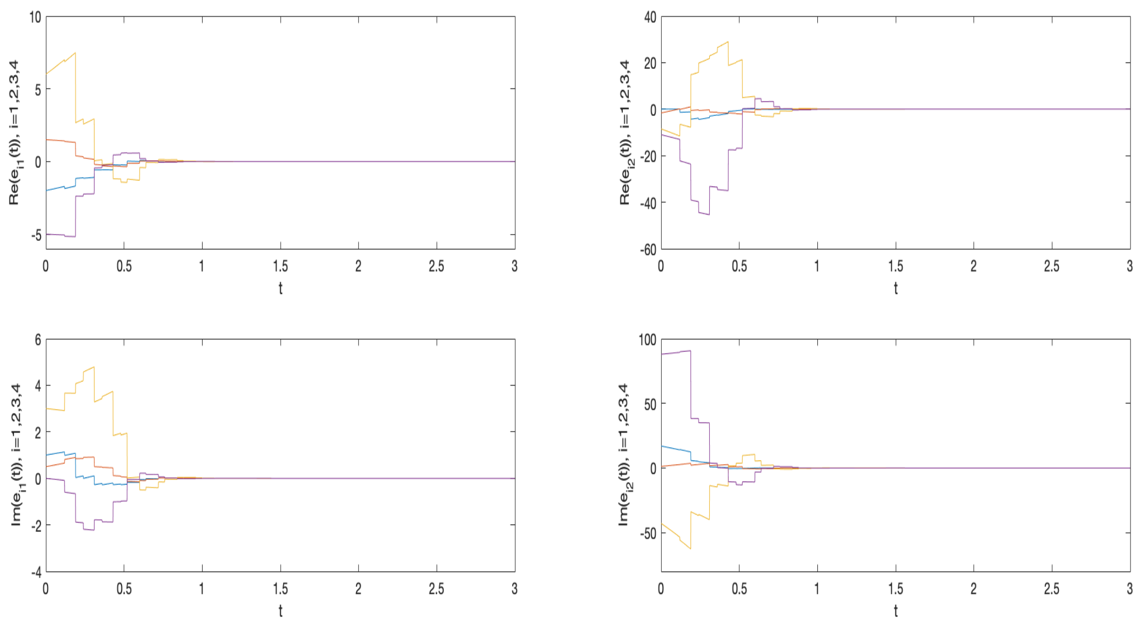

Figure 4.1: The evolution of real and imaginary parts of generalized synchronization error variables $e_{i1}(t)$ and $e_{i2}(t)$ $(i = 1, 2, 3, 4)$ for drive-response CVDNs (4.1) and (4.2) under the hybrid controller (4.3).

Type 2. *(Linear Generalized Synchronization) Let* $\phi_i(\hat{z}_i) \equiv \phi(\hat{z}_i) = P\hat{z}_i + Q$ *for* $i = 1, 2, 3, 4$*, where*

$$P = \begin{pmatrix} 1+2j & 3-j \\ -2+j & -3j \end{pmatrix}, \qquad Q = \begin{pmatrix} -j \\ 2 \end{pmatrix},$$

then we can calculate that

$$P^{-1} = \begin{pmatrix} 0.1297 - 0.1784j & -0.2216 - 0.0703j \\ 0.1622 + 0.0270j & -0.0270 + 0.1622j \end{pmatrix}.$$

Construct the complex-valued response network (4.2) under the hybrid controller (4.16). Choose the impulsive sequence and impulsive gains the same as those in the first scenario, then conditions (4.7)-(4.10) are satisfied with $\delta = 4, \lambda = 20$*. It follows from Corollary 4.2.1 that drive-response CVDNs (4.1) and (4.2) can achieve linear generalized outer synchronization under the hybrid controller (4.16). Define the linear generalized synchronization error as* $e_i(t) = z_i(t) - P\hat{z}_i(t) - Q$*,* $i = 1, 2, 3, 4$*, where* $e_i = (e_{i1}, e_{i2})^T \in \mathbb{C}^2$*. Figure 4.2 shows the time evolution of real and imaginary parts of linear generalized synchronization errors* $e_{i1}(t)$ *and* $e_{i2}(t)$ *for drive-response CVDNs (4.1) and (4.2) under the hybrid controller (4.16). The initial data in Figure 4.2 is chosen the same as that in Figure 4.1. From the results of simulations in Figure 4.2, we can see that linear generalized synchronization of drive-response CVDNs (4.1) and (4.2) is achieved.*

Type 3. *(Projective Synchronization) Choose* $\phi_i(\hat{z}_i) \equiv \phi(\hat{z}_i) = \alpha\hat{z}_i$*, where* $\alpha = 0.5 - j$*. Construct the complex-valued response network (4.2) under the hybrid controller (4.17). Choose the impulsive sequence and impulsive gains the same as those in the first scenario such that conditions (4.7)-(4.10) can be satisfied. Corollary 4.2.2 implies that drive-response CVDNs (4.1) and (4.2) can achieve projective outer synchronization under the hybrid controller (4.17). Define the projective synchronization error as* $e_i(t) = z_i(t) - \alpha\hat{z}_i(t)$*,* $i = 1, 2, 3, 4$*, where* $e_i = (e_{i1}, e_{i2})^T \in \mathbb{C}^2$*. Figure 4.3 shows the time evolution of real and imaginary parts of 2-dimensional projective synchronization errors* $e_{i1}(t)$ *and* $e_{i2}(t)$ *for drive-response CVDNs (4.1) and (4.2) under the hybrid controller (4.17). The initial data in Figure 4.3 is chosen the same as that in Figure 4.1 and Figure 4.2. It is clearly observed from the simulation results in Figure 4.3 that projective outer synchronization of drive-response CVDNs (4.1)-(4.2) is achieved.*

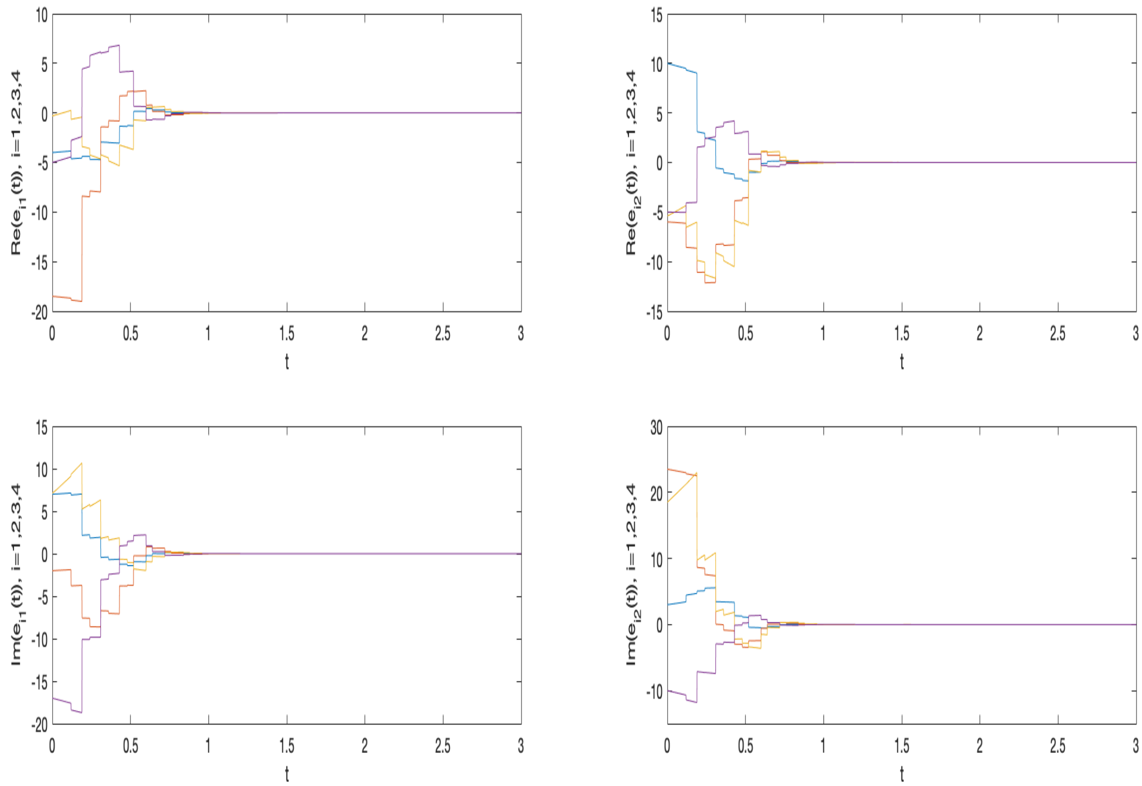

Figure 4.2: The evolution of real and imaginary parts of linear generalized synchronization error variables $e_{i1}(t)$ and $e_{i2}(t)$ $(i = 1, 2, 3, 4)$ for drive-response CVDNs (4.1) and (4.2) under the hybrid controller (4.16).

Remark 4.3.1. *From Example 4.3.1, we can calculate that* $a_{3k-2} < 1$*,* $a_{3k-1} < 1$*, and* $a_{3k} > 1$ *for all* $k \in \mathbb{N}^+$*, which implies that synchronizing impulses and desynchronizing impulses exist simultaneously when constructing the impulsively controlled complex-valued response network. If the hybrid controller is suitably designed such that all the conditions in Theorem 4.2.1 (Corollary 4.2.1, Corollary 4.2.2) are satisfied, it can be seen from the results of simulations in Example 4.3.1 that generalized (linear generalized, projective) outer synchronization of drive-response CVDNs* (4.1) *and* (4.2) *can still be achieved even if the impulsive sequence contains desynchronizing impulses.*

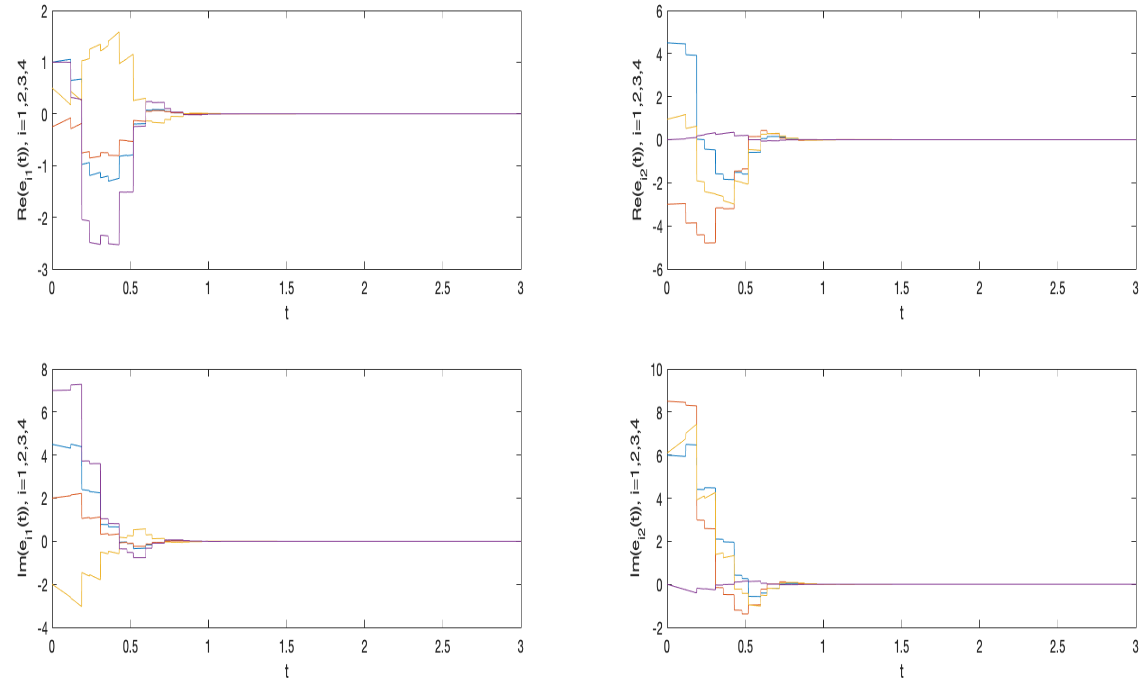

Figure 4.3: The evolution of real and imaginary parts of projective synchronization error variables $e_{i1}(t)$ and $e_{i2}(t)$ ($i = 1, 2, 3, 4$) for drive-response CVDNs (4.1) and (4.2) under the hybrid controller (4.17).

Example 4.3.2. *Consider the drive network (4.1) consisting of 4 coupled nodes. Choose the node dynamics as the following modified time-delay complex-variable Lü system:*

$$\begin{cases} \dot{z}_{i1} = \rho(z_{i2_{\tau 1}} - z_{i1_{\tau 1}}), \\ \dot{z}_{i2} = \nu z_{i2_{\tau 1}} - z_{i1_{\tau 1}} z_{i3_{\tau 1}}, \\ \dot{z}_{i3} = \dfrac{1}{2}\left(\overline{z_{i1}}_{\tau_1} z_{i2_{\tau 1}} + z_{i1_{\tau 1}} \overline{z_{i2}}_{\tau_1}\right) - \mu z_{i3_{\tau 1}} + j\mathfrak{Im}(z_{i1_{\tau 1}})\mathfrak{Re}(z_{i2_{\tau 1}}), \end{cases} \tag{4.25}$$

where $i = 1, 2, 3, 4,\ z_{i1}, z_{i2}, z_{i3} \in \mathbb{C}$ *are complex state variables. The complex-valued nonlinear function* f_i *in* (4.1) *can be described by*

$$f_i(t, z_{i_{\tau 1}}) \equiv f(t, z_{i_{\tau 1}}) = \begin{pmatrix} \rho(z_{i2_{\tau 1}} - z_{i1_{\tau 1}}) \\ \nu z_{i2_{\tau 1}} - z_{i1_{\tau 1}} z_{i3_{\tau 1}} \\ \frac{1}{2}(\overline{z_{i1}}_{\tau_1} z_{i2_{\tau 1}} + z_{i1_{\tau 1}} \overline{z_{i2}}_{\tau_1}) - \mu z_{i3_{\tau 1}} + j\mathfrak{Im}(z_{i1_{\tau 1}})\mathfrak{Re}(z_{i2_{\tau 1}}) \end{pmatrix}$$

for $i = 1, 2, 3, 4$*, where* $z_i = (z_{i1}, z_{i2}, z_{i3})^T$*. Let* $\tau_1 = 0.0005$*, the system* (4.25) *exhibits chaotic attractors when the parameters* $\rho = 21, \nu = 10$ *and* $\mu = 6$*. Denote* $z_{i1} = z_{i1}^R + jz_{i1}^I, z_{i2} = z_{i2}^R + jz_{i2}^I, z_{i3} = z_{i3}^R + jz_{i3}^I$*. Figure* 4.4 *shows the chaotic attractors of the modified time-delay complex-variable Lü system* (4.25) *with initial condition* $z_i(t_0 + s) = \varphi_i(s) = [1 + 2j, 3 + 4j, 5 + 6j]^T$ *for* $s \in [-0.0005, 0]$ *in some domains.*

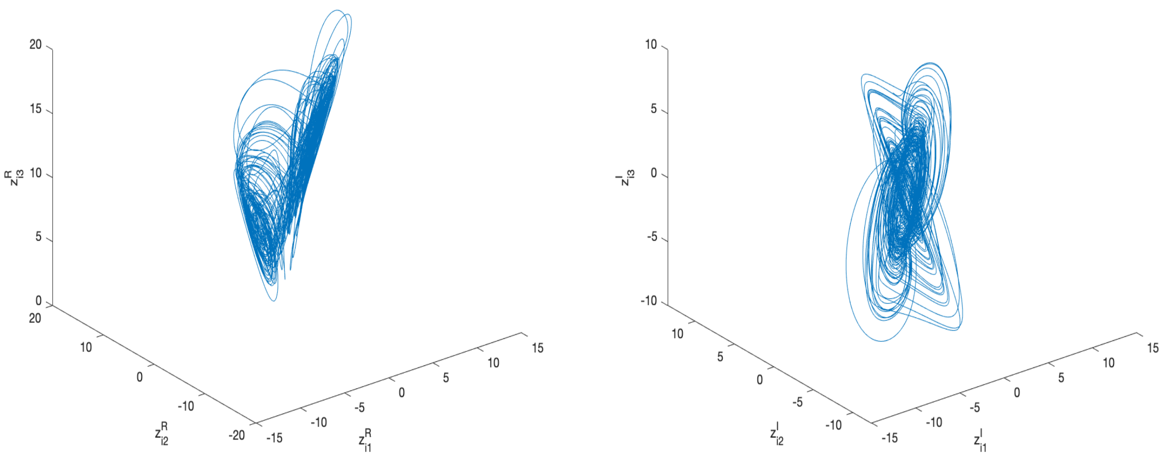

Figure 4.4: Chaotic attractors of the modified time-delay complex-variable Lü system (4.25) with initial condition $\varphi_i(s) = [1 + 2j, 3 + 4j, 5 + 6j]^T$ for $s \in [-\tau_1, 0]$ in some spaces.

It can be clearly seen from Figure 4.4 *that* $|z_{i1}^R| \leq 15, |z_{i2}^R| \leq 20, |z_{i3}^R| \leq 20, |z_{i1}^I| \leq 15, |z_{i2}^I| \leq 15$ *and* $|z_{i3}^I| \leq 10$*. Notice that* $f(t, z_i)$ *can be rewritten as* $f(t, z_i) = Mz_i + h(z_i)$*, where*

$$M = \begin{pmatrix} -\rho & \rho & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -\mu \end{pmatrix}, \quad h(z_i) = \begin{pmatrix} 0 \\ -z_{i1}z_{i3} \\ \frac{1}{2}(\overline{z_{i1}}z_{i2} + z_{i1}\overline{z_{i2}}) + jz_{i1}^I z_{i2}^R \end{pmatrix}.$$

The complex Jacobian of the complex-variable vector-valued function h *can be calculated as*

$$Dh = \begin{pmatrix} 0 & 0 & 0 \\ -z_{i3} & 0 & -z_{i1} \\ z_{i2}^R - j\frac{z_{i2}^I}{2} & \frac{z_{i1}^R}{2} & 0 \end{pmatrix},$$

and we can calculate that $\|Dh\|_\infty = 43.57$*. For any* $u, v \in \mathbb{C}^3$*, we have*

$$\begin{aligned}\|f(t,u)-f(t,v)\|_\infty &\le \|M\|_\infty\|u-v\|_\infty + \|h(u)-h(v)\|_\infty \\ &\le (\|M\|_\infty + \|Dh\|_\infty)\|u-v\|_\infty \\ &\le 85\|u-v\|_\infty,\end{aligned}$$

therefore Assumption 4.1.1 is satisfied with $L = 85$*. Let* $\tau_2 = 0.0003$*, choose the complex inner coupling matrix as* $A = (0.2 - 0.3j)I_3$*, and the non-delayed and delayed complex outer coupling configuration matrices* B *and* C *as*

$$B = [b_{ij}]_{4\times 4} = \begin{pmatrix} -0.8+0.5j & 0.2+0.5j & 0 & 0.6-j \\ 0 & 0.1-0.3j & -0.1+0.3j & 0 \\ 0.4-0.1j & 0 & -0.3-0.5j & -0.1+0.6j \\ 0 & 0.3-0.8j & 0 & -0.3+0.8j \end{pmatrix},$$

$$C = [c_{ij}]_{4\times 4} = \begin{pmatrix} -1+j & 1-j & 0 & 0 \\ 0 & -1+j & 0 & 1-j \\ 1-j & 0 & -1+j & 0 \\ 0 & 0 & 1-j & -1+j \end{pmatrix},$$

respectively. By simple calculation, we have $\alpha = L + 2\|B\otimes A\| + \|C\otimes A\| = 87.4414$*,* $\beta = L + \|C\otimes A\| = 86.0198$*.*

Consider (4.2) as the corresponding response network of the drive network (4.1). Let $\phi_i(\hat{z}_i) \equiv \phi(\hat{z}_i) = (\hat{z}_{i1}+\hat{z}_{i2}^2, \hat{z}_{i2}-\hat{z}_{i3}^2, \hat{z}_{i3})^T$*, where* $\hat{z}_i = (\hat{z}_{i1}, \hat{z}_{i2}, \hat{z}_{i3})^T \in \mathbb{C}^3$*,* $i = 1,2,3,4$*. Then we can get the inverse map* $\phi_i^{-1}(\hat{z}_i) \equiv \phi^{-1}(\hat{z}_i) = \big(\hat{z}_{i1} - (\hat{z}_{i2}+\hat{z}_{i3}^2)^2, \hat{z}_{i2}+\hat{z}_{i3}^2, \hat{z}_{i3}\big)^T$*,*

$$J_{\phi_i} = \begin{pmatrix} 1 & 2\hat{z}_{i2} & 0 \\ 0 & 1 & -2\hat{z}_{i3} \\ 0 & 0 & 1 \end{pmatrix}, \qquad J_{\phi_i}^{-1} = \begin{pmatrix} 1 & -2\hat{z}_{i2} & -4\hat{z}_{i2}\hat{z}_{i3} \\ 0 & 1 & 2\hat{z}_{i3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Construct the complex-valued response network (4.2) under the hybrid controller (4.3). Choose the impulsive sequence ζ *as* $t_{4k-3} = 0.004k - 0.0032, t_{4k-2} = 0.004k - 0.0016, t_{4k-1} = 0.004k - 0.001, t_{4k} = 0.004k$*,* $k \in \mathbb{N}^+$*, and choose* $D_{i,4k-3} = D_{4k-3} = (-0.7+0.5j)I_3$*,* $D_{i,4k-2} = D_{4k-2} = 0.1I_3$*,* $D_{i,4k-1} = D_{4k-1} = (-0.6+0.2j)I_3$*,* $D_{i,4k} = D_{4k} = (0.2+0.1j)I_3$ *for* $i = 1,2,3,4$*,* $k \in \mathbb{N}^+$ *to be the corresponding impulsive gain for each network nodes at each impulsive instant of the impulsive sequence* ζ*. By simple calculation, we can get* $T_a = 0.001$*, and we can choose* $N_0 = 2$ *satisfying the definition of AII. Furthermore, we can calculate that* $a_{4k-3} = 0.34$*,* $a_{4k-2} = 1.21$*,* $a_{4k-1} = 0.2$*, and* $a_{4k} = 1.45$*,* $k \in \mathbb{N}^+$*, which implies that the designed impulsive sequence* ζ *is a hybrid impulsive sequence, and the AIG*

$\mu_a = 0.8$. *Then we can derive* $\alpha + \frac{\ln \mu_a}{T_a} + \mu_a^{-N_0}\beta = -1.2962 < 0$, *and condition* (*4.18*) *in Theorem 4.2.2 is satisfied. Theorem 4.2.2 implies that drive-response CVDNs* (*4.1*) *and* (*4.2*) *can achieve generalized outer synchronization under the hybrid controller* (4.3).

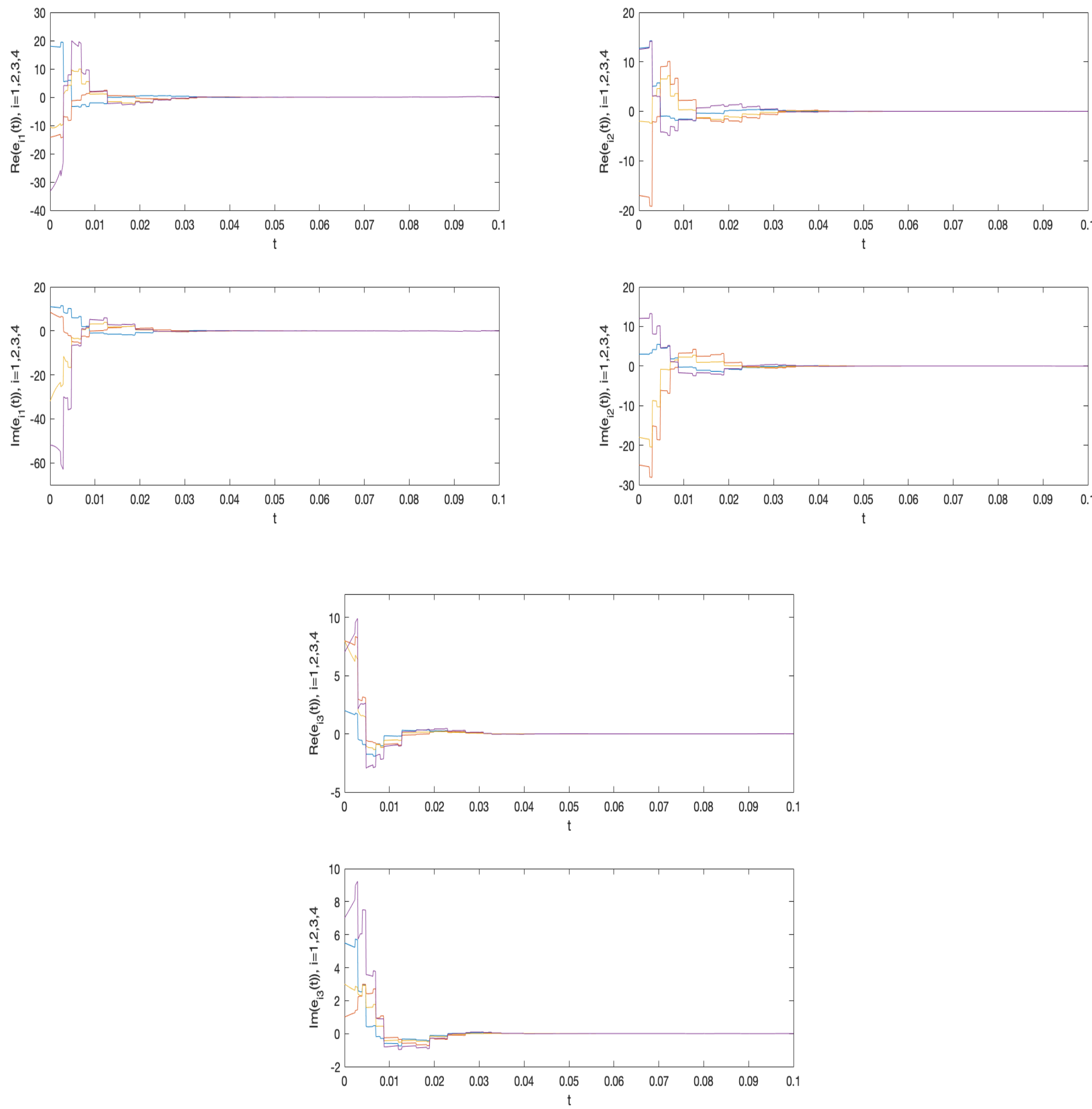

Figure 4.5: Trajectories of real and imaginary parts of generalized synchronization error variables $e_{i1}(t)$, $e_{i2}(t)$ and $e_{i3}(t)$ ($i = 1, 2, 3, 4$) in Example 4.3.2.

The initial condition of the complex-valued drive network (4.1) *is chosen the same as that of system* (4.25)*, and the initial condition of the complex-valued response network* (4.2) *is chosen as* $[\psi_1^T(s), \psi_2^T(s), \psi_3^T(s), \psi_4^T(s)] = [-2-j, -1+4j, 3+0.5j, 0.2+1.5j, 4-j, -3+5j, 2.5-6j, 5+4j, -3+3j, 8+2j, -6.5-4j, -2-j]$ *for* $s \in [-0.0005, 0]$*. Figure 4.5 shows the trajectories of real and imaginary parts of 3-dimensional generalized synchronization errors for drive-response CVDNs (4.1) and (4.2) under the hybrid controller* (4.3)*. It can be seen from the results of simulations in Figure 4.5 that both real and imaginary part of error variables converge to zero as time gets large, which implies that generalized outer synchronization of drive-response CVDNs (4.1) and (4.2) is achieved.*

Chapter 5

Consensus of Complex-Valued Multi-Agent Systems

This chapter studies the average-consensus problem of potential complex-valued multi-agent systems. By considering the continuous-time communication among agents and the instantaneous information exchange at discrete-time instants, a complex-variable hybrid consensus protocol which composed of continuous-time protocol and impulsive protocol is designed for achieving the average-consensus of complex-valued multi-agent systems, and the time-delay is taken into account in both continuous-time and discrete-time protocols. In Section 5.3, we formulate the average-consensus problem of complex-valued multi-agent systems and introduce the complex-variable hybrid consensus protocol. In Section 5.4, delay-dependent sufficient conditions are established to guarantee that average-consensus of potential complex-valued networked multi-agent systems can be achieved under the proposed hybrid consensus protocol. Based on the time-delay size for the continuous-time protocol, our result shows that complex-valued networked multi-agent systems can achieve average-consensus if interaction topologies of continuous-time and discrete-time protocols and impulsive sequences are suitably designed. In Section 5.5, numerical simulations are provided to illustrate the effectiveness of the theoretical results.

5.1 Motivations

Multi-agent systems have recently been intensively studied in the fields of communication networks, mobile robots, intelligent transportation system, and distributed sensor networks (see, [100, 101, 102]). A multi-agent system is a networked system composed of multiple interacting dynamic agents. One of the desired properties in multi-agent systems is consensus among all agents, namely, all agents must reach an agreement upon a common value of a certain quantity of interest.

Recently, many protocols have been proposed to solve the consensus problem of multi-agent systems (see, e.g., [100, 103, 104, 105]), and a few of consensus results are derived by designing protocols based on the impulsive control method (see, e.g., [106, 107, 108]). In [106], an impulsive consensus protocol is proposed for delay-free linear multi-agent systems with fixed and switching topologies. In [107], a hybrid continuous-time and impulsive protocol is proposed to study the consensus problem of multi-agent systems, but time-delay is considered only in the impulsive part of the hybrid protocol. In [108], a hybrid impulsive protocol is designed, and time delays are taken into account in both continuous-time and discrete-time consensus protocols. Nevertheless, consensus results derived in [108] have no information regarding the time-delay in the hybrid protocols. It should be noted that the above consensus results mainly concentrated on multi-agent systems with real variables. Actually, many practical systems in real life can be described more accurately by complex-valued systems, such as the laser system [55], and the reaction-advection-diffusion system [109]. Furthermore, complex-valued chaotic systems and CVNNs have recently been extensively studied because their potential and successful applications are found in many physical and engineering fields. Naturally, there might have potential applications for complex-valued multi-agent systems. Therefore, it is interesting and important to study the consensus problem of complex-valued multi-agent systems.

5.2 Network Topology

In this section, we introduce some preliminary notions in graph theory.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph consisting of N nodes, where $\mathcal{V} = \{v_i|\ i = 1, 2, ..., N\}$ denotes the set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges. An edge of $\mathcal{G}$ is denoted by (v_i, v_j) which means node v_j can receive information from node v_i. The index set $\mathcal{N}_i = \{v_j \in \mathcal{V}|(v_j, v_i) \in \mathcal{E}\}$ represents the set of neighbors of node v_i. A weighted digraph $\mathcal{G}_A = (\mathcal{V}, \mathcal{E}, A)$ is a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ associated with a weighted adjacency matrix $A = [\alpha_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative adjacency elements α_{ij} such that $(v_j, v_i) \in \mathcal{E}$ if and only if $\alpha_{ij} > 0$. It is assumed that $\alpha_{ii} = 0$ for all $i = 1, 2, ..., N$. The in-degree and out-degree of node v_i are defined by $d_{in}(v_i) = \sum_{j\in\mathcal{N}_i} \alpha_{ij}$ and $d_o(v_i) = \sum_{j\in\mathcal{N}_i} \alpha_{ji}$, respectively. The graph Laplacian $\mathcal{L}$ of the weighted digraph $\mathcal{G}_A$ is defined by $\mathcal{L} = \mathcal{D} - A$, where $\mathcal{D} = diag\{d_{in}(v_1), d_{in}(v_2), ..., d_{in}(v_N)\}$. More precisely, $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$, where $l_{ij} = \sum_{j=1}^{N} \alpha_{ij}$ if $i = j$; and $l_{ij} = -\alpha_{ij}$, otherwise. A weighted digraph $\mathcal{G}_A$ is said to be balanced if $d_{in}(v_i) = d_o(v_i)$ for all $i = 1, 2, ..., N$. A digraph $\mathcal{G}$ is said to be strongly connected if any two distinct nodes in $\mathcal{G}$ can be connected by a path that traverses the direction of the edges of $\mathcal{G}$.

5.3 Problem Formulation and Hybrid Consensus Protocols

Consider a graph $\mathcal{G}$ consisting of N nodes, each of which being a complex-valued dynamical agent with integrator dynamics

$$\dot{z}_i(t) = u_i(t), \qquad i = 1, 2, \ldots, N, \tag{5.1}$$

where $z_i \in \mathbb{C}$ denotes the state of the i-th agent, and u_i is the control input. u_i is said to be a protocol of $\mathcal{G}$ if the controller u_i only depends on the state information of node v_i and its neighbors (i.e. $v_j \in \mathcal{N}_i$).

We say that the agents in system (5.1) can achieve consensus if and only if

$$\lim_{t\to\infty} \|z_i(t) - z_j(t)\| = 0, \qquad i, j = 1, 2, \ldots, N.$$

Furthermore, we say that the agents in system (5.1) can achieve average-consensus if and only if

$$\lim_{t\to\infty} \|z_i(t) - Ave(z(0))\| = 0, \qquad i = 1, 2, \ldots, N,$$

where $Ave(z(0)) = \frac{1}{N}\sum_{i=1}^{N} z_i(0)$.

To seek average-consensus for agents in (5.1), we consider the following complex-variable hybrid consensus protocol based on two digraphs $\mathcal{G}_A = (\mathcal{V}, \mathcal{E}, A)$ and $\mathcal{G}_A^{'} = (\mathcal{V}, \mathcal{E}^{'}, A^{'})$:

$$u_i(t) = \sum_{j\in\mathcal{N}_i} \alpha_{ij}[z_j(t-r) - z_i(t-r)] + \sum_{k=1}^{\infty} \sum_{j\in\mathcal{N}_i^{'}} \alpha_{ij}^{'}[z_j(t-\bar{\tau}) - z_i(t-\bar{\tau})]\delta(t-t_k), \tag{5.2}$$

where α_{ij} $(\alpha_{ij}^{'}) \in \mathbb{R}$ is the (i,j)th entry of the weighted adjacent matrix A $(A^{'})$; $\mathcal{N}_i$ $(\mathcal{N}_i^{'})$ denotes the neighbors set of node v_i in digraph $\mathcal{G}_A$ $(\mathcal{G}_A^{'})$; $\delta(\cdot)$ denotes the Dirac delta function; t_k represents the impulsive instant, and impulsive sequence $\{t_k\}$ satisfies $t_0 < t_1 < t_2 < ... < t_k < t_{k+1} < ...$ and $\lim_{k\to\infty} t_k = \infty$; r denotes the time-delay in the continuous-time consensus protocol, and $\bar{\tau}$ represents the time-delay when processing the impulsive information at impulsive instant t_k.

Remark 5.3.1. *The hybrid protocol (5.2) works as follows: on every continuous-time interval (t_{k-1}, t_k), the agents in system (5.1) are connected based on the interaction topology $\mathcal{G}_A$, and at each impulsive instant t_k, the agents exchange information instantaneously based on the digraph $\mathcal{G}_A^{'}$.*

By the definition of $\delta(\cdot)$, the complex-valued multi-agent system (5.1) under the hybrid consensus protocol (5.2) can be described by the following complex-valued impulsive system:

$$\begin{cases} \dot{z}_i(t) = \sum_{j\in\mathcal{N}_i} \alpha_{ij}[z_j(t-r) - z_i(t-r)], & t \geq t_0,\ t \in [t_{k-1}, t_k), \\ \Delta z_i(t_k) = \sum_{j\in\mathcal{N}_i^{'}} \alpha_{ij}^{'}[z_j(t_k-\bar{\tau}) - z_i(t_k-\bar{\tau})], & k \in \mathbb{N}^+, \\ z_i(t_0+s) = \phi_i(s), \quad s \in [-\tau, 0] \end{cases} \tag{5.3}$$

for $i = 1, 2, ..., N$, where $\Delta z_i(t_k) = z_i(t_k^+) - z_i(t_k^-)$, and $\phi_i \in \mathcal{PC}([-\tau, 0], \mathbb{C})$ is the initial function with $\tau = \max\{r, \bar{\tau}\}$. Without loss of generality, we assume that $z_i(t_k) = z_i(t_k^+)$ in the following discussion, which implies solutions of (5.3) are right continuous at each impulsive instant t_k.

Define state vector $z(t) = (z_1, z_2, ..., z_N)^T \in \mathbb{C}^N$, according to interaction topologies $\mathcal{G}_A = (\mathcal{V}, \mathcal{E}, A)$ and $\mathcal{G}_A^{'} = (\mathcal{V}, \mathcal{E}^{'}, A^{'})$, system (5.3) can be rewritten as

$$\begin{cases} \dot{z}(t) = -\mathcal{L}z(t-r), & t \geq t_0,\ t \in [t_{k-1}, t_k), \\ \Delta z(t_k) = -\mathcal{L}^{'} z(t_k - \bar{\tau}), & k \in \mathbb{N}^+, \\ z(t_0+s) = \phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{5.4}$$

where $\mathcal{L}$ and $\mathcal{L}^{'}$ are the graph Laplacians of $\mathcal{G}_A$ and $\mathcal{G}_A^{'}$, respectively, and $\phi = (\phi_1, \phi_2, ..., \phi_N)^T$.

The objective is to derive sufficient conditions on digraphs $\mathcal{G}_A$, $\mathcal{G}_A^{'}$ and the impulsive sequence $\{t_k\}$ to guarantee that complex-variable hybrid consensus protocol (5.2) can lead to the average-consensus for agents in system (5.1).

5.4 Consensus Results

To derive the average-consensus results, we firstly introduce the following disagreement vector

$$e(t) = z(t) - Ave(z(t))\mathbf{1},$$

where $\mathbf{1}$ denotes the column N-vector with all ones, and $Ave(z(t)) = \frac{1}{N}\sum_{i=1}^{N} z_i(t)$. According to [108], if $\mathcal{G}_A$ and $\mathcal{G}_A^{'}$ are balanced, then $Ave(z(t))$ is an invariant quantity for $t \geq 0$, denoted by $Ave(z(t)) = Ave(z(0)) = \frac{1}{N}\sum_{i=1}^{N} z_i(0)$. Furthermore, since the Laplacians $\mathcal{L}$ and $\mathcal{L}^{'}$ have zero row sums, we have $\mathcal{L}Ave(z(0))\mathbf{1} = \mathcal{L}^{'}Ave(z(0))\mathbf{1} = \mathbf{0}$.

According to system (5.4), we can obtain the following impulsive disagreement dynamical system:

$$\begin{cases} \dot{e}(t) = -\mathcal{L}e(t-r), \quad t \geq t_0,\ t \in [t_{k-1}, t_k), \\ \Delta e(t_k) = -\mathcal{L}^{'}e(t_k - \bar{\tau}), \quad k \in \mathbb{N}^+. \end{cases} \tag{5.5}$$

For simplicity, we assume all impulses are uniformly distributed, i.e., $T = t_k - t_{k-1}$ for all $k \in \mathbb{N}^+$, and $\bar{\tau} \leq T$ throughout this section.

Theorem 5.4.1. *Suppose that $\mathcal{G}_A$ is balanced, $\mathcal{G}_A^{'}$ is balanced and strongly connected. Let $\lambda_2(\mathcal{L}_s^{'})$ denote the second smallest eigenvalue of $\mathcal{L}_s^{'} = \frac{\mathcal{L}^{'} + \mathcal{L}^{'T}}{2}$. If there exist constants $\varepsilon > 0$, $0 < \omega \leq 1$ such that*

$$\ln(\alpha + \beta + \omega r) < -cT, \tag{5.6}$$

where $\alpha = (1+\varepsilon)(1 - 2\lambda_2(\mathcal{L}_s^{'}) + \|\mathcal{L}^{'}\|^2)$, $\beta = (1+\frac{1}{\varepsilon})(\bar{\tau}\|\mathcal{L}\|\|\mathcal{L}^{'}\|)^2$, $c = \frac{\|\mathcal{L}\|^2}{\omega} + \omega$, and $T = t_k - t_{k-1}$, then the hybrid consensus protocol (5.2) leads to the average-consensus for agents in system (5.1).

Proof. Construct the Lyapunov functional candidate $V(t) = V_1(t) + V_2(t)$, where $V_1(t) = e^*(t)e(t)$, and $V_2(t) = \omega\int_{t-r}^{t} e^*(s)e(s)ds$. For $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$, calculate the derivative of $V(t)$ along the solution of (5.5), then we can get

$$\begin{aligned} \dot{V}(t) &\leq 2\left|e^*(t)\mathcal{L}e(t-r)\right| + \omega e^*(t)e(t) - \omega e^*(t-r)e(t-r) \\ &\leq 2\|e(t)\|\|\mathcal{L}\|\|e(t-r)\| + \omega e^*(t)e(t) - \omega e^*(t-r)e(t-r) \\ &\leq (\frac{\|\mathcal{L}\|^2}{\omega} + \omega)e^*(t)e(t) \leq (\frac{\|\mathcal{L}\|^2}{\omega} + \omega)\ V(t). \end{aligned}$$

Denote $c = \frac{\|\mathcal{L}\|^2}{\omega} + \omega$. Then we have

$$V(t) \leq V(t_{k-1})\ e^{c(t-t_{k-1})}, \qquad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+. \tag{5.7}$$

For $t = t_k$, by the assumption $\bar{\tau} \leq T$, integrating both sides of (5.5) from $t_k - \bar{\tau}$ to t_k gives $e(t_k - \bar{\tau}) = e(t_k^-) + \int_{t_k-\bar{\tau}}^{t_k} \mathcal{L}e(t-r)dt$. Then it follows from (5.5) that $e(t_k) = X + Y$, where $X = (I - \mathcal{L}^{'})e(t_k^-)$, and $Y = -\mathcal{L}^{'}\mathcal{L}\int_{t_k-\bar{\tau}}^{t_k} e(t-r)dt$. According to Lemma 3.6.1, we have for any $\varepsilon > 0$,

$$V_1(t_k) = (X+Y)^*(X+Y) \leq (1+\varepsilon)X^*X + (1+\frac{1}{\varepsilon})Y^*Y. \tag{5.8}$$

Since $\mathcal{G}_A^{'}$ is balanced and strongly connected, according to [103], we have for $\mathcal{L}_s^{'} = \frac{\mathcal{L}^{'}+\mathcal{L}^{'T}}{2}$,

$$0 = \lambda_1(\mathcal{L}_s^{'}) < \lambda_2(\mathcal{L}_s^{'}) \leq \cdots \leq \lambda_N(\mathcal{L}_s^{'}),$$

hence,

$$\begin{aligned} X^*X &= e^*(t_k^-)(I - \mathcal{L}^{'} - \mathcal{L}^{'T} + \mathcal{L}^{'T}\mathcal{L}^{'})e(t_k^-) \\ &\leq \big(1 - 2\lambda_2(\mathcal{L}_s^{'}) + \lambda_{\max}(\mathcal{L}^{'T}\mathcal{L}^{'})\big)e^*(t_k^-)e(t_k^-) \\ &= (1 - 2\lambda_2(\mathcal{L}_s^{'}) + \|\mathcal{L}^{'}\|^2)V_1(t_k^-). \end{aligned} \tag{5.9}$$

Applying the Cauchy Schwarz inequality for integrable complex-valued functions, we have

$$\begin{aligned} Y^*Y &\leq \|\mathcal{L}^{'}\|^2\|\mathcal{L}\|^2\big(\int_{t_k-\bar{\tau}}^{t_k} e(t-r)dt\big)^*\big(\int_{t_k-\bar{\tau}}^{t_k} e(t-r)dt\big) \\ &\leq \|\mathcal{L}^{'}\|^2\|\mathcal{L}\|^2\bar{\tau}\int_{t_k-\bar{\tau}}^{t_k} e^*(t-r)e(t-r)dt \\ &\leq (\bar{\tau}\|\mathcal{L}\|\|\mathcal{L}^{'}\|)^2 \sup_{s\in[-(\bar{\tau}+r),0]} V_1(t_k^- + s). \end{aligned} \tag{5.10}$$

Then it follows from (5.8), (5.9) and (5.10) that

$$V_1(t_k) \leq \alpha V(t_k^-) + \beta \sup_{s\in[-(\bar{\tau}+r),0]} V(t_k^- + s), \tag{5.11}$$

where $\alpha = (1+\varepsilon)(1 - 2\lambda_2(\mathcal{L}_s^{'}) + \|\mathcal{L}^{'}\|^2)$, and $\beta = (1+\frac{1}{\varepsilon})(\bar{\tau}\|\mathcal{L}\|\|\mathcal{L}^{'}\|)^2$. By the continuity of $V_2(t)$, we can get

$$V_2(t_k) \leq \omega r \sup_{s\in[-r,\ 0]} V(t_k^- + s). \tag{5.12}$$

According to condition (5.6) and the IVT, there exists a unique constant $\lambda > 0$ such that

$$\ln[\alpha + \beta e^{\lambda(\bar{\tau}+r)} + \omega r e^{\lambda r}] = -(\lambda + c)T. \tag{5.13}$$

Since $\lim_{k\to\infty} t_k = \infty$, there exists an integer $p \geq 1$ such that $t_p - \bar{\tau} - r \geq t_0$. For $t \in [t_0, t_p)$, we have

$$V(t) \leq Me^{-\lambda(t-t_0)}, \tag{5.14}$$

where $M = \sup_{t\in[t_0,t_p]} V(t)e^{\lambda(t_p-t_0)}$. In the following, we will use the method of mathematical induction to prove that

$$V(t) \leq Me^{-(\lambda+c)(t_{k+1}-t_0)}e^{c(t-t_0)}, \qquad t\in[t_k,t_{k+1}),\ k\geq p. \tag{5.15}$$

At $t = t_p$, it follows from (5.11), (5.12), (5.13) and (5.14) that

$$\begin{aligned}
V(t_p) &\leq \alpha V(t_p^-) + \beta \sup_{s\in[-(\bar\tau+r),0]} V(t_p^- + s) + \omega r \sup_{s\in[-r,\ 0]} V(t_p^- + s)\\
&\leq \alpha Me^{-\lambda(t_p-t_0)} + \beta Me^{-\lambda(t_p-\bar\tau-r-t_0)} + \omega r Me^{-\lambda(t_p-r-t_0)}\\
&= (\alpha + \beta e^{\lambda(\bar\tau+r)} + \omega r e^{\lambda r})Me^{-\lambda(t_p-t_0)} = e^{-(\lambda+c)(t_{p+1}-t_p)}Me^{-(\lambda+c)(t_p-t_0)}e^{c(t_p-t_0)}\\
&= Me^{-(\lambda+c)(t_{p+1}-t_0)}e^{c(t_p-t_0)},
\end{aligned}$$

which implies that (5.15) holds at $t = t_p$. For $t \in (t_p, t_{p+1})$, it follows from (5.7) that

$$V(t) \leq V(t_p)e^{c(t-t_p)} \leq Me^{-(\lambda+c)(t_{p+1}-t_0)}e^{c(t_p-t_0)}e^{c(t-t_p)} = Me^{-(\lambda+c)(t_{p+1}-t_0)}e^{c(t-t_0)},$$

then we can conclude that (5.15) holds for $t \in [t_p, t_{p+1})$, i.e., (5.15) is true for $k = p$. Suppose that (5.15) is true for $k \leq m$ $(m > p)$, i.e.,

$$V(t) \leq Me^{-(\lambda+c)(t_{k+1}-t_0)}e^{c(t-t_0)}, \qquad t\in[t_k,t_{k+1}) \quad k\leq m\ (m>p), \tag{5.16}$$

we shall prove (5.15) holds for $k = m + 1$. At $t = t_{m+1}$, We will estimate the supremum of $V(t_{m+1}^- + s)$ for $s \in [-(\bar\tau + r), 0]$ by considering the following two cases:

Case 1: If $t_{m+1} + s \in [t_0, t_p)$ for some $s \in [-(\bar\tau + r), 0]$, then from (5.14), we can get

$$V(t_{m+1}^- + s) \leq e^{-\lambda s}Me^{-\lambda(t_{m+1}-t_0)} \leq e^{\lambda(\bar\tau+\bar r)}Me^{-\lambda(t_{m+1}-t_0)}.$$

Case 2: If $t_{m+1}+s \geq t_p$ for some $s \in [-(\bar\tau+r), 0]$, then there exists an integer $\hat k$ $(p \leq \hat k \leq m)$ such that $t_{m+1} + s \in [t_{\hat k}, t_{\hat k+1})$, then according to (5.16),

$$\begin{aligned}
V(t_{m+1}^- + s) &\leq Me^{-(\lambda+c)(t_{\hat k+1}-t_0)}e^{c(t_{m+1}+s-t_0)}\\
&\leq Me^{-\lambda(t_{m+1}+s-t_0)}\\
&\leq e^{\lambda(\bar\tau+r)}Me^{-\lambda(t_{m+1}-t_0)}.
\end{aligned}$$

Combining the case 1 and case 2, we can conclude that $V(t_{m+1}^{-}+s) \leq e^{\lambda(\bar{\tau}+r)}Me^{-\lambda(t_{m+1}-t_0)}$ for all $s \in [-(\bar{\tau}+r), 0]$, which implies that

$$\sup_{s\in[-(\bar{\tau}+r),0]} V(t_{m+1}^{-}+s) \leq Me^{\lambda(\bar{\tau}+r)}e^{-\lambda(t_{m+1}-t_0)}. \tag{5.17}$$

It follows from (5.11), (5.16) and (5.17) that

$$V_1(t_{m+1}) \leq [\alpha + \beta e^{\lambda(\bar{\tau}+r)}]Me^{-\lambda(t_{m+1}-t_0)}.$$

Similarly, from (5.12), (5.14) and (5.16), we can obtain

$$V_2(t_{m+1}) \leq \omega r e^{\lambda r} Me^{-\lambda(t_{m+1}-t_0)}.$$

According to (5.13), we have

$$\begin{aligned} V(t_{m+1}) &\leq [\alpha + \beta e^{\lambda(\bar{\tau}+r)} + \omega r e^{\lambda r}]\ Me^{-\lambda(t_{m+1}-t_0)} \\ &= e^{-(\lambda+c)(t_{m+2}-t_{m+1})}Me^{-(\lambda+c)(t_{m+1}-t_0)}e^{c(t_{m+1}-t_0)} \\ &= M\ e^{-(\lambda+c)(t_{m+2}-t_0)}e^{c(t_{m+1}-t_0)}, \end{aligned}$$

which implies that (5.15) holds at $t = t_{m+1}$. For $t \in (t_{m+1}, t_{m+2})$, it follows from (5.7) that

$$\begin{aligned} V(t) &\leq M\ e^{-(\lambda+c)(t_{m+2}-t_0)}e^{c(t_{m+1}-t_0)}e^{c(t-t_{m+1})} \\ &= M\ e^{-(\lambda+c)(t_{m+2}-t_0)}e^{c(t-t_0)}. \end{aligned}$$

This shows that (5.15) holds for $t \in (t_{m+1}, t_{m+2})$, hence (5.15) is true for $k = m+1$. By mathematical induction, (5.15) holds. Then we can conclude that

$$\begin{aligned} V_1(t) \leq V(t) &\leq M\ e^{-(\lambda+c)(t_{k+1}-t_0)}e^{c(t_{k+1}-t_0)} \\ &= Me^{-\lambda(t_{k+1}-t_0)} \qquad t \in [t_k, t_{k+1}), \quad k \geq p. \end{aligned}$$

Then we have as $k \to \infty$, $t \to \infty$, $V_1(t) \to 0$, which implies that $|z_i(t) - Ave(z(0))| \to 0$ as $t \to \infty$ for any $i = 1, 2, ..., N$. Thus the hybrid consensus protocol (5.2) leads to the average-consensus for agents in complex-valued multi-agent system (5.1). □

Remark 5.4.1. *The parameter $\omega \in (0, 1]$ can adjust the value of $\alpha+\beta+\omega r$ to guanrantee that condition (5.6) in Theorem 5.4.1 can be satisfied for relatively large size of time-delay r. It can be seen from the proof of Theorem 5.4.1 that the impulsive part of the hybrid protocol (5.2) plays control role to accelerate the average-consensus process, while the continuous-time part of protocol (5.2) may either accelerate or decelerate the average-*

consensus process, and condition (5.6) implies that the hybrid protocol (5.2) can establish average-consensus for agents in (5.1) if impulsive distances are suitably designed.

Theorem 5.4.2. *Suppose that $\mathcal{G}_A$ is balanced, $\mathcal{G}_A'$ is balanced and strongly connected. Let $\lambda_2(\mathcal{L}_s')$ be the second smallest eigenvalue of $\mathcal{L}_s' = \frac{\mathcal{L}' + \mathcal{L}'^T}{2}$, and $\rho_{\min} = (\sqrt{1 - 2\lambda_2(\mathcal{L}_s') + \|\mathcal{L}'\|^2} + \bar{\tau}\|\mathcal{L}\|\|\mathcal{L}'\|)^2$. If $\rho_{\min} < 1$, and the impulsive distance T satisfying*

$$\bar{\tau} < T < \begin{cases} -\dfrac{\ln(\rho_{\min} + r)}{\|\mathcal{L}\|^2}, & \text{if } 0 < r < u^* - \rho_{\min}, \\ \dfrac{1 - \rho_{\min}}{\|\mathcal{L}\|^2 \rho_{\min}}, & \text{if } u^* - \rho_{\min} \leq r < \infty, \end{cases} \tag{5.18}$$

where $u^* = e^{W(\rho_{\min} e) - 1}$ and $W(\cdot)$ is the Lambert W function, then the hybrid consensus protocol (5.2) leads to the average-consensus for agents in system (5.1).

Proof. It can be seen from Theorem 5.4.1 that α and β depend on parameter ε, and c depends on parameter ω, where $0 < \omega \leq 1$. Define $\rho = \rho(\varepsilon) = \alpha + \beta = (1+\varepsilon)(1 - 2\lambda_2(\mathcal{L}_s') + \|\mathcal{L}'\|^2) + (1 + \frac{1}{\varepsilon})(\bar{\tau}\|\mathcal{L}\|\|\mathcal{L}'\|)^2$, then condition (5.6) in Theorem 5.4.1 implies that the average-consensus result will be achieved if $T < \frac{-\omega \ln(\rho + \omega r)}{\|\mathcal{L}\|^2}$. To find the upper bound of the length of each impulsive interval T, we will specify the values of parameters ε and ω to maximize $-\omega \ln(\rho + \omega r)$. For any given $0 < \omega \leq 1$, define $F(\rho) = -\omega \ln(\rho + \omega r)$. To maximize $F(\rho)$, we need $\rho + \omega r < 1$, hence $\rho < 1 - \omega r$. By applying the extreme value theory, $\rho = \rho(\varepsilon)$ attains its minimum when $\varepsilon = \frac{\bar{\tau}\|\mathcal{L}\|\|\mathcal{L}'\|}{\sqrt{1 - 2\lambda_2(\mathcal{L}_s') + \|\mathcal{L}'\|^2}}$, and $\rho_{\min} = \min_{\varepsilon > 0} \rho = (\sqrt{1 - 2\lambda_2(\mathcal{L}_s') + \|\mathcal{L}'\|^2} + \bar{\tau}\|\mathcal{L}\|\|\mathcal{L}'\|)^2$. If $\rho_{\min} < 1 - \omega r < 1$, then we can write

$$F(\rho) = -\omega \ln(\rho + \omega r), \qquad \rho \in [\rho_{\min}, 1 - \omega r),$$

and we have $F'(\rho) = \frac{-\omega}{\rho + \omega r} < 0$ for $\rho \in [\rho_{\min}, 1 - \omega r)$. Therefore, $\max F(\rho) = F(\rho_{\min}) = -\omega \ln(\rho_{\min} + \omega r)$. Next, define $G(\omega) = \frac{-\omega \ln(\rho_{\min} + \omega r)}{\|\mathcal{L}\|^2}$, where $0 < \omega < \frac{1 - \rho_{\min}}{r}$, and $\omega \leq 1$. We consider the following two cases depending on the size of delay r.

Case 1: if $0 < \omega < \frac{1-\rho_{\min}}{r} \leq 1$, then $1 - \rho_{\min} \leq r < \infty$, and $G(\omega) = \frac{-\omega \ln(\rho_{\min} + \omega r)}{\|\mathcal{L}\|^2}$, $\omega \in (0, \frac{1-\rho_{\min}}{r})$. Define $u = \rho_{\min} + \omega r$, for $\omega \in (0, \frac{1-\rho_{\min}}{r})$, we have $u \in (\rho_{\min}, 1)$. Then, $G'(\omega) = 0$ implies $\ln u - \frac{\rho_{\min}}{u} + 1 = 0, \; u \in (\rho_{\min}, 1)$. Define

$$f(u) = \ln u - \frac{\rho_{\min}}{u} + 1, \qquad u \in (\rho_{\min}, 1). \tag{5.19}$$

Then we have $f(\rho_{\min}) = \ln(\rho_{\min}) < 0$, $f(1) = 1 - \rho_{\min} > 0$, and $f^{'}(u) = \frac{1}{u} + \frac{\rho_{\min}}{u^2} > 0$ for $u \in (\rho_{\min}, 1)$. By the IVT, there exists a unique $u^* \in (\rho_{\min}, 1)$ such that $f(u^*) = 0$. Let $v = \ln u$, then $u = e^v$, and $f(u) = 0$ implies $e^{v+1}(v+1) = \rho_{\min} \cdot e$. Based on the property of the Lambert W function, we have $v = W(\rho_{\min} e) - 1$, and $u^* = e^v = e^{W(\rho_{\min} e)-1} \in (\rho_{\min}, 1)$. Then, there exists a unique $\omega^* = \frac{u^* - \rho_{\min}}{r}$ such that $G^{'}(\omega^*) = 0$, where $0 < \omega^* = \frac{u^* - \rho_{\min}}{r} < \frac{1-\rho_{\min}}{r} \leq 1$. According to (5.19), if $0 < \omega < \omega^*$, $\rho_{\min} < u < u^*$, then $f(u) < 0$ and $G^{'}(\omega) > 0$; if $\omega^* < \omega < \frac{1-\rho_{\min}}{r}$, $u^* < u < 1$, then $f(u) > 0$ and $G^{'}(\omega) < 0$. Hence, we have for $1 - \rho_{\min} \leq r < \infty$,

$$\max_{\omega \in (0, \frac{1-\rho_{\min}}{r})} G(\omega) = G(\omega^*) = \frac{(u^* - \rho_{\min})^2}{\|\mathcal{L}\|^2 r u^*} \leq \frac{1 - \rho_{\min}}{\|\mathcal{L}\|^2 \rho_{\min}}.$$

Case 2: if $0 < \omega \leq 1 < \frac{1-\rho_{\min}}{r}$, then $0 < r < 1 - \rho_{\min}$, and $G(\omega) = \frac{-\omega \ln(\rho_{\min} + \omega r)}{\|\mathcal{L}\|^2}$, $\omega \in (0, 1]$. Let $u = \rho_{\min} + \omega r$, for $\omega \in (0, 1]$, we have $u \in (\rho_{\min}, \rho_{\min} + r]$. Define

$$h(u) = \ln u - \frac{\rho_{\min}}{u} + 1, \qquad u \in (\rho_{\min}, \rho_{\min} + r]. \tag{5.20}$$

Then, $G^{'}(\omega) = 0$ implies $h(u) = 0$. Furthermore, we have $h(\rho_{\min}) = \ln(\rho_{\min}) < 0$, and $h^{'}(u) = \frac{1}{u} + \frac{\rho_{\min}}{u^2} > 0$ for $u \in (\rho_{\min}, \rho_{\min} + r]$. From the previous discussion, we have $h(u^*) = 0$ with $u^* = e^{W(\rho_{\min} e)-1}$. If $u^* \in (\rho_{\min}, \rho_{\min} + r]$, then $u^* - \rho_{\min} \leq r < 1 - \rho_{\min}$, and there exists a unique $\omega^* = \frac{u^* - \rho_{\min}}{r} \in (0, 1]$ such that $G^{'}(\omega^*) = 0$. According to (5.20), if $0 < \omega < \omega^*$, $\rho_{\min} < u < u^*$, then $h(u) < 0$ and $G^{'}(\omega) > 0$; if $\omega^* < \omega \leq 1$, $u^* < u \leq \rho_{\min} + r$, then $h(u) > 0$ and $G^{'}(\omega) < 0$. Hence, for $u^* - \rho_{\min} \leq r < 1 - \rho_{\min}$, we have

$$\max_{\omega \in (0,1]} G(\omega) = G(\omega^*) = \frac{(u^* - \rho_{\min})^2}{\|\mathcal{L}\|^2 r u^*} \leq \frac{1 - \rho_{\min}}{\|\mathcal{L}\|^2 \rho_{\min}}.$$

On the other hand, if $u^* > \rho_{\min} + r$, then $0 < r < u^* - \rho_{\min}$, and $\omega^* = \frac{u^* - \rho_{\min}}{r} > 1$. According to (5.20), if $0 < \omega \leq 1$, $\rho_{\min} < u \leq \rho_{\min} + r$, then we have $h(u) < 0$ and $G^{'}(\omega) > 0$ for all $\omega \in (0, 1]$. Hence, for $0 < r < u^* - \rho_{\min}$, we have

$$\max_{\omega \in (0,1]} G(\omega) = G(1) = \frac{-\ln(\rho_{\min} + r)}{\|\mathcal{L}\|^2}.$$

According to the above discussion, we can conclude that the hybrid consensus protocol (5.2) leads to average-consensus for agents in complex-valued multi-agent system (5.1) if the length of each impulsive interval T satisfies (5.18), which completes the proof. □

Remark 5.4.2. *Compared with the consensus results in [108], Theorem 5.4.2 establishes delay-dependent conditions for the upper bound of the length of each impulsive interval T to guarantee the average consensus. It can be seen from Theorem 5.4.2 that the upper bound* *of T is closely related to the continuous-time delay size r of the hybrid consensus protocol (5.2) and the value of $\rho_{\min}$. Theorem 5.4.2 shows that if the interaction topologies $\mathcal{G}_A$ and $\mathcal{G}'_A$ of the hybrid consensus protocol (5.2) are suitably designed, then any T less than the upper bound shown in (5.18) can guarantee the proposed hybrid protocol (5.2) leads to the average-consensus of complex-valued multi-agent system (5.1).*

5.5 Numerical Simulations

In this section, we consider an example to illustrate our theoretical results.

Example 5.5.1. *Consider system (5.1) consisting of eight agents under the hybrid consensus protocol (5.2) with digraphs $\mathcal{G}_A$ and $\mathcal{G}'_A$ shown in Figure 5.1. The solid lines denote the edges of the digraph $\mathcal{G}_A$, and the dashed lines represent the edges of the digraph $\mathcal{G}'_A$. $\mathcal{G}_A$ is assumed to have weight 0.12 between the 2^{nd} agent and the 8^{th} agent; weight 0.15 between the 3^{rd} agent and the 7^{th} agent, and weight 0.18 between the 4^{th} agent and the 6^{th} agent; and digraph $\mathcal{G}'_A$ is assumed to have equal weights 0.08.*

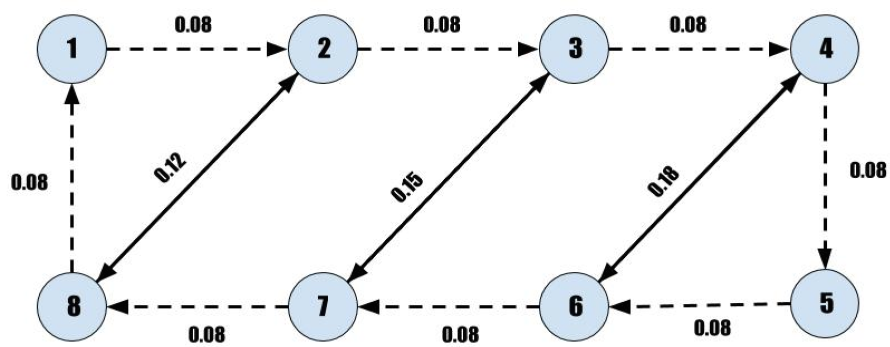

Figure 5.1: Continuous-time and discrete-time interaction topologies $\mathcal{G}_A$ and $\mathcal{G}'_A$.

It can be seen from Figure 5.1 that $\mathcal{G}_A$ is balanced with $\|\mathcal{L}\| = 0.36$, and $\mathcal{G}'_A$ is balanced and strongly connected with $\|\mathcal{L}'\| = 0.16$, and $\lambda_2(\mathcal{L}'_s) = 0.0234$. Let $\bar{\tau} = 0.03$, then we have $\rho_{\min} = 0.982 < 1$, and $u^* = 0.991$. Choose $r = 0.006$, then $r < u^* - \rho_{\min}$. Theorem 5.4.2 implies that protocol (5.2) can establish the average-consensus if the length of each impulsive interval T satisfies $0.03 = \bar{\tau} < T < -\frac{\ln(\rho_{\min}+r)}{\|\mathcal{L}\|^2} = 0.093$. Choose $T = 0.08$, and the initial conditions are chosen as $z(0) = [3-j, -6+2j, 1+6j, 4-3j, -2+4j, 8-5j, -5+j, -3-4j]^T$ such that $Ave(z(0)) = 0$. Figure 5.2 shows that all the eight complex-valued agents finally reach the consent state $Ave(z(0)) = 0$, which implies that the average-consensus is achieved.

On the other hand, if we choose $r = 2$, then we have $r \geq u^* - \rho_{\min}$. It follows from Theorem 5.4.2 that $\bar{\tau} = 0.03 < T < 0.141 = \frac{1-\rho_{\min}}{\|\mathcal{L}\|^2 \rho_{\min}}$ can guarantee the protocol establishes the average-consensus. With the same initial conditions, choose $T = 0.12$, Figure 5.3 shows that the average-consensus can still be confirmed even if $r > T$.

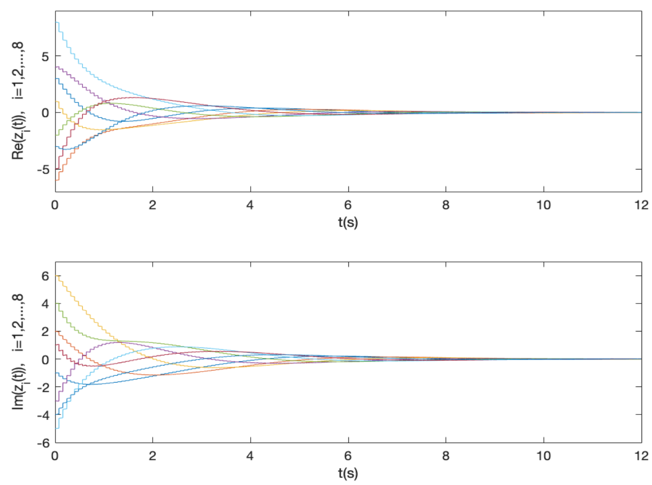

Figure 5.2: Average consensus process of Example 5.5.1 with $\bar{\tau} = 0.03, r = 0.006$, and $T = 0.08$.

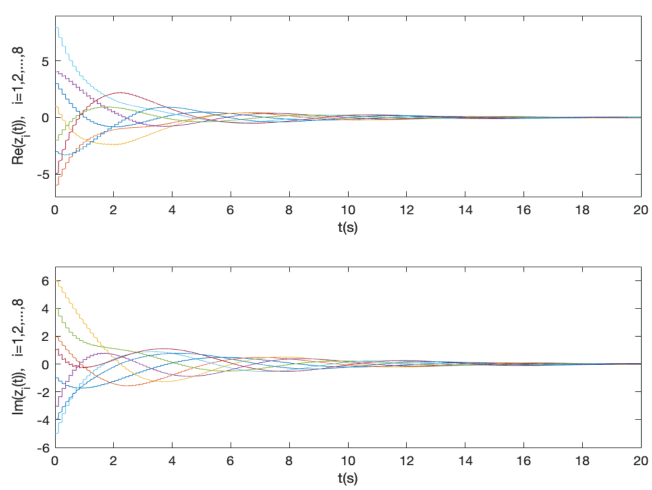

Figure 5.3: Average consensus process of Example 5.5.1 with $\bar{\tau} = 0.03, r = 2$, and $T = 0.12$.

Chapter 6

Applications to Complex-Valued Neural Networks (CVNNs)

As a practical application for complex-valued networked systems, this chapter studies the synchronization problem of master-slave CVNNs. The state estimation problem of CVNNs is also studied in this chapter. In Section 6.1, the CVNN model is introduced. In Section 6.2, we study the synchronization problem of master-slave CVNNs by using a sampled-data-based hybrid control strategy. In Section 6.3, we apply the ETM and the pinning algorithm discussed in Subsection 3.6.1 to propose a delayed ETPIC scheme in the complex domain to study master-salve synchronization of CVNNs. In Section 6.4, we investigate the state estimation problem of CVNNs by designing the adaptive impulsive observer in the complex field.

6.1 CVNN Model

Consider a time-delay CVNN coupled with n neurons, which can be described as follows:

$$\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^{n} a_{ij} f_j(z_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(z_j(t-r)) + J_i, \qquad i = 1, 2, ..., n, \tag{6.1}$$

where $z_i \in \mathbb{C}$ is the state of the i-th neuron; $c_i > 0$ denotes the rate with which the i-th neuron resets its potential to the resting state for isolation when disconnected with other neurons of the network; $a_{ij} \in \mathbb{C}$ and $b_{ij} \in \mathbb{C}$ represent the complex-valued connection weight between neurons i and j at time t and time $t-r$, respectively; $f_j(\cdot) : \mathbb{C} \to \mathbb{C}$ and $g_j(\cdot) : \mathbb{C} \to$

$\mathbb{C}$, $j = 1, 2, ..., n$, denote the non-delayed and delayed complex-valued nonlinear activation functions, respectively; r denotes the transmission delay when processing information from the j-th neuron; $J_i \in \mathbb{C}$ represents the external input to the i-th neuron.

6.2 Master-Slave Synchronization of CVNNs via Hybrid Control

Synchronization, as a crucial and significant collective behavior of neural networks, has recently been received a great deal of research attention. It has been shown that delayed artificial neural networks can exhibit chaotic behavior if parameters of the network and time delays are suitably chosen [110, 111]. Chaos synchronization of neural networks can be applied in many scientific and engineering scenarios, such as secure communication, pattern recognition, associative memory, etc. [112, 113]. In particular, Pecora and Carroll [114] firstly introduced the concept of master-slave (drive-response) synchronization in their pioneering work. Since then, synchronization of master-slave systems has become a hot topic for researchers, especially for nonlinear systems, such as Lur'e systems and neural networks. In recent years, some interesting results for master-slave synchronization of neural networks have been reported (see, e.g., [115, 116, 117, 118]). However, it should be noted that most of the aforementioned synchronization results are about real-valued neural networks. As an extension of real-valued neural networks, CVNNs have higher functionality, learn faster and generalize better than their real-valued counterpart.

Motivated by the above discussions, this section studies the synchronization problem of master-slave CVNNs via hybrid control. In Subsection 6.2.1, we formulate the synchronization problem of master-slave CVNNs, and propose a hybrid controller which consists of sampled-data controller and impulsive controller in the complex field. The synchronization criteria are presented in Subsection 6.2.2 with some discussions. In Subsection 6.2.3, numerical simulations are provided to demonstrate the effectiveness of the obtained results.

6.2.1 Problem Formulation

Consider time-delay CVNN (6.1) as the master system. The initial condition of (6.1) is given by $z_i(t_0+s) = \varphi_i(s)$, for $i = 1, 2, ..., n$, where $\varphi_i \in \mathcal{PC}([-r, 0], \mathbb{C})$. The corresponding slave CVNN of the master system (6.1) is given by:

$$\dot{\hat{z}}_i(t) = -c_i\hat{z}_i(t) + \sum_{j=1}^{n} a_{ij}f_j(\hat{z}_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(\hat{z}_j(t-r)) + J_i + u_i(t), \quad i = 1, 2, ..., n, \tag{6.2}$$

where $u_i(t)$ is the control input of the i-th neuron in the slave CVNN to be designed. The initial condition of (6.2) is given by $\hat{z}_i(t_0 + s) = \psi_i(s)$, for $i = 1, 2, ..., n$, where $\psi_i \in \mathcal{PC}([-r, 0], \mathbb{C})$.

The objective is to synchronize the slave CVNN (6.2) with the master CVNN (6.1), i.e.,

$$\lim_{t\to\infty} \|\hat{z}_i(t) - z_i(t)\| = 0, \quad i = 1, 2, ..., n.$$

For the nonlinear complex-valued activation functions, we make the following assumption:

Assumption 6.2.1. *For any $j = 1, 2, ..., n$, there exist positive constants L_j, F_j, such that*

$$|f_j(u) - f_j(v)| \le L_j|u - v|,$$
$$|g_j(u) - g_j(v)| \le F_j|u - v|$$

for all $u, v \in \mathbb{C}$. Denote $L = diag\{L_1, L_2, \ldots, L_n\}$, and $F = diag\{F_1, F_2, \ldots, F_n\}$.

Let $e_i(t) = \hat{z}_i(t) - z_i(t)$ denote the synchronization error between the i-th neuron of the slave CVNN (6.2) and the master CVNN (6.1). Then, master-slave CVNNs (6.1)-(6.2) can be transformed into the following equivalent error dynamical system:

$$\dot{e}_i(t) = -c_i e_i(t) + \sum_{j=1}^{n} a_{ij}\hat{f}_j(e_j(t)) + \sum_{j=1}^{n} b_{ij}\hat{g}_j(e_j(t-r)) + u_i(t), \quad i = 1, 2, \ldots, n, \tag{6.3}$$

where $\hat{f}_j(e_j(t)) = f_j(\hat{z}_j(t)) - f_j(z_j(t))$, $\hat{g}_j(e_j(t-r)) = g_j(\hat{z}_j(t-r)) - g_j(z_j(t-r))$. The initial condition of the error system (6.3) can be described by $e_i(t_0 + s) = \phi_i(s)$ for $s \in [-r, 0]$, $i = 1, 2, ..., n$, where $\phi_i = \psi_i - \varphi_i$.

Denote $e(t) = (e_1(t), e_2(t), ..., e_n(t))^T$, then we can rewrite error system (6.3) into a matrix-form system:

$$\begin{cases} \dot{e}(t) = -Ce(t) + A\hat{f}(e(t)) + B\hat{g}(e(t-r)) + u(t), \\ e(t_0 + s) = \phi(s), \quad s \in [-r, 0], \end{cases} \tag{6.4}$$

where $C = diag\{c_1, c_2, ..., c_n\}$, $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $\hat{f}(e) = (\hat{f}_1(e_1), \hat{f}_2(e_2), ..., \hat{f}_n(e_n))^T$, $\hat{g}(e) = (\hat{g}_1(e_1), \hat{g}_2(e_2), ..., \hat{g}_n(e_n))^T$, $u = (u_1, u_2, ..., u_n)^T$, and $\phi = (\phi_1, \phi_2, ..., \phi_n)^T$.

For achieving the master-slave synchronization, construct the hybrid controller $u(t)$ as follows:

$$u(t) = K_1 e(t_{k-1}) + K_2 e(t)\delta(t - t_k), \quad t \in [t_{k-1}, t_k), \ k \in \mathbb{N}^+, \tag{6.5}$$

where $K_1 \in \mathbb{C}^{n\times n}$ and $K_2 \in \mathbb{C}^{n\times n}$ are the feedback control gain and the impulsive control gain to be designed, respectively; the impulse sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots$, and $\lim_{k\to\infty} t_k = \infty$; $\delta(\cdot)$ is the Dirac Delta function.

Under (6.5), the error system (6.4) can be rewritten as

$$\begin{cases} \dot{e}(t) = -Ce(t) + A\hat{f}(e(t)) + B\hat{g}(e(t-r)) + K_1 e(t_{k-1}), \quad t \ge t_0,\ t \in [t_{k-1}, t_k), \\ \Delta e(t_k) = K_2 e(t_k^-), \quad k \in \mathbb{N}^+, \\ e(t_0+s) = \phi(s), \quad s \in [-r, 0], \end{cases} \tag{6.6}$$

where $\Delta e(t_k) = e(t_k^+) - e(t_k^-)$. We assume solutions of system (6.6) are right continuous, i.e., $e(t_k) = e(t_k^+)$. Then, the master-slave synchronization problem of CVNNs (6.1) and (6.2) is transformed into the asymptotic stability problem of the zero solution of the error dynamical system (6.6).

The following lemma will be used in the proof of the synchronization result in Subsection 6.2.2.

Lemma 6.2.1. *Consider the following impulsive delay system:*

$$\begin{cases} \dot{x}(t) = f(t, x_t), \quad t \ge t_0,\ t \ne t_k, \\ \Delta x(t_k) = I_k(t_k, x(t_k^-)), \quad k \in \mathbb{N}^+, \\ x(t_0+s) = \phi(s), \quad s \in [-\tau, 0]. \end{cases} \tag{6.7}$$

Assume that there exist constants $p > 0, c_1 > 0, c_2 > 0, c_3 > 0, \alpha \in \mathbb{R}, \beta \ge 0, \rho > 0, \delta \in \mathbb{R}$, and function $V(t,x) : [t_0 - \tau, \infty) \times \mathbb{C}^n \to \mathbb{R}^+$ such that

(i) $c_1\|x(t)\|^p \le V(t,x) \le c_2\|x(t)\|^p + c_3\|x(t_k)\|^p$, $t \in [t_{k-1}, t_k)$;
(ii) $D^+V(t,\varphi) \le \alpha V(t,\varphi(0)) + \beta \sup_{\theta\in[-\tau,0]} V(t+\theta, \varphi(\theta))$ for all $\varphi \in \mathcal{PC}([-\tau,0];\mathbb{C}^n)$, $t \ge t_0, t \ne t_k, k \in \mathbb{N}^+$;
(iii) $V(t_k, \varphi(0) + I_k(t_k, \varphi(0))) \le \rho V(t_k^-, \varphi(0))$ for all $\varphi \in \mathcal{PC}([-\tau,0];\mathbb{C}^n), k \in \mathbb{N}^+$;
(iv) $\ln\rho \le \delta(t_k - t_{k-1})$ for each $k \in \mathbb{N}^+$;
(v) $\delta + \alpha + \gamma\beta < 0$, where $\gamma = \sup_{k\in\mathbb{N}_+}\{e^{\delta(t_k - t_{k-1})}, \frac{1}{e^{\delta(t_k-t_{k-1})}}\}$.
Then, system (6.7) is exponentially stable for any time delay $\tau \in (0,\infty)$, and the convergence rate should not be greater than λ/p, where λ is the unique positive solution of $\lambda + \delta + \alpha + \gamma\beta e^{\lambda\tau} = 0$.

6.2.2 Synchronization Criteria

In this subsection, synchronization criteria for master-slave CVNNs (6.1) and (6.2) are established. Results show that the proposed hybrid controller can greatly improve the overall control efficiency.

Theorem 6.2.1. *Suppose that Assumption 6.2.1 is satisfied. Given constants* $\alpha \in \mathbb{R}, \beta \geq 0$*, and* $\rho > 0$*, if there exist four* $n \times n$ *positive definite Hermitian matrices* P, Q, R, S*, and two complex-valued matrices* U_1, U_2 *with compatible dimensions such that the following LMIs hold:*

$$F^T SF - \beta P \leq 0, \tag{6.8}$$

$$\begin{bmatrix} -PC - C^T P + L^T RL - \alpha P & U_1 & PA & PB \\ \star & -(\alpha+\beta)Q & 0 & 0 \\ \star & \star & -R & 0 \\ \star & \star & \star & -S \end{bmatrix} \leq 0, \tag{6.9}$$

$$\begin{bmatrix} -\rho P & U_2^* \\ \star & -(P+Q) \end{bmatrix} \leq 0, \tag{6.10}$$

and if there exist constant δ *such that the impulsive sequence* $\{t_k\}$ *satisfies*

$$\ln \rho \leq \delta(t_k - t_{k-1}), \quad k \in \mathbb{N}^+, \tag{6.11}$$

and

$$\delta + \alpha + \gamma\beta < 0, \text{ where } \gamma = \sup_{k \in \mathbb{N}^+} \{e^{\delta(t_k - t_{k-1})}, 1/e^{\delta(t_k - t_{k-1})}\}, \tag{6.12}$$

then master-slave synchronization of CVNNs (6.1)-(6.2) can be achieved with the control gains designed by

$$K_1 = P^{-1}U_1, \qquad K_2 = (P+Q)^{-1}U_2 - I.$$

Proof. Consider the following Lyapunov function candidate,

$$V(t) = e^*(t)Pe(t) + e^*(t_{k-1})Qe(t_{k-1}), \qquad t \in [t_{k-1}, t_k),\ k \in \mathbb{N}^+.$$

According to (6.9) and Lemma 3.5.1, we can obtain

$$\begin{bmatrix} -PC - C^T P + L^T RL - \alpha P + PBS^{-1}B^* P & U_1 & PA \\ \star & -(\alpha+\beta)Q & 0 \\ \star & \star & -R \end{bmatrix} \leq 0.$$

Applying Lemma 3.5.1 again, we have

$$\begin{bmatrix} \Omega_{11} & PK_1 \\ \star & -(\alpha+\beta)Q \end{bmatrix} \leq 0, \tag{6.13}$$

where $\Omega_{11} = -PC - C^TP + L^TRL + PAR^{-1}A^*P + PBS^{-1}B^*P - \alpha P$.

For any interval $[t_{k-1}, t_k), k \in \mathbb{N}^+$, differentiate V along the solutions of the error system (6.6), it follows from (6.8), (6.13), Lemma 3.2.1 and Assumption 6.2.1 that

$$\begin{aligned}
&D^+V(t) = \dot{e}^*(t)Pe(t) + e^*(t)P\dot{e}(t) \\
&= -e^*(t)C^TPe(t) + \hat{f}^*(e(t))A^*Pe(t) + \hat{g}^*(e(t-r))B^*Pe(t) + e^*(t_{k-1})K_1^*Pe(t) - e^*(t)PCe(t) \\
&+ e^*(t)PA\hat{f}(e(t)) + e^*(t)PB\hat{g}(e(t-r)) + e^*(t)PK_1e(t_{k-1}) \\
&\leq e^*(t)[-C^TP - PC]e(t) + \hat{f}^*(e(t))R\hat{f}(e(t)) + e^*(t)PAR^{-1}A^*Pe(t) + \hat{g}^*(e(t-r))S\hat{g}(e(t-r)) \\
&+ e^*(t)PBS^{-1}B^*Pe(t) + e^*(t_{k-1})K_1^*Pe(t) + e^*(t)PK_1e(t_{k-1}) \\
&\leq e^*(t)[-C^TP - PC + L^TRL + PAR^{-1}A^*P + PBS^{-1}B^*P]e(t) + e^*(t-r)F^TSFe(t-r) \\
&+ e^*(t_{k-1})K_1^*Pe(t) + e^*(t)PK_1e(t_{k-1}) \\
&\leq \begin{bmatrix} e(t) \\ e(t_{k-1}) \end{bmatrix}^* \begin{bmatrix} -PC - C^TP + PAR^{-1}A^*P + PBS^{-1}B^*P + L^TRL - \alpha P & PK_1 \\ \star & -\alpha Q \end{bmatrix} \begin{bmatrix} e(t) \\ e(t_{k-1}) \end{bmatrix} \\
&+ \begin{bmatrix} e(t) \\ e(t_{k-1}) \end{bmatrix}^* \begin{bmatrix} \alpha P & 0 \\ 0 & \alpha Q \end{bmatrix} \begin{bmatrix} e(t) \\ e(t_{k-1}) \end{bmatrix} + \beta e^*(t-r)Pe(t-r) \\
&= \begin{bmatrix} e(t) \\ e(t_{k-1}) \end{bmatrix}^* \begin{bmatrix} \Omega_{11} & PK_1 \\ \star & -(\alpha+\beta)Q \end{bmatrix} \begin{bmatrix} e(t) \\ e(t_{k-1}) \end{bmatrix} + \alpha V(t) + \beta V(t-r) \\
&\leq \alpha V(t) + \beta V(t-r) \\
&\leq \alpha V(t) + \beta \sup_{\theta \in [-r,0]} V(t+\theta).
\end{aligned} \tag{6.14}$$

On the other hand, it follows from (6.10) and Lemma 3.5.1 that

$$\begin{aligned}
&-\rho P + U_2^*(P+Q)^{-1}U_2 \leq 0 \\
&\Rightarrow -\rho P + U_2^*(P+Q)^{-1}(P+Q)(P+Q)^{-1}U_2 \leq 0 \\
&\Rightarrow -\rho P + (K_2+I)^*(P+Q)(K_2+I) \leq 0.
\end{aligned} \tag{6.15}$$

For $t = t_k, k \in \mathbb{N}^+$, we have from system (6.6) and (6.15) that

$$\begin{aligned}
V(t_k) &= e^*(t_k)(P+Q)e(t_k) \\
&= e^*(t_k^-)(I+K_2)^*(P+Q)(I+K_2)e(t_k^-) \\
&\leq \rho e^*(t_k^-)Pe(t_k^-) \\
&\leq \rho e^*(t_k^-)Pe(t_k^-) + \rho e^*(t_{k-1})Qe(t_{k-1}) \\
&= \rho V(t_k^-).
\end{aligned} \tag{6.16}$$

Then it follows from (6.11), (6.12), (6.14), (6.16) and Lemma 6.2.1 that the trivial solution of error system (6.6) is globally exponentially stable, which implies that the master-slave synchronization of CVNNs (6.1)-(6.2) is achieved with $K_1 = P^{-1}U_1$, and $K_2 = (P + Q)^{-1}U_2 - I$. □

Remark 6.2.1. *In Theorem 6.2.1, some sufficient conditions are derived to design the hybrid controller* (6.5) *with the combination of sampled-data controller and impulsive controller. Compared with the sampled-data control scheme and the impulsive control scheme, the design of the hybrid controller can greatly improve the overall control efficiency. If the sampled-data controller cannot guarantee the stability of the systems, the impulsive controller will be helpful for enhancing the stability of the error system. On the other hand, if the impulses destroy the stability of the error system, the sampled-data controller can still be used to ensure the stability of the error system.*

Suppose that there is no sampled-data control but only impulsive control, that is, for $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$, K_1 in the hybrid controller (6.5) is pre-given, and the sampled-data term $K_1 e(t_k)$ is regarded as the bounded external disturbance which may suppress the stability of the error system. According to Theorem 6.2.1, we can design the impulsive controller to stabilize the error system (6.6).

For convenience, we define the following notations: $T_{\text{sup}} = \sup_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\}, T_{\text{inf}} = \inf_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\}$.

Corollary 6.2.1. *(Impulsive control) Suppose that Assumption 6.2.1 is satisfied. Given constants $\alpha > 0, \beta \geq 0$, and $0 < \rho < 1$, if there exist four $n \times n$ positive definite Hermitian matrices P, Q, R, S, and a complex-valued matrix U_2 with compatible dimension such that* (6.8) *and* (6.10) *hold, and*

$$\begin{bmatrix} -PC - C^TP + L^TRL - \alpha P & PK_1 & PA & PB \\ \star & -(\alpha+\beta)Q & 0 & 0 \\ \star & \star & -R & 0 \\ \star & \star & \star & -S \end{bmatrix} \leq 0, \tag{6.17}$$

and if there exist constant δ such that the impulsive sequence $\{t_k\}$ satisfies

$$\ln \rho \leq \delta T_{\sup}, \tag{6.18}$$

and

$$\delta + \alpha + \frac{\beta}{e^{\delta T_{\sup}}} < 0, \tag{6.19}$$

then master-slave synchronization of CVNNs (6.1)-(6.2) can be achieved with the impulsive control gain designed by

$$K_2 = (P + Q)^{-1} U_2 - I.$$

Proof. For $\alpha > 0, \beta \geq 0, 0 < \rho < 1$, conditions (6.11) and (6.12) in Theorem 6.2.1 can be satisfied for constant $\delta < 0$, then (6.18) implies that condition (6.11) in Theorem 6.2.1 holds, and γ in condition (6.12) of Theorem 6.2.1 is $\frac{1}{e^{\delta T_{\sup}}}$. Consequently, the result follows from Theorem 6.2.1 immediately, and the proof is complete. □

On the other hand, if there is no impulsive control but only sampled-data control, that is, for $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$, K_2 in the hybrid controller (6.5) is pre-given, and impulses are considered to be bounded perturbations. According to Theorem 6.2.1, we can design the sampled-data controller to stabilize the error system (6.6).

Corollary 6.2.2. *(Sampled-data control) Suppose that Assumption 6.2.1 is satisfied. Given constants $\alpha < 0, \beta \geq 0$, and $\rho > 1$, if there exist four $n \times n$ positive definite Hermitian matrices P, Q, R, S, and a complex-valued matrix U_1 with compatible dimension such that (6.8) and (6.9) hold, and*

$$\begin{bmatrix} -\rho P & (I + K_2)^*(P + Q) \\ \star & -(P + Q) \end{bmatrix} \leq 0, \tag{6.20}$$

and if there exist constant δ such that the impulsive sequence $\{t_k\}$ satisfies

$$\ln \rho \leq \delta T_{\inf}, \tag{6.21}$$

and

$$\delta + \alpha + e^{\delta T_{\sup}} \beta < 0, \tag{6.22}$$

then master-slave synchronization of CVNNs (6.1)-(6.2) can be achieved with the sampled-data control gain designed by

$$K_1 = P^{-1} U_1.$$

Proof. For $\alpha < 0, \beta \geq 0, \rho > 1$, conditions (6.11) and (6.12) in Theorem 6.2.1 can be satisfied for constant $\delta > 0$, then (6.21) implies that condition (6.11) in Theorem 6.2.1 holds, and γ in condition (6.12) of Theorem 6.2.1 is $e^{\delta T_{\sup}}$. Then, the conclusion directly follows from Theorem 6.2.1, and the proof is complete. □

6.2.3 Numerical Example

In this subsection, numerical simulations are constructed to demonstrate our theoretical results.

Example 6.2.1. *Consider the following two-neuron CVNN with time-delay as the master system:*

$$\begin{cases} \dot{z}_1(t) = -c_1 z_1(t) + a_{11} f_1(z_1(t)) + a_{12} f_2(z_2(t)) \\ \qquad\qquad + b_{11} g_1(z_1(t-r)) + b_{12} g_2(z_2(t-r)) + J_1, \\ \dot{z}_2(t) = -c_2 z_2(t) + a_{21} f_1(z_1(t)) + a_{22} f_2(z_2(t)) \\ \qquad\qquad + b_{21} g_1(z_1(t-r)) + b_{22} g_2(z_2(t-r)) + J_2, \end{cases} \tag{6.23}$$

where $z_i = z_i^R + j z_i^I$ *for* $i = 1, 2$, $f_i(z_i) = \frac{4}{5} tanh(z_i^R) + j\frac{1}{5} tanh(z_i^I)$, $g_i(z_i) = -|tanh(z_i^I)| + j|tanh(z_i^R)|$, $C = diag\{c_1, c_2\} = diag\{6, 8\}$,

$$A = [a_{ij}]_{2\times 2} = \begin{pmatrix} 3-j & -1+2j \\ 1+5j & 2+4j \end{pmatrix}, \quad B = [b_{ij}]_{2\times 2} = \begin{pmatrix} 1.5+0.5j & -0.2j \\ 0.8-1.5j & 0.6+0.8j \end{pmatrix},$$

$r = 0.5$, *and* $J = [J_1, J_2]^T = [0, 0]^T$. *Note that Assumption 6.2.1 holds with* $L = 0.8I_2$, *and* $F = I_2$. *The initial condition of the master CVNN (6.23) is chosen randomly as* $[\varphi_1(s), \varphi_2(s)]^T = [1+3j, 2-j]^T$ *for* $s \in [-0.5, 0]$. *The corresponding slave CVNN is described by*

$$\begin{cases} \dot{\hat{z}}_1(t) = -c_1 \hat{z}_1(t) + a_{11} f_1(\hat{z}_1(t)) + a_{12} f_2(\hat{z}_2(t)) \\ \qquad\qquad + b_{11} g_1(\hat{z}_1(t-r)) + b_{12} g_2(\hat{z}_2(t-r)) + J_1 + u_1(t), \\ \dot{\hat{z}}_2(t) = -c_2 \hat{z}_2(t) + a_{21} f_1(\hat{z}_1(t)) + a_{22} f_2(\hat{z}_2(t)) \\ \qquad\qquad + b_{21} g_1(\hat{z}_1(t-r)) + b_{22} g_2(\hat{z}_2(t-r)) + J_2 + u_2(t). \end{cases} \tag{6.24}$$

The initial condition of the slave CVNN (6.24) is given by $[\psi_1(s), \psi_2(s)]^T = [-1-j, -3+2j]^T$ *for* $s \in [-0.5, 0]$.

Based on the synchronization results in Subsection 6.2.2, we will study master-slave synchronization of time-delay CVNNs (6.23)-(6.24) by different control methods.

Case 1. *Hybrid control. We firstly consider the hybrid controller* $u(t) = [u_1, u_2]^T$ *in* (6.5) *including sampled-data controller and impulsive controller. Let* $\alpha = -2, \beta = 5 \geq 0, \rho = 0.3 > 0$, *using the MATLAB YALMIP toolbox, LMIs* (6.8)-(6.10) *have the following feasible solutions:*

$$P = \begin{pmatrix} 36.9533 & -0.5447 + 0.8333j \\ -0.5447 - 0.8333j & 24.7409 \end{pmatrix}, \quad Q = \begin{pmatrix} 72.9498 & 1.6420 - 0.8685j \\ 1.6420 + 0.8685j & 78.7720 \end{pmatrix},$$

$$R = \begin{pmatrix} 199.8200 & 13.3662 + 13.7055j \\ 13.3662 - 13.7055j & 166.9579 \end{pmatrix}, \quad S = \begin{pmatrix} 114.9210 & -2.2381 + 4.3874j \\ -2.2381 - 4.3874j & 73.7081 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} -57.1163 & 3.4228 + 1.9049j \\ 3.8273 - 1.9049j & -52.5417 \end{pmatrix}, \quad U_2 = \begin{pmatrix} -26.7446 & -0.0805 - 0.0762j \\ -0.0331 + 0.0762j & -25.0242 \end{pmatrix},$$

and the control gains K_1 *and* K_2 *are designed by*

$$K_1 = \begin{pmatrix} -1.5468 - 0.0046j & 0.0614 + 0.0995j \\ 0.1208 - 0.1292j & -2.1257 + 0.0043j \end{pmatrix}, \quad K_2 = \begin{pmatrix} -1.2434 & 0.0017 - 0.0008j \\ 0.0023 + 0.0008j & -1.2418 \end{pmatrix}.$$

Choose the sampling impulsive sequence as $t_{3n-2} = 0.2n - 0.15, t_{3n-1} = 0.2n - 0.08, t_{3n} = 0.2n$ *for* $n \in \mathbb{N}^+$, *then we can choose* $\delta = -15$ *such that conditions* (6.11) *and* (6.12) *hold. By Theorem 6.2.1, the master-slave synchronization of CVNNs* (6.23)-(6.24) *is achieved. The time evolution of the real and imaginary parts for state variables of the first and the second neuron in master and slave systems* (6.23)-(6.24) *are shown in Figure 6.1 and Figure 6.2, respectively; trajectories of real and imaginary parts of synchronization errors for master-slave CVNNs* (6.23) *and* (6.24) *are plotted in Figure 6.3. It is clearly observed from the simulation results in Figure 6.3 that master-slave synchronization of CVNNs* (6.23) *and* (6.24) *is achieved.*

Case 2. *Impulsive control. Consider the sampled-data term in* (6.5) *as the external disturbance with* $K_1 = \begin{pmatrix} 2+6j & 1+3j \\ 4j & 8+5j \end{pmatrix}$. *It can be seen from Figure 6.4 that both real and imaginary parts of synchronization errors of master-slave CVNNs* (6.23) *and* (6.24) *with external disturbance are unstable.*

Let $\alpha = 1 > 0, \beta = 0.6 \geq 0$, *and* $\rho = 0.1 < 1$, *using the MATLAB LMI toolbox, the LMIs in Corollary 6.2.1 have the following feasible solutions:*

$$P = \begin{pmatrix} 68.2724 & -20.8896 - 6.2204j \\ -20.8896 + 6.2204j & 23.8269 \end{pmatrix}, \quad Q = \begin{pmatrix} 576.4226 & 11.1873 - 6.0005j \\ 11.1873 + 6.0005j & 563.3805 \end{pmatrix},$$

$$R = \begin{pmatrix} 373.0060 & -136.7310 - 26.5562j \\ -136.7310 + 26.5562j & 188.7784 \end{pmatrix}, \quad S = \begin{pmatrix} 38.6228 & -11.3911 - 3.4867j \\ -11.3911 + 3.4867j & 12.3326 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} -32.2779 & 4.0108 + 3.6890j \\ 3.4061 - 3.6890j & -24.4006 \end{pmatrix},$$

and the impulsive control gain K_2 is designed by

$$K_2 = \begin{pmatrix} -1.0499 & 0.0056 + 0.0049j \\ 0.0050 - 0.0052j & -1.0414 \end{pmatrix}.$$

Choose the sampling impulsive sequence as $t_{3n-2} = 0.6n - 0.4, t_{3n-1} = 0.6n - 0.15, t_{3n} = 0.6n$ for $n \in \mathbb{N}^+$, then we have $T_{\sup} = 0.25$. We can choose $\delta = -8$ such that (6.18) and (6.19) are satisfied. Then it follows from Corollary 6.2.1 that the master CVNN (6.23) and the slave CVNN (6.24) can achieve synchronization. The time evolution of real and imaginary parts of synchronization errors for master-slave CVNNs (6.23) and (6.24) under impulsive control are shown in Figure 6.5. It can be seen from the results of simulations in Figure 6.5 that the slave CVNN (6.24) is synchronized with the master CVNN (6.23).

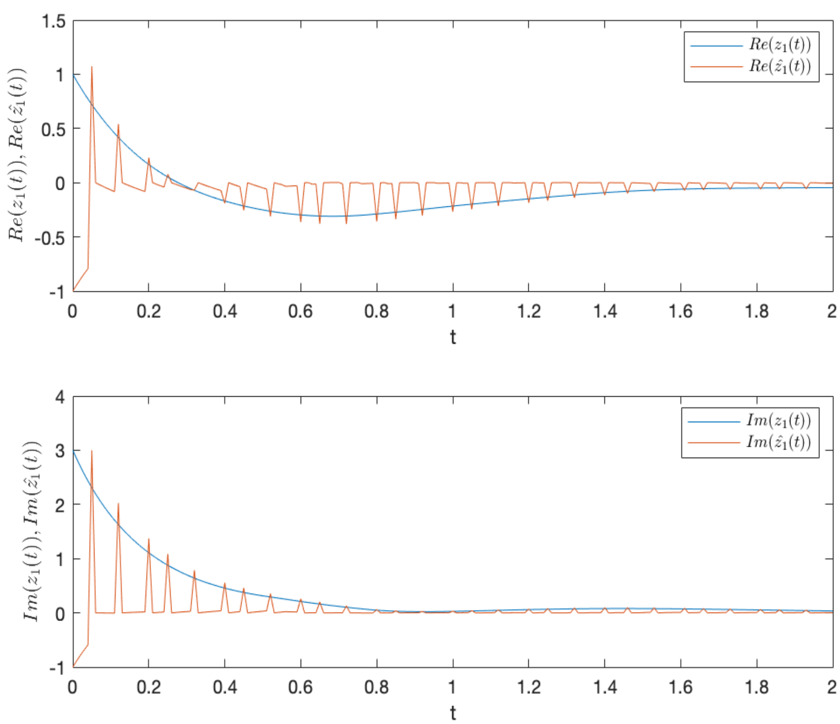

Figure 6.1: Real and imaginary parts of state trajectories for the first neuron in master and slave CVNNs (6.23)-(6.23) under hybrid control.

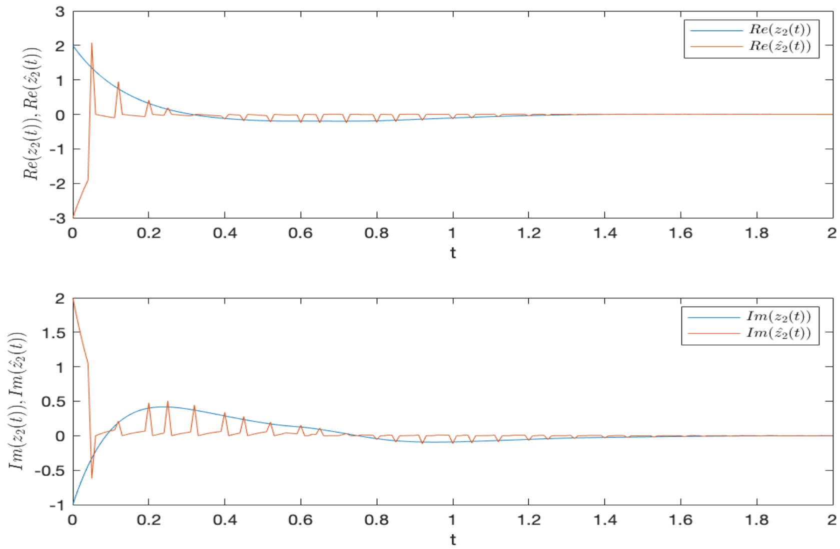

Figure 6.2: Real and imaginary parts of state trajectories for the second neuron in master and slave CVNNs (6.23)-(6.23) under hybrid control.

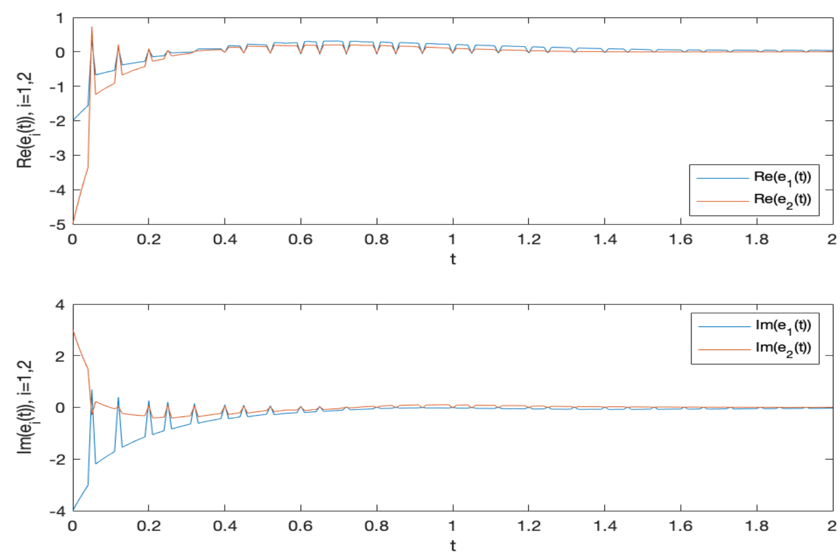

Figure 6.3: Trajectories of real and imaginary parts of synchronization errors for master-slave CVNNs (6.23) and (6.24) via hybrid control.

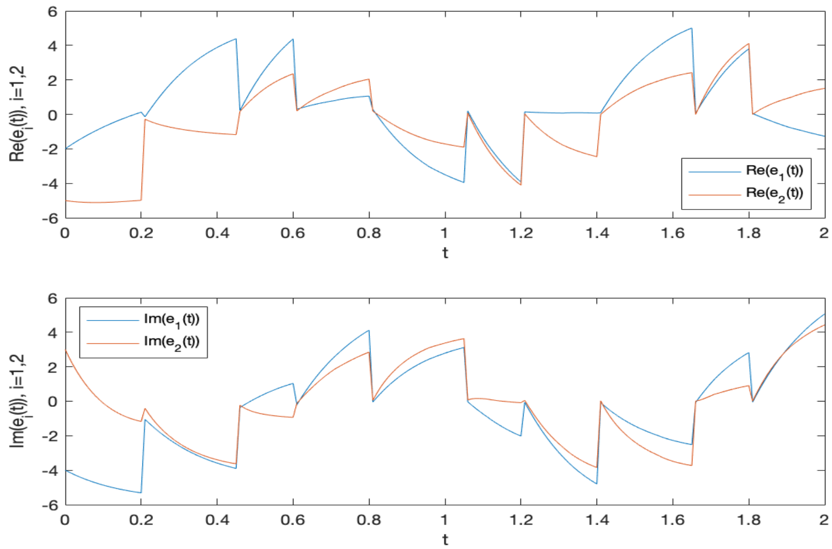

Figure 6.4: Trajectories of real and imaginary parts of synchronization errors of master-slave CVNNs (6.23) and (6.24) with sampled-data disturbance.

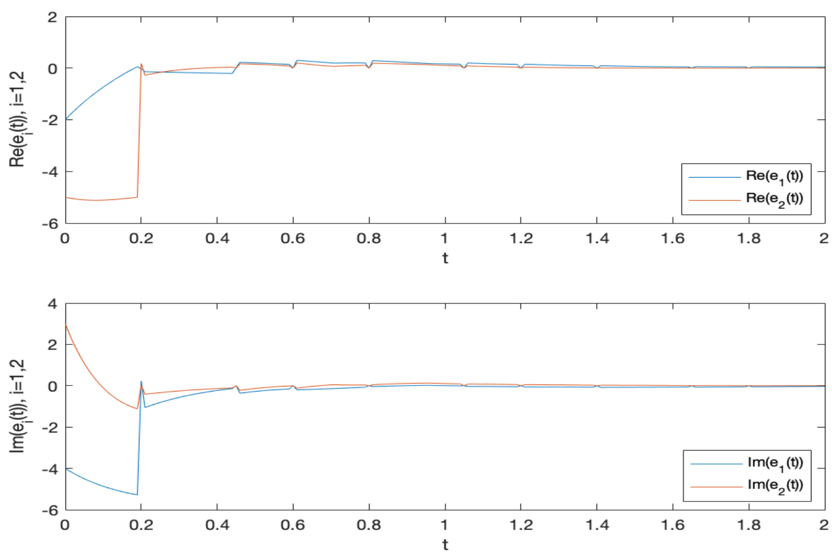

Figure 6.5: Trajectories of real and imaginary parts of synchronization errors of master-slave CVNNs (6.23) and (6.24) via impulsive control.

6.3 Master-Slave Synchronization of CVNNs via Delayed ETPIC

This section studies the synchronization problem of master-slave CVNNs (6.1) and (6.2) via Delayed ETPIC. The outline of this section is as follows. In Subsection 6.3.1, on the basis of the ETPIC scheme introduced in Subsection 3.6.1, we propose an ETPIC scheme with three levels of events in the complex field that takes the sampling delays at impulsive instants into account. The event-based impulsive controllers acting on the pinned neurons depend on the neuron states in recent history. In Subsection 6.3.2, a synchronization criterion for master-slave CVNNs is established. The result shows that the proposed delayed ETPIC scheme can effectively synchronize the slave CVNN with the master CVNN, and the delayed neuron states play a key role in the synchronization process. In Subsection 6.3.3, two numerical examples are presented to demonstrate the theoretical result.

6.3.1 Preliminaries

Consider master-slave time-delay CVNNs (6.1) and (6.2) with initial conditions given in Subsection 6.2.1. Suppose that the complex-valued activation functions in master-slave CVNNs (6.1) and (6.2) satisfy Assumption 6.2.1: For any $j = 1, 2, ..., n$, there exist positive constants L_j, F_j, such that

$$|f_j(u) - f_j(v)| \leq L_j|u - v|, \qquad |g_j(u) - g_j(v)| \leq F_j|u - v|$$

for all $u, v \in \mathbb{C}$. Denote $L = diag\{L_1, L_2, \ldots, L_n\}$, and $F = diag\{F_1, F_2, \ldots, F_n\}$.

According to the ETPIC scheme introduced in Subsection 3.6.1, we will give a novel ETPIC scheme to take the sampling delays at impulsive instants into account to synchronize the slave CVNN (6.2) with the master CVNN (6.1).

Define the synchronization error as $e_i(t) = \hat{z}_i(t) - z_i(t)$, $i = 1, 2, ..., N$, and denote $e(t) = (e_1(t), e_2(t), ..., e_n(t))^T$. Then master-slave CVNNs (6.1) and (6.2) can be transformed into the matrix-form error system (6.4):

$$\begin{cases} \dot{e}(t) = -Ce(t) + A\hat{f}(e(t)) + B\hat{g}(e(t-r)) + u(t), \\ e(t_0 + s) = \phi(s), \quad s \in [-r, 0]. \end{cases}$$

For error system (6.4), consider a Lyapunov functional V as

$$V(t) = V_1(t) + V_2(t) \tag{6.25}$$

with

$$V_1(t) = e^*(t)Pe(t),$$
$$V_2(t) = \omega \int_{t-r}^{t} e^*(s)Qe(s)ds,$$

where $0 < \omega \leq 1$, and P, Q are positive definite diagonal matrices to be determined. Denote $P = diag\{p_1, p_2, \ldots, p_n\}$, and $Q = diag\{q_1, q_2, \ldots, q_n\}$ with $p_i, q_i > 0, i = 1, 2, \ldots, n$.

For achieving the master-slave synchonization, we consider the following delayed ET-PIC scheme with three levels of events.

Delayed ETPIC Scheme: Taking three types of indices: the error threshold-value $\sigma_{\max} > 1$, the control-free index $0 < \sigma_{\min} < 1$, and the error check period Δ satisfying $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r, d\}$ for some $\mu > 0$, where d represents the impulse delay which will be shown later. Choosing the Lyapunov-like functional V as (6.25) with $0 < \omega \leq 1$, and positive definite diagonal matrices P and Q satisfying the following LMI: for constant $\mu > 0$, and positive definite Hermitian matrices R and S,

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & PA & PB \\ \star & F^T SF - \omega Q & 0 & 0 \\ \star & \star & -R & 0 \\ \star & \star & \star & -S \end{bmatrix} \leq 0, \tag{6.26}$$

where $\Omega_{11} = -C^T P - PC + \omega Q + L^T RL - \mu P$. By considering the sampling delays at impulsive instants when constructing the event-based pinning impulsive controller $u_i(t)$, the delayed ETPIC scheme $(u_i(t), \{t_k\})$ with three levels of events is designed as follows:

$$L_1 : \begin{cases} \text{if } \ \Gamma_{1k} := \{\exists t \in (t_{k-1}, t_{k-1} + \Delta] : V(t) \geq \sigma_{\max} V(t_{k-1}^+)\} \neq \emptyset, \\ \text{then, } t_k = \min\{t : t \in \Gamma_{1k}\}, \\ u_i(t) = \begin{cases} \left[\beta_1 e_i(t-d) - e_i(t)\right]\delta(t - t_k), & i \in \mathcal{D}_k^l, \\ 0, \quad i \notin \mathcal{D}_k^l, \quad t \in (t_{k-1}, t_k], \end{cases} \end{cases} \tag{6.27}$$

$$L_2:\begin{cases}\text{if } \ \Gamma_{2k}:=\{\forall t\in(t_{k-1},t_{k-1}+\Delta]:V(t)<\sigma_{\max}V(t_{k-1}^{+}),\\ \qquad\qquad V(t_{k-1}+\Delta)\geq\sigma_{\min}V(t_{k-1}^{+})\}\neq\emptyset,\\ \text{then, } t_k=t_{k-1}+\Delta,\\ u_i(t)=\begin{cases}\big[\beta_2 e_i(t-d)-e_i(t)\big]\delta(t-t_k), & i\in\mathcal{D}_k^l,\\ 0, \quad i\notin\mathcal{D}_k^l, & t\in(t_{k-1},t_k],\end{cases}\end{cases} \tag{6.28}$$

$$L_3:\begin{cases}\text{if } \ \Gamma_{3k}:=\{\forall t\in(t_{k-1},t_{k-1}+\Delta]:V(t)<\sigma_{\max}V(t_{k-1}^{+}),\\ \qquad\qquad V(t_{k-1}+\Delta)<\sigma_{\min}V(t_{k-1}^{+})\}\neq\emptyset,\\ \text{then, } t_k=t_{k-1}+\Delta,\\ u_i(t)=0, \qquad i=1,\dots,n, \quad t\in(t_{k-1},t_k],\end{cases} \tag{6.29}$$

for $k\in\mathbb{N}^+$, where Γ_{1k},Γ_{2k} and Γ_{3k} are the conditions for three levels of events; $\beta_m\in\mathbb{C}$ with $m\in\{1,2\}$ is the impulsive control gain of the event-based pinning impulsive controller $u_i(t)$; d denotes the time delay when impulse sampling takes place; $\delta(\cdot)$ represents the Dirac Delta function. The impulsive sequence $\{t_k\}$ is determined by the three levels of Lyapunov-based event conditions. If the event from level-1 (level-2) occurs at $t=t_k$, let l denote the number of neurons to be pinned at $t=t_k$, then the definition of $\mathcal{D}_k^l$ is similar to the definition of $\mathcal{D}_k^{l_m}$ in ETPIC scheme (3.114)-(3.116). The difference is that no matter the event from level-1 or level-2 occurs at $t=t_k$, the same number of neurons are controlled (i.e., $l_1=l_2=l$).

The objective is to design impulsive control gains β_1,β_2 to synchronize the slave CVNN (6.2) with the master CVNN (6.1).

Let Γ_m be the set of events occur from level-m:

$$\Gamma_m=\bigcup_{k\in\mathbb{N}_+}\Gamma_{mk}, \quad m=1,2,3.$$

According to delayed ETPIC scheme (6.27)-(6.29), the error system (6.4) can be rewritten as the following impulsive system:

$$\begin{cases}\dot e(t)=-Ce(t)+A\hat f(e(t))+B\hat g(e(t-r)), \quad t\neq t_k,\\ e_i(t^+)=\beta_m e_i(t-d), \quad t=t_k, \text{ if } \Gamma_m \text{ occurs at } t=t_k, \ m=1,2, \ i\in\mathcal{D}_k^l,\\ e_i(t^+)=e_i(t), \quad t=t_k, \text{ if } \Gamma_m \text{ occurs at } t=t_k, \ m=1,2, \ i\notin\mathcal{D}_k^l, \text{or } \Gamma_3 \text{ occurs at } t=t_k,\\ e(t_0+s)=\phi(s), \qquad s\in[-\tau,0],\end{cases} \tag{6.30}$$

where $\phi(s)\in\mathcal{PC}([-\tau,0],\mathbb{C}^n)$ is the initial function, and $\tau=\max\{r,d\}$.

Then the master-slave synchronization problem of CVNNs (6.1) and (6.2) is transformed into the stability problem of error system (6.30).

6.3.2 Synchronization Result

In this subsection, the non-Zeno result and the synchronization result for master-slave CVNNs (6.1) and (6.2) are obtained under the delayed ETPIC (6.27)-(6.29). For convenience, we define the following notations:

$$p_{\max} = \max_{1\le i\le n}\{p_i\}, \quad p_{\min} = \min_{1\le i\le n}\{p_i\}, \quad q_{\max} = \max_{1\le i\le n}\{q_i\}, \quad c_{\max} = \max_{1\le i\le n}\{c_i\}.$$

Theorem 6.3.1. *Suppose that Assumption 6.2.1 is satisfied. If there exist two constants $0 < \omega \le 1$, $\mu > 0$, two positive definite diagonal matrices P, Q, and two positive definite Hermitian matrices R, S such that LMI (6.26) holds, then delayed ETPIC* (6.27)-(6.29) *with $V(t)$ in the form of (6.25) is non-Zeno satisfying*

$$\frac{\ln \sigma_{\max}}{\mu} \le t_{k+1} - t_k \le \Delta, \qquad k \in \mathbb{N}. \tag{6.31}$$

Proof. Consider the Lyapunov functional candidate (6.25). It follows from (6.26) and Lemma 3.5.1 that

$$\begin{bmatrix} \Omega_{11} + PBS^{-1}B^*P & 0 & PA \\ \star & F^TSF - \omega Q & 0 \\ \star & \star & -R \end{bmatrix} \le 0.$$

Applying Lemma 3.5.1 again, we have

$$\begin{bmatrix} \Omega_{11} + PAR^{-1}A^*P + PBS^{-1}B^*P & 0 \\ \star & F^TSF - \omega Q \end{bmatrix} \le 0. \tag{6.32}$$

For $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, taking the derivative of V along the trajectory of the error system (6.30), then it follows from Lemma 3.2.1 and Assumption 6.2.1 that

$$\begin{aligned}
&\dot{V}(t) = \dot{e}^*(t)Pe(t) + e^*(t)P\dot{e}(t) + \omega e^*(t)Qe(t) - \omega e^*(t-r)Qe(t-r) \\
&= e^*(t)[-C^TP - PC + \omega Q]e(t) + \hat{f}^*(e(t))A^*Pe(t) + e^*(t)PA\hat{f}(e(t)) + \hat{g}^*(e(t-r))B^*Pe(t) \\
&+ e^*(t)PB\hat{g}(e(t-r)) - \omega e^*(t-r)Qe(t-r) \\
&\le e^*(t)[-C^TP - PC + \omega Q]e(t) + \hat{f}^*(e(t))R\hat{f}(e(t)) + e^*(t)PAR^{-1}A^*Pe(t) \\
&+ \hat{g}^*(e(t-r))S\hat{g}(e(t-r)) + e^*(t)PBS^{-1}B^*Pe(t) - \omega e^*(t-r)Qe(t-r)
\end{aligned}$$

$$\leq e^*(t)[-C^TP - PC + \omega Q + L^TRL + PAR^{-1}A^*P + PBS^{-1}B^*P]e(t)$$
$$+ e^*(t-r)[F^TSF - \omega Q]e(t-r).$$

Define $\eta(t) = (e^T(t), e^T(t-r))^T$, then for $\mu > 0$, it holds that

$$\dot{V}(t) - \mu V_1(t) \leq \eta^*(t) \begin{bmatrix} \Omega_{11} + PAR^{-1}A^*P + PBS^{-1}B^*P & 0 \\ \star & F^TSF - \omega Q \end{bmatrix} \eta(t),$$

where $\Omega_{11} = -C^TP - PC + \omega Q + L^TRL - \mu P$. Then it follows from (6.32) that $\dot{V}(t) \leq \mu V_1(t) \leq \mu V(t)$. Let $\Delta_k = t_{k+1} - t_k$, $k \in \mathbb{N}$, then we have

$$V(t) \leq V(t_k^+)e^{\mu(t-t_k)} \leq V(t_k^+)e^{\mu\Delta_k}, \qquad t \in (t_k, t_{k+1}],\ k \in \mathbb{N}. \tag{6.33}$$

For any interval $(t_k, t_{k+1}], k \in \mathbb{N}$, if the event from L_1 occurs at $t = t_{k+1}$, then by the event-triggered condition of delayed ETPIC (6.27), we have $V(t_{k+1}) = \sigma_{\max}V(t_k^+)$. It follows from (6.33) that $\Delta_k \geq \frac{\ln \sigma_{\max}}{\mu}$. If the event from L_2 or L_3 occurs at $t = t_{k+1}$, then from (6.28) and (6.29), we have $\Delta_k = \Delta$. By choosing $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r, d\}$, it holds that $\Delta_k > \frac{\ln \sigma_{\max}}{\mu}$. Thus, condition (6.31) holds, which implies that delayed ETPIC (6.27)-(6.29) with $V(t)$ in the form of (6.25) is non-Zeno. □

Remark 6.3.1. *Zeno behavior is a phenomenon in which an infinite number of events occur in a finite time interval. Since the triggered instants are implicitly defined based on the Lyapunov-based triggering condition in delayed ETPIC (6.27) if the first level of event occurs, it is necessary and important to guarantee the exclusion of Zeno behavior of error system (6.4) under the proposed delayed ETPIC scheme. By using the linear matrix inequality approach, Theorem 6.3.1 shows the sufficient condition for excluding the Zeno behavior, and the lower bound for inter-execution time is* $\frac{\ln \sigma_{\max}}{\mu}$*, which implies no event will occur within such inter-execution time.*

Theorem 6.3.2. *Suppose that Assumption 6.2.1 is satisfied. If there exist constants* $0 < \omega \leq 1$, $\mu > 0$*, positive definite diagonal matrices* P *and* Q*, and positive definite Hermitian matrices* R *and* S *such that LMI (6.26) holds, and impulsive control gains* β_1, β_2 *are designed to satisfy*

$$|\beta_1|^2\big[\gamma + \sqrt{\kappa_1}\big(d\lambda + (|\beta_1| + 1)\xi + |\beta_2| + 1\big)\big]^2 + (\frac{n}{l} - 1)\gamma^2 + \kappa_2\omega r \leq \frac{\sigma_{\min}}{\sigma_{\max}}, \tag{6.34}$$

$$|\beta_2|^2(\gamma + \sqrt{\kappa_1}d\lambda)^2 + (\frac{n}{l} - 1)\gamma^2 + \kappa_2\omega r \leq \frac{\sigma_{\min}}{\sigma_{\max}}, \tag{6.35}$$

where

$$\kappa_1 = \frac{p_{\max}}{p_{\min}}, \quad \kappa_2 = \frac{q_{\max}}{p_{\min}}, \quad \gamma = \sqrt{\frac{lp_{\max}}{lp_{\min} + (n-l)p_{\max}}},$$
$$\lambda = c_{\max} + \sqrt{nl}\ \big(\max_j\{L_j\}\max_{i,j}\{|a_{ij}|\} + \max_j\{F_j\}\max_{i,j}\{|b_{ij}|\}\big),$$
$$\xi = \lfloor \frac{d\mu}{\ln \sigma_{\max}} \rfloor,$$

then master-slave CVNNs (6.1) and (6.2) can achieve synchronization via delayed ETPIC (6.27)-(6.29) with $V(t)$ in the form of (6.25). Moreover, the convergence rate of synchronization is $\frac{-\ln \sigma_{\min}}{2(h+1)\Delta}$ with $h = \lfloor \frac{(\tau+d)\mu}{\ln \sigma_{\max}} \rfloor + 1$.

Proof. Consider the Lyapunov functional candidate (6.25). By choosing $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r, d\}$, it follows from the proof of Theorem 6.3.1 that (6.31) holds, which implies the number of events that occur on the interval $[t_0, t_0 + \tau + d]$ is at most $\lfloor \frac{\tau+d}{\frac{\ln \sigma_{\max}}{\mu}} \rfloor$. Then there exists a positive integer $h = \lfloor \frac{(\tau+d)\mu}{\ln \sigma_{\max}} \rfloor + 1 \geq 1$ such that $t_h - \tau - d \geq t_0$. For $t \in [t_0, t_{h+1}]$, according to delayed ETPIC (6.27)-(6.29) and (6.33), there exists an integer $\hat{k}$ $(0 \leq \hat{k} \leq h)$ such that

$$V(t) \leq \sup_{t \in [t_0, t_{h+1}]} V(t) \leq \sigma_{\max} V(t_{\hat{k}}^+) \leq \sigma_{\max} V(t_0) e^{\mu(t_{\hat{k}} - t_0)}.$$

Hence, we have

$$V(t) \leq \sigma_{\max} V(t_0) e^{\mu(t_h - t_0)}, \qquad t \in [t_0, t_{h+1}]. \tag{6.36}$$

For any interval $(t_k, t_{k+1}]$, $k \geq h$, we consider the following 3 cases:
Case 1. If the event from L_1 occurs at $t = t_{k+1}$ $(k \geq h)$, and $i \in \mathcal{D}^l_{k+1}$, integrating both sides of error system (6.3) from $t_{k+1} - d$ to t_{k+1}, we have from ETPIC (6.27)-(6.29) that

$$\begin{aligned} e_i(t_{k+1}) - e_i(t_{k+1} - d) &= \int_{t_{k+1}-d}^{t_{k+1}} \big[-c_i e_i(s) + \sum_{j=1}^n a_{ij} \hat{f}_j(e_j(s)) + \sum_{j=1}^n b_{ij} \hat{g}_j(e_j(s-r)) \big] ds \\ &+ \sum_{m \in M_1} \big[\beta_1 e_i(t_{k+1-m} - d) - e_i(t_{k+1-m})\big] + \sum_{m' \in M_2} \big[\beta_2 e_i(t_{k+1-m'} - d) - e_i(t_{k+1-m'})\big], \end{aligned} \tag{6.37}$$

where the sets $M_1 = \{m = 1, 2, \ldots, \xi_{k+1} \big| \ \Gamma_1$ occurs at t_{k+1-m} and $i \in \mathcal{D}^l_{k+1-m}\}$, and $M_2 = \{m' = 1, 2, \ldots, \xi_{k+1} \big| \ \Gamma_2$ occurs at $t_{k+1-m'}$ and $i \in \mathcal{D}^l_{k+1-m'}\}$, and ξ_{k+1} denotes the number of events that occur on the interval $(t_{k+1}-d, t_{k+1})$. From (6.31), we can obtain $\xi_{k+1} \leq \lfloor \frac{d\mu}{\ln \sigma_{\max}} \rfloor = \xi$. Let ν_{k+1} (ν'_{k+1}) denote the number of impulsive instants when Γ_1 (Γ_2) occurs and the i-th neuron is pinned on the interval $(t_{k+1}-d, t_{k+1})$. By choosing $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r, d\}$, it follows from delayed ETPIC (6.27)-(6.29) that if the event from L_1 occurs at $t = t_{k+1}$, then we have $\nu_{k+1} \leq \xi_{k+1} \leq \xi$ and $\nu'_{k+1} \leq 1$. For $i \in \mathcal{D}^l_{k+1}$, it follows from (6.30) and (6.37) that $e_i(t^+_{k+1}) = W_i + X_i + Y_i + Z_i$, where

$$\begin{aligned}
W_i &= \beta_1 e_i(t_{k+1}),\\
X_i &= -\beta_1 \int_{t_{k+1}-d}^{t_{k+1}} \big[- c_i e_i(s) + \sum_{j=1}^{n} a_{ij} \hat{f}_j(e_j(s)) + \sum_{j=1}^{n} b_{ij} \hat{g}_j(e_j(s-r)) \big] ds,\\
Y_i &= -\beta_1 \sum_{m \in M_1} \big[\beta_1 e_i(t_{k+1-m} - d) - e_i(t_{k+1-m}) \big],\\
Z_i &= -\beta_1 \sum_{m' \in M_2} \big[\beta_2 e_i(t_{k+1-m'} - d) - e_i(t_{k+1-m'}) \big].
\end{aligned}$$

According to (6.25), we can rewrite $V_1(t)$ as $V_1(t) = \sum_{i=1}^{n} p_i e_i^*(t) e_i(t)$, then we have for $k \geq h$,

$$V_1(t^+_{k+1}) = \sum_{i \in \mathcal{D}^l_{k+1}} p_i (W_i + X_i + Y_i + Z_i)^* (W_i + X_i + Y_i + Z_i) + \sum_{i \notin \mathcal{D}^l_{k+1}} p_i e_i^*(t_{k+1}) e_i(t_{k+1}).$$

Applying Lemma 3.6.1 three times, we have for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, $k \geq h$,

$$\begin{aligned}
V_1(t^+_{k+1}) \leq p_{\max} \big[&(1+\varepsilon_1) \sum_{i \in \mathcal{D}^l_{k+1}} W_i^* W_i + (1+\frac{1}{\varepsilon_1})(1+\varepsilon_2) \sum_{i \in \mathcal{D}^l_{k+1}} X_i^* X_i + (1+\frac{1}{\varepsilon_1})(1+\frac{1}{\varepsilon_2}) \cdot \\
&(1+\varepsilon_3) \sum_{i \in \mathcal{D}^l_{k+1}} Y_i^* Y_i + (1+\frac{1}{\varepsilon_1})(1+\frac{1}{\varepsilon_2})(1+\frac{1}{\varepsilon_3}) \sum_{i \in \mathcal{D}^l_{k+1}} Z_i^* Z_i \big] + \sum_{i \notin \mathcal{D}^l_{k+1}} p_i e_i^*(t_{k+1}) e_i(t_{k+1}).
\end{aligned} \tag{6.38}$$

Hence we have

$$\sum_{i \in \mathcal{D}^\ell_{k+1}} W_i^* W_i = |\beta_1|^2 \sum_{i \in \mathcal{D}^\ell_{k+1}} e_i^*(t_{k+1}) e_i(t_{k+1}) \leq \frac{|\beta_1|^2}{p_{\min}} \sum_{i \in \mathcal{D}^l_{k+1}} p_i e_i^*(t_{k+1}) e_i(t_{k+1}). \tag{6.39}$$

Applying Cauchy-Schwarz inequality for square-integrable complex-valued functions and

Lemma 3.2.3, and by Assumption 6.2.1, we have for any $\varepsilon_1^{'}, \varepsilon_2^{'} > 0$,

$$\sum_{i\in\mathcal{D}_{k+1}^l} X_i^* X_i = |\beta_1|^2 \sum_{i\in\mathcal{D}_{k+1}^l} \Big| \int_{t_{k+1}-d}^{t_{k+1}} -c_i e_i(s) + \sum_{j=1}^n a_{ij}\hat{f}_j(e_j(s)) + \sum_{j=1}^n b_{ij}\hat{g}_j(e_j(s-r))ds \Big|^2$$

$$\leq |\beta_1|^2 d \sum_{i\in\mathcal{D}_{k+1}^l} \int_{t_{k+1}-d}^{t_{k+1}} \big| -c_i e_i(s) + \sum_{j=1}^n a_{ij}\hat{f}_j(e_j(s)) + \sum_{j=1}^n b_{ij}\hat{g}_j(e_j(s-r))\big|^2 ds$$

$$\leq |\beta_1|^2 d \int_{t_{k+1}-d}^{t_{k+1}} \big[(1+\varepsilon_1^{'}) \sum_{i\in\mathcal{D}_{k+1}^l} c_i^2 |e_i(s)|^2 + (1+\frac{1}{\varepsilon_1^{'}})(1+\varepsilon_2^{'}) \sum_{i\in\mathcal{D}_{k+1}^l} |\sum_{j=1}^n a_{ij}\hat{f}_j(e_j(s))|^2$$

$$+ (1+\frac{1}{\varepsilon_1^{'}})(1+\frac{1}{\varepsilon_2^{'}}) \sum_{i\in\mathcal{D}_{k+1}^l} |\sum_{j=1}^n b_{ij}\hat{g}_j(e_j(s-r))|^2\big] ds$$

$$\leq |\beta_1|^2 d \int_{t_{k+1}-d}^{t_{k+1}} \big[(1+\varepsilon_1^{'}) c_{\max}^2 \sum_{i\in\mathcal{D}_{k+1}^l} |e_i(s)|^2 + (1+\frac{1}{\varepsilon_1^{'}})(1+\varepsilon_2^{'}) n \sum_{i\in\mathcal{D}_{k+1}^l} \sum_{j=1}^n |a_{ij}|^2 |\hat{f}_j(e_j(s))|^2$$

$$+ (1+\frac{1}{\varepsilon_1^{'}})(1+\frac{1}{\varepsilon_2^{'}}) n \sum_{i\in\mathcal{D}_{k+1}^\ell} \sum_{j=1}^n |b_{ij}|^2 |\hat{g}_j(e_j(s-r))|^2\big] ds$$

$$\leq \frac{|\beta_1|^2 d}{p_{\min}} \int_{t_{k+1}-d}^{t_{k+1}} \big[(1+\varepsilon_1^{'}) c_{\max}^2 \sum_{i=1}^n p_i e_i^*(s) e_i(s) + (1+\frac{1}{\varepsilon_1^{'}})(1+\varepsilon_2^{'}) nl \max_j\{L_j\}^2 \max_{i,j}\{|a_{ij}|\}^2 \cdot$$

$$\sum_{i=1}^n p_i e_i^*(s) e_i(s) + (1+\frac{1}{\varepsilon_1^{'}})(1+\frac{1}{\varepsilon_2^{'}}) nl \max_j\{F_j\}^2 \max_{i,j}\{|b_{ij}|\}^2 \sum_{i=1}^n p_i e_i^*(s-r) e_i(s-r)\big] ds$$

$$\leq \frac{|\beta_1|^2 d}{p_{\min}} \Big[\big((1+\varepsilon_1^{'}) c_{\max}^2 + (1+\frac{1}{\varepsilon_1^{'}})(1+\varepsilon_2^{'}) nl \max_j\{L_j\}^2 \max_{i,j}\{|a_{ij}|\}^2\big) \int_{t_{k+1}-d}^{t_{k+1}} V(s) ds$$

$$+ (1+\frac{1}{\varepsilon_1^{'}})(1+\frac{1}{\varepsilon_2^{'}}) nl \max_j\{F_j\}^2 \max_{i,j}\{|b_{ij}|\}^2 \int_{t_{k+1}-d}^{t_{k+1}} V(s-r) ds\Big]$$

$$\leq \frac{|\beta_1|^2 d^2}{p_{\min}} \big[(1+\varepsilon_1^{'}) c_{\max}^2 + (1+\frac{1}{\varepsilon_1^{'}})(1+\varepsilon_2^{'}) nl \max_j\{L_j\}^2 \max_{i,j}\{|a_{ij}|\}^2 + (1+\frac{1}{\varepsilon_1^{'}})(1+\frac{1}{\varepsilon_2^{'}}) nl \cdot$$

$$\max_j\{F_j\}^2 \max_{i,j}\{|b_{ij}|\}^2\big] \sup_{s\in[-d-r,0]} V(t_{k+1}+s).$$

Applying Lemma 3.2.4, we can get

$$\sum_{i\in\mathcal{D}_{k+1}^l} X_i^* X_i \leq \frac{|\beta_1|^2 d^2\lambda^2}{p_{\min}} \sup_{s\in[-d-r,0]} V(t_{k+1}+s)$$

with $(\varepsilon_1^{'},\varepsilon_2^{'}) = \big(\frac{\sqrt{nl}\ (\max_j\{L_j\}\max_{i,j}\{|a_{ij}|\}+\max_j\{F_j\}\max_{i,j}\{|b_{ij}|\})}{c_{\max}}, \frac{\max_j\{F_j\}\max_{i,j}\{|b_{ij}|\}}{\max_j\{L_j\}\max_{i,j}\{|a_{ij}|\}}\big)$.

Furthermore, it holds that $\sup_{s\in[-d-r,0]} V(t_{k+1}+s) \leq \sup_{s\in[-\tau-d,0]} V(t_{k+1}+s)$, and for case 1, the number of events that occur on the interval $[t_{k+1}-\tau-d, t_{k+1})$ is at most $\lfloor\frac{(\tau+d)\mu}{\ln\sigma_{\max}}\rfloor+1 = h$. Then it follows from event-triggered conditions of delayed ETPIC (6.27)-(6.29) that

$$\sup_{s\in[-\tau-d,0]} V(t_{k+1}+s) \leq \sigma_{\max}\cdot \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^{+}), \tag{6.40}$$

where $\mathbb{N}_{-(h-1)} = \{-(h-1), -(h-2), \ldots, -1, 0\}$. Then, we have

$$\sum_{i\in\mathcal{D}_{k+1}^l} X_i^* X_i \leq \frac{|\beta_1|^2 d^2\lambda^2}{p_{\min}}\sigma_{\max}\cdot \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^{+}). \tag{6.41}$$

By Cauchy-Schwarz inequality and Lemma 3.6.1, we have for any $\eta>0$,

$$\begin{aligned}
&\sum_{i\in\mathcal{D}_{k+1}^l} Y_i^* Y_i \leq |\beta_1|^2\nu_{k+1}\sum_{i\in\mathcal{D}_{k+1}^l}\sum_{m\in M_1}\big|\beta_1 e_i(t_{k+1-m}-d)-e_i(t_{k+1-m})\big|^2\\
&\leq |\beta_1|^2\nu_{k+1}\sum_{i\in\mathcal{D}_{k+1}^l}\sum_{m\in M_1}\big[(1+\eta)|\beta_1|^2|e_i(t_{k+1-m}-d)|^2+(1+\frac{1}{\eta})|e_i(t_{k+1-m})|^2\big]\\
&\leq \frac{|\beta_1|^2\nu_{k+1}}{p_{\min}}\big[(1+\eta)|\beta_1|^2\sum_{m\in M_1}\sum_{i=1}^{n} p_i e_i^*(t_{k+1-m}-d)e_i(t_{k+1-m}-d)\\
&+(1+\frac{1}{\eta})\sum_{m\in M_1}\sum_{i=1}^{n} p_i e_i^*(t_{k+1-m})e_i(t_{k+1-m})\big]\\
&= \frac{|\beta_1|^2\nu_{k+1}}{p_{\min}}\big[(1+\eta)|\beta_1|^2\sum_{m\in M_1} V_1(t_{k+1-m}-d)+(1+\frac{1}{\eta})\sum_{m\in M_1} V_1(t_{k+1-m})\big]\\
&\leq \frac{|\beta_1|^2\nu_{k+1}}{p_{\min}}\big[(1+\eta)|\beta_1|^2\nu_{k+1}\sup_{s\in[-2d,0]} V_1(t_{k+1}+s)+(1+\frac{1}{\eta})\nu_{k+1}\sup_{s\in[-d,0]} V_1(t_{k+1}+s)\big]
\end{aligned}$$

$$\leq \frac{|\beta_1|^2\xi^2}{p_{\min}}\big[(1+\eta)|\beta_1|^2 + (1+\frac{1}{\eta})\big] \sup_{s\in[-\tau-d,0]} V(t_{k+1}+s).$$

Then it follows from (6.40) that

$$\sum_{i\in\mathcal{D}_{k+1}^l} Y_i^* Y_i \leq \frac{|\beta_1|^2\xi^2(|\beta_1|+1)^2}{p_{\min}}\sigma_{\max}\cdot \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+) \tag{6.42}$$

with $\eta = \frac{1}{|\beta_1|}$. Similarly, we can obtain

$$\sum_{i\in\mathcal{D}_{k+1}^l} Z_i^* Z_i \leq \frac{|\beta_1|^2(|\beta_2|+1)^2}{p_{\min}}\sigma_{\max}\cdot \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+). \tag{6.43}$$

Combing (6.38), (6.39), (6.41)-(6.43), we can conclude that for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, $k \geq h$,

$$V_1(t_{k+1}^+) \leq \alpha_1' \sum_{i\in\mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1})e_i(t_{k+1}) + \rho_1\sigma_{\max}\cdot \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+) + \sum_{i\notin\mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1})e_i(t_{k+1}), \tag{6.44}$$

where $\alpha_1' = \frac{(1+\varepsilon_1)p_{\max}|\beta_1|^2}{p_{\min}}$, and $\rho_1 = \frac{p_{\max}}{p_{\min}}\big[(1+\frac{1}{\varepsilon_1})(1+\varepsilon_2)|\beta_1|^2 d^2\lambda^2 + (1+\frac{1}{\varepsilon_1})(1+\frac{1}{\varepsilon_2})(1+\varepsilon_3)|\beta_1|^2\xi^2(|\beta_1|+1)^2 + (1+\frac{1}{\varepsilon_1})(1+\frac{1}{\varepsilon_2})(1+\frac{1}{\varepsilon_3})|\beta_1|^2(|\beta_2|+1)^2\big]$. For any $\varepsilon_1 > 0$, letting $\alpha_1 = \frac{lp_{\min}\alpha_1' + (n-l)p_{\max}}{lp_{\min}+(n-l)p_{\max}}$, then according to the pinning algorithm, we have

$$\begin{aligned}
&(1-\alpha_1)\sum_{i\notin\mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1})e_i(t_{k+1}) \leq (1-\alpha_1)p_{\max}(n-l)\max_{i\notin\mathcal{D}_{k+1}^l} |e_i(t_{k+1})|^2 \\
&\leq (1-\alpha_1)p_{\max}(n-l)\min_{i\in\mathcal{D}_{k+1}^l} |e_i(t_{k+1})|^2 \leq (1-\alpha_1)p_{\max}\frac{n-l}{l}\sum_{i\in\mathcal{D}_{k+1}^l} |e_i(t_{k+1})|^2 \\
&\leq \frac{(n-l)p_{\max} - \alpha_1'(n-l)p_{\max}}{lp_{\min}+(n-l)p_{\max}} \sum_{i\in\mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1})e_i(t_{k+1}) \\
&= (\alpha_1 - \alpha_1')\sum_{i\in\mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1})e_i(t_{k+1}).
\end{aligned}$$

Hence, we can obtain

$$\alpha_1' \sum_{i\in\mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1})e_i(t_{k+1}) + \sum_{i\notin\mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1})e_i(t_{k+1}) \le \alpha_1 V_1(t_{k+1}) \le \alpha_1 V(t_{k+1}). \quad (6.45)$$

It follows from (6.44), (6.45) and the event-triggered condition of delayed ETPIC (6.27) that for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, $k \ge h$,

$$\begin{aligned} V_1(t_{k+1}^+) &\le \alpha_1 V(t_{k+1}) + \rho_1\sigma_{\max} \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+) \\ &= \alpha_1\sigma_{\max} V(t_k^+) + \rho_1\sigma_{\max} \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+) \\ &\le (\alpha_1+\rho_1)\sigma_{\max} \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+), \end{aligned} \quad (6.46)$$

where $\alpha_1+\rho_1 = (1+\varepsilon_1)\frac{lp_{\max}|\beta_1|^2}{lp_{\min}+(n-l)p_{\max}} + (1+\frac{1}{\varepsilon_1})(1+\varepsilon_2)\frac{|\beta_1|^2 d^2\lambda^2 p_{\max}}{p_{\min}} + (1+\frac{1}{\varepsilon_1})(1+\frac{1}{\varepsilon_2})(1+\varepsilon_3)\frac{|\beta_1|^2\xi^2(|\beta_1|+1)^2 p_{\max}}{p_{\min}} + (1+\frac{1}{\varepsilon_1})(1+\frac{1}{\varepsilon_2})(1+\frac{1}{\varepsilon_3})\frac{|\beta_1|^2(|\beta_2|+1)^2 p_{\max}}{p_{\min}} + \frac{(n-l)p_{\max}}{lp_{\min}+(n-l)p_{\max}}$.

Let $\kappa_1 = \frac{p_{\max}}{p_{\min}}$, $\gamma = \sqrt{\frac{lp_{\max}}{lp_{\min}+(n-l)p_{\max}}}$. It follows from (6.46) and Lemma 3.6.3 that for $k \ge h$,

$$\begin{aligned} V_1(t_{k+1}^+) \le &\left\{\left[|\beta_1|\gamma + \sqrt{\kappa_1}|\beta_1|\big(d\lambda + (|\beta_1|+1)\xi + |\beta_2|+1\big)\right]^2 + (\frac{n}{l}-1)\gamma^2\right\} \\ &\cdot \sigma_{\max} \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+). \end{aligned} \quad (6.47)$$

By the continuity of $V_2(t)$, we have for $k \ge h$,

$$\begin{aligned} V_2(t_{k+1}^+) &= \omega \int_{t_{k+1}-r}^{t_{k+1}} e^*(s)Qe(s)ds \\ &\le \omega\kappa_2 \int_{t_{k+1}-r}^{t_{k+1}} V_1(s)ds \le \kappa_2\omega r \sup_{s\in[-r,0]} V(t_{k+1}+s). \end{aligned} \quad (6.48)$$

According to (6.31), the number of events that occur on the interval $[t_{k+1}-r, t_{k+1})$ is at most $\lfloor \frac{r\mu}{\ln\sigma_{\max}} \rfloor + 1 = \tilde{h} \le h$. Then, based on event-triggered conditions of delayed ETPIC scheme (6.27)-(6.29), we have

$$\sup_{s\in[-r,0]} V(t_{k+1}+s) \le \sigma_{\max} \max_{\theta\in\mathbb{N}_{-(\tilde{h}-1)}} V(t_{k+\theta}^+) \le \sigma_{\max} \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+).$$

Then from (6.48), we have for $k \geq h$,

$$V_2(t_{k+1}^+) \leq \kappa_2 \omega r \sigma_{\max} \max_{\theta \in \mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+). \tag{6.49}$$

Combining (6.47), (6.49) and condition (6.34), we have for $k \geq h$,

$$\begin{aligned} V(t_{k+1}^+) \leq \Big\{ \big[|\beta_1|\gamma + \sqrt{\kappa_1}|\beta_1| \big(d\lambda + (|\beta_1|+1)\xi + |\beta_2| + 1 \big) \big]^2 + (\frac{n}{l} - 1)\gamma^2 + \kappa_2 \omega r \Big\} \cdot \\ \sigma_{\max} \max_{\theta \in \mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+) \\ \leq \sigma_{\min} \max_{\theta \in \mathbb{N}_{-(h-1)}} V(t_{k+\theta}^+). \end{aligned}$$

Case 2. If the event from L_2 occurs at $t = t_{k+1}$ $(k \geq h)$, by choosing $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r, d\}$, it follows from delayed ETPIC (6.28) that $t_k = t_{k+1} - \Delta < t_{k+1} - d$, and thus there's no event that occur on the interval $(t_{k+1} - d, t_{k+1})$. Integrating both sides of error system (6.3) from $t_{k+1} - d$ to t_{k+1}, gives,

$$e_i(t_{k+1}) - e_i(t_{k+1} - d) = \int_{t_{k+1}-d}^{t_{k+1}} \Big[-c_i e_i(s) + \sum_{j=1}^{n} a_{ij} \hat{f}_j(e_j(s)) + \sum_{j=1}^{n} b_{ij} \hat{g}_j(e_j(s-r)) \Big] ds. \tag{6.50}$$

For $i \in \mathcal{D}_{k+1}^l$, it follows from (6.30) and (6.50) that $e_i(t_{k+1}^+) = \widetilde{W}_i + \widetilde{X}_i$, where

$$\widetilde{W}_i = \beta_2 e_i(t_{k+1}), \quad \widetilde{X}_i = -\beta_2 \int_{t_{k+1}-d}^{t_{k+1}} \Big[-c_i e_i(s) + \sum_{j=1}^{n} a_{ij} \hat{f}_j(e_j(s)) + \sum_{j=1}^{n} b_{ij} \hat{g}_j(e_j(s-r)) \Big] ds.$$

Similar to the proof of case 1, it follows from Lemma 3.6.1 that for any $\tilde{\varepsilon}_1 > 0$, $k \geq h$,

$$\begin{aligned} V_1(t_{k+1}^+) &\leq p_{\max} \Big[(1 + \tilde{\varepsilon}_1) \sum_{i \in \mathcal{D}_{k+1}^l} \widetilde{W}_i^* \widetilde{W}_i + (1 + \frac{1}{\tilde{\varepsilon}_1}) \sum_{i \in \mathcal{D}_{k+1}^l} \widetilde{X}_i^* \widetilde{X}_i \Big] + \sum_{i \notin \mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1}) e_i(t_{k+1}) \\ &\leq \alpha_2^{'} \sum_{i \in \mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1}) e_i(t_{k+1}) + \rho_2 \sup_{s \in [-d-r, 0]} V(t_{k+1} + s) + \sum_{i \notin \mathcal{D}_{k+1}^l} p_i e_i^*(t_{k+1}) e_i(t_{k+1}), \end{aligned}$$

where $\alpha_2^{'} = \frac{(1+\tilde{\varepsilon}_1) p_{\max} |\beta_2|^2}{p_{\min}}$, and $\rho_2 = (1 + \frac{1}{\tilde{\varepsilon}_1}) \frac{p_{\max}}{p_{\min}} |\beta_2|^2 d^2 \lambda^2$. Let $\alpha_2 = \frac{l p_{\min} \alpha_2^{'} + (n-l) p_{\max}}{l p_{\min} + (n-l) p_{\max}}$, using the similar method in the proof of case 1, we have

$$V_1(t_{k+1}^+) \leq \alpha_2 V(t_{k+1}) + \rho_2 \sup_{s \in [-\tau-d, 0]} V(t_{k+1} + s).$$

By the event condition of delayed ETPIC (6.28) and (6.40), we have for any $\tilde{\varepsilon}_1 > 0$, $k \geq h$,

$$V_1(t_{k+1}^+) < \alpha_2\sigma_{\max}V(t_k^+) + \rho_2\sigma_{\max}\max_{\theta\in\mathbb{N}_{-(h-1)}}V(t_{k+\theta}^+) \leq (\alpha_2+\rho_2)\sigma_{\max}\max_{\theta\in\mathbb{N}_{-(h-1)}}V(t_{k+\theta}^+),$$

where $\alpha_2 + \rho_2 = (1+\tilde{\varepsilon}_1)|\beta_2|^2\gamma^2 + (1+\frac{1}{\tilde{\varepsilon}_1})\kappa_1|\beta_2|^2d^2\lambda^2 + (\frac{n}{l}-1)\gamma^2$. Hence we can obtain for $k \geq h$,

$$V_1(t_{k+1}^+) < \big[(|\beta_2|\gamma + \sqrt{\kappa_1}|\beta_2|d\lambda)^2 + (\frac{n}{l}-1)\gamma^2\big]\sigma_{\max}\max_{\theta\in\mathbb{N}_{-(h-1)}}V(t_{k+\theta}^+)$$

with $\tilde{\varepsilon}_1 = \frac{\sqrt{\kappa_1}d\lambda}{\gamma}$.

Similar to case 1, we have $V_2(t_{k+1}^+) \leq \kappa_2\omega r\sup_{s\in[-r,0]}V(t_{k+1}+s)$. According to ETPIC (6.28) and the choice of Δ, we can get $t_k = t_{k+1} - \Delta < t_{k+1} - r$, which implies there's no event that occurs on the interval $[t_{k+1}-r, t_{k+1})$. Then it follows from the event condition of delayed ETPIC (6.28) that

$$\sup_{s\in[-r,0]}V(t_{k+1}+s) < \sigma_{\max}V(t_k^+) \leq \sigma_{\max}\max_{\theta\in\mathbb{N}_{-(h-1)}}V(t_{k+\theta}^+),$$

hence we have for $k \geq h$,

$$V_2(t_{k+1}^+) < \kappa_2\omega r\sigma_{\max}\max_{\theta\in\mathbb{N}_{-(h-1)}}V(t_{k+\theta}^+).$$

According to condition (6.35), we can conclude that for $k \geq h$,

$$\begin{aligned}V(t_{k+1}^+) &< \big[(|\beta_2|\gamma + \sqrt{\kappa_1}|\beta_2|d\lambda)^2 + (\frac{n}{l}-1)\gamma^2 + \kappa_2\omega r\big]\sigma_{\max}\max_{\theta\in\mathbb{N}_{-(h-1)}}V(t_{k+\theta}^+)\\ &\leq \sigma_{\min}\max_{\theta\in\mathbb{N}_{-(h-1)}}V(t_{k+\theta}^+).\end{aligned}$$

Case 3. If the event from L_3 occurs at $t = t_{k+1}$ ($k \geq h$), then it follows from ETPIC (6.29) that the error system is control free. Then by the event condition of (6.29), we have for $k \geq h$,

$$V(t_{k+1}^+) = V(t_{k+1}) < \sigma_{\min}V(t_k^+) \leq \sigma_{\min}\max_{\theta\in\mathbb{N}_{-(h-1)}}V(t_{k+\theta}^+).$$

Let $z(k) = V(t_k^+)$ for $k \in \mathbb{N}$, combining all 3 cases together, we can obtain

$$z(k+1) \leq \sigma_{\min}\bar{z}(k), \qquad k \geq h,$$

where $\bar{z}(k) = \max_{\theta\in\mathbb{N}_{-(h-1)}}\{z(k+\theta)\}$. By $\sigma_{\min} < 1$ and the proof of Theorem 3.3 and Theorem 4.2 in [95], we can get

$$z(k) \leq e^{-\alpha(k-h)} \max_{\theta\in\mathbb{N}_{-(h-1)}} z(h+\theta), \qquad k \geq h \tag{6.51}$$

with $\alpha = \frac{\ln(\frac{1}{\sigma_{\min}})^{\frac{h}{h+1}}}{h} = \frac{-\ln\sigma_{\min}}{h+1}$. For $\forall t > t_h$, there exists integer $\hat{k} \geq h$ such that $t \in (t_{\hat{k}}, t_{\hat{k}+1}]$, and $t - t_h \leq (\hat{k}+1-h)\Delta$, which implies $\hat{k} - h \geq \frac{t-t_h}{\Delta} - 1$. By $e^{-\alpha} < 1$ and (6.51), we have

$$V(t_{\hat{k}}^+) \leq e^{-\alpha(\hat{k}-h)} \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{h+\theta}^+) \leq e^{-(\frac{-\ln\sigma_{\min}}{h+1})(\frac{t-t_h}{\Delta}-1)} \max_{\theta\in\mathbb{N}_{-(h-1)}} V(t_{h+\theta}^+). \tag{6.52}$$

Then for $\forall t > t_h$, there exists $\hat{k} \geq h$ such that $t \in (t_{\hat{k}}, t_{\hat{k}+1}]$, and it follows from delayed ETPIC (6.27)-(6.29), (6.52) and (6.36) that

$$\begin{aligned} V_1(t) &\leq V(t) \leq \sigma_{\max} V(t_{\hat{k}}^+) \leq \sigma_{\max} e^{-(\frac{-\ln\sigma_{\min}}{h+1})(\frac{t-t_h}{\Delta}-1)} \sigma_{\max} V(t_0) e^{\mu(t_h-t_0)} \\ &= \frac{\sigma_{\max}^2}{\sigma_{\min}^{\frac{1}{h+1}}} e^{\left(\mu - \frac{\ln\sigma_{\min}}{(h+1)\Delta}\right)(t_h-t_0)} V(t_0) e^{\frac{\ln\sigma_{\min}}{(h+1)\Delta}(t-t_0)} \\ &\leq \frac{\sigma_{\max}^2}{\sigma_{\min}^{\frac{1}{h+1}}} e^{(\mu h\Delta - \frac{h\ln\sigma_{\min}}{h+1})} V(t_0) e^{\frac{\ln\sigma_{\min}}{(h+1)\Delta}(t-t_0)} = \frac{\sigma_{\max}^2 e^{\mu h\Delta}}{\sigma_{\min}} V(t_0) e^{\frac{\ln\sigma_{\min}}{(h+1)\Delta}(t-t_0)}, \qquad t > t_h. \end{aligned} \tag{6.53}$$

For $\forall t \in [t_0, t_h]$, from (6.36), we have

$$V_1(t) \leq V(t) \leq \sigma_{\max} V(t_0) e^{\mu(t_h-t_0)} \leq \sigma_{\max} V(t_0) e^{\mu h\Delta} \leq \frac{\sigma_{\max}^2 e^{\mu h\Delta}}{\sigma_{\min}} V(t_0) e^{\frac{\ln\sigma_{\min}}{(h+1)\Delta}(t-t_0)}. \tag{6.54}$$

Combining (6.53) with (6.54), we can conclude that

$$V_1(t) \leq V(t) \leq \frac{\sigma_{\max}^2 e^{\mu h\Delta}}{\sigma_{\min}} V(t_0) e^{\frac{\ln\sigma_{\min}}{(h+1)\Delta}(t-t_0)}, \qquad \forall t \geq t_0,$$

which implies that

$$\|e(t)\| \leq M\|e(t_0)\|_\tau e^{\frac{\ln\sigma_{\min}}{2(h+1)\Delta}(t-t_0)}, \quad \forall t \geq t_0,$$

where $M = \sigma_{\max} e^{\frac{\mu h\Delta}{2}} \sqrt{\frac{p_{\max}+q_{\max}r}{p_{\min}\sigma_{\min}}}$. This shows that the zero solution of error system (6.30) is exponentially stable, thus master-slave synchronization of CVNNs (6.1) and (6.2) is achieved with the convergence rate $\frac{-\ln \sigma_{\min}}{2(h+1)\Delta}$, where $h = \lfloor \frac{(\tau+d)\mu}{\ln \sigma_{\max}} \rfloor + 1$. □

Remark 6.3.2. *It can be seen from Theorem 6.3.2 that the size of delays may affect the convergence rate of synchronization, a larger size of time-delay may result in slower convergence speed of synchronization.*

Remark 6.3.3. *In most of the existing work, the usual method for studying the dynamics and control of CVNNs is to separate them into two real-valued neural networks (RVNNs), and apply the common analytical technique and control theory for both RVNNs to analyze the dynamical behaviors of CVNNs. However, an explicit separation of complex-valued activation functions of CVNNs into their real part and imaginary part is needed when applying this kind of method, which is not always expressible in an analytical form. In this section, we retain the complex nature of neural networks and investigate the synchronization criteria on* $\mathbb{C}^n$ *by constructing the Lyapunov functional in complex field.*

Remark 6.3.4. *It should be noted that the above discussions focus on the case that the state of each neuron in the master and slave CVNNs (6.1)-(6.2) is in* 1*-dimensional complex space (i.e.,* $z_i, \hat{z}_i \in \mathbb{C}$ *for* $i = 1, 2, \ldots, n$*). If* $z_i, \hat{z}_i \in \mathbb{C}^m$*, and the complex-valued activation functions* $f_j(\cdot) : \mathbb{C}^m \to \mathbb{C}^m$ *and* $g_j(\cdot) : \mathbb{C}^m \to \mathbb{C}^m$ *satisfy Assumption 6.2.1 with* $u, v \in \mathbb{C}^m$*, by choosing the Lyapunov functional* V *as* $V(t) = e^*(t)(I_n \otimes P)e(t) + \omega \int_{t-r}^{t} e^*(s)(I_n \otimes Q)e(s)ds$*, where* $0 < \omega \leq 1$*, and* $P, Q \in \mathbb{C}^{m\times m}$ *are positive definite Hermitian matrices satisfying the LMI: for constant* $\mu > 0$*, and positive definite Hermitian matrices* $R, S \in \mathbb{C}^{m\times m}$*,*

$$\begin{bmatrix} \tilde{\Omega}_{11} & 0 & A\otimes P & B\otimes P \\ \star & (F^TF)\otimes S - \omega(I_n\otimes Q) & 0 & 0 \\ \star & \star & -(I_n\otimes R) & 0 \\ \star & \star & \star & -(I_n\otimes S) \end{bmatrix} \leq 0,$$

where $\tilde{\Omega}_{11} = -(C^T + C)\otimes P + \omega(I_n\otimes Q) + (L^TL)\otimes R - \mu(I_n \otimes P)$*, and denote* $p_{\max} = \lambda_{\max}(P), p_{\min} = \lambda_{\min}(P), q_{\max} = \lambda_{\max}(Q)$*, then the synchronization result of Theorem 6.3.2 is still applicable under ETPIC (6.27)-(6.29) if impulsive control gains* β_1, β_2 *are designed to satisfy (6.34) and (6.35).*

Remark 6.3.5. *In particular, if there exists positive constant* $\mu < \frac{\ln \sigma_{\max}}{\tau}$ *such that LMI (6.26) has feasible solution, then it follows from Theorem 6.3.1 that* $d < t_{k+1} - t_k$ *for all* $k \in \mathbb{N}$*. Then, (6.37) is reduced to (6.50) when integrating both sides of (6.3) from* $t_{k+1} - d$ *to* t_{k+1}*. We can obtain the following results using the similar method in the proof of Theorem 6.3.2 with* $Y_i = 0$ *and* $Z_i = 0$*.*

Corollary 6.3.1. *Suppose that Assumption 6.2.1 is satisfied. If there exist constants* $0 < \omega \leq 1$, $0 < \mu < \frac{\ln \sigma_{\max}}{\tau}$, *positive definite diagonal matrices* P *and* Q, *and positive definite Hermitian matrices* R *and* S *such that LMI (6.26) holds, and impulsive control gains* β_1, β_2 *are designed to satisfy*

$$|\beta_i| \leq \frac{\sqrt{\frac{\sigma_{\min}}{\sigma_{\max}} - \kappa_2 \omega r - (\frac{n}{l} - 1)\gamma^2}}{\gamma + \sqrt{\kappa_1} d\lambda}, \qquad i = 1, 2, \tag{6.55}$$

where $\kappa_1 = \frac{p_{\max}}{p_{\min}}, \kappa_2 = \frac{q_{\max}}{p_{\min}}, \gamma = \sqrt{\frac{lp_{\max}}{lp_{\min}+(n-l)p_{\max}}}$, *and* $\lambda = c_{\max} + \sqrt{nl}\,(\max\{L_j\}\max\{|a_{ij}|\} + \max\{F_j\}\max\{|b_{ij}|\})$, *then master-slave CVNNs (6.1) and (6.2) can achieve synchronization via delayed ETPIC* (6.27)-(6.29) *with* $V(t)$ *in the form of (6.25), and the convergence rate of synchronization is* $\frac{-\ln \sigma_{\min}}{6\Delta}$. *Moreover, the error system (6.4) does not exhibit Zeno behavior.*

Remark 6.3.6. *If all* n *neurons in the slave CVNN* (6.2) *are controlled at each impulsive instant, then delayed ETPIC scheme (6.27)-(6.29) is changed to delayed event-triggered impulsive control (ETIC) scheme. According to Theorem 6.3.2, the following results can be obtained via delayed ETPIC (6.27)-(6.29) with* $l = n$.

Corollary 6.3.2. *Suppose that Assumption 6.2.1 is satisfied. If there exist constants* $0 < \omega \leq 1$, $\mu > 0$, *positive definite diagonal matrices* P *and* Q, *and positive definite Hermitian matrices* R *and* S *such that LMI (6.26) holds, and impulsive control gains* β_1, β_2 *are designed to satisfy*

$$\kappa_1 |\beta_1|^2 \big[(|\beta_1| + 1)\xi + |\beta_2| + d\lambda + 2\big]^2 + \kappa_2 \omega r \leq \frac{\sigma_{\min}}{\sigma_{\max}}, \tag{6.56}$$

$$\kappa_1 |\beta_2|^2 (1 + d\lambda)^2 + \kappa_2 \omega r \leq \frac{\sigma_{\min}}{\sigma_{\max}}, \tag{6.57}$$

where $\kappa_1 = \frac{p_{\max}}{p_{\min}}, \kappa_2 = \frac{q_{\max}}{p_{\min}}, \lambda = c_{\max} + n(\max_j\{L_j\}\max_{i,j}\{|a_{ij}|\} + \max_j\{F_j\}\max_{i,j}\{|b_{ij}|\})$, *and* $\xi = \lfloor \frac{d\mu}{\ln \sigma_{\max}} \rfloor$, *then master-slave CVNNs (6.1) and (6.2) can achieve synchronization via delayed ETPIC* (6.27)-(6.29) *with* $V(t)$ *in the form of (6.25) and* $l = n$. *Moreover, the convergence rate of synchronization is* $\frac{-\ln \sigma_{\min}}{2(h+1)\Delta}$ *with* $h = \lfloor \frac{(\tau + d)\mu}{\ln \sigma_{\max}} \rfloor + 1$, *and the error system (6.4) does not exhibit Zeno behavior.*

Proof. The result can be directly obtained from Theorem 6.3.2 with $l = n$. □

6.3.3 Numerical Examples

In this subsection, two numerical examples are provided to demonstrate the effectiveness of our theoretical results.

Consider CVNN (6.1) consisting of five neurons as the master system, where $z_i = z_i^R + jz_i^I$, $f_i(z_i) = 0.1(|z_i^R| + j|z_i^I|)$, $g_i(z_i) = \frac{1}{2}\big(tanh(z_i^R) + j \cdot tanh(z_i^I)\big)$ for $i = 1, 2, ..., 5$. Then, Assumption 6.2.1 is satisfied with $L = 0.1I_5$, $F = 0.5I_5$. Choose $C = diag\{c_1, c_2, c_3, c_4, c_5\} = diag\{1, 2, 1, 0.5, 0.8\}$,

$$A = [a_{ij}]_{5\times 5} = \begin{pmatrix} 1+j & 0 & 0.5j & -j & 0 \\ 0 & 1-j & 0 & -2+j & 0 \\ -0.5+j & 2-j & 1+j & 0 & 1+j \\ 1-j & 0 & 0 & 2+0.5j & 1+j \\ 0 & 0 & -0.5j & 1-j & 0 \end{pmatrix},$$

$$B = [b_{ij}]_{5\times 5} = \begin{pmatrix} j & 1+j & -1+j & 0 & 0 \\ 0.8+j & 0.5-j & 0.6+0.2j & 0 & 1+j \\ 0 & 0 & 2+j & 1+j & 0 \\ 1+j & 0.2-0.6j & 0 & 1-j & 0.5+0.3j \\ 1-j & 0 & 0 & 0 & 1+j \end{pmatrix},$$

and $J = (J_1, J_2, J_3, J_4, J_5)^T = (0, 0, 0, 0, 0)^T$. Consider CVNN (6.2) as the corresponding slave system.

Example 6.3.1. *Let $r = 0.1$, $d = 1$. Choose the three indices $\sigma_{\max}, \sigma_{\min}$ and Δ in delayed ETPIC (6.27)–(6.29) as*

$$\sigma_{\max} = 1.2, \quad \sigma_{\min} = 0.8, \quad \Delta = 2.$$

Let $\mu = 15$, $\omega = 0.01$, then LMI (6.26) has feasible solution with

$$P = diag\{3.1502, 2.9886, 2.6596, 3.1448, 3.7714\},$$
$$Q = diag\{77.4985, 76.7963, 77.3929, 76.8210, 76.9119\}.$$

The Lyapunov functional V in delayed ETPIC (6.27)-(6.29) is chosen as (6.25) with ω, P, Q shown above. Then it follows from Theorem 6.3.1 that delayed ETPIC (6.27)-(6.29) *is non-Zeno satisfying*

$$\frac{\ln \sigma_{\max}}{\mu} = 0.012 \leq t_{k+1} - t_k \leq 2 = \Delta, \quad k \in \mathbb{N}.$$

Also, for $\mu = 15$, $r = 0.1$, $d = 1$, *we can verify that the choice of* Δ *satisfies* $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r, d\} = 1$. *Let* $l = 4$ *(i.e., if the event from* L_1 *or* L_2 *occurs at* $t = t_k$*, then four nodes will be impulsively controlled at instant* $t = t_k$*), by simple calculations, we can get* $\kappa_1 = 1.4180, \kappa_2 = 29.1392, \gamma = 1.0232, \lambda = 8$, *and* $\xi = 82$. *Choose the impulsive control gains* $\beta_1 = 0.005 + 0.001j$ *and* $\beta_2 = 0.02 + 0.05j$, *then conditions (6.34) and (6.35) in Theorem 6.3.2 are satisfied. Theorem 6.3.2 implies that master-slave CVNNs (6.1) and (6.2) achieve synchronization via delayed ETPIC* (6.27)-(6.29).

The initial condition for master CVNN (6.1) is randomly chosen as $[\varphi_1(s), \varphi_2(s), \varphi_3(s), \varphi_4(s), \varphi_5(s)]^T = [2+5j, 1-j, -3+j, -1+3j, 0.5+0.2j]^T$ *for* $s \in [-1, 0]$, *and the initial condition of the slave CVNN (6.2) is chosen as* $[\psi_1(s), \psi_2(s), \psi_3(s), \psi_4(s), \psi_5(s)]^T = [-2+2j, 4+3j, -1+3j, 5-3j, -2+j]^T$ *for* $s \in [-1, 0]$. *The attractor of the* 5^{th} *neuron in master CVNN (6.1) and slave CVNN (6.2) are depicted in Figure 6.6 and Figure 6.7, respectively. Figure 6.8 shows the trajectories of real and imaginary parts of synchronization errors of master-slave CVNNs (6.1) and (6.2) under delayed ETPIC* (6.27)-(6.29) *with* $r = 0.1$, $d = 1$, *and the corresponding triggered time instants of three levels of events and release intervals are plotted in Figure 6.9. From the simulation results in Figure 6.8, it is clearly observed that master-slave synchronization of CVNNs (6.1) and (6.2) is achieved.*

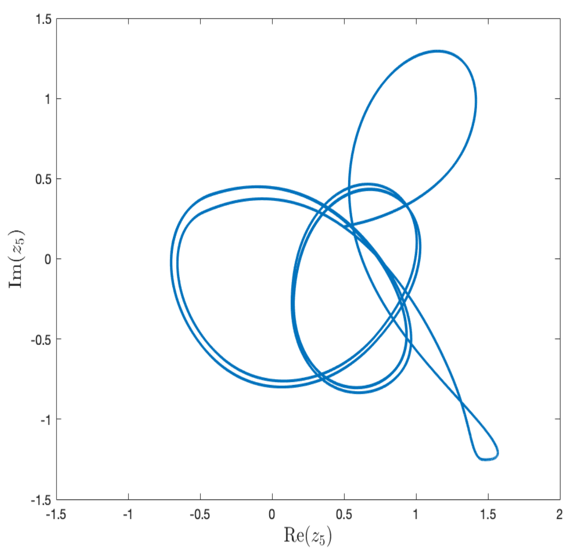

Figure 6.6: The attractor of the 5^{th} neuron in master CVNN (6.1).

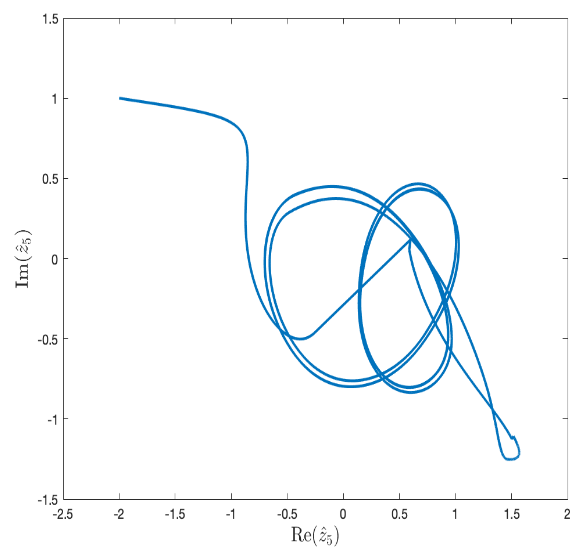

Figure 6.7: The attractor of the 5^{th} neuron in slave CVNN (6.2).

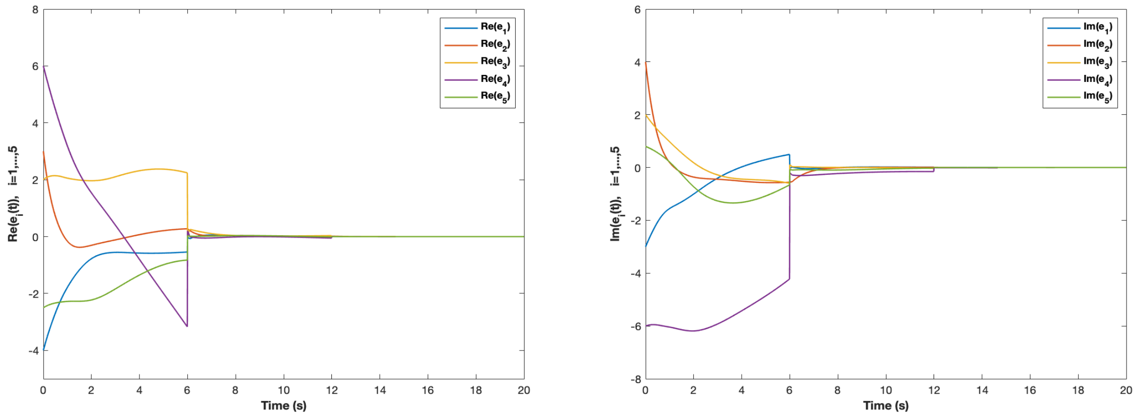

Figure 6.8: Trajectories of real and imaginary parts of synchronization errors for master-slave CVNNs (6.1) and (6.2) via delayed ETPIC (6.27)-(6.29) with $r = 0.1$, $d = 1$.

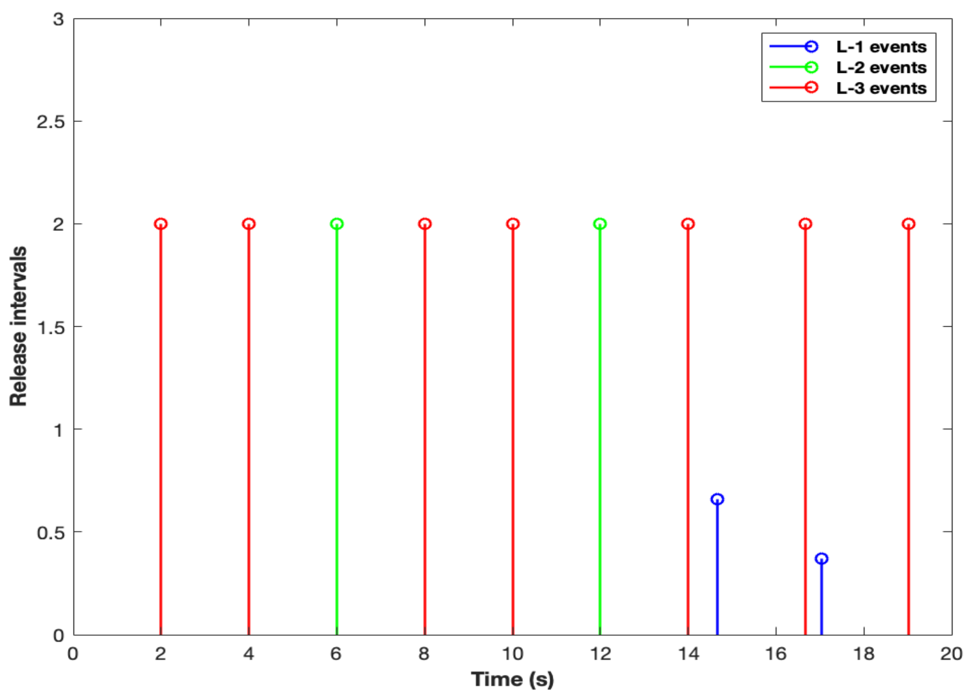

Figure 6.9: Triggered instants of three levels of events and release intervals for Example 6.3.1.

Remark 6.3.7. *It can be seen from Figure 6.9 that the size of the impulse delay in Example 6.3.1 exceeds the length of some of the release intervals, while the master-slave synchronization result can still be confirmed if complex-valued impulsive control gains are suitably designed to satisfy conditions (6.34) and (6.35).*

Example 6.3.2. *Let* $r = 0.04$, $d = 0.06$, *then* $\tau = 0.06$. *The indices* $\sigma_{\max}, \sigma_{\min}$ *and* Δ *in delayed ETPIC (6.27)−(6.29) are chosen the same as those of Example 6.3.1. Let* $\mu = 3 < \frac{\ln \sigma_{\max}}{\tau}$, $\omega = 0.01$, *LMI (6.26) has feasible solution with*

$$P = diag\{0.8338, 0.8633, 0.7913, 0.7041, 1.0834\},$$
$$Q = diag\{86.0887, 83.8486, 85.0877, 81.3350, 83.7953\}.$$

The Lyapunov functional V *in delayed ETPIC (6.27)-(6.29) is chosen as (6.25) with* ω, P, Q *shown above. Also, for* $\mu = 3$, $r = 0.04$, $d = 0.06$, *we can verify that the choice of* Δ *satisfies* $\Delta > \max\{\frac{\ln \sigma_{\max}}{\mu}, r, d\} = 0.061$.

Let $l = 4$, *after calculation, we can get* $\kappa_1 = 1.5387, \kappa_2 = 122.2677, \gamma = 1.0541$, *and* $\lambda = 8$. *Choose the impulsive control gains* $\beta_1 = 0.3{+}0.1j$ *and* $\beta_2 = 0.3{-}0.1j$, *then condition (6.55) in Corollary 6.3.1 is satisfied. Then it follows from Corollary 6.3.1 that master-slave synchronization of CVNNs (6.1) and (6.2) is achieved via ETPIC* (6.27)-(6.29).

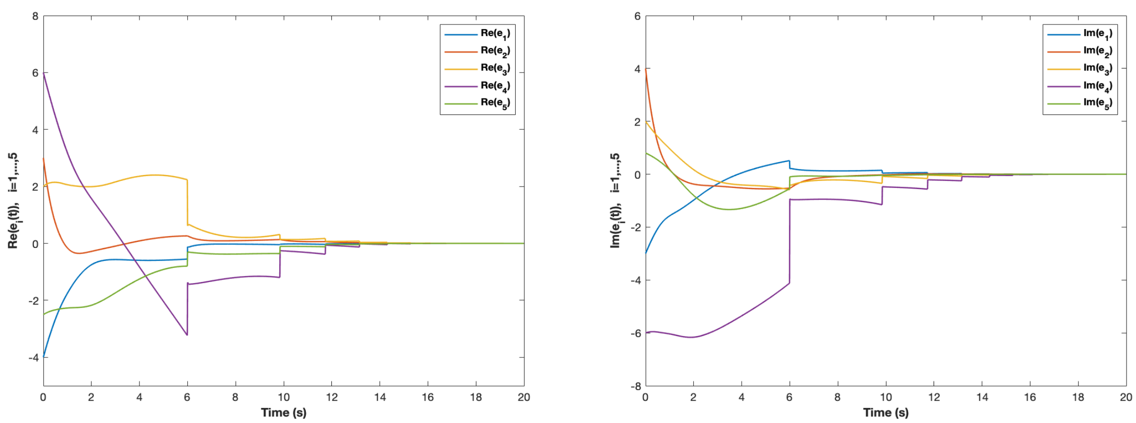

Figure 6.10: Trajectories of real and imaginary parts of synchronization errors for master-slave CVNNs (6.1) and (6.2) via delayed ETPIC (6.27)-(6.29) with $r = 0.04$, $d = 0.06$.

Choosing the same initial conditions as those of Example 6.3.1 for $s \in [-0.06, 0]$. *Figure 6.10 shows the trajectories of real and imaginary parts of synchronization errors of master-slave CVNNs (6.1) and (6.2) under delayed ETPIC* (6.27)-(6.29) *with* $r = 0.04$, $d = 0.06$, *and the corresponding triggered instants of three levels of events and release intervals are plotted in Figure 6.11. It can be seen from the results of simulations in Figure 6.10 that master-slave synchronization of CVNNs (6.1) and (6.2) is achieved.*

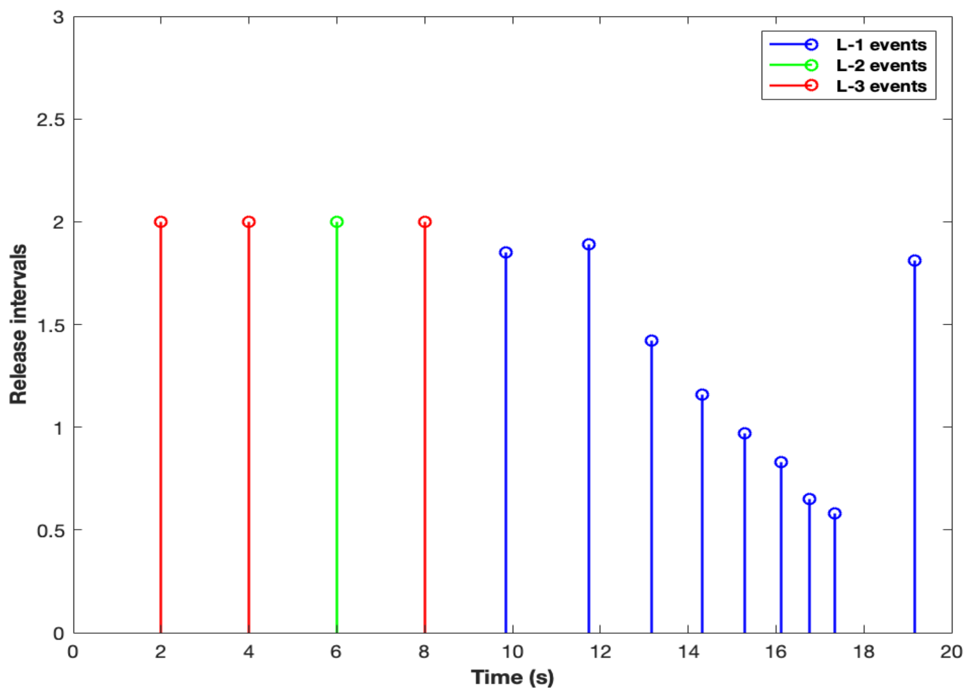

Figure 6.11: Triggered instants of three levels of events and release intervals for Example 6.3.2.

6.4 Adaptive Impulsive Observer Design for State Estimation of CVNNs

This section studies the state estimation problem of CVNNs. Two types of complex-valued activation functions are considered. In Subsection 6.4.1, the state estimation problem of CVNNs is formulated, and the adaptive impulsive observer is constructed. The adaptive updating law is also proposed in the complex domain. In Subsection 6.4.2, sufficient conditions on designing the adaptive impulsive observer are respectively obtained based on the two types of complex-valued activation functions. Numerical simulations are provided to demonstrate our theoretical result.

6.4.1 Problem Formulation

Consider time-delay CVNN (6.1) consisting of n neurons. The initial condition of CVNN (6.1) is given by $z_i(t_0+s)=\varphi_i(s)$ for $i=1,2,...,n$, where $\varphi_i \in \mathcal{PC}([-r,0],\mathbb{C})$.

The objective is to design the following adaptive impulsive observer to estimate the state of CVNN (6.1):

$$\begin{cases} \dot{\hat{z}}_i(t) = -c_i\hat{z}_i(t) + \sum_{j=1}^{n} a_{ij} f_j(\hat{z}_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(\hat{z}_j(t-r)) + J_i - d_i(t)\big(\hat{z}_i(t) - z_i(t)\big), & t \neq t_k, \\ \hat{z}_i(t) = \hat{z}_i(t^-) + L\big(\hat{z}_i(t^-) - z_i(t^-)\big), & t = t_k,\ k \in \mathbb{N}^+, \end{cases} \tag{6.58}$$

for $i=1,2,...,n$, where $\hat{z}_i \in \mathbb{C}$ denotes the estimated state; $d_i \in \mathbb{C}$ represents the adaptive feedback gain; $L \in \mathbb{C}$ represents the impulsive observer gain. The impulsive sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < \ldots < t_k < \ldots$, and $\lim_{k\to\infty} t_k = \infty$. Further, we assume the solutions of adaptive impulsive observer dynamics (6.58) are right continuous at each impulsive instant t_k, i.e., $\hat{z}_i(t_k) = \hat{z}_i(t_k^+)$.

We say that the adaptive impulsive observer (6.58) asymptotically estimates the state of CVNN (6.1) if

$$\lim_{t\to\infty} |\hat{z}_i(t) - z_i(t)| = 0, \qquad i = 1,2,\ldots,n.$$

Let the state estimation error be $e_i(t) = \hat{z}_i(t) - z_i(t)$, $i=1,2,...,n$. To estimate the state of CVNN (6.1), the updating law of the adaptive feedback gain is designed as follows:

$$\dot{d}_i(t) = \gamma_i |e_i(t)|^2, \qquad i=1,2,\ldots,n, \tag{6.59}$$

where $\gamma_i = \gamma_i^R + j\gamma_i^I \in \mathbb{C}$, and $\gamma_i^R > 0$. The initial feedback strength is given by $d_i(t_0) = d_{i0}$ for $i=1,2,...,n$.

For CVNN (6.1), we consider two types of complex-valued activation functions:

Case I. The non-delayed and delayed complex-valued activation functions $f_i(\cdot)$ and $g_i(\cdot)$, $i=1,2,\ldots,n$, in CVNN (6.1) can be decomposed to their real and imaginary parts and satisfy the following assumption:

Assumption 6.4.1. *For $u, v, \hat{u}, \hat{v} \in \mathbb{C}$, denote $u = u^R + ju^I$, $v = v^R + jv^I$, $\hat{u} = \hat{u}^R + j\hat{u}^I$, and $\hat{v} = \hat{v}^R + j\hat{v}^I$. Suppose that the complex-valued activation function $f_i(u)$ and the delayed complex-valued activation function $g_i(v)$ can be separated into their real and imaginary parts as*

$$
\begin{aligned}
f_i(u) &= f_i^R(u^R, u^I) + jf_i^I(u^R, u^I), \\
g_i(v) &= g_i^R(v^R, v^I) + jg_i^I(v^R, v^I)
\end{aligned}
$$

for $i = 1, 2, ..., n$, where $f_i^R(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f_i^I(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g_i^R(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $g_i^I(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy

$$
\begin{aligned}
|f_i^R(\hat{u}^R, \hat{u}^I) - f_i^R(u^R, u^I)| &\le p_i^{RR}|\hat{u}^R - u^R| + p_i^{RI}|\hat{u}^I - u^I|, \\
|f_i^I(\hat{u}^R, \hat{u}^I) - f_i^I(u^R, u^I)| &\le p_i^{IR}|\hat{u}^R - u^R| + p_i^{II}|\hat{u}^I - u^I|, \\
|g_i^R(\hat{v}^R, \hat{v}^I) - g_i^R(v^R, v^I)| &\le q_i^{RR}|\hat{v}^R - v^R| + q_i^{RI}|\hat{v}^I - v^I|, \\
|g_i^I(\hat{v}^R, \hat{v}^I) - g_i^I(v^R, v^I)| &\le q_i^{IR}|\hat{v}^R - v^R| + q_i^{II}|\hat{v}^I - v^I|,
\end{aligned}
$$

where $p_i^{RR}, p_i^{RI}, p_i^{IR}, p_i^{II}, q_i^{RR}, q_i^{RI}, q_i^{IR}, q_i^{II}$ are nonnegative constants for $i = 1, 2, ..., n$. Let $p^{RR} = \max_{1\le i\le n}\{(p_i^{RR})^2\}, p^{RI} = \max_{1\le i\le n}\{(p_i^{RI})^2\}, p^{IR} = \max_{1\le i\le n}\{(p_i^{IR})^2\}$, $p^{II} = \max_{1\le i\le n}\{(p_i^{II})^2\}, q^{RR} = \max_{1\le i\le n}\{(q_i^{RR})^2\}, q^{RI} = \max_{1\le i\le n}\{(q_i^{RI})^2\}$, $q^{IR} = \max_{1\le i\le n}\{(q_i^{IR})^2\}$, and $q^{II} = \max_{1\le i\le n}\{(q_i^{II})^2\}$.

Since $z_i, \hat{z}_i \in \mathbb{C}$, denote $z_i(t) = z_i^R(t) + jz_i^I(t), \hat{z}_i(t) = \hat{z}_i^R(t) + j\hat{z}_i^I(t)$. Then according to Assumption 6.4.1, CVNN (6.1) can be separated into the following two equivalent real-valued systems:

$$
\begin{aligned}
\dot{z_i}^R(t) = &- c_i z_i^R(t) + \sum_{j=1}^{n} a_{ij}^R f_j^R\big(z_j^R(t), z_j^I(t)\big) - \sum_{j=1}^{n} a_{ij}^I f_j^I\big(z_j^R(t), z_j^I(t)\big) \\
&+ \sum_{j=1}^{n} b_{ij}^R g_j^R(z_{j_r}^R, z_{j_r}^I) - \sum_{j=1}^{n} b_{ij}^I g_j^I(z_{j_r}^R, z_{j_r}^I) + J_i^R,
\end{aligned} \tag{6.60}
$$

and

$$
\begin{aligned}
\dot{z_i}^I(t) = &- c_i z_i^I(t) + \sum_{j=1}^{n} a_{ij}^R f_j^I\big(z_j^R(t), z_j^I(t)\big) + \sum_{j=1}^{n} a_{ij}^I f_j^R\big(z_j^R(t), z_j^I(t)\big) \\
&+ \sum_{j=1}^{n} b_{ij}^R g_j^I(z_{j_r}^R, z_{j_r}^I) + \sum_{j=1}^{n} b_{ij}^I g_j^R(z_{j_r}^R, z_{j_r}^I) + J_i^I,
\end{aligned} \tag{6.61}
$$

for $i = 1, 2, ..., n$, where $a_{ij}^R = \mathfrak{Re}(a_{ij}), a_{ij}^I = \mathfrak{Im}(a_{ij}), b_{ij}^R = \mathfrak{Re}(b_{ij}), b_{ij}^I = \mathfrak{Im}(b_{ij}),\ J_i^R = \mathfrak{Re}(J_i)$, and $J_i^I = \mathfrak{Im}(J_i)$. The adaptive impulsive observer (6.58) can be rewritten as the following two real-valued systems:

$$\begin{cases} \dot{\hat{z}}_i^R(t) = -c_i \hat{z}_i^R(t) + \sum_{j=1}^n a_{ij}^R f_j^R\big(\hat{z}_j^R(t), \hat{z}_j^I(t)\big) - \sum_{j=1}^n a_{ij}^I f_j^I\big(\hat{z}_j^R(t), \hat{z}_j^I(t)\big) + \sum_{j=1}^n b_{ij}^R g_j^R(\hat{z}_{j_r}^R, \hat{z}_{j_r}^I) \\ -\sum_{j=1}^n b_{ij}^I g_j^I(\hat{z}_{j_r}^R, \hat{z}_{j_r}^I) + J_i^R - d_i^R(t)\big(\hat{z}_i^R(t) - z_i^R(t)\big) + d_i^I(t)\big(\hat{z}_i^I(t) - z_i^I(t)\big), \quad t \neq t_k, \\ \hat{z}_i^R(t) = \hat{z}_i^R(t^-) + L^R\big(\hat{z}_i^R(t^-) - z_i^R(t^-)\big) - L^I\big(\hat{z}_i^I(t^-) - z_i^I(t^-)\big), \quad t = t_k,\ k \in \mathbb{N}^+, \end{cases} \tag{6.62}$$

$$\begin{cases} \dot{\hat{z}}_i^I(t) = -c_i \hat{z}_i^I(t) + \sum_{j=1}^n a_{ij}^R f_j^I\big(\hat{z}_j^R(t), \hat{z}_j^I(t)\big) + \sum_{j=1}^n a_{ij}^I f_j^R\big(\hat{z}_j^R(t), \hat{z}_j^I(t)\big) + \sum_{j=1}^n b_{ij}^R g_j^I(\hat{z}_{j_r}^R, \hat{z}_{j_r}^I) \\ +\sum_{j=1}^n b_{ij}^I g_j^R(\hat{z}_{j_r}^R, \hat{z}_{j_r}^I) + J_i^I - d_i^R(t)\big(\hat{z}_i^I(t) - z_i^I(t)\big) - d_i^I(t)\big(\hat{z}_i^R(t) - z_i^R(t)\big), \quad t \neq t_k, \\ \hat{z}_i^I(t) = \hat{z}_i^I(t^-) + L^R\big(\hat{z}_i^I(t^-) - z_i^I(t^-)\big) + L^I\big(\hat{z}_i^R(t^-) - z_i^R(t^-)\big), \quad t = t_k,\ k \in \mathbb{N}^+, \end{cases} \tag{6.63}$$

where $d_i^R(t) = \mathfrak{Re}(d_i(t)), d_i^I(t) = \mathfrak{Im}(d_i(t)), L^R = \mathfrak{Re}(L)$, and $L^I = \mathfrak{Im}(L)$.

Denote the state estimation error $e_i(t) = e_i^R(t) + je_i^I(t)$, then we can obtain that $e_i^R(t) = \hat{z}_i^R(t) - z_i^R(t)$, and $e_i^I(t) = \hat{z}_i^I(t) - z_i^I(t)$. According to (6.59)-(6.63), it is easy to derive the following real-valued error dynamical systems:

$$\begin{cases} \dot{e}_i^R(t) = -c_i e_i^R(t) + \sum_{j=1}^n a_{ij}^R \tilde{f}_j^R\big(e_j^R(t), e_j^I(t)\big) - \sum_{j=1}^n a_{ij}^I \tilde{f}_j^I\big(e_j^R(t), e_j^I(t)\big) + \sum_{j=1}^n b_{ij}^R \tilde{g}_j^R(e_{j_r}^R, e_{j_r}^I) \\ \qquad -\sum_{j=1}^n b_{ij}^I \tilde{g}_j^I(e_{j_r}^R, e_{j_r}^I) - d_i^R(t) e_i^R(t) + d_i^I(t) e_i^I(t), \qquad t \neq t_k, \\ e_i^R(t) = (1 + L^R) e_i^R(t^-) - L^I e_i^I(t^-), \quad t = t_k,\ k \in \mathbb{N}^+, \end{cases} \tag{6.64}$$

$$\begin{cases} \dot{e}_i^I(t) = -c_i e_i^I(t) + \sum_{j=1}^n a_{ij}^R \tilde{f}_j^I\big(e_j^R(t), e_j^I(t)\big) + \sum_{j=1}^n a_{ij}^I \tilde{f}_j^R\big(e_j^R(t), e_j^I(t)\big) + \sum_{j=1}^n b_{ij}^R \tilde{g}_j^I(e_{j_r}^R, e_{j_r}^I) \\ \qquad +\sum_{j=1}^n b_{ij}^I \tilde{g}_j^R(e_{j_r}^R, e_{j_r}^I) - d_i^R(t) e_i^I(t) - d_i^I(t) e_i^R(t), \qquad t \neq t_k, \\ e_i^I(t) = (1 + L^R) e_i^I(t^-) + L^I e_i^R(t^-), \quad t = t_k,\ k \in \mathbb{N}^+, \end{cases} \tag{6.65}$$

with adaptive updating laws

$$\begin{aligned} \dot{d}_i^{\,R} &= \gamma_i^R |e_i(t)|^2, \qquad \gamma_i^R > 0, \\ \dot{d}_i^{\,I} &= \gamma_i^I |e_i(t)|^2, \qquad \gamma_i^I \in \mathbb{R}, \end{aligned} \tag{6.66}$$

for $i = 1, 2, ..., n$, where $\tilde{f}_j^R(e_j^R, e_j^I) = f_j^R(\hat{z}_j^R, \hat{z}_j^I) - f_j^R(z_j^R, z_j^I)$, $\tilde{f}_j^I(e_j^R, e_j^I) = f_j^I(\hat{z}_j^R, \hat{z}_j^I) - f_j^I(z_j^R, z_j^I)$, $\tilde{g}_j^R(e_j^R, e_j^I) = g_j^R(\hat{z}_j^R, \hat{z}_j^I) - g_j^R(z_j^R, z_j^I)$, and $\tilde{g}_j^I(e_j^R, e_j^I) = g_j^I(\hat{z}_j^R, \hat{z}_j^I) - g_j^I(z_j^R, z_j^I)$.

Case II. The non-delayed and delayed complex-valued activation functions $f_i(\cdot)$ and $g_i(\cdot)$, $i = 1, 2, ..., n$, in CVNN (6.1) cannot be separated into their real and imaginary parts explicitly. $f_i(\cdot)$ and $g_i(\cdot)$ satisfy the following assumption:

Assumption 6.4.2. *Suppose that there exist positive constants ω_i and ξ_i, $i = 1, 2, ..., n$, such that*

$$\begin{aligned} |f_i(\hat{z}) - f_i(z)| &\le \omega_i |\hat{z} - z|, \\ |g_i(\hat{z}) - g_i(z)| &\le \xi_i |\hat{z} - z|, \end{aligned}$$

for any $z, \hat{z} \in \mathbb{C}$.

Then according to (6.1), (6.58) and (6.59), the state estimation error dynamics can be described as follows:

$$\begin{cases} \dot{e}_i(t) = -c_i e_i(t) + \sum_{j=1}^{n} a_{ij} \bar{f}_j(e_j(t)) + \sum_{j=1}^{n} b_{ij} \bar{g}_j(e_j(t-r)) - d_i(t) e_i(t), \quad t \neq t_k, \\ \dot{d}_i = \gamma_i e_i^*(t) e_i(t), \\ e_i(t) = (1+L) e_i(t^-), \qquad t = t_k, \; k \in \mathbb{N}^+, \end{cases} \tag{6.67}$$

for $i = 1, 2, ..., n$, where $\bar{f}_j(e_j(t)) = f_j(\hat{z}_j(t)) - f_j(z_j(t))$, $\bar{g}_j(e_j(t-r)) = g_j(\hat{z}_j(t-r)) - g_j(z_j(t-r))$.

6.4.2 Observer Design

In this subsection, we present some sufficient conditions for designing the adaptive impulsive observer (6.58) to estimate the state of CVNN (6.1) based on the two types of complex-valued activation functions.

Case I. The complex-valued activation functions in CVNN (6.1) satisfy Assumption 6.4.1.

Theorem 6.4.1. *Suppose that Assumption 6.4.1 holds. Let* $c_{\min} = \min_{1\le i\le n}\{c_i\}$, $\alpha_1 = \max_{i,j}\{(a_{ij}^R)^2\}, \alpha_2 = \max_{i,j}\{(a_{ij}^I)^2\}, \beta_1 = \max_{i,j}\{(b_{ij}^R)^2\}, \beta_2 = \max_{i,j}\{(b_{ij}^I)^2\}$, $p_{\max} = \max\{ p^{RR}+p^{IR}, p^{RI}+p^{II}\}$, *and* $q_{\max} = \max\{q^{RR}+q^{IR}, q^{RI}+q^{II}\}$. *Let* d_{i0}^R *be the real part of the initial feedback strength* d_{i0}, $d = \min_{1\le i\le n}\{d_{i0}^R\}$, $a = -2c_{\min}-2d+4+2n^2p_{\max}(\alpha_1+\alpha_2), b = 2n^2q_{\max}(\beta_1+\beta_2)$. *The real part of the initial feedback strength is limited such that*

$$a > 0, \tag{6.68}$$

then the adaptive impulsive observer (6.58) with updating law (6.59) can estimate the state of CVNN (6.1) if

$$a + \frac{b}{\rho} + \frac{\ln\rho}{\sigma} < 0, \qquad \sigma := \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\}, \tag{6.69}$$

where $\rho = 2\big((1+L^R)^2 + (L^I)^2\big)$.

Proof. Since the updating law (6.59) can be rewritten as (6.66), we have $\dot{d}_i^R \ge 0$, hence $d_i^R(t) \ge d_{i0}^R$ for $t \ge t_0$, $i = 1, 2, ..., n$. Consider the Lyapunov function candidate as follows:

$$V(t) = \frac{1}{2}\sum_{i=1}^{n}[e_i^R(t)]^2 + \frac{1}{2}\sum_{i=1}^{n}[e_i^I(t)]^2. \tag{6.70}$$

For $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, taking the derivative of V along the trajectories of the error dynamical systems (6.64) and (6.65), we have

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{n} e_i^R(t)\dot{e}_i^R(t) + \sum_{i=1}^{n} e_i^I(t)\dot{e}_i^I(t)\\
&= \sum_{i=1}^{n} e_i^R(t)\big[-c_ie_i^R(t) + \sum_{j=1}^{n} a_{ij}^R\tilde{f}_j^R\big(e_j^R(t), e_j^I(t)\big) - \sum_{j=1}^{n} a_{ij}^I\tilde{f}_j^I\big(e_j^R(t), e_j^I(t)\big) + \sum_{j=1}^{n} b_{ij}^R\tilde{g}_j^R(e_{j_r}^R, e_{j_r}^I)\\
&- \sum_{j=1}^{n} b_{ij}^I\tilde{g}_j^I(e_{j_r}^R, e_{j_r}^I) - d_i^R(t)e_i^R(t) + d_i^I(t)e_i^I(t)\big] + \sum_{i=1}^{n} e_i^I(t)\big[-c_ie_i^I(t) + \sum_{j=1}^{n} a_{ij}^R\tilde{f}_j^I\big(e_j^R(t), e_j^I(t)\big)\\
&+ \sum_{j=1}^{n} a_{ij}^I\tilde{f}_j^R\big(e_j^R(t), e_j^I(t)\big) + \sum_{j=1}^{n} b_{ij}^R\tilde{g}_j^I(e_{j_r}^R, e_{j_r}^I) + \sum_{j=1}^{n} b_{ij}^I\tilde{g}_j^R(e_{j_r}^R, e_{j_r}^I) - d_i^R(t)e_i^I(t) - d_i^I(t)e_i^R(t)\big]\\
&\le -c_{\min}\big(\sum_{i=1}^{n}[e_i^R(t)]^2 + \sum_{i=1}^{n}[e_i^I(t)]^2\big) + \frac{1}{2}\sum_{i=1}^{n}\Bigg([e_i^R(t)]^2 + \big[\sum_{j=1}^{n} a_{ij}^R\tilde{f}_j^R\big(e_j^R(t), e_j^I(t)\big)\big]^2\Bigg)
\end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\bigg([e_i^R(t)]^2+\big[\sum_{j=1}^{n}a_{ij}^I\tilde{f}_j^I\big(e_j^R(t),e_j^I(t)\big)\big]^2\bigg)+\frac{1}{2}\sum_{i=1}^{n}\bigg([e_i^R(t)]^2+\big[\sum_{j=1}^{n}b_{ij}^R\tilde{g}_j^R(e_{j_r}^R,e_{j_r}^I)\big]^2\bigg)$$
$$+\frac{1}{2}\sum_{i=1}^{n}\bigg([e_i^R(t)]^2+\big[\sum_{j=1}^{n}b_{ij}^I\tilde{g}_j^I(e_{j_r}^R,e_{j_r}^I)\big]^2\bigg)+\frac{1}{2}\sum_{i=1}^{n}\bigg([e_i^I(t)]^2+\big[\sum_{j=1}^{n}a_{ij}^R\tilde{f}_j^I\big(e_j^R(t),e_j^I(t)\big)\big]^2\bigg)$$
$$+\frac{1}{2}\sum_{i=1}^{n}\bigg([e_i^I(t)]^2+\big[\sum_{j=1}^{n}a_{ij}^I\tilde{f}_j^R\big(e_j^R(t),e_j^I(t)\big)\big]^2\bigg)+\frac{1}{2}\sum_{i=1}^{n}\bigg([e_i^I(t)]^2+\big[\sum_{j=1}^{n}b_{ij}^R\tilde{g}_j^I(e_{j_r}^R,e_{j_r}^I)\big]^2\bigg)$$
$$+\frac{1}{2}\sum_{i=1}^{n}\bigg([e_i^I(t)]^2+\big[\sum_{j=1}^{n}b_{ij}^I\tilde{g}_j^R(e_{j_r}^R,e_{j_r}^I)\big]^2\bigg)-\sum_{i=1}^{n}d_{i0}^R[e_i^R(t)]^2-\sum_{i=1}^{n}d_{i0}^R[e_i^I(t)]^2. \tag{6.71}$$

According to Assumption 6.4.1, we can derive

$$\frac{1}{2}\sum_{i=1}^{n}\big[\sum_{j=1}^{n}a_{ij}^R\tilde{f}_j^R\big(e_j^R(t),e_j^I(t)\big)\big]^2\le\frac{n}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}^R)^2\big[\tilde{f}_j^R\big(e_j^R(t),e_j^I(t)\big)\big]^2$$
$$\le\frac{n}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}^R)^2|f_j^R\big(\hat{z}_j^R(t),\hat{z}_j^I(t)\big)-f_j^R\big(z_j^R(t),z_j^I(t)\big)|^2$$
$$\le\frac{n}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}^R)^2\big(p_j^{RR}|\hat{z}_j^R(t)-z_j^R(t)|+p_j^{RI}|\hat{z}_j^I(t)-z_j^I(t)|\big)^2$$
$$\le n\max_{i,j}\{(a_{ij}^R)^2\}\sum_{i=1}^{n}\bigg(\sum_{j=1}^{n}(p_j^{RR})^2[e_j^R(t)]^2+(p_j^{RI})^2[e_j^I(t)]^2\bigg)$$
$$\le n^2\alpha_1\bigg(p^{RR}\sum_{i=1}^{n}[e_i^R(t)]^2+p^{RI}\sum_{i=1}^{n}[e_i^I(t)]^2\bigg).$$

Similarly, we have

$$\frac{1}{2}\sum_{i=1}^{n}\big[\sum_{j=1}^{n}a_{ij}^I\tilde{f}_j^I\big(e_j^R(t),e_j^I(t)\big)\big]^2\le n^2\alpha_2\bigg(p^{IR}\sum_{i=1}^{n}[e_i^R(t)]^2+p^{II}\sum_{i=1}^{n}[e_i^I(t)]^2\bigg),$$
$$\frac{1}{2}\sum_{i=1}^{n}\big[\sum_{j=1}^{n}b_{ij}^R\tilde{g}_j^R(e_{j_r}^R,e_{j_r}^I)\big]^2\le n^2\beta_1\bigg(q^{RR}\sum_{i=1}^{n}[e_i^R(t-r)]^2+q^{RI}\sum_{i=1}^{n}[e_i^I(t-r)]^2\bigg),$$
$$\frac{1}{2}\sum_{i=1}^{n}\big[\sum_{j=1}^{n}b_{ij}^I\tilde{g}_j^I(e_{j_\tau}^R,e_{j_\tau}^I)\big]^2\le n^2\beta_2\bigg(q^{IR}\sum_{i=1}^{n}[e_i^R(t-r)]^2+q^{II}\sum_{i=1}^{n}[e_i^I(t-r)]^2\bigg),$$

$$\frac{1}{2}\sum_{i=1}^{n}\Big[\sum_{j=1}^{n}a_{ij}^{R}\tilde{f}_j^{I}\big(e_j^R(t),e_j^I(t)\big)\Big]^2 \le n^2\alpha_1\left(p^{IR}\sum_{i=1}^{n}[e_i^R(t)]^2+p^{II}\sum_{i=1}^{n}[e_i^I(t)]^2\right),$$
$$\frac{1}{2}\sum_{i=1}^{n}\Big[\sum_{j=1}^{n}a_{ij}^{I}\tilde{f}_j^{R}\big(e_j^R(t),e_j^I(t)\big)\Big]^2 \le n^2\alpha_2\left(p^{RR}\sum_{i=1}^{n}[e_i^R(t)]^2+p^{RI}\sum_{i=1}^{n}[e_i^I(t)]^2\right),$$
$$\frac{1}{2}\sum_{i=1}^{n}\Big[\sum_{j=1}^{n}b_{ij}^{R}\tilde{g}_j^{I}(e_{j_r}^R,e_{j_r}^I)\Big]^2 \le n^2\beta_1\left(q^{IR}\sum_{i=1}^{n}[e_i^R(t-r)]^2+q^{II}\sum_{i=1}^{n}[e_i^I(t-r)]^2\right),$$
$$\frac{1}{2}\sum_{i=1}^{n}\Big[\sum_{j=1}^{n}b_{ij}^{I}\tilde{g}_j^{R}(e_{j_r}^R,e_{j_r}^I)\Big]^2 \le n^2\beta_2\left(q^{RR}\sum_{i=1}^{n}[e_i^R(t-r)]^2+q^{RI}\sum_{i=1}^{n}[e_i^I(t-r)]^2\right).$$

Then, it follows from (6.71) that for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$,

$$\begin{aligned}
\dot{V}(t) \le\ & (-c_{\min}-d+2)\left(\sum_{i=1}^{n}[e_i^R(t)]^2+\sum_{i=1}^{n}[e_i^I(t)]^2\right)+n^2\alpha_1\bigg((p^{RR}+p^{IR})\sum_{i=1}^{n}[e_i^R(t)]^2 \\
&+(p^{RI}+p^{II})\sum_{i=1}^{n}[e_i^I(t)]^2\bigg)+n^2\alpha_2\left((p^{RR}+p^{IR})\sum_{i=1}^{n}[e_i^R(t)]^2+(p^{RI}+p^{II})\sum_{i=1}^{n}[e_i^I(t)]^2\right) \\
&+n^2\beta_1\left((q^{RR}+q^{IR})\sum_{i=1}^{n}[e_i^R(t-r)]^2+(q^{RI}+q^{II})\sum_{i=1}^{n}[e_i^I(t-r)]^2\right) \\
&+n^2\beta_2\left((q^{RR}+q^{IR})\sum_{i=1}^{n}[e_i^R(t-r)]^2+(q^{RI}+q^{II})\sum_{i=1}^{n}[e_i^I(t-r)]^2\right) \\
&\le \big[-c_{\min}-d+2+n^2p_{\max}(\alpha_1+\alpha_2)\big]\left(\sum_{i=1}^{n}[e_i^R(t)]^2+\sum_{i=1}^{n}[e_i^I(t)]^2\right) \\
&+n^2q_{\max}(\beta_1+\beta_2)\left(\sum_{i=1}^{n}[e_i^R(t-r)]^2+\sum_{i=1}^{n}[e_i^I(t-r)]^2\right) \\
&=2\big[-c_{\min}-d+2+n^2p_{\max}(\alpha_1+\alpha_2)\big]V(t)+2n^2q_{\max}(\beta_1+\beta_2)V(t-r) \\
&\le aV(t)+b\sup_{-r\le s\le 0}V(t+s).
\end{aligned} \tag{6.72}$$

On the other hand, for $t = t_k, k \in \mathbb{N}^+$, it follows from (6.64) and (6.65) that

$$V(t_k)=\frac{1}{2}\sum_{i=1}^{n}[e_i^R(t_k)]^2+\frac{1}{2}\sum_{i=1}^{n}[e_i^I(t_k)]^2$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left[(1+L^R)e_i^R(t_k^-) - L^I e_i^I(t_k^-)\right]^2 + \frac{1}{2}\sum_{i=1}^{n}\left[(1+L^R)e_i^I(t_k^-) + L^I e_i^R(t_k^-)\right]^2$$
$$\leq \sum_{i=1}^{n}(1+L^R)^2[e_i^R(t_k^-)]^2 + (L^I)^2[e_i^I(t_k^-)]^2 + \sum_{i=1}^{n}(1+L^R)^2[e_i^I(t_k^-)]^2 + (L^I)^2[e_i^R(t_k^-)]^2$$
$$= \big((1+L^R)^2 + (L^I)^2\big)\big(\sum_{i=1}^{n}[e_i^R(t_k^-)]^2 + \sum_{i=1}^{n}[e_i^I(t_k^-)]^2\big),$$

hence, we have

$$V(t_k) \leq \rho V(t_k^-), \qquad k \in \mathbb{N}^+. \tag{6.73}$$

Let $\delta = \frac{1}{\rho}$, combining (6.72), (6.73) with conditions (6.68)-(6.69), it follows from Lemma 3.3.2 that

$$V(t) \leq \delta \|V(t_0)\|_r e^{-\lambda(t-t_0)}, \qquad t \geq t_0,$$

where λ is a constant satisfying $0 < \lambda < \frac{\ln \delta}{\sigma} - a - b\delta e^{\lambda r}$. Then we can derive $V(t) \to 0$ as $t \to \infty$, which implies $\sum_{i=1}^{n} |e_i(t)|^2 \to 0$ as $t \to \infty$. Therefore, the adaptive impulsive observer (6.58) with updating law (6.59) estimates the state of CVNN (6.1). □

Next, we will consider the case if complex-valued activation functions in CVNN (6.1) cannot be explicitly separated into their real and imaginary parts.

Case II. The complex-valued activation functions in CVNN (6.1) satisfy Assumption 6.4.2.

To estimate the state of CVNN (6.1) under Assumption 6.4.2, sufficient conditions for designing the adaptive impulsive observer (6.58) will be established by studying the stability of state estimation error system (6.67). For error system (6.67), we consider the following continuous comparison system:

$$\begin{cases} \dot{e}_i(t) = -c_i e_i(t) + \sum_{j=1}^{n} a_{ij}\bar{f}_j(e_j(t)) + \sum_{j=1}^{n} b_{ij}\bar{g}_j(e_j(t-r)) - d_i(t)e_i(t), \\ \dot{d}_i = \gamma_i e_i^*(t)e_i(t) \end{cases} \tag{6.74}$$

for $i = 1, 2, ..., n$. Assume the comparison system (6.74) has the same initial condition as that of (6.67).

Theorem 6.4.2. *Suppose that Assumption 6.4.2 holds. Then the adaptive impulsive observer (6.58) can estimate the state of CVNN (6.1) if the adaptive feedback gain d_i satisfying the updating law (6.59), and the impulsive observer gain is designed to satisfy*

$$|1 + L|^2 < 1. \tag{6.75}$$

Proof. Denote ${d_i}^R = \mathfrak{Re}(d_i), d_i^I = \mathfrak{Im}(d_i)$, then the updating law (6.59) can be rewritten as (6.66). Consider the Lyapunov functional candidate

$$V(t) = \frac{1}{2}\sum_{i=1}^{n} e_i^*(t)e_i(t) + \frac{1}{2}\sum_{i=1}^{n}\frac{({d_i}^R-\alpha)^2}{\gamma_i^R} + \beta\sum_{i=1}^{n}\int_{t-r}^{t} e_i^*(s)e_i(s)ds, \tag{6.76}$$

where $\alpha \in \mathbb{R}, \beta > 0$ are constants to be determined. Taking the derivative of V along the trajectories of the continuous comparison system (6.74) for $t \geq t_0$, we have

$$\begin{aligned}
\dot{V}\big|_{(6.74)} &= \sum_{i=1}^{n}\mathfrak{Re}[e_i^*(t)\dot{e}_i(t)] + \sum_{i=1}^{n}\frac{({d_i}^R-\alpha)}{\gamma_i^R}\dot{d_i}^R + \beta\sum_{i=1}^{n}e_i^*(t)e_i(t) - \beta\sum_{i=1}^{n}e_i^*(t-r)e_i(t-r)\\
&= \sum_{i=1}^{n}\mathfrak{Re}[e_i^*(t)\big(-c_ie_i(t) + \sum_{j=1}^{n}a_{ij}\bar{f}_j(e_j(t)) + \sum_{j=1}^{n}b_{ij}\bar{g}_j(e_j(t-r)) - d_i(t)e_i(t)\big)]\\
&+ \sum_{i=1}^{n}({d_i}^R-\alpha)e_i^*(t)e_i(t) + \beta\sum_{i=1}^{n}e_i^*(t)e_i(t) - \beta\sum_{i=1}^{n}e_i^*(t-r)e_i(t-r)\\
&= \sum_{i=1}^{n}-c_ie_i^*(t)e_i(t) - \alpha\sum_{i=1}^{n}e_i^*(t)e_i(t) + \beta\sum_{i=1}^{n}e_i^*(t)e_i(t) + \mathfrak{Re}\big[\sum_{i=1}^{n}\sum_{j=1}^{n}e_i^*(t)a_{ij}\bar{f}_j(e_j(t))\big]\\
&+ \mathfrak{Re}\big[\sum_{i=1}^{n}\sum_{j=1}^{n}e_i^*(t)b_{ij}\bar{g}_j(e_j(t-r))\big] - \beta\sum_{i=1}^{n}e_i^*(t-r)e_i(t-r).
\end{aligned} \tag{6.77}$$

Denote $E(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T$, $\bar{F}(E(t)) = \big(\bar{f}_1(e_1(t)), \bar{f}_2(e_2(t)), \ldots, \bar{f}_n(e_n(t))\big)^T$, $\bar{G}(E(t-r)) = \big(\bar{g}_1(e_1(t-r)), \bar{g}_2(e_2(t-r)), \ldots, \bar{g}_n(e_n(t-r))\big)^T$, $c_{\min} = \min_{1\leq i\leq n}\{c_i\}$, $A = (a_{ij})_{n\times n}$, and $B = (b_{ij})_{n\times n}$. Then it follows from (6.77) and Assumption 6.4.2 that

$$\begin{aligned}
&\dot{V}\big|_{(6.74)} \leq (\beta - \alpha - c_{\min})E^*(t)E(t) + \mathfrak{Re}\big[E^*(t)A\bar{F}(E(t))\big] + \mathfrak{Re}\big[E^*(t)B\bar{G}(E(t-\tau))\big]\\
&- \beta E^*(t-r)E(t-r)\\
&\leq (\beta - \alpha - c_{\min})E^*(t)E(t) + \big|E^*(t)A\bar{F}(E(t))\big| + \big|E^*(t)B\bar{G}(E(t-r))\big| - \beta E^*(t-r)E(t-r)\\
&\leq (\beta - \alpha - c_{\min})E^*(t)E(t) + \|E(t)\|\|A\|\|\bar{F}(E(t))\| + \|E(t)\|\|B\|\|\bar{G}(E(t-r))\|\\
&- \beta E^*(t-r)E(t-r)\\
&\leq (\beta - \alpha - c_{\min})E^*(t)E(t) + \max_{1\leq j\leq n}\{\omega_j\}\|A\|\|E(t)\|^2 + \|E(t)\|\max_{1\leq j\leq n}\{\xi_j\}\|B\|\|E(t-r)\|\\
&- \beta E^*(t-r)E(t-r)\\
&\leq (\beta - \alpha - c_{\min})E^*(t)E(t) + \max_j\{\omega_j\}\|A\|\|E(t)\|^2 + \frac{1}{2}\big(\|E(t)\|^2 + \max_j\{\xi_j^2\}\|B\|^2\|E(t-r)\|^2\big)
\end{aligned}$$

$$
\begin{aligned}
&- \beta E^*(t-r)E(t-r) \\
&= \Big(\frac{1}{2} + \max_j\{\omega_j\}\|A\| + \beta - \alpha - c_{\min}\Big)E^*(t)E(t) + \Big(\frac{\max_j\{\xi_j^2\}\|B\|^2}{2} - \beta\Big)E^*(t-r)E(t-r).
\end{aligned}
$$

Let $\alpha = \frac{3}{2} + \max_j\{\omega_j\}\|A\| + \frac{\max_j\{\xi_j^2\}\|B\|^2}{2} - c_{\min}$, and $\beta = \frac{\max_j\{\xi_j^2\}\|B\|^2}{2}$, then we have

$$
\dot{V}\big|_{(6.74)} \leq -\sum_{i=1}^{n} e_i^*(t)e_i(t), \qquad t \geq t_0, \tag{6.78}
$$

which implies $\dot{V}\big|_{(6.74)} \leq 0$ for $t \geq t_0$. Furthermore, since $\beta > 0$, it follows from (6.76) that $V(t)\big|_{(6.74)} \geq 0$ for $t \geq t_0$, then we can derive $\lim_{t\to\infty} V(t)\big|_{(6.74)}$ exists. Denote

$$
\lim_{t\to\infty} V(t)\big|_{(6.74)} = \nu. \quad (\nu \geq 0) \tag{6.79}
$$

Similar to the above discussion, for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}^+$, calculating the derivative of V along the trajectories of (6.67), we have

$$
\dot{V}\big|_{(6.67)} = \dot{V}\big|_{(6.74)}, \qquad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}^+. \tag{6.80}
$$

On the other hand, when $t = t_k$, it follows from (6.67) and (6.75) that

$$
\begin{aligned}
V(t_k)\big|_{(6.67)} &= \frac{1}{2}\sum_{i=1}^{n} e_i^*(t_k)e_i(t_k) + \frac{1}{2}\sum_{i=1}^{n}\frac{({d_i}^R(t_k)-\alpha)^2}{\gamma_i^R} + \beta\sum_{i=1}^{n}\int_{t_k-r}^{t_k} e_i^*(s)e_i(s)ds \\
&= \frac{|1+L|^2}{2}\sum_{i=1}^{n} e_i^*(t_k^-)e_i(t_k^-) + \frac{1}{2}\sum_{i=1}^{n}\frac{({d_i}^R(t_k^-)-\alpha)^2}{\gamma_i^R} + \beta\sum_{i=1}^{n}\int_{t_k-r}^{t_k} e_i^*(s)e_i(s)ds \\
&< \frac{1}{2}\sum_{i=1}^{n} e_i^*(t_k^-)e_i(t_k^-) + \frac{1}{2}\sum_{i=1}^{n}\frac{({d_i}^R(t_k^-)-\alpha)^2}{\gamma_i^R} + \beta\sum_{i=1}^{n}\int_{t_k-r}^{t_k} e_i^*(s)e_i(s)ds,
\end{aligned}
$$

which implies

$$
V(t_k)\big|_{(6.67)} < V(t_k^-)\big|_{(6.67)}, \qquad k \in \mathbb{N}^+. \tag{6.81}
$$

Assuming system (6.67) and system (6.74) have the same initial condition, then it follows from (6.80) that

$$
V(t)\big|_{(6.67)} = V(t)\big|_{(6.74)}, \qquad t \in [t_0, t_1). \tag{6.82}
$$

In the following, we shall prove that

$$V(t)\big|_{(6.67)} < V(t)\big|_{(6.74)}, \qquad t \in [t_k, t_{k+1}),\ k \in \mathbb{N}^+. \tag{6.83}$$

At $t = t_1$, we have from (6.81) and (6.82) that

$$V(t_1)\big|_{(6.67)} < V(t_1^-)\big|_{(6.67)} = V(t_1^-)\big|_{(6.74)} = V(t_1)\big|_{(6.74)}. \tag{6.84}$$

Combing (6.84) with (6.80), we can derive

$$V(t)\big|_{(6.67)} < V(t)\big|_{(6.74)}, \qquad t \in [t_1, t_2),$$

which implies (6.83) holds for $k = 1$. Suppose that (6.83) is true for $k = m$ $(m > 1)$, i.e.,

$$V(t)\big|_{(6.67)} < V(t)\big|_{(6.74)}, \qquad t \in [t_m, t_{m+1}), \tag{6.85}$$

then at $t = t_{m+1}$, we can obtain from (6.81) and (6.85) that

$$V(t_{m+1})\big|_{(6.67)} < V(t_{m+1}^-)\big|_{(6.67)} < V(t_{m+1}^-)\big|_{(6.74)} = V(t_{m+1})\big|_{(6.74)}.$$

According to (6.80), we have

$$V(t)\big|_{(6.67)} < V(t)\big|_{(6.74)}, \qquad t \in [t_{m+1}, t_{m+2}),$$

which implies that (6.83) holds for $k = m + 1$. Therefore, (6.83) is true by induction. Combing (6.82) with (6.83), we have

$$V(t)\big|_{(6.67)} \le V(t)\big|_{(6.74)}, \qquad \forall t \ge t_0. \tag{6.86}$$

It follows from (6.79) and (6.86) that

$$\lim_{t\to\infty} V(t)\big|_{(6.67)} \le \lim_{t\to\infty} V(t)\big|_{(6.74)} = \nu. \tag{6.87}$$

We claim

$$\lim_{t\to\infty} |e_i(t)|\big|_{(6.67)} = 0, \qquad i = 1, 2, ..., n. \tag{6.88}$$

If not, then there exists $i^* \in \{1, 2, \ldots, n\}$ such that

$$\lim_{t\to\infty} |e_{i^*}(t)|\big|_{(6.67)} > 0.$$

From (6.66), we have $\dot{d}_{i^*}^R = \gamma_{i^*}^R |e_{i^*}(t)|^2\big|_{(6.67)} > 0$, which implies $\lim_{t\to\infty} d_{i^*}^R(t) = \infty$. Since $\alpha, \gamma_{i^*}^R$ are constants, we can derive $\lim_{t\to\infty} \frac{(d_{i^*}^R(t)-\alpha)^2}{\gamma_{i^*}^R} = \infty$, hence $\lim_{t\to\infty} V(t)\big|_{(6.67)} = \infty$, which contradicts with (6.87). This shows that (6.88) is true, which implies that the zero solution of error system (6.67) is asymptotically stable. Therefore, under the updating law (6.59), the adaptive impulsive observer (6.58) can estimate the state of CVNN (6.1). □

Remark 6.4.1. *From (6.78), we have $\dot{V}\big|_{(6.74)} \leq 0$. The largest invariant set of $\{(E^R, E^I, d^R, d^I) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \dot{V}\big|_{(6.74)} = 0\}$ is $\{(E^R, E^I, d^R, d^I) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : E^R = 0, E^I = 0, \dot{d}^R = 0, \dot{d}^I = 0\}$, where $E^R = \mathfrak{Re}\big(E(t)\big), E^I = \mathfrak{Im}\big(E(t)\big), d^R = ({d_1}^R, {d_2}^R, \ldots, {d_n}^R)^T$, and $d^I = ({d_1}^I, {d_2}^I, \ldots, {d_n}^I)^T$. Based on the LaSalle's invariance principle, for arbitrary initial conditions, we can get $\lim_{t\to\infty} |e_i(t)|\big|_{(6.74)} = 0$, $\lim_{t\to\infty} d_i^R(t)\big|_{(6.74)} = \tilde{d}_i^R$, and $\lim_{t\to\infty} d_i^I(t)\big|_{(6.74)} = \tilde{d}_i^I$, for $i = 1, 2, \ldots, n$, where $\tilde{d}_i^R, \tilde{d}_i^I$ are constants. Therefore, using the updating law (6.59), observer (6.58) can estimate the state of CVNN (6.1) without impulses.*

Corollary 6.4.1. *Suppose that Assumption 6.4.2 holds. The state of CVNN (6.1) can be estimated by using the following adaptive observer and updating laws:*

$$\begin{cases} \dot{\hat{z}}_i(t) = -c_i\hat{z}_i(t) + \sum_{j=1}^{n} a_{ij} f_j(\hat{z}_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(\hat{z}_j(t-r)) + J_i - d_i(t)\big(\hat{z}_i(t) - z_i(t)\big), \\ \dot{d}_i = \gamma_i |e_i(t)|^2, \end{cases} \tag{6.89}$$

for $i = 1, 2, ..., n$, where $\gamma_i = \gamma_i^R + j\gamma_i^I \in \mathbb{C}$, and $\gamma_i^R > 0$.

Remark 6.4.2. *According to Theorem 6.4.2 and Corollary 6.4.1, observers (6.58) and (6.89) can both estimate the state of CVNN* (6.1) *under the adaptive law (6.59), while condition* (6.75) *in Theorem 6.4.2 implies that the impulses would enhance the stability of the error system. Thus compared with the impulse-free adaptive observer (6.89), the convergence speed of the state estimation error would be faster by using the adaptive impulsive observer (6.58).*

6.4.3 Numerical Simulation

Example 6.4.1. *Consider the following three-neuron CVNN*

$$\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^{3} a_{ij} f_j(z_j(t)) + \sum_{j=1}^{3} b_{ij} g_j\big(z_j(t-0.2)\big) + J_i, \quad i = 1, 2, 3, \tag{6.90}$$

where $C = diag\{c_1, c_2, c_3\} = diag\{2, 3, 6\}$, $f_i(z_i) = \frac{1}{1+e^{-\bar{z}_i}}$, $g_i(z_i) = \frac{1-e^{-\bar{z}_i}}{1+e^{-\bar{z}_i}}$ *for* $i = 1, 2, 3$,

$$A = (a_{ij})_{3\times 3} = \begin{pmatrix} -2-3j & 4+0.5j & 1+j \\ 2.4+0.6j & 2+j & -1+3j \\ 1-j & -1+2j & 5-2j \end{pmatrix},$$

$$B = (b_{ij})_{3\times 3} = \begin{pmatrix} 1+j & -1.5+3.5j & -1+j \\ 2-3j & 6+2.5j & 1-j \\ 5+2j & 1+j & -1-2j \end{pmatrix},$$

and $J = [J_1, J_2, J_3]^T = [1+4j, -3+j, 2.5-5j]^T$. *Then it is easy to check that the complex-valued activation functions* $f_i(\cdot)$ *and* $g_i(\cdot)$ *satisfy Assumption 6.4.2.*

Consider the following adaptive impulsive observer:

$$\begin{cases} \dot{\hat{z}}_i(t) = -c_i\hat{z}_i(t) + \sum_{j=1}^{3} a_{ij}f_j(\hat{z}_j(t)) + \sum_{j=1}^{3} b_{ij}g_j\big(\hat{z}_j(t-0.2)\big) + J_i \\ \qquad\qquad - d_i(t)\big(\hat{z}_i(t) - z_i(t)\big), \qquad t \neq t_k, \\ \hat{z}_i(t) = \hat{z}_i(t^-) + L\big(\hat{z}_i(t^-) - z_i(t^-)\big), \qquad t = t_k,\ k \in \mathbb{N}^+, \quad i = 1, 2, 3 \end{cases} \tag{6.91}$$

with updating law

$$\dot{d}_i(t) = \gamma_i|\hat{z}_i(t) - z_i(t)|^2, \quad i = 1, 2, 3, \tag{6.92}$$

where $\gamma_1 = 3-2j, \gamma_2 = 5-j$, *and* $\gamma_3 = 0.8-4j$ *satisfying* $\gamma_i^R > 0$ *for* $i = 1, 2, 3$. *Choose the impulsive observer gain* L *as* $L = -0.6-0.8j$, *then condition (6.75) in Theorem 6.4.2 holds. Theorem 6.4.2 implies that the adaptive impulsive observer (6.91) estimates the state of CVNN (6.90) under the updating law (6.92).*

Choose the length of each impulse interval as $t_k - t_{k-1} = 0.5$, $k \in \mathbb{N}^+$. *The initial conditions of CVNN (6.90) and observer (6.91) are randomly selected as* $[z_1(s), z_2(s), z_3(s)]^T = [-0.5+j, 2-3j, -2+j]^T$ *and* $[\hat{z}_1(s), \hat{z}_2(s), \hat{z}_3(s)]^T = [3-2j, 3+j, 0.2-0.6j]^T$ *for* $s \in [-0.2, 0]$, *respectively, and the the initial feedback strengths* $d_i(0)$, $i = 1, 2, 3$, *are chosen as* $d_{10} = 1-2j, d_{20} = 4+6j$, *and* $d_{30} = 2-5j$. *Figure 6.12 shows the trajectories of the real and imaginary parts for the states of neurons in CVNN (6.90) and the observation states of (6.91). It can be seen from the results of simulations in Figure 6.12 that the adaptive impulsive observer (6.91) estimates the states of neurons in CVNN (6.90).*

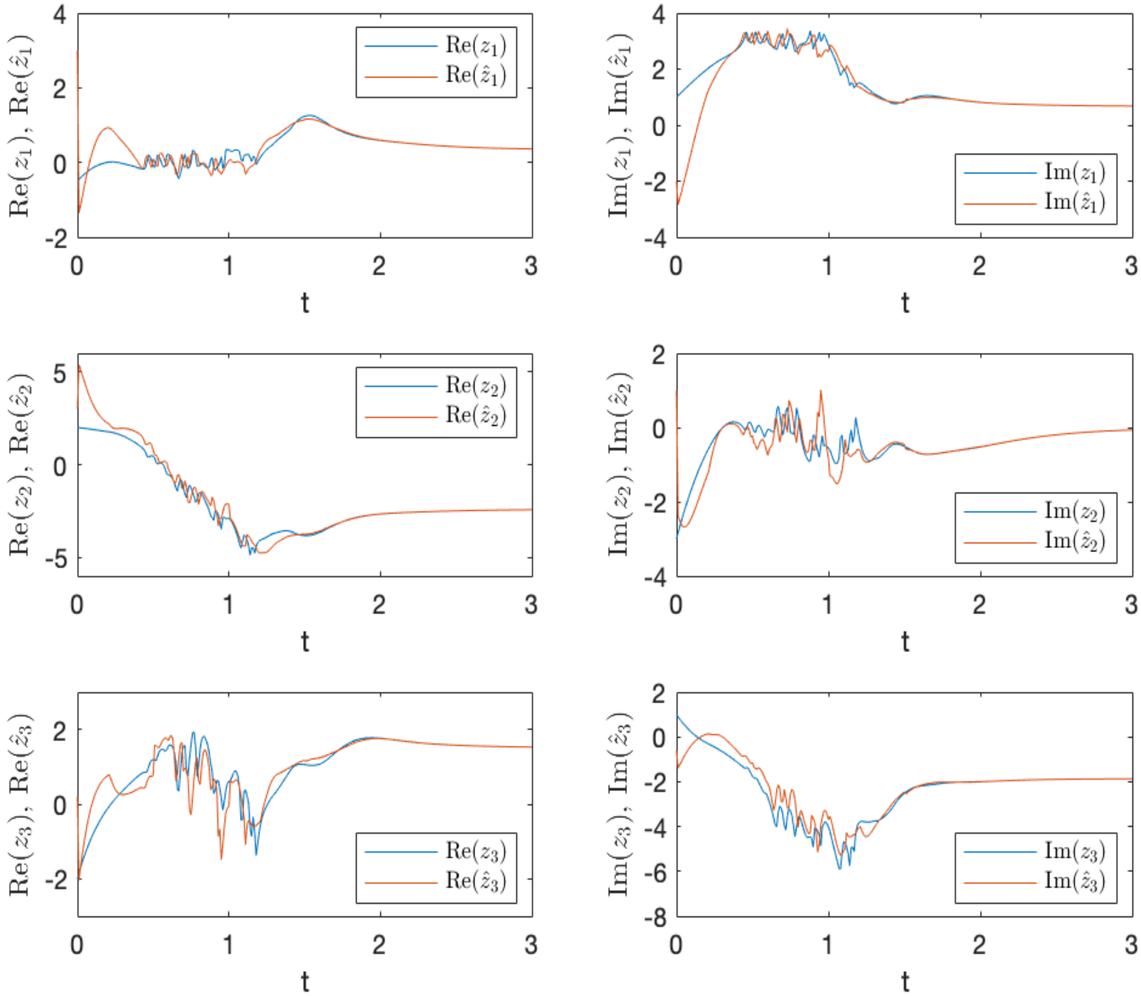

Figure 6.12: Real and imaginary parts of state trajectories of neurons in CVNN (6.90) and state trajectories of the observer (6.91).

Chapter 7

Conclusions and Future Research

In this chapter, we summarize the results of this thesis and suggest possible future work related to the topics that we have studied in the thesis.

In the present thesis, we have investigated the synchronization problem of complex-valued dynamical networks. The average consensus problem related to complex-valued networked systems has also been studied.

Chapter 3 has studied the synchronization problem of a single CVDN. In Section 3.2, we have applied a pinning impulsive controller with delay effects to study the synchronization of time-delay CVDNs. By taking advantage of Lyapunov function/functional in the complex field, some new synchronization criteria for CVDNs have been established, which not only generalize the synchronization results reported in the existing literature but also greatly reduce the complexity of analysis and computation. Furthermore, our results are applicable to CVDNs and pinning impulsive controllers with various sizes of delays. Numerical examples have been provided to demonstrate the theoretical results. In Section 3.3, we have investigated the synchronization of CVDNs with time-varying coupling delays using distributed impulsive control. We have considered two types of time-varying coupling delays: 1) the delay is bounded but has no restriction on the delay derivative; 2) the delay is bounded and its derivative is strictly less than one. By applying a time-varying Lyapunov function/functional approach in the complex domain, some synchronization criteria have been established in terms of complex-valued LMIs. The introduced time-varying Lyapunov function/functional is related to the impulsive sequence, thus it can capture more characteristics of network nodes' dynamics, which leads to less conservative results. In Section 3.5, we have proposed a novel type of memory-based ETIC scheme with three levels of events in the complex field to study synchronization of CVDNs with discrete and distributed time delays. The event-based impulsive controllers depend on the cumulative information of synchronization error states of network nodes in the complex domain. Sufficient conditions for designing event-based impulsive controllers have been established to achieve synchronization among the node states and the objective state. By considering the advantages of pinning control, we have further proposed an ETPIC scheme in Section 3.6 combining the ETIC scheme in Section 3.5 and a pinning algorithm to study the synchronization of CVDNs with discrete and distributed time delays. Results in Section 3.5 and Section 3.6 show that the proposed ETIC/ETPIC scheme can effectively synchronize CVDNs with the desired trajectory by suitably designing impulsive control gains in the complex domain.

Chapter 4 has studied the generalized outer synchronization problem of time-delay CVDNs. A hybrid controller has been designed in the complex domain to construct response complex-valued networks. The proposed hybrid controller can simultaneously permit synchronizing as well as desynchronizing impulses in one impulsive sequence. By using the concepts of average impulsive interval and average impulsive gain, some sufficient conditions have been established which guarantee the generalized outer synchronization of drive-response CVDNs. Results in Chapter 4 show that generalized outer synchronization of CVDNs can be realized under the proposed hybrid controller even if the impulsive sequence contains desynchronizing impulses.

Chapter 5 has investigated the average-consensus problem of potential complex-valued multi-agent systems. By considering the continuous-time communication among agents and the instantaneous information exchange at discrete-time instants, a complex-variable hybrid consensus protocol which composed of continuous-time protocol and impulsive protocol has been designed for achieving the average-consensus of complex-valued multi-agent systems, and the time-delay has been taken into account in both continuous-time and discrete-time protocols. By employing a Lyapunov functional in the complex field and results from graph theory, sufficient conditions on the relation among interaction topologies, the sizes of delays, and the length of impulsive intervals have been established to guarantee the proposed complex-variable hybrid protocol leads to the average-consensus. Based on the delay size of the continuous-time protocol, our results show that complex-valued networked multi-agent systems can achieve average-consensus if network topologies of continuous-time and discrete-time protocols and impulsive sequences are suitably designed.

As a practical application of complex-valued networked systems, CVNN has become an emerging research topic in the most recent years. In Chapter 6, the synchronization problem of CVNNs with time-delay has been studied. In Section 6.2, we have proposed a hybrid controller which consists of sampled-data controller and impulsive controller in the complex field. Some synchronization criteria for CVNNs have been obtained, and the hybrid controller has been designed based on the established complex-valued LMIs. Our results show that the complex-valued hybrid controller can successfully synchronize the slave CVNN with the master CVNN. In Section 6.3, we have proposed a delayed ETPIC scheme by taking into account of time-delay effect when impulse sampling takes place. Some sufficient conditions for designing appropriate event-based pinning impulsive controllers with discrete delay have been established to guarantee the synchronization of time-delay CVNNs can be achieved. Furthermore, the state estimation problem of CVNNs has been investigated in Section 6.4 by designing the adaptive impulsive observer and the updating law in the complex field.

In the present thesis, we have used various types of control strategies to study the synchronization problem of CVDNs, including impulsive control, event-triggered control, pinning control, sampled-data control, and adaptive control. Future work could be done on synchronization of CVDNs via some other kinds of control, such as intermittent control, sliding mode control, etc. On the other hand, in practical engineering, realizing network synchronization in a desired finite time is more valuable. Therefore, future research could be done on studying the finite-time synchronization problem of CVDNs. Furthermore, throughout the thesis, we studied synchronization of CVDNs with fixed topologies, i.e., the outer coupling matrix of CVDNs is assumed to be time-invariant. However, it is more practical to consider network topologies change with time in the synchronization process. In the future, it might be possible to study the synchronization problem of CVDNs with switching topologies and examine how the connection topology influences the synchronization of CVDNs.

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