

GOLDBERG'S CONJECTURE IS TRUE FOR RANDOM MULTIGRAPHS

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ABSTRACT. In the 70s, Goldberg, and independently Seymour, conjectured that for any multigraph G , the chromatic index $\chi'(G)$ satisfies $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \rho(G) \rceil\}$, where $\rho(G) = \max\{\frac{e(G[S])}{\lfloor |S|/2 \rfloor} \mid S \subseteq V\}$. We show that their conjecture (in a stronger form) is true for random multigraphs. Let $M(n, m)$ be the probability space consisting of all loopless multigraphs with n vertices and m edges, in which m pairs from $[n]$ are chosen independently at random with repetitions. Our result states that, for a given $m := m(n)$, $M \sim M(n, m)$ typically satisfies $\chi'(G) = \max\{\Delta(G), \lceil \rho(G) \rceil\}$. In particular, we show that if n is even and $m := m(n)$, then $\chi'(M) = \Delta(M)$ for a typical $M \sim M(n, m)$. Furthermore, for a fixed $\varepsilon > 0$, if n is odd, then a typical $M \sim M(n, m)$ has $\chi'(M) = \Delta(M)$ for $m \leq (1 - \varepsilon)n^3 \log n$, and $\chi'(M) = \lceil \rho(M) \rceil$ for $m \geq (1 + \varepsilon)n^3 \log n$. To prove this result, we develop a new structural characterization of multigraphs with chromatic index larger than the maximum degree.

Keywords: chromatic index, edge colouring, random graphs, random multigraphs.

1. INTRODUCTION

For a (multi)graph $G = (V, E)$, a k -edge-colouring of G is a function $c : E \rightarrow [k]$ where $[k] = \{1, \dots, k\}$, such that $c(e) \neq c(f)$ whenever e and f share a vertex. We denote by $\chi'(G)$ the minimum k such that G has a k -edge-colouring. Since no multigraph with a loop has a k -edge-colouring for any k , we will assume throughout this paper that all our multigraphs are loopless. It is clear that for every multigraph G , the maximum degree $\Delta(G)$ of G is a lower bound on $\chi'(G)$.

For (simple) graphs, Vizing's Theorem [32] tells us that $\chi'(G) \leq \Delta(G) + 1$ for every graph G . We say that a graph G is *Class 1* if $\chi'(G) = \Delta(G)$, and *Class 2* if $\chi'(G) = \Delta(G) + 1$. The problem of determining whether an arbitrary graph is Class 1 is known to be NP-hard [18], and there has been extensive research regarding the conditions under which a graph is Class 1 or Class 2. One of the tools to attack this problem is the following theorem, also due to Vizing.

Theorem 1.1 (Vizing [32]). *If G is a simple graph with maximum degree Δ such that every cycle of G contains a vertex of degree less than Δ , then $\chi'(G) = \Delta$.*

In order to distinguish this result from Vizing's classical theorem that $\chi'(G) \leq \Delta(G) + 1$, we will always refer to this statement as Theorem 1.1.

In the case of multigraphs, the picture is less dichotomous. For a multigraph $G = (V, E)$ let $\mu(e)$ denote the multiplicity of the pair $e = \{u, v\} \in \binom{V}{2}$ in G , and let $\mu(G) = \max\{\mu(e) \mid e \in \binom{V}{2}\}$ be the maximum edge multiplicity of G . Vizing's Theorem [32] for multigraphs states that $\chi'(G) \leq \Delta(G) + \mu(G)$. In some cases, the chromatic index can be quite far from the maximum degree. The classical theorem of Shannon [29] states that $\chi'(G) \leq \lceil 3\Delta/2 \rceil$, and this is best possible as can be seen from the graph with three vertices and $\lceil \Delta/2 \rceil$ or $\lfloor \Delta/2 \rfloor$ edges joining each pair of vertices. In this graph every two edges share a vertex, so we need $\lceil 3\Delta/2 \rceil$ colours to colour the edges of G .

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Since Theorem 1.1 does not apply to multigraphs, it is natural to try and find another graph theoretic parameter (besides Δ and μ) connected to the chromatic index of a multigraph. A famous conjecture due to Goldberg (1973), and also Seymour (1979), is based on the following parameter. For a multigraph $G = (V, E)$ and $S \subseteq V$, let $\rho(S) := \frac{e(G[S])}{\lfloor |S|/2 \rfloor}$. Define $\rho(G) = \max\{\rho(S) \mid S \subseteq V\}$. Then $\chi'(G) \geq \lceil \rho(G) \rceil$, as can be seen from the following argument. For any subset $S \subseteq V(G)$, every matching in $G[S]$ has size at most $\lfloor |S|/2 \rfloor$. Since every colour class forms a matching, we need at least $\frac{e(G[S])}{\lfloor |S|/2 \rfloor}$ colours to colour $G[S]$ alone. This observation gives a lower bound of $\max\{\Delta(G), \lceil \rho(G) \rceil\}$ on the chromatic index of any multigraph G .

The prevalent belief is that the chromatic index of multigraphs should essentially be determined by $\Delta(G)$ and $\rho(G)$. Goldberg [14] and independently Seymour [27] conjectured the following.

Conjecture 1.2. *Let G be a loopless multigraph. Then $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \rho(G) \rceil\}$.*

Goldberg [15] even conjectured that if $\lceil \rho(G) \rceil \leq \Delta(G) - 1$ then $\chi'(G) = \Delta(G)$. On the other hand, for the case $\Delta(G) \leq \rho(G)$, Kahn proved [21] that $\chi'(G) \leq (1 + o(1))\lceil \rho(G) \rceil$. It was also independently conjectured by Andersen [2] and Seymour [28] that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1, \lceil \rho(G) \rceil\}$. In the past few years, there were several improvements for the upper bound on $\chi'(G)$ in terms of $\rho(G)$, see, e.g., [6, 14, 15, 16, 24, 26, 27, 31]. In [6], and independently [26], it was shown that $\chi'(G) \leq \max\{\Delta + \sqrt{\frac{\Delta}{2}}, \lceil \rho(G) \rceil\}$. A very recent breakthrough by Chen, Gao, Kim, Postle, and Shan [5] proves the best known upper bound of $\chi'(G) \leq \max\{\Delta + \sqrt[3]{\frac{\Delta}{2}}, \lceil \rho(G) \rceil\}$. For a more thorough history of this important problem see [30].

One of the main questions in the area of edge colouring is to understand which multigraphs have an upper bound for the chromatic index that matches the trivial lower bound of $\max\{\Delta, \lceil \rho \rceil\}$. This leads to a classification of multigraphs with respect to the chromatic index. Similarly to the graph case, we would like to distinguish between multigraphs whose chromatic index is the same as the trivial lower bound, and those that do not have this property. We say that a multigraph G is *first class* if $\chi'(G) = \max\{\Delta(G), \rho(G)\}$, and otherwise it is *second class*. As proposed in [30], it seems natural to expect that almost all multigraphs are first class (see p.186 in [30] for further discussion).

In this paper we will discuss this problem in the setting of *random graphs* and *random multigraphs*. The two most common random graph models are the Erdős-Rényi model $G(n, m)$ (which is the probability space consisting of all graphs with n labeled vertices and m edges, in which m different elements from $\binom{[n]}{2}$ are chosen uniformly, one by one, without repetitions), and the binomial random graph model $G(n, p)$ (the probability space consisting of all graphs with n labeled vertices, where each one of the $\binom{[n]}{2}$ possible edges is included independently with probability p). For the most important case $p = \frac{1}{2}$, Erdős and Wilson proved [9] that a typical $G \sim G(n, \frac{1}{2})$ is Class 1. They did it by showing that with high probability (w.h.p.) $G \sim G(n, \frac{1}{2})$ contains a unique vertex of maximum degree and then by invoking Vizing's Theorem (Theorem 1.1). Frieze, Jackson, McDiarmid and Reed [11] strengthened this result for every constant p , $0 < p < 1$, and showed that the probability that $G \sim G(n, p)$ is Class 1 tends to one extremely quickly (concretely, is equal to $1 - n^{-\Theta(n)}$).

Here we first show that the Erdős-Wilson result can be extended to sparse random graphs.

Theorem 1.3. *Let n be sufficiently large integer and $p := p(n) = o(1)$. Then w.h.p. $G \sim G(n, p)$ is Class 1.*

A similar proof will lead to the same result for the model $G(n, m)$, where $m = o(n^2)$.

After having dealt with the chromatic index of random graphs, the next question to ask is if the situation remains the same in random multigraphs. In particular, how does the chromatic index of a random multigraph G behave typically with respect to the key parameters $\Delta(G)$ and $\rho(G)$? For this, we need first to set up a probability space for sampling multigraphs. A natural way to define a random multigraph model is to allow edge repetitions in the standard random graph model $G(n, m)$. Let n and m be integers. The model $M(n, m)$ is the probability space consisting of all loopless multigraphs

with n vertices and m edges, in which m elements from $\binom{[n]}{2}$ are chosen independently at random with repetitions.

As far as we know, no work has been done to determine the chromatic index of a random multigraph (in any random multigraph model). For further discussion on $M(n, m)$ and other models of random multigraphs see Section 5. In this paper, we prove that the chromatic index of a typical $M \sim M(n, m)$ is either $\Delta(M)$ or $\lceil \rho(M) \rceil$.

Theorem 1.4. *Let n be an integer and $m := m(n)$. Let $M \sim M(n, m)$. Then w.h.p. $\chi'(M) = \max\{\Delta(M), \lceil \rho(M) \rceil\}$.*

Thus we essentially confirm the informal conjecture stated in [30] (see p.186), and in fact show that Conjecture 1.2 holds for random multigraphs, even in a stronger form.

In order to determine χ' precisely, we distinguish between two cases, the case where n is even and where n is odd.

Theorem 1.5. *Let n be an even integer and let $m := m(n)$. Let $M \sim M(n, m)$, then w.h.p. $\chi'(M) = \Delta(M)$.*

The following is a corollary of Theorem 1.4 (here and later, we denote by $\log n$ the natural logarithm).

Corollary 1.6. *Let $\varepsilon > 0$, and let $M \sim M(n, m)$ where n is an odd integer. Then the following hold.*

- (1) *If $m \leq (1 - \varepsilon)n^3 \log n$ then w.h.p. $\chi'(M) = \Delta(M)$.*
- (2) *If $m \geq (1 + \varepsilon)n^3 \log n$ then w.h.p. $\chi'(M) = \lceil \rho(M) \rceil$.*

For proving the above theorems, we use Vizing's Theorem (Theorem 1.1) and properties of random (multi)graphs. For larger values of m , we make an extensive use of the method of *Tashkinov trees* (see Section 2.1). This method was introduced by Tashkinov [31] in 2000 to address Conjecture 1.2. In particular, we give a new necessary condition, of independent interest, for a multigraph to have a large chromatic index in terms of the two largest degrees and the min/max edge-multiplicities.

Remark 1.7. *Our results are in fact algorithmic in the sense that there exists an efficient (polynomial time) algorithm that w.h.p. finds an optimal edge-colouring in a random multigraph drawn from $M(n, m)$. We refer the reader to Remark 2.4 and Section 4.5 for more details.*

1.1. Notation and terminology. For every positive integer k we use $[k]$ to denote the set $\{1, 2, \dots, k\}$. We also write $x \in y \pm z$ for $x \in [y - z, y + z]$. To avoid confusion, we usually use e for an edge in the (multi)graph, and e for the Euler's constant.

Our graph-theoretic notation is standard and follows that of [33]. In particular, we use the following.

In a multigraph $G = (V, E)$, V is the set of vertices, and E is a multiset of elements from $\binom{V}{2}$. Let $d_G(v) = |\{e \in E(G) \mid v \in e\}|$ denote the degree of v in G (including multiplicities). We denote by $\Delta(G)$ and $\delta(G)$ the maximum degree and the minimum degree in G , respectively. We let $\mu(G) := \mu_{\max}(G)$ be the maximum edge-multiplicity of G , and $\mu_{\min} := \mu_{\min}(G)$ the minimum edge-multiplicity of G .

For a set of vertices $U \subseteq V(G)$, we denote by $G[U]$ the corresponding vertex-induced subgraph of G , we denote by $E_G[U]$ the edges of $G[U]$, and write $e_G(U) = |E_G[U]|$. For two vertex sets $U, W \subseteq V$ we denote by $E_G[U, W]$ all the edges $e \in E$ with both endpoints in $U \cup W$ for which $e \cap U \neq \emptyset$ and $e \cap W \neq \emptyset$. Let $e_G(U, W) = |E_G(U, W)|$. Often, when there is no risk of ambiguity, we omit the subscript G in the above notation.

For a graph G and an edge $e \in E(G)$ (respectively, a vertex $v \in V(G)$) we write $G - e$ (respectively, $G - v$) to refer to the subgraph of G obtained by removing the edge e (respectively, the subgraph of G obtained by removing the vertex v). A multigraph G is said to be *k-critical* if $\chi'(G) = k + 1$ and $G - e$ is k -edge-colourable for every edge e . For an edge colouring $\phi : E \rightarrow C$ of $G = (V, E)$, we say

that a colour c is *missing* at a vertex $v \in V$ if $v \notin e$ for every $e \in E$ such that $\phi(e) = c$. We say that c is a *partial edge colouring* if the domain of ϕ is a (proper) subset of E (that is, not all the edges are coloured according to ϕ). We denote by $E(\phi)$ the set of edges of G that are coloured by ϕ , and write $|\phi| = |E(\phi)|$.

We assume that n is large enough where needed. We say that an event holds *with high probability* (w.h.p.) in an underlying probability space if its probability tends to one as n tends to infinity. We sometimes omit floor and ceiling signs whenever these are not crucial.

1.2. Organization of the paper. In the next section we present some auxiliary results, definitions and technical preliminaries; in particular, in Section 2.1 we present the tool of Tashkinov trees and prove useful lemmas regarding the chromatic index of a (general) multigraph, and in Section 2.3 we prove some properties of random graphs and random multigraphs. In Section 3 we prove Theorem 1.3. In Section 4 we prove Theorems 1.4, 1.5, and Corollary 1.6. In Section 4.5 we discuss the algorithmic issues mentioned in Remark 1.7, and in Section 5 we make some concluding remarks.

2. TOOLS

2.1. Sufficient conditions for k -colourability and Tashkinov trees. The aim of this section is to prove the following theorem that gives a sufficient condition (in terms of the min/max edge-multiplicities, the two largest degrees, and the number of edges) for a multigraph to be k -colourable. Its proof depends heavily on the method of Tashkinov trees.

Theorem 2.1. *Let G be a multigraph with an odd number n of vertices and degree sequence $d_1 \geq d_2 \geq \dots \geq d_n$, minimum multiplicity μ_{\min} and maximum multiplicity μ . Suppose $\chi'(G) > k$. Then one of the following holds.*

- (a) $k \leq d_1 - 1$,
- (b) $k \leq d_2 + 2$,
- (c) $9\mu - 24 > 10\mu_{\min}$,
- (d) $|E(G)| > \frac{(n-1)}{2}k$.

First we describe the method of Tashkinov trees from [31]. Let G be a multigraph, and let ϕ be a partial $(\chi' - 1)$ -edge colouring of G (i.e. a $(\chi' - 1)$ -edge colouring of a subgraph of G). Let $\bar{\phi}(w_j)$ be the set of colours missing at w_j according to ϕ . Let $T = (w_0, e_0, w_1, \dots, w_q)$ be a sequence of distinct vertices w_i and edges e_i of G , such that each e_i has ends w_{i+1} and w_k for some $k \in \{0, \dots, i\}$. Note that T is a tree. We say that T is a ϕ -Tashkinov tree if e_0 is uncoloured, and for all $i > 0$,

$$\phi(e_i) \in \mathcal{M}_{T_i, \phi},$$

where $T_i := (w_0, \dots, w_i)$ and $\mathcal{M}_{T_i, \phi} = \bigcup_{w_j \in T_i} \bar{\phi}(w_j)$ is the set of colours missing at some vertex of T_i . In other words, T is a ϕ -Tashkinov tree if its first edge is uncoloured, and after that, each edge is coloured with a colour that is missing at a previous vertex. We say that a set S of vertices is ϕ -*elementary* if no colour α is missing at two distinct vertices of S . The key property of Tashkinov trees, due to Tashkinov [31], is that if $|\phi|$ is maximum then every ϕ -Tashkinov tree is ϕ -elementary.

In [30] (Theorem 5.1), Tashkinov's Theorem [31] is stated as follows.

Theorem 2.2. *Let G be a multigraph with $\chi'(G) = k + 1$, where $k \geq \Delta(G) + 1$. Suppose e_0 is an edge such that $\chi'(G - e_0) = k$, and let ϕ be a k -edge-colouring of $G - e_0$. If T is a ϕ -Tashkinov tree starting with e_0 then $V(T)$ is ϕ -elementary.*

Here we will use a slightly nonstandard statement of Tashkinov's Theorem that is tailored for our purposes.

Theorem 2.3. *Let G be a multigraph with $\chi'(G) = k + 1$, where $k \geq \Delta(G)$. Suppose e_0 is an edge such that $\chi'(G - e_0) = k$, where the endpoints x and y of e_0 satisfy $d(x) + d(y) \leq 2k - 2$, and suppose that $d(v) \leq k - 1$ for each $v \in V(G) \setminus \{x, y\}$. Let ϕ be a k -edge-colouring of $G - e_0$. If T is a ϕ -Tashkinov tree starting with e_0 then $V(T)$ is ϕ -elementary.*

Here we indicate precisely how to modify the proof of Tashkinov's Theorem as it is written in [30] to obtain Theorem 2.3.

Proof. The condition $k \geq \Delta(G) + 1$ is used in the proof of Theorem 5.1 in [30] only in the following places.

- (1) To ensure that if T is a path then Kierstead's Theorem (Theorem 3.1 in [30]) applies (p.117 line 11). However, the assumptions of Kierstead's Theorem (p.44) only require that $k \geq \Delta(G)$ and that each vertex of T apart from the endpoints of e_0 has degree less than k .
- (2) To ensure that every vertex of G is missing at least one colour under ϕ (p.120 lines 2 and 22).
- (3) To ensure Claim (d) (p.117) in the proof of Theorem 5.1 in [30], which states that if T' is a proper initial segment of T with at least two vertices then there are at least four colours missing and not used on T' . Here T' is itself ϕ -elementary by a minimality assumption on T .

The assumptions in our statement Theorem 2.3 of Tashkinov's Theorem contain the assumptions for Kierstead's Theorem so (1) follows immediately. Statement (2) is also immediate for every $v \in V(G) \setminus \{x, y\}$, and follows for x and y as well since $k \geq \Delta(G)$ and both x and y are incident to the uncoloured edge e_0 . Thus it remains to verify that the assumptions of Theorem 2.3 ensure (3) holds.

Suppose $V(T') = \{w_0, w_1, \dots, w_{t+1}\}$ where $t \geq 0$. By the assumption $d(x) + d(y) \leq 2k - 2$, and since e_0 is uncoloured, there are at least $k - (d(x) - 1) + k - (d(y) - 1) \geq 2k + 2 - (2k - 2) = 4$ colours missing on the first two vertices of T' (which are all different, otherwise e_0 could be coloured). Each w_i with $i \geq 2$ is missing at least one colour by the assumption $d(w_i) \leq k - 1$, and since T' is ϕ -elementary all these $t + 4$ colours are distinct. The number of colours used on T is at most $|E(T)| - 1 = |V(T)| - 2 = t$. Hence at least four colours are missing and not used on T' , verifying (3). \square

Remark 2.4. *In the proof of Theorem 2.3 the condition that $\chi'(G) = k + 1$ can be replaced with the argument that for a colouring ϕ , if $V(T)$ is not ϕ -elementary then one can change the colouring ϕ to get a k -edge-colouring of G . The way to change ϕ is based on an alternating-paths argument and as a result Theorem 2.3 is in fact algorithmic. See Section 4.5 for more details.*

We also need the following simpler statement, which follows easily from known results, e.g. [7] (see also [30], Theorem 2.9). It can also be proved directly, with an argument similar to the proof of Vizing's Theorem. However, as a warm-up we show how to derive it from Theorem 2.3.

Lemma 2.5. *Let k be a positive integer. Let G be a multigraph with maximum degree at most k and maximum edge multiplicity at most μ . Suppose that every vertex of G except one has degree at most $k - t$ where $t \geq \max\{2, \mu\}$. Then $\chi'(G) \leq k$.*

Proof. Suppose on the contrary that $\chi'(G) \geq k + 1$. Note that every subgraph of G also satisfies the conditions of the lemma. Thus by removing edges one by one if necessary, we may assume that $\chi'(G) = k + 1$ but $\chi'(G - e) = k$ for every edge e , i.e. G is k -critical.

The conditions tell us that G has at most one vertex of degree larger than $k - t$. If such a vertex exists, let us call it x and choose an edge e_0 incident to x in G to be the uncoloured edge. Otherwise we can choose e_0 to be an arbitrary edge and set x to be one of its endpoints.

Let ϕ be a k -edge-colouring of $G - e_0$. Let y denote the other endpoint of e_0 . Then the degree assumption implies that $d(x) + d(y) \leq k + k - t \leq 2k - 2$ and $d(v) \leq k - 1$ for each $v \in V(G) \setminus \{x, y\}$. Let T be a maximal ϕ -Tashkinov tree starting with e_0 . Thus the assumptions of Theorem 2.3 are satisfied, and hence T is ϕ -elementary.

Clearly $|V(T)| \geq 2$. Note that since every vertex different from x has degree at most $k - t$, each has at least t missing colours. Since also $y \in V(T - x)$ is incident to the uncoloured edge e_0 in $G[V(T)]$,

the total number of colours missing at vertices of $T - x$ is at least $t(|V(T)| - 1) + 1$. Since T is ϕ -elementary these are all distinct.

Since T is maximal we may conclude that for every colour $\alpha \in \mathcal{M}_{T,\phi}$, no edge e joining $V(T)$ to $V(G) \setminus V(T)$ is such that $\phi(e) = \alpha$. Therefore since T is ϕ -elementary, all colours in $\mathcal{M}_{T-x,\phi}$ appear on edges of $G[V(T)]$ incident to x . Therefore $d_{V(T)}(x) \geq t(|V(T)| - 1) + 1$. But clearly $d_{V(T)}(x) \leq \mu(|V(T)| - 1)$, so we find $\mu > t$, contradicting the assumption of the lemma. Hence $\chi'(G) \leq k$. \square

We apply Lemma 2.5 only in the case $k = \Delta(G)$, but for the purposes of the discussion of algorithmic issues in Section 4.5 it is more convenient to state it in the more general form given here.

Using the previous lemma, we can deduce the following upper bound for the chromatic index, assuming that the difference between the two largest degrees is bigger than the difference between the maximum and the minimum edge multiplicity.

Corollary 2.6. *Let G be a multigraph on n vertices with degree sequence $d_1 \geq d_2 \geq \dots \geq d_n$. Assume that $d_1 - d_2 \geq \mu(G) - \mu_{\min} \geq 2$, where $\mu_{\min} := \mu_{\min}(G)$ is the minimum multiplicity of G . Then $\chi'(G) = \Delta(G)$ if n is even and $\chi'(G) \leq \Delta(G) + \mu_{\min}$ if n is odd.*

Proof. We decompose the graph G into two multigraphs: $G_1 \cup G_2 = G$. Let G_1 be the complete multigraph on n vertices with all edge multiplicities μ_{\min} . If n is even, we have that $\chi'(K_n) = n - 1$ and therefore $\chi'(G_1) = (n - 1)\mu_{\min}$. If n is odd, we have that $\chi'(K_n) = n$ and therefore $\chi'(G_1) \leq n\mu_{\min}$.

Now let $E(G_2) = E(G) \setminus E(G_1)$ and $V(G_2) = V(G)$. Then $\mu(G_2) = \mu(G) - \mu_{\min}$. Denote by $d_1(G_2), d_2(G_2)$ the two largest degrees in G_2 . Then $d_1 - d_2 = d_1(G_2) - d_2(G_2)$. By the assumption we have that $d_1 - d_2 \geq \mu(G) - \mu_{\min}$, that is, $\mu(G_2) \leq d_1(G_2) - d_2(G_2)$. Thus, using Lemma 2.5 applied on G_2 with $k = \Delta(G_2)$ we have that $\chi'(G_2) = \Delta(G_2) = \Delta(G) - (n - 1)\mu_{\min}$.

All in all, we have that

$$\chi'(G) \leq \chi'(G_1) + \chi'(G_2) \leq n\mu_{\min} + \Delta(G) - (n - 1)\mu_{\min} = \Delta(G) + \mu_{\min},$$

when n is odd, and

$$\chi'(G) \leq \chi'(G_1) + \chi'(G_2) = (n - 1)\mu_{\min} + \Delta(G) - (n - 1)\mu_{\min} = \Delta(G),$$

when n is even. Since always $\Delta(G) \leq \chi'(G)$, the claim follows. \square

The remainder of this section is devoted to the proof of Theorem 2.1.

2.1.1. The structure of maximum Tashkinov trees. In order to prove Theorem 2.1 we use the following lemma which gives a more detailed picture of the structure of a multigraph with a maximum Tashkinov tree T . The set Q will be $V(T)$, and the other sets will be described in the discussion following the statement of the lemma.

Lemma 2.7. *Let G be a multigraph with degree sequence $d_1 \geq d_2 \geq \dots \geq d_n$. Suppose $\chi'(G) > k$ where $k \geq d_1$ and $k \geq d_2 + 2$. Then there exist vertex sets X, Z, Q and U in G with the following properties.*

- (1) $Z \subset Q \subseteq U$ and $X = V(G) \setminus U$,
- (2) $|Q|$ is odd and $|Q| \geq 2 + \sum_{z \in Z} (k - d_G(z))$,
- (3)

$$|E[Q]| > \frac{|Q| - 1}{2} (|E[X, Q \setminus Z]| + \sum_{v \in Q} (k - d_G(v)) + 2),$$

- (4) $\sum_{v \in U} (k - d_G(v)) \leq k - 2$. Moreover if n is odd and $|X| \leq 1$ then $\sum_{v \in V(G)} (k - d_G(v)) \leq k - 2$.

The rest of this subsection is devoted to the proof of Lemma 2.7. We begin by recalling some definitions. For a partial edge colouring ϕ of a multigraph G , a set S of vertices is said to be ϕ -elementary if no two vertices in S have a common missing colour under ϕ . A colour β is said to be used on a Tashkinov tree T if some edge of T is coloured β . (Note that any used colour is necessarily

missing at some vertex of T .) A colour γ is said to be *defective* for T if at least two edges coloured γ leave $V(T)$. We write D_T for the set of defective colours for T . For a set S and a colour γ we write $\Gamma_\gamma(S)$ for the set of vertices outside S that are joined to S via an edge coloured γ . A *max-pair* for G is a pair (ϕ, T) such that ϕ is a partial edge colouring with $|\phi|$ as large as possible, T is a ϕ -Tashkinov tree, and for all pairs (ψ, T') where ψ is a partial colouring with the same set of colours as ϕ , $E(\psi) = E(\phi)$, and T' is a ψ -Tashkinov tree, we have $|V(T)| \geq |V(T')|$.

Let G be as in the assumptions of Lemma 2.7. It suffices to prove the statement for $k = \chi'(G) - 1$.

Next we show that it suffices to prove Lemma 2.7 for k -critical multigraphs G . To see this, first note that if G' is a spanning subgraph of G then the assumptions $k \geq d_1$ and $k \geq d_2 + 2$ still hold for G' . Suppose we are able to prove that there exist sets Q, Z, U and X such that the conclusions of Lemma 2.7 hold for G' . Then it is clear that Conclusions (2) and (4) hold for G as well, since $d_G(v) \geq d_{G'}(v)$ for every vertex v . Conclusion (1) is the same for G and G' since $V(G) = V(G')$. For Conclusion (3), let $t = |E_G[X, Q \setminus Z]| - |E_{G'}[X, Q \setminus Z]|$. Then $\sum_{v \in Q} d_G(v) \geq \sum_{v \in Q} d_{G'}(v) + t$, and therefore

$$\begin{aligned} |E_G[Q]| &\geq |E_{G'}[Q]| > \frac{|Q| - 1}{2} (|E_{G'}[X, Q \setminus Z]| + \sum_{v \in Q} (k - d_{G'}(v)) + 2) \\ &\geq \frac{|Q| - 1}{2} (|E_G[X, Q \setminus Z]| - t + \sum_{v \in Q} (k - d_G(v)) + t + 2) \\ &= \frac{|Q| - 1}{2} (|E_G[X, Q \setminus Z]| + \sum_{v \in Q} (k - d_G(v)) + 2). \end{aligned}$$

This shows that we may assume (by removing edges of G one by one if necessary) that G is k -critical. As before, if there still exists a vertex of degree larger than $k - 2$ (which will be unique if it exists by the assumptions of the lemma) then we choose one of its incident edges to be the uncoloured edge e_0 . Otherwise e_0 can be chosen arbitrarily. Then the assumptions of Lemma 2.7 imply that the endpoints x and y of e_0 satisfy $d(x) + d(y) \leq d_1 + d_2 \leq 2k - 2$, and that $d(v) \leq k - 1$ for each $v \in V(G) \setminus \{x, y\}$. Thus at any point in the proof we may apply Theorem 2.3 to any k -edge-colouring ϕ of $G - e_0$ and any ϕ -Tashkinov tree T .

We choose ϕ and T such that (ϕ, T) is a max-pair starting with e_0 . The sets in the conclusion of Lemma 2.7 are defined as follows: $Q = V(T)$, $Z \subset Q$ is the set of vertices v in Q such that every colour missing at v is used on T (note $Z = \emptyset$ is possible), $U = Q \cup (\bigcup_{\gamma \in D_T} \Gamma_\gamma(Q \setminus Z))$, and $X = V(G) \setminus U$.

Our proof will consist of a series of lemmas. We remark that many of the results and ideas used here have appeared in other works (e.g. [6, 26, 30, 31]) but in order to keep this paper self-contained we include all proofs.

Lemma 2.8. *With these definitions, $|Q|$ is odd and $|Q| \geq 2 + \sum_{z \in Z} (k - d_G(z))$. Moreover every defective colour for T occurs on at least three edges leaving Q .*

Proof. Since Q contains the two endpoints of the uncoloured edge we know $|Q| \geq 2$. Since $k \geq d_1$, both endpoints of the uncoloured edge are missing at least one colour (and by maximality of ϕ no colour is missing at both endpoints), thus by definition of Tashkinov tree and by maximality $|Q| > 2$. By maximality of T , no edge coloured with a missing colour β leaves Q , so by Theorem 2.3 the β colour class in $G[Q]$ is a matching that misses exactly one vertex, i.e. $|Q|$ is odd (and hence at least three). By maximality of T , no defective colour is missing on T , which implies that every defective colour occurs on at least three edges leaving $V(T)$.

Each vertex $z \in Z$ is missing at least $k - d_G(z)$ colours, and by Theorem 2.3 these are all distinct. Hence the number of colours appearing on edges of T is at least $\sum_{z \in Z} (k - d_G(z))$. One edge of T is uncoloured, thus T has at least $\sum_{z \in Z} (k - d_G(z)) + 2$ vertices. \square

Lemma 2.9. *The set Q satisfies*

$$|E[Q]| > \frac{|Q|-1}{2} (|E[X, Q \setminus Z]| + \sum_{v \in Q} (k - d_G(v)) + 2).$$

Proof. By definition of U , no edge joining $Q \setminus Z$ to X is coloured with a defective colour (for Q). By maximality of T , no edge joining $Q \setminus Z$ to X is coloured with a colour missing on T . Recall also that the only uncoloured edge is e_0 which is not in $E[Q \setminus Z, X]$. Thus each edge of $E[Q \setminus Z, X]$ is coloured with some δ where the colour class of δ in $G[Q]$ is a matching of size $\frac{|Q|-1}{2}$, and these matchings are all disjoint from the colour classes of missing colours (which are also matchings of size $\frac{|Q|-1}{2}$). Moreover, no colour δ can appear more than once on $E[Q \setminus Z, X]$, otherwise it would be a defective colour.

The number of colours missing on T is at least $\sum_{v \in Q} (k - d_G(v)) + 2$, and these are all distinct by Theorem 2.3. Putting these facts together gives that the number of coloured edges in $E[Q]$ is at least

$$\frac{|Q|-1}{2} (|E[X, Q \setminus Z]| + \sum_{v \in Q} (k - d_G(v)) + 2).$$

The fact that the uncoloured edge e_0 is also in $E[Q]$ gives the strict inequality, as required. \square

The next lemma will imply Conclusion (4) of Lemma 2.7 immediately.

Lemma 2.10. *The set U is ϕ -elementary.*

Proof. Let qa be an edge of G with $q \in Q \setminus Z$ and $a \notin Q$. Suppose qa is coloured with some γ that is defective for T . Since $q \notin Z$ we know that there exists a colour σ missing at q that is not used on T .

Claim 1: Let α be a colour missing at a . Then α is not missing on T .

Proof of Claim 1. Suppose on the contrary that α is missing on T . By maximality of T no edge coloured α or σ leaves T . Hence the (α, σ) -path P from a does not contain any vertex of Q . Form a new colouring by switching on P . Then σ is missing at both a and q , and the tree T is unchanged. Now recolour qa from γ to σ , to get a colouring ψ with $E(\psi) = E(\phi)$. Then T is a ψ -Tashkinov tree since σ was not used on T . Note $\gamma \notin \{\alpha, \sigma\}$. Since γ was defective, there remain at least two edges leaving Q coloured γ , while γ is missing at q . Hence T could be extended under ψ , contradicting that fact that (ϕ, T) was a max-pair.

Claim 2: Let α be a colour missing at a . Then the (α, σ) -path P from a ends at q .

Proof of Claim 2. Suppose on the contrary that P does not end at q . Switch on P to obtain a new colouring. Then since no edge of T was coloured α (using Claim 1) or σ , we know that T is unchanged. But now σ is missing at a as well, so recolouring qa to σ gives a colouring ψ that again contradicts that fact that (ϕ, T) was a max-pair, because T is still a ψ -Tashkinov tree and (as in Claim 1) we can extend T via another edge coloured γ .

Now consider another edge rb of G with $r \in Q \setminus Z$ and $b \notin Q$, where $a \neq b$ ($q = r$ is possible). Suppose rb is coloured with some δ that is defective for T . (Note $\gamma = \delta$ is possible, in which case $q \neq r$.)

Claim 3: No colour α is missing at both a and b .

Proof of Claim 3. Suppose on the contrary that α is such a colour. We know by Claim 2 that the (α, σ) -path P from a ends at q . Switch on P to obtain a new colouring ψ . Since no edge of T was coloured α (using Claim 1) or σ , we know that T is unchanged, except that α is now missing at q instead of σ . The vertex b was not on P and hence α is still missing at b . But now (ψ, T) is a max-pair and δ is a defective colour for T (observe $\delta \notin \{\alpha, \sigma\}$), and α is missing at b and also at $q \in Q$. This contradicts Claim 1 applied to rb and the max-pair (ψ, T) .

Now Claims 1 and 3 together with Theorem 2.3 give us the desired conclusion that U is ϕ -elementary. \square

Lemma 2.10 implies Conclusion (4) in Lemma 2.7 because there are at least $2 + \sum_{v \in U} (k - d_G(v))$ missing colours on the vertices of U (recall that $Q \subseteq U$ contains the uncoloured edge). This is therefore at most the total number k of colours. Moreover if n is odd then no colour can be missing on exactly two vertices of G , hence if $|X| \leq 1$ then in fact the whole set $V(G)$ is ϕ -elementary. This implies that $2 + \sum_{v \in V(G)} (k - d_G(v)) \leq k$ as required.

2.1.2. Conditions for high chromatic index. In this section we show how to use Lemma 2.7 in order to prove Theorem 2.1.

Proof. Let G be as given with $\chi'(G) > k$, and suppose that none of (a)–(c) holds. Then in particular the assumptions of Lemma 2.7 are satisfied. Let X, Z, Q and U be as given by Lemma 2.7.

First we claim that $|Z| \leq \frac{|Q|+1}{3}$. By Lemma 2.7(2) we know $|Q| \geq 2 + \sum_{z \in Z} (k - d_G(z))$. Since (a) and (b) do not hold we find $|Q| \geq 2 + 3(|Z| - 1)$, verifying our claim.

Since $|E[X, Q \setminus Z]| \geq |X|(|Q| - |Z|)\mu_{min}$, we derive from Lemma 2.7(3) that

$$\mu \geq \frac{|E[Q]|}{\binom{|Q|}{2}} > |X|(1 - \frac{|Z|}{|Q|})\mu_{min} + \frac{1}{|Q|}(2 + \sum_{v \in Q} (k - d_G(v))).$$

Since (a) and (b) do not hold we know that $\sum_{v \in Q} (k - d_G(v)) \geq 3(|Q| - 1)$. Therefore $\mu > |X|(1 - \frac{|Z|}{|Q|})\mu_{min} + 3 - \frac{1}{|Q|}$. If $\mu_{min} = 0$, then by the fact that (c) does not hold we have that $\mu \leq 2$. By Lemma 2.7(2) we know that $|Q| \geq 3$ because $|Q|$ is odd, and therefore $\mu > 3 - \frac{1}{|Q|} > 2$. Thus we can assume that $\mu_{min} > 0$. Since $|Z| \leq \frac{|Q|+1}{3}$ we find

$$|X| < \frac{\mu - 3 + \frac{1}{|Q|}}{\mu_{min}(1 - \frac{|Z|}{|Q|})} = \frac{\mu - 3 + \frac{1}{|Q|}}{\mu_{min}} \left(\frac{3}{2 - \frac{1}{|Q|}} \right).$$

Again, by the fact that $|Q| \geq 3$ we conclude (using the fact that (c) does not hold) that $|X| < \frac{\mu - 3 + \frac{1}{3}}{\mu_{min}} \left(\frac{3}{2 - \frac{1}{3}} \right) \leq 2$.

Therefore $|X| \leq 1$. Since n is odd, Lemma 2.7(4) tells us that $\sum_{v \in V(G)} (k - d_G(v)) \leq k - 2$, in other words $kn - 2|E(G)| \leq k - 2$. Thus (d) holds, completing the proof. \square

2.2. Binomial distribution bounds. In many of the probabilistic statements in this paper we use the *binomial distribution*. We say that $X \sim \text{Bin}(m, p)$ if $\Pr[X = k] = \binom{m}{k} p^k (1 - p)^{m-k}$ where $0 \leq k \leq m$ is an integer. We start with four basic observations.

Observation 2.11. *Let d, d', m, m' be integers and let $p, p' \in [0, 1]$. Assume that $m' \leq m$, $d \leq d'$ and $p' \leq p$. Then for $X \sim \text{Bin}(m, p)$ and $X' \sim \text{Bin}(m', p')$ we have $\Pr[X' \geq d'] \leq \Pr[X \geq d]$.*

Observation 2.12. *Let $0 < p < 1$ and let k, m be integers, $0 < k \leq \frac{m}{2}$. Let $X \sim \text{Bin}(m, p)$. Then*

$$\frac{\Pr[X = k]}{\Pr[X = k + 1]} = \frac{(k + 1)(1 - p)}{(m - k) \cdot p} < \frac{4k}{mp}.$$

In addition, $\Pr[X \geq k] = \Pr[X = k] + \Pr[X \geq k + 1] < (1 + \frac{4k}{mp}) \Pr[X \geq k + 1]$.

In the following claim we show that a binomial variable decays relatively slowly around its expectation.

Claim 2.13. *Let $p := p(m) = o(1)$ and $mp = \omega(1)$. Let $h \geq 1$ be such that $d = mp + h$ is an integer and $h = o(mp)$. Then for $X \sim \text{Bin}(m, p)$ and some $\alpha = \Theta\left(\frac{h}{mp}\right)$ we have $\Pr[X = d + 1] = (1 - \alpha) \Pr[X = d]$. If also $h = \Omega(\sqrt{mp})$, then*

$$\Pr[X = mp + h] = O(\alpha) \Pr[X \geq mp + h].$$

Proof. For the first part,

$$\begin{aligned} \frac{\Pr[X = d + 1]}{\Pr[X = d]} &= \frac{m - d}{d + 1} \cdot \frac{p}{1 - p} = \frac{m - mp - h}{mp + h + 1} \cdot \frac{p}{1 - p} \\ &= \frac{mp - mp^2 - hp}{mp - mp^2 - hp + h + 1 - p} \\ &= 1 - \frac{h + 1 - p}{mp - mp^2 - hp + h + 1 - p} := 1 - \alpha, \end{aligned}$$

where $\alpha = \Theta\left(\frac{h}{mp}\right)$.

For the second part, let $d \leq k \leq mp + 2h$, then $\frac{\Pr[X=k+1]}{\Pr[X=k]} \geq 1 - \alpha$, where $\alpha = \Theta\left(\frac{h}{mp}\right)$. Note that $\alpha \cdot h = \Omega(1)$ and thus $1 - (1 - \alpha)^h = \Theta(1)$. Therefore,

$$\begin{aligned} \Pr[X \geq d] &= \sum_{k=d}^{\infty} \Pr[X = k] \geq \sum_{k=d}^{mp+2h} \Pr[X = k] \\ &\geq \sum_{k=d}^{mp+2h} (1 - \alpha)^{k-d} \Pr[X = d] = \Pr[X = d] \sum_{k=0}^h (1 - \alpha)^k \\ &\geq \frac{c}{\alpha} \cdot \Pr[\text{Bin}(m, p) = d], \end{aligned}$$

for some constant $c > 0$. Thus, $\Pr[\text{Bin}(m, p) = d] = O(\alpha) \Pr[\text{Bin}(m, p) \geq d]$ for $\alpha = \Theta\left(\frac{h}{mp}\right)$. \square

One of the most famous bounds on the tails of the binomial distribution, which we use extensively in this paper, is due to Chernoff (see, e.g., [1], [19]).

Lemma 2.14. *Let $X \sim \text{Bin}(n, p)$, $\mu = \mathbb{E}(X)$ and $a \geq 0$, then*

$$\begin{aligned} (1) \Pr[X \leq \mu - a] &\leq \exp\left(-\frac{a^2}{2\mu}\right); \\ (2) \Pr[X \geq \mu + a] &\leq \exp\left(-\frac{a^2}{2(\mu + \frac{a}{3})}\right). \end{aligned}$$

Lemma 2.15. *Let $X \sim \text{Bin}(n, p)$. Then, $\Pr[X \geq k] \leq \left(\frac{enp}{k}\right)^k$.*

Proof. Recall that $\binom{n}{\ell} \leq \left(\frac{en}{\ell}\right)^\ell$. Since $X \sim \text{Bin}(n, p)$ we have $\Pr[X \geq k] \leq \binom{n}{k} p^k \leq \left(\frac{en}{k}\right)^k p^k$. \square

The next lemma is due to DeMoivre-Laplace, and can be found in [4].

Lemma 2.16. *Let m be a sufficiently large integer and let $0 < p := p(m) < 1$ such that $mp(1 - p) \rightarrow \infty$. Let $X \sim \text{Bin}(m, p)$ and let $d = mp + x\sqrt{mp(1 - p)}$ where $x = o(\sqrt[6]{mp(1 - p)})$ and $x \rightarrow \infty$. Then,*

$$\Pr[X \geq d] = (1 + o(1)) \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} = e^{-\frac{x^2}{2} - \log x - \frac{1}{2} \log(2\pi) + o(1)}.$$

In the next two lemmas we show that the tail probabilities of two binomial random variables with close parameters are asymptotically equal.

Lemma 2.17. *Let m be a positive integer and let $p := p(m) = o(1)$ such that $mp(1 - p) \rightarrow \infty$. Let t be an integer and let x be such that:*

- $x = \frac{t - mp}{\sqrt{mp(1 - p)}} \rightarrow \infty$,
- $x = o((mp(1 - p))^{1/6})$,

Let $\alpha := \alpha(m)$ and $\beta := \beta(m)$ be such that $|\alpha| = o(1)$, $|\beta| = o(1)$, $m(1 - \alpha)$ is an integer and $\frac{(1 - \alpha)(1 + \beta)(1 - p)}{1 - p(1 + \beta)} = 1 + o\left(\frac{1}{x^2}\right)$. Then, for $X \sim \text{Bin}(m, p)$ and $X' \sim \text{Bin}(m', p')$,

$$\Pr[X' \geq t'] = (1 + o(1)) \Pr[X \geq t],$$

where $m' = m(1 - \alpha)$, $p' = p(1 + \beta)$, and $t' = \lceil t(1 - \alpha)(1 + \beta) \rceil$

Proof. Set $a = t' - t(1 - \alpha)(1 + \beta) < 1$ and write

$$x' = \frac{t' - m'p'}{\sqrt{m'p'(1-p')}} = \frac{t - mp + \frac{a}{(1-\alpha)(1+\beta)}}{\sqrt{mp(1-p)}} \cdot \sqrt{\frac{(1-\alpha)(1+\beta)}{\frac{1-p(1+\beta)}{1-p}}} = x \cdot \sqrt{\frac{(1-\alpha)(1+\beta)(1-p)}{1-p(1+\beta)}} + o\left(\frac{1}{x^2}\right).$$

Since $x' = (1 + o(1))x$, by Lemma 2.16 we deduce,

$$\Pr[X' \geq t'] = e^{-\frac{x^2}{2} \left(\frac{(1-\alpha)(1+\beta)(1-p)}{1-p(1+\beta)}\right) - \log x - \frac{1}{2} \log\left(\frac{(1-\alpha)(1+\beta)(1-p)}{1-p(1+\beta)}\right) - \frac{1}{2} \log(2\pi) + o(1)}.$$

Note that $\log\left(\frac{(1-\alpha)(1+\beta)(1-p)}{1-p(1+\beta)}\right) = o(1)$ and recall that $\frac{(1-\alpha)(1+\beta)(1-p)}{1-p(1+\beta)} = 1 + o\left(\frac{1}{x^2}\right)$. Thus, again by Lemma 2.16,

$$\Pr[X' \geq t'] = e^{-\frac{x^2}{2} - \log x - \frac{1}{2} \log(2\pi) + o(1)} = (1 + o(1)) \Pr[X \geq t]. \quad \square$$

Lemma 2.18. *Let n be an integer and $n' \leq n$ such that $n' = n(1 - o(1))$. Let $0 \leq p := p(n) \leq 1$ and $p' \leq p$ such that $p' = p(1 - o(1))$. Let $k \leq n'$ be an integer. Let $X \sim \text{Bin}(n, p)$. Assume there exists an integer s such that $k \leq s \leq n'$, $\Pr[X \geq s] = o(\Pr[X \geq k])$, $\frac{n-n'}{n'} = o\left(\frac{1}{s}\right)$, and $\frac{p-p'}{p} = o\left(\frac{1}{s}\right)$. Then for $X' \sim \text{Bin}(n', p')$ we have $\Pr[X \geq k] = (1 + o_n(1)) \Pr[X' \geq k]$.*

Proof. The direction $\Pr[X \geq k] \geq \Pr[X' \geq k]$ follows by Observation 2.11. For showing the other direction, recall that $\Pr[X \geq s] = o(\Pr[X \geq k])$, and thus $\sum_{d=k}^s \binom{n}{d} p^d (1-p)^{n-d} = (1 - o(1)) \sum_{d=k}^n \binom{n}{d} p^d (1-p)^{n-d}$. By the assumptions we also have that $e^{-2s\left(\frac{n-n'}{n'} + \frac{p-p'}{p}\right)} = (1 + o(1))$. Therefore,

$$\begin{aligned} \Pr[X' \geq k] &= \sum_{d=k}^{n'} \binom{n'}{d} (p')^d (1-p')^{n'-d} \geq \sum_{d=k}^{n'} \binom{n'}{d} (p')^d (1-p)^{n-d} \\ &= \sum_{d=k}^{n'} \binom{n}{d} p^d (1-p)^{n-d} \cdot \frac{(n-d)(n-d-1)\dots(n'-d+1)}{n(n-1)\dots(n'+1)} \cdot \left(\frac{p'}{p}\right)^d \\ &\geq \sum_{d=k}^{n'} \binom{n}{d} p^d (1-p)^{n-d} \cdot \left(\frac{n'-d}{n'}\right)^{n-n'} \cdot \left(\frac{p'}{p}\right)^d \\ &\geq \sum_{d=k}^{n'} \binom{n}{d} p^d (1-p)^{n-d} \cdot e^{-2d\frac{n-n'}{n'}} \cdot e^{-2d\frac{p-p'}{p}} \geq \sum_{d=k}^s \binom{n}{d} p^d (1-p)^{n-d} \cdot e^{-2d\left(\frac{n-n'}{n'} + \frac{p-p'}{p}\right)} \\ &\geq e^{-2s\left(\frac{n-n'}{n'} + \frac{p-p'}{p}\right)} \sum_{d=k}^s \binom{n}{d} p^d (1-p)^{n-d} = (1 - o(1)) \sum_{d=k}^s \binom{n}{d} p^d (1-p)^{n-d} \\ &= (1 - o(1)) \sum_{d=k}^n \binom{n}{d} p^d (1-p)^{n-d} = (1 - o(1)) \Pr[X \geq k]. \quad \square \end{aligned}$$

2.3. Properties of random graphs and random multigraphs. We start with two simple properties of random graphs. The first claim shows that in the sparse regime, a typical random graph has no cycles. We leave the proof of this claim as an (easy) exercise.

Claim 2.19. *Let $p = o\left(\frac{1}{n}\right)$ then w.h.p. $G \sim G(n, p)$ is a forest.*

The following theorem is a standard result regarding the maximum degree of a random graph in the relatively dense regime.

Theorem 2.20 (See, e.g., Theorem 3.9 in [4]). *Let $p \leq \frac{1}{2}$ and $p = \omega\left(\frac{\log n}{n}\right)$. Then w.h.p. $G \sim G(n, p)$ has a unique vertex of maximum degree.*

Recall that in the $M(n, m)$ model each multigraph has vertex set $[n]$ and m pairs from $[n]$ are chosen independently, uniformly at random.

Observation 2.21. For any pair $u, v \in [n]$, $\mu(uv) \sim \text{Bin}(m, \frac{1}{\binom{n}{2}})$.

Observation 2.22. For any vertex $v \in [n]$, $d(v) \sim \text{Bin}(m, \frac{n-1}{\binom{n}{2}} = \frac{2}{n})$.

In the following observation one can find a typical upper bound for the maximum degree of a random multigraph.

Observation 2.23. Let $m \geq \frac{n}{\log \log n}$ and $M \sim M(n, m)$. Then w.h.p. $\Delta(M) \leq \frac{2m \log^2 n}{n}$. Moreover, $\Pr \left[d(v) > \frac{2m \log^2 n}{n} \right] < n^{-5}$ for $v \in [n]$.

Proof. Since $d(v) \sim \text{Bin}(m, \frac{2}{n})$ we have by Lemma 2.15,

$$\Pr \left[d(v) > \frac{2m \log^2 n}{n} \right] \leq e^{-\frac{2 \log^2 n}{\log \log n}} < \frac{1}{n^5}.$$

By the union bound the claim follows. \square

The next observation refers to the edge-multiplicities of a random multigraph for different values of m .

Observation 2.24. Let n be an integer and $m := m(n)$. Let $M \sim M(n, m)$. Then the following hold.

- (1) If $m \leq n \log^{100} n$, then the probability that there exists an edge e with $\mu(e) \geq 3$ is at most $O(\frac{\log^{300} n}{n})$.
- (2) If $m < n^2 \log^2 n$, then with probability $1 - o(\frac{1}{n^4})$, for every $e \in \binom{[n]}{2}$, one has $\mu(e) \leq 30 \log^2 n$.
- (3) If $m = \omega(n^2 \log n)$, then with probability $1 - o(\frac{1}{n^4})$, for every $e \in \binom{[n]}{2}$, one has $\mu(e) \in \frac{m}{\binom{n}{2}} \pm 4 \sqrt{\frac{m}{\binom{n}{2}} \log n}$.

Proof. First Recall that $\mu(e) \sim \text{Bin}(m, \binom{n}{2}^{-1})$. Assume that $m \leq n \log^{100} n$. By Lemma 2.15,

$$\Pr[\mu(e) \geq 3] = O\left(\frac{\log^{300} n}{n^3}\right).$$

Then by the union bound, the probability that there exists an edge e with $\mu(e) \geq 3$ is at most $O(\frac{\log^{300} n}{n})$.

Next, assume that $m < n^2 \log^2 n$. Then $\mathbb{E}(\mu(e)) < 3 \log^2 n$, and by Lemma 2.15 we have

$$\Pr[\mu(e) > 30 \log^2 n] \leq e^{-30 \log^2 n} = o\left(\frac{1}{n^6}\right),$$

and by the union bound Item (2) follows.

Finally, assume that $m = \omega(n^2 \log n)$. Take $a = 4 \sqrt{\frac{m}{\binom{n}{2}} \log n}$, by Lemma 2.14,

$$\Pr[\mu(e) \geq \mathbb{E}(\mu(e)) + a] \leq e^{-\frac{a^2}{2(\mathbb{E}(X)+a/3)}} = e^{-\frac{16 \log n \mathbb{E}(X)}{2(\mathbb{E}(X)+o(\mathbb{E}(X)))}} \leq e^{-7 \log n} = o\left(\frac{1}{n^6}\right).$$

By the union bound, with probability $1 - o(\frac{1}{n^4})$, for every $e \in \binom{[n]}{2}$ we have $\mu(e) \leq \mathbb{E}(\mu(e)) + a = \frac{m}{\binom{n}{2}} + 4 \sqrt{\frac{m}{\binom{n}{2}} \log n}$.

Similarly, by Lemma 2.14,

$$\Pr[\mu(e) \leq \mathbb{E}(\mu(e)) - a] \leq e^{-\frac{a^2}{2\mathbb{E}(X)}} = e^{-\frac{16 \log n \mathbb{E}(X)}{2\mathbb{E}(X)}} \leq e^{-7 \log n} = o\left(\frac{1}{n^6}\right).$$

By the union bound, with probability $1 - o(\frac{1}{n^4})$, for every $e \in \binom{[n]}{2}$ we have $\mu(e) \geq \mathbb{E}(\mu(e)) - a = \frac{m}{\binom{n}{2}} - 4\sqrt{\frac{m}{\binom{n}{2}} \log n}$. \square

Corollary 2.25. *Let $m = \omega(n^2 \log n)$ and $M \sim M(n, m)$ and let $\mu(M) = \max\{\mu(e) \mid e \in \binom{[n]}{2}\}$, $\mu_{\min} = \min\{\mu(e) \mid e \in \binom{[n]}{2}\}$. Then w.h.p. $\mu(M) - \mu_{\min} = O\left(\sqrt{\frac{m \log n}{n^2}}\right)$.*

We will use the following claims in the proofs of Theorem 1.4 and Theorem 1.5. The proofs of the claims are rather technicals and can be found in the next subsection. The first claim shows that the typical maximum degree of a random multigraph is not too far from its expectation. Let $\varepsilon \in (0, 1)$ be a constant and define $d_+ = \frac{2m}{n} + (1 + \varepsilon)\sqrt{\frac{4m}{n}(1 - \frac{2}{n}) \log n}$.

Claim 2.26. *Let n be a sufficiently large integer, and let $m = \omega(n \log^3 n)$. Let $M \sim M(n, m)$. Then w.h.p. $\Delta(M) \leq d_+$.*

In the next lemma we show that the typical gap between the two largest degrees in a random multigraph is quite large. This phenomena is already known in the context of random graph (see, e.g., [4]), and here we prove an analogous statement for random multigraphs.

Lemma 2.27. *Let n be sufficiently large integer and let $m = \omega(n \log^5 n)$. Let $M \sim M(n, m)$, and let $d_1 \geq d_2 \geq \dots \geq d_n$ be the (ordered) degree sequence of M . Then w.h.p. $d_1 - d_2 \geq \frac{1}{f(n)}\sqrt{\frac{2m}{n \log n}}$, where $f(n) = o(\log n)$ is a function tending to infinity arbitrarily slowly with n .*

2.3.1. Proofs of Claim 2.26 and Lemma 2.27. We will use the following notation. Let X_d be the number of vertices of degree d in $M \sim M(n, m)$, and let $Y_d = \sum_{\ell \geq d} X_\ell$. Note that $\mathbb{E}(Y_d) = \sum_{v \in [n]} \Pr[d(v) \geq d]$.

In both proofs we will use the following approximation.

Remark 2.28. *Let $\alpha > -1$ be a constant and $Y_s = \sum_{d \geq d_s} X_d$ where $d_s = \left\lceil mp + (1 + \alpha)\sqrt{2mp(1 - p) \log n} \right\rceil$. Note that here $p = \frac{2}{n}$ and $x := \left(1 + \alpha + o\left(\frac{1}{\sqrt{mp}}\right)\right)\sqrt{2 \log n} = o(\sqrt[6]{m/n})$. Then By Lemma 2.16,*

$$\mathbb{E}(Y_s) = n(1 + o(1))\frac{1}{x\sqrt{2\pi}}e^{-x^2/2} = (1 + o(1))\frac{n}{(1 + \alpha)\sqrt{4\pi \log n}}e^{-(1+\alpha)^2 \log n} = \Theta\left(\frac{n^{-2\alpha - \alpha^2}}{\sqrt{\log n}}\right).$$

We first prove Claim 2.26.

Proof of Claim 2.26. Let $Y_+ = \sum_{d \geq d_+} X_d$. By Remark 2.28 we have

$$\mathbb{E}(Y_+) = \Theta\left(\frac{n^{-2\varepsilon - \varepsilon^2}}{\sqrt{\log n}}\right) = o\left(\frac{1}{n^\varepsilon}\right).$$

By Markov's inequality the claim follows. \square

The proof of Lemma 2.27 is more complicated, and we need the following statements.

Let $X \sim \text{Bin}(m, \frac{2}{n})$ and let $\varepsilon \in (0, 1)$. Let d_- be the maximal integer such that $\Pr[X \geq d_-] \geq \frac{f^{1/3}(n)}{n}$ where $f(n)$ is as in Lemma 2.27. Let $Y_- = \sum_{d \geq d_-} X_d$.

Remark 2.29. *The following hold.*

- (1) $\mathbb{E}(Y_-) \geq f^{1/3}(n)$.
- (2) If $d_\varepsilon = \left\lceil \frac{2m}{n} + (1 - \varepsilon)\sqrt{\frac{4m}{n}(1 - \frac{2}{n}) \log n} \right\rceil$ and $Y_\varepsilon = \sum_{d \geq d_\varepsilon} X_d$, then by Remark 2.28, we have that $\mathbb{E}(Y_\varepsilon) \geq n^\varepsilon$, and therefore $d_- > \frac{2m}{n} + (1 - \varepsilon)\sqrt{\frac{4m}{n}(1 - \frac{2}{n}) \log n}$.

- (3) If $d_{up} = \left\lceil \frac{2m}{n} + (1 + \frac{\varepsilon}{2}) \sqrt{\frac{4m}{n} (1 - \frac{2}{n}) \log n} \right\rceil$ and $Y_{up} = \sum_{d \geq d_{up}} X_d$, then by Remark 2.28, we have that $\mathbb{E}(Y_{up}) = o(1)$. Combining it with item (1) and the fact that the tail of the binomial random variable decays slowly (see Claim 2.13) we have $d_- < \frac{2m}{n} + (1 + \frac{\varepsilon}{2}) \sqrt{\frac{4m}{n} (1 - \frac{2}{n}) \log n}$.
- (4) From items (2),(3), we have that $d_+ - d_- \in [\frac{\varepsilon}{2} \sqrt{\frac{4m \log n}{n}}, 2\varepsilon \sqrt{\frac{4m \log n}{n}}]$.

Claim 2.30. Let $m = \omega(n \log^5 n)$. Let $X \sim \text{Bin}(m, \frac{2}{n})$ and $Y \sim \text{Bin}(m - d_-, \frac{2}{n-1})$. Assume that $\mu \leq \max \left\{ 30 \log^2 n, \frac{m}{\binom{n}{2}} + 4 \sqrt{\frac{m}{\binom{n}{2}} \log n} \right\}$ is a positive integer. Then

$$\Pr[Y \geq d_- - \mu] \leq (1 + o(1)) \Pr[X \geq d_-].$$

Proof. Let $\alpha = \frac{d_-}{m}$, $\beta = \frac{1}{n-1}$. Write $d_- = \frac{2m}{n} + x \sqrt{\frac{2m}{n} (1 - \frac{2}{n})}$, then by Remark 2.29, $x = \Theta(\sqrt{\log n})$. Let $d' = \lceil d_-(1 - \alpha)(1 + \beta) \rceil$. By Lemma 2.17 we have that

$$\Pr[Y \geq d'] = (1 + o(1)) \Pr[X \geq d_-].$$

Now, if $d_- - \mu \geq d'$ then we are done. If not, then recall that by Remark 2.29, there exist constants $c_1, c_2 > 0$ such that $\frac{2m}{n} + c_1 \sqrt{\frac{2m \log n}{n}} \leq d_- \leq \frac{2m}{n} + c_2 \sqrt{\frac{2m \log n}{n}}$. Now, let $\ell > 0$ be such that $d_- - \mu + \ell = d'$, and note that ℓ is an integer and that $(1 - \alpha)(1 + \beta) = 1 - \frac{d_- n}{m(n-1)} + \frac{1}{n-1}$. By the condition on μ we have that $\mu \leq 30 \log^2 n + \frac{m}{\binom{n}{2}} + 4 \sqrt{\frac{m}{\binom{n}{2}} \log n}$. Then

$$\begin{aligned} \ell &\leq \mu + d_-(1 - \alpha)(1 + \beta) + 1 - d_- = \mu + d_- \left(-\frac{d_- n}{m(n-1)} + \frac{1}{n-1} \right) + 1 = \mu - \frac{d_-^2 n}{m(n-1)} + \frac{d_-}{n-1} + 1 \\ &\leq \mu - \frac{n}{m(n-1)} \left(\frac{2m}{n} + c_1 \sqrt{\frac{2m \log n}{n}} \right)^2 + \frac{1}{n-1} \left(\frac{2m}{n} + c_2 \sqrt{\frac{2m \log n}{n}} \right) + 1 \\ &= \mu - \frac{2m}{n(n-1)} - \frac{2c_1^2 \log n}{n-1} + (c_2 - 4c_1) \sqrt{\frac{2m \log n}{n(n-1)^2}} + 1 \\ &\leq 30 \log^2 n + \frac{2m}{n(n-1)} + 4 \sqrt{\frac{2m \log n}{n(n-1)}} - \frac{2m}{n(n-1)} - \frac{2c_1^2 \log n}{n-1} + (c_2 - 4c_1) \sqrt{\frac{2m \log n}{n(n-1)^2}} + 1 \\ &= O \left(\sqrt{\frac{m \log n}{n(n-1)}} + \log^2 n \right). \end{aligned}$$

Write $m' = m - d_-$, $p' = \frac{2}{n-1}$. Set $h = d_- - \mu - m'p' = d' - \ell - m'p'$, then $h > 0$ and in fact $h = \Theta \left(\sqrt{\frac{m \log n}{n}} \right)$. Therefore by Claim 2.13 we have that $\Pr[Y = d'] > (1 - \alpha)^\ell \Pr[Y = d' - \ell]$ for $\alpha = O \left(\frac{\sqrt{m \log n}}{\sqrt{nm'p'}} \right)$. Since $\alpha \ell = o(1)$, we have $(1 - \alpha)^\ell = (1 - o(1))$. Thus,

$$\begin{aligned} \Pr(Y \geq d') &= \sum_{k=d'}^{\infty} \Pr(Y = k) \geq \sum_{k=d'}^{\infty} (1 - \alpha)^\ell \Pr(Y = k - \ell) \geq (1 - o(1)) \sum_{k=d'}^{\infty} \Pr(Y = k - \ell) \\ &= (1 - o(1)) \Pr(Y \geq d' - \ell) = (1 - o(1)) \Pr(Y \geq d_- - \mu). \end{aligned}$$

All in all, we have that $\Pr(Y \geq d_- - \mu) \leq (1 + o(1)) \Pr(X \geq d_-)$. \square

Remark 2.31. Let k_1, k_2 be integers. Let $M \sim M(n, m)$ and assume that $u, v \in V(M)$ are two vertices. Recall that $d(u) \sim \text{Bin}(m, \frac{2}{n})$ and that $\mu(uv) \leq \mu(M)$. Also, conditioning on the event $d(v) \geq k_2$, the probability that a new edge is in $E[u, V \setminus \{u, v\}]$ is $\frac{2}{n-1}$ (that is, the probability for

an edge to contain u in the sub(multi)graph $M - v$). Let $d'(u) \sim \text{Bin}(m - k_2, \frac{2}{n-1})$. Then by the monotonicity of the binomial distribution (see Observation 2.11) we have that

$$\Pr[d(u) \geq k_1 \mid d(v) \geq k_2] \leq \Pr[d'(u) \geq k_1 - \mu(M)].$$

Similarly, $\Pr[d(u) = k_1 \mid d(v) = k_2] \leq \Pr[d'(u) = k_1 - \mu(M)]$.

Claim 2.32. Let $m = \omega(n \log^5 n)$, and let $Y_- = \sum_{d \geq d_-} X_d$. Then w.h.p. $Y_- \geq 2$.

Proof. Recall that $Y_- = \sum_{v \in [n]} \mathbf{1}_{d(v) \geq d_-}$, therefore $\mathbb{E}(Y_-) = \sum_{v \in [n]} \Pr(d(v) \geq d_-)$ and $\mathbb{E}(Y_-^2) = \sum_{v, u \in [n]} \Pr(d(v) \geq d_-, d(u) \geq d_-)$. Moreover, by Observation 2.24 we have that $\mu(M) \leq \max\{30 \log^2 n, \frac{m}{\binom{n}{2}} + 4\sqrt{\frac{m}{\binom{n}{2}}} \log n\}$ with probability $1 - o(\frac{1}{n^4})$. Let $X \sim \text{Bin}(m, \frac{2}{n})$ and $Y \sim \text{Bin}(m - d_-, \frac{2}{n-1})$. Thus, by Claim 2.30 and Remark 2.31

$$\begin{aligned} \mathbb{E}(Y_-(Y_- - 1)) &= \mathbb{E}(Y_-^2) - \mathbb{E}(Y_-) = \sum_{v \neq u \in [n]} \Pr[d(v) \geq d_-, d(u) \geq d_-] \\ &= n(n-1) \Pr[d(v_1) \geq d_-, d(v_2) \geq d_-] \\ &= n(n-1) \Pr[d(v_1) \geq d_-] \Pr[d(v_2) \geq d_- \mid d(v_1) \geq d_-] \\ &\leq n(n-1) \Pr[X \geq d_-] \left(\Pr[Y \geq d_- - \mu(M)] + o\left(\frac{1}{n^4}\right) \right) \\ &\leq (1 + o(1))n^2 \Pr[X \geq d_-] \left(\Pr[X \geq d_-] + o\left(\frac{1}{n^4}\right) \right) \\ &= (1 + o(1))(\mathbb{E}(Y_-))^2. \end{aligned}$$

Therefore, $\Pr[Y_- < \frac{1}{2}\mathbb{E}(Y_-)] < \frac{\mathbb{E}(Y_-(Y_- - 1)) + \mathbb{E}(Y_-) - (\mathbb{E}(Y_-))^2}{(\mathbb{E}(Y_-))^2/4} \leq o(1) + \frac{4}{\mathbb{E}(Y_-)} = o(1) + O(\frac{1}{f^{1/3}(n)}) = o(1)$. Thus w.h.p. $Y_- \geq 2$. \square

We are now ready to prove Lemma 2.27.

Proof of Lemma 2.27. We will actually prove something stronger. We will prove that w.h.p. for every $u, v \in \binom{[n]}{2}$ such that $d(v), d(u) \in [d_-, d_+]$ we have $|d(v) - d(u)| \geq \frac{1}{f(n)} \sqrt{\frac{2m}{n \log n}}$. Since by Claim 2.26 and Claim 2.32 we have that w.h.p. $d_1, d_2 \in [d_-, d_+]$, the statement will follow.

Let $X \sim \text{Bin}(m, \frac{2}{n})$ and $Y \sim \text{Bin}(m - d_-, \frac{2}{n-1})$. By Claim 2.30 we have $\Pr[Y \geq d_- - \mu] \leq (1 + o(1)) \Pr[X \geq d_-]$ for $\mu \leq \max\left\{30 \log^2 n, \frac{m}{\binom{n}{2}} + 4\sqrt{\frac{m}{\binom{n}{2}}} \log n\right\}$. Let $\ell_1, \ell_2 \in [d_-, d_+]$, $m' = m - \ell_1$, and $p' = \frac{2}{n-1}$. Let $h = \ell_2 - \mu(M) - m'p' = \Theta\left(\sqrt{\frac{m \log n}{n}}\right)$ and $Y_{\ell_1} \sim \text{Bin}(m - \ell_1, \frac{2}{n-1})$. Thus, by Claim 2.13 we have $\Pr[Y_{\ell_1} = \ell_2 - \mu(M)] \leq \alpha \Pr[Y_{\ell_1} \geq \ell_2 - \mu(M)]$, for $\alpha = \Theta(\frac{nh}{m}) = O\left(\sqrt{\frac{n \log n}{m}}\right)$. Also, note that $\Pr[X \geq d_-] \leq (2 + o(1)) \Pr[X \geq d_- + 1]$. Indeed, by Claim 2.13

$$\begin{aligned} \Pr[X \geq d_-] &= \Pr[X = d_-] + \Pr[X \geq d_- + 1] \\ &\leq \frac{1}{1 - \alpha} \Pr[X = d_- + 1] + \Pr[X \geq d_- + 1] \\ &\leq \left(1 + \frac{1}{1 - \alpha}\right) \Pr[X \geq d_- + 1] = (2 + o(1)) \Pr[X \geq d_- + 1]. \end{aligned}$$

Since by Observation 2.24 we have that $\mu(M) \leq \max \left\{ 30 \log^2 n, \frac{m}{\binom{n}{2}} + 4 \sqrt{\frac{m}{\binom{n}{2}} \log n} \right\}$ with probability $1 - o\left(\frac{1}{n^4}\right)$, and by Lemma 2.26 and Claim 2.32 we have that w.h.p. $d_1, d_2 \in [d_-, d_+]$, using Remark 2.31 and Observation 2.24 we conclude that

$$\begin{aligned}
& \Pr[\exists u, v \in [n], d(u), d(v) \in [d_-, d_+], |d(u) - d(v)| \leq t] \leq n^2 \Pr[d(u), d(v) \in [d_-, d_+], |d(u) - d(v)| \leq t] \\
& \leq (1 + o(1)) n^2 \sum_{\ell_1=d_-}^{d_+} \sum_{\substack{\ell_2=d_- \\ |\ell_1 - \ell_2| \leq t}}^{d_+} \Pr[d(u) = \ell_1, d(v) = \ell_2] \\
& \leq (1 + o(1)) n^2 \sum_{\ell_1=d_-}^{d_+} \sum_{\substack{\ell_2=d_- \\ |\ell_1 - \ell_2| \leq t}}^{d_+} \Pr[d(u) = \ell_1] \Pr[d(v) = \ell_2 \mid d(u) = \ell_1] \\
& \leq (1 + o(1)) n^2 \sum_{\ell_1=d_-}^{d_+} \sum_{\substack{\ell_2=d_- \\ |\ell_1 - \ell_2| \leq t}}^{d_+} \Pr[d(u) = \ell_1] \Pr[Y_{\ell_1} = \ell_2 - \mu(M)] \\
& \leq (1 + o(1)) n^2 \sum_{\ell_1=d_-}^{d_+} \sum_{\substack{\ell_2=d_- \\ |\ell_1 - \ell_2| \leq t}}^{d_+} \Pr[d(u) = \ell_1] \alpha \Pr[Y_{\ell_1} \geq \ell_2 - \mu(M)] \\
& \leq (1 + o(1)) n^2 \alpha \Pr[Y \geq d_- - \mu(M)] \sum_{\ell_1=d_-}^{d_+} \sum_{\substack{\ell_2=d_- \\ |\ell_1 - \ell_2| \leq t}}^{d_+} \Pr[d(u) = \ell_1] \\
& \leq (1 + o(1)) \cdot n^2 (2t + 1) \alpha \left(\Pr[X \geq d_-] + o\left(\frac{1}{n^4}\right) \right) \sum_{\ell_1=d_-}^{d_+} \Pr[d(u) = \ell_1] \\
& \leq (1 + o(1)) \cdot n^2 (2t + 1) \alpha \left(\Pr[X \geq d_-] + o\left(\frac{1}{n^4}\right) \right) \Pr[X \geq d_-] \\
& \leq (1 + o(1)) \cdot n^2 (2t + 1) \alpha \Pr[X \geq d_-]^2 \leq (2 + o(1)) \cdot n^2 (2t + 1) \alpha \Pr[X \geq d_- + 1]^2 \\
& = O\left(n^2 t \alpha \left(\frac{f^{1/3}(n)}{n} \right)^2 \right) = O(t \alpha f^{2/3}(n)).
\end{aligned}$$

All in all, for $t = \frac{1}{f(n)} \sqrt{\frac{2m}{n \log n}}$ and $\alpha = O\left(\sqrt{\frac{n \log n}{m}}\right)$, we have

$$\Pr[d_1 - d_2 < t] = O(t \alpha f^{2/3}(n)) = o(1). \quad \square$$

3. THE CHROMATIC INDEX OF RANDOM GRAPHS

In this section we will prove Theorem 1.3.

The proof of Theorem 1.3 is a simpler version of the proof of Theorem 1.4 for small values of m . For a warm-up, we included the details below.

Let $G \sim G(n, p)$. First we will handle the case where $p = \omega\left(\frac{\log n}{n}\right)$ and $p \leq \frac{1}{2}$. By Theorem 2.20, w.h.p. there exists a unique vertex of maximum degree. Then, by Theorem 1.1 we have that w.h.p. $\chi'(G) = \Delta(G)$.

Next, let us consider the sparse case, where $p = o(\frac{1}{n})$. By Lemma 2.19, $G \sim G(n, p)$ is w.h.p. a forest and therefore by Theorem 1.1 w.h.p. $\chi'(G) = \Delta(G)$.

Thus, we may assume from now on that $\frac{1}{n \log \log n} \leq p \leq \frac{\log^2 n}{n}$. Let $G \sim G(n, p)$, recall that $d(v) \sim \text{Bin}(n-1, p)$ for $v \in [n]$. We will consider the vertices of *very large* degree. Let $X \sim \text{Bin}(n-1, p)$, let $d_0 = \max\{d \mid \Pr[X \geq d] \geq n^{-0.9}\}$, and define $L_0 = \{v \in [n] \mid d(v) \geq d_0\}$. Note that for $v \in [n]$, $\Pr[d(v) \geq d_0 + 1] < n^{-0.9}$ and $\Pr[v \in L_0] \geq n^{-0.9}$.

If we can show that w.h.p. the set L_0 is an independent set and not empty, then clearly w.h.p. the set of vertices of maximum degree forms an independent set, and then by Theorem 1.1 we are done. For proving that w.h.p. every two vertices in L_0 are not neighbours we will use the following two claims.

Observation 3.1. *Since $d(v) \sim \text{Bin}(n-1, p)$ and $d_0 < \log^3 n$, we have by Observation 2.12 that $\Pr[d(v) \geq d_0] \leq (1 + \frac{4d_0}{(n-1)p}) \Pr[d(v) \geq d_0 + 1] < (4 \log^4 n + 1)n^{-0.9}$, and similarly $\Pr[d(v) \geq d_0 - 1] < (1 + \frac{4d_0}{(n-1)p}) \Pr[d(v) \geq d_0] \leq n^{-0.9} \log^8 n$.*

First, we will bound from above the probability that a vertex is in L_0 .

Claim 3.2. *Let $G \sim G(n, p)$, $\frac{1}{n \log \log n} \leq p = o(1)$. Then for $v \in [n]$, $\Pr[v \in L_0] = O(n^{-0.9} \log^4 n)$.*

Proof. By Lemma 2.15, $\Pr[d(v) \geq \log^2 n \cdot \mathbb{E}[d(v)]] \leq e^{-\log^2 n \mathbb{E}[d(v)]} < n^{-0.9}$, where the last inequality comes from the fact that $\mathbb{E}[d(v)] > \frac{1}{2 \log \log n}$. That is, $d_0 < \log^2 n \mathbb{E}[d(v)] \leq \log^2 n \cdot np$. By Observation 3.1 we have $\Pr[v \in L_0] < (4 \log^4 n + 1)n^{-0.9}$. \square

Next, we will show that the set L_0 is not empty.

Claim 3.3. *Let $G \sim G(n, p)$. Then w.h.p. there exists $v \in L_0$.*

Proof. First, let $n' = n - n^{0.95}$, $s = \log^4 n$, and $k = d_0$. Let $X \sim \text{Bin}(n-1, p)$ and $X' \sim \text{Bin}(n - n^{0.95}, p)$. Note that $np \leq \log^3 n$ and thus by Lemma 2.15 we have that $\Pr[X \geq \log^4 n] \leq e^{-\log^4 n} = o(n^{-0.9})$. Thus $d_0 < \log^4 n$. Also, $\frac{n-n'}{n'} = \Theta(\frac{1}{n^{0.05}})$, thus by Lemma 2.18, we have that $\Pr[X \geq d_0] = (1 + o(1)) \Pr[X' \geq d_0]$.

Now, choose a subset $U \subset [n]$ of size $n^{0.95}$. Let X_U be the number of vertices v in U for which $e(\{v\}, [n] \setminus U) \geq d_0$. Then $X_U \sim \text{Bin}(|U|, \tilde{p})$ where $\tilde{p} = \Pr[e(\{v\}, [n] \setminus U) \geq d_0]$ for some $v \in U$. Note that $\tilde{p} = \Pr[e(v, [n] \setminus U) \geq d_0] = \Pr[X' \geq d_0] \geq (1 - o(1)) \Pr[X \geq d_0] \geq n^{-0.9}$. Therefore, $\mathbb{E}(X_U) = n^{0.95} \tilde{p} \geq n^{0.05}$. By Lemma 2.14, $\Pr(X_U \leq \frac{1}{2} n^{0.05}) \leq e^{-\frac{1}{8} n^{0.05}} = o(1)$. Thus w.h.p. $X_U \geq 1$ and $|L_0| \geq 1$. \square

We are now ready to show that w.h.p. L_0 is an independent set.

Claim 3.4. *Let $G \sim G(n, p)$. Then for every $v, u \in L_0$, w.h.p. $u \not\sim v$.*

Proof. Let $X_i \sim \text{Bin}(n-i, p)$, and let v_1, v_2 be vertices and let A be the event that $v_1 \sim v_2$. Then, $\Pr[A] = p \leq \frac{\log^2 n}{n}$. Observe that by Claim 3.2 (and Observations 2.11 and 3.1)

$$\begin{aligned} \Pr[v_1, v_2 \in L_0 \mid A] &= \Pr[d(v_1) \geq d_0 \mid A] \Pr[d(v_2) \geq d_0 \mid A, v_1 \in L_0] \\ &\leq \Pr[X_2 \geq d_0 - 1] \Pr[X_2 \geq d_0 - 1] \\ &\leq \Pr[X_1 \geq d_0 - 1]^2 = O(n^{-1.8} \log^{16} n). \end{aligned}$$

Going over all choices of v_1, v_2 , the probability that there exists an edge between two vertices in L_0 is at most $n^2 \Pr[A] \Pr[v_1, v_2 \in L_0 \mid A] = O\left(n^2 \cdot \frac{\log^2 n}{n} \cdot n^{-1.8} \log^{16} n\right) = o(1)$. \square

4. THE CHROMATIC INDEX OF RANDOM MULTIGRAPHS $M(n, m)$

In this section we prove Theorems 1.4 and 1.5. We divide the proof into three cases. In Section 4.1 we prove that if $m \leq n \log^{100} n$ then w.h.p. $\chi'(M) = \Delta$. In Section 4.2 we show that if $n \log^{100} n \leq m \leq n^2 \log^2 n$ then w.h.p. also here $\chi'(M) = \Delta$. In Section 4.3 we show that if n is even, and $m = \omega(n^2 \log n)$ then w.h.p. $\chi'(M) = \Delta$, completing the proof of Theorem 1.5 and the proof of Theorem 1.4 for the even case. In Section 4.4 we show that if n is odd and $m = \omega(n^2 \log n)$ then w.h.p. $\chi'(M) = \Delta$ or $\lceil \rho(M) \rceil$. Then, in Section 4.5 we discuss the algorithmic aspects of our proofs.

In the entire section we let $d_1 \geq d_2 \geq \dots \geq d_n$ be the (ordered) degree sequence of a multigraph M .

4.1. The case $m \leq n \log^{100} n$. For this case we act as follows. We first show that for very small values of m , say $m = O(\frac{n}{\log \log n})$, a typical multigraph is actually a forest. For the remaining values of m , we will show that the typical edge multiplicity in this case is at most two. Then, the main idea is to remove a matching from the multigraph to create a simple graph with maximum degree $\Delta - 1$ for which the vertices of degree $\Delta - 1$ form an independent set (similarly to the proof of Theorem 1.3). Thus we can colour this graph with $\Delta - 1$ colours, and colour the removed matching with one additional colour.

We start with a claim estimating the probability of a given set of edges to have large multiplicities in $M \sim M(n, m)$.

Claim 4.1. *Let $M \sim M(n, m)$ and let r, k_1, \dots, k_r be integers. Let $A = \{e_1, \dots, e_r\}$ be a set of r pairs from $\binom{[n]}{2}$. Let X be the event that $\mu(e_i) \geq k_i$ for all $i \in [r]$. Then $\Pr[X] \leq \left(\frac{5em}{n^2}\right)^{\sum_{i=1}^r k_i}$.*

Proof. Let $A_0 \subseteq A$. Since $\mu(e) \sim \text{Bin}(m, \frac{2}{n(n-1)})$, we have $\Pr[\mu(e_i) \geq k_i \mid \mu(e_j) \geq k_j \forall e_j \in A_0] \leq \Pr[\mu(e_i) \geq k_i]$ for every i with $e_i \notin A_0$. The inequality follows, for example, from Theorem 10 in [8] or from the main theorem in [22]. By Lemma 2.15, $\Pr[\mu(e) \geq k] \leq \left(\frac{4em}{n(n-1)}\right)^k \leq \left(\frac{5em}{n^2}\right)^k$. Then,

$$\Pr[X] \leq \prod_{i=1}^r \Pr[\mu(e_i) \geq k_i] \leq \left(\frac{5em}{n^2}\right)^{\sum_{i=1}^r k_i}. \quad \square$$

Let us first deal with the case that $m \leq \frac{100n}{\log \log n}$. Let $r \in [n] \setminus \{1\}$ and let v_1, \dots, v_r be r vertices. Denote by X_C the event that the cycle $C = (v_1, \dots, v_r, v_1)$ appears in $M \sim M(n, m)$. Then by Claim 4.1, $\Pr[X_C] \leq (\Pr[\mu(v_1 v_2) \geq 1])^r \leq \left(\frac{5em}{n^2}\right)^r \leq \left(\frac{1500}{n \log \log n}\right)^r$. Thus the probability that there exists a cycle in $M \sim M(n, m)$ is at most

$$\sum_{r=2}^n n^r \Pr[X_C] \leq \sum_{r=2}^n \left(\frac{1500}{\log \log n}\right)^r = o(1).$$

Therefore, for $m \leq \frac{100n}{\log \log n}$, $M \sim M(n, m)$ is w.h.p. a forest and thus by Theorem 1.1 w.h.p. $\chi'(G) = \Delta(G)$.

From now on we will assume that $m \geq \frac{n}{\log \log n}$.

Similarly to the proof of Theorem 1.3, we will consider vertices of very large degrees. Let $X \sim \text{Bin}(m, \frac{2}{n})$, and define

$$d_0 = \max\{d \mid \Pr[X \geq d] \geq n^{-0.9}\}, \quad L_0 = \{v \in V \mid d(v) \geq d_0\},$$

$$d'_0 = \max\{d \mid \Pr[X \geq d] \geq n^{-0.95}\}, \quad L'_0 = \{v \in V \mid d(v) \geq d'_0\}.$$

Note that for $v \in V$ we have

$$\Pr[d(v) \geq d_0 + 1] < n^{-0.9}, \quad \Pr[v \in L_0] \geq n^{-0.9},$$

$$\Pr[d(v) \geq d'_0 + 1] < n^{-0.95}, \quad \Pr[v \in L'_0] \geq n^{-0.95}.$$

Furthermore, by Observation 2.23, $d_0 \leq d'_0 < \frac{2m \log^2 n}{n}$.

We would like to bound from above the probability that a vertex is in L_0 .

Claim 4.2. *Let $M \sim M(n, m)$. Then for $v \in [n]$, $\Pr[v \in L_0] = O(n^{-0.9} \log^2 n)$ and $\Pr[v \in L'_0] = O(n^{-0.95} \log^2 n)$.*

Proof. Since $d(v) \sim \text{Bin}(m, \frac{2}{n})$ and $d_0 \leq d'_0 < \frac{2m \log^2 n}{n}$, by Observation 2.12 we have that

$$\Pr[v \in L_0] = \Pr[d(v) \geq d_0] \leq \left(1 + \frac{4nd_0}{2m}\right) \Pr[d(v) \geq d_0 + 1] < (4 \log^2 n + 1)n^{-0.9},$$

and similarly $\Pr[v \in L'_0] = \Pr[d(v) \geq d'_0] = O(n^{-0.95} \log^2 n)$. \square

Next, we will prove that w.h.p. the vertices in the set L_0 are “far” from each other.

Claim 4.3. *w.h.p. for every $v, u \in L_0$, there is no path of length at most 2 connecting u and v .*

Proof. Let $M \sim M(n, m)$ and let W be a set of ordered vertices $\{v_1, \dots, v_r\}$, $2 \leq r \leq 3$. Let X_P be the event that $P = (v_1, \dots, v_r)$ is a path in M . Then by Claim 4.1 $\Pr[X_P] = O((\frac{m}{n^2})^{r-1})$. Note that $\Pr[d(v) \geq d_0 - 2] = \Pr[d(v) = d_0 - 2] + \Pr[d(v) \geq d_0 - 1]$. By Observation 2.12,

$$\frac{\Pr[d(v) = d_0 - 2]}{\Pr[d(v) = d_0 - 1]} < \frac{2(d_0 - 2)n}{m} \leq 4 \log^2 n,$$

and $\Pr[d(v) \geq d_0 - 2] \leq (4 \log^2 n + 1) \Pr[d(v) \geq d_0 - 1] \leq 20 \log^4 n \Pr[v \in L_0]$.

Recall that by (the proof of) Observation 2.24, for $i \in [r - 1]$, $\mu(v_i v_{i+1}) \leq 2$ with probability $1 - o(\frac{1}{n^2})$. Also, note that conditioning on the existence of the path P (with multiplicities), the degree of v_1 (or v_r) outside P is distributed according to $\text{Bin}(m - \sum_{i=1}^{r-1} \mu(v_i v_{i+1}), p')$ where p' is at most $\frac{n-2}{\binom{n}{2} - r + 1} < \frac{2}{n}$. Then by Observation 2.11, Remark 2.31, and Claim 4.2,

$$\begin{aligned} \Pr[v_1, v_r \in L_0 \mid X_P] &= \Pr[d(v_1) \geq d_0 \mid X_P] \Pr[d(v_r) \geq d_0 \mid X_P, v_1 \in L_0] \\ &\leq \left(\Pr[d(v) \geq d_0 - 2] + o\left(\frac{1}{n^2}\right) \right)^2 \\ &\leq \left(20 \log^4 n \Pr[v \in L_0] + o\left(\frac{1}{n^2}\right) \right)^2 \\ &= O(n^{-1.8} \log^{12} n). \end{aligned}$$

Going over all choices of v_1, \dots, v_r , the probability that there exists a path of length at most 2 with two endpoints in L_0 is at most

$$\sum_{r=1}^3 n^r \Pr[X_P] \Pr[v_1, v_r \in L_0 \mid X_P] = o(1). \quad \square$$

We will show now that the set L_0 contains at least two vertices of different degrees.

Claim 4.4. *w.h.p. there exist $v, u \in V$ such that $v, u \in L_0$ and $d(v) \neq d(u)$.*

Proof. Since $L'_0 \subseteq L_0$, it is enough to prove that w.h.p. $|L_0| > |L'_0| > 0$. First, by Claim 4.2, we have that $\Pr[v \in L'_0] \leq n^{-0.95} \log^3 n$, and $\Pr[v \in L_0] > n^{-0.9}$. Therefore, $\Pr[v \in L_0] = \omega(\Pr[v \in L'_0])$. Recall that by Claim 4.3 w.h.p. every two vertices in L_0 are not connected.

Let $X \sim \text{Bin}(m, \frac{2}{n})$, $X' \sim (m, \frac{2}{n-1})$, and note that $\frac{2m}{n} \leq 2 \log^{100} n$, thus by Lemma 2.15 we have $\Pr[X > \log^{200} n] \leq e^{-\log^{200} n} = o(n^{-0.95})$. Let $(L, d) \in \{(L_0, d_0), (L'_0, d'_0)\}$ and recall that $|L| = \sum_{v \in [n]} \mathbf{1}_{d(v) \geq d}$. By Lemma 2.18, for $s = \log^{200} n$ and $k = d$, we have that $\Pr[X' \geq d] = (1 + o(1)) \Pr[X \geq d]$. Using Remark 2.31 and Observation 2.11 we have that

$$\begin{aligned} \mathbb{E}(|L|^2) &= \sum_{v, u \in [n]} \Pr[d(v) \geq d, d(u) \geq d] = \sum_{\substack{v, u \in [n] \\ u \neq v}} \Pr[d(v) \geq d, d(u) \geq d] + \mathbb{E}(|L|) \\ &= \sum_{\substack{v, u \in [n] \\ u \neq v, u \not\sim v}} \Pr[d(v) \geq d, d(u) \geq d] + \mathbb{E}(|L|) + o(1) \\ &= \sum_{\substack{v, u \in [n] \\ u \neq v, u \not\sim v}} \Pr[d(u) \geq d] \Pr[d(v) \geq d \mid d(u) \geq d] + \mathbb{E}(|L|) + o(1) \\ &\leq n^2 \Pr[X \geq d] \Pr[X' \geq d] + \mathbb{E}(|L|) + o(1) \\ &\leq (1 + o(1)) n^2 \Pr[X \geq d]^2 + \mathbb{E}(|L|) + o(1) \\ &\leq (1 + o(1)) \mathbb{E}(|L|)^2 + \mathbb{E}(|L|) + o(1). \end{aligned}$$

All in all, by Chebyshev inequality we have that

$$\Pr \left[\left| |L| - \mathbb{E}(|L|) \right| > \frac{1}{2} \mathbb{E}(|L|) \right] \leq \frac{\text{Var}(|L|)}{\mathbb{E}(|L|)^2/4} \leq \frac{(1 + o(1)) \mathbb{E}(|L|)^2 + \mathbb{E}(|L|) + o(1) - \mathbb{E}(|L|)^2}{\mathbb{E}(|L|)^2/4} = o(1).$$

Therefore, w.h.p. $|L_0| \geq \frac{1}{2} \mathbb{E}(|L_0|) = \Omega(n^{0.1})$, $|L'_0| \geq \frac{1}{2} \mathbb{E}(|L'_0|) = \Omega(n^{0.05})$, $|L'_0| \leq \frac{3}{2} \mathbb{E}(|L'_0|) = O(n^{0.05} \log^2 n)$, and the claim follows. \square

In the next claim we show that w.h.p. every two edges with multiplicity two are far from each other.

Claim 4.5. *Let $M \sim M(n, m)$. Then for every $3 \leq r \leq 5$ and for every $v_1, \dots, v_r \in [n]$, the probability that $\mu(v_1 v_2), \mu(v_{r-1} v_r) \geq 2$, and $\mu(v_2 v_3), \dots, \mu(v_{r-2} v_{r-1}) \geq 1$ is $o(1)$. In other words, w.h.p. for every pair of edges $e_1, e_2 \in \binom{[n]}{2}$ with multiplicity at least 2, the distance between e_1 and e_2 is at least three.*

Proof. Let r be an integer. For r vertices v_1, \dots, v_r let X_r be the event that there exist $i \neq j \in [r-1]$ such that $\mu(v_i v_{i+1}) \geq 2$, $\mu(v_j v_{j+1}) \geq 2$ and for every $k \in [r-1] \setminus \{i, j\}$, $\mu(v_k v_{k+1}) \geq 1$. Then by Claim 4.1

$$\Pr[X_r] \leq (r-1)(r-2) \left(\frac{5em}{n^2} \right)^{4+(r-3)} \leq r^2 \left(\frac{5em}{n^2} \right)^{r+1}.$$

Therefore, the probability that there exist such r vertices is at most $n^r \Pr[X_r] = O\left(\frac{\log^{100(r+1)} n}{n}\right)$. Thus, the probability that there exist two edges e, e' with $\mu(e), \mu(e') \geq 2$ and distance at most two is at most $\sum_{r=3}^5 n^r \Pr[X_r] = \sum_{r=3}^5 O\left(\frac{\log^{100(r+1)} n}{n}\right) = o(1)$. \square

Now we are ready to prove Theorem 1.4 for the case $\frac{n}{\log \log n} \leq m \leq n \log^{100} n$.

Proof. Let $M \sim M(n, m)$ for $m \leq n \log^{100} n$. Let $d_0 = \max\{d \mid \Pr(\text{Bin}(m, \frac{2}{n}) \geq d) \geq n^{-0.9}\}$, and $L_0 = \{v \in V \mid d(v) \geq d_0\}$. Let Δ be the maximum degree in M . By Claim 4.4 we have that w.h.p. all vertices of degree Δ and $\Delta - 1$ are in L_0 . Recall also that by Observation 2.24, w.h.p. for every $e \in E(M)$, $\mu(e) \leq 2$. Assume that these events occur and furthermore that the assertions of Claims 4.3 and 4.5 hold. These events occur w.h.p., so from now on we argue deterministically and show that $\chi'(M) = \Delta$.

We remove edges from the multigraph in the following way. First we look at all e such that $\mu(e) = 2$, denote this set by E_1 and let $V_1 = \cup_{e \in E_1} e$. We remove from each such e one multiplicity. By Claim 4.5, we removed a matching F_1 . The remaining multigraph is a simple graph. Now, we look at all of the remaining vertices of degree Δ , denote this set by V_0 and note that $V_0 \cap V_1 = \emptyset$. By Claim 4.5, for every vertex v with $d(v) \geq 2$ there exists a neighbour u such that $u \notin V_1$. Therefore, for every vertex $v \in V_0$ we find such neighbour u and remove the edge vu . By Claim 4.3 we removed a matching F_2 . Note that $F = F_1 \cup F_2$ is also a matching. The remaining graph is a simple graph with maximum degree $\Delta - 1$. Furthermore, all of the current vertices of degree $\Delta - 1$ are in L_0 and therefore by Claim 4.3 they form an independent set. By Theorem 1.1, we can colour the edges of the graph with $\Delta - 1$ colours. Colouring the edges of F with the additional colour gives us the desired colouring. \square

4.2. The case $n \log^9 n \leq m \leq n^2 \log^2 n$. In this section we will prove Theorem 1.4 and Theorem 1.5 for the case $n \log^8 n \leq m \leq n^2 \log^2 n$.

Proof. By Observation 2.24, we have that w.h.p. $\mu(M) \leq \log^3 n$. Also, by Lemma 2.27, for, say, $f(n) = \log \log n$, we have that w.h.p. $d_1 - d_2 \geq \frac{1}{f(n)} \sqrt{\frac{2m}{n \log n}}$.

By Lemma 2.5 applied with $k = d_1$, it is enough to show that w.h.p. $d_1 - d_2 \geq \max\{2, \mu(M)\}$. Clearly, $\frac{1}{f(n)} \sqrt{\frac{2m}{n \log n}} \geq 2$, and thus w.h.p. $d_1 - d_2 \geq 2$. For $m \geq n \log^9 n$ we have that w.h.p. $d_1 - d_2 \geq \frac{1}{f(n)} \sqrt{\frac{2m}{n \log n}} > \log^3 n > \mu(M)$. \square

4.3. The case $m = \omega(n^2 \log n)$, and n is even. In this section we complete the proof of Theorem 1.5, and the proof of Theorem 1.4 for the even case.

Proof. By Corollary 2.25, we have that w.h.p. $\mu(M) - \mu_{min} = O\left(\sqrt{\frac{2m}{n^2} \log n}\right)$. Also, by Lemma 2.27 we have that w.h.p. $d_1 - d_2 \geq \frac{1}{f(n)} \sqrt{\frac{2m}{n \log n}}$, where $f(n) \rightarrow \infty$ arbitrarily slowly. That is, w.h.p. $d_1 - d_2 \geq \mu(M) - \mu_{min}$, and thus by Corollary 2.6 we have $\chi'(M) = \Delta(M)$. \square

4.4. The case $m = \omega(n^2 \log n)$, and n is odd. In this section we prove Theorem 1.4 for the case that n is odd and m is large. We then deduce Corollary 1.6.

Before proving Theorem 1.4 and Corollary 1.6, we will present the following observation that indicates the essential difference between small values of m and large values of m , in the case that n is odd. (It is the same phenomenon, as discussed in Section 1, that makes $\lceil \rho(G) \rceil$ a natural lower bound for $\chi'(G)$ for any multigraph G .)

Observation 4.6. *Let $\varepsilon > 0$ and let $m \geq (1 + \varepsilon)n^3 \log n$ where n is odd. Then w.h.p. $\Delta < \chi'(M) = (1 + o(1))\Delta$.*

Proof. Write $n = 2k + 1$. The maximal matching in M is of size at most k , and therefore one needs at least $\frac{m}{k}$ colours to colour the edges of M . Thus $\frac{m}{k} \leq \chi'(M)$. Furthermore, by Claim 2.26 we have that w.h.p. $\Delta \leq \frac{2m}{n} + \sqrt{(1 + \varepsilon)\frac{2m}{n} \log(n^2)}$. Together we have that w.h.p. $\chi'(M) - \Delta \geq \frac{m}{k} - \frac{2m}{n} - \sqrt{(1 + \varepsilon)\frac{2m}{n} \log(n^2)}$. As $\sqrt{m} \geq n^{3/2} \sqrt{(1 + \varepsilon) \log n}$, we get $\frac{m}{k} - \frac{2m}{n} - \sqrt{(1 + \varepsilon)\frac{2m}{n} \log(n^2)} > 0$, that is, w.h.p. $\Delta < \chi'(M)$.

For the upper bound, Note that by Corollary 2.25, we have that w.h.p. $\mu(M) - \mu_{\min} = O(\sqrt{\frac{2m}{n^2} \log n})$. Also, by Lemma 2.27 we have that w.h.p. $d_1 - d_2 \geq \frac{1}{f(n)} \sqrt{\frac{2m}{n \log n}}$. That is, w.h.p. $d_1 - d_2 \geq \mu(M) - \mu_{\min}$, and by Corollary 2.6 w.h.p.

$$\chi'(M) \leq \Delta(M) + \mu_{\min} \leq (1 + o(1))\Delta(M). \quad \square$$

For proving the theorems, we first need the following claim.

Claim 4.7. *Let $m = \omega(n^2 \log n)$, assume that n is odd, and let $M \sim M(n, m)$. Then w.h.p. $\rho(M) = \rho(\lfloor n \rfloor) = \frac{m}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. First, note that since n is odd then $\rho(M)$ is obtained on an odd set. Indeed, assume that $S \subset [n]$ and $|S|$ is even. Take $S' = S \cup \{v\}$ where $v \notin S$ (there is always such vertex since S is even and n is odd). Write $|S| = 2k$, and so $|S'| = 2k + 1$. Then $\lfloor \frac{|S|}{2} \rfloor = \lfloor \frac{|S'|}{2} \rfloor$ and $e(S) \leq e(S')$. Thus $\rho(S) \leq \rho(S')$.

Next, let $S \subsetneq [n]$ be an odd set. Write $n = 2r + 1$ and $|S| = 2k + 1$, then $2k + 1 < 2r$. Note that

$$\begin{aligned} \rho([n]) &= \frac{m}{r} = \frac{e(S) + e([n] \setminus S) + e(S, [n] \setminus S)}{r} \\ &\geq \frac{e(S)}{r} + \frac{\binom{2r-2k}{2} \cdot \mu_{\min} + (2k+1)(2r-2k) \cdot \mu_{\min}}{r} \\ &= \frac{e(S)}{r} + \frac{(r-k)(2r-2k-1) \cdot \mu_{\min} + 2(2k+1)(r-k) \cdot \mu_{\min}}{r} \\ &= \frac{e(S)}{r} + \frac{(2r-2k-1+2(2k+1))(r-k) \cdot \mu_{\min}}{r} \\ &= \frac{e(S)}{r} + \frac{(2r+2k+1)(r-k) \cdot \mu_{\min}}{r}. \end{aligned}$$

Thus it is enough to show that

$$\rho(S) = \frac{e(S)}{k} < \frac{e(S)}{r} + \frac{(2r+2k+1)(r-k) \cdot \mu_{\min}}{r},$$

that is, to show that $e(S)r < e(S)k + k(2r+2k+1)(r-k) \cdot \mu_{\min}$, or simply $e(S) < k(2r+2k+1) \cdot \mu_{\min}$.

Note that $e(S) \leq \binom{2k+1}{2} \cdot \mu = (2k+1)k \cdot \mu$, where $\mu := \mu(M)$. Therefore, we wish to show that $(2k+1)k \cdot \mu < k(2r+2k+1) \cdot \mu_{\min}$, that is, $\frac{\mu}{\mu_{\min}} < \frac{2r+2k+1}{2k+1} = 1 + \frac{2r}{2k+1}$. By Observation 2.24 and the fact that $m = \omega(n^2 \log n)$ we have that w.h.p. $\frac{\mu}{\mu_{\min}} = 1 + o(1)$; on the other hand, we assumed that $2r > 2k+1$, and thus the claim follows. \square

We now ready to prove Theorem 1.4 for the case that n is odd and m is large. In this proof, we will use Theorem 2.1.

Proof. Write $\chi'(M) = k+1$ and let $d_1 \geq d_2 \geq \dots \geq d_n$ be the degree sequence of M . By Theorem 2.1, at least one of the items (a), (b), (c), (d) holds. By Lemma 2.27 and the fact that $m = \omega(n^2 \log n)$, w.h.p. $d_1 - d_2 = \omega(1)$. Therefore, if Item (b) holds, we have that w.h.p. $\chi'(M) \leq d_2 + 3 = d_1 - \omega(1)$, which is impossible since $d_1 \leq \chi'(M)$ always. By Observation 2.24, using $m = \omega(n^2 \log n)$, we have that w.h.p. $\mu_{\min} = (1 - o(1))\mu$, therefore Item (c) is also impossible.

We are left with Items (a) and (d). If Item (a) occurs, then $\chi'(M) \leq d_1$, which gives $\chi'(M) = d_1$. By Claim 4.7, w.h.p. $\rho(M) = \rho(\lfloor n \rfloor) = \frac{e(M)}{\lfloor \frac{n}{2} \rfloor}$. Since n is odd we have that $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$, and thus if Item (d) occurs, then w.h.p. $\lceil \rho(M) \rceil = \lceil \rho(\lfloor n \rfloor) \rceil \geq k+1 = \chi'(M)$. That is, w.h.p. $\lceil \rho(M) \rceil = \chi'(M)$ in the case of Item (d). \square

We conclude this section with the proof of Corollary 1.6.

Proof of Corollary 1.6. For the case $m \leq n^2 \log^2 n$ we already saw in the previous subsections that w.h.p. $\chi'(M) = \Delta(M)$, so we may assume that $m = \omega(n^2 \log n)$.

By Theorem 1.4 we know that w.h.p. $\chi'(M) = \max\{\Delta(M), \lceil \rho(M) \rceil\}$. By Claim 4.7 we have w.h.p. $\rho(M) = \frac{2m}{n-1}$ and by (the proof of) Lemma 2.26 we have that w.h.p. $\Delta(M) \in \frac{2m}{n} + \sqrt{(1 \pm \frac{\varepsilon}{2}) \frac{4m \log n}{n}}$.

Assume that $m \leq (1 - \varepsilon)n^3 \log n$, then w.h.p. we have

$$\begin{aligned} \Delta(M) &\geq \frac{2m}{n} + \sqrt{\left(1 - \frac{\varepsilon}{2}\right) \frac{4m \log n}{n}} \\ &= \frac{2m}{n-1} \left(1 - \frac{1}{n} + \sqrt{\left(1 - \frac{\varepsilon}{2}\right) \frac{(n-1)^2 \log n}{nm}}\right) \\ &\geq \frac{2m}{n-1} \left(1 - \frac{1}{n} + \sqrt{\left(1 - \frac{\varepsilon}{2}\right) \frac{(n-1)^2 \log n}{n(1-\varepsilon)n^3 \log n}}\right) \\ &= \frac{2m}{n-1} \left(1 - \frac{1}{n} + \frac{n-1}{n^2} \sqrt{1 + \frac{\varepsilon}{2(1-\varepsilon)}}\right) > \frac{2m}{n-1} \left(1 + \frac{\Theta_\varepsilon(1)}{n}\right) \\ &> \frac{2m}{n-1} + 1 \geq \lceil \rho(M) \rceil, \end{aligned}$$

where $m = \omega(n^2 \log n)$.

Assume now that $m \geq (1 + \varepsilon)n^3 \log n$, then by Observation 4.6 we have that w.h.p. $\Delta(M) < \chi'(M)$, and since w.h.p. $\chi'(M) = \max\{\Delta(M), \lceil \rho(M) \rceil\}$ we are done. \square

4.5. Algorithmic issues. We conclude this section with a discussion about algorithmic aspects of our results and proofs. We begin by noting that Vizing's theorem (Theorem 1.1) is algorithmic, and could be stated as follows.

Theorem 4.8 (Vizing [32]). *There exists a polynomial-time algorithm that takes as input any simple graph G with the property that every cycle of G contains a vertex of degree less than $\Delta(G)$, and returns an edge-colouring of G with $\Delta(G)$ colours.*

Using this, we get that the proof of Theorem 1.3 is algorithmic, as well as the proof of Theorem 1.4 for the case $m \leq n \log^{100} n$.

We will next (briefly) explain how to show the existence of an efficient (polynomial time) algorithm that w.h.p. finds an optimal edge-colouring for other values of $m := m(n)$. In Remark 2.4 we mentioned that the proof of Theorem 2.3 is algorithmic. For convenience let us say (G, k, e_0, ϕ, T) is a legal 5-tuple if it satisfies the conditions of Theorem 2.3, namely:

- k is a positive integer,
- G is a multigraph with $\Delta(G) \leq k$,
- e_0 is an edge of G such that the endpoints x and y of e_0 satisfy $d(x) + d(y) \leq 2k - 2$,
- $d(v) \leq k - 1$ for each $v \in V(G) \setminus \{x, y\}$,
- ϕ is a k -edge-colouring of $G - e_0$,
- T is a ϕ -Tashkinov tree starting with e_0 .

Then Theorem 2.3 could be restated as follows.

Theorem 4.9. *There exists an algorithm Alg1 that takes as input any legal 5-tuple (G, k, e_0, ϕ, T) with the property that T is not ϕ -elementary, and returns in polynomial time a k -edge-colouring ψ of G .*

The colouring ψ is obtained from ϕ by a series of alternating path switches (see e.g. the proof of Theorem 5.1 in [30] and associated discussion of its algorithmic aspects).

Using Alg1, we obtain an algorithm AlgC that finds either an edge-colouring or an elementary Tashkinov tree. Here we say a multigraph G is (k, t) -bounded if $\Delta(G) \leq k$ and $d(v) \leq k - t$ for all except possibly one vertex of G .

Theorem 4.10. *There exists an algorithm AlgC that takes as input any positive integer $k \geq 2$ and any $(k, 2)$ -bounded multigraph G , and returns in polynomial time either*

- (1) *a k -edge colouring of G , or*
- (2) *a legal 5-tuple (G^*, k, e_0, ϕ, T) where G^* is a subgraph of G and T is a maximal ϕ -elementary Tashkinov tree.*

Moreover, if the subgraph G^ in output (2) has a vertex w with $d(w) \geq k - 1$ (for example $G^* = G$ is possible), then the edge e_0 is incident to w .*

Proof. AlgC proceeds as follows. Given G and k , list the edges of G in an arbitrary order, except that if G has a vertex w with $d(w) \geq k - 1$ then end the list with the set of edges incident to w . Colour the edges greedily in order with k colours until they are all coloured (and so we are done) or there is no available colour for the current edge. In this case call the current edge e_0 , remove all other uncoloured edges and call the resulting subgraph G' . With the current partial colouring ϕ of $G' - e_0$, grow a maximal Tashkinov tree T from e_0 (this is easily seen to be a polynomial procedure). If $V(T)$ is not ϕ -elementary then (G', k, e_0, ϕ, T) is a legal 5-tuple to which Theorem 4.9 applies. Apply Alg1 to obtain a k -edge-colouring of G' . Restore the remaining uncoloured edges in the order and repeat this last step with the next edge in the order until either all of G is coloured, or we stop at a partial colouring ϕ of some subgraph G^* of G , and a maximal ϕ -elementary Tashkinov tree T in G^* . Then (G^*, k, e_0, ϕ, T) is a legal 5-tuple as in output (2). Moreover if G^* contains a vertex of degree at least $k - 1$ then it is w , and by definition of the edge order e_0 is incident to w . \square

For the case that $n \log^9 n \leq m \leq n^2 \log^2 n$, and the case that $m = \omega(n^2 \log n)$ where n is even, we may use the following algorithmic version of Lemma 2.5.

Lemma 4.11. *The algorithm AlgC returns a k -edge-colouring of G , for any (k, t) -bounded multigraph G where $t \geq \max\{2, \mu(G)\}$.*

Proof. Suppose on the contrary that AlgC does not return a k -edge-colouring when applied to G . Then instead it returns a legal 5-tuple (G^*, k, e_0, ϕ, T) where G^* is a subgraph of G and T is a maximal ϕ -elementary Tashkinov tree. Let w denote one endpoint of e_0 , where (as given by AlgC) we may assume that w is the unique vertex in G^* of degree at least $k - 1$, if it exists.

As in the proof of Lemma 2.5 applied to G^* , with e_0 , ϕ and T given by AlgC, we find that $d_T(w) \geq t(|V(T)| - 1) + 1$. But clearly $d_T(w) \leq \mu(|V(T)| - 1)$, so we find $\mu > t$, contradicting the assumption of the lemma. \square

Having Lemma 4.11 (instead of Lemma 2.5) we can simply go over the proofs in Section 4.2 and in Section 4.3 to obtain an efficient algorithm that w.h.p. colours the edges of $M \sim M(n, m)$ with $\max\{\Delta(M), \lceil \rho(M) \rceil\}$ colours.

Finally we give only a brief outline of what happens in the case $m = \omega(n^2 \log n)$ and n is odd. In a very similar way as above (but with more technical details), for $k = \max\{\Delta(G), \lceil \rho(G) \rceil\}$ we show that AlgC will succeed in finding a k -edge-colouring of a multigraph G unless one of (b) or (c) in Theorem 2.1 holds (here (a) and (d) will not hold, by definition of k). The argument involves showing that if on the contrary output (2) is the result when AlgC is applied to G , then sets X , Z , Q and U as in the statement of Lemma 2.7 can be constructed. (In particular Q is the vertex set of the Tashkinov tree T given by output (2).) Then following the proof of Theorem 2.1 tells us that one of (b) or (c) holds. We may then conclude that w.h.p. AlgC succeeds in finding a k -edge-colouring of $M \sim M(n, m)$ when $m = \omega(n^2 \log n)$ and n is odd, since w.h.p. (b) does not hold according to Lemma 2.27, and w.h.p. (c) does not hold according to Observation 2.24.

5. CONCLUDING REMARKS AND FURTHER DISCUSSION

In this paper we considered the chromatic index of random multigraphs. We showed that random multigraphs are w.h.p. first class, verifying the Conjecture 1.2 for the random multigraph case. In particular, in Section 2.1 we gave general tools for bounding the chromatic index of a multigraph (see Theorem 2.1 and Lemma 2.5). We also addressed the informal conjecture mentioned in [30] that almost all multigraphs are first class. To be more formal, Theorem 1.4 proves that if $M \sim M(n, m)$ and $\Pr(M)$ is the probability that M was sampled according to the distribution $M(n, m)$, then $\lim_{n \rightarrow \infty} \sum_{\chi'(M) = \max\{\Delta, \lceil \rho \rceil\}} \Pr(M) = 1$.

Another natural way to generalize the definition of the random graph model $G(n, m)$ to multigraphs is to give each multigraph on n vertices and m edges the same probability (that is, to sample a multigraph with m edges and n vertices uniformly at random). Note that this probability space is not equivalent to the model $M(n, m)$ studied in this paper. Indeed, in $M(n, m)$ multigraphs that have “balanced” multiplicities are chosen with larger probabilities, unlike the uniform model that gives the same probability to a multigraph with one edge with multiplicity m , and a multigraph with many edges and smaller multiplicities. Note in addition that, similarly to the random graph model $G(n, p)$, our model gives a product probability space. For these reasons we found the model $M(n, m)$ to be more natural to consider. However, we believe that using similar methods to those used in this paper, one can show that the Conjecture 1.2 holds also in the uniform model.

Note that our proofs rely heavily on the degree gap, and thus one cannot use the same techniques to prove similar results for multigraphs without this property. Therefore, it would be interesting to study the same problem in the context of d -regular multigraphs in the random case, or for any family of nearly regular multigraphs. We expect this to be a challenging task, as for example the typical value of the chromatic index of a random d -regular graph $G_{n,d}$, for non-constant d , has been established only very recently by Ferber and Jain [10].

An additional (simpler) random multigraph model one can consider is a model corresponding to the random graph model $G(n, p)$. Let $M_p(n, m)$ be the probability space of all multigraphs on n vertices where each pair $e \in \binom{[n]}{2}$ appears independently with multiplicities $\mu(e) \sim \text{Poisson}(\lambda)$, where $\lambda = \frac{m}{\binom{n}{2}}$. This way, the expected number of edges of the multigraph is m , and for a vertex v we have $d(v) \sim \text{Poisson}(\frac{2m}{n})$. Given the well-known connection between the binomial and Poisson distributions, we can expect results similar to those obtained above to hold for this model as well.

Another related question one can also study concerns the list chromatic index in the context of random multigraphs. The *list chromatic index* of a multigraph M , denoted by χ'_ℓ , is the smallest integer k such that for every assignment of lists of size k to each edge, there exists an edge-colouring such that each edge receives a colour from its list. Clearly, $\chi'(M) \leq \chi'_\ell(M)$, and a trivial upper bound for the list chromatic index is $\chi'_\ell < 2\Delta(M)$. Borodin, Kostochka, and Woodall [3] showed that in fact $\chi'_\ell(M) \leq \lceil 3\Delta(M)/2 \rceil$, improving the previously known best upper bound of $9\Delta(M)/5$ due to Hind [17]. The famous list colouring conjecture states that for every multigraph M one has $\chi'(M) = \chi'_\ell(M)$ (see [30], p.260 for a short discussion). This conjecture was verified for the case that M is a bipartite multigraph by Galvin [13]. Later on, Kahn proved [20] that $\chi'(M) = (1 + o(1))\chi'_\ell(M)$. The presently rather limited understanding of edge colouring problems for multigraphs appears to indicate that the list colouring conjecture is very difficult. For this reason, we believe that attacking this problem in the random multigraph setting could be fruitful, as in this case we found the exact form of the chromatic index.

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Note added in proof. Very recently it was announced by G. Chen, G. Jing and W. Zang that they have obtained a proof of the full Goldberg Conjecture.

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