Cliques, Degrees, and Coloring: Expanding the ω, Δ, χ paradigm

by

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Abstract

Many of the most celebrated and influential results in graph coloring, such as Brooks' Theorem and Vizing's Theorem, relate a graph's chromatic number to its clique number or maximum degree. Currently, several of the most important and enticing open problems in coloring, such as Reed's ω, Δ, χ Conjecture, follow this theme.

This thesis both broadens and deepens this classical paradigm.

In Part I, we tackle list-coloring problems in which the number of colors available to each vertex v depends on its degree, denoted $d(v)$, and the size of the largest clique containing it, denoted $\omega(v)$. We make extensive use of the probabilistic method in this part.

We conjecture the "list-local version" of Reed's Conjecture, that is every graph is L-colorable if L is a list-assignment such that

$$|L(v)| \geq \lceil (1-\varepsilon)(d(v)+1) + \varepsilon\omega(v)) \rceil$$

for each vertex v and $\varepsilon \leq 1/2$, and we prove this for $\varepsilon \leq 1/330$ under some mild additional assumptions.

We also conjecture the "mad version" of Reed's Conjecture, even for list-coloring. That is, for $\varepsilon \leq 1/2$, every graph G satisfies

$$\chi_\ell(G) \leq \lceil (1-\varepsilon)(\mathrm{mad}(G)+1) + \varepsilon\omega(G) \rceil,$$

where $\mathrm{mad}(G)$ is the maximum average degree of G. We prove this conjecture for small values of ε, assuming $\omega(G) \leq \mathrm{mad}(G) - \log^{10} \mathrm{mad}(G)$. We actually prove a stronger result that improves bounds on the density of critical graphs without large cliques, a long-standing problem, answering a question of Kostochka and Yancey. In the proof, we use a novel application of the discharging method to find a set of vertices for which any precoloring can be extended to the remainder of the graph using the probabilistic method. Our result also makes progress towards Hadwiger's Conjecture: we improve the best known bound on the chromatic number of K_t-minor free graphs by a constant factor.

We provide a unified treatment of coloring graphs with small clique number. We prove that for Δ sufficiently large, if G is a graph of maximum degree at most Δ with list-assignment L such that for each vertex $v \in V(G)$,

$$|L(v)| \geq 72 \cdot d(v) \min\left\{ \sqrt{\frac{\ln(\omega(v))}{\ln(d(v))}}, \frac{\omega(v)\ln(\ln(d(v)))}{\ln(d(v))}, \frac{\log_2(\chi(G[N(v)])+1)}{\ln(d(v))} \right\}$$

and $d(v) \geq \ln^2 \Delta$, then G is L-colorable. This result simultaneously implies three famous results of Johansson from the 90s, as well as the following new bound on the chromatic number of any graph G with $\omega(G) \leq \omega$ and $\Delta(G) \leq \Delta$ for Δ sufficiently large:

$$\chi(G) \leq 72\Delta\sqrt{\frac{\ln \omega}{\ln \Delta}}.$$

In Part II, we introduce and develop the theory of fractional coloring with local demands. A fractional coloring of a graph is an assignment of measurable subsets of the $[0,1]$-interval to each vertex such that adjacent vertices receive disjoint sets, and we think of vertices "demanding" to receive a set of color of comparatively large measure. We prove and conjecture "local demands versions" of various well-known coloring results in the ω, Δ, χ paradigm, including Vizing's Theorem and Molloy's recent breakthrough bound on the chromatic number of triangle-free graphs.

The highlight of this part is the "local demands version" of Brooks' Theorem. Namely, we prove that if G is a graph and $f : V(G) \to [0,1]$ such that every clique K in G satisfies $\sum_{v\in K} f(v) \leq 1$ and every vertex $v \in V(G)$ demands $f(v) \leq 1/(d(v) + 1/2)$, then G has a fractional coloring ϕ in which the measure of $\phi(v)$ for each vertex $v \in V(G)$ is at least $f(v)$. This result generalizes the Caro-Wei Theorem and improves its bound on the independence number, and it is tight for the 5-cycle.

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Chapter 1

Introduction

In this thesis, we investigate graph coloring and related concepts, particularly list coloring and fractional coloring. In Section 1.1 we provide all of the background and basic definitions needed for the results in this work. In Section 1.2, we introduce the "ω, Δ, χ paradigm," which is central to this thesis. In this paradigm, we seek the best possible bound on the chromatic number χ of graphs of given clique number ω and maximum degree Δ. This thesis focuses more generally on how the presence of cliques and the degrees of vertices in a graph affect the ways in which we can color it. In Sections 1.3 and 1.5, we explore more robust versions of bounds on the list chromatic number and fractional chromatic number, respectively, that we call "local versions." Our results on list coloring sometimes imply new bounds on the chromatic number, as we see in Sections 1.2–1.4. Moreover, in Section 1.4, we present new bounds on the density of critical graphs, proved by applying our local list coloring results and the techniques we developed. Our fractional coloring results often imply new bounds on the independence number.

In Part I we prove the list coloring results that we present in Sections 1.2–1.4, and in Part II we prove the fractional coloring results that we present in Section 1.5.[1]

[1] Part I is primarily based on the papers [23] (with Marthe Bonamy, Peter Nelson, and Luke Postle), [87] (with Luke Postle), and [88] (with Luke Postle). In particular, Chapters 3 and 4 are based on [87], Chapter 5 is based on [88], and Chapter 6 is based on [23]. With the exception of Chapter 9, Part II is based on the preprint [86] (with Luke Postle).

1.1 Graph coloring preliminaries

A *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is a finite set of *vertices* of G and $E(G)$ is a set of unordered pairs of vertices of G, called *edges* of G. If $e \in E(G)$ is an edge and $e = \{u, v\}$ where $u, v \in V(G)$, then we write uv for e and say that u and v are *adjacent* and are *incident* to e. The concept of a *coloring* of a graph, introduced in the following definition, is fundamental to this work.

Definition 1.1.1. Let G be a graph.

- A *coloring* of G is a map $\phi : V(G) \to \mathbb{N}$ such that every edge $uv \in E(G)$ satisfies $\phi(u) \neq \phi(v)$. If the range of ϕ is a subset of $\{1, \ldots, k\}$, then ϕ is a *k-coloring* of G.
- The *chromatic number* of G, denoted $\chi(G)$, is the smallest $k \in \mathbb{N}$ for which there exists a k-coloring of G. If G has chromatic number k, then G is *k-chromatic*.

Graph coloring is one of the oldest and most active branches of graph theory. It can be dated back to the middle of the nineteenth century, when mathematicians became interested in the "Four Color Problem" (now the Four Color Theorem [7, 127]), which states that for any map of the earth, the regions can be colored with at most four colors so that neighboring regions receive different colors. This problem can be reformulated as stating that the chromatic number of any *planar* graph is at most four.

Modern graph coloring still presents many theoretical problems as well as practical applications. Much attention is devoted to proving bounds on the chromatic number. In this work, we supply such bounds on the chromatic number and some of its variants, including the *list chromatic number* and *fractional chromatic number* (introduced in Sections 1.1.2 and 1.1.3), respectively. Using the concepts of list coloring and fractional coloring, we also prove more robust versions of both new and classical bounds that we call "local versions". In Section 1.3, we discuss these local versions for list coloring, and in Section 1.5, we discuss local versions for fractional coloring.

These results are connected by the theme that they all concern colorings of graphs in relation to the degrees of their vertices and the sizes of their cliques.

1.1.1 Graph theory terminology

In this subsection, we introduce the requisite graph theory terminology for this work. For a comprehensive introduction, see the book of Diestel [46].

An *independent set* in a graph is a set of vertices that are pairwise non-adjacent. An independent set is also sometimes called a *stable set*. The *independence number* of a graph G, denoted $\alpha(G)$, is the size of a largest independent set in G. A k-coloring of a graph provides a partition of its vertices into k independent sets. Thus, every graph G satisfies $\chi(G) \cdot \alpha(G) \geq |V(G)|$.

It is convenient to introduce notation for graphs obtained by removing vertices or edges from a graph. To that end, if H and G are graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G and we write $H \subseteq G$. If G is a graph and $X \subseteq V(G)$, we let $G - X$ be the subgraph of G where $V(G - X) = V(G) \setminus X$ and $E(G - X) = \{uv \in E(G) : \{u, v\} \subseteq V(G) \setminus X\}$. If $X = \{v\}$, we may write $G - v$ instead of $G - X$. If $S \subseteq V(G)$ such that $S \cap X = \varnothing$ and $S \cup X = V(G)$, then we let $G[S] = G - X$, and we say $G[S]$ is an *induced subgraph* and that it is the graph *induced by* G *on* S. If G is a graph and $X' \subseteq E(G)$, we let $G - X'$ be the graph where $V(G - X') = V(G)$ and $E(G - X') = E(G) \setminus X'$. In the case when $X' = \{e\}$ where $e = uv$, we always write $G - uv$ to avoid ambiguity with the graph obtained by removing the vertices u and v.

We define a *connected* graph as follows. Any graph with one vertex and no edges is connected, and if G is a graph with a vertex $v \in V(G)$ that is incident to at least one edge and $G - v$ is connected, then G is connected. A *component* of a graph is a connected subgraph that is maximal with respect to the graph relation $\subseteq$. Note that the chromatic number of a graph is the maximum chromatic number of one of its components. A *cut-vertex* of a graph G is a vertex v such that $G - v$ has more components than v. A *block* of a graph G is a maximal connected subgraph that does not contain a cut-vertex.

If G is a graph and v is a vertex of G, then the *neighborhood* of v in G, denoted $N_G(v)$, is the set of vertices adjacent to v. The *closed neighborhood* of v is the set $\{v\} \cup N_G(v)$, and it is denoted $N_G[v]$. The *degree* of v in G, denoted $d_G(v)$, is the size of the set $N_G(v)$. We often omit the subscript G when it is not needed to distinguish between two different graphs.

A *cycle* is a connected graph in which every vertex has degree two, and a *path* is a connected graph in which two vertices have degree one and all other vertices have degree two, or a graph with only one vertex. The *length* of a cycle or path is the number of its edges, and a *k-cycle* is a cycle of length k. A classical observation is that cycles of even length have chromatic number two, while cycles of odd length have chromatic number three. We often call a 3-cycle a *triangle*. The *girth* of a graph is the length of the shortest cycle it contains.

A *complete graph* is a graph in which every pair of vertices is adjacent, and a *clique* in a graph is a set of vertices that are pairwise all adjacent. A *bipartition* of a graph G is a partition (A, B) of $V(G)$ such that A and B are independent sets. A graph G is *bipartite* if its vertices are independent or if it has a bipartition (A, B), and G is *complete bipartite* if it has a bipartition (A, B) where every vertex in A is adjacent to every vertex in B.

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that for every pair of vertices $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. Two graphs G and H are *isomorphic* if there is an isomorphism from one to the other, in which case we write $G \cong H$. Some isomorphism classes of graphs have canonical representations. For example, we use K_t to represent a complete graph on t vertices, C_k to represent a cycle of length k, and $K_{a,b}$ to represent a complete bipartite graph with bipartition (A, B) where $|A| = a$ and $|B| = b$.

1.1.2 List coloring

List coloring was first introduced in the 1970s independently by Vizing [147] and Erdős, Rubin, and Taylor [56]

Definition 1.1.2. Let G be a graph, and let $L = (L(v) \subseteq \mathbb{N} : v \in V(G))$ be a collection of "lists of colors".

- If $L(v)$ is non-empty for each vertex $v \in V(G)$, then L is a *list-assignment* for G, and if $|L(v)| \geq k$ for every $v \in V(G)$, then L is a *k-list-assignment*.
- An *L-coloring* of G is a coloring ϕ of G such that $\phi(v) \in L(v)$ for every vertex $v \in V(G)$, and G is *L-colorable* if there is an L-coloring of G.

The *list chromatic number* of G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable for any k-list-assignment L. If $\chi_\ell(G) \leq k$, then G is *k-list-colorable*.

The list chromatic number is also called the *choice number*, and a k-list-colorable graph is also called *k-choosable*.

If G is a graph with list-assignment L such that $L(v) = \{1, \ldots, k\}$ for each vertex $v \in V(G)$, then an L-coloring of G is also a k-coloring. Hence, every graph G satisfies $\chi(G) \leq \chi_\ell(G)$, but for some graphs this inequality is strict. For example, if $G \cong K_{3,3}$ with bipartition (A, B) and L is a list-assignment for G where the three vertices in A and the three vertices in B have lists of colors $\{1, 2\}, \{1, 3\}$, and $\{2, 3\}$, then G is not L-colorable. Hence, $\chi_\ell(K_{3,3}) > 2$. More generally, Erdős, Rubin, and Taylor [56] showed that $\chi_\ell\left(K_{\binom{2m-1}{m},\binom{2m-1}{m}}\right) > m$, and in fact, $\chi_\ell\left(K_{d,d}\right) = (1 - o(1)) \log_2(d)$. In a similar vein, Alon [4] showed that every graph of minimum degree d has list chromatic number at least

$(1/2 - o(1))\log_2(d)$. Using the container method, Saxton and Thomason [130] recently improved this bound on the list chromatic number by a factor of two, matching the bound of Erdős, Rubin, and Taylor up to lower order terms.

In this thesis, we prove new bounds on the list chromatic number that also provide new bounds on the chromatic number, such as Theorem 1.2.11 and Corollaries 1.3.5, 1.4.6, and 1.4.9. We also prove "local versions" of list coloring results, in which we consider list-assignments where the number of colors available to each vertex depends on local parameters, such as its degree.

We conclude this subsection by presenting a generalization of list coloring called *correspondence coloring*, also known as DP-coloring, first introduced by Dvořák and Postle [49].

Definition 1.1.3. Let G be a graph with list-assignment L.

- If $M = (M_e : e \in E(G))$ where M_e is a matching of $\{u\} \times L(u)$ and $\{v\} \times L(v)$ for each edge $e = uv \in E(G)$, then (L, M) is a *correspondence-assignment* for G. If L is a k-list-assignment for G, then (L, M) is a k-correspondence-assignment.
- An (L, M)-*coloring* of G is a map $\phi : V(G) \to \mathbb{N}$ such that $\phi(v) \in L(v)$ for every $v \in V(G)$ and every edge $e = uv \in E(G)$ satisfies $(u, \phi(u))(v, \phi(v)) \notin M_e$. If G has an (L, M)-coloring, then G is (L, M)-*colorable*.

The *correspondence chromatic number* of G, denoted $\chi_c(G)$, is the minimum k such that G is (L, M)-colorable for every k-correspondence-assignment (L, M).

For convenience, if G is a graph with correspondence-assignment (L, M) with $c_u \in L(u)$ and $c_v \in L(v)$ where $uv \in E(G)$ such that (u, c_u) is matched to (v, c_v) by M_{uv}, then we write $c_u c_v \in M_{uv}$ and we say that the color c_u *corresponds* to c_v. If $cc \in M_{uv}$ for every edge $uv \in E(G)$ and every color $c \in L(u) \cap L(v)$, then an (L, M)-coloring of G is an L-coloring. Thus, every graph G satisfies $\chi_\ell(G) \le \chi_c(G)$.

The correspondence chromatic number is an interesting graph parameter that has received much attention recently [13, 16, 18, 93, 114]. Correspondence coloring has also been useful for studying list coloring problems [16, 49]. Some classical list coloring results naturally extend to correspondence coloring, such as Thomassen's [145] proof that planar graphs are 5-list-colorable. In this thesis, some of our list coloring results such as Theorem 1.3.7 also hold for correspondence coloring, in which case we present them in full generality. Other results, notably Theorem 1.3.4, do not hold for correspondence coloring. However, as we see in Chapters 4 and 5, it is often convenient to work with correspondence coloring for certain applications of the probabilistic method, as in the proof of Theorem 1.3.4.

1.1.3 Fractional coloring

Fractional coloring was first introduced by Hilton, Rado, and Scott [79] in an attempt to prove a relaxation of the Four Color Theorem, which was unknown at the time. As Proposition 1.1.6 will show, there are several possible ways to view fractional coloring. Moreover, notation is not standardized. From the perspective of graph coloring, we find the following definition and notation, first used by Dvořák, Sereni, and Volec [50, 51], to be the most fundamental and natural. This definition is also most amenable to "local versions."

Definition 1.1.4. Let G be a graph.

- A *fractional coloring* of G is a function ϕ with domain $V(G)$ such that for each $v \in V(G)$, the image $\phi(v)$ is a measurable subset of the $[0,1]$-interval such that for each $uv \in E(G)$, we have $\phi(u) \cap \phi(v) = \varnothing$.
- A *demand function* for G is a function $f : V(G) \to [0,1] \cap \mathbb{Q}$.
- If f is a demand function for a graph G, an *f-coloring* is a fractional coloring ϕ such that for every $v \in V(G)$, we have $\mu(\phi(v)) \geq f(v)$, where μ is the Lebesgue measure on the real numbers.
- The *fractional chromatic number* of G, denoted $\chi_f(G)$, is the infimum over all positive real numbers k such that G admits an f-coloring when $f(v) = 1/k$ for each $v \in V(G)$.

The concept of measure and measurable set is not essential in Definition 1.1.4. Instead of measurable subsets of the $[0,1]$-interval, we could use finite unions of open intervals with rational endpoints. The measure of these sets is simply the sum of the length of the intervals. Using measure theory simplifies the definition.

It is sometimes convenient to work with the following discrete analogue of a fractional coloring.

Definition 1.1.5. A *multicoloring* of a graph G is a function ψ with domain $V(G)$ such that for each $v \in V(G)$, the image $\psi(v)$ is a finite subset of $\mathbb{N}$ such that for each $uv \in E(G)$, we have $\psi(u) \cap \psi(v) = \varnothing$.

The fractional chromatic number is sometimes defined as the solution to a linear program, obtained by first constructing an integer program having the chromatic number as the optimal solution, and then taking the fractional relaxation. We avoid this definition in this thesis; however, this polyhedral perspective is occasionally useful. To that end, we use the *stable-set polytope* of a graph G, which is the convex hull of the incidence vectors in $\mathbb{R}^{|V(G)|}$ of independent sets of G.

We now demonstrate several equivalent ways to view fractional coloring.

Proposition 1.1.6 (Dvořák, Sereni, and Volec [50]). *Let G be a graph with demand function f. The following are equivalent.*

(a) The graph G has an f-coloring.
(b) There exists some $N \in \mathbb{N}$ and a multicoloring ψ of G with range $\{1, \ldots, N\}$ such that $|\psi(v)| \geq f(v) \cdot N$ for each $v \in V(G)$.
(c) The vector $(f(v) : v \in V(G))$ is in the stable-set polytope of G.
(d) For every nonnegative weight function $w : V(G) \to \mathbb{R}_+$, the graph G contains an independent set I such that $\sum_{v \in I} w(v) \geq \sum_{v \in V(G)} w(v)f(v)$.

Note that a k-coloring is also a multicoloring, so Proposition 1.1.6 implies that a graph with chromatic number k has an f-coloring when $f(v) = 1/k$ for every vertex v. Hence, every graph G satisfies $\chi_f(G) \leq \chi(G)$. Proposition 1.1.6 also implies that every graph G has an independent set of size at least $|V(G)|/\chi_f(G)$. Therefore we can relate all of the previously mentioned graph coloring parameters in the following inequality:

$$|V(G)|/\alpha(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \leq \chi_c(G). \tag{1.1}$$

Let us describe how we use (a)–(d) of Proposition 1.1.6 in this thesis. When working with fractional coloring, we primarily use (a), introduced in Definition 1.1.4. Multicolorings are usually easier to describe explicitly for small examples, and multicolorings allow us the capability of "permuting" colors. We can use weighted independepence numbers as in (d) as a certificate that a graph does not have an f-coloring. Furthermore, using (d), we obtain many new bounds on the independence number. Using the stable-set polytope allows us to use results from polyhedral combinatorics, primarily Edmond's Matching Polytope Theorem [52]. Using this theorem, we can derive the following result on fractionally coloring odd cycles.

Proposition 1.1.7. *If H is a cycle of length $2k+1$ and g is a demand function for H, then H has a g-coloring if and only if $\sum_{v \in V(H)} g(v) \leq k$ and every edge $uv \in E(H)$ satisfies $g(u) + g(v) \leq 1$.*

Odd cycles are the standard examples of graphs for which the chromatic number and fractional chromatic number are different. Proposition 1.1.7 implies that $\chi_f(C_{2k+1}) = 2 + 1/k$.

There are several notorious open problems regarding the chromatic number that have been proved for the fractional chromatic number. For example, as we see later, Reed's Conjecture [121] is known to be true for the fractional chromatic number, and Kilakos and

Reed [89] proved the fractional relaxation of the Total Coloring Conjecture [12]. Moreover, Reed and Seymour [124] proved that the fractional chromatic number of graphs with no K_{t+1}-minor is at most $2t$, a factor of two away from the bound infamously conjectured by Hadwiger [69]. Other bounds on the fractional chromatic number generalize bounds on the independence number. For example, Dvořák, Sereni, and Volec [50] proved that triangle-free graphs of maximum degree three have fractional chromatic number at most 14/5, generalizing a well-known result of Staton [140] and resolving a conjecture of Heckman and Thomas [78]. In Section 1.5, we see that our results on fractional coloring with local demands both generalize bounds on the independence number and imply new ones, while also providing more robust versions of bounds on the fractional chromatic number.

1.2 ω, Δ, and χ

In this section we introduce the "ω, Δ, χ paradigm," which includes many of the most classical results in graph coloring. Recall that the degree of a vertex in a graph is the size of its neighborhood. For every graph G, we let $\Delta(G)$ denote the maximum degree of a vertex in G. Our starting point is the following bound on the chromatic number of a graph in terms of its maximum degree.

Proposition 1.2.1 (Greedy Bound). *Every graph G satisfies $\chi(G) \leq \Delta(G) + 1$.*

We refer to Proposition 1.2.1 as the "Greedy Bound" because it is proved by considering the number of colors used after coloring the vertices of G greedily. The bound in Proposition 1.2.1 is tight for complete graphs, since $\chi(K_t) = t$ and $\Delta(K_t) = t - 1$. More generally, the chromatic number of a graph is at least the size of a largest clique that it contains. For every graph G, we let $\omega(G)$, the *clique number of G*, denote the size of a largest clique in G. These graph parameters satisfy the following pair of inequalities:

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1. \tag{1.2}$$

In the ω, Δ, χ paradigm, we seek improvements on the bound in Proposition 1.2.1 under certain hypotheses about the clique number. Table 1.1 summarizes the discussion of this section. Note that many of the results and conjectures in this table are for Δ sufficiently large.

1.2.1 Brooks' Theorem

The most classic result of this type is the following theorem of Brooks [30].

Theorem 1.2.2 (Brooks' Theorem [30]). *If G is a graph such that $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.*

Brooks' Theorem can be used to characterize the graphs for which equality holds in Proposition 1.2.1. If G is a connected graph with maximum degree at least three such that $\chi(G) = \Delta(G) + 1$, then by Brooks' Theorem, G is a complete graph. If G is a connected graph with maximum degree at most two such that $\chi(G) = \Delta(G) + 1$, then G is either a complete graph or a cycle of odd length.

Brooks' Theorem improves the Greedy Bound by one. In 1977, Borodin and Kostochka [25] conjectured that if a graph G satisfies $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G) - 1$, then $\chi(G) \leq \Delta(G) - 1$, that is the Greedy Bound can be improved by two. Using the probabilistic method, Reed [122] proved that the Borodin-Kostochka Conjecture holds for graphs of sufficiently large maximum degree. One might wonder if this pattern continues, that is, for any k, does every graph G of sufficiently large maximum degree such that $\omega(G) \leq \Delta(G) + 1 - k$ satisfy $\chi(G) \leq \Delta(G) + 1 - k$? This turns out to be false for $k \geq 3$, as demonstrated by graphs obtained from a complete graph by removing the edges of a 5-cycle. If G is such a graph, then $\omega(G) = \Delta(G) - 2$ and $\chi(G) = \Delta(G) - 1$.

Thus, we can ask what additional conditions guarantee $\chi(G) \leq \Delta(G) + 1 - k$. There are two natural possibilities to consider. As we see in the next subsection, Reed's Conjecture takes a more restrictive assumption on the clique number. In the other direction, one can forbid certain additional subgraphs. Farzad, Molloy, and Reed [59] described precisely which graphs are the minimal obstructions for small values of k. Later, Molloy and Reed [110] proved that if G is a graph such that $\Delta(G)$ is sufficiently large, $(k+1)(k+2) \leq \Delta(G)$, and $\chi(G) > \Delta(G) + 1 - k$, then G contains a subgraph H that is "clique-like" in the following sense: $|V(H)| \leq \Delta(G) + 1$ and $\chi(H) > \Delta(G) + 1 - k$.

1.2.2 Reed's Conjecture

In 1998, Reed [121] made the following conjecture involving ω, Δ, and χ.

Conjecture 1.2.3 (Reed's Conjecture [121]). *Every graph G satisfies*

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil.$$

Reed's Conjecture can be shown to be equivalent to the statement that for every k, if G is a graph such that $\omega(G) \leq \Delta(G)+1-2k$, then $\chi(G) \leq \Delta(G)+1-k$. If Reed's Conjecture is true, then the assumption that $\omega(G) \leq \Delta(G) + 1 - 2k$ is best possible in the following sense. For every positive integer k, there exists a graph G_k satisfying $\omega(G) = \Delta(G)+1-2k$ and $\chi(G) = \Delta(G) + 1 - k$. Let G_k be the graph obtained from a 5-cycle by replacing each vertex with a clique of size $2k$ and each edge with a complete bipartite graph, and note that $\Delta(G_k) = 6k - 1$ and $\omega(G_k) = 4k = \Delta(G_k) + 1 - 2k$. It is straightforward to show that an N-coloring of G_k can be used to obtain a multicoloring of C_5 using N colors in which each vertex receives $2k$ colors. Therefore since $\chi_f(C_5) = 5/2$, we have $\chi(G_k) \geq 5k$. In fact, $\chi(G_k) = 5k = \Delta(G_k) + 1 - k$, as required.

When Reed [121] made his conjecture, he proved that for every k, if G is a graph of maximum degree at least $10^8 \cdot k$ such that $\omega(G) \leq \Delta(G)+1-2k$, then $\chi(G) \leq \Delta(G)+1-k$. It would be a major breakthrough to prove this result for graphs of sufficiently large maximum degree with no dependence on k. Using this result, Reed [121] derived the following relaxation of Conjecture 1.2.3.

Theorem 1.2.4 (Reed [121])**.** *There exists $\varepsilon > 0$ such that every graph G satisfies*

$$\chi(G) \leq (1 - \varepsilon)(\Delta(G) + 1) + \varepsilon\omega(G).$$

We often refer to Theorem 1.2.4 as an "epsilon version" of Reed's Conjecture. Up to rounding, Reed's Conjecture is equivalent to Theorem 1.2.4 when $\varepsilon = 1/2$. Thus, one line of research towards Reed's Conjecture is to prove Theorem 1.2.4 for larger values of ε. For graphs of sufficiently large maximum degree, Bonamy, Perrett, and Postle [24] proved that Theorem 1.2.4 holds with $\varepsilon = 1/26$, and Delcourt and Postle [43] proved it holds with $\varepsilon = 1/13$. We discuss the proof of Theorem 1.2.4 and how these improvements on the value of ε are obtained in greater detail in Chapters 3 and 4.

Another piece of evidence for Reed's Conjecture is that it holds for the fractional chromatic number, as follows.

Theorem 1.2.5. *Every graph G satisfies*

$$\chi_f(G) \leq \frac{\Delta(G) + 1 + \omega(G)}{2}.$$

Note that the rounding in Reed's Conjecture is not needed in Theorem 1.2.5. Theorem 1.2.5 is proved in the book of Molloy and Reed [109]. Theorem 1.2.5 and (1.1) together imply that every graph G satisfies $\alpha(G) \geq 2/(\Delta(G) + 1 + \omega(G))$. This result was proved earlier by Fajtlowicz [57, 58].

For the remainder of this subsection, we discuss various possible strengthenings of Reed's Conjecture. In 2009, King [94] made the following conjecture.

Conjecture 1.2.6 (King [94])**.** *Every graph G satisfies*

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil .$$

King's idea behind Conjecture 1.2.6 was that a strengthened form of Reed's Conjecture may be easier to prove using induction. For certain classes of graphs, this idea has been useful. Using this and the structure theory of claw-free graphs of Chudnovsky and Seymour, King [94] proved that Reed's Conjecture is true for claw-free graphs. The proof also appears in [95]. In 2013, Chudnovsky et al. [36] proved that King's Conjecture holds for quasi-line graphs, and in 2015 King and Reed [95] proved it for claw-free graphs with a three-colorable complement. In the next section, we show how our results imply an "epsilon version" of Conjecture 1.2.6, that is a relaxation of Conjecture 1.2.6 similar to Theorem 1.2.4.

In the next section, we conjecture a "local list version" of Reed's Conjecture. As we will see, this conjecture, if true, implies that Reed's Conjecture holds for the list chromatic number. Thus, we also conjecture the following.

Conjecture 1.2.7 (List coloring version of Reed's Conjecture)**.** *Every graph G satisfies*

$$\chi_\ell(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil .$$

Reed's [121] proof of Theorem 1.2.4 does not work for list coloring. The aforementioned result of Delcourt and Postle [43], however, actually holds for list coloring.

The bound on the chromatic number supplied by Reed's Conjecture can be viewed as the average of the lower and upper bounds provided in (1.2). However, the upper bound in (1.2) can easily be improved by replacing $\Delta(G)$ with $\lfloor \mathrm{mad}(G) \rfloor$, where $\mathrm{mad}(G) = \max_{H \subseteq G} \mathrm{ad}(H)$ and $\mathrm{ad}(H)$ is the average degree of a vertex in H. We call $\mathrm{mad}(G)$ the *maximum average degree* of G. In the spirit of Reed's Conjecture, we conjecture the following which, if true, implies Reed's Conjecture.

Conjecture 1.2.8. *Every graph G satisfies*

$$\chi_\ell(G) \leq \left\lceil \frac{\mathrm{mad}(G) + 1 + \omega(G)}{2} \right\rceil .$$

As we see in Section 1.4, Conjecture 1.2.8 is closely related to the density of critical graphs without large cliques. We use our work on local list versions to obtain an epsilon version of Conjecture 1.2.8, as well as applications to the density of critical graphs, which we present in Section 1.4. A result of Kostochka and Stiebitz [98], discussed further in Section 1.4, implies that if G is triangle-free, then $\chi(G) \leq (1/2 + o(1)) \operatorname{mad}(G)$, confirming Conjecture 1.2.8 asymptotically for this case.

In contrast with Reed's Conjecture, the bound on the chromatic number supplied by Conjecture 1.2.8 is close to tight even for triangle-free graphs. As shown by Kostochka and Nešetřil [101], Tutte [45] (under the pseudonym Blanche Descartes) provided a construction of triangle-free graphs with chromatic number k and average degree at most $2k$. As we see in the next subsection, for triangle-free graphs of large maximum degree, the bound in Reed's Conjecture can be considerably strengthened.

1.2.3 Coloring graphs with small clique number

In this subsection, we discuss how graphs with clique number much smaller than their maximum degree can be colored with significantly fewer colors than is guaranteed by the Greedy Bound.

In 1996, Johansson [81] famously proved that if G is a triangle-free graph of maximum degree at most Δ, then $\chi_\ell(G) = O(\Delta/\ln(\Delta))$, although his result was never published. Determining the best possible value of the leading constant in this bound is of general interest. The best known lower bound, from random Δ-regular graphs, is $\frac{\Delta}{2\ln(\Delta)}$. In 1995, Kim [90] proved that the upper bound holds with a leading constant of $1 + o(1)$ for graphs of girth at least five. In 2015, Pettie and Su [120] improved the leading constant in the upper bound for triangle-free graphs to $4 + o(1)$, and in 2017, Molloy [107], in the following theorem, improved it to $1 + o(1)$, matching the bound of Kim.

Theorem 1.2.9 (Molloy [107])**.** *If G is a triangle-free graph of maximum degree at most Δ, then*

$$\chi_\ell(G) \leq (1 + o(1))\frac{\Delta}{\ln(\Delta)}.$$

Johansson [82] also proved that for any fixed $\omega \geq 4$, if G is a graph of maximum degree at most Δ with no clique of size greater than ω, then $\chi_\ell(G) = O(\Delta \ln(\ln(\Delta))/\ln(\Delta))$; however, the proof was never published. Molloy [107] proved the following stronger result, which holds even when ω is not fixed.

Theorem 1.2.10 (Molloy [107])**.** *If G is a graph of maximum degree at most Δ with no clique of size greater than ω, then*

$$\chi_\ell(G) \leq 200\omega\frac{\Delta\ln(\ln(\Delta))}{\ln(\Delta)}.$$

Improving the leading constant in Theorem 1.2.9 or improving the bound in Theorem 1.2.10 by more than a constant factor would be a major breakthrough. The *Ramsey number* $R(\ell, k)$ is the smallest n such that every graph on at least n vertices contains either a clique of size ℓ or an independent set of size k. In 1980, Ajtai, Komlós, and Szemerédi proved that every triangle-free graph G satisfies $|V(G)|/\alpha(G) \leq 100\frac{\mathrm{ad}(G)}{\ln\mathrm{ad}(G)}$, and Shearer [136] improved the leading constant to $1 + o(1)$. Using Shearer's result, it is straightforward to show that $R(3, k) \leq (1 - o(1))\frac{k^2}{\ln k}$. Any improvement to the leading constant in Theorem 1.2.9 would also improve this bound on the Ramsey number $R(3, k)$. In 1995, Kim [90] proved that $R(3, k) = \Omega(k^2/\ln k)$, and in 2013, Fiz Pontiveros, Griffiths, and Morris [60] and independently Bohman and Keevash [22] proved a lower bound of $(1/4 - o(1))k^2/\ln k$ for $R(3, k)$. Theorem 1.2.10 implies a result of Shearer [138] that for any r, every K_r-free graph satisfies $|V(G)|/\alpha(G) = O\left(\frac{\Delta(G)\ln(\ln(\Delta(G)))}{\ln(\Delta(G))}\right)$. Improving this bound would make progress towards a longstanding problem of Ajtai, Erdős, Komlós, and Szemerédi [1], namely that the $\ln(\ln(\Delta(G)))$ can be omitted from this bound on the independence number.

We provide more detail on these results and their proofs in Chapter 6.

Theorem 1.2.10 only provides a nontrivial bound if $\omega = O(\ln\Delta/\ln\ln\Delta)$. If we relax this assumption on the clique number, then we can still bound the chromatic number by a function that is $o(\Delta)$. This result follows from the following theorem, which we prove in Chapter 6.

Theorem 1.2.11 (Bonamy, Kelly, Nelson, Postle [23])**.** *If G is a graph of maximum degree at most Δ with no clique of size greater than ω and Δ is sufficiently large, then*

$$\chi_c(G) \leq 72\Delta\sqrt{\frac{\ln(\omega)}{\ln(\Delta)}}.$$

Note that Theorem 1.2.11 holds for correspondence coloring. Bernshteyn [14] showed that Theorems 1.2.9 and 1.2.10 also hold for correspondence coloring and found a simpler proof.

Using Theorem 1.2.11, we obtain the following corollary.

Corollary 1.2.12. *Let $f(c) = (72c)^2$. If G is a graph of maximum degree at most Δ such that $\omega(G) \leq \Delta^{1/f(c)}$ and Δ is sufficiently large, then $\chi_c(G) \leq \Delta/c$.*

Corollary 1.2.12 implies that graphs with clique number at most ω and maximum degree at most Δ such that $\ln \omega = o(\ln \Delta)$ have chromatic number bounded by $o(\Delta)$. Corollary 1.2.12 also implies that for some $\varepsilon > 0$, Reed's Conjecture holds for graphs G satisfying $\omega(G) \leq \Delta(G)^{\varepsilon}$, which was not previously known.

Determining the best possible function f in Corollary 1.2.12 would be very interesting. Spencer [139] proved that the Ramsey number $R(c, \omega)$ is at least $\Omega\left((\omega/\ln(\omega))^{\frac{c+1}{2}}\right)$ as $\omega \to \infty$ for fixed $c \geq 3$. Therefore there exists a graph G on n vertices with no independent set of size c (and thus chromatic number at least $n/(c-1)$) and no clique of size ω where n is at least $\omega^{\frac{c+1}{2} - o(1)}$. Since the maximum degree of a graph is at most its number of vertices, it follows that $f(c) \geq c/2 + 1$ if $c \in \mathbb{N}$.

As previously mentioned, the bound of Spencer [139] was improved by Kim [91] for $c = 3$ by a factor of $\ln \omega$ (matching the upper bound of Ajtai, Komlós, and Szemerédi [2] up to a constant factor), and it was improved by Bohman [19] for $c = 4$ by a factor of $\sqrt{\ln \omega}$, and by Bohman and Keevash [21] for $c \geq 5$ by a factor of $\ln^{\frac{1}{c-2}} \omega$. However, these improvements do not change the resulting lower bound on $f(c)$ in Corollary 1.2.12.

Theorem 1.2.11 also implies that for Δ sufficiently large, a graph G with $\omega(G) \leq \omega$ and $\Delta(G) \leq \Delta$ satisfies $|V(G)|/\alpha(G) \leq 72\Delta\sqrt{\ln \omega / \ln \Delta}$, which was proved earlier by Bansal, Gupta, and Guruganesh [9, 10]. Using the above result of Spencer [139], they proved that any bound on $|V(G)|/\alpha(G)$ must be $\Omega(\Delta \ln \omega / \ln \Delta)$.

1.3 Local list versions

In this section, we introduce "local versions" for list coloring. Most of the previously mentioned list coloring results have a natural local analogue. The following is the local version of the Greedy Bound, Proposition 1.2.1.

Proposition 1.3.1 (List-local Greedy Bound)**.** *If G is a graph with list-assignment L such that every vertex $v \in V(G)$ satisfies $|L(v)| \geq d(v) + 1$, then G has an L-coloring.*

Note that Proposition 1.3.1 implies the Greedy Bound. Whereas the Greedy Bound uses the global graph parameter $\Delta(G)$ to bound the list chromatic number, in the local version, each vertex has a list of colors of size according to its degree. Independently Erdős,

ω	$\chi \leq$	
$\leq \Delta$	Δ	Brooks [30]
$\leq \Delta - 1$	$\Delta - 1$	Borodin-Kostochka [25], Reed [121]
$\leq \Delta + 1 - 13k$ $\leq \Delta + 1 - 2k$?	$\Delta + 1 - k$	Delcourt and Postle [43] Reed [121]
$\leq \Delta^{1/(72c)^2}$ $\leq \Delta^{2/(c+2)}$?	Δ/c for $c \geq 2$	Corollary 1.2.12 [23]
	$72\Delta\sqrt{\frac{\ln \omega}{\ln \Delta}}$ $200\Delta\frac{\omega \ln \ln \Delta}{\ln \Delta}$	Theorem 1.2.11 [23] Molloy [107]
$= O(1)$	$O(\Delta\frac{\ln \ln \Delta}{\ln \Delta})$ $O(\Delta/\ln \Delta)$?	Johansson [82]
$= 2$	$(1 + o(1))\Delta/\ln \Delta$ $\frac{\Delta}{2 \ln \Delta}$?	Molloy [107]

Table 1.1: The state of the art in the ω, Δ, χ paradigm for large Δ.

Rubin, and Taylor [56] and Borodin [26, 27] famously proved a local version of Brooks' Theorem, as follows.

Theorem 1.3.2 (Borodin [26, 27]; Erdős, Rubin, and Taylor [56])**.** *If G is a connected graph with list-assignment L such that every vertex $v \in V(G)$ satisfies $|L(v)| \geq d(v)$, then G has an L-coloring unless every block of G is a complete graph or an odd cycle.*

Besides this beautiful result and the "local version" of Galvin's [63] proof of the Dinitz Conjecture, proved by Borodin, Kostochka, and Woodall [28] and extended by Peterson and Woodall [118, 119], "local versions" have not received much attention. In this work, we prove local versions of Theorems 1.2.4, 1.2.9, 1.2.10, and 1.2.11.

1.3.1 A local version of Reed's Conjecture

The following is the local version of Reed's Conjecture.

Conjecture 1.3.3 (List-local Reed's Conjecture [87])**.** *If G is a graph with list-assignment L such that every vertex $v \in V(G)$ satisfies*

$$|L(v)| \geq \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil,$$

where $\omega(v)$ is the size of the largest clique containing v, then G is L-colorable.

In Chapter 4, we prove an epsilon version of Conjecture 1.3.3 under some mild assumptions on the clique number of each vertex's neighborhood, as follows. This result can be viewed as a local version of Theorem 1.2.4.

Theorem 1.3.4. *Let $\varepsilon \leq 1/330$. If Δ is sufficiently large, G is a graph of maximum degree at most Δ, and L is a list-assignment for G such that every vertex $v \in V(G)$ satisfies $|L(v)| \geq \omega(v) + \log^{10}(\Delta)$ and*

$$|L(v)| \geq (1 - \varepsilon)(d(v) + 1) + \varepsilon\omega(v),$$

then G is L-colorable.

Using Theorem 1.3.4, we obtain an epsilon version of Conjecture 1.2.6 for graphs without large cliques, as follows.

Corollary 1.3.5. *Let $\varepsilon < 1/330$. If G is a graph of sufficiently large maximum degree such that $\omega(G) + \log^{10} \Delta(G) \leq (1 - \varepsilon)\Delta(G)$, then*

$$\chi_\ell(G) \leq \max_{v \in V(G)} (1 - \varepsilon)(d(v) + 1) + \varepsilon\omega(v).$$

We present more applications of Theorem 1.3.4 in Section 1.4. We conclude this subsection by introducing some notation that we will frequently use.

Definition 1.3.6. Let G be a graph. For each $v \in V(G)$ we let $\mathrm{Gap}_G(v) = d(v) + 1 - \omega(v)$, and if L is a list-assignment for G, we let $\mathrm{Save}_L(v) = d(v) + 1 - |L(v)|$.

When the graph G or list-assignment L is clear from the context, then we may omit the subscripts G and L in Gap and Save, respectively. Note that Proposition 1.3.1 and Theorem 1.3.2 concern list-assignments such that every vertex v satisfies $\mathrm{Save}(v) \leq 0$ and $\mathrm{Save}(v) \leq 1$, respectively. Conjecture 1.3.3 concerns list-assignments for which every vertex v satisfies $\mathrm{Save}(v) \leq \lfloor \mathrm{Gap}(v)/2 \rfloor$. In Theorem 1.3.4, every vertex v satisfies $\mathrm{Save}(v) \leq \varepsilon\,\mathrm{Gap}(v)$ and $\mathrm{Gap}(v) - \mathrm{Save}(v) \geq \log^{10}(\Delta)$.

1.3.2 A unifying local version

In this subsection, we present local versions of Theorems 1.2.9, 1.2.10, and 1.2.11. We actually prove a local version of all three of these results simultaneously in the following theorem, although we do not match the leading constant in Theorem 1.2.9.

Theorem 1.3.7 (Bonamy, Kelly, Nelson, and Postle [23]). *For all sufficiently large Δ the following holds. Let G be a graph of maximum degree at most Δ with correspondence assignment (L, M). If every vertex $v \in V(G)$ satisfies*

$$|L(v)| \geq 72d(v) \min\left\{\sqrt{\frac{\ln(\omega(v))}{\ln(d(v))}}, \frac{\omega(v)\ln(\ln(d(v)))}{\ln(d(v))}, \frac{\log_2(\chi(G[N(v)])+1)}{\ln(d(v))}\right\},$$

and $d(v) \geq \ln^2(\Delta)$, then G is (L, M)-colorable.

A slightly stronger version of Theorem 1.3.7 actually holds for all graphs if we assume the list sizes are sufficiently large with respect to Δ. In particular, if we let $\delta = \ln^2(\Delta)$, then we can replace the condition $d(v) \geq \ln^2(\Delta)$ with the assumption that $|L(v)| \geq 72\delta/\ln(\delta)$.

Theorem 1.3.7 implies a local version of an unpublished result of Johansson [82] on graphs that are *locally r-colorable*, meaning the neighborhood of every vertex is r-colorable. Note that triangle-free graphs are locally 1-colorable.

Corollary 1.3.8. *For some constant C the following holds. If G is a locally r-colorable graph of maximum degree at most Δ with correspondence assignment (L, M) such that for each $v \in V(G)$,*

$$|L(v)| \geq Cd(v)\frac{\log_2(r+1)}{\ln(d(v))},$$

and $d(v) \geq \ln^2(\Delta)$, then G is (L, M)-colorable.

Theorem 1.3.7 also implies a local version of Theorem 1.2.10, as follows.

Corollary 1.3.9. *For some constant C the following holds. If G is a graph of maximum degree at most Δ with correspondence assignment (L, M) such that for each $v \in V(G)$,*

$$|L(v)| \geq Cd(v)\frac{\omega(v)\ln(\ln(d(v)))}{\ln(d(v))},$$

and $d(v) \geq \ln^2(\Delta)$, then G is (L, M)-colorable.

Of course, Theorem 1.3.7 also implies a "local version" of Theorem 1.2.11, as follows.

Corollary 1.3.10. *For some constant C the following holds. If G is a graph of maximum degree at most Δ with correspondence assignment (L, M) such that for each $v \in V(G)$,*

$$|L(v)| \geq Cd(v)\sqrt{\frac{\ln(\omega(v))}{\ln(d(v))}},$$

and $d(v) \geq \ln^2(\Delta)$, then G is (L, M)-colorable.

Using a standard technique, we can derive Theorem 1.2.11 from Theorem 1.3.7.

Proof of Theorem 1.2.11 assuming Theorem 1.3.7. Let G be a graph of maximum degree at most Δ with no clique of size greater than ω. We may assume that G has minimum degree at least one. If G has minimum degree at least $\ln^2(\Delta)$, then Theorem 1.3.7 implies $\chi_c(G) \leq 72\Delta\sqrt{\frac{\ln(\omega)}{\ln(\Delta)}}$, as desired. Otherwise, we use the following standard procedure to obtain a graph of larger minimum degree containing G as a subgraph. We duplicate the graph G, and we add an edge between each vertex of minimum degree and its duplicate. Note that the minimum degree is increased by one, and that for every vertex v, the size of a largest clique containing v in the new graph does not increase. We repeat this procedure until we obtain a graph G', having G as a subgraph, and with minimum degree at least $\ln^2(\Delta)$. The result now follows by applying Theorem 1.3.7 to G'. $\square$

Although we can not match the leading constant in Theorem 1.2.9 in our "local version," we can get the leading constant within a factor of $4\ln(2)$, as follows.

Theorem 1.3.11. *For every $\xi > 0$, if Δ is sufficiently large and G is a graph of maximum degree at most Δ with correspondence assignment (L, M) such that for each $v \in V(G)$,*

$$|L(v)| \geq (4 + \xi)\frac{d(v)}{\log_2(d(v))}$$

and $d(v) \geq \ln^2(\Delta)$, then G is (L, M)-colorable.

Davies, de Joannis de Verclos, Kang, and Pirot [42] proved that the constant in Theorem 1.3.11 can be improved to match Theorem 1.2.9 if one assumes a more restrictive bound on the minimum degree.

Theorem 1.3.12 (Davies et al. [42]). *For every $\xi > 0$, if Δ is sufficiently large and G is a triangle-free graph of maximum degree at most Δ with list-assignnment L such that for each $v \in V(G)$,*

$$|L(v)| \geq (1+\xi)\frac{d(v)}{\ln d(v)}$$

and $d(v) \geq (192 \ln \Delta)^{2/\xi}$, then G is L-colorable.

It would be interesting to improve the constant in Theorem 1.3.11 to match Theorem 1.2.9 or improve the minimum degree bound in Theorem 1.3.12 to match Theorem 1.3.11. Davies et al. [42, Proposition 11] proved that in these theorems it is necessary to bound the degree of each vertex by some function of Δ; however, this function is much smaller than the logarithm.

In Chapter 6, we prove Theorems 1.3.7, 1.3.11, and 1.3.12. Our proof of Theorem 1.3.12 slightly simplifies the one given in [42]. We actually prove a more general result, Theorem 6.1.2, that functions as a sort of "black box" that reduces the task of proving a coloring result to proving a Ramsey-theoretic type result, and we use this theorem in turn to prove Theorems 1.3.7 and 1.3.11. Our proof of Theorems 6.1.2 and 1.3.12 employ the novel approach of Molloy's [107] proof of Theorem 1.2.9.

1.4 Critical graphs

A graph G is *k-critical* if G is not $(k-1)$-colorable but every proper induced subgraph of G is, and if L is a list-assignment for G, then G is *L-critical* if G is not L-colorable but every proper induced subgraph of G is. Recall that we denote the average degree of a graph G by $\mathrm{ad}(G)$.

The average degree of critical graphs has been extensively studied. Note that a k-critical graph has no vertex of degree less than $k-1$, so the average degree of a k-critical graph is trivially at least $k-1$. Much work has been devoted to improving this bound. Brooks' Theorem [30] implies that for $k \geq 4$ this inequality is strict, unless the graph is complete. Dirac [47] proved that if G is an n-vertex k-critical graph for $k \geq 4$ and $n \geq k+2$, then $\mathrm{ad}(G) \geq k - 1 + \frac{k-3}{n}$, and Gallai [61, 62] proved that under the same hypotheses, $\mathrm{ad}(G) \geq k-1+\frac{k-3}{k^2-3}$, which is an improvement when n is large. Subsequent improvements were made by Krivelevich [103] and Kostochka and Stiebitz [102]. In a breakthrough result from 2014, Kostochka and Yancey [99] proved the following bound, which is tight for every $n \equiv 1 \mod k-1$ as shown by Ore [116].

Theorem 1.4.1 (Kostochka and Yancey [99]). *If $k \geq 4$ and G is k-critical, then*

$$\mathrm{ad}(G) \geq k - \frac{2}{k-1} - \frac{k^2 - 3k}{n(k-1)}.$$

Kostochka and Yancey [99] asked how their bound may be improved for graphs without large cliques. Kostochka and Stiebitz [98] proved such a bound earlier for graphs of bounded clique number, as follows.

Theorem 1.4.2 (Kostochka and Stiebitz [98]). *For any fixed $r \in \mathbb{N}$, if G is an L-critical graph where L is a list-assignment for G such that $|L(v)| = k - 1$ for each vertex $v \in V(G)$ and $\omega(G) \leq r$, then*

$$\mathrm{ad}(G) \geq 2k - o(k).$$

The bound in Theorem 1.4.2 is tight up to lower order terms as demonstrated by the construction of Tutte [45] mentioned earlier. As alluded to previously, Theorem 1.4.2 implies that if G is triangle-free, then $\chi(G) \leq (1/2 + o(1))\,\mathrm{mad}(G)$.

Our results on critical graphs also improve the bound of Kostochka and Yancey with a significantly less restrictive assumption on the size of the largest clique in G.

Theorem 1.4.3 (Kelly and Postle [87]). *For every $\alpha > 0$, if $\varepsilon \leq \alpha^2/1350$ then the following holds. If G is an L-critical graph for some list-assignment L such that $\omega(G) < (\frac{1}{2} - \alpha)k$, $|L(v)| = k - 1$ for each vertex $v \in V(G)$, and k is sufficiently large, then*

$$\mathrm{ad}(G) > (1 + \varepsilon)k.$$

Our next result generalizes Theorem 1.4.3 to graphs with clique number at most $k - \log^{10} k$ at the expense of a worse value of ε.

Theorem 1.4.4 (Kelly and Postle [88]). *Let $\varepsilon \leq 2.6 \cdot 10^{-10}$. If G is an L-critical graph for some list-assignment L where k is sufficiently large, $|L(v)| = k - 1$ for each vertex $v \in V(G)$, and $\omega(G) \leq k - \log^{10} k$, then*

$$\mathrm{ad}(G) > (1 + \varepsilon)k - \varepsilon\omega(G) - 1. \tag{1.3}$$

Recall the definition of Gap and Save from Definition 1.3.6; if G is a graph with list-assignment L and $v \in V(G)$, then $\mathrm{Gap}(v) = d(v) + 1 - \omega(v)$ and $\mathrm{Save}(v) = d(v) + 1 - |L(v)|$. It is convenient to use this notation in the context of Theorem 1.4.4, and it allows us to prove a "local version", as follows.

Theorem 1.4.5. *Let $\varepsilon \leq 2.6 \cdot 10^{-10}$. If G is an L-critical graph for some list-assignment where k is sufficiently large, $|L(v)| = k-1$ for each vertex $v \in V(G)$, and $\omega(G) \leq k - \log^{10} k$, then*

$$\sum_{v \in V(G)} \mathrm{Save}_L(v) > \sum_{v \in V(G)} \varepsilon\, \mathrm{Gap}(v).$$

Theorem 1.4.5 implies that Theorem 1.4.4 actually holds with $\omega(G)$ replaced with $\sum_{v \in V(G)} \omega(v)/|V(G)|$.

Recall that the list-assignment in Conjecture 1.3.3, the list-local version of Reed's Conjecture, satisfies $\mathrm{Save}(v) \leq \lfloor \mathrm{Gap}(v) \rfloor / 2$ for each vertex v, and the list-assignment in Theore 1.3.4 satisfies $\mathrm{Save}(v) \leq \varepsilon\, \mathrm{Gap}(v)$ for each vertex v. It would be very interesting to prove a common generalization of Theorem 1.4.5 and Theorem 1.3.4 by proving Theorem 1.4.5 for any list-assignment L satisfying $|L(v)| \leq k - 1$ for each vertex $v \in V(G)$.

We prove Theorem 1.4.5 in Chapter 5 and use it to deduce Theorem 1.4.4. We actually prove Theorem 1.4.3 in Chapter 5.1 by combining Theorem 1.3.4 with a simple discharging argument. The proof of Theorem 1.4.5 requires significantly more effort. The proofs of both Theorem 1.3.4 and 1.4.5 heavily utilize the probabilistic method, of which the bulk of the details are presented in Chapter 3. Most of Chapter 5 is devoted to a novel discharging argument which can be used to find a subset of vertices such that any precoloring can be extended to the rest of the graph using the results of Chapter 3.

We also show how to derive the following epsilon version of Conjecture 1.2.8.

Corollary 1.4.6. *There exists $\varepsilon > 0$ such that the following holds. Every graph G such that $\omega(G) \leq \mathrm{mad}(G) - \log^{10} \mathrm{mad}(G)$ satisfies*

$$\chi_\ell(G) \leq \lceil (1-\varepsilon)(\mathrm{mad}(G) + 1) + \varepsilon\omega(G) \rceil.$$

1.4.1 An application to Hadwiger's Conjecture

We use our results on the density of critical graphs to make progress towards Hadwiger's Conjecture [69], which is widely considered one of the most important open problems in graph theory. Let G and H be graphs. The graph H is a *minor* of G if there exists a partition $(V_v : v \in V(H))$ of $V(G)$ such that for each $v \in V(H)$, the induced subgraph $G[V_v]$ is connected and for each edge $uv \in E(H)$, there exists $x \in V_v$ and $y \in V_v$ such that $xy \in E(G)$. In this case, we say G has an *H-minor*, and H is obtained from a subgraph of G by *contracting* the edges in $E(G[V_v])$ for each $v \in V(H)$.

Conjecture 1.4.7 (Hadwiger's Conjecture [69])**.** *If G is a graph with no K_{t+1}-minor, then*

$$\chi(G) \leq t.$$

The case $t = 4$ of Hadwiger's Conjecture can easily be shown to imply the Four Color Theorem, since a planar graph does not contain a K_5-minor. In fact, Wagner [148] showed that they are equivalent, and Robertson, Seymour, and Thomas [128] showed that the case $t = 5$ of Hadwiger's Conjecture is also equivalent to the Four Color Theorem. For large values of t, not much is known about Hadwiger's Conjecture. The previous best known upper bound on the chromatic number of graphs with no K_{t+1}-minor came from the following theorem bounding their maximum average degree.

Theorem 1.4.8 (Thomason [144])**.** *If G is a graph with no K_t-minor, then*

$$\mathrm{ad}(G) \leq (\gamma + o(1))t\sqrt{\log t},$$

where $\gamma = 0.63817...$ is an explicit constant.

By combining Theorem 1.4.3 with Theorem 1.4.8, we improve the previous best known bound by a factor of .99982, as follows.

Corollary 1.4.9. *If G is a graph with no K_t-minor, then*

$$\chi_\ell(G) \leq (.99982 \cdot \gamma + o(1))t\sqrt{\log t},$$

where γ is the explicit constant from Theorem 1.4.8.

Proof. It suffices to show that for every $\xi > 0$, if $k_t = .99982(\gamma + \xi)t\sqrt{\log t}$, then for sufficiently large t, every K_t-minor free graph is k_t-list-colorable. Suppose not. Then there exists a graph G with no K_t-minor that is L-critical for some k_t-list-assignment L where $k_t \geq 1000t$. Using Theorem 1.4.8, we may assume $\mathrm{ad}(G) \leq k_t/.99982$.

Let $\alpha = 499/1000$ and $\varepsilon = \alpha^2/1350$. Since $\omega(G) < t$, we have $\omega(G) < (\frac{1}{2} - \alpha)k_t$. Since G is L-critical, by Theorem 1.4.3, $\mathrm{ad}(G) > (1 + \varepsilon)k_t$. But $(1 + \varepsilon) \geq 1/.99982$, a contradiction. $\square$

1.5 Fractional coloring with local demands

In this section, we investigate local versions in the setting of fractional coloring. In particular, we study fractional colorings with respect to demand functions where the "demand" for each vertex is based on local parameters. The results in this section are proved in Part II. The archetypal example of a result of this type is the following "local demands" version of the Greedy Bound, Proposition 1.2.1.

Theorem 1.5.1 (Local Fractional Greedy Bound). *If G is a graph with demand function f such that $f(v) \leq 1/(d(v)+1)$ for each $v \in V(G)$, then G has an f-coloring.*

Theorem 1.5.1 implies the famous Caro-Wei Theorem [33, 149], which states that every graph G satisfies $\alpha(G) \geq \sum_{v\in V(G)} 1/(d(v)+1)$. The dual formulation of Theorem 1.5.1 was proved in [129]. Theorem 1.5.1 also implies the "fractional relaxation" of the Greedy Bound, that is that every graph G satisfies $\chi_f(G) \leq \Delta(G)+1$. We provide three different proofs of Theorem 1.5.1 in Chapter 7.2, all inspired by proofs of the Caro-Wei Theorem.

Bounds on the independence number in terms of the degree sequence like in the Caro-Wei Theorem have been extensively studied [29, 64, 70, 71, 72, 73, 134, 137], even for hypergraphs [34, 143]. It is natural to seek generalizations of many of these results in the setting of fractional coloring with local demands. Moreover, the perspective of coloring guides us to many interesting new problems for both fractional coloring and independence numbers.

1.5.1 Local demands version of Brooks' Theorem

The Caro-Wei Theorem and Theorem 1.5.1, as well as the greedy bound on the chromatic number, are all tight for complete graphs. Considering the setting of fractional coloring with local demands and the ω, Δ, χ paradigm, one might ask whether for some $\varepsilon > 0$ it is possible to prove that every graph has an f-coloring whenever $f(v) \leq 1/(d(v)+1-\varepsilon)$ for each vertex v, under some assumptions about the cliques in G. There are two natural restrictions to impose on the cliques; the first restriction is that no vertex is *simplicial*, that is, no clique contains a vertex and all of its neighborhood. The second, less strict restriction, which is actually a necessary condition, is that there is no clique K such that $\sum_{v\in K} f(v) > 1$. In either case, one could not do better than $\varepsilon = 1/2$ because of the 5-cycle.

Our first main result is an affirmative answer to this question, even with this less strict restriction. This result yields the "local demands" version of Brooks' Theorem, as follows.

Theorem 1.5.2 (Local Fractional Brooks' [86]). *If G is a graph with demand function f such that $f(v) \leq 1/(d(v)+1/2)$ for each $v \in V(G)$ and $\sum_{v\in K} f(v) \leq 1$ for each clique $K \subseteq V(G)$, then G has an f-coloring.*

If f is a demand function such that $f(v) \leq 1/(d(v)+1)$ for each vertex v, then every clique satisfies $\sum_{v\in K} f(v) \leq 1$. Thus, Theorem 1.5.2 generalizes Theorem 1.5.1. If f is a demand function for a graph with no simplicial vertex such that $f(v) \leq 1/d(v)$ for each vertex v, then every clique K satisfies $\sum_{v\in K} f(v) \leq 1$. Therefore Theorem 1.5.2 implies the following.

Corollary 1.5.3. *If G is a graph with demand function f such that $f(v) \leq 1/(d(v)+1/2)$ for each $v \in V(G)$ and no vertex of G is simplicial, then G has an f-coloring.*

Using Proposition 1.1.6, this result implies the following bound on the independence number.

Corollary 1.5.4. *If G is a graph with no simplicial vertex, then G has an independent set of size at least $\sum_{v\in V(G)} 1/(d(v)+1/2)$.*

Even with this less strict restriction on the cliques, this bound on the independence number of $\sum_{v\in V(G)} 1/(d(v)+1-\varepsilon)$ in Corollary 1.5.4 was not previously known for any $\varepsilon > 0$. As we discuss in Chapter 10, the proof for any $\varepsilon > 0$ already requires some ingenuity; however, considerably more effort is required in our proof with $\varepsilon = 1/2$.

Recall that the fractional chromatic number of a cycle of length $2k+1$ is $2+1/k$, and thus, the fractional chromatic number of any graph of maximum degree at most two with no triangle is at most $3/2$. Hence, Brooks' Theorem implies that if $\chi_f(G) > \Delta(G)+1/2$, then G has a clique of size $\Delta(G)+1$; this result also follows easily from Theorem 1.5.2. Moreover, Brooks' Theorem implies that if $\chi_f(G) > \Delta(G)$, then either G contains a clique of size $\Delta(G)+1$ or $\Delta(G) = 2$ and G contains an odd cycle. Thus, it is tempting to conjecture a strengthening of Theorem 1.5.2 by allowing $\varepsilon \in [0,1]$ and excluding odd cycle components. However, as we explain in Chapter 10, we also need to exclude *blowups* of odd cycles, where a blowup of a graph is obtained by replacing some vertices with cliques and replacing edges with maximal complete bipartite graphs. It is also necessary to exclude the wheel on six vertices. A *wheel graph* is obtained from a cycle by adding a vertex adjacent to every other vertex; we let W_n represent a wheel graph on n vertices. We believe these are the only obstructions to finding such a coloring; thus, we conjecture the following.

Conjecture 1.5.5 (Local Fractional Brooks' – Extended [86]). *Let $\varepsilon \in [0,1]$, and let G be a graph with demand function f such that $f(v) \leq 1/(d(v)+1-\varepsilon)$ for each $v \in V(G)$. If G has no subgraph H such that*

(i) H is a clique and $\sum_{v\in V(H)} f(v) > 1$,
(ii) for some integer $k \leq \lfloor 1/(1-\varepsilon)\rfloor$, H is a blowup of a cycle C_{2k+1} and $\sum_{v\in V(H)} f(v) > k$, or
(iii) H is a component isomorphic to W_6 and $\sum_{v\in V(C)} f(v) > 2(1-f(u))$, where C is an induced 5-cycle in H and u is the vertex not in C,

then G has an f-coloring.

Note that Theorem 1.5.2 confirms Conjecture 1.5.5 for $\varepsilon \leq 1/2$. We note that condition (iii) is only necessary in Conjecture 1.5.5 when $\varepsilon > (\sqrt{89}-9)/4 \approx 0.8915$. It would be very interesting to confirm Conjecture 1.5.5 for any $\varepsilon > 1/2$ or for graphs of large minimum degree. As we see in Chapter 10, if H is an "unbalanced" blowup of C_{2k+1} with large minimum degree and demand function f such that $f(v) \leq 1/(d(v)+1-\varepsilon)$ for each vertex $v \in V(G)$, then $\sum_{v\in V(H)} f(v) > k$ only if either $k = 2$ and ε is not much smaller than $3/4$ or $k = 3$ and ε is close to one. In light of the discussion from Section 1.2.1, Reed's [122] proof of the Borodin-Kostochka Conjecture [25] for large maximum degree implies that the largest k for which there exists a Δ_0 such that for every $\Delta \geq \Delta_0$, graphs of maximum degree at most Δ and clique number at most ω have chromatic number at most $\max\{\omega, \Delta+1-k\}$, is 2. We ask the analogous question in the setting of local demands, and we believe the answer is $3/4$, as in the following conjecture.

Conjecture 1.5.6 (Kelly and Postle [86]). *For every $\varepsilon < 3/4$ there exists $\delta \in \mathbb{N}$ such that the following holds. If G is a graph of minimum degree at least δ with demand function f such that $f(v) \leq 1/(d(v)+1-\varepsilon)$ for each $v \in V(G)$ and $\sum_{v\in K} f(v) \leq 1$ for each clique $K \subseteq V(G)$, then G has an f-coloring.*

We present a few more Conjectures like Conjecture 1.5.6 in Chapter 10. In particular, we consider the analogue of [110].

Reed's Conjecture has the following natural analogue in the setting of fractional coloring with local demands.

Conjecture 1.5.7 (Local Fractional Reed's [86]). *If G is a graph with demand function f such that $f(v) \leq 2/(d(v)+\omega(v)+1)$ for each $v \in V(G)$, where $\omega(v) = \omega(G[N[v]])$, then G has an f-coloring.*

Note that Theorem 1.5.2 implies Conjecture 1.5.7 for any graph G satisfying $\omega(v) \geq d(v)$ for each $v \in V(G)$. If true, Conjecture 1.5.7 can be shown to imply Theorem 1.5.2. On the other hand, Reed's Conjecture does not imply Brooks' Theorem, which is the case with

Conjectures 1.5.7 and 1.5.5. We think it is appropriate for the local fractional analogue of Brooks' Theorem to follow from the local fractional analogue of Reed's Conjecture, because of the tendency of fractional coloring to smooth the intricacies and complications that arise in ordinary coloring.

1.5.2 Edge-coloring and χ-boundedness

Conjecture 1.5.7 generalizes a conjecture of Brause et al. [29] on the independence number. In Chapter 8, we prove that Conjecture 1.5.7 holds in a stronger sense for *perfect* graphs, which strengthens one of the main results in [29]. A graph is perfect if every subgraph H satisfies $\chi(H) \le \omega(H)$. The celebrated Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour and Thomas [38] states that a graph is perfect if and only if neither it nor its complement contains an induced odd cycle of length at least five. We prove the following in Chapter 8.

Theorem 1.5.8 (Kelly and Postle [86])**.** *If G is a perfect graph with demand function f such that $f(v) \le 1/\omega(v)$ for each $v \in V(G)$, then G has an f-coloring.*

A class of graphs $\mathcal{G}$ is *χ-bounded* if there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \le g(\omega(G))$. We actually derive Theorem 1.5.8 by proving a more general result about χ-bounded classes of graphs with a linear χ-bounding function.

The *line graph* of a graph G, denoted $L(G)$, is the graph with vertices $E(G)$ in which $e, f \in E(G)$ are adjacent if and only if there is a vertex incident to both e and f. One of the most classical results in graph coloring, Vizing's Theorem [146], obtained independently by Gupta [66], states that every graph G satisfies $\chi(L(G)) \le \Delta(G) + 1$. Thus, since every graph G that is not a triangle satisfies $\omega(L(G)) = \Delta(G)$, line graphs also form a χ-bounded class of graphs. We also prove the "local demands" version of the generalized Vizing's Theorem [146] in Chapter 8, as follows. A *multigraph* is a graph such that the edges actually form a multiset.

Theorem 1.5.9 (Local Fractional Generalized Vizing's [86])**.** *If G is a multigraph and f is a demand function for $L(G)$ such that each $e \in V(L(G))$ with $e = uv \in E(G)$ satisfies $f(e) \le 1/(\max\{d(u), d(v)\} + |uv|)$, where $|uv|$ is the number of edges of G incident to both u and v, then $L(G)$ has an f-coloring.*

The *total graph* of a graph G, denoted $T(G)$, is the graph with vertices $V(G) \cup E(G)$ and edges prescribed as follows:

- $u, v \in V(G)$ are adjacent in $T(G)$ if and only if $uv \in E(G)$,
- $v \in V(G)$ is adjacent to $e \in E(G)$ in $T(G)$ if and only if v is incident to e in G, and
- $e, f \in E(G)$ are adjacent in $T(G)$ if and only if there is some vertex in G incident to both e and f.

The notorious Total Coloring Conjecture [12] states that every graph G satisfies $\chi(T(G)) \leq \Delta(G) + 2$. Kilakos and Reed [89] proved the fractional relaxation of the Total Coloring Conjecture. We conjecture the local demands version of this conjecture, as follows.

Conjecture 1.5.10 (Local Fractional Total Coloring Conjecture [86])**.** *If G is a graph and f is a demand function for $T(G)$ such that each $v \in V(G)$ satisfies $f(v) \leq 1/(d(v)+2)$ and each $uv \in E(G)$ satisfies $f(uv) \leq 1/(\max\{d(u), d(v)\} + 2)$, then $T(G)$ has an f-coloring.*

It would also be interesting to consider total colorings of graphs of high girth, as in [84, 85].

It is natural to consider local versions of these χ-boundedness and edge-coloring problems in the list coloring setting as well, but we do not focus on these problems in this thesis. Since bipartite graphs are perfect and also have arbitrarily large list chromatic number, there is no direct list coloring analogue of Theorem 1.5.8. A list-local version of Theorem 1.5.9 is possibly true, but it is very closely related to the infamous List Edge Coloring Conjecture, which states that every graph G satisfies $\chi_\ell(L(G)) = \chi(L(G))$. It would be interesting to prove a list-local version of Kahn's [83] result that the List Edge Coloring Conjecture is asymptotically correct, that is every graph G satisfies $\chi_\ell(G) \leq (1+o(1))\Delta(G)$. Similar results for total coloring would also be interesting.

1.5.3 Small clique number

Lastly, we consider local demands versions of the results discussed in Section 1.2.3. The most interesting problem among these is the following conjecture, which if true, simultaneously implies the fractional relaxation of Theorem 1.2.9 and the bound on the independence number proved by Shearer [137].

Conjecture 1.5.11 (Local Fractional Shearer's/Molloy's)**.** *If G is a triangle-free graph with demand function f such that $f(v) \leq ((1 - o(1)) \log d(v))/d(v)$ for each $v \in V(G)$, then G has an f-coloring.*

In the same vein as Conjecture 1.5.11, we believe local demands versions of Theorems 1.2.10 and Theorem 1.2.11 hold as well.

As we show in Chapter 9, Conjecture 1.5.11, if true, implies a recent conjecture of Cames van Batenburg et al. [32, Conjecture 4.3].

We make partial progress towards these conjectures by proving that they are true if we have vertices demand slightly less color.

Theorem 1.5.12. *If G is a triangle-free graph with demand function f such that*

$$f(v) \leq (2e + o(1))^{-1} \frac{\ln d(v)}{d(v) \ln \ln d(v)}$$

for each $v \in V(G)$, then G has an f-coloring.

Theorem 1.5.13. *For any fixed ω, if G is a graph with no clique of size greater than ω with demand function f such that*

$$f(v) = O\left(\frac{\ln d(v)}{d(v)(\ln \ln d(v))^2}\right)$$

for each $v \in V(G)$, then G has an f-coloring.

Note that Theorem 1.5.13 implies Conjecture 1.5.7 for graphs of bounded clique number and sufficiently large minimum degree, which was not previously known, even for triangle-free graphs.

We prove Theorems 1.5.12 and 1.5.13 in Chapter 9. We also discuss how the probabilistic method may potentially be used to improve these results.

Chapter 2

Probabilistic method preliminaries

The probabilistic method has become one of the most powerful tools in combinatorics since it was pioneered by Erdős in the middle of the last century. In this thesis, we heavily utilize the probabilistic method in Part I to obtain our results on list coloring. In particular, we use this method in Chapters 6 and 3 to prove Theorems 1.3.7 and 3.1.8, and we use Theorem 3.1.8 in turn in Chapters 4 and 5 to prove Theorems 1.3.4 and 1.4.5, respectively. In Part II, we see how notions from probability theory are relevant to fractional coloring and how the probabilistic method may be potentially useful for resolving some of our open problems in the setting of fractional coloring with local demands.

In all of our applications of the probabilistic method, we randomly assign each vertex of a graph a color from a list of available colors, with the ultimate goal of finding a proper coloring. In most situations, this random assignment is unlikely to be a proper coloring of the whole graph. However, using the probabilistic method, we can find a proper coloring of a subgraph that we can extend to a proper coloring of the whole graph. Since the 1990s, numerous major advances in graph coloring have been made using this approach. The book of Molloy and Reed [109] provides an excellent treatment of these results.

In this chapter, we introduce the tools from the probabilistic method that we need in Part I. Moreover, we develop a new "concentration inequality," Theorem 2.3.2, that we apply in Chapter 3. For a more complete introduction to the probabilistic method, see either the aforementioned book of Molloy and Reed [109] or the book of Alon and Spencer [5].

2.1 The basics

First, we need a few basic definitions. A *probability space* is a triple $(\Omega, \Sigma, \mathbb{P})$ where Ω is a set of *outcomes*, Σ is a set of *events*, and $\mathbb{P}$ is a *probability distribution*. An event is simply a subset of the outcomes, and the events Σ in a probability space satisfy some basic axioms:

- $\Omega \in \Sigma$,
- if $A \in \Sigma$ is an event, then $(\Omega \setminus A) \in \Sigma$, that is, "not A" is also an event, and
- if $A_i \in \Sigma$ is an event for $i \in I \subseteq \mathbb{N}$, then $\cup_{i \in I} A_i \in \Sigma$.

A σ*-algebra* is a collection of subsets satisfying these axioms. The probability distribution $\mathbb{P}$ is an assignment of probabilities to the events. Formally, $\mathbb{P}$ is a map from Σ to the $[0,1]$-interval satisfying these basic axioms:

- if $A_i \in \Sigma$ for $i \in I \subseteq \mathbb{N}$ is a collection of pairwise disjoint events, then $\mathbb{P}[\cup_{i \in I} A_i] = \sum_{i \in I} \mathbb{P}[A_i]$, and
- $\mathbb{P}[\Omega] = 1$.

In Part I, we only consider probability spaces $(\Omega, \Sigma, \mathbb{P})$ in which Ω is a finite set and Σ is simply the collection of *all* subsets of Ω. We can essentially ignore the σ-algebra Σ in this context and define a *finite probability space* as a pair $(\Omega, \mathbb{P})$, where $\mathbb{P}$ assigns a value in the $[0,1]$-interval to each outcome $\omega \in \Omega$ such that $\sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1$ and for any $A \subseteq \Omega$ we define $\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$. We choose to provide the complete definition of a probability space here because we prove Theorem 2.3.2 in this section in full generality.

If $(\Omega, \mathbb{P})$ is a finite probability space, then a *random variable* is simply a map $X : \Omega \to \mathbb{R}$. In this case, the *expected value of* X is $\mathbb{E}[X] = (\sum_{\omega \in \Omega} X(\omega))/|\Omega|$. It is possible to extend these definitions for any probability space, but we do not consider the details to be important for this thesis.

If $\mathbb{P}$ is a probability distribution and A and B are events, then $\mathbb{P}[A|B] = \mathbb{P}[A \cap B] / \mathbb{P}[B]$ is the *conditional probability* of A given B, which can be thought of as the probability that A occurs if we assume that B occurs. Two events A and B are *independent* if $\mathbb{P}[A|B] = \mathbb{P}[A]$.

2.2 Our toolkit

In this section we present the tools we need to apply the probabilistic method and discuss how we use them in Part I.

2.2.1 The Lovász Local Lemma

In typical applications of the probabilistic method, we prove that a certain combinatorial object exists by finding a probability space in which the probability of randomly choosing a desired object is nonzero. Often times, particularly in early applications of the probabilistic method, one finds that the probability of randomly choosing a desired object is very close to one. For example, in Erdős' [54] celebrated proof that there exist graphs of arbitrarily large girth and chromatic number, we find a graph G on n vertices such that G contains at most $n/2$ cycles of length at most k and no independent set of size at least $n/(2k)$, and assuming the graph G is chosen randomly in an appropriate way, the probability that G has each property tends to one as n tends to infinity.

In our applications of the probabilistic method, we randomly sample a proper coloring of a subgraph and show that with nonzero probability this random coloring can be extended to the whole graph. If we can show, for example, that with nonzero probability every uncolored vertex has more available colors than it does remaining uncolored neighbors, then by Proposition 1.3.1, our random coloring can be extended. To that end, we use our first tool, the Lovász Local Lemma, to prove that with nonzero probability, every vertex satisfies such a local condition.

In all of the tools we present in this section, there is an implicit probability space $(\Omega, \Sigma, \mathbb{P})$. If $B \in \Sigma$ is a random event, we let $\overline{B} = \Omega \setminus B$, that is the event "not B".

Lemma 2.2.1 (Lovász Local Lemma)**.** *Let I be a finite set, and for each $i \in I$, let B_i be a random event. Suppose that for every $i \in I$, there is a set $\Gamma(i) \subseteq I$ such that $|\Gamma(i)| \leq d$ and for all $Z \subseteq I \setminus \Gamma(i)$,*

$$\mathbb{P}\left[B_i | \bigcap_{j \in Z} \overline{B_j}\right] \leq p.$$

If $4pd \leq 1$, then with nonzero probability none of the events B_i occur.

The Lovász Local Lemma actually has many forms; Lemma 2.2.1 is usually called the Lopsided Lovász Local Lemma. When we apply Lemma 2.2.1 to a random coloring of a graph G, the index set I is usually $V(G) \times \{1, \ldots, k\}$ for some k, that is we have k "bad events" for each vertex v. If B_i is a bad event for a vertex v, then the set $\Gamma(i)$ corresponds to events that are not independent of B_i. The dependency d is tied to the degree of v and its neighbors. Thus, in order to apply the Local Lemma, our goal is to show that the probability of a bad event for each vertex is small compared to the vertex's degree.

Let us briefly discuss algorithmic aspects of the Local Lemma. Lemma 2.2.1 implies that there is an outcome in the probability space not contained in any of the bad events B_i; the problem of constructing an efficient algorithm to find one of these outcomes has attracted considerable attention. The original proof of Erdős and Lovász [55] is nonconstructive. In 1991, Beck [11] first provided an efficient algorithmic version of the Local Lemma in the special case of hypergraph 2-coloring in which each edge shares vertices with a bounded number of edges, which is one of the most classical applications of the Local Lemma. Beck's algorithm produces a 2-coloring with no monochromatic edge efficiently so long as each edge has at least k vertices and each edge shares vertices with at most $2^{k/48}$ edges – a straightforward application of Lemma 2.2.1 guarantees such a coloring exists as long as each edge shares vertices with at most $2^k/4$ other edges. Molloy and Reed [108] generalized Beck's result to the so-called "variable model" of the Local Lemma, in which the probability space is determined by a set of independent random trials, each bad event B_i is determined by a subset T_i of these trials, and the set $\Gamma(i)$ is precisely the events B_j for which $T_i \cap T_j \neq \varnothing$. In this model, each event B_i is in fact mutually independent of all events not in $\Gamma(i)$, so it suffices to show that $\mathbb{P}[B_i] \leq p$ for each i in order to apply Lemma 2.2.1. Molloy and Reed's [108] result essentially provides an algorithmic version of Lemma 2.2.1 in this setting in which we require $512pd^9 \leq 1$ rather than $4pd \leq 1$. In 2009, Moser [112] achieved a breakthrough by inventing the "entropy compression" technique and using it to prove an algorithmic version of the Local Lemma in the hypergraph 2-coloring setting with a near-optimal bound on the dependencies; Moser and Tardos [113] subsequently generalized this result to the variable model, and if the number of dependencies is bounded by a constant, then the algorithm can be "derandomized", that is there is a deterministic algorithm with polynomial running time. In Chapter 3, we can use this result to obtain an algorithmic version of Theorem 3.1.8, and in turn we obtain efficient algorithms to find the colorings guaranteed by Theorem 1.3.4 and Corollary 1.4.6 in Chapters 4 and 5, respectively. In Chapter 6, our application of Lemma 2.2.1 does not fit the variable model, so we do not obtain an algorithmic version of Theorem 1.2.11 or 1.3.7. Molloy's [107] proof of Theorem 1.2.9 using entropy compression does in fact yield an efficient algorithm to find the coloring in the case of triangle-free graphs, though, but this does not extend to Theorems 1.2.10 or 1.2.11.

2.2.2 Basic concentration inequalities

The rest of our toolkit is used for bounding the probabilities of these bad events in Lemma 2.2.1. When we randomly sample a proper coloring, we consider random variables for each vertex v such as the number of remaining colors available for v. Our bad events in our application of Lemma 2.2.1 are always that one of these random variables is too small (or large). To bound the probability of these bad events, we first show that the random variables are large (or small) in expectation. Next, we use a "concentration inequality" to show that the random variables are unlikely to deviate significantly from their expectation.

The first such concentration inequality that we present is the Chernoff Bound. This shows that *binomial random variables* are concentrated around their expectation. The binomial random variable $\mathrm{BIN}(n,p)$ is the sum of n independent random variables, each equal to 1 with probability p and 0 otherwise. Note that $\mathbb{E}[\mathrm{BIN}(n,p)] = pn$. The basic Chernoff Bound is that for any $0 < t \le np$, we have $\mathbb{P}[|\mathrm{BIN}(n,p) - np| > t] \le 2\exp(-t^2/(3np))$. We need a generalization of this bound.

Definition 2.2.2. A random variable $X : \Omega \to \mathbb{R}$ is *boolean* if $X(\omega) \in \{0,1\}$ for every outcome $\omega \in \Omega$. Boolean random variables $X_1, \ldots, X_m$ are *negatively correlated* if for every $I \subseteq \{1, \ldots, m\}$,

$$\mathbb{P}[\cap_{i\in I}(X_i = 1)] \le \prod_{i\in I} \mathbb{P}[X_i = 1].$$

We use the following generalization of the Chernoff Bound for negatively correlated boolean random variables in Chapter 6. See [107, Lemma 3].

Lemma 2.2.3 (Chernoff Bound [107])**.** *Let $X_1, \ldots, X_m$ be negatively correlated boolean random variables, and let $X = \sum_{i=1}^m X_i$. Then for any $0 < t \le \mathbb{E}[X]$,*

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] \le 2\exp\left(-\frac{t^2}{3\mathbb{E}[X]}\right).$$

We also need a simple "one-sided" form of this bound. If $X_1, \ldots, X_m$ are boolean random variables and $X = \sum_{i=1}^m X_i$, then for any $\delta > 0$, we have $\mathbb{P}[X \ge (1+\delta)\mathbb{E}[X]] \le \exp(-\delta^2\mathbb{E}[X]/(2+\delta))$.

In Chapter 3, we are not able to apply Lemma 2.2.3 to the random variables that we want to concentrate. Instead we use a form of Talagrand's Inequality [142] that we can apply to a broader range of random variables. As we will see in the next section, it is not straightforward to apply Talagrand's Inequality directly. Molloy and Reed [109, Talagrand's Inequality II] derived the following version of Talagrand's Inequality that is easier to apply to most of the random variables we consider in Chapter 3.

Theorem 2.2.4 ("Talagrand's Inequality II" [109])**.** *Let X be a non-negative random variable, not identically 0, which is determined by n independent trials $T_1, \ldots, T_n$, and satisfying the following for some $c, r > 0$:*

1. *changing the outcome of any one trial can affect X by at most c, and*
2. *for any s, if $X \geq s$ then there is a set of at most rs trials whose outcomes certify that $X \geq s$,*

then for any $0 \leq t \leq \mathbb{E}[X]$,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > t + 60c\sqrt{r\mathbb{E}[X]}\right] \leq 4\exp\left(-\frac{t^2}{8c^2 r\mathbb{E}[X]}\right).$$

Note that $\mathrm{BIN}(n,p)$ satisfies the hypotheses of Theorem 2.2.4 with $c = r = 1$. When we apply Theorem 2.2.4 in Chapter 3, the trials $T_1, \ldots, T_n$ correspond to randomly assigning each vertex of a graph a color from its list.

2.3 A new concentration inequality

Unfortunately Theorem 2.2.4 is not applicable in every situation we need in Chapter 3. In our case, changing the outcome of a trial corresponds to changing the color randomly assigned to a vertex, and this can affect the value of our random variables significantly if many vertices are randomly assigned the same color. Fortunately for us, these events are unlikely, and we are thus still able to show that the random variables are concentrated. To that end, in this section we develop a new form of Talagrand's Inequality that we use in Chapter 3 and which may be of independent interest. Similar variations of Talagrand's Inequality that incorporate so-called exceptional outcomes already exist: for example [109, Talagrand's Inequality V] and [31, Theorem 12]. However, as we explain later, these are not suitable for our purposes.

First, we need some definitions. If $(\Omega_i, \Sigma_i, \mathbb{P}_i)$ for $i \in \{1,2\}$ are probability spaces, then the *product space* $(\Omega_1, \Sigma_1, \mathbb{P}_1) \times (\Omega_2, \Sigma_2, \mathbb{P}_2)$ is the probability space $(\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2, \mathbb{P})$ where $\mathbb{P}[(A,B)] = \mathbb{P}_1[A] \cdot \mathbb{P}_2[B]$. When we randomly sample a coloring of a graph in Chapter 3, we have a probability space for each vertex, and we consider the product space of these probability spaces.

Definition 2.3.1. Let $((\Omega_i, \Sigma_i, \mathbb{P}_i))_{i=1}^n$ be probability spaces, let $(\Omega, \Sigma, \mathbb{P})$ be their product space, let $\Omega^* \subseteq \Omega$ be a set of *exceptional outcomes*, and let $X : \Omega \to \mathbb{R}_{\geq 0}$ be a non-negative random variable. Let $r, d \geq 0$.

Section 2.3 is based on the appendix of the paper [87] (with Luke Postle).

- If $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$ and $s > 0$, an (r, d)-*certificate* for X, ω, s, and Ω^* is an index set $I \subseteq \{1, \ldots, n\}$ of size at most rs such that for all $k \geq 0$, we have that

$$X(\omega') \geq s - kd,$$

for all $\omega' = (\omega'_1, \ldots, \omega'_n) \in \Omega \setminus \Omega^*$ such that $\omega_i \neq \omega'_i$ for at most k values of $i \in I$.
- If for every $s > 0$ and $\omega \in \Omega \setminus \Omega^*$ such that $X(\omega) \geq s$, there exists an (r, d)-certificate for X, ω, s, and Ω^*, then X is (r, d)-*certifiable* with respect to Ω^*.

Note that if $\Omega^* = \varnothing$, then a random variable being (r, d)-certifiable with respect to Ω^* is similar to it satisfying the conditions of Theorem 2.2.4 with $c = d$ (we use d because later we use c to denote a color). We introduce k into the definition of (r, d)-certificates rather than consider changing the outcome of only one trial because it is necessary in order to apply the original form of Talagrand's Inequality, for reasons we will see later.

Now we state our concentration inequality, which is also proved in [87, Theorem 6.3].

Theorem 2.3.2 (Kelly and Postle [87])**.** *Let $((\Omega_i, \Sigma_i, \mathbb{P}_i))_{i=1}^n$ be probability spaces, let $(\Omega, \Sigma, \mathbb{P})$ be their product space, let $\Omega^* \subseteq \Omega$ be a set of exceptional outcomes, and let $X : \Omega \to \mathbb{R}_{\geq 0}$ be a non-negative random variable. Let $r, d \geq 0$.*

If X is (r, d)-certifiable with respect to Ω^*, then for any $t > 96d\sqrt{r\mathbb{E}[X]} + 128rd^2 + 8\mathbb{P}[\Omega^*](\sup X)$,

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] \leq 4\exp\left(\frac{-t^2}{8d^2r(4\mathbb{E}[X] + t)}\right) + 4\mathbb{P}[\Omega^*].$$

Theorem 2.3.2 is similar to Theorem 12 of Bruhn and Joos [31]. Bruhn and Joos defined *upward (s, c)-certificates*. If a random variable is (r, d)-certifiable with respect to a set of exceptional outcomes Ω^*, then it has upward (s, c)-certificates with $c = d$ and $s = r \cdot \sup X$, and for the random variables with which we are concerned, they have upward (s, c)-certificates only if $s \geq \sup X$. The important difference between the bound supplied by their result and Theorem 2.3.2 is that we have $r(4\mathbb{E}[X]+t)$ in the denominator of the exponential concentration function whereas they simply have s. In [31], Bruhn and Joos apply their concentration inequality to random variables for which $\mathbb{E}[X] = \Omega(\sup X)$, so this difference does not concern them. However, in our situation it is possible that $\sup X = \Delta$ and yet $\mathbb{E}[X] = \log^{10}\Delta$, where Δ is a bound on the maximum degree of the graph in consideration. Thus, we are unable to use the result of Bruhn and Joos for all of the random variables in Chapter 3. "Talagrand's Inequality V" in [109] has essentially the same problem, with D taking the role of s.

The remainder of this chapter is devoted to the proof of Theorem 2.3.2; our proof is similar to the proof of Bruhn and Joos.

In order to prove Theorem 2.3.2, we first prove the following theorem which yields concentration around the median under the same conditions. If X is a random variable, then $Med(X) = \sup\{t \in \mathbb{R} : \mathbb{P}[X \geq t] \leq 1/2\}$.

Theorem 2.3.3. *If X is (r, c)-certifiable with respect to Ω^*, then for any $t > 0$,*

$$\mathbb{P}[|X - Med(X)| > t] \leq 4 \exp\left(-\frac{t^2}{4c^2 r(Med(X) + t)}\right) + 4\mathbb{P}[\Omega^*]$$

We then prove that the expectation and median are close as in the following lemma.

Lemma 2.3.4. *If X is (r, c)-certifiable with respect to Ω^* and $M = \sup X$, then*

$$|\mathbb{E}[X] - Med(X)| \leq 48c\sqrt{r\mathbb{E}[X]} + 64rc^2 + 4M\mathbb{P}[\Omega^*].$$

Proof. Let $Y = X + \mathbb{E}[X]$. Note that $\mathbb{E}[Y] - Med(Y) = \mathbb{E}[X] - Med(X)$, that $Med(Y) \geq \mathbb{E}[X] > 0$, and $\mathbb{E}[Y] \leq 2\mathbb{E}[X]$. Note also that

$$|\mathbb{E}[Y] - Med(Y)| \leq \mathbb{E}[|Y - Med(Y)|].$$

Let $L = \lfloor M/(c\sqrt{rMed(Y)})\rfloor$, and note that $|Y - Med(Y)| \leq (L+1)c\sqrt{rMed(Y)}$. By partitioning the possible values of $|Y - Med(Y)|$ into intervals of length $c\sqrt{rMed(Y)}$, we obtain

$$\begin{aligned}\mathbb{E}[|Y - Med(Y)|] \leq \sum_{\ell=0}^{L} c\sqrt{rMed(Y)}(\ell+1)\Big(&\mathbb{P}\left[|Y - Med(Y)| \geq \ell c\sqrt{rMed(Y)}\right] \\ &-\mathbb{P}\left[|Y - Med(Y)| \geq (\ell+1)c\sqrt{rMed(Y)}\right]\Big). \\ = \sum_{\ell=0}^{L} c\sqrt{rMed(Y)}\Big(&\mathbb{P}\left[|Y - Med(Y)| \geq \ell c\sqrt{rMed(Y)}\right]\Big).\end{aligned}$$

By applying Theorem 2.3.3 with $t = \ell c\sqrt{rMed(Y)}$ to every summand,

$$\mathbb{E}[|Y - Med(Y)|] \leq 4c\sqrt{rMed(Y)}\sum_{\ell=0}^{L}\left(\exp\left(-\frac{\ell^2 c^2 rMed(Y)}{4c^2 r(Med(Y) + \ell c\sqrt{rMed(Y)})}\right) + \mathbb{P}[\Omega^*]\right).$$

Note that

$$\frac{\ell^2 c^2 r Med(Y)}{4c^2 r(Med(Y) + \ell c\sqrt{rMed(Y)})} \leq \frac{\ell^2 c^2 r Med(Y)}{8c^2 r \max\{Med(Y), \ell c\sqrt{rMed(Y)}\}}$$
$$\leq \frac{\ell^2 c^2 r Med(Y)}{8c^2 r Med(Y)} + \frac{\ell^2 c^2 r Med(Y)}{8c^3 r \ell\sqrt{rMed(Y)}\}} = \ell^2/8 + \frac{\ell\sqrt{Med(Y)}}{8c\sqrt{r}}.$$

Note also that

$$4c\sqrt{rMed(Y)}\sum_{\ell=0}^{L}\mathbb{P}\left[\Omega^*\right] \leq 4M\mathbb{P}\left[\Omega^*\right].$$

Therefore

$$\mathbb{E}\left[|Y - Med(Y)|\right] \leq 4c\sqrt{rMed(Y)}\sum_{\ell=0}^{\infty}\left(\exp\left(-\ell^2/8\right) + \exp\left(-\frac{\ell\sqrt{Med(Y)}}{8c\sqrt{r}}\right)\right) + 4M\mathbb{P}\left[\Omega^*\right].$$

Note that $\sum_{\ell=0}^{\infty} e^{-\ell x} = \frac{1}{1-e^{-x}}$. Note also that $\frac{x}{2} \leq 1 - e^{-x}$ if $x < \frac{3}{2}$. Since $\frac{1}{1-e^{-x}} < 2$ when $x \geq \frac{3}{2}$, we have $\frac{1}{1-e^{-x}} \leq \max\{2, \frac{2}{x}\}$. Therefore

$$\sum_{\ell=0}^{\infty}\exp\left(-\frac{\ell\sqrt{Med(Y)}}{8c\sqrt{r}}\right) \leq \max\left\{2, \frac{16c\sqrt{r}}{\sqrt{Med(Y)}}\right\}.$$

Note that $\sum_{\ell=0}^{\infty} e^{-\ell^2/8} < 4$. Therefore

$$\mathbb{E}\left[|Y - Med(Y)|\right] \leq 4c\sqrt{rMed(Y)}\left(4 + \max\left\{2, \frac{16c\sqrt{r}}{\sqrt{Med(Y)}}\right\}\right) + 4M\mathbb{P}\left[\Omega^*\right].$$

Since the maximum of two numbers is at most their sum,

$$\mathbb{E}\left[|Y - Med(Y)|\right] \leq 24c\sqrt{rMed(Y)} + 64rc^2 + 4M\mathbb{P}\left[\Omega^*\right].$$

Since $Med(Y) \leq 2\mathbb{E}\left[Y\right] \leq 4\mathbb{E}\left[X\right]$,

$$\mathbb{E}\left[Y - Med(Y)|\right] \leq 48c\sqrt{r\mathbb{E}\left[X\right]} + 64rc^2 + 4M\mathbb{P}\left[\Omega^*\right],$$

as desired. $\square$

Lemma 2.3.4 is similar to Fact 20.1 in [109]. However, the proof of Fact 20.1 is flawed, as we now describe. Molloy and Reed upper bound $\mathbb{P}\left[|X - Med(X)| > ic\sqrt{rMed(X)}\right]$ by $4e^{-i^2/8}$ for every positive integer i using Talagrand's Inequality I; however, Talagrand's Inequality I only applies if $0 \leq ic\sqrt{rMed(X)} \leq Med(X)$. Our proof of Lemma 2.3.4 avoids this flaw, since Theorem 2.3.3 has no restriction on t.

Now we can prove Theorem 2.3.2 assuming Theorem 2.3.3.

Proof of Theorem 2.3.2. Since $t > 96c\sqrt{r\mathbb{E}[X]} + 128rd^2 + 8M\mathbb{P}[\Omega^*]$,

$$\frac{t}{2} > 48c\sqrt{r\mathbb{E}[X]} + 64rd^2 + 4M\mathbb{P}[\Omega^*]. \tag{2.1}$$

By applying Lemma 2.3.4 and then (2.1),

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] \leq \mathbb{P}\left[|X - Med(X)| > \frac{t}{2}\right].$$

Since $Med(X) \leq 2\mathbb{E}[X]$, Theorem 2.3.3 implies that

$$\begin{aligned}\mathbb{P}\left[|X - Med(X)| > \frac{t}{2}\right] &\leq 4\exp\left(-\frac{(t/2)^2}{4d^2r(2\mathbb{E}[X] + (t/2))}\right) + 4\mathbb{P}[\Omega^*],\\ &= 4\exp\left(-\frac{t^2}{8d^2r(4\mathbb{E}[X] + t)}\right) + 4\mathbb{P}[\Omega^*]\end{aligned}$$

as desired. $\square$

It remains to prove Theorem 2.3.3.

Let $((\Omega_i, \Sigma_i, \mathbb{P}_i))_{i=1}^n$ be probability spaces and $(\Omega, \Sigma, \mathbb{P})$ their product space. For a set $A \subseteq \Omega$ and event $\omega \in \Omega$, let

$$d(\omega, A) = \sup_{||\alpha||=1}\left\{\tau : \sum_{i:\omega_i \neq \omega'_i} \alpha_i \geq \tau \text{ for all } \omega' \in \mathrm{A}\right\}. \tag{2.2}$$

We use the original version of Talagrand's Inequality.

Theorem 2.3.5 (Talagrand's Inequality [142])**.** *If $A, B \subseteq \Omega$ are measurable sets such that for all $\omega \in B$, we have $d(\omega, A) \geq \tau$, then*

$$\mathbb{P}[A]\,\mathbb{P}[B] \leq e^{-\tau^2/4}.$$

We can now prove Theorem 2.3.3.

Proof of Theorem 2.3.3. It suffices to show that

$$\mathbb{P}\left[X \leq Med(X) - t\right] \leq 2\exp\left(-\frac{t^2}{8rc^2(Med(X)+t)}\right) + 2\mathbb{P}\left[\Omega^*\right] \tag{2.3}$$

and

$$\mathbb{P}\left[X \geq Med(X) + t\right] \leq 2\exp\left(-\frac{t^2}{8rc^2(Med(X)+t)}\right) + 2\mathbb{P}\left[\Omega^*\right]. \tag{2.4}$$

Let

$$\begin{aligned} A &= \{\omega \in \Omega \backslash \Omega^* : X(\omega) \geq Med(X) + t\}, \text{ and} \\ B &= \{\omega \in \Omega \backslash \Omega^* : X(\omega) \leq Med(X)\}. \end{aligned}$$

We need to show the following.

Claim 2.3.5.1. *For all $\omega \in B$, $d(\omega, A) \geq \frac{t}{c\sqrt{r(Med(X)+t)}}$.*

To that end, let $\omega' \in A$. Since X is (r, c)-certifiable, there exists an (r, c)-certificate, I, for $X, \omega', Med(X) + t$, and Ω^*. Thus, the outcomes ω and ω' differ in at least t/c coordinates of I. Therefore if we set $\alpha = 1/\sqrt{|I|} \cdot \mathbf{1}_I$ where $\mathbf{1}_I$ is the characteristic vector of I, then ω and ω' have α-hamming distance at least $t/(c\sqrt{r(Med(X)+t)})$. Hence, the claim follows.

Now (2.4) follows from Claim 2.3.5.1 and Theorem 2.3.5. The proof of (2.3) is similar, so we omit it. $\square$

The proof of Claim 2.3.5.1 demonstrates why we introduce k into the definition of (r, d)-certificates, rather than considering changing the outcome of only one trial. We may change the outcome of one trial and obtain an exceptional outcome, in which case we need that changing the outcome of yet another trial does not greatly affect X, or else the outcomes ω and ω' may differ for only two trials.

Part I

List coloring

This part is devoted to proving our results on list coloring. In particular, in Chapter 4, we prove Theorem 1.3.4, the "local list version" of the "epsilon version" of Reed's Conjecture (Theorem 1.2.4), in Chapter 5 we prove our bound on the density of critical graphs without large cliques (Theorem 1.4.5) and use it to derive Corollary 1.4.6, the "epsilon version" of our proposed generalization of Reed's Conjecture for maximum average degree (Conjecture 1.2.8), and in Chapter 6, we prove our "unifying local version" for coloring graphs with small clique number (Theorem 1.3.7) as well as the local versions of bounds on the chromatic number of triangle-free graphs (Theorems 1.3.11 and 1.3.12).

We prove all of these results using the probabilistic method. In Chapters 4 and 5 we use a novel variation of the "naive coloring procedure" [109] that we call the "local naive random coloring procedure." Chapter 3 is devoted to describing this procedure and presenting some of its applications. In particular, we prove Theorem 3.1.8, which essentially provides sufficient conditions for our random coloring procedure to produce a valid coloring. Moreover, in Section 3.4, we discuss applications of Theorem 3.1.8 to coloring graphs in which the neighborhood of each vertex is missing many edges, as in Theorem 3.4.4, which we prove in Chapter 4.

We apply Theorem 3.1.8 in turn in Chapters 4 and 5. In Chapter 4, we use Theorem 3.1.8 in conjunction with a structural result to prove Theorem 1.3.4. We obtain this structural result from Theorem 4.2.1, a new result concerning critical graphs that may be of independent interest. In Chapter 5, we combine Theorems 3.1.8 and 4.2.1 with a unique application of the discharging method to prove Theorem 1.4.5.

In Chapter 6 we use an approach inspired by Molloy's [107] ingenious proof of Theorem 1.2.9. In effect, we analyze a partial coloring of a graph chosen uniformly at random among all partial colorings and prove Theorem 6.1.2, a more general result about coloring graphs in which the average size of an independent set in each vertex's neighborhood is large compared to the number of these independent sets. In order to obtain Theorem 1.3.7 using this theorem, we prove a Ramsey-theoretic type result, Lemma 6.3.3. This approach is quite different from previous ones such as the method employed Kim [90] and Johansson [81, 82], sometimes called the "nibble method," which we discuss in Section 6.1.1.

Chapter 3

The local naive random coloring procedure

In this chapter, we discuss how we apply the probabilistic method to obtain the results proved in Chapters 4 and 5. The main result of this chapter is Theorem 3.1.8, which provides sufficient conditions for coloring a graph using the probabilistic method. We use Theorem 3.1.8 in turn in Chapters 4 and 5.

We use a variant of a technique called the "naive coloring procedure," given its name in [109]. Essentially, we analyze a random partial coloring of a graph and prove that with nonzero probability this partial coloring can be extended deterministically to a coloring of the whole graph. The random partial coloring is described formally in Definition 3.1.5. After the random partial coloring, we let G' be the subgraph induced by G on the vertices that are not colored, and we let L' be a list-assignment for G so that any L'-coloring of G' can be combined with the random partial coloring to obtain an L-coloring of G. We prove that with nonzero probability G' is L'-colorable. To do this, we would like to show that with high probability, every vertex $v \in V(G')$ satisfies $|L'(v)| > d_{G'}(v)$, i.e. that $\mathrm{Save}_{L'}(v) \leq 0$. However, this is not necessarily the case for the graphs that we consider. In fact, it may be likely that $\mathrm{Save}_{L'}(v) = \mathrm{Save}_L(v)$. For example, the neighborhood of a vertex v may form $\sqrt{d(v)}$ cliques, while for the list-assignment L, the vertices in each clique have the same list of available colors and vertices in different cliques have disjoint lists of available colors (see Figure 3.1). Nevertheless, if a vertex v has many neighbors with at least as many available colors, we are able to show that $\mathrm{Save}_{L'}(v) < \mathrm{Save}_L(v)$. This motivates the following definitions.

With the exception of Section 3.4, this chapter is based on the paper [87] (with Luke Postle).

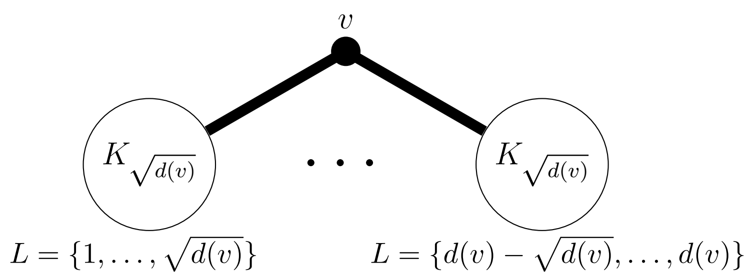

Figure 3.1: A pathological case: $\text{Save}_{L'}(v) \approx \text{Save}_L(v)$.

Definition 3.0.1. Let α be some constant to be determined later. Let G be a graph with list-assignment L, let $v \in V(G)$, and let $u \in N(v)$.

- If $|L(u)| < |L(v)|$, then we say u is a *subservient neighbor* of v.
- If $|L(u)| \in [|L(v)|, (1 + \alpha)|L(v)|)$, then we say u is an *egalitarian neighbor* of v.
- If $|L(u)| \geq (1 - \sigma)|L(v)|$, then we say u is a *σ-egalitarian neighbor* of v.
- If $|L(u)| \geq (1 + \alpha)|L(v)|$, then we say u is a *lordlier neighbor* of v.

For convenience, we will let $\text{Lord}(v)$ denote the set of lordlier neighbors of v, $\text{Egal}(v)$ denote the set of egalitarian neighbors of v, $\text{Egal}_\sigma(v)$ denote the set of σ-egalitarian neighbors of v, and $\text{Subserv}(v)$ denote the set of subservient neighbors of v. In Chapter 4, we do not need to consider the σ-egalitarian neighbors of v. In Chapter 5, we slightly modify the definition of egalitarian and subservient neighbors.

Definition 3.0.2. Let G be a graph with list-assignment L, let $v \in V(G)$, and let u be an egalitarian neighbor of v.

- If $|L(u)| < |L(v)| + \alpha \,\text{Gap}(v)$, then we say u is a *strongly egalitarian neighbor* of v.
- If $|L(u)| \geq |L(v)| + \alpha \,\text{Gap}(v)$, then we say u is a *weakly egalitarian neighbor* and also a *slightly lordlier neighbor* of v.

For convenience, we will let $\text{SEgal}(v)$ denote the set of strongly egalitarian neighbors of v, $\text{WEgal}(v)$ denote the set of weakly egalitarian neighbors of v, and $\text{NEgal}(v) = N(v) - \text{Egal}(v)$.

If a vertex v has many subservient neighbors, then we say v is *lordly*. The names "subservient", "egalitarian", and "lordlier" neighbors are evocative of feudalism in medeival Europe, where power is analogous to list size. As Figure 3.1 indicates, if v is a lordly vertex, we are unable to guarantee that $\text{Save}_{L'}(v) < \text{Save}_L(v)$ for certain list-assignments for v's subservient neighbors. We resolve this issue by coloring vertices before their subservient neighbors when finding an L'-coloring, thus giving "priority" to the lordly vertices.

A lordlier neighbor also has the power to choose from more colors. If v has many lordlier neighbors or weakly egalitarian neighbors, then it is likely that after the random partial coloring v has many neighbors receiving a color not in $L(v)$. We call such vertices *aberrant*. If v has many pairs of non-adjacent egalitarian neighbors, then it is likely that after the random partial coloring there are many colors assigned to multiple neighbors of v. In both cases, $\text{Save}_{L'}(v) < \text{Save}_L(v)$.

A common technique in coloring is to attempt to greedily color a vertex of smallest degree, since fewer neighbors means fewer potential color conflicts. However, for our "local version," this technique is not so useful because vertices of lower degree also have fewer available colors. Our trick to finding an L'-coloring of G' in Chapter 4 is to order the vertices of G' by the size of their list in L, from greatest to least, and color greedily, which may seem counterintuitive. This works because we are able to guarantee for every vertex $v \in V(G')$, that $\text{Save}_{L'}(v)$ is smaller than the number of neighbors of v in G' that will be colored after v in this ordering, and thus $|L'(v)|$ is larger than the number of neighbors of v in G' that will be colored before v in this ordering. In Chapter 5 we take a similar approach, but we color the vertices of G' in a different order.

For each vertex v, after an application of our naive coloring procedure, we refer to the number of neighbors of v receiving a color not in $L(v)$, plus the multiplicity less 1 of each color in $L(v)$ assigned to multiple neighbors, plus the number of uncolored subservient neighbors of v as the "savings" for v. The main result of this chapter is Theorem 3.1.8, which essentially says that it suffices to show that the expected savings for each vertex is at least $\text{Save}_L(v)$ and is sufficiently large. We apply Theorem 3.1.8 in both Chapter 4 and Chapter 5.

In Section 3.1, we compile all of the requisite definitions and propositions for stating Theorem 3.1.8. In Section 3.2, we prove Theorem 3.1.8, deferring some concentration details to Section 3.3 where we apply our exceptional outcomes versions of Talagrand's Inequality, Theorem 2.3.2. In Section 3.4, we discuss the naive coloring procedure in greater detail. We also provide a simple application of Theorem 3.1.8 from [109, Theorem 10.5] as well as more potential applications.

3.1 A metatheorem

In this section we introduce Theorem 3.1.8, which gives sufficient conditions for our naive coloring procedure to be extended to a coloring of the whole graph. Namely, we need that the expected "savings" for each vertex is at least $\text{Save}_L(v)$ and is sufficiently large. Before we can state Theorem 3.1.8, we need to formalize our naive coloring procedure.

In this section and in Sections 3.2 and 3.3, we let G be a graph with correspondence-assignment (L, M), we let $\prec$ be a partial ordering of $V(G)$, and we let $\varepsilon, \sigma \in [0, 1)$. Using correspondence coloring helps improve the value of ε in Theorem 1.3.4, because we can assume egalitarian neighbors of a vertex v have at least $|L(v)|$ colors in common, thus making it more likely that a color is assigned to more than one of them. After applying our random coloring procedure to obtain a partial coloring, we will complete the coloring greedily in the ordering specified by $\prec$. When we apply Theorem 3.1.8 to prove Theorem 1.3.4 in Chapter 4, we let σ be 0 and for $u, v \in V(G)$, we have $u \prec v$ if $|L(u)| < |L(v)|$. In Chapter 5, we apply Theorem 3.1.8 with $\sigma > 0$ and a different ordering.

Definition 3.1.1.

- We say a *naive partial* (L, M)*-coloring* of G is a pair (ϕ, U) where $\phi : V(G) \to \mathbb{N}$ such that $\phi(u) \in L(u)$ for every $u \in V(G)$ and $U \subseteq V(G)$ is a set of *uncolored vertices* such that $\phi|_{V(G)\setminus U}$ is an (L, M)-coloring of $G - U$.
- If (ϕ, U) is a naive partial (L, M)-coloring of G, for each $v \in U$, let

$$L^{\phi,U}(v) = L(v)\backslash\{c \in L(v) : \exists u \in N(v)\backslash U, c\phi(u) \in M_{vu}\}$$

and for each $uv \in E(G[U])$, let $M^{\phi,U}_{uv}$ be the matching induced by M_{uv} on $\{u\} \times L^{\phi,U}(u)$ and $\{v\} \times L^{\phi,U}(v)$.

If (ϕ, U) is a naive partial (L, M)-coloring of G, then we call a vertex v *uncolored* if it is in U, and otherwise we call it *colored*.

We consider the following proposition to be self-evident.

Proposition 3.1.2. *If* (ϕ, U) *is a naive partial* (L, M)*-coloring of* G *and* $G[U]$ *is* $(L^{\phi,U}, M^{\phi,U})$*-colorable, then* G *is* (L, M)*-colorable.* □

The following is a variant of the naive coloring procedure, but it is not the one we use in Theorem 3.1.8.

Definition 3.1.3. The *local naive random coloring procedure* samples a random naive partial (L, M)-coloring in the following way. For each $v \in V(G)$,

1. choose $\phi(v) \in L(v)$ uniformly at random, and
2. let $v \in U$ if there exists $u \in N(v)$ such that $|L(u)| \le |L(v)|$ and $\phi(u)\phi(v) \in M_{uv}$.

Recall that $\varepsilon \in [0, 1)$. Let $K_\varepsilon = .999e^{\frac{-1}{1-\varepsilon}}$. We need the following proposition.

Proposition 3.1.4. *If (ϕ, U) is a random naive partial (L, M)-coloring sampled using the local naive random coloring procedure, then*

$$\mathbb{P}\left[v \notin U\right] = \prod_{\{u \in N(v): |L(u)| \le |L(v)|\}} \left(1 - \frac{1}{|L(v)|}\right).$$

Moreover, there exists $\delta = \delta(\varepsilon)$ such that the following holds. If each $v \in V(G)$ satisfies $|L(v)| \ge (1-\varepsilon)d(v)$ and G has minimum degree at least δ, then for each $v \in V(G)$,

$$\mathbb{P}\left[v \notin U\right] \ge K_\varepsilon.$$

Proof. The vertex v is in U if and only if there exists $u \in N(v)$ such that $|L(u)| \le |L(v)|$ and $\phi(u)\phi(v) \in M_{uv}$, and for each such $u \in N(v)$, the probability that $\phi(u)\phi(v) \in M_{uv}$ is at most $1/|L(v)|$. Since these events are independent, the first equation follows.

Since $|L(v)| \ge (1-\varepsilon)d(v)$,

$$\mathbb{P}\left[v \notin U\right] \ge \left(1 - \frac{1}{(1-\varepsilon)d(v)}\right)^{d(v)} \ge \left(1 - \frac{1}{(1-\varepsilon)^2 d(v)}\right) e^{-\frac{1}{1-\varepsilon}}.$$

We let $\delta(\varepsilon) = 1000/(1-\varepsilon)^2$, and the result follows. ☐

The local naive random coloring procedure may be useful for some applications, but it is not sufficient for our purposes. Using Proposition 3.1.2, we want to find a naive partial (L, M)-coloring (ϕ, U) of G such that $G[U]$ is $(L^{\phi,U}, M^{\phi,U})$-colorable. To do this, it suffices to show that for every vertex v, $\mathrm{Save}_{L^{\phi,U}}(v)$ is less than the number of uncolored neighbors u of v such that $u \prec v$. However, it is possible that for each $u \prec v$, $\mathbb{P}\left[u \in U\right] = 0$ and $\mathrm{Save}_{L^{\phi,U}}(v) > 0$. For this reason, we modify the local naive random coloring procedure so that $\mathbb{P}\left[u \in U\right] > 0$.

This new procedure can only be applied to graphs of sufficiently large minimum degree such that $\mathrm{Save}(v)$ is at most a fraction of $d(v)$ for each vertex v. To that end, we say that the pair (G, L) is (Δ, ε)-*bounded* if

- $\Delta(G) \le \Delta$,
- G has minimum degree at least $\delta(\varepsilon)$ (as in Proposition 3.1.4), and
- $\Delta \ge |L(v)| \ge (1-\varepsilon)d(v)$ for each vertex $v \in V(G)$.

We can now define our random coloring procedure, as follows.

Definition 3.1.5. If (G, L) is (Δ, ε)-bounded, then the *local naive random coloring procedure with ε-equalizing coin-flips* samples a random naive partial (L, M)-coloring (ϕ, U) in the following way.

1. Let (ϕ, U') be sampled using the local naive random coloring procedure,
2. for each $v \in V(G) - U'$, let $v \in U''$ with probability $1 - K_\varepsilon/\mathbb{P}[v \notin U']$, and
3. let $U = U' \cup U''$.

The following proposition shows why the ε-equalizing coin-flips are useful.

Proposition 3.1.6. *If (ϕ, U) is sampled using the local naive random coloring procedure with ε-equalizing coin fips, then for each $v \in V(G)$,*

$$\mathbb{P}[v \notin U] = K_\varepsilon.$$

Proof. Let $U = U' \cup U''$ as in Definition 3.1.5. Note that $\mathbb{P}[v \notin U] = \mathbb{P}[v \notin U'' | v \notin U'] \cdot \mathbb{P}[v \notin U']$. By the choice of U'', we have $\mathbb{P}[v \notin U'' | v \notin U'] = K_\varepsilon/\mathbb{P}[v \notin U']$, and the result follows. □

Recall that a vertex u is a σ-egalitarian neighbor of a vertex v if $|L(u)| \geq (1-\sigma)|L(v)|$. Recall also that $\prec$ is a partial ordering of $V(G)$. We can now formalize what we mean by the "savings" for each vertex, as follows.

Definition 3.1.7. For each $v \in V(G)$, we define the following random variables.

- Let $\mathbf{unmatched}_v((\phi, U))$ count the number of colored neighbors u of v such that $\phi(u)$ is not matched by M_{uv}.
- Let $\mathbf{pairs}_{v,\sigma}((\phi, U))$ and $\mathbf{trips}_{v,\sigma}((\phi, U))$ count the number of nonadjacent pairs and triples respectively of colored σ-egalitarian neighbors of v that receive colors that are matched to the same color in $L(v)$.
- Let $\mathbf{uncolored}_{v,\prec}((\phi, U))$ count the number of uncolored neighbors u of v such that $u \prec v$.
- Let $\mathbf{savings}_{v,\sigma,\prec}((\phi, U)) = \mathbf{unmatched}_v((\phi, U)) + \mathbf{uncolored}_{v,\prec}((\phi, U))$
$+ \mathbf{pairs}_{v,\sigma}((\phi, U)) - \mathbf{trips}_{v,\sigma}((\phi, U))$.

More precisely, letting $T(H)$ denote the set of triangles of a graph H, we have that

$$\begin{aligned}
&\mathbf{unmatched}_v((\phi, U)) = |\{u \in N(v)\backslash U : \phi(u) \notin V(M_{uv})\}|,\\
&\mathbf{pairs}_{v,\sigma}((\phi, U)) = |\{xy \in E(\overline{G[\mathrm{Egal}_\sigma(v)]}), c \in L(v) : x, y \notin U,\\
&\qquad \phi(x)c \in M_{xv} \text{ and } \phi(y)c \in M_{uv}\}|,\\
&\mathbf{trips}_{v,\sigma}((\phi, U)) = |\{xyz \in T(\overline{G[\mathrm{Egal}_\sigma(v)]}), c \in L(v) : x, y, z \notin U,\\
&\qquad \phi(x)c \in M_{xv}, \phi(y)c \in M_{yv}, \text{ and } \phi(z)c \in M_{zv}\}|, \text{ and}\\
&\mathbf{uncolored}_{v,\prec}((\phi, U)) = |\{u \in U : u \prec v\}|.
\end{aligned}$$

We are now prepared to state Theorem 3.1.8.

Theorem 3.1.8 (Kelly and Postle [87]). *For every $\xi_1, \xi_2 > 0$ and $\varepsilon, \sigma \in [0, 1)$, there exists Δ_0 such that the following holds. If G is a graph with correspondence-assignment (L, M) such that (G, L) is (Δ, ε)-bounded where $\Delta \geq \Delta_0$, and $\prec$ is a partial ordering of $V(G)$ such that*

$$\mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] \geq \max\{(1+\xi_1)\,\mathrm{Save}_L(v), \xi_2 \log^{10}\Delta\}, \tag{3.1}$$

then G is (L, M)-colorable

3.2 Proof of Theorem 3.1.8

In order to prove Theorem 3.1.8, we need the following lemma.

Lemma 3.2.1. *Under the conditions of Theorem 3.1.8, if (ϕ, U) is sampled using the local naive random coloring procedure with ε-equalizing coin flips, then with nonzero probability every $v \in V(G)$ satisfies*

$$\mathrm{Save}_{L_{\phi,U}}(v) \leq \mathbf{uncolored}_{v,\prec}((\phi, U)). \tag{3.2}$$

Observe that by the inclusion-exclusion principle, if we let the *repetitiveness* of color $c \in L(v)$ be one less than the number of colored neighbors $u \in N(v)$ such that $\phi(u)c \in M_{uv}$, then $\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma}$ undercounts the total repetitiveness of colors assigned to neighbors of v. Therefore

$$\mathrm{Save}_L(v) - \mathrm{Save}_{L_{\phi,U}}(v) \geq \mathbf{unmatched}_v((\phi, U)) + \mathbf{pairs}_{v,\sigma}((\phi, U)) - \mathbf{trips}_{v,\sigma}((\phi, U)). \tag{3.3}$$

We need to show that with high probability, these random variables are close to their expectation. We make this precise in the following definition.

Definition 3.2.2. We say a random variable X is Δ-*concentrated* if

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq 2\max\{\mathbb{E}[X]^{5/6}, \log^9\Delta\}\right] < \frac{\Delta^{-4}}{16}.$$

We will use the following lemma to prove Lemma 3.2.1.

Lemma 3.2.3. *If (G, L) is (Δ, ε)-bounded and Δ is sufficiently large, then for each $v \in V(G)$, the random variables $\mathbf{unmatched}_v, \mathbf{uncolored}_{v,\prec}, \mathbf{pairs}_{v,\sigma}$, and $\mathbf{trips}_{v,\sigma}$ are Δ-concentrated.*

We defer the proof of Lemma 3.2.3 to Section 3.3. Lemma 3.2.3 is the reason why we need to include the parameter σ.

Now we are ready to prove Lemma 3.2.1.

Proof of Lemma 3.2.1. For each $v \in V(G)$, let $\mathcal{A}_v$ be the event that (3.2) does not hold, and let $\mathcal{A} = \{\mathcal{A}_v : v \in V(G)\}$. Note that for each $v \in V(G)$, $\mathcal{A}_v$ depends only on trials at vertices at distance at most two from v, so if $u \in V(G)$ has distance at least five to v, then $\mathcal{A}_u$ and $\mathcal{A}_v$ do not depend on any of the same trials. Therefore each $\mathcal{A}_v$ is mutually independent of a set of all but at most Δ^4 events in $\mathcal{A}$.

By Lemma 2.2.1, it suffices to show that for each $v \in V(G)$, we have $\mathbb{P}[\mathcal{A}_v] \leq \Delta^{-4}/4$. Let

$$Z_v = 2(\max\{\mathbb{E}[\mathbf{unmatched}_v]^{5/6}, \log^9 \Delta\} + \max\{\mathbb{E}[\mathbf{uncolored}_{v,\prec}]^{5/6}, \log^9 \Delta\} + \max\{\mathbb{E}\left[\mathbf{pairs}_{v,\sigma}\right]^{5/6}, \log^9 \Delta\} - \max\{\mathbb{E}\left[\mathbf{trips}_{v,\sigma}\right]^{5/6}, \log^9 \Delta\}),$$

and let $\mathcal{A}'_v$ be the event that

$$\mathbf{savings}_{v,\sigma,\prec} \leq \mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] - Z_v.$$

By Lemma 3.2.3 and the Union Bound, $\mathbb{P}[\mathcal{A}'_v] < \Delta^{-4}/4$. We claim that $\mathcal{A}_v \subseteq \mathcal{A}'_v$, which completes the proof. By (3.3), it suffices to show

$$\mathrm{Save}_L(v) \leq \mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] - Z_v. \tag{3.4}$$

By the assumption that $\mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] \geq \xi_2 \log^{10} \Delta$, we have $Z_v = o(\mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right])$. Since Δ is sufficiently large, we may assume that $Z_v \leq \xi_1 \mathrm{Save}_L(v)$. Since $\mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] \geq (1 + \xi_1) \mathrm{Save}_L(v)$, (3.4) holds, which completes the proof. $\square$

We conclude this section with the proof of Theorem 3.1.8.

Proof of Theorem 3.1.8. By Proposition 3.1.2, it suffices to show that $G[U]$ is $(L^{\phi,U}, M^{\phi,U})$-colorable with nonzero probability. Thus it suffices to show that for some instance of (ϕ, U), for each $v \in U$,

$$|L^{\phi,U}(v)| - 1 \geq |\{u \in N(v) \cap U : u \not\prec v\}|, \tag{3.5}$$

because then we can color $G[U]$ greedily in the ordering provided by $\prec$, breaking ties arbitrarily. By Lemma 3.2.1, we may consider the instance (ϕ, U) in which each $v \in V(G)$ satisfies (3.2).

Note that for each $v \in V(G)$,

$$|\{u \in N(v) \cap U : u \not\prec v\}| = d_U(v) - \mathbf{uncolored}_{v,\prec}((\phi, U)).$$

Therefore if (3.2) holds, then (3.5) holds. Thus, $G[U]$ is $(L^{\phi,U}, M^{\phi,U})$-colorable, as desired. □

Per the discussion in Section 2.2.1, we can replace the application of Lemma 2.2.1 in the proof of Lemma 3.2.1 with the main result of [113] to obtain a randomized algorithm that finds the naive partial (L, M)-coloring (ϕ, U) satisfying (3.2) with polynomial expected running time. Hence, we obtain an algorithmic version of Theorem 3.1.8. Moreover, if we treat Δ as a fixed parameter in Theorem 3.1.8, then we have a deterministic algorithm with polynomial running time. With this algorithmic version in hand, the proofs of Theorems 1.3.4 and 1.4.6 in the following sections naturally provide a polynomial-time algorithm to obtain the colorings guaranteed by these results.

3.3 Concentrations

In this section we prove Lemma 3.2.3. Recall that $\varepsilon, \sigma \in [0, 1)$, G is a graph with correspondence-assignment (L, M) such that (G, L) is (Δ, ε)-bounded, and Δ is sufficiently large.

3.3.1 Exceptional outcomes

Using Theorem 2.3.2, we show that each of the random variables that contribute to the savings for each vertex are Δ-concentrated. The exceptional outcomes we consider when applying Theorem 2.3.2 will involve many neighbors of a vertex v receiving the same color (from v's perspective), so we need this to be unlikely. In our coloring procedure, when adjacent vertices u and v received the same color, we uncolored v if $|L(v)| \geq |L(u)|$. If we instead uncolored both u and v (as in [109]) or flipped a coin to decide which to uncolor (as in [31]), then changing the color of a vertex may be likely to greatly affect the number of uncolored subservient neighbors, because subservient neighbors, having fewer available colors, are more likely to be assigned the same color. Luckily, our trick of uncoloring a vertex only if it is in conflict with a neighbor having equal or fewer available colors resolves this issue.

This also explains why we need $\sigma < 1$ to apply Theorem 2.3.2 to $\textbf{pairs}_{v,\sigma}$ and $\textbf{trips}_{v,\sigma}$. In the extreme case, a vertex v could have many neighbors with only two available colors, one of which corresponds to the same color for v. Switching the color of one of these neighbors will significantly affect the number of pairs or triples if many neighbors of v receive the same color, and this is likely. However, it is unlikely that many σ-egalitarian neighbors of v receive the same color, as long as $|L(v)|$ is large, as demonstrated by the following proposition.

Proposition 3.3.1. *For each $v \in V(G)$, let $\Omega^*_{v,\sigma}$ be the set of events where there exists $u \in V(G), c \in L(u)$, and a set $X \subset (\mathrm{Egal}_\sigma(u) \cap N(v))$ of size at least $\log \Delta$ such that for each $w \in X$, we have that $\phi(w)c \in M_{wu}$. Now*

$$\mathbb{P}\left[\Omega^*_{v,\sigma}\right] \leq \Delta^4 \left(\frac{e}{(1-\sigma)(1-\varepsilon)\log \Delta}\right)^{\log \Delta}.$$

Proof. For each $u \in V(G)$ and $c \in L(u)$, let

$$Y_{u,c} = |\{w \in (N(v) \cap \mathrm{Egal}_\sigma(u)) : \phi(w)c \in M_{wv}\}|.$$

Now

$$\mathbb{P}\left[Y_{u,c} \geq \log \Delta\right] \leq \sum_{i=\lceil \log \Delta \rceil}^{d(u)} \binom{d(u)}{i} \frac{1}{((1-\sigma)|L(u)|)^i}.$$

By applying the bound $\binom{d(u)}{i} < \left(\frac{e \cdot d(u)}{i}\right)^i$ and using the fact that $\frac{1}{|L(u)|} \leq \frac{1}{(1-\varepsilon)d(u)}$,

$$\mathbb{P}\left[Y_{u,c} \geq \log \Delta\right] \leq \sum_{i=\lceil \log \Delta \rceil}^{d(u)} \left(\frac{e \cdot d(u)}{i}\right)^i \frac{1}{(1-\varepsilon)^i d(u)^i} = \sum_{i=\lceil \log \Delta \rceil}^{d(u)} \left(\frac{e}{(1-\sigma)(1-\varepsilon)i}\right)^i.$$

Since each term in the sum is at most $\left(\frac{e}{(1-\sigma)(1-\varepsilon)\log \Delta}\right)^{\log \Delta}$ and there are at most Δ terms, it follows that

$$\mathbb{P}\left[Y_{u,c} \geq \log \Delta\right] \leq \Delta \left(\frac{e}{(1-\sigma)(1-\varepsilon)\log \Delta}\right)^{\log \Delta}.$$

Since $|N(v) \cap \mathrm{Egal}_\sigma(u)| = 0$ for all but Δ^2 vertices u, and each has at most Δ available colors, by the Union Bound,

$$\mathbb{P}\left[\Omega^*_{v,\sigma}\right] \leq \Delta^4 \left(\frac{e}{(1-\sigma)(1-\varepsilon)\log \Delta}\right)^{\log \Delta},$$

as desired. ∎

Observe that $\mathbb{P}\left[\Omega^*_{v,\sigma}\right] = o(\Delta^{-4})$.

3.3.2 Proof of Lemma 3.2.3

We always apply Theorem 2.3.2 with $t = \max\{\mathbb{E}\left[X\right]^{5/6}, \log^9 \Delta\}$, $r \leq 4$, and $d \leq \log^3 \Delta$. Note that, assuming Δ is sufficiently large and $\mathbb{P}\left[\Omega^*\right]$ is sufficiently small, t is large enough to apply Theorem 2.3.2.

The following proposition will be useful.

Proposition 3.3.2. *If X is a non-negative random variable and $t = \max\{\gamma \cdot \mathbb{E}\left[X\right]^{5/6}, \log^9 \Delta\}$ where $\gamma > 0$, then*

$$\frac{t^2}{4\mathbb{E}\left[X\right] + t} \geq \frac{\log^{36/5} \Delta}{1 + 4/\gamma^{6/5}}.$$

Proof. Since $\mathbb{E}\left[X\right] \leq (t/\gamma)^{6/5}$,

$$\frac{t^2}{4\mathbb{E}\left[X\right] + t} \geq \frac{t^2}{4(t/\gamma)^{6/5} + t} \geq \frac{t^{4/5}}{1 + 4/\gamma^{6/5}}.$$

Since $t \geq \log^9 \Delta$, the result follows. $\square$

Now we can prove Lemma 3.2.3. For each $v \in V(G)$, let $(\Omega_{v,1}, \Sigma_{v,1}, \mathbb{P}_{v,1})$ be the probability space where $\Omega_{v,1} = L(v)$, the sigma-algebra $\Sigma_{v,1}$ is the discrete sigma-algebra, and $\mathbb{P}_{v,1}$ is the uniform distribution (i.e. this probability space corresponds to assigning v a color from $L(v)$ uniformly at random) and let $(\Omega_{v,2}, \Sigma_{v,2}, \mathbb{P}_{v,1})$ be the probability space where $\Omega_{v,2} = \{\text{heads}, \text{tails}\}$, the sigma-algebra $\Sigma_{v,2}$ is again discrete, and $\mathbb{P}_{v,2}[\text{heads}] = 1 - K_\varepsilon/p$, where p is the probability that v is not uncolored after an application of the local naive random coloring procedure (i.e. this probability space corresponds to an ε-equalizing coin-flip for v). Let $(\Omega, \Sigma, \mathbb{P})$ be the product space of $(\Omega_{v,i}, \Sigma_{v,i}, \mathbb{P}_{v,i})_{v \in V(G), i \in \{1,2\}}$. In order to sample a naive partial coloring using the local naive random coloring procedure with ε-equalizing coin flips, we sample from Ω. If ω is an outcome in Ω, then we let (ϕ_ω, U_ω) be the corresponding naive partial coloring.

We prove each random variable is Δ-concentrated individually. In order to apply Theorem 2.3.2, we need to show that these random variables are (r,d)-certifiable. Recall that a random variable X is (r,d)-certifiable with respect to exceptional outcomes Ω^* if for every $s > 0$ and outcome ω there is a set I of size at most rs indexing the trials such that for all $k \geq 0$, we have that $X(\omega') \geq s - kd$ for all outcomes ω' that differ from ω for at most k of the trials indexed by I.

Proof that $\mathbf{uncolored}_{v,\prec}$ *is* Δ*-concentrated.* We claim that $\mathbf{uncolored}_{v,\prec}$ is (r,d)-certifiable with respect to $\Omega^*_{v,0}$ from Proposition 3.3.1, where $r=2$ and $d=\log\Delta$. Let $s>0$ and let $\omega\notin\Omega^*_{v,0}$ such that $\mathbf{uncolored}_{v,\prec}(\omega)\geq s$. We show that there is an (r,d)-certificate, I, for $\mathbf{uncolored}_{v,\prec},\omega,s$, and $\Omega^*_{v,0}$.

Since $\mathbf{uncolored}_{v,\prec}(\omega)\geq s$, there is a set S_1 of s uncolored neighbors u of v such that $u\prec v$. Each such vertex $u\in S_1$ either has a neighbor u' such that $|L(u')|\leq|L(u)|$ and $\phi_\omega(u)\phi_\omega(u')\in M_{uu'}$ or is uncolored by an ε-equalizing coin-flip. In the former case, we choose precisely one such neighbor u' of u, let u' be in the set S_2, and let u be in the set $S_{u'}$. In the latter case, we let $u\in S_1'$. By the definition of these sets,

$$\mathbf{uncolored}_{v,\prec}(\omega)\geq|S_1'|+\sum_{u\in S_2}|S_u|=s. \tag{3.6}$$

We let I index the trials determining $\phi_\omega(u)$ for the $u\in S_1$, and for each $u\in S_1\setminus S_1'$, there exists $u'\in N(u)\cap S_2$, and we also let I index the trial determining $\phi_\omega(u')$. For each $u\in S_1'$, u is uncolored by an ε-equalizing coin flip, and we let I index this trial. Note that $|I|\leq 2s$.

We claim that I is an (r,d)-certificate for $\mathbf{uncolored}_{v,\prec},\omega,s$, and $\Omega^*_{v,0}$. To that end, let $\omega'\in\Omega\setminus\Omega^*_{v,0}$ and $k\geq 0$ such that ω and ω' differ for at most k trials indexed by I. We say a vertex *keeps its color* if $\phi_\omega(u)=\phi_{\omega'}(u)$. Let T_1' be the set of vertices in S_1' that are also uncolored by an ε-equalizing coin-flip in the outcome ω', let T_2 be the set of vertices in S_2 that keep their color, and for each $u\in T_2$, let T_u be the set of vertices in S_u that keep their color. Note that

$$\mathbf{uncolored}_{v,\prec}(\omega')\geq|T_1'|+\sum_{u\in T_2}|T_u|. \tag{3.7}$$

Moreover, the sets in the above inequality are pairwise disjoint. Since ω and ω' differ in at most k trials indexed by I,

$$|S_1'\setminus T_1'|+|S_2\setminus T_2|+|\cup_{u\in T_2}S_u\setminus T_u|\leq k. \tag{3.8}$$

Since $\omega\notin\Omega^*_{v,0}$, for each $u\in S_2$, we have that $|S_u|\leq\log\Delta=d$. Therefore by (3.6), (3.7), and (3.8), $\mathbf{uncolored}_{v,\prec}(\omega')\geq s-kd$, so I is an (r,d)-certificate for $\mathbf{uncolored}_{v,\prec},\omega,s$, and $\Omega^*_{v,0}$, as claimed. Thus, $\mathbf{uncolored}_{v,\prec}$ is (r,d)-certifiable with respect to $\Omega^*_{v,0}$, as claimed, and we can apply Theorem 2.3.2. We choose $t=\max\{\mathbb{E}\left[\mathbf{uncolored}_{v,\prec}\right]^{5/6},\log^9\Delta\}$, so by Proposition 3.3.2 and Theorem 2.3.2, for some constant $\gamma_1>0$,

$$\mathbb{P}\left[|\mathbf{uncolored}_{v,\prec}-\mathbb{E}\left[\mathbf{uncolored}_{v,\prec}\right]|>t\right]\leq 4\exp(-\gamma_1(\log^{26/5}(\Delta)))+4\mathbb{P}\left[\Omega^*_{v,0}\right].$$

Since Δ is sufficiently large, the result follows. $\square$

Proof that $\mathbf{unmatched}_v$ *is* Δ*-concentrated.* We can not show that $\mathbf{unmatched}_v$ is (r,d)-certifiable with respect to any appropriate set of exceptional outcomes, but we can express $\mathbf{unmatched}_v$ as the difference of two random variables that are. To that end, we define the following random variables in which (ϕ, U) is a random naive partial coloring:

$$\mathbf{unmatched}_v^{\text{tot}} = |\{u \in N(v) : \phi(u) \notin V(M_{uv})\}|, \text{ and}$$
$$\mathbf{unmatched}_v^{\text{uncol}} = |\{u \in N(v) \cap U : \phi(u) \notin V(M_{uv})\}|.$$

Note that $\mathbf{unmatched}_v^{\text{tot}}$ is (r,d)-certifiable with respect to $\Omega^* = \varnothing$, where $r, d = 1$. Note also that $\mathbb{E}[\mathbf{unmatched}_v] = K_\varepsilon \cdot \mathbb{E}\left[\mathbf{unmatched}_v^{\text{tot}}\right] = K_\varepsilon \cdot \mathbb{E}\left[\mathbf{unmatched}_v^{\text{uncol}}\right]/(1 - K_\varepsilon)$. We choose $t = \max\{\mathbb{E}[\mathbf{unmatched}_v]^{5/6}, \log^9 \Delta\}$, so by Proposition 3.3.2 and Theorem 2.3.2, for some constant $\gamma_2 > 0$,

$$\mathbb{P}\left[|\mathbf{unmatched}_v^{\text{tot}} - \mathbb{E}\left[\mathbf{unmatched}_v^{\text{tot}}\right]| > t\right] \leq 4\exp(-\gamma_2(\log^{36/5}(\Delta))). \tag{3.9}$$

By the same argument as in the proof that $\mathbf{uncolored}_{v,\prec}$ is Δ-concentrated, $\mathbf{unmatched}_v^{\text{uncol}}$ is (r,d)-certifiable with exceptional outcomes $\Omega^*_{v,0}$ from Proposition 3.3.1 with $r = 2$ and $d = \log \Delta$. By Proposition 3.3.2 and Theorem 2.3.2, for some constant $\gamma_3 > 0$,

$$\mathbb{P}\left[|\mathbf{unmatched}_v^{\text{uncol}} - \mathbb{E}\left[\mathbf{unmatched}_v^{\text{uncol}}\right]| > t\right] \leq 4\exp(-\gamma_3(\log^{26/5}(\Delta))) + 4\mathbb{P}\left[\Omega_v^*\right]. \tag{3.10}$$

Since $\mathbf{unmatched}_v = \mathbf{unmatched}_v^{\text{tot}} - \mathbf{unmatched}_v^{\text{uncol}}$, it follows from (3.9), (3.10), and Proposition 3.3.1 that $\mathbf{unmatched}_v$ is Δ-concentrated, as desired. $\square$

Proof that $\mathbf{pairs}_{v,\sigma}$ *and* $\mathbf{trips}_{v,\sigma}$ *are* Δ*-concentrated.* As in the proof that $\mathbf{unmatched}_v$ is Δ-concentrated, we do not show that $\mathbf{pairs}_{v,\sigma}$ and $\mathbf{trips}_{v,\sigma}$ are (r,d)-certifiable with respect to some set of exceptional outcomes. Instead, we express $\mathbf{pairs}_{v,\sigma}$ and $\mathbf{trips}_{v,\sigma}$ as differences of such random variables and apply Theorem 2.3.2 to each of these new random variables. If H is a graph, recall that $T(H)$ denotes the set of triangles in H. We define the following random variables in which (ϕ, U) is a random naive partial coloring:

$$\mathbf{pairs}_{v,\sigma}^{\text{tot}} = |\{xy \in E(\overline{G[\text{Egal}_\sigma(v)]}), c \in L(v) : \phi(x)c \in M_{xv} \text{ and } \phi(y)c \in M_{uv}\}|,$$
$$\mathbf{trips}_{v,\sigma}^{\text{tot}} = |\{xyz \in T(\overline{G[\text{Egal}_\sigma(v)]}), c \in L(v) : \phi(x)c \in M_{xv}, \phi(y)c \in M_{yv}, \text{ and } \phi(z)c \in M_{zv}\}|,$$
$$\mathbf{pairs}_{v,\sigma}^{\text{uncol}} = |\{xy \in E(\overline{G[\text{Egal}_\sigma(v)]}), c \in L(v) : \{x,y\} \cap U \neq \varnothing,$$
$$\phi(x)c \in M_{xv} \text{ and } \phi(y)c \in M_{uv}\}|, \text{ and}$$
$$\mathbf{trips}_{v,\sigma}^{\text{uncol}} = |\{xyz \in T(\overline{G[\text{Egal}_\sigma(v)]}), c \in L(v), c \in L(v) : \{x,y,z\} \cap U \neq \varnothing,$$
$$\phi(x)c \in M_{xv}, \phi(y)c \in M_{yv}, \text{ and } \phi(z)c \in M_{zv}\}|.$$

Note that $\mathbf{pairs}_{v,\sigma} = \mathbf{pairs}_{v,\sigma}^{\text{tot}} - \mathbf{pairs}_{v,\sigma}^{\text{uncol}}$ and $\mathbf{trips}_{v,\sigma} = \mathbf{trips}_{v,\sigma}^{\text{tot}} - \mathbf{trips}_{v,\sigma}^{\text{uncol}}$. Note also that $\mathbb{E}\left[\mathbf{pairs}_{v,\sigma}\right] = K_\varepsilon^2 \cdot \mathbb{E}\left[\mathbf{pairs}_{v,\sigma}^{\text{tot}}\right] = K_\varepsilon^2 \cdot \mathbb{E}\left[\mathbf{pairs}_{v,\sigma}^{\text{uncol}}\right]/(1-K_\varepsilon^2)$ and $\mathbb{E}\left[\mathbf{trips}_{v,\sigma}\right] = K_\varepsilon^3 \cdot \mathbb{E}\left[\mathbf{trips}_{v,\sigma}^{\text{tot}}\right] = K_\varepsilon^3 \cdot \mathbb{E}\left[\mathbf{trips}_{v,\sigma}^{\text{uncol}}\right]/(1-K_\varepsilon^3)$.

We claim that $\mathbf{pairs}_{v,\sigma}^{\text{tot}}$ and $\mathbf{pairs}_{v,\sigma}^{\text{uncol}}$ are (r,d)-certifiable with respect to exceptional outcomes $\Omega_{v,\sigma}^*$ from Proposition 3.3.1, where $r = 4$ and $d = \log^2 \Delta$. We only provide a proof for $\mathbf{pairs}_{v,\sigma}^{\text{uncol}}$, since the proof for $\mathbf{pairs}_{v,\sigma}^{\text{tot}}$ is easier. Let $s > 0$ and let $\omega \notin \Omega_{v,\sigma}^*$ such that $\mathbf{pairs}_{v,\sigma}^{\text{uncol}}(\omega) \geq s$. We show that there is an (r,d)-certificate, I, for $\mathbf{pairs}_{v,\sigma}^{\text{uncol}}, \omega, s$, and $\Omega_{v,\sigma}^*$.

For each $c \in L(v)$, define $S_{c,1}$ as follows. If the set of uncolored σ-egalitarian neighbors u of v such that $\phi_\omega(u)c \in M_{uv}$ has size at least two, then let that set be $S_{c,1}$. Otherwise, let $S_{c,1} = \varnothing$. For each $c \in L(v)$, each vertex $u \in S_{c,1}$ either has a neighbor u' such that $|L(u')| \leq |L(u)|$ and $\phi_\omega(u)\phi_\omega(u') \in M_{uu'}$ or is uncolored by an ε-equalizing coin-flip. In the former case, we choose precisely one such neighbor u' of u, let u' be in the set $S_{c,2}$, and let u be in the set $S_{c,u'}$. In the latter case, we let $u \in S'_{c,1}$. By the definition of these sets,

$$\mathbf{pairs}_{v,\sigma}^{\text{uncol}}(\omega) = \sum_{c \in L(v)} \binom{|S_{c,1}|}{2} \tag{3.11}$$

and

$$S_{c,1} = S'_{c,1} \bigcup \left(\cup_{u \in S_{c,2}} S_{c,u}\right).$$

Since $\omega \notin \Omega_{v,\sigma}^*$, for each $c \in L(v)$, we have that $|S_{c,1}| \leq \log \Delta$, and for each $u \in S_{c,2}$, we have that $|\cup_{c' \in L(v)} S_{c',u}| \leq \log \Delta$.

For each $c \in L(v)$, we let I_c index the trials determining $\phi_\omega(u)$ for the $u \in S_{c,1} \cup S_{c,2}$, and for each $c \in L(v)$ and $u \in S'_c$, the vertex u is uncolored by an ε-equalizing coin flip, and we also let I_c index this trial. We let $I = \cup_{c \in L(v)} I_c$.

We claim that I is an (r,d)-certificate for $\mathbf{pairs}_{v,\sigma}^{\text{uncol}}, \omega, s$, and $\Omega_{v,\sigma}^*$. To that end, let $\omega' \in \Omega \setminus \Omega_{v,\sigma}^*$ and $k \geq 0$ such that ω and ω' differ for at most k trials indexed by I. We say a vertex u *keeps its color* if $\phi_\omega(u) = \phi_{\omega'}(u)$. For each $c \in L(v)$, let $T'_{c,1}$ be the set of vertices in $S'_{c,1}$ that keep their color and are also uncolored by an ε-equalizing coin-flip in the outcome ω', let $T_{c,2}$ be the set of vertices in $S_{c,2}$ that keep their color, and for each $u \in T_{c,2}$, let $T_{c,u}$ be the set of vertices in $S_{c,u}$ that keep their color. For each $c \in L(v)$, let $T_{c,1} = T'_{c,1} \bigcup \left(\cup_{u \in T_{c,2}} T_{c,u}\right)$. Note that

$$\mathbf{pairs}_{v,\sigma}^{\text{uncol}}(\omega') \geq \sum_{c \in L(v)} \binom{|T_{c,1}|}{2}. \tag{3.12}$$

and

$$\sum_{c \in L(v)} \left(\binom{|S_{c,1}|}{2} - \binom{|T_{c,1}|}{2} \right) = \sum_{c \in L(v)} |S_{c,1} \setminus T_{c,1}|(|S_{c,1}| + |T_{c,1}| - 1)/2. \tag{3.13}$$

Recall that for each $c \in L(v)$ and $u \in S_{c,2}$, we have that $| \cup_{c' \in L(v)} S_{c',u}| \leq \log \Delta$. Since ω and ω' differ for at most k trials indexed by I, it follows that $\sum_{c \in L(v)} |S_{c,1} \setminus T_{c,1}| \leq k \log \Delta$. Also note that $|S_{c,1}| + |T_{c,1}| - 1)/2 \leq \log \Delta$. Therefore by (3.11), (3.12), and (3.13), $\mathbf{pairs}^{\mathrm{uncol}}_{v,\sigma}(\omega') \geq s - kd$, as required.

Note that for each $c \in L(v)$, we have that $|I_c| \leq 2|S_{c,1}|$, and hence $|I_c| \leq 4\binom{|S_{c,1}|}{2}$. Therefore $|I| \leq 4s$, as required. It follows that I is an (r, d)-certificate for $\mathbf{pairs}^{\mathrm{uncol}}_{v,\sigma}, \omega, s$, and $\Omega^*_{v,\sigma}$, and hence $\mathbf{pairs}^{\mathrm{uncol}}_{v,\sigma}$ is (r, d)-certifiable with respect to $\Omega^*_{v,\sigma}$, as claimed. Therefore we can apply Theorem 2.3.2.

We choose $t = \max\{\mathbb{E}\left[\mathbf{pairs}_{v,\sigma}\right]^{5/6}, \log^9 \Delta\}$, so by Proposition 3.3.2 and Theorem 2.3.2, for some constant $\gamma_4 > 0$,

$$\mathbb{P}\left[|\mathbf{pairs}^{\mathrm{tot}}_{v,\sigma} - \mathbb{E}\left[\mathbf{pairs}^{\mathrm{tot}}_{v,\sigma}\right]| > t\right] \leq 4\exp(-\gamma_4(\log^{16/5}(\Delta))) + 4\mathbb{P}\left[\Omega^*_v\right], \tag{3.14}$$

and

$$\mathbb{P}\left[|\mathbf{pairs}^{\mathrm{uncol}}_{v,\sigma} - \mathbb{E}\left[\mathbf{pairs}^{\mathrm{uncol}}_{v,\sigma}\right]| > t\right] \leq 4\exp(-\gamma_4(\log^{16/5}(\Delta))) + 4\mathbb{P}\left[\Omega^*_v\right]. \tag{3.15}$$

It follows from (3.14), (3.15), and Proposition 3.3.1 that $\mathbf{pairs}_{v,\sigma}$ is Δ-concentrated, as desired.

Similarly, we can apply Theorem 2.3.2 to $\mathbf{trips}^{\mathrm{tot}}_{v,\sigma}$ and $\mathbf{trips}^{\mathrm{uncol}}_{v,\sigma}$ with exceptional outcomes $\Omega^*_{v,\sigma}$, $r = 6$, and $d = \log^3 \Delta$. Letting $t = \max\{\mathbb{E}\left[\mathbf{trips}_{v,\sigma}\right]^{5/6}, \log^9 \Delta\}$, we observe that for some constant $\gamma_5 > 0$,

$$\mathbb{P}\left[|\mathbf{trips}^{\mathrm{tot}}_{v,\sigma} - \mathbb{E}\left[\mathbf{trips}^{\mathrm{tot}}_{v,\sigma}\right]| > t\right] \leq 4\exp(-\gamma_5(\log^{6/5}(\Delta))) + 4\mathbb{P}\left[\Omega^*_v\right], \tag{3.16}$$

and

$$\mathbb{P}\left[|\mathbf{trips}^{\mathrm{uncol}}_{v,\sigma} - \mathbb{E}\left[\mathbf{trips}^{\mathrm{uncol}}_{v,\sigma}\right]| > t\right] \leq 4\exp(-\gamma_5(\log^{6/5}(\Delta))) + 4\mathbb{P}\left[\Omega^*_v\right]. \tag{3.17}$$

It follows from (3.16), (3.17), and Proposition 3.3.1 that $\mathbf{trips}_{v,\sigma}$ is Δ-concentrated, as desired. $\square$

3.4 Sparsity

We close this chapter with an application of Theorem 3.1.8 to coloring graphs in which the neighborhood of each vertex is missing many edges. We also provide more background on the naive coloring procedure from [109].

The following is one of the first applications of the naive coloring procedure given by Molloy and Reed [109, Theorem 10.5] in their book on graph coloring with the probabilistic method.

Theorem 3.4.1 (Molloy and Reed [109]). *If Δ is sufficiently large, $B \geq \Delta \log^3 \Delta$, and G is a graph such that $\Delta(G) \leq \Delta$ and every vertex $v \in V(G)$ satisfies $|E(G[N(v)])| \leq \binom{\Delta}{2} - B$, then*

$$\chi_\ell(G) \leq \Delta + 1 - \frac{B}{e^6 \Delta}.$$

Note that the bound on the list chromatic number provided by Theorem 3.4.1 implies that every vertex $v \in V(G)$ satisfies $\mathrm{Save}_L(v) \leq e^{-6} B/\Delta$.

3.4.1 Improving the constants

Molloy and Reed [109] actually only prove Theorem 3.4.1 for the chromatic number, and they leave it as an exercise to modify the proof to generalize this result to list-coloring. As we now demonstrate, if we use correspondence coloring instead, the proof requires essentially no modification. That is, the proof is simpler in this more general framework. We say a correspondence-assignment (L, M) is *total* if for every edge $uv \in E(G)$, the matching M_{uv} saturates one of $\{u\} \times L(u)$ and $\{v\} \times L(v)$. Counting repeated colors for a total correspondence assignment is the same as if every vertex had the same list of available colors, as in the following lemma.

Lemma 3.4.2. *If G is a graph with total correspondence-assignment (L, M) such that $|L(v)| = k$ for each $v \in V(G)$ and (G, L) is (Δ, ε)-bounded for some Δ and ε, then for each $v \in V(G)$,*

$$\mathbb{E}\left[\mathbf{pairs}_{v,0}\right] \geq K_\varepsilon^2 \cdot \frac{|E(\overline{G[N(v)]})|}{k}$$

and

$$\mathbb{E}\left[\mathbf{trips}_{v,0}\right] \leq K_\varepsilon \cdot \frac{(2|E(\overline{G[N(v)]})|)^{3/2}}{6k^2}.$$

We do not prove Lemma 3.4.2; the proof is straightforward, and it is implicit in the proof of Lemma 4.1.3 in Chapter 4. We use Lemma 3.4.2 in conjunction with Theorem 3.1.8 to obtain the following variation of Theorem 3.4.1.

Theorem 3.4.3. *For every $\varepsilon, \xi \in (0,1)$, the following holds for sufficiently large Δ. Suppose $B \geq \Delta \log^{10} \Delta$ and*

$$(1+\xi)\varepsilon \frac{B}{\Delta} \leq K^2_{\varepsilon B/\Delta^2} \frac{B}{\Delta + 1 - \varepsilon B/\Delta} - K_{\varepsilon B/\Delta^2} \frac{(2B)^{3/2}}{6(\Delta + 1 - \varepsilon B/\Delta)^2}. \tag{3.18}$$

If G is a graph such that $\Delta(G) \leq \Delta$ and every vertex $v \in V(G)$ satisfies $|E(G[N(v)])| \leq \binom{\Delta}{2} - B$, then

$$\chi_c(G) \leq \Delta + 1 - \varepsilon B/\Delta.$$

Proof. By possibly embedding G as a subgraph in a graph of larger minimum degree without increasing the number of edges in the neighborhood of any vertex (as in the proof of Theorem 1.2.11), we may assume G is Δ-regular, that is each vertex in G has degree Δ. Hence, every vertex $v \in V(G)$ satisfies $\text{Save}_L(v) \leq \varepsilon B/\Delta$. By possibly removing colors from the lists of some vertices, we may assume every vertex $v \in V(G)$ satisfies $|L(v)| = \lceil \Delta + 1 - \varepsilon B/\Delta \rceil$. Note that (G, L) is $(\Delta, \varepsilon B/\Delta^2)$-bounded. By Lemma 3.4.2 with $k = \lceil \Delta + 1 - \varepsilon B/\Delta \rceil$, for any ordering $\prec$, we have that $\mathbb{E}\left[\mathbf{savings}_{v,0,\prec}\right]$ is (up to rounding) at least the right side of (3.18). Hence, the result follows from Theorem 3.1.8 with $\xi_1 = \xi/2, \xi_2 = \varepsilon$, and $\sigma = 0$. $\square$

Theorem 3.4.1 implies Theorem 3.4.3 without the constraint (3.18) for $\varepsilon = e^{-6}$. If $B = o(\Delta^2)$ in Theorem 3.4.3, then $\varepsilon = .999e^{-2} - o(1)$ satisfies (3.18), which improves the value of ε for this consequence of Theorem 3.4.1 by a factor of roughly e^4. Note that $B \leq \binom{\Delta}{2}$ and thus $(2B)^{3/2} \leq 2B\Delta$, so (3.18) is satisfied if

$$(1+\xi)\varepsilon \leq \frac{K_{\varepsilon B/\Delta^2}\Delta}{\Delta + 1 - \varepsilon B/\Delta} \left(K_{\varepsilon B/\Delta^2} - \frac{\Delta/3}{\Delta + 1 - \varepsilon B/\Delta} \right).$$

The previous inequality holds if $(1+\xi)\varepsilon \leq K_\varepsilon \left(K_\varepsilon - \frac{1}{3(1-\varepsilon)} \right)$, which is satisfied for $\varepsilon = e^{-6}$ and $\xi \leq .045$. Therefore Theorem 3.4.3 strengthens Theorem 3.4.1 at the expense of the requirement that $B \geq \Delta \log^{10} \Delta$ rather than $B \geq \Delta \log^3 \Delta$. This improvement is most significant when B is $o(\Delta^2)$ or at most some small fraction of $\binom{\Delta}{2}$. We are able to provide a better value of ε in Theorem 3.4.3 because we count the number of repeated colors in each vertex's neighborhood after the random coloring in a more refined way. Molloy and

Reed [109] count the number of colors assigned to at least two neighbors that are retained by *all* of the neighbors assigned that color. Bruhn and Joos [31] first introduced the idea to instead count pairs and triples. Concentrating these random variables is more difficult, however, and it requires the lower bound on B in terms of Δ to be slightly more restrictive.

If $B = \delta\binom{\Delta}{2}$, then, ignoring lower order terms for large Δ, the constraint (3.18) essentially becomes

$$(1+\xi)\varepsilon \cdot \delta \leq K_{\varepsilon\cdot\delta/2}^2 \frac{\delta}{2-\varepsilon\cdot\delta} - K_{\varepsilon\cdot\delta/2} \frac{\delta^{3/2}}{6(1-\varepsilon\cdot\delta/2)^2}.$$

Letting $\gamma = \varepsilon \cdot \delta/2$ in [31, Lemma 5], Bruhn and Joos obtain a variant of Theorem 3.4.3 with the constraint (3.18) replaced by the following:

$$\varepsilon \cdot \delta < K_{\varepsilon\cdot\delta/2} \frac{\delta}{2-\varepsilon\cdot\delta} - K_{\varepsilon\cdot\delta/2}^{7/8} \frac{\delta^{3/2}}{6(1-\varepsilon\cdot\delta/2)}.$$

Note the similarities between the right sides of the previous two inequalities. In particular, the first term of the right side of the previous inequality differs from the respective term in the preceding inequality by a factor of $K_{\varepsilon\cdot\delta/2}$, and the second term of the right side of the previous inequality differs from the respective term in the preceding inequality by a factor of $K_{\varepsilon\cdot\delta/2}^{-1/8}$. These differences result from the fact that in the variant of the naive coloring procedure used by Bruhn and Joos [31], when adjacent vertices are assigned the same color, only one of the two is uncolored, decided by a coin flip. Thus, vertices are more likely to retain their color, and we expect more pairs and triples of repeated colors.

Bruhn and Joos remark that in the regime of $\delta \in [0, .09)$, it suffices to have $\varepsilon < .3654 - .1556\sqrt{\delta}$. Bonamy, Perrett, and Postle [24, Theorem 1.6] asked about the best possible value of ε in terms of δ and proved that in the same range, $\varepsilon < .60256 - .2566\sqrt{\delta}$ suffices, an improvement by a factor of roughly $\sqrt{e}$. To obtain this improvement, they show that the naive random coloring procedure can be iterated. After applying the procedure, one can assume that the uncolored subgraph still has the property that every vertex is missing many edges in its neighborhood. As we discuss in Chapter 4, this improvement is a key step in improving the value of ε in Theorem 1.2.4, the "epsilon version" of Reed's Conjecture. In Chapter 6, we also discuss how iterating a random coloring procedure has been used for coloring graphs with small clique number.

It is likely that the local naive random coloring procedure can also be iterated in this way to obtain a nontrivial improvement on the value of ε in Theorem 1.3.4. However, doing so decreases the probability that vertices are uncolored, and consequently the expected value of $\textbf{uncolored}_{v,\prec}$ would be lowered for each vertex v. Nevertheless, as we see in

Chapter 4, the "bottleneck" for the value of ε in Theorem 1.3.4 is rather $\mathbf{unmatched}_v$ and $\mathbf{pairs}_{v,0} - \mathbf{trips}_{v,0}$, so we do expect our result can be improved by iterating the procedure.

3.4.2 Local sparsity, cascading, and more potential applications

Theorem 3.4.1 is in some sense a prototypical application of the naive coloring procedure. Note that in our proof of Theorem 3.4.3, we did not use a specific ordering $\prec$ when we applied Theorem 3.1.8. The fact that vertices with "sparse neighborhoods" have large expected savings without specifying an ordering is crucial to many major advances in graph coloring that have been made using the probabilistic method. In these situations, we can choose the ordering so that these "sparse" vertices go last. If a vertex v has many sparse neighbors, then the expected value of $\mathbf{uncolored}_{v,\prec}$ is large. These vertices can be chosen next to last in the ordering, and by exploiting the structure of the "dense" vertices, we may be able to show that these savings "cascade" through the whole graph. Variations of this approach were notably used by Reed [121] to prove Theorem 1.2.4, by Molloy and Reed [111] to prove that the total chromatic number of graphs of maximum degree Δ is at most Δ plus a constant, by Reed [122] to prove the Borodin-Kostochka Conjecture for graphs of large maximum degree, by Havet, Reed, and Sereni [75, 76] to resolve Griggs and Yeh's Conjecture on $L(2,1)$-labelings for graphs of large maximum degree, and by Molloy and Reed [110] to extend Reed's result on the Borodin-Kostochka Conjecture as discussed in Section 1.2.1.

Indeed, we also use this approach in our proof of Theorem 1.4.5 in Chapter 5, and it is in some sense also the case in our proof of Theorem 1.3.4 in Chapter 4. As mentioned, when we apply Theorem 3.1.8 in Chapter 4, we order the vertices by the size of their list. We are able to show that vertices with the smallest list size are either "egalitarian-sparse" or have many "lordlier" neighbors and consequently have enough savings irrespective of the ordering. Since these vertices are last in the ordering, the vertices with the next smallest list of available colors also have enough savings, and the savings cascade through the graph.

Similarly, using Theorem 3.1.8, we prove the following local version of Theorem 3.4.1, at the expense of a factor two loss in the bound on Save_L and a slightly stronger requirement on the number of missing edges in each vertex's neighborhood.

Theorem 3.4.4. *If Δ is sufficiently large, G is a graph with correspondence-assignment (L, M) such that (G, L) is $(\Delta, e^{-6}/4)$-bounded, and for each vertex $v \in V(G)$ there exists $B_v \geq d(v)\log^{10}\Delta$ such that $|E(G[N(v)])| \leq \binom{d(v)}{2} - B_v$ satisfying*

$$\mathrm{Save}_L(v) \leq \frac{B_v}{2e^6 d(v)},$$

then G is (L, M)-colorable.

The proof of Theorem 3.4.4 is in some sense a simplified version of the proof of Theorem 1.3.4, so we postpone it until Chapter 4.

Theorem 3.4.4 could be considered a proof of concept that the local naive random coloring procedure or some variation of it would be useful for proving local versions of results proved using the naive coloring procedure, such as the ones already mentioned in this section.

Chapter 4

The local version of Reed's Conjecture

In this chapter, we prove Theorems 1.3.4 and 3.4.4. As alluded to in the previous chapter, the proof of Theorem 1.3.4 combines Theorem 3.1.8 with a structural result for the "dense vertices." In particular, we prove Theorem 4.2.3, which essentially implies that dense vertices are either lordly or aberrant, that is, every vertex v has either $\Omega(\mathrm{Gap}(v)d(v))$ non-edges in its neighborhood between neighbors with lists similar to v's or $\Omega(\mathrm{Gap}(v))$ neighbors with a smaller list of available colors than v, or v expects $\Omega(\mathrm{Gap}(v))$ neighbors to receive a color not in $L(v)$ after applying the local naive random coloring procedure. In order to prove Theorem 4.2.3, we first prove Theorem 4.2.1, which we use in Chapter 5 and which may also be of independent interest.

This dichotomy between the dense and sparse vertices is also present in proofs of Theorem 1.2.4, the "epsilon version" of Reed's Conjecture. Reed's [121] original proof uses a decomposition of the dense vertices into "clique-like" structures. King and Reed [96] found a shorter proof by handling the dense vertices in a different way. First, they use a result of King [97] that if G is a graph such that $\omega(G) > 2(\Delta(G) + 1)/3$, then G contains an independent set hitting every maximal clique and every vertex's closed neighborhood. By possibly choosing such an independent set to be a color class, this result allows them to assume that $\omega(G) \leq 2(\Delta(G) + 1)/3$. Note that this method does not work when proving Theorem 1.2.4 for the list chromatic number. Next, they prove that in a minimum counterexample with this bound on the clique number, every vertex has a sparse neighborhood. Bonamy, Perrett, and Postle [24, Theorem 1.5] improve this argument to show that in a

This chapter is based on the paper [87] (with Luke Postle).

$\lfloor(1-\varepsilon)(\Delta(G)+1)\rfloor$-list-critical graph with $\omega(G) \leq (1-\alpha)(\Delta(G)+1)$, the number of non-edges in each vertex's neighborhood is at least $(\alpha-\varepsilon)^2\binom{\Delta}{2}/2$. Theorem 1.2.4 now follows from Theorem 3.4.1, and one can improve the value of ε by improving either the lower bound on the number of non-edges in each vertex's neighborhood or by improving the analysis in Theorem 3.4.1. In the case of regular graphs with a uniform list-assignment, our Theorem 4.2.3 shows that the number of non-edges in the neighborhood of a vertex v is $\Omega(\mathrm{Gap}(v)d(v))$, which is why Theorem 1.3.4 extends Theorem 1.2.4 to the list chromatic number in the regime $\omega \leq \Delta - \log^{10}\Delta$.

4.1 Local sparsity

In this section, we prove Theorem 3.4.4. Many of the lemmas will be used in Section 4.3 in the proof of Theorem 1.3.4.

Recall that if v is a vertex of a graph G and $u \in N(v)$, we say u is a subservient neighbor of v if $|L(u)| < |L(v)|$, a strongly egalitarian neighbor of v if $|L(u)| \in [|L(v)|, |L(v)| + \alpha\,\mathrm{Gap}(v))$, a weakly egalitarian neighbor of v if $|L(u)| \in [|L(v)| + \alpha\,\mathrm{Gap}(v), (1+\alpha)|L(v)|)$, and a lordlier neighbor of v if $|L(u)| \geq (1+\alpha)|L(v)|$. Recall also that this partitions the neighbors of v into the sets $\mathrm{Subserv}(v), \mathrm{SEgal}(v), \mathrm{WEgal}(v)$, and $\mathrm{Lord}(v)$, the sets of subservient, strongly egalitarian, weakly egalitarian, and lordlier neighbors of v, respectively, and that we let $\mathrm{NEgal}(v) = N(v) - \mathrm{Egal}(v)$.

For the remainder of this section, unless specified otherwise, G is a graph with list-assignment L such that (G, L) is (Δ, ε)-bounded. Recall that we let $K_\varepsilon = .999e^{\frac{-1}{1-\varepsilon}}$. For convenience, we let $K = K_\varepsilon$.

First, we need to lower bound the expected savings for each vertex, as in the following lemmas.

Lemma 4.1.1. *If $u \prec v$ for every $u, v \in V(G)$ such that $|L(u)| < |L(v)|$, then for each $v \in V(G)$,*

$$\mathbb{E}\left[\mathbf{uncolored}_{v,\prec}\right] = (1-K)|\mathrm{Subserv}(v)|.$$

Proof. By Proposition 3.1.6, each neighbor of v is uncolored with probability $1-K$. Thus the lemma follows by linearity of expectation. □

Lemma 4.1.2. *For each $v \in V(G)$,*

$$\mathbb{E}\left[\mathbf{unmatched}_v\right] \geq K\left(\frac{\alpha}{1+\alpha}|\mathrm{Lord}(v)| + \frac{\alpha\,\mathrm{Gap}(v)}{d(v)+\alpha\,\mathrm{Gap}(v)}|\mathrm{WEgal}(v)|\right).$$

Proof. Let

$$\mathbf{unmatched}_v^{\text{tot}} = |\{u \in N(v) : \phi(u) \notin V(M_{uv})\}|,$$

and note that $\mathbb{E}\left[\mathbf{unmatched}_v\right] = K \cdot \mathbb{E}\left[\mathbf{unmatched}_v^{\text{tot}}\right]$. For each $u \in \mathrm{Lord}(v)$,

$$\mathbb{P}\left[\phi(u) \notin V(M_{uv})\right] \geq \frac{\alpha}{1+\alpha},$$

and for each $u \in \mathrm{WEgal}(v)$,

$$\mathbb{P}\left[\phi(u) \notin V(M_{uv})\right] \geq \frac{\alpha \,\mathrm{Gap}(v)}{|L(v)| + \alpha \,\mathrm{Gap}(v)} \geq \frac{\alpha \,\mathrm{Gap}(v)}{d(v) + \alpha \,\mathrm{Gap}(v)}.$$

Therefore it follows that

$$\mathbb{E}\left[\mathbf{unmatched}_v^{\text{tot}}\right] \geq \frac{\alpha}{1+\alpha}|\mathrm{Lord}(v)| + \frac{\alpha \,\mathrm{Gap}(v)}{d(v) + \alpha \,\mathrm{Gap}(v)}|\mathrm{WEgal}(v)|.$$

Since $\mathbb{E}\left[\mathbf{unmatched}_v\right] = K \cdot \mathbb{E}\left[\mathbf{unmatched}_v^{\text{tot}}\right]$, the result follows. □

Recall that we apply Theorem 3.1.8 with $\sigma = 0$. Thus we need to bound $\mathbf{pairs}_{v,0} - \mathbf{trips}_{v,0}$, as in the following lemma.

Lemma 4.1.3. *If (L, M) is a total correspondence assignment for G, then for each $v \in V(G)$,*

$$\mathbb{E}\left[\mathbf{pairs}_{v,0} - \mathbf{trips}_{v,0}\right] \geq \left(\frac{K|E(\overline{G[\mathrm{Egal}(v)]})|}{d(v)}\right)\left(\frac{K}{(1+\alpha)^2} - \frac{\left(2|E(\overline{G[\mathrm{Egal}(v)]})|\right)^{1/2}}{3(1-\varepsilon)^2 d(v)}\right).$$

Proof. Recall that we let $T(H)$ denote the set of triangles in a graph H. We define the following random variables:

$\mathbf{pairs}_v^{\text{tot}} = |\{xy \in E(\overline{G[\mathrm{Egal}(v)]}), c \in L(v) : \phi(x)c \in M_{xv} \text{ and } \phi(y)c \in M_{uv}\}|,$ and
$\mathbf{trips}_v^{\text{tot}} = |\{xyz \in T(\overline{G[\mathrm{Egal}(v)]}), c \in L(v) : \phi(x)c \in M_{xv}, \phi(y)c \in M_{yv}, \text{ and } \phi(z)c \in M_{zv}\}|.$

Note that for any pair $x, y \in V(G)$ of non-adjacent vertices, $\mathbb{P}\left[x, y \notin U\right] \geq K^2$. Moreover, for any triple $x, y, z \in V(G)$ of pairwise non-adjacent vertices, $\mathbb{P}\left[x, y, z \notin U\right] \leq \mathbb{P}\left[x \notin U\right] \leq K$. Hence,

$$\mathbb{E}\left[\mathbf{pairs}_{v,0} - \mathbf{trips}_{v,0}\right] \geq K^2\mathbb{E}\left[\mathbf{pairs}_v^{\text{tot}}\right] - K\mathbb{E}\left[\mathbf{trips}_v^{\text{tot}}\right].$$

In 2002, Rivin [126] proved that

$$|T(H)| \leq \frac{(2|E(H)|)^{\frac{3}{2}}}{6}. \tag{4.1}$$

Let $H = \overline{G[\mathrm{Egal}(v)]}$. Observe that since (L, M) is total,

$$\mathbb{E}\left[\mathbf{pairs}_v^{\mathrm{tot}}\right] = \sum_{xy \in E(H)} \frac{|L(v)|}{|L(x)||L(y)|}.$$

By the definition of $\mathrm{Egal}(v)$, if $x, y \in \mathrm{Egal}(v)$,

$$\frac{|L(v)|}{|L(x)||L(y)|} \geq \frac{1}{(1+\alpha)^2 |L(v)|}.$$

Therefore

$$\mathbb{E}\left[\mathbf{pairs}_v^{\mathrm{tot}}\right] \geq \frac{|E(H)|}{(1+\alpha)^2 d(v)}. \tag{4.2}$$

Similarly,

$$\mathbb{E}\left[\mathbf{trips}_v^{\mathrm{tot}}\right] = \sum_{xyz \in T(H)} \frac{|L(v)|}{|L(x)||L(y)||L(z)|},$$

and

$$\frac{|L(v)|}{|L(x)||L(y)||L(z)|} \leq \frac{1}{|L(v)|^2} \leq \frac{1}{(1-\varepsilon)^2 d(v)^2}.$$

Therefore

$$\mathbb{E}\left[\mathbf{trips}_v^{\mathrm{tot}}\right] \leq \frac{|T(H)|}{(1-\varepsilon)^2 d(v)^2}. \tag{4.3}$$

By (4.1) and (4.3),

$$\mathbb{E}\left[\mathbf{trips}_v^{\mathrm{tot}}\right] \leq \frac{\sqrt{8}|E(H)|^{3/2}}{6(1-\varepsilon)^2 d(v)^2}. \tag{4.4}$$

It follows from (4.2) and (4.4) that

$$\mathbb{E}\left[\mathbf{pairs}_v - \mathbf{trips}_v\right] \geq \left(\frac{K|E(H)|}{d(v)}\right)\left(\frac{K}{(1+\alpha)^2} - \frac{\sqrt{2|E(H)|}}{3(1-\varepsilon)^2 d(v)}\right),$$

as desired. $\square$

Using Lemmas 4.1.1, 4.1.2, and 4.1.3 and Theorem 3.1.8, we can now prove Theorem 3.4.4.

Proof of Theorem 3.4.4. Let $u \prec v$ if $|L(u)| \prec |L(v)|$, let $\alpha = 1/50$, and let $\varepsilon = e^{-6}/4$. We may assume the correspondence-assignment (L, M) is total. We will apply Theorem 3.1.8 with $\xi_1 = .36, \xi_2 = 1.36 \cdot (2\varepsilon)$, and $\sigma = 0$ to show that G is (L, M)-colorable.

Note that

$$|E(\overline{G[\mathrm{Egal}(v)]})| \geq B_v - d(v)|\mathrm{NEgal}(v)|$$

and

$$\frac{\left(2|E(\overline{G[\mathrm{Egal}(v)]})|\right)^{1/2}}{d(v)} \leq 1.$$

Thus, by Lemmas 4.1.1, 4.1.2, and 4.1.3,

$$\begin{aligned} \mathbb{E}\left[\mathbf{savings}_{v,0,\prec}\right] \geq \left(1 - K - K\left(\frac{K}{(1+\alpha)^2} - \frac{1}{3(1-\varepsilon)^2}\right)\right)|\mathrm{Subserv}(v)| \\ + \left(\frac{K\alpha}{1+\alpha} - K\left(\frac{K}{(1+\alpha)^2} - \frac{1}{3(1-\varepsilon)^2}\right)\right)|\mathrm{Lord}(v)| + K\left(\frac{K}{(1+\alpha)^2} - \frac{1}{3(1-\varepsilon)^2}\right)B_v/d(v). \end{aligned} \tag{4.5}$$

The first two terms of (4.5) are positive, and the last term is at least $1.36(2\varepsilon)B_v/d(v)$. Therefore (3.1) holds, and by Theorem 3.1.8, G is (L, M)-colorable, as desired. $\square$

The proof of Theorem 1.3.4 is similar, but we need to first prove that in a minimum counterexample vertices satisfy some sort of "local sparsity" condition.

4.2 Density

The main result of this section is Theorem 4.2.3, which lower bounds the number of non-adjacent egalitarian neighbors of a vertex in terms of the number of its neighbors that are lordlier, subservient, or weakly egalitarian.

4.2.1 A density lemma

First we need to prove the following theorem, which may be of independent interest. It bounds the number of edges of a critical graph in terms of the size of a matching in the complement. Recall that a graph G with list-assignment L is L-*critical* if G is not L-colorable but every proper induced subgraph of G is. If G is a graph with list-assignment L that satisfies the conditions of Theorem 1.3.4, then so does any subgraph of G. Therefore in proving Theorem 1.3.4, we may assume G is L-critical, and hence we can apply Theorem 4.2.3.

Theorem 4.2.1. *If G is L-critical, H is an induced subgraph of G, and M is a matching in $\overline{H}$, then*

$$|E(\overline{H})| \geq |M|(|V(H)| - |M|) - \sum_{u \in V(H)} \mathrm{Save}_L(u).$$

We will apply Theorem 4.2.1 to an appropriate subset of the neighborhood of each vertex. In order to prove Theorem 4.2.1, we need an improved version of a classic result of Erdős, Rubin, and Taylor [56] about list-coloring a complete graph with a matching removed, proved by Delcourt and Postle [43].

Lemma 4.2.2 (Delcourt and Postle [43])**.** *If $G = K_n - M$, where M is a matching and L is a list-assignment for G such that*

1. *for all $ab \in M$, $|L(a)|, |L(b)| \geq |M|$ and $|L(a)| + |L(b)| \geq n$,*
2. *for all $v \in V(G - V(M))$, $|L(v)| \geq n - |M|$,*

then G is L-colorable.

Proof. We proceed by induction on n. If $n \leq 1$, then $M = \varnothing$ and by 2, $|L(v)| \geq n$ for all $v \in G$. So we may assume $n \geq 2$.

Suppose there exists $ab \in M$ such that $L(a) \cap L(b) \neq \varnothing$. Let $c \in L(a) \cap L(b)$, and for all $v \in V(G) \backslash \{a, b\}$, let $L'(v) = L(v) \backslash \{c\}$. Let $G' = G - a - b$ and $M' = M - ab$. Then G', M', and L' satisfy conditions 1 and 2. By induction, G' has an L'-coloring. Therefore G has an L-coloring, obtained from an L'-coloring G' by coloring a and b with color c, as desired.

Therefore we may assume that for all $ab \in M$, $L(a) \cap L(b) = \varnothing$. Since $|L(a)| + |L(b)| \geq n$, $|L(a) \cup L(b)| \geq n$. We claim for all $X \subseteq V(G)$, $|\bigcup_{v \in X} L(v)| \geq |X|$. If there exists $ab \in M$ such that $a, b \in X$, then $|\bigcup_{v \in X} L(v)| \geq n \geq |X|$, as claimed. Therefore we may assume that $|X| \leq n - |M|$. If $X \backslash V(M) \neq \varnothing$, then $|\bigcup_{v\in X} L(v)| \geq n - |M| \geq |X|$, as claimed. Hence, we may assume that $X \subseteq V(M)$. But then $|\bigcup_{v\in X} L(v)| \geq |M| \geq |X|$, as claimed.

Therefore $|\bigcup_{v\in X} L(v)| \geq |X|$ for all $X \subseteq V(G)$. By Hall's Theorem, there is a matching from $V(G)$ to $\cup_v L(v)$, and thus G has an L-coloring, as desired. $\square$

Now we prove Theorem 4.2.1.

Proof of Theorem 4.2.1. We proceed by induction on $|V(H)|$. Since G is L-critical, $G - V(H)$ has an L-coloring ϕ. For all $v \in V(H)$, let $L'(v) = L(v) \backslash \{\phi(u) : u \in N(v) \cap V(G - V(H))\}$. Since G does not have an L-coloring, H does not have an L'-coloring. By possibly adding edges to H, we may assume without loss of generality that H is isomorphic to $K_n - M$. Thus, by Lemma 4.2.2, either there exists $ab \in M$ such that $|L(a)| < |M|$ or $|L(a)|+|L(b)| < |V(H)|$ or there exists $v \in V(H - V(M))$ such that $|L'(v)| < |V(H)| - |M|$. Note that for all $v \in V(H)$,

$$\begin{aligned} |L'(v)| &\geq |L(v)| - d_{G-V(H)}(v) \\ &= d_H(v) + 1 - \mathrm{Save}_L(v). \end{aligned} \tag{4.6}$$

If there exists $ab \in M$ such that $|L'(a)| < |M|$, then let $H' = H - a$ and $M' = M - ab$. By (4.6), $d_H(a) + 1 - \mathrm{Save}_L(a) < |M|$. Hence,

$$\begin{aligned} d_{\overline{H}}(a) &= |V(H)| - 1 - d_H(a) \\ &> |V(H)| - |M| - \mathrm{Save}_L(a). \end{aligned}$$

By induction, $|E(\overline{H}')| \geq |M'|(|V(H')| - |M'|) - \sum_{u\in V(H')} \mathrm{Save}_L(u)$. Therefore,

$$\begin{aligned} |E(\overline{H})| &= |E(\overline{H}')| + d_{\overline{H}}(a) \\ &> |M'|(|V(H')| - |M'|) - \sum_{u\in V(H')} \mathrm{Save}_L(u) + |V(H)| - |M| - \mathrm{Save}_L(a) \\ &= (|M| - 1)(|V(H)| - |M|) + |V(H)| - |M| - \sum_{u\in V(H)} \mathrm{Save}_L(u) \\ &= |M|(|V(H)| - |M|) - \sum_{u\in V(H)} \mathrm{Save}_L(u), \end{aligned}$$

as desired.

If there exists $ab \in M$ such that $|L'(a)| + |L'(b)| < |V(H)|$ then let $H' = H - a - b$ and $M' = M - ab$. By (4.6), $d_H(a) + 1 - \mathrm{Save}_L(a) + d_H(b) + 1 - \mathrm{Save}_L(b) < |V(H)|$. Hence,

$$\begin{aligned}
|\delta_{\overline{H}}(\{a,b\})| &= 2(|V(H)| - 2) - d_H(a) - d_H(b) \\
&> 2(|V(H)| - 2) - |V(H)| + 2 - \mathrm{Save}_L(a) - \mathrm{Save}_L(b) \\
&= |V(H)| - 2 - \mathrm{Save}_L(a) - \mathrm{Save}_L(b),
\end{aligned}$$

where $\delta_{\overline{H}}(\{a,b\})$ is the set of edges in $\overline{H}$ incident to precisely one of a and b.

By induction, $|E(\overline{H}')| \geq |M'|(|V(H')| - |M'|) - \sum_{u \in V(H')} \mathrm{Save}_L(u)$. Therefore,

$$\begin{aligned}
|E(\overline{H})| &= |E(\overline{H}')| + \delta_{\overline{H}}(\{a,b\}) + 1 \\
&\geq |M'|(|V(H')| - |M'|) - \sum_{u \in V(H')} \mathrm{Save}_L(u) + |V(H)| - \mathrm{Save}_L(a) - \mathrm{Save}_L(b) \\
&= (|M| - 1)(|V(H)| - |M| - 1) + |V(H)| - \sum_{u \in V(H)} \mathrm{Save}_L(u) \\
&> |M|(|V(H)| - |M|) - \sum_{u \in V(H)} \mathrm{Save}_L(u),
\end{aligned}$$

as desired.

Otherwise, there exists some $v \in V(H - V(M))$ such that $|L'(v)| < |V(H)| - |M|$, so let $H' = H - v$. By (4.6), $d_{H'}(v) + 1 - \mathrm{Save}_L(v) < |V(H)| - |M|$. Hence,

$$\begin{aligned}
d_{\overline{H}}(v) &= |V(H)| - 1 - d_H(v) \\
&> |M| - \mathrm{Save}_L(v).
\end{aligned}$$

By induction, $|E(\overline{H}')| \geq |M|(|V(H')| - |M|) - \sum_{u \in V(H')} \mathrm{Save}_L(u)$. Therefore,

$$\begin{aligned}
|E(\overline{H})| &= |E(\overline{H}')| + d_{\overline{H}}(v) \\
&> |M|(|V(H')| - |M|) - \sum_{u \in V(H')} \mathrm{Save}_L(u) + |M| - \mathrm{Save}_L(v) \\
&= |M|(|V(H)| - |M| - 1) + |M| - \sum_{u \in V(H)} \mathrm{Save}_L(u) \\
&= |M|(|V(H)| - |M|) - \sum_{u \in V(H)} \mathrm{Save}_L(u),
\end{aligned}$$

as desired. $\square$

4.2.2 The egalitarian neighborhood is sparse

The following is the main result of this section.

Theorem 4.2.3. *Let $\varepsilon \in (0,1)$. If G is an L-critical graph for some list-assignment L such that for every $v \in V(G)$, $|L(v)| \geq \varepsilon\omega(v) + (1-\varepsilon)(d(v)+1)$, then for all $v \in V(G)$,*

$$|E(\overline{G[\mathrm{Egal}(v)]})| \geq \left(\frac{1}{4} - \frac{\varepsilon(4+3\alpha)}{2(1-\varepsilon)}\right)\mathrm{Gap}(v)d(v) - \left(\frac{1}{2} - \frac{\varepsilon(1+\alpha)}{2(1-\varepsilon)}\right)d(v)|\mathrm{NEgal}(v)| \\ - \left(\frac{1}{4} - \frac{\varepsilon(2+\alpha)}{2(1-\varepsilon)}\right)\mathrm{Gap}(v)|\mathrm{WEgal}(v)|.$$

For the remainder of this section, we assume that G is a graph with list-assignment L satisfying the conditions of Theorem 4.2.3.

Theorem 4.2.3 is useful because it implies that if v does not have many lordlier or subservient neighbors, or many weakly egalitarian neighbors, then it has many non-adjacent egalitarian neighbors. We prove Theorem 4.2.3 by considering a maximum antimatching M among v's egalitarian neighbors and applying Theorem 4.2.1 with $H = G[V(M) \cup \mathrm{SEgal}(v)]$. If u is a strongly egalitarian neighbor of v, then $\mathrm{Save}_L(u)$ is close to $\mathrm{Save}_L(v)$. If u is a weakly egalitarian neighbor of v, then we can not bound $\mathrm{Save}_L(u)$ well enough, so we do not include u in H unless u is in the antimatching.

We will use the following propositions to prove Theorem 4.2.3. First, we need to bound the size of a maximum antimatching taken among the egalitarian neighbors, as in the following proposition.

Proposition 4.2.4. *If M is a maximum matching in $\overline{G[\mathrm{Egal}(v)]}$, then*

$$\frac{\mathrm{Gap}(v) - |\mathrm{NEgal}(v)|}{2} \leq |M| \leq \mathrm{Gap}(v).$$

Proof. Since M is maximum, $G[\mathrm{Egal}(v) - V(M)]$ is a clique, so $2|M| \geq |\mathrm{Egal}(v)| - \omega(G[\mathrm{Egal}(v)]) \geq |\mathrm{Egal}(v)| - \omega(v) = \mathrm{Gap}(v) - |\mathrm{NEgal}(v)|$, as desired.

Since no clique in $G[\mathrm{Egal}(v)]$ contains an edge in M, $\omega(G[\mathrm{Egal}(v)]) \leq |\mathrm{Egal}(v)| - |M|$. Note that for any $H \subseteq G[N(v) \cup \{v\}]$, $|V(H)| - \omega(H) \leq \mathrm{Gap}(v)$. Hence, $|M| \leq \mathrm{Gap}(v)$, as desired. □

Proposition 4.2.5. *If u is an egalitarian neighbor of a vertex v (i.e. $u \in \mathrm{Egal}(v)$), then*

$$\mathrm{Gap}(u) \leq \frac{1+\alpha}{1-\varepsilon}d(v).$$

Proof. Since G is L-critical, $|L(v)| \leq d(v)$. Since $u \in \text{Egal}(v)$, $|L(u)| \leq (1+\alpha)|L(v)|$. Hence, $|L(u)| \leq (1+\alpha)d(v)$. Since $|L(u)| \geq (1-\varepsilon)d(u)$, $d(u) \leq \frac{1+\alpha}{1-\varepsilon}d(v)$. Since $\text{Gap}(u) \leq d(u)$, the result follows. $\square$

Since we will apply Theorem 4.2.1, we will need to upper bound $\text{Save}_L(u)$ for egalitarian neighbors u of v. Since $\text{Save}_L(u) \leq \varepsilon\,\text{Gap}(u)$, it suffices to upper bound $\text{Gap}(u)$. Proposition 4.2.5 provides a rough bound on $\text{Gap}(u)$ that we will use for the egalitarian neighbors in the antimatching. The next proposition provides an improved bound on $\text{Gap}(u)$ if u is a strongly egalitarian neighbor that is not in the antimatching.

Proposition 4.2.6. *If M is a maximum matching in $\overline{G[\text{Egal}(v)]}$ and $u \in \text{SEgal}(v) - V(M)$, then*

$$\text{Gap}(u) \leq \frac{(2+\alpha)\,\text{Gap}(v) + |\text{NEgal}(v)|}{1-\varepsilon}.$$

Proof. Since M is maximum, $G[\text{Egal}(v) - V(M)]$ is a clique, so

$$\omega(u) \geq |\text{Egal}(v)| - 2|M|. \tag{4.7}$$

Since $u \in \text{SEgal}(v)$, $|L(u)| \leq |L(v)| + \alpha\,\text{Gap}(v)$. Since G is L-critical, $|L(v)| \leq d(v)$. Hence, $|L(u)| \leq d(v) + \alpha\,\text{Gap}(v)$. Since $d(u) \leq \frac{|L(u)| - \varepsilon\omega(u)}{1-\varepsilon}$,

$$d(u) \leq \frac{d(v) + \alpha\,\text{Gap}(v) - \varepsilon\omega(u)}{1-\varepsilon}. \tag{4.8}$$

Now the result follows from (4.7), (4.8), and Proposition 4.2.4. $\square$

Now we are ready to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. Let M be a maximum matching in $\overline{G[\text{Egal}(v)]}$, and let $\text{WEgal}'(v) = \text{WEgal}(v) - V(M)$. Let $H = G[V(M) \cup \text{SEgal}(v)])$. By Theorem 4.2.1,

$$|E(\overline{H})| \geq |M|(|V(H)| - |M|) - \sum_{u \in V(H)} \text{Save}(u). \tag{4.9}$$

By Proposition 4.2.6,

$$\begin{aligned}\sum_{u \in V(H - V(M))} \text{Save}(u) &\leq \sum_{u \in V(H - V(M))} \varepsilon\,\text{Gap}(u) \\ &\leq (|V(H)| - |M|)\left(\frac{\varepsilon}{1-\varepsilon}\right)((2+\alpha)\,\text{Gap}(v) + |\text{NEgal}(v)|). \end{aligned} \tag{4.10}$$

By Proposition 4.2.5 and 4.2.4.

$$\sum_{u \in V(M)} \mathrm{Save}(u) \leq \frac{\varepsilon(1+\alpha)}{1-\varepsilon} d(v)|M| \leq \frac{\varepsilon(1+\alpha)\,\mathrm{Gap}(v) d(v)}{1-\varepsilon}. \tag{4.11}$$

By (4.9), (4.10), and (4.11),

$$|E(\overline{H})| \geq (|V(H)| - |M|) \left(|M| - \frac{\varepsilon((2+\alpha)\,\mathrm{Gap}(v) + |\mathrm{NEgal}(v)|)}{1-\varepsilon} \right) - \frac{\varepsilon(1+\alpha)\,\mathrm{Gap}(v) d(v)}{1-\varepsilon}. \tag{4.12}$$

Note that $|M| \leq |V(H)|/2$, so $|V(H)| - |M| \geq |V(H)|/2$. Therefore by Proposition 4.2.4 and (4.12),

$$\begin{aligned} |E(\overline{H})| \geq \left(\frac{|V(H)|}{2} \right) \left(\mathrm{Gap}(v) \left(\frac{1}{2} - \frac{\varepsilon(2+\alpha)}{1-\varepsilon} \right) - |\mathrm{NEgal}(v)| \left(\frac{1}{2} + \frac{\varepsilon}{1-\varepsilon} \right) \right) \\ - \frac{\varepsilon}{1-\varepsilon}(1+\alpha)\,\mathrm{Gap}(v) d(v). \end{aligned} \tag{4.13}$$

Since $|V(H)| = d(v) - |\mathrm{NEgal}(v)| - |\mathrm{WEgal}'(v)|$, by combining terms in (4.13) and ignoring some positive terms, we have that

$$\begin{aligned} |E(\overline{H})| \geq \mathrm{Gap}(v) d(v) \left(\frac{1}{4} - \frac{\varepsilon(4+3\alpha)}{2(1-\varepsilon)} \right) - d(v)|\mathrm{NEgal}(v)| \left(\frac{1}{4} + \frac{\varepsilon}{2(1-\varepsilon)} \right) \\ - \mathrm{Gap}(v)|\mathrm{NEgal}(v)| \left(\frac{1}{4} - \frac{\varepsilon(2+\alpha)}{2(1-\varepsilon)} \right) - \mathrm{Gap}(v)|\mathrm{WEgal}'(v)| \left(\frac{1}{4} - \frac{\varepsilon(2+\alpha)}{2(1-\varepsilon)} \right). \end{aligned}$$

Since $\mathrm{Gap}(v) \leq d(v)$, $|\mathrm{WEgal}'(v)| \leq |\mathrm{WEgal}(v)|$, and $|E(\overline{G[\mathrm{Egal}(v)]})| \geq |E(\overline{H})|$,

$$\begin{aligned} |E(\overline{G[\mathrm{Egal}(v)]})| \geq \left(\frac{1}{4} - \frac{\varepsilon(4+3\alpha)}{2(1-\varepsilon)} \right) \mathrm{Gap}(v) d(v) - \left(\frac{1}{2} - \frac{\varepsilon(1+\alpha)}{2(1-\varepsilon)} \right) d(v)|\mathrm{NEgal}(v)| \\ - \left(\frac{1}{4} - \frac{\varepsilon(2+\alpha)}{2(1-\varepsilon)} \right) \mathrm{Gap}(v)|\mathrm{WEgal}(v)|, \end{aligned}$$

as desired. □

Note that we could take a maximum antimatching among the strongly egalitarian neighbors of a vertex v and follow the same proof strategy of Theorem 4.2.3 to obtain a bound of

$$|E(\overline{G[\mathrm{SEgal}(v)]})| = \Omega(\mathrm{Gap}(v) d(v)) - O(d(v))|N(v) - \mathrm{SEgal}(v)|.$$

However, this is not a good enough bound, because if there are $\Omega(\mathrm{Gap}(v))$ weakly egalitarian neighbors of v, we do not have enough non-adjacent strongly egalitarian neighbors to expect many colors assigned to multiple neighbors of v, and we do not expect enough weakly egalitarian neighbors to receive a color not in $L(v)$.

4.3 Proof of Theorem 1.3.4

In this section, we prove Theorem 1.3.4. For the remainder of this section, G, L, ε, and Δ are assumed to satisfy the conditions of Theorem 1.3.4, and we assume that G is L-critical. For each edge $e \in E(G)$, we let M_e be a matching of $\{u\} \times L(u)$ and $\{v\} \times L(v)$ such that (L, M) is a total correspondence assignment for G where every (L, M)-coloring of G is an L-coloring. Note that we are assuming G is L-critical before assuming the correspondence assignment is total, since Theorem 4.2.3 does not hold for correspondence coloring. For $u, v \in V(G)$, let $u \prec v$ if $|L(u)| < |L(v)|$.

Combining Theorem 4.2.3 with Lemmas 4.1.1, 4.1.2, and 4.1.3, we prove that the expected savings for each vertex v is larger than $\varepsilon\,\mathrm{Gap}(v)$. The proof is similar to the proof of Theorem 3.4.4, but we instead use Theorem 4.2.3 to obtain a lower bound on the number of non-edges in the egalitarian neighborhood. We also make a greater effort to optimize the value of ε.

Lemma 4.3.1. *Let $\alpha = 1/50$. For each vertex $v \in V(G)$,*

$$\mathbb{E}\left[\mathbf{savings}_{v,0,\prec}\right] \geq 1.01\varepsilon\,\mathrm{Gap}(v).$$

Proof. By Lemmas 4.1.1, 4.1.2, and 4.1.3,

$$\begin{aligned}\mathbb{E}\left[\mathbf{savings}_{v,0,\prec}\right] \geq{}& (1-K)|\mathrm{Subserv}(v)| \\ &+ K\left(\frac{\alpha}{1+\alpha}|\mathrm{Lord}(v)| + \frac{\alpha\,\mathrm{Gap}(v)}{d(v)+\alpha\,\mathrm{Gap}(v)}|\mathrm{WEgal}(v)|\right) \\ &+ \frac{K}{d(v)}\left(\frac{K}{(1+\alpha)^2} - \frac{\left(2|E(\overline{G[\mathrm{Egal}(v)]})|\right)^{1/2}}{3(1-\varepsilon)^2 d(v)}\right)|E(\overline{G[\mathrm{Egal}(v)]})|. \end{aligned} \tag{4.14}$$

By Theorem 4.2.3,

$$|E(\overline{G[\mathrm{Egal}(v)]})| \geq \left(\frac{1}{4} - \frac{\varepsilon(4+3\alpha)}{2(1-\varepsilon)}\right) \mathrm{Gap}(v) d(v) - \left(\frac{1}{2} - \frac{\varepsilon(1+\alpha)}{2(1-\varepsilon)}\right) d(v) |\mathrm{NEgal}(v)| \\ - \left(\frac{1}{4} - \frac{\varepsilon(2+\alpha)}{2(1-\varepsilon)}\right) \mathrm{Gap}(v) |\mathrm{WEgal}(v)|, \quad (4.15)$$

Note that for any constants a and b, the function $(a - b\sqrt{x})x$ is increasing for $0 \leq x < (2a/(3b))^2$ and decreasing for $x > (2a/(3b))^2$. Letting $a = K/(1+\alpha)^2$ and $b = \sqrt{2}/(3(1-\varepsilon)^2 d(v))$, this fact implies that the right side of (4.14) is maximized when $|E(\overline{G[\mathrm{Egal}(v)]})| = \left(\sqrt{2}K(1-\varepsilon)^2 d(v)/(1-\alpha)^2\right)^2$. Since this value is larger than the right side of (4.15) (indeed, it is larger than $d(v)^2/4$), we have that $\mathbb{E}\left[\mathbf{savings}_{v,0,\prec}\right]$ is at least as large as the minimum of two values: the right side of (4.14) when $|E(\overline{G[\mathrm{Egal}(v)]})|$ is either at most the right side of (4.15) or simply $\binom{d(v)}{2}$.

Consider the former case. By Lemma 4.1.1 and 4.1.2, we may assume that

$$(1-K)|\mathrm{Subserv}(v)| + \frac{K\alpha}{1+\alpha}\left(|\mathrm{Lord}(v)| + \frac{\mathrm{Gap}(v)}{d(v)}|\mathrm{WEgal}(v)|\right) \leq 1.01\varepsilon \cdot \mathrm{Gap}(v).$$

Subject to this inequality, since $1 - K \geq \frac{K\alpha}{1+\alpha}$ and $\frac{1}{2} - \frac{\varepsilon(1+\alpha)}{2(1-\varepsilon)} \geq \frac{1}{4} - \frac{\varepsilon(2+\alpha)}{2(1-\varepsilon)}$, the right side of (4.15) is at least as large as the case when $|\mathrm{Subserv}(v)| = |\mathrm{WEgal}(v)| = 0$ and $|\mathrm{Lord}(v)| \leq 1.01\varepsilon(1+\alpha)\,\mathrm{Gap}(v)/(K\alpha)$, that is

$$|E(\overline{G[\mathrm{Egal}(v)]})| \geq \mathrm{Gap}(v) d(v) \left(\frac{1}{4} - \frac{\varepsilon(4+3\alpha)}{2(1-\varepsilon)} - 1.01\varepsilon\frac{1+\alpha}{\alpha K}\left(\frac{1}{2} - \frac{\varepsilon(1+\alpha)}{2(1-\varepsilon)}\right)\right).$$

Therefore, since $\mathrm{Gap}(v) \leq d(v)$,

$$\mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] \geq \min_{i\in\{1,2\}} K\left(\frac{K}{(1+\alpha)^2} - \frac{(2\cdot \mathrm{sparsity}_i(\alpha,\varepsilon))^{1/2}}{3(1-\varepsilon)^2}\right)\mathrm{Gap}(v)\cdot \mathrm{sparsity}_i(\alpha,\varepsilon), \tag{4.16}$$

where $\mathrm{sparsity}_1(\alpha,\varepsilon) = \frac{1}{4} - \frac{\varepsilon(4+3\alpha)}{2(1-\varepsilon)} - 1.01\varepsilon\frac{1+\alpha}{\alpha K}\left(\frac{1}{2} - \frac{\varepsilon(1+\alpha)}{2(1-\varepsilon)}\right)$ and $\mathrm{sparsity}_2(\alpha,\varepsilon) = 1/2$.

Since $\alpha = 1/50$, $\varepsilon = 1/330$, and $K = .999e^{-330/329}$, the right side of (4.16) is at least $1.01\varepsilon\,\mathrm{Gap}(v)$, as required. □

Finally we can prove Theorem 1.3.4.

Proof of Theorem 1.3.4. Recall that we assume G is L-critical, and we assume (L, M) is a total correspondence assignment for G such that an (L, M)-coloring is an L-coloring. We will apply Theorem 3.1.8 with $\xi_1 = 1.01$, $\xi_2 = \varepsilon$, and $\sigma = 0$ to show that G is (L, M)-colorable, contradicting that G is L-critical.

Let $v \in V(G)$. Since G is L-critical, $d(v) \geq |L(v)| \geq \log^{10}(\Delta)$. Hence we may assume that G has minimum degree at least $\delta(\varepsilon)$. By Lemma 4.3.1, since $\mathrm{Gap}(v) - \mathrm{Save}(v) \geq \log^{10}(\Delta)$,

$$\mathbb{E}\left[\mathbf{savings}_{v,0,\prec}\right] \geq \xi_2 \log^{10} \Delta,$$

and since $\mathrm{Save}(v) \leq \varepsilon\, \mathrm{Gap}(v)$,

$$\mathbb{E}\left[\mathbf{savings}_{v,0,\prec}\right] \geq 1.01\, \mathrm{Save}(v).$$

Therefore by Theorem 3.1.8, G is (L, M)-colorable, contradicting that G is L-critical. $\square$

Chapter 5

The density of critical graphs with no large cliques

In this chapter we prove our results on the density of critical graphs with no large cliques. In particular, in Section 5.1 we prove Theorem 1.4.3 using Theorem 1.3.4. The rest of the chapter is devoted to the proof of Theorem 1.4.5. For technical reasons explained in Section 5.2, we actually prove the following stronger version.

Theorem 5.0.1. *Let $\varepsilon \le 2.6 \cdot 10^{-10}$. If G is an L-critical graph for some list-assignment L such that for each vertex $v \in V(G)$ we have $|L(v)| \le k$ where k is sufficiently large, then*

$$\sum_{v \in V(G)} (\mathrm{Save}_L(v) + \varepsilon \log^{10} k) > \sum_{v \in V(G)} (2\varepsilon\ \mathrm{Gap}(v) - 7\varepsilon(k - |L(v)|)). \tag{5.1}$$

In Section 5.5, we show how to derive Theorem 1.4.5 from Theorem 5.0.1.

5.1 Applying Theorem 1.3.4 when $\omega < k/2$

In this section we prove Theorem 1.4.3. We prove Theorem 1.4.3 by finding an appropriate induced subgraph G' of the graph G, using the criticality of G to L-color $G - V(G')$, and then using Theorem 1.3.4 to extend this coloring to an L-coloring of G, contradicting

With the exception of Section 5.1, which appears in [87], this chapter is based on the paper [88] (with Luke Postle).

the criticality of G. In order to extend the L-coloring of $G - V(G')$ to one of G using Theorem 1.3.4, the vertices of G' need to have few neighbors in $G - V(G')$. The following lemma provides the existence of such a subgraph. Recall that $\mathrm{ad}(H)$ denotes the average degree of a graph H.

Lemma 5.1.1. *For every $1 \geq \alpha > \varepsilon > 0$, every graph H with $\mathrm{ad}(H) \leq (1+\varepsilon)\delta(H)$ contains a nonempty induced subgraph $H' \subseteq H$ such that every $v \in V(H')$ satisfies*

1. $d_{H'}(v) \geq \left(\frac{1-\alpha}{2}\right)\delta(H)$ *and*
2. $d_H(v) \leq \left(1 + \frac{1+\alpha}{\alpha-\varepsilon}\varepsilon\right)\delta(H)$.

Proof. We use the discharging method. For each $v \in V(G)$, let the *charge* of v be $ch(v) = d(v) - \mathrm{ad}(H)$. Note that $\sum_{v\in V(G)} ch(v) = 0$. Let X denote the set of vertices of H with degree greater than $\left(1 + \frac{1+\alpha}{\alpha-\varepsilon}\varepsilon\right)\delta(H)$. Note that X is a proper subset of the vertices of H since $\mathrm{ad}(H) \leq (1+\varepsilon)\delta(H)$. We may assume $\delta(H - X) < \left(\frac{1-\alpha}{2}\right)\delta(H)$ or else $H - X$ is the desired induced subgraph.

We redistribute the charges in the following way. Let every $v \in X$ send $ch(v)/d(v)$ charge to each of its neighbors. Note that for every $v \in X$,

$$\frac{ch(v)}{d(v)} = 1 - \frac{\mathrm{ad}(H)}{d(v)} > 1 - \frac{\mathrm{ad}(H)}{\left(1 + \frac{1+\alpha}{\alpha-\varepsilon}\varepsilon\right)\delta(H)} \geq \frac{\varepsilon}{\alpha}.$$

Therefore every vertex in X has zero charge, and every $v \in V(H - X)$ has charge at least $d_H(v) - \mathrm{ad}(H) + \frac{\varepsilon}{\alpha}(d_H(v) - d_{H-X}(v))$. If $d_H(v) - d_{H-X}(v) > 0$, then the inequality is strict.

Now we claim we can iteratively remove vertices from $H - X$ of minimum degree to obtain a nonempty graph of minimum degree at least $\left(\frac{1-\alpha}{2}\right)\delta(H)$. When we remove a vertex of $H - X$, we add it to a new set X', and we let it send charge ε/α to every neighbor not in $X \cup X'$. It suffices to show that every vertex in X' has nonnegative charge and that at least one vertex in X' has positive charge, because then the sum of the charges taken over vertices in $H - (X \cup X')$ is negative, and thus $H' = H - (X \cup X')$ is nonempty.

Note that if $v \notin X \cup X'$ has degree at most $\left(\frac{1-\alpha}{2}\right)\delta(H)$ in $H - (X \cup X')$, then v has at least $\left(\frac{1+\alpha}{2}\right)\delta(H)$ neighbors in $X \cup X'$. Therefore v receives at least $\frac{\varepsilon}{\alpha}(\frac{1+\alpha}{2})\delta(H)$ charge and sends at most $\frac{\varepsilon}{\alpha}\left(\frac{1-\alpha}{2}\right)\delta(H)$ charge. Hence the difference in charge received and sent is at least $\varepsilon\delta(H)$, and if v has a neighbor in X, the inequality is strict. Therefore v has nonnegative charge, and since at least one vertex of X' has a neighbor in X, there is a vertex of X' with positive charge, as desired. $\square$

Now we can prove Theorem 1.4.3.

Proof of 1.4.3. Let $\alpha > 0$, and let $\varepsilon \leq \alpha^2/1350$. Let G be an L-critical graph for some k-list-assignment L such that $\omega(G) \leq (\frac{1}{2} - \alpha)k$. Note then that $\alpha < \frac{1}{2}$. Suppose for a contradiction that $\mathrm{ad}(G) \leq (1+\varepsilon)k$. Since G is L-critical, G has minimum degree at least k. By Lemma 5.1.1, there exists $G' \subseteq G$ such that for every $v \in V(G')$,

1. $d_{G'}(v) \geq \left(\frac{1-\alpha}{2}\right)\delta(G)$, and
2. $d_G(v) \leq \left(1 + \frac{1+\alpha}{\alpha-\varepsilon}\varepsilon\right)\delta(G)$.

Since G is L-critical, $G - V(G')$ is L-colorable. Let ϕ be an L-coloring of $G - V(G')$, and for each $v \in V(G')$, let

$$L'(v) = L(v) \setminus \{c \in L(v) : \exists u \in N(v) \setminus V(G') : \phi(u) = c\}.$$

Note that G' is not L'-colorable, because we can combine an L'-coloring of G' with ϕ to obtain an L-coloring of G.

Since $d_{G'}(v) \geq \left(\frac{1-\alpha}{2}\right)\delta(G)$, $\delta(G) \geq k$, and $\omega(v) \leq \omega(G) \leq (\frac{1}{2} - \alpha)k$ for each $v \in V(G')$,

$$\mathrm{Gap}_{G'}(v) \geq \frac{\alpha}{2}k.$$

Since each $v \in V(G')$ has at most $d_G(v) - d_{G'}(v)$ neighbors in $V(G) \setminus V(G')$,

$$\mathrm{Save}_{L'}(v) \leq d_G(v) - k \leq \left(\left(1 + \frac{1+\alpha}{\alpha-\varepsilon}\varepsilon\right)(1+\varepsilon) - 1\right)k.$$

Since $\varepsilon \leq \frac{\alpha^2}{1350}$ and $\alpha < \frac{1}{2}$,

$$\frac{1+\alpha}{\alpha-\varepsilon}\varepsilon(1+\varepsilon) + \varepsilon \leq \frac{\alpha}{1350}\left(\frac{(1+\alpha)(1+\alpha^2/1350)}{1-\alpha/1350} + \alpha\right) \leq \frac{\alpha}{660}.$$

Therefore $\mathrm{Save}_{L'}(v) \leq \frac{\alpha}{660}k$. Now for every vertex $v \in V(G')$, $\mathrm{Save}_{L'}(v) \leq \frac{1}{330}\mathrm{Gap}_{G'}(v)$ and for sufficiently large k, $\mathrm{Gap}_{G'}(v) - \mathrm{Save}_{L'}(v) \geq \log^{10}(\Delta(G'))$. Thus, by Theorem 1.3.4, G' is L'-colorable, a contradiction. □

5.2 Overview of the proof of Theorem 5.0.1

The remainder of this chapter is devoted to the proof of Theorem 5.0.1. In this section, we provide an overview of the proof.

5.2.1 The savings

In order to apply Theorem 3.1.8, we need to show that for each vertex v, the "savings" for v is larger than $\mathrm{Save}_L(v)$. Recall that the "savings" for v (defined in Definition 3.1.7) is the sum of several random variables counting roughly the following after an application of the local naive random coloring procedure: the number of repeated colors in $N(v)$, the number of uncolored neighbors u of v such that $u \prec v$, and the number of colored neighbors of v receiving a color not in $L(v)$. Recall also that if v is a vertex of a graph G and $u \in N(v)$, we say u is a subservient neighbor of v if $|L(u)| < |L(v)|$, a strongly egalitarian neighbor of v if $|L(u)| \in [|L(v)|, |L(v)| + \alpha\,\mathrm{Gap}(v))$, a weakly egalitarian neighbor of v if $|L(u)| \in [|L(v)|+\alpha\,\mathrm{Gap}(v), (1+\alpha)|L(v)|)$, and a lordlier neighbor of v if $|L(u)| \geq (1+\alpha)|L(v)|$. Recall also that this partitions the neighbors of v into the sets $\mathrm{Subserv}(v), \mathrm{SEgal}(v), \mathrm{WEgal}(v)$, and $\mathrm{Lord}(v)$, the sets of subservient, strongly egalitarian, weakly egalitarian, and lordlier neighbors of v, respectively, and that we let $\mathrm{NEgal}(v) = N(v) - \mathrm{Egal}(v)$.

In this chapter, the "savings" for a vertex v is large in any of the following situations (defined formally in Definition 5.3.3):

- many neighbors of v have many colors in their list that are not in $L(v)$, in which case we say v is *aberrant* (or *slightly aberrant*),
- many pairs of non-adjacent neighbors of v have lists of colors of size close to $|L(v)|$, in which case we say v is *egalitarian-sparse*, or
- many neighbors u of v satisfy $u \prec v$, where $\prec$ is the ordering of $V(G)$ in which we greedily extend the partial coloring obtained by the local naive random coloring procedure.

In Chapter 4, we implicitly showed that if G is a graph with list-assignment L such that $\mathrm{Save}_L(v) \leq \varepsilon\,\mathrm{Gap}(v)$ for *every* $v \in V(G)$, then every vertex v is either aberrant, slightly-aberrant, egalitarian-sparse, or has many neighbors u such that $|L(u)| < |L(v)|$ (in which case we say v is *lordly*). By setting $u \prec v$ whenever $|L(u)| < |L(v)|$, under some additional technical assumptions on G and L, the savings for each vertex is large enough to apply Theorem 3.1.8.

In this chapter, we only require $\mathrm{Save}_L(v) \leq \varepsilon\,\mathrm{Gap}(v)$ *on average*. However, we need the list-assignment L to be close to uniform (i.e. $|L(v)|$ is close to k for each vertex v), so Theorem 5.0.1 is incomparable to Theorem 1.3.4. In this new setting, we refine the set of outcomes for each vertex (see Lemma 5.4.11). Now, a vertex v may have $\mathrm{Save}_L(v)$ comparatively larger than $\varepsilon\,\mathrm{Gap}(v)$, in which case we say v is *heavy*, or v may have many heavy neighbors, in which case we say v is *sponsored* (see Definition 5.4.10). A lordly vertex v may be *very lordly* (see Definition 5.4.5), in which case it has considerably many neighbors with a much smaller list of colors, or its neighborhood contains the complement of a bipartite subgraph with partition (A, B) called a *half-egalitarian bipartition* (see Definition 5.3.2) where vertices in A have lists of colors of size close to $|L(v)|$ and vertices in B have many non-neighbors in A. In the latter case, we say v is *bipartite-sparse* (Definition 5.3.3). The savings for a bipartite-sparse vertex is large enough to apply Theorem 3.1.8. Thus, if every vertex is either aberrant, slightly aberrant, egalitarian-sparse, bipartite-sparse, or has many neighbors appearing later in the ordering, then we say the graph is "saved" (see Definition 5.3.8). Using Theorem 3.1.8, we prove Theorem 5.3.9, which says that we can color a saved graph.

5.2.2 The dense vertices

We also will use Theorem 4.2.1 from Chapter 4, so we introduce the following definition.

Definition 5.2.1. If H is an induced subgraph of G and M is a matching in $\overline{H}$ such that

$$|E(\overline{H})| < |M|(|V(H)| - |M|) - \sum_{u \in V(H)} \mathrm{Save}_L(u),$$

then we say H is *dense with respect to L*.

Theorem 4.2.1 implies that L-critical graphs do not contain subgraphs that are dense with respect to L.

Note that for any graph H, a maximum matching in $\overline{H}$ has size at least $(|V(H)| - \omega(H))/2$. As we saw in Chapter 4, if the neighborhood of a vertex v is not dense with respect to L, then either v has sparsity on the order of $\mathrm{Gap}(v)d(v)$ or v has many neighbors u for which $\mathrm{Save}_L(u)$ is $\Omega(\mathrm{Gap}(u))$. In the former case, v is egalitarian-sparse, and we expect there to be many repeated colors after an application of the local naive random coloring procedure. If v has many neighbors for which Save_L is large, then v is either aberrant, slightly aberrant, or sponsored, as we will see in Lemma 5.4.11 (c) and (d).

5.2.3 The discharging

Using the discharging method as in the proof of Lemma 5.1.1, in Section 5.4 we prove Lemma 5.4.1, which implies that if a hypothetical L-critical graph G does not satisfy (5.1) and does not contain a subgraph that is dense with respect to L, then G contains a subgraph H such that after coloring $V(G) \setminus V(H)$, the graph H with the list-assignment consisting of the remaining available colors is “saved”. To that end, we let the *charge* of a vertex v be

$$ch(v) = \mathrm{Save}_L(v) - 2\varepsilon\,\mathrm{Gap}(v) + \varepsilon \log^{10} k + 7\varepsilon(k - |L(v)|).$$

Since (5.1) does not hold, the total sum of the charges of the vertices is at most zero. In Section 5.4, we prove Lemma 5.4.2, which says that if G is not saved, then we can find a set $D \subseteq V(G)$ such that each vertex in D can send 9ε charge to each of its neighbors not in D, so that the sum of the remaining charges of the vertices in D is positive. Therefore the sum of the charges of vertices in $V(G) \setminus D$ is negative, which implies that D is a proper subset of $V(G)$. If $G - D$ is not the desired saved subgraph H, then we can apply Lemma 5.4.2 iteratively to obtain Lemma 5.4.1, as shown in Section 5.4.5. We prove the stronger form of Theorem 5.0.1 because it allows us to iterate this lemma.

To find this set D, we let S_0 be the vertices of G that are aberrant, slightly aberrant, egalitarian-sparse, or bipartite-sparse, and for $i \geq 1$, we let S_i be the vertices with many neighbors in $\cup_{j=0}^{i-1} S_j$. If $\cup_i S_i = V(G)$, then by construction, the graph G is saved. If not, we let $\mathcal{L}$ be the very lordly vertices not in S_∞, and we let $D = V(G) \setminus (\mathcal{L} \cup \bigcup_i S_i)$. Lemma 5.4.11 implies that the vertices in D are either heavy or sponsored, which in turn implies that the total charges of vertices in D is large. Since vertices in D do not have many neighbors in $\cup S_i$, they can afford to send 9ε charge to each neighbor there. Lemma 5.4.12 implies that heavy and sponsored vertices with many very lordly neighbors are aberrant. Thus, the vertices in D do not have many very lordly neighbors and can afford to send them each 9ε charge as well.

5.3 Coloring a saved graph with Theorem 3.1.8

The main result of this section is Theorem 5.3.9, which we prove using Theorem 3.1.8. First, we need several definitions. Note that some of the definitions are slightly different from those in Chapters 3 and 4.

5.3.1 Ways to save

We need to partition the neighborhood of each vertex according to the size of each neighbor's list of colors, as follows.

Definition 5.3.1. Let $\sigma = 2/3$, and let $\alpha, \delta, \varepsilon' \in (0, 1)$ be some constants to be determined later. Let $K = .999e^{-1/(1-\varepsilon')}$. Assume

$$\delta < 1 - \sigma, \tag{5.2}$$

$$\varepsilon' \leq 1/2, \tag{5.3}$$

$$\frac{1-2\delta}{(1+\alpha)^2} > \frac{K}{3(1-\delta)^2(1-\varepsilon')^2}, \text{ and} \tag{5.4}$$

$$\frac{\delta - \varepsilon'}{1-\varepsilon'} > \frac{15\delta}{16}. \tag{5.5}$$

Let G be a graph with list-assignment L, let $v \in V(G)$, and let $u \in N(v)$.

- If $|L(u)| < (1 - \delta)|L(v)|$, then we say u is a *subservient neighbor* of v.
- If $|L(u)| \in [(1 - \delta)|L(v)|, (1 + \alpha)|L(v)|)$, then we say u is an *egalitarian neighbor* of v.
- If $|L(u)| \geq (1 + \alpha)|L(v)|$, then we say u is a *lordlier neighbor* of v.
- If $|L(u)| \geq |L(v)| + \alpha \operatorname{Gap}(v)$, then we say u is a *slightly lordlier neighbor* of v.

For convenience, we will let Slightly-Lord(v) denote the set of slightly lordlier neighbors of v, Lord(v) denote the set of lordlier neighbors of v, Egal(v) denote the set of egalitarian neighbors of v, and Subserv(v) denote the set of subservient neighbors of v.

The main difference between these definitions and those in the previous chapters is that neighbors of a vertex with a slightly smaller list are now considered to be egalitarian rather than subservient.

Recall that $\mathrm{Egal}_\sigma(v)$ is the set of σ-egalitarian neighbors of v, which are neighbors u of v with at least $(1 - \sigma)|L(v)|$ available colors.

The following definitions provide sufficient conditions for $\mathbf{savings}_{v,\sigma,\prec}$ to be sufficiently large in expectation.

Definition 5.3.2. A pair (A, B) of disjoint subsets of $N(v)$ is a *half-egalitarian bipartition* for v if

- $B \subseteq \mathrm{Egal}(v)$,
- $A \subseteq \mathrm{Egal}_\sigma(v) \cap \mathrm{Subserv}(v)$, and
- each vertex $u \in A$ has at least $\left(\frac{\delta-\varepsilon'}{1-\varepsilon'} - \frac{15\delta}{16}\right) d(v)$ non-neighbors in B.

Definition 5.3.3. Let aber(α, ε'), egal-sparse($\alpha, \delta, \varepsilon'$), and bipart-sparse($\alpha, \delta, \varepsilon', \sigma$) be constants to be determined later, and let G be a graph with list-assignment L. We say a vertex $v \in V(G)$ is

- *aberrant* with respect to L and k if

$$|\mathrm{Lord}(v)| \geq \left(\mathrm{Save}_L(v) + 11\varepsilon' \log^{10} k\right) / \mathrm{aber}(\alpha, \varepsilon'),$$

- *slightly aberrant* with respect to L and k if

$$|\text{Slightly-Lord}(v)| \geq \frac{d(v)}{\mathrm{Gap}(v)} \left(\mathrm{Save}_L(v) + 11\varepsilon' \log^{10} k\right) / \mathrm{aber}(\alpha, \varepsilon'),$$

- *egalitarian-sparse* with respect to L and k if

$$|E(\overline{G[\mathrm{Egal}(v)]})| \geq d(v) \left(\mathrm{Save}_L(v) + 11\varepsilon' \log^{10} k\right) / \text{egal-sparse}(\alpha, \delta, \varepsilon'),$$

- and *bipartite-sparse* with respect to L and k if v has a half-egalitarian bipartition (A, B) such that

$$|A| \geq \left(\mathrm{Save}_L(v) + 11\varepsilon' \log^{10} k\right) / \text{bipart-sparse}(\alpha, \delta, \varepsilon', \sigma).$$

As Lemma 5.3.4 shows, an aberrant or slightly aberrant vertex v has large expected savings because $\mathbf{unmatched}_v$ is large in expectation. Each lordlier or slightly lordlier neighbor of v has a good chance to receive a color not in $L(v)$.

As Lemma 5.3.5 shows, an egalitarian-sparse vertex v has large expected savings because $\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma}$ is large in expectation. Every pair of non-adjacent egalitarian neighbors of v has a good chance to receive the same color. Here it is important to consider correspondence coloring, rather than list-coloring. As in Chapter 4, correspondence coloring allows us to essentially assume that two neighboring vertices' lists of colors have as many colors in common as possible. As long as $\sigma < 1/2$, two σ-egalitarian neighbors of v are forced to have some colors that correspond to the same color in $L(v)$. Since $\delta < 1/3$, egalitarian neighbors have a nontrivial amount of colors that correspond to the same color in $L(v)$.

Lemma 5.3.6 shows that a bipartite-sparse vertex v also has large expected savings because $\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma}$ is large in expectation. If (A, B) is a half-egalitarian bipartition for v, then each vertex in A has a good chance to receive the same color as many of its non-neighbors in B. Here we also use correspondence coloring to force a vertex in A and a vertex in B to have some colors that correspond to the same color in $L(v)$.

5.3.2 Expectations

In this subsection, we let G be a graph with list-assignment L, and we let (L, M) be a correspondence assignment for G. We assume (L, M) is *total*, meaning for each $uv \in E(G)$, the matching M_{uv} saturates at least one of $\{u\} \times L(u)$ or $\{v\} \times L(v)$.

We prove a series of lemmas that shows that $\mathbf{savings}_{v,\sigma,\prec}$ is sufficiently large if a vertex v satisfies one of the properties defined in Definition 5.3.3. To that end, we let (ϕ, U) be randomly sampled according to the local naive random coloring procedure with ε-equalizing coin-flips.

The first such lemma will be applied to vertices that are either aberrant or slightly aberrant.

Lemma 5.3.4. *Let* $\text{aber}(\alpha, \varepsilon') = \frac{K\alpha(1-\varepsilon')}{1+\alpha}$*. For each* $v \in V(G)$ *such that* $|L(v)| \leq d(v)$*,*

$$\mathbb{E}\left[\mathbf{unmatched}_v\right] \geq \frac{\text{aber}(\alpha, \varepsilon')}{1-\varepsilon'} \max\left\{|\text{Lord}(v)|, \frac{\text{Gap}(v)}{d(v)}|\text{Slightly-Lord}(v)|\right\}.$$

Proof. Let

$$\mathbf{unmatched}_v^{\text{tot}} = |\{u \in N(v) : \phi(u) \notin V(M_{uv})\}|,$$

and note that $\mathbb{E}\left[\mathbf{unmatched}_v\right] = K \cdot \mathbb{E}\left[\mathbf{unmatched}_v^{\text{tot}}\right]$. For each $u \in \text{Lord}(v)$,

$$\mathbb{P}\left[\phi(u) \notin V(M_{uv})\right] \geq \frac{\alpha}{1+\alpha},$$

and for each $u \in \text{Slightly-Lord}(v)$, since $d(v) \geq |L(v)|$ and $d(v) \geq \text{Gap}(v)$,

$$\mathbb{P}\left[\phi(u) \notin V(M_{uv})\right] \geq \frac{\alpha\, \text{Gap}(v)}{|L(v)| + \alpha\, \text{Gap}(v)} \geq \frac{\alpha}{1+\alpha}\frac{\text{Gap}(v)}{d(v)}.$$

Therefore it follows that

$$\mathbb{E}\left[\mathbf{unmatched}_v^{\text{tot}}\right] \geq \frac{\alpha}{1+\alpha} \max\left\{|\text{Lord}(v)|, \frac{\text{Gap}(v)}{d(v)}|\text{Slightly-Lord}(v)|\right\}.$$

Since $\mathbb{E}\left[\mathbf{unmatched}_v\right] = K \cdot \mathbb{E}\left[\mathbf{unmatched}_v^{\text{tot}}\right]$, the result follows. ◻

The next lemma will be applied to the vertices that are egalitarian-sparse.

Lemma 5.3.5. *Let* $\text{egal-sparse}(\alpha, \delta, \varepsilon') = K(1-\varepsilon')\left(\frac{(1-2\delta)K}{(1+\alpha)^2} - \frac{1}{3(1-\delta)^2(1-\varepsilon')^2}\right)$*. For each* $v \in V(G)$ *such that* $d(v) \geq |L(v)| \geq (1-\varepsilon')d(v)$*,*

$$\mathbb{E}\left[\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma}\right] \geq \frac{\text{egal-sparse}(\alpha, \delta, \varepsilon')}{1-\varepsilon'} \frac{|E(\overline{G[\text{Egal}(v)]})|}{d(v)}.$$

Proof. Recall that we let $T(H)$ denote the set of triangles in a graph H. Let $H = G[\mathrm{Egal}(v)]$. We define the following random variables:

$\mathbf{pairs}_v^{\mathrm{tot}}((\phi, U)) = |\{xy \in E(H), c \in L(v) : \phi(x)c \in M_{xv} \text{ and } \phi(y)c \in M_{yv}\}|$, and
$\mathbf{trips}_v^{\mathrm{tot}}((\phi, U)) = |\{xyz \in T(H), c \in L(v) : \phi(x)c \in M_{xv}, \phi(y)c \in M_{yv}, \text{ and } \phi(z)c \in M_{zv}\}|$.

Note that

$$\mathbb{E}\left[\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma}\right] \geq K^2 \mathbb{E}\left[\mathbf{pairs}_v^{\mathrm{tot}}\right] - K \mathbb{E}\left[\mathbf{trips}_v^{\mathrm{tot}}\right]. \tag{5.6}$$

Let $C_{x,y}$ be the set of colors $c \in L(v)$ for which there exist colors $c_x \in L(x)$ and $c_y \in L(y)$ such that c corresponds to both c_x and c_y. We claim that $|C_{x,y}| \geq (1-2\delta)|L(v)|$ for each $x, y \in \mathrm{Egal}(v)$. Suppose $|L(y)| < |L(v)|$, or else $|C_{x,y}| \geq (1-\delta)|L(v)| \geq (1-2\delta)|L(v)|$ since (L, M) is total, as claimed. Now $|L(x)| + |L(y)| - |C_{x,y}| \leq |L(v)|$. Hence, $|C_{x,y}| \geq 2(1-\delta)|L(v)| - |L(v) = (1-2\delta)|L(v)|$, as claimed. Now

$$\mathbb{E}\left[\mathbf{pairs}_v^{\mathrm{tot}}\right] = \sum_{xy \in E(H)} \frac{|C_{x,y}|}{|L(x)|\,|L(y)|} \geq \frac{1-2\delta}{(1+\alpha)^2 |L(v)|}|E(H)| \geq \frac{1-2\delta}{(1+\alpha)^2 d(v)}|E(H)|. \tag{5.7}$$

Similarly,

$$\mathbb{E}\left[\mathbf{trips}_v^{\mathrm{tot}}\right] \leq \sum_{xyz \in T(H)} \frac{1}{|L(x)||L(y)|} \leq \frac{1}{(1-\delta)^2 |L(v)|^2}|T(H)| \leq \frac{1}{(1-\delta)^2(1-\varepsilon')^2 d(v)^2}|T(H)|. \tag{5.8}$$

By (5.8) and (4.1),

$$\mathbb{E}\left[\mathbf{trips}_v^{\mathrm{tot}}\right] \leq \frac{\sqrt{8}|E(H)|^{3/2}}{6(1-\delta)^2(1-\varepsilon')^2 d(v)^2}. \tag{5.9}$$

By (5.6), (5.7) and (5.9),

$$\mathbb{E}\left[\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma}\right] \geq \left(\frac{K|E(H)|}{d(v)}\right)\left(\frac{K(1-2\delta)}{(1+\alpha)^2} - \frac{\sqrt{8|E(H)|}}{6(1-\delta)^2(1-\varepsilon')^2 d(v)}\right).$$

Since $|E(H)| \leq \binom{d(v)}{2}$,

$$\frac{\sqrt{8|E(H)|}}{d(v)} \leq 2.$$

By the previous two inequalities,

$$\mathbb{E}\left[\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma}\right] \geq \left(\frac{K|E(H)|}{d(v)}\right)\left(\frac{K(1-2\delta)}{(1+\alpha)^2} - \frac{1}{3(1-\delta)^2(1-\varepsilon')^2}\right),$$

as desired. $\square$

The next lemma will be applied to the vertices that are bipartite-sparse.

Lemma 5.3.6. *Let* bipart-sparse$(\alpha, \delta, \varepsilon', \sigma) = K^2(1-\varepsilon')\left(\frac{\delta-\varepsilon'}{1-\varepsilon'} - \frac{15\delta}{16}\right)\left(\frac{1-\sigma-\delta}{(1+\alpha)(1-\delta)}\right)$. *If* $v \in V(G)$ *has a half-egalitarian bipartition* (A, B), *then*

$$\mathbb{E}\left[\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma}\right] \geq |A| \, \text{bipart-sparse}(\alpha, \delta, \varepsilon', \sigma)/(1-\varepsilon')$$

Proof. Let H be a bipartite subgraph of $\overline{G[A \cup B]}$ with bipartition (A, B) such that each vertex $u \in A$ has at least $\left(\frac{\sigma-\varepsilon'}{1-\varepsilon'} - \frac{15\delta}{16}\right) d(v)$ neighbors in B. Define the random variable

$$\mathbf{pairs}_v = |\{xy \in E(H), c \in L(v) : \phi(x)c \in M_{xv} \text{ and } \phi(y)c \in M_{yv}\}|.$$

Note that

$$\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma} \geq K^2 \, \mathbf{pairs}_v. \tag{5.10}$$

For each $xy \in E(H)$, let $C_{x,y}$ be the set of colors $c \in L(v)$ for which there exist colors $c_x \in L(x)$ and $c_y \in L(y)$ such that c corresponds to both c_x and c_y. We claim that $|C_{x,y}| \geq (1-\delta-\sigma)|L(v)|$ for each $xy \in E(H)$. Suppose $|L(y)| < |L(v)|$, or else $|C_{x,y}| \geq (1-\sigma)|L(v)| \geq (1-\delta-\sigma)|L(v)|$ since (L, M) is total, as claimed. Now $|L(x)| + |L(y)| - |C_{x,y}| \leq |L(v)|$. Hence, $|C_{x,y}| \geq (1-\sigma)|L(v)| + (1-\delta)|L(v)| - |L(v)| = (1-\sigma-\delta)|L(v)|$, as claimed.

Now

$$\begin{aligned}\mathbb{E}\left[\mathbf{pairs}_v\right] = \sum_{xy \in E(H)} \frac{|C_{x,y}|}{|L(x)||L(y)|} &\geq \frac{1-\sigma-\delta}{(1+\alpha)(1-\delta)|L(v)|}|E(H)| \\ &\geq \left(\frac{\sigma-\varepsilon'}{1-\varepsilon'} - \frac{15\delta}{16}\right)\left(\frac{1-\sigma-\delta}{(1+\alpha)(1-\delta)}\right)|A|.\end{aligned}$$

The previous inequality, together with (5.10), implies the lemma. ☐

Remark 1. By equations (5.2), (5.4) and (5.5), the constants egal-sparse$(\alpha, \delta, \varepsilon')$ and bipart-sparse$(\alpha, \delta, \varepsilon', \sigma)$ are both positive.

Vertices that are not aberrant, slightly aberrant, egalitarian-sparse, or bipartite-sparse will have many neighbors that appear prior in the ordering $\prec$ in a saved graph. We will apply the following lemma to these vertices.

Lemma 5.3.7. *If* $\prec$ *is an ordering of* $V(G)$ *and* $v \in V(G)$, *then*

$$\mathbb{E}\left[\mathbf{uncolored}_{v,\prec}\right] = (1-K)|\{u \in N(v) : u \prec v\}|.$$

Proof. Since $\mathbb{P}\left[u \in U\right] = 1 - K$ for each $u \in N(v)$, the result follows by Linearity of Expectation. ☐

5.3.3 Applying Theorem 3.1.8

We are finally ready to state the definition of a saved graph.

Definition 5.3.8. We say a graph G with list-assignment L is *saved with respect to* L *and* k if (G, L) is $(2k, \varepsilon')$-bounded and there exists an ordering $\prec$ of $V(G)$ such that every vertex v is either aberrant, slightly aberrant, egalitarian-sparse, or bipartite-sparse with respect to L and k or has at least $(\mathrm{Save}_L(v) + \varepsilon' \log^{10} k)/((1-K)(1-\varepsilon'))$ neighbors u such that $u \prec v$.

We say a subgraph $H \subseteq G$ is *saved with respect to* L *and* k if for any L-coloring ϕ of $G - V(H)$, the graph H is saved with respect to L' and k where $L'(v) = L(v) \setminus (\cup_{u \in N(v) \setminus V(H)} \phi(u))$.

The following is the main result of this section. It implies that an L-critical graph does not contain a saved subgraph.

Theorem 5.3.9. *If k is sufficiently large and G is saved with respect to a list-assignment L and k, then G is L-colorable.*

Proof. We apply Theorem 3.1.8 with $\Delta = 2k$, $\sigma = 2/3$, $\varepsilon = 11\varepsilon'$, $\xi_1 = \varepsilon'/(1-\varepsilon')$, and $\xi_2 = .99\varepsilon/(1-\varepsilon')$. We assume that (L, M) is a total correspondence-assignment for G such that any (L, M)-coloring of G is an L-coloring. We assume k is large enough so that $.99 \log^{10} 2k \le \log^{10} k$.

Let $v \in V(G)$. If v is aberrant or slightly aberrant with respect to L and k, then by Lemma 5.3.4,

$$\mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] \ge \mathbb{E}\left[\mathbf{unmatched}_v\right] \ge \frac{\mathrm{Save}_L(v) + \varepsilon' \log^{10} k}{1-\varepsilon'} \ge (1+\xi_1)\,\mathrm{Save}_L(v) + \xi_2 \log^{10} 2k.$$

If v is egalitarian-sparse or bipartite-sparse with respect to L and k, then by Lemmas 5.3.5 and 5.3.6,

$$\mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] \ge \mathbb{E}\left[\mathbf{pairs}_{v,\sigma} - \mathbf{trips}_{v,\sigma}\right] \ge \frac{\mathrm{Save}_L(v) + \varepsilon' \log^{10} k}{1-\varepsilon'} \ge (1+\xi_1)\,\mathrm{Save}_L(v) + \xi_2 \log^{10} 2k.$$

If v is neither aberrant, slightly aberrant, egalitarian-sparse, nor bipartite-sparse with respect to L and k, then since G is saved, v has at least $(\mathrm{Save}_L(v) + \varepsilon' \log^{10} k)/((1 - K)/(1-\varepsilon'))$ neighbors u such that $u \prec v$. Hence, by Lemma 5.3.7,

$$\mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] \ge \mathbb{E}\left[\mathbf{uncolored}_{v,\prec}\right] \ge \frac{\mathrm{Save}_L(v) + \varepsilon' \log^{10} k}{1-\varepsilon'} \ge (1+\xi_1)\,\mathrm{Save}_L(v) + \xi_2 \log^{10} 2k.$$

Therefore $\mathbb{E}\left[\mathbf{savings}_{v,\sigma,\prec}\right] \geq \max\{(1+\xi_1)\operatorname{Save}_L(v), \xi_2 \log^{10} 2k\}$, as required. Now by Theorem 3.1.8, G is (L, M)-colorable, as desired. □

5.4 Finding a saved subgraph

The main result of this section is the following lemma, which states that if G is a graph with list-assignment L not satisfying (5.1), then G contains either a dense or saved subgraph.

Lemma 5.4.1. *Let G be a graph with list-assignment L such that for each vertex $v \in V(G)$ we have $|L(v)| \leq \min\{d(v), k\}$ and*

$$\sum_{v \in V(G)} (\operatorname{Save}_L(v) + \varepsilon \log^{10} k) \leq \sum_{v \in V(G)} (2\varepsilon \operatorname{Gap}(v) - 7\varepsilon(k - |L(v)|)).$$

If G has no dense subgraph with respect to L and k is sufficiently large, then there exists a subgraph $H \subseteq G$ such that H is saved with respect to L and k.

5.4.1 A stronger version

For inductive purposes, we actually prove the following stronger result and show that it implies Lemma 5.4.1.

Lemma 5.4.2. *Let G be a graph with list-assignment L such that (5.1) does not hold and for each vertex $v \in V(G)$, we have $|L(v)| \leq \min\{d(v), k\}$. If G has no dense subgraph with respect to L and k is sufficiently large, then either*

(a) G is saved with respect to L and k, or
(b) there is a nonempty set $D \subsetneq V(G)$ such that

$$\sum_{v \in V(G-D)} (\operatorname{Save}_L(v) + \varepsilon \log^{10} k) < \sum_{v \in V(G-D)} (2\varepsilon \operatorname{Gap}_{G-D}(v) - 7\varepsilon(k - |L(v)| + |N(v) \cap D|)).$$

For each $v \in V(G)$, let the *charge* of v be

$$ch(v) = \operatorname{Save}_L(v) - 2\varepsilon \operatorname{Gap}(v) + \varepsilon \log^{10} k + 7\varepsilon(k - |L(v)|).$$

Now, $\sum_{v \in V(G)} ch(v) < 0$. As mentioned in Section 5.2, we think of D in Lemma 5.4.2 as the "discharging set", that is the vertices in D will send charge to their neighbors. When we redistribute the charges in Section 5.4.4, each vertex not in D receives 9ε charge from each neighbor in D, and each vertex in D still has positive charge.

Definition 5.4.3. A vertex $v \in V(G)$ is *heavy* if $ch(v) \geq (36\varepsilon/\delta)\operatorname{Gap}(v)$ and normal otherwise. A vertex $v \in V(G)$ is *extremely heavy* if $ch(v) \geq 9\varepsilon \cdot d(v)$.

The next lemma implies that if v is an extremely heavy vertex, then $D = \{v\}$ satisfies (b) in Lemma 5.4.2. Thus, we can essentially assume there are no extremely heavy vertices.

Lemma 5.4.4. *Let G be a graph with list-assignment L such that (5.1) does not hold. If $u \in V(G)$ is extremely heavy, then*

$$\sum_{v\in V(G-u)} (\operatorname{Save}_L(v) + \varepsilon \log^{10} k) < \sum_{v\in V(G-u)} (2\varepsilon \operatorname{Gap}_{G-v}(v) - 7\varepsilon(k - |L(v)| + |N(v)\cap\{u\}|)).$$

Proof. Let u send charge 9ε to each of its neighbors, and denote the resulting charge ch_*. Since u is extremely heavy, $ch_*(u) \geq 0$. Hence,

$$\sum_{v\in V(G-u)} ch_*(v) \leq \sum_{v\in V(G)} ch(v) < 0.$$

For each vertex $v \in N(u)$, we have $\operatorname{Gap}_{G-u}(v) \geq \operatorname{Gap}_G(v) - 1$. Hence,

$$\begin{aligned}\sum_{v\in V(G-u)} (\operatorname{Save}_L(v) + \varepsilon \log^{10} k) - \sum_{v\in V(G-u)} (2\varepsilon \operatorname{Gap}_{G-u}(v) - 7\varepsilon(k - |L(v)| + |N(v)\cap\{u\}|)) \leq \\ \sum_{v\in V(G-u)} (\operatorname{Save}_L(v) + \varepsilon \log^{10} k) - \sum_{v\in V(G)\setminus N[u]} (2\varepsilon \operatorname{Gap}_G(v) - 7\varepsilon(k - |L(v)|)) \\ - \sum_{v\in N(u)} (2\varepsilon(\operatorname{Gap}_G(v) - 1) - 7\varepsilon(k - |L(v)| + 1)) = \sum_{v\in V(G-u)} ch_*(v).\end{aligned}$$

Now the lemma follows from the previous two inequalities. □

By combining Lemma 5.4.4 with the next lemma, Lemma 5.4.6, we obtain Lemma 5.4.2. First we need the following definition.

Definition 5.4.5. A vertex $v \in V(G)$ is *very lordly* if $\operatorname{Gap}(v) \geq (3\delta/4)d(v)$ and $|\operatorname{Subserv}(v)| > \operatorname{Gap}(v)/4$.

Lemma 5.4.6. *Let G be a graph with list-assignment L not satisfying (5.1) such that for each vertex $v \in V(G)$ we have $|L(v)| \leq \min\{d(v), k\}$. Let S_0 be the vertices of G that are* *either aberrant, slightly aberrant, egalitarian-sparse, or bipartite-sparse. For $i \geq 1$, let S_i be the vertices with at least*

$$\frac{\mathrm{Save}_L(v) + \varepsilon \log^{10} k}{(1-K)(1-\varepsilon')}$$

neighbors in $\cup_{j=0}^{i-1} S_i$, and define $S_\infty = \cup_{i \geq 1} S_i$. Let $\mathcal{L}$ be the very lordly vertices not in S_∞, and let $D = V(G) \setminus (S_\infty \cup \mathcal{L})$. If G has no extremely heavy vertex and no dense subgraph with respect to L, then

$$\sum_{v \in V(G-D)} (\mathrm{Save}_L(v) + \varepsilon \log^{10} k) < \sum_{v \in V(G-D)} (2\varepsilon \,\mathrm{Gap}_{G-D}(v) - 7\varepsilon(k - |L(v)| + |N(v) \cap D|)).$$

In Section 5.4.5, we prove Lemma 5.4.1 using Lemma 5.4.2, and we prove Lemma 5.4.2 using Lemmas 5.4.4 and 5.4.6. Sections 5.4.2, 5.4.3, and 5.4.4 are devoted to the proof of Lemma 5.4.6.

5.4.2 Preliminaries

Since Sections 5.4.2, 5.4.3, and 5.4.4 are devoted to the proof of Lemma 5.4.6, we assume in these sections that G is a graph with list-assignment L not satisfying (5.1) such that $|L(v)| \leq \min\{d(v), k\}$ for each vertex v, and moreover G does not contain a dense subgraph or any extremely heavy vertices. Using this assumption, we prove several useful propositions in this subsection.

In various places throughout this section, we need $\varepsilon, \alpha, \delta$, and σ to satisfy certain inequalities. In order to make it easier to check that our parameters satisfy all of these requirements, we collect them below:

$$\varepsilon \leq \frac{\mathrm{aber}(\alpha, \varepsilon')}{4(36/\delta + 2)} \tag{5.11}$$

$$\varepsilon \leq \frac{\text{bipart-sparse}(\alpha, \delta, \varepsilon', \sigma)}{4(36/\delta + 2)} \tag{5.12}$$

$$\varepsilon \leq \frac{\delta}{11(16 - 15\delta)} \tag{5.13}$$

$$\varepsilon < \delta/(36 + 2\delta) \tag{5.14}$$

$$\varepsilon \leq \text{egal-sparse}(\alpha, \delta, \varepsilon') \frac{\delta/64 - 11\varepsilon((\delta/8) + 1)((1+\alpha)/(1-\varepsilon') + 1)}{36/\delta + 2} \tag{5.15}$$

$$\varepsilon \leq \left(\frac{\text{egal-sparse}(\alpha,\delta,\varepsilon')}{36/\delta+2}\right)\left(\frac{1-\delta}{16} - \varepsilon\left(\frac{5/2-\alpha}{\delta(36+2\delta)-\varepsilon} + 22\frac{2+\alpha-\varepsilon'}{1-\varepsilon'}\right)\right), \tag{5.16}$$

$$\alpha \leq \frac{\delta(2+\varepsilon')-4\varepsilon'}{4-3\delta}, \tag{5.17}$$

$$\varepsilon \leq \left(\frac{1}{36(1+\delta/18)}\right)\Big/\left(\frac{1}{(1-K)(1-\varepsilon')} + \frac{1}{\text{aber}(\alpha,\delta,\varepsilon')}\right), \tag{5.18}$$

$$\delta \leq \frac{\left(\frac{1}{2} - \frac{\varepsilon(36+2\delta)}{4(1-K)(1-\varepsilon')}\right)\left(\frac{(5/2-\alpha)(1-\varepsilon')(36+2\delta)}{(1-\varepsilon(36+2\delta)/\delta)(2(1+\delta/18))(1+\alpha\delta/4)}\right)}{2+9\left(1+\varepsilon(36/\delta+2)\left(\frac{1}{(1-K)(1-\varepsilon')} + \text{aber}(\alpha,\delta,\varepsilon')^{-1}\right)\right)}. \tag{5.19}$$

We need the following proposition about the sizes of vertices' lists of available colors. In this proposition, we need that there are no extremely heavy vertices.

Proposition 5.4.7. *If $v \in V(G)$, then*

(a) $|L(v)| \geq (1-11\varepsilon)d(v)$ *and* $\text{Save}_L(v) < 11\varepsilon d(v)$,
(b) and if $\varepsilon \leq 3/154$, *then* $|L(v)| > k/3$.

Proof. First we prove (a). Since v is not extremely heavy, $9\varepsilon d(v) > ch(v) > \text{Save}_L(v) - 2\varepsilon\,\text{Gap}(v)$. Hence, since $\text{Gap}(v) \leq d(v)$, we have $11\varepsilon d(v) > \text{Save}_L(v)$, as desired. Therefore $d(v)+1-|L(v)| < 11\varepsilon d(v)$, so $|L(v)| > (1-11\varepsilon)d(v)$, as desired.

Now we prove (b). Since v is not extremely heavy, $9\varepsilon d(v) > ch(v) > 7\varepsilon(k-|L(v)|) - 2\varepsilon\,\text{Gap}(v)$. Hence, since $\text{Gap}(v) \leq d(v)$, we have $11d(v) + 7|L(v)| > 7k$. By (a), $d(v) \leq |L(v)|/(1-11\varepsilon)$, and since $\varepsilon \leq 3/154$, we have $d(v) \leq 14|L(v)|/11$. Therefore $7k < 11d(v) + 7|L(v)| \leq 21|L(v)$, so $|L(v)| > k/3$, as desired. □

Proposition 5.4.7 (b) reveals why we need to add the term $7\varepsilon(k-|L(v)|)$ in Theorem 5.0.1. Note that Proposition 5.4.7 (b) implies that all neighbors of a vertex are σ-egalitarian. This fact will be crucial in Lemma 5.4.11 (b).

The next proposition provides useful facts about the heavy vertices.

Proposition 5.4.8. *If $v \in V(G)$ is heavy, then*

(a) $\text{Gap}(v) \leq (\delta/4)d(v)$, *and*
(b) $ch(v) > \frac{\text{Save}_L(v)+\varepsilon\log^{10}k}{1+\delta/18}$.

Proof. First we prove (a). Since v is not extremely heavy, $ch(v) \leq 9\varepsilon \cdot d(v)$. Since $ch(v) \geq (36\varepsilon/\delta)\,\text{Gap}(v)$, we have $\text{Gap}(v) \leq (\delta/4)d(v)$, as desired.

Now we prove (b). Since v is heavy, $ch(v) > (36\varepsilon/\delta)\,\mathrm{Gap}(v)$. Hence, $2\varepsilon\,\mathrm{Gap}(v) < \delta ch(v)/18$. Therefore

$$ch(v) > \mathrm{Save}_L(v) - \delta ch(v)/18 + \varepsilon \log^{10} k + 7\varepsilon(k - |L(v)|),$$

and the result follows by rearranging terms. $\square$

The heavy vertices in D will send charge to their neighbors in S_∞. Assuming ε is small enough, Proposition 5.4.8 (b) implies that these vertices will have plenty of charge to send to these neighbors. Proposition 5.4.8 (a), in conjunction with Lemma 5.4.12, implies that heavy vertices with many very lordly neighbors are aberrant. Thus, heavy vertices in D do not have to send too much charge to very lordly neighbors.

The next proposition implies that if v is a normal vertex, then $\mathrm{Save}_L(v) + \varepsilon \log^{10} k$ is a fraction of $\mathrm{Gap}(v)$. Thus, the main result in [87, Theorem 1.7] implies that if every vertex is normal, then for ε small enough, the graph is colorable.

Proposition 5.4.9. *If $v \in V(G)$ is normal, then*

$$\mathrm{Gap}(v) \geq \frac{\mathrm{Save}_L(v) + \varepsilon \log^{10} k}{\varepsilon(36/\delta + 2)}.$$

Proof. Since v is normal, $ch(v) \leq (36\varepsilon/\delta)\,\mathrm{Gap}(v)$, and since $|L(v)| \leq k$, we have $ch(v) \leq \mathrm{Save}_L(v) - 2\varepsilon\,\mathrm{Gap}(v) + \varepsilon \log^{10} k$. Therefore $\mathrm{Save}_L(v) + \varepsilon \log^{10} k \leq (36\varepsilon/\delta)\,\mathrm{Gap}(v) + 2\varepsilon\,\mathrm{Gap}(v)$, and the result follows by rearranging terms. $\square$

5.4.3 Structure

In this subsection we prove Lemma 5.4.11, which implies that every normal vertex not in S_∞ is either very lordly or has many heavy neighbors. In the latter case the vertex is in D, and the charge it receives from its heavy neighbors compensates for the charge it sends to its neighbors not in D. We also prove Lemma 5.4.12, which implies that that a vertex in D does not have too many very lordly neighbors. In Section 5.4.4, we use these two lemmas to show that after redistributing charges, the vertices in D all have positive charge.

As mentioned, we show that normal vertices in D have many heavy neighbors. The following makes this precise.

Definition 5.4.10. A vertex $v \in V(G)$ is *sponsored* if it has at least $d(v)/2$ heavy neighbors u with $\mathrm{Save}_L(u) \geq \frac{\varepsilon(5/2-\alpha)}{\delta/(36+2\delta)-\varepsilon}\,\mathrm{Gap}(v)$ and $d(u) \leq \frac{1+\alpha\delta/4}{1-\varepsilon'} d(v)$.

Lemma 5.4.11. *Let $v \in V(G)$ be a normal vertex.*

(a) If $\mathrm{Gap}(v) \geq (3\delta/4)d(v)$ and ε satisfies (5.11) and (5.15), then v is either aberrant, egalitarian-sparse, or very lordly.
(b) If $\mathrm{Gap}(v) \in [(\delta/4)d(v), (3\delta/4)d(v))$ and ε satisfies (5.11), (5.12), (5.13), and (5.15), then v is either aberrant, bipartite-sparse, or egalitarian-sparse.
(c) If $\mathrm{Gap}(v) \leq (\delta/4)d(v)$ and ε satisfies (5.11), (5.12), (5.13), and (5.16), then v is either aberrant, bipartite-sparse, egalitarian-sparse, or sponsored.

The proof of Lemma 5.4.11 comprises most of this subsection. First, we state the other important lemma of this subsection.

Lemma 5.4.12. *Let $v \in V(G)$ be a vertex such that $\mathrm{Gap}(v) \leq (\delta/4)d(v)$ and v has at least*

$$\frac{\mathrm{Save}_L(v) + \varepsilon \log^{10} k}{\mathrm{aber}(\alpha, \delta, \varepsilon')} + \mathrm{Gap}(v)$$

very lordly neighbors. If α satisfies (5.17), then v is aberrant.

The following lemma will be used to prove Lemma 5.4.11.

Lemma 5.4.13. *Let $v \in V(G)$ be a normal vertex.*

(a) If ε satisfies (5.11) and either $|\mathrm{Lord}(v)| \geq \mathrm{Gap}(v)/4$ or $|\text{Slightly-Lord}(v)| \geq d(v)/4$, then v is aberrant or slightly aberrant.
(b) If $\mathrm{Gap}(v) < (3\delta/4)d(v)$, $|\mathrm{Lord}(v)| < \mathrm{Gap}(v)/4$, ε satisfies (5.12) and (5.13) and $|\mathrm{Subserv}(v)| \geq \mathrm{Gap}(v)/4$, then v is bipartite-sparse.
(c) If $\mathrm{Gap}(v) \geq (\delta/4)d(v)$, $|\mathrm{Subserv}(v) \cup \mathrm{Lord}(v)| < \mathrm{Gap}(v)/2$, and ε satisfies (5.14) and (5.15), then v is egalitarian-sparse.
(d) If $\mathrm{Gap}(v) \leq (\delta/4)d(v)$, $|\mathrm{Subserv}(v) \cup \mathrm{Lord}(v)| < \mathrm{Gap}(v)/2$, and ε satisfies (5.14) and (5.16), then v is either egalitarian-sparse or sponsored.

Proof of Lemma 5.4.13 (a). First, suppose $|\mathrm{Lord}(v)| \geq \mathrm{Gap}(v)/4$. Hence, since v is normal, by Proposition 5.4.9,

$$|\mathrm{Lord}(v)| \geq \frac{\mathrm{Save}_L(v) + \varepsilon \log^{10} k}{4\varepsilon(36/\delta + 2)}.$$

Since ε satisfies (5.11), the previous inequality implies that

$$|\mathrm{Lord}(v)| \geq \frac{\mathrm{Save}_L(v) + \varepsilon \log^{10} k}{\mathrm{aber}(\alpha, \varepsilon')},$$

so v is aberrant, as desired.

Therefore we may assume $|\text{Slightly-Lord}(v)| \geq d(v)/4 = (\text{Gap}(v)d(v))/(4\,\text{Gap}(v))$. By Proposition 5.4.9,

$$|\text{Slightly-Lord}(v)| \geq \left(\frac{d(v)}{\text{Gap}(v)}\right)\left(\frac{\text{Save}_L(v) + \varepsilon \log^{10} k}{4\varepsilon(36/\delta + 2)}\right).$$

Since ε satisfies (5.11), the previous inequality implies that

$$|\text{Slightly-Lord}(v)| \geq \left(\frac{d(v)}{\text{Gap}(v)}\right)\left(\frac{\text{Save}_L(v) + \varepsilon \log^{10} k}{\text{aber}(\alpha, \varepsilon')}\right),$$

so v is slightly aberrant, as desired. □

Proof of Lemma 5.4.13 (b). Let B be a maximum cardinality clique in $G[N(v) \setminus \text{Lord}(v)]$. Note that $|B| \geq \omega(v) - 1 - |\text{Lord}(v)|$. Since $\text{Gap}(v) < (3\delta/4)d(v)$, we have $\omega(v) - 1 \geq (1 - 3\delta/4)d(v)$, and since $|\text{Lord}(v)| < \text{Gap}(v)/4 < (3\delta/16)d(v)$, we have

$$|B| \geq (1 - 15\delta/16)d(v). \tag{5.20}$$

Let $A = \text{Subserv}(v)$. We claim that (A, B) is a half-egalitarian bipartition for v. By Proposition 5.4.7 (b), $A \subseteq \text{Egal}_\sigma(v)$. Since $\varepsilon \leq \frac{\delta}{11(16-15\delta)}$, by Proposition 5.4.7 (a) and (5.20), for each $u \in B$, we have $|L(u)| \geq |B|/(1 - 11\varepsilon) \geq (1 - \delta)d(v) \geq (1 - \delta)|L(v)|$. Hence, $B \subseteq \text{Egal}(v)$. By Proposition 5.4.7 (a), for each $u \in A$, we have $d(u) \leq (1 - \delta)|L(v)|/(1 - \varepsilon') \leq (1 - \delta)d(v)/(1 - \varepsilon')$. Hence, by (5.20), each $u \in A$ has at least $(1 - 15\delta/16 - (1 - \delta)/(1 - \varepsilon'))d(v) = (\frac{\delta - \varepsilon'}{1 - \varepsilon'} - \frac{15\delta}{16})d(v)$ non-neighbors in B, as required. Therefore (A, B) is a half-egalitarian bipartition for v, as claimed.

Since $|A| = |\text{Subserv}(v)| \geq \text{Gap}(v)/4$, by Proposition 5.4.9,

$$|A| \geq \frac{\text{Save}_L(v) + \varepsilon \log^{10} k}{4\varepsilon(36/\delta + 2)}.$$

Since $\varepsilon \leq \frac{\text{bipart-sparse}(\alpha,\delta,\varepsilon',\sigma)}{4(36/\delta+2)}$, the previous inequality implies that

$$|A| \geq \left(\frac{\text{Save}_L(v) + 11\varepsilon' \log^{10} 2k}{K^2(1 - \varepsilon')}\right) \Big/ \left(\left(\frac{\delta - \varepsilon'}{1 - \varepsilon'} - \frac{15\delta}{16}\right)\left(\frac{1 - \sigma - \delta}{(1 + \alpha)(1 - \delta)}\right)\right).$$

Therefore v is bipartite-sparse, as desired. □

Lemma 5.4.13 (a) and (b) together imply that if a vertex $v \in D$ satisfies $\text{Gap}(v) < (3\delta/4)d(v)$, then v has many egalitarian neighbors. Our next goal is to prove that since these vertices are not egalitarian-sparse, they have many heavy neighbors. We use the fact that the egalitarian neighbors of a vertex do not induce a dense subgraph with respect to L, so it will be useful to bound the value of Save_L, as in the next two propositions.

Proposition 5.4.14. *If u is an egalitarian neighbor of a vertex v (i.e. $u \in \text{Egal}(v)$), then*

$$\text{Save}_L(u) \leq \varepsilon' \left(\frac{1+\alpha}{1-\varepsilon'} d(v) + 1 \right).$$

Proof. Since $u \in \text{Egal}(v)$, by Proposition 5.4.7 (a), $d(u) \leq (1+\alpha)|L(v)|/(1-\varepsilon') \leq (1+\alpha)d(v)/(1-\varepsilon')$ and $\text{Save}_L(v) \leq \varepsilon'((1+\alpha)d(v)/(1-\varepsilon')+1)$, as desired. □

Proposition 5.4.15. *Let $v \in V(G)$ such that $|\text{Subserv}(v) \cup \text{Lord}(v)| < \text{Gap}(v)/2$. If M is a maximum matching in $\overline{G[\text{Egal}(v)]}$, then $|M| \geq \text{Gap}(v)/4$. Furthermore, if $u \in \text{Egal}(v) \setminus (V(M) \cup \text{Slightly-Lord}(v))$ is normal and ε satisfies (5.14), then*

$$\text{Save}_L(u) \leq \frac{\varepsilon(5/2-\alpha)}{\delta/(36+2\delta)-\varepsilon} \text{Gap}(v).$$

Proof. Let M be a maximum matching in $\overline{G[\text{Egal}(v)]}$. By the choice of M, the vertices of $\text{Egal}(v) \setminus V(M)$ form a clique. Therefore

$$2|M| + |\text{Subserv}(v) \cup \text{Lord}(v)| + \omega(v) - 1 \geq d(v).$$

Similarly, since $u \in \text{Egal}(v) \setminus V(M)$,

$$\omega(u) \geq d(v) + 1 - 2|M| - |\text{Subserv}(v) \cup \text{Lord}(v)|. \tag{5.21}$$

Hence, since $|\text{Subserv}(v) \cup \text{Lord}(v)| < \text{Gap}(v)/2$,

$$|M| \geq \text{Gap}(v)/4,$$

as desired.

Since no clique in $G[\text{Egal}(v)]$ contains an edge in M, $\omega(G[\text{Egal}(v)]) \leq |\text{Egal}(v)| - |M|$. Note that for any $H \subseteq G[N(v) \cup \{v\}]$, $|V(H)| - \omega(H) \leq \text{Gap}(v)$. Hence,

$$|M| \leq \text{Gap}(v). \tag{5.22}$$

By (5.21) and (5.22), since $|\mathrm{Subserv}(v) \cup \mathrm{Lord}(v)| < \mathrm{Gap}(v)/2$,

$$\omega(u) \geq d(v) + 1 - (5/2)\,\mathrm{Gap}(v). \tag{5.23}$$

Since $|L(u)| = d(u) + 1 - \mathrm{Save}_L(u) = \mathrm{Gap}(u) + \omega(u) - \mathrm{Save}_L(u)$, by (5.23),

$$|L(u)| \geq \mathrm{Gap}(u) - \mathrm{Save}_L(u) + d(v) + 1 - (5/2)\,\mathrm{Gap}(v).$$

Since $u \in \text{Slightly-Lord}(v)$, we have $|L(u)| \leq |L(v)| + \alpha\,\mathrm{Gap}(v)$. Hence,

$$|L(v)| + (5/2 - \alpha)\,\mathrm{Gap}(v) \geq \mathrm{Gap}(u) - \mathrm{Save}_L(u) + d(v) + 1.$$

Since $d(v) + 1 - |L(v)| = \mathrm{Save}_L(v)$, we have

$$\mathrm{Gap}(u) - \mathrm{Save}_L(u) \leq (5/2 - \alpha)\,\mathrm{Gap}(v) - \mathrm{Save}_L(v). \tag{5.24}$$

Since u is normal, by Proposition 5.4.9,

$$\mathrm{Gap}(u) \geq \frac{\mathrm{Save}_L(v)}{\varepsilon(36/\delta + 2)}. \tag{5.25}$$

Combining (5.24) and (5.25), we obtain

$$\mathrm{Save}_L(u)\left(\frac{1}{\varepsilon(36/\delta + 2)} - 1\right) \leq (5/2 - \alpha)\,\mathrm{Gap}(v) - \mathrm{Save}_L(v) \leq (5/2 - \alpha)\,\mathrm{Gap}(v).$$

By rearranging terms in the previous inequality, since (5.14) holds, we obtain the desired bound on $\mathrm{Save}_L(u)$. □

Now we can prove Lemma 5.4.13 (c) and (d).

Proof of Lemma 5.4.13 (c). Let $H = G[\mathrm{Egal}(v)]$. By Proposition 5.4.15, there is a maximum matching M in H such that $|M| \geq \mathrm{Gap}(v)/8$. Since G contains no dense subgraph with respect to L,

$$|E(\overline{H})| \geq |M|(|V(H)| - |M|) - \sum_{u \in V(H)} \mathrm{Save}_L(u). \tag{5.26}$$

Therefore by (5.26) and Proposition 5.4.14,

$$\begin{aligned} |E(\overline{H})| &\geq |M|(|V(H)| - |M|) - |V(H)|\varepsilon'((1+\alpha)d(v)/(1-\varepsilon') + 1) \\ &= |V(H)|(|M| - \varepsilon'((1+\alpha)d(v)/(1-\varepsilon') + 1)) - |M|^2. \end{aligned}$$

Since $|V(H)| \geq d(v) - \mathrm{Gap}(v)/2$ and $|M| \geq \mathrm{Gap}(v)/4$, the previous inequality implies that

$$\begin{aligned}|E(\overline{H})| &\geq (d(v) - \mathrm{Gap}(v)/2)(\mathrm{Gap}(v)/4 - \varepsilon'((1+\alpha)d(v)/(1-\varepsilon')+1)) - (\mathrm{Gap}(v)/4)^2 \\ &\geq \mathrm{Gap}(v)d(v)\left(1/4 - (1/2)\varepsilon'((1+\alpha)/(1-\varepsilon')+1)\right) - (3/16)\,\mathrm{Gap}(v)^2 \\ &\quad - d(v)^2\left(\varepsilon'((1+\alpha)/(1-\varepsilon')+1)\right).\end{aligned}$$

Since $\mathrm{Gap}(v) \leq d(v)$, we have $(3/16)\,\mathrm{Gap}(v)^2 \leq (3/16)\,\mathrm{Gap}(v)d(v)$. Therefore since $\mathrm{Gap}(v) \geq (\delta/4)d(v)$,

$$\begin{aligned}|E(\overline{H})| &\geq \mathrm{Gap}(v)d(v)\left(1/16 - (1/2)\varepsilon'((1+\alpha)/(1-\varepsilon')+1)\right) - d(v)^2\left(\varepsilon'((1+\alpha)/(1-\varepsilon')+1)\right) \\ &\geq d(v)^2\left((\delta/4)\left(1/16 - (1/2)\varepsilon'((1+\alpha)/(1-\varepsilon')+1)\right) - (\varepsilon'((1+\alpha)/(1-\varepsilon')+1))\right) \\ &= d(v)^2\left(\delta/64 - 11\varepsilon((\delta/8)+1)((1+\alpha)/(1-\varepsilon')+1)\right).\end{aligned}$$

Hence, by Proposition 5.4.9,

$$|E(\overline{H})| \geq d(v)\left(\frac{\mathrm{Save}_L(v) + \varepsilon\log^{10} k}{\varepsilon(36/\delta)+2}\right)\left(\delta/64 - 11\varepsilon((\delta/8)+1)((1+\alpha)/(1-\varepsilon')+1)\right).$$

Since ε satisfies (5.15), v is egalitarian-sparse, as desired. □

Proof of Lemma 5.4.13 (d). By Proposition 5.4.15, there is a maximum matching M in $\overline{G[\mathrm{Egal}(v)]}$ such that $|M| \geq \mathrm{Gap}(v)/4$. Let H be the graph induced by $\overline{G}$ on vertices in $V(M)$ and vertices $\mathrm{Egal}(v) \setminus (V(M) \cup \text{Slightly-Lord}(v))$ such that $\mathrm{Save}_L(u) \leq \frac{\varepsilon(5/2-\alpha)}{\delta(36+2\delta)-\varepsilon}\mathrm{Gap}(v)$. By Proposition 5.4.15, since (5.14) holds, if $u \in \mathrm{Egal}(v) \setminus V(H)$, then either $u \in \text{Slightly-Lord}(v)$ or u is heavy. By the choice of H, if $u \in \mathrm{Egal}(v) \setminus V(H)$ is heavy, then $\mathrm{Save}_L(u) \geq \frac{\varepsilon(5/2-\alpha)}{\delta/(36+2\delta)-\varepsilon}\mathrm{Gap}(v)$. Hence, since $|\text{Slightly-Lord}(v)| < d(v)/4$,

$$|V(H)| \geq d(v)/4, \tag{5.27}$$

or else v has at least $d(v)/2$ heavy egalitarian neighbors with $\mathrm{Save}_L(u) \geq \frac{\varepsilon(5/2-\alpha)}{\delta(36+2\delta)-\varepsilon}\mathrm{Gap}(v)$ and $d(u) \leq |L(u)|/(1-\varepsilon') \leq \frac{|L(v)|+\alpha\,\mathrm{Gap}(v)}{1-\varepsilon'} \leq \frac{1+\alpha\delta/4}{1-\varepsilon'}d(v)$, as desired.

Since G contains no dense subgraph with respect to L,

$$|E(\overline{H})| \geq |M|(|V(H)| - |M|) - \sum_{u\in V(H)} \mathrm{Save}_L(u). \tag{5.28}$$

By the choice of H,

$$\sum_{u\in V(H)\setminus V(M)} \mathrm{Save}_L(u) \leq (|V(H)| - 2|M|)\left(\frac{\varepsilon(5/2-\alpha)}{\delta(36+2\delta)-\varepsilon}\right)\mathrm{Gap}(v). \tag{5.29}$$

By Proposition 5.4.14,

$$\sum_{u\in V(M)} \mathrm{Save}(u) \le \varepsilon' 2|M| \left(\frac{1+\alpha}{1-\varepsilon'} d(v) + 1\right). \tag{5.30}$$

By (5.28), (5.29), and (5.30),

$$|E(\overline{H})| \ge (|V(H)| - |M|)\left(|M| - \left(\frac{\varepsilon(5/2-\alpha)}{\delta(36+2\delta)-\varepsilon}\right)\mathrm{Gap}(v)\right) - \varepsilon' 2|M|\left(\frac{1+\alpha}{1-\varepsilon'}d(v)+1\right). \tag{5.31}$$

By (5.27) and (5.31), since $\mathrm{Gap}(v)/4 \le |M| \le \mathrm{Gap}(v) \le (\delta/4)d(v)$,

$$\begin{aligned}|E(\overline{H})| &\ge \left(\left(\frac{1-\delta}{4}\right)\left(\frac{1}{4} - \frac{\varepsilon(5/2-\alpha)}{\delta(36+2\delta)-\varepsilon}\right) - 22\varepsilon\left(\frac{1+\alpha}{1-\varepsilon'}+1\right)\right)\mathrm{Gap}(v)d(v)\\ &= \left(\frac{1-\delta}{16} - \varepsilon\left(\frac{5/2-\alpha}{\delta(36+2\delta)-\varepsilon} + 22\frac{2+\alpha-\varepsilon'}{1-\varepsilon'}\right)\right)\mathrm{Gap}(v)d(v).\end{aligned}$$

Hence, by Proposition 5.4.9, since ε satisfies (5.16),

$$|E(\overline{H})| \ge d(v)\frac{\mathrm{Save}_L(v) + \varepsilon\log^{10} k}{\text{egal-sparse}(\alpha,\delta,\varepsilon')}.$$

Therefore v is egalitarian-sparse, as desired. ☐

Now we use Lemma 5.4.13 to prove Lemma 5.4.11.

Proof of Lemma 5.4.11. First we prove (a). Assume v is not very lordly. Hence, $|\mathrm{Subserv}(v)| \le \mathrm{Gap}(v)/4$. By Lemma 5.4.13 (a), since (5.11) holds, we may assume $|\mathrm{Lord}(v)| < \mathrm{Gap}(v)/4$, or else v is aberrant, as desired. Hence, $|\mathrm{Subserv}(v) \cup \mathrm{Lord}(v)| < \mathrm{Gap}(v)/2$. Therefore, by Lemma 5.4.13 (c), since (5.15) holds, v is egalitarian-sparse, as desired.

Next we prove (b). By Lemma 5.4.13 (a), since (5.11) holds, we may assume $|\mathrm{Lord}(v)| < \mathrm{Gap}(v)/4$, or else v is aberrant, as desired. By Lemma 5.4.13 (b), since ε satisfies (5.12) and (5.13), we may assume $|\mathrm{Subserv}(v)| \le \mathrm{Gap}(v)/4$, or else v is bipartite-sparse, as desired. Therefore $|\mathrm{Subserv}(v) \cup \mathrm{Lord}(v)| < \mathrm{Gap}(v)/2$. Hence, by Lemma 5.4.13 (c), since (5.15) holds, v is egalitarian-sparse, as desired.

Now we prove (c). By Lemma 5.4.13 (a), since (5.11) holds, we may assume that $|\mathrm{Lord}(v)| < \mathrm{Gap}(v)/4$ and also $|\text{Slightly-Lord}(v)| < d(v)/4$, or else v is aberrant or slightly aberrant, as desired. By Lemma 5.4.13 (b), since ε satisfies (5.12) and (5.13), we may assume $|\mathrm{Subserv}(v)| \le \mathrm{Gap}(v)/4$, or else v is bipartite-sparse, as desired. Therefore $|\mathrm{Subserv}(v) \cup \mathrm{Lord}(v)| < \mathrm{Gap}(v)/2$. By Lemma 5.4.13 (d), since ε satisfies (5.16), v is either egalitarian-sparse or sponsored, as desired. ☐

In the remainder of this subsection, we prove Lemma 5.4.12. First we need the following proposition, which implies that many very lordly neighbors of a vertex v that is heavy or sponsored are lordlier.

Proposition 5.4.16. *Let $v \in V(G)$, and let $u \in N(v)$ be a very lordly vertex such that $\omega(u) \geq (1-\delta/4)d(v)+1$. If α satisfies (5.17), then $u \in \mathrm{Lord}(v)$.*

Proof. Since $d(u)+1 = \mathrm{Gap}(u)+\omega(u)$, $\mathrm{Gap}(u) \geq (3\delta/4)d(u)$, and $\omega(u) \geq (1-\delta/4)d(v)$, we have $d(u)+1 \geq (3\delta/4)d(u)+(1-\delta/4)d(v)+1$. Hence,

$$(1-3\delta/4)d(u) \geq (1-\delta/4)d(v) \geq (1-\delta/4)|L(v)|.$$

By Proposition 5.4.7 (a) and the previous inequality,

$$|L(u)| \geq \frac{(1-\varepsilon')(1-\delta/4)}{1-3\delta/4}|L(v)|.$$

Note that

$$\frac{(1-\varepsilon')(1-\delta/4)}{1-3\delta/4} = 1 + \frac{\delta(2+\varepsilon')-4\varepsilon'}{4-3\delta}.$$

Since α satisfies (5.17), by the previous two inequalities, $|L(u)|(1+\alpha)|L(v)|$, so $u \in \mathrm{Lord}(v)$, as desired. □

Proof of Lemma 5.4.12. Since $\mathrm{Gap}(v) \leq (\delta/4)d(v)$, we have $\omega(v) \geq (1-\delta/4)d(v)+1$. By assumption, v has at least $\frac{\mathrm{Save}_L(v)+\varepsilon\log^{10}k}{\mathrm{aber}(\alpha,\delta,\varepsilon')}$ very lordly neighbors u such that $\omega(u) \geq \omega(v)$. Hence, by Proposition 5.4.16, each such very lordly neighbor u is in $\mathrm{Lord}(v)$. Therefore v is aberrant, as desired. □

5.4.4 Discharging

In this subsection we prove Lemma 5.4.6 using discharging. We redistribute the charges sequentially according to the following rules. Let $v \in D$.

(R1) If v is heavy, then v sends 9ε charge to each neighbor not in D. Denote the new charges ch_1.

(R2) If v is heavy, then v sends $ch(v)/(2(|N(v) \cap D|))$ to each neighbor in D. Denote the new charges ch_2.

If v is normal, then

(R3) v sends 9ε to each neighbor in S_∞, and

(R4) v sends 9ε to each neighbor in $\mathcal{L}$.

Denote the final charges ch_*.

Proposition 5.4.17. *If $v \in S_\infty \cup \mathcal{L}$, then the final charge is*

$$ch_*(v) = ch(v) + 9\varepsilon(|N(v) \cap D|).$$

Proof. If $v \in S_\infty$, then v receives 9ε charge from each neighbor in D under R1 and R3. If $v \in \mathcal{L}$, then v receives 9ε charge from each neighbor in D under R1 and R4. □

Our aim is to show that for each vertex v in D, we have $ch_*(v) > 0$. The next lemma implies this result for heavy vertices in D.

Lemma 5.4.18. *Let $v \in D$ be heavy. If α satisfies (5.17) and ε satisfies (5.18), then $ch_1(v) > ch(v)/2$.*

Proof. It suffices to show that v sends less than $ch(v)/2$ charge under R1. Since $v \notin S_\infty$, at most $\frac{\mathrm{Save}_L(v)+\varepsilon\log^{10} k}{(1-K)(1-\varepsilon')}$ neighbors of v are in S_∞, and since α satisfies (5.17), by Proposition 5.4.8 (a) and Lemma 5.4.12, v has at most $\frac{\mathrm{Save}_L(v)+\varepsilon\log^{10} k}{\mathrm{aber}(\alpha,\delta,\varepsilon')} + \mathrm{Gap}(v)$ neighbors in L. Therefore v sends at most

$$9\varepsilon\left(\left(\mathrm{Save}_L(v) + \varepsilon\log^{10} k\right)\left(\frac{1}{(1-K)(1-\varepsilon')} + \frac{1}{\mathrm{aber}(\alpha,\delta,\varepsilon')}\right) + \mathrm{Gap}(v)\right)$$

charge under R1.

Since $ch(v) \geq (36\varepsilon/\delta)\,\mathrm{Gap}(v)$, we have $9\varepsilon\,\mathrm{Gap}(v) \leq (\delta/4)ch(v) < ch(v)/4$. By Proposition 5.4.8 (b) and (5.18),

$$9\varepsilon\left(\mathrm{Save}_L(v) + \varepsilon\log^{10} k\right)\left(\frac{1}{(1-K)(1-\varepsilon')} + \frac{1}{\mathrm{aber}(\alpha,\delta,\varepsilon')}\right) \leq \frac{\mathrm{Save}_L(v) + \varepsilon\log^{10} k}{4(1+\delta/18)} < ch(v)/4.$$

Therefore v sends at most $ch(v)/2$ charge under R1, as desired. □

Now we show that normal vertices in D also have positive final charge.

Lemma 5.4.19. *Let $v \in D$ be normal. If ε satisfies (5.11), (5.12), (5.13), (5.14), (5.15), and (5.16), and δ satisfies (5.19), then $ch_*(v) > 0$.*

Proof. By Lemma 5.4.11, $\mathrm{Gap}(v) < (\delta/4)d(v)$ and v is sponsored, that is there is set X of at least $d(v)/2$ heavy neighbors u with $\mathrm{Save}_L(u) \geq \frac{\varepsilon(5/2-\alpha)}{\delta/(36+2\delta)-\varepsilon}\,\mathrm{Gap}(v)$.

Since $v \notin S_\infty$, at most $\frac{\mathrm{Save}_L(v)+\varepsilon\log^{10}k}{(1-K)(1-\varepsilon')}$ neighbors of v are in S_∞. In particular,

$$|X \cap S_\infty| \leq \frac{\mathrm{Save}_L(v)+\varepsilon\log^{10}k}{(1-K)(1-\varepsilon')}. \tag{5.32}$$

By Proposition 5.4.9, since $\mathrm{Gap}(v) < (\delta/4)d(v)$,

$$\mathrm{Save}_L(v)+\varepsilon\log^{10}k < \varepsilon(36/\delta+2)(\delta/4)d(v). \tag{5.33}$$

Combining (5.32) and (5.33), since $|X| \geq d(v)/2$,

$$|X \setminus S_\infty| > \left(\frac{1}{2} - \frac{\varepsilon(36+2\delta)}{4(1-K)(1-\varepsilon')}\right)d(v). \tag{5.34}$$

By Proposition 5.4.8 (b), each vertex $u \in X \setminus S_\infty$ sends at least

$$\begin{aligned}\frac{\mathrm{Save}_L(u)}{2(1+\delta/18)d(u)} &\geq \left(\frac{\varepsilon(5/2-\alpha)}{\delta/(36+2\delta)-\varepsilon}\right)\left(\frac{\mathrm{Gap}(v)}{2(1+\delta/18)d(u)}\right)\\ &\geq \left(\frac{\varepsilon(5/2-\alpha)(1-\varepsilon')}{(\delta/(36+2\delta)-\varepsilon)(2(1+\delta/18))(1+\alpha\delta/4)}\right)\left(\frac{\mathrm{Gap}(v)}{d(v)}\right) \quad (5.35)\end{aligned}$$

charge under R2.

By (5.34) and (5.35), v receives greater than

$$\left(\frac{1}{2} - \frac{\varepsilon(36+2\delta)}{4(1-K)(1-\varepsilon')}\right)\left(\frac{\varepsilon(5/2-\alpha)(1-\varepsilon')(36+2\delta)/\delta}{(1-\varepsilon(36+2\delta)/\delta)(2(1+\delta/18))(1+\alpha\delta/4)}\right)\mathrm{Gap}(v)$$

charge under R2. Since $ch(v) \geq -2\varepsilon\,\mathrm{Gap}(v)$,

$$ch_2(v) > \varepsilon\left(\left(\frac{1}{2} - \frac{\varepsilon(36+2\delta)}{4(1-K)(1-\varepsilon')}\right)\left(\frac{(5/2-\alpha)(1-\varepsilon')(36+2\delta)/\delta}{(1-\varepsilon(36+2\delta)/\delta)(2(1+\delta/18))(1+\alpha\delta/4)}\right) - 2\right)\mathrm{Gap}(v). \tag{5.36}$$

Since α satisfies (5.17), by Proposition 5.4.8 (a) and Lemma 5.4.12, v has at most $\frac{\mathrm{Save}_L(v)+\varepsilon\log^{10}k}{\mathrm{aber}(\alpha,\delta,\varepsilon')} + \mathrm{Gap}(v)$ neighbors in L. Since v at most $\frac{\mathrm{Save}_L(v)+\varepsilon\log^{10}k}{(1-K)(1-\varepsilon')}$ neighbors of v are in S_∞, v sends at most

$$9\varepsilon\left(\left(\mathrm{Save}_L(v)+\varepsilon\log^{10}k\right)\left(\frac{1}{(1-K)(1-\varepsilon')} + \frac{1}{\mathrm{aber}(\alpha,\delta,\varepsilon')}\right) + \mathrm{Gap}(v)\right)$$

charge under R3 and R4. Hence, by Proposition 5.4.9,

$$ch_2(v) - ch_*(v) \le 9\varepsilon\left(1 + \varepsilon(36/\delta + 2)\left(\frac{1}{(1-K)(1-\varepsilon')} + \mathrm{aber}(\alpha,\delta,\varepsilon')^{-1}\right)\right)\mathrm{Gap}(v). \tag{5.37}$$

By combining (5.36) and (5.37),

$$\begin{aligned}\frac{ch_*(v)}{\varepsilon\,\mathrm{Gap}(v)} > \left(\frac{1}{2} - \frac{\varepsilon(36+2\delta)}{4(1-K)(1-\varepsilon')}\right)\left(\frac{(5/2-\alpha)(1-\varepsilon')(36+2\delta)/\delta}{(1-\varepsilon(36+2\delta)/\delta)(2(1+\delta/18))(1+\alpha\delta/4)}\right)\\ -2-9\left(1+\varepsilon(36/\delta+2)\left(\frac{1}{(1-K)(1-\varepsilon')}+\mathrm{aber}(\alpha,\delta,\varepsilon')^{-1}\right)\right).\end{aligned} \tag{5.38}$$

By (5.19) and (5.38), $ch_*(v) > 0$, as desired. □

We can finally prove Lemma 5.4.6.

Proof of Lemma 5.4.6. Let $\varepsilon = 2.6 \cdot 10^{-10}$. Recall that $\sigma = 2/3$, $\varepsilon' = 11\varepsilon$ and $K = .999 \cdot e^{-1/(1-\varepsilon')}$. Let $\delta = \frac{1}{100}$ and $\alpha = \frac{\delta(2+\varepsilon')-4\varepsilon'}{4-3\delta}$. Note that $\alpha, \delta, \varepsilon, \varepsilon', K$, and σ satisfy (5.2)-(5.5) and (5.11)-(5.19).

Recall that

$$\sum_{v\in V(G)} ch_*(v) = \sum_{v\in V(G)} ch(v) \le 0. \tag{5.39}$$

By Lemmas 5.4.18 and 5.4.19, if $v \in D$, then $ch_*(v) > 0$. Therefore $\sum_{v\in V(G)\setminus D} ch_*(v) > 0$, and thus $D \subsetneq V(G)$. Note that for each $v \in V(G)\setminus D$, we have

$$\begin{aligned}ch_*(v) &= ch(v) + 9\varepsilon|N(v)\cap D|\\ &\ge \mathrm{Save}_L(v) + \varepsilon\log^{10} k - 2\varepsilon\,\mathrm{Gap}_{G-D}(u) + 7\varepsilon(k - |L(v)| + |N(v)\cap D|).\end{aligned} \tag{5.40}$$

Combining (5.39) and (5.40), we have

$$\sum_{v\in V(G)\setminus D}(\mathrm{Save}_L(v) + \varepsilon\log^{10} k) \le \sum_{v\in V(G)\setminus D}(2\varepsilon\,\mathrm{Gap}_{G-D} - 7\varepsilon(k - |L(v)| + |N(v)\cap D|),$$

as desired. □

5.4.5 Proof of Lemma 5.4.1

We conclude this section with the proof of Lemma 5.4.1. First we prove Lemma 5.4.2 using Lemmas 5.4.4 and 5.4.6.

Proof of Lemma 5.4.2. Let $D, \mathcal{L}$, and S_∞ be as defined in Lemma 5.4.6. By Lemma 5.4.4, we may assume G has no extremely heavy vertices, or else (b) holds, as desired. Therefore, by Lemma 5.4.6, $D = \varnothing$, or else (b) holds, as desired.. We claim that $\mathcal{L} = \varnothing$. Suppose not, and let $v \in \mathcal{L}$ such that $|L(v)|$ is minimum. By the choice of v, $\mathrm{Subserv}(v) \subseteq S_\infty$. Since v is very lordly, $|\mathrm{Subserv}(v)| \geq \mathrm{Gap}(v)/4$. By Proposition 5.4.9, v has at least $\frac{\mathrm{Save}_L(v)+\varepsilon\log^{10}k}{4\varepsilon(36/\delta+2)} \geq \frac{\mathrm{Save}_L(v)+\varepsilon\log^{10}k}{(1-K)(1-\varepsilon')}$ neighbors in S_∞, so $v \in S_\infty$, a contradiction. Hence, $\mathcal{L} = \varnothing$, as claimed. Therefore $S_\infty = V(G)$.

Define an ordering $\prec$ of $V(G)$ as follows. If $u \in S_i$ and $v \in S_j$ such that $i < j$, let $u \prec v$. By the construction of the sets S_i, every vertex $v \in V(G)$ is either aberrant, slightly aberrant, egalitarian-sparse, bipartite-sparse, or has at least $\frac{\mathrm{Save}_L(v)+\varepsilon\log^{10}k}{(1-K)(1-\varepsilon')}$ neighbors u such that $u \prec v$. By Proposition 5.4.7 (b), $d(v) \geq k/3$ for each vertex v, and since $|L(v)| \leq k$ and $(1-\varepsilon')d(v) \leq |L(v)| \leq d(v)$ for each vertex v, (G, L) is $(2k, \varepsilon')$-bounded. Thus, G is saved with respect to L and k, as desired. □

Proof of Lemma 5.4.1. We may assume G is not saved with respect to L and k or else the lemma holds. Hence, by Lemma 5.4.2, there is a nonempty set $D \subsetneq V(G)$ such that

$$\sum_{v\in V(G-D)} \mathrm{Save}_L(v) \leq \sum_{v\in V(G-D)} 2\varepsilon\,\mathrm{Gap}_{G-D}(v) - \varepsilon\log^{10}k.$$

Subject to that, we choose D to have maximum cardinality.

We claim that $G - D$ is saved with respect to L and k. To that end, suppose ϕ is an L-coloring of $G[D]$, and let $L'(v) = L(v) \setminus (\cup_{u\in N(v)\cap D}\phi(u))$ for each vertex $v \in V(G-D)$. Note that $\mathrm{Save}_{L'}(v) \leq \mathrm{Save}_L(v)$ for each vertex v. We assume that equality holds for each vertex, by possibly removing colors from $L'(v)$ arbitrarily. Now $G - D$ and L' satisfy the hypotheses of Lemma 5.4.2. By the choice of D, (b) does not hold. Hence, by Lemma 5.4.6, $G - D$ is saved with respect to L' and k and thus with respect to L and k, as claimed. □

5.5 Putting it all together

In this section we prove the main technical result of this chapter, Theorem 5.0.1.

Proof of Theorem 5.0.1. Let G be an L-critical graph for a list-assignment L such that for each vertex $v \in V(G)$, we have $|L(v)| \leq k$ where k is sufficiently large as in Theorem 5.3.9, and suppose for a contradiction that (5.1) does not hold. Since G is L-critical, by Theorem 4.2.1, G does not contain a subgraph that is dense with respect to L. Moreover, for each vertex $v \in V(G)$, we have $|L(v)| \leq d(v)$. Therefore, by Lemma 5.4.1, there exists a subgraph $H \subseteq G$ such that H is saved with respect to L and K. Since G is L-critical, there is an L-coloring ϕ of $G - V(H)$. Since H is saved with respect to L' and k where $L'(v) = L(v) \setminus (\cup_{u \in N(v)\setminus V(H)} \phi(u))$, by Theorem 5.3.9, H is L'-colorable. By combining an L'-coloring of H with an L-coloring of $G - V(H)$, we obtain an L-coloring of G, contradicting that G is L-critical. □

Next we show how Theorem 1.4.5 follows from Theorem 5.0.1.

Proof of Theorem 1.4.5. Let G be an L-critical graph for some $(k-1)$-uniform list-assignment where $\omega(G) \leq k - \log^{10} k$ and $k-1$ is sufficiently large to apply Theorem 5.0.1. Since G is L-critical, $d(v) \geq k-1$ for each $v \in V(G)$. Therefore for each vertex v, since $\omega(v) \leq \omega(G) \leq k - \log^{10} k$, we have $\mathrm{Gap}(v) \geq \log^{10} k$. By Theorem 5.0.1 applied with $k-1$, we have

$$\sum_{v \in V(G)} (\mathrm{Save}_L(v) + \varepsilon \log^{10}(k-1)) \geq \sum_{v \in V(G)} 2\varepsilon\,\mathrm{Gap}(v) - 7\varepsilon(k-1-|L(v)|).$$

Since $\mathrm{Gap}(v) \geq \log^{10}(k-1)$ and $|L(v)| = k-1$, the result follows. □

We conclude by proving Theorems 1.4.4 and 1.4.6.

Proof of Theorem 1.4.4. Let G be an L-critical graph for some $(k-1)$-uniform list-assignment where $\omega(G) \leq k - \log^{10} k$ and k is sufficiently large to apply Corollary 1.4.5. Let $\varepsilon' = \varepsilon/(1+\varepsilon)$, and note that $\varepsilon' \leq 2.6 \cdot 10^{-10}$. By Corollary 1.4.5,

$$\sum_{v \in V(G)} \mathrm{Save}_L(v) \geq \sum_{v \in V(G)} \varepsilon'\,\mathrm{Gap}(v).$$

Since $\mathrm{Save}_L(v) = d(v) + 1 - (k-1)$ and $\mathrm{Gap}(v) = d(v) + 1 - \omega(v)$, by rearranging terms in the previous inequality, we obtain $\sum_{v \in V(G)} (1-\varepsilon')(d(v)+1) + \varepsilon'\omega(v) \geq (k-1)|V(G)|$. Rearranging terms again, we have

$$\mathrm{ad}(G) \geq \frac{k - 1 - \varepsilon' \sum_{v \in V(G)} \omega(v)/|V(G)|}{1-\varepsilon'} - 1 = (1+\varepsilon)(k-1) - \varepsilon \sum_{v \in V(G)} \omega(v)/|V(G)| - 1.$$

Since $\sum_{v \in V(G)} \omega(v)/|V(G)| \leq \omega(G)$, the result follows. □

Proof of Theorem 1.4.6. We let $\varepsilon > 0$ be some constant chosen to satisfy certain inequalities throughout the proof. Let G be a graph with list-assignment L such that $\omega(G) \leq \operatorname{mad}(G) - \log^{10} \operatorname{mad}(G)$, and let

$$k = \lceil (1-\varepsilon)(\operatorname{mad}(G)+1) + \varepsilon\omega(G) \rceil .$$

First we prove that there exists an integer k_0 such that if $\operatorname{mad}(G) \geq k_0$, then $\chi_\ell(G) \leq k$. We choose k_0 such that k is large enough to apply Corollary 1.4.5. By assumption,

$$\omega(G) \leq \operatorname{mad}(G) - \log^{10} \operatorname{mad}(G) \leq \frac{k - \varepsilon\omega(G)}{1-\varepsilon} - \log^{10} \frac{k}{1-\varepsilon}.$$

Since $\omega(G) \leq k$, the above inequality implies that $\omega(G) \leq k - \log^{10} k$.

We may assume there exists a k-list-assignment L for G such that G is not L-colorable, or else $\chi_\ell(G) \leq k$, as desired. Therefore G contains an L-critical subgraph G', and by Corollary 1.4.5, $\sum_{v\in V(G')} \operatorname{Save}_L(v) \geq \sum_{v \in V(G')} \varepsilon' \operatorname{Gap}(v)$ for $\varepsilon' = 2.6 \cdot 10^{-10}$. Rearranging terms, we have

$$\operatorname{ad}(G') \geq \frac{k - \varepsilon'\omega(G)}{1-\varepsilon'} - 1.$$

However, if $\varepsilon < \varepsilon'$, we obtain a contradiction, since $\operatorname{mad}(G) \geq \operatorname{ad}(G')$. Therefore G is L-colorable, as desired.

It remains to show that $\chi_\ell(G) \leq k$ if $\operatorname{mad}(G) < k_0$. If we choose $\varepsilon < \frac{1}{k_0+2}$, then

$$k \geq \left\lceil \left(1 - \frac{1}{\operatorname{mad}(G)+2}\right)(\operatorname{mad}(G)+1)\right\rceil = \left\lceil \operatorname{mad}(G) + \frac{1}{\operatorname{mad}(G)+2}\right\rceil \geq \lfloor \operatorname{mad}(G) \rfloor + 1.$$

Therefore we can obtain an L-coloring of G for any k-list-assignment L by coloring greedily. Thus, $\chi_\ell(G) \leq k$, as desired. $\square$

Chapter 6

Coloring graphs with small clique number

In this Chapter, we prove the results presented in Section 1.3.2, Theorems 1.3.7, 1.3.11, and 1.3.12, and we further discuss the results of Section 1.2.3.

6.1 More background

Mathematicians have been fascinated by the chromatic number of triangle-free graphs for a long time. Tutte [44, 45], (under the pseudonym Blanche Descartes), was the first to provide examples of triangle-free graphs of arbitrarily large chromatic number. Zykov [150] and Mycielski [115] also came up with constructions of such graphs. As mentioned in Chapter 2, Erdős [54] extended these results using the probabilistic method by proving the existence of graphs with arbitrarily large girth and chromatic number.

These graphs all have vertices of large degree, which led Grünbaum [65] to conjecture that for every Δ, there exist graphs of arbitrarily large girth with chromatic number and maximum degree Δ. Theorem 1.2.9 demonstrates that Grünbaum's Conjecture is dramatically false for graphs of large maximum degree. In the late 1970s, Borodin and Kostochka [25], Lawrence [104], and Catlin [35] independently proved that every K_4-free graph of maximum degree at most Δ has chromatic number at most $\lceil 3(\Delta + 1)/4\rceil$, which disproves Grünbaum's Conjecture for $\Delta \geq 7$. Kostochka later proved that every triangle-free graph of maximum degree at most Δ has chromatic number at most $2\Delta/3 + 2$, and

This chapter is based on the preprint [23] (with Marthe Bonamy, Peter Nelson, and Luke Postle).

he [100] also proved for $\Delta \geq 5$ that graphs of sufficiently large girth of maximum degree at most Δ have chromatic number at most $\Delta/2 + 2$, which disproves Grünbaum's Conjecture for $\Delta \geq 5$. Note that this confirms Reed's Conjecture (Conjecture 1.2.3) for these graphs; however, Reed's Conjecture is still open for triangle-free graphs. Wormald (see [80]) conjectured that graphs of maximum degree at most four of sufficiently large girth are 3-colorable, that is that Grünbaum's Conjecture is false for $\Delta = 4$.

6.1.1 The nibble method and the finishing blow

Johansson [81] and Kim [90] in the mid-1990s obtained the dramatic improvements on these bounds for graphs of large maximum degree and girth at least 4 and 5 using the probabilistic method. In particular, they used an iterative version of a random procedure,, which is sometimes called the "semi-random method" or the "nibble method." In these procedures, vertices are randomly assigned a color only with a probability on the order of $1/\ln \Delta$. Since vertices expect to have at most roughly $\Delta/\ln \Delta$ colored neighbors, a calculation like the one in Proposition 3.1.4 shows that these vertices can retain their color with some nontrivial probability, despite the fact that there are much fewer colors available. This random coloring is repeated roughly $\text{poly} \log \Delta$ times.

In Kim's [90] original proof of the $(1 + o(1))\Delta/\ln \Delta$-bound on the chromatic number of girth five graphs, he tracked the degree of each vertex, the number of remaining available colors for each vertex, and the *color degree* for each vertex and color in its list, where the color degree for a vertex v and $c \in L(v)$, denoted $d(v, c)$, is the number of neighbors of v with c in their list of colors. The notion of color degree naturally extends to the setting of correspondence coloring; if (L, M) is a correspondence-assignment for a graph G, then for each $v \in V(G)$ and $c \in L(v)$ we let $d_{(L,M)}(v, c)$ denote the number of neighbors of v with a color in their list that corresponds to c at v. It is crucial to the proof that after each iteration the color degrees decrease faster than both the degrees and list sizes. After the semi-random coloring procedure is complete, the graph can be colored greedily as in Proposition 1.2.1. Each iteration of Kim's proof is fairly similar to the naive random coloring procedure, with the modification that initially only a small proportion of the vertices are assigned a color.

Molloy and Reed [109, Chapter 12] adapted Kim's [90] random coloring procedure and simplified the proof. Rather than remove a color from a vertex's list if a colored neighbor retained that color, Molloy and Reed's procedure removes any color assigned to a neighbor, even if it is not retained. Because of this modification, they called their procedure the "wasteful coloring procedure." In the analysis of this procedure, they only need to track list sizes and color degrees. Once the procedure is finished, they complete the coloring with the following result, the so-called "finishing blow," which is proved with a simple application of the Lovász Local Lemma.

Theorem 6.1.1. *If G is a graph with a k-correspondence-assignment (L, M) such that each $v \in V(G)$ and $c \in L(v)$ satisfies $d_{(L,M)}(v, c) \leq k/(2e)$, then G is (L, M)-colorable.*

Theorem 6.1.1 was first proved by Reed [123] for list coloring. Using topological methods, Haxell [77] showed that the e in the denominator of the constraint on color degrees is not needed. Both of these results actually hold in the more general setting of correspondence coloring and even for multigraphs, where Haxell's result is tight. Using the nibble method as well, Reed and Sudakov [125] improved Theorem 6.1.1 by showing it holds with the $2e$ replaced with $1 + o(1)$ for large k. Their result was proved only for list coloring, but the proof can be generalized to the setting of correspondence coloring simple graphs (see [105]).

Johansson's [81] proof of the $O(\Delta/\ln\Delta)$-bound on the chromatic number is a bit more complicated than Molloy and Reed's simplified proof of Kim's result. This result of Johansson [81], as well as his proof of Theorem 1.2.10 for fixed ω and his result on locally r-colorable graphs (see Corollary 1.3.8) [82] were never published. However, Bansal, Gupta, and Guruganesh [10] obtained a copy of Johansson's [82] manuscript and include complete proofs of the latter two results in the appendix of their paper. Molloy and Reed [109, Chapter 13] provide a proof of Johansson's bound for triangle-free graphs.

Rather than choosing colors from a vertex's list uniformly at random as in Kim's proof, a key feature of Johansson's proof is to maintain a probability distribution over the colors that is initially uniform but changes in each iteration. The *entropy* of this distribution remains high, which means that the assignment probabilities do not differ too much. As a consequence, most of the colors in each vertex's list are not *too* likely to be chosen, which could otherwise disrupt the procedure. The proof does not track color degrees, but it uses a finishing blow similar to Theorem 6.1.1. Molloy and Reed's [109] proof tracks degrees of each vertex as well as the probability that adjacent vertices are assigned the same color. If these probabilities are sufficiently low compared to the degrees, then a method similar to the proof of Theorem 6.1.1 completes the coloring.

In 2015, Pettie and Su [120] improved Johansson's bound for triangle-free graphs by showing that a leading constant of $4 + o(1)$ suffices, and the complexity of their proof is on par with that of Molloy and Reed's simplified proof of Kim's bound. Instead of tracking color degrees, Pettie and Su cleverly track the *average* color degree for each vertex, taken over the colors in its list. Note that if a vertex is contained in many 4-cycles, its color degree with a color in its list is not necessarily concentrated around its expectation. For example, if the graph is complete bipartite, then the color degrees are either 0 or the degree of the vertex. However, in graphs of girth at least five, color degrees are concentrated, and in triangle-free graphs, the average color degrees are concentrated. The fact that at least half of the colors in a vertex's list have color degree at most twice the average explains the factor four difference in the bounds of Kim and Pettie and Su. Although these results do not imply Theorem 1.2.9, the $(1+o(1))\Delta/\ln\Delta$-bound for triangle-free graphs, they have the advantage that they hold when Δ is replaced by the maximum color degree.

Molloy's [107] proof of Theorem 1.2.9 is quite different from these previous approaches. However, Molloy applied Theorem 6.1.1 in a similar way, and we use Theorem 1.2.9 in the proof of Theorem 1.3.12 in Section 6.4. Our proof of this result is simpler than the one in [42] because we use Theorem 6.1.1 directly, whereas Davies et al. [42, Lemma 6] prove and apply a generalization of it.

6.1.2 A more general Theorem

As we mentioned in Section 1.5.3, Bernshteyn [14] proved that Theorems 1.2.9 and 1.2.10 hold for the correspondence chromatic number. Many aspects of Bernshteyn's proofs are similar to those of Molloy's [107]; however, Bernshteyn's proof is much shorter and simpler. Molloy used a proof technique known as "entropy compression," which proves that a random algorithm terminates. Entropy compression can be used to prove an algorithmic version of the Lovász Local Lemma and sometimes leads to improved results when used in its stead. Bernshteyn realized that the use of entropy compression in Molloy's proof can be replaced with the Lopsided Lovász Local Lemma, resulting in a substantial simplification of the proof. As we will see, rather than considering many applications of a random coloring procedure, it suffices to analyze a single random partial coloring. In fact, the random partial coloring is simply chosen uniformly at random. Molloy's brilliant insight is in the analysis of this random coloring.

Both proofs can be applied in the more general setting of graphs in which the average size of an independent set in each vertex's neighborhood is somewhat large in comparison to the number of such independent sets. We make this precise by extracting a more general theorem from their proofs, and we actually prove a "local version" of it, as follows.

For a graph H, let $\overline{\alpha}(H)$ and $i(H)$ denote the average size of an independent set and the number of independent sets in H respectively.

Theorem 6.1.2 (Bonamy, Kelly, Nelson, and Postle [23]). *Let G be a graph of maximum degree at most Δ with correspondence-assignment (L, M), and $\varepsilon \in (0, 1/2)$. Let ℓ, t :*

$V(G) \to \mathbb{N}$, and for each $v \in V(G)$, let $\alpha_{min}(v)$ be the minimum of $\overline{\alpha}(H)$ taken over all induced subgraphs $H \subseteq G[N(v)]$ such that $i(H) \geq t(v)$. If for each $v \in V(G)$,

$$|L(v)| \geq \max\left\{(1+2\varepsilon)\frac{d(v)}{\alpha_{min}(v)}, \frac{2t(v)\ell(v)}{\varepsilon(\varepsilon - 2\varepsilon^2)}\right\},$$

and

1. $\ell(v) \geq 18\ln(3\Delta)$,
2. $\binom{d(v)}{\ell(v)}/\ell(v)! < \Delta^{-3}/8$,

then G is (L, M)-colorable.

We think that proving Theorem 6.1.2 separately makes the proof easier to understand, and we think that Theorem 6.1.2 may have applications not listed in this paper.

In order to apply Theorem 6.1.2, one needs to find a lower bound on $\alpha_{\min}(v)$. We do this by proving a general bound on $\overline{\alpha}(H)$ for a graph H in terms of $i(H)$ and $\omega(H)$. For large values of $\omega(H)$, our bound is better than the bound used by Molloy [107], and this yields the improvement in Theorem 1.2.11. The condition that $|L(v)| \geq (1+2\varepsilon)\frac{d(v)}{\alpha_{\min}(v)}$ in Theorem 1.3.7 corresponds to the bound on $|L(v)|$ in Theorem 1.3.7. The condition that $|L(v)| \geq \frac{2t(v)\ell(v)}{\varepsilon(\varepsilon-2\varepsilon^2)}$ restricts the choice of functions ℓ and t.

In Section 6.2, we prove Theorem 6.1.2. The proof is similar to Bernshteyn's proof of Theorem 1.2.10 from [14]; however, we prove the more general theorem, and some changes are necessary in order to prove the "local version" of it.

In Section 6.3, we prove Theorems 1.3.7 and 1.3.11 using Theorem 6.1.2. Unfortunately, Theorem 6.1.2 does not imply Theorem 1.2.9. However, the proofs of these two results are very similar. One of the main differences is that in the proof of Theorem 6.1.2, we complete the random coloring greedily with Proposition 1.2.1, whereas in the proof of Theorem 1.2.9, we apply Theorem 6.1.1. In Section 6.4, we prove Theorem 1.3.12 in this way, which can easily be shown to imply Theorem 1.2.9.

6.2 Proof of Theorem 6.1.2

In this section, we prove Theorem 6.1.2. We assume $G, (L, M), \Delta, \ell,$ and t satisfy the conditions of Theorem 6.1.2 throughout the section.

6.2.1 Partial colorings

The proof of Theorem 6.1.2 relies on analyzing a "partial coloring" of the graph chosen uniformly at random. In this subsection, we define some notation that will be useful for this analysis.

Definition 6.2.1. Let G be a graph with correspondence-assignment (L, M), and let BLANK be a color not in $L(v)$ for any $v \in V(G)$.

- A *partial* (L, M)*-coloring* of G is a mapping ϕ with domain $V(G)$ such that every vertex $v \in V(G)$ satisfies $\phi(v) \in L(v) \cup \{\text{BLANK}\}$.
- If $\phi(v) = \text{BLANK}$, we say v is ϕ*-uncolored*.
- For each ϕ-uncolored vertex v, we let $L_\phi(v)$ denote the set of colors $c \in L(v)$ such that for every neighbor u of v, $\phi(u)$ does not correspond to c, and we let M_ϕ denote the restriction of M to edges between ϕ-uncolored vertices.
- If ϕ' is a partial (L_ϕ, M_ϕ)-coloring of the ϕ-uncolored vertices of G, we let

$$(\phi + \phi')(v) = \begin{cases} \phi(v) & \text{if } \phi(v) \neq \text{BLANK}, \\ \phi'(v) & \text{otherwise.} \end{cases}$$

We will show that with nonzero probability the random partial coloring can be extended to a coloring of the whole graph. Using the following proposition, it will suffice to show that if ϕ is a partial coloring of G chosen uniformly at random, then the ϕ-uncolored vertices induce a subgraph that is (L_ϕ, M_ϕ)-colorable.

Proposition 6.2.2. *If G is a graph with correspondence-assignment (L, M), ϕ is a partial (L, M) coloring of G, and ϕ' is a (L_ϕ, M_ϕ)-coloring of the graph induced by G on the ϕ-uncolored vertices, then $\phi + \phi'$ is an (L, M)-coloring of G.* □

6.2.2 Completing a partial coloring

We prove Theorem 6.1.2 by finding a partial (L, M)-coloring of G that we can greedily extend to an (L, M)-coloring. The following lemma provides the existence of such a partial (L, M)-coloring.

Lemma 6.2.3. *There exists a partial (L, M)-coloring ϕ of G such that for every ϕ-uncolored vertex v,*

1. *$|L_\phi(v)| \geq \ell(v)$, and*
2. *v has fewer than $\ell(v)$ ϕ-uncolored neighbors u such that $\ell(u) \geq \ell(v)$.*

Lemma 6.2.3 generalizes Lemma 4.1 in the proof of Bernshteyn [14], and the partial (L, M)-coloring in Lemma 6.2.3 generalizes the "flaw-free" coloring output by the random algorithm of Molloy [107]. When the function ℓ is not constant, our second condition is slightly weaker, so we are not necessarily able to complete the partial coloring greedily in any order as in their proofs.

Proof of Theorem 6.1.2 assuming Lemma 6.2.3. Let ϕ be the partial (L, M)-coloring of G satisfying (1) and (2) of Lemma 6.2.3. By Proposition 6.2.2, it suffices to show that the ϕ-uncolored vertices induce a graph that is (L_ϕ, M_ϕ)-colorable. This follows by ordering the ϕ-uncolored vertices v by $\ell(v)$ from greatest to least, breaking ties arbitrarily, and coloring greedily. □

6.2.3 Analyzing a random partial coloring

We prove Lemma 6.2.3 by analyzing a partial (L, M)-coloring of G chosen uniformly at random and using the Local Lemma to show that with nonzero probability, the random partial coloring satisfies Lemma 6.2.3. Instead of using the Local Lemma, Molloy [107] used the entropy compression technique. The key insight of Bernshteyn [14] was that a clever application of the Lopsided Local Lemma is sufficient, and this greatly simplified the proof. In order to apply the Local Lemma to prove Lemma 6.2.3, we will need the following lemma.

Lemma 6.2.4. *Let $v \in V(G)$ and fix a partial (L, M)-coloring ϕ_1 of $G - N[v]$. Let ϕ_2 be a partial (L_{ϕ_1}, M_{ϕ_1}) coloring of $G[N(v)]$ chosen uniformly at random, and let $\phi = \phi_1 + \phi_2$. Let*

1. *A_{v,ϕ_1} be the event that $|L_\phi(v)| < \ell(v)$ and*
2. *B_{v,ϕ_1} be the event that v has at least $\ell(v)$ ϕ-uncolored neighbors u such that $|L_\phi(u)| \geq \ell(u) \geq \ell(v)$.*

Then $\mathbb{P}[A_{v,\phi_1}], \mathbb{P}[B_{v,\phi_1}] \leq \Delta^{-3}/8$.

Lemma 6.2.4 generalizes Lemma 4.2 in the proof of Bernshteyn [14] and Lemma 12 in the proof of Molloy [107].

To bound $\mathbb{P}[A_{v,\phi_1}]$ in Lemma 6.2.4, we show that $|L_\phi(v)|$ is large in expectation and with high probability is concentrated around its expectation, as in the following lemma.

Lemma 6.2.5. *Under the assumptions of Lemma 6.2.4,*

$$\mathbb{E}\left[|L_\phi(v)|\right] \geq \varepsilon(\varepsilon - 2\varepsilon^2)\frac{|L(v)|}{t(v)}, \tag{6.1}$$

and

$$\mathbb{P}\left[\left||L_\phi(v)| - \mathbb{E}\left[|L_\phi(v)|\right]\right| > \frac{1}{2}\mathbb{E}\left[|L_\phi(v)|\right]\right] \leq 2\exp\left(\frac{-\mathbb{E}\left[|L_\phi(v)|\right]}{12}\right). \tag{6.2}$$

Before proving Lemma 6.2.5, we need some definitions. For the remainder of this subsection, we assume v, ϕ_1, ϕ_2, and ϕ are as in Lemma 6.2.4.

Definition 6.2.6. For each $c \in L(v)$, let the random variable

$$\mathbf{appear}_c = |\{u \in N(v) : c\phi_2(u) \in M_{vu}\}|,$$

i.e. the number of neighbors u of v such that $\phi_2(u)$ corresponds to c.

Note that

$$\mathbb{E}\left[|L_\phi(v)|\right] = \sum_{c \in L(v)} \mathbb{P}\left[\mathbf{appear}_c = 0\right]. \tag{6.3}$$

By (6.3), in order to prove Lemma 6.2.5, we need some bounds on $\mathbb{P}\left[\mathbf{appear}_c = 0\right]$ for the colors $c \in L(v)$. These bounds will depend on the average size and number of independent sets of certain subgraphs induced by neighbors u of v such that $L(u)$ contains a color corresponding to c.

Definition 6.2.7. Fix $c \in L(v)$ and a partial (L_{ϕ_1}, M_{ϕ_1}) coloring ϕ_2' of $G[N(v)]$ such that for no neighbor u of v, the color $\phi_2'(u)$ corresponds to c. Let $\mathrm{col}(c, \phi_2')$ denote the ϕ_2'-uncolored neighbors u of v such that $L_{\phi_1+\phi_2'}(u)$ contains c, i.e. the ϕ_2'-uncolored neighbors of v that can be colored c without creating conflicts.

Definition 6.2.8. For each $c \in L(v)$, let ϕ_2^c be the partial coloring obtained from ϕ_2 by uncoloring any neighbor u of v such that $\phi_2(u)$ corresponds to c.

Proposition 6.2.9. *Fix $c \in L(v)$ and a partial (L_{ϕ_1}, M_{ϕ_1})-coloring ϕ_2' such that for no neighbor u of v, the color $\phi_2'(u)$ corresponds to c. Then*

(i) $\mathbb{E}\left[\mathbf{appear}_c | \phi_2^c = \phi_2'\right] = \overline{\alpha}(G[\mathrm{col}(c, \phi_2')])$, *and*
(ii) $\mathbb{P}\left[\mathbf{appear}_c = 0 | \phi_2^c = \phi_2'\right] = i(\mathrm{col}(c, \phi_2'))^{-1}$.

Definition 6.2.10. Let $\mathbf{col}(c)$ denote the random set of neighbors u of v such that $L_{\phi_1}(u)$ contains c and $\phi_2(u) \in \{c, \text{BLANK}\}$.

We can now prove Lemma 6.2.5.

Proof of Lemma 6.2.5. First we prove that (6.1) holds. We divide $L(v)$ into two parts in the following way. For each $c \in L(v)$, we let $c \in L_1(v)$ if $\mathbb{P}\left[i(\mathbf{col}(c)) \leq t(v)\right] \geq \varepsilon$, and otherwise we let $c \in L_2(v)$.

First we claim that $|L_2(v)| \leq |L(v)|/(1+\varepsilon-2\varepsilon^2)$. If $c \in L_2(v)$, by Proposition 6.2.9 and the definition of $\alpha_{\min}(v)$,

$$\mathbb{E}\left[\mathbf{appear}_c\right] \geq (1-\varepsilon)\alpha_{\min}(v). \tag{6.4}$$

However,

$$\sum_{c \in L_2(v)} \mathbb{E}\left[\mathbf{appear}_c\right] \leq d(v). \tag{6.5}$$

By (6.4) and (6.5), $(1-\varepsilon)\alpha_{\min}(v)|L_2(v)| \leq d(v)$. Since $|L(v)| \geq (1+2\varepsilon)\frac{d(v)}{\alpha_{\min}(v)}$, by rearranging terms,

$$|L_2(v)| \leq \frac{d(v)}{(1-\varepsilon)\alpha_{\min}(v)} \leq \frac{|L(v)|}{(1-\varepsilon)(1+2\varepsilon)},$$

as claimed. Since $|L(v)| = |L_1(v)| + |L_2(v)|$, this implies $|L_1(v)| \geq (\varepsilon - 2\varepsilon^2)|L(v)|$.

If $c \in L_1(v)$, by Proposition 6.2.9 and the definition of $L_1(v)$,

$$\mathbb{P}\left[\mathbf{appear}_c = 0\right] \geq \frac{\varepsilon}{t(v)}. \tag{6.6}$$

By (6.3) and (6.6),

$$\mathbb{E}\left[|L_\phi(v)|\right] \geq \frac{\varepsilon}{t(v)}|L_1(v)| \geq \varepsilon(\varepsilon - 2\varepsilon^2)\frac{|L(v)|}{t(v)},$$

as desired.

Now we prove that (6.2) holds. By (6.3), $\mathbb{E}\left[|L_\phi(v)|\right]$ is a sum of Boolean random variables

$$X_c = \begin{cases} 1 & \text{if } \mathbf{appear}_c = 0, \\ 0 & \text{if } \mathbf{appear}_c > 0, \end{cases}$$

for each $c \in L(v)$. Since the random variables X_c are negatively correlated, it follows from Lemma 2.2.3 with $t = \mathbb{E}\left[L_\phi(v)\right]/2$. □

We can now prove Lemma 6.2.4.

Proof of Lemma 6.2.4. First we prove that $\mathbb{P}\left[A_{v,\phi_1}\right] \leq \Delta^{-3}/8$. Since $|L(v)| \geq \frac{2\ell(v)t(v)}{\varepsilon(\varepsilon-2\varepsilon^2)}$, Lemma 6.2.5 implies that

$$\mathbb{E}\left[|L_\phi(v)|\right] \geq 2\ell(v). \tag{6.7}$$

Therefore by the definition of A_{v,ϕ_1},

$$\mathbb{P}\left[A_{v,\phi_1}\right] \leq \mathbb{P}\left[\left||L_\phi(v)| - \mathbb{E}\left[|L_\phi(v)|\right]\right| > \frac{1}{2}\mathbb{E}\left[|L_\phi(v)|\right]\right].$$

Now by Lemma 6.2.5, (6.7), and the hypothesis that $\ell(v) \geq 18\ln(3\Delta)$,

$$\mathbb{P}\left[A_{v,\phi_1}\right] \leq 2\exp\left(-\ell(v)/6\right) \leq \Delta^{-3}/8,$$

as desired.

It remains to bound $\mathbb{P}\left[B_{v,\phi_1}\right]$. Let X be any set of $\ell(v)$ neighbors u of v such that $|L_\phi(u)| \geq \ell(u) \geq \ell(v)$. For any partial (L_{ϕ_1}, M_{ϕ_1}) coloring ϕ_2' of $N(v)$ such that the vertices in X are ϕ_2'-uncolored, there are at least $\ell(v)!$ partial $(L_{\phi_1+\phi_2'}, M_{\phi_1+\phi_2'})$ colorings of X, and in only one of them all of X is uncolored. Therefore the probability that every vertex of X is ϕ-uncolored is at most $\frac{1}{\ell(v)!}$. By the Union Bound and the assumption that $\binom{d(v)}{\ell(v)}/\ell(v)! \leq \Delta^{-3}/8$, this implies $\mathbb{P}\left[B_{v,\phi_1}\right] \leq \Delta^{-3}/8$, as desired. $\square$

6.2.4 Finding the partial coloring

In this subsection, we prove Lemma 6.2.3. Recall that Lemma 6.2.3 implies Theorem 6.1.2.

Proof of Lemma 6.2.3. Let ϕ be a partial (L, M)-coloring of G chosen uniformly at random. For each $v \in V(G)$, let A_v be the event that $|L_\phi(v)| < \ell(v)$, let B_v be the event that v has at least $\ell(v)$ ϕ-uncolored neighbors u such that $|L_\phi(u)| \geq \ell(u) \geq \ell(v)$, and let $\Gamma(v)$ denote the set of vertices of distance at most three from v in G. Note that for all $v \in V(G)$, $|\Gamma(v)| \leq \Delta^3$.

First we claim that with nonzero probability, none of the events $(A_v \cup B_v)$ occur in the random partial coloring ϕ. By the Local Lemma (Lemma 2.2.1), it suffices to show that for each $v \in V(G)$ and $Z \subseteq V(G)\backslash\Gamma(v)$, $\mathbb{P}\left[A_v | \bigcap_{u\in Z} \overline{A_u \cup B_u}\right] \leq \Delta^{-3}/8$ and $\mathbb{P}\left[B_v | \bigcap_{u\in Z} \overline{A_u \cup B_u}\right] \leq \Delta^{-3}/8$. This follows from Lemma 6.2.4.

Therefore there is a partial (L, M)-coloring ϕ' for which none of the events $(A_v \cup B_v)$ occur. We claim that ϕ' satisfies Lemma 6.2.3. Suppose not. If for some $v \in V(G)$, condition (1) is not satisfied, then A_v holds, a contradiction. Therefore we may assume for some $v \in V(G)$, condition (2) is not satisfied, that is v has fewer than $\ell(v)$ ϕ'-uncolored neighbors u such that $\ell(u) \geq \ell(v)$. Since B_v holds, for some neighbor u, $|L_{\phi'}(u)| < \ell(u)$, contradicting that A_u does not hold. Therefore ϕ' satisfies Lemma 6.2.3, as claimed. □

6.3 Proofs of Theorems 1.3.7 and 1.3.11

In this section we prove Theorems 1.3.7 and 1.3.11. In this section, log means the base 2 logarithm.

6.3.1 Bounding the average size of an independent set

We will use Theorem 6.1.2, so we will need a lower bound on $\alpha_{\min}(v)$. We do this by bounding the average size of an independent set in terms of the total number. In the proof of Theorem 1.2.10, Molloy [107] and Bernshteyn [14] use the following result of Shearer [138], which we will also need.

Lemma 6.3.1 ([138]). *If H is a graph with no clique of size greater than ω, then*

$$\overline{\alpha}(H) \geq \frac{\log(i(H))}{2\omega \log(\log(i(H)))}.$$

We will also need the following result of Alon [3].

Lemma 6.3.2 ([3]). *If H is a graph on n vertices, then*

$$\overline{\alpha}(H) \geq \frac{\log(i(H))}{10 \log\left(n/\log(i(H)) + 1\right)}.$$

Since $\log(i(H)) \geq \alpha(H)$, we can replace the $\log(i(H))$ in the denominator of the bound in Lemma 6.3.2 with $\alpha(H)$ to get a suitable bound if H contains a large independent set.

The following lemma provides an improvement over Lemma 6.3.1 for larger values of ω.

Lemma 6.3.3. *If H is a graph on n vertices with no clique of size greater than ω and n is sufficiently large, then*

$$\overline{\alpha}(H) \geq \frac{1}{24}\sqrt{\frac{\log(i(H))}{\log(\omega)}}.$$

We will actually use Lemma 6.3.2 to prove Lemma 6.3.3. To apply Lemma 6.3.2, we need to upper bound $\log(n)$ in terms of $\log(i(H))$ and $\omega(H)$, as in the following lemma.

Lemma 6.3.4. *If H is a graph on n vertices with no clique of size greater than ω and n is sufficiently large, then*

$$\log(i(H)) \geq \frac{\log^2(n)}{2e\log(\omega)}.$$

Proof. We may assume $\omega \geq 3$, or else H has an independent set of size at least $\sqrt{n}$, and the result follows.

Let α be some positive integer to be determined later, and let $s = R(\alpha, \omega + 1)$, the Ramsey number. We will actually prove there are at least $2^{\frac{\log^2(n)}{2e\log(\omega)}}$ independent sets of size α.

By the definition of s, every subset of $V(H)$ of size s has an independent set of size α. Since every independent set of size α is contained in at most $\binom{n-\alpha}{s-\alpha}$ subsets of $V(H)$ of size s, there are at least

$$\binom{n}{s} \Big/ \binom{n-\alpha}{s-\alpha} \geq \left(\frac{n-\alpha}{s}\right)^{\alpha} = 2^{\alpha(\log(n-\alpha)-\log(s))} \tag{6.8}$$

independent sets of size α.

We let $\alpha = \frac{\log(n)}{e\log(\omega)} + 1$. By (6.8), it suffices to show that $\log(n-\alpha) - \log(s) \geq \log(n)/2$. It is well-known that $R(\alpha, \omega+1) \leq \binom{\alpha+\omega-1}{\alpha-1} \leq \left(\frac{\alpha+\omega-1}{\alpha-1}\cdot e\right)^{\alpha-1}$. Therefore

$$\log(s) \leq (\alpha - 1)\log\left(\frac{\alpha+\omega-1}{\alpha-1}\cdot e\right) \leq \frac{\log(n)}{e\log(\omega)}\log\left(e + \omega\frac{\log(\omega)}{\log(n)}\right).$$

Since $\alpha = o(n)$ and $\omega \geq 3$, for n sufficiently large, $\log(n-\alpha) - \log(s) \geq \log(n)/2$, as desired. $\square$

Now we can prove Lemma 6.3.3.

Proof of Lemma 6.3.3. By Lemma 6.3.2,

$$\overline{\alpha}(H) \geq \frac{\log(i(H))}{10\log(n)}.$$

By Lemma 6.3.4,

$$\log(n) \leq \sqrt{2e\log(i(H))\log(\omega)},$$

and the result follows. $\square$

Remark 2. Recall that Theorem 1.2.11 generalizes a result of Bansal, Gupta, and Guruganesh [10, Theorem 1.2] on the independence ratio. The proofs of these results share some similarities. In particular, Lemma 6.3.4 resembles [10, Theorem 3.4], and the derivation of Lemma 6.3.3 is implicit in the proof of [10, Theorem 1.2]. When we initially proved Theorem 1.2.11, we were unaware of their result, and these lemmas were obtained independently.

6.3.2 The proofs

Lemmas 6.3.1, 6.3.2, and 6.3.3 provide good enough bounds for $\alpha_{\min}(v)$. Now we are able to prove Theorems 1.3.7 and 1.3.11. For convenience, for each $v \in V(G)$, let

$$f(v) = \min\left\{\sqrt{\frac{\ln(\omega(v))}{\ln(d(v))}}, \frac{\omega(v)\ln(\ln(d(v)))}{\ln(d(v))}, \frac{\log_2(\chi(G[N(v)])+1)}{\ln(d(v))}\right\},$$

and note that in Theorem 1.3.7 the list-assignment L satisfies $|L(v)| \geq 72d(v)\cdot f(v)$ for each vertex $v \in V(G)$.

Proof of Theorem 1.3.7. Let $\varepsilon = 1/4$, and let $v \in V(G)$. We show that v satisfies the conditions of Theorem 6.1.2. Let $\ell(v) = d(v)^{5/8}$, and let $t(v) = d(v)^{1/4}$. By Lemma 6.3.3,

$$\alpha_{\min}(v) \geq \frac{1}{48}\sqrt{\frac{\log(d(v))}{\log(\omega(v))}}.$$

By Lemma 6.3.1,

$$\alpha_{\min}(v) \geq \frac{\log(d(v))}{8\omega(v)\log(\log(d(v)))}.$$

Note that $\log(i(H)) \geq \alpha(H)$ for any graph H. Hence if $H \subseteq G[N(v)]$, then $|V(H)|/\log(i(H)) \leq \chi(H) \leq \chi(v)$. Therefore by Lemma 6.3.2,

$$\alpha_{\min}(v) \geq \frac{\log(d(v))}{40\log(\chi(v)+1)}.$$

Since $|L(v)| \geq 72d(v)f(v)$, it follows that $|L(v)| \geq (1+2\varepsilon)\frac{d(v)}{\alpha_{\min}(v)}$, as desired. Note that $f(v)^8 = o(d(v))$. Since Δ is sufficiently large and $d(v) \geq \log^2(\Delta)$, we may assume $f(v) \geq 168d(v)^{-1/8}$. Since $t(v)\ell(v) = d(v)^{7/8}$, $|L(v)| \geq \frac{2t(v)\ell(v)}{\varepsilon(\varepsilon-2\varepsilon^2)}$, as desired.

Since $\ell(v) \geq \ln^{5/4}(\Delta)$ and Δ is sufficiently large, $\ell(v) \geq 18\ln(3\Delta)$, as desired. It remains to show that $\binom{d(v)}{\ell(v)}/\ell(v)! < \Delta^{-3}/8$. We will use the following form of Stirling's approximation:

$$n! \geq \sqrt{2\pi}n^{n+1/2}e^{-n}.$$

Therefore

$$\binom{d(v)}{\ell(v)}/\ell(v)! < \frac{d(v)^{\ell(v)}}{(\ell(v)!)^2} \leq \left(\frac{e^2 d(v)}{2\pi\ell(v)^{2+1/\ell(v)}}\right)^{\ell(v)}. \tag{6.9}$$

Since $d(v)/\ell(v)^{2+1/\ell(v)} \leq \ell(v)^{-2/5}$, by taking the logarithm of the bound in (6.9), it suffices to show that

$$\ell(v)\ln\left(\frac{2\pi\ell(v)^{2/5}}{e^2}\right) \geq 3\ln(8\Delta).$$

Since $\ell(v) \geq \ln^{5/4}(\Delta)$ and Δ is sufficiently large, this follows. □

Proof of Theorem 1.3.11. Let $\varepsilon = \xi/10$ and $\varepsilon' > 0$ be some constant to be chosen later, and let $v \in V(G)$. Let $\ell(v) = d(v)^{(1+\varepsilon')/2}$, and let $t(v) = d(v)^{(1-2\varepsilon')/2}$. Since G is triangle-free,

$$\alpha_{\min}(v) \geq \frac{\log(t(v))}{2} = (1-2\varepsilon')\log(d(v))/4.$$

Since $|L(v)| \geq (4+\xi)d(v)/\log(d(v))$, it follows that $|L(v)| \geq (1+\xi/4)(1-2\varepsilon')\frac{d(v)}{\alpha_{\min}(v)}$. We choose ε' sufficiently small so that $(1+\xi/4)(1-2\varepsilon') \geq 1+2\varepsilon$.

Note that $t(v)\ell(v) = d(v)^{1-\varepsilon'/2}$. Hence we may assume $d(v)$ is sufficiently large so that $|L(v)| \geq \frac{2t(v)\ell(v)}{\varepsilon(2-2\varepsilon^2)}$, as desired.

Note also that $\ell(v) \geq \ln^{1+\varepsilon'}(\Delta)$. The rest of the proof is similar to the proof of Theorem 1.3.7, so we omit it. □

6.4 Proof of Theorem 1.3.12

In this section, we show how to simplify Davies et al.'s [42] proof of Theorem 1.3.12. In particular, their proof uses a generalization of Theorem 6.1.1 ([42, Lemma 6]), and we show that it is not necessary. Using a more general form of the Lovász Local Lemma than Lemma 2.2.1, they prove that if G is a graph with correspondence-assignment (L, M) such that every vertex $v \in V(G)$ and $c \in L(v)$ satisfies $d_{(L,M)}(v, c) \le \min_{u \in N(v)} |L(u)|/8$, then G is L-colorable.

For the remainder of this section, let G be a graph with correspondence-assignment (L, M) satisfying the conditions of Theorem 1.3.12. For each $v \in V(G)$, let $\ell(v) = d(v)^{\xi/2}$. The function ℓ plays a similar role as in Theorem 6.1.2. We need the following variation of Lemma 6.2.3, which is essentially the same as [42, Lemma 9].

Lemma 6.4.1. *There exists a partial (L, M)-coloring ϕ of G such that for every ϕ-uncolored vertex v,*

1. $|L_\phi(v)| \ge \ell(v)$, *and*
2. $d_{(L_\phi, M_\phi)}(v, c) \le 24 \log \Delta$ *for all* $c \in L(v)$.

The proof of Lemma 6.4.1 is essentially the same as the proof of Lemma 6.2.3, so we omit it. It follows in a straightforward way from the following variation of Lemma 6.2.4, which is essentially the same as [42, Lemma 10]. The proof of this lemma is very similar to the proof of [14, Lemma 3.6].

Lemma 6.4.2. *Let $v \in V(G)$ and fix a partial (L, M)-coloring ϕ_1 of $G - N[v]$. Let ϕ_2 be a partial (L_{ϕ_1}, M_{ϕ_1}) coloring of $G[N(v)]$ chosen uniformly at random, and let $\phi = \phi_1 + \phi_2$. Let*

1. A_{v,ϕ_1} *be the event that* $|L_\phi(v)| < \ell(v)$ *and*
2. B_{v,ϕ_1} *be the event that there is a color* $c \in L_\phi(v)$ *such that* $d_{L_\phi, M_\phi}(v, c) > 24 \log \Delta$.

Then $\mathbb{P}[A_{v,\phi_1}], \mathbb{P}[B_{v,\phi_1}] \le \Delta^{-3}/8$.

Proof. First we prove 1. We show that $\mathbb{E}[|L_\phi(v)|]$ is larger than $\ell(v)$ and then we use Lemma 2.2.3 to show that $|L_\phi(v)|$ is close to its expectation.

For each $c \in L(v)$, let $N(v, c)$ be the set of vertices $u \in N(v)$ for which there exists $c' \in L_{\phi_1}(u)$ such that c' corresponds to c at v. Now for each $c \in L(v)$,

$$\mathbb{P}[c \in L_\phi(v)] = \prod_{u \in N(v,c)} \left(1 - \frac{1}{|L_{\phi_1}(u)| + 1}\right).$$

Hence, since $\exp(-1/\alpha) \le 1 - 1/(\alpha+1) \le \exp(-1/(\alpha+1))$ for every $\alpha > 0$, we have

$$\exp\left(-\sum_{u\in N(v,c)} \frac{1}{|L_{\phi_1}(u)|}\right) \le \mathbb{P}\left[c \in L_\phi(v)\right] \le \exp\left(-\sum_{u\in N(v,c)} \frac{1}{|L_{\phi_1}(u)|+1}\right). \tag{6.10}$$

Thus, by Linearity of Expectation,

$$\mathbb{E}\left[|L_\phi(v)|\right] = \sum \mathbb{P}\left[c \in L_\phi(v)\right] \ge \sum_{c\in L(v)} \exp\left(-\sum_{u\in N(v,c)} \frac{1}{|L_{\phi_1}(u)|}\right). \tag{6.11}$$

Since $\sum_{c\in L(v)} \sum_{u\in N(v,c)} 1/|L_{\phi_1}(u)| \le \sum_{u\in N(v):|L_{\phi_1}|>0} \sum_{c'\in L_{\phi_1}(u)} 1/|L_{\phi_1}(u)| \le d(v)$, by convexity, the right side of (6.11) is at least

$$|L(v)| \exp\left(-\frac{d(v)}{|L(v)|}\right) \ge \frac{d(v)}{\ln d(v)} \exp\left(-\frac{\ln d(v)}{1+\xi}\right) = \frac{d(v)^{\xi/(1+\xi)}}{\ln d(v)}.$$

Therefore since $d(v) = \Omega(\ln \Delta^{2/\xi})$ and Δ is sufficiently large, we have that $\mathbb{E}\left[|L_\phi(v)|\right] \ge 2\ell(v)$. As in Lemma 6.2.4, the indicator random variables for the events $c \notin L_\phi(v)$ are negatively correlated. Therefore by Lemma 2.2.3,

$$\mathbb{P}\left[A_{v,\phi_1}\right] \le \mathbb{P}\left[|L_\phi(v)| < \mathbb{E}\left[|L_\phi(v)|\right]/2\right] \le 2\exp(-\mathbb{E}\left[|L_\phi(v)|\right]/12) = O(\exp(-d(v)^{\xi/2}).$$

Since $d(v) \ge (192 \ln \Delta)^{2/\xi}$, the right side of the previous inequality is at most $\Delta^{-3}/8$ for sufficiently large Δ, so 1 holds.

For each $c \in L(v)$, let $B_{v,\phi_1,c}$ be event that $c \in L_\phi(v)$ and $d_{L_\phi,M_\phi}(v,c) > 24 \log \Delta$. We claim that $\mathbb{P}\left[B_{v,\phi_1,c}\right] \le \Delta^{-4}$ for each $c \in L(v)$. Suppose not. Since $\mathbb{P}\left[B_{v,\phi_1,c}\right] \le \mathbb{P}\left[c \in L_\phi(v)\right]$, the right side of (6.10) is at least Δ^{-4}. Therefore

$$\mathbb{E}\left[d_{(L_\phi,M_\phi)}(v,c)\right] = \sum_{u\in N(v,c)} \mathbb{P}\left[\phi_2(u) = \text{BLANK}\right] = \sum \frac{1}{|L_{\phi_1}(u)|+1} \le 4 \ln \Delta.$$

The indicator random variables for the events $\phi_2(u) = \text{BLANK}$ are mutually independent, so by the one-sided Chernoff Bound from Chapter 2 with $\delta = 5$, we have $\mathbb{P}\left[B_{v,\phi_1,c}\right] \le \Delta^{-4}$, as claimed. Therefore by the Union Bound, $\mathbb{P}\left[B_{v,\phi_1}\right] \le \sum_{c\in L(v)} \mathbb{P}\left[B_{v,\phi_1,c}\right] \le \Delta^{-3}/8$, as desired, so 2 holds. $\square$

Proof of Theorem 1.3.12. Let ϕ be the partial (L, M)-coloring of G satisfying (1) and (2) of Lemma 6.4.1. By Proposition 6.2.2, it suffices to show that the ϕ-uncolored vertices induce a graph that is (L_ϕ, M_ϕ)-colorable. Since $d(v) \geq (192 \ln \Delta)^{2/\xi}$ for each vertex v, by (1) we have $|L_\phi(v)| \geq 192 \ln \Delta$. Letting $k = 192 \ln \Delta$, by (2) every $c \in L_\phi(v)$ satisfies $d_{(L_\phi, M_\phi)}(v, c) \leq k/8$. Therefore the ϕ-uncolored vertices induce a graph that is (L_ϕ, M_ϕ)-colorable by Theorem 6.1.1, as desired. □

Part II

Fractional coloring

In this part we prove our results on fractional coloring with local demands. First, in Chapter 7, we establish some preliminary results that will be useful in Chapters 9 and 10. We also prove a "local version" of a result of Alon, Tuza, and Voigt [6] that the fractional list chromatic number of every graph equals its fractional chromatic number, and we provide three different proofs of Theorem 1.5.1, the "local fractional greedy bound."

We prove analogues of χ-boundedness results in Chapter 8. Using an equivalence between multicoloring and coloring blowups of a graph, in Section 8.1 we prove Theorem 8.1.2 and use it to derive a number of results about χ-bounded classes of graphs with a linear χ-binding function, including perfect graphs. Using Edmonds' Matching Polytope Theorem [52], in Section 8.2 we prove results about edge-coloring, including Theorem 1.5.9, the local demands version of the generalized Vizing's Theorem.

In Chapter 9, we discuss local demands for graphs of small clique number and prove Theorems 1.5.12 and 1.5.13, which are approximations of the local demands analogues of Theorems 1.2.9 and 1.2.10, respectively. We actually use Theorems 1.2.9 and 1.2.10 to obtain the fractional colorings of Theorems 1.5.12 and 1.5.13. We refine this approach and reduce Conjecture 1.5.11, the local demands version of Shearer's [137] and Molloy's [107] bound on the independence number and chromatic number, respectively, of triangle-free graphs, to Conjecture 9.3.2, a "list-local color degree" version of Theorem 1.2.9.

Finally, in Chapter 10, we prove the local demands version of Brooks' Theorem, Theorem 1.5.2, which is by far the most difficult result of this part. We also discuss extensions of Theorem 1.5.2 for graphs of large minimum degree in Section 10.1, which could be considered analogues of the results of [59, 110].

With the exception of Chapter 9, this part is based on the preprint [86] (with Luke Postle).

Chapter 7

Fractional coloring preliminaries

7.1 List versions

It is natural to formulate a list coloring version of fractional coloring, but there are a few different possible ways of doing this. Using Proposition 1.1.6 (b) and the notion of multicoloring, one could formulate a definition of the *fractional list chromatic number*. However, as we discuss in Section 7.1.2, Alon, Tuza, and Voigt [6] showed that this invariant is always equal to the fractional chromatic number, and we generalize their result to the setting of local demands with Theorem 7.1.11. In Section 7.1.1, we instead consider the notion of a *fractional list-assignment* and present some applications that will be useful for us in Section 7.2 and Chapters 9 and 10.

7.1.1 Fractional list-assignments

In this subsection, we discuss analogues of list coloring in the setting of "local demands" for fractional coloring. Some of the results in this subsection are needed in Section 7.2 and Chapter 9 and 10.

Definition 7.1.1. Let G be a graph with demand function f.

- If L is a function with domain $V(G)$ such that $L(v)$ is a measurable subset of the $[0, 1]$-interval for each $v \in V(G)$, then L is a *fractional list-assignment* for G. If there exists some constant c such that each $v \in V(G)$ satisfies $\mu(L(v)) = c$, then L is *c-uniform*.

- A fractional coloring ϕ of G such that every vertex $v \in V(G)$ satisfies $\phi(v) \subseteq L(v)$ is called a *fractional L-coloring.*
- A *fractional (f, L)-coloring* of G is a fractional L-coloring of G that is also an f-coloring.

In Section 7.2 and Chapter 9 and 10, we often find a fractional coloring of an induced subgraph of a graph G and try to extend it to all of G. To that end, we introduce the following notation.

Definition 7.1.2. If ϕ is a fractional coloring of $G[S]$, then we let L_ϕ be a fractional list-assignment for $G - S$ where for each $v \in V(G - S)$,

$$L_\phi(v) = [0,1] \setminus \bigcup_{u \in S \cap N(v)} \phi(u).$$

We consider the following proposition to be self-evident.

Proposition 7.1.3. *Let G be a graph with demand function f. If for some $S \subseteq V(G)$, ϕ is an $f|_S$ coloring of $G[S]$ such that $G - S$ has a fractional $(f|_{V(G)\setminus S}, L_\phi)$-coloring, then G has an f-coloring.* □

If ϕ is a fractional coloring of $G[S]$ for some $S \subseteq V(G)$ and $u \in V(G)$, then v *sees the color* $\bigcup_{u \in S \cap N(v)} \phi(u)$, and if $\mu(\bigcup_{u \in S \cap N(v)} \phi(u)) \leq \alpha$, then v *sees at most α color.* If ϕ' is a fractional coloring of $G[S']$ for some $S' \subseteq V(G)$ such that $S \subseteq S'$ and $\phi = \phi|_S$, then ϕ' *extends* ϕ.

In fractional coloring, one can partition the $[0,1]$-interval as finely as needed and find fractional colorings in each part separately. The following lemma makes use of this idea.

Lemma 7.1.4. *Let G be a graph with demand functions f and g and fractional list-assignment L such that each $v \in V(G)$ satisfies $g(v) \leq f(v)\mu(L(v))$. If $G[S]$ has an f-coloring for each $S \subseteq V(G)$ such that $\mu\left(\cap_{v \in S} L(v)\right) > 0$, then G has a fractional (g, L)-coloring.*

Proof. For each $S \subseteq V(G)$, let

$$C_S = (\cap_{v \in S} L(v)) \setminus (\cup_{v \in V(G)\setminus S} L(v)),$$

and for each $v \in S$, let $L_S(v) = C_S$ and $f_S = f(v) \cdot \mu\left(C_S\right)$. Note that if $S \neq S'$, then $C_s \cap C_{S'} = \varnothing$. By assumption, $G[S]$ has an f-coloring, so $G[S]$ has an (f_S, L_S)-coloring ϕ_S. For each $v \in V(G)$, let $\phi(v) = \cup_{S \ni v} \phi_S(v)$. Now ϕ is a fractional (g, L)-coloring, as desired. □

Lemma 7.1.4 implies that if a graph G has an (f/c)-coloring for some $c \in (0,1)$, then G has an (f,L)-coloring for any c-uniform fractional list-assignment L. That is, the "worst" uniform fractional list-assignment is the one which assigns the same list to every vertex. In Chapter 10 in the proof of Theorem 1.5.2, we encounter a uniform fractional list-assignment, and we go to considerable length to ensure that vertices have different lists.

We will often use the following lemma of Edwards and King [53, Lemma 3], which is proved using Hall's Theorem.

Lemma 7.1.5 (Edwards and King [53]). *If H is a graph with demand function g and fractional list-assignment L such that for each $S \subseteq V(H)$,*

$$\sum_{v\in S} g(v) \leq \mu\left(\bigcup_{v\in S} L(v)\right),$$

then H has a (g,L)-coloring.

Using Lemma 7.1.5, we prove the following lemma, which may be of independent interest.

Lemma 7.1.6. *Let $H \cong K_n - M$ where M is a matching, and let g be a demand function for H. If L is a fractional list-assignment for H such that*

(i) for each $v \in V(H)\backslash V(M)$, we have $\mu(L(v)) \geq \sum_{u\in V(H)\backslash V(M)} g(u) + \sum_{uw\in M} \max\{g(u), g(w)\}$,

(ii) for each $v \in V(M)$, we have $\mu(L(v)) \geq g(v) + \sum_{uw\in M, v\notin\{u,w\}} \max\{g(u), g(w)\}$, and

(iii) for each $uv \in M$, we have $\mu(L(u)) + \mu(L(v)) \geq \sum_{w\in V(H)} g(w)$,

then H has a fractional (g,L)-coloring.

Proof. Suppose not. Choose H, g, and L such that H has no fractional (g,L)-coloring and the number of edges $uv \in M$ such that $\mu(L(u)\cap L(v)) \neq \varnothing$ is minimum, and subject to that, the number of vertices $u \in V(M)$ such that $g(u) = 0$ is maximum.

First, suppose $\mu(L(u)\cap L(v)) = \varnothing$ for each $uv \in M$. By Lemma 7.1.5, there exists $S \subseteq V(H)$ such that $\sum_{v\in S} g(v) < \mu(\cup_{v\in S} L(v))$. By (i), if $S \setminus V(M) \neq \varnothing$, then there exists $uw \in M$ such that $\{u,w\} \subseteq S$, and by (ii), if $S \cap V(M) \neq \varnothing$, then there exists $uw \in M$ such that $\{u,w\} \subseteq S$. Hence, there exists $uw \in M$ such that $\{u,w\} \subseteq S$. Since $\mu(L(u)\cap L(v)) = \varnothing$, we have $\mu(\cup_{v\in S} L(v)) \geq \mu(L(u)) + \mu(L(w))$. Therefore by (iii), $\mu(\cup_{v\in S} L(v)) \geq \sum_{v\in V(H)} g(v)$, contradicting that $\mu(\cup_{v\in S} L(v)) < \sum_{v\in S} g(v)$.

Therefore we may assume there exists $xy \in M$ such that $\mu(L(x)\cap L(y)) \neq \varnothing$. We may assume without loss of generality that $g(x) \leq g(y)$. Let C be a maximal subset of $L(x)\cap L(y)$ of measure at most $g(x)$. Now

- for each $v \in \{x, y\}$, let $g'(v) = g(v) - \mu(C)$ and $L'(v) = L(v) \setminus C$, and
- for each $v \in V(H) \setminus \{x, y\}$, let $g'(v) = g(v)$ and $L'(v) = L(v) \setminus C$.

Note that either $L'(x) \cap L'(y) = \varnothing$ or $g'(x) = 0$. Now we claim that H has a fractional (g', L')-coloring. By (i), for each $v \in V(H) \setminus V(M)$,

$$\mu(L'(v)) \geq \mu(L(v)) - \mu(C) \geq \sum_{u \in V(H) \setminus V(M)} g'(u) + \sum_{uw \in M} \max\{g'(u), g'(w)\}.$$

By (ii), for each $v \in V(M)$,

$$\mu(L'(v)) \geq \mu(L(v)) - \mu(C) \geq g'(v) + \sum_{uw \in M, v \notin \{u,w\}} \max\{g'(u), g'(w)\}.$$

By (iii), for each $uv \in M$,

$$\mu(L'(u)) + \mu(L'(v)) \geq \mu(L(u)) + \mu(L(v)) - 2\mu(C) \geq \sum_{w \in V(H)} g'(w).$$

By the choice of H, g and L, the graph H has a fractional (g', L')-coloring, as claimed, contradicting that H has no fractional (g, L)-coloring. □

Lemma 7.1.6 provides an example of an interesting problem concerning fractional list-assignments that are not necessarily uniform. For a recent example concerning triangle-free graphs, see [42, Theorem 2].

We apply Lemma 7.1.6 in the special case when $|M| = 1$ in Chapter 10, as in the following corollary.

Corollary 7.1.7. *Let $H \cong K_n - xy$ where $xy \in E(K_n)$, and let g be a demand function for H. If L is a fractional list-assignment for H such that*

(i) for each $v \in V(H) \setminus \{x, y\}$, we have $\mu(L(v)) \geq \max\{g(x) + g(y)\} + \sum_{u \in V(H) \setminus \{x,y\}} g(u)$,
(ii) $\mu(L(x)) \geq g(x)$ and $\mu(L(y)) \geq g(y)$, and
(iii) $\mu(L(x)) + \mu(L(y)) \geq \sum_{v \in V(H)} g(v)$,

then H has a fractional (g, L)-coloring.

7.1.2 The fractional list chromatic number

When Erdős, Rubin, and Taylor [56] introduced list coloring, they also introduced the following definition that is related to fractional coloring: a graph G is $(a : b)$-*choosable* if for every a-list-assignment L, there is a multicoloring ϕ that assigns to each vertex $v \in V(G)$ a b-subset of $L(v)$. If G is $(a : b)$-choosable, then G has a multicoloring using b colors such that each vertex receives $f(v) \cdot b$ colors where f is a demand function for G such that $f(v) = a/b$ for each $v \in V(G)$. Erdős, Rubin, and Taylor [56] posed the following question regarding $(a : b)$-choosability.

Problem 7.1.8 (Erdős, Rubin, and Taylor [56])**.** *If a graph G is (a, b)-list-colorable, is it (am, bm)-list-colorable for every positive integer m?*

Recently, Dvořák, Hu, and Sereni [48] resolved Problem 7.1.8 with an answer of "no": they proved the existence of a 4-choosable graph that is not $(8 : 2)$-choosable. However, Alon, Tuza, and Voigt [6] proved that Problem 7.1.8 is true in a strong sense for large m, as follows.

Theorem 7.1.9 (Alon, Tuza, and Voigt [6])**.** *If a graph G is (a, b)-colorable, then there exists an integer $m > 1$ such that G is (am, bm)-list-colorable.*

As alluded to previously, one can define the fractional list chromatic number, as follows.

Definition 7.1.10. Let G be a graph with demand function f, and let $N \in \mathbb{N}$.

- If L is an N-list-assignment for G, an *f-fold L-coloring* of G is an assignment ψ of subsets of $L(v)$ to the vertices of G such that for every $uv \in E(G)$, $\psi(u) \cap \psi(v) = \varnothing$ and for every $v \in V(G)$, $|\psi(v)| \geq N \cdot f(v)$.
- The graph G is *(f, N)-list-colorable* if it has an f-fold L-coloring for every N-list-assignment L.

The *fractional list chromatic number* is the infimum over all positive real numbers k such that G is (f, N)-list-colorable for some $N \in \mathbb{N}$ where f is a demand function for G such that $f(v) = 1/k$ for each $v \in V(G)$.

Proposition 1.1.6 and Theorem 7.1.9 imply that the fractional list chromatic number is equal to the fractional chromatic number. It is natural to wonder if this situation changes if we consider demand functions that are not constant. In this section, we show that Theorem 7.1.9 holds for any demand function, as follows.

Theorem 7.1.11. *Let G be a graph with demand function f, and let $N \in \mathbb{N}$. If G has an (f, N)-coloring, then there exists an $M > N$ such that G is (f, M)-list-colorable.*

The proof of Theorem 7.1.11 is similar to the proof of Theorem 7.1.9. We need the following two lemmas from [6].

Lemma 7.1.12 (Alon, Tuza, and Voigt [6]). *Let $(n_i : i \in I)$ be a sequence of positive integers, where each n_i is at most k. Let M and N be two integers and suppose that $\sum_{i\in I} n_i = M$, that M/N is divisible by all integers up to k, and that $k \cdot \mathrm{lcm}(2, 3, \ldots, k) \leq M/N$, where $\mathrm{lcm}(2, 3, \ldots, k)$ denotes the least common multiple of $2, 3, \ldots, k$. Then there is a partition $I = I_1 \cup I_2 \cup I_N$ of I into N pairwise disjoint sets such that for every $j \in \{1, \ldots, N\}$, $\sum_{i\in I_j} n_i = M/N$.*

Lemma 7.1.13 (Alon, Tuza, and Voigt [6]). *If $H = (X, F)$ is a uniform hypergraph with n edges, then there is a partition $X = \cup_{i\in I} X_i$ of X into pairwise disjoint sets such that H_{X_i} is n_i-uniform and $n_i \leq (n + 1)^{(n+1)/2}$ for every $i \in I$.*

Proof of Theorem 7.1.11. By possibly choosing a larger N, we may assume $f(v) \cdot N$ is an integer for each $v \in V(G)$. Let $n = |V(G)|$, let $k = (n + 1)^{(n+1)/2}$, and choose M to be divisible by all integers up to $\max\{k^4, N^4\}$.

We show that G is (f, M)-list-colorable. To that end, let L be an M-list-assignment for G. It suffices to show that G has an f-fold L-coloring. Let H be the hypergraph with vertex set $X = \cup_{v\in V(G)} L(v)$ and edges $F = (L(v) : v \in V(G))$. By Lemma 7.1.13, there is a partition $(X_i : i \in I)$ of the colors X so that for each i and each $v \in V(G)$,

$$|L(v) \cap X_i| = n_i,$$

where $n_i \leq k$. Note that $\sum_{i\in I} n_i = M$. By Lemma 7.1.12, there is a partition $I_1, \ldots, I_N$ of I such that for each $j \in \{1, \ldots, N\}$, $\sum_{i\in I_j} n_i = M/N$.

By assumption, G has an (f, N)-coloring ψ. For each $v \in V(G)$, let

$$\psi'(v) = \bigcup_{j\in\psi(v)} \cup_{i\in I_j} \{L(v) \cap X_i\}.$$

Now,

$$|\psi'(v)| = \sum_{j\in\psi(v)} \sum_{i\in I_j} n_i = \sum_{j\in\psi(v)} M/N = |\psi(v)|M/N = f(v) \cdot M,$$

and $\psi'(u)$ and $\psi'(v)$ are disjoint when u and v are adjacent. Hence, ψ' is an f-fold L-coloring, as desired. □

7.2 The local fractional greedy bound

In this section we present three different proofs of Theorem 1.5.1. The first proof uses Definition 1.1.4, and it is inspired by Wei's [149] original proof of the Caro-Wei Theorem. The proof is also suggestive of our approach to Theorem 1.5.2 in Chapter 10.

First Proof of Theorem 1.5.1. Suppose G is a minimum counterexample to Theorem 1.5.1, that is a graph with the fewest number of vertices having no f-coloring where f is a demand function satisfying $f(v) \leq 1/(d(v)+1)$ for each v. Let $v \in V(G)$ have minimum degree. Since G is a minimum counterexample, $G - v$ has an f-coloring. Note that for each $u \in N(v)$, $f(u) \leq 1/(d(v)+1)$. Therefore v sees at most $d(v)/(d(v)+1)$ color, so $\mu\left(L_\phi(v)\right) \geq f(v)$. Thus, G is f-colorable, contradicting that G is a counterexample. $\square$

The second proof of Theorem 1.5.1 is due to Alon and Spencer [5]. We use the fact that (c) in Proposition 1.1.6 is equivalent to the existence of a finite probability space $(\mathcal{I}, \mathbb{P})$ where $\mathcal{I}$ is the set of independent sets in G such that if $\mathbf{I} \in \mathcal{I}$ is sampled according to the distribution then $\mathbb{P}\left[v \in \mathbf{I}\right] \geq f(v)$ for each vertex $v \in V(G)$.

Second Proof of Theorem 1.5.1. Let I be a random independent set of G selected according to the following distribution. Choose a total ordering $\prec$ of $V(G)$ uniformly at random, and let $v \in I$ if $v \prec u$ for all $u \in N(v)$. Note that each vertex is in I with probability $1/(d(v)+1)$. Therefore by Proposition 1.1.6 (c), G has an f-coloring, as desired. $\square$

The last proof of Theorem 1.5.1 that we present uses the concept of a fractional list-assignment as discussed in Section 1.1.2. It is inspired by a proof of the Caro-Wei Theorem due to Griggs [64].

Third Proof of Theorem 1.5.1. Suppose G is a minimum counterexample. Let $v \in V(G)$ such that $f(v)$ is minimum, and let $\phi(v) \subseteq [0,1]$ have measure at least $f(v)$. Let f' be the demand function for $G-v$ such that for each $u \in V(G-v)$, $f'(u) = 1/(d_{G-v}(u)+1)$. Since G is a minimum counterexample, $G - v$ has an f'-coloring. Note that for each $u \in N(v)$,

$$f'(u)\mu\left(L_\phi(v)\right) = \frac{1 - f(v)}{d_{G-v}(u)+1} \geq \frac{1 - f(u)}{d_G(u)} \geq \frac{1 - 1/(d_G(u)+1)}{d_G(u)} \geq \frac{1}{d_G(u)+1} \geq f(u).$$

Therefore by Lemma 7.1.4, $G - v$ has an (f, L_ϕ)-coloring, contradicting Proposition 7.1.3. $\square$

It is plausible that one could prove Theorem 1.5.2 using the approach of either the second or third proof of Theorem 1.5.1, but we were unable to do so.

Chapter 8

χ-boundedness and edge-coloring

Recall that a class of graphs is χ-bounded if the chromatic number of every graph in the class is bounded by a function of its clique number. In this case, such a function is called a *χ-binding* function. There is an enormous amount of research devoted to studying χ-boundedness. In the 1980s, Gyárfás [68] posed several beautiful conjectures that were critical in popularizing the subject. A *hole* in a graph is an induced cycle of length at least four, and an *antihole* is an induced subgraph whose complement is a hole in the complement graph. The Strong Perfect Graph Theorem [38] implies that graphs with no odd hole or odd antihole are χ-bounded by the identity function. At that time, it was not known if this family of graphs is χ-bounded at all, so Gyárfás [68, Conjecture 3.2] proposed the "Weakened Strong Perfect Graph Conjecture." He conjectured three strengthenings of this problem: that graphs with no odd holes, graphs with no long holes (that is, holes of length at least ℓ, for any ℓ), and graphs with no long odd holes are χ-bounded. Note that the third case includes the former two. The first of these conjectures was recently proved by Scott and Seymour [131], the second by Chudnovsky, Scott, and Seymour [39], and the third by Chudnovsky, Scott, Seymour, and Spirkl [40]. Finally, Scott and Seymour [132] generalized all of these results by proving that graphs with no holes of a specific residue are χ-bounded.

Gyárfás [68] also asked for which graphs H is the class of *H-free* graphs, that is the graphs with no induced subgraph isomorphic to H, χ-bounded. The Gyárfás-Sumner Conjecture [67, 141] states that if H is a tree, then the class of H-free graphs is χ-bounded. Note that since there exist graphs of arbitrarily large girth and chromatic number as proved by Erdős [54], it is necessary that H not contain a cycle. The conjecture has been proved for various families of trees, but it remains open.

Finding the best possible χ-binding functions for various graph classes has also been extensively studied. In addition to classes of graphs characterized by excluded induced subgraphs, geometric graphs have received much attention. Many of these graphs are described by the intersections of a collection of geometric objects. Each object corresponds to a vertex and vertices are adjacent if the corresponding objects intersect. For example, Asplund and Grünbaum [8] proved that the family of intersection graphs of axis-parallel rectangles is χ-bounded, and determining the optimum χ-binding function remains an interesting open problem. Peeters [117] proved that intersection graphs of unit disks are χ-bounded with function $3\omega - 2$, Malesińska, Piskorz, and Weißenfels [106], proved that intersection graphs of disks are χ-bounded with function $6\omega - 6$. Kim, Kostochka, and Nakprasit [92] generalized these results for any fixed convex compact set.

There are many more interesting results and problems in the study of χ-boundedness. For more information, see the recent survey of Scott and Seymour [133].

In this chapter we prove Theorem 1.5.8 in Section 8.1 and Theorem 1.5.9 in Section 8.2. In Section 8.1 we introduce the notion of *local-fractional χ-boundedness*, and we actually derive Theorem 1.5.8 from Theorem 8.1.2, a more general result about fractional coloring graphs with a linear χ-binding function. We derive a number of other results as well. Since Vizing's Theorem [146] implies that the class of line graphs is χ-bounded with function $\omega + 1$, our results in Section 8.2 on fractional edge-coloring also fit into the paradigm of local-fractional χ-boundedness.

8.1 χ-boundedness

First, we introduce the concept of local-fractional χ-boundedness.

Definition 8.1.1. Let $\mathcal{G}$ be a class of graphs.

- If there exists a function $g : \mathbb{N} \to \mathbb{R}$ such that for all $G \in \mathcal{G}$, and all induced subgraphs H of G, H has an f-coloring for every demand function f such that for each $v \in V(H)$, $f(v) \leq 1/g(\omega(v))$, then $\mathcal{G}$ is *local-fractionally χ-bounded*;
- in this case, the class $\mathcal{G}$ is *local-fractionally χ-bounded* by g and g is a *local-fractional χ-binding function* for $\mathcal{G}$.

The following is the main result of this subsection.

Theorem 8.1.2. *If $\mathcal{G}$ is a χ-bounded class of graphs with χ-binding function $g(n) = c \cdot n$ for some $c \in \mathbb{R}$ and $\mathcal{G}$ is closed under taking blowups, then $\mathcal{G}$ is local-fractionally χ-bounded by g.*

Theorem 8.1.2 has a number of corollaries. In particular, note that Theorem 1.5.8 follows from Theorem 8.1.2. In [29, Theorem 2.6], Brause et al. proved that the weaker version of Conjecture 1.5.7 for the independence number holds for perfect graphs. Theorem 1.5.8 generalizes this result to its weighted version and also improves the bound. It also confirms Conjecture 1.5.7 for perfect graphs. Theorem 1.5.8 also implies that if G is a bipartite graph, then the line graph of G has an f-coloring if $f(uv) \leq 1/\max\{d(u), d(v)\}$ for each $uv \in E(G)$, which could be considered the local demands version of the fractional relaxation of König's Line Coloring Theorem.

A graph is *quasiline* if the neighborhood of every vertex is the union of two cliques. Combining Theorem 8.1.2 with the main result of [37], we obtain the following.

Corollary 8.1.3. *If G is a quasiline graph with demand function f such that for each $v \in V(G)$, $f(v) \leq 2/(3\omega(v))$, then G has an f-coloring.*

The *claw* is the graph $K_{1,3}$. Combining Theorem 8.1.2 with the main result of [41], we obtain the following.

Corollary 8.1.4. *If G is a claw-free graph with independence number at least three, and if f is a demand function for G such that for each $v \in V(G)$, $f(v) \leq 1/(2\omega(v))$, then G has an f-coloring.*

Since duplicating a geometric object corresponds to blowing up a vertex in the intersection graph, we can combine Theorem 8.1.2 with the aforementioned result of [92] to obtain the following.

Corollary 8.1.5. *If G is the intersection graph of homothets of a fixed convex compact set in the plane with demand function f such that each $v \in V(G)$ satisfies $f(v) \leq 1/(6\omega(v))$, then G has an f-coloring. Moreover, if G is the intersection graph of translates of a fixed convex compact set in the plane and each $v \in V(G)$ satisfies $f(v) \leq 1/(3\omega(v))$, then G has an f-coloring.*

It is interesting to ask if there are other natural local-fractionally χ-bounded classes of graphs, and if so, what is their optimal local-fractional χ-binding function. In particular, the discussion at the beginning of this chapter provides many examples to consider. In Section 8.2, we prove such results about line graphs, and Conjecture 1.5.10 concerns total graphs.

8.1.1 Proof of Theorem 8.1.2

The remainder of this section is devoted to the proof of Theorem 8.1.2. First we need the following lemma.

Lemma 8.1.6. *If G is a graph with demand function f such that for each $v \in V(G)$, we have $f(v) = 1/(c\omega(v))$ for some constant c, and $f(v) \cdot N \in \mathbb{N}$ for each vertex $v \in V(G)$, then the graph obtained from G by replacing each vertex with a clique of size $N \cdot f(v)$ contains no clique of size greater than N/c.*

Proof. Let G' be the graph obtained from G by replacing each vertex with a clique of size $N \cdot f(v)$, and for each $v \in V(G)$, let K_v denote the clique in G' replacing v. Let $K \subseteq V(G')$ be a clique in G', and let $X = \{v \in V(G) : K_v \cap K \neq \varnothing\}$. Since K is a clique in G', we have that X is a clique in G. Hence, for each $v \in X$, we have that $|K_v| \leq N/(c|X|)$. Note that $|K| \leq \sum_{v \in X} |K_v|$. Therefore $|K| \leq N/c$, so G' contains no clique of size greater than N/c, as desired. □

Now the proof of Theorem 8.1.2 follows easily.

Proof of Theorem 8.1.2. Let H be an induced subgraph of a graph in $\mathcal{G}$, and let f be a demand function for H such that $f(v) = 1/g(\omega(v))$ for each $v \in V(H)$. It suffices to show that H has an f-coloring. Note that by Proposition 1.1.6 (b), it suffices to show that the graph H' obtained from H by replacing each vertex with a clique of size $N \cdot f(v)$ has chromatic number at most N. By Lemma 8.1.6, $\omega(H') \leq N/c$, and since $\mathcal{G}$ is closed under taking blowups, $H' \in \mathcal{G}$. Since $\mathcal{G}$ is χ-bounded with χ-binding function $g(n) = c \cdot n$, it follows that $\chi(H') \leq N$, as desired. □

8.2 Edge-Coloring

In this section we consider fractional edge-coloring with local demands. Vizing's Theorem [146] states that every graph G can be edge-colored with at most $\Delta(G) + 1$ colors. Equivalently, it states that every graph G satisfies $\chi(L(G)) \leq \Delta(G) + 1$, where $L(G)$ is the line graph of G. We prove the following, which is the local demands version of this result.

Theorem 8.2.1 (Local Fractional Vizing's)**.** *If G is a graph and f is a demand function for $L(G)$ such that each $e \in V(L(G))$ where $e = uv \in E(G)$ satisfies $f(e) \leq 1/(\max\{d(u), d(v)\} + 1)$, then $L(G)$ has an f-coloring.*

Vizing's Theorem can be generalized to multigraphs, as follows. Every multigraph G can be edge-colored with at most $\Delta(G) + \max_{uv \in E(G)} |uv|$ colors, where $\Delta(G)$ is the maximum degree of the underlying simple graph and $|uv|$ is the *multiplicity* of the edge uv, or the number of edges in G incident to both u and v. In this section, if G is a multigraph and $v \in V(G)$, we use $|N(v)|$ to denote the number of neighbors of v and let $d(v) = \sum_{u \in N(v)} |uv|$. In this section, we prove Theorem 1.5.9, which is the local demands version of this generalization of Vizing's Theorem. Note that Theorem 8.2.1 follows from Theorem 1.5.9, so we only prove Theorem 1.5.9.

We also prove the following local demands version of a theorem of Shannon [135].

Theorem 8.2.2 (Local Fractional Shannon's)**.** *If G is a multigraph and f is a demand function for $L(G)$ such that each $e \in V(L(G))$ with $e = uv \in E(G)$ satisfies $f(e) \leq 2/(3\max\{d(u), d(v)\})$, then $L(G)$ has an f-coloring.*

Since line graphs of multigraphs are quasiline, Theorem 8.2.2 follows from Corollary 8.1.3; however, since Corollary 8.1.3 relies on the results of [37], we provide a more direct proof of Theorem 8.2.2 in this section.

In order to prove each of these theorems, we will need Edmonds' Matching Polytope Theorem [52], which, in light of Proposition 1.1.6, characterizes all of the fractional colorings of a line graph.

Theorem 8.2.3 (Edmonds' Matching Polytope Theorem [52])**.** *If G is a simple graph and f a demand function for $L(G)$, then $L(G)$ has an f-coloring if and only if for all $v \in V(G)$,*

$$\sum_{u \in N(v)} f(uv) \leq 1, \tag{8.1}$$

and for every $S \subseteq V(G)$,

$$\sum_{e \in E(G[S])} f(e) \leq \lfloor |S|/2 \rfloor. \tag{8.2}$$

In order to show that (8.2) holds, we need the following lemma.

Lemma 8.2.4. *If G is a simple graph, then*

$$\sum_{v \in V(G)} \frac{d(v)}{d(v)+1} \leq |V(G)| - 1.$$

Proof. We use induction on $|V(G)|$. If $|V(G)| = 1$, then there are no edges and the lemma follows. Therefore we may assume $|V(G)| > 1$.

Let $v \in V(G)$ have minimum degree. By induction,

$$\sum_{u \in V(G-v)} \frac{d_{G-v}(u)}{d_{G-v}(u)+1} \leq |V(G)| - 2.$$

Therefore it suffices to show that

$$\frac{d_G(v)}{d_G(v)+1} + \sum_{u \in N(v)} \frac{d_G(u)}{d_G(u)+1} - \frac{d_{G-v}(u)}{d_{G-v}(u)+1} \leq 1. \tag{8.3}$$

If $u \in N(v)$, then $d_{G-v}(u) = d_G(u) - 1$. Hence,

$$\frac{d_G(u)}{d_G(u)+1} - \frac{d_{G-v}(u)}{d_{G-v}(u)+1} = \frac{1}{d_G(u)(d_G(u)+1)} \leq \frac{1}{d_G(v)(d_G(v)+1)}.$$

Therefore

$$\sum_{u \in N(v)} \frac{d_G(u)}{d_G(u)+1} - \frac{d_{G-v}(u)}{d_{G-v}(u)+1} \leq \frac{1}{d_G(v)+1},$$

and (8.3) follows, as required. □

Before proving Theorem 1.5.9, we show that the proof essentially reduces to the case of simple graphs using the following lemma.

Lemma 8.2.5. *If G is a multigraph and $v \in V(G)$,*

$$\sum_{u \in N(v)} \frac{|uv|}{d(v)+|uv|} \leq \frac{|N(v)|}{1+|N(v)|}.$$

Proof. Note that by definition, $d(v) = \sum_{u \in N(v)} |uv|$. Note also that $\frac{x}{d(v)+x}$ is concave as a function of x, so by Jensen's Inequality,

$$\frac{\sum_{u \in N(v)} |uv|/(d(v)+|uv|)}{|N(v)|} \leq \frac{d(v)/|N(v)|}{d(v)+d(v)/|N(v)|} = \frac{1/|N(v)|}{1+1/|N(v)|}.$$

Rearranging terms,

$$\sum_{u \in N(v)} |uv|/(d(v)+|uv|) \leq \frac{|N(v)|}{1+|N(v)|},$$

as desired. □

We can now prove Theorem 1.5.9.

Proof of Theorem 1.5.9. Let G' be the underlying simple graph of G, and let f' be a demand function for $L(G')$ such that for each $e \in V(L(G'))$ where $e = uv \in E(G')$, we have $f'(e) = |uv|/(\max\{d_G(u), d_G(v) + |uv|)$. It suffices to show that $L(G')$ has an f'-coloring, because then $L(G)$ has an f-coloring, as desired. By Theorem 8.2.3, it suffices to show that (8.1) and (8.2) hold for G' and f'.

For each $v \in V(G)$ and $u \in N(v)$, we have $f'(uv) \leq |uv|/(d_G(v) + 1)$. Hence, for each $v \in V(G)$, we have $\sum_{u \in N(v)} f'(uv) \leq d_G(v)/(d_G(v) + 1) \leq 1$, so (8.1) holds, as desired.

Let $S \subseteq V(G)$, and note that

$$2 \sum_{e \in E(G'[S])} f'(e) = \sum_{v \in S} \sum_{u \in N(v) \cap S} f'(uv) \leq \sum_{v \in S} \sum_{u \in N(v) \cap S} \frac{|uv|}{d_{G[S]}(v) + 1}.$$

By Lemma 8.2.5 applied to each $v \in S$,

$$\sum_{v \in S} \sum_{u \in N(v) \cap S} \frac{|uv|}{d_{G[S]}(v) + 1} \leq \sum_{v \in S} \frac{|N(v) \cap S|}{1 + |N(v) \cap S|} = \sum_{v \in S} \frac{d_{G'[S]}(v)}{1 + d_{G'[S]}(v)}.$$

By Lemma 8.2.4, the right side of the previous inequality is at most $|S| - 1$, so the previous two inequalities imply that

$$\sum_{e \in E(G[S])} f(e) \leq \frac{|S| - 1}{2} \leq \left\lfloor \frac{|S|}{2} \right\rfloor,$$

as required. $\square$

We conclude this section with the proof of Theorem 8.2.2.

Proof of Theorem 8.2.2. Let G' be the underlying simple graph of G, and let f' be a demand function for $L(G')$ such that for each $e \in V(L(G'))$ where $e = uv \in E(G')$, we have $f'(e) = (2|uv|)/(3\max\{d_G(u), d_G(v)\})$. It suffices to show that $L(G')$ has an f'-coloring, because then $L(G)$ has an f-coloring, as desired. By Theorem 8.2.3, it suffices to show that (8.1) and (8.2) hold for G' and f'.

For each $v \in V(G)$ and $u \in N(v)$, we have $f'(uv) \leq (2|uv|)/(3d_G(v))$. Hence, for each $v \in V(G)$, we have $\sum_{u \in N(v)} f'(uv) \leq (2d_G(v))/(3d_G(v)) = 2/3 \leq 1$, so (8.1) holds, as desired. Moreover, for any $S \subseteq V(G)$,

$$2 \sum_{e \in E(G'[S])} f'(e) \leq \sum_{v \in S} \sum_{u \in N(v)} f'(uv) \leq \sum_{v \in S} 2/3 = 2|S|/3.$$

Therefore $\sum_{e \in E(G'[S])} f'(e) \leq |S|/3$, so if $|S| \geq 2$, then $\sum_{e \in E(G'[S])} f'(e) \leq \lfloor |S|/2 \rfloor$. Hence, (8.2) holds, as required. $\square$

Chapter 9

Local demands for graphs with small clique number

In this chapter, we discuss fractionally coloring triangle-free graphs and graphs with small clique number. In Section 9.1, we show that Conjecture 1.5.11 implies a recent conjecture of Cames van Batenburg et al. [32] on the fractional chromatic number of triangle-free graphs. In Section 9.2, we prove Theorems 1.5.12 and 1.5.13. We actually prove Theorem 9.2.1, a more general result that can function as a blackbox to obtain a result about fractional coloring with local demands using a bound on the chromatic number, and we use this result in conjunction with Theorems 1.2.9 and 1.2.10 to prove Theorems 1.5.12 and 1.5.13, respectively. We refine this approach in Section 9.3 and reduce Conjecture 1.5.11 to Conjecture 9.3.2, a "list-local" version of the color degree problems mentioned in Chapter 6.

9.1 The fractional chromatic number of triangle-free graphs

Beyond being independently interesting, Conjecture 1.5.11 also has theoretical applications. Recently, Cames van Batenburg et al. [32, Conjecture 4.3] conjectured the following.

Conjecture 9.1.1 (Cames van Batenburg et al. [32]). *If G is a triangle-free graph on n vertices, then $\chi_f(G) \leq (\sqrt{2} + o(1))\sqrt{n/\log n}$.*

If true, Conjecture 9.1.1 implies the best known upper bound on the Ramsey number

$R(3,k)$ discussed in Section 1.2.3. The following proposition shows that if true, Conjecture 1.5.11 implies their conjecture.

Proposition 9.1.2. *For every $\varepsilon, c > 0$, the following holds for sufficiently large n. Let G be a triangle-free graph on n vertices with demand function f such that $f(v) \leq c \log d(v)/d(v)$ for each $v \in V(G)$. If G has an f-coloring, then $\chi_f(G) \leq (\sqrt{2/c} + \varepsilon)\sqrt{n/\log n}$.*

Proof. Let g be the demand function for G where $g(v) = d(v)/n$ for each $v \in V(G)$. Since G is triangle-free, it has a g-coloring, by assigning all vertices in the neighborhood of each vertex an interval of measure $1/n$ that is disjoint from the others. If G has an f-coloring, then by combining an f-coloring and a g-coloring, we obtain a fractional coloring of G such that each vertex receives at least

$$\frac{1}{2}\left(\frac{c \log d(v)}{d(v)} + \frac{d(v)}{n}\right)$$

color. Using calculus, we see that the vertices receiving the least amount of color satisfy $cn \approx \frac{d(v)^2}{\log(d(v)) - 1}$, i.e. $d(v) \approx \sqrt{cn \log n/2}$. Therefore each vertex receives at least roughly

$$\begin{aligned}\frac{1}{2}\left(\frac{c \log \sqrt{cn \log n/2}}{\sqrt{cn \log n/2}} + \frac{\sqrt{cn \log n/2}}{n}\right) &\approx \frac{1}{2}\left(\frac{\sqrt{c} \log n/2}{\sqrt{n \log n/2}} + \sqrt{\frac{c \log n}{2n}}\right)\\ &= \frac{1}{2}\left(\sqrt{\frac{c \log n}{2n}} + \sqrt{\frac{c \log n}{2n}}\right)\end{aligned}$$

color. Thus, for n sufficiently large, each vertex receives at least $\sqrt{\log n/n}/(\sqrt{2/c} + \varepsilon)$ color, so $\chi_f(G) \leq (\sqrt{2/c} + \varepsilon)\sqrt{n/\log n}$, as desired. □

Cames van Batenburg et al. [32] proved a weaker form of their conjecture with the $\sqrt{2}$ replaced with a 2. Using Proposition 9.1.2, this result can be improved by proving a weaker form of Conjecture 1.5.11 in which the demands for each vertex are within a factor less than two of the conjectured value.

Conjecture 1.5.11 appears very similar to the following conjecture of Harris [74, Conjecture 6.2], although they are incomparable. A graph is *d-degenerate* if every subgraph contains a vertex of degree at most d.

Conjecture 9.1.3 (Harris [74])**.** *If G is a d-degenerate triangle-free graph, then $\chi_f(G) = O(d/\log d)$.*

Note that if G is d-degenerate, then $\text{mad}(G) \leq 2d$, and if $\text{mad}(G) \leq d$, then G is d-degenerate. Therefore Conjecture 9.1.3 is equivalent to the following: every triangle-free graph G satisfies $\chi_f(G) = O(\text{mad}(G)/\log \text{mad}(G))$. As we discussed in Section 1.2.2 in regards to the mad version of Reed's Conjecture, Harris' Conjecture does not hold if we replace χ_f with χ.

Conjecture 9.1.3 is the fractional analogue of Shearer's [136] bound on the independence number of triangle-free graphs in terms of the average degree, while Conjecture 1.5.11 is the fractional analogue of Shearer's [137] bound on the independence number of triangle-free graphs in terms of the degree sequence. Although the later result of Shearer [137] implies the earlier one [136] using Jensen's Inequality, this implication does not hold in the fractional coloring setting. Nevertheless, we believe progress towards one of these two conjectures should provide insight into the other.

9.2 An approximate version of Conjecture 1.5.11

In this section, we prove Theorems 1.5.12 and 1.5.13 using Theorems 1.2.9 and 1.2.10 of Molloy [107]. First, we prove the following result, which we then use as a "black box."

Theorem 9.2.1. *Let $r : \mathbb{N} \to \mathbb{R}$ be increasing and tending to infinity such that for any $\varepsilon > 0$, if d is sufficiently large, then*

$$\frac{r(d \cdot r(d))}{e\, r(d) \ln(r(d \cdot r(d)))} \leq \varepsilon, \tag{9.1}$$

and let $\mathcal{G}$ be a hereditary class of graphs such that $\chi(G) \leq \Delta/r(\Delta)$ for every sufficiently large Δ and every graph $G \in \mathcal{G}$ with $\Delta(G) \leq \Delta$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. If $G \in \mathcal{G}$ has demand function f such that

$$f(v) \leq \min\{(2e + \varepsilon)^{-1} \frac{r(d(v))}{d(v) \ln r(d(v) \cdot r(d(v)))}, \delta\}$$

for each $v \in V(G)$, then G has an f-coloring.

We prove this theorem by partitioning the vertices of G and coloring each part in turn using the assumption that $\chi(G) \leq \Delta/r(\Delta)$. To that end, we introduce the following definition.

Definition 9.2.2. Let H be a graph with demand function f, and let $\varepsilon > 0$. We say H is a *ladder with rungs* $R_0, \ldots, R_k$ with respect to f and ε if $(R_0, \ldots, R_k)$ is a partition of $V(H)$ such that for every $i \in \{0, \ldots, k-1\}$, if $v \in R_i$ and $u \in R_j$ for $j > i$, then $f(u) \le \varepsilon / d(v)$.

Lemma 9.2.3. *If H is a ladder with rungs $R_0, \ldots, R_k$ with respect to a demand function f and ε such that $H[R_i]$ has an $f/(1-\varepsilon)$-coloring for each $i \in \{0, \ldots, k\}$, then H has an f-coloring.*

Proof. Let ϕ be an f-coloring of $H[S]$ for some $S \subseteq V(H)$ chosen in the following way. If $R_i \cap S \ne \varnothing$, then $R_i \subseteq S$, and if $R_i \subseteq S$ for $i < k$, then $R_{i+1} \subseteq S$. That is, ϕ is an f-coloring of as many rungs of H as possible, from top to bottom. Suppose for a contradiction that $S \ne V(H)$. Let i be maximum such that $R_i \cap S = \varnothing$, and let $v \in R_i$. Since H is a ladder with respect to f and ε, the vertex v sees at most $\sum_{u \in N(v) \cap \cup_{j=i+1}^{k} R_j} f(u) \le \varepsilon$ color from ϕ. By assumption, $H[R_i]$ has a fractional coloring in which each vertex receives at least $f(v)/(1-\varepsilon)$ color, so by Lemma 7.1.4, ϕ can be extended to an f-coloring of $H[R_i]$, contradicting the choice of S. Therefore H has an f-coloring, as desired. □

In the proof of Theorem 9.2.1, we partition the vertices of G into two ladders and color each with half of the available color. We apply Lemma 9.2.3 to both of these ladders to obtain this coloring. In the next section, we also use Lemma 9.2.3 in a similar way.

In order to apply Lemma 9.2.3, we need to show that each rung in one of our ladders has the desired coloring. We need the following definition.

Definition 9.2.4. A graph H is *(r, M)-stratified by* $S_1, \ldots, S_M$ if $(S_0, \ldots, S_M)$ is a partition of $V(H)$ such that for every $i \in \{0, \ldots, M\}$, if $u, v \in S_i$, then $d(u) \le d(v) \cdot r(d(v))^{1/M}$.

We use the following lemma to color each rung of the ladders in the proof of Theorem 9.2.1.

Lemma 9.2.5. *Suppose a graph H is (r, M)-stratified by $S_0, \ldots, S_k$ such that for every $i \in \{0, \ldots, k\}$ and every Δ, if $\Delta(G[S_i]) \le \Delta$, then $\chi(G[R_i]) \le \Delta / r(\Delta)$. If $g : \mathbb{R} \to R$ satisfies*

$$g(x) \le \frac{r(x)}{xM \cdot r(x)^{1/M}} \tag{9.2}$$

and f is a demand function for G where $f(v) = g(d(v))$ for each vertex v, then the graph $G[R_i]$ has an f-coloring.

Proof. We color each graph $G[S_i]$ separately with disjoint sets of color of measure $1/M$. By assumption, for each $i \in \{1, \ldots, M\}$, we have $\chi(G[R_i]) \leq \Delta(G[R_i])/r(\Delta(G[R_i])$. Therefore for each $i \in \{1, \ldots, M\}$ there is a fractional coloring ϕ_i of $G[R_i]$ such that each vertex in R_i receives at least $r(\Delta(G[R_i]))/(M\Delta(G[R_i]))$ color and the range of ϕ_i and ϕ_j is disjoint if $i \neq j$. Since H is (r, M)-stratified, for each $i \in \{1, \ldots, M\}$ and $v \in S_i$, we have $\Delta(G[R_i]) \leq d(v) \cdot r(d(v))^{1/M}$. Since r is increasing, each vertex v receives at least $r(d(v))/(Md(v) \cdot r(d(v))^{1/M})$ color. By (9.2), v receives at least $g(d(v))$ color, as desired. □

Combining Lemmas 9.2.3 and 9.2.5, we now prove Theorem 9.2.1.

Proof of Theorem 9.2.1. We may assume that every vertex in G has degree at least $\delta^{-1}-1$, since $f(v) \leq \delta$ for every $v \in V(G)$. We choose δ sufficiently small so that every vertex in G has large enough degree to satisfy certain inequalities throughout the proof.

Let $M_1, M_2, \ldots$ and $M'_1, M'_2, \ldots$ be increasing sequences of positive integers to be determined later. We partition the vertices of G into two parts that induce ladders H and H' with rungs $R_0, \ldots, R_k$ and $R'_0, \ldots, R'_{k'}$ such that the rungs R_i and R'_i induce (r, M_i)-stratified and (r, M'_i)-stratifed graphs,respectively, as follows.

Let $\delta_{0,0}$ be the minimum degree of a vertex in G, and let $\delta'_{i,0}$ for $i \geq 0$ and $\delta_{i,0}$ for $i \geq 1$ be determined later. For each $i \geq 0$ and each $j \in \{1, \ldots, M_i\}$, let $\delta_{i,j} = \delta_{i,j-1} \cdot r(\delta_{i,j-1})^{1/M_i}$. Similarly, for each $i \geq 0$ and each $j \in \{1, \ldots, M'_i\}$, let $\delta'_{i,j} = \delta_{i,j-1} \cdot r(\delta_{i,j-1})^{1/M'_i}$. For each $i \geq 0$, let $\delta'_{i,0} = \delta_{i,M_i}$, and for each $i \geq 1$, let $\delta_{i,0} = \delta'_{i-1,M'_{i-1}}$. For each $i \geq 0$, let R_i be the set of vertices of G with degree at least $\delta_{i,0}$ and less than $\delta'_{i,0}$, and let R'_i be the set of vertices of G with degree at least $\delta'_{i,0}$ and at most $\delta_{i+1,0}$. Let H be the graph induced by G on $\cup R_i$, and let H' be the graph induced by G on $\cup R'_i$. Since G is finite, we let k be the largest integer such that there is a vertex in $R_k \cup R'_k$. We may assume without loss of generality that there is a vertex in R'_k.

Since r is increasing, for each $i \in \{0, \ldots, k\}$, $\delta'_{i,0} \geq \delta_{i,0} \cdot r(\delta_{i,0})$. Similarly, for each $i \geq \{1, \ldots, k\}$, $\delta_{i,0} \geq \delta'_{i-1,0}$. Therefore for each $i \in \{1, \ldots, k\}$, the graphs $H[R_i]$ and $H[R'_i]$ are (r, M_i)-stratified and (r, M'_i)-stratified, respectively. For each $i \in \{1, \ldots, k\}$, let $g_i(x) = \frac{r(x)}{xM_i \cdot r(x)^{1/M_i}}$, and let $g'_i(x) = \frac{r(x)}{xM'_i \cdot r(x)^{1/M'_i}}$. Let g and g' be demand functions for H and H' respectively, where $g(v) = (1-\varepsilon/2)g_i(d(v))$ if $v \in R_i$ and $g'(v) = (1-\varepsilon/2)g'_i(d(v))$ if $v \in R'_i$. We claim that H is a ladder with rungs $(R_0, \ldots, R_k)$ with respect to g and $\varepsilon/2$ and H' is a ladder with rungs $(R'_0, \ldots, R'_k)$ with respect to g' and $\varepsilon/2$. For each $u \in \cup_{j=i+1}^{k} R_j$, we have

$$g(u) \leq \frac{r(d(v) \cdot r(d(v)))}{d(v)r(d(v))M_j \cdot r(d(v) \cdot r(d(v))^{1/M_j}}.$$

Note that for any x, the function $M \cdot x^{1/M}$ is maximized when $M = \ln x$ and is thus at most $e \ln x$. Letting $M = M_j$ and $x = r(d(v) \cdot r(d(v)))$, we have for each $u \in R_j$ for $j > i$,

$$f(u) \leq \frac{r(d(v) \cdot r(d(v)))}{d(v)r(d(v))e \ln(r(d(v) \cdot r(d(v))))}.$$

Therefore by (9.1), we may assume the right side of the previous inequality is at most $\varepsilon/(2d(v))$, as required. Hence, H is a ladder as claimed, and the proof for H' is the same.

Let f be the demand function for G where $f(v) = g(v)/2$ if $v \in R_i$ and $f(v) = g'(v)/2$ if $v \in R'_i$. Note that for $v \in R_i$, since r is increasing, $f(v) \geq \frac{(1-\varepsilon/2)r(d(v))}{2d(v)M_i \cdot r(\delta'_{i,0})^{1/M_i}}$. Letting $M_i = \lceil \log(\delta'_{i,0}) \rceil$ and $M'_i = \lceil \ln(\delta_{i+1,0}) \rceil$, assuming $\varepsilon < 1$, we have $f(v) \geq (2e+\varepsilon)^{-1} \frac{r(d(v))}{d(v)\ln(r(\delta'_{i,0}))}$. Since $d(v)r(d(v)) \geq \delta'_{i,0}$, we have

$$f(v) \geq (2e+\varepsilon)^{-1} \frac{r(d(v))}{d(v) \ln(r(d(v) \cdot r(d(v))))}.$$

Therefore it suffices to show that G has an f-coloring. If H and H' have g and g'-colorings respectively, then G has an f-coloring, obtained by averaging g and g'. Thus, we only need to show that H has a g-coloring, since the proof that H' has a g'-coloring is the same. By Lemma 9.2.5, $H[R_i]$ has a fractional coloring in which each vertex receives at least $g_i(d(v))$ color, so by Lemma 9.2.3, ϕ can be extended to a g coloring of $H[R_i]$, contradicting the choice of S. Therefore H has a g-coloring, so G has an f-coloring, as required. $\square$

Now we show how to obtain Theorems 1.5.12 and 1.5.13 using Theorem 9.2.1.

Proof of Theorem 1.5.12. For each $d \in \mathbb{N}$, let $r(d) = (1 + o(1))^{-1} \ln d$, and note that r satisfies (9.1) and is increasing and tending to infinity, as required. Let $\mathcal{G}$ be the class of triangle-free graphs, and note that $\mathcal{G}$ is hereditary and every graph $G \in \mathcal{G}$ of maximum degree at most Δ satisfies $\chi(G) \leq \Delta/r(\Delta)$ by Theorem 1.2.9, as required. Since $r(d(v))/(d(v) \ln r(d(v) \cdot r(d(v)))) = (1 + o(1))^{-1} \ln d(v)/(d(v) \ln \ln d(v))$, the result follows from Theorem 9.2.1. $\square$

Proof of Theorem 1.5.13. For each $d \in \mathbb{N}$, let $r(d) = \ln d/(200\omega \ln \ln d)$, and note that r satisfies (9.1) and is increasing and tending to infinity, as required. Let $\mathcal{G}$ be the class of graphs with clique number at most ω, and note that $\mathcal{G}$ is hereditary and every graph $G \in \mathcal{G}$ of maximum degree at most Δ satisfies $\chi(G) \leq \Delta/r(\Delta)$ by Theorem 1.2.10, as required. Since $r(d(v))/(d(v) \ln r(d(v) \cdot r(d(v)))) = O(\ln d(v)/(d(v)(\ln \ln d(v))^2)$, the result follows from Theorem 9.2.1. $\square$

9.3 Color degrees

In this section, we reduce Conjecture 1.5.11 to a problem involving color degrees. Recall the following definition, which we discussed in Chapter 6.

Definition 9.3.1. Let G be a graph with list-assignment L.

- For each $v \in V(G)$ and $c \in L(v)$, the *color degree* of v and c, denoted $d_{G,L}(v, c)$, is the number of neighbors $u \in N(v)$ such that $c \in L(u)$.
- We let $\Delta(G, L)$ denote the *maximum color degree*, the maximum over $v \in V(G)$ and $c \in L(v)$ of $d(v, c)$.

We believe the techniques discussed in Section 6.1.1 may be useful for proving the following conjecture.

Conjecture 9.3.2. *For every $\varepsilon > 0$, there exists $\alpha > 0$ and Δ sufficiently large such that the following holds. If G is a triangle-free graph with list-assignment L such that each $v \in V(G)$ and $c \in L(v)$ satisfies*

$$|L(v)| \geq (1+\varepsilon)\frac{d(v,c)}{\log d(v,c)}$$

and $\Delta^{1-\alpha} \leq d(v, c) \leq \Delta$, then G is L-colorable.

As is the case with Theorems 1.3.11 and 1.3.12, a result of Davies et al. [42, Proposition 11] implies that some lower bound on the color degrees in Conjecture 9.3.2 is necessary; however, we believe that this bound could be lowered to $\operatorname{poly}\log \Delta$.

In the remainder of the section, we show that Conjecture 9.3.2, if true, implies Conjecture 1.5.11. The following definition is crucial.

Definition 9.3.3. Let G be a graph with demand function f, and let $N \in \mathbb{N}$ such that $f(v) \cdot N$ and $f(v) \cdot N/(1 + f(v))$ is an integer for each vertex $v \in V(G)$. If G' is a graph obtained from G by replacing each vertex v with an independent set $\{v_1, \ldots, v_{f(v)\cdot N/(1+f(v))}\}$ and if L is a list-assignment for G' such that the lists $L(v_i)$ partition $\{1, \ldots, N\}$ and are each of size at least $\lceil f(v)^{-1}\rceil$, then (G, L) is an *N-color-partitioned blowup* of G with respect to f and the vertex v is the *progenitor* of the vertices in $\{v_1, \ldots, v_{f(v)\cdot N/(1+f(v))}\}$.

Note that in this definition, every vertex $v \in V(G)$ satisfies $\lceil f(v)^{-1}\rceil \cdot (f(v) \cdot N/(1 + f(v))) \leq N$. Hence, N-color-partitioned blowups of G indeed exist for every such N.

The following proposition reveals the connection between color degrees and fractional coloring, using the notion of the N-color-partitioned blowup.

Proposition 9.3.4. *Let G be a graph with demand function f, and let (G', L) be an N-color-partitioned blowup of G with respect to f.*

1. *If G' is L-colorable, then G has an $f/(1+f)$-coloring,*
2. *and for each $v' \in V(G')$ and $c \in L(v)$, if $v \in V(G)$ is the progenitor of v', then $d(v', c) = d(v)$.*

Proof. Let ϕ be an L-coloring of G'. For each $v \in V(G)$, let

$$\psi(v) = \cup_{i \in \{1,\ldots,f(v)\cdot N/(1+f(v))\}} \phi(v_i),$$

where v is the progenitor for each vertex in $\{v_1, \ldots, v_{f(v)\cdot N/(1+f(v))}\}$. Since the lists $L(v_i)$ partition $\{1, \ldots, N\}$, the colors $\phi(v_i)$ are distinct. Hence, $|\psi(v)| = f(v) \cdot N/(1 + f(v))$. Moreover, $\psi(v) \cap \psi(u) = \varnothing$ for every $uv \in E(G)$. Therefore ψ is an $(f/(1+f), N)$-coloring of G. By Proposition 1.1.6, G has an $f/(1 + f)$-coloring, as desired.

Since the lists $L(v_i)$ partition $\{1, \ldots, N\}$, for every $c \in L(v)$ and $u \in N(v)$, there is precisely one vertex u' of which u is the progenitor such that $c \in L(u')$. Therefore $d(v', c) = d(v)$, as claimed. □

Let $r : \mathbb{N} \to \mathbb{R}$ and $a, b \in \mathbb{R}$. A graph G is *r-locally color-degree list-colorable for color-degrees in* $[a, b]$ if G is L-colorable for any list-assignment L such that each $v \in V(G)$ and $c \in L(v)$ satisfies

$$|L(v)| \geq d(v, c)/r(d(v, c))$$

and $a \leq d(v, c) \leq b$. A class of graphs $\mathcal{G}$ is *closed under duplicating vertices* if for every $G \in \mathcal{G}$ and $v \in V(G)$, the graph obtained from G by adding a vertex adjacent to each vertex in $N(v)$ is in $\mathcal{G}$. Note that if $\mathcal{G}$ is closed under duplicating vertices, $G \in \mathcal{G}$, and (G', L) is an N-color-partitioned blowup of G, then $G' \in \mathcal{G}$.

The following theorem is the main result of this section.

Theorem 9.3.5. *Let $r : \mathbb{R} \to \mathbb{R}$ be increasing and tending to infinity such that for any $\varepsilon > 0$, if x is sufficiently large, then $r(3\varepsilon^{-1}x \cdot r(x)) \leq (1 - \varepsilon)3r(x)/2$ and $r(x)^{1/\varepsilon} = o(x^\alpha)$ for every $\alpha > 0$. Let $\mathcal{G}$ be a hereditary class of graphs closed under duplicating vertices such that for some $\alpha > 0$ and every sufficiently large δ, every graph $G \in \mathcal{G}$ is r-locally color-degree list-colorable for color-degrees in $[\Delta^{1-\alpha}, \Delta]$.*

For every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. If $G \in \mathcal{G}$ has demand function f such that

$$f(v) \leq \min\left\{(1 + \varepsilon)^{-1}\frac{r(d(v))}{d(v)}, \delta\right\}$$

for each $v \in V(G)$, then G has an f-coloring.

Proof. It suffices to prove the result for $\varepsilon = 1/n$ for $n \in \mathbb{N}$ sufficiently large. We may assume that every vertex has degree at least $\delta^{-1} - 1$, since $f(v) \leq \delta$ for every $v \in V(G)$. We choose n sufficiently large and δ sufficiently small so that every vertex in G has large enough degree to satisfy certain inequalities throughout the proof. We separate the vertices of G into n different ladders such that each vertex is contained in $n-1$ of them, as follows.

Let $\delta_{0,0} < 1/\delta$ be some constant to be determined later. For $i \geq 0$ and $j \in \{1, \dots, n\}$, let $\delta_{i,j} = 3\varepsilon^{-1}\delta_{i,j-1} \cdot r(\delta_{i,j-1})$, and for $i \geq 1$, let $\delta_{i,0} = \delta_{i-1,n}$. For each $i \geq 0$ and $j \in \{1, \dots, n\}$, let $R_{i,j}$ be the set of vertices of G with degree at least $\delta_{i,j}$ and less than $\delta_{i+1,j-1}$. For each $j \in \{1, \dots, n\}$, let H_j be the graph induced by G on $\cup_i R_{i,j}$, and for each $i \geq 0$, let $H_{i,j}$ be the graph induced by G on $R_{i,j}$.

We claim that H_ℓ is a ladder with rungs $(R_{i,\ell})_{i\geq 0}$ with respect to f and $(1-\varepsilon)(1+\varepsilon)^{-1}\varepsilon/2$. For each $v \in R_{j,\ell}$ and $u \in \cup_{i\geq j+1} R_{i,\ell}$, since r is increasing, we have

$$f(u) \leq (1+\varepsilon)^{-1}\frac{r(3\varepsilon^{-1}d(v) \cdot r(d(v)))}{3\varepsilon^{-1}d(v) \cdot r(d(v))}.$$

Since $r(3\varepsilon^{-1}x \cdot r(x)) \leq (1-\varepsilon)3r(x)/2$, the right side of the previous inequality is at most $\varepsilon(1-\varepsilon)(1+\varepsilon)^{-1}/(2d(v))$. Therefore $f(u)/ \leq \varepsilon(1-\varepsilon)(1+\varepsilon)^{-1}/(2d(v))$, as required.

Choose N such that $f(v) \cdot N$ and $f(v) \cdot N/(1+f(v))$ are integers for each $v \in V(G)$, and for each $i \geq 0$ and $j \in \{1, \dots, n\}$, let $(H'_{i,j}, L_{i,j})$ be an N-color-partitioned blowup up H_j. Since $\mathcal{G}$ is hereditary, each graph $H_j \in \mathcal{G}$, and since $\mathcal{G}$ is closed under taking blowups, each graph $H'_{i,j} \in \mathcal{G}$.

We claim that for each $i \geq 0$ and each $j \in \{1, \dots, n\}$, we have that $\delta_{i+1,j-1} \leq (3/2)^{\binom{n}{2}}(3\varepsilon^{-1}r(\delta_{i,j}))^{n-1}\delta_{i,j}$. We prove this claim for the case $j = n$, and the proof for other j is the same. Since $\delta_{i+1,0} = \delta_{i,n}$, it suffices to show that for each $i \geq 0$ we have $\delta_{i,n-1} \leq (3/2)^{\binom{n}{2}}(3\varepsilon^{-1}r(\delta_{i,0}))^n\delta_{i,0}$. We actually prove by induction the stronger statement that for each $i \geq 0$ and $\ell \in \{0, \dots, n-1\}$, we have $\delta_{i,\ell} \leq (3/2)^{\binom{\ell+1}{2}}(3\varepsilon^{-1}r(\delta_{i,0}))^\ell\delta_{i,0}$. Suppose this statement is true for $\ell' \leq \ell$, where $\ell \geq 0$. By the definition of $\delta_{i,\ell+1}$ and the inductive hypothesis, we have

$$\delta_{i,\ell+1} = 3\varepsilon^{-1}\delta_{i,\ell} \cdot r(\delta_{i,\ell}) \leq 3\varepsilon^{-1}(3/2)^{\binom{\ell}{2}}(3\varepsilon^{-1}r(\delta_{i,0}))^\ell\delta_{i,0} \cdot r(\delta_{i,\ell}). \tag{9.3}$$

Since r is increasing and r satisfies $r(3\varepsilon^{-1}x \cdot r(x)) \leq 3r(x)/2$, assuming $\delta_{0,0}^{-1}$ is sufficiently large, we have

$$r(\delta_{i,\ell}) \leq (3/2)^\ell r(\delta_{i,0}) \tag{9.4}$$

Combining (9.3) and (9.4), we have $\delta_{i,\ell+1} \leq (3/2)^{\binom{\ell+1}{2}}(3\varepsilon^{-1}r(\delta_{i,0})^\ell \cdot \delta_{i,0}$, as required. Since the case $\ell = 0$ trivially holds, the claim follows.

Therefore, since $r(x)^{1/\varepsilon} = r(x)^n = o(x^\alpha)$, for δ^{-1} sufficiently large we have $\delta_{i+1,j-1} \leq \delta_{i,j}^{1+\alpha}$. Since every graph in $\mathcal{G}$ is r-locally color-degree list-colorable for color degrees in $[\Delta^{1-\alpha}, \Delta]$, for each j, by Proposition 9.3.4 (b), we have that $H'_{i,j}$ is $L_{i,j}$-colorable for each N-color-partitioned blowup (H'_j, L_j). By Proposition 9.3.4 (a), each graph $H_{i,j}$ has a $(1+\varepsilon)f/(1+f)$-coloring. Since $f(v) \leq \delta$ for each vertex v, we may assume $(1+\varepsilon)f/(1+f) \geq (1+\varepsilon)f/(1+\delta) \geq f/(1-\varepsilon/2)$. Therefore by applying Lemma 9.2.3 with $(1+\varepsilon)f$ and $\varepsilon(1-\varepsilon)(1+\varepsilon)^{-1}/2$, since $(1+\varepsilon)f/(1-\varepsilon(1-\varepsilon)(1+\varepsilon)^{-1}/2) = f/(1-\varepsilon/2)$, each graph H_j has a $(1+\varepsilon)f$-coloring. Since $\varepsilon = 1/n$ and each vertex is in $n-1$ of the graphs H_j, averaging these $(1+\varepsilon)f$-colorings yields an f-coloring of G, as desired. □

Note that the function $r(x) = \ln x$ and the class $\mathcal{G}$ of triangle-free graphs satisfy the hypotheses of Theorem 9.3.5. Therefore by Theorem 9.3.5, Conjecture 9.3.2, if true, implies Conjecture 1.5.11.

Chapter 10

The local demands version of Brooks' Theorem

In this chapter, we prove Theorem 1.5.2. Sections 10.2–10.7 are devoted to the proof. First, in Section 10.1, we discuss several intriguing possible strengthenings of Theorem 1.5.2 for graphs of large minimum degree.

10.1 Beyond Brooks' Theorem

In this section we discuss possible ways of improving Theorem 1.5.2 and their relation to results about coloring graphs for which χ is close to Δ. We present several more conjectures in this vein.

For each $k \in \mathbb{N}$, define ε_k^* to be the supremum over all values of ε for which the following holds. There exists $\delta \in \mathbb{N}$ and k graphs $H_0, \dots, H_{k-1}$ such that if G is a graph of minimum degree at least δ with demand function f such that $f(v) \leq 1/(d(v) + 1 - \varepsilon)$ and every subgraph $H \subseteq G$ isomorphic to a blowup of H_i for $i \in \{0, \dots, k-1\}$ has an f-coloring, then G has an f-coloring. It is an interesting problem to determine each value in the sequence $(\varepsilon_k^*)_{k=1}^{\infty}$ and the graphs $H_0, \dots, H_{k-1}$.

The analogous problem for the ordinary chromatic number is the following. For each $k \in \mathbb{N}$ and Δ_k sufficiently large, determine the $(\Delta + 1 - k)$-critical graphs of maximum degree Δ where $\Delta \geq \Delta_k$. Brooks' Theorem [30] implies that for $k = 1$, these critical graphs are complete graphs. The Borodin-Kostochka Conjecture [25] states that for $k = 2$ and $\Delta_2 = 9$, these critical graphs are still only complete graphs. Since we only ask for

Δ_2 to be sufficiently large, Reed's [122] proof of the Borodin-Kostochka Conjecture for graphs of sufficiently large maximum degree confirms that these critical graphs for $k = 2$ are complete graphs. Farzad, Molloy, and Reed [59] investigated this problem for small values of k. They determined that for $k = 3$ these critical graphs are complete graphs and blowups of W_6. For $k = 4$ they determined that these critical graphs are complete graphs, blowups of W_6, and blowups of two additional graphs, and for $k = 5$, they determined that they are blowups of an additional 22 graphs, one of which is W_8. The result of Molloy and Reed [110] mentioned in Section 1.2.1 provides a qualitative description of these obstructions to $(\Delta - k)$-coloring, and in particular it implies that there are finitely many of them for each k and Δ when Δ is large.

As we see in this section, this result does not carry over to the setting of local demands. We show that $\limsup_{k\to\infty} \varepsilon_k^* \le 3/2$. However, we conjecture that equality actually holds. Theorem 1.5.2 implies that $\varepsilon_1^* \ge 1/2$, with $H_0 = K_1$. In this section we show that $\varepsilon_1^* \le 3/4$, and Conjecture 1.5.6, if true, implies that $\varepsilon_1^* = 3/4$. We propose several conjectures regarding the values of ε_k^* for small values of k and provide upper bounds.

10.1.1 The obstructions

We begin with a characterization of the fractional colorings of blowups of odd cycles.

Proposition 10.1.1. *If H is a blowup of a cycle of length $2k + 1$ and g is a demand function for H, then H has a g-coloring if and only if $\sum_{v\in V(H)} g(v) \le k$ and every clique $K \subseteq V(H)$ satisfies $\sum_{v\in K} g(v) \le 1$.*

We do not provide a proof of Proposition 10.1.1; however, it is easy to reduce Proposition 10.1.1 to Proposition 1.1.7, the case when H is an odd cycle. It is easy to observe that $\sum_{v\in V(H)} g(v) \le k$ is a necessary condition in order for H to have a g-coloring, since H has independence number k.

Wheels are natural candidates as obstacles to improving Theorem 1.5.2. Note that the graphs described in Section 1.2.1 with max degree Δ, clique number $\Delta - 2$, and chromatic number $\Delta - 1$, are blowups of the wheel on six vertices. Thus, $\varepsilon_1^* \le 2$. Farzad, Molloy, and Reed [59, Lemma 2.5] showed that critical graphs with chromatic number close to their maximum degree that are not cliques are essentially obtained from smaller critical graphs by adding dominating vertices. The wheel is obtained in this way. It will be useful for us to characterize the fractional colorings of wheel-blowups, as follows.

Proposition 10.1.2. *If H is a blowup of W_{2k+2} and g is a demand function for H, then H has a g-coloring if and only if every clique K satisfies $\sum_{v\in K} g(v) \leq 1$ and*

$$\sum_{v\in C} g(v) \leq k\left(1 - \sum_{v\in V(H)\setminus C} g(v)\right), \tag{10.1}$$

where $H[C]$ is a blowup of a $(2k+1)$-cycle and every vertex in $H \setminus C$ is adjacent to every other vertex in H.

Proof. First we show that (10.1) is a necessary condition. Let w be the weight function for H where $w(v) = 1$ if $v \in C$ and $w(v) = k$ otherwise. We claim that $\sum_{v\in I} w(v) \leq k$ for every independent set I in H. If $I \subseteq C$, then this inequality holds since $\alpha(H[C]) \leq k$. Hence, we may assume I contains a vertex not in C; however, in this case, $|I| = 1$. Thus, $\sum_{v\in I} w(v) \leq k$, as claimed. Therefore, by Proposition 1.1.6, if H has a g-coloring, then

$$\sum_{v\in C} g(v) + \sum_{v\in V(H)\setminus C} k\cdot g(v) = \sum_{v\in V(H)} w(v)g(v) \leq k.$$

Rearranging terms in the previous inequality, we obtain (10.1), as desired.

Now we suppose (10.1) holds, and we show that H has a g-coloring. Since $\sum_{v\in K} g(v) \leq 1$ for each clique K, by Lemma 7.1.5, $H - C$ has a g-coloring, ϕ. Let f be the demand function for $H[C]$ where $f(u) = g(u)/\left(1 - \sum_{v\in V(H)\setminus C} g(v)\right)$, and note that every clique $K \subseteq C$ satisfies $\sum_{v\in K} f(v) \leq 1$. By (10.1) and Proposition 10.1.1, $H[C]$ has an f-coloring. By Lemma 7.1.4, $H[C]$ has a (g, L_ϕ)-coloring. Therefore by Proposition 7.1.3, H has a g-coloring, as desired. □

The blowups of wheels that are most problematic are blowups of 5-cycles with precisely one vertex that is adjacent to all of the others. We call such a graph a *5-cycle blowup dominated by u*, where u is the unique vertex adjacent to every other vertex in the graph.

Now we show that there are in fact cycle-blowups that do not satisfy the hypotheses of Conjecture 1.5.5. Moreover, these graphs provide upper bounds on ε_k^*.

Proposition 10.1.3. *For every $\delta \in \mathbb{N}$ and $k \in \{1, 2, 3\}$, there exists a graph H_k of minimum degree at least δ with demand function f such that $f(v) \leq 1/(d(v) + 1 - \varepsilon_k)$ for each $v \in V(G)$ and*

(i) H_1 is a 5-cycle blowup, $\varepsilon_1 < 3/4$, and $\sum_{v\in V(H_1)} f(v) > 2$,

(ii) H_2 *is a 7-cycle blowup,* $\varepsilon_2 < 1$*, and* $\sum_{v \in V(H_2)} f(v) > 3$*, and*

(iii) H_3 *is a blowup of* W_6*,* $\varepsilon_3 < 5/4$*, and*

$$\sum_{v \in C} \frac{1}{d(v) + 1 - \varepsilon} > 2\left(1 - \frac{|V(H)| - |C|}{|V(H)| - \varepsilon}\right),$$

where $H_3[C]$ *is a 5-cycle blowup dominated by a vertex in* H*.*

Proof. For $k \in \{1, 2\}$, let H_k be obtained from the $(2k+3)$-cycle by blowing up each vertex of an independent set of size $k+1$ to a clique of size $\delta - 1$, and choose $\varepsilon_k < (3/2) \cdot (k/(k+1))$ to satisfy $\varepsilon_k > (3/2 - (1+\varepsilon)(2+\varepsilon)/(\delta k)) \cdot (k/(k+1))$. Let H_3 be obtained from W_6 by blowing up two non-adjacent vertices each to a clique of size $\delta - 2$, and choose $\varepsilon < 5/4$ such that $\varepsilon > 5/4 - (8\varepsilon^2 - 21\varepsilon + 14)/(8\delta) + (2\varepsilon^3 - 5\varepsilon^2 + 5\varepsilon + 4)/\delta^2$. For each $k \in \{1, 2, 3\}$, let $f(v) = 1/(d(v) + 1 - \varepsilon_k)$. It is straightforward to verify the desired inequalities. $\square$

Proposition 10.1.4. *For every* $\delta, k \in \mathbb{N}$*,* $k \geq 4$*, there exists a graph* H_k *of minimum degree at least* δ *that is a blowup of* C_{2k+1} *with demand function* f *such that* $f(v) \leq 1/(d(v) + 1 - \varepsilon_k)$ *for each* $v \in V(H_k)$ *where* $\varepsilon_k < (3k-2)/(2k)$*, each clique* $K \subseteq V(H_k)$ *satisfies* $\sum_{v \in K} f(v) \leq 1$*, and* $\sum_{v \in V(H_k)} f(v) > k + 1$*. In particular* H_4 *is a blowup of* C_9 *and* $\varepsilon_4 \leq 5/4$*.*

Proof. Let H_k be obtained from the $(2k+1)$-cycle by blowing up each vertex of an independent set of size k to a clique of size $\delta - 1$, and choose $\varepsilon_k = (3k-2)/(2k) - 1/\delta$.

If $v \in V(H_k)$ has degree larger than δ or is adjacent to more than one vertex of degree δ, then let $f(v) = 1/(d(v) + 1 - \varepsilon_k)$, and otherwise, let $f(v) = 1/d(v)$. There are only two maximal cliques in H_k that contain only vertices of degree δ, and each vertex v in one of these cliques has at most one neighbor of degree larger than δ and thus $f(v) = 1/\delta$. Hence, if K is one of these cliques, then $\sum_{v \in K} f(v) \leq 1$, as required. Every other maximal clique in H_k contains a vertex of degree $2\delta - 2$, and since $\varepsilon_k \leq 3/2 - 1/\delta$, if K is one of these cliques, then

$$\begin{aligned}\sum_{v \in K} f(v) = \frac{\delta - 1}{\delta + 1 - \varepsilon_k} + \frac{1}{2\delta - 1 - \varepsilon_k} &\leq \frac{\delta - 1}{\delta + 1 - 3/2 + 1/\delta} + \frac{1}{2\delta - 1 - 3/2 + 1/\delta} \\ &= \frac{(2-\delta)(5\delta - 2)}{(\delta^2 - \delta + 2)(4\delta^2 - 5\delta + 2)} + 1 \leq 1,\end{aligned}$$

as required. Finally, note that $\varepsilon_k > (3k-2)/(2k) - (2\delta)^{-1}(3 - \varepsilon^2 - (2 + \varepsilon_k - \varepsilon_k^2)/(k+1)) \cdot (k+1)/k)$, in which case it is straightforward to verify that $\sum_{v \in V(H)} f(v) > k + 1$, as desired. $\square$

k	$\varepsilon_k^* \leq$	new obstructions
1	$3/4$	K_1
2	1	C_5
$3, 4$	$5/4$	C_7
5	$13/10$	C_9, W_6
$k \geq 6$	$(3k-2)/(2k)$	C_{2k-1}

Table 10.1: Obstructions to $1/(d(v) + 1 - \varepsilon)$-coloring graphs of large minimum degree.

Note that in Proposition 10.1.3, for each $k \in \{1, 2, 3\}$, every clique $K \subseteq V(H_k)$ satisfies $\sum_{v \in K} f(v) \leq 1$. Hence, together with Proposition 10.1.1 and 10.1.2, Proposition 10.1.3 implies that that $\varepsilon_1^* \leq 3/4$, $\varepsilon_2^* \leq 1$, and $\varepsilon_3^* \leq 5/4$. Moreover, Proposition 10.1.4 implies that for every $k \geq 4$, we have $\varepsilon_k^* \leq (3k-2)/(2k)$. These bounds are summarized in Table 10.1. Since the sequence $(\varepsilon_k^*)_{k=1}^{\infty}$ is monotonically increasing and bounded by $3/2$, we can define $\varepsilon_\infty^* = \lim_{k\to\infty} \varepsilon_k^*$. Proposition 10.1.4 implies that $\varepsilon_\infty^* \leq 3/2$. We conjecture that equality actually holds for all of these inequalities.

Note that the blowups considered in Proposition 10.1.4 are highly *unbalanced*, that is they have vertices of degree much larger than the minimum degree, even though they are blowups of regular graphs (or close to regular, in the case of W_6). For this reason, their chromatic number is not close enough to their maximum degree to be relevant in [59, 110]. However, one can show that these blowups maximize the sum over the vertices of their demands for the demand functions that we are considering, and thus in light of Proposition 10.1.1, are essentially the most difficult to color.

10.1.2 $\varepsilon \leq 1$

In this subsection, we conjecture sufficient conditions for graphs of large minimum degree to have an f-coloring where f is a demand function such that $f(v) \leq 1/(d(v) + 1 - \varepsilon)$ for each vertex v and $\varepsilon \leq 1$. These conjectures, as well as Conjecture 1.5.6, are all weak versions of Conjecture 1.5.5.

Conjecture 10.1.5. *For every $\varepsilon < 1$ there exists $\delta \in \mathbb{N}$ such that the following holds. If G is a graph of minimum degree at least δ with demand function f such that $f(v) \leq 1/(d(v) + 1 - \varepsilon)$ for each $v \in V(G)$,*

(i) $\sum_{v \in K} f(v) \leq 1$ for each clique $K \subseteq V(G)$, and
(ii) $\sum_{v \in V(H)} f(v) \leq 2$ for every $H \subseteq G$ isomorphic to a blowup of a 5-cycle,

then G has an f-coloring.

Conjecture 10.1.5, if true, implies that $\varepsilon_2^* = 1$ where $H_0 = K_1$ and $H_1 = C_5$. The next conjecture is the special case of Conjecture 1.5.5 for graphs of sufficiently large minimum degree when $\varepsilon = 1$.

Conjecture 10.1.6. *There exists $\delta \in \mathbb{N}$ such that the following holds. If G is a graph of minimum degree at least δ with demand function f such that $f(v) \leq 1/d(v)$ for each $v \in V(G)$,*

(i) $\sum_{v \in K} f(v) \leq 1$ for each clique $K \subseteq V(G)$,
(ii) $\sum_{v \in V(H)} f(v) \leq 2$ for every $H \subseteq G$ isomorphic to a blowup of a 5-cycle, and
(iii) $\sum_{v \in V(H)} f(v) \leq 3$ for every $H \subseteq G$ isomorphic to a blowup of a 7-cycle,

then G has an f-coloring.

Conjecture 10.1.6 may be true for $\delta = 4$, in which case it would be tight for the graph obtained from the 9-cycle by blowing up each vertex in an independent set of size four to a clique of size three.

10.1.3 Finitely many obstructions

Next, we conjecture the values of ε_k^* up to the point in which we need to consider W_6.

Conjecture 10.1.7. *For every $\varepsilon < 5/4$ there exists $\delta \in \mathbb{N}$ such that the following holds. If G is a graph of minimum degree at least δ with demand function f such that $f(v) \leq 1/(d(v) + 1 - \varepsilon)$ for each $v \in V(G)$,*

(i) $\sum_{v \in K} f(v) \leq 1$ for each clique $K \subseteq V(G)$, and
(ii) $\sum_{v \in V(H)} f(v) \leq k$ for every $H \subseteq G$ isomorphic to a blowup of a C_{2k+1} cycle for $k \in \{2, 3\}$,

then G has an f-coloring.

Note that the above conjectures in this section are increasing in strength. Next, we conjecture that $\varepsilon_\infty^* = 3/2$, which is incomparable to any of the previous conjectures.

Conjecture 10.1.8. *For every $\varepsilon < 3/2$, there exists $\delta, k \in \mathbb{N}$ and graphs $H_0, \ldots, H_{k-1}$ such that the following holds. If G is a graph of minimum degree at least δ with demand function f such that $f(v) \leq 1/(d(v) + 1 - \varepsilon)$ for each $v \in V(G)$ and every $H \subseteq G$ isomorphic to a blowup of H_i for some $i \in \{0, \ldots, k-1\}$ has an f-coloring, then G has an f-coloring.*

We actually believe Conjecture 10.1.8 can be strengthened by specifying the graphs $H_0, \ldots, H_{k-1}$. We believe that these graphs are precisely K_1, W_6, and the odd-cycles. Moreover, for $\varepsilon < 3/2$, the only blowups of W_6 that are problematic are 5-cycle blowups dominated by a single vertex. The following conjecture, if true, implies all of the conjectures in this section, as well as Conjecture 1.5.6.

Conjecture 10.1.9. *For every $\varepsilon < 3/2$, there exists $\delta \in \mathbb{N}$ such that the following holds. If G is a graph of minimum degree at least δ with demand function f such that $f(v) \leq 1/(d(v)+1-\varepsilon)$ for each $v \in V(G)$,*

(i) $\sum_{v \in K} f(v) \leq 1$ for each clique $K \subseteq V(G)$,
(ii) $\sum_{v \in V(H)} f(v) \leq k$ for each $H \subseteq G$ isomorphic to a blowup of a $(2k+1)$-cycle, and
(iii) $2f(u) + \sum_{v \in V(H)} f(v) \leq 2$ for every $u \in V(G)$ and $H \subseteq G$ such that H is a blowup of a 5-cycle dominated by u,

then G has an f-coloring.

10.2 Proving Theorem 1.5.2

In this section, we provide an overview of the proof of Theorem 1.5.2. We actually prove Theorem 1.5.2 assuming some key lemmas which we prove in later sections. Let G be a minimum counterexample to Theorem 1.5.2, that is a graph that has no f-coloring, where f is a demand function such that $f(v) \leq 1/(d(v)+1-\varepsilon)$ for each $v \in V(G)$ where $\varepsilon \leq 1/2$, and for each clique K in G, we have $\sum_{v \in K} f(v) \leq 1$. We denote the minimum degree of a vertex in G by δ.

The following definitions are crucial to the proof. A *base clique* of a graph G is a maximum cardinality set of vertices of minimum degree that forms a clique in G.

Definition 10.2.1. Let K be a base clique of G. If u is a vertex adjacent to every vertex of K, then u *apexes* K. Now,

- let A_K be the set of vertices not in K that apex K,
- let U_K be the subset of vertices in $V(G) \setminus (A_K \cup K)$ with a neighbor in K,
- let $\ell_K = \delta + 1 - |K| - |A_K|$ denote the number of neighbors each vertex in K has in U_K, and
- let $D_K = \max\{|K \cap N(u)| : u \in U_K\}$.

The proof mainly focuses on base cliques of G. In Section 10.3, we prove some lemmas that will be used frequently in the proof. For example, we prove that neighbors of minimum degree vertices have bounded degree, and minimum degree vertices have many neighbors of minimum degree. In Section 10.5, we prove some important structural properties of the base cliques. For example, Lemma 10.5.1 implies that every minimum degree vertex of G is in a unique base clique and is not adjacent to any vertex in a different base clique. As a consequence, we obtain a lower bound on the size of base cliques. The important information is summarized in the following lemma.

Lemma 10.2.2. *If K is a base clique in G and $\varepsilon \leq 1/2$, then $D_k \leq \ell_K \leq (\delta + 1)/2$. Moreover, $\ell_K \geq 2$ and $\delta \geq 3$.*

In the proof that $\ell_K > 1$ in Lemma 10.2.2, we remove some vertices of a base clique, add some edges, find a fractional coloring by induction, and try to extend it to the uncolored vertices. However, adding the edges may create a clique that is not colorable with respect to the demands we desire, in which case G contains a 5-cycle blowup. Section 10.4 is devoted to showing that G does not contain any of these blowups. The argument for proving $\ell_K \neq 0$ is different. In this argument, we remove the base clique K, identify a pair of non-adjacent vertices in A_K, find a fractional coloring with respect to carefully chosen demands by induction, and extend the coloring to the vertices that demand more color than they have.

In Section 10.6, we show that the complements of base cliques admit f-colorings with nice properties. In particular, the resulting fractional list-assignment for the vertices in the base clique is far from uniform. Since G is a counterexample, these colorings do not extend to the base clique. We prove the following lemma by showing that if the conclusion does not hold, then these colorings extend to the base clique.

Lemma 10.2.3. *If K is a base clique of G and $\varepsilon \leq 1/2$, then*

$$\ell_K + 1 - \varepsilon - \frac{|K|}{\delta + 1 - \varepsilon} < \sum_{u \in U_K} \frac{|K \cap N(u)|(\delta + 2 - \varepsilon - |K|) + \varepsilon|K|}{|K|(\delta + 2 - |K \cap N(u)|)},$$

Combining Lemmas 10.2.2 and 10.2.3, we can prove Theorem 1.5.2. We essentially use calculus to show that a base clique can not have the structure prescribed by Lemma 10.2.2 and simultaneously satisfy the inequality in Lemma 10.2.3. These calculus arguments appear several times in the proof, so we defer their proofs to Section 10.7.

Proof of Theorem 1.5.2. It suffices to prove the result for $\varepsilon = 1/2$. Let K be a base clique of G. For convenience, let $\ell = \ell_K$, $D = D_K$, and $U = U_K$. By Lemma 10.2.2, $D \leq \ell \leq (\delta + 1)/2$ and $\ell \geq 2$. Therefore for each $u \in U$, we have $|K \cap N(u)| \leq \ell$.

Let

$$lc(x_1, \ldots, x_{\ell_K|K|}) = \sum_{i=1}^{\ell_K|K|} \frac{x_i(\delta + 2 - \varepsilon - |K|) + \varepsilon|K|}{|K|(\delta + 2 - x_i)}.$$

Note that if $x_2 \geq x_1$, then

$$lc(x_1 - 1, x_2 + 1, x_3, \ldots, x_{\ell|K|}) - lc(x_1, \ldots, x_{\ell|K|}) = \frac{(\delta + 2)(\delta + 2 - \varepsilon - |K|)}{|K|} \left(\frac{1}{(\delta + 2 - x_2)(\delta + 1 - x_2)} - \frac{1}{(\delta + 3 - x_1)(\delta + 2 - x_1)} \right) > 0,$$

Therefore if $1 \leq x_1 \leq x_2 \leq D_K - 1$, then

$$lc(x_1 - 1, x_2 + 1, x_3, \ldots, x_{\ell|K|}) > lc(x_1, \ldots, x_{\ell|K|}). \tag{10.2}$$

Let

$$x_i = \begin{cases} \ell & i \in \{1, \ldots, |K|\}, \\ 0 & \text{otherwise.} \end{cases}$$

By (10.2), since $\sum_{u \in U} |K \cap N(u)| = \ell|K|$, the right side of the inequality in Lemma 10.2.3 is at most $lc(x_1, \ldots, x_{\ell|K|})$, so by Lemma 10.2.3,

$$\ell + .5 - \frac{|K|}{\delta + .5} < \frac{\ell(\delta + 1.5 - |K|) + .5|K|}{\delta + 2 - \ell}.$$

We need the following claim, which we prove in Section 10.7.

Claim 10.2.3.1. *Let*

$$q_\delta(\ell, k) = \ell + .5 - \frac{k}{\delta + .5} - \frac{\ell(\delta + 1.5 - k) + .5k}{\delta + 2 - \ell}.$$

If $\ell \in [2, \delta/2]$, $k \geq (\delta + 1)/2$, and $\delta \geq 4$, then $q_\delta(\ell, k) \geq 0$.

Note that the difference of the left and right side of the above inequality is $q_\delta(\ell, |K|)$ from Claim 10.2.3.1. Hence, by Claim 10.2.3.1, we may assume either $\delta \leq 3$ or $\ell = (\delta + 1)/2$.

First, suppose $\delta \leq 3$. By Lemma 10.5.6, $\delta = 3$, so $\ell = 2$. Hence, $|K| = 2$ and $D = 1$. Now the right side of Lemma 10.2.3 is 1.75 and the left side is $2.5 - 2/3.5 = 27/14$, a contradiction.

Therefore we may assume $\ell = (\delta + 1)/2$. Hence, $|K| = (\delta + 1)/2$ and $D \leq (\delta - 1)/2$. Moreover, δ is odd, and since $\delta \geq 4$, we have $\delta \geq 5$. By (10.2), the right side of Lemma 10.2.3 is at most

$$\left\lceil \frac{\ell|K|}{.5(\delta - 1)} \right\rceil \left(\frac{.5(\delta - 1)(\delta + 1.5 - |K|) + .5|K|}{|K|(\delta + 2 - .5(\delta - 1))} \right).$$

Note that

$$\left\lceil \frac{\ell|K|}{.5(\delta-1)} \right\rceil = \left\lceil \frac{(\delta+1)^2}{2(\delta-1)} \right\rceil = \left\lceil \frac{(\delta+3)(\delta-1)+4}{2(\delta-1)} \right\rceil \leq \lceil .5(\delta+3)+.5 \rceil = .5(\delta+5).$$

Combining the previous two inequalities and Lemma 10.2.3,

$$\ell + .5 - \frac{|K|}{\delta+.5} - .5(\delta+5)\left(\frac{.5(\delta-1)(\delta+1.5-|K|)+.5|K|}{|K|(\delta+2-.5(\delta-1)}\right) < 0.$$

However, the left side of the previous inequality is

$$\begin{aligned} .5(\delta+1)+.5-\frac{.5(\delta+1)}{\delta+.5} - .5(\delta+5)\left(\frac{.5(\delta-1)(\delta+1.5-.5(\delta+1))+.25(\delta+1)}{.5(\delta+1)(\delta+2-.5(\delta-1))}\right) \\ = \frac{0.75(\delta+1/3)}{(\delta+0.5)(\delta+1)} > 0, \end{aligned}$$

a contradiction. □

Without using the results of Section 10.4 and some of the results of Section 10.5 one can prove Theorem 1.5.2 with ε slightly smaller than $1/3$ in a similar way.

10.3 Useful lemmas

In this section we prove some useful simple properties of a hypothetical minimal counterexample to Theorem 1.5.2. The following basic fact actually holds for any minimum counterexample, regardless of the demand function.

Proposition 10.3.1. *For each $v \in V(G)$,*

$$f(v) + \sum_{u \in N(v)} f(u) > 1.$$

Proof. Suppose to the contrary that there exists $v \in V(G)$ such that $f(v)+\sum_{u\in N(v)} f(u) \leq 1$. Since G is a minimum counterexample, $G-v$ has an f-coloring ϕ. Since $f(v) + \sum_{u\in N(v)} f(u) \leq 1$, $\mu(L_\phi(v)) \geq f(v)$. Hence, G has an f-coloring, a contradiction. □

Recall that a vertex is *simplicial* if $\omega(v) = d(v)+1$. Lemma 10.3.1 easily implies the following.

Lemma 10.3.2. *There are no simplicial vertices in G. In particular, $\delta \geq 2$.*

Proof. Suppose to the contrary that v is a simplicial vertex in G. Now $G[N[v]]$ is a clique, so $f(v) + \sum_{u \in N(v)} f(u) \leq 1$, contradicting Lemma 10.3.1. $\square$

Lemma 10.3.3. *Suppose $X \subseteq V(G)$ such that $\sum_{v \in X} f(v) > 1$. If $X' \subseteq X$ and $u \in X \setminus X'$ such that*

- *every vertex $v \in X'$ has degree at least $|X| - 1$ and*
- *every vertex $v \in X \setminus X'$ has degree at least $|X|$,*

then

$$d(u) < \frac{|X| + 1 - \varepsilon}{1 + \varepsilon - |X'|/(|X| - \varepsilon)} + 1 - \varepsilon.$$

Proof. Since each vertex $v \in X'$ has degree at least $|X| - 1$, if $v \in X'$, then $f(v) \leq 1/(|X| - \varepsilon)$. Since each vertex $v \in X \setminus X'$ has degree at least $|X|$, if $v \in X \setminus X'$, then $f(v) \leq 1/(|X| + 1 - \varepsilon)$. Therefore

$$1 < \sum_{v \in X} f(v) \leq \frac{|X'|}{|X| - \varepsilon} + \frac{|X| - |X'| - 1}{|X| + \varepsilon} + \frac{1}{d(u) + 1 - \varepsilon},$$

so

$$d(u) < \left(1 - \frac{|X'|}{|X| - \varepsilon} + \frac{|X| - |X'| - 1}{|X| + \varepsilon}\right)^{-1} + 1 - \varepsilon.$$

Note that

$$1 - \frac{|X'|}{|X| - \varepsilon} + \frac{|X| - |X'| - 1}{|X| + \varepsilon} = \frac{(1 + \varepsilon)(|X| - .5) - |X'|}{(|X| - \varepsilon)(|X| + 1 - \varepsilon)}$$

Combining the previous two expressions yields the desired inequality. $\square$

We frequently apply Lemma 10.3.3 to the neighborhoods of vertices of minimum degree, so the following lemma is useful.

Lemma 10.3.4. *If $d(v) \leq d(u)$ for all $u \in N(v)$ and $X' = \{u \in N[v] : d(u) = d(v)\}$, then for each $u \in N(v)$,*

$$d(u) < \frac{d(v) + 2 - \varepsilon}{1 + \varepsilon - |X'|/(d(v) + 1 - \varepsilon)} + 1 - \varepsilon.$$

Proof. By Lemma 10.3.1, if $X = N[v]$, then $\sum_{v \in X} f(v) > 1$. The result follows by applying Lemma 10.3.3 with $X = N[v]$ and X'. $\square$

Lemma 10.3.4 implies that a vertex of minimum degree does not have neighbors of very large degree, and the bound on the degree of the neighbors is stronger when the minimum degree vertex has fewer neighbors of minimum degree. We will often use the weakest form of this bound as in the following lemma.

Lemma 10.3.5. *If $d(v) \le d(u)$ for all $u \in N(v)$, then $d(u) < \frac{d(v)}{1-\varepsilon} + \varepsilon$ for all $u \in N(v)$.*

Proof. Let $u \in N(v)$. Since $f(u) \le 1/(d(u)+1-\varepsilon$ and $f(w) \le 1/(d(v)+1-\varepsilon)$ for each vertex $w \in N(v)$, by Lemma 10.3.1,

$$\frac{1}{d(u)+1-\varepsilon} + \frac{d(v)}{d(v)+1-\varepsilon} \ge f(u) + \sum_{w \in N[v]\setminus\{u\}} f(w) > 1.$$

By rearranging terms in the previous inequality,

$$d(u) \le \frac{d(v)}{1-\varepsilon} + \varepsilon,$$

as desired. □

The following lemma is similar to Lemma 10.3.3. Instead of bounding the degree of one vertex, it bounds the number of vertices of large degree.

Lemma 10.3.6. *Suppose $X \subseteq V(G)$ such that $\sum_{v \in X} f(v) > 1$. If $d(v) \ge |X|-1$ for every $v \in X$, then fewer than $\varepsilon(|X|+1-\varepsilon)$ vertices in X have degree at least $|X|$.*

Proof. Let $X' \subseteq X$ be the vertices in X of degree $|X|-1$ in G. Now

$$1 < \sum_{v \in X} f(v) \le \frac{|X'|}{|X|-\varepsilon} + \frac{|X|-|X'|}{|X|+1-\varepsilon} = \frac{|X|(|X|-\varepsilon)+|X'|}{(|X|-\varepsilon)(|X|+1-\varepsilon)}.$$

Therefore

$$|X'| > (1-\varepsilon)(|X|-\varepsilon),$$

so fewer than

$$|X| - (1-\varepsilon)(|X|-\varepsilon) = \varepsilon(|X|+1-\varepsilon)$$

vertices have degree at least $|X|$, as desired. □

We will often need to apply Lemma 10.3.6 to the neighborhood of a minimum degree vertex, so the following lemma is useful.

Lemma 10.3.7. *If $d(v) \le d(u)$ for all $u \in N(v)$, then fewer than $\varepsilon(d(v)+2-\varepsilon)$ neighbors of v have degree greater than $d(v)$.*

Proof. Let $X = N[v]$. By assumption, $d(w) \ge |X| - 1$ for every $w \in X$, and by Lemma 10.3.1, $\sum_{v \in X} f(v) > 1$. The result follows by applying Lemma 10.3.6. $\square$

The final lemma in this section is the most technical. We apply it twice: once in Section 10.4 to part of a 5-cycle blowup, and once in Section 10.5 to a base clique.

Lemma 10.3.8. *Suppose K is a clique of G such that for some d, every vertex $v \in K$ has degree d and for every $w \in N(v) \setminus K$, we have $d(w) \ge d$. Suppose also that u and u' are distinct non-adjacent vertices that are adjacent to every vertex in K. If $\varepsilon \le 1/2$ and ϕ is an f-coloring of $G - (K \cup \{u, u'\})$ such that $\mu(L_\phi(u)) \ge f(u)$ and $\mu(L_\phi(u')) \ge f(u')$, then*

$$\mu(L_\phi(u) \cap L_\phi(u')) < \frac{.5}{\max\{d(u) + d(u') - d - .5, \delta + .5\}}.$$

Proof. Suppose to the contrary that $G-(K \cup \{u, u'\})$ has an f-coloring such that $\mu(L_\phi(u)) \ge f(u)$, $\mu(L_\phi(u')) \ge f(u')$, and $\mu(L_\phi(u) \cap L_\phi(u')) \ge \frac{.5}{\max\{d(u)+d(u')-d-.5,\delta+.5\}}$. Hence there exists an f-coloring ϕ' of $G-K$ such that $\mu(\phi(u) \cap \phi(u')) \ge .5/\max\{d(u) + d(u') - d - .5, \delta + .5\}$. Thus, for each $v \in K$, the vertex v sees at most

$$\frac{d-1-|K|}{d+1-\varepsilon} + \frac{1}{d(u)+1-\varepsilon} + \frac{1}{d(u')+1-\varepsilon} - \frac{.5}{\max\{d(u)+d(u')-d-.5, \delta+.5\}}.$$

color in ϕ', so, since $\varepsilon \le 1/2$,

$$\mu(L_{\phi'}(v)) \ge \frac{|K|+1.5}{d+.5} - \frac{1}{d(u)+.5} - \frac{1}{d(u')+.5} + \frac{.5}{\max\{d(u)+d(u')-d-.5, \delta+.5\}}. \tag{10.3}$$

We need the following claim, which we prove in Section 10.7.

Claim 10.3.8.1. *Let*

$$q_\delta(d, d_u, d_{u'}) = \frac{1.5}{d+.5} - \frac{1}{d_u+.5} - \frac{1}{d_{u'}+.5} + \min\left\{\frac{.5}{d_u + d_{u'} - d - .5}, \frac{.5}{\delta+.5}\right\}.$$

For $d \ge \delta$, and $d_u, d_{u'} \ge d$, we have $q_\delta(d, d_u, d_{u'}) \ge 0$.

Since G is a counterexample, by Proposition 7.1.3, $G[K]$ has no fractional $(f|_K, L_{\phi'})$-coloring. Therefore by Lemma 7.1.5, there exists $S \subseteq K$ such that $\mu\left(\cup_{v\in S} L_{\phi'}(v)\right) < \sum_{v\in S} g(v)$. However, since $\varepsilon \le 1/2$, we have $\sum_{v\in S} g(v) \le |K|/(d+.5)$, so by (10.3),

$$\frac{1.5}{d+.5} - \frac{1}{d(u)+.5} - \frac{1}{d(u')+.5} + \frac{.5}{\max\{d(u)+d(u')-d-.5, \delta+.5\}} < 0.$$

The left side of the previous inequality is $q_\delta(d, d(u), d(u'))$, contradicting Claim 10.3.8.1. □

10.4 Handling 5-cycle blowups

In this section we show that G does not contain certain 5-cycle blowups that appear in Section 10.5. This section is not needed to prove Theorem 1.5.2 for ε smaller than roughly $1/3$, so in this section, we assume $\varepsilon = 1/2$. The main result of this section is Lemma 10.4.4. First, we need the following definitions.

Definition 10.4.1. We say $(V_0, V_1, V_2, V_3, V_4)$ is a *δ-based 5-cycle blowup* if the following holds:

- for $i \in \{0, \dots, 4\}$, we have $V_i \subseteq V(G)$ and these sets are pairwise disjoint,
- for $i \in \{0, \dots, 4\}$, the set $V_i \cup V_{i+1}$ forms a clique in G (where addition is modulo 5),
- every $v \in V_0 \cup V_1$ has degree δ,
- every $v \in V_2 \cup V_4$ has degree at least $\delta + 1$,
- $|V_1| = |V_4| = 1$, and
- $|V_2| \le |V_0|$.

If $\sum_{v\in V_2\cup V_3\cup V_4} f(v) > 1$, then $(V_0, V_1, V_2, V_3, V_4)$ is *dangerous*.

In Section 10.5, dangerous δ-based 5-cycle blowups appear when we attempt to remove a base clique containing $V_0 \cup V_1$ and add edges between the vertex in V_4 and the vertices in V_2. Since the blowup is dangerous, we are unable to find an f-coloring of the resulting graph. We handle this by showing that we can remove part of the blowup, find an f-coloring, and extend it to G, unless the blowup has some specific structure. Hence, we need the following definitions.

Definition 10.4.2. We say (V_0, v, u, u', X) is an *essential restriction* of a δ-based 5-cycle blowup $(V_i)_{i=0}^4$ if the following holds:

- $V_1 = \{v\}$, $V_4 = \{u\}$,

- $u' \in V_2$ is not adjacent to u, and
- $X = \{w \in V_2 \cup V_3 \cup V_4 : d(w) = |V_2 \cup V_3 \cup V_4| - 1\}$.

Note that if $(V_i)_{i=0}^4$ is a dangerous δ-based 5-cycle blowup, then it has an essential restriction, because otherwise $V_2 \cup V_3 \cup V_4$ is a clique of G such that $\sum_{w \in V_2 \cup V_3 \cup V_4} f(w) > 1$.

Definition 10.4.3. Suppose $\delta = 2$ and (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$ such that $|X| = 2$. If $d(u) = d(u') = 4$ and u and u' both have a neighbor of degree two not in $V_0 \cup V_1$, then $(V_i)_{i=0}^4$ is a *turtle*. If $d(u) = d(u') = 3$, then $(V_i)_{i=0}^4$ is a *skew-turtle*.

Lemma 10.4.4. *If $(V_i)_{i=0}^4$ is a dangerous δ-based 5-cycle blowup, then $|V_0| = |V_2| = \delta/2$. Moreover, if $\delta = 2$, then $(V_i)_{i=0}^4$ is a turtle or a skew-turtle.*

We prove Lemma 10.4.4 by showing that we can remove an essential restriction, find an f-coloring, and extend it to G, unless the blowup has the structure described in the lemma. In Section 10.5, we show that G does not contain any turtles or skew-turtles.

In order to prove Lemma 10.4, we need the following lemmas, which rely heavily on the results of Section 10.3.

Lemma 10.4.5. *If (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$, then for each $x \in (V_2 \cup V_3 \cup V_4) \setminus X$,*

$$d(x) \leq \frac{|V_2 \cup V_3 \cup V_4| + .5}{1.5 - |X|/(|V_2 \cup V_3 \cup V_4| - .5)} - .5.$$

Proof. The result follows by applying Lemma 10.3.3 to $V_2 \cup V_3 \cup V_4$ and X. □

Lemma 10.4.6. *If (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$, then $\delta + 1 \leq |V_2 \cup V_3 \cup V_4| \leq 2\delta + 1$ and $|X| \geq |V_2 \cup V_3 \cup V_4|/2$.*

Proof. Since $(V_i)_{i=0}^4$ is dangerous, $\sum_{w \in V_2 \cup V_3 \cup V_4} f(w) > 1$. Since $f(w) \leq 1/(\delta + .5)$ for each w, we have $|V_2 \cup V_3 \cup V_4| \geq \lceil \delta + .5 \rceil = \delta + 1$, as desired. By Lemma 10.3.5, $d(u') \leq 2\delta$. Therefore $|V_2 \cup V_3 \cup V_4| \leq 2\delta + 1$, as desired.

Since $|V_2| \leq |V_0|$, we have $d(u) \geq |V_2 \cup V_3 \cup V_4| - 1$. Since each $w \in V_2$ is adjacent to every vertex in $V_1 \cup V_2 \cup V_3$ besides itself, $d(w) \geq |V_2 \cup V_3 \cup V_4| - 1$. Note that also for each $w \in V_3$, we have $d(w) \geq |V_2 \cup V_3 \cup V_4| - 1$. Hence, by Lemma 10.3.6,

$$|X| > |V_2 \cup V_3 \cup V_4| - (|V_2 \cup V_3 \cup V_4| + .5)/2 = |V_2 \cup V_3 \cup V_4|/2 - 1/4.$$

Since $|X|$ is an integer, the previous inequality implies that $|X| \geq |V_2 \cup V_3 \cup V_4|/2$, as desired. □

Lemma 10.4.7. *If (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$ and $u \in X$, then $|V_0| = |V_2|$ and u has no neighbors in V_2.*

Proof. By definition, if $u \in X$, then $d(u) = |V_2 \cup V_3 \cup V_4| - 1 = |V_2| + |V_3|$. However, $d(u) \geq |V_0| + |V_3|$. Since $|V_2| \leq |V_0|$, equality actually holds, and u has no neighbors in V_2, as desired. □

Lemma 10.4.8. *Suppose (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$. If $X \cap V_3 = \varnothing$, then $|V_0| = |V_2| = \delta/2$.*

Proof. Since v is a vertex of degree δ adjacent to every vertex in $V_0 \cup V_2$, we have $|V_0| + |V_2| \leq \delta$. Hence, since $|V_2| \leq |V_0|$, we have $|V_2| \leq \delta/2$.

Since every vertex of $V_2 \cup V_4$ has degree at least $\delta + 1$ and $(V_2 \cup V_4) \cap X \neq \varnothing$, we have $|V_2 \cup V_3 \cup V_4| \geq \delta + 2$. By Lemma 10.4.6, $|X| \geq \delta/2 + 1$. Since $X \cap V_3 = \varnothing$, we have $|X \cap V_2| \geq \delta/2$. It follows that $|V_0| = |V_2| = \delta/2$, as desired. □

The next lemma shows that we can remove an essential restriction and find a particularly nice f-coloring.

Lemma 10.4.9. *If (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$, then there exists an f-coloring ϕ of $G - (V_0 \cup X \cup \{v, u, u'\})$ such that*

$$\mu\left(L_\phi(u) \cap L_\phi(u')\right) \geq \frac{.5}{\max\{d(u) + d(u') - |V_2 \cup V_3 \cup V_4| + .5, \delta + .5\}}.$$

Proof. Let G' be the graph obtained from $G - ((V_0 \cup X \cup \{v\}) \setminus \{u, u'\})$ by identifying u and u' into a new vertex, say z. Let f' be a demand function for G' such that $f'(z) = \min\{.5/(d_G(u) + d_G(u') - 2|X| + .5), .5/(\delta + .5)\}$ and for each $w \in V(G') \setminus \{z\}$, we have $f'(w) = f(w)$. We claim that G' has an f'-coloring. First we show that

$$d_{G'}(z) \leq \max\{d(u) + d(u') - 2|X|, |V_2 \cup V_3 \cup V_4| - |X|\}. \tag{10.4}$$

We may assume $d_{G'}(z) > |V_2 \cup V_3 \cup V_4| - |X|$, or else (10.4) holds, as claimed. Therefore $\{u, u'\} \setminus X \neq \varnothing$.

Note that $d_{G'}(z) \leq d(u) + d(u') - |N_G(u) \cap (V_0 \cup X)| - |N_G(u') \cap (X \cup \{v\})|$. Therefore it suffices to show that

$$|N_G(u) \cap (V_0 \cup X)| + |N_G(u') \cap (X \cup \{v\})| \geq 2|X|. \tag{10.5}$$

For $x \in \{u, u'\}$, since $\{u, u'\} \setminus X \neq \varnothing$, we have $|N_G(x) \cap X| \geq |X| - 1$. Moreover, if $u \notin X$, then $|N_G(u) \cap (V_0 \cup X)| \geq |X|$ and $|N_G(u') \cap (X \cup \{v\})| \geq |X|$. Similarly, if $u' \notin X$, then $|N_G(u') \cap (X \cup \{v\})| \geq |X|$ and $|N_G(u) \cap (V_0 \cup X)| \geq |X|$. Therefore, (10.5) holds, as desired. Now (10.4) follows, as claimed.

We claim that for each clique K' in G', we have $\sum_{w \in K'} f'(w) \leq 1$. Note that for each $w \in V(G')$, we have $f'(w) \leq 1/(d_{G'}(w) + .5)$. If $z \notin K'$, then $\sum_{w \in K'} f'(w) \leq 1$, as claimed, because K' is a clique in G and $f'(w) = f(w)$ for all $w \in K'$. Therefore we may assume that $z \in K'$. Since $f'(z) \leq .5/(\delta + .5)$, we have $|K'| \geq \delta + 2$.

First suppose $d_{G'}(z) \leq d(u) + d(u') - 2|X|$. Hence, $f'(z) \leq .5/(d_{G'}(z) + .5)$. For each $w \in K'$, we have $d_G(w) \geq |K'| - 1$. Therefore $f'(w) \leq (|K'| - .5)^{-1}$ and $f'(z) \leq .5(|K'| - .5)^{-1}$, so $\sum_{w \in K'} f'(w) \leq (|K'| - 1)/(|K'| - .5) + .5/(|K'| - .5) \leq 1$, as claimed.

Therefore we may assume that $d_{G'}(z) > d(u) + d(u') - 2|X|$. By (10.4), $d_{G'}(z) \leq |V_2 \cup V_3 \cup V_4| - |X|$. Hence, $K' \subseteq ((V_2 \cup V_3 \cup V_4) \setminus X) \cup \{z\}$. By Lemma 10.4.6, $|K'| \leq (2\delta + 1)/2 + 1 < \delta + 2$, a contradiction.

Hence, since G is a minimum counterexample, G' has an f'-coloring, say ϕ', as claimed. Let $\phi = \phi|_{V(G) \setminus (V_0 \cup X \cup \{v, u, u'\})}$. Since $\phi'(z) \subseteq L_\phi(u) \cap L_\phi(u')$, by Lemma 10.4.6, we have

$$\mu\left(L_\phi(u) \cap L_\phi(u')\right) \geq f'(z) \geq \frac{.5}{\max\{d(u) + d(u') - |V_2 \cup V_3 \cup V_4| + .5, \delta + .5\}},$$

as desired. $\square$

Our objective in proving Lemma 10.4.4 is to take an f-coloring from Lemma 10.4.9, show that it can be extended to $\{v, u, u'\} \cup V_0 \cup (X \cap V_2)$, and then show that it can be extended to $X \cap V_3$ using Lemma 10.3.8. However, we need to be careful about the order in which we extend this coloring. The following lemma allows us to extend the coloring first to either u or u'.

Lemma 10.4.10. *Suppose (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$ and ϕ is an f-coloring of $G - (V_0 \cup X \cup \{v, u, u'\})$. If $x \in \{u, u'\}$, then $\mu\left(L_\phi(x)\right) \geq f(x)$.*

Proof. First, suppose $x \in X$. Since $\delta \geq 2$, we have $d(x) \geq 3$, so by Lemma 10.4.6, $|X| \geq 2$. Now x sees at most $\frac{|V_2 \cup V_3 \cup V_4| - |X|}{|V_2 \cup V_3 \cup V_4| + .5}$ color in ϕ. Hence, $\mu\left(L_\phi(x)\right) \geq \frac{|X| + .5}{|V_2 \cup V_3 \cup V_4| + .5} \geq \frac{|X| - .5}{d(x) + .5} \geq f(x)$, as desired.

Therefore we may assume $x \notin K'$. We claim that x sees at most

$$\frac{|V_2 \cup V_3| - |X|}{|V_2 \cup V_3 \cup V_4| + .5} + \frac{d(x) - |V_2 \cup V_3|}{\delta + .5}$$

color. Note that x has at most $|V_2 \cup V_3| - |X|$ neighbors in $V_2 \cup V_3 \cup V_4 \setminus X$. Note also that u has at most $d(u) - |V_0 \cup V_3|$ neighbors not in $\cup_{i=0}^4 V_i$, and u' has at most $d(u') - (|V_2 \cup V_3| - 1|) - 1$ neighbors not in $\cup_{i=0}^4 V_i$. Since $|V_2| \le |V_0|$, x has at most $d(x) - |V_2 \cup V_3|$ neighbors not in $\cup_{i=0}^4 V_i$. Therefore x sees at most $\frac{|V_2 \cup V_3| - |X|}{|V_2 \cup V_3 \cup V_4| + .5}$ color among vertices in $V_2 \cup V_3 \cup V_4 \setminus X$ and at most $\frac{d(x) + |V_2 \cup V_3|}{\delta + .5}$ color among the remaining vertices. The claim follows.

Hence, it suffices to show that

$$\frac{|V_2 \cup V_3| - |X|}{|V_2 \cup V_3 \cup V_4| + .5} + \frac{d(x) - |V_2 \cup V_3|}{\delta + .5} + f(x) \le 1. \tag{10.6}$$

We need the following claims, which will be proved in Section 10.7.

Claim 10.4.10.1. *Let*

$$q_\delta(k, d_x) = 1 - \frac{d_x - k}{\delta + .5} - \frac{1}{d_x + .5}.$$

For $\hat{k} \ge \delta + 1$, $k \in [\hat{k}/2, \delta - 1]$ *and* $d_x \in [\delta + 1, \frac{\hat{k} + .5}{1.5 - k/(\hat{k} - .5)} - .5]$, *we have* $q_\delta(k, d_x) \ge 0$.

Claim 10.4.10.2. *Let*

$$q_\delta(k', \hat{k}, d_x) = 1 - \frac{\hat{k} - k' - 1}{\hat{k} + .5} - \frac{d_x + 1 - \hat{k}}{\delta + .5} - \frac{1}{d_x + .5}.$$

If $k' \ge \delta$, $\hat{k} \ge \max\{k' + 1, \delta + 2\}$, *and* $d_x \in [\hat{k}, 2\delta]$, *then* $q_\delta(k', \hat{k}, d_x) \ge 0$.

Note that

$$\frac{|V_2 \cup V_3| - |X|}{|V_2 \cup V_3 \cup V_4| + .5} + \frac{d(x) - |V_2 \cup V_3|}{\delta + .5} \le \frac{d(x) - |X|}{\delta + .5}.$$

Therefore (10.6) holds if

$$1 - \frac{d(x) - |X|}{\delta + .5} - \frac{1}{d(x) + .5} \ge 0.$$

Now suppose $|X| \le \delta - 1$. The left side of the above inequality is $q_\delta(|X|, d(x))$ from Claim 10.4.10.1. By Claim 10.4.10.1 and Lemma 10.4.5, (10.6) holds, as desired.

Therefore we may assume $|X| \ge \delta$. First we show that $|V_2 \cup V_3 \cup V_4| \ge \delta + 2$. If $\{u, u'\} \in V_2 \cup V_3 \cup V_4 \setminus X$, then $|V_2 \cup V_3 \cup V_4| \ge |X| + 2 \ge \delta + 2$, as desired. If one of u and u' is in X, since both have degree at least $\delta + 1$, we have $|V_2 \cup V_3 \cup V_4| \ge \delta + 2$, as desired. Note that (10.6) holds if $q_\delta(|X|, |V_2 \cup V_3 \cup V_4|, d(x)) \ge 0$, where q_δ is the function from Claim 10.4.10.2. Hence, by Claim 10.4.10.2, (10.6) holds, as desired. $\square$

After using Lemma 10.4.10 to extend our coloring to one of u and u', it is fairly easy to show that it can be extended to $V_0 \cup V_1$. Our next objective is to show that this coloring can be extended to whichever of u and u' remains uncolored. The following lemma says that we can do this in the case when neither of u and u' are in X, under certain assumptions.

Lemma 10.4.11. *Suppose (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$ such that $u, u' \notin X$. Suppose also that $\{x, x'\} = \{u, u'\}$ where $d(x) \le d(x')$, and if $d(x) = d(x')$, then $d(x)$ has at most as many neighbors of degree δ as x does. If $(V_i)_{i=0}^4$ is not a turtle and ϕ is an f-coloring of $G - (\{x\} \cup X)$, then $\mu(L_\phi(x)) \ge f(x)$.*

Proof. Since $u, u' \notin X'$, x has at most $|V_2 \cup V_3| - 1 - |X|$ neighbors in $(V_2 \cup V_3 \cup V_4) \setminus X$. Also, x has at most $d(x) + 1 - |V_2 \cup V_3|$ neighbors not in $V_2 \cup V_3 \cup V_4$. Therefore x sees at most

$$\frac{|V_2 \cup V_3 \cup V_4| - |X| - 2}{|V_2 \cup V_3 \cup V_4| + .5} + \frac{d(x) + 2 - |V_2 \cup V_3 \cup V_4|}{\delta + .5}$$

color. Therefore it suffices to show that

$$\frac{|V_2 \cup V_3 \cup V_4| - |X| - 2}{|V_2 \cup V_3 \cup V_4| + .5} + \frac{d(x) + 2 - |V_2 \cup V_3 \cup V_4|}{\delta + .5} + \frac{1}{d(x) + .5} \le 1. \tag{10.7}$$

Suppose $|X| \le \delta - 1$. Note that

$$\frac{|V_2 \cup V_3 \cup V_4| - |X| - 2}{|V_2 \cup V_3 \cup V_4| + .5} + \frac{d(x) + 2 - |V_2 \cup V_3 \cup V_4|}{\delta + .5} \le \frac{d(x) - |X|}{\delta + .5}.$$

Therefore (10.7) holds if

$$1 - \frac{d(x) - |X|}{\delta + .5} - \frac{1}{d(x) + .5} \ge 0.$$

The left side of the above inequality is $q_\delta(|X|, d(x))$ from Claim 10.4.10.1. By Claim 10.4.10.1 and Lemma 10.4.5, (10.7) holds, as desired.

Therefore we may assume $|X| \ge \delta$. We need the following claims, which will be proved in Section 10.7.

Claim 10.4.11.1. *Let*

$$q_\delta(k', \hat{k}, d_{u'}) = 1 - \frac{\hat{k} - k' - 2}{\hat{k} + .5} - \frac{d_{u'} + 2 - \hat{k}}{\delta + .5} - \frac{1}{d_{u'} + .5}.$$

If $k' \ge \delta$, $\hat{k} \ge \max\{k' + 2, \delta + 3\}$, and $d_{u'} \in [\hat{k}, 2\delta]$, then $q_\delta(k', \hat{k}, d_{u'}) \ge 0$.

Claim 10.4.11.2. *Let*

$$q_\delta(k', \hat{k}, d_{u'}) = 1 - \frac{\hat{k} - k' - 2}{\hat{k} + .5} - \frac{d_{u'} + 2 - \hat{k}}{\delta + .5} - \frac{1}{d_{u'} + .5}.$$

If $k' \geq \delta$, $\hat{k} \geq k' + 2$, and $d_{u'} \in [\hat{k}, 2\delta - 1]$, then $q_\delta(k', \hat{k}, d_{u'}) \geq 0$.

By Claim 10.4.11.1, (10.7) holds if $|V_2 \cup V_3 \cup V_4| \geq \delta + 3$. By Claim 10.4.11.2, (10.7) holds if $d(x) \leq 2\delta - 1$. Therefore we may assume that $|V_2 \cup V_3 \cup V_4| = \delta + 2$ and $d(x) = 2\delta$.

By assumption, $d(x) \leq d(x')$, so $d(x') = 2\delta$. Since $(V_i)_{i=0}^4$ is dangerous,

$$\begin{aligned} 1 < \sum_{w \in V_2 \cup V_3 \cup V_4} f(w) &\leq \frac{2}{2\delta + .5} + \frac{|V_2 \cup V_3 - 1|}{|V_2 \cup V_3 \cup V_4| - .5} = \frac{2}{2\delta + .5} + \frac{\delta}{\delta + 1.5} \\ &= \frac{\delta - 2.25}{(\delta + .25)(\delta + 1.5)} = 1 - \frac{.5(\delta - 2.25)}{(\delta + .25)(\delta + 1.5)}, \end{aligned}$$

so $\delta = 2$. By Lemma 10.4.6, $|X| = 2$. Since $(V_i)_{i=0}^4$ is not a turtle, by the choice of x and x', the vertex x has only one neighbor of degree δ, and it is in $V_0 \cup V_1$. Hence, x sees at most $1/2.5 + 1/3.5$ color, so

$$\mu\left(L_\phi(x)\right) - f(x) \geq 1 - \frac{1}{2.5} - \frac{1}{3.5} - \frac{1}{4.5} = \frac{29}{315} > 0,$$

as desired. □

In the case when $u \in X$ and $u' \notin X$, we color u' first. The following lemma will allow us to extend our coloring to u in this case.

Lemma 10.4.12. *Suppose (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$ such that $u' \notin X$ and $u \in X$. If ϕ is an f-coloring of $G - (\{u\} \cup X)$, then $\mu\left(L_\phi(u)\right) \geq f(u)$.*

Proof. Note that u sees at most $\frac{|V_2 \cup V_3 \cup V_4| - 1 - |X|}{|V_2 \cup V_3 \cup V_4| + .5} + \frac{|V_0|}{\delta + .5}$ color in ϕ. By Lemma 10.4.6, $|V_2 \cup V_3 \cup V_4| - 1 - |X| \geq (d(u) - 1)/2$, and by Lemma 10.4.7, $|V_0| = |V_2| \leq \delta/2$. Therefore u sees at most $\frac{.5(d(u)-1)}{d(u)+1.5} + \frac{.5\delta}{\delta + .5}$ color in ϕ. Now we need the following claim, which will be proved in Section 10.7.

Claim 10.4.12.1. *Let*

$$q_\delta(d) = 1 - \frac{.5(d - 1)}{d + 1.5} - \frac{.5\delta}{\delta + .5} - \frac{1}{d + .5}.$$

If $d \in [\delta, 2\delta]$, then $q_\delta(d) \geq 0$.

By Claim 10.4.12.1, since $\mu\left(L_\phi(u)\right) - f(u) \geq q_\delta(d(u))$, we have $\mu\left(L_\phi(u)\right) \geq f(u)$, as desired. $\square$

In the case when $u \notin X$ and $u' \in X$, we color u first. The following two lemmas will allow us to extend our coloring to u' in this case. It will also allow us to extend it to $X \cap V_2$, even if $u' \notin X$. The first of these two lemmas considers the special case when $|X \cap V_3| = 1$ separately.

Lemma 10.4.13. *Suppose (V_0, v, u, u', X) is an essential restriction of a dangerous δ-based 5-cycle blowup $(V_i)_{i=0}^4$ such that $|X \cap V_3| = 1$ and $u \notin X$. If $x \in V_2 \cap X$, and ϕ is an f-coloring of $G - (\{x\} \cup (X \cap V_3))$, then $\mu\left(L_\phi(x)\right) \geq f(x)$.*

Proof. First we claim that $|V_2 \cup V_3 \cup V_4| \leq \delta + 2$. Suppose not. By Lemma 10.4.6, $|X| \geq (\delta+3)/2$. Since $u \in X$ and $|X \cap V_3| = 1$, we know $|V_2 \cap X| \geq (\delta+1)/2$, a contradiction. Hence, $|V_2 \cup V_3 \cup V_4| \leq \delta + 2$, as claimed.

The vertex x sees at most

$$\frac{|V_2 \cup V_3 \cup V_4| - |X| - 1}{|V_2 \cup V_3 \cup V_4| + .5} + \frac{|X| - 2}{|V_2 \cup V_3 \cup V_4| - .5} + \frac{1}{\delta + .5}$$

color in ϕ. Therefore

$$\mu\left(L_\phi(x)\right) - f(x) \geq 1 - \frac{|V_2 \cup V_3 \cup V_4| - |X| - 1}{|V_2 \cup V_3 \cup V_4| + .5} - \frac{|X| - 1}{|V_2 \cup V_3 \cup V_4| - .5} - \frac{1}{\delta + .5}. \quad (10.8)$$

Since $\frac{|X|-1}{|V_2 \cup V_3 \cup V_4| - .5} \leq \frac{|X|}{|V_2 \cup V_3 \cup V_4| + .5}$, we have

$$\mu\left(L_\phi(x)\right) - f(x) \geq 1 - \frac{|V_2 \cup V_3 \cup V_4| - 1}{|V_2 \cup V_3 \cup V_4| + .5} - \frac{1}{\delta + .5}.$$

If $|V_2 \cup V_3 \cup V_4| = \delta + 1$, then

$$\mu\left(L_\phi(x)\right) - f(x) \geq 1 - \frac{\delta}{\delta + 1.5} - \frac{1}{\delta + .5} = \frac{.5(\delta - 1.5)}{(\delta + 0.5)(\delta + 1.5)} > 0,$$

as desired. Therefore we may assume $|V_2 \cup V_3 \cup V_4| = \delta + 2$. Hence,

$$\mu\left(L_\phi(x)\right) - f(x) \geq 1 - \frac{\delta + 1}{\delta + 2.5} - \frac{1}{\delta + .5} = \frac{.5(\delta - 3.5)}{(\delta + .5)(\delta + 2.5)}.$$

If $\delta \geq 4$, then the previous expression is positive, as desired. Therefore we may assume $\delta \leq 3$. Hence, $|V_0| + |V_2| \leq 3$ and $|V_2| \leq |V_0|$, so $|V_2| = 1$. Since $u \notin X$ and $|X \cap V_3| = 1$, we have $|X| = 2$. By Lemma 10.4.6, $|V_2 \cup V_3 \cup V_4| \leq 4$. Since $x \in X$, we have $d(x) = 3$, so $\delta = 2$. Now the right side of (10.8) is $1 - 1/4.5 - 1/3.5 - 1/2.5 = 29/315 > 0$, as desired. □

Lemma 10.4.14. *Suppose* (V_0, v, u, u', X) *is an essential restriction of a dangerous* δ*-based 5-cycle blowup* $(V_i)_{i=0}^4$ *such that* $|X \cap V_3| \geq 2$. *If* $x \in V_2 \cap X$ *and* ϕ *is an* f*-coloring of* $G - (\{x\} \cup (X \cap V_3))$, *then* $\mu(L_\phi(x)) \geq f(x)$

Proof. Since $x \in X$, we know x sees at most

$$\frac{|V_2 \cup V_3 \cup V_4| - |X| - 1}{|V_2 \cup V_3 \cup V_4| + .5} + \frac{|X| - 3}{|V_2 \cup V_3 \cup V_4| - .5} + \frac{1}{\delta + .5}$$

color in ϕ. Since $f(x) \leq 1/(|V_2 \cup V_3 \cup V_4| - .5)$,

$$\mu(L_\phi(x)) - f(x) \geq 1 - \frac{|V_2 \cup V_3 \cup V_4| - |X| - 1}{|V_2 \cup V_3 \cup V_4| + .5} - \frac{|X| - 2}{|V_2 \cup V_3 \cup V_4| - .5} - \frac{1}{\delta + .5}.$$

Since $\frac{|X|-2}{|V_2 \cup V_3 \cup V_4| - .5} \leq \frac{|X|-1}{|V_2 \cup V_3 \cup V_4| + .5}$, we have

$$\mu(L_\phi(x)) - f(x) \geq 1 - \frac{|V_2 \cup V_3 \cup V_4| - 2}{|V_2 \cup V_3 \cup V_4| + .5} - \frac{1}{\delta + .5}.$$

Since $\frac{|V_2 \cup V_3 \cup V_4| - 2}{|V_2 \cup V_3 \cup V_4| + .5} \leq \frac{2\delta - 1}{2\delta + 1.5}$, we have

$$\mu(L_\phi(x)) - f(x) \geq 1 - \frac{2\delta - 1}{2\delta + 1.5} - \frac{1}{\delta + .5} = \frac{2.5}{2\delta + 1.5} - \frac{1}{\delta + .5} \geq 0,$$

as desired. □

Now we can finally prove Lemma 10.4.4.

Proof of Lemma 10.4.4. Suppose $(V_i)_{i=0}^4$ is a dangerous δ-based 5-cycle blowup, and let (V_0, v, u, u', X) be an essential restrction of $(V_i)_{i=0}^4$. If $\delta = 2$, assume $(V_i)_{i=0}^4$ is not a turtle or a skew-turtle.

First, suppose $|X \cap V_3| = 0$. By Lemma 10.4.8, we may assume $\delta = 2$, since $|V_0| = |V_2| = \delta/2$, as required. Since $\delta = 2$, we have $|V_2| = 1$, and since $|X \cap V_3| = 0$, we have

$X = V_1 \cup V_2$. Therefore $d(u) = d(u') = 3$, so $(V_i)_{i=0}^4$ is a skew-turtle, a contradiction. Thus, we may assume $|X \cap V_3| \geq 1$.

By Lemma 10.4.9, there is an f-coloring ϕ_0 of $G - (V_0 \cup X \cup \{v, u, u'\})$ and $C \subseteq L_{\phi_0}(u) \cap L_{\phi_0}(u')$ such that

$$\mu(C) = \frac{.5}{\max\{d(u) + d(u') - |V_2 \cup V_3 \cup V_4| + .5, \delta + .5\}}. \tag{10.9}$$

Choose u_1 and u_2 such that $\{u_1, u_2\} = \{u, u'\}$ as follows. If $u' \in X$, let $u_2 = u'$ and $u_1 = u$. If $\{u, u'\} \cap X = \varnothing$ and $d(u') < d(u)$, then let $u_2 = u'$ and $u_1 = u$. If $\{u, u'\} \cap X = \varnothing$ and $d(u') = d(u)$ such that u' has at most as many neighbors of degree δ as u does, then let $u_2 = u'$ and $u_1 = u$. Otherwise let $u_2 = u$ and $u_1 = u'$. By Lemma 10.4.10, $\mu(L_{\phi_0}(u_1)) \geq f(u_1)$. Therefore, we can extend ϕ_0 to an f-coloring ϕ_1 of $G - ((V_0 \cup \{v, u_2\} \cup X) \setminus \{u_1\})$ such that $C \subseteq \phi_1(u_1)$.

We show that ϕ_1 can be extended to $V_0 \cup V_1$ without using color from C. Let $v' \in V_0 \cup V_1$ be a neighbor of u_2. Let $w \in (V_0 \cup V_1) \setminus \{v'\}$. Note that w sees at most $(\delta - |V_0|)/(\delta + .5)$ color in ϕ_1, so

$$\mu(L_{\phi_1}(w) \setminus C) \geq 1 - \frac{\delta - |V_0|}{\delta + .5} - \frac{.5}{\delta + .5} = \frac{|V_0|}{\delta + .5}.$$

Note also that v' sees at most $(\delta - 1 - |V_0|)/(\delta + .5)$ color in ϕ_1, so

$$\mu(L_{\phi_1}(v') \setminus C) \geq 1 - \frac{\delta - 1 - |V_0|}{\delta + .5} - \frac{.5}{\delta + .5} = \frac{|V_0| + 1}{\delta + .5}.$$

By Lemma 7.1.5 and the previous two inequalities, there exists an f-coloring ϕ_2 of $G - (\{u_2\} \cup X \setminus \{u_1\})$ such that $C \subseteq \phi_2(u_1) \cap L_{\phi_2}(u_2)$.

We claim that $\mu(L_{\phi_2}(u_2)) \geq f(u_2)$. First, suppose $u_2 = u$. If $u \notin X$, then by the choice of u_1 and u_2, we have $d(u) < d(u')$. Hence, by Lemma 10.4.11, $\mu(L_{\phi_2}(u_2)) \geq f(u_2)$, as required. If $u \in X$, then by the choice of u_1 and u_2, we have $u' \notin X$. Hence, by Lemma 10.4.12, $\mu(L_{\phi_2}(u_2)) \geq f(u_2)$, as required. Now suppose $u_2 = u'$. If $u' \in X$, then by Lemmas 10.4.13 and 10.4.14, $\mu(L_{\phi_2}(u_2)) \geq f(u_2)$, as required. If $u \notin X$, then by the choice of u_1 and u_2, we have $d(u) \leq d(u')$. Hence, since $(V_i)_{i=0}^4$ is not a turtle, by Lemma 10.4.11, $\mu(L_{\phi_2}(u_2)) \geq f(u_2)$, as required. Therefore, $\mu(L_{\phi_2}(u_2)) \geq f(u_2)$, as claimed, and we can extend ϕ_2 to an f-coloring ϕ_3 of $G - (X \setminus \{u, u'\})$ such that $C \subseteq \phi_2(u) \cap \phi_2(u')$.

By Lemmas 10.4.13, 10.4.14, and 7.1.5, ϕ_2 can be extended to an f-coloring ϕ_3 of $G - (X \cap V_3)$. Let $\phi_4 = \phi_3|_{V(G) \setminus ((X \cap V_3) \cup \{u, u'\})}$. By applying Lemma 10.3.8 with K being $X \cap V_3$, $d = |V_2 \cup V_3 \cup V_4| - 1$, u, u', and $\phi = \phi_4$, we conclude that $\mu(L_{\phi_4}(u) \cap L_{\phi_4}(u')) < .5/\max\{d(u) + d(u') - |V_2 \cup V_3 \cup V_4|.5, \delta + .5\}$. However, $C \subseteq L_{\phi_4}(u) \cap L_{\phi_4}(u')$, contradicting (10.9). $\square$

10.5 Structure around base cliques

We continue proving properties of a minimum counterexample to Theorem 1.5.2 in this section. In particular, we prove Lemma 10.2.2.

Lemma 10.5.1. *If $\varepsilon \leq 1/2$, then no two non-adjacent vertices of minimum degree share a common neighbor of minimum degree.*

Proof. Suppose to the contrary that u and w are non-adjacent vertices of minimum degree with a common neighbor v of minimum degree. By the minimality of G, the graph $G - \{u, v, w\}$ has an f-coloring ϕ. Note that u and w see at most $\frac{\delta-1}{\delta+1-\varepsilon}$ color and v sees at most $\frac{\delta-2}{\delta+1-\varepsilon}$. Therefore $\mu\left(L_\phi(u)\right), \mu\left(L_\phi(w)\right) \geq \frac{2-\varepsilon}{\delta+1-\varepsilon}$ and $\mu\left(L_\phi(v)\right) \geq \frac{3-\varepsilon}{\delta+1-\varepsilon}$. Since $\varepsilon \leq 1/2$,

$$\mu\left(L_\phi(u)\right) + \mu\left(L_\phi(w)\right) \geq \frac{4-2\varepsilon}{\delta+1-\varepsilon} \geq f(v) + f(u) + f(w).$$

Hence by Lemma 7.1.7, $G[\{u, v, w\}]$ has an (f, L_ϕ)-coloring. By Proposition 7.1.3, G has an f-coloring, a contradiction. □

Lemma 10.5.1 implies that every vertex of minimum degree is contained in a unique base clique. Moreover, vertices in different base cliques are not adjacent.

Lemma 10.5.2. *If $\varepsilon \leq 1/2$ and K is a base clique, then*

$$|K| > (1-\varepsilon)(\delta+1-\varepsilon).$$

Moreover, $|K| \geq (\delta+1)/2$.

Proof. By Lemmas 10.3.7 and 10.5.1,

$$|K| > \delta + 1 - \varepsilon(\delta + 2 - \varepsilon) = (1-\varepsilon)(\delta+1-\varepsilon),$$

as desired. Moreover, since $\varepsilon \leq 1/2$ and $|K|$ is an integer, we have $|K| \geq \lceil(\delta + .5)/2\rceil \geq (\delta+1)/2$, as desired. □

Lemma 10.5.2 implies that $\ell_K \leq (\delta+1)/2$ in Lemma 10.2.2. As mentioned, we need to prove that $\ell_K \neq 0$ separately, as follows.

Lemma 10.5.3. *If K is a base clique and $\varepsilon \leq 1/2$, then $\ell_K > 0$.*

Proof. For convenience, let $A = A_K$ and $\ell = \ell_K$. Suppose to the contrary that $\ell = 0$. Now $|K \cup A| = \delta + 1$. By Lemma 10.3.2, A is not a clique, so there exists a pair of non-adjacent vertices $u, w \in A$.

Let G' be the graph obtained from $G - K$ by identifying u and w into a new vertex, say z. Define a new demand function f' for G' in the following way. For each $v \in V(G') \setminus \{z\}$, let $f'(v) = f(v)$, and let $f'(z) = .5/(d_G(u) + d_G(w) - 2|K| + .5)$.

Note that for each $v \in V(G')$, we have $f'(v) \leq 1/(d_{G'}(v) + .5)$, and moreover, $f'(z) \leq .5/(d_{G'}(z) + .5)$. We claim that for each clique K' in G', we have $\sum_{v \in K'} f'(v) \leq 1$. If $z \notin K'$, this holds because K' is a clique in G and $f'(v) = f(v)$ for all $v \in K'$. If $z \in K'$, then for each $v \in K'$, we have $d_G(v) \geq |K'| - 1$. Therefore $f'(v) \leq (|K'| - .5)^{-1}$ and $f'(z) \leq .5(|K'| - .5)^{-1}$, so $\sum_{v \in K'} f'(v) \leq (|K'| - 1)/(|K'| - .5) + (1 - .5)/(|K'| - .5) \leq 1$, as claimed. Hence, since G is a minimum counterexample, G' has an f'-coloring ϕ'. Let $\phi = \phi|_{G - (K \cup \{u,w\})}$. We will apply Lemma 10.3.8.

We claim that for $x \in \{u, w\}$, we have $\mu\left(L_\phi(x)\right) \geq f(x)$. Since x sees at most $\frac{d(x) - |K|}{\delta + .5}$ color,

$$\mu\left(L_\phi(x)\right) \geq \frac{\delta + .5 + |K| - d(x)}{\delta + .5}. \tag{10.10}$$

By applying Lemma 10.3.4 to any vertex in K,

$$d(x) \leq \frac{\delta + 1.5}{1.5 - |K|/(\delta + .5)} + .5. \tag{10.11}$$

By (10.10), we have $\mu\left(L_\phi(x)\right) - f(x) \geq q_\delta(|K'|, d(x))$ from Claim 10.4.10.1. Therefore (10.11) and Claim 10.4.10.1 with $\hat{k} = \delta + 1$ imply that $\mu\left(L_\phi(x)\right) \geq f(x)$, as claimed.

Therefore by Lemma 10.3.8 applied to K,

$$\mu\left(L_\phi(u) \cap L_\phi(w)\right) < \frac{.5}{d(u) + d(w) - 2|K| + .5} = f'(z),$$

a contradiction.

□

The remainder of this section is needed to prove Theorem 1.5.2 for $\varepsilon = 1/2$. The following lemma describes how dangerous δ-based 5-cycle blowups appear.

Lemma 10.5.4. *Let K be a base clique. If $u \in U_K$ and $v \in K \backslash N(u)$ such that $|N(u) \cap K| \geq \ell_K$, then for some $V_3 \subseteq V(G)$, we have $(N(u) \cap K, \{v\}, N(v) \cap U_K, V_3, \{u\})$ is a dangerous δ-based 5-cycle blowup.*

Proof. For convenience, let $U = U_K$ and $\ell = \ell_K$. Let $X = \{v\} \cup (N(u) \cap K)$, and let $v' \in N(u) \cap K$. First we claim that for every f-coloring ϕ of $G - X$, we have

$$\mu\left(\cup_{w \in U \cap N(v)} \phi(w) \cap \cup_{w \in U \cap N(v')} \phi(w)\right) > (\ell - \varepsilon) f(v). \tag{10.12}$$

To that end, suppose ϕ is an f-coloring of $G - X$, and let $C = \cup_{w \in U \cap N(v)} \phi(w)$ and $C' = \cup_{w \in U \cap N(v')} \phi(w)$. By Proposition 7.1.3, $G[X]$ does not have an (f, L_ϕ)-coloring. By Lemma 7.1.5, there exists $S \subseteq X$ such that $\sum_{x \in S} f(x) > \mu\left(\cup_{x \in S} L(x)\right)$. Each vertex $x \in X$ sees at most $\frac{\delta+1-|X|}{\delta+1-\varepsilon}$ color. Hence, for each $x \in X$, we have $\mu\left(L_\phi(x)\right) \geq (|X| - \varepsilon) f(x)$, so $S = X$. Furthermore,

$$L_\phi(v) \cup L_\phi(v') = [0, 1] \setminus (\cup_{w \in (A \cup K) \setminus X} \phi(w) \bigcup (C \cap C')).$$

Hence,

$$\mu\left(L_\phi(v) \cup L_\phi(v')\right) \geq 1 - (\delta + 1 - \ell - |X|) f(v) - \mu\left(C \cap C'\right) = (\ell + |X| - \varepsilon) f(v) - \mu\left(C \cap C'\right).$$

Since $\mu\left(L_\phi(v) \cup L_\phi(v')\right) < |X| f(v)$, we have

$$\mu\left(C \cap C'\right) > (\ell - \varepsilon) f(v),$$

as claimed.

Let G' be the graph obtained from $G - X$ by adding an edge between u and each vertex $w \in U \cap N(v')$ if one was not already present. Now if G' has an f-coloring ϕ, then $\phi(u) \cap \cup_{w \in N(v) \cap U} \phi(w) = \varnothing$. Therefore $\mu\left(\cup_{w \in U \cap N(v)} \phi(w) \cap \cup_{w \in U \cap N(v')} \phi(w)\right) \leq (\ell - 1) f(v)$, a contradiction. Since $|N(u) \cap K| \geq \ell$, for each $w \in V(G')$, we have $d_{G'}(w) \leq d_G(w)$. Therefore there exists a clique K' in G' such that $\sum_{w \in K'} f(w) > 1$. Let $V_3 = K' \setminus (\{u\} \cup (N(v) \cap U))$. Now $(N(u) \cap K, \{v\}, N(v) \cap U, V_3, \{u\})$ is the desired dangerous δ-based 5-cycle blowup. □

We will use Lemma 10.5.4 to show that $\delta \geq 3$. First, we need to handle skew-turtles and turtles.

Lemma 10.5.5. *If v is a vertex of degree two, then there is a turtle $(V_i)_{i=0}^4$ such that $v \in V_0 \cup V_1$.*

Proof. Since v has degree two, $\delta = 2$ and v is contained in a base clique K. By Lemma 10.5.2, $|K| = 2$, and by Lemma 10.5.3, $\ell_K = 1$. Let $u \in U_K$ be adjacent to $v' \in K$, let $v \in K$ be the vertex in K not adjacent to u, and let u' be the neighbor of v in U_K. By Lemma 10.5.4, for some $V_3 \subseteq V(G)$, we have $(V_0 = \{v'\}, V_1 = \{v\}, V_2 = \{u'\}, V_3, V_4 = \{u\})$ is a dangerous δ-based 5-cycle blowup. By Lemma 10.4.4, it is a turtle or a skew-turtle.

It suffices to show that $(V_i)_{i=0}^4$ is not a skew-turtle. Suppose not. By the definition of a skew-turtle, $|V_3| = 2$ and each vertex in V_3 has degree at least four. Let $\{u_1, u_2\} = V_3$, and note that u_1 and u_2 have neighbors not in $\cup_{i=0}^4 V_i$, whereas every other vertex in $\cup_{i=0}^4 V_i$ has no neighbors not in $\cup_{i=0}^4$. Since G is a minimum counterexample, by Proposition 1.1.6, for some N, the graphs $G[\cup_{i=0}^4 V_i]$ and $G - (V_0 \cup V_1 \cup V_2 \cup V_4)$ have (f, N)-colorings ψ_1 and ψ_2 respectively. Since u_1 and u_2 are not adjacent, for $i \in \{1, 2\}$, we have $\psi_i(u_1) \cap \psi_i(u_2) = \varnothing$. By permuting colors, we may assume without loss of generality that $\psi_1(u_i) = \psi_2(u_i)$ for $i \in \{1, 2\}$. By combining ψ_1 and ψ_2, we obtain an (f, N)-coloring of G, and by Proposition 1.1.6, G has an f-coloring, a contradiction. □

Lemma 10.5.6. *If $\varepsilon \leq 1/2$, then $\delta \geq 3$.*

Proof. Suppose to the contrary that $\delta = 2$. By Lemma 10.5.5, there exists an essential restriction $(\{v'\}, v, u, u', X)$ of a turtle $(V_i)_{i=0}^4$. By the definition of a turtle, u and u' have neighbors of degree two, say v_1 and v_2, respectively, not in $V_0 \cup V_1$. Morover, $v_1, v_2 \notin X$. By Lemma 10.5.2, v_1 and v_2 are distinct and both have neighbors of degree two. By Lemma 10.5.5, there is a turtle $(V_i')_{i=0}^4$ such that $v_1 \in V_0' \cup V_1'$. We may assume without loss of generality that $v_1 \in V_0'$, by the symmetry of the turtle. Now $V_4' = V_4$, so $V_3' = V_3$ and $V_2' = V_2$. Hence, $V_1' = \{v_2\}$, and G is a graph on eight vertices. Since G is a minimum counterexample, $G - \{v_1, v_2\}$ has an f-coloring ϕ. However, ϕ can be extended to an f-coloring of G by coloring v_1 with $\phi(v')$ and v_2 with $\phi(v)$, contradicting that G is a counterexample. □

Finally, we can prove Lemma 10.2.2.

Proof of Lemma 10.2.2. For convenience, let $\ell = \ell_K, U = U_K$, and $D = D_K$. Note that Lemma 10.5.2 implies that $\ell \leq (\delta+1)/2$ and Lemma 10.5.6 implies that $\delta \geq 3$, as required. Suppose that $\ell \leq D$. By Lemma 10.5.3, $\ell \geq 1$, so $U \neq \varnothing$. Let $u \in U$ have D neighbors in K, and let $v \in K \setminus N(u)$. Now, by Lemma 10.5.4, for some $V_3 \subseteq V(G)$, we have $(N(u) \cap K, \{v\}, N(v) \cap U, V_3, \{u\})$ is a dangerous δ-based 5-cycle blowup. By Lemma 10.4.4, $D = |N(u) \cap K| = |N(v) \cap U| = \ell = \delta/2$. Therefore, $D \leq \ell$, as desired. Moreover, if $\ell = 1$, then $\ell \leq D$, and the previous argument implies that $\delta = 2$, contradicting Lemma 10.5.6. □

10.6 Overcoloring the complements of base cliques

The most important result of this section is Lemma 10.2.3. In order to prove it, we need to find particularly nice colorings of the complements of base cliques, as in the following definition.

Definition 10.6.1. Let K be a base clique of G, and let f_K be the demand function for $G - K$ defined by

$$f_K(u) = 1/\max\{\omega_{G-K}(u), d_{G-K}(u) + 1 - \varepsilon\}$$

for each $u \in V(G - K)$. If ψ is an f_K-coloring of $G - K$, then ψ is an *overcoloring* with respect to K. An f-coloring ϕ of $G - K$ is an *optimized reduction* of ψ if for each $u \in V(G - K)$,

1. $\phi(u) \subseteq \psi(u)$ and subject to that,
2. $\mu\left(\cup_{u\in K} L_\phi(u)\right)$ is maximum, and subject to that,
3. $\sum_{u\in K} \mu\left(L_\phi(u)\right)$ is maximum.

Since G is a minimum counterexample, for any base clique K, there is an overcoloring with respect to K. By Lemma 10.3.2, if $u \in V(G) \setminus (K \cup U_K \cup A_K)$, then $f_K(u) = 1/(d(u) + 1 - \varepsilon) \geq f(u)$. If $u \in V(G) \cap (U_K \cup A_K)$, then $\omega_{G-K}(u) \leq d_G(u)$, and thus $f_K(u) \geq f(u)$. Hence, for every overcoloring ψ, there is an optimized reduction of ψ.

For convenience, if L is a fractional list-assignment for a base clique K, then for any $S \subseteq K$, we let $L(S) = \bigcup_{v\in S} L(v)$.

Lemma 10.6.2. *If K is a base clique of G and ϕ is an f-coloring of $G - K$, then for each $v \in K$,*

$$\mu\left(L_\phi(v)\right) \geq \frac{|K| + 1 - \varepsilon}{\delta + 2 - \varepsilon} \geq (|K| - \varepsilon)f(v).$$

Moreover,

$$\mu\left(L_\phi(K)\right) < \frac{|K|}{\delta + 1 - \varepsilon}. \tag{10.13}$$

Proof. Each $v \in K$ sees at most $\frac{\delta+1-|K|}{\delta+2-\varepsilon}$ color used by ϕ, and the first inequality follows. Since G is a counterexample, using Proposition 7.1.3, we may assume $G[K]$ has no (f, L)-coloring. By Lemma 7.1.5 and the first inequality, (10.13) follows. □

Lemma 10.6.2 shows that for any f-coloring of the complement of a base clique, the vertices in the base clique see roughly the same color. An optimized reduction of an overcoloring is designed so that the vertices in the base clique see color that is as different as possible. The following makes this more precise.

Definition 10.6.3. Let ψ be an overcoloring with respect to a base clique K, and let ϕ be an optimized reduction of ψ. Let $u \in V(G-K)$ be a vertex with a neighbor v in K.

- We say $\psi(u) \setminus \phi(u)$ is the *lost color* of u, and we say $L_\phi(K \cap N(u)) \setminus L_\phi(K \setminus N(u))$ is the *special color* of u.
- We say
$$\phi(u) \cap \left(\bigcup_{u' \in N(v) \setminus (K \cup \{u\})} \phi(u') \right)$$
is the *repeated color* of u for v.
- The *switchable color* of u for v is the subset of $\phi(u)$ obtained by removing $L_\phi(K)$ and the repeated color of v at u.

Lemma 10.6.4. *Let ψ be an overcoloring with respect to a base clique K, and let ϕ be an optimized reduction of ψ. If $\varepsilon \le 1/2$, then for each u with a neighbor $v \in K$, the switchable color of u for v has non-zero measure.*

Proof. Let C denote the switchable color of u for v, and let α denote the measure of the repeated color of u for v. Note that

$$\mu\left(L_\phi(K)\right) \ge \mu\left(L_\phi(v)\right) + \alpha + \mu\left(L_\phi(K \setminus N(u))\right).$$

By Lemma 10.6.2, $\alpha + \mu\left(L_\phi(K \setminus N(u))\right) < \varepsilon f(v)$. Therefore $\mu(C) \ge f(u) - \varepsilon f(v)$. By Lemma 10.3.5, $f(u) > \frac{1-\varepsilon}{\delta + \varepsilon(1-\varepsilon)} > (1-\varepsilon)f(v)$. Therefore $\mu(C) > (1-2\varepsilon)f(v) \ge 0$, as desired. □

Lemma 10.6.5. *Let ψ be an overcoloring with respect to a base clique K, and let ϕ be an optimized reduction of ψ. For each u with a neighbor in K, all but possibly a measure zero subset of the lost color of u is contained in the special color of u.*

Proof. Suppose C is a subset of the lost color of u that is disjoint from the special color of u. We show that $\mu(C) = 0$, which proves the Lemma. Let $v \in N(u) \cap K$. By Lemma 10.6.4, there exists a non-zero measure subset C' of switchable color of u for v. By possibly choosing subsets of C or C', we may assume without loss of generality that $\mu(C) = \mu(C')$. Let ϕ' be the fractional coloring such that $\phi'(u) = \phi(u) \cup C' \setminus C$ and for each $u' \neq u$, $\phi'(u') = \phi'(u)$. Note that since $C' \cap L_\phi(K) = \varnothing$,

$$\mu\left(\bigcup_{w \in K} L_\phi(w)\right) \ge \mu\left(\left(L_\phi(K \cap N(u)) \setminus C\right) \cup L_\phi(K \setminus N(u))\right) + \mu(C').$$

Since C is disjoint from the special color of U,

$$\mu\left((L_\phi(K \cap N(u)) \setminus C) \cup L_\phi(K \setminus N(u))\right) = \mu\left(L_\phi(K)\right).$$

Therefore by the choice of ϕ, we have $\mu(C') = 0$, so $\mu(C) = 0$, as claimed. ☐

Lemma 10.6.6. *Let ψ be an overcoloring with respect to a base clique K, and let ϕ be an optimized reduction of ψ. If u and u' are distinct vertices with neighbors in K, then the lost color of u and u' has a measure-zero intersection.*

Proof. Suppose to the contrary that C is a non-zero measure subset of the lost color of u and u'. By Lemma 10.6.5, u and u' have a common neighbor in K, say v, and by possibly choosing a non-zero measure subset of C, we may assume without loss of generality that $C \subseteq L_\phi(v)$. By Lemma 10.6.4, there exists non-zero measure subsets C_1 and C_2 of switchable color of u and u' for v. By definition, C_1 and C_2 are disjoint. By possibly choosing subsets, we may assume without loss of generality that $\mu(C_1) = \mu(C_2) = \mu(C)$. Let ϕ' be the fractional coloring such that $\phi'(u) = \phi(u) \cup C \setminus C_1$, $\phi'(u') = \phi(u') \cup C \setminus C_2$, and for each vertex $w \notin \{u, u'\}$, we have $\phi'(w) = \phi(w)$. Note that

$$\mu\left(\bigcup_{w \in K} L_{\phi'}(w)\right) \geq \mu\left(L_\phi(K) \setminus C\right) + \mu(C_1) + \mu(C_2) > \mu\left(L_\phi(K)\right),$$

contradicting the choice of ϕ. ☐

Finally, we prove Lemma 10.2.3.

Proof of Lemma 10.2.3. Let ψ be an overcoloring with respect to K, and let ϕ be an optimized reduction of ψ. For convenience, let $\ell = \ell_K$, $U = U_K$, and $A = A_K$. For each $u \in U$, let α_u denote the measure of the lost color of u, and note that $\alpha_u = f_K(u) - f(u)$. By Lemmas 10.6.5 and 10.6.6, for each $v \in K$,

$$\mu\left(L_\phi(K)\right) \geq \mu\left(L_\phi(v)\right) + \sum_{u \in U \setminus N(v)} \alpha_u.$$

By (10.13) and the previous inequality, $\mu\left(L_\phi(v)\right) + \sum_{u \in U \setminus N(v)} \alpha_u < |K|/(\delta + 1 - \varepsilon)$ for each $v \in K$. Since each $v \in K$ sees at most $\sum_{u \in N(v) \setminus K} f(u)$ color, we have

$$1 - \sum_{u \in N(v) \setminus K} f(u) + \sum_{u \in U \setminus N(v)} (f_K(u) - f(u)) < \frac{|K|}{\delta + 1 - \varepsilon}.$$

Rearranging terms, we have

$$1 - \frac{|K|}{\delta + 1 - \varepsilon} - \sum_{u \in N(v) \cap A} f(u) < \sum_{u \in U} f(u) - \sum_{u \in U \setminus N(v)} f_K(u).$$

Since $f(u) \le 1/(\delta + 2 - \varepsilon)$ for each $u \in A$, the left side of the previous inequality is at least $\frac{\delta+2-\varepsilon-|A|}{\delta+2-\varepsilon} - \frac{|K|}{\delta+1-\varepsilon}$. Since $\delta + 1 - |A| = |K| + \ell$, we have for each $v \in K$,

$$\frac{\ell + 1 - \varepsilon}{\delta + 2 - \varepsilon} - \frac{|K|}{(\delta + 2 - \varepsilon)(\delta + 1 - \varepsilon)} < \sum_{u \in U} f(u) - \sum_{u \in U \setminus N(v)} f_K(u).$$

Since the previous inequality holds for each $v \in K$, the left side is at most the average of $\sum_{u \in U} f(u) - \sum_{u \in U \setminus N(v)} f_K(u)$ taken over all $v \in K$. Therefore

$$\frac{\ell + 1 - \varepsilon}{\delta + 2 - \varepsilon} - \frac{|K|}{(\delta + 2 - \varepsilon)(\delta + 1 - \varepsilon)} < \sum_{u \in U} \left(f(u) - \frac{(|K| - |K \cap N(u)|) f_K(u)}{|K|} \right).$$

For each $u \in U$,

$$\begin{aligned} f(u) - \frac{(|K| - |K \cap N(u)|) f_K(u)}{|K|} &\le \frac{1}{d(u) + 1 - \varepsilon} - \frac{|K| - |K \cap N(u)|}{|K|(d(u) + 1 - |K \cap N(u)|)} \\ &= \frac{|K \cap N(u)|(d(u) + 1 - \varepsilon - |K|) + \varepsilon |K|}{|K|(d(u) + 1 - |K \cap N(u)|)(d(u) + 1 - \varepsilon)}. \end{aligned}$$

Since the above expression is decreasing as a function of $d(u)$ if $d(u) \ge \delta$, the right side of the above inequality is at least

$$\frac{|K \cap N(u)|(\delta + 2 - \varepsilon - |K|) + \varepsilon |K|}{|K|(\delta + 2 - |K \cap N(u)|)(\delta + 2 - \varepsilon)}.$$

for each $u \in U$. Hence

$$\ell + 1 - \varepsilon - \frac{|K|}{\delta + 1 - \varepsilon} < \sum_{u \in U} \frac{|K \cap N(u)|(\delta + 2 - \varepsilon - |K|) + \varepsilon |K|}{|K|(\delta + 2 - |K \cap N(u)|)},$$

as desired. $\square$

10.7 Proving the claims

Having proved Lemmas 10.2.2 and 10.2.3, it only remains to prove the technical claims used throughout the proof of Theorem 1.5.2. All of these results state that a multivariate rational function evaluates to something positive in a certain region. The proofs reduce the problem to determining that a univariate rational function is always positive over a certain region, which we prove by computing its roots. In some cases, these roots are irrational, so we use an approximation and use the symbol $\approx$. These approximations are precise enough to determine that the function is positive.

Proof of Claim 10.2.3.1. Note that

$$\frac{\partial}{\partial k} = \frac{-1}{\delta + .5} + \frac{\ell - .5}{\delta + 2 - \ell} \geq \frac{-1}{\delta + .5} + \frac{1.5}{\delta} > 0.$$

Therefore $q_\delta(\ell, k) \geq q_\delta(\ell, (\delta+1)/2)$. For convenience, let $q'_\delta(\ell) = q_\delta(\ell, (\delta+1)/2))$, and note that

$$\frac{\partial}{\partial \ell} q'_\delta(\ell) = 1 - \frac{(.5\delta + 1)(\delta + 2) + .25(\delta + 1)}{(\delta + 2 - \ell)^2}.$$

Hence, $\frac{\partial}{\partial \ell} q'_\delta(\ell) \geq 0$ if and only if

$$\ell^2 - 2(\delta + 2)\ell + (\delta + 2)^2 - ((.5\delta + 1)(\delta + 2) + .25(\delta + 1)) \geq 0.$$

By the quadratic formula applied to ℓ, $\frac{\partial}{\partial \ell} q'_\delta(\ell) \leq 0$ if and only if

$$\begin{aligned}\frac{1}{2}\left(2(\delta + 2) - \sqrt{(2(\delta + 2))^2 - 4((\delta + 2)^2 - ((.5\delta + 1)(\delta + 2) + .25(\delta + 1)))}\right) &\leq \ell \\ \leq \frac{1}{2}\left(2(\delta + 2) + \sqrt{(2(\delta + 2))^2 - 4((\delta + 2)^2 - ((.5\delta + 1)(\delta + 2) + .25(\delta + 1)))}\right)&.\end{aligned}$$

Note that

$$(2(\delta + 2))^2 - 4((\delta + 2)^2 - ((.5\delta + 1)(\delta + 2) + .25(\delta + 1))) = 2\delta^2 + 9\delta + 9 \geq 0.$$

Therefore $\frac{\partial}{\partial \ell} q'_\delta(\ell) \leq 0$ if

$$\delta + 2 - .5\sqrt{2\delta^2 + 9\delta + 9} \leq \ell \leq \delta + 2$$

and $\frac{\partial}{\partial \ell} q'_\delta(\ell) \geq 0$ if $\ell \leq \delta + 2 - .5\sqrt{2\delta^2 + 9\delta + 0}$. Therefore $q'_\delta(\ell) \geq \min\{q'_\delta(2), q'_\delta(\delta/2)\}$. Note that

$$q'_\delta(2) = -\frac{1.25\,\delta + 2.25}{\delta} + \frac{-0.5\,\delta - 0.5}{\delta + 0.5} + 2.5 \approx \frac{(\delta - 3.28935)(\delta + 0.456017)}{\delta(\delta + 0.5)} > 0,$$

and

$$q'_\delta(\delta/2) = 0.5\,\delta - \frac{\delta(0.25\,\delta + 0.5) + 0.25\,\delta + 0.25}{0.5\,\delta + 2} + \frac{-0.5\,\delta - 0.5}{\delta + 0.5} + 0.5$$
$$\approx \frac{0.5(\delta - 2.15831)(\delta + 1.15831)}{(\delta + 0.5)(\delta + 4)} > 0,$$

as desired. □

Proof of Claim 10.3.8.1. Since $d_u, d_{u'} \geq d$, we have

$$\frac{1.5}{d + .5} - \frac{1}{d_u + .5} - \frac{1}{d_{u'} + .5} \geq \frac{-.5}{d + .5}.$$

Thus, if $\frac{.5}{\delta + .5} \leq \frac{.5}{d_u + d_{u'} - d - .5}$, we have

$$q_\delta(d, d_u, d_{u'}) \geq -\frac{.5}{d + .5} + \frac{.5}{\delta + .5} \geq 0,$$

as desired. Therefore it suffices to show that

$$q'_\delta(d, d_u, d_{u'}) = \frac{.5}{d_u + d_{u'} - d - .5} + \frac{1.5}{d + .5} - \frac{1}{d_u + .5} - \frac{1}{d_{u'} + .5} \geq 0.$$

Since $\frac{1.5}{d+.5} - \frac{1}{d_u+.5} \geq 0$, we may assume $\frac{.5}{d_u+d_{u'}-d-.5} \leq \frac{1}{d_{u'}+.5}$. Therefore $\frac{\partial}{\partial d_{u'}} q'_\delta(d, d_u, d_{u'}) > 0$. By a symmetrical argument, $\frac{\partial}{\partial d_u} q'_\delta(d, d_u, d_{u'}) > 0$. Therefore $q'_\delta(d, d_u, d_{u'}) \geq q'_\delta(d, d, d)$, and

$$q'_\delta(d, d, d) = \frac{.5}{2d - d - .5} - \frac{.5}{d + .5} \geq \frac{.5}{d - .5} - \frac{.5}{d + .5} > 0,$$

as desired. □

Proof of Claim 10.4.10.1. Note that

$$\frac{\partial}{\partial d_x} q_\delta(k, d_x) = \frac{-1}{\delta + .5} + \frac{1}{(d_x + .5)^2} < 0. \tag{10.14}$$

Hence, since $d_x \leq \frac{\hat{k}+.5}{1.5-k/(\hat{k}-.5)} - .5$, we have $q_\delta(k, d_x) \geq q_\delta(k, d(k, \hat{k}))$ where $d(k, \hat{k}) = \frac{\hat{k}+.5}{1.5-k/(\hat{k}-.5)} - .5$. Let $q'_\delta(k, \hat{k}) = q_\delta(k, d(k, \hat{k}))$, and observe that

$$q'_\delta(k, \hat{k}) = \frac{\delta + k + 1}{\delta + .5} - \frac{\hat{k} + .5}{(\delta + .5)(1.5 - k/(\hat{k} - .5))} - \frac{1.5 - k/(\hat{k} - .5)}{\hat{k} + .5}.$$

Note that

$$\frac{\partial}{\partial k} q'_\delta(k, \hat{k}) = \frac{1}{\delta + .5}\left(1 - \frac{(\hat{k} + .5)/(\hat{k} - .5)}{(1.5 - k/(\hat{k} - .5))^2}\right) + \frac{1}{(\hat{k} - .5)(\hat{k} + .5)}$$

Since $k \geq \hat{k}/2$,

$$-\frac{(\hat{k} + .5)/(\hat{k} - .5)}{(1.5 - k/(\hat{k} - .5))^2} \leq -\frac{\hat{k} + .5}{\hat{k} - .5}.$$

Combining the previous two expressions,

$$\frac{\partial}{\partial k} q'_\delta(k, \hat{k}) \leq \frac{1}{\delta + .5}\left(1 - \frac{\hat{k} + .5}{\hat{k} - .5}\right) + \frac{1}{(\hat{k} - .5)(\hat{k} + .5)} = \frac{1}{\hat{k} - .5}\left(\frac{1}{\hat{k} + .5} - \frac{1}{\delta + .5}\right) < 0.$$

Therefore since $k \leq \delta - 1$, we have $q'_\delta(k, \hat{k}) \geq q'_\delta(\delta - 1, \hat{k})$.

Let $d(\hat{k}) = d(\delta - 1, \hat{k})$. We claim that $d(\hat{k}) \leq \max\{d(\delta + 1), d(2(\delta - 1))\}$. Note that

$$\begin{aligned}\frac{\partial}{\partial \hat{k}} d(\hat{k}) &= \frac{1.5 - (\delta - 1)/(\hat{k} - .5) - (\hat{k} + .5)(\delta - 1)/(\hat{k} - .5)^2}{(1.5 - (\delta - 1)/(\hat{k} - .5))^2} \\ &= \frac{1.5(\hat{k} - .5)^2 - (\delta - 1)(\hat{k} - .5) - (\hat{k} + .5)k'}{((\hat{k} - .5)(1.5 - (\delta - 1)/(\hat{k} - .5)))^2}.\end{aligned}$$

Note that the numerator of the right side of the above expression is $1.5(\hat{k} - .5)^2 - 2(\delta - 1)(\hat{k} - .5) - (\delta - 1)$, which, by the quadratic formula, is zero if and only if

$$\hat{k} = .5 + \frac{1}{3}\left(2(\delta - 1) \pm \sqrt{4(\delta - 1)^2 + 3(\delta - 1)}\right).$$

Therefore $\frac{\partial}{\partial \hat{k}} d(\hat{k}) \leq 0$ if $\delta + 1 \leq \hat{k} \leq .5 + \frac{1}{3}\left(2(\delta - 1) + \sqrt{4(\delta - 1)^2 + 3k'}\right)$ and $\frac{\partial}{\partial \hat{k}} d(\hat{k}) \geq 0$ if $\hat{k} \geq .5 + \frac{1}{3}\left(2k' + \sqrt{4k'^2 + 3k'}\right)$. It follows that $d(\hat{k}) \leq \max\{d(\delta + 1), d(2(\delta - 1))\}$, as claimed.

Now $q_\delta(k, d_x) \geq \min\{q'_\delta(\delta - 1, \delta + 1), q'_\delta(\delta - 1, 2(\delta - 1))\}$. Note that

$$q'_\delta(\delta - 1, \delta + 1) = \frac{2\delta}{\delta + .5} - \frac{\delta + 1.5}{.5\delta + 1.75} - \frac{1.5 - (\delta - 1)/(\delta + .5)}{\delta + 1.5} \approx \frac{2.5(\delta - 1.93303)(\delta + 1.73303)}{(\delta + 0.5)(\delta + 1.5)(\delta + 3.5)} \geq 0,$$

and

$$q'_\delta(\delta-1,2(\delta-1)) = \frac{\delta + \frac{2\,\delta-1.5}{\frac{\delta-1}{2\,\delta-2.5}-1.5} - 0.5}{\delta+0.5} + \frac{\frac{\delta-1}{2\,\delta-2.5}-1.5}{2\,\delta-1.5} + 1$$

$$\approx \frac{0.75(\delta-1.76045)(\delta^2-2.57289\delta+1.68932)}{(\delta-1.375)(\delta-1.25)(\delta-0.75)(\delta+0.5)} > 0.$$

Therefore $q_\delta(k,d_x) \geq 0$, as required. □

Proof of Claim 10.4.10.2. Note that

$$\frac{\partial}{\partial d_x} q_\delta(k',\hat{k},d_x) = \frac{-1}{\delta+.5} + \frac{1}{(d_x+.5)^2} < 0,$$

$$\frac{\partial}{\partial k'} q_\delta(k',\hat{k},\delta) = \frac{1}{\hat{k}+.5} > 0,$$

and

$$\frac{\partial}{\partial \hat{k}} q_\delta(k',\hat{k},\delta) = \frac{1}{\delta+.5} - \frac{k'+1.5}{(\hat{k}+.5)^2} \geq \frac{1}{\delta+.5} - \frac{1}{\hat{k}+.5} > 0.$$

Hence, since $d_x \leq 2\delta$, $k' \geq \delta$, and $\hat{k} \geq \delta+2$, we have $q_\delta(k',\hat{k},d_x) \geq q_\delta(\delta,\delta+2,2\delta)$. Now

$$q_\delta(\delta,\delta+2,2\delta) = 1 - \frac{1}{\delta+2.5} - \frac{\delta-1}{\delta+.5} - \frac{1}{2\delta+.5} = \frac{1.875(\delta+0.1)}{(\delta+0.25)(\delta+0.5)(\delta+2.5)} > 0,$$

as required. □

Proof of Claim 10.4.11.1. Note that

$$\frac{\partial}{\partial d_{u'}} q_\delta(k',\hat{k},d_{u'}) = \frac{-1}{\delta+.5} + \frac{1}{(d_{u'}+.5)^2} < 0,$$

$$\frac{\partial}{\partial k'} q_\delta(k',\hat{k},\delta) = \frac{1}{\hat{k}+.5} > 0,$$

and

$$\frac{\partial}{\partial \hat{k}} q_\delta(k',\hat{k},\delta) = \frac{1}{\delta+.5} - \frac{k'+1.5}{(\hat{k}+.5)^2} \geq \frac{1}{\delta+.5} - \frac{1}{\hat{k}+.5} > 0.$$

Hence, since $d_{u'} \leq 2\delta$, $k' \geq \delta$, and $\hat{k} \geq \delta+3$, we have $q_\delta(k',\hat{k},d_{u'}) \geq q_\delta(\delta,\delta+3,2\delta)$. Now

$$q_\delta(\delta,\delta+3,2\delta) = 1 - \frac{1}{\delta+3.5} - \frac{\delta-1}{\delta+.5} - \frac{1}{2\delta+.5} \approx \frac{2.875(\delta+0.108696)}{(\delta+0.25)(\delta+0.5)(\delta+3.5)} > 0,$$

as required. □

Proof of Claim 10.4.11.2. By the same arguments as in the previous two claims, we have $q_\delta(k', \hat{k}, d_{u'}) \geq q_\delta(\delta, \delta + 2, 2\delta - 1)$. Note that

$$q_\delta(\delta, \delta + 2, 2\delta - 1) = 1 - \frac{\delta - 1}{\delta + .5} - \frac{1}{2\delta - .5} = \frac{\delta - .675}{(\delta - .25)(\delta + .5)} > 0,$$

as required. $\square$

Proof of Claim 10.4.12.1. Note that

$$q_\delta(2\delta) = 1 - \frac{0.5\delta}{\delta + 0.5} - \frac{\delta - .5}{2\delta + 1.5} - \frac{1}{2\delta + 0.5} \approx \frac{0.375(\delta - 0.301956)(\delta + 0.551956)}{(\delta + 0.25)(\delta + 0.5)(\delta + 0.75)} > 0$$

and

$$q_\delta(\delta) = 1 - \frac{.5\delta - .5}{\delta + 1.5} - \frac{.5\delta + 1}{\delta + .5} = \frac{0.5(\delta - 1)}{(\delta + 0.5)(\delta + 1.5)} > 0.$$

Note also that

$$\frac{\partial}{\partial d} q_\delta(d) = \frac{1}{(d + .5)^2} - \frac{1.25}{(d + 1.5)^2} \approx -\frac{0.25(d - 7.97214)(d - 0.972136)}{(d + 0.5)^2(d + 1.5)^2}.$$

Therefore $\frac{\partial}{\partial d} q_\delta(d) \geq 0$ if and only if $d \leq 7.97214$. Hence, if $d \leq 7.97214$, then $q_\delta(d) \geq q_\delta(\delta) \geq 0$, and if $d > 8.97214$, then $q_\delta(d) \geq q_\delta(2\delta) \geq 0$, as required. $\square$

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