Accepted Manuscript

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PII: S0165-1889(18)30037-X
DOI: 10.1016/j.jedc.2018.01.023
Reference: DYNCON 3528

To appear in: *Journal of Economic Dynamics & Control*

Received date: 28 February 2017
Revised date: 12 January 2018
Accepted date: 13 January 2018

Please cite this article as: Yan Zeng, Danping Li, Zheng Chen, Zhou Yang, Ambiguity aversion and optimal derivative-based pension investment with stochastic income and volatility, *Journal of Economic Dynamics & Control* (2018), doi: 10.1016/j.jedc.2018.01.023

Ambiguity aversion and optimal derivative-based pension investment with stochastic income and volatility

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Abstract

This paper provides a derivative-based optimal investment strategy for an ambiguity-averse pension investor who faces not only risks from time-varying income and market return volatility but also uncertain economic conditions over a long time horizon. We derive a robust dynamic derivative strategy and show that the optimal strategy under ambiguity aversion reduces the exposures to market return risk and volatility risk and that the investor holds opposite positions for the two risk exposures. In the presence of a derivative, ambiguity has distinct effects on the optimal investment strategy. More important, we demonstrate the utility improvement when considering ambiguity and exploiting derivatives and show that ambiguity aversion and derivative trading significantly improve utility when return volatility increases. This improvement becomes more significant under ambiguity aversion over a long investment horizon.

JEL classification: C61; G11; G22

Key words: Robust portfolio choice; DC pension plan; Ambiguity; Derivative; Stochastic volatility; Stochastic salary

1. Introduction

Pension funds hold a significant share of the global market portfolio. Global institutional pension fund assets in 22 major markets are approximately $36.4 trillion and increased 4.3% in 2016, and the total pension assets in these countries amount to 62% of their GDP[1].

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[1]The data are from Willis Towers Watson's Global Pension Assets Study 2017: https://www.willistowerswatson.com/en/insights/2017/01/global-pensions-asset-study-2017.

Therefore, pension investment has become increasingly important. Moreover, derivatives are increasingly popularity in pension investment and investors are often ambiguity averse. In this paper, we combine these two features and provide a derivative-based optimal investment strategy for an ambiguity-averse pension investor. The investor considers a market with stochastic volatility and faces uncertainties concerning both salary income and economic conditions over a long time horizon. We show that ambiguity aversion reduces the exposures to market return and volatility risks. In the presence of a derivative, i.e., taking a call/put option or a straddle option as an example, the investor buys stocks and simultaneously shorts call and straddle options or shorts both the stock and put option. These trading strategies incentivize the investor to reduce portfolio risk. For each type of option, there are distinct effects of the ambiguity over the market return risk and the stochastic volatility risk on the optimal investment strategy. For example, ambiguity concerning market return risk always reduces the investment in both the stock and the straddle option, while ambiguity concerning volatility risk reduces the investment in the straddle option while increasing the investment in the stock. Our analysis further shows that ambiguity aversion and derivative trading significantly improve investors' utility, especially when the return volatility is high and/or the time horizon is long.

Motivated by recent studies on pension investment, this paper provides an integrated framework for studying an optimal derivative-based pension investment problem. There are two types of pension funds: defined benefit (DB) and defined contribution (DC) pension plans. Due to demographic change and financial market development, many countries have shifted their pension schemes from DB to DC plans to ease the pressure on social security programs and have therefore transferred the investment risk to investors (Poterba et al., 2007). As DC pension plans are playing an increasingly important role, more and more individuals who build their own DC pension funds have been exposed to the investment risk.

This paper explores various aspects of intertemporal portfolio choices regarding risk and uncertainty in DC pension plans, including market return and stochastic volatility risks and income and economic uncertainties. In particular, wealth accumulation depends on financial return and investors' contribution which is related to their salary income. Over a long horizon, investors face model instability (structural change of the model economy) and asset return variability. The experimental studies (Bossaerts et al., 2010) demonstrate that investors are averse not only to risk (the known probability distribution) but also to ambiguity (the unknown probability distribution). Also, as we all know, the expected returns

are extremely difficult to estimate, and investors are skeptical of the reliability of standard historical estimates. Therefore, it becomes increasingly important to take ambiguity aversion into account, see Anderson et al. (1999), Merton (1980) and the references therein. Moreover, long-term pension investments need to incorporate the risks of salary and the stochastic volatility of stock returns, which are well documented in the empirical literature. On the one hand, salary has significant effects on the optimal long-term portfolio choice of investors. Munk and Sørensen (2010) show that the relation between salary growth and interest rate remains a significant factor determining the optimal investment strategy. On the other hand, as an important improvement of the Black-Scholes model, stochastic volatility has been developed in the literature of option pricing, portfolio selection and related statistics (e.g., Heston, 1993; Kim et al., 1998; Fernndez-Villaverde et al., 2015; Campbell et al., 2016). In this paper, we also take stochastic salary and stochastic volatility into account and study the effects on the optimal investment decisions.

This paper is also related to the use of derivatives for optimal investment. Liu and Pan (2003) develop an optimal investment strategy of using derivatives with stochastic volatility and price jumps. They find that derivatives help to improve investors' utility. In practice, the derivative market is well developed and provides abundant opportunities for pension funds to cope with volatility risk. Derivatives are becoming increasingly popular for pension funds in many countries. For example, the second and third pillars of the UK pension funds are invested not only in capital markets such as stocks and bonds, but also in foreign option markets. In this paper, we follow this trend and consider the optimal investment strategy for a DC pension investor who is ambiguity averse and is able to invest in bond, stock, and derivative markets.

This paper is the first, to our knowledge, to explore the joint effect of ambiguity aversion and derivative trading on optimal pension investment and to examine their roles in improving utility. The main contributions of this paper are as follows. First, we provide a proof showing that the optimization problem is well posed, and also present the verification theorems to guarantee the validity of the results. Second, we derive an optimal investment strategy for the underlying asset and its derivative in a DC pension plan. As noted by Liu and Pan (2003), derivative trading is essential for improving investors' utility. We investigate two models, one with and one without the derivative. By comparing the results of the two models, we find that trading in derivatives leads to utility improvement by offering additional investment opportunities. Third, after explicitly solving the model, we show that ambiguity

aversion affects an investor's risk sharing in both the myopic and hedging components. Moreover, the risk exposures to market return and volatility risks decrease with respect to (w.r.t.) ambiguity. However, for the explicit investment strategies with the straddle option, ambiguity concerning market return risk always reduces the investment in both the stock and the derivative; ambiguity concerning volatility risk reduces the investment in the derivative while increasing the investment in the stock. Finally, in DC pension investment, we find that the optimal investment strategy has an additional hedging component that addresses salary risk. In our model, salary risk generates different effects on an investor's exposures to market return and volatility risks.

This paper is related to three strands of the literature. The first strand is on the asset allocation of DC pension funds. Given the widespread use of DC pension plans in practice, there is extensive literature addressing the asset allocation problems of DC pension funds. The existing literature adopts a variety of objectives, such as the expected utility maximization (see Blake et al., 2013, 2014; Chen et al., 2017; Deelstra et al., 2004; Emms, 2012; Giacinto et al., 2011) and the mean-variance criterion (see He and Liang, 2013; Sun et al., 2016; Wu and Zeng, 2015).[2] In a DC pension plan, human capital constitutes an indispensable part of investors' wealth. Therefore, the uncertainty regarding the future salary is considered to be a typical background risk. Several scholars have conducted research on portfolio choices with salary risk (e.g., Bodie et al., 1992; Bodie et al., 2004). To explore the effect of stochastic salary on an investor's investment behavior, we assume that the salary process follows a general stochastic process, and then explicitly derive an optimal strategy. We find that the correlation between the salary and market return/volatility risks results in distinct effects: as salary risk increases, the investor always shorts more derivatives, but she may reduce or increase stock investment for different types of options.

[2]These papers explore different aspects of factors involved in the investment of DC pension plans. In the utility maximization framework, Deelstra et al. (2004) study the optimal design of guarantees in DC plans. Giacinto et al. (2011) investigate a model of optimal allocation for a DC pension plan with a minimum guarantee. Blake et al. (2013, 2014) use numerical algorithms to solve optimal investment problems under S-shaped utility and Epstein-Zin utility, respectively. Chen et al. (2017) adopt an S-shaped utility to describe an investor's preferences and obtain the optimal investment strategy in closed-form. Under the mean-variance criterion, He and Liang (2013) study a portfolio model for a DC pension plan during the accumulation phase and derive a time-consistent investment strategy within the game theoretic framework. Wu and Zeng (2015) consider the effects of mortality risk on equilibrium strategies. Sun et al. (2016) use a jump-diffusion model to investigate an optimal investment problem for DC pensions.

The second strand of the literature explores certain potentials and roles of derivative trading in managing stochastic volatility in DC pension plans. There is considerable empirical evidence on time-varying stock return volatility (see Taylor, 1994, for a survey). Following Ílhan et al. (2005) and Liu and Pan (2003)[3], Hsuku (2007) studies a dynamic consumption and asset allocation problem with a derivative under a recursive utility function. Jalal (2013) derives dynamic option-based investment strategies for an investor who exhibits downside loss aversion. Recently, Escobar et al. (2015) consider an optimal investment strategy for an ambiguity-averse investor who can invest in stock and derivative markets. However, there are very limited results on dynamic asset allocation with derivatives in pension investment, despite the increasing popularity of using derivatives in the pension investment market. According to a report by the Singapore Exchange (SGX) from January 6, 2015, the value of securities trading fell 25%, while derivative trading volume rose to a record high in 2014. In the pension investment market, derivatives have been increasingly used over the past decade. The 2012 NAPF Annual Survey shows that 57% of member schemes include derivatives. Moreover, the Global Pension Assets Study 2016 reports that at the end of 2015, the average global asset allocation of the seven largest markets (Australia, Canada, Japan, the Netherlands, Switzerland, the UK and the US) is 44% equities, 29% bonds, 3% cash and 24% other assets, which are mainly derivatives. In this paper, we assume that the DC pension investor is allowed to invest in a derivative market. By examining cases with and without a derivative, we find that the use of a derivative always improves investor's utility.

The third strand is on ambiguity in portfolio selection. Ellsberg (1961) is the first to state that most people are ambiguity averse. Then there are numerous theoretical and empirical studies that explore the significance of ambiguity in affecting investor behavior (Bossaerts et al., 2010; Cao et al., 2005; Dimmock et al., 2016, etc). Recent studies consider investment problems with ambiguity and robust decisions.Anderson et al. (2003) develop a constrained worst-case model and derive a robust decision. The model helps the decision maker to assess the fragility of any given decision rule. Maenhout (2004, 2006) also derive the optimal

[3]Specifically, Liu and Pan (2003) study the optimal investment strategies when an investor has access not only to bond and stock markets but also to a derivative market and provide an example of the role of derivatives in the presence of volatility risk. They find that derivative trading helps to improve investors' utility. Ílhan et al. (2005) investigate an optimal investment problem for an investor who maximizes the expected exponential utility from terminal wealth, combining a static position in derivatives with a traditional dynamic trading strategy in stocks.

investment strategy for an investor who is ambiguity averse w.r.t. expected market returns. Following Maenhout (2004), some studies address the implications of ambiguity for portfolio choice. For example, Liu (2010) examines an optimal consumption and investment problem for an ambiguity-averse investor with time-varying investment opportunities. Branger and Larsen (2013) consider the optimal portfolio choice under different degrees of ambiguity aversion concerning jump and diffusion risks. Flor and Larsen (2014) consider an optimal investment strategy for an ambiguity-averse investor in the context of a stochastic interest rate. Munk and Rubtsov (2014) study a portfolio management problem for an ambiguity-averse investor under stochastic interest risk and inflation risk. Zheng et al. (2016) consider a robust optimal investment-reinsurance problem using a constant elasticity of variance (CEV) model. They also explicitly solve the case of an exponential utility function. Luo (2016) studies the strategic consumption-portfolio rules with information frictions and salary risk. Our work is related to these works and makes several extensions to address ambiguity and portfolio choice.

By considering ambiguity aversion, this paper provides a theoretical explanation of the portfolio choice puzzle of "low portfolio fractions allocated to equity" in the empirical literature (Dimmock et al., 2016). We further explore the distinct effects of different ambiguity attitudes toward market return and volatility risks on the risk exposures and investment proportions. In the presence of a derivative, we show that ambiguity always reduces the derivative investment (in absolute terms), while its effect on stock investment is uncertain. By considering salary risk, our model of DC pension investment is much richer than the classical type of deterministic contribution model. A stochastic salary stipulates an exogenous income stream, which makes it difficult to solve the optimization problem. In this paper, we derive a closed-form of the robust investment strategy for DC pension plans (with a stochastic salary). As in Anderson et al. (2003) and Maenhout (2004), the discrepancy between the reference model and the alternative models is defined in terms of relative entropy, which serves as a penalty and quantifies the investor's degree of ambiguity aversion about the reference model. The aim of the investor is to maximize the expected utility from the terminal wealth at retirement. Using the robust control approach, the robust optimal investment strategy is derived in closed-form.

This paper provides some insights into the efficient investment of DC pension plans. First, derivatives can provide an efficient way to diversify various risk factors to improve pension funds' investment performance. Because the DC pension investment horizon is long,

volatility risk has a significant effect on portfolio selection, and therefore, derivatives can be very useful to manage such risk. We show that utility is always improved by using the derivative, regardless of ambiguity aversion. Second, if an investor experiences uncertainty concerning her reference model, she usually reduces her exposures to market return risk and volatility risk. Moreover, there are distinct effects of ambiguity on the stock and derivative investments. Third, different levels of the pension's salary process, i.e., the different parameters in the salary process, result in different investment behaviors and have a significant effect on the investment strategy. Paying attention to the salary process is necessary for the design of a DC pension plan.

The paper is organized as follows. Section 2 describes the model. Section 3 derives the explicit expressions of the robust optimal risk exposures, investment strategies and the corresponding optimal value function when the derivative is available. Section 4 provides the solutions without derivatives trading. Section 5 presents several numerical examples to illustrate the effects of the model parameters on the robust optimal investment strategy and utility improvements generated by considering ambiguity aversion and derivative trading. Section 6 concludes the paper.

2. Investment under ambiguity

We study the optimal investment strategy of a DC pension investor who can invest in a financial market consisting of a bond, a stock and a derivative of the stock. The stock price follows a stochastic volatility process. We assume that there are no transaction costs or taxes in the financial market and that trading occurs continuously. In addition to undertaking financial risk, the investor also receives a stochastic salary stream and faces salary risk during her working period. Moreover, she is ambiguity averse regarding both the dynamics of the stock and its stochastic volatility. Throughout this paper, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ is a filtered complete probability space on which the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by a two-dimensional Brownian motion $(W_S(t), W_V(t))$, where $T > 0$ is a finite constant representing the investment time horizon (retirement date); $\mathcal{F}_t$ denotes the information available until time t; and $\mathbb{P}$ is a reference measure.

2.1. Financial market

The financial market consists of a risk-free bond, a stock and a derivative. The risk-free bond evolves according to

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t, \qquad S_0(0) = 1, \tag{1}$$

where $r > 0$ represents the risk-free interest rate. The stock price follows

$$\mathrm{d}S(t) = S(t)\left[(r + \lambda_1 V(t))\,\mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_S(t)\right], \qquad S(0) = s_0, \tag{2}$$

while the stock return variance $V(t)$ is governed by

$$\mathrm{d}V(t) = \kappa(\delta - V(t))\mathrm{d}t + \sigma_V\sqrt{V(t)}\left(\rho_V \mathrm{d}W_S(t) + \sqrt{1-\rho_V^2}\mathrm{d}W_V(t)\right), \qquad V(0) = v_0, \tag{3}$$

where $W_S(t)$ and $W_V(t)$ are independent Brownian motions on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. In this model, the instantaneous variance process $V(t)$ is a stochastic process with long-run mean $\delta > 0$, mean-reversion rate $\kappa > 0$, and volatility coefficient $\sigma_V > 0$. The price and volatility are correlated, which is captured by the coefficient $\rho_V \in (-1, 1)$ and represents an important feature of the real data. λ_1 is a constant capturing the market price of the risk factor $W_S(t)$.

In addition to investing in the risk-free bond and the stock, the pension investor also has the opportunity to invest in the derivative with the risky asset as the underlying asset. Following Liu and Pan (2003), we consider the derivative with price $O(t, S(t), V(t))$, (or $O(t)$ for short) at time t; this depends on the underlying price of the stock $S(t)$ and its volatility $V(t)$, and its payoff structure at the expiration time τ is defined by $O(\tau) = f(S(\tau), V(\tau))$ for some function f. [4] Inspired by Liu and Pan (2003) and Escobar et al.(2015), we assume that the price process of derivative $O(t, S(t), V(t))$ satisfies

$$\begin{cases} \mathrm{d}O(t) = rO(t)\mathrm{d}t + (O_s S(t) + \sigma_V \rho_V O_v)\left(\lambda_1 V(t)\mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_S(t)\right) \\ \qquad\qquad + \sigma_V\sqrt{1-\rho_V^2}O_v\left(\lambda_2 V(t)\mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_V(t)\right), \quad t \le \tau, \\ O(\tau) = f(S(\tau), V(\tau)), \end{cases} \tag{4}$$

where λ_2 is a constant capturing the market price of stochastic volatility risk $W_V(t)$; O_s and O_v are the partial derivatives of O w.r.t. $S(t)$ and $V(t)$, respectively. We can show that given a physical measure, there exists a unique risk-neutral measure in the extended financial market (S_0, S, O) which is given by (1), (2) and (4), and prove that the financial market in our paper is complete and, furthermore, there is only one pricing kernel (see Appendix A).

[4] As in the literature, such as Liu and Pan (2003), the derivative includes most traded option types. As shown in Liu and Pan (2003), the expiration date τ of the derivative does not need to match the investment horizon T. They present some examples of derivative types. For instance, a derivative with a linear payoff structure $f(S(\tau), V(\tau)) = S(\tau)$ becomes the stock itself. However, for some strike price $K > 0$, a derivative with a non-linear payoff structure $f(S(\tau), V(\tau)) = (S(\tau) - K)^+$ corresponds to a European-style call option, while that with $f(S(\tau), V(\tau)) = (K - S(\tau))^+$ corresponds to a European-style put option.

In a DC pension plan, the investor contributes part of her salary to the pension fund before retirement. The salary process is essential when considering a DC pension plan. In this paper, we assume that the dynamics of the investor's salary are described by

$$\begin{cases} \mathrm{d}L(t) = L(t)\left[\mu_L \mathrm{d}t + \sigma_L\rho_L\left(\lambda_1 V(t)\mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_S(t)\right)\right. \\ \qquad\qquad \left. +\sigma_L\sqrt{1-\rho_L^2}\left(\lambda_2 V(t)\mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_V(t)\right)\right], \\ L(0) = l_0, \end{cases} \tag{5}$$

where $\mu_L \geq 0$ is the appreciation rate, $\sigma_L \geq 0$ is the volatility and $\rho_L \in [-1, 1]$ is the coefficient parameter.

Remark 2.1. The salary process plays an important role in pension plans and is analyzed in several studies (Bodie et al., 2004; Chen et al., 2017; Deelstra et al., 2004; Dybvig and Liu, 2010; Guan and Liang, 2014, 2015). Among these contributions, Bodie et al. (2004) and Dybvig and Liu (2010) assume that the salary process is spanned by the stocks in the financial market, which reflects the fact that salary is related to the profitability of the company. Guan and Liang (2014) furthermore assume that the salary process is correlated with the volatility of the stock. In those cases, salary risk is insurable in the stock market. Because the stochastic volatility contains some other risks faced by the investor in our model, we assume the salary to be related to stochastic volatility. It would be interesting and more realistic to introduce an independent random process on the stochastic salary. In this case, the part related to l^2 (the salary variable) cannot be separated in the Hamilton-Jacobi-Bellman (HJB) equation. It becomes difficult to derive closed-form solutions to the optimization problems, which significantly complicates the analysis of the problems.

2.2. Ambiguity

The above-mentioned framework is a traditional portfolio choice model in the DC pension plan, where the investor is assumed to be ambiguity neutral. However, in reality, the investor is usually ambiguity averse and wants to guard herself against worst-case scenarios. To incorporate ambiguity aversion into the investor's investment problem, we assume that the reference model capturing the knowledge of the investor's ambiguity is described by the probability measure $\mathbb{P}$, but she is skeptical of this reference model and is willing to consider some alternative models, which are defined by a class of probability measures equivalent to $\mathbb{P}$ as follows (cf. Anderson et al., 2003; Maenhout, 2004):

$$\mathcal{Q} := \{\mathbb{Q} | \mathbb{Q} \sim \mathbb{P}\}.$$

Define $\Phi := \{\phi(t) := (\phi_S(t), \phi_V(t))\}_{t\in[0,T]}$, which satisfies three conditions: (i) $\phi_S(t)$ and $\phi_V(t)$ are $\mathcal{F}_t$-measurable for each $t \in [0,T]$; (ii) $\mathrm{E}\Big\{ \exp\Big\{\frac{1}{2}\int_0^T [(\phi_S(t))^2 dt + (\phi_V(t))^2]\, dt\Big\}\Big\} < \infty$; and (iii) $|\phi(t)|^2 \le \underline{\kappa}^2 V(t)$ for a.s. $(t,\omega) \in [0,T]\times\Omega$, with constant $\underline{\kappa} \in [\max(\underline{\phi}, \underline{\phi}_3), \kappa/\sigma_V)$, where $\underline{\phi}$, and $\underline{\phi}_3$ are defined in (20) and (41), respectively. We will explain $\underline{\phi}$ in footnote 7 and $\underline{\phi}_3$ in footnote 14 below. We denote Θ for the space of all such processes Φ. Furthermore, we define a real-valued process $\{\Lambda^\Phi(t)|t \in [0,T]\}$ as

$$\Lambda^\Phi(t) = \exp\left\{-\int_0^t \phi_S(s)\mathrm{d}W_S(s) - \frac{1}{2}\int_0^t (\phi_S(s))^2\mathrm{d}s - \int_0^t \phi_V(s)\mathrm{d}W_V(s) - \frac{1}{2}\int_0^t (\phi_V(s))^2\mathrm{d}s\right\}. \tag{6}$$

Accordingly, $\Lambda^\Phi(t)$ is a $\mathbb{P}$-martingale. For each Φ, a new alternative measure $\mathbb{Q}$ that is absolutely continuous with $\mathbb{P}$ on $\mathcal{F}_T$ is defined by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = \Lambda^\Phi(T).$$

By Girsanov's Theorem, under the alternative measure $\mathbb{Q}$, we have

$$\mathrm{d}W_S^\Phi(t) = \mathrm{d}W_S(t) + \phi_S(t)\mathrm{d}t,$$

$$\mathrm{d}W_V^\Phi(t) = \mathrm{d}W_V(t) + \phi_V(t)\mathrm{d}t,$$

where $W_S^\Phi(t)$ and $W_V^\Phi(t)$ are one-dimensional standard Brownian motions. Furthermore, the price and volatility of the stock, the price of the derivative and the stochastic salary under $\mathbb{Q}$ can be written as

$$\mathrm{d}S^\Phi(t) = S^\Phi(t)\left[\left(r + \lambda_1 V^\Phi(t) - \phi_S(t)\sqrt{V^\Phi(t)}\right)\mathrm{d}t + \sqrt{V^\Phi(t)}\mathrm{d}W_S^\Phi(t)\right], \tag{7}$$

$$\begin{aligned}\mathrm{d}V^\Phi(t) = &\left[\kappa(\delta - V^\Phi(t)) - \sigma_V\sqrt{V^\Phi(t)}(\rho_V\phi_S(t) + \sqrt{1-\rho_V^2}\phi_V(t))\right]\mathrm{d}t\\ &+ \sigma_V\sqrt{V^\Phi(t)}(\rho_V\mathrm{d}W_S^\Phi(t) + \sqrt{1-\rho_V^2}\mathrm{d}W_V^\Phi(t)),\end{aligned} \tag{8}$$

$$\begin{aligned}\mathrm{d}O^\Phi(t) = &rO^\Phi(t)\mathrm{d}t + (O_s S^\Phi(t) + \sigma_V\rho_V O_v)\left[\lambda_1 V^\Phi(t)\mathrm{d}t - \phi_S(t)\sqrt{V^\Phi(t)}\mathrm{d}t + \sqrt{V^\Phi(t)}\mathrm{d}W_S^\Phi(t)\right]\\ &+ \sigma_V\sqrt{1-\rho_V^2}O_v\left[\lambda_2 V^\Phi(t)\mathrm{d}t - \phi_V(t)\sqrt{V^\Phi(t)}\mathrm{d}t + \sqrt{V^\Phi(t)}\mathrm{d}W_V^\Phi(t)\right],\end{aligned} \tag{9}$$

$$\begin{aligned}\mathrm{d}L^\Phi(t) = &L^\Phi(t)\Big[\mu_L\mathrm{d}t + \sigma_L\rho_L(\lambda_1 V^\Phi(t)\mathrm{d}t - \phi_S(t)\sqrt{V^\Phi(t)}\mathrm{d}t + \sqrt{V^\Phi(t)}\mathrm{d}W_S^\Phi(t))\\ &+ \sigma_L\sqrt{1-\rho_L^2}(\lambda_2 V^\Phi(t)\mathrm{d}t - \phi_V(t)\sqrt{V^\Phi(t)}\mathrm{d}t + \sqrt{V^\Phi(t)}\mathrm{d}W_V^\Phi(t))\Big].\end{aligned} \tag{10}$$

2.3. Wealth process

Let $\mathbf{u} := \{u(t) := (u_S(t), u_O(t))\}_{t\in[0,T]}$ be a trading strategy, and $X^\mathbf{u}(t)$ is the wealth process under strategy $\mathbf{u}$, where $u_S(t)$, $u_O(t)$ and $1 - u_S(t) - u_O(t)$ are the proportions of the

wealth invested in the stock, derivative and risk-free bond, respectively. Then, the wealth process $X^{\mathbf{u}}(t)$ under probability measure $\mathbb{P}$ follows

$$\begin{cases} \mathrm{d}X^{\mathbf{u}}(t) = X^{\mathbf{u}}(t)\left[(1-u_S(t)-u_O(t))\dfrac{\mathrm{d}S_0(t)}{S_0(t)} + u_S(t)\dfrac{\mathrm{d}S(t)}{S(t)} + u_O(t)\dfrac{\mathrm{d}O(t)}{O(t)}\right] + \xi L(t)\mathrm{d}t \\ \qquad\qquad = X^{\mathbf{u}}(t)\left[r\mathrm{d}t + \theta_S(t)\left(\lambda_1 V(t)\mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_S(t)\right)\right. \\ \qquad\qquad \left. +\theta_V(t)\left(\lambda_2 V(t)\mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_V(t)\right)\right] + \xi L(t)\mathrm{d}t, \\ X^{\mathbf{u}}(0) = x_0, \end{cases} \tag{11}$$

where

$$\theta(t) = \begin{pmatrix} \theta_S(t) \\ \theta_V(t) \end{pmatrix} = \begin{pmatrix} 1 & \frac{O_s S(t) + \sigma_V \rho_V O_v}{O(t)} \\ 0 & \frac{\sigma_V\sqrt{1-\rho_V^2}O_v}{O(t)} \end{pmatrix} \begin{pmatrix} u_S(t) \\ u_O(t) \end{pmatrix} \tag{12}$$

represent the investor's exposures to market return risk $W_S(t)$ and additional volatility risk $W_V(t)$, respectively. Here, we consider the exposures instead of portfolio weights to simplify the analysis.[5] As shown in Liu and Pan (2003), the exposure stems from the dynamics of asset prices and the specific portfolio.

In addition, we assume that the contribution rate of the salary is $\xi \in [0,1]$. Then under the ambiguity framework, the wealth process $X^{\Phi,\mathbf{u}}(t)$ under probability measure $\mathbb{Q}$ follows

$$\begin{aligned} \mathrm{d}X^{\Phi,\mathbf{u}}(t) = X^{\Phi,\mathbf{u}}(t)\Big[r\mathrm{d}t + \theta_S(t)\Big(\lambda_1 V^{\Phi}(t)\mathrm{d}t - \phi_S(t)\sqrt{V^{\Phi}(t)}\mathrm{d}t + \sqrt{V^{\Phi}(t)}\mathrm{d}W_S^{\Phi}(t)\Big) \\ +\theta_V(t)\Big(\lambda_2 V^{\Phi}(t)\mathrm{d}t - \phi_V(t)\sqrt{V^{\Phi}(t)}\mathrm{d}t + \sqrt{V^{\Phi}(t)}\mathrm{d}W_V^{\Phi}(t)\Big)\Big] + \xi L^{\Phi}(t)\mathrm{d}t. \end{aligned} \tag{13}$$

Definition 2.2. *A strategy* $\boldsymbol{u} = \{u(t) := (u_S(t), u_O(t))\}_{t\in[0,T]}$ *is said to be admissible if*
(i) $u_S(t)$ *and* $u_O(t)$ *are* $\mathcal{F}_t$*-progressively measurable processes;*
(ii) Eq. (13) has a pathwise-unique solution $\left\{X^{\Phi,\mathbf{u}}(t)\right\}_{t\in[0,T]}$*, for any* $(t,x,v,l) \in O := [0,T]\times\mathbb{R}^3$*;*
(iii) $\mathrm{E}^{\Phi}_{t,x,v,l}\left\{\int_t^T [V^{\Phi}(s)\left((u_S(s))^2 + (u_O(s))^2\right)]\mathrm{d}s\right\} < \infty$ *and* $\mathrm{E}^{\Phi}_{t,x,v,l}\left[|U(X^{\Phi,\mathbf{u}}(T))|\right] < \infty$*, for any* $(t,x,v,l) \in O$*, where* $\mathrm{E}^{\Phi}_{t,x,v,l}[\cdot] = \mathrm{E}^{\Phi}[\cdot|(X^{\Phi,\mathbf{u}}(t), V^{\Phi}(t), L^{\Phi}(t)) = (x,v,l)]$.

Denote by Π *the set of all admissible strategies.*

2.4. Optimization problem

In this paper, the pension investor is assumed to be risk averse with a constant relative risk aversion (CRRA) utility function and seeks to derive an investment strategy during

[5] We also provide the non-redundant condition as shown in Eq. (3.3) in Escobar et al. (2015) and Eq. (12) in Liu and Pan (2003). Because we have only one derivative in the model and the relationship between risk exposure and the portfolio weight is shown by Eq. (12), the non-redundant condition becomes $\sqrt{1-\rho_V^2}O_v \neq 0$.

the time interval $[0, T]$ to maximize the expected utility from terminal wealth under the ambiguity framework. Then, the optimization problem for the investor can be written as[6]

$$\sup_{\mathbf{u}\in\Pi}\inf_{\Phi\in\Theta} \mathrm{E}^{\Phi}\left[U(X^{\Phi,\mathbf{u}}(T)) + \int_0^T \left(\frac{(\phi_S(s))^2}{2\Psi_S(s,x,v,l)} + \frac{(\phi_V(s))^2}{2\Psi_V(s,x,v,l)}\right) \mathrm{d}s\right], \tag{14}$$

where

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \tag{15}$$

and γ is the coefficient of relative risk aversion. We assume that $\gamma > 1$ for practical relevance (see Branger and Larsen, 2013; Escobar et al., 2015; Flor and Larsen, 2014). The perturbations $\phi_S(t)$ and $\phi_V(t)$ in the penalty term are scaled by $\Psi_S(t,x,v,l)$ and $\Psi_V(t,x,v,l)$, respectively. $\Psi_S(t,x,v,l)$ and $\Psi_V(t,x,v,l)$ represent the preference parameters for ambiguity aversion and measure the degree of confidence in the reference model $\mathbb{P}$ at time t; and deviations from the reference measure are penalized by the last integral term in the expectation, which depends on the relative entropy arising from the diffusion risks. According to Maenhout (2004), the larger $\Psi_S(t,x,v,l)$ and $\Psi_V(t,x,v,l)$ are, the less the deviations from the reference model are penalized. Furthermore, the pension investor has less faith in the reference model, such that she is more likely to consider alternative models. Hence, the pension investor's ambiguity aversion is increasing w.r.t. $\Psi_S(t,x,v,l)$ and $\Psi_V(t,x,v,l)$.

Proposition 2.3. *There exists a unique function $H(t,x,v,l)$ satisfying*

$$H(t,x,v,l) = \sup_{\boldsymbol{u}\in\Pi} H^{\Phi^*,\boldsymbol{u}}(t,x,v,l), \tag{16}$$

$$\begin{aligned} H^{\Phi^*,\boldsymbol{u}}(t,x,v,l) &= \inf_{\Phi\in\Theta} H^{\Phi,\boldsymbol{u}}(t,x,v,l) \\ &= \inf_{\Phi\in\Theta} \mathrm{E}^{\Phi}_{t,x,v,l}\left[U(X^{\Phi,\boldsymbol{u}}(T)) + \int_t^T \left(\frac{(\phi_S(s))^2}{2\Psi_S(s,x,v,l)} + \frac{(\phi_V(s))^2}{2\Psi_V(s,x,v,l)}\right) \mathrm{d}s\right], \end{aligned} \tag{17}$$

$$\Psi_S(t,x,v,l) = \frac{\beta_S}{(1-\gamma)H(t,x,v,l)}, \qquad \Psi_V(t,x,v,l) = \frac{\beta_V}{(1-\gamma)H(t,x,v,l)}, \tag{18}$$

and (8), (10), (13).

Proof. See Appendix B. □

[6]Following Anderson et al. (2003) and Maenhout (2004), the alternative models considered by the investor are difficult to distinguish statistically from the reference model. To take this issue into account, the value function includes a penalty term for deviating excessively from the reference model in the sense of relative entropy (the last integral term in the expectation in Eq. (14)), which arises from diffusion risk.

Based on Proposition 2.3, we define $H(t,x,v,l)$ as the optimal value function of our optimization problem.

According to Pathak (2002), Branger and Larsen (2013), Escobar et al. (2015), Flor and Larsen (2014) and Maenhout (2004), we assume the forms of $\Psi_S(t,x,v,l)$ and $\Psi_V(t,x,v,l)$ given by (18) for analytical tractability. In (18), β_S and β_V are positive constants and called ambiguity aversion parameters; these are used to describe the investor's attitude toward ambiguity. We allow the level of ambiguity concerning the stock price to differ from that concerning the stock's volatility. For convenience, we abuse the notation slightly and interpret β_S as ambiguity aversion regarding market return risk and β_V as ambiguity aversion regarding additional volatility risk.

3. Optimal investment strategy with a derivative

This section is devoted to deriving the optimal investment strategy for the DC pension investor in the presence of a derivative. We first provide a closed-form solution to the case in which the investor is ambiguity averse in general and then analyze a special case without ambiguity aversion.

For convenience, we introduce some notations. Let

$$\begin{aligned} C^{1,2,2,2}(O) = \; & \{\psi(t,x,v,l) | \psi(t,\cdot,\cdot,\cdot) \textit{ is once continuously differentiable on } [0,T] \\ & \textit{and } \psi(\cdot,x,v,l) \textit{ is twice continuously differentiable on } \mathbb{R}^3\}. \end{aligned}$$

Let $u=(u_S,u_O)$, $\theta=(\theta_S,\theta_V)$ and $\phi=(\phi_S,\phi_V)$ denote the values taken by $u(t)=(u_S(t),u_O(t))$, $\theta(t)=(\theta_S(t),\theta_V(t))$ and $\phi(t)=(\phi_S(t),\phi_V(t))$, respectively. For any $(t,x,v,l)\in O$ and $\psi(t,x,v,l)\in C^{1,2,2,2}(O)$, we define an infinitesimal generator as

$$\begin{aligned} \mathcal{A}^{\phi,u}\psi(t,x,v,l) = & \psi_t + [rx + x\theta_S\lambda_1 v + x\theta_V\lambda_2 v - x\theta_S\phi_S\sqrt{v} - x\theta_V\phi_V\sqrt{v} + \xi l]\,\psi_x \\ & + \left[\kappa(\delta - v) - \sigma_V\sqrt{v}\rho_V\phi_S - \sigma_V\sqrt{v}\sqrt{1-\rho_V^2}\phi_V\right]\psi_v \\ & + \left[\mu_L l + l\sigma_L\lambda_1 v\rho_L - l\sigma_L\sqrt{v}\phi_S\rho_L + l\sigma_L\lambda_2 v\sqrt{1-\rho_L^2} - l\sigma_L\sqrt{v}\phi_V\sqrt{1-\rho_L^2}\right]\psi_l \\ & + \frac{1}{2}x^2 v(\theta_S^2+\theta_V^2)\psi_{xx} + \frac{1}{2}\sigma_v^2 v\psi_{vv} + \frac{1}{2}l^2\sigma_L^2 v\psi_{ll} + l\sigma_L v\sigma_V\left(\rho_V\rho_L + \sqrt{1-\rho_V^2\rho_L^2}\right)\psi_{lv} \\ & + (x\sigma_V\theta_S v\rho_V + x\sigma_V\theta_V v\sqrt{1-\rho_V^2})\psi_{xv} + (x\theta_S l\sigma_L v\rho_L + x\theta_V l\sigma_L v\sqrt{1-\rho_L^2})\psi_{xl}, \end{aligned}$$

where ψ_t, ψ_x, ψ_v, ψ_l, ψ_{xx}, ψ_{vv}, ψ_{ll}, ψ_{lv}, ψ_{xv} and ψ_{xl} represent the partial derivatives of ψ w.r.t. the corresponding variables.

According to the principle of dynamic programming, the HJB equation with ambiguity aversion can be derived as (see Escobar et al., 2015; Maenhout, 2006; Yi et al., 2013)

$$\sup_{u\in\mathbb{R}^2}\inf_{||\phi||\leq\sqrt{\underline{\kappa}v}}\left\{\mathcal{A}^{\phi,u}J(t,x,v,l)+\frac{\phi_S^2}{2\Psi_S}+\frac{\phi_V^2}{2\Psi_V}\right\}=0 \tag{19}$$

with the boundary condition $J(T,x,v,l)=U(x)$.

The following proposition presents the conditions under which the solution of the HJB equation is indeed the value function, and the control is the optimal strategy.

Proposition 3.1. *If there exists a function $J(t,x,v,l)\in C^{1,2,2,2}(O)$ and a control $(\boldsymbol{u}^*,\Phi^*):=\{(u^*(t),\phi^*(t))\}_{t\in[0,T]}\in\Pi\times\Theta$ such that*
(1) for any $||\phi||\leq\sqrt{\underline{\kappa}v}$, $\mathcal{A}^{\phi,u^*}J(t,x,v,l)+\frac{\phi_S^2}{2\Psi_S}+\frac{\phi_V^2}{2\Psi_V}\geq 0$;
(2) for any $u\in\mathbb{R}^2$, $\mathcal{A}^{\phi^*,u}J(t,x,v,l)+\frac{(\phi_S^*)^2}{2\Psi_S}+\frac{(\phi_V^*)^2}{2\Psi_V}\leq 0$;
(3) $\mathcal{A}^{\phi^*,u^*}J(t,x,v,l)+\frac{(\phi_S^*)^2}{2\Psi_S}+\frac{(\phi_V^*)^2}{2\Psi_V}=0$, with $J(T,x,v,l)=U(x)$; and
(4) $\{J(\tau,x,v,l)\}_{\tau\in\mathfrak{T}}$ and $\{\frac{(\phi_S^*(\tau))^2}{2\Psi_S(\tau,x,v,l)}+\frac{(\phi_V^*(\tau))^2}{2\Psi_V(\tau,x,v,l)}\}_{\tau\in\mathfrak{T}}$ are uniformly integrable, where $\mathfrak{T}$ denotes the set of stopping times $\tau\leq T$, $u^*=(u_S^*,u_O^*)$ and $\phi^*=(\phi_S^*,\phi_V^*)$ denote the values taken by $u^*(t)=(u_S^*(t),u_O^*(t))$ and $\phi^*(t)=(\phi_S^*(t),\phi_V^*(t))$, respectively. Then $J(t,x,v,l)=H(t,x,v,l)$, and $(\boldsymbol{u}^*,\Phi^*)$ is an optimal control.

Proof. See Appendix C. □

According to Proposition 3.1, we know that the optimal investment strategy is $\mathbf{u}^*$, the optimal risk exposure is

$$\theta^*(t):=(\theta_S^*(t),\theta_V^*(t))=\begin{pmatrix}1 & \frac{O_sS(t)+\sigma_V\rho_VO_v}{O(t)}\\ 0 & \frac{\sigma_V\sqrt{1-\rho_V^2}O_v}{O(t)}\end{pmatrix}u^*(t),$$

the worst-case measure is Φ^*, and the corresponding optimal value function is $J(t,x,v,l)$ if Novikov's condition is satisfied, which is given below.

Theorem 3.2. *For the robust portfolio choice problem (16) with wealth process (13), if the*

parameters satisfy certain technical conditions [7]*, the optimal risk exposure is*

$$\begin{aligned}\theta_S^*(t) &= m(t)\left(1+\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right)-\sigma_L\rho_L\bar{h}(t)\frac{L(t)}{X^{u^*}(t)},\\ \theta_V^*(t) &= n(t)\left(1+\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right)-\sigma_L\sqrt{1-\rho_L^2}\bar{h}(t)\frac{L(t)}{X^{u^*}(t)};\end{aligned} \tag{22}$$

the optimal investment strategy is

$$u_S^*(t)=\theta_S^*(t)-\frac{O_sS(t)+\sigma_V\rho_VO_v}{O(t)}u_O^*(t),\qquad u_O^*(t)=\frac{O(t)\theta_V^*(t)}{\sigma_V\sqrt{1-\rho_V^2}O_v}; \tag{23}$$

the corresponding optimal value function is

$$J(t,x,v,l)=\frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\bar{g}(t)v+\hat{g}(t)); \tag{24}$$

and the worst-case measure is given by

$$\phi_S^*(t)=\frac{\beta_S(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_S+\gamma)},\qquad \phi_V^*(t)=\frac{\beta_V(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2}\bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_V+\gamma)}, \tag{25}$$

where $\{X^*(t)\}_{t\in[0,T]}$ *is the wealth process under the corresponding optimal strategy, and*

$$m(t)=\frac{\lambda_1(1-\gamma)+(1-(\beta_S+\gamma))\sigma_V\rho_V\bar{g}(t)}{(1-\gamma)(\beta_S+\gamma)}, \tag{26}$$

$$n(t)=\frac{\lambda_2(1-\gamma)+(1-(\beta_V+\gamma))\sigma_V\sqrt{1-\rho_V^2}\bar{g}(t)}{(1-\gamma)(\beta_V+\gamma)}, \tag{27}$$

$$\bar{g}(t)=\frac{\nu_1\nu_2-\nu_1\nu_2\mathrm{e}^{\alpha_2(\nu_1-\nu_2)(T-t)}}{\nu_2-\nu_1\mathrm{e}^{\alpha_2(\nu_1-\nu_2)(T-t)}}, \tag{28}$$

$$\hat{g}(t)=\int_t^T[r(1-\gamma)+\kappa\delta\bar{g}(s)]\,\mathrm{d}s, \tag{29}$$

[7] The technical conditions are $\underline{\phi}<\kappa^2/\sigma_V^2$ with

$$\underline{\phi}\triangleq\max\left\{\frac{\beta_S^2\lambda_1^2}{(\beta_S+\gamma)^2},\frac{\beta_S^2(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}(0))^2}{(1-\gamma)^2(\beta_S+\gamma)^2}\right\}+\max\left\{\frac{\beta_V^2\lambda_2^2}{(\beta_V+\gamma)^2},\frac{\beta_V^2(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2}\bar{g}(0))^2}{(1-\gamma)^2(\beta_V+\gamma)^2}\right\}, \tag{20}$$

and for $\bar{g}(t)\in[\bar{g}(0),0]$,

$$[64(1-\gamma)^2-4(1-\gamma)][(m(t))^2+(n(t))^2]+8(1-\gamma)A(t)\le\frac{\kappa^2}{2\sigma_V^2}, \tag{21}$$

which are needed in the verification theorem. According to Dotsis et al. (2007) and Sepp (2008), who give the parameter estimates of the Heston model using the S&P500 index, we know that the value of κ^2/σ_V^2 in the technique conditions is very large (approximately 375.39). Therefore, more parameters can satisfy conditions (20) and (21).

$$\bar{h}(t) = \frac{\xi}{\mu_L - r}(\mathrm{e}^{(\mu_L - r)(T-t)} - 1), \tag{30}$$

$$\alpha_1 = -\kappa + \frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V\rho_V}{\beta_S+\gamma} + \frac{\lambda_2(1-(\beta_V+\gamma))\sigma_V\sqrt{1-\rho_V^2}}{\beta_V+\gamma}, \tag{31}$$

$$\begin{aligned}\alpha_2 = {} & \frac{\sigma_V^2}{2} - \frac{\beta_S\sigma_V^2\rho_V^2 + \beta_V\sigma_V^2(1-\rho_V^2)}{2(1-\gamma)} \\ & + \frac{(1-(\beta_S+\gamma))^2\sigma_V^2\rho_V^2}{2(\beta_S+\gamma)(1-\gamma)} + \frac{(1-(\beta_V+\gamma))^2\sigma_V^2(1-\rho_V^2)}{2(\beta_V+\gamma)(1-\gamma)},\end{aligned} \tag{32}$$

$$\alpha_3 = \frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)} + \frac{\lambda_2^2(1-\gamma)}{2(\beta_V+\gamma)}, \qquad \nu_{1,2} = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_2\alpha_3}}{-2\alpha_2}, \tag{33}$$

$$A(t) = \gamma(m(t))^2 - \frac{\sigma_V\rho_V\bar{g}(t)}{\beta_S+\gamma}m(t) + \gamma(n(t))^2 - \frac{\sigma_V\sqrt{1-\rho_V^2}\bar{g}(t)}{\beta_V+\gamma}n(t). \tag{34}$$

Proof. See Appendix D. □

Theorem 3.2 presents three features of our results. First, the components $m(t)$ and $n(t)$ in optimal risk exposures $\theta_S^*(t)$ and $\theta_V^*(t)$ consist of traditional components involving the myopic and hedging components. Taking exposure to market return risk $\theta_S^*(t)$ as an example, the myopic component $\frac{\lambda_1}{\beta_S+\gamma}$ is constant and decreases in the ambiguity aversion parameter β_S for stock risk but does not depend on the ambiguity aversion parameter β_V for additional volatility risk. This shows that a myopic investor concentrates solely on the ambiguity aversion parameter β_S w.r.t. market return risk. The hedging component $\frac{(1-(\beta_S+\gamma))\sigma_V\rho_V\bar{g}(t)}{(1-\gamma)(\beta_S+\gamma)}$ is time dependent, and for a non-myopic investor, this component depends on β_V, as $\bar{g}(t)$ depends on β_V. That is, the investor is concerned not only with β_S but also with β_V w.r.t. market return risk. The case of exposure to additional volatility risk $\theta_V^*(t)$ is easily analyzed in a similar manner. Second, from the remaining components of optimal risk exposure, we find that the salary process exists in the portfolio and generates a new hedging component w.r.t. salary risk. Due to the assumption that the risk factors $W_S(t)$ and $W_V(t)$ are contained in the salary process, this component is affected by both β_S and β_V. Third, the worst-case measure is chosen by Eq. (25), which is proportional to volatility $\sqrt{V(t)}$. The case of $\phi_S^*(t)$ is affected by both the ambiguity regarding market return risk β_S and the ambiguity regarding additional volatility risk β_V.

Remark 3.3. In our results, $m(t)$ and $n(t)$ in the optimal risk exposure are consistent with the previous studies on ambiguity, such as Branger and Larsen (2013) and Escobar et al. (2015). However, they do not consider the salary risk, which is very important in a DC pension plan. In this model, the worst-case measure here takes a form similar to that in Escobar et al. (2015).

Theorem 3.4. *For problem (16), if there exists a function $J(t,x,v,l) \in C^{1,2,2,2}(O)$ that is a solution to the HJB equation (19) with boundary condition $J(T,x,v,l) = U(x)$ and if its parameters satisfy conditions (20) and (21), then the optimal value function is $H(t,x,v,l) = J(t,x,v,l)$, and the optimal strategy is $\boldsymbol{u}^* = \{(u_S^*(t), u_O^*(t))\}_{t\in[0,T]}$ given in Theorem 3.2.*

Proof. See Appendix E. □

Remark 3.5. We present several special cases to show the relationships between $\theta_S^*(t)$, $\theta_V^*(t)$ and β_S, β_V and γ. It is obvious that the effects of σ_L on $\theta_S^*(t)$ and $\theta_V^*(t)$ depend on the value of ρ_L. When $\rho_L = 0$, the optimal risk exposure in this case, denoted $\theta_{1S}^*(t)$ and $\theta_{1V}^*(t)$, can be written as $\theta_{1S}^*(t) = m(t)\left(1 + \bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right)$ and $\theta_{1V}^*(t) = n(t)\left(1 + \bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right) - \sigma_L\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case, denoted $J_1(t,x,v,l)$, can be written as $J_1(t,x,v,l) = \frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\bar{g}_1(t)v + \hat{g}_1(t))$.[8] Moreover, as $\bar{h}(t) > 0$, $\bar{g}(t) < 0$ and $\gamma > 1$, following simple calculations, when $\rho_V = 0$, we have $\frac{\partial\theta_{1S}^*(t)}{\partial(\beta_S+\gamma)} < 0$, which implies that the optimal risk exposure decreases w.r.t. the sum of aversion to ambiguity and risk in some cases, which implies that the investor decreases her exposure to market return risk when she is more ambiguity averse and risk averse.

Remark 3.6. If $\sigma_L = 0$, the salary process is non-stochastic; then the optimal risk exposure in this case, denoted $\theta_{2S}^*(t)$ and $\theta_{2V}^*(t)$, can be written as $\theta_{2S}^*(t) = m(t)(1 + \frac{\hat{h}(t)}{X^{u^*}(t)})$ and $\theta_{2V}^*(t) = n(t)(1 + \frac{\hat{h}(t)}{X^{u^*}(t)})$, and the optimal value function in this case, denoted $J_2(t,x,v)$, can be written as $J_2(t,x,v) = \frac{(x+\hat{h}(t))^{1-\gamma}}{1-\gamma}\exp(\bar{g}(t)v + \hat{g}(t))$, where

$$\hat{h}(t) = \frac{\xi l_0}{\mu_L - r}[\exp(\mu_L T - r(T-t)) - \exp(\mu_L t)], \tag{35}$$

and $m(t)$, $n(t)$, $\bar{g}(t)$, $\hat{g}(t)$ are given by Eqs. (26), (27), (28) and (29).[9] In this case, we find that the optimal risk exposures are proportional to $m(t)$ and $n(t)$.

---

[8] The optimal investment strategy when $\rho_L = 0$, denoted $u_{1S}^*(t)$ and $u_{1O}^*(t)$, can be written as $u_{1S}^*(t) = \theta_{1S}^*(t) - \frac{O_s S(t) + \sigma_V\rho_V O_v(t)}{O(t)}u_{1O}^*(t)$ and $u_{1O}^*(t) = \frac{O(t)\theta_{1V}^*(t)}{\sigma_V\sqrt{1-\rho_V^2}O_v}$, and the worst-case measure in this case, denoted $\phi_{1S}^*(t)$ and $\phi_{1V}^*(t)$, can be written as $\phi_{1S}^*(t) = \frac{\beta_S(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}_1(t))\sqrt{V(t)}}{(1-\gamma)(\beta_S+\gamma)}$ and $\phi_{1V}^*(t) = \frac{\beta_V(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2}\bar{g}_1(t))\sqrt{V(t)}}{(1-\gamma)(\beta_V+\gamma)}$, where

$$\bar{g}_1(t) = \frac{\nu_{11}\nu_{21} - \nu_{11}\nu_{21}e^{\alpha_{21}(\nu_{11}-\nu_{21})(T-t)}}{\nu_{21} - \nu_{11}e^{\alpha_{21}(\nu_{11}-\nu_{21})(T-t)}}, \quad \hat{g}_1(t) = \int_t^T [r(1-\gamma) + \kappa\delta\bar{g}_1(s)]\,\mathrm{d}s,$$
$$\alpha_{11} = -\kappa + \frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V\rho}{\beta_S+\gamma} + \frac{\lambda_2(1-(\beta_V+\gamma))\sigma_V\sqrt{1-\rho_V^2}}{\beta_V+\gamma},$$
$$\alpha_{21} = \frac{\sigma_V^2}{2} - \frac{\beta_S^2\sigma_V^2\rho_V^2 + \beta_V\sigma_V^2(1-\rho_V^2)}{2(1-\gamma)} + \frac{(1-(\beta_S+\gamma))^2\sigma_V^2\rho_V^2}{2(\beta_S+\gamma)(1-\gamma)} + \frac{(1-(\beta_V+\gamma))^2\sigma_V^2(1-\rho_V^2)}{2(\beta_V+\gamma)(1-\gamma)},$$
$$\alpha_{31} = \frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)} + \frac{\lambda_2^2(1-\gamma)}{2(\beta_V+\gamma)}, \quad \nu_{11,21} = \frac{\alpha_{11} \pm \sqrt{\alpha_{11}^2 - 4\alpha_{21}\alpha_{31}}}{-2\alpha_{21}},$$

and $\bar{h}(t)$ is given by Eq. (30). By derivation, we obtain $\alpha_{11}^2 - 4\alpha_{21}\alpha_{31} \geq 0$.

[9] The optimal investment strategy when $\sigma_L = 0$, denoted $u_{2S}^*(t)$ and $u_{2O}^*(t)$, can be written as

Furthermore, if there is no salary in our model, i.e., $\xi = 0$ or $L(t) = 0$, our problem reduces to a portfolio selection problem. The optimal risk exposure in this case, denoted $\theta^*_{3S}(t)$ and $\theta^*_{3V}(t)$, can be written as $\theta^*_{3S}(t) = m(t)$ and $\theta^*_{3V}(t) = n(t)$, and the optimal value function in this case, denoted $J_3(t,x,v)$, can be written as $J_3(t,x,v) = \frac{x^{1-\gamma}}{1-\gamma}\exp(\bar{g}(t)v + \hat{g}(t))$, where $m(t)$, $n(t)$, $\bar{g}(t)$ and $\hat{g}(t)$ are given by Eqs. (26), (27), (28) and (29), respectively.[10] Correspondingly, the optimal risk exposure is independent of wealth x. It is worth noting that the optimal investment strategy obtained in the case without stochastic salary is the same as that given in Escobar et al. (2015) without jumps.

Remark 3.7. If the pension investor is ambiguity neutral, i.e., both ambiguity aversion parameters β_S and β_V equal 0, the optimal risk exposure in this case, denoted $\theta^*_{4S}(t)$ and $\theta^*_{4V}(t)$, can be written as $\theta^*_{4S}(t) = \frac{\lambda_1+\sigma_V\rho_V\bar{g}_2(t)}{\gamma}\left(1+\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right) - \sigma_L\rho_L\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}$ and $\theta^*_{4V}(t) = \frac{\lambda_2+\sigma_V\sqrt{1-\rho_V^2}\bar{g}_2(t)}{\gamma}\left(1+\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right) - \sigma_L\sqrt{1-\rho_L^2}\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case, denoted $J_2(t,x,v,l)$, can be written as $J_4(t,x,v,l) = \frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\bar{g}_2(t)v + \hat{g}_2(t))$, where

$$\bar{g}_2(t) = \frac{\nu_{12}\nu_{22} - \nu_{12}\nu_{22}\mathrm{e}^{\alpha_{22}(\nu_{12}-\nu_{22})(T-t)}}{\nu_{22} - \nu_{12}\mathrm{e}^{\alpha_{22}(\nu_{12}-\nu_{22})(T-t)}}, \quad \hat{g}_2(t) = \int_t^T \left[r(1-\gamma) + \kappa\delta\bar{g}_2(s)\right]\mathrm{d}s, \quad (36)$$

and $\bar{h}(t)$ is given by Eq.(30). By derivation, we obtain $\alpha_{12}^2 - 4\alpha_{22}\alpha_{32} \geq 0$.[11]

Similarly, the following remark provides the optimal investment strategy in the case of no ambiguity and no stochastic salary.

$u^*_{2S}(t) = \theta^*_{2S}(t) - \frac{O_sS(t)+\sigma_V\rho_VO_v}{O(t)}u^*_{2O}(t)$ and $u^*_{2O}(t) = \frac{O(t)\theta^*_{2V}(t)}{\sigma_V\sqrt{1-\rho_V^2}O_v}$, and the worst-case measure in this case, denoted $\phi^*_{2S}(t)$ and $\phi^*_{2V}(t)$, can be written as $\phi^*_{2S}(t) = \frac{\beta_S(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_S+\gamma)}$ and $\phi^*_{2V}(t) = \frac{\beta_V(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2}\bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_V+\gamma)}$.

[10]The optimal investment strategy without stochastic salary, denoted $u^*_{3S}(t)$ and $u^*_{3O}(t)$, can be written as $u^*_{3S}(t) = \theta^*_{3S}(t) - \frac{O_sS(t)+\sigma_V\rho_VO_v}{O(t)}u^*_{3O}(t)$ and $u^*_{3O}(t) = \frac{O(t)\theta^*_{3V}(t)}{\sigma_V\sqrt{1-\rho_V^2}O_v}$, and the worst-case measure in this case, denoted $\phi^*_{3S}(t)$ and $\phi^*_{3V}(t)$, can be written as $\phi^*_{3S}(t) = \frac{\beta_S(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_S+\gamma)}$ and $\phi^*_{3V}(t) = \frac{\beta_V(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2}\bar{g}(t))\sqrt{V(t)}}{(1-\gamma)(\beta_V+\gamma)}$.

[11]The optimal investment strategy without ambiguity, denoted $u^*_{4S}(t)$ and $u^*_{4O}(t)$, can be written as $u^*_{4S}(t) = \theta^*_{4S}(t) - \frac{O_sS(t)+\sigma_V\rho_VO_v}{O(t)}u^*_{4O}(t)$ and $u^*_{4O}(t) = \frac{O(t)\theta^*_{4V}(t)}{\sigma_V\sqrt{1-\rho_V^2}O_v}$. In Eq. (36),

$$\alpha_{12} = -\kappa + \frac{\lambda_1(1-\gamma)\sigma_V\rho_V}{\gamma} + \frac{\lambda_2(1-\gamma)\sigma_V\sqrt{1-\rho_V^2}}{\gamma}, \quad \alpha_{22} = \frac{\sigma_V^2}{2\gamma},$$
$$\alpha_{32} = \frac{(\lambda_1^2+\lambda_2^2)(1-\gamma)}{2\gamma}, \quad \nu_{12,22} = \frac{\alpha_{12} \pm \sqrt{\alpha_{12}^2 - 4\alpha_{2a_{32}}}}{-2\alpha_{22}}.$$

Remark 3.8. If the pension investor is ambiguity neutral and $\sigma_L = 0$, the salary process is non-stochastic, and the optimal risk exposure in this case, denoted $\theta^*_{5S}(t)$ and $\theta^*_{5V}(t)$, can be written as $\theta^*_{5S}(t) = \frac{\lambda_1+\sigma_V\rho_V\bar{g}_2(t)}{\gamma}(1+\frac{\hat{h}(t)}{X^{u^*}(t)})$ and $\theta^*_{5V}(t) = \frac{\lambda_2+\sigma_V\sqrt{1-\rho_V^2}\bar{g}_2(t)}{\gamma}(1+\frac{\hat{h}(t)}{X^{u^*}(t)})$, and the optimal value function in this case, denoted $J_5(t,x,v)$, can be written as $J_5(t,x,v) = \frac{(x+\hat{h}(t))^{1-\gamma}}{1-\gamma}\exp(\bar{g}_2(t)v+\hat{g}_2(t))$, where $\hat{h}(t)$, $\bar{g}_2(t)$ and $\hat{g}_2(t)$ are given by Eqs. (35)-(36).[12]

Furthermore, if there is no salary and no ambiguity in our model, the optimization problem becomes a portfolio selection problem for an ambiguity-neutral investor; the optimal risk exposure in this case, denoted $\theta^*_{6S}(t)$ and $\theta^*_{6V}(t)$, can be written as $\theta^*_{6S}(t) = \frac{\lambda_1+\sigma_V\rho_V\bar{g}_2(t)}{\gamma}$ and $\theta^*_{6V}(t) = \frac{\lambda_2+\sigma_V\sqrt{1-\rho_V^2}\bar{g}_2(t)}{\gamma}$, and the optimal value function in this case, denoted $J_6(t,x,v)$, can be written as $J_6(t,x,v) = \frac{x^{1-\gamma}}{1-\gamma}\exp(\bar{g}_2(t)v+\hat{g}_2(t))$, where $\bar{g}_2(t)$ and $\hat{g}_2(t)$ are given by Eq. (36).[13] In this case, the result reduces to that of the optimal portfolio problem in the case without jumps in Liu and Pan (2003).

4. Optimal investment strategy without a derivative

In this section, to illustrate the significant role of the derivative, we seek the solution to the case without a derivative and compare it to the result with a derivative.

If there is no derivative security in the financial market, the optimal investment strategy equals the optimal risk exposure to $W_S(t)$, and the surplus process of an ambiguity-averse pension investor under measure $\mathbb{Q}$ becomes

$$\begin{aligned}\mathrm{d}X^{\tilde{\Phi},\tilde{\mathbf{u}}}(t) = \; & X^{\tilde{\Phi},\tilde{\mathbf{u}}}(t)\left[r\mathrm{d}t+\tilde{u}(t)\left(\lambda_1 V^{\tilde{\Phi}}(t)\mathrm{d}t-\tilde{\phi}_S(t)\sqrt{V^{\tilde{\Phi}}(t)}\mathrm{d}t+\sqrt{V^{\tilde{\Phi}}(t)}\mathrm{d}W_S^{\tilde{\Phi}}(t)\right)\right] \\ & +\xi L^{\tilde{\Phi}}(t)\mathrm{d}t,\end{aligned} \tag{37}$$

where $\tilde{\mathbf{u}} := \{\tilde{u}(t)\}_{t\in[0,T]}$, $\tilde{\Phi} := \{\tilde{\phi}(t) := (\tilde{\phi}_S(t),\tilde{\phi}_V(t))\}_{t\in[0,T]}$, and the risk exposure equals the investment strategy, i.e., $\tilde{\theta}_S(t) = \tilde{u}(t)$. The optimization problem becomes

$$\sup_{\tilde{\mathbf{u}}\in\Pi}\inf_{\tilde{\Phi}\in\Theta}\left\{\mathrm{E}^{\tilde{\Phi}}_{t,x,v,l}\left[U(X^{\tilde{\Phi},\tilde{\mathbf{u}}}(T))+\int_t^T\left(\frac{(\tilde{\phi}_S(s))^2}{2\tilde{\Psi}_S(s,x,v,l)}+\frac{(\tilde{\phi}_V(s))^2}{2\tilde{\Psi}_V(s,x,v,l)}\right)\mathrm{d}s\right]\right\}, \tag{38}$$

and the corresponding HJB equation becomes

$$\sup_{\tilde{u}\in\mathbb{R}}\inf_{||\tilde{\phi}||\le\sqrt{\kappa v}}\left\{\tilde{\mathcal{A}}^{\tilde{\phi},\tilde{u}}\tilde{J}(t,x,v,l)+\frac{\tilde{\phi}_S^2}{2\tilde{\Psi}_S}+\frac{\tilde{\phi}_V^2}{2\tilde{\Psi}_V}\right\}=0, \tag{39}$$

[12]The optimal investment strategy when $\sigma_L = 0$ for an ambiguity-neutral pension investor, denoted $u^*_{5S}(t)$ and $u^*_{5O}(t)$, can be written as $u^*_{5S}(t) = \theta^*_{5S}(t) - \frac{O_sS(t)+\sigma_V\rho_VO_v}{O(t)}u^*_{5O}(t)$ and $u^*_{5O}(t) = \frac{O(t)\theta^*_{5V}(t)}{\sigma_V\sqrt{1-\rho_V^2}O_v}$.

[13]The optimal investment strategy without stochastic salary and ambiguity, denoted $u^*_{6S}(t)$ and $u^*_{6O}(t)$, can be written as $u^*_{6S}(t) = \theta^*_{6S}(t) - \frac{O_sS(t)+\sigma_V\rho_VO_v}{O(t)}u^*_{6O}(t)$ and $u^*_{6O}(t) = \frac{O(t)\theta^*_{6V}(t)}{\sigma_V\sqrt{1-\rho_V^2}O_v}$.

$$\tilde{\Psi}_S(t,x,v,l)=\frac{\beta_S}{(1-\gamma)\tilde{J}(t,x,v,l)},\qquad \tilde{\Psi}_V(t,x,v,l)=\frac{\beta_V}{(1-\gamma)\tilde{J}(t,x,v,l)},\tag{40}$$

with the boundary condition $\tilde{J}(T,x,v,l)=U(x)$, where $\tilde{u}$ and $\tilde{\phi}=(\tilde{\phi}_S,\tilde{\phi}_V)$ denote the values that $\tilde{u}(t)$ and $\tilde{\phi}(t)=(\tilde{\phi}_S(t),\tilde{\phi}_V(t))$ take, respectively, and

$$\begin{aligned}\tilde{\mathcal{A}}^{\tilde{\phi},\tilde{u}}\psi(t,x,v,l)&=\psi_t+[rx+x\tilde{u}\lambda_1 v-x\tilde{u}\tilde{\phi}_S\sqrt{v}+\xi l]\psi_x+\frac{1}{2}x^2v\tilde{u}^2\psi_{xx}\\&+[\kappa(\delta-v)-\sigma_V\sqrt{v}\rho_V\tilde{\phi}_S-\sigma_V\sqrt{v}\sqrt{1-\rho_V^2}\tilde{\phi}_V]\psi_v+\frac{1}{2}\sigma_v^2v\psi_{vv}\\&+\left[\mu_L l+l\sigma_L\lambda_1 v\rho_L-l\sigma_L\sqrt{v}\tilde{\phi}_S\rho_L+l\sigma_L\lambda_2 v\sqrt{1-\rho_L^2}-l\sigma_L\sqrt{v}\tilde{\phi}_V\sqrt{1-\rho_L^2}\right]\psi_l\\&+\frac{1}{2}l^2\sigma_L^2v\psi_{ll}+l\sigma_Lv\sigma_V(\rho_V\rho_L+\sqrt{1-\rho_V^2}\sqrt{1-\rho_L^2})\psi_{lv}+x\sigma_V\tilde{u}v\rho_V\psi_{xv}+x\tilde{u}l\sigma_Lv\rho_L\psi_{xl}.\end{aligned}$$

The following theorem presents the optimal investment strategy and optimal value function for the DC pension investor without a derivative.

Theorem 4.1. *For the robust portfolio choice problem (38) without a derivative, if the parameters satisfy certain technical conditions,* [14] *the optimal investment strategy and risk exposure are*

$$\tilde{u}^*(t)=\tilde{\theta}_S^*(t)=\tilde{m}(t)\left(1+\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right)-\sigma_L\rho_L\bar{h}(t)\frac{L(t)}{X^{u^*}(t)};\tag{43}$$

the corresponding optimal value function is

$$\tilde{J}(t,x,v,l)=\frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\bar{g}_3(t)v+\hat{g}_3(t));\tag{44}$$

and the worst-case measure is given by

$$\tilde{\phi}_S^*(t)=\frac{\beta_S\sqrt{V(t)}(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}_3(t))}{(1-\gamma)(\beta_S+\gamma)},\qquad \tilde{\phi}_V^*(t)=\frac{\beta_V\sqrt{V(t)}(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2}\bar{g}_3(t))}{(1-\gamma)(\beta_V+\gamma)},\tag{45}$$

[14] The technical conditions are $\underline{\phi}_3<\kappa/\sigma_V$, where

$$\underline{\phi}_3\triangleq\max\left\{\frac{\beta_S^2\lambda_1^2}{(\beta_S+\gamma)^2},\frac{\beta_S^2(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}_3(0))^2}{(1-\gamma)^2(\beta_S+\gamma)^2}\right\}+\max\left\{\frac{\beta_V^2\lambda_2^2}{(\beta_V+\gamma)^2},\frac{\beta_V^2(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2}\bar{g}_3(0))^2}{(1-\gamma)^2(\beta_V+\gamma)^2}\right\},\tag{41}$$

and for $\bar{g}_3(t)\in[\bar{g}_3(0),0]$,

$$[64(1-\gamma)^2-4(1-\gamma)](\tilde{m}(t))^2+8(1-\gamma)\gamma(\tilde{m}(t))^2-8(1-\gamma)\frac{\sigma_V\rho\bar{g}_3(t)}{\beta_S+\gamma}\tilde{m}(t)\le\frac{\kappa^2}{2\sigma_V^2}.\tag{42}$$

Similar to conditions $\underline{\phi}<\kappa/\sigma_V$ and (21), conditions $\underline{\phi}_3<\kappa/\sigma_V$ and (42) are also technical conditions and easily satisfied.

where $\{X^*(t)\}_{t\in[0,T]}$ *is the wealth process under the corresponding optimal strategy, and*

$$\begin{aligned}
&\tilde{m}(t) = \frac{\lambda_1(1-\gamma) + (1-(\beta_S+\gamma))\sigma_V\rho_V\bar{g}_3(t)}{(1-\gamma)(\beta_S+\gamma)}, \qquad \bar{g}_3(t) = \frac{\tilde{\nu}_1\tilde{\nu}_2 - \tilde{\nu}_1\tilde{\nu}_2 \mathrm{e}^{\tilde{\alpha}_2(\tilde{\nu}_1-\tilde{\nu}_2)(T-t)}}{\tilde{\nu}_2 - \tilde{\nu}_1 \mathrm{e}^{\tilde{\alpha}_2(\tilde{\nu}_1-\tilde{\nu}_2)(T-t)}},\\
&\hat{g}_3(t) = \int_t^T \left[r(1-\gamma) + \kappa\delta\bar{g}_3(s)\right] \mathrm{d}s, \qquad \tilde{\alpha}_1 = -\kappa + \frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V\rho_V}{\beta_S+\gamma},\\
&\tilde{\alpha}_2 = \frac{\sigma_V^2}{2} - \frac{\beta_S\sigma_V^2\rho_V^2}{2(1-\gamma)} - \frac{\beta_V\sigma_V^2(1-\rho_V^2)}{2(1-\gamma)} + \frac{(1-(\beta_S+\gamma))^2\sigma_V^2\rho_V^2}{2(\beta_S+\gamma)(1-\gamma)},\\
&\tilde{\alpha}_3 = \frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)}, \qquad \tilde{\nu}_{1,2} = \frac{\tilde{\alpha}_1 \pm \sqrt{\tilde{\alpha}_1^2 - 4\tilde{\alpha}_2\tilde{\alpha}_3}}{-2\tilde{\alpha}_2},
\end{aligned} \tag{46}$$

and $\bar{h}(t)$ *is given by Eq.(30). By derivation, we obtain* $\tilde{\alpha}_1^2 - 4\tilde{\alpha}_2\tilde{\alpha}_3 \geq 0$.

The proof of Theorem 4.1 is similar to that of Theorem 3.2, and thus, we omit it here.

Theorem 4.2. *For problem (38), if there exists a function* $\tilde{J}(t,x,v,l) \in C^{1,2,2,2}(O)$ *that is a solution to the HJB equation (65) with boundary condition* $\tilde{J}(T,x,v,l) = U(x)$*, and the parameters satisfy conditions (41) and (42), then the optimal value function is* $\tilde{J}(t,x,v,l)$*, and the optimal strategy is* $\tilde{\boldsymbol{u}}^* = \{\tilde{u}^*(t)\}_{t\in[0,T]}$ *given in Theorem 4.1.*

The proof of Theorem 4.2 is similar to that of Theorem 3.4, and thus, we omit it here.

From Theorem 4.1, we find that the optimal investment strategy and risk exposure are both given by Eq. (43). Compared with the former case and optimal exposure to market return risk (22), the difference lies in the form of $\tilde{m}(t)$, particularly, the values of $\nu_{1,2}$ and $\tilde{\nu}_{1,2}$. Here, because the market is incomplete and the investor has only one stock to invest in and obtains one risk premium, the equity premium λ_2 for additional volatility risk is disappearing; as a result, hedging w.r.t. additional volatility risk is less efficient. This quantitative influence depends on the chosen parameters of the model, as illustrated in the following numerical examples. We find that the utility that the pension investor gains is substantially improved when investing in the derivative. Similar results are also found in Escobar et al. (2015). Similar to the case of investment with the derivative, we also provide some special cases in Appendix F if the pension investor has no access to the derivative.

5. Numerical analysis

In this section, we provide several numerical examples to illustrate the effects of model parameters on the robust optimal risk exposures and investment strategies. We also illustrate the utility improvements by considering ambiguity aversion and derivative trading. To improve the credibility of our empirical results, we fix a set of base-case parameters for our

model (Table 1) using data from existing empirical studies. For details, refer to Liu and Pan (2003) and Escobar et al. (2015).[15]

Table 1: Values of model parameters in the numerical examples.

r	κ	δ	ξ	λ_1	λ_2	μ_L	σ_L	σ_V	γ	β_S	β_V
0.05	5	0.13^2	0.2	4	-6	0.08	0.5	0.25	4	3	1
ρ_V	ρ_L	x	l	v	S	K	τ	T	t		
-0.4	0.3	1	1	0.15^2	100	100	0.1	5	0		

5.1. Effects of model parameters on risk exposures

Risk exposures θ_S^* and θ_V^* more intuitively describe the exposures to risks W_S and W_V, and the risk exposures are independent of the types of options. Other related studies also consider the performance of risk exposures; please see Escobar et al. (2015). Therefore, in this subsection, we first consider the effects of model parameters on the risk exposures.

Figure 1 shows the effects of the ambiguity aversion parameters β_S and β_V on the optimal market return risk exposure θ_S^* and volatility risk exposure θ_V^*, respectively. We find that θ_S^* decreases in β_S, consistent with Escobar et al. (2015). Another main result is that θ_V^* significantly decreases (in absolute terms) in β_V. These results show that in an ambiguous environment, the investor becomes less aggressive. We now focus on one specific risk exposure and show how the two ambiguity aversion parameters have distinct effects on it. Taking θ_S^* as an example, we find that the stock ambiguity aversion parameter β_S has a relatively greater effect than the volatility ambiguity aversion parameter β_V. This is consistent with the case of θ_V^*. Compared to β_V (β_S), β_S (β_V) represents a direct way to affect market return risk exposure (volatility risk exposure).

[15] According to Liu and Pan (2003), the empirical properties of the stochastic volatility model have been extensively examined using either the time-series data on the S&P 500 index alone (Andersen et al., 2002; Eraker et al., 2003) or the joint time-series data on the S&P 500 index and options (Chernov and Ghysels, 2000; Pan, 2002). Because of different sample periods or empirical approaches in those studies, the exact model estimates may differ from one paper to another. Our chosen model parameters agree with the cases studied by Liu and Pan (2003) and Escobar et al. (2015).

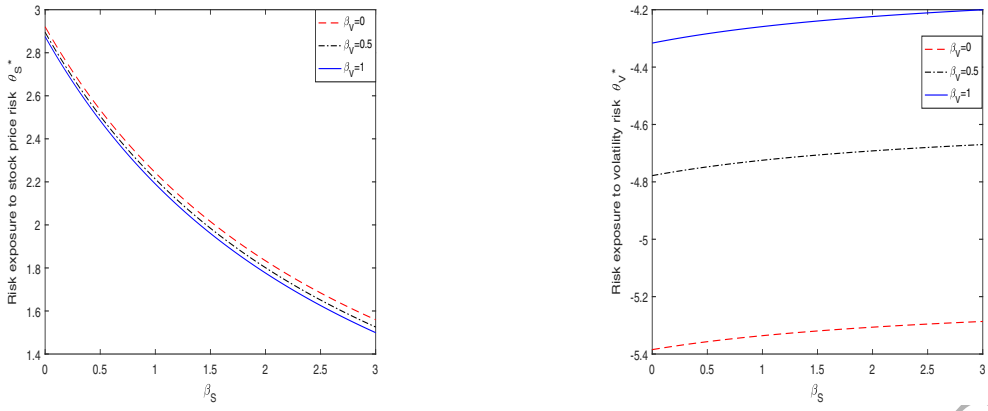

Figure 1: Effects of β_S and β_V on θ_S^* and θ_V^*.

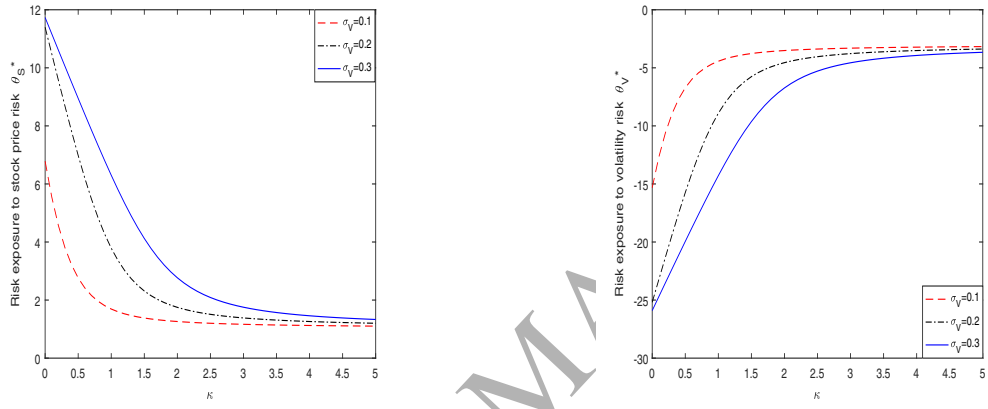

Figure 2: Effects of κ and σ_V on θ_S^* and θ_V^*.

Figure 2 shows the effects of the mean-reversion rate κ and volatility coefficient σ_V on the optimal market return risk exposure θ_S^* and volatility risk exposure θ_V^*, respectively. In the stock return variance process, a lower mean-reversion rate κ and higher volatility σ_V usually imply greater additional volatility risk. As a result, θ_V^* decreases and increases (in absolute terms) in κ and σ_V, respectively. The case of θ_S^* is similar to that of θ_V^*, as there is a diversification effect (benefit from risk diversification).

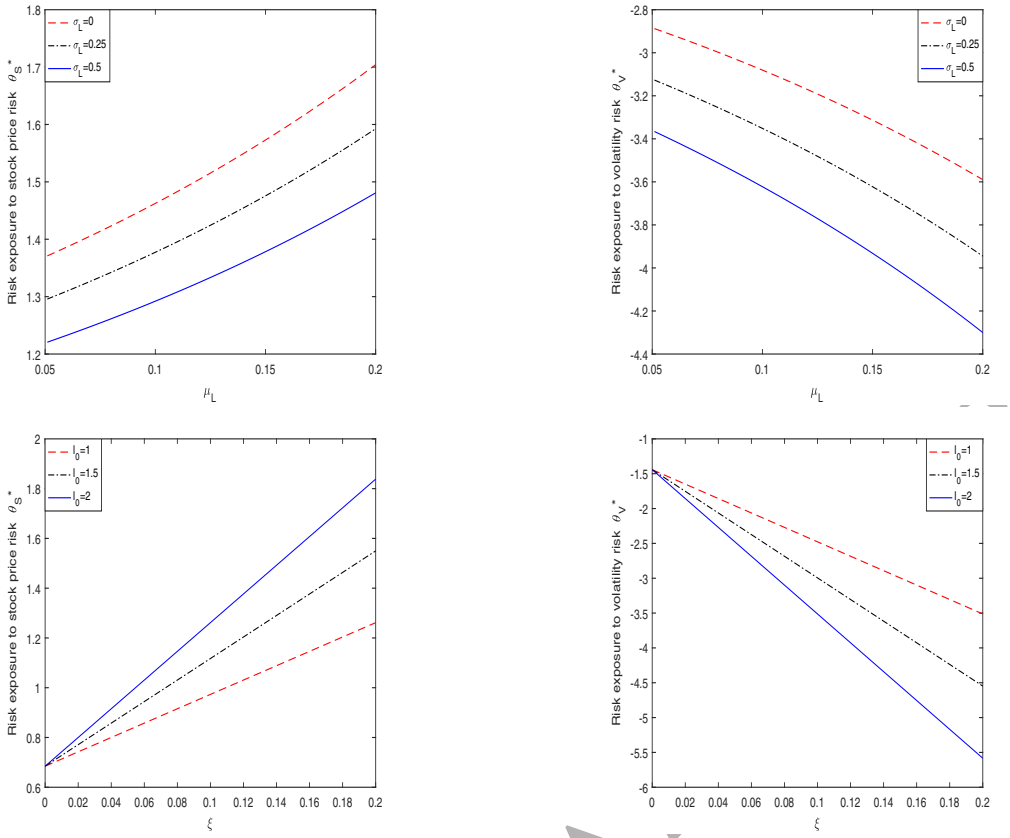

Figure 3: Effects of salary parameters μ_L, σ_L, ξ and l_0 on θ_S^* and θ_V^*.

Figure 3 shows the effects of the salary parameters, appreciation rate μ_L, volatility coefficient σ_L, contribution rate ξ and initial salary l_0 on the optimal market return risk exposure θ_S^* and volatility risk exposure θ_V^*. We find that both θ_S^* and θ_V^* increase (in absolute terms) in μ_L, ξ and l_0. When μ_L, ξ and l_0 increase, more pension funds are accumulated. Therefore, the investor prefers to undertake more risks to earn more profits. In addition, θ_S^* decreases in σ_L and θ_V^* increases (in absolute terms) in σ_L.

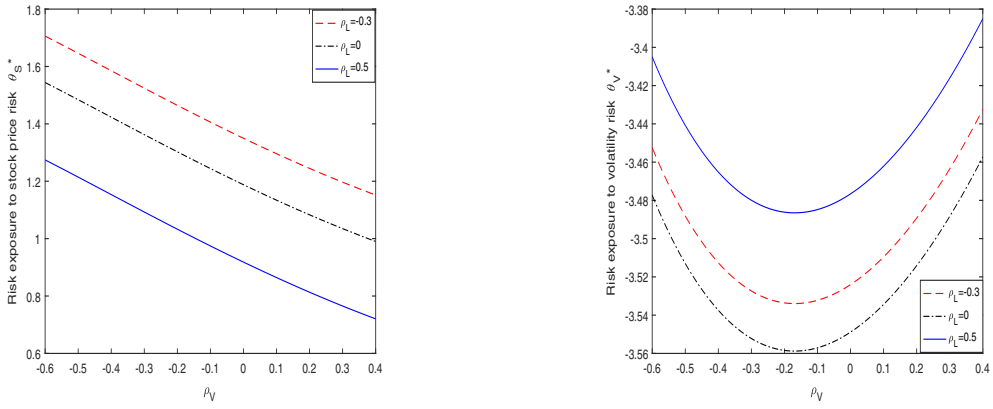

Figure 4: Effects of ρ_V and ρ_L on θ_S^* and θ_V^*.

Figure 4 shows the effects of the correlation coefficients ρ_V and ρ_L on the optimal market return risk exposure θ_S^* and volatility risk exposure θ_V^*, respectively. This figure shows that θ_S^* decreases in ρ_V and ρ_L, while θ_V^* first increases (in absolute terms) and then decreases in ρ_V and ρ_L. This behavior stems from the assumption of our model. From Eqs. (22), (26) and (27), ρ_V and $\sqrt{1-\rho_V^2}$ (ρ_L and $\sqrt{1-\rho_L^2}$) reflect different properties of a sensitivity analysis for ρ_V (ρ_L). ρ_V (ρ_L) may be negative or non-negative, and $\sqrt{1-\rho_V^2}$ ($\sqrt{1-\rho_L^2}$) is non-negative. Therefore, the risk exposure to W_S decreases in ρ_V and ρ_L, and the risk exposure to W_V decreases (in absolute terms) in $|\rho_V|$ and $|\rho_L|$.

5.2. Effects of model parameters on investment strategies

In this subsection, we take the straddle option[16] as an example to demonstrate the effects of model parameters on investment strategies. The result further illustrates the significant role of the derivative on the optimal investment strategy.

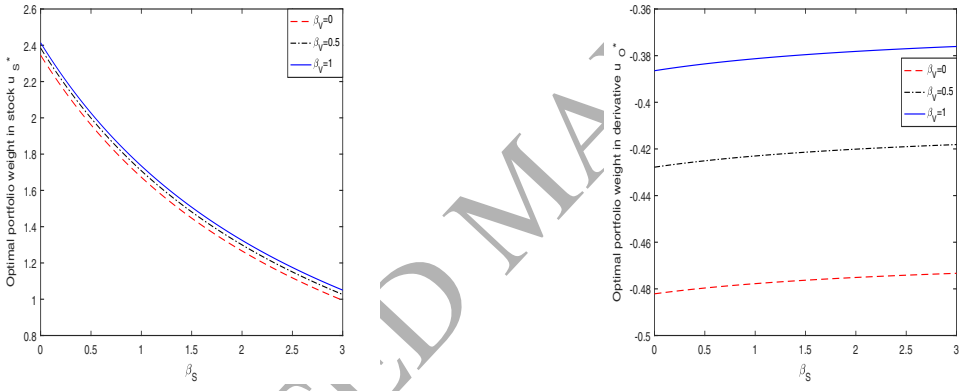

Figure 5: Effects of β_S and β_V on u_S^* and u_O^*.

Figure 5 shows the effects of the ambiguity aversion parameters β_S and β_V on the optimal proportions invested in the stock u_S^* and derivative u_O^*, respectively. We find that both u_S^* and u_O^* decrease (in absolute terms) in β_S. Compared to those in stock investment, the changes in derivative investment are relatively small. When β_S increases, the investor becomes more ambiguity averse to the return of the stock. Therefore, she tends to invest less in the stock. Moreover, u_O^* decreases (in absolute terms) in β_V in a similar way. As ambiguity reduces the

[16]The straddle is a portfolio comprising a call option and a put option with the same underlying strike price, time to maturity, and market volatility, and its price is given in Appendix G. We assume that the initial stock price is 100, and the strike price is chosen in a way that makes the straddle "delta-neutral". For details, refer to Liu and Pan (2003) and Cui et al. (2017). The analyses with other types of options, such as call options and put options are similar. To save space, we do not include these results in our paper.

volatility risk premium, the derivative investment becomes less attractive to the ambiguous-averse investor. Therefore, she shorts the straddle option less. However, u_S^* increases in β_V. As ambiguity hampers the investor's judgement regarding the variation in the stock's volatility, the investor holding the short straddle may worry about the substantial increase in the stock price. Hence, at this time, she invests more wealth in the stock to reduce the total risk of the portfolio.

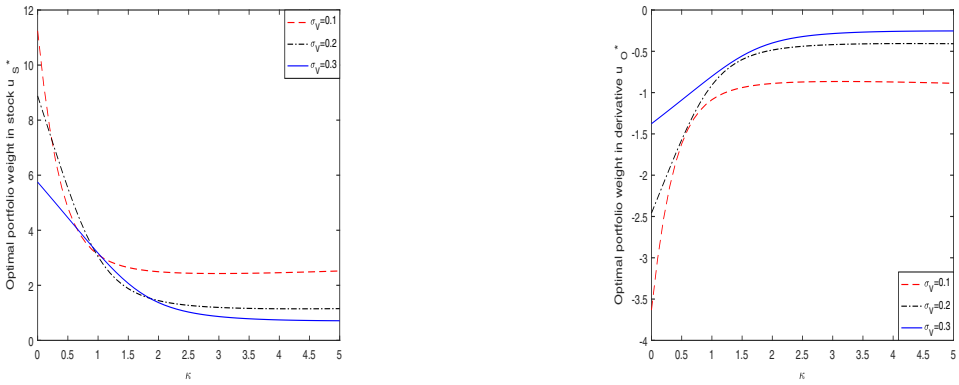

Figure 6: Effects of κ and σ_V on u_S^* and u_O^*.

Figure 6 shows the effects of the mean-reversion rate κ and volatility coefficient σ_V on the optimal proportions invested in the stock u_S^* and derivative u_O^*, respectively. As κ increases, both u_S^* and u_O^* decrease (in absolute terms). As the correlation ρ_V is negative, the uncertainties of the stock price and its volatility change in different ways. Although $V(t)$ will be stable as κ increases, there is an increased probability of a decrease in the stock price. The decrease affects the investment strategies in the stock and the derivative. Moreover, when $\kappa < 2$, the effects of σ_V on the optimal investment strategies are not monotone; when $\kappa \geq 2$, u_S^* and u_O^* decrease (in absolute terms) as σ_V increases. In other words, the larger σ_V is, the more risk the stock has. Therefore, the investor will invest less in the stock and the derivative.

Figure 7 shows the effects of the salary parameters, appreciation rate μ_L, volatility coefficient σ_L, contribution rate ξ and initial salary l_0 on the optimal proportions invested in the stock u_S^* and derivative u_O^*. We find that both u_S^* and u_O^* increase (in absolute terms) in μ_L, ξ and l_0: the increasing μ_L, ξ and l_0 imply that there will be greater pension fund accumulation. Therefore, the investor prefers to undertake more risks to earn more. In addition, u_S^* decreases in σ_L, and u_O^* increases (in absolute terms) in σ_L. The investor now both shorts the straddle option more and buys less stock to reduce the portfolio risk. The

results here are consistent with the results on risk exposures in Figure 3.

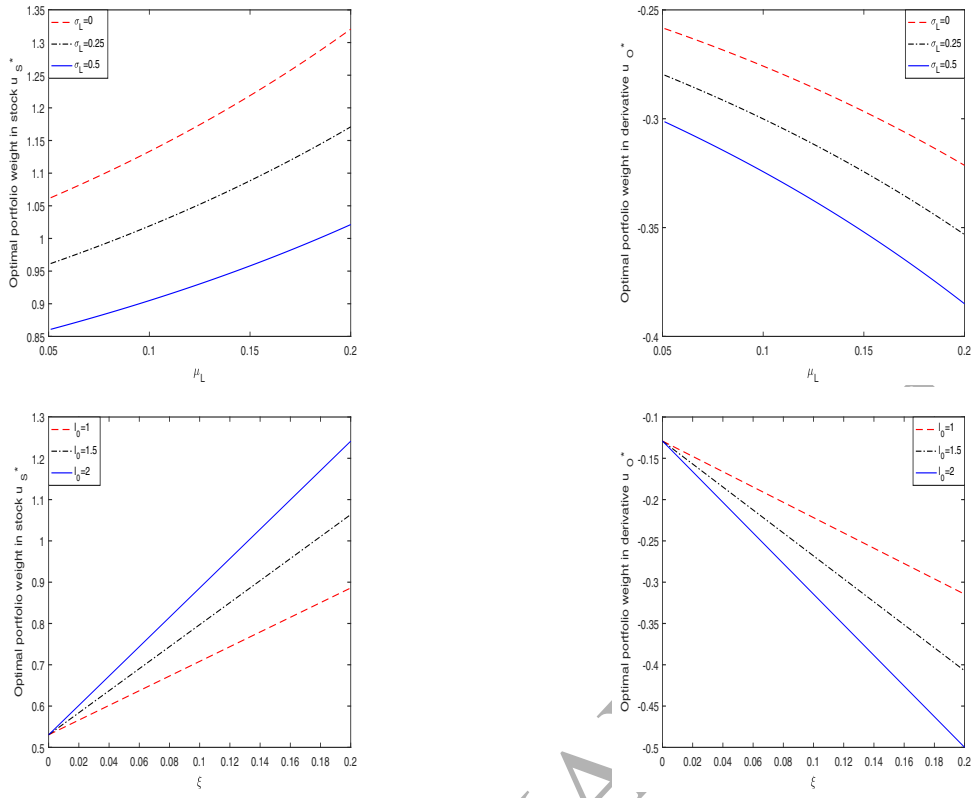

Figure 7: Effects of salary parameters μ_L, σ_L, ξ and l_0 on u_S^* and u_O^*.

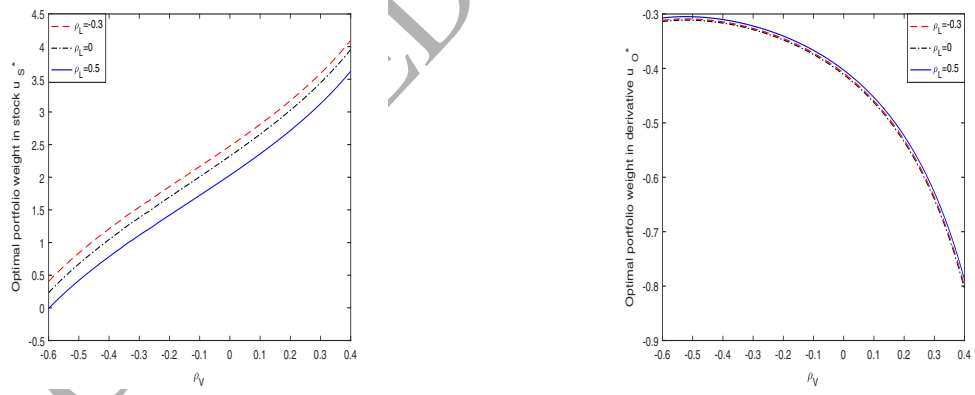

Figure 8: Effects of ρ_V and ρ_L on u_S^* and u_O^*.

Figure 8 shows the effects of correlation coefficients ρ_V and ρ_L on the optimal proportions invested in the stock u_S^* and derivative u_O^*, respectively. On the one hand, both u_S^* and u_O^* increase (in absolute terms) in ρ_V. When the risks of the financial market increase, the investor goes long on more stocks and shorts more derivative to reduce her portfolio risk. On the other hand, both u_S^* and u_O^* decrease (in absolute terms) in ρ_L. As it is difficult to

reduce salary risk by managing the portfolio of the stock and the derivative, investment in the stock and the derivative will decrease.

Moreover, we find that the derivative type has no effect on the value function in our model. This is because that in our paper, the financial market is complete and we can treat the risk exposure instead of the investment strategy as the control variable in the investor's wealth process, which makes the value function independent of the derivative type. The same is true for the optimal terminal wealth level X^{Ψ^*,u^*}. However, the strategy needed to replicate this optimal terminal wealth depends on the form of the derivative since the form of the derivative specifies the terminal condition of the BSDE in (4).

Since the derivative type has an important effect on the investment strategy, we demonstrate this argument by theoretical and numerical analysis as follows. From Theorem 3.2, the optimal investment strategy $\{(u_S^*(t),)u_O^*(t)\}_{t\in[0,T]}$ is

$$u_S^*(t) = \theta_S^*(t) - \frac{S(t)\theta_V^*(t)}{\sigma_V\sqrt{1-\rho_V^2}}\cdot\frac{O_s}{O_v} - \frac{\rho_V\theta_V^*(t)}{\sqrt{1-\rho_V^2}},$$
$$u_O^*(t) = \frac{\theta_V^*(t)}{\sigma_V\sqrt{1-\rho_V^2}}\cdot\frac{O(t)}{O_v}.$$

$\theta_S^*(t)$ and $\theta_V^*(t)$ are independent of the derivative type (see Eq. (22) in Theorem 3.2), while $\frac{O_s}{O_v}$ and $\frac{O(t)}{O_v}$ affect $u_S^*(t)$ and $u_O^*(t)$, respectively. In other words, $O(t)$, O_s and O_v have important effects on the investment strategy. Using the parameters in Table 1, we have $-\frac{S(t)\theta_V^*(t)}{\sigma_V\sqrt{1-\rho_V^2}} > 0$ and $\frac{\theta_V^*(t)}{\sigma_V\sqrt{1-\rho_V^2}} < 0$, showing that $u_S^*(t)$ increases in $\frac{O_s}{O_v}$, and $u_O^*(t)$ decreases in $\frac{O(t)}{O_v}$. In particular, without loss of generality, we provide the numerical analysis for the cases of call option, put option and straddle option at $t = 0$. We have $\theta_S^*(t) = 1.8383$ and $\theta_V^*(t) = -5.5831$, and the other values are given in Table 2. We find that $\frac{O(t)}{O_v}$ is positive under the three options, i.e., the pension investor shorts the three options. However, their qualitative effects are different. The investor who chooses the straddle option will short fewer than the inyestor who chooses the call option, while shorting more than the investor who chooses the put option. Furthermore, the effect of the derivative type on the strategy invested in the stock is complicated. The positions invested in the stock for the cases of the straddle and call options are long, while a short position is adopted for the case of the put option. The percentage that the pension investor longs (or shorts) for the case of the call (or put) option is larger than that for the case of the straddle option.

Table 2: Comparison of three derivative types.

	$-\frac{S(t)\theta_V^*(t)}{\sigma_V\sqrt{1-\rho_V^2}}$	$\frac{O_s}{O_v}$	$\frac{\rho_V\theta_V^*(t)}{\sqrt{1-\rho_V^2}}$	$\frac{\theta_V^*(t)}{\sigma_V\sqrt{1-\rho_V^2}}$	$\frac{O(t)}{O_v}$	$u_S^*(t)$	$u_O^*(t)$
call option	0.0024	0.0061	2.4367	-24.3667	0.0232	14.3194	-0.5651
straddle option	0.0024	0.0008	2.4367	-24.3667	0.0205	1.2415	-0.4999
put option	0.0024	-0.0046	2.4367	-24.3667	0.0178	-11.8364	-0.4347

5.3. Utility improvement

In this subsection, we study the utility improvement obtained by considering ambiguity aversion and derivative trading. We focus on two cases of utility improvement for a DC pension investor. One is the utility improvement delivered by considering ambiguity aversion, the other is the utility improvement delivered by allowing the investor to trade in the derivative.

For the first case, we calculate the utility improvement delivered by considering the ambiguity aversion case compared with the case in which ambiguity is ignored. In particular, we assume that the investor does not adopt the optimal strategy $\mathbf{u}^* = \{(u_S^*(t), u_O^*(t)\}_{t\in[0,T]}$ given in Theorem 3.2 but instead makes the decision as if she were ambiguity neutral, i.e., the pension investor follows the strategy $\mathbf{u}_4^* = \{(u_{4S}^*(t), u_{4O}^*(t))\}_{t\in[0,T]}$ given in Remark 3.7. The value function for the pension investor in this case is defined by

$$\bar{J}(t,x,v,l) = \inf_{\bar{\Phi}\in\Theta}\left\{\mathrm{E}_{t,x,v,l}^{\bar{\Phi}}\left[U(X^{\bar{\Phi},\mathbf{u}_4^*}(T)) + \int_t^T\left(\frac{(\bar{\phi}_S(s))^2}{2\bar{\Psi}_S(s,x,v,l)} + \frac{(\bar{\phi}_V(s))^2}{2\bar{\Psi}_V(s,x,v,l)}\right)\mathrm{d}s\right]\right\},$$

where

$$\bar{\Psi}_S(t,x,v,l) = \frac{\beta_S}{(1-\gamma)\bar{J}(t,x,v,l)}, \qquad \bar{\Psi}_V(t,x,v,l) = \frac{\beta_V}{(1-\gamma)\bar{J}(t,x,v,l)}.$$

Similar to the above derivation, we derive the optimal value function under the suboptimal strategy

$$\bar{J}(t,x,v,l) = \frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\bar{g}_9(t)v + \hat{g}_9(t)).^{17} \tag{47}$$

[17] In Eq. (47),

$$\bar{g}_9(t) = \frac{\bar{\nu}_1\bar{\nu}_2 - \bar{\nu}_1\bar{\nu}_2\mathrm{e}^{\bar{\alpha}_2(\bar{\nu}_1-\bar{\nu}_2)(T-t)}}{\bar{\nu}_2 - \bar{\nu}_1\mathrm{e}^{\bar{\alpha}_2(\bar{\nu}_1-\bar{\nu}_2)(T-t)}}, \qquad \hat{g}_9(t) = \int_t^T\left[r(1-\gamma) + \kappa\delta\bar{g}_9(s)\right]\mathrm{d}s,$$

$$\bar{\alpha}_1 = -\kappa + \frac{(\lambda_1+\sigma_V\rho_V\bar{g}_2(t))(1-(\beta_S+\gamma))\sigma_V\rho_V}{\gamma} + \frac{(\lambda_2+\sigma_V\sqrt{1-\rho_V^2}\bar{g}_2(t))(1-(\beta_V+\gamma))\sigma_V\sqrt{1-\rho_V^2}}{\gamma},$$

$$\bar{\alpha}_2 = \frac{\sigma_V^2}{2} - \frac{\beta_S\sigma_V^2\rho_V^2}{2(1-\gamma)} - \frac{\beta_V\sigma_V^2(1-\rho_V^2)}{2(1-\gamma)}, \qquad \bar{\nu}_{1,2} = \frac{\bar{\alpha}_1 \pm \sqrt{\bar{\alpha}_1^2 - 4\bar{\alpha}_2\bar{\alpha}_3}}{-2\bar{\alpha}_2},$$

$$\bar{\alpha}_3 = -\frac{(\lambda_1+\sigma_V\rho_V\bar{g}_2(t))^2(1-\gamma)\beta_S}{2\gamma^2} - \frac{(\lambda_2+\sigma_V\sqrt{1-\rho_V^2}\bar{g}_2(t))^2(1-\gamma)\beta_V}{2\gamma^2} + \frac{(1-\gamma)(\lambda_1^2+\lambda_2^2-\sigma_V^2(\bar{g}_2(t))^2)}{2\gamma},$$

and $\bar{h}(t)$ and $\bar{g}_2(t)$ are given by Eqs. (30) and (36). After some calculations, we have $\bar{\alpha}_1^2 - 4\bar{\alpha}_2\bar{\alpha}_3 \geq 0$.

Furthermore, we define the utility improvement obtained by considering the ambiguity aversion given by

$$UI_1(t,x,v,l) := 1 - \frac{J(t,x,v,l)}{\bar{J}(t,x,v,l)} = 1 - \exp((\bar{g}(t) - \bar{g}_9(t))v + \hat{g}(t) - \hat{g}_9(t)), \tag{48}$$

where $J(t,x,v,l)$ and $\bar{J}(t,x,v,l)$ are given by Eqs. (24) and (47).

For the second case, we calculate the utility improvement obtained by considering derivative trading compared with the case in which a derivative is inaccessible. In particular, it is defined by

$$UI_2(t,x,v,l) := 1 - \frac{J(t,x,v,l)}{\tilde{J}(t,x,v,l)} = 1 - \exp((\bar{g}(t) - \bar{g}_3(t))v + \hat{g}(t) - \hat{g}_3(t)), \tag{49}$$

where $J(t,x,v,l)$ and $\tilde{J}(t,x,v,l)$ are given by Eqs. (24) and (44).

Remark 5.1. From the expressions of $\bar{g}_9(t)$, $\hat{g}_9(t)$, $\bar{g}_3(t)$, utility improvements UI_1 and UI_2 are independent of the salary process.

Remark 5.2. Liu and Pan (2003) state that in a setting without ambiguity, trading in the derivative can significantly improve an investor's utility. Here, we further show that when the investor is ambiguity averse, there is also a utility improvement obtained from gaining access to the derivative market. The quantitative improvement is shown in the following numerical examples, which also reveal that the utility improvement delivered by having access to the derivative is large. This implies that the derivative plays a crucial role in providing investment opportunities and improving the efficiency of the market.

Figure 9 shows the effects of the ambiguity aversion parameters β_S and β_V on utility improvements. UI_1 is the utility improvement from considering ambiguity aversion, and we find that it increases in β_S and β_V. Intuitively, when the investor is more uncertain about the reference model, considering ambiguity aversion may deliver greater utility improvements. Furthermore, the ambiguity aversions w.r.t. stock and volatility have different effects on the degree of utility improvement. UI_2 is the utility improvement from allowing derivative trading. The effects of β_S and β_V on UI_2 are different from those on UI_1, which shows that when the investor has no access to the derivative, the effects of β_S and β_V on UI_2 are much less obvious than those on UI_1, and even in the absence of ambiguity aversion ($\beta_S = \beta_V = 0$), there is still a high degree of utility improvement for the investor. From this, we reiterate that it is suboptimal to exclude the derivative. The derivative improves the investment efficiency and helps the investor to pursue good investment performance.

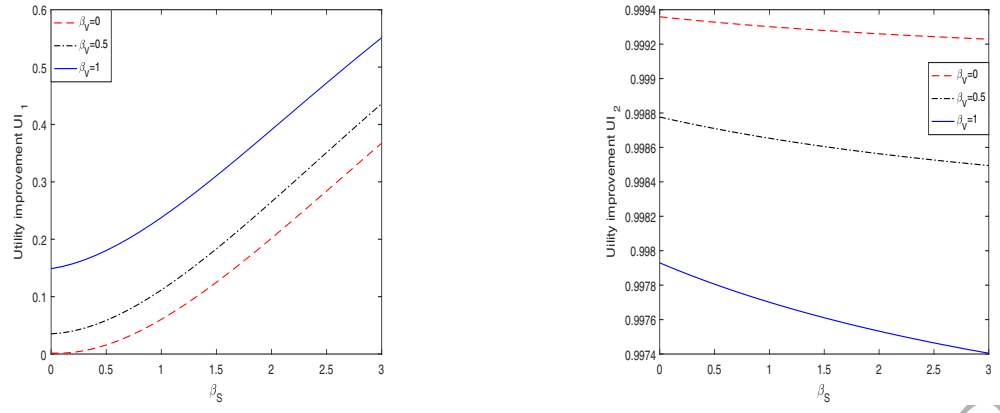

Figure 9: Effects of β_S and β_V on utility improvements.

Figure 10 shows the effects of the mean-reversion rate κ and volatility coefficient σ_V on utility improvements. In the stock return variance process, a larger mean-reversion rate κ and smaller volatility σ_V indicate less uncertainty in the variance process. That is, the investor faces low volatility risk. We find that both types of utility improvements decrease in κ and increase in σ_V. Furthermore, in both two cases, when the investor faces lower volatility risk, her utility improvement is smaller.[18]

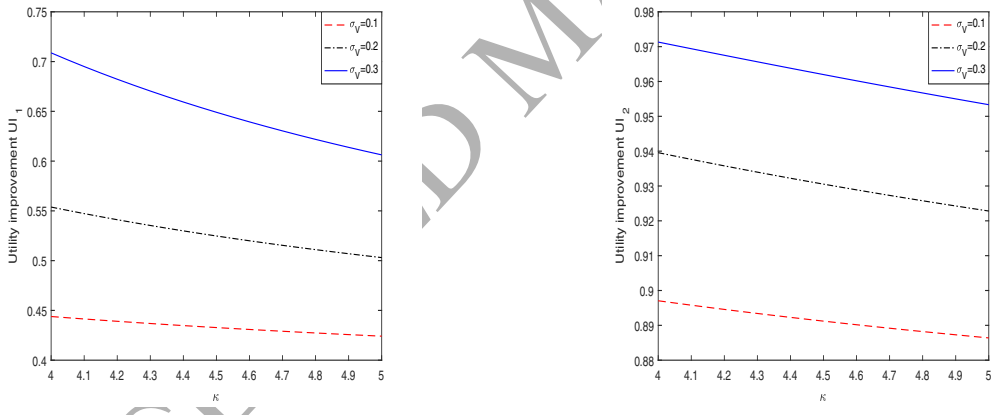

Figure 10: Effects of κ and σ_V on utility improvements.

Figure 11 shows the effects of the time horizon T and correlation $\rho_V \in (-1, 1)$ on utility improvements. The figure shows that the utility improvements UI_1 and UI_2 increase in the time horizon T. It is therefore necessary to incorporate ambiguity aversion and derivative trading in a DC pension plan over a long investment period. The case of the correlation ρ_V is interesting. Due to the specific parametrization of the model, the utility improvements

[18]This is because there is ambiguity aversion toward the volatility risk and the derivative investment opportunity exists; as a result, when the volatility risk is low, the investor's optimal behavior will lead to less utility improvement than in the case where volatility risk is high.

(both UI_1 and UI_2) first increase and then decrease in the correlation ρ_V. Note that when $\rho_V \to \pm 1$, two risky assets are almost fully correlated; then, the role of the derivative is weakened when utility improvements are relatively small.

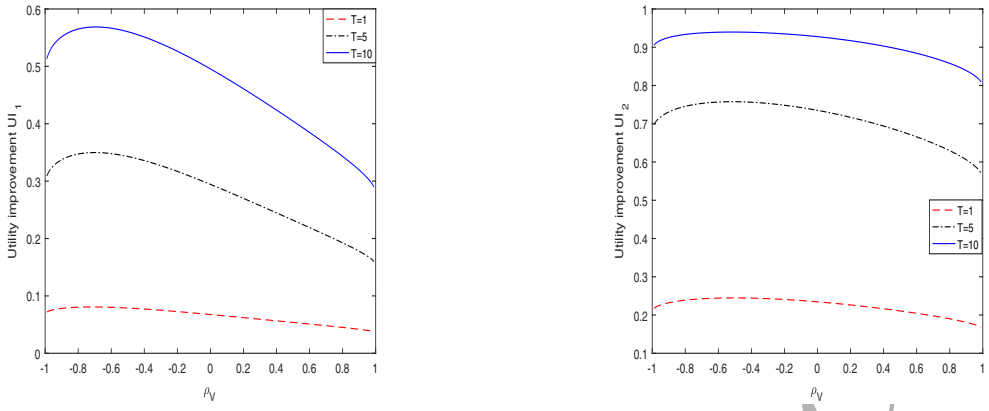

Figure 11: Effects of ρ_V and T on utility improvements.

6. Conclusion

In this paper, we consider a robust optimal investment problem for a DC pension investor facing a stochastic salary. The stock price exhibits stochastic volatility, and the investor has different levels of uncertainty regarding the diffusion component of the stock and its volatility. To cope with volatility risk, she is able to invest her wealth in a derivative. We first solve an optimal investment problem with both ambiguity aversion and a derivative in closed-form and provide verification theorems to guarantee the validity of the solution. Next, we obtain the solutions without the derivative, ambiguity, or salary for some interesting special cases. We also discuss the utility improvements for an investor who considers ambiguity aversion or has access to the derivative. Finally, we explore several detailed conclusions in numerical examples.

We find that three factors play significant roles in the optimal investment strategy in the DC pension plan. The first factor is ambiguity aversion. When an investor experiences uncertainty concerning her reference model, she usually reduces her exposures to the market return risk and volatility risk, because in an uncertain environment, it is optimal to adopt a conservative strategy. Moreover, the investor adopts different investment strategies for the stock and the derivative and there are distinct effects of ambiguity on the stock and derivative investments. The second factor is the derivative. Derivatives have the convenient properties of providing frequent trading opportunities and improving market efficiency. Investment in derivatives may deliver a large utility improvement. The third factor is salary.

In a DC pension plan, the salary and the contribution thereof are essential and generate additional wealth for the investor. More important, the salary has an important effect on her investment strategy, and the investor has a new hedge demand in her portfolio to address salary risk. In the numerical examples, we verify the results and find that different model parameters generate distinct properties and that different degrees of ambiguity aversion lead to complicated cases. It is necessary to determine a more accurate relationship among the key factors; this is an interesting problem left for future research.

Acknowledgments

The authors are very grateful to Lijun Bo, Zhenyu Cui, Xuezhong He, Yanchu Liu, Yang Shen, Pei Wang, Pengyu Wei, Xiaole Xue, Jieming Zhou and Yunfan Zhu for their helpful suggestions. This research was supported by the National Natural Science Foundation of China under Grant (Nos. 71771220, 71721001, 71571195, 11371155, 11326199), For Ying Tung Education Foundation for Young Teachers in the Higher Education Institutions of China (No. 151081), Guangdong Natural Science Foundation for Research Team (No. 2014A030312003), Guangdong Natural Science Funds for Distinguished Young Scholars (No. 2015A030306040), Natural Science Foundation of Guangdong Province of China (No. 2014A030310195) and Insurance Society of China (No. ISCKT2016-N-l-07).

Appendix A.

Proof of market completeness.

In this proof, we show that given a physical measure, there exists a unique risk-neutral measure in our paper. We prove that the financial market in our paper is complete. The proof includes three steps.

First, we present the following result on the existence and uniqueness of risk-neutral equivalent martingale measure $\hat{\mathbb{P}}$ in the extended financial market (S_0, S, O) which is given by (1), (2) and (4).

Theorem A.1. Let $\mathcal{N} \triangleq \{(t,s,v) : s>0,\ v>0,\ 0 \le t < T\}$. Suppose that the following PDE admits a classical solution $O \in C^{2,1}(\mathcal{N} \cap C(\mathcal{N} \cup \{t=T\})$,

$$\begin{cases} -\partial_t O - \mathcal{L}^{\lambda_2} O = 0 \ \text{ in } \ \mathcal{N}, \\ O(T,s,v) = f(s,v), \ \ \forall\, (s,v) \in (0,+\infty)^2, \end{cases} \tag{50}$$

where the differential operator $\mathcal{L}^{\lambda_2}$ takes the following form

$$\mathcal{L}^{\lambda_2}O \triangleq \frac{vs^2}{2}\partial_{ss}O + \frac{\sigma_V^2 v}{2}\partial_{vv}O + \rho_V\sigma_V vs\partial_{sv}O + rs\partial_s O + \big[\kappa(\delta - v) - \sigma_V\rho_V\lambda_1 v - \sigma_V\lambda_2 v\sqrt{1-\rho_V^2}\big]\partial_v O - rO. \tag{51}$$

Here, λ_2 is a constant capturing the market price of stochastic volatility risk $W_V(t)$ and satisfies that

$$\mathbb{E}\left[\exp\left\{\frac{\lambda_1^2+\lambda_2^2}{2}\int_0^T V(t)\mathrm{d}t\right\}\right] < +\infty. \tag{52}$$

If there exists a derivative O with terminal payoff $f(S(T), V(T))$ in the financial market, whose price function is given by $O(t, S(t), V(t))$, then there exists a risk-neutral equivalent martingale measure $\hat{\mathbb{P}}$ in the financial market (S_0, S, O). More precisely, it admits the form given by

$$d\hat{\mathbb{P}} \triangleq \exp\left\{-\int_0^T \lambda_1\sqrt{V(t)}\mathrm{d}W_S(t) - \int_0^T \lambda_2\sqrt{V(t)}\mathrm{d}W_V(t) - \frac{\lambda_1^2+\lambda_2^2}{2}\int_0^T V(t)\mathrm{d}t\right\}\mathrm{d}\mathbb{P}. \tag{53}$$

In other words, for the value process X of the portfolio (S_0, S, O) satisfying that $\{X(\tau)\}$ is uniformly integrable under the measure $\hat{\mathbb{P}}$ for all $\mathbb{F}$-stopping times, $\{e^{-rt}X(t)\}_{t=0}^T$ is an $\mathbb{F}$-martingale under the measure $\hat{\mathbb{P}}$. Moreover, if $\partial_V O \neq 0$ in $\mathcal{N}$, then the risk-neutral equivalent martingale measure $\hat{\mathbb{P}}$ is unique.

Proof. Let Δ^{S_0}, Δ^S and Δ^O be the shares invested in the riskless bond, stock and the derivative, respectively, which are $\mathbb{F}$-adapted processes. Then the portfolio value process X is given by

$$X = \Delta^{S_0}S_0 + \Delta^S S + \Delta^O O.$$

Using the self-financing trading strategy, it follows from (7) and (8) that

$$\begin{aligned} dX(t) &= \big(X(t) - \Delta^S(t)S(t) - \Delta^O(t)O(t)\big)\,\mathrm{d}S_0(t) + \Delta^S(t)\mathrm{d}S(t) + \Delta^O(t)\mathrm{d}O(t) \\ &= \mathcal{X}^1(t)\mathrm{d}t + \mathcal{X}^2(t)\mathrm{d}W_S(t) + \mathcal{X}^3(t)\mathrm{d}W_V(t) \\ &= rX(t)\mathrm{d}t + \mathcal{X}^2(t)\mathrm{d}\widehat{W}_S(t) + \mathcal{X}^3(t)\mathrm{d}\widehat{W}_V(t), \end{aligned} \tag{54}$$

where we used Itô's formula in the second equality and PDE (50) in the third equality, and

$$\begin{aligned}\mathcal{X}^1(t) =& r\left(X(t) - \Delta^S(t)S(t) - \Delta^O(t)O(t)\right) + \Delta^S(t)S(t)\left(r + \lambda_1 V(t)\right)\\ &+\Delta^O(t)\Big[\partial_t O(\cdot) + \frac{1}{2}V(t)(S(t))^2\partial_{ss}O(\cdot) + \frac{1}{2}\sigma_V^2 V(t)\partial_{vv}O(\cdot)\\ &+\rho_V\sigma_V V(t)S(t)\partial_{sv}O(\cdot) + \left(r + \lambda_1 V(t)\right)S(t)\partial_s O(\cdot) + \kappa(\delta - V(t))\partial_v O(\cdot)\Big],\\ \mathcal{X}^2(t) =& \left[\Delta^S(t)S(t) + \Delta^O(t)\left(\partial_s O(\cdot)S(t) + \rho_V\sigma_V\partial_v O(\cdot)\right)\right]\sqrt{V(t)}, \qquad (55)\\ \mathcal{X}^3(t) =& \sqrt{1-\rho_V^2}\sigma_V\Delta^O(t)\partial_v O(\cdot)\sqrt{V(t)}, \qquad (56)\\ \mathrm{d}\widehat{W}_S(t) =& \mathrm{d}W_S(t) + \lambda_1\sqrt{V(t)}\,\mathrm{d}t, \quad \mathrm{d}\widehat{W}_V(t) = \mathrm{d}W_V(t) + \lambda_2\sqrt{V(t)}\,\mathrm{d}t, \qquad (57)\end{aligned}$$

and $(\cdot)$ in $\mathcal{X}^1(t), \mathcal{X}^2(t), \mathcal{X}^3(t)$ represents $(t, S(t), V(t))$. Recalling (52), we have that the density process satisfies the so-called Novikov's condition (see Karatzas and Shreve (1991)), and $\hat{\mathbb{P}}$ is an equivalent martingale measure of $\mathbb{P}$, and $(\widehat{W}_S, \widehat{W}_V)$ is a Brownian motion under the measure $\hat{\mathbb{P}}$. From (54), we deduce that for any $0 \leq t < s \leq T$ and $\mathbb{F}-$stopping time $\tau \in [\,t, s\,]$,

$$e^{-r\tau}X(\tau) = e^{-rt}X(t) + \int_t^\tau e^{-ru}\mathcal{X}^2(u)\mathrm{d}\widehat{W}_S(u) + \int_t^\tau e^{-ru}\mathcal{X}^3(u)\mathrm{d}\widehat{W}_V(u).$$

Let $\tau = \tau_n = \inf\{u \in [\,t, s\,] : |\mathcal{X}^2(u)| + |\mathcal{X}^3(u)| \geq n\}$ for any $n = 1, 2, \cdots$, and take the conditional expectation with respect to $\mathcal{F}_t$ under measure $\hat{\mathbb{P}}$ in the above equality, we have

$$\mathbb{E}^{\hat{\mathbb{P}}}\left[e^{-r\tau_n}X(\tau_n)\Big|\mathcal{F}_t\right] = e^{-rt}X(t).$$

Let $n \to +\infty$, it follows from $\{e^{-r\tau}X(\tau)\}$ being uniformly integrable that $\{e^{-rt}X(t)\}_{t=0}^T$ is an $\mathbb{F}$-martingale under the measure $\hat{\mathbb{P}}$.

If $\partial_v O \neq 0$ a.e. in $\mathcal{N}$, then the second equality in (54) implies that $\{e^{-rt}X(t)\}_{t=0}^T$ is an $\mathbb{F}$-local martingale under $\hat{\mathbb{P}}$ for any portfolio satisfying the assumption in this theorem only if $(\widehat{W}_S, \widehat{W}_V)$ takes the form (57). Thus, the risk-neutral equivalent martingale measure $\hat{\mathbb{P}}$ in the financial market (S_0, S, O) is unique. □

Next, we present a result regarding the existence and uniqueness of the classical solution of PDE (50).

Lemma A.2. Suppose that $|\rho_V| < 1$ and the coefficients r, λ_1, λ_2 are bounded and satisfy $\kappa \geq \sigma_V\rho_V\lambda_1 + \sigma_V\sqrt{1-\rho_V^2}\lambda_2$. Moreover, assume that the terminal payoff function f is a continuous function, and that there exist a positive constant C and a nonnegative constant k such that

$$|f(s,v)| \leq C\left(1 + s + e^{k\sqrt{v+1}}\right), \ \forall\ (s,v) \in (0,+\infty)^2. \qquad (58)$$

Then PDE (50) has a unique classical solution.

Proof. Define $\mathcal{N}_n \triangleq \{(t,s,v) : 1/n < s < n,\ 1/n < v < n,\ 0 \le t < T\}$, and we use the following PDEs in bounded domain $\mathcal{N}_n$ with the uniformly parabolic differential operator to approximate PDE (50) in unbounded domain $\mathcal{N}$ with the degenerate parabolic differential operator given by

$$\begin{cases} -\partial_t O_n - \mathcal{L}^{\lambda_2} O_n = 0 \ \text{ in } \ \mathcal{N}_n, \\ O_n(t,s,v) = f_n(s,v) \text{ on } \partial_p \mathcal{N}_n, \end{cases} \tag{59}$$

where $n \in \mathbb{N}$, $\partial_p \mathcal{N}_n$ is the parabolic boundary of $\mathcal{N}_n$, and $\{f_n\}$ is a smooth function sequence such that f_n converges to f in $C([\,1/m, m\,]^2)$ for any $m \in \mathbb{N}$, and satisfies

$$|f_n(t,s,v)| \le C\left(2 + s + e^{k\sqrt{v+1}}\right).$$

Since $|\rho_V| < 1$, the differential operator $\mathcal{L}^{\lambda_2}$ satisfies the uniformly elliptic condition and the coefficient functions and the terminal function are smooth, the theory for PDEs implies that there exists a classical solution $O_n \in C^{2+\alpha,1+\alpha/2}(\mathcal{N}_n)$ for PDE (59) (see Lieberman (1996)).

Next, we establish a uniform estimate on the maximum of the solutions $|O_n|$. Denote

$$\overline{O} = Ce^{K(T-t)}\left(2 + s + e^{k\sqrt{V+1}}\right),$$

where C is the constant in (58) and K is a positive constant defined later. Then we have

$$\begin{aligned} & -\partial_t \overline{O} - \mathcal{L}^{\lambda_2}\overline{O} \\ \ge\ & Ce^{K(T-t)}\left[K\left(2 + s + e^{k\sqrt{v+1}}\right) - \frac{\sigma_V^2 v}{2} e^{k\sqrt{v+1}} \frac{k^2}{v+1} - rs - \kappa\delta e^{k\sqrt{v+1}} \frac{k}{2\sqrt{v+1}}\right. \\ & \left. + r\left(2 + s + e^{k\sqrt{V+1}}\right)\right] \ge 0, \end{aligned}$$

provided that K is large enough, where we have used the fact that $\kappa \ge \sigma_V \rho_V \lambda_1 + \sigma_V\sqrt{1-\rho_V^2}\lambda_2$ in the first inequality . Moreover, it is clear that $\overline{O} \ge f_n = O_n$ on $\partial_p \mathcal{N}_n$. Thus, the comparison principle (see Lieberman (1996)) implies that $O_n \le \overline{O}$ in $\overline{\mathcal{N}_n}$, which implies that $\{O_n\}$ has a uniform upper bound $\overline{O}$. Repeating the same argument, we can derive that $\{O_n\}$ has a uniform lower bound $-\overline{O}$.

Thus far, we have shown that $|O_n| \le \overline{O}$, which is bounded in any bounded domain $\mathcal{N}_m$. Applying the Hölder interior estimate, we deduce that there exists a function O defined in $\mathcal{N}_m$ such that O_n converges to O in $C^{2,1}(\mathcal{N}_m)$. By the standard method for Cauchy problem, we can deduce that O can be uniquely extended in $\mathcal{N}$, and $O \in C^{2,1}(\mathcal{N}) \cap C(\mathcal{N} \cup \{t = T\})$ is the unique classical solution of PDE (50). □

Finally, we show the completeness result of the financial market (S_0, S, O).

Theorem A.3. Suppose that $|\rho_V| < 1$ and r, λ_1, λ_2 are bounded and satisfy $\kappa \geq \sigma_V \rho_V \lambda_1 + \sigma_V \sqrt{1-\rho_V^2}\lambda_2$, and the deterministic continuous function $f(s,v)$ satisfies (58). Assume that there exists a derivative O with terminal payoff $f(S(T), V(T))$ in financial market, whose price function is $O(t, S(t), V(t))$, where O is the unique classical solution of PDE (50). If $\partial_v O \neq 0$ in $\mathcal{N}$, then the financial market (S_0, S, O) is complete. More precisely, for any contingent claim ξ satisfying that ξ is measurable with respect to $\mathcal{F}_T$ and $\mathbb{E}^{\hat{\mathbb{P}}}(|\xi|) < +\infty$, there exists a unique portfolio $(\Delta^{S_0}, \Delta^S, \Delta^O)$ to replicate ξ, where the measure $\hat{\mathbb{P}}$ is defined in (53). Moreover, the price process of the contingent claim is given as

$$O(t) = \mathbb{E}^{\hat{\mathbb{P}}}\left[e^{rt-rT}\xi|\mathcal{F}_t\right] = \frac{1}{\pi(t)}\mathbb{E}^{\mathbb{P}}\left[\pi(T)\xi|\mathcal{F}_t\right], \tag{60}$$

where π is the pricing kernel satisfing the following SDE:

$$\frac{d\pi(t)}{\pi(t)} = -r\mathrm{d}t - \lambda_1\sqrt{V(t)}\mathrm{d}W_S(t) - \lambda_2\sqrt{V(t)}\mathrm{d}W_V(t). \tag{61}$$

Proof. From Theorem A.1 and Lemma A.2, we know that there exists a unique risk-neutral equivalent martingale measure $\hat{\mathbb{P}}$ in the financial market (S_0, S, O). If ξ is measurable with respect to $\mathcal{F}_T$ and $\mathbb{E}^{\hat{\mathbb{P}}}(|\xi|) < +\infty$, then $\mathbb{E}^{\hat{\mathbb{P}}}\left(e^{-rT}\xi|\mathcal{F}_\cdot\right)$ is a martingale under the unique risk-neutral equivalent martingale measure $\hat{\mathbb{P}}$. Using the martingale representation theorem, it follows that there exists an $\mathbb{F}$-adapted stochastic process (ζ^1, ζ^2) such that

$$\mathbb{E}^{\hat{\mathbb{P}}}\left(e^{-rT}\xi\middle|\mathcal{F}_t\right) = \int_0^t \zeta^1(u)\mathrm{d}\widehat{W}_S(u) + \int_0^t \zeta^2(u)\mathrm{d}\widehat{W}_V(u).$$

Let $\mathcal{X}^2(t) = \zeta^1(t)e^{rt}$ and $\mathcal{X}^3(t) = \zeta^2(t)e^{rt}$. Since

$$\sqrt{1-\rho_V^2}\sigma_V\partial_v O(t, S(t), V(t))\sqrt{V(t)} \neq 0, \quad S(t)\sqrt{V(t)} \neq 0 \text{ a.s. in } \Omega,$$

we can solve Δ^S and Δ^O as

$$\begin{aligned}
\Delta^O(t) &= \frac{\zeta^2(t)e^{rt}}{\sqrt{1-\rho_V^2}\sigma_V\partial_v O(t, S(t), V(t))\sqrt{V(t)}}, \\
\Delta^S(t) &= \frac{\zeta^1(t)e^{rt} - \Delta^O(t)\left(\partial_s O(t, S(t), V(t))S(t) + \rho_V\sigma_V\partial_v O(t, S(t), V(t))\right)\sqrt{V(t)}}{S(t)\sqrt{V(t)}}.
\end{aligned}$$

Let

$$X(t) = \mathbb{E}^{\hat{\mathbb{P}}}\left(e^{r(t-T)}\xi\middle|\mathcal{F}_t\right), \qquad \Delta^{S_0}(t) = X(t) - \Delta^S(t)S(t) - \Delta^O(t)O(t),$$

then (54) still holds. Thus, $(\Delta^{S_0}, \Delta^S, \Delta^O)$ is self-financing, and $X(T) = \xi$, which implies that ξ can be replicated by the portfolio $X = (\Delta^{S_0}, \Delta^S, \Delta^O)$. The pricing formula (60) follows the expression of X, and (61) can be deduced from (54). □

Appendix B.

Proof of Proposition 2.3. We will use the contraction mapping principle (see Theorem 5.1 in Gilbarg and Trudinger, 2001) to prove the conclusion. (If the mapping $\mathcal{T}$ from Banach space $\mathcal{B}$ onto itself satisfies that there exists a constant $\theta < 1$ such that $\|\mathcal{T}J_1 - \mathcal{T}J_2\| \leq \theta\|J_1 - J_2\|$ for all $J_1, J_2 \in \mathcal{B}$, then, there exists a unique solution $J \in \mathcal{B}$ such that $\mathcal{T}J = J$.)

Restrict the initial state (x, v, l) in a compact set $A \subset \mathbb{R}^3$, choose a small enough positive constant δ, defined below, and let $\mathcal{B} = L^\infty(B)$ with $B = [\,T-\delta, T\,] \times A$, where $L^\infty(B)$ is the space of Borel-measurable functions with norm esssup$\{|J(t,x,v,l)| : (t,x,v,l) \in B\}$. Next, we first consider the optimal control problem on the time interval $[\,T-\delta, T\,]$. Fix a function $J \in \mathcal{B}$; then, we denote

$$\Psi_S^J(s,x,v,l) = \frac{\beta_S}{(1-\gamma)J(s,x,v,l)}, \qquad \Psi_V^J(s,x,v,l) = \frac{\beta_V}{(1-\gamma)J(s,x,v,l)},$$

and

$$H^{\mathbf{u};J}(t,x,v,l) = \inf_{\Phi\in\Theta} \mathrm{E}^{\Phi}_{t,x,v,l}\left[U(X^{\Phi,\mathbf{u}}(T)) + \int_t^T \left(\frac{(\phi_S(s))^2}{2\Psi_S^J(s,x,v,l)} + \frac{(\phi_V(s))^2}{2\Psi_V^J(s,x,v,l)} \right) \mathrm{d}s \right]$$

subject to (13), (8) and (10).

Consider the optimal control problem

$$H^J(t,x,v,l) = \sup_{\mathbf{u}\in\Pi} H^{\mathbf{u};J}(t,x,v,l), \;\; \forall \;\; (t,x,v,l) \in B.$$

It is clear that there exists a unique value function $H^J \in \mathcal{B}$ (see Yong and Zhou, 1999) for the above optimal control problem. Thus, we define a mapping $\mathcal{T} : J \to H^J$ from $\mathcal{B}$ onto itself. Suppose that J_1, J_2 are two functions in $\mathcal{B}$; then, we compute that for any $\Phi \in \Theta$, $\mathbf{u} \in \Pi$,

$$\begin{aligned}
&\|\mathcal{T}(J_1) - \mathcal{T}(J_2)\|_{\mathcal{B}} = \sup_{(t,x,v,l)\in B} |H^{J_1}(t,x,v,l) - H^{J_2}(t,x,v,l)| \\
\leq &\sup_{\mathbf{u}\in\Pi,\Phi\in\Theta,(t,x,v,l)\in B} \left| \mathrm{E}^{\Phi}_{t,x,v,l}\left[\int_t^T \left(\frac{(\phi_S(s))^2}{2\Psi_S^{J_1}(s,x,v,l)} + \frac{(\phi_V(s))^2}{2\Psi_V^{J_1}(s,x,v,l)} \right.\right.\right. \\
&\left.\left.\left. - \frac{(\phi_S(s))^2}{2\Psi_S^{J_2}(s,x,v,l)} - \frac{(\phi_V(s))^2}{2\Psi_V^{J_2}(s,x,v,l)} \right) \mathrm{d}s \right] \right| \\
\leq \;& \frac{1-\gamma}{2} \sup_{\Phi\in\Theta,(t,x,v,l)\in B} \mathrm{E}^{\Phi}\left[\int_{T-\delta}^T |(J_1 - J_2)(s,x,v,l)| \left(\frac{(\phi_S(s))^2}{\beta_S} + \frac{(\phi_V(s))^2}{\beta_V} \right) \mathrm{d}s \right] \\
\leq \;& \frac{(1-\gamma)\|J_1 - J_2\|_{\mathcal{B}}}{2\min\{\beta_S, \beta_V\}} \sup_{\Phi\in\Theta} \mathrm{E}^{\Phi}\left[\int_{T-\delta}^T ||\phi(s)||^2 \, \mathrm{d}s \right]. \qquad (62)
\end{aligned}$$

It is not difficult to compute that

$$\begin{aligned}
&\sup_{\Phi\in\Theta}\mathrm{E}^{\Phi}\left[\int_{T-\delta}^{T}||\phi(s)||^2\,\mathrm{d}s\right]=\sup_{\Phi\in\Theta}\mathrm{E}\left[\Lambda^{\Phi}(T)\int_{T-\delta}^{T}||\phi(s)||^2\,\mathrm{d}s\right]\\
=&\sup_{\Phi\in\Theta}\mathrm{E}\left[\left[\Lambda^{\kappa\Phi/\underline{\kappa}\sigma_V}(T)\right]^{\underline{\kappa}\sigma_V/\kappa}\exp\left\{\frac{\kappa-\underline{\kappa}\sigma_V}{2\underline{\kappa}\sigma_V}\int_0^T||\phi(s)||^2\,\mathrm{d}s\right\}\int_{T-\delta}^{T}||\phi(s)||^2\,\mathrm{d}s\right]\\
\leq&\sup_{\Phi\in\Theta}\underline{\kappa}^2\mathrm{E}\left[\left[\Lambda^{\kappa\Phi/\underline{\kappa}\sigma_V}(T)\right]^{\underline{\kappa}\sigma_V/\kappa}\exp\left\{\frac{\underline{\kappa}(\kappa-\underline{\kappa}\sigma_V)}{2\sigma_V}\int_0^T V(s)\,\mathrm{d}s\right\}\int_{T-\delta}^{T}V(s)\,\mathrm{d}s\right]\\
\leq&\sup_{\Phi\in\Theta}\underline{\kappa}^2\,\mathrm{E}\left[\Lambda^{\kappa\Phi/\underline{\kappa}\sigma_V}(T)\right]^{\underline{\kappa}\sigma_V/\kappa}\,\mathrm{E}\left[\exp\left\{\frac{\kappa^2}{2\sigma_V^2}\int_0^T V(s)\,\mathrm{d}s\right\}\right]^{\underline{\kappa}\sigma_V(\kappa-\underline{\kappa}\sigma_V)/\kappa^2}\\
&\mathrm{E}\left[\int_{T-\delta}^{T}(V(s))^{\kappa^2/(\kappa-\underline{\kappa}\sigma_V)^2}\,\mathrm{d}s\right]^{(\kappa-\underline{\kappa}\sigma_V)^2/\kappa^2}\delta^{1-(\kappa-\underline{\kappa}\sigma_V)^2/\kappa^2},
\end{aligned}\tag{63}$$

where we use Assumption (iii) in footnote 6 in the first equality and Holder's inequality in the second inequality.

From Theorem 5.1 in Taksar and Zeng (2009), we conclude that

$$\mathrm{E}\left[\exp\left(\frac{\kappa^2}{2\underline{\kappa}^2\sigma_V^2}\int_0^T||\phi(s)||^2\mathrm{d}s\right)\right]\leq\mathrm{E}\left[\exp\left(\frac{\kappa^2}{2\sigma_V^2}\int_0^T V(s)\mathrm{d}s\right)\right]<\infty,$$

and $\Lambda^{\kappa\Phi/\underline{\kappa}\sigma_V}$ is an exponential martingale. Moreover, the regularity result for SDE implies that

$$\mathrm{E}\left[\int_0^T(V(s))^{\kappa^2/(\kappa-\underline{\kappa}\sigma_V)^2}\,\mathrm{ds}\right]<+\infty.$$

Thus, combining (62) and (63), we can choose a small enough $\delta>0$ such that

$$\sup_{\Phi\in\Theta}\mathrm{E}^{\Phi}\left[\int_{T-\delta}^{T}||\phi(s)||^2\,\mathrm{d}s\right]\leq\frac{\min\{\beta_S,\beta_V\}}{1-\gamma},$$

and

$$\|\mathcal{T}(J_1)-\mathcal{T}(J_2)\|_{\mathcal{B}}\leq\frac{1}{2}\|J_1-J_2\|_{\mathcal{B}}.$$

Hence, the mapping $\mathcal{T}$ is a contraction mapping. According to the contraction mapping principle, the mapping $\mathcal{T}$ has a unique fixed point. This means that there exists a unique value function $H(t,x,v,l)$ of the optimal control problem if $t\in[\,T-\delta,T\,]$ and $(x,v,l)\in A$, which consists of (16), (17) and (18) subject to (13), (8) and (10).

Next, we extend the result into the total time interval $[\,0,T\,]$. Suppose that we have proven that there exists a unique value function $H(t,x,v,l)$ of the optimal control problem if $t\in[\,\widehat{T},T\,]$ and $(x,v,l)\in A$.

Then, we choose a small enough positive number δ such that

$$\sup_{\Phi\in\Theta}\mathrm{E}^{\Phi}\left[\int_{\widehat{T}-\delta}^{\widehat{T}}||\phi(s)||^2\,\mathrm{d}s\right]=\frac{\min\{\beta_S,\beta_V\}}{1-\gamma}.$$

Moreover, let $\mathcal{B} = L^\infty(B)$ with $B = [\,\widehat{T}-\delta, \widehat{T}\,] \times A$. Fix a function $J \in \mathcal{B}$; then, we denote

$$\Psi_S^J(s,x,v,l) = \frac{\beta_S}{(1-\gamma)J(s,x,v,l)I_{\{s\in[\,\widehat{T}-\delta,\widehat{T}\,]\}} + (1-\gamma)H(s,x,v,l)I_{\{s\in(\widehat{T},T\,]\}}},$$
$$\Psi_V^J(s,x,v,l) = \frac{\beta_V}{(1-\gamma)J(s,x,v,l)I_{\{s\in[\,\widehat{T}-\delta,\widehat{T}\,]\}} + (1-\gamma)H(s,x,v,l)I_{\{s\in(\widehat{T},T\,]\}}},$$

and

$$H^{\mathbf{u};J}(t,x,v,l) = \inf_{\Phi\in\Theta} \mathrm{E}^{\Phi}_{t,x,v,l}\left[U(X^{\Phi,\mathbf{u}}(T)) + \int_t^T \left(\frac{(\phi_S(s))^2}{2\Psi_S^J(s,x,v,l)} + \frac{(\phi_V(s))^2}{2\Psi_V^J(s,x,v,l)}\right) \mathrm{d}s\right]$$

subject to (13), (8) and (10).

Consider the optimal control problem

$$H^J(t,x,v,l) = \sup_{\mathbf{u}\in\Pi} H^{\mathbf{u};J}(t,x,v,l), \quad \forall \ (t,x,v,l) \in B.$$

Repeating the same argument as above, we can prove that there exists a unique value function $H(t,x,v,l)$ of the optimal control problem if $(t,x,v,l) \in B$. Repeating the same argument in the domain $[\,T-2\delta, T-\delta\,] \times A$, $[\,T-3\delta, T-2\delta\,] \times A, \cdots$, we can prove that there exists a unique value function $H(t,x,v,l)$ of the optimal control problem if $(t,x,v,l) \in [\,0,T\,] \times A$. Since the set A is arbitrary, and the compatibility in different compact sets is obvious, then we have proven that there exists a unique value function $H(t,x,v,l)$ of the optimal control problem for any $(t,x,v,l) \in [\,0,T\,] \times \mathbb{R}^3$. □

Appendix C.

Proof of Proposition 3.1. We know that $\Psi_S(t,x,v,l)$, $\Psi_V(t,x,v,l)$ in Proposition 2.3 are $\Psi_S^J(t,x,v,l)$, $\Psi_V^J(t,x,v,l)$, respectively. Consider the optimal control problem

$$H^J(t,x,v,l) = \sup_{\mathbf{u}\in\Pi}\inf_{\Phi\in\Theta} \mathrm{E}^{\Phi}_{t,x,v,l}\left[U(X^{\Phi,\mathbf{u}}(T)) + \int_t^T g(s,x,v,l,\phi_S,\phi_V)\,ds\right]$$

subject to (13), (8) and (10) for any $(t,x,v,l) \in O$, where

$$g(s,x,v,l,\phi_S,\phi_V) = \frac{\phi_S^2}{2\Psi_S^J(s,x,v,l)} + \frac{\phi_V^2}{2\Psi_V^J(s,x,v,l)}.$$

Note that in this optimal control problem, J in Ψ_S^J and Ψ_V^J is the function given in the assumptions rather than the value function. Thus, g is a given function w.r.t. (s,x,v,l,ϕ), independent of the value function H^J, and the optimal control problem is standard.

Repeating a proof similar to that in Theorem 3.2 in Mataramvura and Øksendal (2008), we deduce that J is the value function of the above optimal control problem. Since the

value function and J in Ψ_S^J and Ψ_V^J are the same, J is the value function of the optimal control problem, consisting of (16), (17) and (18) subject to (13), (8) and (10). Thus, by Proposition 2.3, the uniqueness of the value function implies that $H(t,x,v,l) = J(t,x,v,l)$ for any $(t,x,v,l) \in O$, and $(\mathbf{u}^*, \Phi^*)$ is an optimal control. □

Appendix D.

Proof of Theorem 3.2. According to the first-order optimality conditions, the functions ϕ_S^* and ϕ_V^*, which realize the infimum part of Eq. (19), are given by

$$\begin{aligned}
\phi_S^* &= \frac{\beta_S\sqrt{v}}{(1-\gamma)J}\left[x\theta_S J_x + \sigma_V\rho J_v + l\sigma_L\rho_L J_l\right], \\
\phi_V^* &= \frac{\beta_V\sqrt{v}}{(1-\gamma)J}\left[x\theta_V J_x + \sigma_V\sqrt{1-\rho_L^2}J_v + l\sigma_L\sqrt{1-\rho^2}J_l\right].
\end{aligned} \tag{64}$$

Substituting Eq. (64) into Eq. (19), we have

$$\begin{aligned}
&J_t + (rx + x\theta_S\lambda_1 v + x\theta_V\lambda_2 v + \xi l)J_x + \kappa(\delta - v)J_v + (\mu_L l + l\sigma_L\lambda_1 v\rho_L + l\sigma_L\lambda_2 v\sqrt{1-\rho_L^2})J_l \\
&+\frac{1}{2}x^2 v(\theta_S^2 + \theta_V^2)J_{xx} + \frac{1}{2}\sigma_V^2 v J_{vv} + \frac{1}{2}l^2\sigma_L^2 v J_{ll} + (x\sigma_V\theta_S v\rho_V + x\sigma_V\theta_V v\sqrt{1-\rho_V^2})J_{xv} \\
&+(x\theta_S l\sigma_L v\rho_L + x\theta_V l\sigma_L v\sqrt{1-\rho_L^2})J_{xl} + l\sigma_L v\sigma_V(\rho_V\rho_L + \sqrt{1-\rho_V^2}\sqrt{1-\rho_L^2})J_{lv} \\
&-\frac{\beta_S v}{2(1-\gamma)J}[x\theta_S J_x + \sigma_V\rho_V J_v + l\sigma_L\rho_L J_l]^2 \\
&-\frac{\beta_V v}{2(1-\gamma)J}[x\theta_V J_x + \sigma_V\sqrt{1-\rho_V^2}J_v + l\sigma_L\sqrt{1-\rho_L^2}J_l]^2 = 0.
\end{aligned} \tag{65}$$

Differentiating Eq. (65) w.r.t. (θ_S, θ_V) implies

$$\begin{aligned}
\theta_S^* &= \frac{\lambda_1 J_x - \frac{\beta_S}{(1-\gamma)J}(\sigma_V\rho_V J_x J_v + l\sigma_L\rho_L J_x J_l) + \sigma_V\rho_V J_{xv} + l\sigma_L\rho_L J_{xl}}{x\left[\frac{\beta_S}{(1-\gamma)J}J_x^2 - J_{xx}\right]}, \\
\theta_V^* &= \frac{\lambda_2 J_x - \frac{\beta_V}{(1-\gamma)J}(\sigma_V\sqrt{1-\rho_V^2}J_x J_v + l\sigma_L\sqrt{1-\rho_L^2}J_x J_l) + \sigma_V\sqrt{1-\rho_V^2}J_{xv} + l\sigma_L\sqrt{1-\rho_L^2}J_{xl}}{x\left[\frac{\beta_V}{(1-\gamma)J}J_x^2 - J_{xx}\right]}.
\end{aligned} \tag{66}$$

Plugging Eq. (66) into Eq. (65) implies

$$
\begin{aligned}
&J_t + (rx+\xi l)J_x + \kappa(\delta - v)J_v + (\mu_L l + l\sigma_L\lambda_1 v\rho_L + l\sigma_L\lambda_2 v\sqrt{1-\rho_L^2})J_l + \frac{1}{2}\sigma_V^2 v J_{vv} + \frac{1}{2}l^2\sigma_L^2 v J_{ll} \\
&+l\sigma_L v\sigma_V(\rho_V\rho_L + \sqrt{1-\rho_V^2}\sqrt{1-\rho_L^2})J_{lv} - \frac{\beta_S v}{2(1-\gamma)J}\left(\sigma_V^2\rho_V^2 J_v^2 + l^2\sigma_L^2\rho_L^2 J_l^2 + 2\sigma_V\sigma_L\rho_V\rho_L l J_v J_l\right) \\
&-\frac{\beta_V v}{2(1-\gamma)J}\left(\sigma_V^2(1-\rho_V^2)J_v^2 + l^2\sigma_L^2(1-\rho_L^2)J_l^2 + 2\sigma_V\sigma_L\sqrt{1-\rho_V^2}\sqrt{1-\rho_L^2} l J_v J_l\right) \\
&+\frac{v\left[\lambda_1 J_x - \frac{\beta_S}{(1-\gamma)J}(\sigma_V\rho_V J_x J_v + l\sigma_L\rho_L J_x J_l) + \sigma_V\rho_V J_{xv} + l\sigma_L\rho_L J_{xl}\right]^2}{2\left[\frac{\beta_S}{(1-\gamma)J}J_x^2 - J_{xx}\right]} \\
&+\frac{v[\lambda_2 J_x - \frac{\beta_V}{(1-\gamma)J}(\sigma_V\sqrt{1-\rho_V^2}J_x J_v + l\sigma_L\sqrt{1-\rho_L^2}J_x J_l) + \sigma_V\sqrt{1-\rho_V^2}J_{xv} + l\sigma_L\sqrt{1-\rho_L^2}J_{xl}]^2}{2\left[\frac{\beta_V}{(1-\gamma)J}J_x^2 - J_{xx}\right]} = 0.
\end{aligned}
\tag{67}
$$

To solve Eq. (67), we attempt to conjecture the solution in the following form:

$$
J(t,x,v,l) = \frac{(x+h(t,l))^{1-\gamma}}{1-\gamma}g(t,v), \quad h(T,l) = 0, \quad g(T,v) = 1, \tag{68}
$$

the partial derivatives of which are

$$
\begin{aligned}
&J_t = g_t\frac{(x+h)^{1-\gamma}}{1-\gamma} + g(x+h)^{-\gamma}h_t, \quad J_x = g(x+h)^{-\gamma}, \quad J_{xx} = -\gamma g(x+h)^{-\gamma-1}, \\
&J_v = g_v\frac{(x+h)^{1-\gamma}}{1-\gamma}, \quad J_{vv} = g_{vv}\frac{(x+h)^{1-\gamma}}{1-\gamma}, \quad J_l = g(x+h)^{-\gamma}h_l, \quad J_{lv} = g_v(x+h)^{-\gamma}h_l \\
&J_{ll} = -\gamma g(x+h)^{-\gamma-1}h_l^2 + g(x+h)^{-\gamma}h_{ll}, \quad J_{xv} = g_v(x+h)^{-\gamma}, \quad J_{xl} = -\gamma g(x+h)^{-\gamma-1}h_l.
\end{aligned}
\tag{69}
$$

Substituting Eqs. (68)-(69) into Eq. (67), we have

$$
\begin{aligned}
&g_t\frac{(x+h)^{1-\gamma}}{1-\gamma} + g(x+h)^{-\gamma}h_t + rxg(x+h)^{-\gamma} + \xi l g(x+h)^{-\gamma} + \kappa(\delta-v)g_v\frac{(x+h)^{1-\gamma}}{1-\gamma} \\
&+(\mu_L l + l\sigma_L\lambda_1 v\rho_L + l\sigma_L\lambda_2 v\sqrt{1-\rho_L^2})g(x+h)^{-\gamma}h_l + \frac{1}{2}\sigma_V^2 v g_{vv}\frac{(x+h)^{1-\gamma}}{1-\gamma} \\
&+\frac{1}{2}l^2\sigma_L^2 v[-\gamma g(x+h)^{-\gamma-1}h_l^2 + g(x+h)^{-\gamma}h_{ll}] + l\sigma_L v\sigma_V(\rho_V\rho_L + \sqrt{1-\rho_V^2}\sqrt{1-\rho_L^2})g_v(x+h)^{-\gamma}h_l \\
&-\frac{\beta_S v}{2g}\left[\sigma_V^2\rho_V^2 g_v^2\frac{(x+h)^{1-\gamma}}{(1-\gamma)^2} + l^2\sigma_L^2\rho_L^2 g^2(x+h)^{-\gamma-1}h_l^2 + 2\sigma_L\sigma_V\rho_V\rho_L l g_v\frac{(x+h)^{-\gamma}}{1-\gamma}gh_l\right] \\
&-\frac{\beta_V v}{2g}\Bigg[\sigma_V^2(1-\rho_V^2)g_v^2\frac{(x+h)^{1-\gamma}}{(1-\gamma)^2} + l^2\sigma_L^2(1-\rho_L^2)g^2(x+h)^{-\gamma-1}h_l^2 \\
&+2\sigma_L\sigma_V\sqrt{1-\rho_V^2}\sqrt{1-\rho_L^2}l g_v\frac{(x+h)^{-\gamma}}{1-\gamma}gh_l\Bigg] \\
&+\frac{v\left[\lambda_1 g(x+h)^{-\gamma} + \frac{1-(\beta_S+\gamma)}{1-\gamma}\sigma_V\rho_V g_v(x+h)^{-\gamma} - (\beta_S+\gamma)l\sigma_L\rho_L g(x+h)^{-\gamma-1}h_l\right]^2}{2(\beta_S+\gamma)g(x+h)^{-\gamma-1}} \\
&+\frac{v\left[\lambda_2 g(x+h)^{-\gamma} + \frac{1-(\beta_V+\gamma)}{1-\gamma}\sigma_V\sqrt{1-\rho_V^2}g_v(x+h)^{-\gamma} - (\beta_V+\gamma)l\sigma_L\sqrt{1-\rho_L^2}g(x+h)^{-\gamma-1}h_l\right]^2}{2(\beta_V+\gamma)g(x+h)^{-\gamma-1}} = 0.
\end{aligned}
\tag{70}
$$

Furthermore, let

$$\begin{aligned} &g(t,v)=\mathrm{e}^{\bar{g}(t)v+\hat{g}(t)}, \quad \bar{g}(T)=\hat{g}(T)=0, \\ &h(t,l)=\bar{h}(t)l+\hat{h}(t), \quad \bar{h}(T)=\hat{h}(T)=0. \end{aligned} \tag{71}$$

Then,

$$g_t=g(\bar{g}_t v+\hat{g}_t), \quad g_v=g\bar{g}, \quad g_{vv}=g\bar{g}^2, \quad h_t=\bar{h}_t l+\hat{h}_t, \quad h_l=\bar{h}, \quad h_{ll}=0. \tag{72}$$

Inserting Eqs. (71)-(72) into Eq. (70) implies

$$\begin{aligned} &\frac{x+h}{1-\gamma}\left\{v\left[\bar{g}_t+\left(-\kappa+\frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V\rho}{\beta_S+\gamma}+\frac{\lambda_2(1-(\beta_V+\gamma))\sigma_V\sqrt{1-\rho^2}}{\beta_V+\gamma}\right)\bar{g}\right.\right. \\ &+\left(\frac{\sigma_V^2}{2}-\frac{\beta_S\sigma_V^2\rho^2}{2(1-\gamma)}-\frac{\beta_V\sigma_V^2(1-\rho^2)}{2(1-\gamma)}+\frac{(1-(\beta_S+\gamma))^2\sigma_V^2\rho^2}{2(\beta_S+\gamma)(1-\gamma)}+\frac{(1-(\beta_V+\gamma))^2\sigma_V^2(1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)}\right)\bar{g}^2 \\ &\left.\left.+\frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)}+\frac{\lambda_2^2(1-\gamma)}{2(\beta_V+\gamma)}\right]+\hat{g}_t+r(1-\gamma)+\kappa\delta\bar{g}\right\}+l\left\{\bar{h}_t+(\mu_L-r)\bar{h}+\xi\right\}+\hat{h}_t-r\hat{h}=0. \end{aligned} \tag{73}$$

By separating the variables with and without x, v and l, we can derive the following equations:

$$\begin{aligned} &\bar{g}_t+\left(-\kappa+\frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V\rho}{\beta_S+\gamma}+\frac{\lambda_2(1-(\beta_V+\gamma))\sigma_V\sqrt{1-\rho^2}}{\beta_V+\gamma}\right)\bar{g} \\ &+\left(\frac{\sigma_V^2}{2}-\frac{\beta_S\sigma_V^2\rho^2}{2(1-\gamma)}-\frac{\beta_V\sigma_V^2(1-\rho^2)}{2(1-\gamma)}+\frac{(1-(\beta_S+\gamma))^2\sigma_V^2\rho^2}{2(\beta_S+\gamma)(1-\gamma)}+\frac{(1-(\beta_V+\gamma))^2\sigma_V^2(1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)}\right)\bar{g}^2 \\ &+\frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)}+\frac{\lambda_2^2(1-\gamma)}{2(\beta_V+\gamma)}=0, \\ &\hat{g}_t+r(1-\gamma)+\kappa\delta\bar{g}=0, \\ &\bar{h}_t+(\mu_L-r)\bar{h}+\xi=0, \quad \hat{h}_t-r\hat{h}=0. \end{aligned}$$

Considering the boundary conditions, we have

$$\begin{aligned} &\bar{g}(t)=\frac{\nu_1\nu_2-\nu_1\nu_2\mathrm{e}^{\alpha_2(\nu_1-\nu_2)(T-t)}}{\nu_2-\nu_1\mathrm{e}^{\alpha_2(\nu_1-\nu_2)(T-t)}}, \quad \hat{g}(t)=\int_t^T\left[r(1-\gamma)+\kappa\delta g_1(s)\right]\mathrm{d}s, \\ &\bar{h}(t)=\frac{\xi}{\mu_L-r}(\mathrm{e}^{(\mu_L-r)(T-t)}-1), \quad \hat{h}(t)=0, \end{aligned} \tag{74}$$

where

$$\begin{aligned} &\alpha_1=-\kappa+\frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V\rho}{\beta_S+\gamma}+\frac{\lambda_2(1-(\beta_V+\gamma))\sigma_V\sqrt{1-\rho^2}}{\beta_V+\gamma}, \\ &\alpha_2=\frac{\sigma_V^2}{2}-\frac{\beta_S\sigma_V^2\rho^2}{2(1-\gamma)}-\frac{\beta_V\sigma_V^2(1-\rho^2)}{2(1-\gamma)}+\frac{(1-(\beta_S+\gamma))^2\sigma_V^2\rho^2}{2(\beta_S+\gamma)(1-\gamma)}+\frac{(1-(\beta_V+\gamma))^2\sigma_V^2(1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)}, \\ &\alpha_3=\frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)}+\frac{\lambda_2^2(1-\gamma)}{2(\beta_V+\gamma)}, \\ &\nu_{1,2}=\frac{\alpha_1\pm\sqrt{\alpha_1^2-4\alpha_2\alpha_3}}{-2\alpha_2}. \end{aligned}$$

Substituting $\bar{g}(t)$, $\hat{g}(t)$, $\bar{h}(t)$ and $\hat{h}(t)$ into Eqs. (64) and (66), we can derive $\theta_S^*(t)$, $\theta_V^*(t)$, $\phi_S^*(t)$ and $\phi_V^*(t)$.

As $\beta_S, \beta_V > 0$, $\gamma > 1$, we have $\alpha_2 > 0$ and $\alpha_3 < 0$. By calculations, we obtain

$$\begin{aligned}
\alpha_2 &= \frac{\sigma_V^2}{2} - \frac{\beta_S\sigma_V^2\rho^2}{2(1-\gamma)} - \frac{\beta_V\sigma_V^2(1-\rho^2)}{2(1-\gamma)} + \frac{(1-(\beta_S+\gamma))^2\sigma_V^2\rho^2}{2(\beta_S+\gamma)(1-\gamma)} + \frac{(1-(\beta_V+\gamma))^2\sigma_V^2(1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)},\\
&= \frac{\sigma_V^2}{2} - \frac{\beta_S\sigma_V^2\rho^2}{2(1-\gamma)} - \frac{\beta_V\sigma_V^2(1-\rho^2)}{2(1-\gamma)} + \frac{\sigma_V^2\rho^2}{2(\beta_S+\gamma)(1-\gamma)} - \frac{\sigma_V^2\rho^2}{1-\gamma} + \frac{(\beta_S+\gamma)\sigma_V^2\rho^2}{2(1-\gamma)}\\
&+ \frac{\sigma_V^2(1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)} - \frac{\sigma_V^2(1-\rho^2)}{1-\gamma} + \frac{(\beta_V+\gamma)\sigma_V^2(1-\rho^2)}{2(1-\gamma)},\\
&= \frac{\sigma_V^2}{2} + \frac{\sigma_V^2\rho^2}{2(\beta_S+\gamma)(1-\gamma)} + \frac{\sigma_V^2(1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)} - \frac{\sigma_V^2}{1-\gamma} + \frac{\gamma\sigma_V^2}{2(1-\gamma)}.
\end{aligned}$$

As $\gamma > 1$, we have

$$\frac{\sigma_V^2\rho^2}{2(\beta_S+\gamma)(1-\gamma)} + \frac{\sigma_V^2(1-\rho^2)}{2(\beta_V+\gamma)(1-\gamma)} > \frac{\sigma_V^2\rho^2}{2(1-\gamma)} + \frac{\sigma_V^2(1-\rho^2)}{2(1-\gamma)} = \frac{\sigma_V^2}{2(1-\gamma)}.$$

Therefore,

$$\alpha_2 > \frac{\sigma_V^2}{2} + \frac{\sigma_V^2}{2(1-\gamma)} - \frac{\sigma_V^2}{1-\gamma} + \frac{\gamma\sigma_V^2}{2(1-\gamma)} = 0.$$

Because $\alpha_3 < 0$, $\alpha_1^2 - 4\alpha_2\alpha_3 > 0$. The proof of Theorem 3.2 is completed. □

Appendix E

This appendix mainly provides the proof of Theorem 3.4. Before giving the proof, we present some lemmas, which are used in the proof of Theorem 3.4.

Lemma E.1. $\bar{g}(t)$ given by Eq. (28) is an increasing function of t and $\bar{g}(t) \leq 0$, $\forall t \in [0, T]$.

Proof. The direct calculation shows that

$$\bar{g}_t(t) = \frac{-\nu_1\nu_2(\nu_1-\nu_2)^2\alpha_2 \mathrm{e}^{\alpha_2(\nu_1-\nu_2)(T-t)}}{(\nu_2 - \nu_1 \mathrm{e}^{\alpha_2(\nu_1-\nu_2)(T-t)})^2}.$$

It is obvious that $\nu_2 > 0 > \nu_1$ and $\alpha_2 > 0$, which implies that $\bar{g}_t(t) > 0$, i.e., $\bar{g}(t)$ is an increasing function of t. As $\bar{g}(T) = 0$, then $\bar{g}(t) \leq 0$, $\forall t \in [0, T]$. □

In Theorem 3.2, we have already derived the optimal risk exposure and the optimal investment strategy. However, we should guarantee that the Radon-Nikodym derivative $\Lambda^*(t)$ of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ corresponding to the optimal worst-case scenario drifts $\phi_S^*(t)$ and $\phi_V^*(t)$, i.e., the expression $\Lambda(t)$ with $\phi_S^*(t)$, $\phi_V^*(t)$ instead of $\phi_S(t)$ and $\phi_V(t)$, is indeed a $\mathbb{P}$-martingale, which ensures a well-defined $\mathbb{Q}^*$. The following lemma states sufficient conditions for this scenario based on Novikov's condition and Theorem 5.1 in Taksar and Zeng (2009).

Lemma E.2. Novikov's condition

$$\mathrm{E}\left[\exp\left(\int_0^T\left(\frac{1}{2}(\phi_S^*(s))^2+\frac{1}{2}(\phi_V^*(s))^2\right)\mathrm{d}s\right)\right]<\infty$$

holds for $\phi_S^*(t)$ and $\phi_V^*(t)$ if the parameters satisfy that for $\forall\bar{g}(t)\in[\bar{g}(0),0]$,

$$\frac{\beta_S^2(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}(t))^2}{(1-\gamma)^2(\beta_S+\gamma)^2}+\frac{\beta_V^2(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2}\bar{g}(t))^2}{(1-\gamma)^2(\beta_V+\gamma)^2}<\frac{\kappa^2}{\sigma_V^2}. \tag{75}$$

Proof. From Theorem 3.2, we have

$$\phi_S^*(t)=\frac{\beta_S\sqrt{V(t)}(\lambda_1(1-\gamma)+\sigma_V\rho_V\bar{g}(t))}{(1-\gamma)(\beta_S+\gamma)},\quad \phi_V^*(t)=\frac{\beta_V\sqrt{V(t)}(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho_V^2}\bar{g}(t))}{(1-\gamma)(\beta_V+\gamma)}.$$

Then

$$\begin{aligned}&\mathrm{E}\left[\exp\left(\int_0^T\left(\frac{1}{2}(\phi_S^*(s))^2+\frac{1}{2}(\phi_V^*(s))^2\right)\mathrm{d}s\right)\right]\\&=\mathrm{E}\left[\exp\left(\int_0^T\left(\frac{\beta_S^2(\lambda_1(1-\gamma)+\sigma_V\rho\bar{g}(s))^2}{2(1-\gamma)^2(\beta_S+\gamma)^2}+\frac{\beta_V^2(\lambda_2(1-\gamma)+\sigma_V\sqrt{1-\rho^2}\bar{g}(s))^2}{2(1-\gamma)^2(\beta_V+\gamma)^2}\right)V(t)\mathrm{d}s\right)\right].\end{aligned}$$

With condition (75), we can verify that $\Phi^*:=\{\phi^*(t):=(\phi_S^*(t),\phi_V^*(t))\}_{t\in[0,T]}$ satisfies Novikov's condition as follows.

$$\begin{aligned}&\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T||\phi^*(s)||^2\mathrm{d}s\right)\right]=\mathrm{E}\left[\exp\left(\int_0^T\left(\frac{1}{2}(\phi_S^*(s))^2+\frac{1}{2}(\phi_V^*(s))^2\right)\mathrm{d}s\right)\right]\\&\le\mathrm{E}\left[\exp\left(\frac{\kappa^2}{2\sigma_V^2}\int_0^T V(s)\mathrm{d}s\right)\right]<\infty.\end{aligned}$$

The first estimate follows from condition (75) because of the property of quadratic functions, and the second is from Theorem 5.1 in Taksar and Zeng (2009). □

To verify condition (4) in Proposition 3.1, we present another lemma.

Lemma E.3. For problem (16), if $J(t,x,v,l)$ is the solution to the HJB equation (19) and the parameters satisfy that for $\bar{g}(t)\in[\bar{g}(0),0]$,

$$[64(1-\gamma)^2-4(1-\gamma)][(m(t))^2+(n(t))^2]+8(1-\gamma)A(t)\le\frac{\kappa^2}{2\sigma_V^2}, \tag{76}$$

we have

$$\mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}|J(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))|^4\right]<\infty,$$

and

$$\mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}\left|\frac{(\phi_S^*(t))^2}{2\Psi_S(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))}+\frac{(\phi_V^*(t))^2}{2\Psi_V(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))}\right|^2\right]<\infty,$$

where

$$A(t)=\gamma(m(t))^2-\sigma_V\rho_V\bar{g}(t)m(t)+\gamma(n(t))^2-\sigma_V\sqrt{1-\rho_V^2}\bar{g}(t)n(t), \tag{77}$$

and $m(t)$, $n(t)$ are given by Eqs. (26) and (27).

Proof. Step 1. Proof of $\mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}|J(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))|^4\right]<\infty$.

Substituting Eqs. (22) and (25) into Eq. (13), we have

$$\frac{\mathrm{d}(X^{\Phi^*,\mathbf{u}^*}(t)+\bar{h}(t)L(t))}{X^{\Phi^*,\mathbf{u}^*}+\bar{h}(t)L(t)}=(r+A(t)V(t))\mathrm{d}t+m(t)\sqrt{V(t)}\mathrm{d}W_S^{\Phi^*}(t)+n(t)\sqrt{V(t)}\mathrm{d}W_V^{\Phi^*}(t), \tag{78}$$

where $m(t)$, $n(t)$ and $A(t)$ are given by Eqs. (26), (27) and (77). It is easy to obtain that Eq. (78) has a unique positive solution

$$\begin{aligned}X^{\Phi^*,\mathbf{u}^*}(t)+\bar{h}(t)L(t)= &\ (x_0+\bar{h}(0)l_0)\exp\left\{\int_0^t r\mathrm{d}s+\int_0^t\underbrace{\left(A(s)-\frac{1}{2}(m(s))^2-\frac{1}{2}(n(s))^2\right)}_{\bar{A}(s)}V(s)\mathrm{d}s\right.\\ &\left.+\int_0^t m(s)\sqrt{V(s)}\mathrm{d}W_S^{\Phi^*}(s)+\int_0^t n(s)\sqrt{V(s)}\mathrm{d}W_V^{\Phi^*}(s)\right\}.\end{aligned}$$

Because

$$J(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))=\frac{(X^{\Phi^*,\mathbf{u}^*}(t)+\bar{h}(t)L(t))^{1-\gamma}}{1-\gamma}\exp(\bar{g}(t)V(t)+\hat{g}(t)),$$

$\bar{g}(t)\in[\bar{g}(0),0]$, and $\hat{g}(t)$ is bounded, we obtain the following estimate with the appropriate constant $K_1>0$,

$$\begin{aligned}|J(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))|^4 &=\left|\frac{(X^{\Phi^*,\mathbf{u}^*}(t)+\bar{h}(t)L(t))^{1-\gamma}}{1-\gamma}\exp(\bar{g}(t)V(t)+\hat{g}(t))\right|^4\\ &\le K_1\left|(X^{\Phi^*,\mathbf{u}^*}(t)+\bar{h}(t)L(t))^{1-\gamma}\right|^4.\end{aligned}$$

Next, we focus on $|(X^{\Phi^*,\mathbf{u}^*}(t)+\bar{h}(t)L(t))^{1-\gamma}|^4$.

$$\begin{aligned}&\left|(X^{\Phi^*,\mathbf{u}^*}(t)+\bar{h}(t)L(t))^{1-\gamma}\right|^4\\ &\le K_2\exp\left\{\int_0^t 4(1-\gamma)\bar{A}(s)V(s)\mathrm{d}s+\int_0^t 4(1-\gamma)m(s)\sqrt{V(s)}\mathrm{d}W_S^{\Phi^*}(s)\right.\\ &\left.+\int_0^t 4(1-\gamma)n(s)\sqrt{V(s)}\mathrm{d}W_V^{\Phi^*}(s)\right\}\\ &=\underbrace{K_2\exp\left\{\int_0^t\left[32(1-\gamma)^2(m(s))^2+32(1-\gamma)^2(n(s))^2+4(1-\gamma)\bar{A}(s)\right]V(s)\mathrm{d}s\right\}}_{F_1(t)}\\ &\underbrace{\cdot\exp\left\{\int_0^t-32(1-\gamma)^2(m(s))^2V(s)\mathrm{d}s+\int_0^t 4(1-\gamma)m(s)\sqrt{V(s)}\mathrm{d}W_S^{\Phi^*}(s)\right\}}_{F_2(t)}\\ &\underbrace{\cdot\exp\left\{\int_0^t-32(1-\gamma)^2(n(s))^2V(s)\mathrm{d}s+\int_0^t 4(1-\gamma)n(s)\sqrt{V(s)}\mathrm{d}W_V^{\Phi^*}(s)\right\}}_{F_3(t)},\end{aligned}$$

where K_2 is a constant. For the term $F_2(t)$, we can find an estimate as

$$\begin{aligned}
&\mathrm{E}^{\Phi^*}[(F_2(t))^4] \\
&= \mathrm{E}^{\Phi^*}\left[\exp\left(\int_0^t -128(1-\gamma)^2(m(s))^2V(s)\mathrm{d}s + \int_0^t 16(1-\gamma)m(s)\sqrt{V(s)}\mathrm{d}W_S^{\Phi^*}(s)\right)\right] < \infty.
\end{aligned}$$

Because $(F_2(t))^4$ is a non-negative local martingale, it is a supermartingale. In fact, $(F_2(t))^4$ is a martingale due to bounded function $16(1-\gamma)m(t)$ on $[0,T]$ (see Lemma 4.3 in Taksar and Zeng, 2009). Similarly, we have $\mathrm{E}^{\Phi^*}[(F_3(t))^4] < \infty$, and $(F_3(t))^4$ is also a martingale.

For the term $F_1(t)$, we estimate $\mathrm{E}^{\Phi^*}[(F_1(t))^2]$ as

$$\mathrm{E}^{\Phi^*}[(F_1(t))^2] = \mathrm{E}^{\Phi^*}\left[\exp\left(\int_0^t \left(64(1-\gamma)^2(m(s))^2 + 64(1-\gamma)^2(n(s))^2 + 8(1-\gamma)\bar{A}(s)\right)V(s)\mathrm{d}s\right)\right].$$

Again applying Theorem 5.1 in Taksar and Zeng (2009), we obtain $\mathrm{E}^{\Phi^*}[(F_1(t))^2] < \infty$ if for $\bar{g}(t) \in [\bar{g}(0), 0]$, the following condition holds:

$$64(1-\gamma)^2(m(s))^2 + 64(1-\gamma)^2(n(s))^2 + 8(1-\gamma)\bar{A}(s) \leq \frac{\kappa^2}{2\sigma_V^2},$$

i.e.,

$$[64(1-\gamma)^2 - 4(1-\gamma)][(m(s))^2 + (n(s))^2] + 8(1-\gamma)A(s) \leq \frac{\kappa^2}{2\sigma_V^2}.$$

Applying the Cauchy-Schwartz inequality, we can arrive at

$$\begin{aligned}
&\mathrm{E}^{\Phi^*}|J(t, X^{\Phi^*,\mathbf{u}^*}(t), V(t), L(t))|^4 \leq K_3\mathrm{E}^{\Phi^*}\left[|(X^{\Phi^*,\mathbf{u}^*}(t) + \bar{h}(t)L(t))^{1-\gamma}|^4\right] \leq K_4\mathrm{E}^{\Phi^*}\left[F_1(t)F_2(t)F_3(t)\right] \\
&\leq K_4\left\{\mathrm{E}^{\Phi^*}[(F_1(t))^2]\mathrm{E}^{\Phi^*}[(F_2(t)F_3(t))^2]\right\}^{\frac{1}{2}} \\
&\leq K_4\left\{\mathrm{E}^{\Phi^*}[F_1(t)^2](\mathrm{E}^{\Phi^*}[(F_2(t))^4]\mathrm{E}^{\Phi^*}[(F_3(t))^4])^{\frac{1}{2}}\right\}^{\frac{1}{2}} < \infty,
\end{aligned}$$

where K_3 and K_4 are appropriate positive constants.

Step 2. Proof of $\mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}|\frac{(\phi_S^*(t))^2}{2\Psi_S(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))} + \frac{(\phi_V^*(t))^2}{2\Psi_V(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))}|^2\right] < \infty$. Inserting Eq. (18) into $\mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}|\frac{(\phi_S^*(t))^2}{2\Psi_S(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))} + \frac{(\phi_V^*(t))^2}{2\Psi_V(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))}|^2\right]$ yields

$$\begin{aligned}
&\mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}\left|\frac{(\phi_S^*(t))^2}{2\Psi_S(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))} + \frac{(\phi_V^*(t))^2}{2\Psi_V(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))}\right|^2\right] \\
&= \mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}\left|\frac{(1-\gamma)J(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))(\phi_S^*(t))^2}{2\beta_S}\right.\right. \\
&\left.\left.+\frac{(1-\gamma)J(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))(\phi_V^*(t))^2}{2\beta_V}\right|^2\right] \\
&\leq \mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}\left|\frac{(1-\gamma)(\phi_S^*(t))^2}{2\beta_S} + \frac{(1-\gamma)(\phi_V^*(t))^2}{2\beta_V}\right|^2|J(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))|^2\right] \\
&\leq \mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}\left|\frac{(1-\gamma)(\phi_S^*(t))^2}{2\beta_S} + \frac{(1-\gamma)(\phi_V^*(t))^2}{2\beta_V}\right|^4\right]^{\frac{1}{2}}\mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}|J(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))|^4\right]^{\frac{1}{2}} \\
&< \infty.
\end{aligned}$$

□

Based on Lemmas E.2 and E.3, we can prove the verification theorem.

Proof of Theorem 3.4. Following the process of solving the HJB equation, conditions (1) and (2) of the admissible strategy hold, and condition (3) of the admissible strategy can be obtained by $\mathrm{E}^{\Phi^*}\left[\sup_{t\in[0,T]}|J(t,X^{\Phi^*,\mathbf{u}^*}(t),V(t),L(t))|^4\right]<\infty$ in Lemma E.3. Thus, $\mathbf{u}^*$ is an admissible strategy. For Lemmas E.2 and E.3, we can simply apply Proposition 3.1 to prove that $\mathbf{u}^*$ is the optimal strategy for problem (16) and $J(t,x,v,l)$ is the corresponding optimal value function. □

Appendix F

This appendix provides some special cases when the pension investor has no access to the derivative.

Remark F.1. We present several special cases to show the relationships between $\tilde{u}^*(t)$ and β_S, β_V and γ. It is obvious that the effect of σ_L on $\tilde{u}^*(t)$ depends on the value of ρ_L. When $\rho_L=0$, the optimal investment strategy in this case, denoted $\tilde{u}_1^*(t)$, can be written as $\tilde{u}_1^*(t)=\tilde{m}_1(t)\left(1+\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right)$, and the optimal value function in this case, denoted $\tilde{J}_1(t,x,v,l)$, can be written as $\tilde{J}_1(t,x,v,l)=\frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\bar{g}_4(t)v+\hat{g}_4(t))$, where

$$\tilde{m}_1(t)=\frac{\lambda_1(1-\gamma)+(1-(\beta_S+\gamma))\sigma_V\rho_V\bar{g}_3(t)}{(1-\gamma)(\beta_S+\gamma)},\quad \bar{g}_4(t)=\frac{\tilde{\nu}_{11}\tilde{\nu}_{21}-\tilde{\nu}_{11}\tilde{\nu}_{21}\mathrm{e}^{\tilde{\alpha}_{21}(\tilde{\nu}_{11}-\tilde{\nu}_{21})(T-t)}}{\tilde{\nu}_{21}-\tilde{\nu}_{11}\mathrm{e}^{\tilde{\alpha}_{21}(\tilde{\nu}_{11}-\tilde{\nu}_{21})(T-t)}},$$
$$\hat{g}_4(t)=\int_t^T\left[r(1-\gamma)+\kappa\delta\bar{g}_4(s)\right]\mathrm{d}s,\quad \tilde{\alpha}_{11}=-\kappa+\frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V\rho_V}{\beta_S+\gamma},$$
$$\tilde{\alpha}_{21}=\frac{\sigma_V^2}{2}-\frac{\beta_S\sigma_V^2\rho_V^2}{2(1-\gamma)}-\frac{\beta_V\sigma_V^2(1-\rho_V^2)}{2(1-\gamma)}+\frac{(1-(\beta_S+\gamma))^2\sigma_V^2\rho_V^2}{2(\beta_S+\gamma)(1-\gamma)},$$
$$\tilde{\alpha}_{31}=\frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)},\quad \tilde{\nu}_{11,21}=\frac{\tilde{\alpha}_{11}\pm\sqrt{\tilde{\alpha}_{11}^2-4\tilde{\alpha}_{21}\tilde{\alpha}_{31}}}{-2\tilde{\alpha}_{21}},$$

and $\bar{h}(t)$ is given by Eq. (30). By derivation, we obtain $\tilde{\alpha}_{11}^2-4\tilde{\alpha}_{21}\tilde{\alpha}_{31}\geq 0$. As $\bar{h}(t)>0$, $\bar{g}_4(t)<0$, $\rho_V=0$ and $\gamma>1$, following simple calculations, we have $\frac{\partial\tilde{u}_1^*(t)}{\partial(\beta_S+\gamma)}<0$, which implies that the optimal investment strategy decreases w.r.t. aversion to ambiguity and risk in some cases. This result is intuitive and similar to the case involving the derivative. When $\rho_L=1$, the optimal investment strategy in this case, denoted $\tilde{u}_2^*(t)$, can be written as $\tilde{u}_2^*(t)=\tilde{m}(t)\left(1+\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right)-\sigma_L\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case,

denoted $\tilde{J}_2(t,x,v,l)$, can be written as $\tilde{J}_2(t,x,v,l)=\frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\bar{g}_5(t)v+\hat{g}_5(t))$, where

$$\begin{aligned}
\bar{g}_5(t)&=\frac{\tilde{\nu}_{12}\tilde{\nu}_{22}-\tilde{\nu}_{12}\tilde{\nu}_{22}\mathrm{e}^{\tilde{\alpha}_{22}(\tilde{\nu}_{12}-\tilde{\nu}_{22})(T-t)}}{\tilde{\nu}_{22}-\tilde{\nu}_{12}\mathrm{e}^{\tilde{\alpha}_{22}(\tilde{\nu}_{12}-\tilde{\nu}_{22})(T-t)}},\\
\hat{g}_5(t)&=\int_t^T[r(1-\gamma)+\kappa\delta\bar{g}_5(s)]\,\mathrm{d}s,\\
\tilde{\alpha}_{12}&=-\kappa+\frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V}{\beta_S+\gamma},\\
\tilde{\alpha}_{22}&=\frac{\sigma_V^2}{2}-\frac{\beta_S\sigma_V^2}{2(1-\gamma)}+\frac{(1-(\beta_S+\gamma))^2\sigma_V^2}{2(\beta_S+\gamma)(1-\gamma)},\\
\tilde{\alpha}_{32}&=\frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)},\qquad \tilde{\nu}_{12,22}=\frac{\tilde{\alpha}_{12}\pm\sqrt{\tilde{\alpha}_{12}^2-4\tilde{\alpha}_{22}\tilde{\alpha}_{32}}}{-2\tilde{\alpha}_{22}},
\end{aligned}$$

and $\bar{h}(t)$ is given by Eq. (30). By derivation, we obtain $\tilde{\alpha}_{12}^2-4\tilde{\alpha}_{22}\tilde{\alpha}_{32}\geq 0$. When $\rho_L=-1$, the optimal investment strategy in this case, denoted $\tilde{u}_3^*(t)$, can be written as $\tilde{u}_3^*(t)=\tilde{m}(t)\left(1+\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right)+\sigma_L\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case, denoted $\tilde{J}_3(t,x,v,l)$, can be written as $\tilde{J}_3(t,x,v,l)=\frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\bar{g}_6(t)v+\hat{g}_6(t))$, where

$$\begin{aligned}
\bar{g}_6(t)&=\frac{\tilde{\nu}_{13}\tilde{\nu}_{23}-\tilde{\nu}_{13}\tilde{\nu}_{23}\mathrm{e}^{\tilde{\alpha}_{23}(\tilde{\nu}_{13}-\tilde{\nu}_{23})(T-t)}}{\tilde{\nu}_{23}-\tilde{\nu}_{13}\mathrm{e}^{\tilde{\alpha}_{23}(\tilde{\nu}_{13}-\tilde{\nu}_{23})(T-t)}},\\
\hat{g}_6(t)&=\int_t^T[r(1-\gamma)+\kappa\delta\bar{g}_6(s)]\,\mathrm{d}s,\\
\tilde{\alpha}_{13}&=-\kappa-\frac{\lambda_1(1-(\beta_S+\gamma))\sigma_V}{\beta_S+\gamma},\\
\tilde{\alpha}_{23}&=\frac{\sigma_V^2}{2}-\frac{\beta_S\sigma_V^2}{2(1-\gamma)}+\frac{(1-(\beta_S+\gamma))^2\sigma_V^2}{2(\beta_S+\gamma)(1-\gamma)},\\
\tilde{\alpha}_{33}&=\frac{\lambda_1^2(1-\gamma)}{2(\beta_S+\gamma)},\qquad \tilde{\nu}_{13,23}=\frac{\tilde{\alpha}_{13}\pm\sqrt{\tilde{\alpha}_{13}^2-4\tilde{\alpha}_{23}\tilde{\alpha}_{33}}}{-2\tilde{\alpha}_{23}},
\end{aligned}$$

and $\bar{h}(t)$ is given by Eq. (30). By derivation, we obtain $\tilde{\alpha}_{13}^2-4\tilde{\alpha}_{23}\tilde{\alpha}_{33}\geq 0$.

Compared with Remark 3.5, we find that when the investor has no access to the derivative, the non-redundant condition is unnecessary. Therefore, we analyze the case of $\rho=\pm 1$ here and provide related explicit results. From the previous results, we find that the equity premium λ_2 for additional volatility risk is now 0; the investor has no way to cope with the volatility risk. She may increase her wealth invested in the stock (the second part in Eq. (23) is dropped), which causes her to undertake more risk than in the case with the derivative, and decrease her utility at retirement. The following special cases can be studied in a similar way. For a detailed comparison, we list related explicit results below.

Remark F.2. If $\sigma_L=0$, the salary process is non-stochastic; then, the optimal investment in this case, denoted $\tilde{u}_4^*(t)$, can be written as $\tilde{u}_4^*(t)=\tilde{m}(t)(1+\frac{\hat{h}(t)}{X^{u^*}(t)})$, and the optimal value function in this case, denoted $\tilde{J}_4(t,x,v)$, can be written as $\tilde{J}_4(t,x,v)=\frac{(x+\hat{h}(t))^{1-\gamma}}{1-\gamma}\exp(\bar{g}_3(t)v+\hat{g}_3(t))$, where $\hat{h}(t)$, $\tilde{m}(t)$, $\bar{g}_3(t)$ and $\hat{g}_3(t)$ are given by Eqs. (35) and (46). In this case, we find that the optimal investment strategy is proportional to $\tilde{m}(t)$.

Furthermore, if there is no salary and no derivative, our model reduces to a portfolio selection problem for an ambiguity-averse investor. The optimal investment strategy in this case, denoted $\tilde{u}_5^*(t)$, can be written as $\tilde{u}_5^*(t) = \frac{\lambda_1(1-\gamma)+(1-(\beta_S+\gamma))\sigma_V\rho_V\bar{g}_3(t)}{(1-\gamma)(\beta_S+\gamma)}$, and the optimal value function in this case, denoted $\tilde{J}_5(t,x,v)$, can be written as $\tilde{J}_5(t,x,v) = \frac{x^{1-\gamma}}{1-\gamma}\exp(\bar{g}_3(t)v + \hat{g}_3(t))$, where $\bar{g}_3(t)$ and $\hat{g}_3(t)$ are given by Eq. (46).

Remark F.3. If there is no derivative in the financial market and if the pension investor is ambiguity neutral, then the optimal investment strategy, denoted $\tilde{u}_6^*(t)$, can be written as $\tilde{u}_6^*(t) = \frac{\lambda_1+\sigma_V\rho_V\bar{g}_7(t)}{\gamma}\left(1+\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}\right) - \sigma_L\rho_L\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}$, and the optimal value function, denoted $\tilde{J}_6(t,x,v,l)$, can be written as $\tilde{J}_6(t,x,v,l) = \frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\bar{g}_7(t)v + \hat{g}_7(t))$, where

$$\bar{g}_7(t) = \frac{\tilde{\nu}_{14}\tilde{\nu}_{24} - \tilde{\nu}_{14}\tilde{\nu}_{24}\mathrm{e}^{\tilde{\alpha}_{24}(\tilde{\nu}_{14}-\tilde{\nu}_{24})(T-t)}}{\tilde{\nu}_{24} - \tilde{\nu}_{14}\mathrm{e}^{\tilde{\alpha}_{24}(\tilde{\nu}_{14}-\tilde{\nu}_{24})(T-t)}}, \tag{79}$$

$$\hat{g}_7(t) = \int_t^T \left[r(1-\gamma) + \kappa\delta\bar{g}_7(s)\right]\mathrm{d}s, \tag{80}$$

$$\tilde{\alpha}_{14} = -\kappa + \frac{\lambda_1(1-\gamma)\sigma_V\rho_V}{\gamma}, \quad \tilde{\alpha}_{24} = \frac{\sigma_V^2}{2} + \frac{(1-\gamma)\sigma_V^2\rho_V^2}{2\gamma},$$

$$\tilde{\alpha}_{34} = \frac{\lambda_1^2(1-\gamma)}{2\gamma}, \quad \tilde{\nu}_{14,24} = \frac{\tilde{\alpha}_{14} \pm \sqrt{\tilde{\alpha}_{14}^2 - 4\tilde{\alpha}_{24}\tilde{\alpha}_{34}}}{-2\tilde{\alpha}_{24}},$$

and $\bar{h}(t)$ is given by Eq.(30). By derivation, we obtain $\tilde{\alpha}_{14}^2 - 4\tilde{\alpha}_{24}\tilde{\alpha}_{34} \geq 0$.

Remark F.4. If there is no derivative in the financial market, the pension investor is ambiguity neutral and $\sigma_L = 0$, the salary process is non-stochastic; then in this case, the optimal investment strategy, denoted $\tilde{u}_{7S}^*(t)$, can be written as $\tilde{u}_{7S}^*(t) = \frac{\lambda_1+\sigma_V\rho_V\bar{g}_7(t)}{\gamma}(1 + \frac{\hat{h}(t)}{X^{u^*}(t)})$, and the optimal value function in this case, denoted $\tilde{J}_7(t,x,v)$, can be written as $\tilde{J}_7(t,x,v) = \frac{(x+\hat{h}(t))^{1-\gamma}}{1-\gamma}\exp(\bar{g}_7(t)v + \hat{g}_7(t))$, where $\hat{h}(t)$, $\bar{g}_7(t)$ and $\hat{g}_7(t)$ are given by Eqs. (35), (79)-(80).

Furthermore, if there is no salary, no ambiguity and no derivative in our model, the optimization problem becomes a portfolio selection problem for an ambiguity-neutral investor; the optimal investment in this case, denoted $\tilde{u}_{8S}^*(t)$, can be written as $\tilde{u}_{8S}^*(t) = \frac{\lambda_1+\sigma_V\rho_V\bar{g}_7(t)}{\gamma}$, and the optimal value function in this case, denoted $\tilde{J}_8(t,x,v)$, can be written as $\tilde{J}_8(t,x,v) = \frac{x^{1-\gamma}}{1-\gamma}\exp(\bar{g}_7(t)v + \hat{g}_7(t))$, where $\bar{g}_7(t)$ and $\hat{g}_7(t)$ are given by Eqs. (79) and (80).

Remark F.5. If $\sigma_V = 0$, the volatility of the risky asset is non-stochastic, and as noted above, the derivative is indeed redundant. The optimal investment strategy in this special case, denoted $\tilde{u}_9^*(t)$, can be written as $\tilde{u}_9^*(t) = \frac{\lambda_1}{\beta_S+\gamma}(1 + \bar{h}(t)\frac{L(t)}{X^{u^*}(t)}) - \sigma_L\rho_l\bar{h}(t)\frac{L(t)}{X^{u^*}(t)}$, and the optimal value function in this case, denoted $\tilde{J}_9(t,x,l)$, can be written as $\tilde{J}_9(t,x,l) =$

$\frac{(x+\bar{h}(t)l)^{1-\gamma}}{1-\gamma}\exp(\hat{g}_8(t))$, where

$$\hat{g}_8(t)=\left(r(1-\gamma)+\frac{\lambda_1^2(1-\gamma)\delta}{2(\beta_S+\gamma)}\right)(T-t)+\frac{\lambda_1^2(1-\gamma)(v_0-\delta)}{2(\beta_S+\gamma)\kappa}\left(\exp(-\kappa t)-\exp(-\kappa T)\right),$$

and $\bar{h}(t)$ is given by Eq.(30).

Appendix G

This appendix provides the optimal strategy under two special cases, European-style call and put options. Option pricing for the stochastic volatility model adopted here refers to Liu and Pan (2003) and Cui et al. (2017). We derive the prices of European-style call and put options with time τ to expiration and striking at K as follows

$$C(t)=c(t,\tau,S,V;K);\qquad P(t)=p(t,\tau,S,V;K),$$

where S is the spot price and V is the market volatility at time t, and the call and put options' prices are, respectively,

$$c(t,\tau,S,V;K)=S\mathcal{P}_1(t,\tau,S,V;K)-\mathrm{e}^{-r\tau}K\mathcal{P}_2(t,\tau,S,V;K),$$

$$p(t,\tau,S,V;K)=\mathrm{e}^{-r\tau}K(1-\mathcal{P}_2(t,\tau,S,V;K))-S(1-\mathcal{P}_1(t,\tau,S,V;K)),$$

where the risk-neutral probabilities $\mathcal{P}_1$ and $\mathcal{P}_2$ are recovered from inverting the respective characteristic functions

$$\mathcal{P}_1(t,\tau,S,V;K)=\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathbf{Im}\left[\frac{\mathrm{e}^{iz(\ln K-\ln S-r\tau)}\mathrm{e}^{A(1-iz)+B(1-iz)V}}{z}\right]\mathrm{d}z,$$

$$\mathcal{P}_2(t,\tau,S,V;K)=\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathbf{Im}\left[\frac{\mathrm{e}^{iz(\ln K-\ln S-r\tau)}\mathrm{e}^{A(-iz)+B(-iz)V}}{z}\right]\mathrm{d}z,$$

where **Im** denotes the imaginary component of a complex number, and $A(y)$, $B(y)$ are given by

$$B(y)=-\frac{a(1-\mathrm{e}^{-q\tau})}{2q-(q+b)(1-\mathrm{e}^{-q\tau})},$$

$$A(y)=-\frac{\kappa^*\delta^*}{\sigma_V^2}\left((q+b)\tau+2\ln\left(1-\frac{q+b}{2q}(1-\mathrm{e}^{-q\tau})\right)\right),$$

$$a=y(1-y),\qquad b=\rho_V\sigma_V y-\kappa^*,$$

$$q=\sqrt{b^2+a\sigma_V^2},\qquad \kappa^*=\kappa+\sigma_V(\rho_V\lambda_1+\sqrt{1-\rho_V^2}\lambda_2),\qquad \delta^*=\frac{\kappa\delta}{\kappa^*}.$$

The price of the straddle option used in our numerical examples is given by

$$O(t)=c(t,\tau,S,V;K)+p(t,\tau,S,V;K).$$

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